Modelling of flow and transport of non-Newtonian fluids interacting with poroelastic media

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We propose and analyze a model for solving the coupled problem arising in the interaction of a free fluid with a poroelastic structure. The flow in the fluid region is described by Stokes equations and in the poroelastic medium by the quasi-static Biot model. The focus of the model is on the quasi- Newtonian fluids that exhibit a shear-thinning property. We establish existence and uniqueness of the solution for two alternative formulations of the proposed model. Then we establish and show the existence and uniqueness of the solution of semidiscrete continuous-in-time formulation. We present complete stability and error analysis, as well as results of numerical simulations showing optimal rates of convergence for all variables. After that, the modeling of a transport equation in a non-linear Biot-Stokes flow will be analyzed. We use discontinuous Galerkin method to solve the transport equation. Several numerical tests are presented illustrating theoretical results and the capabilities of the method.

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Preface

The interaction of a free fluid with a deformable porous medium is a challenging multiphysics problem that has a wide range of applications, including processes arising in gas and oil extraction from naturally or hydraulically fractured reservoirs, designing industrial filters, and blood-vessel interactions. The free fluid region can be modeled by the Stokes or the Navier-Stokes equations, while the flow through the deformable porous medium is mod-eled by the quasi-static Biot system of poroelasticity [12]. The two regions are coupled via dynamic and kinematic interface conditions, including balance of forces, continuity of normal velocity, and a no slip or slip with friction tangential velocity condition. These multiphysics models exhibit features of coupled Stokes-Darcy flows and fluid-structure interaction (FSI). There is extensive literature on modeling these separate couplings, see e.g. [35, 55, 70] for Stokes-Darcy flows and [41, 40, 45] for FSI. More recently there has been growing interest in modeling Stokes-Biot couplings, which can be referred to as fluid-poroelastic structure interaction (FPSI). The well-posedness of the mathematical model is studied in [77]. A variational multiscale stabilized finite element method for the Navier-Stokes-Biot problem is developed in [7]. In [19] a non-iterative operator-splitting method is developed for the Navier-Stokes-Biot model with pressure Darcy formulation. The well posedness of a related model is studied in [24]. The Stokes-Biot problem with a mixed Darcy formulation is studied in [18] and [5] using Nitsche's method and a Lagrange multiplier, respectively, to impose the continuity of normal velocity on the interface. An optimization-based iterative algorithm with Neumann control is proposed in [25]. A reduced-dimension fracture model coupling Biot and an averaged Brinkman equation is developed in [21]. Alternative fracture models using the Reynolds lubrication equation coupled with Biot have also been studied, see e.g.[51].

All of the above mentioned works are based on Newtonian fluids. In this work, we develop FPSI with non-Newtonian fluids, which, to the best of our knowledge, has not been studied in the literature. In many applications the fluid exhibits properties that cannot be captured by a Newtonian fluid assumption. For instance, during water flooding in oil extraction, polymeric solutions are often added to the aqueous phase to increase its viscosity, resulting in a more stable displacement of oil by the injected water [59]. In hydraulic fracturing, proppant particles are mixed with polymers to maintain high permeability of the fractured media [57]. In blood flow simulations of small vessels or for patients with a cardiovascular disease, where the arterial geometry has been altered to include regions of re-circulation, one needs to consider models that can capture the sheer-thinning property of the blood [54].

In this work we use nonlinear Stokes equations to model the free fluid in the flow region and a nonlinear Biot model for the fluid in the poroelastic region. Our model is built on the nonlinear Stokes-Darcy model presented in [39] and the linear Stokes-Biot model considered in [5]. Our Biot model is based on a linear stress-strain constitutive relationship and a nonlinear Darcy flow. We neglect the inertia terms in both the fluid and solid regions. Such assumption is justified in many applications with low flow and displacement rates, including, for example, subsurface modeling, due to the low permeability and high stiffness of the media. The coupling conditions between the two subdomains include mass conservation, conservation of momentum and the Beavers-Joseph-Saffman slip with friction condition. We focus on fluids that possess the sheer thinning property, i.e., the viscosity decreases under shear strain, which is typical for polymer solutions and blood. Viscosity models for such non-Newtonian fluids include the Power law, the Cross model and the Carreau model [13, 26, 66, 59, 67]. The Power law model is popular because it only contains two parameters, and it is possible to derive analytical solutions in various flow conditions [13]. On the other hand, it implies that in the flow region the viscosity goes to infinity if the deformation goes to zero, which may not be representative in certain applications. The Cross and Carreau models have been deduced empirically as alternatives of the Power law model. They have three parameters, and in some parameter regimes, the viscosity is strictly greater than zero and bounded. We assume that the viscosity in each subdomain satisfies one such model, with dependence on the magnitude of the deformation tensor and the magnitude of Darcy velocity in the fluid and poroelastic regions, respectively. We further assume that along the interface the fluid viscosity is a function of the fluid and structure interface velocities. We consider both unbounded and bounded parameter regimes. In the former case, the analysis is done in an appropriate Sobolev space setting, using spaces such as $W^{1,r}$, where $1 < r < 2$ is the viscosity shear thinning parameter. In the latter case, the analysis reduces to the Hilbert space setting. Nonlinear Stokes-Darcy models with bounded viscosity have been studied in [38, 36, 23], while the unbounded case is considered in [39].

Following the approach in [5], we enforce the continuity of normal velocity on the interface through the use of a Lagrange multiplier. The resulting weak formulation is a nonlinear time-dependent system, which is difficult to analyze, due to to the presence of the time derivative of the displacement in some non-coercive terms. We consider an alternative mixed elasticity formulation with the structure velocity and elastic stress as primary variables, see also [77]. In this case we obtain a system with a degenerate evolution in time operator and a nonlinear saddle-point type spatial operator. The structure of the problem is similar to the one analyzed in [78], see also [16] in the linear case. However, the analysis in [78] is restricted to the Hilbert space setting and needs to be extended to the Sobolev space setting. Furthermore, the analysis in [78] is for monotone operators, see [76], and as a result requires certain right hand side terms to be zero, while in typical applications these terms may not be zero. Here we explore the coercivity of the operators to reformulate the problem as a parabolic-type system for the pressure and stress in the poroelastic region. We show well posedness for this system for general source terms and that the solution satisfies the original formulation. We also prove that the solution to the original formulation is unique and provide a stability bound. We then consider a semidiscrete finite element approximation of the system and carry out stability and error analysis. For this purpose we establish a discrete inf-sup condition, which involves a non-conforming Lagrange multiplier discretization that allows for non-matching grids across the Stokes-Biot interface.

In the second chapter, we study the a transport equation with flow from the Biot-Stokes system in chapter 1. Adopting idea from [79], we will use discontinuous Galerkin method to handle our trans- port problem. However we made some improvements from the scheme set up in [79]. We noticed that the dispersion tensor in transport equation is a nonlinear function of velocity. And they used cut-off operator to handle this difficulty. We avoid using the cut-off operator to do analysis by showing that $\|\nabla \cdot \mathbf{u}_h\|_{L^\infty(\Omega)}$ is bounded. Hence, the computed velocity do not have to be modified when used for the transport equation. The key idea for such improvement is that we arrange terms in error equation differently from [79], and use property of interpolation to bound the term $[\Pi c - c]$ in (2.77) by $\mathcal{O}(h)$. Several numerical tests are presented illustrating theoretical results and the capabilities of the method.

1.0 A nonlinear Biot-Stokes model for the interaction of a non-Newtonian fluid with poroelastic media

1.1 Introduction and model problem

This chapter is devoted to investigate the Biot-Stokes flow system. In the following, we introduce the governing equations in both regions, as well as the coupling conditions along the interface. Section 1.2 is devoted to the weak formulation of interest, upon which we base the numerical method, and an alternative formulation, which is needed for the purpose of the analysis. In Section 1.3 we begin by proving the well-posedness of the alternative formulation and then show how this translates to the well-posedness of the other formulation.

Our model is built upon those presented in [39] and [5]. In [39] the authors focused on coupling generalized nonlinear Stokes and Darcy equations, and [5] deals with the effects of the deformation of the poroelastic region. As a result, we obtain a nonlinear time-dependent system, where the operator corresponding to evolution in time is degenerate and the one corresponding to dynamics in space is of saddle-point type. In the general Sobolev space setting of the weak formulation we establish existence and uniqueness of the solution of the modeling equations. We consider a coupled problem for the interaction of flow and porous and deformable media. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a Lipschitz domain with boundary $\Gamma = \partial\Omega$, which is subdivided into a union of non-overlapping and possibly non-connected regions Ω_f and Ω_p. Here Ω_f stands for a fluid region with flow and Ω_p is a poroelasticity region. We further assume that $\overline{\Omega}_f \cup \overline{\Omega}_p = \overline{\Omega}$ and $\partial\Omega_f \cap \partial\Omega_p = \Gamma_{fp}$ denotes the (nonempty) interface between these regions. We denote by $\mathbf{n}_f$ the unit normal vector which points outward from $\partial\Omega_f$, and by $\mathbf{n}_p$ the outward unit normal vector to $\partial\Omega_p$. Note that in this case $\mathbf{n}_f = -\mathbf{n}_p$ on Γ_{fp}.

Let $(\mathbf{u}_\star, p_\star)$ be the velocity-pressure pairs in $\Omega_\star$, $\star = f$, p, and let $\boldsymbol{\eta}_p$ be the displacement in Ω_p. We assume that the flow in Ω_f is governed by the nonlinear generalized Stokes equations with homogeneous boundary conditions on $\partial\Omega_f \setminus \Gamma_{fp}$:

$$-\nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = \mathbf{f}_f, \qquad \nabla \cdot \mathbf{u}_f = q_f \quad \text{in } \Omega_f, \qquad \mathbf{u}_f = \mathbf{0} \quad \text{on } \partial\Omega_f \setminus \Gamma_{fp}, \tag{1.1}$$

where $\mathbf{D}(\mathbf{u}_f)$ and $\boldsymbol{\sigma}_f(\mathbf{u}_f, p_f)$ denote the deformation and the stress tensors, respectively:

$$\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla \mathbf{u}_f^T), \quad \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = -p_f \mathbf{I} + 2\nu \mathbf{D}(\mathbf{u}_f),$$

where $\mathbf{I}$ stands for the identity operator.

Our model problem assumes a generalized Newtonian fluids, that have a non-constant viscosity ν. Instead, ν is a function of the magnitude of the deformation tensor. While different generalized models correspond to different specifications of the viscosity function, we will focus on the power law fluids, i.e. the fluids that possess a shear-thinning property. More precisely, we assume that as the magnitude of $\mathbf{D}(\mathbf{u}_f)$ increases, the viscosity decreases. Models for such viscosity functions include the following [26, 66],

Carreau model.

$$\nu(\mathbf{D}(\mathbf{u}_f)) = \nu_\infty + (\nu_0 - \nu_\infty)/(1 + K_f |\mathbf{D}(\mathbf{u}_f)|^2)^{(2-r)/2}, \tag{1.2}$$

where $r > 1,\ \nu_0,\ \nu_\infty$, and $K_f > 0$ are constants.

Cross model.

$$\nu(\mathbf{D}(\mathbf{u}_f)) = \nu_\infty + (\nu_0 - \nu_\infty)/(1 + K_f |\mathbf{D}(\mathbf{u}_f)|^{2-r}), \tag{1.3}$$

where $r > 1,\ \nu_0,\ \nu_\infty$, and $K_f > 0$ are constants.

Power law model.

$$\nu(\mathbf{D}(\mathbf{u}_f)) = K_f |\mathbf{D}(\mathbf{u}_f)|^{r-2}, \tag{1.4}$$

where $r > 1$ and $K_f > 0$ are constants.

In turn, in Ω_p we consider the quasi-static Biot system [12]

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) &= \mathbf{f}_p \quad \text{in } \Omega_p, & (1.5)\\ \nu_{eff} K^{-1} \mathbf{u}_p + \nabla p_p = 0, \quad \frac{\partial}{\partial t}(s_0 p_p + \alpha_p \nabla \cdot \boldsymbol{\eta}_p) + \nabla \cdot \mathbf{u}_p &= q_p \quad \text{in } \Omega_p, & (1.6)\\ \mathbf{u}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^N \times (0,T], p_p = 0 \text{ on } \Gamma_p^D \times (0,T], \quad \boldsymbol{\eta}_p &= \mathbf{0} \quad \text{on } \partial\Omega_p \setminus \Gamma_{fp} \times (0,T] & (1.7) \end{aligned}$$

where $\Gamma_p = \Gamma_p^D \cup \Gamma_p^N$, and we assume that $|\Gamma_p^D| > 0$, $\text{dist}(\Gamma_p^D, \Gamma_{fp}) > 0$. We define the terms $\boldsymbol{\sigma}_e(\boldsymbol{\eta})$ and $\boldsymbol{\sigma}_p(\boldsymbol{\eta}, p)$ to be the elasticity and poroelasticity stress tensors, respectively as below:

$$\boldsymbol{\sigma}_e(\boldsymbol{\eta}) = \lambda_p(\nabla\cdot\boldsymbol{\eta})\mathbf{I} + 2\mu_p\mathbf{D}(\boldsymbol{\eta}), \quad \boldsymbol{\sigma}_p(\boldsymbol{\eta}, p_p) = \boldsymbol{\sigma}_e(\boldsymbol{\eta}) - \alpha_p p_p \mathbf{I}, \tag{1.8}$$

and α_p is the Biot-Willis constant, $\lambda_p,\ \mu_p$ are Lamè coefficients, s_0 is a storage coefficient and K is the symmetric and uniformly positive definite permeability tensor. The above system of equations is complemented by a set of initial conditions:

$$p_p(0,x) = p_{p,0}(x), \quad \boldsymbol{\eta}_b(0,x) = \boldsymbol{\eta}_{p,0}(x) \text{ in } \Omega_p.$$

The initial data $p_{p,0}$ and $\boldsymbol{\eta}_{p,0}$ need to satisfy a compatibility condition. In particular, given initial pressure $p_{p,0}$, the initial displacement $\boldsymbol{\eta}_{p,0}$ is determined from (1.5) and the boundary and interface conditions. The details are discussed in Section (1.3).

In a porous medium, two models for the effective viscosity ν_{eff} are as follow, [59, 67],

Cross model.

$$\nu_{eff}(\mathbf{u}_p) = \nu_\infty + (\nu_0 - \nu_\infty)/(1 + K_p|\mathbf{u}_p|^{2-r}), \tag{1.9}$$

where $r > 1,\ \nu_0,\ \nu_\infty$, and $K_p > 0$ are constants.

Power law model.

$$\nu_{eff}(\mathbf{u}_p) = K_p(|\mathbf{u}_p|/(\sqrt{\kappa}m_c))^{r-2}, \tag{1.10}$$

where $r > 1$ and $K_p > 0$ are constants, and m_c is a constant that depends on the internal structure of the porous media.

For the rest of the paper we restrict $r \in (1, 2]$, where for $r \in (1, 2)$ the fluids possesses a shear thinning property, and $r = 2$ corresponds to the special case of a Newtonian fluid. We will also write ν or ν_{eff} keeping in mind that ν, ν_{eff} are functions of $\mathbf{D}(\mathbf{u}_f)$ or $\mathbf{u}_p$.

The *interface conditions* on the fluid-poroelasticity interface Γ_{fp}, are *mass conservation*, *balance of normal stress*, and the Beavers-Joseph-Saffman (BJS) law [11, 73] modeling *slip with friction* [77, 7]:

$$\mathbf{u}_f\cdot\mathbf{n}_f + \left(\frac{\partial\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot\mathbf{n}_p = 0, \tag{1.11}$$

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p, \tag{1.12}$$

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{t}_{f,j} = \nu_I \, \alpha_{BJS} \sqrt{K_j^{-1}} \left(\mathbf{u}_f - \frac{\partial \boldsymbol{\eta}_p}{\partial t} \right) \cdot \mathbf{t}_{f,j} \quad \text{on } \Gamma_{fp}, \tag{1.13}$$

as well as *conservation of momentum*:

$$\boldsymbol{\sigma}_f \mathbf{n}_f = -\boldsymbol{\sigma}_p \mathbf{n}_p \quad \text{on } \Gamma_{fp}, \tag{1.14}$$

where $\mathbf{t}_{f,j}$, $1 \leq j \leq d-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $K_j = \mathbf{t}_{f,j} \cdot K \cdot \mathbf{t}_{f,j}$ and $\alpha_{BJS} > 0$ is an experimentally determined friction coefficient. We note that the continuity of flux takes into account the normal velocity of the solid skeleton, while the BJS condition accounts for its tangential velocity. We assume that along the interface the fluid viscosity ν_I is a nonlinear function of $|\sum_{j=1}^{d-1}((\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j}|$ given by Cross model (1.9) or Power law model (1.10). Due to technical analysis, we simplified the problem to the case $\mathbf{f}_f = \mathbf{f}_p = \mathbf{0}$ and $q_f = 0$, we allow only q_p can be nonzero.

We then establish and show the existence and uniqueness of the solution of semidiscrete continuous in time formulation. We present complete stability and error analysis, as well as results of numerical simulations showing optimal rates of convergence for all variables. After that, we give some results from experiments of actual numerical method, based on one of these formulations.

In the second chapter, the solution of the flow equations is used to set up our transport equation, as in (2.17). Let $\mathbf{u}(t)$ be a velocity field over $\Omega = \Omega_f \cup \Omega_p$, such that $\mathbf{u}(t)|_{\Omega_f} = \mathbf{u}_f(t), \mathbf{u}(t)|_{\Omega_p} = \mathbf{u}_p(t)$. Where $(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \boldsymbol{\eta}_p)$ is the solution to the Biot-Stokes systems consisting of (1.1), (1.5), (1.6) and interface condition. Let T be a terminal time, and $J = (0, T]$. Then the Biot-Stokes flow with transport has equation on $\Omega = \Omega_f \cup \Omega_p$:

$$\phi c_t + \nabla \cdot (c\mathbf{u}(t) - \mathbf{D}(\mathbf{u})\nabla c) = qc^*, \quad \forall (\mathbf{x}, t) \in \Omega \times J \tag{1.15}$$

where $c(\mathbf{x}, t)$ is the concentration of some chemical component, q is the imposed external total flow rate, the sum of sources and sinks, c^* is the injected concentration c_w if $q > 0$ and is the resident concentration c if $q < 0$. $0 < \phi_* \leq \phi(\mathbf{x}) \leq \phi^*$ is the porosity of the medium in Ω_p (it is set to 1 in Ω_f), $\mathbf{D}(\mathbf{u})$ is the diffusion dispersion tensor.

$$\mathbf{D}(\mathbf{u}) = d\mathbf{I} + |\mathbf{u}|(\alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t(\mathbf{I} - \mathbf{E}(\mathbf{u}))) \tag{1.16}$$

where $(\mathbf{E}(\mathbf{u}))_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}$, $d = \phi\tau D_m$, ϕ is the porosity presenting the fraction of the volume of the medium occupied by pores. τ is the tortuosity coefficient. D_m is the molecular diffusivity. α_l, α_t are the longitudinal and transverse dispersivities, respectively, and $s(\mathbf{x}, t)$ is a source term. The initial condition for the concentration is

$$c(\mathbf{x}, 0) = c^0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega \tag{1.17}$$

and the boundary conditions

$$(c\mathbf{u} - \mathbf{D}\nabla c) \cdot \mathbf{n} = (c_{in}\mathbf{u}) \cdot \mathbf{n} \text{ on } \Gamma_{in}, \tag{1.18}$$

$$(\mathbf{D}\nabla c) \cdot \mathbf{n} = 0 \text{ on } \Gamma_{out}. \tag{1.19}$$

Where, $\Gamma_{in} := \{\mathbf{x} \in \partial\Omega : \mathbf{u} \cdot \mathbf{n} < 0\}$, $\Gamma_{out} := \{\mathbf{x} \in \partial\Omega : \mathbf{u} \cdot \mathbf{n} \geq 0\}$, and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

Before doing analysis of the problem, we make some assumption regarding viscosity functions. Adopting the approach from [39, 38], we assume that the viscosity functions satisfy one of the two sets of assumptions (A1)–(A2) or (B1)–(B2) below. Let $g(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}^+ \cup \{0\}$ and let $\mathbf{G}(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}^d$ be given by $\mathbf{G}(\mathbf{x}) = g(\mathbf{x})\mathbf{x}$. For $\mathbf{x}, \mathbf{h} \in \mathbb{R}^d$, let $\mathbf{G}(\mathbf{x})$ satisfy, for constants $C_1, \ldots, C_4 > 0$ and $c \geq 0$,

$$(\mathbf{G}(\mathbf{x}+\mathbf{h}) - \mathbf{G}(\mathbf{x})) \cdot \mathbf{h} \geq C_1|\mathbf{h}|^2, \tag{A1}$$

$$|\mathbf{G}(\mathbf{x}+\mathbf{h}) - \mathbf{G}(\mathbf{x})| \leq C_2|\mathbf{h}|, \tag{A2}$$

or

$$(\mathbf{G}(\mathbf{x}+\mathbf{h}) - \mathbf{G}(\mathbf{x})) \cdot \mathbf{h} \geq C_3 \frac{|\mathbf{h}|^2}{c + |\mathbf{x}|^{2-r} + |\mathbf{x}+\mathbf{h}|^{2-r}}, \tag{B1}$$

$$|\mathbf{G}(\mathbf{x}+\mathbf{h}) - \mathbf{G}(\mathbf{x})| \leq C_4 \frac{|\mathbf{h}|}{c + |\mathbf{x}|^{2-r} + |\mathbf{x}+\mathbf{h}|^{2-r}}, \tag{B2}$$

with the convention that $\mathbf{G}(\mathbf{x}) = \mathbf{0}$ if $\mathbf{x} = \mathbf{0}$, and $|\mathbf{h}|/(c + |\mathbf{x}| + |\mathbf{h}|) = 0$ if $c = 0$ and $\mathbf{x} = \mathbf{h} = \mathbf{0}$. From (B1)–(B2) it follows that there exist constants $C_5, C_6 > 0$ such that for $\mathbf{s}, \mathbf{t}, \mathbf{w} \in (L^r(G))^d$ [74]

$$(\mathbf{G}(\mathbf{s}) - \mathbf{G}(\mathbf{t}), \mathbf{s} - \mathbf{t})_G \geq C_5 \frac{\|\mathbf{s} - \mathbf{t}\|^2_{L^r(G)}}{c + \|\mathbf{s}\|^{2-r}_{L^r(G)} + \|\mathbf{t}\|^{2-r}_{L^r(G)}}, \tag{1.20}$$

$$(\mathbf{G}(\mathbf{s}) - \mathbf{G}(\mathbf{t}), \mathbf{w})_G \leq C_6 \left\| \frac{|\mathbf{s} - \mathbf{t}|}{c + |\mathbf{s}| + |\mathbf{t}|} \right\|^{\frac{2-r}{r}}_{L^\infty(G)} (|\mathbf{G}(\mathbf{s}) - \mathbf{G}(\mathbf{t})|, |\mathbf{s} - \mathbf{t}|)_G^{1/r'} \|\mathbf{w}\|_{L^r(G)}. \tag{1.21}$$

Remark 1.1.1. *It is shown in [36] that conditions* (A1)–(A2) *are satisfied for* $g(\mathbf{D}(\mathbf{u}_f)) = \nu(\mathbf{D}(\mathbf{u}_f))$ *given in the Carreau model* (1.2) *with* $\nu_\infty > 0$*, in which case* $\nu_\infty \le g(\mathbf{x}) \le \nu_0$*. A similar argument can be applied to show that* (A1)–(A2) *hold for the Cross model, with* $g(\mathbf{D}(\mathbf{u}_f)) = \nu(\mathbf{D}(\mathbf{u}_f))$ *given in* (1.3) *for Stokes and* $g(\mathbf{u}_p) = \nu_{eff}(\mathbf{u}_p)$ *given in* (1.9) *for Darcy, in the case of* $\nu_\infty > 0$*. Furthermore, it is shown in [74] that conditions* (B1)–(B2) *with* $c > 0$ *hold in the case of the Carreau model* (1.2) *with* $\nu_\infty = 0$*, and that conditions* (B1)–(B2) *with* $c = 0$ *hold for the Power law model* (1.4) *and* (1.10)*.*

1.2 Variational formulation

We complement the Biot Stokes flow system given in (1.1), (1.5), (1.6) and (1.7) with the following set of initial conditions:

$$p_p(0,\mathbf{x}) = p_{p,0}(\mathbf{x}),\ \boldsymbol{\eta}_p(0,\mathbf{x}) = \boldsymbol{\eta}_{p,0}(\mathbf{x}) \text{ in } \Omega_p.$$

For a given $r > 1$ its conjugate is r', satisfying $r^{-1} + (r')^{-1} = 1$. Let

$$\mathbf{V}_f = \{\mathbf{v}_f \in W^{1,r}(\Omega_f)^d : \mathbf{v}_f = \mathbf{0} \text{ on } \partial\Omega_f \setminus \Gamma_{fp}\}, \qquad W_f = L^{r'}(\Omega_f), \tag{1.22}$$

with the corresponding norms

$$\|\mathbf{v}_f\|_{\mathbf{V}_f} := \|\mathbf{v}_f\|_{(W^{1,r}(\Omega_f))^d}, \quad \|w_f\|_{W_f} := \|w_f\|_{L^{r'}(\Omega_f)}, \qquad \forall \mathbf{v}_f \in \mathbf{V}_f,\, w_f \in W_f.$$

Next, let

$$L^r(\mathrm{div}\,;\Omega_p) := \{\mathbf{v}_p \in (L^r(\Omega_p))^d : \nabla\cdot\mathbf{v}_p \in L^r(\Omega_p)\}.$$

Additionally, define:

$$\mathbf{V}_p = \{\mathbf{v}_p \in L^r(\mathrm{div}\,;\Omega_p) : \mathbf{v}_p\cdot\mathbf{n}_p = 0 \text{ on } \partial\Omega_p \setminus \Gamma_{fp}\}, \qquad W_p = L^{r'}(\Omega_p),$$
$$\mathbf{X}_p = \{\boldsymbol{\xi}_p \in H^1(\Omega_p)^d : \boldsymbol{\xi}_p = \mathbf{0} \text{ on } \partial\Omega_p \setminus \Gamma_{fp}\}.$$

with the norms

$$\|\mathbf{v}_p\|^2_{\mathbf{V}_p} := \|\mathbf{v}_p\|^2_{(L^r(\Omega_p))^d} + \|\nabla\cdot\mathbf{v}_p\|^2_{(L^r(\Omega_p))}, \quad \|w_p\|_{W_p} := \|w_p\|_{L^{r'}(\Omega_p)}, \quad \forall \mathbf{v}_p \in \mathbf{V}_p,\ w_p \in W_p,$$
$$\|\boldsymbol{\eta}_p\|_{\mathbf{V}_p} := \|\boldsymbol{\eta}_p\|_{(W^{1,r}(\Omega_p))^d}, \qquad \forall \boldsymbol{\eta}_p \in \mathbf{X}_p.$$

In the case of (A1)–(A2), we consider Hilbert spaces, with the above definitions replaced by

$$\mathbf{V}_f = \{\mathbf{v}_f \in H^1(\Omega_f)^d : \mathbf{v}_f = \mathbf{0} \text{ on } \Gamma_f\}, \qquad W_f = L^2(\Omega_f), \tag{1.23}$$
$$\mathbf{V}_p = \{\mathbf{v}_p \in H(\mathrm{div};\Omega_p) : \mathbf{v}_p\cdot\mathbf{n}_p = 0 \text{ on } \Gamma_p^N\}, \qquad W_p = L^2(\Omega_p). \tag{1.24}$$

The global spaces are products of the subdomain spaces. For simplicity we assume that each region consists of a single subdomain.

Remark 1.2.1. *For simplicity of the presentation, for the rest of the paper we focus on the case* (B1)–(B2)*, which is the technically more challenging case. The arguments apply directly to the case* (A1)–(A2)*.*

1.2.1 Lagrange multiplier formulation

To derive the weak formulation we multiply (1.1)-(1.6) by the appropriate test functions and integrate each over the corresponding region, utilizing boundary and interface conditions (2.7)-(1.14). Note that the integration by parts of the first equation in (1.1), (1.5) and the first equation in (1.6) leads to the interface term

$$I_{\Gamma_{fp}} = -\langle \boldsymbol{\sigma}_f\mathbf{n}_f, \mathbf{v}_f\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\sigma}_p\mathbf{n}_p, \boldsymbol{\xi}_p\rangle_{\Gamma_{fp}} + \langle p_p, \mathbf{v}_p\cdot\mathbf{n}_p\rangle_{\Gamma_{fp}}.$$

As in [5], this term will be incorporated into the weak formulation by introducing a Lagrange multiplier which has a meaning of Darcy pressure on the interface:

$$\lambda = -(\boldsymbol{\sigma}_f\mathbf{n}_f)\cdot\mathbf{n}_f = p_p, \qquad \text{on } \Gamma_{fp}.$$

With λ introduced, we have using (2.8), (2.9) and (1.14),

$$I_{\Gamma_{fp}} = a_{BJS}(\mathbf{u}_f, \partial_t\boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda),$$

where

$$a_{BJS}(\mathbf{u}_f,\boldsymbol{\eta}_p;\mathbf{v}_f,\boldsymbol{\xi}_p)=\sum_{j=1}^{d-1}\left\langle \nu_I\,\alpha_{BJS}\sqrt{K_j^{-1}}(\mathbf{u}_f-\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j},(\mathbf{v}_f-\boldsymbol{\xi}_p)\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}},$$

$$b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\mu)=\langle\mathbf{v}_f\cdot\mathbf{n}_f+(\boldsymbol{\xi}_p+\mathbf{v}_p)\cdot\mathbf{n}_p,\mu\rangle_{\Gamma_{fp}}.$$

and $\partial_t\phi:=\partial\phi/\partial t$. For the term $b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\lambda)$ to be well-defined, we need to choose the Lagrange multiplier space as $\Lambda=W^{1/r,r'}(\Gamma_{fp})$ [39].

Finally we introduce the functionals related to Stokes, Darcy and the elasticity operators, respectively, as follows:

$$\begin{aligned}
a_f(\cdot,\cdot)\,:\,\mathbf{V}_f\times\mathbf{V}_f\;&\longrightarrow\;\mathbb{R}, \qquad a_f(\mathbf{u}_f,\mathbf{v}_f):=(2\nu\mathbf{D}(\mathbf{u}_f),\mathbf{D}(\mathbf{v}_f))_{\Omega_f},\\
a_p^d(\cdot,\cdot)\,:\,\mathbf{V}_p\times\mathbf{V}_p\;&\longrightarrow\;\mathbb{R}, \qquad a_p^d(\mathbf{u}_p,\mathbf{v}_p):=(\nu_{eff}K^{-1}\mathbf{u}_p,\mathbf{v}_p)_{\Omega_p},\\
a_p^e(\cdot,\cdot)\,:\,\mathbf{X}_p\times\mathbf{X}_p\;&\longrightarrow\;\mathbb{R}, \qquad a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p):=(2\mu_p\mathbf{D}(\boldsymbol{\eta}_p),\mathbf{D}(\boldsymbol{\xi}_p))_{\Omega_p}+(\lambda_p\nabla\cdot\boldsymbol{\eta}_p,\nabla\cdot\boldsymbol{\xi}_p)_{\Omega_p}
\end{aligned}$$

and also

$$b_\star(\cdot,\cdot)\,:\,\mathbf{V}_\star\times W_\star\;\longrightarrow\;\mathbb{R}, \qquad b_\star(\mathbf{v},w):=-(\nabla\cdot\mathbf{v},w)_{\Omega_\star}.$$

Then the Lagrange multiplier variational formulation reads: *Given* $p_p(0)=p_{p,0}\in W_p$, $\boldsymbol{\eta}_p(0)=\boldsymbol{\eta}_{p,0}\in\mathbf{X}_p$, *for* $t\in(0,T]$, *find* $(\mathbf{u}_f(t),p_f(t),\mathbf{u}_p(t),p_p(t),\boldsymbol{\eta}_p(t),\lambda(t))\in L^\infty(0,T;\mathbf{V}_f)\times L^\infty(0,T;W_f)\times L^\infty(0,T;\mathbf{V}_p)\times W^{1,\infty}(0,T;W_p)\times W^{1,\infty}(0,T;\mathbf{X}_p)\times L^\infty(0,T;\Lambda)$, *such that for all* $\mathbf{v}_f\in\mathbf{V}_f$, $w_f\in W_f$, $\mathbf{v}_p\in\mathbf{V}_p$, $w_p\in W_p$, $\boldsymbol{\xi}_p\in\mathbf{X}_p$, *and* $\mu\in\Lambda$,

$$\begin{aligned}
&a_f(\mathbf{u}_f,\mathbf{v}_f)+a_p^d(\mathbf{u}_p,\mathbf{v}_p)+a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p)+a_{BJS}(\mathbf{u}_f,\partial_t\boldsymbol{\eta}_p;\mathbf{v}_f,\boldsymbol{\xi}_p)+b_f(\mathbf{v}_f,p_f)+b_p(\mathbf{v}_p,p_p)\\
&\qquad+\alpha_p b_p(\boldsymbol{\xi}_p,p_p)+b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\lambda)=0, &(1.25)\\
&(s_0\partial_t p_p,w_p)_{\Omega_p}-\alpha_p b_p\left(\partial_t\boldsymbol{\eta}_p,w_p\right)-b_p(\mathbf{u}_p,w_p)-b_f(\mathbf{u}_f,w_f)\\
&\qquad=(q_p,w_p)_{\Omega_p}, &(1.26)\\
&b_\Gamma\left(\mathbf{u}_f,\mathbf{u}_p,\partial_t\boldsymbol{\eta}_p;\mu\right)=0. &(1.27)
\end{aligned}$$

As it was shown in [39], for a given $\mathbf{v}_f\in W^{1,r}(\Omega_f)$, $\mathbf{v}_p\in L^r(\mathrm{div}\,,\Omega_p)$, $\lambda\in W^{1/r,r'}(\Gamma_{fp})$ and $\boldsymbol{\xi}_p\in H^1(\Omega_p)\subset W^{1,r}(\Omega_p)$ the integrals corresponding to the interface:

$$\int_{\Gamma_{fp}}\mathbf{v}_f\cdot\mathbf{n}_f\lambda\,ds, \qquad \int_{\Gamma_{fp}}\mathbf{v}_p\cdot\mathbf{n}_p\lambda\,ds \quad\text{and}\quad \int_{\Gamma_{fp}}\boldsymbol{\xi}_p\cdot\mathbf{n}_p\,\lambda\,ds$$

have a well-defined interpretation. Due to assumption $r > 1$, we have that $r' > 2$ and $L^{r'}(\Omega_p) \subset L^2(\Omega_p)$, thus the term $(s_0 \partial_t p_p, w_p)_{\Omega_p}$ is well-defined. Finally, since for given $\mathbf{v}_f \in W^{1,r}(\Omega_f)$ we have $\mathbf{v}_f\big|_{\partial\Omega_f} \in W^{1/r',r}(\partial\Omega_f)$ and for given $\boldsymbol{\xi}_p \in H^1(\Omega_p)$ we have $\boldsymbol{\xi}_p\big|_{\partial\Omega_p} \in H^{1/2}(\partial\Omega_p)$, so that the term arising from the BJS coupling conditions:

$$\sum_{j=1}^{n-1} \int_{\Gamma_{fp}} (\nu_I \, \alpha_{BJS} \sqrt{K_j^{-1}} (\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})(\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{t}_{f,j}) \, ds$$

is also well-defined.

Although many models for the fluid-structure interaction problem have been analyzed previously, e.g. the well-posedness of non-Newtonian Stokes-Darcy model was investigated in [39] and solvability of Newtonian dynamic Stokes-Biot model was shown in [77], the question of existence and uniqueness of solution for (1.25)-(1.27) should still be addressed. However, the presence of the time derivative of displacement, $\partial_t \boldsymbol{\eta}_p$ in non-coercive terms significantly complicates the analysis. Therefore, we will introduce an alternative formulation, show that it is well-posed and then prove that two formulations are equivalent.

1.2.2 Alternative formulation

Our goal is to obtain a system of evolutionary saddle point type, which fits the general framework studied in [78]. Following the approach from [77], we do this by considering a mixed elasticity formulation with the structure velocity and elastic stress as primary variables.

Recall that the elasticity stress tensor $\boldsymbol{\sigma}_e$ is connected to the structure displacement $\boldsymbol{\eta}_p$ through the relation [17]:

$$A\boldsymbol{\sigma}_e = \mathbf{D}(\boldsymbol{\eta}_p). \tag{1.28}$$

Here A is a bounded, symmetric and positive definite compliance tensor, which in the isotropic case has the form:

$$A\boldsymbol{\sigma}_e := \frac{1}{2\mu_p}\left(\boldsymbol{\sigma}_e - \frac{\lambda_p}{2\mu_p + d\lambda_p}\mathrm{tr}(\boldsymbol{\sigma}_e)\mathbf{I}\right), \quad \text{with} \;\; A^{-1}\boldsymbol{\sigma}_e = 2\mu_p \, \boldsymbol{\sigma}_e \, + \, \lambda_p \mathrm{tr}(\boldsymbol{\sigma}_e)\mathbf{I}. \tag{1.29}$$

To derive a new variational formulation, we start by multiplying the first equation in (1.1) and the first equation in (1.6) by test functions $\mathbf{v}_f \in \mathbf{V}_f$ and $\mathbf{v}_p \in \mathbf{V}_p$, respectively, and integrating by parts to obtain:

$$\int_{\Omega_f} (2\nu \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f) - p_f \nabla \cdot \mathbf{v}_f) \, dA + \int_{\Omega_p} \left(\nu_{eff} K^{-1} \mathbf{u}_p \cdot \mathbf{v}_p - p_p \nabla \cdot \mathbf{v}_p\right) \, dA \\ + \int_{\Gamma_{fp}} (-\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f + p_p \mathbf{v}_p \cdot \mathbf{n}_p) \, ds = \int_{\Omega_f} \mathbf{f}_f \cdot \mathbf{v}_f \, dA. \tag{1.30}$$

Decomposing the stress term into its normal and tangential components, and using the balance of normal stress condition (2.8), we obtain:

$$\int_{\Gamma_{fp}} -\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f \, ds = \int_{\Gamma_{fp}} -(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f \mathbf{v}_f \cdot \mathbf{n}_f \, ds - \sum_{j=1}^{n-1} \int_{\Gamma_{fp}} ((\boldsymbol{\sigma}_f \cdot \mathbf{n}_f) \cdot \mathbf{t}_{f,j})(\mathbf{v}_f \cdot \mathbf{t}_{f,j}) \, ds \\ = \int_{\Gamma_{fp}} p_p \mathbf{v}_f \cdot \mathbf{n}_f \, ds + \sum_{j=1}^{n-1} \int_{\Gamma_{fp}} (\nu_I \alpha_{BJS} \sqrt{K_j^{-1}} (\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})(\mathbf{v}_f \cdot \mathbf{t}_{f,j}) \, ds. \tag{1.31}$$

We multiply (1.5) by $\mathbf{v}_s \in \mathbf{X}_p$ and integrate by parts, using the fact that $\boldsymbol{\sigma}_e = \boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}$:

$$\int_{\Omega_p} ((\boldsymbol{\sigma}_e - \alpha_p p_p \mathbf{I}) : \mathbf{D}(\mathbf{v}_s)) \, dA + \int_{\Gamma_{fp}} (\alpha_p p_p \mathbf{v}_s \cdot \mathbf{n}_p - \boldsymbol{\sigma}_e \mathbf{n}_p \cdot \mathbf{v}_s) \, ds = 0. \tag{1.32}$$

For the elastic stress, conservation of momentum (1.14) reads:

$$(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = (\boldsymbol{\sigma}_e \mathbf{n}_p) \cdot \mathbf{n}_p - \alpha_p p_p, \quad (\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{t}_{f,j} = -(\boldsymbol{\sigma}_e \mathbf{n}_p) \cdot \mathbf{t}_{f,j} \quad \text{on } \Gamma_{fp}.$$

We use this modified condition to rewrite the interface terms in (1.32), similarly to how it was done for the fluid stress in (1.31)

$$\int_{\Gamma_{fp}} -(\boldsymbol{\sigma}_e \mathbf{n}_p) \cdot \mathbf{v}_s \, ds \\ = \int_{\Gamma_{fp}} (-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f \mathbf{v}_s \cdot \mathbf{n}_p - \alpha_p p_p \mathbf{v}_s \cdot \mathbf{n}_p) \, ds - \sum_{j=1}^{n-1} \int_{\Gamma_{fp}} ((\boldsymbol{\sigma}_e \cdot \mathbf{n}_p) \cdot \mathbf{t}_{f,j})(\mathbf{v}_s \cdot \mathbf{t}_{f,j}) \, ds \\ = \int_{\Gamma_{fp}} (1 - \alpha_p) p_p \mathbf{v}_s \cdot \mathbf{n}_p \, ds + \sum_{j=1}^{n-1} \int_{\Gamma_{fp}} (-\nu_I \, \alpha_{BJS} \sqrt{K_j^{-1}} (\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})(\mathbf{v}_s \cdot \mathbf{t}_{f,j}) \, ds. \tag{1.33}$$

Therefore, (1.30)-(1.33) can be combined as follows:

$$\int_{\Omega_f} \left(2\nu \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f) - p_f \nabla \cdot \mathbf{v}_f\right) dA \tag{1.34}$$

$$+ \int_{\Omega_p} \big(\nu_{eff} K^{-1} \mathbf{u}_p \cdot \mathbf{v}_p - p_p \nabla \cdot \mathbf{v}_p + (\boldsymbol{\sigma}_e - \alpha_p p_p) : \mathbf{D}(\mathbf{v}_s)\big) \, dA$$

$$+ \sum_{j=1}^{n-1} \int_{\Gamma_{fp}} (\nu_I \alpha_{BJS} \sqrt{K_j^{-1}} (\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})((\mathbf{v}_f - \mathbf{v}_s) \cdot \mathbf{t}_{f,j}) \, ds$$

$$+ \int_{\Gamma_{fp}} \left((\mathbf{v}_f \cdot \mathbf{n}_f + \mathbf{v}_s \cdot \mathbf{n}_p + \mathbf{v}_p \cdot \mathbf{n}_p) \, p_p\right) ds = 0. \tag{1.35}$$

We note that we can eliminate the displacement, $\boldsymbol{\eta}_p$, from the system by differentiating (1.28) and introducing a new variable $\mathbf{u}_s := \partial_t \boldsymbol{\eta}_p \in \mathbf{X}_p$, which has a meaning of structure velocity. Now, multiplying second equation in (1.1), (1.28) and second equation in (1.6) by corresponding test functions and adding the result, we obtain:

$$\int_{\Omega_p} \left(A \partial_t \boldsymbol{\sigma}_e : \boldsymbol{\tau}_e - \mathbf{D}(\mathbf{u}_s) : \boldsymbol{\tau}_e + s_0 \partial_t p_p w_p + \alpha_p \nabla \cdot \mathbf{u}_s w_p + \nabla \cdot \mathbf{u}_p w_p\right) dA + \int_{\Omega_f} \left(\nabla \cdot \mathbf{u}_f w_f\right) dA$$
$$= \int_{\Omega_p} q_p w_p \, dA. \tag{1.36}$$

As in the first formulation we use a Lagrange multiplier to impose the mass conservation interface condition (2.7). Finally, we introduce the space for the elastic stress $\Sigma_e = L^2_{sym}(\Omega_p)^{d \times d}$ with the norm

$$\|\boldsymbol{\sigma}_e\|^2_{\Sigma_e} := \sum_{i,j=1}^{d} \|(\boldsymbol{\sigma}_e)_{i,j}\|^2_{L^2(\Omega_p)}.$$

Then, the weak formulation reads: *given* $p_p(0) = p_{p,0} \in W_p$, $\boldsymbol{\sigma}_e(0) = A^{-1}\mathbf{D}(\boldsymbol{\eta}_{p,0}) \in \Sigma_e$, *for* $t \in (0, T]$, *find* $(\mathbf{u}_f(t), p_f(t), \mathbf{u}_p(t), p_p(t), \mathbf{u}_s(t), \boldsymbol{\sigma}_e(t), \lambda(t)) \in L^\infty(0, T; \mathbf{V}_f) \times L^\infty(0, T; W_f) \times L^\infty(0, T; \mathbf{V}_p) \times W^{1,\infty}(0, T; W_p) \times L^\infty(0, T; \mathbf{X}_p) \times W^{1,\infty}(0, T; \Sigma_e) \times L^\infty(0, T; \Lambda)$, *such that for all* $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in W_f$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in W_p$, $\mathbf{v}_s \in \mathbf{X}_p$, $\boldsymbol{\tau}_e \in \Sigma_e$, *and* $\mu \in \Lambda$,

$$\int_{\Omega_p} \big(\boldsymbol{\sigma}_e : \mathbf{D}(\mathbf{v}_s) - \alpha_p p_p \nabla \cdot \mathbf{v}_s + \nu_{eff} K^{-1} \mathbf{u}_p \cdot \mathbf{v}_p - p_p \nabla \cdot \mathbf{v}_p + A \partial_t \boldsymbol{\sigma}_e : \boldsymbol{\tau}_e - \mathbf{D}(\mathbf{u}_s) : \boldsymbol{\tau}_e\big) \, dA$$
$$+ \int_{\Omega_p} \left(s_0 \partial_t p_p w_p + \alpha_p \nabla \cdot \mathbf{u}_s w_p + \nabla \cdot \mathbf{u}_p w_p\right) dA$$

$$
\begin{aligned}
&+\int_{\Omega_f} \left(2\nu \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f) - p_f \nabla\cdot\mathbf{v}_f + \nabla\cdot\mathbf{u}_f w_f\right)\, dA \\
&+\sum_{j=1}^{n-1}\int_{\Gamma_{fp}} (\nu_I\, \alpha_{BJS}\sqrt{K_j^{-1}}(\mathbf{u}_f-\mathbf{u}_s)\cdot\mathbf{t}_{f,j})((\mathbf{v}_f-\mathbf{v}_s)\cdot\mathbf{t}_{f,j})\, ds \\
&+\int_{\Gamma_{fp}} \left((\mathbf{v}_f\cdot\mathbf{n}_f + \mathbf{v}_s\cdot\mathbf{n}_p + \mathbf{v}_p\cdot\mathbf{n}_p)\,\lambda\right)\, ds - \int_{\Gamma_{fp}} \left((\mathbf{u}_f\cdot\mathbf{n}_f + \mathbf{u}_s\cdot\mathbf{n}_p + \mathbf{u}_p\cdot\mathbf{n}_p)\,\mu\right)\, ds \\
&= \int_{\Omega_p} (q_p w_p)\, dA.. \qquad (1.37)
\end{aligned}
$$

We introduce the bilinear forms $b_s(\cdot,\cdot) : \mathbf{X}_p \times \Sigma_e \longrightarrow \mathbb{R}$ and $a_p^s(\cdot,\cdot) : \Sigma_e \times \Sigma_e \longrightarrow \mathbb{R}$ defined by

$$
b_s(\mathbf{v}_s, \boldsymbol{\tau}_e) := (\mathbf{D}(\mathbf{v}_s), \boldsymbol{\tau}_e)_{\Omega_p}, \qquad a_p^s(\boldsymbol{\sigma}_e, \boldsymbol{\tau}_e) := (A\boldsymbol{\sigma}_e, \boldsymbol{\tau}_e)_{\Omega_p}.
$$

Hence, we can rewrite (1.37) in a more compact form:

$$
\begin{aligned}
&a_f(\mathbf{u}_f,\mathbf{v}_f) + a_p^d(\mathbf{u}_p,\mathbf{v}_p) + a_{BJS}(\mathbf{u}_f,\mathbf{u}_s;\mathbf{v}_f,\mathbf{v}_s) + b_f(\mathbf{v}_f,p_f) + b_p(\mathbf{v}_p,p_p) \\
&\qquad + \alpha_p b_p(\mathbf{v}_s,p_p) + b_s(\mathbf{v}_s,\boldsymbol{\sigma}_e) + b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\mathbf{v}_s;\lambda) = 0, \qquad (1.38) \\
&(s_0\partial_t p_p, w_p)_{\Omega_p} + a_p^s(\partial_t\boldsymbol{\sigma}_e,\boldsymbol{\tau}_e) - \alpha_p b_p(\mathbf{u}_s,w_p) - b_p(\mathbf{u}_p,w_p) - b_s(\mathbf{u}_s,\boldsymbol{\tau}_e) - b_f(\mathbf{u}_f,w_f) \\
&\qquad = (q_p,w_p)_{\Omega_p}, \qquad (1.39) \\
&b_\Gamma(\mathbf{u}_f,\mathbf{u}_p,\mathbf{u}_s;\mu) = 0. \qquad (1.40)
\end{aligned}
$$

On the other hand, we can write (1.37) in a more general, operator notation:

$$
\frac{\partial}{\partial t}\mathcal{E}_1\mathbf{q}(t) + \mathcal{A}\mathbf{q}(t) + \mathcal{B}'s(t) = \mathbf{0} \qquad \text{in } \mathbf{Q}', \qquad (1.41)
$$

$$
\frac{\partial}{\partial t}\mathcal{E}_2 s(t) - \mathcal{B}\mathbf{q}(t) + \mathcal{C}s(t) = g(t) \qquad \text{in } S'. \qquad (1.42)
$$

where we define $\mathbf{Q}$, the space of generalized displacement variables, as follows

$$
\begin{aligned}
\mathbf{Q} = \Big\{ &\mathbf{q} = (\mathbf{v}_p,\mathbf{v}_s,\mathbf{v}_f) \in \mathbf{V}_p\times\mathbf{X}_p\times\mathbf{V}_f \text{ such that} \\
&\mathbf{v}_p\cdot\mathbf{n}_p = 0 \text{ on } \partial\Omega_p\setminus\Gamma_{fp},\ \mathbf{v}_s = \mathbf{0} \text{ on } \partial\Omega_p\setminus\Gamma_{fp},\ \mathbf{v}_f = \mathbf{0} \text{ on } \partial\Omega_f\setminus\Gamma_{fp} \Big\}
\end{aligned}
$$

and, similarly, the space S, consisting of generalized stress variables:

$$
S = \{ s = (w_p, \boldsymbol{\tau}_e, w_f, \mu) \in W_p\times\Sigma_p\times W_f\times\Lambda \}.
$$

and $g = (q_p, \mathbf{0}, 0, 0)$. The spaces $\mathbf{Q}$ and S are equipped with norms:

$$\|\mathbf{q}\|_{\mathbf{Q}} = \|\mathbf{v}_p\|_{\mathbf{V}_p} + \|\mathbf{v}_s\|_{\mathbf{X}_p} + \|\mathbf{v}_f\|_{\mathbf{V}_f},$$
$$\|s\|_S = \|w_p\|_{W_p} + \|\boldsymbol{\tau}_e\|_{\Sigma_e} + \|w_f\|_{W_f} + \|\mu\|_{\Lambda}.$$

We define the operators $\mathcal{A} : \mathbf{Q} \to \mathbf{Q}'$, $\mathcal{B} : \mathbf{Q} \to S'$, $\mathcal{C} : S \to S'$ as follows:

$$\mathcal{A} = \begin{pmatrix} \nu_{eff} K^{-1} & 0 & 0 \\ 0 & \alpha_{BJS}\gamma_T' \nu_I \sqrt{K^{-1}}\gamma_T & -\alpha_{BJS}\gamma_T' \nu_I \sqrt{K^{-1}}\gamma_T \\ 0 & -\alpha_{BJS}\gamma_T' \nu_I \sqrt{K^{-1}}\gamma_T & 2\nu \mathbf{D} : \mathbf{D} + \alpha_{BJS}\gamma_T' \nu_I \sqrt{K^{-1}}\gamma_T \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} \nabla\cdot & \alpha_p \nabla\cdot & 0 \\ 0 & -D & 0 \\ 0 & 0 & \nabla\cdot \\ \gamma_n & \gamma_n & \gamma_n \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where γ_T and γ_n denote the tangential and normal trace operators, respectively, and γ_T' the adjoint operator of γ_T.

And the operators $\mathcal{E}_1 : \mathbf{Q} \to \mathbf{Q}'$, $\mathcal{E}_2 : S \to S'$ are given by:

$$\mathcal{E}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} s_0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

1.3 Well-posedness of the model

As both methods have been introduced, we are ready to derive the conditions on the viscosity models, as well as the given data and initial conditions, that will be sufficient for solvability of (1.25)-(1.27). We start with the analysis of the alternative formulation, (1.37).

1.3.1 Existence and uniqueness of solution of the alternative formulation

First we explore important properties of the operators introduced at the end of Section 3.

Lemma 1.3.1. *The operator $\mathcal{B}$ and its adjoint $\mathcal{B}'$ are bounded and continuous. Moreover, there exist constants β_1, $\beta_2 > 0$ such that*

$$\inf_{\mathbf{0}\neq(\mathbf{0},\mathbf{v}_s,\mathbf{0})\in\mathbf{Q}} \sup_{(0,\boldsymbol{\tau}_e,0,0)\in S} \frac{b_s(\mathbf{v}_s,\boldsymbol{\tau}_e)}{\|(\mathbf{0},\mathbf{v}_s,\mathbf{0})\|_{\mathbf{Q}}\|(0,\boldsymbol{\tau}_e,0,0)\|_S} \geq \beta_1, \tag{1.43}$$

$$\inf_{0\neq(w_p,0,w_f,\mu)\in S} \sup_{(\mathbf{v}_p,\mathbf{0},\mathbf{v}_f)\in\mathbf{Q}} \frac{b_f(\mathbf{v}_f,w_f)+b_p(\mathbf{v}_p,w_p)+b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0};\lambda)}{\|(\mathbf{v}_p,\mathbf{0},\mathbf{v}_f)\|_{\mathbf{Q}}\|(w_p,0,w_f,\mu)\|_S} \geq \beta_2. \tag{1.44}$$

Proof. We recall that operator $\mathcal{B}$ is linear and satisfies for all $\mathbf{q} = (\mathbf{v}_p,\mathbf{v}_s,\mathbf{v}_f) \in \mathbf{Q}$ and $s = (w_p,\boldsymbol{\tau}_e,w_f,\mu) \in S$

$$\begin{aligned}
\mathcal{B}(\mathbf{q})(s) &= b_f(\mathbf{v}_f,w_f) + b_p(\mathbf{v}_p,w_p) + \alpha_p b_p(\mathbf{v}_s,w_p) - b_s(\mathbf{v}_s,\boldsymbol{\tau}_e) + b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\mathbf{v}_s;\mu)\\
&\leq \|\nabla\cdot\mathbf{v}_f\|_{L^r(\Omega_f)}\|w_f\|_{L^{r'}(\Omega_f)} + \|\nabla\cdot\mathbf{v}_p\|_{L^r(\Omega_p)}\|w_p\|_{L^{r'}(\Omega_p)} + \|\mathbf{D}(\mathbf{v}_s)\|_{L^2(\Omega_p)}\|\boldsymbol{\tau}_e\|_{L^2(\Omega_p)}\\
&+ \alpha_p\|\nabla\cdot\mathbf{v}_s\|_{L^r(\Omega_p)}\|w_p\|_{L^{r'}(\Omega_p)} + \|\mathbf{v}_f\cdot\mathbf{n}_f + (\mathbf{v}_p+\mathbf{v}_s)\cdot\mathbf{n}_p\|_{W^{-1/r,r}(\Gamma_{fp})}\|\mu\|_{W^{1/r,r'}(\Gamma_{fp})}\\
&\leq C\Big(\|\mathbf{v}_f\|_{W^{1,r}(\Omega_f)}\|w_f\|_{L^{r'}(\Omega_f)} + \|\mathbf{v}_p\|_{r(\mathrm{div};\Omega_p)}\|w_p\|_{L^{r'}(\Omega_p)} + \|\mathbf{v}_s\|_{H^1(\Omega_p)}\|\boldsymbol{\tau}_e\|_{L^2(\Omega_p)}\\
&+ \|\mathbf{v}_s\|_{H^1(\Omega_p)}\|w_p\|_{L^{r'}(\Omega_p)} + \|\mathbf{v}_f\|_{W^{1,r}(\Omega_f)}\|\mu\|_{W^{1/r,r'}(\Gamma_{fp})} + \|\mathbf{v}_p\|_{r(\mathrm{div};\Omega_p)}\|\mu\|_{W^{1/r,r'}(\Gamma_{fp})}\\
&+ \|\mathbf{v}_s\|_{H^1(\Omega_p)}\|\mu\|_{W^{1/r,r'}(\Gamma_{fp})}\Big) \leq C\|\mathbf{q}\|_{\mathbf{Q}}\|s\|_S,
\end{aligned}$$

which implies that $\mathcal{B}$ and $\mathcal{B}'$ are bounded and continuous.

Next, let $\mathbf{0} \neq (\mathbf{0},\mathbf{v}_s,\mathbf{0}) \in \mathbf{Q}$ be given. We choose $\boldsymbol{\tau}_e = \mathbf{D}(\mathbf{v}_s)$ and, using Korn's inequality, we obtain

$$\frac{b_s(\mathbf{v}_s,\boldsymbol{\tau}_e)}{\|\boldsymbol{\tau}_e\|_{L^2(\Omega_p)}} = \frac{\|\mathbf{D}(\mathbf{v}_s)\|^2_{L^2(\Omega_p)}}{\|\mathbf{D}(\mathbf{v}_s)\|_{L^2(\Omega_p)}} = \|\mathbf{D}(\mathbf{v}_s)\|_{L^2(\Omega_p)} \geq C_2\|\mathbf{v}_s\|_{H^1(\Omega_p)}.$$

Therefore, (1.43) holds.

Finally, we note that (1.44) was proven in [39] in case of velocity boundary conditions with restricted mean value of $W_f \times W_p$ and presence of an inflow and outflow boundaries, Γ_{in} and Γ_{out}. However, it can be shown that the result holds if $|\Gamma_{in}| = |\Gamma_{out}| = 0$ and, since $|\Gamma_D| > 0$, no restriction on $W_f \times W_p$ is required. $\square$

Lemma 1.3.2. *The operators $\mathcal{A}$ and $\mathcal{E}_2$ are bounded, continuous, and monotone. In addition, the following continuity and coercivity estimates hold with constants c_f, $\bar{c}_f$, C_f, c_p, $\bar{c}_p$, C_p, c_I, $\bar{c}_I$, $C_I > 0$ for all $\mathbf{u}_f, \mathbf{v}_f \in \mathbf{V}_f$, $\mathbf{u}_p, \mathbf{v}_p \in \mathbf{V}_p$ and $\mathbf{u}_s, \mathbf{v}_s \in \mathbf{X}_p$,*

$$c_f\|\mathbf{v}_f\|^r_{W^{1,r}(\Omega_f)} - c * \bar{c}_f \leq a_f(\mathbf{v}_f, \mathbf{v}_f), \quad a_f(\mathbf{u}_f, \mathbf{v}_f) \leq C_f\|\mathbf{u}_f\|^{r/r'}_{W^{1,r}(\Omega_f)}\|\mathbf{v}_f\|_{W^{1,r}(\Omega_f)}, \tag{1.45}$$

$$c_p\|\mathbf{v}_p\|^r_{L^r(\Omega_p)} - c * \bar{c}_p \leq a_p^d(\mathbf{v}_p, \mathbf{v}_p), \quad a_p^d(\mathbf{u}_p, \mathbf{v}_p) \leq C_p\|\mathbf{u}_p\|^{r/r'}_{L^r(\Omega_f)}\|\mathbf{v}_p\|_{L^r(\Omega_f)}, \tag{1.46}$$

$$c_I|\mathbf{v}_f - \mathbf{v}_s|^r_{BJS} - c * \bar{c}_I \leq a_{BJS}(\mathbf{v}_f, \mathbf{v}_s; \mathbf{v}_f, \mathbf{v}_s), \tag{1.47}$$

$$a_{BJS}(\mathbf{u}_f, \mathbf{u}_s; \mathbf{v}_f, \mathbf{v}_s) \leq C_I|\mathbf{u}_f - \mathbf{u}_s|^{r/r'}_{BJS}\|\mathbf{v}_f - \mathbf{v}_s\|_{L^r(\Gamma_{fp})}, \tag{1.48}$$

where c is the constant from (B1)–(B2).

Proof. The operator $\mathcal{E}_2$ is linear and, using (1.29), it satisfies

$$\mathcal{E}_2(s)(t) = (s_0 p_p, w_p)_{\Omega_p} + (A\boldsymbol{\sigma}_e, \boldsymbol{\tau}_e)_{\Omega_p} \leq C\left(\|p_p\|_{L^2(\Omega_p)}\|w_p\|_{L^2(\Omega_p)} + \|\boldsymbol{\sigma}_e\|_{L^2(\Omega_p)}\|\boldsymbol{\tau}_e\|_{L^2(\Omega_p)}\right),$$
$$\mathcal{E}_2(s)(s) = (s_0 p_p, p_p)_{\Omega_p} + (A\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_e)_{\Omega_p} \geq C\left(\|p_p\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\sigma}_e\|^2_{L^2(\Omega_p)}\right), \quad \forall s, t \in S,$$

which imply that $\mathcal{E}_2$ is bounded, continuous and monotone. The continuity and monotonicity of the operator $\mathcal{A}$ follow from (B1)–(B2), see [39] and [76, Example 5.a, p.59].

For the continuity of $a_f(\cdot,\cdot)$, we apply (1.21) with $\mathbf{G}(\mathbf{x}) = \nu(\mathbf{x})\mathbf{x}$, $\mathbf{s} = \mathbf{D}(\mathbf{u}_f)$, $\mathbf{t} = \mathbf{0}$ and $\mathbf{w} = \mathbf{D}(\mathbf{v}_f)$:

$$a_f(\mathbf{u}_f, \mathbf{v}_f) \leq 2\,C_6\left\|\frac{|\mathbf{D}(\mathbf{u}_f)|}{c + |\mathbf{D}(\mathbf{u}_f)|}\right\|^{\frac{2-r}{r}}_{L^\infty(\Omega_f)}(|\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f)|, |\mathbf{D}(\mathbf{u}_f)|)^{1/r'}_{\Omega_f}\|\mathbf{D}(\mathbf{v}_f)\|_{L^r(\Omega_f)}.$$

Using (B2) with $\mathbf{x} = \mathbf{0}$, $\mathbf{h} = \mathbf{D}(\mathbf{u}_f)$, we also have

$$|\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f)| \leq C_4\frac{|\mathbf{D}(\mathbf{u}_f)|}{c + |\mathbf{D}(\mathbf{u}_f)|^{2-r}} \leq C_4\frac{|\mathbf{D}(\mathbf{u}_f)|^{r-1}}{c|\mathbf{D}(\mathbf{u}_f)|^{r-2} + 1} \leq C_4|\mathbf{D}(\mathbf{u}_f)|^{r-1}.$$

Combining the above two estimates, we obtain

$$a_f(\mathbf{u}_f, \mathbf{v}_f) \leq C\|\mathbf{D}(\mathbf{u}_f)\|^{r/r'}_{L^r(\Omega_f)}\|\mathbf{D}(\mathbf{v}_f)\|_{L^r(\Omega_f)} \leq C_f\|\mathbf{u}_f\|^{r/r'}_{W^{1,r}(\Omega_f)}\|\mathbf{v}_f\|_{W^{1,r}(\Omega_f)}.$$

To establish the coercivity bound for $a_f(\cdot,\cdot)$ given in (1.45) we consider three cases.

(i) $c = 0$. From (1.20) we have

$$a_f(\mathbf{v}_f, \mathbf{v}_f) \geq 2\,C_5\frac{\|\mathbf{D}(\mathbf{v}_f)\|^2_{L^r(\Omega_f)}}{\|\mathbf{D}(\mathbf{v}_f)\|^{2-r}_{L^r(\Omega_f)}} = 2\,C_5\|\mathbf{D}(\mathbf{v}_f)\|^r_{L^r(\Omega_f)} \geq 2\,C_5 C^r_{K,f}\,\|\mathbf{v}_f\|^r_{W^{1,r}(\Omega_f)}, \tag{1.49}$$

where $C_{K,f}$ is the constant arising in Korn's inequality, $\|\mathbf{D}(\mathbf{w})\|_{L^r(\Omega_f)} \geq C_{K,f}\|\mathbf{w}\|_{W^{1,r}(\Omega_f)}$, for $\mathbf{w} \in \mathbf{V}_f$.

(ii) $c \neq 0$ and $\mathbf{v}_f \in \mathbf{V}_f$ with $\|\mathbf{D}(\mathbf{v}_f)\|^{2-r}_{L^r(\Omega_f)} \geq c$. Then from (1.20) we have

$$a_f(\mathbf{v}_f, \mathbf{v}_f) \geq 2\, C_5 \frac{\|\mathbf{D}(\mathbf{v}_f)\|^2_{L^r(\Omega_f)}}{c + \|\mathbf{D}(\mathbf{v}_f)\|^{2-r}_{L^r(\Omega_f)}} \geq C_5 \|\mathbf{D}(\mathbf{v}_f)\|^r_{L^r(\Omega_f)} \geq C_5 C_K^r \, \|\mathbf{v}_f\|^r_{W^{1,r}(\Omega_f)}. \tag{1.50}$$

(iii) $c \neq 0$ and $\mathbf{v}_f \in \mathbf{V}_f$ with $\|\mathbf{D}(\mathbf{v}_f)\|^{2-r}_{L^r(\Omega_f)} < c$. Then $C_K^r \|\mathbf{v}_f\|^r_{W^{1,r}(\Omega_f)} \leq \|\mathbf{D}(\mathbf{v}_f)\|^r_{L^r(\Omega_f)}$ $\leq c^{r/(2-r)}$. Denote the coercivity constant from (1.50) as $c_f = C_5 C_K^r$ and let $\bar{c}_f = C_5 c^{(2r-2)/(2-r)}$. Now,

$$c_f \|\mathbf{v}_f\|^r_{W^{1,r}(\Omega_f)} \leq C_5 \|\mathbf{D}(\mathbf{v}_f)\|^r_{L^r(\Omega_f)} \leq C_5 c^{r/(2-r)} = c\bar{c}_f,$$

hence

$$c_f \|\mathbf{v}_f\|^r_{W^{1,r}(\Omega_f)} - c\bar{c}_f \leq 0 \leq a_f(\mathbf{v}_f, \mathbf{v}_f). \tag{1.51}$$

Combining (1.49)-(1.51) yields the coercivity estimate given in (1.45). The reader is also referred to [64], where a similar result is proven under slightly different assumptions, which are satisfied by the Carreau model with $\nu_\infty = 0$.

The continuity and coercivity bounds (1.46) and (1.48) follow in the same way. $\square$

We introduce the following operators and prove some of their properties. Let $R_s : X_p \longrightarrow X_p'$, $R_p : V_p \longrightarrow V_p'$, $L_f : W_f \longrightarrow W_f'$, $L_p : W_p \longrightarrow W_p'$ be defined by

$$R_s(\mathbf{u}_s)(\mathbf{v}_s) := r_s(\mathbf{u}_s, \mathbf{v}_s) = (\mathbf{D}(\mathbf{u}_s), \mathbf{D}(\mathbf{v}_s))_{\Omega_p}, \tag{1.52}$$

$$R_p(\mathbf{u}_p)(\mathbf{v}_p) := r_p(\mathbf{u}_p, \mathbf{v}_p) = (|\nabla \cdot \mathbf{u}_p|^{r-2} \nabla \cdot \mathbf{u}_p, \nabla \cdot \mathbf{v}_p)_{\Omega_p}\,, \tag{1.53}$$

$$L_f(p_f)(w_f) := l_f(p_f, w_f) = (|p_f|^{r'-2} p_f, w_f)_{\Omega_f}, \tag{1.54}$$

$$L_p(p_p)(w_p) := l_p(p_p, w_p) = (|p_p|^{r'-2} p_p, w_p)_{\Omega_p}\,. \tag{1.55}$$

Lemma 1.3.3. *The operators R_s, R_p, L_f, and L_p are bounded, continuous, coercive, and monotone.*

Proof. The operators satisfy the following continuity and coercivity bounds:

$$
\begin{aligned}
&R_s(\mathbf{u}_s)(\mathbf{v}_s) \le \|\mathbf{u}_s\|_{H^1(\Omega_p)}\|\mathbf{v}_s\|_{H^1(\Omega_p)}, && R_s(\mathbf{u}_s)(\mathbf{u}_s) \ge C_{K,p}\|\mathbf{u}_s\|^2_{H^1(\Omega_p)}, \quad \forall \mathbf{u}_s, \mathbf{v}_s \in \mathbf{X}_p,\\
&R_p(\mathbf{u}_p)(\mathbf{v}_p) \le \|\nabla\cdot\mathbf{u}_p\|^{r/r'}_{L^r(\Omega_p)}\|\nabla\cdot\mathbf{v}_p\|_{L^r(\Omega_p)}, && R_p(\mathbf{u}_p)(\mathbf{u}_p) \ge \|\nabla\cdot\mathbf{u}_p\|^r_{L^r(\Omega_p)}, \quad \forall \mathbf{u}_p, \mathbf{v}_p \in \mathbf{V}_p,\\
&L_f(p_f)(w_f) \le \|p_f\|^{r'/r}_{L^{r'}(\Omega_f)}\|w_f\|_{L^{r'}(\Omega_f)}, && L_f(p_f)(p_f) \ge \|p_f\|^{r'}_{L^{r'}(\Omega_f)}, \quad \forall p_f, w_f \in W_f,\\
&L_p(p_p)(w_p) \le \|p_p\|^{r'/r}_{L^{r'}(\Omega_p)}\|w_p\|_{L^{r'}(\Omega_p)}, && L_p(p_p)(p_p) \ge \|p_p\|^{r'}_{L^{r'}(\Omega_p)}, \quad \forall p_p, w_p \in W_p.
\end{aligned}
$$

The coercivity bounds follow directly from the definitions, using Korn's inequality for R_s. The continuity bounds follow from the Cauchy-Schwarz or Hölder's inequalities. The above bounds imply that the operators are bounded, continuous, and coercive. Monotonicity follows from bounds similar to (1.20), which can be established in a way similar to the Power law model [74]. □

It was shown in [39] that there exists a bounded extension of λ from $W^{1/r,r'}(\Gamma_{fp})$ to $W^{1/r,r'}(\partial\Omega_p)$, defined as $E_\Gamma\lambda = \gamma\phi(\lambda)$, where γ is the trace operator from $W^{1,r}(\Omega_p)$ to $W^{1/r,r'}(\partial\Omega_p)$ and $\phi(\lambda) \in W^{1,r'}(\Omega_p)$ is the weak solution of

$$-\nabla\cdot|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda) = 0, \quad \text{in } \Omega_p, \tag{1.56}$$

$$\phi(\lambda) = \lambda, \quad \text{on } \Gamma_{fp}, \tag{1.57}$$

$$|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n} = 0, \quad \text{on } \partial\Omega_p \setminus \Gamma_{fp}\,. \tag{1.58}$$

We prove the following equivalent of norms.

Lemma 1.3.4. *For $\lambda \in W^{1/r,r'}(\Gamma_{fp})$ and $\phi(\lambda)$ defined by* (1.136)–(1.58), *there exists c_1, $c_2 > 0$ such that*

$$c_1\|\phi(\lambda)\|_{W^{1,r'}(\Omega_p)} \le \|\lambda\|_{W^{1/r,r'}(\Gamma_{fp})} \le c_2\|\phi(\lambda)\|_{W^{1,r'}(\Omega_p)}. \tag{1.59}$$

Proof. For $\phi \in W^{1,r'}(\Omega)$, $|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda) \in L^{r'}(\mathrm{div};\Omega)$ and, therefore, from (1.136)–(1.58), we have

$$\begin{aligned}(|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda), \nabla\phi(\lambda))_{\Omega_p} &= \langle|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}, E_\Gamma\lambda\rangle_{\partial\Omega_p} \\ &\le \||\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}\|_{W^{-1/r,r}(\partial\Omega_p)}\|E_\Gamma\lambda\|_{W^{1/r,r'}(\partial\Omega_p)} \\ &\le C\,\||\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}\|_{W^{-1/r,r}(\partial\Omega_p)}\|\lambda\|_{W^{1/r,r'}(\Gamma_{fp})}. \quad (1.60)\end{aligned}$$

Now, for $\psi \in W^{1,r'}(\Omega_p)$,

$$\begin{aligned}\int_{\partial\Omega_p}|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}\,\psi\,ds &= \int_{\Omega_p}\nabla\cdot|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\,\psi\,d\mathbf{x} \\ &\quad+\int_{\Omega_p}|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\nabla\psi\,d\mathbf{x} \\ &\le \||\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\|_{L^r(\Omega_p)}\,\|\psi\|_{W^{1,r'}(\Omega_p)} \qquad \text{(using (1.136))} \\ &= \|\nabla\phi\|^{r'/r}_{L^{r'}(\Omega_p)}\,\|\psi\|_{W^{1,r'}(\Omega_p)}\,. \quad (1.61)\end{aligned}$$

Using the fact the trace operator, $\gamma(\cdot)$, is a bounded, linear, bijective operator for the quotient space $W^{1,q}(\Omega_p)/W^{1,q}_0(\Omega_p)$ onto $W^{1-\frac{1}{q},q}(\partial\Omega_p)$ [46], we have

$$\begin{aligned}&\||\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}\|_{W^{-1/r,r}(\partial\Omega_p)} \quad (1.62)\\ &= \sup_{\xi\in W^{1/r,r'}(\partial\Omega_p)}\frac{\langle|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}\,,\xi\rangle_{W^{-1/r,r}(\partial\Omega_p),W^{1/r,r'}(\partial\Omega_p)}}{\|\xi\|_{W^{1/r,r'}(\partial\Omega_p)}} \\ &\le C\sup_{\psi\in W^{1,r'}(\Omega_p)}\frac{\int_{\partial\Omega_p}|\nabla\phi(\lambda)|^{r'-2}\nabla\phi(\lambda)\cdot\mathbf{n}\,\gamma(\psi)\,ds}{\|\psi\|_{W^{1,r'}(\Omega_p)}} \\ &\le C\,\|\nabla\phi\|^{r'/r}_{L^{r'}(\Omega_p)}\,,\ \text{(using (1.61))}. \quad (1.63)\end{aligned}$$

Combining (1.60) and (1.63) with the Poincare inequality implies that

$$\|\phi(\lambda)\|_{W^{1,r'}(\Omega)} \le C\|\lambda\|_{W^{1/r,r'}(\Gamma_{fp})}. \quad (1.64)$$

On the other hand, due to (1.138) and the trace inequality, we have

$$\|\lambda\|_{W^{1/r,r'}(\Gamma_{fp})} \le C\|\phi(\lambda)\|_{W^{1,r'}(\Omega)}. \quad (1.65)$$

Combining (1.64) and (1.65), we obtain (1.59). □

Introduce $L_\Gamma : \Lambda \longrightarrow \Lambda'$ defined by

$$L_\Gamma(\lambda)(\mu) := l_\Gamma(\lambda, \mu) = \left(|\nabla\phi(\lambda)|^{r-2}\, \nabla\phi(\lambda), \nabla\phi(\mu)\right)_{\Omega_p}. \tag{1.66}$$

Lemma 1.3.5. *The operator L_Γ is bounded, continuous, coercive, and monotone.*

Proof. The result can be obtained in a similar manner to the proof of lemma 1.3.3, using the equivalence of norms proved in lemma 1.3.4. □

Denote by $W_{p,2}$ and $\boldsymbol{\Sigma}_{e,2}$ the closure of the spaces W_p and $\boldsymbol{\Sigma}_e$ with respect to the norms

$$\|w_p\|^2_{W_{p,2}} := (s_0 w_p, w_p)_{L^2(\Omega_p)}, \quad \|\boldsymbol{\tau}_e\|^2_{\boldsymbol{\Sigma}_{e,2}} := (A\boldsymbol{\tau}_e, \boldsymbol{\tau}_e)_{L^2(\Omega_p)}.$$

Note that $W_{p,2} = L^2(\Omega_p)$, and $\boldsymbol{\Sigma}_{e,2} = \boldsymbol{\Sigma}_e$.

Lemma 1.3.6. *For every $\bar{g}_p \in W'_{p,2}, \bar{g}_e \in \boldsymbol{\Sigma}'_{e,2}$, let $g = (s_0\bar{g}_p, A\bar{g}_e, 0, 0)$ there exists a solution of*

$$\begin{aligned} &\mathbf{q} \in \mathbf{Q},\ s \in S: \\ &\qquad \mathcal{A}\mathbf{q} + \mathcal{B}'s = \boldsymbol{0}, \ \text{in } \mathbf{Q}' \qquad (1.67) \\ &\qquad -\mathcal{B}\mathbf{q} + \mathcal{E}_2 s = g, \ \text{in } S'. \qquad (1.68) \end{aligned}$$

Proof. Consider the following functionals:

$$\begin{aligned} r_s(\mathbf{u}_s, \mathbf{v}_s) &= (\mathbf{D}(\mathbf{u}_s), \mathbf{D}(\mathbf{v}_s))_{\Omega_s}, \mathbf{r}_p(\mathbf{u}_p, \mathbf{v}_p) = (|\nabla\cdot\mathbf{u}_p|^{r-2}\nabla\cdot\mathbf{u}_p, \nabla\cdot\mathbf{v}_p)_{\Omega_p}, \\ &\qquad \forall \mathbf{u}_s, \mathbf{v}_s \in \mathbf{X}_p, \mathbf{u}_p, \mathbf{v}_p \in \mathbf{V}_p, \\ l_f(p_f, w_f) &= (|p_f|^{r'-2}p_f, w_f)_{\Omega_f}, l_p(p_p, w_p) = (|p_p|^{r'-2}p_p, w_p)_{\Omega_f}, \forall p_f, w_f \in W_f,\ p_p, w_p \in W_p. \end{aligned}$$

Clearly, r_s, l_f and l_p are bounded, monotone and coercive over $\mathbf{X}_p$, W_f and W_p, respectively. Next let $\lambda \in \Lambda$ be given. Define the operator $T : W^{1,r'(\Omega_p)} \to \left(W^{1,r'}(\Omega_p)\right)'$ as

$$(T\phi, \psi) := \int_\Omega \left(|\nabla\phi|^{r'-2}\nabla\phi\cdot\nabla\psi\right)\, dA.$$

We define the corresponding functional as follows:

$$l_\lambda(\lambda, \mu) = (T\phi(\lambda), \phi(\mu))_\Omega, \qquad \forall \lambda, \mu \in \Lambda.$$

Then

$$
\begin{aligned}
l_\lambda(\lambda,\lambda) &= (T\phi(\lambda),\phi(\lambda))_\Omega = \|\phi(\lambda)\|^{r'}_{W^{1,r'}(\Omega)} \ge C\|\lambda\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})},\\
l_\lambda(\lambda,\mu) &= (T\phi(\lambda),\phi(\mu))_\Omega \le \|\phi(\lambda)\|^{r'-1}_{W^{1,r'}(\Omega)}\|\phi(\mu)\|_{W^{1,r'}(\Omega)}\\
&\qquad\qquad = \|\phi(\lambda)\|^{r'/r}_{W^{1,r'}(\Omega)}\|\phi(\mu)\|_{W^{1,r'}(\Omega)} \le C\|\lambda\|^{r'/r}_{W^{1/r,r'}(\Gamma_{fp})}\|\mu\|_{W^{1/r,r'}(\Gamma_{fp})},
\end{aligned}
$$

which implies that r_λ is bounded, coercive and continuous over $W^{1/r,r'}(\Gamma_{fp})$.

We define the corresponding operators $\mathcal{R}:\mathbf{Q}\to\mathbf{Q}'$ and $\mathcal{L}:S\to S'$ via:

$$
\begin{aligned}
\mathcal{R}\mathbf{q}_1(\mathbf{q}_2) &= r_s(\mathbf{v}_{s,1},\mathbf{v}_{s,2}) + r_p(\mathbf{v}_{p,1},\mathbf{v}_{p,2}),\\
\mathcal{L}\mathbf{s}_1(s_2) &= l_f(w_{f,1},w_{f,2}) + l_p(w_{p,1},w_{p,2}) + l_\Gamma(\mu_1,\mu_2).
\end{aligned}
$$

For $\epsilon > 0$, consider a regularized problem:

$$
\begin{aligned}
&\mathbf{q}_\epsilon \in \mathbf{Q},\ s_\epsilon \in S:\\
&\qquad \epsilon\mathcal{R}\mathbf{q}_\epsilon + \mathcal{A}\mathbf{q}_\epsilon + \mathcal{B}'s_\epsilon = \mathbf{0}, \qquad (1.69)\\
&\qquad (\mathcal{E}_2 + \epsilon\mathcal{L})s_\epsilon - \mathcal{B}\mathbf{q}_\epsilon = g. \qquad (1.70)
\end{aligned}
$$

Denote the resulting operator by $\mathcal{O}:\mathbf{Q}\times S\to(\mathbf{Q}\times S)'$:

$$
\mathcal{O}\begin{pmatrix}\mathbf{q}\\ s\end{pmatrix} = \begin{pmatrix}\epsilon\mathcal{R}+\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \epsilon\mathcal{L}+\mathcal{E}_2\end{pmatrix}\begin{bmatrix}\mathbf{q}\\ s\end{bmatrix}.
$$

Let $\mathbf{q}^{(1)},\mathbf{q}^{(2)}\in\mathbf{Q}$ and $s^{(1)},s^{(2)}\in S$ be given, then

$$
\begin{aligned}
&\left(\mathcal{O}\begin{pmatrix}\mathbf{q}^{(1)}\\ s^{(1)}\end{pmatrix} - \mathcal{O}\begin{pmatrix}\mathbf{q}^{(2)}\\ s^{(2)}\end{pmatrix}\right)\left(\begin{pmatrix}\mathbf{q}^{(1)}\\ s^{(1)}\end{pmatrix} - \begin{pmatrix}\mathbf{q}^{(2)}\\ s^{(2)}\end{pmatrix}\right)\\
&= ((\epsilon\mathcal{R}+\mathcal{A})\mathbf{q}^{(1)} - (\epsilon\mathcal{R}+\mathcal{A})\mathbf{q}^{(2)})(\mathbf{q}^{(1)}-\mathbf{q}^{(2)}) + ((\mathcal{E}_2+\epsilon\mathcal{L})s^{(1)} - (\mathcal{E}_2+\epsilon\mathcal{L})s^{(2)})(s^{(1)}-s^{(2)}).
\end{aligned}
$$

From Lemmas 1.3.1, 1.3.2, 1.3.3, and 1.3.5 we have that $\mathcal{O}$ is a bounded, continuous, and monotone operator. Moreover, using the coercivity bounds from (1.45)–(1.48), we also have

$$
\begin{aligned}
&\mathcal{O}\begin{pmatrix}\mathbf{q}\\ s\end{pmatrix}\left(\begin{pmatrix}\mathbf{q}\\ s\end{pmatrix}\right) = (\epsilon\mathcal{R}+\mathcal{A})\mathbf{q}(\mathbf{q}) + (\mathcal{E}_2+\epsilon\mathcal{L})s(s)\\
&= \epsilon r_s(\mathbf{v}_s,\mathbf{v}_s) + \epsilon r_p(\mathbf{v}_p,\mathbf{v}_p) + a_f(\mathbf{v}_f,\mathbf{v}_f) + a_p^d(\mathbf{v}_p,\mathbf{v}_p) + a_{BJS}(\mathbf{v}_f,\mathbf{v}_s;\mathbf{v}_f,\mathbf{v}_s)
\end{aligned}
$$

$$
\begin{aligned}
&+ (s_0 w_p, w_p)_{\Omega_p} + a_p^e(\boldsymbol{\tau}_e, \boldsymbol{\tau}_e) + \epsilon l_f(w_f, w_f) + \epsilon l_p(w_p, w_p) + \epsilon l_\lambda(\mu, \mu) \\
&\geq C \left(\epsilon \|\mathbf{D}(\mathbf{v}_s)\|^2_{L^2(\Omega_p)} + \epsilon \|\nabla \cdot \mathbf{v}_p\|^r_{L^r(\Omega_p)} + \|\mathbf{D}(\mathbf{v}_f)\|^r_{L^r(\Omega_f)} + \|\mathbf{v}_p\|^r_{L^r(\Omega_p)} + \|\mathbf{v}_f - \mathbf{v}_s\|^r_{L^r(\Gamma_{fp})} \right) \\
&+ C \left(s_0 \|w_p\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\tau}_e\|^2_{L^2(\Omega_p)} + \epsilon \|w_f\|^{r'}_{L^{r'}(\Omega_f)} + \epsilon \|w_p\|^{r'}_{L^{r'}(\Omega_p)} + \epsilon \|\mu\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})} \right). \qquad (1.71)
\end{aligned}
$$

In the case of (B1)–(B2) with $c > 0$, we have an extra term $-c(\bar{c}_f + \bar{c}_p + \bar{c}_I)$ on the right-hand side of (1.71) due to the coercivity estimates from (1.45)–(1.48). The argument in this case doesn't change and we omit this term for simplicity. It follows from (1.71) that $\mathcal{O}$ is coercive. Thus, an application of the Browder-Minty theorem [68] establishes the existence of a solution $(\mathbf{q}_\epsilon, s_\epsilon) \in \mathbf{Q} \times S$ of (1.69)–(1.70), where $\mathbf{q}_\epsilon = (\mathbf{u}_{p,\epsilon}, \mathbf{u}_{s,\epsilon}, \mathbf{u}_{f,\epsilon})$ and $s_\epsilon = (p_{p,\epsilon}, \boldsymbol{\sigma}_{e,\epsilon}, p_{f,\epsilon}, \lambda_\epsilon)$.

From (1.71) and (1.69) - (1.70), we have

$$
\begin{aligned}
&\epsilon \|\mathbf{u}_{s,\epsilon}\|^2_{H^1(\Omega_p)} + \epsilon \|\nabla \cdot \mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + \|\mathbf{u}_{f,\epsilon}\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + \|\mathbf{u}_{f,\epsilon} - \mathbf{u}_{s,\epsilon}\|^r_{L^r(\Gamma_{fp})} \\
&\quad + s_0 \|p_{p,\epsilon}\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\sigma}_{e,\epsilon}\|^2_{L^2(\Omega_p)} + \epsilon \|p_{f,\epsilon}\|^{r'}_{L^{r'}(\Omega_f)} + \epsilon \|p_{p,\epsilon}\|^{r'}_{L^{r'}(\Omega_p)} + \epsilon \|\lambda_\epsilon\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})} \\
&\qquad \leq C \Big(\|\bar{q}_p\|_{L^r(\Omega_p)} \|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)} + \|\bar{g}_e\|_{L^2(\Omega_p)} \|\boldsymbol{\sigma}_{e,\epsilon}\|_{L^2(\Omega_p)} \Big). \qquad (1.72)
\end{aligned}
$$

From (1.70), $\boldsymbol{\sigma}_{e,\epsilon}$ and $\mathbf{u}_{s,\epsilon}$ satisfy

$$
a_p^s(\boldsymbol{\sigma}_{e,\epsilon}, \boldsymbol{\tau}_e) - b_s(\mathbf{u}_{s,\epsilon}, \boldsymbol{\tau}_e) = (A\bar{g}_e, \boldsymbol{\tau}_e)_{\Omega_p}, \quad \forall \boldsymbol{\tau}_e \in \boldsymbol{\Sigma}_e.
$$

Therefore, applying the inf-sup condition (1.43), we obtain:

$$
\begin{aligned}
\|\mathbf{u}_{s,\epsilon}\|_{H^1(\Omega_p)} &\leq C \sup_{0 \neq (0,\boldsymbol{\tau}_e,0,0) \in S} \frac{b_s(\mathbf{u}_{s,\epsilon}, \boldsymbol{\tau}_e)}{\|(0, \boldsymbol{\tau}_e, 0, 0)\|_S} = \sup_{0 \neq (0,\boldsymbol{\tau}_e,0,0) \in S} \frac{a_p^s(\boldsymbol{\sigma}_{e,\epsilon}, \boldsymbol{\tau}_e) + (A\bar{g}_e, \boldsymbol{\tau}_e)_{\Omega_p}}{\|(0, \boldsymbol{\tau}_e, 0, 0)\|_S} \\
&\leq C \left(\|\boldsymbol{\sigma}_{e,\epsilon}\|_{L^2(\Omega_p)} + \|\bar{g}_e\|_{L^2(\Omega_p)} \right). \qquad (1.73)
\end{aligned}
$$

Combining (1.73) and (1.72), and using Young's inequality, for $a, b \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\delta > 0$,

$$
ab \leq \frac{\delta^p a^p}{p} + \frac{b^q}{\delta^q q}, \qquad (1.74)
$$

we obtain

$$
\begin{aligned}
&\|\mathbf{u}_{s,\epsilon}\|^2_{H^1(\Omega_p)} + \epsilon \|\nabla \cdot \mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + \|\mathbf{u}_{f,\epsilon}\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + |\mathbf{u}_{f,\epsilon} - \mathbf{u}_{s,\epsilon}|^r_{BJS} + \epsilon \|\mathbf{u}_{s,\epsilon}\|^2_{H^1(\Omega_p)} \\
&\quad + s_0 \|p_{p,\epsilon}\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\sigma}_{e,\epsilon}\|^2_{L^2(\Omega_p)} + \epsilon \|p_{f,\epsilon}\|^{r'}_{L^{r'}(\Omega_f)} + \epsilon \|p_{p,\epsilon}\|^{r'}_{L^{r'}(\Omega_p)} + \epsilon \|\lambda_\epsilon\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})} \\
&\leq C \Big(\|\bar{q}_p\|_{L^r(\Omega_p)} \|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)} + \|\bar{g}_e\|^2_{L^2(\Omega_p)} \Big) + \frac{1}{2} \left(\|\boldsymbol{\sigma}_{e,\epsilon}\|^2_{L^2(\Omega_p)} \right), \qquad (1.75)
\end{aligned}
$$

from which it follows that

$$\begin{aligned}\|\mathbf{u}_{s,\epsilon}\|^2_{H^1(\Omega_p)} + \epsilon\|\nabla\cdot\mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + \|\mathbf{u}_{f,\epsilon}\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)}\\ + \|\boldsymbol{\sigma}_{e,\epsilon}\|^2_{L^2(\Omega_p)} + |\mathbf{u}_{f,\epsilon} - \mathbf{u}_{s,\epsilon}|^r_{BJS}\\ \le C\left(\|\bar g_e\|^2_{L^2(\Omega_p)} + \|\bar g_p\|_{L^r(\Omega_p)}\|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)}\right).\end{aligned} \tag{1.76}$$

To obtain bounds for $p_{p,\epsilon}$, $p_{f,\epsilon}$, and λ_ϵ we use (1.44). With $s = (p_{p,\epsilon}, \mathbf{0}, p_{f,\epsilon}, \lambda_\epsilon) \in S$, we have

$$\begin{aligned}&\|p_{f,\epsilon}\|_{L^{r'}(\Omega_f)} + \|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)} + \|\lambda_\epsilon\|_{W^{1/r,r'}(\Gamma_{fp})}\\ &\quad\le C \sup_{(\mathbf{v}_p,\mathbf{0},\mathbf{v}_f)\in\mathbf{Q}} \frac{b_f(\mathbf{v}_f, p_{f,\epsilon}) + b_p(\mathbf{v}_p, p_{p,\epsilon}) + b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0};\lambda_\epsilon)}{\|(\mathbf{v}_p,\mathbf{0},\mathbf{v}_f)\|_{\mathbf{Q}}}\\ &\quad\le C\sup_{\mathbf{q}\in\mathbf{Q}} \frac{-\epsilon\, r_p(\mathbf{u}_{p,\epsilon},\mathbf{v}_p) - a_f(\mathbf{u}_{f,\epsilon},\mathbf{v}_f) - a^d_p(\mathbf{u}_{p,\epsilon},\mathbf{v}_p) - a_{BJS}(\mathbf{u}_{f,\epsilon},\mathbf{u}_{s,\epsilon};\mathbf{v}_f,\mathbf{0})}{\|(\mathbf{v}_p,\mathbf{0},\mathbf{v}_f)\|_{\mathbf{Q}}}\\ &\quad\le C\left(\epsilon\|\nabla\cdot\mathbf{u}_{p,\epsilon}\|^{r/r'}_{L^r(\Omega_p)} + \|\mathbf{u}_{f,\epsilon}\|^{r/r'}_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_{p,\epsilon}\|^{r/r'}_{L^r(\Omega_p)} + |\mathbf{u}_{f,\epsilon} - \mathbf{u}_{s,\epsilon}|^{r/r'}_{BJS}\right).\end{aligned} \tag{1.77}$$

Using Young's inequality, (1.76) and (1.77), we obtain

$$\begin{aligned}\|\mathbf{u}_{s,\epsilon}\|^2_{H^1(\Omega_p)} + \epsilon\|\nabla\cdot\mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + \|\mathbf{u}_{f,\epsilon}\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_{p,\epsilon}\|^r_{L^r(\Omega_p)} + \|\boldsymbol{\sigma}_{e,\epsilon}\|^2_{L^2(\Omega_p)}\\ + |\mathbf{u}_{f,\epsilon} - \mathbf{u}_{s,\epsilon}|^r_{BJS} + \|p_{f,\epsilon}\|^{r'}_{L^{r'}(\Omega_f)} + \|p_{p,\epsilon}\|^{r'}_{L^{r'}(\Omega_p)} + \|\lambda_\epsilon\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})}\\ \le C\left(\|\bar g_p\|^r_{L^r(\Omega_p)} + \|\bar g_e\|^2_{L^2(\Omega_p)}\right),\end{aligned} \tag{1.78}$$

hence $\|\mathbf{u}_{s,\epsilon}\|_{H^1(\Omega_p)}$, $\|\mathbf{u}_{f,\epsilon}\|_{W^{1,r}(\Omega_f)}$, $\|\boldsymbol{\sigma}_{e,\epsilon}\|_{L^2(\Omega_p)}$, $\|p_{f,\epsilon}\|_{L^{r'}(\Omega_f)}$, $\|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)}$ and $\|\lambda_\epsilon\|_{W^{1/r,r'}(\Gamma_{fp})}$ are bounded independently of ϵ.

Also, as $\nabla\cdot\mathbf{V}_p = (W_p)'$, we have from (1.70) and the continuity of L_p stated in Lemma 1.3.3:

$$\begin{aligned}\|\nabla\cdot\mathbf{u}_{p,\epsilon}\|_{L^r(\Omega_p)} &\le \|\bar g_p\|_{L^r(\Omega_p)} + s_0\|p_{p,\epsilon}\|_{L^r(\Omega_p)} + \alpha_p\|\nabla\cdot\mathbf{u}_{s,\epsilon}\|_{L^r(\Omega_p)} + \epsilon\|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)}\\ &\le \|\bar g_p\|_{L^r(\Omega_p)} + s_0\|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)} + \alpha_p\|\mathbf{u}_{s,\epsilon}\|_{H^1(\Omega_p)} + \epsilon\|p_{p,\epsilon}\|_{L^{r'}(\Omega_p)}.\end{aligned}$$

Therefore $\|\mathbf{u}_{p,\epsilon}\|_{L^r(\mathrm{div};\Omega_p)}$ is also bounded independently of ϵ.

Since $\mathbf{Q}$ and S are reflexive Banach spaces, as $\epsilon\to 0$ we can extract weakly convergent subsequences $\{\mathbf{q}_{\epsilon,n}\}_{n=1}^\infty$, $\{s_{\epsilon,n}\}_{n=1}^\infty$, and $\{\mathcal{A}\mathbf{q}_{\epsilon,n}\}_{n=1}^\infty$, such that $\mathbf{q}_{\epsilon,n} \rightharpoonup \mathbf{q}$ in $\mathbf{Q}$, $s_{\epsilon,n}\rightharpoonup s$ in S, $\mathcal{A}\mathbf{q}_{\epsilon,n}\rightharpoonup\zeta$ in $\mathbf{Q}'$, and

$$\mathbf{q}_\epsilon \xrightarrow{w} \mathbf{q} \text{ in } \mathbf{Q},\ s_\epsilon \xrightarrow{w} s \text{ in } S, \mathcal{A}\mathbf{q}_\epsilon \xrightarrow{w} \zeta \text{ in } \mathbf{Q}'$$

as $\epsilon \to 0$, which satisfies

$$\zeta + \mathcal{B}'s = \mathbf{f} \text{ in } \mathbf{Q}',$$
$$\mathcal{E}_2 s - \mathcal{B}\mathbf{q} = g \text{ in } S'.$$

Moreover, from (1.69) and (1.70), we have

$$\begin{aligned} \limsup_{\epsilon\to 0} \mathcal{A}\mathbf{q}_\epsilon(\mathbf{q}_\epsilon) &= \limsup_{\epsilon\to 0}(-\epsilon\mathcal{R}\mathbf{q}_\epsilon(\mathbf{q}_\epsilon) - (\mathcal{E}_2 + \epsilon\mathcal{L})s_\epsilon(s_\epsilon) + \mathbf{f}(\mathbf{q}_\epsilon) + g(s_\epsilon)) \\ &\le -\mathcal{E}_2 s(s) + \mathbf{f}(\mathbf{q}) + g(s) = \zeta(\mathbf{q}) \end{aligned}$$

Since $\mathcal{A}$ is monotone and continuous, it follows, see [76] p.38, $\mathcal{A}\mathbf{q} = \zeta$. Hence, $\mathbf{q}$ and s solve (1.67)-(1.68). □

We will use theorem (6.1) part b in [76] to conclude that the alternative formulation has a solution. The result in [76] can be restated as follow.

Theorem 1.3.7. *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given for the real vector space E to its algebraic dual E^*, and let E'_b be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b \;=\; \mathcal{N}x\,(x)^{1/2}\,, \quad x \in E\,.$$

Let $M \subset E \times E'_b$ be a relation with domain $D = \{x \in E \,:\, M(x) \neq \emptyset\}$.
Assume $\mathcal{M}$ is monotone and $Rg(\mathcal{N} + \mathcal{M}) = E'_b$. Then, for each $u_0 \in D$ and for each $f \in W^{1,1}(0, T; E'_b)$, there is a solution u of

$$\frac{d}{dt}\,(\mathcal{N}u(t)) \;+\; \mathcal{M}\,(u(t)) \ni f(t)\,, \quad 0 < t < T\,,$$

with

$$\mathcal{N}u \in W^{1,\infty}(0, T; E'_b)\,, \quad u(t) \in D\,, \quad \text{for all } 0 \le t \le T\,, \qquad \text{and } \mathcal{N}u(0) \;=\; \mathcal{N}u_0\,.$$

To use the above theorem, first we need to prove the lemma 1.3.8 below.

Let $p_{p,0} \in W^{1,r'}(\Omega_p)$ be given, from (1.46), we have a_p^d is coercive. Hence, by Browder-Minty theorem, there exists a solution $\mathbf{u}_{p,0} \in L^r(\Omega_p)$ to

$$a_p^d(\mathbf{u}_{p,0}, \mathbf{v}_p) = -(\nabla p_{p,0}, \mathbf{v}_p), \quad \forall \mathbf{v}_p \in L^r(\Omega_p). \tag{1.79}$$

Lemma 1.3.8. *Assume $p_{p,0} \in W^{1,r'}$ and that the solution to (1.79) satisfies $\mathbf{u}_{p,0} \in \mathbf{V}_p$. There exists $(\mathbf{u}_{p,0},\ \mathbf{u}_{s,0},\ \mathbf{u}_{f,0}) \in \mathbf{Q}$, and $(p_{f,0}, \lambda_0, \boldsymbol{\sigma}_{e,0}) \in W_f \times \Lambda \times \boldsymbol{\Sigma}_p$ such that*

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_0 \\ s_0 \end{pmatrix} \in \begin{pmatrix} \mathbf{Q}'_{\mathcal{E}_1} \\ S'_{\mathcal{E}_2} \end{pmatrix} \tag{1.80}$$

where $\mathbf{q}_0 = (\mathbf{u}_{p,0}, \mathbf{u}_{s,0}, \mathbf{u}_{f,0})$ and $s_0 = (p_{p,0}, \boldsymbol{\sigma}_{e,0}, p_{f,0}, \lambda_0)$.

Proof. First, we will show that there exists $(\mathbf{u}_{p,0}, \mathbf{u}_{s,0}, \mathbf{u}_{f,0}) \in \mathbf{Q}$ and $(p_{f,0}, \lambda_0, \boldsymbol{\sigma}_{e,0}) \in W_f \times \Lambda \times \boldsymbol{\Sigma}_p$ such that

$$\begin{aligned} a_f(\mathbf{u}_{f,0}, \mathbf{v}_f) + a_p^d(\mathbf{u}_{p,0}, \mathbf{v}_p) + a_{BJS}(\mathbf{u}_{f,0}, \mathbf{u}_{s,0}; \mathbf{v}_f, \mathbf{v}_s) + b_f(\mathbf{v}_f, p_{f,0}) \qquad\qquad (1.81)\\ + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \mathbf{v}_s; \lambda_0) + b_s(\mathbf{v}_s, \boldsymbol{\sigma}_{e,0}) = -b_p(\mathbf{v}_p, p_{p,0}) - \alpha_p b_p(\mathbf{v}_s, p_{p,0}) \\ b_f(\mathbf{u}_{f,0}, w_f) + b_\Gamma(\mathbf{u}_{f,0}, \mathbf{u}_{p,0}, \mathbf{u}_{s,0}; \mu) = 0. \qquad\qquad (1.82) \end{aligned}$$

Define $\lambda_0 = p_{p,0}|_{\Gamma_{fp}} \in \Lambda$. Taking $\mathbf{v}_p \in \mathbf{V}_p$ in (1.79) and integrating by parts, we implies (1.81) with test function $\mathbf{v}_p$.

Define $(\mathbf{u}_{f,0}, p_{f,0})$ from (1.81) with $\mathbf{v}_f$, and (1.82) with w_f, and taking $\mathbf{u}_{s,0} \cdot \mathbf{t}_{f,j} = 0$ in a_{BJS}. This is a well defined problem, since it corresponds to the weak solution of the Stokes system with the given boundary on Γ_f and the boundary conditions

$$-(\boldsymbol{\sigma}_{f,0}\mathbf{n}_f) \cdot \mathbf{n}_f = \lambda_0, \quad -(\boldsymbol{\sigma}_{f,0}\mathbf{n}_f) \cdot \mathbf{t}_{f,j} = \nu_I \alpha_{BJS} \sqrt{\kappa_j^{-1}} \mathbf{u}_{f,0} \mathbf{t}_{f,j} \text{ on } \Gamma_{fp}.$$

Note that λ_0 is datum for this problem.

If we couple the equation (1.81) with test function $\mathbf{v}_s$, with the equation

$$a_p^s(\boldsymbol{\sigma}_{e,0}, \boldsymbol{\tau}_e) - b_s(\boldsymbol{\eta}_{p,0}, \boldsymbol{\tau}_e) = 0, \forall \boldsymbol{\tau}_e \in \boldsymbol{\Sigma}_e,$$

we will have a well posed problem, since it correspond to solving a mixed elasticity problem with the given boundary conditions on Γ_p and the boundary conditions

$$-(\boldsymbol{\sigma}_{p,0}\mathbf{n}_p)\cdot\mathbf{n}_p = \lambda_0, \quad -(\boldsymbol{\sigma}_{p,0}\mathbf{n}_p)\cdot\mathbf{t}_{p,j} = \nu_I\alpha_{BJS}\sqrt{\kappa^{-1}}\mathbf{u}_{f,0}\cdot\mathbf{t}_{f,j} \text{ on } \Gamma_{fp}.$$

Let $\mathbf{u}_{s,0}\in\mathbf{X}_p$ satisfies (1.82) with test function μ and $\mathbf{u}_{s,0}\cdot\mathbf{t}_{p,j}=0$ on Γ_{fp}. Note that $\mathbf{u}_{p,0}$ and $\mathbf{u}_{f,0}$ are data for this problem.

From the above construction, and with the assumption $\mathbf{u}_{p,0}\in\mathbf{V}_p$, we conclude that the system (1.81), (1.82) have a solution. Now, by doing algebra, we have

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{q}_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ g_s^* \end{pmatrix} \tag{1.83}$$

where $g_s^*\big((w_p,\boldsymbol{\tau}_e,w_f,\mu)\big) = -b_p(\mathbf{u}_{p,0},w_p)-\alpha_p b_p(\mathbf{u}_{s,0},w_p)-b_s(\mathbf{u}_{s,0},\boldsymbol{\tau}_e)$, hence $g_s^*\in S_2'$. So we get the desired result. $\square$

Let $\mathbf{Q}_1, S_2$ be the closer of $\mathbf{Q}, S$ with respect to the scalar products $\mathcal{E}_1,\mathcal{E}_2$. One can see that $\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix} = \{(\mathbf{q},s) : \mathbf{q}=\mathbf{0}, s=(q_p,\boldsymbol{\sigma}_e,0,0), q_p\in W_{p,2}', \boldsymbol{\sigma}_e\in\boldsymbol{\Sigma}_{e,2}'\}$. From the lemma 1.3.6 we have $\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}\subset Rg\Big[\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2\end{pmatrix}+\begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}\Big]$, and this is usually a proper inclusion, in other words, $\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}\neq Rg\Big[\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2\end{pmatrix}+\begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}\Big]$. Hence, to use the theorem 1.3.7, we need to restrict the domain of operator $\begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}$ in order to have the desired equality in the assumption of theorem 1.3.7. In order to do this, let $\mathcal{O}_1 := \begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}$, we define

$$D := \{(\mathbf{q},s)\in(\mathbf{Q},S) \;:\; \mathcal{O}_1\begin{pmatrix}\mathbf{q}\\ s\end{pmatrix}\in\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}\}.$$

Now, we restrict the domain of $\mathcal{O}_1$ to be D, from now without saying anything, we mean that operators is restricted to D. Then we have $Rg(\mathcal{O}_1)\subset\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}$, so we have $Rg\Big[\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2\end{pmatrix}+\begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}\Big]\subset\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}$. In addition, by lemma 1.3.6, for every $\bar{g}_p\in W_{p,2}', \bar{g}_e\in\boldsymbol{\Sigma}_{e,2}'$, let $g=(s_0\bar{g}_p, A\bar{g}_e,0,0)$ there exists a solution of (1.67), (1.68): $\mathbf{q}^*=(\mathbf{u}_{p,0},\mathbf{u}_{s,0},\mathbf{u}_{q,0}), s^*=(p_{p,0},\boldsymbol{\sigma}_{e,0},p_{f,0},\lambda_0)$. Looking at the equations (1.67), (1.68) and the form of elements in $\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}$, we see that $\begin{pmatrix}\mathbf{q}^*\\ s^*\end{pmatrix}\in D$, so $\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}\subset Rg\Big(\begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathcal{E}_2\end{pmatrix}\Big)$, thus $Rg\Big(\begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathcal{E}_2\end{pmatrix}\Big)=\begin{pmatrix}\mathbf{Q}_1'\\ S_2'\end{pmatrix}$.

Remark 1.3.9. *In the lemma 1.3.8, we need assumption about* $p_{p,0}$. *The data* $(p_{p,0},\boldsymbol{\eta}_{p,0})$ *also need to satisfies* (1.5), *and notice that we have* $\boldsymbol{\sigma}_{e,0}=A^{-1}\mathbf{D}(\boldsymbol{\eta}_{p,0})$. *Thus, in the following by saying the data is compatible, we mean these conditions.*

By combining lemma 1.3.8 with theorem 1.3.7 we get the following result.

Theorem 1.3.10. *For each* $p_p(0) \in L^2(\Omega_p)$, $\boldsymbol{\sigma}_e(0) \in \Sigma_p$ *and* $q_p \in W^{1,1}(0,T; L^r(\Omega_p))$, *where* $(p_{p,0}, \boldsymbol{\sigma}_{e,0})$ *are compatible data, there exists a solution of* (1.38)-(1.40) *with initial conditions:* $s_0 p_p(0) = s_0 p_{p,0}$, $A\boldsymbol{\sigma}_e(0) = A\boldsymbol{\sigma}_{e,0}$, *and satisfy* $(\mathbf{u}_f(t), p_f(t), \mathbf{u}_p(t), p_p(t), \mathbf{u}_s(t), \boldsymbol{\sigma}_e(t), \lambda(t)) \in L^\infty(0,T;\mathbf{V}_f) \times L^\infty(0,T;W_f) \times L^\infty(0,T;\mathbf{V}_p) \times W^{1,\infty}(0,T;W_p) \times L^\infty(0,T;\mathbf{X}_p) \times W^{1,\infty}(0,T;\Sigma_e) \times L^\infty(0,T;\Lambda)$.

Proof. From lemma 1.3.8, there exists $(\mathbf{u}_{p,0}, \mathbf{u}_{s,0}, \mathbf{u}_{f,0}) \in \mathbf{Q}$ and $(p_{f,0}, \lambda_0) \in W_f \times \Lambda$ such that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_0 \\ s_0 \end{pmatrix} \in \begin{pmatrix} \mathbf{Q}_1' \\ S_2' \end{pmatrix}$$

where $\mathbf{q}_0 = (\mathbf{u}_{p,0}, \mathbf{u}_{s,0}, \mathbf{u}_{f,0})$ and $s_0 = (p_{p,0}, \boldsymbol{\sigma}_{e,0}, p_{f,0}, \lambda_0)$.

We have $\begin{pmatrix} \mathbf{q}_0 \\ s_0 \end{pmatrix} \in D$, where D is the domain of operator $\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix}$ as above. It is obvious that $\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix}$ is monotone. We now prove that $\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix}$ is monotone. We have

$$\left(\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ s_1 \end{pmatrix} - \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_2 \\ s_2 \end{pmatrix} \right) \begin{pmatrix} \mathbf{q}_1 - \mathbf{q}_2 \\ s_1 - s_2 \end{pmatrix} = (\mathcal{A}(\mathbf{q}_1) - \mathcal{A}(\mathbf{q}_2))(\mathbf{q}_1 - \mathbf{q}_2) \geq 0.$$

Hence, $\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix}$ is monotone. We already established the range condition, thus by theorem 1.3.7 we have the desired result. □

1.3.2 Existence and uniqueness of solution of the Lagrange multiplier formulation

Recall, that the variable $\mathbf{u}_s$ has the meaning of structure velocity and, therefore, the displacement solution can be recovered using the relation:

$$\boldsymbol{\eta}_p(t) = \boldsymbol{\eta}_{p,0} + \int_0^t \mathbf{u}_s(s)\, ds, \quad \forall t \in (0,T]. \tag{1.84}$$

Since $\mathbf{u}_s(t) \in L^\infty(0,T;\mathbf{X}_p)$, $\boldsymbol{\eta}_p(t) \in W^{1,\infty}(0,T;\mathbf{X}_p)$ for any $\boldsymbol{\eta}_{p,0} \in W^{1,\infty}(0,T;\mathbf{X}_p)$.

Unfortunately, the numerical method based on formulation (1.38) -(1.40) is rather difficult to implement and expensive to use, since the stress space is required to consist of symmetric matrices [17]. In this section we discuss how the well-posedness of the Lagrange multiplier formulation follows from the existence of solution of (1.38)-(1.40).

Theorem 1.3.11. *For each $p_{p,0} \in W^{1,\infty}(0,T;W_p)$ and $\boldsymbol{\eta}_{p,0} \in W^{1,\infty}(0,T;\mathbf{X}_p)$, where ($p_{p,0}$, $\boldsymbol{\eta}_{p,0}$) are compatible data, there exists a unique solution $(\mathbf{u}_f(t), p_f(t), \mathbf{u}_p(t), p_p(t), \boldsymbol{\eta}_p(t), \lambda(t) \in L^\infty(0,T;\mathbf{V}_f) \times L^\infty(0,T;W_f) \times L^\infty(0,T;\mathbf{V}_p) \times W^{1,\infty}(0,T;W_p) \times W^{1,\infty}(0,T;\mathbf{X}_p) \times L^\infty(0,T;\Lambda)$ of* (1.25)-(1.27).

Proof. We begin by using the existence of solution of the alternative formulation (1.38) -(1.40) to help establish solvability of the Lagrange multiplier formulation (1.25)-(1.27). We note that the solution of (1.38) -(1.40) satisfies:

$$a_f(\mathbf{u}_f, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_f) + a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, 0) + \langle \mathbf{v}_f \cdot \mathbf{n}_f, \lambda \rangle_{\Gamma_{fp}} = 0, \tag{1.85}$$

$$a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) + \langle \mathbf{v}_p \cdot \mathbf{n}_p, \lambda \rangle_{\Gamma_{fp}} = 0, \tag{1.86}$$

$$(\boldsymbol{\sigma}_e, \mathbf{D}(\mathbf{v}_s))_{\Omega_p} + \alpha_p b_p(\mathbf{v}_s, p_p) - a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; 0, \mathbf{v}_s) + \langle \mathbf{v}_s \cdot \mathbf{n}_p, \lambda \rangle_{\Gamma_{fp}} = 0, \tag{1.87}$$

$$(A \partial_t \boldsymbol{\sigma}_e, \boldsymbol{\tau}_e)_{\Omega_p} - (\partial_t \mathbf{D}(\boldsymbol{\eta}_p), \boldsymbol{\tau}_e)_{\Omega_p} = 0, \tag{1.88}$$

$$(s_0\, \partial_t p_p, w_p)_{\Omega_p} - \alpha_p b_p(\partial_t \boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) = (q_p, w_p)_{\Omega_p}, \tag{1.89}$$

$$-b_f(\mathbf{u}_f, w_f) = 0, \tag{1.90}$$

$$b_\Gamma(\mathbf{u}_f, \mathbf{u}_p, \partial_t \boldsymbol{\eta}_p; \mu) = 0. \tag{1.91}$$

where $\boldsymbol{\eta}_p$ is given as in (1.84) (in particular, $\partial_t \boldsymbol{\eta}_p = \mathbf{u}_s$). We integrate equation (1.88) in time from $s = 0$ to an arbitrary $s = t \in (0, T]$:

$$0 = \int_0^t \big(\partial_t (A\boldsymbol{\sigma}_e - \mathbf{D}(\boldsymbol{\eta}_p), \boldsymbol{\tau}_e)\big) = (A\boldsymbol{\sigma}_e(t) - \mathbf{D}(\boldsymbol{\eta}_p(t)), \boldsymbol{\tau}_e) - (A\boldsymbol{\sigma}_e(0) - \mathbf{D}(\boldsymbol{\eta}_p(0)), \boldsymbol{\tau}_e).$$

Since $\boldsymbol{\sigma}_e(0) = A^{-1}\mathbf{D}(\boldsymbol{\eta}_p(0))$, we have for any $t \in (0, T]$

$$(A\boldsymbol{\sigma}_e(t) - \mathbf{D}(\boldsymbol{\eta}_p(t)), \boldsymbol{\tau}_e) = 0. \tag{1.92}$$

Since (1.92) holds for any $\boldsymbol{\tau}_e \in \Sigma_e$ and $\mathbf{D}(\mathbf{X}_p) \subset \Sigma_e$, we can choose $\boldsymbol{\tau}_e = A\boldsymbol{\sigma}_e(t) - \mathbf{D}(\boldsymbol{\eta}_p(t))$ to conclude that $\boldsymbol{\sigma}_e(t) = A^{-1}\mathbf{D}(\boldsymbol{\eta}_p(t))$. Therefore, with (1.29),

$$(\boldsymbol{\sigma}_e, \mathbf{D}(\mathbf{v}_s)) = (A^{-1}\mathbf{D}(\boldsymbol{\eta}_p), \mathbf{D}(\mathbf{v}_s)) = a_p^e(\boldsymbol{\eta}_p, \mathbf{v}_s). \tag{1.93}$$

Combining (1.85)-(1.91) and (1.92)-(1.93), we conclude that if $(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \mathbf{u}_s, \boldsymbol{\sigma}_e, \lambda) \in L^\infty(0, T; \mathbf{V}_f) \times L^\infty(0,T;W_f) \times L^\infty(0,T;\mathbf{V}_p) \times W^{1,\infty}(0,T;W_p) \times L^\infty(0,T;\mathbf{X}_p) \times W^{1,\infty}(0,T;\Sigma_e) \times L^\infty(0, T; \Lambda)$ solves (1.37), then $(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \boldsymbol{\eta}_{p,0} + \int_0^t \mathbf{u}_s(s)\, ds, \lambda) \in L^\infty(0,T;$

$\mathbf{V}_f) \times L^\infty(0,T;W_f) \times L^\infty(0,T;\mathbf{V}_p) \times W^{1,\infty}(0,T;W_p) \times W^{1,\infty}(0,T;\mathbf{X}_p) \times L^\infty(0,T;\Lambda)$ is a solution of the Lagrange multiplier formulation (1.25)-(1.27).

Now, assume that the solution of (1.25)-(1.27) is not unique. Indeed, let $(\mathbf{u}_f^1, p_f^1, \mathbf{u}_p^1, p_p^1, \boldsymbol{\eta}_p^1, \lambda^1)$ and $(\mathbf{u}_f^2, p_f^2, \mathbf{u}_p^2, p_p^2, \boldsymbol{\eta}_p^2, \lambda^2)$ be two solutions corresponding to the same data.

We use the monotonicity property (1.20) with $G(\mathbf{u}) = \nu(\mathbf{u})\mathbf{u}$, $\mathbf{s} = \mathbf{D}(\mathbf{u}_f^1)$ and $\mathbf{t} = \mathbf{D}(\mathbf{u}_f^2)$:

$$\begin{aligned}
&C\Big(\frac{\|\mathbf{D}(\mathbf{u}_f^1) - \mathbf{D}(\mathbf{u}_f^2)\|^2_{L^2(\Omega_f)}}{c + \|\mathbf{D}(\mathbf{u}_f^1)\|^{2-r}_{L^r(\Omega_f)} + \|\mathbf{D}(\mathbf{u}_f^2)\|^{2-r}_{L^r(\Omega_f)}} \\
&\quad + \int_{\Omega_f} |\nu(\mathbf{D}(\mathbf{u}_f^1))\mathbf{D}(\mathbf{u}_f^1) - \nu(\mathbf{D}(\mathbf{u}_f^2)\mathbf{D}(\mathbf{u}_f^2))||\mathbf{D}(\mathbf{u}_f^1) - \mathbf{D}(\mathbf{u}_f^2)|dA\Big) \\
&\leq \int_{\Omega_f} \big(\nu(\mathbf{D}(\mathbf{u}_f^1))\mathbf{D}(\mathbf{u}_f^1) - \nu(\mathbf{D}(\mathbf{u}_f^2))\mathbf{D}(\mathbf{u}_f^2)\big) : \big(\mathbf{D}(\mathbf{u}_f^1) - \mathbf{D}(\mathbf{u}_f^2)\big)\, dA \\
&= \frac{1}{2}\left(a_f(\mathbf{u}_f^1,\, \mathbf{u}_f^1 - \mathbf{u}_f^2) \ - \ a_f(\mathbf{u}_f^2,\, \mathbf{u}_f^1 - \mathbf{u}_f^2)\right) \ =: \frac{1}{2} I_1.
\end{aligned} \tag{1.94}$$

Similarly, we use (1.20) with $G(\mathbf{u}) = K^{-1}\nu_{eff}(\mathbf{u})\mathbf{u}$, $\mathbf{s} = \mathbf{u}_p^1$ and $\mathbf{t} = \mathbf{u}_p^2$:

$$\begin{aligned}
&C\left(\frac{\|\mathbf{u}_p^1 - \mathbf{u}_p^2\|^2_{L^r(\Omega_p)}}{c + \|\mathbf{u}_p^1\|^{2-r}_{L^r(\Omega_p)} + \|\mathbf{u}_p^2\|^{2-r}_{L^r(\Omega_p)}} + \int_{\Omega_p} \frac{1}{k_M}|\nu_{eff}(\mathbf{u}_p^1)\mathbf{u}_p - \nu_{eff}(\mathbf{u}_p^2)\mathbf{u}_p^2||\mathbf{u}_p^1 - \mathbf{u}_p^2|dA\right) \\
&\qquad \leq \int_{\Omega_p} K^{-1}\left(\nu_{eff}(\mathbf{u}_p^1)\mathbf{u}_p^1 - \nu_{eff}(\mathbf{u}_p^2)\mathbf{u}_p^2\right) : (\mathbf{u}_p^1 - \mathbf{u}_p^2)dA \\
&\qquad = a_p^d(\mathbf{u}_f^1,\, \mathbf{u}_f^1 - \mathbf{u}_f^2) \ - \ a_p^d(\mathbf{u}_f^2,\, \mathbf{u}_f^1 - \mathbf{u}_f^2) \ =: I_2,
\end{aligned} \tag{1.95}$$

where k_M is the largest eigenvalue of K.

We apply (1.20) one more time to bound the terms coming from Beavers-Joseph-Saffman condition. Set $G(\mathbf{u}) = K\nu_{eff}(\mathbf{u})\mathbf{u}$, $\mathbf{s} = \mathbf{u}_f^1 - \partial_t\boldsymbol{\eta}_p^1$ and $\mathbf{t} = \mathbf{u}_f^2 - \partial_t\boldsymbol{\eta}_p^2$, then

$$\begin{aligned}
&C\frac{\|\mathbf{u}_f^1 - \partial_t\boldsymbol{\eta}_p^1 - (\mathbf{u}_f^2 - \partial_t\boldsymbol{\eta}_p^2)\|^2_{L^r(\Gamma_{fp})}}{c + \|\mathbf{u}_f^1 - \partial_t\boldsymbol{\eta}_p^1\|^{2-r}_{L^r(\Gamma_{fp})} + \|\mathbf{u}_f^2 - \partial_t\boldsymbol{\eta}_p^2\|^{2-r}_{\Gamma_{fp}}} \\
&+C\int_{\Gamma_{fp}} \frac{1}{k_M}\,|\nu_I(\mathbf{u}_f^1 - \partial_t\boldsymbol{\eta}_p^1)(\mathbf{u}_f^1 - \partial_t\boldsymbol{\eta}_p^1) - \nu_I(\mathbf{u}_f^2 - \partial_t\boldsymbol{\eta}_p^2)(\mathbf{u}_f^2 - \partial_t\boldsymbol{\eta}_p^2)| \cdot |\mathbf{u}_f^1 - \partial_t\boldsymbol{\eta}_p^1 - (\mathbf{u}_f^2 - \partial_t\boldsymbol{\eta}_p^2)|ds \\
&\quad \leq a_{BJS}(\mathbf{u}_f^1, \partial_t\boldsymbol{\eta}_p^1; \mathbf{u}_f^1 - \mathbf{u}_f^2, \partial_t\boldsymbol{\eta}_p^1 - \partial_t\boldsymbol{\eta}_p^2) - a_{BJS}(\mathbf{u}_f^2, \partial_t\boldsymbol{\eta}_p^2; \mathbf{u}_f^1 - \mathbf{u}_f^2, \partial_t\boldsymbol{\eta}_p^1 - \partial_t\boldsymbol{\eta}_p^2) =: I_3.
\end{aligned} \tag{1.96}$$

Then from (1.25) we have

$$I_1 + I_2 + I_3 + a_p^e(\boldsymbol{\eta}_p^1 - \boldsymbol{\eta}_p^2, \partial_t\boldsymbol{\eta}_p^1 - \partial_t\boldsymbol{\eta}_p^2) = -b_f(\mathbf{u}_f^1 - \mathbf{u}_f^2, p_f^1 - p_f^2) - b_p(\mathbf{u}_p^1 - \mathbf{u}_p^2, p_p^1 - p_p^2)$$

$$-\alpha_p b_p(\partial_t\boldsymbol{\eta}_p^1 - \partial_t\boldsymbol{\eta}_p^2, p_p^1 - p_p^2) - b_\Gamma(\mathbf{u}_f^1 - \mathbf{u}_f^2, \mathbf{u}_p^1 - \mathbf{u}_p^2, \partial_t\boldsymbol{\eta}_p^1 - \partial_t\boldsymbol{\eta}_p^2; \lambda^1 - \lambda^2). \tag{1.97}$$

On the other hand, it follows from (1.26) and (1.27), with $w_f = p_f^1 - p_f^2$, $w_p = p_p^1 - p_p^2$, $\mu = \lambda^1 - \lambda^2$, that

$$\begin{aligned}(s_0\,\partial_t\left(p_p^1 - p_p^2\right), p_p^1 - p_p^2) - \alpha b_p(\partial_t\left(\boldsymbol{\eta}_p^1 - \boldsymbol{\eta}^2 - p\right), p_p^1 - p_p^2) - b_p(\mathbf{u}_p^1 - \mathbf{u}_p^2, p_p^1 - p_p^2)\\ -b_f(\mathbf{u}_f^1 - \mathbf{u}_f^2, p_f^1 - p_f^2) - b_\Gamma(\mathbf{u}_f^1 - \mathbf{u}_f^2, \mathbf{u}_p^1 - \mathbf{u}_p^2, \partial_t\left(\boldsymbol{\eta}_p^1 - \boldsymbol{\eta}_p^2\right); \lambda^1 - \lambda^2) = 0\,.\end{aligned} \tag{1.98}$$

Combining (1.97) and (1.98), we obtain

$$\begin{aligned}I_1 + I_2 + I_3 + a_p^e(\boldsymbol{\eta}_p^1 - \boldsymbol{\eta}_p^2, \partial_t\boldsymbol{\eta}_p^1 - \partial_t\boldsymbol{\eta}_p^2) &= -(s_0\,\partial_t\left(p_p^1 - p_p^2\right), p_p^1 - p_p^2)\\ &= -\frac{1}{2}\frac{d}{dt}\|p_p^1 - p_p^2\|_{L^2(\Omega_p)}\end{aligned}$$

$$\text{i.e., } \frac{1}{2}\partial_t\left(a_p^e(\boldsymbol{\eta}_p^1 - \boldsymbol{\eta}_p^2, \boldsymbol{\eta}_p^1 - \boldsymbol{\eta}_p^2) + s_0\|p_p^1 - p_p^2\|^2_{L^2(\Omega_p)}\right) + I_1 + I_2 + I_3 = 0.$$

Integrating in time from 0 to $t \in (0, T]$, and using $p_p^1(0) = p_p^2(0)$, $\boldsymbol{\eta}_p^1(0) = \boldsymbol{\eta}_p^2(0)$, we obtain

$$\frac{1}{2}\left(a_p^e(\boldsymbol{\eta}_p^1(t) - \boldsymbol{\eta}_p^2(t), \boldsymbol{\eta}_p^1(t) - \boldsymbol{\eta}_p^2(t)) + s_0\|p_p^1(t) - p_p^2(t)\|^2_{L^2(\Omega_p)}\right) + \int_0^t (I_1 + I_2 + I_3)\; ds = 0.$$

Hence, using (1.94)-(1.95), we have

$$\begin{aligned}\frac{1}{2}\left(a_p^e(\boldsymbol{\eta}_p^1(t) - \boldsymbol{\eta}_p^2(t), \boldsymbol{\eta}_p^1(t) - \boldsymbol{\eta}_p^2(t)) + s_0\|p_p^1(t) - p_p^2(t)\|^2_{L^2(\Omega_p)}\right)\\ + \int_0^t C\left(\frac{\|\mathbf{D}(\mathbf{u}_f^1) - \mathbf{D}(\mathbf{u}_f^2)\|^2_{L^2(\Omega_f)}}{c + \|\mathbf{D}(\mathbf{u}_f^1)\|^{2-r}_{L^r(\Omega_f)} + \|\mathbf{D}(\mathbf{u}_f^2)\|^{2-r}_{L^r(\Omega_f)}} + \frac{\|\mathbf{u}_p^1 - \mathbf{u}_p^2\|^2_{L^r(\Omega_p)}}{c + \|\mathbf{u}_p^1\|^{2-r}_{L^r(\Omega_p)} + \|\mathbf{u}_p^2\|^{2-r}_{L^r(\Omega_p)}}\right) ds \le 0.\end{aligned} \tag{1.99}$$

As $a_p^e(\boldsymbol{\xi}_p, \boldsymbol{\xi}_p) > 0$ for any $\boldsymbol{\xi}_p \neq 0$, it follow from (1.99) that $\mathbf{u}_f^1(t) = \mathbf{u}_f^2(t), \mathbf{u}_p^1(t) = \mathbf{u}_p^2, \boldsymbol{\eta}^1(t) = \boldsymbol{\eta}_p^2$, $\forall t \in (0, T]$.

Finally, we use the inf-sup condition (1.44) for $p_f^1 - p_f^2, p_p^1 - p_p^2, \lambda^1 - \lambda^2$ together with (1.25):

$$\begin{aligned}&\|(p_f^1 - p_f^2, p_p^1 - p_p^2, \lambda^1 - \lambda^2)\|_{W_f\times W_p\times\Lambda}\\ &\qquad\le C \sup_{(\mathbf{v}_f,\mathbf{v}_p)\in\mathbf{V}_f\times\mathbf{V}_p} \frac{b_f(\mathbf{v}_f, p_f^1 - p_f^2) + b_p(\mathbf{v}_p, p_p^1 - p^2) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, 0; \lambda^1 - \lambda^2)}{\|(\mathbf{v}_f, \mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}}\end{aligned}$$

$$= C \sup_{(\mathbf{v}_f,\mathbf{v}_p)\in\mathbf{V}_f\times\mathbf{V}_p} \Big(\frac{a_f(\mathbf{u}_f^2,\mathbf{v}_f) - a_f(\mathbf{u}_f^1,\mathbf{v}_f) + a_p^d(\mathbf{u}_p^2,\mathbf{v}_p) - a_p^d(\mathbf{u}_p^1,\mathbf{v}_p)}{\|(\mathbf{v}_f,\mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}}$$
$$+ \frac{a_{BJS}(\mathbf{u}_f^2,\partial_t\boldsymbol{\eta}_p^2;\mathbf{v}_f,0) - a_{BJS}(\mathbf{u}_f^1,\partial_t\boldsymbol{\eta}_p^1;\mathbf{v}_f,0)}{\|(\mathbf{v}_f,\mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}}\Big) = 0.$$

Therefore, for all $t \in (0,T]$, $p_f^1 = p_f^2,\ p_p^1 = p_p^2,\ \lambda^1 = \lambda^2$, and we can conclude that (1.25)-(1.27) has a unique solution. $\square$

1.3.3 Stability analysis

We will prove the following stability bound for the solution of (1.25)–(1.27).

Theorem 1.3.12. *For the solution of* (1.25)–(1.27)*, assuming sufficient regularity of the data, there exists* $C > 0$ *such that*

$$\begin{aligned}
&\|\mathbf{u}_f\|^r_{L^r(0,T;W^{1,r}(\Omega_f))} + \|\mathbf{u}_p\|^r_{L^r(0,T;L^r(\Omega_p))} + |\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p|^r_{L^r(0,T;BJS)} + \|p_f\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_f))} \\
&\quad + \|p_p\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_p))} + \|\lambda\|^{r'}_{L^{r'}(0,T;W^{1/r,r'}(\Gamma_{fp}))} + \|\boldsymbol{\eta}_p\|^2_{L^\infty(0,T;H^1(\Omega_p))} + s_0\|p_p\|^2_{L^\infty(0,T;L^2(\Omega_p))} \\
&\le C\exp(T)\Big(\|\boldsymbol{\eta}_p(0)\|^2_{H^1(\Omega_p)} + s_0\|p_p(0)\|^2_{L_2(\Omega_p)} + \|q_p\|^r_{L^r(0,T;L^r(\Omega_f))} + c(\bar{c}_f + \bar{c}_p + \bar{c}_I)\Big).
\end{aligned}$$

Proof. We first note that the term $c(\bar{c}_f + \bar{c}_p + \bar{c}_I)$ appears due to the use of the coercivity bounds in (1.45)–(1.48) in the general case $c > 0$. For simplicity, we present the proof for $c = 0$, noting that the extra term appears in (1.101) and the last inequality in the proof. We choose $(\mathbf{v}_f, w_f, \mathbf{v}_p, w_p, \boldsymbol{\xi}_p, \mu) = (\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \partial_t\boldsymbol{\eta}_p, \lambda)$ in (1.25)–(1.27) to get

$$\begin{aligned}
&\frac{1}{2}\partial_t\left[(s_0 p_p, p_p)_{\Omega_p} + a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\eta}_p)\right] + a_f(\mathbf{u}_f,\mathbf{u}_f) + a_p^d(\mathbf{u}_p,\mathbf{u}_p) + a_{BJS}(\mathbf{u}_f,\partial_t\boldsymbol{\eta}_p;\mathbf{u}_f,\partial_t\boldsymbol{\eta}_p) \\
&\quad = (q_p, p_p)_{\Omega_p}.
\end{aligned} \tag{1.100}$$

Next, we integrate (1.100) from 0 to $t \in (0,T]$ and use the coercivity bounds in (1.45)–(1.48) and note that $a_p^e(\cdot,\cdot)$ satisfies the bound

$$c_e\|\boldsymbol{\xi}_p\|^2_{H^1(\Omega_p)} \le a_p^e(\boldsymbol{\xi}_p,\boldsymbol{\xi}_p),\ a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p) \le C_e\|\boldsymbol{\eta}_p\|_{H^1(\Omega_p)}\|\boldsymbol{\xi}_p\|_{H^1(\Omega_p)}.$$

We have

$$s_0\|p_p(t)\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\eta}_p(t)\|^2_{H^1(\Omega_p)} + \int_0^t \Big(\|\mathbf{u}_f\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_p\|^r_{L^r(\Omega_p)} + |\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p|^r_{BJS}\Big)\, ds$$

$$
\begin{aligned}
&\le C\Big(\int_0^t \big((q_p, p_p)_{\Omega_p}\big)\, ds + s_0\|p_p(0)\|^2_{L_2(\Omega_p)} + \|\boldsymbol{\eta}_p(0)\|^2_{H^1(\Omega_p)}\Big) \\
&\le C\left(\|\boldsymbol{\eta}_p(0)\|^2_{H^1(\Omega_p)} + s_0\|p_p(0)\|^2_{L_2(\Omega_p)} + \int_0^t \|q_p\|^r_{L^r(\Omega_p)} ds\right) \\
&\quad + \epsilon_1 \int_0^t \left(\|p_f\|^{r'}_{L^{r'}(\Omega_f)} + \|p_p\|^{r'}_{L^{r'}(\Omega_p)}\right) ds, \qquad (1.101)
\end{aligned}
$$

using Young's inequality (1.74) for the last inequality. We next apply the inf-sup condition (1.44) for (p_f, p_p, λ) to obtain

$$
\begin{aligned}
\|(p_f, p_p, \lambda)\|_{W_f\times W_p\times\Lambda} &\le C \sup_{(\mathbf{v}_f,\mathbf{v}_p)\in\mathbf{V}_f\times\mathbf{V}_p} \frac{b_f(\mathbf{v}_f, p_f) + b_p(\mathbf{v}_p, p_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \mathbf{0}; \lambda)}{\|(\mathbf{v}_f, \mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}} \\
&= C \sup_{(\mathbf{v}_f,\mathbf{v}_p)\in\mathbf{V}_f\times\mathbf{V}_p} \frac{-a_f(\mathbf{u}_f, \mathbf{v}_f) - a_p^d(\mathbf{u}_p, \mathbf{v}_p) - a_{BJS}(\mathbf{u}_f, \partial_t\boldsymbol{\eta}_p; \mathbf{v}_f, 0)}{\|(\mathbf{v}_f, \mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}}. \qquad (1.102)
\end{aligned}
$$

Using the continuity bounds in (1.45)–(1.48), we have from (1.102),

$$
\|(p_f, p_p, \lambda)\|_{W_f\times W_p\times\Lambda} \le C\left(\|\mathbf{u}_f\|^{r/r'}_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_p\|^{r/r'}_{L^r(\Omega_p)} + |\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p|^{r/r'}_{BJS}\right),
$$

implying

$$
\begin{aligned}
&\epsilon_2 \int_0^t \big(\|p_f\|^{r'}_{L^{r'}(\Omega_f)} + \|p_p\|^{r'}_{L^{r'}(\Omega_p)} + \|\lambda\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})}\big)\, ds \\
&\qquad \le C\epsilon_2 \int_0^t \left(\|\mathbf{u}_f\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_p\|^r_{L^r(\Omega_p)} + |\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p|^r_{BJS}\right) ds. \qquad (1.103)
\end{aligned}
$$

Adding (1.101) and (1.103) and choosing ϵ_2 small enough, and then ϵ_1 small enough, implies

$$
\begin{aligned}
s_0\|p_p(t)\|^2_{L^2(\Omega_p)} &+ \|\boldsymbol{\eta}_p(t)\|^2_{H^1(\Omega_p)} + \int_0^t \left(\|\mathbf{u}_f\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_p\|^r_{L^r(\Omega_p)} + |\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p|^r_{BJS}\right) ds \\
&+ \int_0^t \left(\|p_f\|^{r'}_{L^{r'}(\Omega_f)} + \|p_p\|^{r'}_{L^{r'}(\Omega_p)} + \|\lambda\|^{r'}_{W^{1/r,r'}(\Gamma_{fp})}\right) ds \\
&\le C\Big(\|\boldsymbol{\eta}_p(0)\|^2_{H^1(\Omega_p)} + s_0\|p_p(0)\|^2_{L_2(\Omega_p)} + \int_0^t \left(\|q_p\|^r_{L^r(\Omega_p)}\right) ds\Big).
\end{aligned}
$$

The assertion of the theorem now follows from applying Gronwall's inequality. □

1.4 Well-posedness of the semidiscrete problem

We consider a shape-regular and quasi-uniform simplicial partitions $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ of Ω_f and Ω_p, respectively, not necessarily matching along the interface Γ_{fp}. We define the finite element spaces $\mathbf{V}_{f,h} \subset \mathbf{V}_f$, $W_{f,h} \subset W_f$, $\mathbf{V}_{p,h} \subset \mathbf{V}_p$, $W_{p,h} \subset W_p$ and $\mathbf{X}_{p,h} \subset \mathbf{X}_p$. We assume that $\mathbf{V}_{f,h}$, $W_{f,h}$ is any inf-sup stable pair (e.g., Taylor-Hood, MINI elements). We choose $\mathbf{V}_{p,h}$, $W_{p,h}$ to be any of well-known inf-sup stable mixed finite element spaces (e.g., Raviart-Thomas or the Brezzi-Douglas-Marini spaces). The global spaces are

$$\mathbf{V}_h = \{\mathbf{v}_h = (\mathbf{v}_{f,h}, \mathbf{v}_{p,h}) \in \mathbf{V}_{f,h} \times \mathbf{V}_{p,h}\}, \quad W_h = \{w_h = (w_{f,h}, w_{p,h}) \in W_{f,h} \times W_{p,h}\}.$$

We employ a conforming Lagrangian finite element spaces $\mathbf{X}_{p,h} \subset \mathbf{X}_p$ to approximate the structure displacement. Note that the finite element spaces $\mathbf{V}_{f,h}$, $\mathbf{V}_{p,h}$, and $\mathbf{X}_{p,h}$ satisfy the prescribed homogeneous boundary conditions on the external boundaries Γ_f and Γ_p. Finally, following [5], we choose a nonconforming approximation for the Lagrange multiplier:

$$\Lambda_h = \mathbf{V}_{p,h} \cdot \mathbf{n}_p|_{\Gamma_{fp}}.$$

We equip Λ_h with a discrete version of the $W^{1/r,r'}(\Gamma_{fp})$-norm:

$$\|\mu_h\|_{\Lambda_h} = \|\mu_h\|_{L^{r'}(\Gamma_{fp})}.$$

The semi-discrete continuous-in-time problem reads: for $t \in (0,T]$, find $(\mathbf{u}_{f,h}(t), p_{f,h}(t), \mathbf{u}_{p,h}(t), p_{p,h}(t), \boldsymbol{\eta}_{p,h}(t), \lambda_h(t)) \in L^\infty(0,T;\mathbf{V}_{f,h}) \times L^\infty(0,T;W_{f,h}) \times L^\infty(0,T;\mathbf{V}_{p,h}) \times W^{1,\infty}(0,T; W_{p,h}) \times W^{1,\infty}(0,T;\mathbf{X}_{p,h}) \times L^\infty(0,T;\Lambda_h)$, such that for all $\mathbf{v}_{f,h} \in \mathbf{V}_{f,h}$, $w_{f,h} \in W_{f,h}$, $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}$, $w_{p,h} \in W_{p,h}$, $\boldsymbol{\xi}_{p,h} \in \mathbf{X}_p$, and $\mu_h \in \Lambda_h$,

$$\begin{aligned} a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_p^d(\mathbf{u}_{p,h}, \mathbf{v}_{p,h}) + a_{BJS}(\mathbf{u}_{f,h}, \partial_t \boldsymbol{\eta}_{p,h}; \mathbf{v}_{f,h}, \boldsymbol{\xi}_{p,h}) + a_p^e(\boldsymbol{\eta}_{p,h}, \boldsymbol{\xi}_{p,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}) \\ + b_p(\mathbf{v}_{p,h}, p_{p,h}) + \alpha b_p(\boldsymbol{\xi}_{p,h}, p_{p,h}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \boldsymbol{\xi}_{p,h}; \lambda_h) = 0, \quad (1.104) \\ (s_0 \partial_t p_{p,h}, w_{p,h})_{\Omega_p} - \alpha b_p(\partial_t \boldsymbol{\eta}_{p,h}, w_{p,h}) - b_p(\mathbf{u}_{p,h}, w_{p,h}) \\ - b_f(\mathbf{u}_{f,h}, w_{f,h}) = (q_{p,h}, w_{p,h})_{\Omega_p}, \quad (1.105) \\ b_\Gamma(\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \partial_t \boldsymbol{\eta}_{p,h}; \mu_h) = 0. \quad (1.106) \end{aligned}$$

We assume that the initial conditions for the semi-discrete problem (2.22)-(2.24) are chosen as suitable approximations of $p_{p,0}$ and $\boldsymbol{\eta}_{p,0}$.

In order to prove that the semi-discrete formulation (2.22) -(2.24) is well-posed, we will follow the same strategy as in the fully continuous case. For the analysis purposes only, we consider a conforming discretization of the weak formulation (1.38)-(1.40). Let the spaces $\mathbf{V}_h$, W_h, $\mathbf{X}_{p,h}$ and Λ_h be as described above. Let $\mathbf{X}_{p,h}$ consist of polynomials of degree at most k_s, then we introduce the stress space $\Sigma_{e,h} \subset \Sigma_e$ as discontinuous symmetric polynomials of degree at most $k_s - 1$:

$$\Sigma_{e,h} = \{\boldsymbol{\sigma}_e \in \Sigma_e \big| \quad \boldsymbol{\sigma}_e|_{T \in \mathcal{T}_h^p} \in \mathcal{P}_{k_s-1}^{\mathrm{sym}}(T)\}$$

Then the corresponding semi-discrete formulation reads: for $t \in (0,T]$, find $(\mathbf{u}_{f,h}(t)$, $p_{f,h}(t)$, $\mathbf{u}_{p,h}(t)$, $p_{p,h}(t)$, $\mathbf{u}_{s,h}(t), \boldsymbol{\sigma}_{e,h}(t), \lambda_h(t)) \in L^\infty(0,T;\mathbf{V}_{f,h}) \times L^\infty(0,T;W_{f,h}) \times L^\infty(0,T;$ $\mathbf{V}_{p,h})$ $\times W^{1,\infty}(0,T;$ $W_{p,h}) \times$ $L^\infty(0,T;\mathbf{X}_{p,h})$ $\times W^{1,\infty}(0,T;\Sigma_{e,h}) \times L^\infty(0,T;\Lambda_h)$, such that for all $\mathbf{v}_{f,h} \in \mathbf{V}_{f,h}$, $w_{f,h} \in W_{f,h}$, $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}$, $w_{p,h} \in W_{p,h}$, $\mathbf{v}_{s,h} \in \mathbf{X}_{p,h}$, $\boldsymbol{\tau}_{e,h} \in \Sigma_{e,h}$, and $\mu_h \in \Lambda_h$,

$$\begin{aligned} a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_p^d(\mathbf{u}_{p,h}, \mathbf{v}_{p,h}) + a_{BJS}(\mathbf{u}_{f,h}, \mathbf{u}_{s,h}; \mathbf{v}_{f,h}, \mathbf{v}_{s,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}) + b_p(\mathbf{v}_{p,h}, p_{p,h}) \\ +\alpha_p b_p(\mathbf{v}_{s,h}, p_{p,h}) + b_s(\mathbf{v}_{s,h}, \boldsymbol{\sigma}_{e,h}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \mathbf{v}_{s,h}; \lambda_h) = 0, \quad (1.107) \\ (s_0 \partial_t p_{p,h}, w_{p,h})_{\Omega_p} + a_p^s(\partial_t \boldsymbol{\sigma}_{e,h}, \boldsymbol{\tau}_{e,h}) - \alpha_p b_p\left(\mathbf{u}_{s,h}, w_{p,h}\right) - b_p(\mathbf{u}_{p,h}, w_{p,h}) - b_s(\mathbf{u}_{s,h}, \boldsymbol{\tau}_{e,h}) \\ -b_f(\mathbf{u}_{f,h}, w_{f,h}) = (q_p, w_{p,h})_{\Omega_p}, \quad (1.108) \\ b_\Gamma\left(\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \mathbf{u}_{s,h}; \mu_h\right) = 0. \quad (1.109) \end{aligned}$$

The initial conditions $p_{p,h}(0)$ and $\boldsymbol{\sigma}_{e,h}(0)$ are suitable approximations of $p_{p,0}$ and $\boldsymbol{\sigma}_{e,0} = A^{-1}\mathbf{D}(\boldsymbol{\eta}_{p,0})$. We define the spaces of generalized velocities and pressures, $\mathbf{Q}_h = \mathbf{V}_{p,h} \times \mathbf{X}_{p,h} \times \mathbf{V}_{f,h}$ and $S_h = W_{p,h} \times \Sigma_{e,h} \times W_{f,h} \times \Lambda_h$, respectively, equipped with the corresponding norms:

$$\begin{aligned} \|\mathbf{q}_h\|_{\mathbf{Q}_h} = \|\mathbf{v}_{p,h}\|_{\mathbf{V}_p} + \|\mathbf{v}_{s,h}\|_{\mathbf{X}_{p,h}} + \|\mathbf{v}_{f,h}\|_{\mathbf{V}_f}, \|s_h\|_{S_h} \\ = \|w_{p,h}\|_{W_p} + \|\boldsymbol{\tau}_{e,h}\|_{\Sigma_e} + \|w_{f,h}\|_{W_f} + \|\mu_h\|_{\Lambda_h}. \end{aligned}$$

1.4.1 The inf-sup condition

We first recall the inf-sup conditions for the individual Stokes and Darcy problems [39]. Since $|\Gamma_p^D| > 0$, it is sufficient to consider $\mathbf{v}_{p,h} \in \mathbf{V}^0_{p,h,\Gamma_{fp}} = \{\mathbf{v}_{p,h} \in \mathbf{V}_{p,h} : \mathbf{v}_{p,h} \cdot \mathbf{n}_p\big|_{\Gamma_{fp}} = 0\}$. There exist constant $C_{p,1} > 0$ and $C_{f,1} > 0$ independent of h such that

$$\inf_{w_{p,h}\in W_{p,h}} \sup_{\mathbf{v}_{p,h}\in \mathbf{V}^0_{p,h,\Gamma_{fp}}} \frac{b_p(\mathbf{v}_{p,h}, w_{p,h})}{\|\mathbf{v}_{p,h}\|_{\mathbf{V}_p}\|w_{p,h}\|_{W_p}} \geq C_{p,1}, \quad \inf_{w_{f,h}\in W_{f,h}} \sup_{\mathbf{v}_{f,h}\in \mathbf{V}_{f,h}} \frac{b_f(\mathbf{v}_{f,h}, w_{f,h})}{\|\mathbf{v}_{f,h}\|_{\mathbf{V}_f}\|w_{f,h}\|_{W_f}} \geq C_{f,1}. \tag{1.110}$$

We next prove inf-sup condition for $b_\Gamma(\cdot;\cdot)$. We recall the mixed finite element interpolant $\Pi_{p,h}$ onto $\mathbf{V}_{p,h}$ [14], which satisfies for all $\mathbf{v}_p \in \mathbf{V}_p \cap (W^{s,r}(\Omega_p))^d$, $s > 0$,

$$(\nabla\cdot \Pi_{p,h}\mathbf{v}_p, w_{p,h})_{\Omega_p} = (\nabla\cdot \mathbf{v}_p, w_{p,h})_{\Omega_p}, \qquad \forall w_{p,h} \in W_{p,h}, \tag{1.111}$$

$$\langle \Pi_{p,h}\mathbf{v}_p\cdot\mathbf{n}_p, \mathbf{v}_{p,h}\cdot\mathbf{n}_p\rangle_{\Gamma_{fp}} = \langle \mathbf{v}_p\cdot\mathbf{n}_p, \mathbf{v}_{p,h}\cdot\mathbf{n}_p\rangle_{\Gamma_{fp}}, \qquad \forall \mathbf{v}_{p,h} \in \mathbf{V}_{p,h}, \tag{1.112}$$

as well as the continuity bound [1, 37]

$$\|\Pi_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} \leq C\left(\|\mathbf{v}_p\|_{W^{s,r}(\Omega_p)} + \|\nabla\cdot\mathbf{v}_p\|_{L^r(\Omega_p)}\right). \tag{1.113}$$

Let $\mathbf{V}^0_{p,h} = \{\mathbf{v}_{p,h} \in \mathbf{V}_{p,h} : \nabla\cdot\mathbf{v}_{p,h} = 0\}$.

Lemma 1.4.1. *There exists a constant $C_2 > 0$ independent of h such that*

$$\inf_{\mu_h\in\Lambda_h} \sup_{\mathbf{v}_{p,h}\in\mathbf{V}^0_{p,h}} \frac{b_\Gamma(\mathbf{v}_{p,h}, \mathbf{0}, \mathbf{0}; \mu_h)}{\|\mathbf{v}_{p,h}\|_{\mathbf{V}_p}\|\mu_h\|_{\Lambda_h}} \geq C_2. \tag{1.114}$$

Proof. Let $\mu_h \in \Lambda_h$ be given. Consider the auxiliary problem

$$\nabla\cdot\nabla\phi = 0, \qquad \text{in } \Omega_p, \tag{1.115}$$

$$\phi = 0 \qquad \text{on } \Gamma_p^D, \tag{1.116}$$

$$\nabla\phi\cdot\mathbf{n}_p = \mu_h^{r'-1}, \qquad \text{on } \Gamma_{fp}, \tag{1.117}$$

$$\nabla\phi\cdot\mathbf{n}_p = 0, \qquad \text{on } \Gamma_p^N. \tag{1.118}$$

Let $\mathbf{v} = \nabla\phi$. Elliptic regularity for (1.115)–(1.118) [31], [52] implies that

$$\|\mathbf{v}\|_{W^{1/r,r}(\Omega_p)} \leq C\|\mu_h^{r'-1}\|_{L^r(\Gamma_{fp})}. \tag{1.119}$$

Let $\mathbf{v}_{p,h} = \Pi_{p,h}\mathbf{v}$. Note that, due to (1.111), $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}^0$. We have

$$\frac{b_\Gamma(\mathbf{v}_{p,h}, 0, 0; \mu_h)}{\|\mathbf{v}_{p,h}\|_{\mathbf{V}_p}} = \frac{\langle \Pi_{p,h}\mathbf{v}\cdot\mathbf{n}_p, \mu_h\rangle_{\Gamma_{fp}}}{\|\Pi_{p,h}\mathbf{v}\|_{\mathbf{V}_p}} = \frac{\langle \mathbf{v}\cdot\mathbf{n}_p, \mu_h\rangle_{\Gamma_{fp}}}{\|\Pi_{p,h}\mathbf{v}\|_{\mathbf{V}_p}} = \frac{\|\mu_h\|_{L^{r'}(\Gamma_{fp})}^{r'}}{\|\Pi_{p,h}\mathbf{v}\|_{L^r(\Omega_p)}},$$

and, using (1.113) with $s = 1/r$ and (1.119),

$$\|\Pi_{p,h}\mathbf{v}\|_{L^r(\Omega_p)} \le C\|\mathbf{v}\|_{W^{1/r,r}(\Omega_p)} \le C\|\mu_h^{r'-1}\|_{L^r(\Gamma_{fp})} = C\|\mu_h\|_{L^{r'}(\Gamma_{fp})}^{r'-1}.$$

The proof is completed by combining the above two inequalities. $\square$

We next prove the inf-sup conditions for the formulation (1.107)–(1.109).

Theorem 1.4.2. *There exist constants β_1, $\beta_2 > 0$ independent of h such that*

$$\inf_{(w_{p,h},\mathbf{0},w_{f,h},\mu_h)\in S_h} \sup_{(\mathbf{v}_{p,h},\mathbf{0},\mathbf{v}_{f,h})\in\mathbf{Q}_h} \frac{b(\mathbf{q}_h; s_h) + b_\Gamma(\mathbf{q}_h; s_h)}{\|(\mathbf{v}_{p,h},\mathbf{0},\mathbf{v}_{f,h})\|_{\mathbf{Q}_h}\|(w_{p,h},0,w_{f,h},\mu_h)\|_{S_h}} \ge \beta_1, \tag{1.120}$$

$$\inf_{(\mathbf{0},\mathbf{v}_{s,h},\mathbf{0})\in\mathbf{Q}_h} \sup_{(0,\boldsymbol{\tau}_{e,h},0,0)\in S_h} \frac{b_s(\mathbf{v}_{s,h},\boldsymbol{\tau}_{e,h})}{\|(\mathbf{0},\mathbf{v}_{s,h},\mathbf{0})\|_{\mathbf{Q}}\|(0,\boldsymbol{\tau}_{e,h},0,0)\|_{S_h}} \ge \beta_2, \tag{1.121}$$

where

$$b(\mathbf{q}_h; s_h) = b_f(\mathbf{v}_{f,h}, w_{f,h}) + b_p(\mathbf{v}_{p,h}, w_{p,h}), \quad b_\Gamma(\mathbf{q}_h; s_h) = b_\Gamma(\mathbf{v}_{p,h},\mathbf{0},\mathbf{v}_{f,h};\mu_h).$$

Proof. Let $s_h = (w_{p,h},\mathbf{0},w_{f,h},\mu_h) \in S_h$ be given. It follows from (1.110) and (1.114), respectively, that there exist $\mathbf{q}_h^1 = (\mathbf{v}_{p,h}^1,\mathbf{0},\mathbf{v}_{f,h}^1) \in \mathbf{Q}_h$ with $\|\mathbf{v}_{p,h}^1\|_{\mathbf{V}_p} = 1$, $\|\mathbf{v}_{f,h}^1\|_{\mathbf{V}_f} = 1$, as well as $\mathbf{q}_h^2 = (\mathbf{v}_{p,h}^2,\mathbf{0},\mathbf{0}) \in \mathbf{Q}_h$ with $\|\mathbf{v}_{p,h}^2\|_{\mathbf{V}_p} = 1$ such that

$$b_p(\mathbf{v}_{p,h}^1, w_{p,h}) \ge \frac{C_{p,1}}{2}\|w_{p,h}\|_{W_p}, b_f(\mathbf{v}_{f,h}^1, w_{f,h}) \ge \frac{C_{f,1}}{2}\|w_{f,h}\|_{W_f}, b_\Gamma(\mathbf{v}_{p,h}^2,\mathbf{0},\mathbf{0};\mu_h) \ge \frac{C_2}{2}\|\mu_h\|_{\Lambda_h}.$$

Since $\mathbf{v}_{p,h}^1\cdot\mathbf{n}_p\big|_{\Gamma_{fp}} = 0$, we have

$$\begin{aligned} b_\Gamma(\mathbf{q}_h^1; s_h) &= \langle \mathbf{v}_{f,h}^1\cdot\mathbf{n}_f + \mathbf{v}_{p,h}^1\cdot\mathbf{n}_p, \mu_h\rangle_{\Gamma_{fp}} = \langle \mathbf{v}_{f,h}^1\cdot\mathbf{n}_f, \mu_h\rangle_{\Gamma_{fp}} \\ &\le C\|\mathbf{v}_{f,h}^1\|_{L^r(\Gamma_{fp})}\|\mu_h\|_{L^{r'}(\Gamma_{fp})} \\ &\le C\|\mathbf{v}_{f,h}^1\|_{W^{1-1/r,r}(\partial\Omega_f)}\|\mu_h\|_{L^{r'}(\Gamma_{fp})} \\ &\le C_\Gamma\|\mathbf{v}_{f,h}^1\|_{W^{1,r}(\Omega_f)}\|\mu_h\|_{L^{r'}(\Gamma_{fp})} = C_\Gamma\|\mathbf{v}_{f,h}^1\|_{\mathbf{V}_f}\|\mu_h\|_{\Lambda_h}, \end{aligned}$$

where we used the trace inequality. Let $\mathbf{r}_h = \mathbf{q}_h^1 + (1 + 2C_\Gamma C_2^{-1})\mathbf{q}_h^2$. Since $\nabla \cdot \mathbf{v}_{p,h}^2 = 0$, we obtain

$$
\begin{aligned}
b(\mathbf{r}_h; s_h) &= b_f(\mathbf{v}_{f,h}^1, w_{f,h}) + b_p(\mathbf{v}_{p,h}^1, w_{p,h}) + \left(1 + 2C_\Gamma C_2^{-1}\right) b_p(\mathbf{v}_{p,h}^2, w_{p,h}) \\
&= b_f(\mathbf{v}_{f,h}^1, w_{f,h}) + b_p(\mathbf{v}_{p,h}^1, w_{p,h}) \geq \frac{\min(C_{f,1}, C_{p,1})}{2}(\|(w_{p,h}\|_{W_p} + \|w_{f,h}\|_{W_f}), \\
b_\Gamma(\mathbf{r}_h; s_h) &= b_\Gamma(\mathbf{q}_h^1; s_h) + \left(1 + 2C_\Gamma C_2^{-1}\right) b_\Gamma(\mathbf{q}_h^2; s_h) \\
&\geq -C_\Gamma \|\mu_h\|_{\Lambda_h} + \frac{C_2}{2}\left(1 + 2C_\Gamma C_2^{-1}\right) \|\mu_h\|_{\Lambda_h} = \frac{C_2}{2}\|\mu_h\|_{\Lambda_h}.
\end{aligned}
$$

Hence, using that $\|\mathbf{r}_h\|_{\mathbf{Q}_h} \leq 3 + 2C_\Gamma C_2^{-1}$, we obtain

$$
b(\mathbf{r}_h; s_h) + b_\Gamma(\mathbf{r}_h; s_h) \geq \frac{\min(C_{f,1}, C_{p,1}, C_2)}{2}\|s_h\|_{S_h} \geq \frac{\min(C_{f,1}, C_{p,1}, C_2)}{6 + 4C_\Gamma C_2^{-1}}\|\mathbf{r}_h\|_{\mathbf{Q}_h}\|s_h\|_{S_h},
$$

which completes the proof of (1.120). To show (1.121), let $(\mathbf{0}, \mathbf{v}_{s,h}, \mathbf{0}) \in \mathbf{Q}_h$ be given. We choose $\boldsymbol{\tau}_{e,h} = \mathbf{D}(\mathbf{v}_{s,h}) \in \boldsymbol{\Sigma}_{e,h}$ and, using Korn's inequality, we obtain

$$
\frac{b_s(\mathbf{v}_{s,h}, \boldsymbol{\tau}_{e,h})}{\|\boldsymbol{\tau}_{e,h}\|_{L^2(\Omega_p)}} = \frac{\|\mathbf{D}(\mathbf{v}_{s,h})\|_{L^2(\Omega_p)}^2}{\|\mathbf{D}(\mathbf{v}_{s,h})\|_{L^2(\Omega_p)}} = \|\mathbf{D}(\mathbf{v}_{s,h})\|_{L^2(\Omega_p)} \geq \beta_2\|\mathbf{v}_{s,h}\|_{H^1(\Omega_p)}.
$$

□

1.4.2 Existence and uniqueness of the solution

To prove that the semidiscrete problem (2.22) - (2.24) has a solution, we proceed in a similar way as in the continuous case. One thing we need to have is that the initial approximation data $(p_{p,h}(0), \boldsymbol{\eta}_{p,h}(0))$, $(p_{p,h}(0), \boldsymbol{\sigma}_{e,h}(0))$ compatible. In the following, we will discuss about how to achieve the approximation data $(p_{p,h}(0), \boldsymbol{\eta}_{p,h}(0))$, $(p_{p,h}(0), \boldsymbol{\sigma}_{e,h}(0))$ that is compatible.

We define

$$
\mathcal{A}_h : \mathbf{Q} \to \mathbf{Q}_h', \quad \mathcal{B}_h : \mathbf{Q} \to S_h'
$$

to be the discrete counterparts of the operators in the continuous case. Where $\mathbf{Q}_h = \mathbf{V}_{p,h} \times \mathbf{V}_{s,h}, \times \mathbf{V}_{f,h}$, and $S_h = W_{p,h} \times \boldsymbol{\Sigma}_{e,h} \times W_{f,h} \times \Lambda_h$.

From the lemma 1.3.8, with suitable data $p_{p,0}$, there exists $\mathbf{u}_{s,0}, \mathbf{u}_{f,0}, p_{p,0}, p_{f,0}, \lambda_0$, $\boldsymbol{\sigma}_{e,0}$ such that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_0 \\ s_0 \end{pmatrix} \in \begin{pmatrix} \mathbf{Q}'_{\mathcal{E}_1} \\ S'_{\mathcal{E}_2} \end{pmatrix} \tag{1.122}$$

where $q_0 = (\mathbf{u}_{p,0}, \mathbf{u}_{s,0}, \mathbf{u}_{f,0})$, and $s_0 = (p_{p,0}, \boldsymbol{\sigma}_{e,0}, p_{f,0}, \lambda_0)$. We define $(q_{0,h}, s_{0,h})$ to be the elliptic projection of (q_0, s_0),

$$\begin{pmatrix} \mathcal{A}_h & \mathcal{B}'_h \\ -\mathcal{B}_h & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_{0,h} \\ s_{0,h} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_h & \mathcal{B}'_h \\ -\mathcal{B}_h & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_0 \\ s_0 \end{pmatrix}.$$

From there, we will have

$$\begin{pmatrix} \mathcal{A}_h & \mathcal{B}'_h \\ -\mathcal{B}_h & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_{0,h} \\ s_{0,h} \end{pmatrix} \in \begin{pmatrix} \mathbf{Q}'_{h,\mathcal{E}_1} \\ S'_{h,\mathcal{E}_2} \end{pmatrix}.$$

Similar to the continuous case, we define the domain D_h as follow, $D_h := \{(\mathbf{q}_h, s_h) \in \mathbf{Q}_h \times S_h : \left(\begin{smallmatrix} \mathcal{A}_h & \mathcal{B}'_h \\ -\mathcal{B}_h & \mathbf{0} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{q}_h \\ s_h \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathbf{Q}'_{h,\mathcal{E}_1} \\ S'_{h,\mathcal{E}_2} \end{smallmatrix}\right)\}$. Then we have $\left(\begin{smallmatrix} \mathbf{q}_{0,h} \\ s_{0,h} \end{smallmatrix}\right) \in D_h$. By doing the same argument as the continuous case, we will have $Rg\left(\left(\begin{smallmatrix} \mathcal{A}_h & \mathcal{B}'_h \\ -\mathcal{B}_h & \mathcal{E}_{h,2} \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \mathbf{Q}'_{h,\mathcal{E}_1} \\ S'_{h,\mathcal{E}_2} \end{smallmatrix}\right)$ and establish the compatibility of $(p_{p,h}(0), \boldsymbol{\sigma}_{e,h}(0))$, or $(p_{p,h}(0), \boldsymbol{\eta}_{p,h}(0))$. Hence, the theorems about the existence of solutions can be done similarly to the continuous case. We state the theorems as follow.

Theorem 1.4.3. *For each $q_f \in W^{1,1}(0,T; W'f)$, $q_p \in W^{1,1}(0,T; W'_p)$, $\boldsymbol{\sigma}_{e,0} = A^{-1}\mathbf{D}(\boldsymbol{\eta}_{p,0}) \in \Sigma_e$, $p_{p,0} \in W_p$, with compatible data $(p_{p,h}(0), \boldsymbol{\sigma}_{p,h}(0))$, there exists a solution of the alternative problem (1.107) - (1.109), with $(\mathbf{u}_{f,h},\ p_{f,h},\ \mathbf{u}_{p,h},\ p_{p,h},\ \mathbf{u}_{s,h},\ \boldsymbol{\sigma}_{e,h},\ \lambda_h) \in L^\infty(0,T; \mathbf{V}_{f,h}) \times L^\infty(0,T; W_{f,h}) \times L^\infty(0,T; \mathbf{V}_{p,h}) \times W^{1,\infty}(0,T; W_{p,h}) \times L^\infty(0,T; \mathbf{X}_{p,h}) \times W^{1,\infty}(0,T; \boldsymbol{\Sigma}_{e,h}) \times L^\infty(0,T; \Lambda_h)$.*

We already established the inf-sup conditions (1.120), (1.121) for finite element spaces, therefore the above theorem can be proved similarly as in the continuous setting. Similar to the continuous case, as a corollary of theorem 1.4.3, we have the following result.

Theorem 1.4.4. *For each $q_f \in W^{1,1}(0,T; W'_f)$, $q_p \in W^{1,1}(0,T; W'_p)$ and $p_{p,0} \in W_{p,h}$, $\boldsymbol{\eta}_{p,0} \in \mathbf{X}_{p,h}$, with compatible data $(p_{p,h}(0), \boldsymbol{\sigma}_{p,h}(0))$, there exists a unique solution $(\mathbf{u}_{f,h}(t), p_{f,h}(t), \mathbf{u}_{p,h}(t), p_{p,h}(t), \boldsymbol{\eta}_{p,h}(t), \lambda_h(t)) \in L^\infty(0,T; \mathbf{V}_{f,h}) \times L^\infty(0,T; W_{f,h}) \times L^\infty(0,T; \mathbf{V}_{p,h}) \times W^{1,\infty}(0,T; W_{p,h}) \times W^{1,\infty}(0,T; \mathbf{X}_{p,h}) \times L^\infty(0,T; \Lambda_h)$ of (2.22)–(2.24).*

The following result about stability estimate is also can be proved similar to the theorem 1.3.12.

Theorem 1.4.5. *There exists* $0 < C$ *such that the solutions of* (2.22), (2.23) *and* (2.24) *satisfy the following bound*

$$\begin{aligned}
&\|\mathbf{u}_{f,h}\|^r_{L^r(0,T;W^{1,r}(\Omega_f))} + \|\mathbf{u}_{p,h}\|^r_{L^r(0,T;L^r(\Omega_p))} + |\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}|^r_{L^r(0,T;BJS)} + \|p_{f,h}\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_f))} \\
&\quad + \|p_{p,h}\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_p))} + \|\lambda_h\|^{r'}_{L^{r'}(0,T;\Lambda_h)} + \|\boldsymbol{\eta}_{p,h}\|^2_{L^\infty(0,T;H^1(\Omega_p))} + s_0\|p_{p,h}\|^2_{L^\infty(0,T;L^2(\Omega_p))} \\
&\leq C\exp(T)\Big(\|\boldsymbol{\eta}_{p,h}(0)\|^2_{H^1(\Omega_p)} + s_0\|p_{p,h}(0)\|^2_{L_2(\Omega_p)} + \|q_p\|^r_{L^r(0,T;L^r(\Omega_f))} + c(\bar{c}_f + \bar{c}_p + \bar{c}_I)\Big).
\end{aligned}$$

1.5 Error analysis

1.5.1 Preliminaries

We introduce $Q_{f,h}$ $Q_{p,h}$ $Q_{\lambda,h}$ as the L^2 projection operators onto $W_{f,h}$, $W_{p,h}$ and Λ_h, respectively, satisfying:

$$(p_f - Q_{f,h}p_f, \psi_h)_{\Omega_f} = 0, \qquad \forall \psi_h \in W_{f,h}, \tag{1.123}$$

$$(p_p - Q_{p,h}p_p, \phi_h)_{\Omega_p} = 0, \qquad \forall \phi_h \in W_{p,h}, \tag{1.124}$$

$$\langle \lambda - Q_{\lambda,h}\lambda, \nu_h\rangle_{\Gamma_{fp}} = 0, \qquad \forall \nu_h \in \Lambda_h. \tag{1.125}$$

Our finite element spaces satisfy the approximation properties reported below [34].

Lemma 1.5.1. *Let* $Q_{f,h}, Q_{p,h}, Q_{\lambda,h}$ *be projectors, defined above. Then*

$$\|p_f - Q_{f,h}p_f\|_{L^{r'}(\Omega_f)} \leq Ch^{s_f+1}\|p_f\|_{W^{s_f+1,r'}(\Omega_f)}, \tag{1.126}$$

$$\|p_p - Q_{p,h}p_p\|_{L^{r'}(\Omega_p)} \leq Ch^{s_p+1}\|p_p\|_{W^{s_p+1,r'}(\Omega_p)}, \tag{1.127}$$

$$\|\lambda - Q_{\lambda,h}\lambda\|_{L^{r'}(\Gamma_{fp})} \leq Ch^{k_p+1}\|\lambda\|_{W^{k_p+1,r'}(\Gamma_{fp})}. \tag{1.128}$$

In order to deal with nonconformity of Lagrange multiplier approximation, we would like to use an operator $I_h = (I_{f,h}, I_{p,h}, I_{s,h}) : \mathbf{U} \to \mathbf{U}_h$, where

$$\mathbf{U} = \left\{(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p) \in \mathbf{V}_f \times \mathbf{V}_p \times \mathbf{X}_p : b_\Gamma\left(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \mu\right) = 0, \forall \mu \in \Lambda\right\}$$

and $\mathbf{U}_h$ is its discrete analogue.

We recall that the MFE interpolant $\Pi_{p,h}$ satisfies for all $\mathbf{v}_p \in \mathbf{V}_p \cap (W^{1,r}(\Omega_p))^d$

$$(\nabla \cdot \Pi_{p,h}\mathbf{v}_p, \psi_h)_{\Omega_p} = (\nabla \cdot \mathbf{v}_p, \psi_h)_{\Omega_p}, \qquad \forall \psi_h \in W_{p,h}, \tag{1.129}$$

$$\langle \Pi_{p,h}\mathbf{v} \cdot \mathbf{n}_p, \psi_h \rangle_{\Gamma_{fp}} = \langle \mathbf{v}_p \cdot \mathbf{n}_p, \psi_h \rangle_{\Gamma_{fp}}, \qquad \forall \psi_h \in W_{p,h}. \tag{1.130}$$

We will make use of the following bounds on $\Pi_{p,h}$ [1, 37]:

$$\|\mathbf{v}_p - \Pi_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} \le Ch^{k_p+1}\|\mathbf{v}_p\|_{W^{k_p+1,r}(\Omega_p)}, \tag{1.131}$$

$$\|\Pi_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} \le C\left(\|\mathbf{v}_p\|_{L^r(\Omega_p)} + h\|\nabla \mathbf{v}_p\|_{L^r(\Omega_p)}\right). \tag{1.132}$$

We also consider $S_{f,h}$, $S_{s,h}$ to be the Scott-Zhang interpolation operators onto $\mathbf{V}_{f,h}$ and $\mathbf{X}_{p,h}$, respectively, satisfying [75]

$$\|\mathbf{v}_f - S_{f,h}\mathbf{v}_p\|_{L^r(\Omega_f)} + h\|\nabla(\boldsymbol{\xi}_p - S_{s,h}\boldsymbol{\xi}_p)\|_{L^r(\Omega_f)} \le Ch^{k_f+1}\|\boldsymbol{\xi}_p\|_{W^{k_f+1,r}(\Omega_f)}, \tag{1.133}$$

$$\|\boldsymbol{\xi}_p - S_{s,h}\boldsymbol{\xi}_p\|_{L^2(\Omega_p)} + h\|\nabla(\boldsymbol{\xi}_p - S_{s,h}\boldsymbol{\xi}_p)\|_{L^2(\Omega_p)} \le Ch^{k_s+1}\|\boldsymbol{\xi}_p\|_{H^{k_s+1}(\Omega_p)}. \tag{1.134}$$

Using the construction from [50], we set $I_{f,h} = S_{f,h} + \delta_{f,h}$ and $I_{s,h} = S_{s,h} + \delta_{s,h}$, where the corrections $\delta_{f,h}$, $\delta_{s,h}$ are such that

$$\int_{\Gamma_{fp}} (I_{f,h}\mathbf{v}_f - \mathbf{v}_f) \cdot \mathbf{n}_f = \int_{\Gamma_{fp}} \left(I_{s,h}\boldsymbol{\xi}_p - \boldsymbol{\xi}_p\right) \cdot \mathbf{n}_p = 0. \tag{1.135}$$

To construct $I_{p,h}$, we first consider an axillary problem:

$$\nabla \cdot \nabla \phi = 0, \qquad \text{in } \Omega_p, \tag{1.136}$$

$$\phi = 0 \qquad \text{on } \Gamma_p^D, \tag{1.137}$$

$$\nabla \phi \cdot \mathbf{n}_p = (\mathbf{v}_f - I_{f,h}\mathbf{v}_f) \cdot \mathbf{n}_f + \partial_t(\boldsymbol{\xi}_p - I_{s,h}\boldsymbol{\xi}_p) \cdot \mathbf{n}_p, \qquad \text{on } \Gamma_{fp}, \tag{1.138}$$

$$\nabla \phi \cdot \mathbf{n}_p = 0, \qquad \text{on } \Gamma_p^N. \tag{1.139}$$

Let $\mathbf{z} = \nabla\phi$ and define $\mathbf{w} = \mathbf{z} + \mathbf{v}_p$ for any test function $\mathbf{v}_p \in \mathbf{V}_p$. Since $\nabla \cdot \mathbf{z} = 0$, we notice that $\mathbf{w} \in \mathbf{V}_p$ too. Then, thanks to our construction,

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{z} + \nabla \cdot \mathbf{v}_p = \nabla \cdot \mathbf{v}_p, \qquad \text{in } \Omega_p \tag{1.140}$$

$$\begin{aligned} \mathbf{w} \cdot \mathbf{n}_p &= \mathbf{z} \cdot \mathbf{n}_p + \mathbf{v}_p \cdot \mathbf{n}_p \\ &= \mathbf{v}_p \cdot \mathbf{n}_p + \mathbf{v}_f \cdot \mathbf{n}_f - I_{f,h}\mathbf{v}_f \cdot \mathbf{n}_f + \partial_t \boldsymbol{\xi}_p \cdot \mathbf{n}_p - \partial_t I_{s,h}\boldsymbol{\xi}_p \cdot \mathbf{n}_p, \qquad \text{on } \Gamma_{fp}. \end{aligned} \tag{1.141}$$

Finally, we define $I_{p,h}\mathbf{v}_p$ as the MFE interpolant of $\mathbf{w}$:

$$I_{p,h}\mathbf{v}_p = \Pi_{p,h}\mathbf{w} = \Pi_{p,h}\mathbf{v}_p + \Pi_{p,h}\mathbf{z}. \tag{1.142}$$

Since $\Pi_{p,h}$ satisfies (1.129), we have

$$(\nabla \cdot I_{p,h}\mathbf{v}_p, w_{p,h})_{\Omega_p} = (\nabla \cdot \Pi_{p,h}\mathbf{w}, w_{p,h})_{\Omega_p} = (\nabla \cdot \mathbf{w}, w_{p,h})_{\Omega_p} = (\nabla \cdot \mathbf{v}_p, w_{p,h})_{\Omega_p}, \quad \forall w_{p,h} \in W_{p,h}. \tag{1.143}$$

Moreover, using (1.141) and (1.130), we also have for all $\mu_h \in \Lambda_h$

$$\begin{aligned} \langle I_{p,h}\mathbf{v}_p \cdot \mathbf{n}_p, \mu_h \rangle_{\Gamma_{fp}} &= \langle \Pi_{p,h}\mathbf{w} \cdot \mathbf{n}_p, \mu_h \rangle_{\Gamma_{fp}} \\ &= \left\langle \mathbf{v}_p \cdot \mathbf{n}_p + \mathbf{v}_f \cdot \mathbf{n}_f - I_{f,h}\mathbf{v}_f \cdot \mathbf{n}_f + \partial_t \boldsymbol{\xi}_p \cdot \mathbf{n}_p - \partial_t I_{s,h}\boldsymbol{\xi}_p \cdot \mathbf{n}_p, \mu_h \right\rangle_{\Gamma_{fp}}. \end{aligned}$$

Rearranging the terms, we obtain

$$\left\langle I_{p,h}\mathbf{v}_p \cdot \mathbf{n}_p + \partial_t I_{s,h}\boldsymbol{\xi}_p \cdot \mathbf{n}_p + I_{f,h}\mathbf{v}_f \cdot \mathbf{n}_f, \mu_h \right\rangle_{\Gamma_{fp}} = \left\langle \mathbf{v}_p \cdot \mathbf{n}_p + \partial_t \boldsymbol{\xi}_p \cdot \mathbf{n}_p + \mathbf{v}_f \cdot \mathbf{n}_f, \mu_h \right\rangle_{\Gamma_{fp}}.$$

Therefore, $I_h : \mathbf{U} \mapsto \mathbf{U}_h$ satisfies

$$\left\langle I_{p,h}\mathbf{v}_p \cdot \mathbf{n}_p + \partial_t I_{s,h}\boldsymbol{\xi}_p \cdot \mathbf{n}_p + I_{f,h}\mathbf{v}_f \cdot \mathbf{n}_f, \mu_h \right\rangle_{\Gamma_{fp}} = 0, \qquad \forall \mu_h \in \Lambda_h. \tag{1.144}$$

As our construction is complete, we summarize the properties of all components of I_h in the following lemma.

Lemma 1.5.2. *For all* $(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p) \in \mathbf{U}$*, the interpolation operator* $I_h : \mathbf{U} \to \mathbf{U}_h$ *satisfies*

$$\|\mathbf{v}_f - I_{f,h}\mathbf{v}_f\|_{L^r(\Omega_f)} + h\|\nabla(\mathbf{v}_f - I_{f,h}\mathbf{v}_f)\|_{L^r(\Omega_f)} \le Ch^{k_f+1}\|\mathbf{v}_f\|_{W^{k_f+1,r}(\Omega_f)}, \tag{1.145}$$

$$\|\boldsymbol{\xi}_p - I_{s,h}\boldsymbol{\xi}_p\|_{L^2(\Omega_p)} + h\|\nabla(\boldsymbol{\xi}_p - I_{s,h}\boldsymbol{\xi}_p)\|_{L^2(\Omega_p)} \le Ch^{k_s+1}\|\boldsymbol{\xi}_p\|_{H^{k_s+1}(\Omega_p)}, \tag{1.146}$$

$$\|\mathbf{v}_p - I_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} \le C\left(h^{k_p+1}\|\mathbf{v}_p\|_{W^{k_p+1,r}(\Omega_p)} + h^{k_f}\|\mathbf{v}_f\|_{W^{k_f+1,r}(\Omega_f)} + h^{k_s}\|\partial_t\boldsymbol{\xi}_p\|_{H^{k_s+1}(\Omega_p)}\right). \tag{1.147}$$

Proof. The first two estimates (2.37)-(1.146) follow immediately from (1.133)-(1.134) and the fact that the corrections $\delta_{f,h}$, $\delta_{s,h}$ are of optimal order [50].

Next, by (1.142),

$$\|\mathbf{v}_p - I_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} = \|\mathbf{v}_p - \Pi_{p,h}\mathbf{v}_p - \Pi_{p,h}\mathbf{z}\|_{L^r(\Omega_p)} \le \|\mathbf{v}_p - \Pi_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} + \|\Pi_{p,h}\mathbf{z}\|_{L^r(\Omega_p)}.$$

Recall that $z = \nabla\phi$, where ϕ is the solution of (1.139). Therefore, by the elliptic regularity [52] and (1.132)

$$\begin{aligned}
\|\Pi_{p,h}\mathbf{z}\|_{L^r(\Omega_p)} &\le C\|\mathbf{z}\|_{W^{1,r}(\Omega_p)} \le C\|(\mathbf{v}_f - I_{f,h}\mathbf{v}_f)\cdot\mathbf{n}_f + \partial_t\left(\boldsymbol{\xi}_p - I_{s,h}\boldsymbol{\xi}_p\right)\cdot\mathbf{n}_p\|_{W^{1-1/r,r}(\Gamma_{fp})} \\
&\le C\left(\|(\mathbf{v}_f - I_{f,h}\mathbf{v}_f)\cdot\mathbf{n}_f\|_{W^{1-1/r,r}(\Gamma_{fp})} + \|\partial_t\left(\boldsymbol{\xi}_p - I_{s,h}\boldsymbol{\xi}_p\right)\cdot\mathbf{n}_p\|_{W^{1-1/r,r}(\Gamma_{fp})}\right) \\
&\le C\left(\|\mathbf{v}_f - I_{f,h}\mathbf{v}_f\|_{W^{1,r}(\Omega_f)} + \|\partial_t\left(\boldsymbol{\xi}_p - I_{s,h}\boldsymbol{\xi}_p\right)\|_{H^1(\Omega_p)}\right) \\
&\le C\left(h^{k_f}\|\mathbf{v}_f\|_{W^{k_f+1,r}(\Omega_f)} + h^{k_s}\|\partial_t\boldsymbol{\xi}_p\|_{H^{k_s+1}(\Omega_p)}\right).
\end{aligned}$$

Finally, by (1.131)

$$\begin{aligned}
\|\mathbf{v}_p - I_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} &\le \|\mathbf{v}_p - \Pi_{p,h}\mathbf{v}_p\|_{L^r(\Omega_p)} + \|\Pi_{p,h}\mathbf{z}\|_{L^r(\Omega_p)} \\
&\le Ch^{k_p+1}\|\mathbf{v}_p\|_{W^{k_p+1,r}(\Omega_p)} + C\left(h^{k_f}\|\mathbf{v}_f\|_{W^{k_f+1,r}(\Omega_f)} + h^{k_s}\|\partial_t\boldsymbol{\xi}_p\|_{H^{k_s+1}(\Omega_p)}\right).
\end{aligned}$$

□

1.5.2 Error estimates

For $\mathbf{u} = (\mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\eta}_p)$ and $\mathbf{u}_h = (\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \boldsymbol{\eta}_{p,h})$, let's define

$$
\begin{aligned}
\mathcal{E}(\mathbf{u}, \mathbf{u}_h) &= \left\| \frac{|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|}{c + |\mathbf{D}(\mathbf{u}_f)| + |\mathbf{D}(\mathbf{u}_{f,h})|} \right\|_{L^\infty(\Omega_f)}^{\frac{2-r}{r}} + \left\| \frac{|\mathbf{u}_p - \mathbf{u}_{p,h}|}{c + |\mathbf{u}_p| + |\mathbf{u}_{p,h}|} \right\|_{L^\infty(\Omega_p)}^{\frac{2-r}{r}} \\
&+ \left\| \frac{|(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j} - (\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}) \cdot \mathbf{t}_{f,j}|}{c + |(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j}| + |(\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}) \cdot \mathbf{t}_{f,j}|} \right\|_{L^\infty(\Gamma_{fp})}^{\frac{2-r}{r}} \quad \text{and} \\
\mathcal{G}(\mathbf{u}, \mathbf{u}_h) &= \int_{\Omega_f} |\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - \nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h})||\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})| dA \\
&+ \int_{\Omega_p} (1/k_M)|\nu_{eff}(\mathbf{u}_p)\mathbf{u}_p - \nu_{eff}(\mathbf{u}_{p,h})\mathbf{u}_{p,h}||\mathbf{u}_p - \mathbf{u}_{p,h}| dA \\
&+ \int_{\Gamma_{fp}} \frac{a_{BJS}}{k_M^{1/2}} |\nu_I(((\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j}((\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j} \\
&- \nu_I(((\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j})((\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j}| \\
&\cdot |((\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j} - ((\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h}) \cdot \mathbf{t}_{f,j})\mathbf{t}_{f,j}| ds
\end{aligned} \tag{1.148}
$$

Theorem 1.5.3. *Let* $(\mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\eta}_p, p_f, p_p, \lambda)$ *be the solution of* (1.25)-(1.27) *and* $(\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \boldsymbol{\eta}_{p,h}, p_{f,h}, p_{p,h}, \lambda_h)$ *be the solution of* (2.22)-(2.24)*. There exists a constant* $C > 0$ *such that*

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{L^2(0,T;W^{1,r}(\Omega_f))} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|^2_{L^2(0,T;L^r(\Omega_p))} + \sum_{j=1}^{d-1} |\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p - (\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h})|^2_{L^2(0,T;BJS,j)} \\
&+ \|p_f - p_{f,h}\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_f))} + \|p_p - p_{p,h}\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_p))} + \|Q_{\lambda,h}\lambda - \lambda_h\|^{r'}_{L^{r'}(0,T;L^{r'}(\Gamma_{fp}))} \\
&+ \|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|^2_{L^\infty(0,T;H^1(\Omega_p))} + s_0\|p_p - p_{p,h}\|^2_{L^\infty(0,T;L^2(\Omega_p))} + \|\mathcal{G}(\mathbf{u}, \mathbf{u}_h)\|_{L^1(0,T)} \\
&\le C \exp(T) \Big[h^{2k_f} \|\mathbf{u}_f\|^2_{L^2(0,T;W^{k_f+1,r}(\Omega_f))} + h^{rk_f} \|\mathbf{u}_f\|^r_{L^r(0,T;W^{k_f+1,r}(\Omega_f))} \\
&+ h^{2(s_f+1)} \|p_f\|^2_{L^2(0,T;W^{s_f+1,r'}(\Omega_f))} + h^{r'(s_f+1)} \|p_f\|^{r'}_{L^{r'}(0,T;W^{s_f+1,r'}(\Omega_f))} \\
&+ h^{r(k_p+1)} \|\mathbf{u}_p\|^r_{L^r(0,T;W^{k_p+1,r}(\Omega_p))} + h^{r'(s_p+1)} \|p_p\|^{r'}_{L^{r'}(0,T;W^{s_p+1,r'}(\Omega_p))} \\
&+ h^{2(s_p+1)} \left(\|\partial_t p_p\|^2_{L^2(0,T;W^{s_p+1,r'}(\Omega_p))} + \|p_p\|^2_{L^\infty(0,T;W^{s_p+1,r'}(\Omega_p))} \right) \\
&+ h^{2k_s} \left(\|\boldsymbol{\eta}_p\|^2_{L^2(0,T;H^{k_s+1}(\Omega_p))} + \|\partial_t \boldsymbol{\eta}_p\|^2_{L^2(0,T;H^{k_s+1}(\Omega_p))} + \|\boldsymbol{\eta}_p\|^2_{L^\infty(0,T;H^{k_s+1}(\Omega_p))} \right) \\
&+ h^{rk_s} \|\partial_t \boldsymbol{\eta}_p\|^r_{L^r(0,T;H^{k_s+1}(\Omega_p))} + h^{r'(k_p+1)} \|\lambda\|^{r'}_{L^{r'}(0,T;W^{k_p+1,r'}(\Gamma_{fp}))} \\
&+ h^{2(k_p+1)} \left(\|\lambda\|^2_{L^2(0,T;W^{k_p+1,r'}(\Gamma_{fp}))} + \|\partial_t \lambda\|^2_{L^2(0,T;W^{k_p+1,r'}(\Gamma_{fp}))} + \|\lambda\|^2_{L^\infty(0,T;W^{k_p+1,r'}(\Gamma_{fp}))} \right) \\
&+ \|\boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_{p,h}(0)\|^2_{H^1(\Omega_p)} + \|p_p(0) - p_{p,h}(0)\|^2_{L^{r'}(\Omega_p)} \Big].
\end{aligned} \tag{1.149}
$$

Proof. At first, we will achieve a bound for for $\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{W^{1,r}(\Omega_f)}$ and $\|\mathbf{u}_p - \mathbf{u}_{p,h}\|_{L^r(\Omega_p)}$ using the the monotonicity (1.20) and continuity (1.21) assumptions.

Using (1.20) with $\mathbf{G}(\mathbf{x}) = \nu(\mathbf{x})\mathbf{x}$, $\mathbf{s} = \mathbf{D}(\mathbf{u}_f)$ and $\mathbf{t} = \mathbf{D}(\mathbf{u}_{f,h})$:

$$\begin{aligned}
C\Big(&\frac{\|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})\|^2_{L^r(\Omega_f)}}{c + \|\mathbf{D}(\mathbf{u}_f)\|^{2-r}_{L^r(\Omega_f)} + \|\mathbf{D}(\mathbf{u}_{f,h})\|^{2-r}_{L^r(\Omega_f)}} \\
&\qquad + (|\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - \nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h})|, |\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|_{\Omega_f}\Big) \\
&\leq (2\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - 2\nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h}), \mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h}))_{\Omega_f} \qquad (1.150) \\
&= (2\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - 2\nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h}), \mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{v}_{f,h}))_{\Omega_f} \\
&\qquad + (2\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - 2\nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h}), \mathbf{D}(\mathbf{v}_{f,h}) - \mathbf{D}(\mathbf{u}_{f,h}))_{\Omega_f} \\
&=: J_1 + J_2, \qquad \forall \mathbf{v}_{f,h} \in \mathbf{V}_{f,h}, \qquad (1.151)
\end{aligned}$$

where we used the factor 2ν in (1.150) in order that the term J_2 may be expressed in terms of $a_f(\cdot,\cdot)$. The term J_1 can be bounded using (1.21) with $\mathbf{s} = \mathbf{D}(\mathbf{u}_f)$, $\mathbf{t} = \mathbf{D}(\mathbf{u}_{f,h})$ and $\mathbf{w} = \mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{v}_{f,h})$:

$$\begin{aligned}
J_1 &\leq C\left(|\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - \nu(\mathbf{D}(u_{f,h}))\mathbf{D}(\mathbf{u}_{f,h})|, |\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|\right)^{1/r'}_{\Omega_f} \\
&\qquad \times \left\| \frac{|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|}{c + |\mathbf{D}(\mathbf{u}_f)| + |\mathbf{D}(\mathbf{u}_{f,h})|} \right\|^{\frac{2-r}{r}}_{L^\infty(\Omega_f)} \|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{v}_{f,h})\|_{L^r(\Omega_f)} \\
&\leq \epsilon(|\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - \nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h})|, |\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|)_{\Omega_f} \\
&\qquad + C\left\| \frac{|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|}{c + |\mathbf{D}(\mathbf{u}_f)| + |\mathbf{D}(\mathbf{u}_{f,h})|} \right\|^{2-r}_{L^\infty(\Omega_f)} \|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{v}_{f,h})\|^r_{L^r(\Omega_f)}, \qquad (1.152)
\end{aligned}$$

where we used Young's inequality (1.74). We choose ϵ small enough and combine (1.151)–(1.152) to obtain

$$\begin{aligned}
&\frac{\|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})\|^2_{L^r(\Omega_f)}}{c + \|\mathbf{D}(\mathbf{u}_f)\|^{2-r}_{L^r(\Omega_f)} + \|\mathbf{D}(\mathbf{u}_{f,h})\|^{2-r}_{L^r(\Omega_f)}} \\
&\quad + (|\nu(\mathbf{D}(\mathbf{u}_f))\mathbf{D}(\mathbf{u}_f) - \nu(\mathbf{D}(\mathbf{u}_{f,h}))\mathbf{D}(\mathbf{u}_{f,h})|, |\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|)_{\Omega_f} \\
&\qquad \leq C\left(\left\| \frac{|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{u}_{f,h})|}{c + |\mathbf{D}(\mathbf{u}_f)| + |\mathbf{D}(\mathbf{u}_{f,h})|} \right\|^{2-r}_{L^\infty(\Omega_f)} \|\mathbf{D}(\mathbf{u}_f) - \mathbf{D}(\mathbf{v}_{f,h})\|^r_{L^r(\Omega_f)} + J_2\right). \qquad (1.153)
\end{aligned}$$

Similarly, to bound the error in the Darcy velocity we use (1.20) and (1.21) with $\mathbf{G}(\mathbf{x}) = \nu_{eff}(\mathbf{x})\mathbf{x}$, $\mathbf{s} = \mathbf{u}_p$, $\mathbf{t} = \mathbf{u}_{p,h}$, and $\mathbf{w} = \mathbf{u}_p - \mathbf{v}_{p,h}$, $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}$, to obtain

$$\begin{aligned}\frac{\|\mathbf{u}_p - \mathbf{u}_{p,h}\|^2_{L^r(\Omega_p)}}{c + \|\mathbf{u}_p\|^{2-r}_{L^r(\Omega_p)} + \|\mathbf{u}_{p,h}\|^{2-r}_{L^r(\Omega_p)}} &+ (|\nu_{eff}(\mathbf{u}_p)\mathbf{u}_p - \nu_{eff}(\mathbf{u}_{p,h})\mathbf{u}_{p,h}|, |\mathbf{u}_p - \mathbf{u}_{p,h}|)_{\Omega_p} \\ &\leq C\left(\left\|\frac{|\mathbf{u}_p - \mathbf{u}_{p,h}|}{c + |\mathbf{u}_p| + |\mathbf{u}_{p,h}|}\right\|^{2-r}_{L^\infty(\Omega_p)} \|\mathbf{u}_p - \mathbf{v}_{p,h}\|^r_{L^r(\Omega_p)} + J_4\right),\end{aligned} \tag{1.154}$$

where

$$J_4 := (\nu_{eff}(\mathbf{u}_p)\kappa^{-1}\mathbf{u}_p - \nu_{eff}(\mathbf{u}_{p,h})\kappa^{-1}\mathbf{u}_{p,h}, \mathbf{v}_{p,h} - \mathbf{u}_{p,h})_{\Omega_p}.$$

The factor κ^{-1} is introduced in the definition of J_4 in order that it may be expressed in terms of $a_p^d(\cdot,\cdot)$. Similarly, to bound the terms coming from the BJS condition, we set in (1.20) and (1.21), $\mathbf{G}(\mathbf{x}) = \nu_I(\mathbf{x})\mathbf{x}$, $\mathbf{s} = ((\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j}$, $\mathbf{t} = ((\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j}$ and $\mathbf{w} = ((\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j} - ((\mathbf{v}_{f,h} - \boldsymbol{\xi}_{p,h})\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j}$, $\mathbf{v}_{f,h} \in \mathbf{V}_{f,h}$, $\boldsymbol{\xi}_{p,h} \in \mathbf{X}_{p,h}$, to obtain

$$\begin{aligned}C\sum_{j=1}^{d-1} &\frac{\|(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j} - (\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j}\|^2_{L^r(\Gamma_{fp})}}{c + \|(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j}\|^{2-r}_{L^r(\Gamma_{fp})} + \|(\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j}\|^{2-r}_{L^r(\Gamma_{fp})}} \\ &+ C\sum_{j=1}^{d-1}\alpha_{BJS}\langle|\nu_I(((\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j})((\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j} \\ &\qquad - \nu_I(((\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j})((\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j}|, \\ &\qquad |((\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j} - ((\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j}|\rangle_{\Gamma_{fp}} \\ &\leq \sum_{j=1}^{d-1}\left\|\frac{|(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j} - (\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j}|}{c + |(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j}| + |(\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j}|}\right\|^{2-r}_{L^\infty(\Gamma_{fp})} \\ &\qquad \times \|(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j} - (\mathbf{v}_{f,h} - \boldsymbol{\xi}_{p,h})\cdot\mathbf{t}_{f,j}\|^r_{L^r(\Gamma_{fp})} + J_6,\end{aligned} \tag{1.155}$$

where

$$\begin{aligned}J_6 := \sum_{j=1}^{d-1}&\alpha_{BJS}\langle\sqrt{\kappa^{-1}}(\nu_I(((\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j})(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p)\cdot\mathbf{t}_{f,j} \\ &- \nu_I(((\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j})\mathbf{t}_{f,j})(\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j}), (\mathbf{v}_{f,h} - \boldsymbol{\xi}_{p,h})\cdot\mathbf{t}_{f,j} \\ &- (\mathbf{u}_{f,h} - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}}.\end{aligned}$$

Combining (1.153)–(1.155) together with the regularity of the solution from Theorems 1.3.11 and 1.4.4, we obtain

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|^2_{L^r(\Omega_p)} + |(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) - (\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h})|^2_{BJS} + \mathcal{G}(\mathbf{u}, \mathbf{u}_h) \\
&\quad \le C[\mathcal{E}(\mathbf{u}, \mathbf{u}_h)^r (\|\mathbf{u}_f - \mathbf{v}_{f,h}\|^r_{W^{1,r}(\Omega_f)} + \|\mathbf{u}_p - \mathbf{v}_{p,h}\|^r_{L^r(\Omega_p)} + \|\partial_t \boldsymbol{\eta}_p - \boldsymbol{\xi}_{p,h}\|^r_{H^1(\Omega_p)}) \\
&\quad + J_2 + J_4 + J_6],
\end{aligned} \tag{1.156}
$$

where we used the trace inequality. To bound the last three terms above, note that

$$
\begin{aligned}
J_2 &= a_f(\mathbf{u}_f, \mathbf{v}_{f,h} - \mathbf{u}_{f,h}) - a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h} - \mathbf{u}_{f,h}), \\
J_4 &= a_p^d(\mathbf{u}_p, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}) - a_p^d(\mathbf{u}_{p,h}, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}), \\
J_6 &= a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_{f,h} - \mathbf{u}_{f,h}, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}) - a_{BJS}(\mathbf{u}_{f,h}, \partial_t \boldsymbol{\eta}_{p,h}; \mathbf{v}_{f,h} - \mathbf{u}_{f,h}, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}).
\end{aligned}
$$

Second, we will get a bound of $\|\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_{p,h}(t)\|_{H^1(\Omega_p)}$ and $\|p_p(t) - p_{p,h}(t)\|_{L^2(\Omega_p)}$.

We subtract (2.22) from (1.25) and test with $(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h})$, to obtain

$$
\begin{aligned}
&J_2 + J_4 + J_6 = a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}) \\
&+ b_f(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, p_{f,h} - p_f) + \alpha b_p(\boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}, p_{p,h} - p_p) \\
&\quad + b_p(\mathbf{v}_{p,h} - \mathbf{u}_{p,h}, p_{p,h} - p_p) + b_\Gamma(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}; \lambda_h - \lambda) \\
&\quad = a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_p) + a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, \partial_t \boldsymbol{\eta}_p - \partial_t \boldsymbol{\eta}_{p,h}) + b_f(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, p_{f,h} - Q_{f,h} p_f) \\
&\quad + b_f(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, Q_{f,h} p_f - p_f) + \alpha b_p(\boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}, p_{p,h} - Q_{p,h} p_p) \\
&\quad + \alpha b_p(\boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}, Q_{p,h} p_p - p_p) \\
&\quad + b_p(\mathbf{v}_{p,h} - \mathbf{u}_{p,h}, p_{p,h} - Q_{p,h} p_p) + b_p(\mathbf{v}_{p,h} - \mathbf{u}_{p,h}, Q_{p,h} p_p - p_p) \\
&\quad + b_\Gamma(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}; \lambda_h - Q_{\lambda,h} \lambda) \\
&\quad + b_\Gamma(\mathbf{v}_{f,h} - \mathbf{u}_{f,h}, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}, \boldsymbol{\xi}_{p,h} - \partial_t \boldsymbol{\eta}_{p,h}; Q_{\lambda,h} \lambda - \lambda).
\end{aligned} \tag{1.157}
$$

Since $\nabla \cdot \mathbf{V}_{p,h} = W_{p,h}$ and $\mathbf{V}_{p,h} \cdot \mathbf{n}_p|_{\Gamma_{fp}} = \Lambda_h$, (2.32) and (2.33) imply that

$$
b_p(\mathbf{v}_{p,h} - \mathbf{u}_{p,h}, Q_{p,h} p_p - p_p) = 0, \quad b_\Gamma(0, \mathbf{v}_{p,h} - \mathbf{u}_{p,h}, 0; Q_{\lambda,h} \lambda - \lambda) = 0.
$$

Now we take $(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \boldsymbol{\xi}_{p,h}) = (I_{f,h}\mathbf{u}_f, I_{p,h}\mathbf{u}_p, I_{s,h}\partial_t\boldsymbol{\eta}_p)$. Then (1.157) can be written as follows:

$$
\begin{aligned}
J_2 + J_4 + J_6 + a_p^e(\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}, \partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}) &= a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p) \\
&+ b_f(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, p_{f,h} - Q_{f,h}p_f) + b_f(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, Q_{f,h}p_f - p_f) \\
&+ \alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, p_{p,h} - Q_{p,h}p_p) + \alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_p) \\
&+ b_\Gamma(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, I_{p,h}\mathbf{u}_p - \mathbf{u}_{p,h}, I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}; \lambda_h - Q_{\lambda,h}\lambda) + b_p(I_{p,h}\mathbf{u}_p - \mathbf{u}_{p,h}, p_{p,h} - Q_{p,h}p_p) \\
&+ b_\Gamma(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, 0, I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}; Q_{\lambda,h}\lambda - \lambda). \quad (1.158)
\end{aligned}
$$

Note that due to (2.24) and (1.144), we have

$$
b_\Gamma(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, I_{p,h}\mathbf{u}_p - \mathbf{u}_{p,h}, I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}; \lambda_h - Q_{\lambda,h}\lambda) = 0. \quad (1.159)
$$

We next subtract (2.23) from (1.26) with the choice $(w_{f,h}, w_{p,h}) = (Q_{f,h}p_f - p_{f,h}, Q_{p,h}p_p - p_{p,h})$:

$$
\begin{aligned}
& s_0(\partial_t p_p - Q_{p,h}\partial_t p_p, Q_{p,h}p_p - p_{p,h})_{\Omega_p} + s_0(Q_{p,h}\partial_t p_p - \partial_t p_{p,h}, Q_{p,h}p_p - p_{p,h})_{\Omega_p} \\
&\quad - \alpha b_p(\partial_t\boldsymbol{\eta}_p - I_{s,h}\partial_t\boldsymbol{\eta}_p, Q_{p,h}p_p - p_{p,h}) - \alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_{p,h}) \\
&\quad - b_p(\mathbf{u}_p - I_{p,h}\mathbf{u}_p, Q_{p,h}p_p - p_{p,h}) - b_p(I_{p,h}\mathbf{u}_p - \mathbf{u}_{p,h}, Q_{p,h}p_p - p_{p,h}) \\
&\quad - b_f(\mathbf{u}_f - I_{f,h}\mathbf{u}_f, Q_{f,h}p_f - p_{f,h}) - b_f(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, Q_{f,h}p_f - p_{f,h}) = 0. \quad (1.160)
\end{aligned}
$$

By (2.32) and (2.30), we have

$$
s_0(\partial_t p_p - Q_{p,h}\partial_t p_p, Q_{p,h}p_p - p_{p,h})_{\Omega_p} = 0, \quad b_p(\mathbf{u}_p - I_{p,h}\mathbf{u}_p, Q_{p,h}p_p - p_{p,h}) = 0.
$$

Then (1.160) becomes

$$
\begin{aligned}
s_0(Q_{p,h}\partial_t p_p - \partial_t p_{p,h}, Q_{p,h}p_p - p_{p,h})_{\Omega_p} &= \alpha b_p(\partial_t\boldsymbol{\eta}_p - I_{s,h}\partial_t\boldsymbol{\eta}_p, Q_{p,h}p_p - p_{p,h}) \\
&+ \alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_{p,h}) + b_p(I_{p,h}\mathbf{u}_p - \mathbf{u}_{p,h}, Q_{p,h}p_p - p_{p,h}) \\
&+ b_f(\mathbf{u}_f - I_{f,h}\mathbf{u}_f, Q_{f,h}p_f - p_{f,h}) + b_f(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, Q_{f,h}p_f - p_{f,h}). \quad (1.161)
\end{aligned}
$$

We now combine (1.158), (1.159), and (1.161), to obtain

$$
\begin{aligned}
& J_2 + J_4 + J_6 + a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, \partial_t\boldsymbol{\eta}_{p,h} - \partial_t\boldsymbol{\eta}_p) + s_0(Q_{p,h}\partial_t p_p - \partial_t p_{p,h}, Q_{p,h}p_p - p_{p,h})_{\Omega_p} \\
&\quad = a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p) + b_f(\mathbf{u}_f - I_{f,h}\mathbf{u}_f, Q_{f,h}p_f - p_{f,h})
\end{aligned}
$$

$$
\begin{aligned}
&+ b_f(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, Q_{f,h}p_f - p_f) \\
&\quad + \alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p, Q_{p,h}p_p - p_{p,h}) + \alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_p) \\
&\quad + \langle (I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h})\cdot\mathbf{n}_f, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}} + \langle (I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}}.
\end{aligned}
\tag{1.162}
$$

We next bound the first four and the sixth terms of the right using Young's inequality (1.74). We note that the velocity and displacement errors are controlled in $L^2(0,T)$, so the terms involving such errors are bounded using (1.74) with $p = q = 2$. The pressure and Lagrange multiplier errors are controlled in $L^{r'}(0,T)$, so for such terms we use (1.74) with $p = r'$ and $q = r$. We have

$$
\begin{aligned}
&a_p^e(\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p, I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p) \le C(\|\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)} + \|I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)}), \\
&b_f(\mathbf{u}_f - I_{f,h}\mathbf{u}_f, Q_{f,h}p_f - p_{f,h}) \le \epsilon_1\|p_{f,h} - Q_{f,h}p_f\|^{r'}_{L^{r'}(\Omega_f)} + C\|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^r_{W^{1,r}(\Omega_f)}, \\
&b_f(I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}, Q_{f,h}p_f - p_f) \le \epsilon_2\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{W^{1,r}(\Omega_f)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + C(\|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^2_{W^{1,r}(\Omega_f)} + \|Q_{f,h}p_f - p_f\|^2_{L^{r'}(\Omega_f)}), \\
&\alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p, Q_{p,h}p_p - p_{p,h}) \le \epsilon_1\|p_{p,h} - Q_{p,h}p_p\|^{r'}_{L^{r'}(\Omega_p)} + C\|I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p\|^r_{H^1(\Omega_p)}, \\
&\langle (I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h})\cdot\mathbf{n}_f, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}} \le \epsilon_2\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{W^{1,r}(\Omega_f)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + C(\|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^2_{W^{1,r}(\Omega_f)} + \|Q_{\lambda,h}\lambda - \lambda\|^2_{L^{r'}(\Gamma_{fp})}).
\end{aligned}
\tag{1.163}
$$

We combine (1.162) and (1.163) and integrate in time from 0 to $t \in (0,T]$:

$$
\begin{aligned}
&\frac{1}{2}(a_p^e(\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_{p,h}(t), \boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_{p,h}(t)) \\
&+ s_0\|Q_{p,h}p_p(t) - p_{p,h}(t)\|^2_{L^2(\Omega_p)}) + \int_0^t (J_2 + J_4 + J_6)\, ds \\
&\le \int_0^t \Big(\epsilon_1\|p_{f,h} - Q_{f,h}p_f\|^{r'}_{L^{r'}(\Omega_f)} + \epsilon_1\|p_{p,h} - Q_{p,h}p_p\|^{r'}_{L^{r'}(\Omega_p)} + \epsilon_2\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{W^{1,r}(\Omega_f)}\Big)\, ds \\
&\quad + \frac{1}{2}\Big(a_p^e(\boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_{p,h}(0), \boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_{p,h}(0)) + s_0\|Q_{p,h}p_p(0) - p_{p,h}(0)\|^2_{L^2(\Omega_p)}\Big) \\
&\quad + C\int_0^t \Big(\|\boldsymbol{\eta}_{p,h} - \boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)} + \|I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)} + \|I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p\|^r_{H^1(\Omega_p)} \\
&\quad + \|Q_{f,h}p_f - p_f\|^2_{L^{r'}(\Omega_f)} + \|Q_{\lambda,h}\lambda - \lambda\|^2_{L^{r'}(\Gamma_{fp})} \\
&+ \|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^2_{W^{1,r}(\Omega_f)} + \|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^r_{W^{1,r}(\Omega_f)}\Big) ds
\end{aligned}
\tag{1.164}
$$

$$+\int_0^t \left(\alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_p) + \langle (I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}}\right) ds. \tag{1.165}$$

For the last two terms on the right hand side we use integration by parts:

$$\begin{aligned}
&\int_0^t \left(\alpha b_p(I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_p) + \langle (I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}}\right) ds \\
&\quad = \alpha b_p(I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_p)\Big|_0^t + \langle (I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}}\Big|_0^t \\
&\qquad - \int_0^t \left(\alpha b_p(I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}, Q_{p,h}\partial_t p_p - \partial_t p_p) + \langle (I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\partial_t\lambda - \partial_t\lambda\rangle_{\Gamma_{fp}}\right) ds
\end{aligned} \tag{1.166}$$

and bound the terms on the right hand side above as follows:

$$\begin{aligned}
&\alpha b_p(I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}, Q_{p,h}p_p - p_p)\Big|_0^t + \langle (I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\lambda - \lambda\rangle_{\Gamma_{fp}}\Big|_0^t \\
&\le \epsilon_2\|\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_{p,h}(t)\|^2_{H^1(\Omega_p)} \\
&+ C\Big(\|I_{s,h}\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_p(t)\|^2_{H^1(\Omega_p)} + \|Q_{p,h}p_p(t) - p_p(t)\|^2_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\lambda(t) - \lambda(t)\|^2_{L^{r'}(\Gamma_{fp})} \\
&+ \|I_{s,h}\boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_{p,h}(0)\|^2_{H^1(\Omega_p)} + \|Q_{p,h}p_p(0) - p_p(0)\|^2_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\lambda(0) - \lambda(0)\|^2_{L^{r'}(\Gamma_{fp})}\Big),
\end{aligned} \tag{1.167}$$

$$\begin{aligned}
&\int_0^t \left(\alpha b_p(I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}, Q_{p,h}\partial_t p_p - \partial_t p_p) + \langle (I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\cdot\mathbf{n}_p, Q_{\lambda,h}\partial_t\lambda - \partial_t\lambda\rangle_{\Gamma_{fp}}\right) ds \\
&\qquad \le C\int_0^t \Big(\|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|^2_{H^1(\Omega_p)} + \|I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)} \\
&\qquad\qquad + \|Q_{p,h}\partial_t p_p - \partial_t p_p\|^2_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\partial_t\lambda - \partial_t\lambda\|^2_{L^{r'}(\Gamma_{fp})}\Big) ds.
\end{aligned} \tag{1.168}$$

Combining (1.165)–(1.168), we obtain

$$\begin{aligned}
&\|\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_{p,h}(t)\|^2_{H^1(\Omega_p)} + s_0\|Q_{p,h}p_p(t) - p_{p,h}(t)\|^2_{L^2(\Omega_p)} + \int_0^t (J_2 + J_4 + J_6)\, ds \\
&\quad \le \epsilon_2\left(\|\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_{p,h}(t)\|^2_{H^1(\Omega_p)} + \int_0^t \|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{W^{1,r}(\Omega_f)}\right) + C\int_0^t \|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|^2_{H^1(\Omega_p)} ds \\
&\qquad + \epsilon_1\int_0^t \left(\|p_{f,h} - Q_{f,h}p_f\|^{r'}_{L^{r'}(\Omega_f)} + \|p_{p,h} - Q_{p,h}p_p\|^{r'}_{L^{r'}(\Omega_p)}\right) ds \\
&\qquad + C\int_0^t \Big(\|I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)} + \|I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p\|^2_{H^1(\Omega_p)} + \|I_{s,h}\partial_t\boldsymbol{\eta}_p - \partial_t\boldsymbol{\eta}_p\|^{r}_{H^1(\Omega_p)}
\end{aligned}$$

$$
\begin{aligned}
&+ \|Q_{\lambda,h}\lambda - \lambda\|^2_{L^{r'}(\Gamma_{fp})} + \|Q_{p,h}\partial_t p_p - \partial_t p_p\|^2_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\partial_t\lambda - \partial_t\lambda\|^2_{L^{r'}(\Gamma_{fp})} \\
&+ \|Q_{f,h}p_f - p_f\|^2_{L^{r'}(\Omega_f)} + \|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^2_{W^{1,r}(\Omega_f)} + \|I_{f,h}\mathbf{u}_f - \mathbf{u}_f\|^r_{W^{1,r}(\Omega_f)}\Big)\, ds \\
&+ C\Big(\|I_{s,h}\boldsymbol{\eta}_p(t) - \boldsymbol{\eta}_p(t)\|^2_{H^1(\Omega_p)} + \|Q_{p,h}p_p(t) - p_p(t)\|^2_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\lambda(t) - \lambda(t)\|^2_{L^{r'}(\Gamma_{fp})} \\
&+ \|I_{s,h}\boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_p(0)\|^2_{H^1(\Omega_p)} + \|Q_{p,h}p_p(0) - p_p(0)\|^2_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\lambda(0) - \lambda(0)\|^2_{L^{r'}(\Gamma_{fp})} \\
&+ \|\boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_{p,h}(0)\|^2_{H^1(\Omega_p)} + \|p_p(0) - p_{p,h}(0)\|^2_{L^{r'}(\Omega_p)}\Big). \qquad (1.169)
\end{aligned}
$$

Next, bounds for $\|p_f - p_{f,h}\|_{L^{r'}(\Omega_f)}$ in the following argument. $\|p_p - p_{p,h}\|_{L^{r'}(\Omega_p)}$ and $\|\lambda - \lambda_h\|_{L^{r'}(\Gamma_{fp})}$.

Using the inf-sup condition (1.120), we obtain

$$
\begin{aligned}
&\|(p_{f,h} - Q_{f,h}p_f, p_{p,h} - Q_{p,h}p_p, \lambda_h - Q_{\lambda,h}\lambda)\|_{W_f\times W_p\times\Lambda_h} \\
&\le C \sup_{(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\in\mathbf{V}_{f,h}\times\mathbf{V}_{p,h}} \Big[\frac{b_f(\mathbf{v}_{f,h}, p_{f,h} - Q_{f,h}p_f)}{\|(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\|_{\mathbf{V}_f\times\mathbf{V}_p}} \\
&\quad + \frac{b_p(\mathbf{v}_{p,h}, p_{p,h} - Q_{p,h}p_p) + b_\Gamma(\mathbf{v}_{f,h},\mathbf{v}_{p,h},\mathbf{0};\lambda_h - Q_{\lambda,h}\lambda)}{\|(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\|_{\mathbf{V}_f\times\mathbf{V}_p}}\Big] \\
&= C \sup_{(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\in\mathbf{V}_{f,h}\times\mathbf{V}_{p,h}} -\Big[\frac{a_f(\mathbf{u}_{f,h},\mathbf{v}_{f,h}) - a_f(\mathbf{u}_f,\mathbf{v}_{f,h})}{\|(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\|_{\mathbf{V}_f\times\mathbf{V}_p}} + \frac{a_p^d(\mathbf{u}_{p,h},\mathbf{v}_{p,h}) - a_p^d(\mathbf{u}_p,\mathbf{v}_{p,h})}{\|(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\|_{\mathbf{V}_f\times\mathbf{V}_p}} \\
&\quad + \frac{a_{BJS}(\mathbf{u}_{f,h},\partial_t\boldsymbol{\eta}_{p,h};\mathbf{v}_{f,h},\mathbf{0}) - a_{BJS}(\mathbf{u}_f,\partial_t\boldsymbol{\eta}_p;\mathbf{v}_{f,h},\mathbf{0})}{\|(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\|_{\mathbf{V}_f\times\mathbf{V}_p}} \\
&\quad + \frac{b_f(\mathbf{v}_{f,h}, Q_{f,h}p_f - p_f) + b_p(\mathbf{v}_{p,h}, Q_{p,h}p_p - p_p) + b_\Gamma(\mathbf{v}_{f,h},\mathbf{v}_{p,h},\mathbf{0}; Q_{\lambda,h}\lambda - \lambda)}{\|(\mathbf{v}_{f,h},\mathbf{v}_{p,h})\|_{\mathbf{V}_f\times\mathbf{V}_p}}\Big] \\
&\le C\big[\mathcal{E}(\mathbf{u},\mathbf{u}_h)\mathcal{G}(\mathbf{u},\mathbf{u}_h)^{1/r'} + \|Q_{f,h}p_f - p_f\|_{L^{r'}(\Omega_f)} \\
&\qquad + \|Q_{p,h}p_p - p_p\|_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\lambda - \lambda\|_{L^{r'}(\Gamma_{fp})}\big],
\end{aligned}
$$

we have used (1.21) for the last inequality. Hence, as $\mathcal{E}(\mathbf{u},\mathbf{u}_h) \le (d+1)$,

$$
\begin{aligned}
&\epsilon_1\int_0^t \Big(\|p_{f,h} - Q_{f,h}p_f\|^{r'}_{L^{r'}(\Omega_f)} + \|p_{p,h} - Q_{p,h}p_p\|^{r'}_{L^{r'}(\Omega_p)} + \|\lambda_h - Q_{\lambda,h}\lambda\|^{r'}_{L^{r'}(\Gamma_{fp})}\Big) \\
&\le \epsilon_1 C\int_0^t \Big(\mathcal{G}(\mathbf{u},\mathbf{u}_h) + \|Q_{f,h}p_f - p_f\|^{r'}_{L^{r'}(\Omega_f)} + \|Q_{p,h}p_p - p_p\|^{r'}_{L^{r'}(\Omega_p)} + \|Q_{\lambda,h}\lambda - \lambda\|^{r'}_{L^{r'}(\Gamma_{fp})}\Big)\, ds. \qquad (1.170)
\end{aligned}
$$

Finally, we can combine all the above bounds to get the result. We now integrate (1.156) in time, combine it with (1.169) and (1.170), take ϵ_1 small enough, then ϵ_2 small enough, and apply Gronwall's inequality, to obtain

$$
\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{f,h}\|^2_{L^2(0,T;W^{1,r}(\Omega_f))} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|^2_{L^2(0,T;L^r(\Omega_p))} \\
&\quad + |(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) - (\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h})|^2_{L^2(0,T;BJS)} + \|Q_{f,h} p_f - p_{f,h}\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_f))} \\
&\quad + \|Q_{p,h} p_p - p_{p,h}\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_p))} + \|Q_{\lambda,h}\lambda - \lambda_h\|^{r'}_{L^{r'}(0,T;L^{r'}(\Gamma_{fp}))} \\
&\quad + \|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|^2_{L^\infty(0,T;H^1(\Omega_p))} + s_0\|Q_{p,h} p_p - p_{p,h}\|^2_{L^\infty(0,T;L^2(\Omega_p))} + \|\mathcal{G}(\mathbf{u},\mathbf{u}_h)\|_{L^1(0,T)} \\
&\leq C\exp(T)\Big(\|\mathbf{u}_f - I_{f,h}\mathbf{u}_f\|^2_{L^2(0,T;W^{1,r}(\Omega_f))} + \|\mathbf{u}_f - I_{f,h}\mathbf{u}_f\|^r_{L^r(0,T;W^{1,r}(\Omega_f))} \\
&\quad + \|\boldsymbol{\eta}_p - I_{s,h}\boldsymbol{\eta}_p\|^2_{L^2(0,T;H^1(\Omega_p))} + \|\mathbf{u}_p - I_{p,h}\mathbf{u}_p\|^r_{L^r(0,T;L^r(\Omega_p))} \\
&\quad + \|\partial_t\boldsymbol{\eta}_p - I_{s,h}\partial_t\boldsymbol{\eta}_p\|^r_{L^r(0,T;H^1(\Omega_p))} + \|\partial_t\boldsymbol{\eta}_p - I_{s,h}\partial_t\boldsymbol{\eta}_p\|^2_{L^2(0,T;H^1(\Omega_p))} \\
&\quad + \|Q_{f,h}p_f - p_f\|^2_{L^2(0,T;L^{r'}(\Omega_f))} + \|Q_{\lambda,h}\lambda - \lambda\|^2_{L^2(0,T;L^{r'}(\Gamma_{fp}))} \\
&\quad + \|Q_{p,h}\partial_t p_p - \partial_t p_p\|^2_{L^2(0,T;L^{r'}(\Omega_p))} + \|Q_{\lambda,h}\partial_t\lambda - \partial_t\lambda\|^2_{L^2(0,T;L^{r'}(\Gamma_{fp}))} \\
&\quad + \|\boldsymbol{\eta}_p - I_{s,h}\boldsymbol{\eta}_p\|^2_{L^\infty(0,T;H^1(\Omega_p))} + \|Q_{p,h}p_p - p_p\|^2_{L^\infty(0,T;L^{r'}(\Omega_p))} + \|Q_{\lambda,h}\lambda - \lambda\|^2_{L^\infty(0,T;L^{r'}(\Gamma_{fp}))} \\
&\quad + \|Q_{f,h}p_f - p_f\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_f))} + \|Q_{p,h}p_p - p_p\|^{r'}_{L^{r'}(0,T;L^{r'}(\Omega_p))} + \|Q_{\lambda,h}\lambda - \lambda\|^{r'}_{L^{r'}(0,T;L^{r'}(\Gamma_{fp}))} \\
&\quad + \|\boldsymbol{\eta}_p(0) - \boldsymbol{\eta}_{p,h}(0)\|^2_{H^1(\Omega_p)} + \|p_p(0) - p_{p,h}(0)\|^2_{L^{r'}(\Omega_p)}\Big).
\end{aligned}
$$

The assertion of the theorem follows from the approximation bounds (2.34)–(2.36) and (2.37)–(1.147) and the use of the triangle inequality for the pressure error terms. □

1.6 Numerical results

1.6.1 Convergence test

In this section we discuss numerical results that verify the theoretical bound (1.149). The numerical experiments in this section are from [3] and were performed by Ilona Ambartsumyan as part of our collaboration.

We discretize the problem (2.22)-(2.24) in time using Backward Euler scheme. Let T denote the final time and τ the length of time step, then for each $n = 1, \ldots, N$ the n-th time step is $t_n = n\tau$. To approximate the time derivatives we use:

$$d_\tau \phi = \frac{\phi^n - \phi^{n-1}}{\tau}, \qquad n = 1, \ldots, N.$$

For the spacial discretization in fluid domain we will use $\mathcal{P}_1 b - \mathcal{P}_1 b$ MINI elements, we will also use $\mathcal{RT}_0 - \mathcal{P}_0$ for $\mathbf{V}_{p,h} \times W_{p,h}$, continuous piecewise linears $\mathcal{P}_1$ for $\mathbf{X}_{p,h}$ and piecewise constants $\mathcal{P}_0$ for Λ_h. We handle nonlinearity in Stokes and Darcy terms using Picard iterations and we assume that the constant in the Beavers-Joseph-Saffman condition (2.9) does not depend on fluid viscosity.

We consider a computational domain $\Omega = [0,2] \times [0,1]$, where $\Omega_f = [0,1] \times [0,1]$ represents the fluid region and $\Omega_p = [1,2] \times [0,1]$ the solid region. The flow is driven by the pressure drop: on the left boundary of Ω_f we set $p_{in} = 1$ kPa and on the right boundary of Ω_p $p_{out} = 0$ kPa, which is also chosen as initial condition for Darcy pressure. Along the top and bottom boundaries, we impose a no-slip boundary condition for the Stokes flow and a no-flow boundary condition for the Darcy flow. We also set zero displacement boundary condition on top, bottom and right parts of boundary of structure subdomain, as well as zero initial condition for the displacement. We set $\lambda_p = \mu_p = s_0 = \alpha = \alpha_{BJS} = 1.0$ and $K = \mathbf{I}$.

We assume that the fluid viscosity in Stokes region satisfies the Cross model:

$$\nu_f(|D(\mathbf{u}_f)|) = \nu_{f,\infty} + \frac{\nu_{f,0} - \nu_{f,\infty}}{1 + K_f |D(\mathbf{u}_f)|^{2-r_f}}.$$

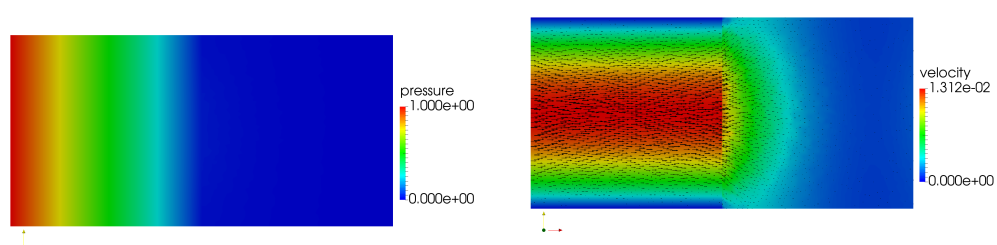
And the effective viscosity in Darcy region also satisfies the Cross model:

$$\nu_p(|\mathbf{u}_p|) = \nu_{p,\infty} + \frac{\nu_{p,0} - \nu_{p,\infty}}{1 + K_p |\mathbf{u}_p|^{2-r_p}},$$

where the parameters are chosen as follows: $K_f = K_p = 1$, $\nu_{f,\infty} = \nu_{p,\infty} = 1$, $\nu_{f,0} = \nu_{p,0} = 10$, $r_f = r_p = 1.35$. The simulation time is $T = 1.0$ s and the time step $\tau = 0.01$ s. To verify the convergence estimate (1.149), we compute a reference solution, obtained on the mesh with characteristic size $h = 1/320$. Table 1 shows the relative errors and rates of convergence for the solutions computed with discretization steps $h = 1/20, 1/40, 1/80$ and $1/160$ for the case of lowest order elements. Since we use bounded functions to model viscosity in both regions, we compute the norms of the errors using $r = r' = 2$. As we can see, the results agree with theory, i.e. we observe at least first convergence rate for all variables.

	$\frac{\|\mathbf{u}^{ref}_{f,h}-\mathbf{u}_{f,h}\|_{l^2(0,T;H^1(\Omega_f))}}{\|\mathbf{u}^{ref}_{f,h}\|_{l^2(0,T;H^1(\Omega_f))}}$		$\frac{\|\mathbf{u}^{ref}_{p,h}-\mathbf{u}_{p,h}\|_{l^2(0,T;L^2(\Omega_p))}}{\|\mathbf{u}^{ref}_{p,h}\|_{l^2(0,T;L^2(\Omega_p))}}$		$\frac{\|p^{ref}_{f,h}-p_{f,h}\|_{l^2(0,T;L^2(\Omega_f))}}{\|p^{ref}_{f,h}\|_{l^2(0,T;L^2(\Omega_f))}}$	
h	error	order	error	order	error	order
1/20	4.83E-03	–	1.55E-01	–	2.75E-02	–
1/40	2.31E-03	1.06	8.63E-02	0.85	1.03E-02	1.41
1/80	1.04E-03	1.16	4.08E-02	1.08	4.62E-03	1.16
1/160	3.94E-04	1.40	2.07E-02	0.98	2.14E-04	1.11
	$\frac{\|p^{ref}_{p,h}-p_{p,h}\|_{l^2(0,T;L^2(\Omega_p))}}{\|p^{ref}_{p,h}\|_{l^2(0,T;L^2(\Omega_p))}}$		$\frac{\|p^{ref}_{p,h}-p_{p,h}\|_{l^\infty(0,T;L^2(\Omega_p))}}{\|p^{ref}_{p,h}\|_{l^\infty(0,T;L^2(\Omega_p))}}$		$\|\frac{\boldsymbol{\eta}^{ref}_{p,h}-\boldsymbol{\eta}_{p,h}\|_{l^\infty(0,T;H^1(\Omega_p))}}{\boldsymbol{\eta}^{ref}_{p,h}\|_{l^\infty(0,T;H^1(\Omega_p))}}$	
h	error	order	error	order	error	order
1/20	4.10E-02	–	1.15E-01	–	4.98E-02	–
1/40	1.92E-02	1.10	5.28E-02	1.12	2.88E-02	0.79
1/80	8.24E-03	1.22	2.25E-02	1.23	1.61E-02	0.84
1/160	2.75E-03	1.58	7.48E-03	1.59	6.59E-03	1.29

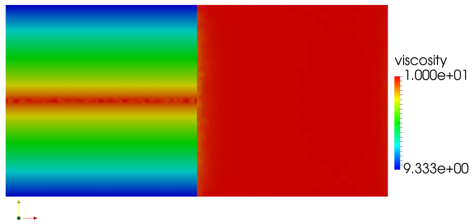
Table 1: Convergence for $\mathcal{P}_1 b \times \mathcal{P}_1 b \times \mathcal{RT}_0 \times \mathcal{P}_0 \times \mathcal{P}_1 \times \mathcal{P}_0$ elements



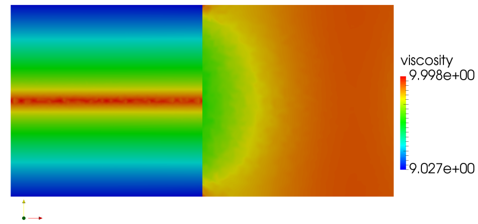
(a) pressure

(b) velocity vector (arrows) and magnitude (color)

Figure 1: Nonlinear pressure and velocity solutions at time $t = 1$

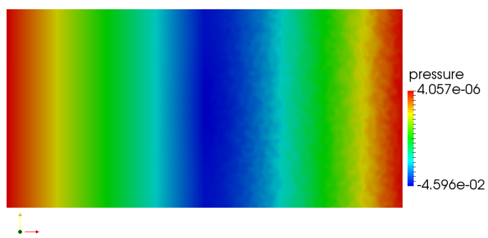


(a) at $t = 0.01$

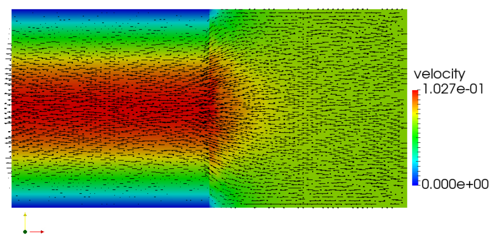


(b) at $t = 1$

Figure 2: Nonlinear viscosity

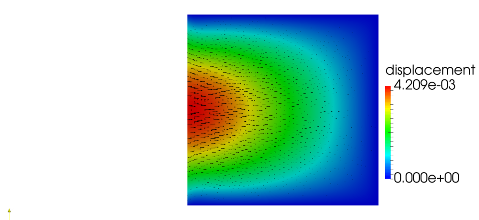


(a) pressure

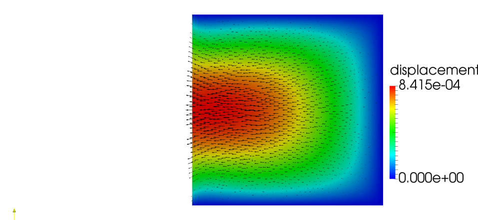


(b) velocity

Figure 3: Difference between non-Newtonian and Newtonian solutions at time $t = 1$



(a) nonlinear solution



(b) difference

Figure 4: Non-Newtonian displacement solution and difference at time $t = 1$

We also investigate the behavior of solution visually and compare it to the solution of linear analogue of the method (2.22)-(2.24). For visualization we use the solutions corresponding to the mesh size $h = 1/40$. All plots are presented at the first and final time steps. For a fair comparison between models, we calculate the viscosity in linear case as $\nu_f^{lin} = \nu_f\big|_{r_f=2} = 5.5$ and $\nu_p^{lin} = \nu_p\big|_{r_p=2} = 5.5$. Figures with difference between velocity and displacement solutions are obtained by plotting $\mathbf{u}_{f,h}^{nonlin} - \mathbf{u}_{f,h}^{lin}$, $\mathbf{u}_{p,h}^{nonlin} - \mathbf{u}_{p,h}^{lin}$ and $\boldsymbol{\eta}_{p,h}^{nonlin} - \boldsymbol{\eta}_{p,h}^{lin}$, where colors represent the magnitude of the corresponding difference and arrows represent the direction.

As we can see, in nonlinear case the viscosity is high in the middle of the fluid domain and it decreases towards the boundary, which is due to the fact that the strain rate increases towards the boundary. On the other hand, the viscosity does not vary as much in the solid domain due to almost uniform velocity profile (see Figure 2). We note that these observations agree with conclusions in [38]. Moreover, use of non-Newtomian model results in lower Stokes velocity, as shown on Figure 3(b), which in turn entails lower displacement, Figure 4(b).

1.6.2 Example 2: application to hydraulic fracturing

We next present an example motivated by hydraulic fracturing. We study the interaction between a stationary fracture filled with fluid and the surrounding reservoir. The units in this example are meters for length, seconds for time, and kPa for pressure. We consider a reference domain $\hat{\Omega} = [0,1] \times [-1,1]$ and a fracture domain $\hat{\Omega}_f$, which is located in the middle with a boundary

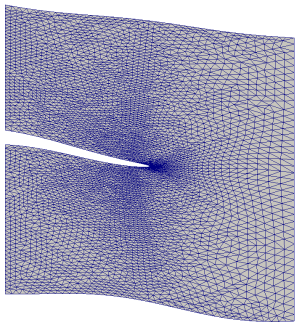
$$\hat{x}^2 = 200(0.05 - \hat{y})(0.05 + \hat{y}), \quad \hat{y} \in [-0.05, 0.05].$$

The reference poroelastic domain is $\hat{\Omega}_p = \hat{\Omega} \setminus \hat{\Omega}_f$. The computational domain, shown in Figure 5 (left), is obtained from the reference domain via the mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} (\hat{x}, \hat{y}) = \begin{bmatrix} x \\ (5\cos(\frac{\hat{x}+\hat{y}}{100})\cos(\frac{\pi\hat{x}+\hat{y}}{100})^2 + \hat{y}/2 - \hat{x}/10) \end{bmatrix}.$$

We enforce an inflow rate $\mathbf{u}_f \cdot \mathbf{n}_f = 10$ m/s, $\mathbf{u}_f \cdot \boldsymbol{\tau}_f = 0$ m/s on the left part of $\partial\Omega_f$ and no flow $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ m/s and no stress $\boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0}$ kPa on the left part of $\partial\Omega_p$. On the top, bottom, and right boundaries we set $p_p = 1000$ kPa, $\boldsymbol{\eta}_p \cdot \mathbf{n}_p = 0$ m/s, and $\boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\tau}_p = 0$ kPa. The initial conditions are $p_p = 1000$ kPa and $\boldsymbol{\eta} = \mathbf{0}$ m/s. The poroelastic parameters, which are typical for hydraulic fracturing and are similar to the ones used in [51], are given in Figure 5 (right). The nonlinear viscosity model in Stokes and Darcy is from [59] for a polymer used in hydraulic fracturing, see Figure 6 (left) for the viscosity dependence on the shear rate. We match the curve using the Cross model with parameters $K_f = K_p = 7$, $\nu_{f,\infty} = \nu_{p,\infty} = 3.0 \times 10^{-6}$ kPa s, $\nu_{f,0} = \nu_{p,0} = 1.0 \times 10^{-2}$ kPa s, and $r_f = r_p = 1.35$.

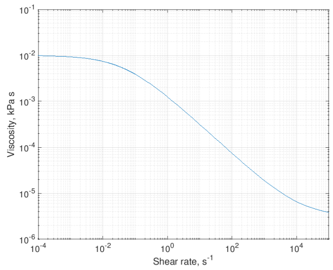
We run the simulation for 300s with time step $\tau = 1$ s and compare the results of the linear and nonlinear models. For the linear model we use the viscosity for water, $\nu_f^{lin} = \nu_p^{lin} = 1.0 \times 10^{-6}$ kPa s, which is slightly lower than the minimum value of the nonlinear viscosity. We present the simulation results at the final time for both models in Figures 6–8. We note that the scales in the plots are different for the two models, due to significant differences in the solution values. The computed Stokes and Darcy velocities are shown in Figure 7. We observe channel-like flow in the fracture with both models. However, the higher nonlinear viscosity results in smaller velocity, especially near the fracture tip. The nonlinear viscosity in the fracture is shown in Figure 6 (middle). We note the significant shear-thinning effect, especially along the wall of the fracture, where the viscosity is reduced to values in the order of $\nu_{f,\infty}$. Comparing the Darcy velocity fields in Figure 7, we observe that the combination of very small permeability and high fluid viscosity in the nonlinear case results in very little fluid penetration into the reservoir. This is an expected behavior in hydraulic fracturing. Correspondingly, the nonlinear viscosity in the poroelastic region, as shown in Figure 6 (right), is significantly reduced in a close vicinity of the fracture, but is equal to the maximum value $\nu_{p,0}$ away from the fracture. In the linear case, the Darcy velocity is larger and the fluid penetrates further into the reservoir. The behavior for both models is consistent with the computed pressure fields shown in Figure 8. For both models we observe increase in pressure near the fracture. In the linear case the pressure gradient extends away from the fracture. In the nonlinear case, since the fluid cannot penetrate further into the reservoir, we observe a significant pressure buildup along the fracture, about 100 times larger than in the linear case. This in turn results in about 100 times larger displacement in the nonlinear case. This includes larger opening of the fracture, all the way



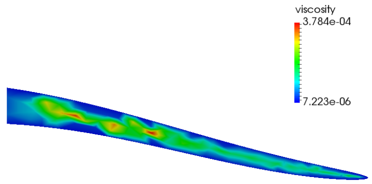
Parameter		Units	Values
Young's modulus	E	(kPa)	10^7
Poisson's ratio	σ		0.2
Lame coefficient	μ_p	(kPa)	$5/12 \times 10^7$
Lame coefficient	λ_p	(kPa)	$5/18 \times 10^7$
Permeability	K	(m^2)	$(200, 50) \times 10^{-12}$
Mass storativity	s_0	(kPa^{-1})	6.89×10^{-2}
Biot-Willis const.	α		1.0
BJS coeff.	α_{BJS}		1.0
Total time	T	(s)	300

Figure 5: Computational domain (left) and parameters (right) for Example 2

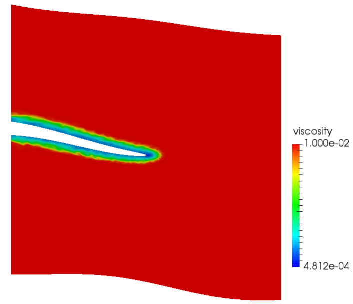
to the tip. We note that our models are for stationary fractures, but the large displacement and corresponding stress near the fracture tip in the nonlinear case may result in practice in fracture propagation, as would be expected in hydraulic fracturing. To summarize, this is a numerically very challenging test case, due to the large stiffness and small permeability of the rock. The numerical difficulty for the non-Newtonian fluid is further increased due to the model nonlinearity and the larger viscosity. We observe that the model is capable of handling parameters in this challenging range and produce results that are qualitatively similar to practical hydraulic fracturing applications.



(a) Viscosity model

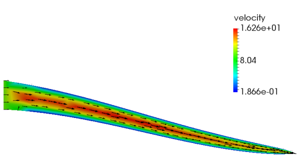


(b) Stokes viscosity

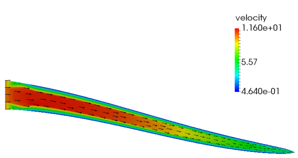


(c) Darcy viscosity

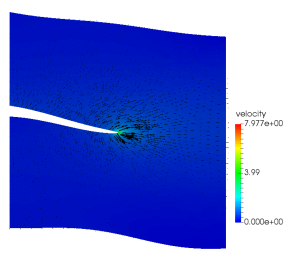
Figure 6: Nonlinear viscosity model and computed Stokes and Darcy viscosity at $t = 300$s



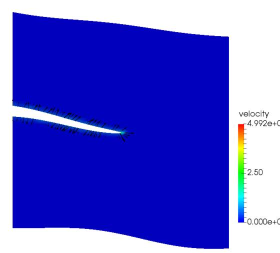
(a) Stokes, linear



(b) Stokes, nonlinear

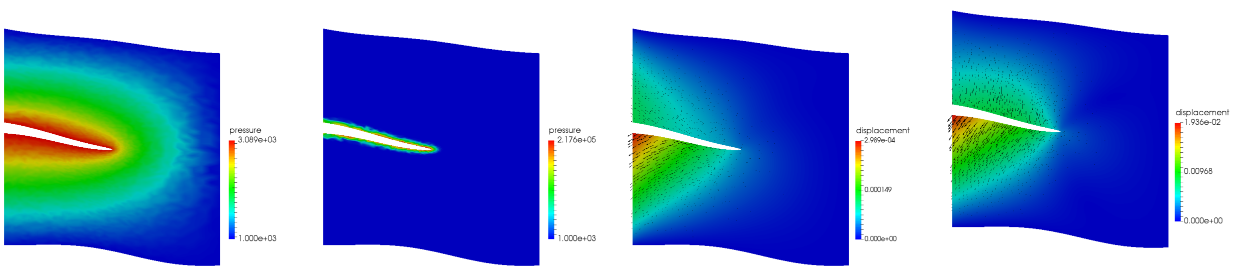


(c) Darcy, linear



(d) Darcy, nonlinear

Figure 7: Stokes and Darcy velocity at time $t = 300$s



(a) pressure, linear (b) pressure, nonlinear (c) displacement, linear (d) displacement, nonlinear

Figure 8: Poroelastic pressure and displacement at time $t = 300$s

2.0 Coupling Biot-Stokes flow with transport

2.1 Introduction and model problem

In this chapter, we will investigate the transport equation with the flows that are solutions of the Biot-Stokes system in chapter 1. At first, we use the Stokes equations to model the free fluid in the fractures and the Biot poroelasticity model [12] for the fluid in the poroelastic region. The latter is based on a linear stress-strain constitutive relationship for the porous solid, and Darcy's law, which describes the average velocity of the fluid in the pores. The interaction across the fracture-matrix interfaces exhibits features of both Stokes-Darcy coupling [35, 49, 55, 70, 82] and fluid-structure interaction (FSI) [45, 9, 22, 40, 8, 69]. We refer to the Stokes-Biot coupling considered in this chapter as fluid-poroelastic structure interaction (FPSI). There has been growing interest in such models in the literature. The well-posedness of the mathematical model was studied in [77]. Numerical studies include variational multiscale methods for the monolithic system and iterative partitioned scheme [7], a non-iterative operator-splitting method [19], a partitioned method based on Nitsche's coupling [18], and a Lagrange multiplier formulation for the continuity of flux [5].

In this chapter we employ a monolithic scheme for the full-dimensional Stokes-Biot problem to model flow in fractured poroelastic media. We note that an alternative approach is based on a reduced-dimension fracture model, including the Reynolds lubrication equation [48, 51, 56, 61] and an averaged Brinkman equation [20]. Works that do not account for elastic deformation of the media include averaged Darcy models [60, 42, 62, 30, 44], Forchheimer models [43], Brinkman models [58], and an averaged Stokes model that results in a Brinkman model for the fracture flow [63].

For the discretization of the full-dimensional Stokes-Biot problem we consider the mixed formulation for Darcy flow in the Biot system, which provides a locally mass conservative flow approximation and an accurate Darcy velocity. This formulation results in the continuity of normal velocity condition being of essential type, which is enforced through a Lagrange multiplier [5]. The discretization allows for the use of any stable Stokes spaces in the fracture region and any stable mixed Darcy spaces [15]. For the elasticity equation we employ a displacement formulation with continuous Lagrange elements.

The Stokes-Biot system is coupled with an advection diffusion equation for modeling transport of chemical species within the fluid. The transport equation is discretized by a discontinuous Galerkin (DG) method. DG methods [65, 71, 6, 29, 81] exhibit local mass conservation, reduced numerical diffusion, variable degrees of approximation, and accurate approximations for problems with discontinuous coefficients. Due to their low numerical diffusion, DG methods are especially suited for advection-diffusion problems [29, 28, 81, 33, 84, 2]. Coupled Darcy flow and transport problems utilizing DG for transport have been studied in [80, 79, 32, 84]. Coupling of Stokes-Darcy flow with transport using a local discontinuous Galerkin scheme was developed in [83]. A coupled phase field-transport model for proppant-filled fractures is studied in [57]. To the best of our knowledge, the coupled Stokes-Biot-transport problem has not been studied in the literature. Here we follow the approach from [79] for miscible displacement in porous media and employ the non-symmetric interior penalty Galerkin (NIPG) method for the transport problem. We note that the dispersion tensor in the transport equation is a nonlinear function of the velocity. The work in [79] handles this difficulty by utilizing a cut-off operator. However, adopting idea from [79], we will use discontinuous Galerkin method to handle our transport problem. We avoid using the "cut-off" function to do analysis. Hence, the computed velocity do not have to be modified when used for the transport equation. The chapter is organized as follow, in section 2.1 we set up the transport equation. Due to the estimation of error of transport equation related to error of flow in Hilbert spaces, in section 2.2, we give some results of the error of flow in the simplified case. In section 2.3, we give the discrete scheme for the transport problem. Then we give a stability estimate in section 2.4 and error estimate in section 2.5. The last section is devoted to give numerical results.

Because the error analysis of transport equation is related to the error of Biot-Stokes flow in the Hilbert norm only, so within this section, for simplicity, we consider the case where the spaces of Biot-Stokes flows are Hilbert spaces. We assume the the viscosities are constants in this section. We restate the problem as follow.

We consider a simulation domain $\Omega \in \mathbb{R}^d, d = 2, 3$, where $\Omega = \Omega_f \cup \Omega_p$. The interface between Ω_f, Ω_p is $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$.

The flow in the fracture region Ω_f is governed by the Stokes equations:

$$-\nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = \mathbf{f}_f \quad \text{in } \Omega_f \times (0,T], \tag{2.1}$$

$$\nabla \cdot \mathbf{u}_f = q_f \quad \text{in } \Omega_f \times (0,T], \tag{2.2}$$

where $\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\sigma}_f$ are defined as follow

$$\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla \mathbf{u}_f^T), \quad \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = -p_f \mathbf{I} + 2\nu \boldsymbol{\epsilon}(\mathbf{u}_f). \tag{2.3}$$

Let $\boldsymbol{\eta}$ be the displacement in Ω_p, $\boldsymbol{\sigma}_e(\boldsymbol{\eta})$ and $\boldsymbol{\sigma}_p(\boldsymbol{\eta}, p)$ are the elasticity and poroelasticity stress tensors, respectively:

$$\boldsymbol{\sigma}_e(\boldsymbol{\eta}) = \lambda_p(\nabla \cdot \boldsymbol{\eta})\mathbf{I} + 2\mu_p \mathbf{D}(\boldsymbol{\eta}), \quad \boldsymbol{\sigma}_p(\boldsymbol{\eta}, p_p) = \boldsymbol{\sigma}_e(\boldsymbol{\eta}) - \alpha_p p_p \mathbf{I}. \tag{2.4}$$

The poroelasticity region Ω_p is governed by the quasi-static Biot system:

$$-\nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) = \mathbf{f}_p \quad \text{in } \Omega_p \times (0,T], \tag{2.5}$$

$$\nu_{eff} K^{-1} \mathbf{u}_p + \nabla p_p = 0, \quad \frac{\partial}{\partial t}(s_0 p_p + \alpha_p \nabla \cdot \boldsymbol{\eta}_p) + \nabla \cdot \mathbf{u}_p = q_p \quad \text{in } \Omega_p \times (0,T], \tag{2.6}$$

where s_0 is a storage coefficient and K is a symmetric and uniformly positive definite permeability tensor. Following [5, 7, 77], on the fluid-poroelasticity interface Γ_{fp} we prescribe the following *interface conditions*: *mass conservation*, *balance of normal stress*, *conservation of momentum*, and the Beavers-Joseph-Saffman (BJS) condition modeling *slip with friction* [10, 72]:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \left(\frac{\partial \boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right) \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0,T], \tag{2.7}$$

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p, \quad \boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0,T], \tag{2.8}$$

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_{f,j} = \nu \alpha_{BJS} \sqrt{K_j^{-1}} \left(\mathbf{u}_f - \frac{\partial \boldsymbol{\eta}_p}{\partial t}\right) \cdot \boldsymbol{\tau}_{f,j} \quad \text{on } \Gamma_{fp} \times (0,T], \tag{2.9}$$

where $\mathbf{n}_f$ and $\mathbf{n}_p$ are the outward unit normal vectors to $\partial\Omega_f$ and $\partial\Omega_p$, respectively, $\boldsymbol{\tau}_{f,j}$, $1 \le j \le d-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $K_j = (K\boldsymbol{\tau}_{f,j}) \cdot \boldsymbol{\tau}_{f,j}$ and $\alpha_{BJS} > 0$ is an experimentally determined friction coefficient.

The above system of equations is complemented by a set of boundary and initial conditions. Let $\Gamma_f = \partial\Omega_f \backslash \Gamma_{fp}$, $\Gamma_p = \partial\Omega_p \backslash \Gamma_{fp} = \Gamma_p^N \cup \Gamma_p^D$. For simplicity we assume homogeneous boundary conditions

$$\mathbf{u}_f = 0 \text{ on } \Gamma_f \times (0, T], \quad \mathbf{u}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^N \times (0, T],$$

$$p_p = 0 \text{ on } \Gamma_p^D \times (0, T], \quad \boldsymbol{\eta}_p = 0 \text{ on } \Gamma_p \times (0, T].$$

We further set the initial conditions

$$p_p(\mathbf{x}, 0) = p_{p,0}(\mathbf{x}), \ \boldsymbol{\eta}_p(\mathbf{x}, 0) = \boldsymbol{\eta}_{p,0}(\mathbf{x}) \text{ in } \Omega_p.$$

We define the following vector spaces.

$$\mathbf{V}_f := \{\mathbf{v}_f \in H^1(\Omega_f)^d : \mathbf{v}_f = \mathbf{0} \text{ on } \partial\Omega_f \setminus \Gamma_{fp}\}, \quad W_f := L^2(\Omega_f). \tag{2.10}$$

Next, let

$$H(div; \Omega_p) := \{\mathbf{v}_p \in (L^2(\Omega_p))^d : \nabla \cdot \mathbf{v}_p \in L^2(\Omega_p)\} \tag{2.11}$$

and we define

$$\mathbf{V}_p := \{\mathbf{v}_p \in H(div, \Omega_p) : \mathbf{v}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^N \times (0, T]\}, \quad W_p := L^2(\Omega_p), \tag{2.12}$$

$$\mathbf{X}_p := \{\boldsymbol{\xi} \in H^1(\Omega_p)^d : \boldsymbol{\xi}_p = \mathbf{0} \text{ on } \partial\Omega_p \setminus \Gamma_{fp}\}. \tag{2.13}$$

And also, $\Lambda := H^{1/2}(\Gamma_{fp})$.

The weak formulation is obtained by multiplying the equations in each region by the corresponding test functions, integrating by parts the second order terms in space, and utilizing the interface and boundary conditions. The integration by parts in (2.1) and (2.5) leads to the bilinear forms, corresponding to the Stokes, Darcy and the elasticity operators:

$$\begin{aligned}
a_f(\cdot,\cdot) \,:\, \mathbf{V}_f \times \mathbf{V}_f \,\longrightarrow\, \mathbb{R}, \qquad & a_f(\mathbf{u}_f, \mathbf{v}_f) := (2\nu\boldsymbol{\epsilon}(\mathbf{u}_f), \boldsymbol{\epsilon}(\mathbf{v}_f))_{\Omega_f}, \\
a_p^d(\cdot,\cdot) \,:\, \mathbf{V}_p \times \mathbf{V}_p \,\longrightarrow\, \mathbb{R}, \qquad & a_p^d(\mathbf{u}_p, \mathbf{v}_p) := (\nu K^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}, \\
a_p^e(\cdot,\cdot) \,:\, \mathbf{X}_p \times \mathbf{X}_p \,\longrightarrow\, \mathbb{R}, \qquad & a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) := (2\mu_p\boldsymbol{\epsilon}(\boldsymbol{\eta}_p), \boldsymbol{\epsilon}(\boldsymbol{\xi}_p))_{\Omega_p} + (\lambda_p \nabla \cdot \boldsymbol{\eta}_p, \nabla \cdot \boldsymbol{\xi}_p)_{\Omega_p},
\end{aligned}$$

the bilinear forms

$$b_\star(\cdot,\cdot)\,:\,\mathbf{V}_\star\times W_\star \longrightarrow \mathbb{R}, \qquad b_\star(\mathbf{v},w) := -(\nabla\cdot\mathbf{v},w)_{\Omega_\star}, \quad \star = f,p,$$

and the interface term

$$I_{\Gamma_{fp}} = -\langle \boldsymbol{\sigma}_f\mathbf{n}_f,\mathbf{v}_f\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\sigma}_p\mathbf{n}_p,\boldsymbol{\xi}_p\rangle_{\Gamma_{fp}} + \langle p_p,\mathbf{v}_p\cdot\mathbf{n}_p\rangle_{\Gamma_{fp}}.$$

To handle the interface term, we introduce a Lagrange multiplier λ with a meaning of Darcy pressure on the interface [5]

$$\lambda = -(\boldsymbol{\sigma}_f\mathbf{n}_f)\cdot\mathbf{n}_f = p_p \quad \text{on } \Gamma_{fp}.$$

Using (2.8)–(2.9), we obtain

$$I_{\Gamma_{fp}} = a_{BJS}(\mathbf{u}_f,\partial_t\boldsymbol{\eta}_p;\mathbf{v}_f,\boldsymbol{\xi}_p) + b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\lambda),$$

where

$$\begin{aligned} a_{BJS}(\mathbf{u}_f,\boldsymbol{\eta}_p;\mathbf{v}_f,\boldsymbol{\xi}_p) &= \sum_{j=1}^{d-1}\left\langle \nu_I\,\alpha_{BJS}\sqrt{K^{-1}}(\mathbf{u}_f-\boldsymbol{\eta}_p)\cdot\boldsymbol{\tau}_{f,j},(\mathbf{v}_f-\boldsymbol{\xi}_p)\cdot\boldsymbol{\tau}_{f,j}\right\rangle_{\Gamma_{fp}}, \\ b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\mu) &= \langle \mathbf{v}_f\cdot\mathbf{n}_f + (\boldsymbol{\xi}_p+\mathbf{v}_p)\cdot\mathbf{n}_p,\mu\rangle_{\Gamma_{fp}}. \end{aligned}$$

We get the weak formulation for the problem: given $p_p(0) = p_{p,0}\in W_p$, $\boldsymbol{\eta}_p(0) = \boldsymbol{\eta}_{p,0}\in\mathbf{X}_p$, find, for $t\in(0,T]$, $\mathbf{u}_f(t)\in\mathbf{V}_f$, $p_f(t)\in W_f$, $\mathbf{u}_p(t)\in\mathbf{V}_p$, $p_p(t)\in W_p$, $\boldsymbol{\eta}_p(t)\in\mathbf{X}_p$, and $\lambda(t)\in\Lambda$ such that for all $\mathbf{v}_f\in\mathbf{V}_f$, $w_f\in W_f$, $\mathbf{v}_p\in\mathbf{V}_p$, $w_p\in W_p$, $\boldsymbol{\xi}_p\in\mathbf{X}_p$, and $\mu\in\Lambda$,

$$\begin{aligned} & a_f(\mathbf{u}_f,\mathbf{v}_f) + a_p^d(\mathbf{u}_p,\mathbf{v}_p) + a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p) + a_{BJS}(\mathbf{u}_f,\partial_t\boldsymbol{\eta}_p;\mathbf{v}_f,\boldsymbol{\xi}_p) + b_f(\mathbf{v}_f,p_f) + b_p(\mathbf{v}_p,p_p) \\ & \qquad + \alpha_p b_p(\boldsymbol{\xi}_p,p_p) + b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\lambda) = (\mathbf{f}_f,\mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p,\boldsymbol{\xi}_p)_{\Omega_p}, && (2.14) \\ & (s_0\partial_t p_p,w_p)_{\Omega_p} - \alpha_p b_p\left(\partial_t\boldsymbol{\eta}_p,w_p\right) - b_p(\mathbf{u}_p,w_p) - b_f(\mathbf{u}_f,w_f) = (q_f,w_f)_{\Omega_f} + (q_p,w_p)_{\Omega_p}, && (2.15) \\ & b_\Gamma\left(\mathbf{u}_f,\mathbf{u}_p,\partial_t\boldsymbol{\eta}_p;\mu\right) = 0. && (2.16) \end{aligned}$$

The Stokes-Biot problem is coupled with the transport equation in Ω:

$$\phi c_t + \nabla\cdot(c\mathbf{u} - \mathbf{D}\nabla c) = q\tilde{c}, \qquad \text{in } \Omega\times(0,T], \tag{2.17}$$

where $c(\mathbf{x},t)$ is the concentration of some chemical component, $0 < \phi_* \leq \phi(\mathbf{x}) \leq \phi^*$ is the porosity of the medium in Ω_p (it is set to 1 in Ω_f), $\mathbf{u}$ is the velocity field over $\Omega = \Omega_f \cup \Omega_p$, defined as $\mathbf{u}|_{\Omega_f} = \mathbf{u}_f, \mathbf{u}|_{\Omega_p} = \mathbf{u}_p$, q is the source term given by $q\big|_{\Omega_f} = q_f$ and $q\big|_{\Omega_p} = q_p$, and

$$\tilde{c} = \begin{cases} \text{injected concentration } c_w, & q > 0, \\ \text{resident concentration } c, & q < 0. \end{cases}$$

The diffusion/dispersion tensor $\mathbf{D}$, which combines the effects of molecular diffusion and mechanical dispersion, is a nonlinear function of the velocity, given by

$$\mathbf{D}(\mathbf{u}) = d_m\mathbf{I} + |\mathbf{u}|\{\alpha_l\mathbf{E}(\mathbf{u}) + \alpha_t(\mathbf{I} - \mathbf{E}(\mathbf{u}))\}, \tag{2.18}$$

where $d_m = \phi\tau D_m$, τ is the tortuosity coefficient, D_m is the molecular diffusivity, $\mathbf{E}(\mathbf{u})$ is the tensor that projects onto the $\mathbf{u}$ direction with $(\mathbf{E}(\mathbf{u}))_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}$, and α_l, α_t are the longitudinal and transverse dispersion, respectively. The model is complemented by the initial condition

$$c(\mathbf{x},0) = c_0(\mathbf{x}) \quad \text{in } \Omega, \tag{2.19}$$

and the boundary conditions

$$(c\mathbf{u} - \mathbf{D}\nabla c)\cdot\mathbf{n} = (c_{in}\mathbf{u})\cdot\mathbf{n} \qquad \text{on } \Gamma_{in} \times (0,T], \tag{2.20}$$

$$(\mathbf{D}\nabla c)\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_{out} \times (0,T], \tag{2.21}$$

where $\Gamma_{in} := \{\mathbf{x} \in \partial\Omega : \mathbf{u}\cdot\mathbf{n} < 0\}$, $\Gamma_{out} := \{\mathbf{x} \in \partial\Omega : \mathbf{u}\cdot\mathbf{n} \geq 0\}$ and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

Remark 2.1.1. *We note that the coupling between the flow and transport problems is one way. In particular, the transport equation uses the Stokes-Biot velocity, but the flow problem does not depend on the concentration.*

2.2 Some numerical estimations for velocity

The stability estimate and error estimate for transport problems in the next sections depend on the error of the flow problem. Hence, within this section, we will present some necessary results about error estimation of velocity.

First, we set up the finite element problem for the velocity. Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular and quasi-uniform partitions of Ω_f and Ω_p, respectively, both consisting of affine elements with maximal element diameter h. The two partitions may be non-matching at the interface Γ_{fp}. For the discretization of the fluid velocity and pressure we choose finite element spaces $\mathbf{V}_{f,h} \subset \mathbf{V}_f$ and $W_{f,h} \subset W_f$, which are assumed to be inf-sup stable. Examples of such spaces include the MINI elements, the Taylor-Hood elements and the conforming Crouzeix-Raviart elements. For the discretization of the porous medium problem we choose $\mathbf{V}_{p,h} \subset \mathbf{V}_p$ and $W_{p,h} \subset W_p$ to be any of well-known inf-sup stable mixed finite element spaces, such as the Raviart-Thomas or the Brezzi-Douglas-Marini spaces. The global spaces are

$$\mathbf{V}_h = \{\mathbf{v}_h = (\mathbf{v}_{f,h}, \mathbf{v}_{p,h}) \in \mathbf{V}_{f,h} \times \mathbf{V}_{p,h}\}, \quad W_h = \{w_h = (w_{f,h}, w_{p,h}) \in W_{f,h} \times W_{p,h}\}.$$

We employ a conforming Lagrangian finite element spaces $\mathbf{X}_{p,h} \subset \mathbf{X}_p$ and $\Lambda_h \subset \Lambda$ to approximate the structure displacement and Lagrange multiplier. Note that the finite element spaces $\mathbf{V}_{f,h}$, $\mathbf{V}_{p,h}$, and $\mathbf{X}_{p,h}$ satisfy the prescribed homogeneous boundary conditions on the external boundaries $\partial\Omega_f$ and $\partial\Omega_p$.

Semi-discrete Stokes-Biot problem: given $p_{p,h}(0)$ and $\boldsymbol{\eta}_{p,h}(0)$, for $t \in (0,T]$, find $\mathbf{u}_{f,h}(t) \in \mathbf{V}_{f,h}$, $p_{f,h}(t) \in W_{f,h}$, $\mathbf{u}_{p,h}(t) \in \mathbf{V}_{p,h}$, $p_{p,h}(t) \in W_{p,h}$, $\boldsymbol{\eta}_{p,h}(t) \in \mathbf{X}_{p,h}$, and $\lambda_h(t) \in \Lambda_h$ such that for all $\mathbf{v}_{f,h} \in \mathbf{V}_{f,h}$, $w_{f,h} \in W_{f,h}$, $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}$, $w_{p,h} \in W_{p,h}$, $\boldsymbol{\xi}_{p,h} \in \mathbf{X}_{p,h}$, and $\mu_h \in \Lambda_h$,

$$\begin{aligned} &a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_p^d(\mathbf{u}_{p,h}, \mathbf{v}_{p,h}) + a_p^e(\boldsymbol{\eta}_{p,h}, \boldsymbol{\xi}_{p,h}) + a_{BJS}(\mathbf{u}_{f,h}, \partial_t \boldsymbol{\eta}_{p,h}; \mathbf{v}_{f,h}, \boldsymbol{\xi}_{p,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}) \\ &\quad + b_p(\mathbf{v}_{p,h}, p_{p,h}) + \alpha b_p(\boldsymbol{\xi}_{p,h}, p_{p,h}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \boldsymbol{\xi}_{p,h}; \lambda_h) = (\mathbf{f}_f, \mathbf{v}_{f,h})_{\Omega_f} + (\mathbf{f}_p, \boldsymbol{\xi}_{p,h})_{\Omega_p}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &(s_0 \partial_t p_{p,h}, w_{p,h})_{\Omega_p} - \alpha b_p(\partial_t \boldsymbol{\eta}_{p,h}, w_{p,h}) - b_p(\mathbf{u}_{p,h}, w_{p,h}) - b_f(\mathbf{u}_{f,h}, w_{f,h}) \\ &\quad = (q_f, w_{f,h})_{\Omega_f} + (q_p, w_{p,h})_{\Omega_p}, \end{aligned} \tag{2.23}$$

$$b_\Gamma(\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \partial_t \boldsymbol{\eta}_{p,h}; \mu_h) = 0. \tag{2.24}$$

We take $p_{p,h}(0) = Q_{p,h} p_{p,0}$ and $\boldsymbol{\eta}_{p,h}(0) = I_{s,h} \boldsymbol{\eta}_{p,0}$, where the operators $Q_{p,h}$ and $I_{s,h}$ are defined in the following section.

We denote by $k_f \geq 1$ and $s_f \geq 1$ the degrees of polynomials in the spaces $\mathbf{V}_{f,h}$ and $W_{f,h}$ respectively. Let $k_p \geq 0$ and $s_p \geq 0$ be the degrees of polynomials in the spaces $\mathbf{V}_{p,h}$ and $W_{p,h}$ respectively. Finally, let $k_s \geq 1$ be the polynomial degree in $\mathbf{X}_{p,h}$.

It was shown in [5] that the above problem has a unique solution satisfying

$$\begin{aligned}
&\|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|_{L^\infty(0,T;H^1(\Omega_p))} + \sqrt{s_0}\|p_p - p_{p,h}\|_{L^\infty(0,T;L^2(\Omega_p))} + \|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(0,T;H^1(\Omega_f))} \\
&\quad + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|_{L^2(0,T;L^2(\Omega_p))} + \left|(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) - (\mathbf{u}_{f,h} - \partial_t \boldsymbol{\eta}_{p,h})\right|_{L^2(0,T;a_{BJS})} \\
&\quad + \|p_f - p_{f,h}\|_{L^2(0,T;L^2(\Omega_f))} + \|p_p - p_{p,h}\|_{L^2(0,T;L^2(\Omega_p))} + \|\lambda - \lambda_h\|_{L^2(0,T;\Lambda_h)} \\
&\leq C\Bigg(h^{r_{k_f}} \|\mathbf{u}_f\|_{L^2(0,T;H^{r_{k_f}+1}(\Omega_f))} + h^{r_{s_f}} \|p_f\|_{L^2(0,T;H^{r_{s_f}}(\Omega_f))} + h^{r_{k_p}} \|\mathbf{u}_p\|_{L^2(0,T;H^{r_{k_p}}(\Omega_p))} \\
&\quad + h^{\tilde{r}_{k_p}} \left(\|\lambda\|_{L^2(0,T;H^{\tilde{r}_{k_p}}(\Gamma_{fp}))} + \|\lambda\|_{L^\infty(0,T;H^{\tilde{r}_{k_p}}(\Gamma_{fp}))} + \|\partial_t \lambda\|_{L^2(0,T;H^{\tilde{r}_{k_p}}(\Gamma_{fp}))} \right) \\
&\quad + h^{r_{s_p}} \left(\|p_p\|_{L^\infty(0,T;H^{r_{s_p}}(\Omega_p))} + \|p_p\|_{L^2(0,T;H^{r_{s_p}}(\Omega_p))} + \|\partial_t p_p\|_{L^2(0,T;H^{r_{s_p}}(\Omega_p))} \right) \\
&\quad + h^{r_{k_s}} \left(\|\boldsymbol{\eta}_p\|_{L^\infty(0,T;H^{r_{k_s}+1}(\Omega_p))} + \left\|\boldsymbol{\eta}_p\right\|_{L^2(0,T;H^{r_{k_s}+1}(\Omega_p))} + \left\|\partial_t \boldsymbol{\eta}_p\right\|_{L^2(0,T;H^{r_{k_s}+1}(\Omega_p))} \right) \Bigg),
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&0 \leq r_{k_f} \leq k_f,\ 0 \leq r_{s_f} \leq s_f + 1,\ 1 \leq \{r_{k_p}, \tilde{r}_{k_p}\} \leq k_p + 1, \\
&0 \leq r_{s_p} \leq s_p + 1,\ 0 \leq r_{k_s} \leq k_s,
\end{aligned}$$

where, for $\mathbf{v}_f \in \mathbf{V}_f$, $\boldsymbol{\xi}_p \in \mathbf{X}_p$,

$$|\mathbf{v}_f - \boldsymbol{\xi}_p|^2_{a_{BJS}} = a_{BJS}(\mathbf{v}_f, \boldsymbol{\xi}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) = \sum_{j=1}^{d-1} \mu \alpha_{BJS} \|K_j^{-1/4}(\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_{f,j}\|^2_{L^2(\Gamma_{fp})}.$$

The following result gives an error estimate for the fluid velocity in $L^\infty(0,T)$, it is given in [4]. The result requires control of $\mathbf{u}_{f,h}(0)$ and $\mathbf{u}_{p,h}(0)$. To simplify the analysis, we assume that the initial pressure $p_{p,0}$ and displacement $\boldsymbol{\eta}_{p,0}$ are constants.

Lemma 2.2.1. *Assume that* $p_{p,0}$ *and* $\boldsymbol{\eta}_{p,0}$ *are constants. If the solution of* (2.14)–(2.16) *is sufficiently regular, there exists a positive constant* C *independent of* h *such that*

$$\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^\infty(0,T;H^1(\Omega_f))} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|_{L^\infty(0,T;L^2(\Omega_p))} + \|\partial_t(\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\|_{L^2(0,T;H^1(\Omega_p))} \\
&+ \|\partial_t(p_p - p_{p,h})\|_{L^2(0,T;L^2(\Omega_p))} \\
&\quad \le C\Bigg[h^{r_{k_f}}\left(\|\mathbf{u}_f\|_{L^2(0,T;H^{r_{k_f}+1}(\Omega_f))} + \|\mathbf{u}_f\|_{L^\infty(0,T;H^{r_{k_f}+1}(\Omega_f))} + \|\partial_t\mathbf{u}_f\|_{L^2(0,T;H^{r_{k_f}+1}(\Omega_f))}\right) \\
&\qquad + h^{r_{s_f}}\left(\|p_f\|_{L^2(0,T;H^{r_{s_f}}(\Omega_f))} + \|p_f\|_{L^\infty(0,T;H^{r_{s_f}}(\Omega_f))} + \|\partial_t p_f\|_{L^2(0,T;H^{r_{s_f}}(\Omega_f))}\right) \\
&\qquad + h^{r_{k_p}}\left(\|\mathbf{u}_p\|_{L^2(0,T;H^{r_{k_p}}(\Omega_p))} + \|\mathbf{u}_p\|_{L^\infty(0,T;H^{r_{k_p}}(\Omega_p))} + \|\partial_t\mathbf{u}_p\|_{L^2(0,T;H^{r_{k_p}}(\Omega_p))}\right) \\
&\qquad + h^{\tilde{r}_{k_p}}\left(\|\lambda\|_{L^2(0,T;H^{\tilde{r}_{k_p}}(\Gamma_{fp}))} + \|\lambda\|_{L^\infty(0,T;H^{\tilde{r}_{k_p}}(\Gamma_{fp}))} + \|\partial_t\lambda\|_{L^2(0,T;H^{\tilde{r}_{k_p}}(\Gamma_{fp}))}\right) \\
&\qquad + h^{r_{s_p}}\left(\|p_p\|_{L^\infty(0,T;H^{r_{s_p}}(\Omega_p))} + \|p_p\|_{L^2(0,T;H^{r_{s_p}}(\Omega_p))} + \|\partial_t p_p\|_{L^2(0,T;H^{r_{s_p}}(\Omega_p))}\right) \\
&\qquad + h^{r_{k_s}}\Big(\|\boldsymbol{\eta}_p\|_{L^\infty(0,T;H^{r_{k_s}+1}(\Omega_p))} + \|\boldsymbol{\eta}_p\|_{L^2(0,T;H^{r_{k_s}+1}(\Omega_p))} + \|\partial_t\boldsymbol{\eta}_p\|_{L^2(0,T;H^{r_{k_s}+1}(\Omega_p))} \\
&\qquad\qquad + \|\partial_t\boldsymbol{\eta}_p\|_{L^\infty(0,T;H^{r_{k_s}+1}(\Omega_p))} + \|\partial_{tt}\boldsymbol{\eta}_p\|_{L^2(0,T;H^{r_{k_s}+1}(\Omega_p))}\Big)\Bigg]. \qquad (2.26)
\end{aligned}$$

$$\begin{aligned}
&0 \le r_{k_f} \le k_f,\ 0 \le r_{s_f} \le s_f + 1,\ 1 \le \{r_{k_p},\, \tilde{r}_{k_p}\} \le k_p + 1, \\
&0 \le r_{s_p} \le s_p + 1,\ 0 \le r_{k_s} \le k_s.
\end{aligned}$$

Proof. We introduce the errors for all variables and split them into approximation and discretization errors:

$$\begin{aligned}
\mathbf{e}_f &:= \mathbf{u}_f - \mathbf{u}_{f,h} = (\mathbf{u}_f - I_{f,h}\mathbf{u}_f) + (I_{f,h}\mathbf{u}_f - \mathbf{u}_{f,h}) := \boldsymbol{\chi}_f + \boldsymbol{\phi}_{f,h}, \\
\mathbf{e}_p &:= \mathbf{u}_p - \mathbf{u}_{p,h} = (\mathbf{u}_p - I_{p,h}\mathbf{u}_p) + (I_{p,h}\mathbf{u}_p - \mathbf{u}_{p,h}) := \boldsymbol{\chi}_p + \boldsymbol{\phi}_{p,h}, \\
\mathbf{e}_s &:= \boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h} = (\boldsymbol{\eta}_p - I_{s,h}\boldsymbol{\eta}_p) + (I_{s,h}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}) := \boldsymbol{\chi}_s + \boldsymbol{\phi}_{s,h}, \\
e_{fp} &:= p_f - p_{f,h} = (p_f - Q_{f,h}p_f) + (Q_{f,h}p_f - p_{f,h}) := \chi_{fp} + \phi_{fp,h}, \\
e_{pp} &:= p_p - p_{p,h} = (p_p - Q_{p,h}p_p) + (Q_{p,h}p_p - p_{p,h}) := \chi_{pp} + \phi_{pp,h}, \\
e_\lambda &:= \lambda - \lambda_h = (\lambda - Q_{\lambda,h}\lambda) + (Q_{\lambda,h}\lambda - \lambda_h) := \chi_\lambda + \phi_{\lambda,h}, \qquad (2.27)
\end{aligned}$$

where the operator $I = (I_{f,h}, I_{p,h}, I_{s,h})$ satisfies, see [5] for details,

$$b_\Gamma\left(I_{f,h}\mathbf{v}_f, I_{p,h}\mathbf{v}_p, I_{s,h}\boldsymbol{\xi}_p; \mu_h\right) = 0, \qquad \forall \mu_h \in \Lambda_h, \qquad (2.28)$$

$$b_f(I_{f,h}\mathbf{v}_f - \mathbf{v}_f, w_{f,h}) = 0, \qquad \forall w_{f,h} \in W_{f,h}, \qquad (2.29)$$

$$b_p(I_{p,h}\mathbf{v}_p - \mathbf{v}_p, w_{p,h}) = 0, \qquad \forall w_{p,h} \in W_{p,h}, \tag{2.30}$$

and $Q_{f,h}$, $Q_{p,h}$ and $Q_{\lambda,h}$ are the L^2–projection operators such that

$$(p_f - Q_{f,h}p_f, w_{f,h})_{\Omega_f} = 0, \qquad \forall\, w_{f,h} \in W_{f,h}, \tag{2.31}$$

$$(p_p - Q_{p,h}p_p, w_{p,h})_{\Omega_p} = 0, \qquad \forall\, w_{p,h} \in W_{p,h}, \tag{2.32}$$

$$\langle \lambda - Q_{\lambda,h}\lambda, \mu_h\rangle_{\Gamma_{fp}} = 0, \qquad \forall\, \mu_h \in \Lambda_h. \tag{2.33}$$

The operators have the following approximation properties:

$$\|p_f - Q_{f,h}p_f\|_{L^2(\Omega_f)} \leq Ch^{r_{s_f}}\|p_f\|_{H^{r_{s_f}}(\Omega_f)}, \qquad 0 \leq r_{s_f} \leq s_f + 1, \tag{2.34}$$

$$\|p_p - Q_{p,h}p_p\|_{L^2(\Omega_p)} \leq Ch^{r_{s_p}}\|p_p\|_{H^{r_{s_p}}(\Omega_p)}, \qquad 0 \leq r_{s_p} \leq s_p + 1, \tag{2.35}$$

$$\|\lambda - Q_{\lambda,h}\lambda\|_{L^2(\Gamma_{fp})} \leq Ch^{r_{k_p}}\|\lambda\|_{H^{r_{k_p}}(\Gamma_{fp})}, \qquad 0 \leq \tilde{r}_{k_p} \leq k_p + 1, \tag{2.36}$$

$$\|\mathbf{v}_f - I_{f,h}\mathbf{v}_f\|_{H^1(\Omega_f)} \leq Ch^{r_{k_f}}\|\mathbf{v}_f\|_{H^{r_{k_f}+1}(\Omega_f)}, \qquad 0 \leq r_{k_f} \leq k_f, \tag{2.37}$$

$$\|\boldsymbol{\xi}_p - I_h^s\boldsymbol{\xi}_p\|_{H^m(\Omega_p)} \leq Ch^{r_{k_s}-m}\|\boldsymbol{\xi}_p\|_{H^{r_{k_s}}(\Omega_p)}, \quad m = 0,1, \quad 1 \leq r_{k_s} \leq k_s + 1, \tag{2.38}$$

$$\begin{aligned}\|\mathbf{v}_p - I_{p,h}\mathbf{v}_p\|_{L^2(\Omega_p)} &\leq C\left(h^{r_{k_p}}\|\mathbf{v}_p\|_{H^{r_{k_p}}(\Omega_p)} + h^{r_{k_f}}\|\mathbf{v}_f\|_{H^{r_{k_f}+1}(\Omega_f)} + h^{r_{k_s}}\|\boldsymbol{\xi}_p\|_{H^{r_{k_s}+1}(\Omega_p)}\right), \\ &1 \leq r_{k_p} \leq k_p + 1,\ 0 \leq r_{k_f} \leq k_f,\ 0 \leq r_{k_s} \leq k_s.\end{aligned} \tag{2.39}$$

To obtain a velocity bound in $L^\infty(0,T)$, we differentiate (2.14) and (2.22) in time, and then subtract (2.22)–(2.23) from (2.14)–(2.15) to form the error equation

$$\begin{aligned}&a_f(\partial_t\mathbf{e}_f, \mathbf{v}_{f,h}) + a_p^d(\partial_t\mathbf{e}_p, \mathbf{v}_{p,h}) + a_p^e(\partial_t\mathbf{e}_s, \boldsymbol{\xi}_{p,h}) + a_{BJS}(\partial_t\mathbf{e}_f, \partial_{tt}\mathbf{e}_s; \mathbf{v}_{f,h}, \boldsymbol{\xi}_{p,h}) + b_f(\mathbf{v}_{f,h}, \partial_t e_{fp}) \\ &\quad + b_p(\mathbf{v}_{p,h}, \partial_t e_{pp}) + \alpha b_p(\boldsymbol{\xi}_{p,h}, \partial_t e_{pp}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \boldsymbol{\xi}_{p,h}; \partial_t e_\lambda) + (s_0\, \partial_t e_{pp}, w_{p,h}) \\ &\quad - \alpha b_p(\partial_t e_s, w_{p,h}) - b_p(\mathbf{e}_p, w_{p,h}) - b_f(\mathbf{e}_f, w_{f,h}) = 0.\end{aligned}$$

Setting $\mathbf{v}_{f,h} = \boldsymbol{\phi}_{f,h}, \mathbf{v}_{p,h} = \boldsymbol{\phi}_{p,h}, \boldsymbol{\xi}_{p,h} = \partial_t\boldsymbol{\phi}_{s,h}, w_{f,h} = \partial_t\phi_{fp,h}$, and $w_{p,h} = \partial_t\phi_{pp,h}$, we have

$$\begin{aligned}&a_f(\partial_t\boldsymbol{\chi}_f, \boldsymbol{\phi}_{f,h}) + a_f(\partial_t\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{f,h}) + a_p^d(\partial_t\boldsymbol{\chi}_p, \boldsymbol{\phi}_{p,h}) + a_p^d(\partial_t\boldsymbol{\phi}_{p,h}, \boldsymbol{\phi}_{p,h}) + a_p^e\left(\partial_t\boldsymbol{\chi}_s, \partial_t\boldsymbol{\phi}_{s,h}\right) \\ &\quad + a_p^e\left(\partial_t\boldsymbol{\phi}_{s,h}, \partial_t\boldsymbol{\phi}_{s,h}\right) + a_{BJS}\left(\partial_t\boldsymbol{\chi}_f, \partial_{tt}\boldsymbol{\chi}_s; \boldsymbol{\phi}_{f,h}, \partial_t\boldsymbol{\phi}_{s,h}\right) + a_{BJS}\left(\partial_t\boldsymbol{\phi}_{f,h}, \partial_{tt}\boldsymbol{\phi}_{s,h}; \boldsymbol{\phi}_{f,h}, \partial_t\boldsymbol{\phi}_{s,h}\right) \\ &\quad + b_f(\boldsymbol{\phi}_{f,h}, \partial_t\chi_{fp}) + b_f(\boldsymbol{\phi}_{f,h}, \partial_t\phi_{fp,h}) \\ &\quad + b_p(\boldsymbol{\phi}_{p,h}, \partial_t\chi_{pp}) + b_p(\boldsymbol{\phi}_{p,h}, \partial_t\phi_{pp,h}) + \alpha b_p\left(\partial_t\boldsymbol{\phi}_{s,h}, \partial_t\chi_{pp}\right) \\ &\quad + \alpha b_p\left(\partial_t\boldsymbol{\phi}_{s,h}, \partial_t\phi_{pp,h}\right) + b_\Gamma\left(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{p,h}, \partial_t\boldsymbol{\phi}_{s,h}; \partial_t\chi_\lambda\right) + b_\Gamma\left(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{p,h}, \partial_t\boldsymbol{\phi}_{s,h}; \partial_t\phi_{\lambda,h}\right)\end{aligned}$$

$$+ (s_0\, \partial_t \chi_{pp}, \partial_t \phi_{pp,h}) + (s_0\, \partial_t \phi_{pp,h}, \partial_t \phi_{pp,h}) - \alpha b_p\left(\partial_t \boldsymbol{\chi}_s, \partial_t \phi_{pp,h}\right) - \alpha b_p\left(\partial_t \boldsymbol{\phi}_{s,h}, \partial_t \phi_{pp,h}\right)$$
$$- b_p(\boldsymbol{\chi}_p, \partial_t \phi_{pp,h}) - b_p(\boldsymbol{\phi}_{p,h}, \partial_t \phi_{pp,h}) - b_f(\boldsymbol{\chi}_f, \partial_t \phi_{fp,h}) - b_f(\boldsymbol{\phi}_{f,h}, \partial_t \phi_{fp,h}) = 0. \tag{2.40}$$

The following terms simplify, due to the projection operators properties (2.32),(2.33), (2.29), and (2.30):

$$\begin{aligned} b_f(\boldsymbol{\chi}_f, \partial_t \phi_{fp,h}) = b_p(\boldsymbol{\chi}_p, \partial_t \phi_{pp,h}) = b_p(\boldsymbol{\phi}_{p,h}, \partial_t \chi_{pp}) = 0, \\ (s_0\, \partial_t \chi_{pp}, \partial_t \phi_{pp,h}) = \langle \boldsymbol{\phi}_{p,h} \cdot \mathbf{n}_p, \partial_t \chi_\lambda \rangle_{\Gamma_{fp}} = 0. \end{aligned} \tag{2.41}$$

Where we have also used that $\Lambda_h = \mathbf{V}_{p,h} \cdot \mathbf{n}_p|_{\Gamma_{fp}}$ for the last equality. We also have

$$\begin{aligned} b_\Gamma\left(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{p,h}, \partial_t \boldsymbol{\phi}_{s,h}; \partial_t \phi_{\lambda,h}\right) = 0, \\ b_\Gamma\left(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{p,h}, \partial_t \boldsymbol{\phi}_{s,h}; \partial_t \chi_\lambda\right) = \left\langle \boldsymbol{\phi}_{f,h} \cdot \mathbf{n}_f + \partial_t \boldsymbol{\phi}_{s,h} \cdot \mathbf{n}_p, \partial_t \chi_\lambda \right\rangle_{\Gamma_{fp}}. \end{aligned}$$

Where we have used (2.28) and (2.24), and (2.41). Now, the error equation (2.40) becomes

$$\begin{aligned}
&\frac{1}{2}\partial_t \left(a_f(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{f,h}) + a_p^d(\boldsymbol{\phi}_{p,h}, \boldsymbol{\phi}_{p,h}) + \left|\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}\right|^2_{a_{BJS}} \right) \\
&\quad + a_p^e(\partial_t \boldsymbol{\phi}_{s,h}, \partial_t \boldsymbol{\phi}_{s,h}) + s_0 \|\partial_t \phi_{pp,h}\|^2_{L^2(\Omega_p)} \\
&\quad = a_f(\partial_t \boldsymbol{\chi}_f, \boldsymbol{\phi}_{f,h}) + a_p^d(\partial_t \boldsymbol{\chi}_p, \boldsymbol{\phi}_{p,h}) + a_p^e\left(\partial_t \boldsymbol{\chi}_s, \partial_t \boldsymbol{\phi}_{s,h}\right) \\
&\quad + \sum_{j=1}^{d-1} \left\langle \nu \alpha_{BJS} \sqrt{K_j^{-1}} \partial_t(\boldsymbol{\chi}_f - \partial_t \boldsymbol{\chi}_s) \cdot \boldsymbol{\tau}_{f,j}, (\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}) \cdot \boldsymbol{\tau}_{f,j} \right\rangle_{\Gamma_{fp}} - b_f(\boldsymbol{\phi}_{f,h}, \partial_t \chi_{fp}) \\
&\quad - \alpha b_p(\partial_t \boldsymbol{\phi}_{s,h}, \partial_t \chi_{pp}) + \alpha b_p(\partial_t \boldsymbol{\chi}_s, \partial_t \phi_{pp,h}) - \langle \boldsymbol{\phi}_{f,h} \cdot \mathbf{n}_f + \partial_t \boldsymbol{\phi}_{s,h} \cdot \mathbf{n}_p, \partial_t \chi_\lambda \rangle_{\Gamma_{fp}} \\
&\le C\left(\|\boldsymbol{\phi}_{f,h}\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\phi}_{p,h}\|^2_{L^2(\Omega_p)} + \left|\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}\right|^2_{a_{BJS}} \right) + \epsilon \|\partial_t \boldsymbol{\phi}_{s,h}\|^2_{H^1(\Omega_p)} \\
&+ C\Big(\|\partial_t \boldsymbol{\chi}_f\|^2_{H^1(\Omega_f)} + \|\partial_t \boldsymbol{\chi}_p\|^2_{L^2(\Omega_p)} + \|\partial_t \boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)} + \|\partial_{tt} \boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)} \\
&+ \alpha b_p(\partial_t \boldsymbol{\chi}_s, \partial_t \phi_{pp,h}) + \|\partial_t \chi_{fp}\|^2_{L^2(\Omega_f)} + \|\partial_t \chi_{pp}\|^2_{L^2(\Omega_p)} + \|\partial_t \chi_\lambda\|^2_{L^2(\Gamma_{fp})} \Big).
\end{aligned} \tag{2.42}$$

Where we have used the Cauchy-Schwartz, Young's and trace inequalities. Using the coercivity of the bilinear forms $a_f(\cdot,\cdot)$, $a_p^d(\cdot,\cdot)$, and $a_p^e(\cdot,\cdot)$, choosing ϵ small enough, and integrating (2.42) in time from 0 to an arbitrary $t \in (0, T]$ gives

$$\begin{aligned}
&\|\boldsymbol{\phi}_{f,h}(t)\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\phi}_{p,h}(t)\|^2_{L^2(\Omega_p)} + \left|\boldsymbol{\phi}_{f,h}(t) - \partial_t \boldsymbol{\phi}_{s,h}(t)\right|^2_{a_{BJS}} \\
&\qquad + \int_0^t \left(\|\partial_t \boldsymbol{\phi}_{s,h}\|^2_{H^1(\Omega_p)} + s_0 \|\partial_t \phi_{pp,h}\|^2_{L^2(\Omega_p)} \right) ds \\
&\quad \le \|\boldsymbol{\phi}_{f,h}(0)\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\phi}_{p,h}(0)\|^2_{L^2(\Omega_p)} + \left|\boldsymbol{\phi}_{f,h}(0) - \partial_t \boldsymbol{\phi}_{s,h}(0)\right|^2_{a_{BJS}}
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^t \Big(\|\boldsymbol{\phi}_{f,h}\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\phi}_{p,h}\|^2_{L^2(\Omega_p)} + \big|\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}\big|^2_{a_{BJS}} \\
&\qquad + \|\partial_t \boldsymbol{\chi}_f\|^2_{H^1(\Omega_f)} + \|\partial_t \boldsymbol{\chi}_p\|^2_{L^2(\Omega_p)} + \|\partial_t \boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)} + \|\partial_{tt} \boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)} \\
&\qquad + \|\partial_t \chi_{fp}\|^2_{L^2(\Omega_f)} + \|\partial_t \chi_{pp}\|^2_{L^2(\Omega_p)} + \|\partial_t \chi_\lambda\|^2_{L^2(\Gamma_{fp})} + \alpha b_p(\partial_t \boldsymbol{\chi}_s, \partial_t \phi_{pp,h}) \Big)\, ds.
\end{aligned} \tag{2.43}
$$

Using integration by parts for the last term, we get

$$
\begin{aligned}
&\int_0^t \alpha b_p(\partial_t \boldsymbol{\chi}_s, \partial_t \phi_{pp,h})\, ds = \alpha b_p(\partial_t \boldsymbol{\chi}_s(t), \phi_{pp,h}(t)) - \alpha b_p(\partial_t \boldsymbol{\chi}_s(0), \phi_{pp,h}(0)) \\
&- \int_0^t \alpha b_p(\partial_{tt} \boldsymbol{\chi}_{s,h}, \phi_{pp,h})\, ds \le \epsilon \left(\|\phi_{pp,h}(t)\|^2_{L^2(\Omega_p)} + \int_0^t \|\phi_{pp,h}\|^2_{L^2(\Omega_p)} \right) \\
&+ C\left(\|\partial_t \boldsymbol{\chi}_s(t)\|^2_{H^1(\Omega_p)} + \|\phi_{pp,h}(0)\|^2_{L^2(\Omega_p)} + \|\partial_t \boldsymbol{\chi}_s(0)\|^2_{H^1(\Omega_p)} + \int_0^t \|\partial_{tt} \boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)}\, ds \right).
\end{aligned} \tag{2.44}
$$

Next, using an inf-sup condition for the Stokes-Darcy problem [47, 5] and the error equation obtained by subtracting (2.22) from (2.14) and taking $\boldsymbol{\xi}_{p,h} = 0$, we obtain

$$
\begin{aligned}
&\|(\phi_{fp,h}, \phi_{pp,h}, \phi_{\lambda,h})\|_{W_f \times W_p \times \Lambda_h} \\
&\le C \sup_{0 \ne \mathbf{v}_h \in \mathbf{V}_h} \frac{b_f(\mathbf{v}_{f,h}, \phi_{fp,h}) + b_p(\mathbf{v}_{p,h}, \phi_{pp,h}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, 0; \phi_{\lambda,h})}{\|\mathbf{v}_h\|_{\mathbf{V}}} \\
&\quad = C \sup_{0 \ne \mathbf{v}_h \in \mathbf{V}_h} \Bigg(\frac{-a_f(\mathbf{e}_f, \mathbf{v}_{f,h}) - a_p^d(\mathbf{e}_p, \mathbf{v}_{p,h}) - a_{BJS}(\mathbf{e}_f, \partial_t \mathbf{e}_s; \mathbf{v}_{f,h}, 0)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \\
&\qquad\qquad + \frac{-b_f(\mathbf{v}_{f,h}, \chi_{fp}) - b_p(\mathbf{v}_{p,h}, \chi_{pp}) - b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, 0; \chi_\lambda)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \Bigg).
\end{aligned}
$$

We have $b_p(\mathbf{v}_{p,h}, \chi_{pp}) = 0$ and $\langle \mathbf{v}_{p,h} \cdot \mathbf{n}_p, \chi_\lambda \rangle_{\Gamma_{fp}} = 0$. Then, using the continuity of the bilinear forms and the trace inequality, we get

$$
\begin{aligned}
&\epsilon(\|\phi_{fp,h}\|^2_{L^2(\Omega_f)} + \|\phi_{pp,h}\|^2_{L^2(\Omega_p)} + \|\phi_{\lambda,h}\|^2_{L^2(\Gamma_{fp})}) \\
&\quad \le C\epsilon \Big(\|\boldsymbol{\phi}_{f,h}\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\phi}_{p,h}\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\phi}_{s,h}\|^2_{H^1(\Omega_p)} + \big|\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}\big|^2_{a_{BJS}} \\
&\qquad + \|\boldsymbol{\chi}_f\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\chi}_p\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)} + \|\partial_t \boldsymbol{\chi}_s\|^2_{H^1(\Omega_p)} \\
&\qquad + \|\chi_{fp}\|^2_{L^2(\Omega_f)} + \|\chi_{pp}\|^2_{L^2(\Omega_p)} + \|\chi_\lambda\|_{L^2(\Gamma_{fp})} \Big).
\end{aligned} \tag{2.45}
$$

Finally, to control the error at $t = 0$, we note that the assumed solution regularity on the right hand side of (2.26) implies that (2.14)–(2.16) and (2.22)–(2.24) hold at $t = 0$.

We subtract (2.22)–(2.23) from (2.14)–(2.15) at $t = 0$, sum the two equations, and take $\mathbf{v}_{f,h} = \boldsymbol{\phi}_{f,h}, \mathbf{v}_{p,h} = \boldsymbol{\phi}_{p,h}, \boldsymbol{\xi}_{p,h} = \partial_t \boldsymbol{\phi}_{s,h}, w_{f,h} = \phi_{fp,h}$, and $w_{p,h} = \phi_{pp,h}$, to obtain

$$
\begin{aligned}
& a_f(\boldsymbol{\phi}_{f,h}(0), \boldsymbol{\phi}_{f,h}(0)) + a_p^d(\boldsymbol{\phi}_{p,h}(0), \boldsymbol{\phi}_{p,h}(0)) + \left|\boldsymbol{\phi}_{f,h}(0) - \partial_t \boldsymbol{\phi}_{s,h}(0)\right|^2_{a_{BJS}} \\
&\quad = -a_p^e(\boldsymbol{\phi}_{s,h}(0), \partial_t \boldsymbol{\phi}_{s,h}(0)) - s_0(\partial_t \phi_{pp,h}(0), \phi_{pp,h}(0))_{\Omega_p} \\
&\qquad + a_f(\boldsymbol{\chi}_f(0), \boldsymbol{\phi}_{f,h}(0)) + a_p^d(\boldsymbol{\chi}_p(0), \boldsymbol{\phi}_{p,h}(0)) + a_p^e\left(\boldsymbol{\chi}_s(0), \partial_t \boldsymbol{\phi}_{s,h}(0)\right) \\
&\qquad + \sum_{j=1}^{d-1} \left\langle \mu \alpha_{BJS} \sqrt{K_j^{-1}} (\boldsymbol{\chi}_f(0) - \partial_t \boldsymbol{\chi}_s(0)) \cdot \boldsymbol{\tau}_{f,j}, (\boldsymbol{\phi}_{f,h}(0) - \partial_t \boldsymbol{\phi}_{s,h}(0)) \cdot \boldsymbol{\tau}_{f,j} \right\rangle_{\Gamma_{fp}} \\
&\qquad - b_f(\boldsymbol{\phi}_{f,h}(0), \chi_{fp}(0)) \\
&\qquad + \alpha b_p(\partial_t \boldsymbol{\phi}_{s,h}(0), \chi_{pp}(0)) + \alpha b_p(\partial_t \boldsymbol{\chi}_s(0), \phi_{pp,h}(0)) + \langle \boldsymbol{\phi}_{f,h}(0) \cdot \mathbf{n}_f + \partial_t \boldsymbol{\phi}_{s,h}(0) \cdot \mathbf{n}_p, \chi_\lambda(0) \rangle_{\Gamma_{fp}}.
\end{aligned}
$$

Since $p_{p,h}(0) = Q_{p,h} p_{p,0}$ and $\boldsymbol{\eta}_{p,h}(0) = I_{s,h} \boldsymbol{\eta}_{p,0}$, we have that $\phi_{pp,h}(0) = 0$ and $\boldsymbol{\phi}_{s,h}(0) = 0$. Since $p_{p,0}$ and $\boldsymbol{\eta}_{p,0}$ are constants, we also have that $\boldsymbol{\chi}_s = 0$, $\chi_{pp} = 0$, and $\chi_\lambda = 0$. It is then easy to see that

$$
\begin{aligned}
\|\boldsymbol{\phi}_{f,h}(0)\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\phi}_{p,h}(0)\|^2_{L^2(\Omega_p)} + \left|\boldsymbol{\phi}_{f,h}(0) - \partial_t \boldsymbol{\phi}_{s,h}(0)\right|^2_{a_{BJS}} \\
\leq C(\|\boldsymbol{\chi}_f\|^2_{H^1(\Omega_f)} + \|\boldsymbol{\chi}_p\|^2_{L^2(\Omega_p)} + \|\chi_{fp}\|^2_{L^2(\Omega_f)}).
\end{aligned} \tag{2.46}
$$

The assertion of the lemma follows from combining (2.43)–(2.46) and using Gronwall's inequality, the triangle inequality, and the approximation properties (2.34)–(2.39). □

Now, we will prove the following lemma.

Lemma 2.2.2. *With the assumption similar to the lemma 2.2.1, we have the following estimation,*

$$
\|\nabla \cdot (\mathbf{u}_p - \mathbf{u}_{p,h})\|_{L^2(0,T;L^2(\Omega_p))} \leq C h^{\min\{k_f, k_s, s_f+1, s_p+1, k_p+1\}}.
$$

Proof. From (2.15), we have

$$
(s_0 \partial_t p_p, w_p)_{\Omega_p} - \alpha_p b_p(\partial_t \boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) = (q_p, w_p)_{\Omega_p}.
$$

We subtract this equation with the corresponding finite element equation we get,

$$
(s_0 \partial_t e_{pp}, w_{p,h})_{\Omega_p} - \alpha_p b_p(\partial_t \mathbf{e}_s, w_{p,h}) - b_p(\mathbf{e}_p, w_{p,h}) = 0. \tag{2.47}
$$

Where $e_{pp} = p_p - p_{p,h}, \mathbf{e}_s = \boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}$ and $\mathbf{e}_p = \mathbf{u}_p - \mathbf{u}_{p,h}$. Let $\Pi_{p,h}$ be the mixed finite element interpolation, we write

$$\mathbf{e}_p = \mathbf{u}_p - \mathbf{u}_{p,h} = (\mathbf{u}_p - \Pi_{p,h}\mathbf{u}_p) + (\Pi_{p,h} - \mathbf{u}_{p,h}) := \boldsymbol{\chi}_p + \Phi_{p,h}.$$

Now we chose $w_{p,h} = \nabla \cdot \Phi_{p,h}$ in the equation (2.47), we get

$$\begin{aligned} \|\nabla \cdot \Phi_{p,h}\|^2_{\Omega_p} = -b_p(\Phi_{p,h}, \nabla \cdot \Phi_{p,h}) = &- (s_0 \partial_t e_{pp}, \nabla \cdot \Phi_{p,h})_{\Omega_p} + \alpha_p b_p(\partial_t \mathbf{e}_s, \nabla \cdot \Phi_{p,h})_{\Omega_p} \\ &+ b_p(\boldsymbol{\chi}_p, \nabla \cdot \Phi_{p,h}). \end{aligned} \tag{2.48}$$

By Cauchy-Schwarz inequality, we have $-(s_0 \partial_t e_{pp}, \nabla \cdot \Phi_{p,h})_{\Omega_p} \leq C\|\partial_t e_{pp}\|^2_{\Omega_p} + \epsilon \|\nabla \cdot \Phi_{p,h}\|^2_{\Omega_p}$, and similarly for the two other terms in (2.48), we deduce that

$$\|\nabla \cdot \Phi_{p,h}\|^2_{\Omega_p} \leq C(\|\partial_t e_{pp}\|^2_{\Omega_p} + \|\partial_t \mathbf{e}_s\|^2_{\Omega_p} + \|\nabla \cdot \boldsymbol{\chi}_p\|^2_{\Omega_p}) \tag{2.49}$$

Or we can have $\|\nabla \cdot \Phi_{p,h}\|_{\Omega_p} \leq C(\|\partial_t e_{pp}\|_{\Omega_p} + \|\partial_t \mathbf{e}_s\|_{\Omega_p} + \|\nabla \cdot \boldsymbol{\chi}_p\|_{\Omega_p})$. We have $\|\nabla \cdot \mathbf{e}_p\|_{\Omega_p} \leq \|\nabla \cdot \boldsymbol{\chi}_p\|_{\Omega_p} + \|\nabla \cdot \Phi_{p,h}\|_{\Omega_p}$, therefore

$$\|\nabla \cdot \mathbf{e}_p\|_{\Omega_p} \leq C(\|\partial_t e_{pp}\|_{\Omega_p} + \|\partial_t \mathbf{e}_s\|_{\Omega_p} + \|\nabla \cdot \boldsymbol{\chi}_p\|_{\Omega_p}).$$

By the property of the MFE interpolation, we have $\|\nabla \cdot \boldsymbol{\chi}_p\|_{\Omega_p} \leq Ch^{k_p+1}$, together with lemma (2.2.1), we get the desired result. □

Lemma 2.2.3. *Under assumption of lemma* (2.2.1)*, for any choice of stable spaces when d = 2, and for* $f_f \geq 2, k_p \geq 1, s_p \geq 1$*, and* $k_s \geq 2$ *when* $d = 3$*, there exists a positive constant* $M = M(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \boldsymbol{\eta}_p, \lambda)$*, such that for* $t \in (0, T]$ *we have the estimate*

$$\|\nabla \cdot \mathbf{u}_{p,h}\|_{L^2(0,T;L^\infty(\Omega_p))} \leq M.$$

Proof. We do the following estimations

$$\|\nabla \cdot \mathbf{u}_{p,h}\|_{L^\infty(\Omega_p)} \leq \|\nabla \cdot (\mathbf{u}_{p,h} - \Pi_{p,h}\mathbf{u}_p)\|_{L^\infty(\Omega_p)} + \|\nabla \cdot \Pi_{p,h}\mathbf{u}_p\|_{L^\infty(\Omega_p)}.$$

With Piola's transformation, we have $\nabla \cdot \mathbf{v} = \frac{1}{J_E}\hat{\nabla}\cdot\hat{\mathbf{v}}$, where J_E is the Jacobian of the map from reference triangle to FE triangle. So for any finite element vector $\mathbf{v}_{p,h}$ we have the estimation:

$$\|\nabla \cdot \mathbf{v}_{p,h}\|_{\infty,E} \leq \frac{1}{h^d}\|\hat{\nabla}\cdot\hat{\mathbf{v}}_{p,h}\|_{\infty,\hat{E}} \leq \frac{C}{h^d}\|\hat{\nabla}\cdot\hat{\mathbf{v}}_{p,h}\|_{\hat{E}} \leq C\frac{h^{d/2}}{h^d}\|\nabla \cdot \mathbf{v}_{p,h}\|_E = Ch^{-d/2}\|\nabla \cdot \mathbf{v}_{p,h}\|_E.$$

Therefore,

$$\begin{aligned}
&\|\nabla \cdot \mathbf{u}_{p,h}\|_{L^\infty(\Omega_p)} \\
&\leq h^{-d/2}\|\nabla \cdot (\mathbf{u}_{p,h} - \Pi_{p,h}\mathbf{u}_p)\|_{L^2(\Omega_p)} + \|\nabla \cdot \Pi_{p,h}\mathbf{u}_p\|_{L^\infty(\Omega_p)} \\
&\leq h^{-d/2}(\|\nabla \cdot (\mathbf{u}_{p,h} - \mathbf{u}_p)\|_{L^2(\Omega_p)} + \|\nabla \cdot (\mathbf{u}_p - \Pi_{p,h}\mathbf{u}_p)\|_{L^2(\Omega_p)}) + \|\nabla \cdot \Pi_{p,h}\mathbf{u}_p\|_{L^\infty(\Omega_p)}
\end{aligned} \tag{2.50}$$

The MFE interpolation $\Pi_{p,h}$ from $\mathbf{V}_p$ to $\mathbf{V}_{p,h}$ satisfies [1]

$$\|\nabla \cdot (\mathbf{u}_p - \Pi_{p,h}\mathbf{u}_p)\|_{L^2(\Omega_p)} \leq Ch^{r_{k_p}}\|\nabla \cdot \mathbf{u}_p\|_{r_{k_p}}, \quad 1 \leq r_{k_p} \leq k_p + 1.$$

Because $\nabla \cdot \Pi_{p,h}\mathbf{u}_p$ is the L^2-projection of $\nabla \cdot \mathbf{u}_p$, we have the estimation

$$\|\nabla \cdot \Pi_{p,h}\mathbf{u}_p\|_{L^\infty(\Omega_p)} \leq C\|\nabla \cdot \mathbf{u}_p\|_{L^\infty(\Omega_p)} < \infty. \tag{2.51}$$

Hence, from (2.50), by taking integral over t, we get the desired result. □

In the next section, we need the fact that $\|\nabla\cdot\mathbf{u}_{f,h}\|_{L^\infty(\Omega_f)}$, and $\|\nabla\cdot\mathbf{u}_{p,h}\|_{L^\infty(\Omega_p)}$ is bounded to do analysis. From the lemma 2.2.1, we get the bound for $\|\nabla\cdot\mathbf{u}_{f,h}\|_{L^\infty(\Omega_f)}$, and from lemma 2.2.3, we at least get the bound of $\|\nabla \cdot \mathbf{u}_{p,h}\|_{L^2(0,T;L^\infty(\Omega_p))}$.

2.3 Weak formulation for transport

Let $\mathcal{T}_h$ be a quasi-uniform family of triangle partition of Ω, where h is the maximal element diameter. We denote by E_h all interior edges, and $E_{h,out}, E_{h,in}$ to be the set of edges on $\Gamma_{out}, \Gamma_{in}$ respectively. On each edges, a unit normal vector $\mathbf{n}_e$ is arbitrarily fixed. On the boudary, $\mathbf{n}_e$ coindides with the outward unit normal vector. Following [79], we adopt the DG scheme known as the non-symmetric interior penalty Galerkin (NIPG) [71].

For $s \geq 0$, define

$$H^s(\mathcal{T}_h) = \{\phi \in L^2(\Omega) : \phi \in H^s(E), E \in \mathcal{T}_h\}.$$

Now we define the jump and average for $\phi \in H^s(\mathcal{T}_h), s > 1/2$. Let $E_i, E_j \in H^s(\mathcal{T}_h)$ and $e = \partial E_i \cap \partial E_j \in E_h$, with $\mathbf{n}_e$ exterior to E_i. Denote the jump to be

$$[\phi] = (\phi|_{E_i})|_e - (\phi|_{E_j})|_e,$$

and the average

$$\{\phi\} = \frac{1}{2}((\phi|_{E_i})|_e + (\phi|_{E_j})|_e).$$

The usual Sobolev norm on each element is denoted by $\|\cdot\|_{m,E}$, we equip the the space $H^s(\mathcal{T}_h)$ with the norm

$$|||\phi|||_{m,\Omega} = (\sum_{E\in\mathcal{T}_h} \|\phi\|^2_{m,E})^{1/2}.$$

The finite element space is taken to be

$$\mathcal{D}_r(\mathcal{T}_h) = \{\phi \in L^2(\Omega : \phi|_E \in \mathcal{P}_r(E)), E \in \mathcal{T}_h\},$$

where $\mathcal{P}_r(E)$ denotes the space of polynomials of degree less than or equal to r on E.

Now, we are ready to set up the DG scheme adopting the idea in [79]. First, let us define the bilinear form $B_{\mathbf{u}_h}(c, \psi)$ and the linear functional $L_h(\psi)$ as follow.

$$B_{\mathbf{u}_h}(c,\psi) = \sum_{E\in\mathcal{T}_h}\int_E (\mathbf{D}(\mathbf{u}_h)\nabla c - c\mathbf{u}_h)\cdot\nabla\psi - \sum_{e\in E_h}\int_e \{\mathbf{D}(\mathbf{u}_h)\nabla c\cdot\mathbf{n}_e\}[\psi] \tag{2.52}$$

$$+ \sum_{e\in E_h}\int_e \{\mathbf{D}(\mathbf{u}_h)\nabla\psi\cdot\mathbf{n}_e\}[c] + \sum_{e\in E_h}\int_e c^*\mathbf{u}_h\cdot\mathbf{n}_e[\psi] + \sum_{e\in E_{h,out}}\int_e c\mathbf{u}_h\cdot\mathbf{n}_e\psi \tag{2.53}$$

$$- \int_\Omega c q^- \psi + J_0^{\sigma,\beta}(c,\psi) \tag{2.54}$$

where, $c^*|_e$ is the upwind value of concentration

$$c^*|_e = \begin{cases} c|_{E_1} \text{ if } \mathbf{u}_h \cdot \mathbf{n}_e > 0 \\ c|_{E_2} \text{ if } \mathbf{u}_h \cdot \mathbf{n}_e < 0. \end{cases}$$

for $\mathbf{n}_e$ is the outward unit normal vector to E_1, and

$$q^+ = \max(q, 0)$$
$$q^- = \min(q, 0).$$

$J_0^{\sigma,\beta}(c,\psi)$ is the interior penalty term, defined as follow

$$J_0^{\sigma,\beta}(c,\psi) := \sum_{e \in E_h} \frac{\sigma_e}{h_e^\beta} \int_e [c][\psi]$$

where, σ is a discrete positive function that takes constant value σ_e on the edge, and bounded below by $\sigma_* > 0$ and above σ^*, h_e is the side of e and $\beta \geq 0$ is a real number. The linear functional $L_h(\psi)$ is defined as

$$L_h(\psi) = \int_\Omega c_w q^+ \psi - \sum_{e \in E_{h,in}} \int_e c_{in} \mathbf{u}_h \cdot \mathbf{n}_e \psi. \tag{2.55}$$

The the DG method for the transport problem is stated as follow: find $c_h \in L^\infty(J, \mathcal{D}_r(\mathcal{T}_h))$ such that

$$(\phi \frac{\partial c_h}{\partial t}, \psi) + B_{\mathbf{u}_h}(c_h, \psi) = L_h(\psi), \forall \psi \in \mathcal{D}_r(\mathcal{T}_h), \forall t \in J, \tag{2.56}$$

$$(c_h, \psi) = (c_0, \psi), \forall \psi \in \mathcal{D}_r(\mathcal{T}_h), t = 0. \tag{2.57}$$

Let $\hat{P}_h$ denote the L^2 projection of $H^s(\mathcal{T}_h)$ onto $\mathcal{D}_r(\mathcal{T}_h)$, and define the interpolation error, finite element error as:

$$E_c^I = \hat{P}_h c - c, \; E_c = c - c_h.$$

In this section we discuss the stability and error estimates for the transport problem (2.56). We note that a similar scheme has been used and analyzed in details in [79]. The main difference and improvement in this work is the fact that the numerically computed velocity field $\mathbf{u}_h$ is directly incorporated into the scheme for transport (2.56), while in [79] the authors used a special "cut-off" operator in order to ensure optimal properties of the method.

2.4 Stability analysis

First, we introduce some notation for norms as follow

$$\|\mathbf{u}\|_{(L^2(\Omega))^d} = \|(|\mathbf{u}|_2)\|_{L^2(\Omega)}, \tag{2.58}$$

$$\|\mathbf{u}\|_{(L^\infty(\Omega))^d} = \|(|\mathbf{u}|_2)\|_{L^\infty(\Omega)}. \tag{2.59}$$

Where $|\cdot|_2$ is the usual Euclidean norm for vectors. In [79], there given the following properties of dispenser matrix $\mathbf{D}$.

Lemma 2.4.1. *Let* $\mathbf{D}(\mathbf{u})$ *defined as in equation* (2.18)*, where,* $d_m(x) \geq 0, \alpha_l(x) \geq 0$ *and* $\alpha_t(x) \geq 0$ *are nonnegative functions of* $x \in \Omega$*. Then*

$$\mathbf{D}(\mathbf{u})\nabla c \cdot \nabla c \geq (d_m + \min(\alpha_l, \alpha_t)|\mathbf{u}|)|\nabla c|_2^2. \tag{2.60}$$

In particular, if $d_m(x) \geq d_{m,*} > 0$ *uniformly in the domain* Ω*, then* $\mathbf{D}(\mathbf{u})$ *is uniformly positive definite and for all* $x \in \Omega$*, we have,*

$$\mathbf{D}(\mathbf{u})\nabla c \cdot \nabla c \geq d_{m,*}|\nabla c|_2^2. \tag{2.61}$$

Lemma 2.4.2. *Let* $\mathbf{D}(\mathbf{u})$ *defined as in equation* (2.18)*, where,* $d_m(x) \geq 0, \alpha_l(x) \geq 0$ *and* $\alpha_t(x) \geq 0$ *are nonnegative function of* $x \in \Omega$*, and the dispersivity* α_l *and* α_t *are uniformly bounded, i.e.* $\alpha_l(x) \leq \alpha_l^*$ *and* $\alpha_t(x) \leq \alpha_t^*$*.*

Then

$$\|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})\|_{(L^2(\Omega))^{d\times d}} \leq k_D\|\mathbf{u} - \mathbf{v}\|_{(L^2(\Omega))^d} \tag{2.62}$$

where, $k_D = (4\alpha_t^* + 3\alpha_l^*)d^{3/2}$ *is a fixed number (*$d = 2$ *or* 3 *is the dimension of domain* Ω*.)*

In addition, let $|\mathbf{D}(\mathbf{u})|_2$ be the matrix norm of $\mathbf{D}(\mathbf{u})$ induced by the usual Euclidean norm, one can show that $|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_2 \leq C|\mathbf{u} - \mathbf{v}|_2$.

In the lemmas about stability estimate and also error estimate below, we need an assumption that $\|\nabla \cdot \mathbf{u}_h\|_{L^\infty(\Omega)}$ is bounded. From lemma (2.2.1), we have $\|\mathbf{u}_{p,f}\|_{L^\infty(\Omega_f)}$ is bounded, and by lemma (2.2.3) we have $\|\nabla \cdot \mathbf{u}_{p,h}\|_{L^2(0,T;L^\infty(\Omega_p))}$ is bounded, which is weaker than what we need. Another approach to the problem is to prove that $\mathbf{u}_h$ is bounded, it is accomplished in [4].

Lemma 2.4.3. *Assume that* $\|\nabla\cdot\mathbf{u}_h\|_{L^\infty(\Omega)}$ *is bounded, and let* c_h *be a solution of* (2.56). *Then for all* $t\in[0,T]$, *we have*

$$\begin{aligned} &|||c_h(T)|||^2_{0,\Omega}+\int_0^T|||\nabla c_h|||^2_{0,\Omega}ds \\ &\le Ce^T\Big(|||c_h(0)|||^2_{0,\Omega}+\int_0^T\big[\int_\Omega c_w^2(q^+)^2+\sum_{E_{h,in}}\int_e|\mathbf{u}_h\cdot\mathbf{n}_e|c_{in}^2\big]\Big)\end{aligned}\tag{2.63}$$

Proof. At each time t, we take $\psi=c_h(t)$ in the equation (2.56), then we get

$$\begin{aligned}(\phi\frac{\partial c_h}{\partial t},c_h)&+\sum_{E\in\mathcal{T}_h}\int_E(\mathbf{D}(\mathbf{u}_h)\nabla c_h-c_h\mathbf{u}_h)\cdot\nabla c_h-\sum_{e\in E_h}\int_e\{\mathbf{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\}[c_h]\\ &+\sum_{e\in E_h}\int_e\{\mathbf{D}(\mathbf{u}_h)\nabla c_h\cdot\mathbf{n}_e\}[c_h]+\sum_{e\in E_h}\int_e c_h^*\mathbf{u}_h\cdot\mathbf{n}_e[c_h]+\sum_{e\in E_{u,out}}\int_e\mathbf{u}_h\cdot\mathbf{n}_ec_h^2\\ &-\int_\Omega q^-c_h^2+J_0^{\sigma,\beta}(c_h,c_h)\\ &=\int_\Omega c_wq^+c_h-\sum_{e\in E_{h,in}}\int_e c_{in}\mathbf{u}_h\cdot\mathbf{n}_ec_h.\end{aligned}\tag{2.64}$$

We have

$$\begin{aligned}-\sum_{E\in\mathcal{T}_h}\int_E(c_h\mathbf{u}_h)\cdot\nabla c_h&=\frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E(\nabla\cdot\mathbf{u}_h)c_h^2-\frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_{\partial E}(\mathbf{u}_h\cdot\mathbf{n}_E)c_h^2\\ =\frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E(\nabla\cdot\mathbf{u}_h)c_h^2&-\frac{1}{2}\sum_{e\in E_h}\int_e(\mathbf{u}_h\cdot\mathbf{n}_e)[c_h^2]-\frac{1}{2}\sum_{e\in E_{h,in,out}}(\mathbf{u}_h\cdot\mathbf{n}_e)c_h^2\end{aligned}$$

Now, we make the following abbreviation

$$J_1:=(\phi\frac{\partial c_h}{\partial t},c_h)_\Omega+\sum_{E\in\mathcal{T}_h}\int_E\mathbf{D}(\mathbf{U}_h)\nabla c_h\cdot\nabla c_h\tag{2.65}$$

$$J_2:=\frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E(\nabla\cdot\mathbf{u}_h)c_h^2-\frac{1}{2}\sum_{e\in E_h}\int_e(\mathbf{u}_h\cdot\mathbf{n}_e)[c_h^2]+\sum_{e\in E_h}\int_e c_h^*\mathbf{u}_h\cdot\mathbf{n}_e[c_h]\tag{2.66}$$

$$\begin{aligned}J_3&:=\frac{1}{2}\sum_{e\in E_{h,out}}\langle c_h\mathbf{u}_h\cdot\mathbf{n}_e,c_h\rangle_e-\frac{1}{2}\sum_{e\in E_{h,in}}\langle c_h\mathbf{u}_h\cdot\mathbf{n}_e,c_h\rangle_e\\ &=\frac{1}{2}\sum_{e\in E_{h,out}}\langle c_h\mathbf{u}\cdot\mathbf{n}_e,c_h\rangle_e-\frac{1}{2}\sum_{e\in E_{h,in}}\langle c_h\mathbf{u}\cdot\mathbf{n}_e,c_h\rangle_e\end{aligned}\tag{2.67}$$

$$J_4:=-\int_\Omega q^-c_h^2+J_0^{\sigma,\beta}(c_h,c_h).\tag{2.68}$$

We have the last equality for J_3 because on Γ_f, Γ_p, the velocity $\mathbf{u}_h$ and the normal component $\mathbf{u}_h \cdot \mathbf{n}$ are the L^2 -projection of the true velocity. The equation (2.64) become

$$J_1 + J_2 + J_3 + J_4 = L_h(c_h).$$

With each term, we do the following estimate.

$$(\phi \frac{\partial c_h}{\partial t}, c_h)_\Omega = \frac{\phi}{2} \frac{\partial}{\partial t} |||c_h|||^2_{0,\Omega},$$

using lemma (2.4.1), we have

$$\sum_{E \in \mathcal{T}_h} \int_E \mathbf{D}(\mathbf{u}_h) \nabla c_h \cdot \nabla c_h \geq d_{m,*} |||\nabla c_h|||^2_{0,\Omega}$$

Therefore, $J_1 \geq C(\frac{\partial}{\partial t} |||c_h|||^2_{0,\Omega} + |||\nabla c_h|||^2_{0,\Omega})$ for some constant C. With the second term,

$$\begin{aligned} J_2 &= \frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E (\nabla \cdot \mathbf{u}_h) c_h^2 + \sum_{e \in E_h} \int_e (c_h^* - \{c_h\})(\mathbf{u}_h \cdot \mathbf{n}_e)[c_h] \\ &= \frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E (\nabla \cdot \mathbf{u}_h) c_h^2 + \frac{1}{2} \sum_{e \in E_h} \int_e |\mathbf{u}_h \cdot n_e| [c_h]^2. \end{aligned}$$

We have the estimate $\frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E (\nabla \cdot \mathbf{u}_h) c_h^2 \leq C |||c_h|||^2_{0,\Omega}$, where $C = \frac{1}{2} \|\nabla \cdot \mathbf{u}_h\|_{L^\infty(\Omega)}$. The second term in J_2 is positive.

Clearly, we have the two terms J_3, J_4 are also positive. From the above arguments, one can deduce that

$$\begin{aligned} (\phi \frac{\partial c_h}{\partial t}, c_h) + B_{\mathbf{u}_h}(c_h, c_h) &\geq \phi_* \frac{\partial}{\partial t} |||c_h|||^2_{0,\Omega} + d_{m,*} |||\nabla c_h|||^2_{0,\Omega} \\ &+ \frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E (\nabla \cdot \mathbf{u}_h) c_h^2 + \frac{1}{2} \sum_{e \in E_h} \int_e |\mathbf{u}_h \cdot \mathbf{n}_e| [c_h]^2 \\ + \frac{1}{2} \sum_{e \in E_{h,out}} \langle c_h \mathbf{u}_h \cdot \mathbf{n}_e, c_h \rangle_e &- \frac{1}{2} \sum_{e \in E_{h,in}} \langle c_h \mathbf{u}_h \cdot \mathbf{n}_e, c_h \rangle_e - \int_\Omega q^- c_h^2 + J_0^{\sigma,\beta}(c_h, c_h). \end{aligned} \tag{2.69}$$

Notice that, except for the term $\frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E (\nabla \cdot \mathbf{u}_h) c_h^2$, all other terms of the right hand side are positive. We have,

$$\int_\Omega c_w q^+ c_h \leq \int_\Omega c_w^2 (q^+)^2 + \int_\Omega c_h^2.$$

and,

$$-\sum_{e\in E_{h,in}}\int_e c_{in}\mathbf{u}_h\cdot\mathbf{n}_e c_h \le \sum_{e\in E_{h,in}} |\mathbf{u}_h\cdot\mathbf{n}_e|(\epsilon c_h^2+\frac{1}{4\epsilon}c_{in}^2)$$

The term $\sum_{e\in E_{h,in}}|\mathbf{u}_h\cdot\mathbf{n}_e|\epsilon c_h^2$ can be hiddden by the term J_3. Combining all the above estimates, and from the equation (2.64), we get to the inequality,

$$\frac{\partial}{\partial t}|||c_h|||_{0,\Omega}^2+|||\nabla c_h|||_{0,\Omega}^2 \le C\big(|||c_h|||_{0,\Omega}^2+\int_\Omega c_w^2(q^+)^2+\sum_{E_{h,in}}\int_e|\mathbf{u}_h\cdot\mathbf{n}_e|c_{in}^2\big)$$

for some constant C. Integrating the above equation from 0 to $t\in[0,T]$, we have

$$\begin{aligned}&|||c_h(t)|||_{0,\Omega}^2+\int_0^t|||\nabla c_h|||_{0,\Omega}^2\\ \le\ &|||c_h(0)|||_{0,\Omega}^2+C\int_0^t\big[|||c_h|||_{0,\Omega}^2+\int_\Omega c_w^2(q^+)^2+\sum_{E_{h,in}}\int_e|\mathbf{u}_h\cdot\mathbf{n}_e|c_{in}^2\big].\end{aligned}$$

Thus, by Gronwall's lemma, for all $t\in[0,T]$, we have

$$\begin{aligned}&|||c_h(T)|||_{0,\Omega}^2+\int_0^T|||\nabla c_h|||_{0,\Omega}^2\\ \le\ &Ce^T\Big(|||c_h(0)|||_{0,\Omega}^2+\int_0^T\big[\int_\Omega c_w^2(q^+)^2+\sum_{E_{h,in}}\int_e|\mathbf{u}_h\cdot\mathbf{n}_e|c_{in}^2\big]\Big).\end{aligned}$$

We just completed the lemma of stability estimate. □

2.5 Error estimate

Now, we will do analysis about error estimate. Let $\bar{\Pi}\mathbf{u} \in \mathbf{V}_h$ denote the L^2 projection of $\mathbf{u}$, that is

$$(\mathbf{u} - \bar{\Pi}\mathbf{u}, \mathbf{v}_h)_\Omega = 0, \forall \mathbf{v}_h \in \mathbf{V}_h.$$

The L^2 projection has the approximation property [27],

$$\|\mathbf{u} - \bar{\Pi}\mathbf{u}\|_{m,\Omega} \le Kh^{l-m}\|\mathbf{u}\|_{l,\Omega}, \quad 0 \le m \le l \le \min\{k_f, k_p, k_s\} + 1$$

From the above approximation property and the trace inequality:

$$\forall e \in \partial E, \|\mathbf{u}\|_{L^2(e)} \le K(h^{-1/2}\|\mathbf{u}\|_{L^2(E)} + h^{1/2}|\mathbf{u}|_{1,E}), \quad \forall \mathbf{u} \in (H^1(E))^d \tag{2.70}$$

by summing all over elements, we can deduce that

$$\sum_{E\in\mathcal{T}_h}\sum_{e\in\partial E} \|\mathbf{u} - \bar{\Pi}\mathbf{u}\|_{L^2(e)} \le Kh^{l-1/2}\|\mathbf{u}\|_{l,\Omega}, 1 \le l \le \min\{k_f, k_p, k_s\} + 1. \tag{2.71}$$

Let Πc be the Scott Zhang interpolation of c. First, we prove the following lemma.

Lemma 2.5.1. *Assume $\Pi c \in \mathcal{D}_r(\mathcal{T}_h)$. For any point p on any edge, we have $|\Pi c(p) - c(p)| < Ch^{r+1}$. Where C depends only on c and independent of h.*

Proof. If $r = 0$, assume E is the element that contains e. Let g be the centroid of E, then $\Pi c(g) = c(g)$. For any $p \in e$, we have $\Pi c(p) = \Pi c(g) = c(g)$.
Consider the function $f(t) := c((1-t)g + tp)$, we have $c(p) - c(g) = f(1) - f(0) = f'(\xi), \xi \in (0,1)$. We have $f'(\xi) = c_x(x_p - x_g) + c_y(y_p - y_g)$, where x_p, y_p are the x, y coordinates of p respectively. Notice that $|x_p - x_g| \le h, |y_p - y_g| \le h$, hence $|c(p) - \Pi c(p)| \le Ch$, where $C = 2\|c\|_{W_1^\infty(\Omega)}$.

If $0 < r$, then by restricting on each edge, Πc become one dimension Lagrange interpolation of c on such edge. Thus, by interpolation theory we have $|\Pi c(p) - c(p)| < Ch^{r+1}$. $\square$

We define $\theta_c := c - \Pi c$, and $\delta_c := c_h - \Pi c$. By the approximation property in [27], we have

$$\|\|\theta_c\|\|_\Omega \le Ch^{r+1}, \text{ and } \|\|\nabla\theta_c\|\|_\Omega \le Ch^r. \tag{2.72}$$

We also define a bi-linear form,

$$\begin{aligned} B_{\mathbf{u}}(c,\psi) &:= \sum_{E\in\mathcal{T}_h}\int_E (\mathbf{D}(\mathbf{u})\nabla c - c\mathbf{u})\cdot\nabla\psi - \sum_{e\in E_h}\int_e \{\mathbf{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_e\}[\psi] \\ &+ \sum_{e\in E_h}\int_e \{\mathbf{D}(\mathbf{u})\nabla\psi\cdot\mathbf{n}_e\}[c] + \sum_{e\in E_h}\int_e c^*\mathbf{u}\cdot\mathbf{n}_e[\psi] + \sum_{e\in E_{h,out}}\int_e c\mathbf{u}\cdot\mathbf{n}_e\psi \\ &- \int_\Omega cq^-\psi + J_0^{\sigma,\beta}(c,\psi) \end{aligned} \tag{2.73}$$

Also, the linear form

$$L(\psi) := \int_\Omega c_w q^+\psi - \sum_{e\in E_{h,in}}\int_e c_{in}\mathbf{u}\cdot\mathbf{n}_e\psi.$$

Becasue c is the true solution, c satisfies the equation

$$(\phi\partial_t c,\psi) + B_{\mathbf{u}}(c,\psi) = L(\psi), \forall\psi\in\mathcal{D}_r(\mathcal{T}_h). \tag{2.74}$$

We take (2.56) subtract (2.74), we get

$$\begin{aligned} &(\phi\partial_t(c_h-\Pi c),\psi)_\Omega + B_{\mathbf{u}_h}(c_h-\Pi c,\psi) \\ &= (\phi\partial_t(c-\Pi c),\psi)_\Omega + B_{\mathbf{u}}(c-\Pi c,\psi) + B_{\mathbf{u}}(\Pi c,\psi) - B_{\mathbf{u}_h}(\Pi c,\psi) + L_h(\psi) - L(\psi), \\ &\forall\psi\in\mathcal{D}_r(\mathcal{T}_h). \end{aligned}$$

If we chose $\psi = \delta_c$ in the above equation, then it becomes

$$\begin{aligned} &(\phi\partial_t\delta_c,\delta_c)_\Omega + B_{\mathbf{u}_h}(\delta_c,\delta_c) \\ &= (\phi\partial_t\theta_c,\delta_c)_\Omega + B_{\mathbf{u}}(\theta_c,\delta_c) + B_{\mathbf{u}}(\Pi c,\delta_c) - B_{\mathbf{u}_h}(\Pi c,\delta_c) + L_h(\delta_c) - L(\delta_c). \end{aligned} \tag{2.75}$$

Similar to previous section, we denote $\theta_c := c - \Pi c$, and $\delta_c := c_h - \Pi c$. By replacing c_h by δ_c in (2.69), then from there we can deduce that

$$(\phi\partial_t\delta_c,\delta_c)_\Omega + B_{\mathbf{u}_h}(\delta_c,\delta_c) \ge \phi_*\frac{\partial}{\partial t}\|\|\delta_c\|\|^2_{0,\Omega} + d_{m,*}\|\|\nabla\delta_c\|\|^2_{0,\Omega} + \frac{1}{2}\int_\Omega(\nabla\cdot\mathbf{u}_h)\delta_c^2$$

$$+\frac{1}{2}\sum_{e\in E_h}\int_e |\mathbf{u}_h\cdot\mathbf{n}_e|[\delta_c]^2+\frac{1}{2}\sum_{e\in E_{h,out}}\langle\delta_c\mathbf{u}_h\cdot\mathbf{n}_e,\delta_c\rangle_e-\frac{1}{2}\sum_{e\in E_{h,in}}\langle\delta_c\mathbf{u}_h\cdot\mathbf{n}_e,\delta_c\rangle_e$$
$$-\int_\Omega q^-\delta_c^2+J_0^{\sigma,\beta}(\delta_c,\delta_c)$$

Hence,

$$(\phi\partial_t\delta_c,\delta_c)_\Omega+B_{\mathbf{u}_h}(\delta_c,\delta_c)+M\int_\Omega\delta_c^2\geq\phi_*\frac{\partial}{\partial t}|||\delta_c|||_{0,\Omega}^2+d_{m,*}|||\nabla\delta_c|||_{0,\Omega}^2$$
$$+\frac{1}{2}\sum_{e\in E_h}\int_e |\mathbf{u}_h\cdot\mathbf{n}_e|[\delta_c]^2+\frac{1}{2}\sum_{e\in E_{h,out}}\langle\delta_c\mathbf{u}_h\cdot\mathbf{n}_e,\delta_c\rangle_e-\frac{1}{2}\sum_{e\in E_{h,in}}\langle\delta_c\mathbf{u}_h\cdot\mathbf{n}_e,\delta_c\rangle_e$$
$$-\int_\Omega q^-\delta_c^2+J_0^{\sigma,\beta}(\delta_c,\delta_c)$$

where $M=\frac{1}{2}\|\nabla\cdot\mathbf{u}_h\|_\infty$. Now, we are going to do analysis for the right hand side of (2.75). We have

$$(\phi\partial_t\theta_c,\delta_c)_\Omega\leq\frac{(\phi^*)^2}{4}|||\partial_t\theta_c|||_\Omega^2+|||\delta_c|||_\Omega^2. \tag{2.76}$$

And,

$$B_{\mathbf{u}}(\Pi c,\delta_c)-B_{\mathbf{u}_h}(\Pi c,\delta_c)=\sum_{E\in\mathcal{T}_h}\int_E((\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u}_h))\nabla\Pi c)\cdot\nabla\delta_c$$
$$-\sum_{E\in\mathcal{T}_h}\int_E\Pi c(\mathbf{u}-\mathbf{u}_h)\cdot\nabla\delta_c-\sum_{e\in E_h}\int_e\{(\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u}_h))\nabla\Pi c\cdot\mathbf{n}_e\}[\delta_c]$$
$$+\sum_{e\in E_h}\int_e\{(\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u}_h))\nabla\delta_c\cdot\mathbf{n}_e\}[\Pi c]+\sum_{e\in E_h}\int_e(\Pi c)^*(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e[\delta_c]$$
$$+\sum_{e\in E_{h,out}}\int_e\Pi c(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e\delta_c=:T_1+T_2+...+T_6.$$

and,

$$B_{\mathbf{u}}(\theta_c,\delta_c)=\sum_{E\in\mathcal{T}_h}(\mathbf{D}(\mathbf{u})\nabla\theta_c,\nabla\delta_c)_\Omega-\sum_{E\in\mathcal{T}_h}\theta_c\mathbf{u}\cdot\nabla\delta_c-\sum_{e\in E_h}\int_e\{\mathbf{D}(\mathbf{u})\nabla\theta_c\cdot\mathbf{n}_e\}[\delta_c]$$
$$+\sum_{e\in E_h}\int_e\{\mathbf{D}(\mathbf{u})\nabla\delta_c\cdot\mathbf{n}_e\}[\theta_c]+\sum_{e\in E_h}\int_e(\theta_c)^*\mathbf{u}\cdot\mathbf{n}_e[\delta_c]+\sum_{e\in E_{h,out}}\int_e\theta_c\mathbf{u}\cdot\mathbf{n}_e\delta_c$$
$$-\int_\Omega q^-\theta_c\delta_c+J_0^{\sigma,\beta}(\theta_c,\delta_c)=:H_1+H_2+...+H_8.$$

Now, we will give estimate for each term T_i above. Using ∇c is bounded and $\|\nabla \Pi c\|_\infty \leq C\|\nabla c\|_\infty$, together with $\|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h)\|_2 \leq M\|\mathbf{u} - \mathbf{u}_h\|_2$, we have

$$T_1 \leq C \sum_{E\in\mathcal{T}_h} \|\mathbf{u} - \mathbf{u}_h\|_{L^2(E)} \|\nabla\delta_c\|_{L^2(E)} \leq C(\frac{1}{4\epsilon}\|\!|\mathbf{u} - \mathbf{u}_h\|\!|^2_{0,\Omega} + \epsilon\|\!|\nabla\delta_c\|\!|^2_{0,\Omega}).$$

By using Πc is bounded, we also have similar estimate for the second term

$$T_2 \leq C(\frac{1}{4\epsilon}\|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(\Omega)} + \epsilon\|\!|\nabla\delta_c\|\!|^2_{0,\Omega}).$$

We will use the penalty term to handle the third term as follow,

$$\begin{aligned}
T_3 &= -\sum_{e\in E_h}\int_e \{(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla\Pi c \cdot \mathbf{n}_e\}[\delta_c] \\
&\leq \sum_{e\in E_h}\int_e \frac{h}{4\epsilon}\{(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla\Pi c \cdot \mathbf{n}_e\}^2 + \frac{\epsilon}{h}[\delta_c]^2 \\
&\leq \sum_{e\in E_h}\int_e \frac{h}{4\epsilon}|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h)|_2^2(|(\nabla\Pi c)^+|_2^2 + |(\nabla\Pi c)^-|_2^2) + \frac{\epsilon}{h}[\delta_c]^2 \\
&\leq \sum_{e\in E_h}\frac{Ch}{4\epsilon}\|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(e)} + \sum_{e\in E_h}\int_e \frac{\epsilon}{h}[\delta_c]^2
\end{aligned}$$

Where, in the third step, we have used the fact that $\nabla\Pi c$ is bounded. Now, we use $[c] = 0$ in interior edges, and lemma (2.5.1) for the following estimate.

$$\begin{aligned}
T_4 &= \sum_{e\in E_h}\int_e \{(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla\delta_c \cdot \mathbf{n}_e\}[\Pi c - c] \qquad (2.77)\\
&\leq C\sum_{e\in E_h} h\int_e |\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h)|_2(|(\nabla\delta_c)^+|_2 + |(\nabla\delta_c)^-|_2) \\
&\leq C\sum_{e\in E_h} h\int_e |\mathbf{u} - \mathbf{u}_h|_2(|(\nabla\delta_c)^+|_2 + |(\nabla\delta_c)^-|_2) \\
&\leq C\sum_{e\in E_h}\int_e \frac{h}{4\epsilon}|\mathbf{u} - \mathbf{u}_h|_2^2 + h\epsilon|(\nabla\delta_c)^+|_2^2 + h\epsilon|(\nabla\delta_c)^-|_2^2 \\
&\leq C\sum_{e\in E_h}\frac{h}{4\epsilon}\|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(e)} + C\epsilon\sum_{E\in\mathcal{T}_h}\|\nabla\delta_c\|^2_{L^2(E)}
\end{aligned}$$

Where in the last step, we have used the trace and inverse inequality. We can also use penalty term to handle the next term.

$$T_5 = \sum_{e\in E_h}\int_e (\Pi c)^*(\mathbf{u} - \mathbf{u}_h)\cdot\mathbf{n}_e[\delta_c] \leq \sum_{e\in E_h}\int_e \frac{h}{4\epsilon}\big((\Pi c)^*(\mathbf{u} - \mathbf{u}_h)\cdot\mathbf{n}_e\big)^2 + \frac{\epsilon}{h}[\delta_c]^2 \qquad (2.78)$$

$$\leq \sum_{e\in E_h} \frac{C}{4\epsilon} h\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(e)} + \sum_{e\in E_h} \int_e \frac{\epsilon}{h}[\delta_c]^2$$

The term T_6 can be done as follow.

$$T_6 = \frac{1}{2}\sum_{E_{h,out}} \int_e \Pi c(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e\delta_c \leq \frac{1}{2}\int_\Gamma \frac{1}{4h\epsilon}|\Pi c(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e|_2^2 + h\epsilon\delta_c^2 \tag{2.79}$$
$$\leq C\int_\Gamma \frac{1}{h\epsilon}|(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e|_2^2 + \epsilon\int_\Omega \delta_c^2.$$

Similarly,

$$L_h(\delta_c) - L(\delta_c) = \sum_{e\in E_{h,in}} \int_e c_{in}(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e\delta_c \leq C\int_\Gamma \frac{1}{h\epsilon}|(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e|_2^2 + \epsilon\int_\Omega \delta_c^2. \tag{2.80}$$

Now, we will give estimate for H_i terms,

$$H_1 = \sum_{E\in\mathcal{T}_h} (\mathbf{D}(\mathbf{u})\nabla\theta_c, \nabla\delta_c)_E \leq \sum_{E\in\mathcal{T}_h} \int_E \frac{1}{4\epsilon}|\mathbf{D}(\mathbf{u})\nabla\theta_c|^2 + \epsilon\int_E |\nabla\delta_c|^2 \tag{2.81}$$
$$\leq \frac{C}{4\epsilon}|||\nabla\theta_c|||_{0,\Omega} + \epsilon|||\nabla\delta_c|||_{0,\Omega}$$

$$H_2 = -\sum_{E\in\mathcal{T}_h} \int_E \theta_c\mathbf{u}\cdot\nabla\delta_c \leq \sum_{E\in\mathcal{T}_h} \int_E \frac{1}{4\epsilon}|\theta_c\mathbf{u}|^2 + \epsilon|\nabla\delta_c|^2 \leq \frac{C}{4\epsilon}|||\theta_c|||_{0,\Omega} + \epsilon|||\nabla\delta_c|||_{0,\Omega} \tag{2.82}$$

$$H_3 = -\sum_{e\in E_h} \int_e \{\mathbf{D}(\mathbf{u})\nabla\theta_c\cdot\mathbf{n}_e\}[\delta_c] \leq \sum_{e\in E_h} \int_e \frac{h}{4\epsilon}\{\mathbf{D}(\mathbf{u})\nabla\theta_c\cdot\mathbf{n}_e\}^2 + \frac{\epsilon}{h}[\delta_c]^2 \tag{2.83}$$
$$\leq \frac{C}{4\epsilon}\sum_{e\in E_h} \int_e h|\nabla\theta_c|^2 + \sum_{e\in E_h} \int_e \frac{\epsilon}{h}[\delta_c]^2$$

$$H_4 = \sum_{e\in E_h} \int_e \{\mathbf{D}(\mathbf{u})\nabla\delta_c\cdot\mathbf{n}_e\}[\theta_c] \leq C\sum_{e\in E_h} \int_e \epsilon h|\nabla\delta_c|^2 + \frac{h^{-1}}{\epsilon}[\theta_c]^2 \tag{2.84}$$
$$\leq C\Big(\sum_{E\in\mathcal{T}_h} \int_E \epsilon(\nabla\delta_c)^2 + \sum_{e\in E_h} \frac{h^{-1}}{\epsilon}[\theta_c]^2\Big)$$

Where we have used the trace and inverse inequality for the second estimate.

$$H_5 = \sum_{e\in E_h} \int_e (\theta_c)^*\mathbf{u}\cdot\mathbf{n}_e[\delta_c] \leq \sum_{e\in E_h} C\int_e h((\theta_c)^*)^2 + \frac{\epsilon}{h}[\delta_c]^2 \tag{2.85}$$

$$\leq \sum_{E\in\mathcal{T}_h} C\int_E \theta_c^2 + h^2|\nabla\theta_c|^2 + \sum_{e\in E_h}\frac{\epsilon}{h}[\delta_c]^2$$

$$H_6 = \sum_{e\in E_{h,out}}\int_e \theta_c \mathbf{u}\cdot\mathbf{n}_e\delta_c \leq \sum_{e\in E_{h,out}}\int_e Ch^{-1}\theta_c^2 + h\delta_c^2 \tag{2.86}$$
$$\leq \sum_{e\in E_{h,out}}\int_e Ch^{-1}\theta_c^2 + \sum_{E\in\mathcal{T}_h}\int_E \delta_c^2$$

$$H_7 = \int_\Omega (-q^-)\theta_c\delta_c \leq \int_\Omega (-q^-)\frac{1}{4\epsilon}\theta_c^2 + \int_\Omega (-q^-)\epsilon\delta_c^2 \leq \int_\Omega C\theta_c^2 + \int_\Omega (-q^-)\epsilon\delta_c^2 \tag{2.87}$$

Notice that $-q^-$ is positive, and the second term in H_7 can be hidden in the left hand side. The term H_8 can be handled similarly. We take $\beta = 1$ in the penalty term.

$$H_8 = J_0^{\sigma,\beta}(\theta_c,\delta_c) = \sum_{e\in E_h}\frac{\sigma}{h}\int_e [\theta_c][\delta_c] \leq \sum_{e\in E_h}\frac{\sigma}{h}\int_e \frac{1}{4\epsilon}[\theta_c]^2 + \sum_{e\in E_h}\frac{\sigma}{h}\int_e \epsilon[\delta_c]^2 \tag{2.88}$$
$$\leq \sum_{E\in\mathcal{T}_h}\frac{C\sigma}{h^2}\int_E \theta_c^2 + h^2|\nabla\theta_c|^2 + \sum_{e\in E_h}\frac{\sigma}{h}\int_e \epsilon[\delta_c]^2.$$

Integrating (2.75) over $[0,\tau]$ with $\tau\in(0,T]$, and by combining all the above estimate, we deduce the following estimate using Gronwall's lemma.

$$|||\delta_c(\tau)|||_{0,\Omega}^2 + \int_0^\tau |||\nabla\delta_c|||_{0,\Omega}^2 + \sum_{e\in E_h}\int_e [\delta_c]^2/h$$
$$\leq Ce^\tau \int_0^\tau \Big(\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}^2 + h\|\mathbf{u}-\mathbf{u}_h\|_{E_h}^2 + h^{-1}\|(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e\|_\Gamma^2$$
$$+|||\partial_t\theta_c|||_{0,\Omega}^2 + |||\nabla\theta_c|||_{0,\Omega}^2 + h^{-2}|||\theta_c|||_{0,\Omega}^2 + h^{-1}\|\theta_c\|_{\Gamma_{out}}^2\Big)ds$$

Where $\|\mathbf{u}\|_{E_h} := (\sum_{e\in E_h}\int_e \mathbf{u}^2)^{1/2}$, $\|\mathbf{u}\|_\Gamma := (\int_\Gamma \mathbf{u}^2)^{1/2}$, and $\|\mathbf{u}\|_{\Gamma_{out}} := (\int_{\Gamma_{out}}\mathbf{u}^2)^{1/2}$. We have the following estimation

$$h^{1/2}\|\mathbf{u}-\mathbf{u}_h\|_{E_h} \leq h^{1/2}(\|\mathbf{u}-\bar{\Pi}\mathbf{u}\|_{E_h} + \|\bar{\Pi}\mathbf{u}-\mathbf{u}_h\|_{E_h})$$
$$\leq h^{1/2}\|\mathbf{u}-\bar{\Pi}\mathbf{u}\|_{E_h} + C\|\bar{\Pi}\mathbf{u}-\mathbf{u}_h\|_\Omega$$
$$\leq h^{1/2}\|\mathbf{u}-\bar{\Pi}\mathbf{u}\|_{E_h} + C\|\bar{\Pi}\mathbf{u}-\mathbf{u}\|_\Omega + C\|\mathbf{u}-\mathbf{u}_h\|_\Omega$$

One can deduce that,

$$\|\delta_c\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla\delta_c\|_{L^2(0,T,L^2(\Omega))} \le Ce^T\Big(\|\mathbf{u}-\mathbf{u}_h\|_{L^2(0,T,L^2(\Omega))}$$

$$+h^{1/2}\|\mathbf{u}-\bar{\Pi}\mathbf{u}\|_{L^2(0,T,L^2(E_h))} + \|\mathbf{u}-\bar{\Pi}\mathbf{u}\|_{L^2(0,T,L^2(\Omega))} + h^{-1/2}\|(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e\|_{L^2(0,T,L^2(\Gamma))}$$

$$+\|\partial_t\theta_c\|_{L^2(0,T,L^2(\Omega))} + \|\nabla\theta_c\|_{L^2(0,T,L^2(\Omega))} + h^{-1}\|\theta_c\|_{L^2(0,T,L^2(\Omega))} + h^{-1/2}\|\theta_c\|_{L^2(0,T,L^2(\Gamma_{out}))}\Big).$$

Where, with any normed space X, $\|x(t)\|_{L^\infty(0,T,X)} := \sup_{t\in[0,T]}\|x(t)\|_X$ and $\|x(t)\|_{L^2(0,T,X)} := (\int_0^T \|x(t)\|_X^2 ds)^{1/2}$. Notice that we have $(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_e|_\Gamma = 0$, so from (2.25), (2.71) and (2.72), by using triangle inequality we get the following result.

Lemma 2.5.2. *Assume $\|\nabla\cdot\mathbf{u}_h\|_{L^\infty(\Omega)}$ is bounded, and let $E_c = c - c_h$ be the error of the concentration and its finite element approximation. Then*

$$\|E_c\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla E_c\|_{L^2(0,T,L^2(\Omega))} \le Ce^T\Big(h^{\min\{k_f,k_s,k_p+1,s_f+1,s_p+1,r\}}\Big).$$

2.6 Numerical result

In this section, we present results from several computational experiments in two dimensions. The method is implemented using the finite element package FreeFem++ [53].

2.6.1 Convergent test

In this test we study the convergence of the spatial discretization using analytical solution. We take the region for fluid is $\Omega_f = [0,1]\times[0,1]$, and region for the porous is $\Omega_p = [1,2]\times[0,1]$. We will use the Backward Euler scheme to approximate the time derivative, specifically, we will approximate $\frac{\partial c}{dt}$ by

$$\frac{\partial c(t_n)}{\partial t} \approx \frac{c_n - c_{n-1}}{\delta_t}, \quad n = 1, ..., N.$$

Where N is the final time step. If we set $\alpha_p = 2$, then the following set of functions become a true solution for the Biot-Stokes system.

$$\mathbf{u}_f(t) = cos(t)\begin{pmatrix} 1+2x \\ 0 \end{pmatrix},$$

$$\mathbf{u}_p(t) = cos(t)\begin{pmatrix} 1+x \\ 0 \end{pmatrix},$$

$$\boldsymbol{\eta}(t) = sin(t)\begin{pmatrix} x \\ 0 \end{pmatrix},$$

$$p_p(t) = (\lambda_p + 2\mu_p)sin(t) + 2\mu_f cos(t),$$

$$p_f(t) = (\lambda_p + 2\mu_p)sin(t) + 6\mu_f cos(t).$$

We can see that they satisfy the boundary conditions. The right hand sides of the Biot-

parameter	ϕ	q	$c_0(x,y)$	d_m	α_l	α_t	c_{in}	σ_e	β
value	1.0	1.0	0.0	1.0	0.0	0.0	0.0	50	1.0

Table 2: Table of parameters

Stokes system are computed correspondingly. To avoid the effect of time discretization error, we chose small $\delta t, T$: $T = 0.00001$, and $\delta t = 0.1 * T$. We take the true concentration as follow,

$$c(t) = \frac{t}{T}y^2(y-1)^2x^2(x-2)^2. \tag{2.89}$$

We take the diffusion tensor $D = \mathbb{I}$, porosity $\phi = 1$, and information about finite element spaces is in table 3. By chosing the above functions, we always have $\mathbf{D}\nabla c \cdot \mathbf{n} = 0$, on the whole boundary. Hence from the boundary condition

$$(c\mathbf{u} - \mathbf{D}\nabla c)\cdot \mathbf{n} = (c_{in}\mathbf{u})\cdot\mathbf{n} \text{ on } \Gamma_{in}, \tag{2.90}$$

$$(\mathbf{D}\nabla c)\cdot\mathbf{n} = 0 \text{ on } \Gamma_{out}. \tag{2.91}$$

We deduce that $c_{in} = 0.0$. The values of constants are taken as in table 2. The right hand side is to be made to equal the left hand side, we take $q = 1$, and the function c_w is computed accordingly. We get the below error table with convergence rate. We have intensionally chose dt to be small in order to avoid error contributed by approximation of time derivative $\frac{\partial c}{\partial t}$. With the above chosen finite element spaces, from the lemma (2.5.2), we expect a convergence of order 1 for the concentration, and from the lemma (2.2.1), we expect a convergence of order 1 for velocity. The numerical results are in tables 4 and 5.

paprameter	value
$\mathbf{V}_{f,h}$	$[P1b, P1b]$
$W_{p,h}$	$P1$
$\mathbf{V}_{p,h}$	$RT0$
$W_{p,h}$	$P0$
$\mathbf{X}_h$	$[P1, P1]$
Λ_h	$P0$
C_h	$P1dc$

Table 3: Table of finite element spaces

2.6.2 Experiments with filter

Again, we consider a computational domain $\Omega = [0, 2] \times [0, 1]$, where $\Omega_f = [0, 1] \times [0, 1]$ represents the fluid region and $\Omega_p = [1, 2] \times [0, 1]$ the porous region. The flow is driven by the pressure drop: on the left boundary of Ω_f we set $p_{in} = 10kPa$ and on the right boundary of Ω_p, $p_{out} = 0kPa$, which is also chosen as initial condition for Darcy pressure. Along the top and bottom boundaries, we impose a no-slip boundary condition for the Stokes

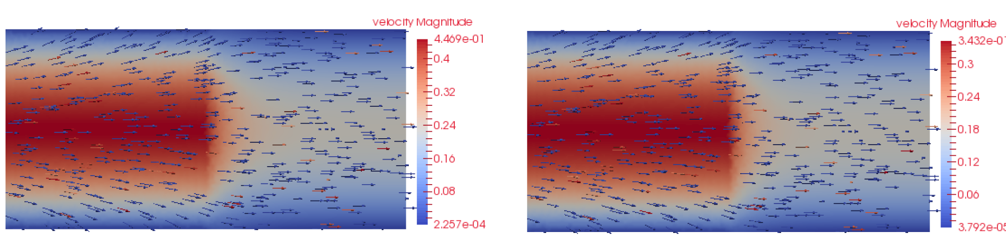
h	$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^\infty(0,T;H^1(\Omega_f))}$		$\|\mathbf{u}_p - \mathbf{u}_{p,h}\|_{L^\infty(0,T;L^2(\Omega_p))}$	
	error	rate	error	rate
1/16	1.24E-1	—-	1.56E-2	—-
1/32	7.28E-2	0.77	7.75E-3	1.00
1/64	3.62E-2	1.00	3.84E-3	1.01

Table 4: Table of error for velocity

h	$\|c-c_h\|_{L^\infty(0,T;L^2(\Omega))}+\|\|\nabla(c-c_h)\|\|_{L^2(0,T;L^2(\Omega))}$	
	error	rate
1/16	1.46E-4	—-
1/32	5.30E-5	1.40
1/64	2.50E-5	1.08

Table 5: Table of error for concentration

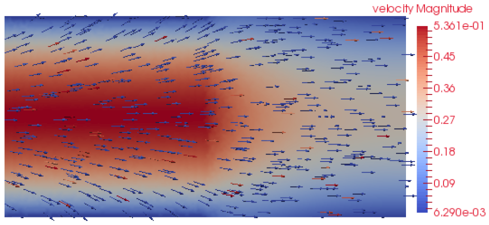
flow and a no-flow boundary condition for the Darcy flow. We also set zero displacement boundary condition on top, bottom and right parts of boundary of structure subdomain, as well as zero initial condition for the displacement. We set $\lambda_p = \mu_p = s_0 = \alpha = \alpha_{BJS} = 1.0$ and $K = I$. We assume that the fluid viscosity in Stokes region satisfies the Cross model: $\nu_f(|D(\mathbf{u}_f)|) = \nu_{f,\infty} + \frac{\nu_{f,0}-\nu_{f,\infty}}{1+K_f|D(\mathbf{u})_f)|^{2-r_f}}$. And the effective viscosity in Darcy region also satisfies the Cross model: $\nu_p(\mathbf{u}_p) = \nu_{p,\infty} + \frac{\nu_{p,0}-\nu_{p,\infty}}{1+K_p|\mathbf{u}_p|^{2-r_p}}$. Where we chose $K_f = K_p = 1, \nu_{f,\infty} = \nu_{p,\infty} = 1, \nu_{f,0} = \nu_{p,0} = 10$, $r_f = r_p = 1.35$. We do the experiments for both velocity fields from linear case and non-linear case. In the case of linear problem, we chose $r_f = r_p = 2$. Taking $T = 10, \delta_t = 0.1$, the velocity fields are as below. In this case, tangential velocity is not allow. We see that the velocity field of the linear case is a little bit higher than the case of non-linear, it is because the non-linear case has higher viscosity.



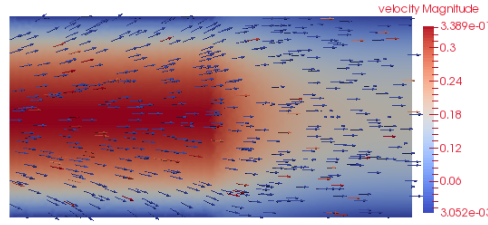
(a) Linear velocity field

(b) Nonlinear velocity field

Figure 9: Velocity fields at $t = 0.1$



(a) Linear velocity field



(b) Nonlinear velocity field

Figure 10: Velocity fields at $t = 0.3$

For the following experiments, first we keep injecting concentration from the left and in the second test, we take a plume of concentration to see how it moves over time. We run the case that the horizontal velocity is caused by pressure drop with P_{in}, P_{out} are the pressure of the left and right boundary respectively. The parameters of the Biot-Stokes system are the same in the experiment that we did in chapter 1, and the parameters of transport equation are taken as the table below. We see that the concentration is moving from the left to the right corresponding to the velocity. In the later time, we see that the concentration accumulates along the top and the bottom boundary. It is due to the fact that the velocity is very small near the top and bottom boundary. It appears to be an appropriate behavior.

Below are images of concentration with the case D = 0.01, and with injecting concentration.

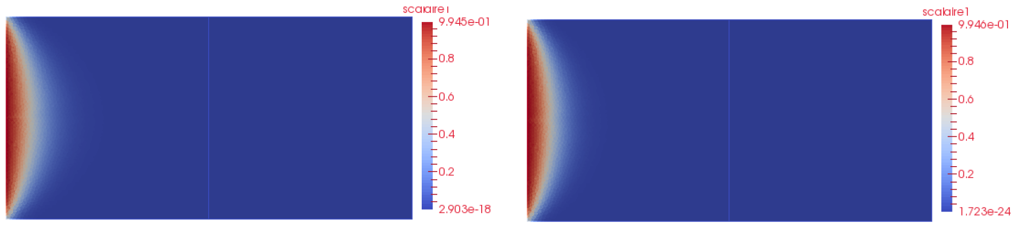
(a) Linear concentration

(b) Nonlinear concentration

Figure 11: Concentration at $t = 0.1$

paprameter	value
fespace of C	$P1dc$
ϕ	$\begin{cases} 0.5 \text{ on } \Omega_p, \text{ i.e } x \geq 1 \\ 1 \text{ on } \Omega_f, \text{ i.e } x < 1. \end{cases}$
qc^*	0
$c_0(x,y)$	either $\begin{cases} 1, \text{ if } (x-0.5)^2 + (y-0.5)^2 \leq 0.1^2 \\ 0, \text{ otherwise} \end{cases}$ or 0
ρ_f, ρ_p	1
c_{in}	0.0
T	10
dt	0.1

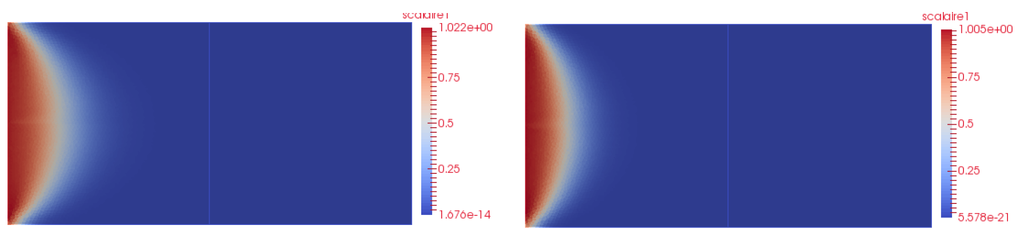
Table 6: Table of parameters



(a) Linear concentration

(b) Nonlinear concentration

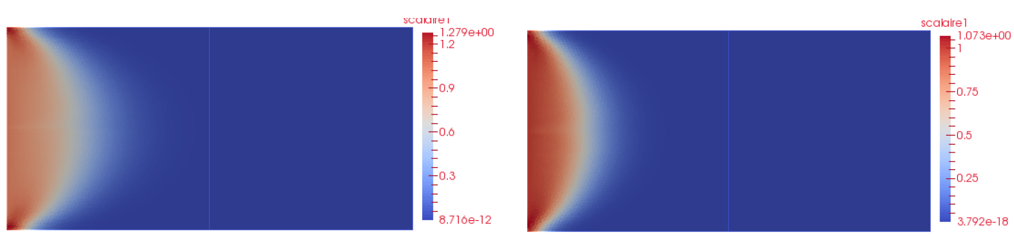
Figure 12: Concentration at $t = 0.2$



(a) Linear concentration

(b) Nonlinear concentration

Figure 13: Concentration at $t = 0.3$



(a) Linear concentration

(b) Nonlinear concentration

Figure 14: Concentration at $t = 0.4$

(a) Linear concentration

(b) Nonlinear concentration

Figure 15: Concentration at $t = 0.5$

(a) Linear concentration

(b) Nonlinear concentration

Figure 16: Concentration at $t = 0.6$

(a) Linear concentration

(b) Nonlinear concentration

Figure 17: Concentration at $t = 0.7$

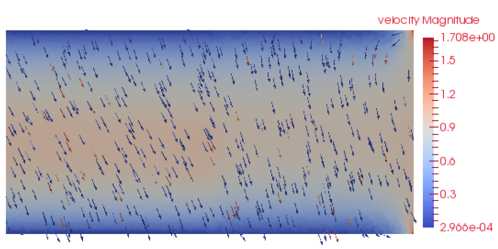
(a) Linear concentration

(b) Nonlinear concentration

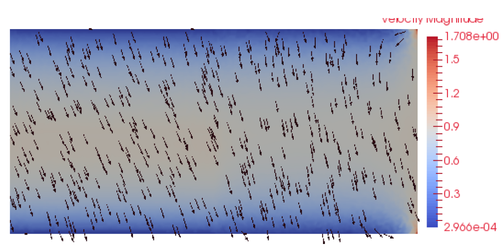
Figure 18: Concentration at $t = 0.8$

In the following, we will do some experiments to see some behavior of the concentration moving under the effect of velocity. For these experiments, we will take into account the effect of gravity, hence equation (1.1) and (1.6) now become $-\nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = \rho_f g \nabla \mathbf{Z}$, in $\mathbf{\Omega_f}$, and $\nu_{\text{eff}} K^{-1} \mathbf{u}_p + \nabla p_p + \rho_p g \nabla \mathbf{Z} = 0$, in Ω_p. Where ρ_f, ρ_p is the density, $g = 9.8$ is the gravity constant, $\mathbf{Z}$ is the depth, hence $\nabla \mathbf{Z} = (0, -1)^T$. We set $\rho = 0.5$ and $p_{in} = 1, p_{out} = 0$ in this experiment.

Before showing the concentration, we will show the velocity fields. We witness that with the effect of gravity, we have a velocity field moving to the right and down. Also, we see that there exists an inflow velocity in the top right of the filter. It is also due to the effect of gravity. The concentration, moving toward the bottom boundary and then it cumulates there.

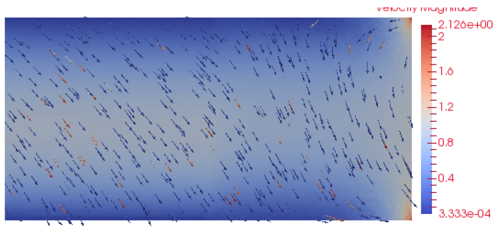


(a) Linear velocity field

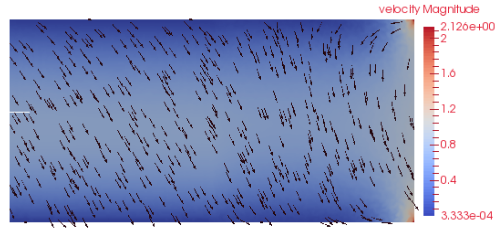


(b) Nonlinear velocity field

Figure 19: Velocity field at $t = 0.1$

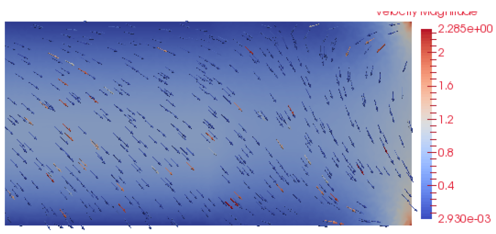


(a) Linear velocity field

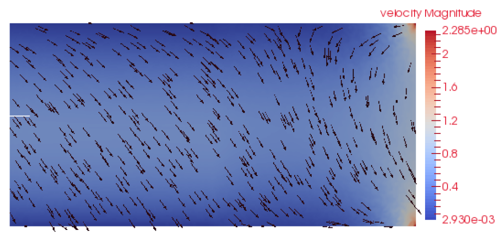


(b) Nonlinear velocity field

Figure 20: Velocity field at $t = 0.2$

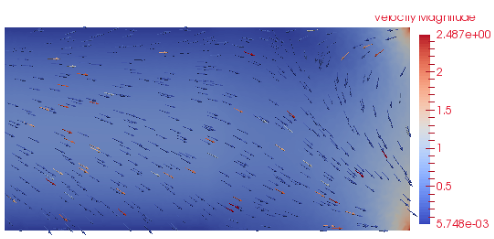


(a) Linear velocity field

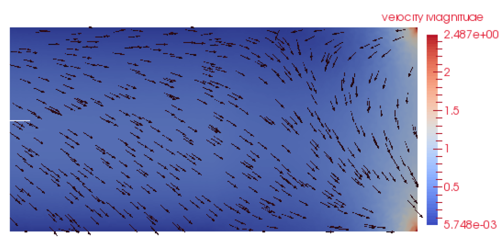


(b) Nonlinear velocity field

Figure 21: Velocity field at $t = 0.3$



(a) Linear velocity field



(b) Nonlinear velocity field

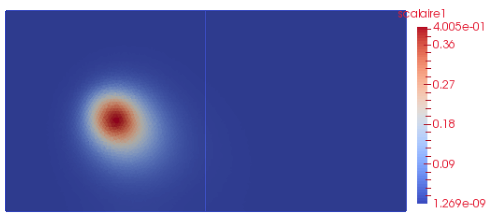
Figure 22: Velocity field at $t = 0.7$

Below are the images from running the case with the effect of gravity.

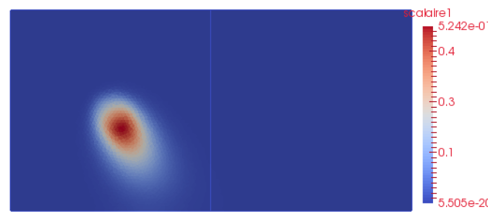
(a) Linear concentration

(b) Nonlinear concentration

Figure 23: Concentration at $t = 0.1$

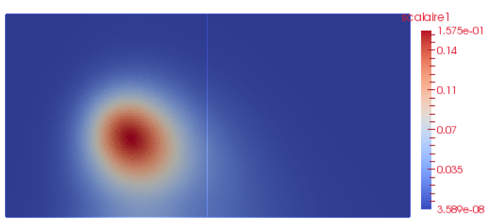


(a) Linear concentration

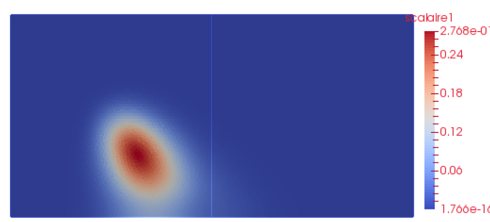


(b) Nonlinear concentration

Figure 24: Concentration at $t = 0.2$

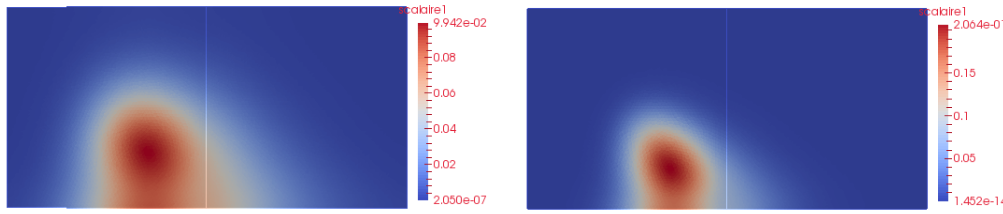


(a) Linear concentration



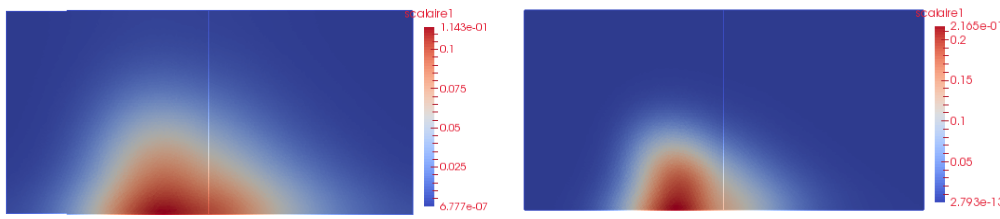
(b) Nonlinear concentration

Figure 25: Concentration at $t = 0.3$



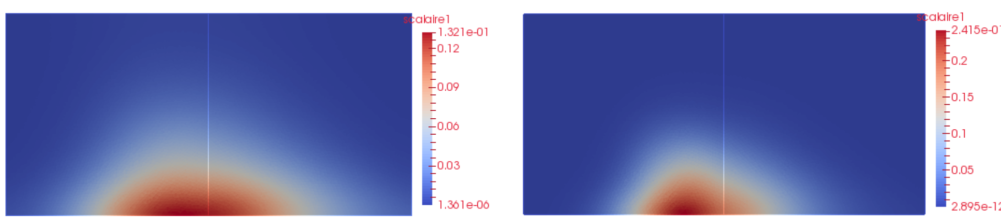
(a) Linear concentration (b) Nonlinear concentration

Figure 26: Concentration at $t = 0.4$



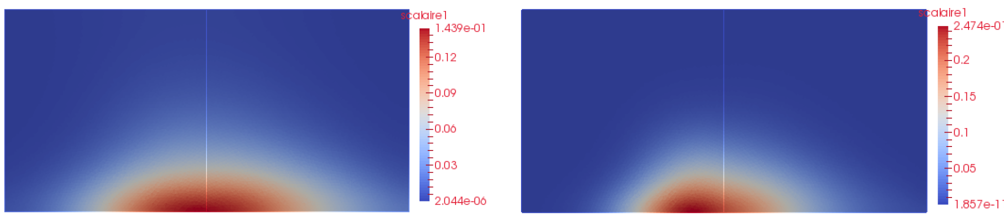
(a) Linear concentration (b) Nonlinear concentration

Figure 27: Concentration at $t = 0.5$



(a) Linear concentration (b) Nonlinear concentration

Figure 28: Concentration at $t = 0.6$

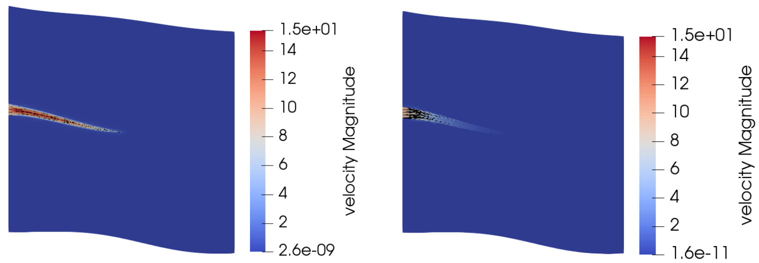


(a) Linear concentration (b) Nonlinear concentration

Figure 29: Concentration at $t = 0.7$

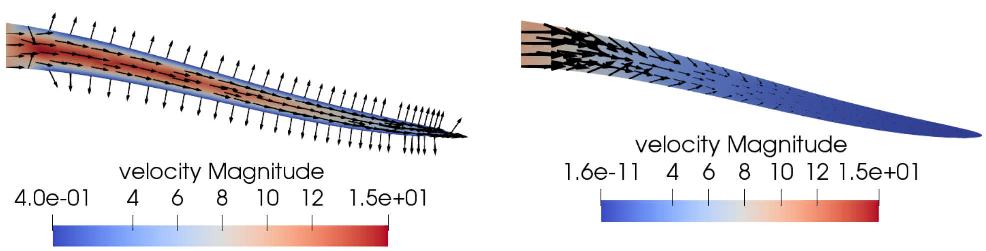
2.6.3 Flow and transport through a fractured reservoir

In this example, we use the domain and the velocity field from section (1.6.2). The porosity function ϕ is 1 in the fluid region, and 0.4 in the porous region. The initial value of the concentration is 0 over all the region. We inject the concentration on the left boundary of the fluid region. We take $d_m = 5 * 10^{-2}$, $\alpha_l = \alpha_t = 10^{-4}$. The whole time length will be $T = 20$. The velocity field is shown in figures 30 and 31. In the images, we can see that tracer come from the left boundary of the fluid and propagates along the fracture following the Stokes velocity. The tracer also diffuses into the poroelastic region, however the form of the concentration still resembles the fluid domain. In both cases, the concentration has the highest value at the tip of the fracture. However, in the case of linear velocity, we witness the velocity go toward the surrounding along the border of fracture. Hence, we see the concentration diffuse to the surrounding. While, in the nonlinear case, the velocity, flow along the fracture, hence the concentration is transmitted to the tip of the fracture and accumulates there.



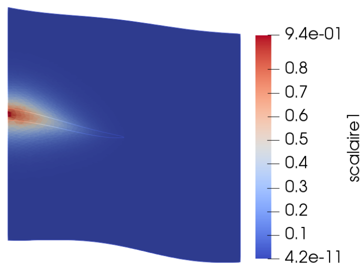
(a) Linear velocity (b) Nonlinear velocity

Figure 30: Velocity field at time $t = 2$

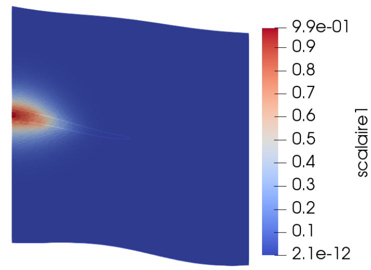


(a) Linear velocity (b) Nonlinear velocity

Figure 31: Velocity field of the fluid region at time $t = 2$

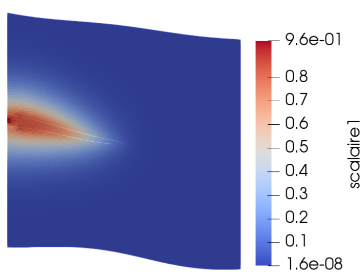


(a) Linear velocity

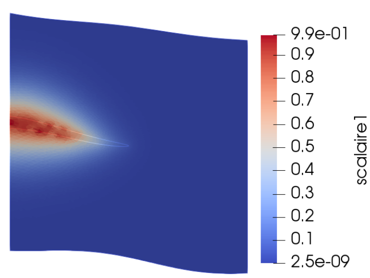


(b) Nonlinear velocity

Figure 32: Concentration at time $t = 2$

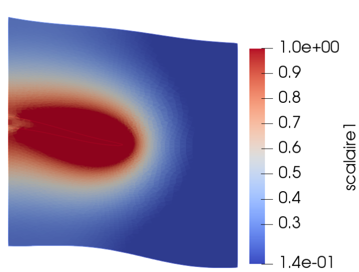


(a) Linear velocity

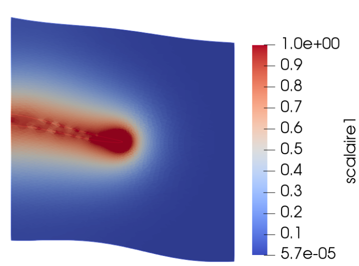


(b) Nonlinear velocity

Figure 33: Concentration at time $t = 4$



(a) Linear velocity



(b) Nonlinear velocity

Figure 34: Concentration at time $t = 20$

2.7 Conclusion

In the first chapter, we study the coupling of Biot-Stokes equations that is given by (1.1), (1.5), (1.6) and (1.7), with viscosity models given by *Carreau model, Cross model* and *Power law model*. We formulate the problem two way: Lagrange multiplier formulation and alternative formulation. The reason we need to have the alternative formulation is because the term $\partial_t \boldsymbol{\eta}_p$ in the Lagrange formulation that make it difficult to prove the existence of the solution in this formulation. By setting $\mathbf{u}_s = \partial_t \boldsymbol{\eta}_p$, and using the operator A: $A\boldsymbol{\sigma}_e = \mathbf{D}(\boldsymbol{\eta}_p)$, we can set up the alternative formulation. Then we can use the theorem 1.3.7 to show that there exists solution for the alternative formulation. Then we can come back to prove that the Lagrange multiplier formulation has solution. However, the alternative formulation is difficult and expensive to implement because of the term $\boldsymbol{\sigma}_e$ belongs to a vector space of matrix. Thus, we use the Lagrange formulation to implement. We do two experiments for the problem. The first one is about convergent test and the second one is an application to hydraulic fracturing. Due to technical problem when using theorem 1.3.7, in this work we have to assume that $\mathbf{f}_f = \mathbf{f}_p = \mathbf{0}$, and $q_f = 0$. In the work of the paper [3], with different approach, we can prove the existence of the solution without assuming that $\mathbf{f}_f = \mathbf{f}_p = \mathbf{0}$, and $q_f = 0$. It may possible to extend this work to Navier-Stokes - Biot models, models with mixed elasticity formulations, multiphase flow in porous media, and multirate time-stepping schemes.

In the second chapter, we investigate the transport equation (2.17) with the velocity field from the Stokes-Biot problem. Following [79], we set up the DG scheme (2.73). We note that the dispersion tensor in the transport equation is a nonlinear function of the velocity. The work in [79] handles this difficulty by utilizing a cut-off operator. Here we can avoid the need of using cut-off operator after showing that $\nabla \cdot \mathbf{u}_h$ is bounded. We then do numerical experiments about convergent test and experiment with filter.

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