# TWISTOR CR MANIFOLD BY NON-RIEMANNIAN CONNECTIONS

by

**Ho Chi Low**

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This dissertation was presented

by

Ho Chi Low

It was defended on

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and approved by

G. A. J. Sparling, Ph. D., Associate Professor

J. DeBlois, Ph. D., Associate Professor

P. Hajlasz, Ph. D., Professor

C. LeBrun, Ph. D., Professor

Dissertation Director: G. A. J. Sparling, Ph. D., Associate Professor

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Ho Chi Low, PhD

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Developed by LeBrun, twistor CR manifold is a 5-dimensional CR manifold foliated by Riemann spheres. The CR structure is determined by both the complex structure on the Riemann sphere and the geometric information of the space of leaves, which is a 3-manifold endowed with a conformal class and a trace-free symmetric(1,1)-tensor.

When the (1,1)-tensor is zero, the *twistor CR structure of zero torsion*, named as the *rival* CR structure on LeBrun's paper "*Foliated CR Manifolds*", is obtained. These CR structures are embeddable to a complex 3-manifold if and only if the metric tensor is conformal to a real analytic metric.

We try to understand twistor CR structures through the corresponding Fefferman metric defined on the canonical circle-bundle of the given CR manifold. The conformal class of the Fefferman metric is preserved over the choice of contact forms of the CR structure, so it makes possible to classify CR structures by the confomal curvature tensor of the Fefferman metric.

Our main results include representing the Weyl tensor of the Fefferman metric in terms of the Cotton tensor on the 3-manifold when the twistor CR structure is of zero torsion. Moreover, we obtain conditions for vanishing Weyl tensor when the space of leaves is under a flat metric.

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## PREFACE

I would like to express my deepest gratitude to every professor in the dissertation committee. In particular, thank you very much to Professor LeBrun for his wonderful work on the subject of the twistor CR manifold, combined with his advice and suggestions on our research. Last but not least, special thanks to Professor Sparling as this article would have never appeared without his help.

# 1.0 INTRODUCTION

We would begin with an overview of the thesis, followed by background knowledge in CR geometry which are essential to the research. The background materials are adopted mainly from early chapters of [3] and [4]. Meanwhile, the books [7] and [19] provides fundamental concepts in differential geometry.

## 1.1 OVERVIEW

The concept of twistor CR manifold originates from Penrose's work on the twistor theory [14]. The twistor space ($\mathbb{T}$) of the four dimensional Minkowski space ($\mathbf{M}$) is defined to be a 4-dimensional complex vector space such that every null twistor inside $\mathbb{T}$ could represent a null geodesic on $\mathbf{M}$. The space of null geodesics would then form a hypersurface inside the projective twistor space ($P(\mathbb{T})$), which is diffeomorphic to $\mathbb{C}P^3$.

A natural CR structure is then introduced to the space of null geodesics by Penrose on [15]. On that article, the incident correspondence is defined to relate points of $\mathbf{M}$ to null twistors on $\mathbb{T}$. In this approach, every equivalence class $[W]$ of null twistors on $P(\mathbb{T})$, outside an exceptional set $\mathscr{I}$, could be identified with a unique null geodesic on the Minkowski space. The space of null geodesics, denoted by $P(\mathbb{T}_0)\backslash\mathscr{I}$, carries a natural CR structure from $P(\mathbb{T})$.

As a generalization, over a globally hyperbolic Lorentzian 4-manifold, the space of null geodesics could be equipped with a CR structure depending on the choice of a hypersurface which meets every null geodesic on the manifold [15].

LeBrun developed the concept of twistor CR manifold much in his work: [8], [9] and [10]. On [10], LeBrun proves that a twistor CR manifold in 5 dimension, is exactly a foliation of Riemann spheres equipped with a non-degenerate CR structure. The space of leaves is a real 3-manifold $M$. Moreover, the CR structure itself can be characterized by the *first and second fundamental forms* of $M$, which represent a prescribed conformal class and a prescribed trace-free symmetric $(1,1)$-tensor respectively.

When $M$ is a hypersurface of the twistor space of a self-dual Riemannian 4-manifold, which happens to be a complex 3-manifold, the above *first* and *second fundamental forms* become the conformal part of the first and second fundamental forms of $M$ in their usual definitions correspondingly under the twistor construction.

A particular class of twistor CR manifolds can be obtained if the second fundamental form vanishes. It is named the *rival CR structure* on [10] and elaborated in details on [9]. LeBrun also shows that this special twistor CR manifold of a smooth 3-manifold $M$ equipped with the metric $g$, is embeddable to a complex 3-manifold if and only if its conformal class $[g]$ contains a real analytic metric on $M$. This property doesn't hold in general for a twistor CR manifold.

The *rival CR structure* is also named as the *twistor CR manifold of Hamiltonian distribution* or that of *zero torsion* in this article.

Our research initiates from LeBrun's work on the twistor CR manifold and borrow many of his definitions and results as foundations. We try to develop from his work and classify these CR structures by investigating the Fefferman metric on the Fefferman bundle. Since the conformal class of the Fefferman metric of a CR structure is a CR invariant, the conformal curvature of the Fefferman metric conveys information of the geometry of the CR structure.

The theoretical tools we adopt, come from the theory of CR geometry and that of differential geometry. We build the local model of the twistor CR structure, with or without torsion, and analyze the CR structure in local variables in order to capture properties of the Weyl tensor. This process would demand a lot of computational work, so computer programming in MATLAB [12] is also an essential part in our research.

The remaining part of the beginning chapter is a summary of definitions and theorems in CR geometry. Then, in Chapter 2, we will elaborate LeBrun's work on the twistor CR manifold of Hamiltonian distribution extensively and try to translate his concepts and results to our local model of twistor CR structures. The *second fundamental form*, which is equivalent to the trace-free torsion tensor of a metric connection on the 3-manifold [8], is introduced in Chapter 3, where we also complete the construction of the local model of the twistor CR structure (with torsion).

Chapter 4 and 5, combined as one unit, contains the theoretical picture of both the Tanaka-Webster connection and the Fefferman metric to support the computational model. The Weyl curvature tensor of the twistor CR structure, as the key subject in our research, will be discussed in Chapter 6 and 7, followed by the main findings.

By characterizing the coefficients of the Weyl tensor, we could obtain important results such as representing the Weyl tensor of the Fefferman metric in terms of the Cotton tensor on the 3-manifold when the twistor CR structure is of zero torsion. Moreover, we obtain conditions for vanishing Weyl tensor when the space of leaves is under a flat metric.

## 1.2 CAUCHY-RIEMANN GEOMETRY

Let $N$ be a smooth manifold and $\mathbb{L}$ be a smooth complex distribution on $N$. We define $\overline{\mathbb{L}}$ to be the complex conjugation of $\mathbb{L}$ in $\mathbb{C}TN$.

**Definition.** *$\mathbb{L}$ is a CR structure on $N$ if both conditions (1) and (2) are satisfied.*

*(1)* $\mathbb{L} \cap \overline{\mathbb{L}} = \{\mathbf{0}\}$.

*(2)* *$\mathbb{L}$ is integrable: for any open set $U$ on $N$, smooth sections of $\mathbb{L}$ over $U$ are involutive. That is,* $\big[\Gamma^\infty(U, \mathbb{L}),\ \Gamma^\infty(U, \mathbb{L})\big] \subseteq \Gamma^\infty(U, \mathbb{L})$.

We specify that $\mathbb{L}$ is the holomorphic bundle and $\overline{\mathbb{L}}$ is the antiholomorphic bundle of the CR structure. If $\mathbb{L}$ is of complex rank $\nu$ and $N$ is a manifold of real dimension $2\nu + d$, then we say $\mathbb{L}$ is of type $(\nu, d)$. $\nu$ is called the CR dimension, and $d$ the CR codimension of $\mathbb{L}$. When $d = 1$, $\mathbb{L}$ is of hypersurface type.

The pair $(N, \mathbb{L})$ is then called a CR manifold (of type $(n, d)$). Given that $N'$ is another CR manifold equipped with the CR structure $\mathbb{L}'$, a smooth function $f$ from $N$ to $N'$ is a CR map whenever $df(\mathbb{L}) \subseteq \mathbb{L}'$. Moreover, we say that $N$ is CR equivalent (or CR isomorphic) to $N'$ when there is a diffeomorphism $f : N \to N'$ such that $df(\mathbb{L}) = \mathbb{L}'$.

The real subbundle of rank $2\nu$, $H(N)$, consists of tangent vectors to $N$ in the form of $X + \overline{X}$, where $X$ is on $\mathbb{L}$. $H(N)$ is named the Levi distribution of the CR structure.

There is a natural almost complex structure $(J)$ defined on $H(N)$. We first impose that $J(X) = iX$ and $J(\overline{X}) = -iX$ when $X$ is a holomorphic vector. Then $J$ is characterized by

$$J\big(X + \overline{X}\big) = i\big(X - \overline{X}\big) \quad \text{for} \quad X \in \mathbb{L}.$$

Assume that the CR manifold $N$ is always of hypersurface type $(d = 1)$ from here.

**Definition.** *A pseudo-hermitian structure of $N$ is a smooth real 1-form $\alpha$ annihilating $H(N)$. The Levi form $\mathcal{L}$ associated with $\alpha$ is a complex bilinear map defined by*

$$\mathcal{L} : \mathbb{L} \times \overline{\mathbb{L}} \to \mathbb{C}, \quad \mathcal{L}(X, \overline{Y}) \;=\; -\, id\alpha(X, \overline{Y})$$

*for $X$ on $\mathbb{L}$ and $Y$ on $\overline{\mathbb{L}}$.*

The 1-form $\alpha$ could be replaced by $f\alpha$ for any non-vanishing function $f$ on $N$. In this case, we let $\mathcal{L}_1$ be the Levi form associated with $f\alpha$. It makes

$$\mathcal{L}_1(X, \overline{Y}) \;=\; f \cdot \mathcal{L}(X, \overline{Y}) \quad \text{for any} \quad X \in \mathbb{L},\ Y \in \overline{\mathbb{L}}.$$

We say that the Levi form $\mathcal{L}$ is non-degenerate, if for any $v$ on $\mathbb{L}$, there is a vector $\overline{w}$ on $\overline{\mathbb{L}}$ such that $\mathcal{L}(v, \overline{w}) \neq 0$. Moreover, we say that $\mathcal{L}$ is positive definite when $\mathcal{L}(v, \overline{v}) > 0$ for any nonzero vector $v$ on $\mathbb{L}$.

If $\mathcal{L}$ is non-degenerate, then any other Levi form of $\mathbb{L}$ is also non-degenerate. So we could say that the CR structure $\mathbb{L}$ is non-degenerate when any of its Levi forms is non-degenerate. In this case, we also say $\alpha$ is a contact form on $N$ since $\alpha \wedge (d\alpha)^n \neq 0$ at every point of $N$.

**Definition.** *Let $N$ be a CR manifold of hypersurface type. Suppose $\mathbb{L}$ is the CR structure of $N$ and $\mathbb{L}$ is non-degenerate.*

*(1) $N$ is strictly pseudoconvex, if for some choice of pseudo-hermitian form $\alpha$, its associated Levi form is positively definite.*

*(2) $N$ is anticlastic if the Levi form, associated to any pseudo-hermitian form, consists of eigenvalues of opposite signs.*

## 1.3 EMBEDDABLE CR MANIFOLDS

A huge class of CR manifolds is the collection of CR submanifolds of $\mathbb{C}^m$. It means that the manifold $N$ is a submanifold of $\mathbb{C}^m$ and it inherits a CR structure from the standard complex structure $(J_0)$ of $\mathbb{C}^m$. The Levi distribution is the $J_0$-invariant subspace of $TN$, i.e.

$$H(N) \;=\; TN \cap J_0(TN).$$

We define the holomorphic bundle and the antiholomorphic bundle of the CR structure by

$$T^{1,0}N = T^{1,0}\mathbb{C}^m \cap \mathbb{C}TN \quad \text{and} \quad T^{0,1}N = T^{0,1}\mathbb{C}^m \cap \mathbb{C}TN$$

In general, an embedded manifolds in $\mathbb{C}^m$ obtains a natural CR structure from $\mathbb{C}^m$ as long as the real dimension of $H(N)$ keeps constant. We would mainly look at the real hypersurface in $\mathbb{C}^m$ since they correspond to the case that CR codimension is 1.

An example of real hypersurface is the graph of function.

Let $z = x + iy$ be the first coordinate of $\mathbb{C}^m$ and let $w = (w_1, \cdots, w_{m-1})$ be the remaining complex coordinates. We define a smooth function $h : \mathbb{R} \times \mathbb{C}^{m-1} \to \mathbb{R}$ by $h = h(x, w)$. Its graph is a hypersurface $M = \big\{(x + iy, w) \in \mathbb{C}^m \mid y = h(x, w)\big\}$. Furthermore, we assume

$$h(0,0) \;=\; 0, \quad \frac{\partial h}{\partial x} \;=\; \frac{\partial h}{\partial w_j} \;=\; \frac{\partial h}{\partial \overline{w}_j} \;=\; 0$$

at $(0, 0)$ for $j = 1, \cdots, m - 1$.

Since the direct sum of $TM$ and $J_0(TM)$ spans the entire $\mathbb{C}^m$, we get

$$\dim_{\mathbb{R}}(H(M)) \;=\; \dim_{\mathbb{R}}(TM) + \dim_{\mathbb{R}}(J_0(TM)) - \dim_{\mathbb{R}}(\mathbb{C}^m) \;=\; 2m - 2.$$

So the CR codimension of $M$ is 1, equal to its geometric codimension. In general, we say that a CR submanifold of $\mathbb{C}^m$ is generic if its geometric codimension equal to its CR codimension.

The holomorphic bundle $T^{1,0}M$ of the real hypersurface $M$ is spanned by the vectors

$$T_j = \frac{\partial}{\partial w_j} + \frac{2i}{(1-ih_x)}\frac{\partial h}{\partial w_j}\frac{\partial}{\partial z}$$

at $(x+iy, w)$ for $j = 1, \cdots, m-1$. The antiholomorphic bundle is spanned by

$$T_{\bar{j}} = \frac{\partial}{\partial \overline{w}_j} - \frac{2i}{(1+ih_x)}\frac{\partial h}{\partial \overline{w}_j}\frac{\partial}{\partial \overline{z}}.$$

A pseudo-hermitian structure of $M$ could be found by

$$\theta = \frac{(1-ih_x)}{2}dz + \frac{(1+ih_x)}{2}d\overline{z} - i\frac{\partial h}{\partial w_j}dw_j + i\frac{\partial h}{\partial \overline{w}_j}d\overline{w}_j.$$

All real analytic CR manifolds are locally embeddable to $\mathbb{C}^m$ for some $m$. We quote the *real analytic embedding theorem* (Theorem 1.1) here. Readers may refer to Chapter 11 of [3] for its proof and more details.

**Definition.** *A CR manifold $N$ is real analytic if $N$ is a real analytic manifold, and the CR structure $\mathbb{L}$ is a real analytic subbundle of the complex tangent bundle of $N$.*

**Theorem 1.1.** *Suppose $N$ is a real analytic CR manifold of real dimension $2m-d$. Let $\mathbb{L}$ be its CR structure, and the CR codimension of $\mathbb{L}$ be $d \geq 1$. Then, given any point $p$ on $N$, there is a neighborhood $U$ of $p$ such that the CR structure $(U, \mathbb{L})$ is CR equivalent to a generic real analytic CR submanifold of $\mathbb{C}^m$ with CR codimension equal to $d$.*

If $\phi = (z_1, \cdots, z_m)$ defines such a local embedding from $N$ to $\mathbb{C}^m$, then all the component functions $z_j$'s of $\phi$ must be CR functions (complex-valued CR maps). That is, $\overline{Y}(z_j) = 0$ for any vector field $\overline{Y}$ in $\overline{\mathbb{L}}$.

## 1.4 TANAKA-WEBSTER CONNECTION

Let $N$ be a CR manifold of dimension 5. The CR structure of $N$ is denoted by $\mathfrak{D}$, which also represents its antiholomorphic bundle. Over the course of the thesis, we would like to reserve any capital letter $D$ (or $\mathfrak{D}$) to represent the $T^{0,1}$-part of the CR structure. The holomorphic bundle would be denoted by $\bar{D}$ or $\bar{\mathfrak{D}}$.

Suppose $\alpha$ is a pseudo-hermitian structure on $N$. When $d\alpha$ is non-degenerate, there exists a unique tangent vector field $T$, such that $\alpha(T) = 1$ and $\iota_T d\alpha = 0$ at every point. We may call $T$ the characteristic vector field associated with $\alpha$. If $\alpha$ is also a contact form, then $T$ is the Reeb vector field associated with $\alpha$.

Also, we let $T_1$, $T_2$ be a basis for $\bar{\mathfrak{D}}$. So $T_{\bar{1}} = \overline{T}_1$ and $T_{\bar{2}} = \overline{T}_2$ is a basis for $\mathfrak{D}$. The collection of $T_1$, $T_2$, $T_{\bar{1}}$, $T_{\bar{2}}$ and $T$ form a moving frame for $\mathbb{C}TN$.

Under our notation, the Levi form $\mathcal{L}$ of $\alpha$ maps from $\bar{\mathfrak{D}} \times \mathfrak{D}$ to $\mathbb{C}$. Let $h_{\alpha\bar{\beta}} = \mathcal{L}(T_\alpha, T_{\bar{\beta}})$. We let $h = \left[h_{\alpha\bar{\beta}}\right]$ be a $2 \times 2$ matrix and denote its inverse by $h^{-1} = \left[h^{\bar{\alpha}\beta}\right]$, i.e.

$$\sum_\beta h_{\alpha\bar{\beta}} \cdot h^{\bar{\beta}\sigma} = \delta_{\alpha\sigma}.$$

$h$ is hermitian so that $\overline{h_{\alpha\bar{\beta}}} = h_{\beta\bar{\alpha}}$ and $\overline{h^{\bar{\beta}\alpha}} = h^{\bar{\alpha}\beta}$.

**Definition.** *The Webster metric $g$ associated with the pseudo-hermitian structure $\alpha$, is a pseudo-Riemannian metric on $N$ defined by*

$$g(T,T) = 1, \quad g(T_\alpha, T_{\bar{\beta}}) = h_{\alpha\bar{\beta}} \quad \textit{and} \quad g(T_\alpha, T_\beta) = g(T_{\bar{\alpha}}, T_{\bar{\beta}}) = g(T_\alpha, T) = g(T_{\bar{\beta}}, T) = 0,$$

*for $\alpha, \beta = 1, 2$.*

The Webster metric is Riemannian if the CR structure is strictly pseudo-convex and the metric itself is defined by a positive definite Levi form. However, since our work is on anticlastic CR manifolds, the signature of $g$ would be $(+++--)$.

Rather than the Levi-Civita connection, we study the Tanaka-Webster connection of $g$ referring to Webster's paper [20] or in Chapter 1 of [4].

**Definition.** *The Tanaka-Webster connection $\nabla$ associated with $\alpha$, is an affine connection on $N$ uniquely defined by the following properties.*

*(1)* $\nabla$ *is a metric connection with respect to* $g$.

*(2)* $\nabla_X T_\alpha$ *belongs to* $T^{1,0}N$, *and* $\nabla_X T_{\bar\beta}$ *belongs to* $T^{0,1}N$, *for any* $X$ *on* $\mathbb{C}TN$.

*(3)* $\nabla_X T = 0$ *for any* $X$ *on* $\mathbb{C}TN$.

*(4)* *Let tor be the torsion tensor of* $\nabla$, $tor(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.
*Then,* $tor(T_\alpha, T_{\bar\beta})$ *belongs to the linear span of* $T$.

*(5)* *Let* $\tau$ *be the operator* $\tau(X) = tor(T,X)$. *Then,* $\tau$ *sends* $T^{1,0}N$ *to* $T^{0,1}N$ *and vice versa.*

By the definition above, for $m, n = 1, 2$, we let

$$\nabla_{T_m} T_n = \Gamma^k_{mn} T_k, \quad \nabla_{T_{\bar m}} T_n = \Gamma^k_{\bar m n} T_k \quad \text{and} \quad \nabla_T T_n = \Gamma^k_{0n} T_k.$$

The coefficients of the Tanaka-Webster connection are then given by

$$\begin{cases} \Gamma^k_{mn} &= h^{\bar l k}\Big(dh_{n\bar l}(T_m) - g\big(T_n, [T_m, T_{\bar l}]\big)\Big), \\ \Gamma^k_{\bar m n} &= h^{\bar l k} \cdot g\big([T_{\bar m}, T_n], T_{\bar l}\big), \\ \Gamma^k_{0n} &= h^{\bar l k} \cdot g\big([T, T_n], T_{\bar l}\big). \end{cases} \tag{1.1}$$

Let $R$ be the curvature tensor field of the Tanaka-Webster connection $\nabla$. We follow the usual definition that $R(Y,Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$. In our study, we assume that $Y$ lies on $\bar{\mathfrak{D}}$ and $Z$ lies on $\mathfrak{D}$. In particular,

$$R(T_k, T_{\bar{l}})T_m = \nabla_{T_k} \nabla_{T_{\bar{l}}} T_m - \nabla_{T_{\bar{l}}} \nabla_{T_k} T_m - \nabla_{[T_k, T_{\bar{l}}]} T_m.$$

Let $R(T_k, T_{\bar{l}})T_m = R_m{}^n{}_{k\bar{l}}\, T_n$. Explicitly,

$$R_m{}^n{}_{k\bar{l}} \;=\; d\Gamma^n_{\bar{l}m}(T_k) - d\Gamma^n_{km}(T_{\bar{l}}) - \Gamma^p_{km}\Gamma^n_{\bar{l}p} + \Gamma^p_{\bar{l}m}\Gamma^n_{kp} + \Gamma^p_{\bar{l}k}\Gamma^n_{pm} - \Gamma^{\bar{p}}_{k\bar{l}}\Gamma^n_{\bar{p}m} + 2ih_{k\bar{l}}\Gamma^n_{0m}. \tag{1.2}$$

As a remark, the Christoffel symbol $\Gamma^{\bar{p}}_{k\bar{l}}$ equals $\overline{(\Gamma^p_{\bar{k}l})}$.

The lower index of $R$ is defined by $R_{m\bar{n}k\bar{l}} = g\big(R(T_k, T_{\bar{l}})T_m, T_{\bar{n}}\big) = R_m{}^p{}_{k\bar{l}}\; h_{p\bar{n}}$.

We also define the Ricci tensor (ric) and the scalar curvature ($\rho$) specific to the Tanaka-Webster connection on $N$. These two variables are more often named as the *pseudo-hermitian Ricci tensor* and the *pseudo-hermitian scalar curvature* respectively. To keep everything simple, we would call them by the shorter names. They are defined by

$$\mathrm{ric}(T_\lambda, T_{\bar{\mu}}) = R_{\lambda\bar{\mu}} = R_\lambda{}^\alpha{}_{\alpha\bar{\mu}} \quad \text{and} \quad \rho = h^{\bar{\mu}\lambda} \cdot R_{\lambda\bar{\mu}}. \tag{1.3}$$

The raise-index of ric becomes an operator from $\bar{\mathfrak{D}}$ to itself. We would call it by $\mathrm{ric}^\sharp$.

$$\mathrm{ric}^\sharp(T_m) = R^n_m T_n \quad \text{with} \quad R^n_m = R_{m\bar{p}} \cdot h^{\bar{p}n}.$$

The Chern-Moser curvature tensor field ($C$) is the analogue of the Weyl curvature tensor in Riemannian geometry. We would introduce it briefly here, and for more details, readers may refer to [2] and [20]. Let $\nu$ be the CR dimension of $\mathfrak{D}$. We have

$$\begin{aligned} C(T_k, T_{\bar{l}})T_m \;=\; & R(T_k, T_{\bar{l}})T_m - \frac{1}{\nu+2}\Big(h_{k\bar{l}}\,\mathrm{ric}^\sharp(T_m) + h_{m\bar{l}}\,\mathrm{ric}^\sharp(T_k) + R_{k\bar{l}}\,T_m + R_{m\bar{l}}\,T_k\Big) \\ & + \frac{\rho}{(\nu+1)(\nu+2)}\Big(h_{k\bar{l}}\,T_m + h_{m\bar{l}}\,T_k\Big). \end{aligned}$$

Writing $C(T_k, T_{\bar{l}})T_m = C_m{}^n{}_{k\bar{l}}\, T_n$ and $\nu = 2$, we have

$$C_m{}^n{}_{k\bar{l}} \;=\; R_m{}^n{}_{k\bar{l}} - \frac{1}{4}\Big(R^n_m\; h_{k\bar{l}} + R^n_k\; h_{m\bar{l}} + R_{k\bar{l}}\;\delta_{mn} + R_{m\bar{l}}\;\delta_{kn}\Big) + \frac{\rho}{12}\Big(h_{k\bar{l}}\;\delta_{mn} + h_{m\bar{l}}\;\delta_{kn}\Big). \quad (1.4)$$

The lower-index of $C$ is then given by $C_{m\bar{n}k\bar{l}} = g\big(C(T_k, T_{\bar{l}})T_m, T_{\bar{n}}\big)$. Explicitly,

$$C_{m\bar{n}k\bar{l}} \;=\; R_{m\bar{n}k\bar{l}} - \frac{1}{4}\Big(R_{m\bar{n}}h_{k\bar{l}} + R_{k\bar{l}}h_{m\bar{n}} + R_{m\bar{l}}h_{k\bar{n}} + R_{k\bar{n}}h_{m\bar{l}}\Big) + \frac{\rho}{12}\Big(h_{m\bar{n}}h_{k\bar{l}} + h_{m\bar{l}}h_{k\bar{n}}\Big). \quad (1.5)$$

The Chern tensor behaves similarly to the Weyl tensor in the way that if it remains unchanged when the pseudo-hermitian form $\alpha$ is replaced by another pseudo-hermitian form.

**Proposition 1.2.** *Let $\alpha$ be a pseudo-hermitian form of $\mathfrak{D}$, and $C$ be the (1,3)-Chern tensor of the Tanaka-Webster connection of $\alpha$. Suppose $\tilde{\alpha} = e^{2f}\alpha$ for some smooth real-valued function $f$ on $N$, and $\tilde{C}$ is the (1,3)-Chern tensor of the Tanaka-Webster connection of $\tilde{\alpha}$. Then, $C = \tilde{C}$ on $N$.*

If we fix the basis of $T_1$, $T_2$, $T_{\bar{1}}$, $T_{\bar{2}}$ for the Levi distribution $\bar{\mathfrak{D}} \oplus \mathfrak{D}$, then the above theorem means $C(T_k, T_{\bar{l}})T_m = \tilde{C}(T_k, T_{\bar{l}})T_m$. Explicitly, $\tilde{C}_m{}^n{}_{k\bar{l}} = C_m{}^n{}_{k\bar{l}}$ and $\tilde{C}_{m\bar{n}k\bar{l}} = e^{2f}\, C_{m\bar{n}k\bar{l}}$ for every $m$, $n$, $k$ and $l$.

## 1.5 THE FEFFERMAN BUNDLE AND THE FEFFERMAN METRIC

The approach from [11] is adopted to construct the Fefferman bundle and the Fefferman metric of the CR manifold $N$. We are in the case that the CR dimension $\nu$ is 2, so the Fefferman bundle is of real dimension $2\nu + 2 = 6$. For better understanding on the subject of Fefferman metrics, especially when $\nu = 1$, readers may also refer to [13].

A pseudo-hermitian structure $\alpha$ on $N$ is fixed in the following so that the Levi-form $(\mathcal{L},\, h_{\alpha\bar{\beta}})$ and the Tanaka-Webster connection $(\nabla)$ are well-defined.

Consider the moving frame $\{T_1, T_2, T_{\bar{1}}, T_{\bar{2}}, T\}$ for $TN$. Let $\{\theta^1, \theta^2, \theta^{\bar{1}}, \theta^{\bar{2}}, \alpha\}$ be the dual coframe. It means that $\theta^i(T_j) = \theta^{\bar{i}}(T_{\bar{j}}) = \delta_{ij}$, $\theta^i(T_{\bar{j}}) = \theta^{\bar{i}}(T_j) = \alpha(T_j) = \alpha(T_{\bar{j}}) = 0$ and $\alpha(T) = 1$. We say that a 1-form $\eta$ on $N$ is of type $(0,1)$ if

$$\eta(T) = \eta(T_j) = 0 \quad \text{for } j = 1, 2.$$

As the complement, we say that $\eta$ is of type $(1,0)$ if

$$\eta(T_{\bar{j}}) = 0 \quad \text{for } j = 1, 2.$$

Our notation of differential forms of type $(0,1)$ and type $(1,0)$, is adopted from [3]. Note that the pseudo-hermitian structure $\alpha$ is of type $(1,0)$ by default.

The connection forms of the Tanaka-Webster connection, are the 1-forms $\omega_m^n$ on $N$ with $m, n = 1, 2$, defined by $\nabla T_m = \omega_m^n \otimes T_n$. In terms of the Christoffel symbols,

$$\omega_m^n = \Gamma_{km}^n\, \theta^k + \Gamma_{\bar{k}m}^n\, \theta^{\bar{k}} + \Gamma_{0m}^n\, \alpha. \tag{1.6}$$

We then define the canonical bundle $K(N) = \Lambda^{3,0}(N)$ to be the complex line bundle of differential forms of type $(3,0)$ on $N$. For example, $K(N)$ is locally spanned by $\alpha \wedge \theta^1 \wedge \theta^2$. We then let $C(N)$ be the quotient space,

$$C(N) = \big(K(N) - \{\mathbf{0}\}\big)/\mathbb{R}_+.$$

It means that we exclude the zero section of $K(N)$, and then every nonzero element $\epsilon_1$ in $K(N)$ is identified with $k\,\epsilon_1$ for any positive real number $k$.

$C(N)$ defines a principal $\mathbb{S}^1$-bundle over $N$ and it is called the Fefferman bundle of $N$. Let $\pi : C(N) \to N$ be the projection map. Also, we introduce $\gamma$ to be the real parameter of $\mathbb{S}^1$ on $C(N)$, which represents the equivalence class of $e^{i\gamma}\,\alpha \wedge \theta^1 \wedge \theta^2$.

The collection $\big\{T_1, T_2, T_{\bar{1}}, T_{\bar{2}}, T, \frac{\partial}{\partial\gamma}\big\}$ forms a basis for the complex tangent bundle of $C(N)$. Corresponding to this basis, we define a 1-form $\sigma$ on $C(N)$ by ($\nu = 2$)

$$\sigma = \frac{1}{\nu+2}\Big[d\gamma + \pi^*\big(i\omega^m_m - \frac{i}{2}h^{\bar{n}m}dh_{m\bar{n}} - \frac{1}{4(\nu+1)}\rho\ \alpha\big)\Big].$$

The Fefferman metric (associated with $\alpha$) of the CR manifold $N$, is a pseudo-Riemannian metric on $C(N)$ given by

$$F \ = \ \pi^*\big(g|_{\bar{\mathfrak{D}}\oplus\mathfrak{D}}\big) + 2\big(\pi^*\alpha \odot \sigma\big). \tag{1.7}$$

The tensor $g|_{\bar{\mathfrak{D}}\oplus\mathfrak{D}}$ is restriction of the Webster metric, $2\,h_{\alpha\bar{\beta}}\,\theta^\alpha \odot \theta^{\bar{\beta}}$. Here the symmetric product $\odot$ between two $(0,1)$-tensors $A$ and $B$ is obtained by

$$A \odot B = \frac{1}{2}\big(A \otimes B + B \otimes A\big).$$

The conformal class of the Fefferman metric is a CR invariant.

**Theorem 1.3.** [11] *Let $N$ be a CR manifold and $F_\alpha$ be the Fefferman metric on $C(N)$ associated with the pseudo-hermitian structure $\alpha$. Suppose that the Levi form of $N$ is non-degenerate. Let $\hat{\alpha} = e^{2f}\alpha$ be another pseudo-hermitian structure, for some real function $f$ on $N$. Let $F_{\hat{\alpha}}$ be the corresponding Fefferman metric. Then, $F_{\hat{\alpha}} = e^{2f\circ\pi}F_\alpha$.*

Denote the Levi-Civita connection of $F$ on $C(N)$ by $\hat{\nabla}$. We use the notation that $u_1 = T_1$, $u_2 = T_{\bar{1}}$, $u_3 = T_2$, $u_4 = T_{\bar{2}}$, $u_5 = T$ and $u_6 = \frac{\partial}{\partial\gamma}$ for simplicity.

Let $\hat{\nabla}_{u_i}u_j = \hat{\Gamma}^k_{ij}u_k$ and $[u_i, u_j] = \mathcal{A}^k_{ij}u_k$. The Koszul formula gives,

$$\begin{aligned} 2F\big(\hat{\nabla}_{u_i}u_j, u_k\big) \ &= \ u_i\big(F(u_j,u_k)\big) + u_j\big(F(u_i,u_k)\big) - u_k\big(F(u_i,u_j)\big) \\ &\quad -F\big([u_i,u_k],u_j\big) - F\big([u_j,u_k],u_i\big) + F\big([u_i,u_j],u_k\big). \end{aligned}$$

So, we have

$$\hat{\Gamma}_{ij,k} \;=\; F\big(\hat{\nabla}_{u_i}u_j, u_k\big) \;=\; \frac{1}{2}\Big[dF_{jk}(u_i) + dF_{ik}(u_j) - dF_{ij}(u_k) - \mathcal{A}^l_{ik}F_{jl} - \mathcal{A}^l_{jk}F_{il} + \mathcal{A}^l_{ij}F_{kl}\Big]$$

and also

$$\hat{\Gamma}^k_{ij} = \frac{1}{2}F^{kl}\Big[dF_{jl}(u_i) + dF_{il}(u_j) - dF_{ij}(u_l) - \mathcal{A}^p_{il}F_{jp} - \mathcal{A}^p_{jl}F_{ip} + \mathcal{A}^p_{ij}F_{lp}\Big]. \tag{1.8}$$

Let $\hat{R}$ be the Riemann curvature tensor of $\hat{\nabla}$ on $C(N)$.

$$\begin{aligned}\hat{R}(u_i, u_j)u_k \;&=\; \hat{\nabla}_{u_i}\hat{\nabla}_{u_j}u_k - \hat{\nabla}_{u_j}\hat{\nabla}_{u_i}u_k - \hat{\nabla}_{[u_i,u_j]}u_k \\ &=\; \Big(d\hat{\Gamma}^l_{jk}(u_i) - d\hat{\Gamma}^l_{ik}(u_j) + \hat{\Gamma}^p_{jk}\hat{\Gamma}^l_{ip} - \hat{\Gamma}^p_{ik}\hat{\Gamma}^l_{jp} - \mathcal{A}^p_{ij}\hat{\Gamma}^l_{pk}\Big)u_l\end{aligned}$$

We write $\hat{R}(u_i, u_j)u_k = \hat{R}^l_{ijk}u_l$, and it means

$$\hat{R}^l_{ijk} = d\hat{\Gamma}^l_{jk}(u_i) - d\hat{\Gamma}^l_{ik}(u_j) + \hat{\Gamma}^p_{jk}\hat{\Gamma}^l_{ip} - \hat{\Gamma}^p_{ik}\hat{\Gamma}^l_{jp} - \mathcal{A}^p_{ij}\hat{\Gamma}^l_{pk}. \tag{1.9}$$

Contracting with the metric $F$, we also have

$$\hat{R}_{ijkl} = F\big(\hat{R}(u_i, u_j)u_k, u_l\big) = F_{ml}\Big(d\hat{\Gamma}^m_{jk}(u_i) - d\hat{\Gamma}^m_{ik}(u_j) + \hat{\Gamma}^p_{jk}\hat{\Gamma}^m_{ip} - \hat{\Gamma}^p_{ik}\hat{\Gamma}^m_{jp} - \mathcal{A}^p_{ij}\hat{\Gamma}^m_{pk}\Big). \tag{1.10}$$

Accordingly, the Ricci tensor are defined by

$$\hat{R}_{ij} \;=\; \mathrm{Ric}(u_i, u_j) \;=\; \hat{R}^k_{kij} \;=\; F^{kl}\hat{R}_{kijl}. \tag{1.11}$$

And the scalar curvature is given by

$$S \;=\; F^{np}\hat{R}_{np} \;=\; F^{np}F^{mq}\hat{R}_{mnpq}. \tag{1.12}$$

Let $\mathrm{Ric}^\sharp$ be the raise-index of Ric. We let $\mathrm{Ric}^\sharp(u_i) = \hat{R}^j_i u_j$ so $\hat{R}^j_i = \hat{R}_{ik}F^{kj}$.

Using J. M. Lee's theorem, the term $S$ could be found easily by the scalar curvature of the Tanaka-Webster connection.

**Theorem 1.4.** [11] *The scalar curvature of the Fefferman metric $F$ is given by*

$$S = \frac{2\nu + 1}{\nu + 1}\rho,$$

*where $\rho$ is the scalar curvature of the Tanaka-Webster connection, and $\nu$ is the CR dimension of the underlying CR structure.*

In terms of the connection forms of $\hat{\nabla}$ on $C(N)$, we have

$$\hat{\nabla} u_i = \hat{\omega}_i^j \otimes u_j \quad \text{for any} \quad i, j = 1, \cdots, 6. \tag{1.13}$$

We may also find out the coefficients of $\hat{R}$ using the connection forms above. We obtain $\hat{R}\big(u_i, u_j\big)u_m = 2\hat{\Omega}_m^n(u_i, u_j)u_n$ with $\hat{\Omega}_m^n = d\hat{\omega}_m^n - \hat{\omega}_m^k \wedge \hat{\omega}_k^l$. In other words,

$$\hat{R}_{ijk}^l \;=\; 2\,\big(d\hat{\omega}_k^l - \hat{\omega}_k^p \wedge \hat{\omega}_p^l\big)\big(u_i, u_j\big). \tag{1.14}$$

As a remark, we will always use the convention that $\phi \wedge \psi = \frac{1}{2}\big(\phi \otimes \psi - \psi \otimes \phi\big)$ and $d\phi(u, v) = \frac{1}{2}\Big[u\big(\phi(v)\big) - v\big(\phi(u)\big) - \phi\big([u, v]\big)\Big]$ for any 1-forms $\phi$ and $\psi$.

The Weyl curvature tensor of $\hat{\nabla}$ in terms of a $(1, 3)$-tensor, is defined by

$$\begin{aligned}\mathcal{W}(u_i, u_j)u_k \;&=\; \hat{R}(u_i, u_j)u_k + \frac{1}{4}F(u_i, u_k)\mathrm{Ric}^\sharp(u_j) - \frac{1}{4}F(u_j, u_k)\mathrm{Ric}^\sharp(u_i) \\ &\quad + \frac{1}{4}\mathrm{Ric}(u_i, u_k)u_j - \frac{1}{4}\mathrm{Ric}(u_j, u_k)u_i + \frac{S}{20}\Big(F(u_j, u_k)u_i - F(u_i, u_k)u_j\Big).\end{aligned}$$

for every $u_i$, $u_j$ and $u_k$. Let $\mathcal{W}(u_i, u_j)u_k = \mathcal{W}_{ijk}^l u_l$. We have

$$\mathcal{W}_{ijk}^l = \hat{R}_{ijk}^l + \frac{1}{4}\hat{R}_j^l F_{ik} - \frac{1}{4}\hat{R}_i^l F_{jk} + \frac{1}{4}\hat{R}_{ik}\delta_{jl} - \frac{1}{4}\hat{R}_{jk}\delta_{il} + \frac{S}{20}\Big(F_{jk}\delta_{il} - F_{ik}\delta_{jl}\Big). \tag{1.15}$$

The lower-index of $\mathcal{W}$ is then defined by $\mathcal{W}_{ijkl} = F\big(\mathcal{W}(u_i, u_j)u_k, u_l\big)$, with

$$\mathcal{W}_{ijkl} = \hat{R}_{ijkl} - \frac{1}{4}\hat{R}_{il}F_{jk} - \frac{1}{4}\hat{R}_{jk}F_{il} + \frac{1}{4}\hat{R}_{ik}F_{jl} + \frac{1}{4}\hat{R}_{jl}F_{ik} + \frac{S}{20}\Big(F_{il}F_{jk} - F_{ik}F_{jl}\Big). \tag{1.16}$$

The Weyl tensor shares the same symmetries with the Riemann tensor $\hat{R}$, including

$$\mathcal{W}_{ijkl} = -\mathcal{W}_{jikl} = -\mathcal{W}_{ijlk}, \quad \mathcal{W}_{ijkl} = \mathcal{W}_{klij}, \quad \text{and} \quad \mathcal{W}_{ijkl} + \mathcal{W}_{jkil} + \mathcal{W}_{kijl} = 0.$$

Moreover, $\mathcal{W}$ is trace-free because $\mathcal{W}_{ijkl}F^{il} = 0$.

As a remark. $\mathcal{W}$ is conformally invariant with respect to the metric. That is, if the metric $F$ is replaced by $e^{2\lambda}F$, then the (1,3)-Weyl tensor remains the same, and the (0,4)-Weyl tensor of $e^{2\lambda}F$ becomes $e^{2\lambda}\mathcal{W}$. Moreover, $F$ is conformally flat if and only if the Weyl tensor vanishes.

## 2.0 TWISTOR CR MANIFOLD OF HAMILTONIAN DISTRIBUTION

On LeBrun's paper [9], for every 3-dimensional Riemannian manifold $M$ equipped with the metric $\mathsf{g}$, the twistor CR manifold of $(M, \mathsf{g})$ is constructed by the Hamiltonian distribution on the complex cotangent bundle of $M$. This definition of a 5-dimensional twistor CR manifold is invariant over the conformal class of $\mathsf{g}$.

The twistor CR manifold mentioned in this chapter, refers to the twistor CR structures of zero torsion in later context.

### 2.1 THE TWISTOR CR MANIFOLD $N$ OF $(M, \mathsf{g})$

Let $M$ be a 3-dimensional real manifold. Let $\mathbb{C}T^*M$ be the complex cotangent bundle of $M$. Suppose $\mathsf{g}$ is a Riemannian metric on $M$. Let $(x_1, x_2, x_3)$ be a coordinate system on $M$. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame on $M$ over the local chart. We can define a coordinate system $(x, \mu) = \big(x_1, x_2, x_3, \mu_1, \mu_2, \mu_3\big)$ on $\mathbb{C}T^*M$ to represent the covector $\mu = \mu_i e^i$ at the point $x = (x_1, x_2, x_3)$.

Let $\mathsf{g}^{-1}$ be the cometric of $\mathsf{g}$. If $\mu = \mu_i e^i$ is at the point $x$, then $\mathsf{g}^{-1}(\mu, \mu) = \sum_{i=1}^{3} \mu_i^2$. The 7-dimensional submanifold $\hat{N}$ of $\mathbb{C}T^*M$ consists of all null covectors,

$$\hat{N} = \Big\{(x, \mu) \in \mathbb{C}T^*M \mid \mathsf{g}_x^{-1}(\mu, \mu) = 0,\ \mu \neq 0\Big\}.$$

Let $\pi$ be the projection map from $\hat{N}$ to $M$ .

Let $\theta$ be the canonical 1-form on $\mathbb{C}T^*M$. At $(x,\mu)$, $\theta = \mu_i e^i$. The Hamiltonian form on $\mathbb{C}T^*M$ is the derivative of $\theta$, $\omega = d\theta$. Denote the Riemannian connection of $\mathsf{g}$ by $\bar{\nabla}$ and its connection form by $\omega_{ij}$. We write $\bar{\nabla}_{e_i} e_j = G^k_{ij} e_k$ and $\omega_{jk} = G^k_{ij} e^i$. Therefore, the Hamiltonian form is $\omega = D\mu_i \wedge e^i$, where $D\mu_i = d\mu_i + \mu_j \omega_{ji}$ is the covariant differential of $\mu_i$ on $\mathbb{C}T^*M$.

Define the Hamiltonian distribution $D$ by the kernel of $\omega$ restricted on $\mathbb{C}T\hat{N}$. It is an involutive distribution of complex 3-planes on $\hat{N}$. At the point $(x,\mu)$, $D$ is spanned by the horizontal vector field $\mu^h$ with

$$\mu^h = \mu_j\, e_j - \mu_m\, G^l_{mk}\, \mu_k\, \frac{\partial}{\partial \mu_l} - \mu_m\, G^l_{mk}\, \overline{\mu}_k\, \frac{\partial}{\partial \overline{\mu}_l} \tag{2.1}$$

and other vertical vectors $\displaystyle\sum_{i=1}^{3} c_i \frac{\partial}{\partial \overline{\mu}_i}$ such that $\displaystyle\sum_{i=1}^{3} c_i \overline{\mu}_i = 0$.

**Proposition 2.1.** [9] *$D$ is a CR structure on $\hat{N}$ of type $(3,1)$.*

Let $(x,\mu)$ be a point on the 7-manifold $\hat{N}$. The covector $\mu$ is on the fibre $\hat{N}_x$, so $c\mu$ is also on $\hat{N}_x$ for any $c \in \mathbb{C}^*$. We may define a 5-manifold $N = \hat{N}/\mathbb{C}^*$ in this way. With respect to the orthonormal frame coordinates $(x,\mu)$, we let $[\mu] = [\mu_1 : \mu_2 : \mu_3]$. Therefore,

$$N = \Big\{ (x,[\mu]) \in \mathbb{P}T^*M \mid \mu_1^2 + \mu_2^2 + \mu_3^2 = 0 \ \text{ and } \ \mu \neq 0 \Big\}.$$

Every fibre $N_x$ is biholomorphic to a Riemann sphere.

Let $P : \hat{N} \to N$ be the quotient map. Given $x \in M$ and $c \in \mathbb{C}^*$, the left multiplication on $\hat{N}_x$ is defined by $m_c(\mu) = c\mu$. We have $D_{c\mu} = dm_c(D_\mu)$ and so $dP_{c\mu}(D_{c\mu}) = dP_\mu(D_\mu)$. Therefore, the Hamiltonian distribution $D$ descends to a complex 2-plane distribution $\mathfrak{D}$ on $N$. Namely, $\mathfrak{D}_{[\mu]} = dP_\mu(D_\mu)$ at every $(x,[\mu]) \in N$.

**Proposition 2.2.** [9] *$\mathfrak{D}$ is a CR structure on $N$ of type $(2,1)$.*

The above CR 5-manifold $(N, \mathfrak{D})$ is called the twistor CR manifold of $(M, \mathsf{g})$. Note that $\mathfrak{D}$ only depends on the conformal structure $[\mathsf{g}]$ of $\mathsf{g}$. The set of null covectors remain the same for all metrics conformal to $\mathsf{g}$, so $\hat{N}$ and $N$ are uniquely defined by $[\mathsf{g}]$. The fact that $\mathfrak{D}$ remains unchanged will be discussed in Section 5.1.

## 2.2 THE RATIONAL PARAMETRIZATION OF $\hat{N}$ AND $N$

Let $(x, \mu)$ be a covector on $\hat{N}$ under the coordinate system corresponding to the frame $\{e_1, e_2, e_3\}$ on $M$. On this coordinate neighborhood, we define the rational parametrization, which is also found in [8], of $\hat{N}$ through the map $\hat{f} : \mathbb{C}^2 \backslash \mathbf{0} \to \hat{N}_x$,

$$\mu = \hat{f}(s,t) : \qquad (\mu_1, \mu_2, \mu_3) = \Big(s^2 - t^2,\ 2st,\ i(s^2 + t^2)\Big), \tag{2.2}$$

on every fibre $\hat{N}_x$. Similarly the rational parametrization of $N$ is the map $f : \mathbb{C}P^1 \to N_x$,

$$[\mu] = f([s:t]) : \qquad [\mu_1 : \mu_2 : \mu_3] = \Big[s^2 - t^2 :\ 2st :\ i(s^2 + t^2)\Big]. \tag{2.3}$$

**Proposition 2.3.**
*The map* $\hat{f} : (s,t) \mapsto \Big(s^2 - t^2, 2st, i(s^2 + t^2)\Big)$ *is a 2:1 covering map on every fibre.*
*The map* $f : [s:t] \mapsto \Big[s^2 - t^2 : 2st : i(s^2 + t^2)\Big]$ *is a biholomorphism on every fibre.*

Consider the covector $v = i\mu \times \overline{\mu}$ on $\mathbb{R}T^*M$. Explicitly,

$$v = i\Big(\mu_2\overline{\mu}_3 - \mu_3\overline{\mu}_2\Big)e^1 + i\Big(\mu_3\overline{\mu}_1 - \mu_1\overline{\mu}_3\Big)e^2 + i\Big(\mu_1\overline{\mu}_2 - \mu_2\overline{\mu}_1\Big)e^3.$$

By equation (2.2),

$$v = 2(|s|^2 + |t|^2)\Big((s\overline{t} + \overline{s}t)e^1 + (|t|^2 - |s|^2)e^2 + i(s\overline{t} - \overline{s}t)e^3\Big).$$

**Proposition 2.4.**
*Let* $\mu = \Big(s^2 - t^2, 2st, i(s^2+t^2)\Big)$ *in the rational parametrization. Let* $v = i\mu \times \overline{\mu}$.

*(1)* $\quad |\mu| = \sqrt{2}\ (|s|^2 + |t|^2)$.

*(2)* $\quad |v| = |\mu|^2 = 2\ (|s|^2 + |t|^2)^2$.

In particular, the covectors $\mathrm{Re}(\mu)$, $\mathrm{Im}(\mu)$ and $v$ form an orthogonal basis for $\mathbb{R}T^*M$. Note that $|\mathrm{Re}(\mu)| = |\mathrm{Im}(\mu)| = |s|^2 + |t|^2$.

The rational parametrization $\hat{f}$ induces the following two vector fields on $\hat{N}$,

$$R = d\hat{f}\Big(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\Big) \quad \text{and} \quad Q = d\hat{f}\Big(-\overline{t}\frac{\partial}{\partial s} + \overline{s}\frac{\partial}{\partial t}\Big)$$

which are always transverse to each other at nonzero $(s,t)$. In terms of coordinates $(x,\mu)$,

$$R = 2\sum_{k=1}^{3}\mu_k\frac{\partial}{\partial \mu_k} \quad \text{and} \quad Q = \frac{-\sqrt{2}}{|\mu|}\sum_{k=1}^{3} v_k\frac{\partial}{\partial \mu_k}.$$

The Hamiltonian distribution $D$ at $\mu = \hat{f}(s,t)$ is then spanned by $\mu^h$ (2.1), $\overline{R}$ and $\overline{Q}$.

Meanwhile on the manifold $N$, we introduce a complex parameter $u$ to represent the point $[s:t] = [u:1]$ in (2.3). We write $[\mu] = f(u) = [u^2 - 1 : 2u : i(u^2+1)]$. The CR distribution $\mathfrak{D}$ at $(x,u)$ is spanned by $dP_{(x,\mu)}(\mu^h)$ and $\dfrac{\partial}{\partial \overline{u}}$. We are going to describe the former vector field $dP_{(x,\mu)}(\mu^h)$ in the coordinates $(x,u)$.

Consider this commutative diagram. On the left, $P_0$ is the map $u = \dfrac{s}{t}$.

$$\begin{array}{ccccc}
(\overline{X}_0) & \mathbb{C}^2_{(s,t)} & \xrightarrow{\hat{f}} & \hat{N} & (\mu^h) \\
 & P_0 \downarrow & & \downarrow P & \\
(\overline{X}_1) & \mathbb{C}_u & \xrightarrow{f} & N & (dP(\mu^h))
\end{array}$$

Let $\overline{X}_1$ be a vector field on $T\mathbb{C}$ such that $df(\overline{X}_1) = dP(\mu^h)$. Suppose $\overline{X}_1$ comes from a vector field $\overline{X}_0$ on $T\mathbb{C}^2$, i.e. $dP_0(\overline{X}_0) = \overline{X}_1$. Then we have $df\Big(dP_0(\overline{X}_0)\Big) = dP(\mu^h)$. By the commutative diagram, $dP\Big(d\hat{f}(\overline{X}_0)\Big) = dP(\mu^h)$. Therefore, $d\hat{f}(\overline{X}_0) \equiv \mu^h$ modulus $R$ and $\overline{R}$.

At the point $(x, \mu)$ of $\hat{N}$, we may rewrite the formula (2.1) of $\mu^h$ in terms of $R$, $Q$ and their complex conjugates. It leads to

$$\mu^h = \mu_j e_j - \frac{\mu_m G^l_{mk}\mu_k\overline{\mu}_l}{4(|s|^2+|t|^2)^2}R + \frac{\mu_m G^l_{mk}\mu_k v_l}{4(|s|^2+|t|^2)^3}Q - \frac{\mu_m G^l_{mk}\overline{\mu}_k\mu_l}{4(|s|^2+|t|^2)^2}\overline{R} + \frac{\mu_m G^l_{mk}\overline{\mu}_k v_l}{4(|s|^2+|t|^2)^3}\overline{Q}.$$

We would shorten this expression by letting $\mu^h = \mu_j e_j - K_1 R + K_2 Q - K_3\overline{R} + K_4\overline{Q}$. Since $dP(R) = dP(\overline{R}) = 0$, we get to

$$dP(\mu^h) = \mu_j e_j + K_2 dP(Q) + K_4 dP(\overline{Q}).$$

$\overline{X}_0$ is then defined by

$$\overline{X}_0 = \mu_j e_j - \overline{t}K_2\frac{\partial}{\partial s} + \overline{s}K_2\frac{\partial}{\partial t} - tK_4\frac{\partial}{\partial \overline{s}} + sK_4\frac{\partial}{\partial \overline{t}}.$$

Using $dP_0\big(\frac{\partial}{\partial s}\big) = \frac{\partial}{\partial u}$ and $dP_0\big(\frac{\partial}{\partial t}\big) = -u\frac{\partial}{\partial u}$ when $s = u$ and $t = 1$,

$$\overline{X}_1 = dP_0(\overline{X}_0) = \mu_j e_j - (1+|u|^2)K_2\frac{\partial}{\partial u} - (1+|u|^2)K_4\frac{\partial}{\partial \overline{u}}.$$

Note that the projection map $P$ is holomorphic. If the coefficient of $\mu^h$ with respect to $\frac{\partial}{\partial \mu_l}$ is holomorphic in $\mu$, then the coefficient of $\overline{X}_1$ with respect to $\frac{\partial}{\partial u}$ is holomorphic in $u$. Indeed we have

$$\overline{X}_1 = \mu_j e_j - \frac{i\mu_m}{2}\Big(G^2_{m1}\mu_3 + G^1_{m3}\mu_2 + G^3_{m2}\mu_1\Big)\frac{\partial}{\partial u} + \frac{i\mu_m}{2}\Big(G^2_{m1}\overline{\mu}_3 + G^1_{m3}\overline{\mu}_2 + G^3_{m2}\overline{\mu}_1\Big)\frac{\partial}{\partial \overline{u}}. \quad (2.4)$$

with $\mu_1 = u^2 - 1$, $\mu_2 = 2u$ and $\mu_3 = i(u^2+1)$ at $(x, u)$. In the following context, we would just say $\overline{X}_1 = dP(\mu^h)$ without any ambiguity.

## 2.3 THE LEVI FORM OF $(\hat{N}, D)$ AND $(N, \mathfrak{D})$

A pseudo-hermitian structure of the CR structure $D$ is given by

$$\hat{\alpha} = \frac{v}{|v|} = \frac{v_k}{|v|} e^k.$$

By the identity $de^i = \omega_{ij} \wedge e^j$ for every $i$, its exterior derivative is

$$d\hat{\alpha} = D\Big(\frac{v_k}{|v|}\Big) \wedge e^k = \bigg(d\Big(\frac{v_k}{|v|}\Big) + \frac{v_j}{|v|}\omega_{jk}\bigg) \wedge e^k.$$

Let $\hat{\mathcal{L}}$ be the Levi form of $\hat{D}$ associated with $\hat{\alpha}$, so $\hat{\mathcal{L}} = -i\, d\hat{\alpha}$. We may show that $\hat{\mathcal{L}}$ is a degenerate bilinear form.

Recall that $v_k = i(\mu \times \overline{\mu})_k$ for every $k$. We define the horizontal vector field $v^h$ at a point $(x, \mu)$ of $\hat{N}$, just as the way $\mu^h$ being defined. Explicitly,

$$v^h = v_j\, e_j - v_m\, G^l_{mk}\, \mu_k\, \frac{\partial}{\partial \mu_l} - v_m\, G^l_{mk}\, \overline{\mu}_k\, \frac{\partial}{\partial \overline{\mu}_l}. \tag{2.5}$$

Important properties of the differential $dv_k$ are listed as follows.

**Proposition 2.5.** *For every $j = 1, 2, 3$,*

*(1)* $\quad dv_j\big(\mu_l \dfrac{\partial}{\partial \mu_l}\big) = v_j,$

*(2)* $\quad dv_j\big(\overline{\mu}_l \dfrac{\partial}{\partial \mu_l}\big) = 0,$

*(3)* $\quad dv_j(\mu^h) = \mu_m\, G^k_{mj}\, v_k,$

*(4)* $\quad dv_j(v^h) = v_m\, G^k_{mj}\, v_k.$

*Proof.* Let $\epsilon_{jkl}$ be the sign of the permutation $(j, k, l)$, where $j, k, l = 1, 2, 3$. We say $\epsilon_{jkl} = 0$ when $(j, k, l)$ is not any permutation of numbers 1, 2 and 3. For example $\epsilon_{123} = 1$ and $\epsilon_{213} = -1$. We may then write $v_j = i\big(\epsilon_{jkl}\, \mu_k\, \overline{\mu}_l\big)$ and so

$$dv_j = i\, \epsilon_{jkl}\big(\overline{\mu}_l\, d\mu_k + \mu_k\, d\overline{\mu}_l\big).$$

Therefore,

$$dv_j\big(\mu_k \frac{\partial}{\partial \mu_k}\big) \;=\; i\big(\epsilon_{jkl}\,\overline{\mu}_l\,\mu_k\big) \;=\; v_j.$$

Similarly, we could obtain $dv_j\big(\overline{\mu}_k \frac{\partial}{\partial \mu_k}\big) = 0$. For the item (3), we have

$$\begin{aligned}
dv_j(\mu^h) &= i\,\epsilon_{jkl}\Big(\overline{\mu}_l\,d\mu_k + \mu_k\,d\overline{\mu}_l\Big)\Big(\mu_p\,e_p - \mu_m\,G^q_{mp}\,\mu_p\,\frac{\partial}{\partial \mu_q} - \mu_m\,G^q_{mp}\,\overline{\mu}_p\,\frac{\partial}{\partial \overline{\mu}_q}\Big)\\
&= -i\,\mu_m\,\epsilon_{jkl}\Big(\overline{\mu}_l\,G^k_{mp}\,\mu_p + \mu_k\,G^l_{mp}\,\overline{\mu}_p\Big)\\
&= -i\,\mu_m\,\epsilon_{jkl}\,G^k_{mp}\big(\mu_p\,\overline{\mu}_l - \mu_l\,\overline{\mu}_p\big)\\
&= -i\,\mu_m\,\epsilon_{jkl}\,G^k_{mj}\big(\mu_j\,\overline{\mu}_l - \mu_l\,\overline{\mu}_j\big)\\
&= -i\,\mu_m\,\epsilon_{jkl}\,G^k_{mj}\big(-\,i\,\epsilon_{jlk}\,v_k\big) \;=\; \mu_m\,G^k_{mj}\,v_k.
\end{aligned}$$

Replacing $\mu_m$ by $v_m$, we obtain the item (4) as well. □

By Proposition 2.5, we could see that $d\hat{\alpha}(R,\overline{Y}) = 0$ for any $\overline{Y}$ in $D$. It is because

$$\begin{aligned}
d\hat{\alpha}(R,\overline{Y}) &= \frac{1}{2}\,d\Big(\frac{v_k}{|v|}\Big)(R)\cdot e^k(\overline{Y})\\
&= \Big(-\frac{v_k\,v_p}{|v|^{3/2}}\,dv_p + \frac{1}{|v|}\,dv_k\Big)\big(\mu_l\,\frac{\partial}{\partial \mu_l}\big)\cdot e^k(\overline{Y})\\
&= \Big(-\frac{v_k\,|v|^2}{|v|^{3/2}} + \frac{v_k}{|v|}\Big)\cdot e^k(\overline{Y}) \;=\; 0.
\end{aligned}$$

Therefore, $\hat{\mathcal{L}}$ is degenerate and so $D$ is a degenerate CR structure on $\hat{N}$. We may also show that $d\hat{\alpha}(v^h,Y) = d\hat{\alpha}(v^h,\overline{Y}) = 0$ for any $Y$ in $\bar{\mathfrak{D}}$. Using (2.5),

$$\begin{aligned}
d\hat{\alpha}(v^h,\overline{R}) &= -\frac{1}{2}\,d\Big(\frac{v_k}{|v|}\Big)\big(\overline{R}\big)\cdot e^k(v^h) \;=\; 0\\
d\hat{\alpha}\big(v^h,\mu^h\big) &= \frac{\mu_k}{2}\,D\Big(\frac{v_k}{|v|}\Big)\big(v^h\big) - \frac{v_k}{2}\,D\Big(\frac{v_k}{|v|}\Big)\big(\mu^h\big) \;=\; 0,\\
d\hat{\alpha}\big(v^h,\overline{Q}\big) &= -\frac{v_k}{2}\,d\Big(\frac{v_k}{|v|}\Big)\big(\overline{Q}\big) \;=\; 0 \quad \text{since} \quad \sum_{k=1}^{3} v_k\,d\Big(\frac{v_k}{|v|}\Big) = 0.
\end{aligned}$$

Since both $\hat{\alpha}$ and $v^h$ are real-valued, we could consider the complex conjugation of the above formulas. As a result, $\iota_{v^h} d\hat{\alpha} = 0$. Moreover, we have the fact that $\hat{\alpha}(v^h) = 1$.

On the other hand, a pseudo-hermitian structure of the CR manifold $(N, \mathfrak{D})$ is given by

$$\alpha = \frac{u+\overline{u}}{1+|u|^2} e^1 + \frac{1-|u|^2}{1+|u|^2} e^2 + \frac{i(u-\overline{u})}{1+|u|^2} e^3 \tag{2.6}$$

in $(x, u)$. Note that $P^*\alpha = \hat{\alpha}$. Its exterior derivative $d\alpha$ is given by

$$\begin{aligned} d\alpha \;=\; & \frac{1}{(1+|u|^2)^2} \begin{pmatrix} (1-\overline{u}^2)\; du \wedge e^1 + (1-u^2)\; d\overline{u} \wedge e^1 - 2\overline{u}\; du \wedge e^2 \\ -2u\; d\overline{u} \wedge e^2 + i(1+\overline{u}^2)\; du \wedge e^3 - i(1+u^2)\; d\overline{u} \wedge e^3 \end{pmatrix} \\ & + \frac{1}{1+|u|^2} \Big( (u+\overline{u})\; de^1 + (1-|u|^2)\; de^2 + i(u-\overline{u})\; de^3 \Big). \end{aligned}$$

The associated Levi form $\mathcal{L}_N : \overline{\mathfrak{D}} \times \mathfrak{D} \to \mathbb{C}$ is non-degenerate, for

$$\mathcal{L}_N\Big(\frac{\partial}{\partial u}, \overline{X}_1\Big) \;=\; -\, i d\alpha\Big(\frac{\partial}{\partial u}, \overline{X}_1\Big) \;=\; \frac{i}{2(1+|u|^2)^2} \cdot 2(1+|u|^2)^2 \;=\; i$$

at every $(x, u)$ in $N$. Here $\overline{X}_1$ follows from the equation (2.4)

**Proposition 2.6.** [9] *The CR structure $(N, \mathfrak{D})$ is non-degenerate and anticlastic.*

$\alpha$ is then a contact form of the Levi distribution $\mathfrak{D} \oplus \overline{\mathfrak{D}}$. Indeed, we have

$$\alpha \wedge (d\alpha)^2 = \frac{-4i}{(1+|u|^2)^2}\; du \wedge d\overline{u} \wedge e^1 \wedge e^2 \wedge e^3.$$

The Reeb vector field $T$ associated with $\alpha$ is given by

$$T \;=\; dP\Big(\frac{v^h}{|v|}\Big) \;=\; \frac{v_j}{|v|}\, e_j - \frac{1}{2}\frac{v_m}{|v|}\Big(G^l_{mk}\, \mu_k\, \frac{v_l}{|v|}\Big)\frac{\partial}{\partial u} - \frac{1}{2}\frac{v_m}{|v|}\Big(G^l_{mk}\, \overline{\mu}_k\, \frac{v_l}{|v|}\Big)\frac{\partial}{\partial \overline{u}} \tag{2.7}$$

at any point $(x, u)$. For any $\overline{Y} = dP(\overline{Y}_0)$ in $\mathfrak{D}$ with $\overline{Y}_0$ in $D$,

$$d\alpha(T, \overline{Y}) = P^* d\alpha\Big(\frac{v^h}{|v|}, \overline{Y}_0\Big) = d\hat{\alpha}\Big(\frac{v^h}{|v|}, \overline{Y}_0\Big) = 0.$$

## 2.4 CR STRUCTURE ON THE SPHERE BUNDLE OF $M$

The twistor CR manifold $N$ of $(M, \mathsf{g})$ is diffeomorphic to the sphere bundle $(S)$ of $M$. We may construct a CR structure $\Pi$ on $S$ such that $\mathfrak{D}$ is CR equivalent to $\Pi$. This process will make use of the horizontal and vertical spaces of the tangent bundle of $M$ [16].

Corresponding to the orthonormal frame $\{e_1, e_2, e_3\}$, a unit tangent vector $\lambda = \lambda_i e_i$ at $x \in M$ is represented by $(x, \lambda) = \big(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3\big)$ with $\sum_{i=1}^{3} \lambda_i^2 = 1$. Let $X = \xi_i e_i$ be a tangent vector on $T_x M$. The horizontal lift of $X$ at $(x, \lambda)$ on $TS$ is

$$X^h = \xi_i \, e_i - \xi_j \, G^l_{jk} \, \lambda_k \, \frac{\partial}{\partial \lambda_l}, \tag{2.8}$$

The horizontal bundle $\mathcal{H}$ of $S$ is defined by

$$\mathcal{H}_{(x,\lambda)} = \Big\{ X^h \in T_{(x,\lambda)} S \mid \mathsf{g}(X, \lambda) = 0 \Big\}$$

at every point $(x, \lambda)$. On the other hand, the vertical lift of $X$ at $(x, \lambda)$ is

$$X^v = \xi_i \, \frac{\partial}{\partial \lambda_i}. \tag{2.9}$$

We define the vertical bundle $\mathcal{V}$ of $S$ by

$$\mathcal{V}_{(x,\lambda)} = \Big\{ X^v \in T_{(x,\lambda)} S \mid \mathsf{g}(X, \lambda) = 0 \Big\}.$$

at every point $(x, \lambda)$ on $S$.

An almost complex structure $J$ is defined on the 4-dimensional distribution $\mathcal{V} \oplus \mathcal{H}$. Given $\lambda = \lambda_i e_i$ and $\eta = \eta_i e_i$ on $T_x M$, the cross product $\lambda \times w$ is given by

$$\lambda \times \eta = \big(\lambda_2 \eta_3 - \lambda_3 \eta_2\big) e_1 + \big(\lambda_3 \eta_1 - \lambda_1 \eta_3\big) e_2 + \big(\lambda_1 \eta_2 - \lambda_2 \eta_1\big) e_3.$$

At the point $(x, \lambda)$, for every $X^h \in \mathcal{H}$ and $X^v \in \mathcal{V}$, we let

$$JX^h = (\lambda \times X)^h \quad \text{and} \quad JX^v = (\lambda \times X)^v.$$

Explicitly, if $X$ is on $T_xM$ with $\mathsf{g}(X,\lambda)=0$, then by (2.8) and (2.9)

$$JX^h = (\lambda\times X)_i\, e_i - (\lambda\times X)_j\, G^l_{jk}\,\lambda_k\,\frac{\partial}{\partial\lambda_l} \quad \text{and} \quad JX^v = (\lambda\times X)_i\,\frac{\partial}{\partial\lambda_i}$$

at $(x,\lambda)$.

Let $\Pi$ be the antiholomorphic bundle over $S$ corresponding to $\mathcal{V}\oplus\mathcal{H}$ and $J$. At every $(x,\lambda)$, the complex 2-plane $\Pi$ is spanned by the vectors in the form of $X^h+iJX^h$ and $X^v+iJX^v$, given that $\mathsf{g}(X,\lambda)=0$.

$\Pi$ is integrable and therefore $\Pi$ defines a CR structure on the sphere bundle $S$.

This CR manifold $(S,\Pi)$ can be identified with the twistor CR manifold $(N,\mathfrak{D})$ of $M$. The identification $\Phi: N\to S$ is defined as follows. At any point $x\in M$, $\Phi$ maps from the fibre $N_x$ to the fibre $S_x$ such that

$$\Phi([\mu]) = \frac{v}{|v|} = \frac{i\mu\times\overline{\mu}}{|\mu|^2},$$

where $\mu$ is any representative in the equivalence class $[\mu]$. In terms of $(x,u)$, we have

$$\Phi(u) = \frac{u+\overline{u}}{1+|u|^2}\; e_1 + \frac{1-|u|^2}{1+|u|^2}\; e_2 + \frac{i(u-\overline{u})}{1+|u|^2}\; e_3. \tag{2.10}$$

We also let the composite function $\hat{\Phi}:\hat{N}\to S$ be defined by $\hat{\Phi}=\Phi\circ P$.

**Proposition 2.7.** [9] *$\Phi: N\to S$ is a CR isomorphism between $\mathfrak{D}$ and $\Pi$.*

*Proof.* The differential map $d\Phi$ at $(x,u)$ is described as follows. First of all, we have

$$\begin{aligned} d\Phi\Big(\frac{\partial}{\partial u}\Big) &= d\Big(\frac{v_k}{|v|}\Big)\Big(\frac{\partial}{\partial u}\Big)\;\frac{\partial}{\partial\lambda_k} \\ &= \frac{1}{(1+|u|^2)^2}\Big((1-\overline{u}^2)\frac{\partial}{\partial\lambda_1} - 2\overline{u}\frac{\partial}{\partial\lambda_2} + i(1+\overline{u}^2)\frac{\partial}{\partial\lambda_3}\Big) \\ &= \frac{-1}{(1+|u|^2)^2}\Big(\overline{\mu}_l\frac{\partial}{\partial\lambda_l}\Big). \end{aligned}$$

By taking the complex conjugate, we have $d\Phi\Big(\dfrac{\partial}{\partial\overline{u}}\Big) = \dfrac{-1}{(1+|u|^2)^2}\cdot\mu_l\dfrac{\partial}{\partial\lambda_l}$. Moreover,

$$\begin{aligned} d\Phi\big(\overline{X}_1\big) &= d\hat{\Phi}\big(\mu^h\big) = \mu_j e_j + d\big(\frac{v_k}{|v|}\big)\big(\mu^h\big)\ \frac{\partial}{\partial\lambda_k} \\ &= \mu_j e_j + \Big(\frac{1}{|v|}dv_k - \frac{v_k}{|v|^3}v_j dv_j\Big)\big(\mu^h\big)\ \frac{\partial}{\partial\lambda_k} \\ &= \mu_j e_j + \frac{1}{|v|}\mu_m G^l_{mk} v_l\frac{\partial}{\partial\lambda_k} - \frac{v_k}{|v|^3}v_j\cdot\mu_m G^l_{mj}v_l\frac{\partial}{\partial\lambda_k} \\ &= \mu_j e_j - \mu_m G^k_{ml}\frac{v_l}{|v|}\frac{\partial}{\partial\lambda_k}. \end{aligned}$$

Therefore, $d\Phi_{(x,u)}$ sends $\mathfrak{D}_{(x,u)}$ isomorphically to $\Pi_{(x,\Phi(u))}$. □

## 2.5 EMBEDDING INTO COMPLEX 3-MANIFOLD

If $M$ is a real analytic 3-manifold, then $\mathfrak{D}$ is a real analytic CR structure on $N$. By Theorem 1.1, $N$ is locally embeddable to $\mathbb{C}^3$ and it could be globally embedded to a complex 3-manifold. LeBrun showed that the converse also holds in [9].

**Theorem 2.8.** [9] *Let $M$ be a smooth 3-manifold equipped with the conformal structure $[\mathsf{g}]$. Let $N$ be the twistor CR manifold of $M$ equipped with the CR structure $\mathfrak{D}$. Then, $(N,\mathfrak{D})$ is embeddable into a complex 3-manifold if and only if $M$ admits a real analytic atlas on which there is a real analytic metric $\mathsf{g}$ in class $[\mathsf{g}]$.*

When $M$ is equipped with a flat metric, and $(x,u)$ are coordinates on $N$, $\mathfrak{D}$ is spanned by

$$\overline{X}_1 = (u^2-1)\frac{\partial}{\partial x_1} + 2u\frac{\partial}{\partial x_2} + i(u^2+1)\frac{\partial}{\partial x_3} \quad\text{and}\quad \overline{X}_2 = \frac{\partial}{\partial\overline{u}}.$$

Let $f: N\to\mathbb{C}$ be a CR function on $N$. Then we must have $f_{\overline{u}}=0$ and

$$(u^2-1)f_{x_1} + 2u\,f_{x_2} + i(u^2+1)f_{x_3} = 0 \tag{2.11}$$

We may first let $\gamma(t) = \Big(x_1(t), x_2(t), x_3(t), u(t)\Big)$ be a characteristic curve of (2.11), i.e.

$$x_1' = u^2 - 1, \quad x_2' = 2u, \quad x_3' = i(u^2+1) \quad \text{and} \quad u' = 0.$$

Immediately we have $u = u_0$ for some constant $u_0$, and so $x_1' = u_0^2 - 1$.

$$\frac{dx_2}{dx_1} = \frac{x_2'(t)}{x_1'(t)} = \frac{2u}{u^2-1} \qquad \Longrightarrow \qquad x_2 = \frac{2u\,x_1}{u^2-1} + c_0$$

for some constant $c_0$. It could be written as $(u^2-1)x_2 - 2ux_1 \;=\; c_0(u_0^2-1)$. Similarly,

$$\frac{dx_3}{dx_1} = \frac{x_3'(t)}{x_1'(t)} = \frac{i(u^2+1)}{u^2-1} \qquad \Longrightarrow \qquad x_3 = \frac{i(u^2+1)x_1}{(u^2-1)} + c_1$$

for some constant $c_1$. Therefore, $(u^2-1)x_3 - i(u^2+1)x_1 = (u_0^2-1)c_1$.

Hence, the solution $f$ to equation (2.11) is a function of

$$u, \quad 2u\,x_1 - (u^2-1)x_2 \quad \text{and} \quad i(u^2+1)x_1 - (u^2-1)x_3.$$

We make use of these three basic solutions and let

$$\begin{cases} w_1 &= u, \\ w_2 &= 2u\,x_1 - (u^2-1)x_2, \\ w_3 &= i(u^2+1)x_1 - (u^2-1)x_3. \end{cases}$$

In order to obtain an algebraic relation between $w_1$, $w_2$ and $w_3$, we note that

$$\begin{aligned} (\overline{u}^2-1)w_3 - (u^2-1)\overline{w}_3 &= 2i\,x_1\big(|u|^4-1\big), \\ (\overline{u}^2-1)w_2 - (u^2-1)\overline{w}_2 &= 2x_1(\overline{u}-u)(1+|u|^2). \end{aligned}$$

This implies

$$\frac{(\overline{w}_1^2-1)w_3 - (w_1^2-1)\overline{w}_3}{2i(|w_1|^4-1)} \;=\; x_1 \;=\; \frac{(\overline{w}_1^2-1)w_2 - (w_1^2-1)\overline{w}_2}{2(\overline{w}_1-w)(1+|w_1|^2)}.$$

As a result, $(w_1, w_2, w_3)$ satisfies the relation

$$(|w_1|^2-1)\Big((\overline{w}_1^2-1)w_2 - (w_1^2-1)\overline{w}_2\Big) \;=\; i(w_1-\overline{w}_1)\Big((\overline{w}_1^2-1)w_3 - (w_1^2-1)\overline{w}_3\Big).$$

We could then simplify the relation by setting

$$\begin{cases} y_1 &= w_1 &= u, \\ y_2 &= \dfrac{-w_2 - i\,w_1\,w_3}{w_1^2 - 1} &= u\,x_1 + x_2 + i\,u\,x_3, \\ y_3 &= \dfrac{w_3 - i\,w_1\,w_2}{w_1^2 - 1} &= -i\,x_1 + i\,u\,x_2 - x_3. \end{cases}$$

The relation between $y_1$, $y_2$ and $y_3$ defines a hyperquadric $Q$ in $\mathbb{C}^3$,

$$Q = \Big\{ \big(y_1, y_2, y_3\big) \mid y_2 - \overline{y}_2 \;=\; -i\big(y_1\,\overline{y}_3 + y_3\,\overline{y}_1\big) \Big\}.$$

Let $[\xi] = [\xi_0 : \xi_1 : \xi_2 : \xi_3]$ be the homogeneous coordinates on $\mathbb{C}P^3$. $\mathbb{C}^3$ is embedded to $\mathbb{C}P^3$ in the way that $(y_1, y_2, y_3)$ is mapped to $[1 : y_1 : y_2 : y_3]$. That means, $y_j = \xi_j/\xi_0$ for $\xi_0 \neq 0$. Then, $Q$ is embedded to a hyperquadric $Q'$ (in $\mathbb{C}P^3$),

$$Q' = \Big\{ \big[\xi_0 : \xi_1 : \xi_2 : \xi_3\big] \mid \xi_2\,\overline{\xi}_0 - \overline{\xi}_2\,\xi_0 \;=\; -i\,\big(\xi_1\,\overline{\xi}_3 + \overline{\xi}_1\,\xi_3\big). \Big\}$$

Let $N_0$ be the coordinate chart of $(x, u)$ on $N$. We may identify $N_0$ with $Q$, and map the CR manifold $N$ to an open subset of $Q'$.

**Proposition 2.9.** *Let $\phi(x, u) = (y_1, y_2, y_3)$ be defined as above.*

*(1) $\phi$ defines a CR isomorphism from $N_0$ to $Q$.*

*(2) $\phi$ could be extended to a CR isomorphism $\Phi$ from $N$ to an open subset of $Q'$,*

$$U' \;=\; Q' \cap \Big\{ [\xi] \in \mathbb{C}P^3 \mid \xi_0 \neq 0 \text{ or } \xi_1 \neq 0 \Big\}.$$

## 3.0 CR STRUCTURE BY AFFINE CONNECTIONS

Fix $N$ to be the twistor CR manifold of $(M, [\mathsf{g}])$. In Chapter 2, we mentioned that the CR structure $\mathfrak{D}$ depends on the conformal class of $\mathsf{g}$ only. It means that we may replace the Riemannian connection of $\mathsf{g}$ by that of $e^{2\lambda}\mathsf{g}$ to obtain the same CR structure. This idea would be generalized to any Weyl connection of $\mathsf{g}$.

Moreover, we may consider any metric connections with nonzero torsion on $M$. If $M$ is embedded to a 4-manifold, then we could define such a connection by the second fundamental form of $M$. In this case, we could get to different CR structures than $\mathfrak{D}$ on $N$.

### 3.1 WEYL CONNECTION ON $(M, \mathsf{g})$

**Definition.** [1] *Suppose* $[\mathsf{g}]$ *is a conformal structure on* $M$*. A Weyl structure on a manifold* $M$ *is a map* $F : [\mathsf{g}] \to \Omega^1(M)$*, satisfying the condition* $F(e^{\lambda}\mathsf{g}) = F(\mathsf{g}) - d\lambda$ *for all* $\lambda$ *in* $C^{\infty}(M)$*.*

Given a metric $\mathsf{g}$ and a 1-form $\alpha$ on $M$, a Weyl structure is determined by the equations $F(\mathsf{g}) = -\alpha$ and $F(e^{\lambda}\mathsf{g}) = -\alpha - d\lambda$. For this Weyl structure, there is a unique torsion-free affine connection $\nabla$ on $M$, characterized by (1) $\nabla\mathsf{g} = \alpha \otimes \mathsf{g}$ and (2) $\nabla$ is torsion free. It is called the Weyl connection of the Weyl structure determined by $\mathsf{g}$ and $\alpha$.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame on $(M, \mathsf{g})$. Write $\alpha = \alpha_k e^k$. The Christoffel symbols of the Weyl connection $\nabla$ on $M$ are given by $\mathcal{G}_{ij}^k = \mathsf{g}(\nabla_{e_i} e_j, e_k)$.

To distinguish $\nabla$ from the Riemannian connection of $(M, \mathsf{g})$, we denote the later by $\bar{\nabla}$ and write $G_{ij}^k = \mathsf{g}(\bar{\nabla}_{e_i} e_j, e_k)$. The relation between $\nabla$ and $\bar{\nabla}$ is given by the identity,

$$\nabla_{e_i} e_j = \bar{\nabla}_{e_i} e_j - \mathcal{S}_{ij}^k e_k.$$

Here $\mathcal{S}_{ij}^k = \frac{1}{2}\Big(\alpha_i \delta_{jk} + \alpha_j \delta_{ik} - \alpha_k \delta_{ij}\Big)$. In terms of the Christoffel symbols of $\nabla$ and $\bar{\nabla}$,

$$\mathcal{G}_{ij}^k = G_{ij}^k - \mathcal{S}_{ij}^k.$$

Suppose $X = \xi_i e_i$ is a tangent vector on $T_x M$. Let $(x, \lambda)$ be a point on $N$, regarded as the sphere bundle of $M$ here, with $\lambda = \lambda_i e_i$ being a unit vector on $T_x M$. Assume $\mathsf{g}(X, \lambda) = 0$. Similar to (2.8), the horizontal lift of $X$ by $\nabla$ at $(x, \lambda)$ is

$$\begin{aligned}
X^H &= \xi_j\, e_j - \xi_j\, \mathcal{G}_{jk}^l\, \lambda_k \frac{\partial}{\partial \lambda_l} \\
&= \xi_j\, e_j - \xi_j\, G_{jk}^l\, \lambda_k \frac{\partial}{\partial \lambda_l} + \xi_j\, \mathcal{S}_{jk}^l\, \lambda_k \frac{\partial}{\partial \lambda_l} \\
&= X^h + \xi_j \Big(\frac{1}{2}(\alpha_j\, \delta_{kl} + \alpha_k\, \delta_{jl} - \alpha_l\, \delta_{jk})\Big) \lambda_k \frac{\partial}{\partial \lambda_l} \\
&= X^h + \frac{1}{2}\alpha(X)\lambda^v + \frac{1}{2}\alpha(\lambda) X^v.
\end{aligned}$$

Here $X^h$ is the horizontal lift of $X$ by $\bar{\nabla}$ at $(x, \lambda)$. Since $\lambda^v$ is in the radial direction to the sphere bundle, we omit this term and define the horizontal lift of $X$ at $(x, \lambda)$ by

$$X^H = X^h + \frac{1}{2}\alpha(\lambda) X^v. \tag{3.1}$$

The horizontal bundle $\mathcal{H}_{(x,\lambda)}$ is then the space of all horizontal vectors $X^H$ at $(x, \lambda)$ with $\mathsf{g}(X, \lambda) = 0$. The vertical lift of $X$ at $(x, \lambda)$ is again defined by (2.9),

$$X^v = \xi_j \frac{\partial}{\partial \lambda_j},$$

and the vertical bundle $\mathcal{V}_{(x,\lambda)}$ consists of vertical vectors $X^v$ at $(x,\lambda)$ given $\mathsf{g}(X,\lambda)=0$. Note that the rank-4 bundle $\mathcal{V}\oplus\mathcal{H}$ is exactly the one we get from $\bar{\nabla}$. We define an almost complex structure $J$ on $\mathcal{V}\oplus\mathcal{H}$ by

$$JX^H=(\lambda\times X)^H \quad \text{and} \quad JX^v=(\lambda\times X)^v$$

at $(x,\lambda)$. The almost complex structure $J$ on $\mathcal{V}\oplus\mathcal{H}$ is the same almost complex structure we define in Section 2.4 on $\mathcal{V}\oplus\mathcal{H}$, for

$$\begin{aligned} JX^h &= J\Big(X^H-\frac{1}{2}\alpha(\lambda)X^v\Big) \\ &= (\lambda\times X)^h+\frac{1}{2}\alpha(\lambda)(\lambda\times X)^v-\frac{1}{2}\alpha(\lambda)(\lambda\times X)^v \\ &= (\lambda\times X)^h. \end{aligned}$$

Therefore, by any Weyl connection $\nabla$ on $M$ with respect to $\mathsf{g}$ and $\alpha$, we define the same CR structure $\mathfrak{D}$ on $N$.

**Proposition 3.1.** *Let $\mathsf{g}$ be a metric and $\alpha$ be a 1-form on the 3-manifold $M$. Let $\nabla$ be the Weyl connection on $M$ determined by $\mathsf{g}$ and $\alpha$. Then, the CR structure defined on the 5-manifold $N$ by $\nabla$ coincides with $\mathfrak{D}$.*

As a remark, $\alpha$ could be a complex 1-form on $M$, and we define the horizontal lift of vector $X$ at $(x,\lambda)$ by (3.1). The almost complex structure is also defined by $JX^H=(\lambda\times X)^H$ and $JX^v=(\lambda\times X)^v$. We may see that the linear span of $X^H$ and $X^v$ with $\mathsf{g}(X,\lambda)=0$ is a subspace of the complexified $\mathcal{V}\oplus\mathcal{H}$. The corresponding antiholomorphic bundle coincides with $\mathfrak{D}$.

Therefore, we get to the same CR structure $\mathfrak{D}$ on $N$ when $\alpha$ is complex-valued.

## 3.2 METRIC CONNECTION WITH TORSION

The description of the torsion tensor in this section is quoted from [17], where readers may find out more details and applications of the torsion tensor.

Let $\nabla$ be a metric connection with nonzero torsion tensor on the 3-manifold $(M, \mathsf{g})$. For any vector field $X$ and $Y$ on $M$, we have $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. Given an orthonormal frame $\{e_1, e_2, e_3\}$ on $M$, we let $T_{ij}^k$'s be the coefficients of $T$, i.e.

$$T(e_i, e_j) = T_{ij}^k e_k.$$

We denote the Riemannian connection of $\mathsf{g}$ by $\bar{\nabla}$ and its Christoffel symbols by $G_{ij}^k$ as in Section 3.1. The Christoffel symbols of $\nabla$ are defined by $\mathcal{G}_{ij}^k = \mathsf{g}\big(\nabla_{e_i} e_j, e_k\big)$ with

$$\mathcal{G}_{ij}^k = G_{ij}^k + \frac{1}{2}\big(T_{ij}^k - T_{ik}^j - T_{jk}^i\big). \tag{3.2}$$

The torsion tensor $T$ can be decomposed to three components in a sum,

$$T_{ij}^k = \frac{1}{2}\Big(T_i \delta_{kj} - T_j \delta_{ki}\Big) + \frac{\tau}{6}\,\epsilon_{ijk} + q_{ijk}. \tag{3.3}$$

The terms $T_i$, $\tau$ and $q_{ijk}$ are defined in the below context. The term $\epsilon_{ijk}$ is the same as in Proposition 2.5. We would introduce these three components of (3.3) separately.

**(I) The trace component of $T$:** $\frac{1}{2}\Big(T_i \delta_{kj} - T_j \delta_{ki}\Big)$

Let $T_i = \sum_{k=1}^{3} T_{ik}^k$ be the trace of $T$ acting on $e_i$. If we write $P_{ij}^k = \frac{1}{2}\Big(T_i \delta_{kj} - T_j \delta_{ki}\Big)$, then we have $P_{ijk} = -P_{jik}$ and $P_{ijk} + P_{jki} + P_{kij} = 0$ for every $i, j, k$.

**(II) The scalar function $\tau$ on $M$:** $\frac{\tau}{6}\,\epsilon_{ijk}$

We may say that $\epsilon_{ijk} = 6\, e^1 \wedge e^2 \wedge e^3$. So $\epsilon_{123} = 1$, $\epsilon_{213} = -1$ and so on. We define

$$\tau = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \cdot T_{ij}^k$$

$\tau$ is independent of the choice of the positively oriented orthonormal frame.

**(III) The trace-free cocyclic component of $T$:** $q_{ijk}$

The term $q_{ijk}$ is defined by the difference,

$$q_{ijk} = T^k_{ij} - \frac{1}{2}\Big(T_i\,\delta_{kj} - T_j\,\delta_{ki}\Big) - \frac{\tau}{6}\,\epsilon_{ijk}.$$

We have the following properties about $q_{ijk}$'s.

**Proposition 3.2.**

*(1)* $q_{ijk} = -q_{jik}$

*(2)* $\displaystyle\sum_{j=1}^{3} q_{ijj} = 0$

*(3)* $\displaystyle\sum_{i,j,k=1}^{3} \epsilon_{ijk}\,q_{ijk} = 0$

The item (2) comes from the contraction of the torsion tensor $T$.

$$\sum_{j=1}^{3} q_{ijj} \;=\; \sum_{j=1}^{3} T^j_{ij} - \frac{1}{2}\sum_{j=1}^{3}\big(T_i - T_j\,\delta_{ji}\big) \;=\; T_i - \frac{1}{2}\big(3T_i - T_i\big) \;=\; 0$$

For the item (3), we multiply $\epsilon_{ijk}$ by $q_{ijk}$ and obtain

$$\sum_{i,j,k} \epsilon_{ijk}\,q_{ijk} \;=\; \tau - \Big(\sum_{i,j,k} \epsilon_{ijk}\,P_{ijk}\Big) - \frac{\tau}{6}\sum_{i,j,k} \epsilon^2_{ijk} \;=\; \tau - 0 - \tau \;=\; 0.$$

Let $q$ be a $(1,1)$-tensor on $M$ using the coefficients $q_{ijk}$'s. Namely,

$$q(e_k) = q^l_k e_l \quad \text{and} \quad q_{ijk} = \sum_{l=1}^{3} \epsilon_{ijl}\,q^l_k.$$

The second identity means that $q^l_k = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{3} \epsilon_{ijl}\,q_{ijk}$. Note $tr(q) = q^k_k = 0$. When $k \neq l$,

$$q^l_k = \frac{1}{2}\sum_{i,j} \epsilon_{ijl}\,q_{ijk} = q_{mkk} = -q_{mll} = q^k_l$$

for $(k,l,m)$ being a positive permutation within $\{1,2,3\}$. So $q^l_k = q^k_l$ for every $k$, $l$.

We then turn to the horizontal lift of vector fields from $M$ to $N$ by $\nabla$. Let $(x, \lambda)$ be a point on $N$ which represents the unit vector $\lambda_i e_i$ at $x$. Let $X = \xi_i e_i$ be a vector on $T_x M$ orthogonal to $\lambda$. By (3.2), the horizontal lift of $X$ at $(x, \lambda)$ by $\nabla$ is given as

$$X^H = X^h - \frac{1}{2}\,\xi_j\,\left(T^l_{jk} - T^k_{jl} - T^j_{kl}\right)\,\lambda_k\,\frac{\partial}{\partial \lambda_l}. \tag{3.4}$$

We would then examine the effect of each linear component in (3.4) to $X^H$. To begin with, we set $T^k_{ij} = \frac{1}{2}\big(T_i\,\delta_{kj} - T_j\,\delta_{ki}\big)$, $\tau = 0$ and $q_{ijk} = 0$. That is,

$$T^l_{jk} - T^k_{jl} - T^j_{kl} = -T_k\,\delta_{jl} + T_l\,\delta_{jk}.$$

It leads to

$$\begin{aligned} X^H &= X^h - \frac{1}{2}\xi_j\Big(-T_k\,\delta_{jl} + T_l\,\delta_{jk}\Big)\lambda_k\,\frac{\partial}{\partial \lambda_l} \\ &= X^h + \frac{1}{2}(T_k\lambda_k)\,\xi_l\,\frac{\partial}{\partial \lambda_l} - \frac{1}{2}(\xi_j\lambda_j)\,T_l\,\frac{\partial}{\partial \lambda_l} \\ &= X^h + \frac{1}{2}\,(\mathrm{trT})(\lambda)\,X^v. \end{aligned}$$

Next, we set $T^k_{ij} = \frac{\tau}{6}\,\epsilon_{ijk}$ with $\tau \neq 0$.

$$T^l_{jk} - T^k_{jl} - T^j_{kl} \;=\; \frac{\tau}{6}\Big(\epsilon_{jkl} - \epsilon_{jlk} - \epsilon_{klj}\Big) \;=\; \frac{\tau}{6}\,\epsilon_{jkl}$$

It implies that

$$\begin{aligned} X^H &= X^h - \frac{1}{2}\,\xi_j\,\left(\frac{\tau}{6}\,\epsilon_{jkl}\right)\lambda_k\,\frac{\partial}{\partial \lambda_l} \\ &= X^h - \frac{\tau}{12}\Big(\xi_j\,\epsilon_{jkl}\,\lambda_k\Big)\,\frac{\partial}{\partial \lambda_l} \\ &= X^h + \frac{\tau}{12}\,\big(\lambda \times X\big)^v. \end{aligned}$$

Both the first and second components of $T$ would result in the same CR structure $\mathfrak{D}$ on $N$, under the construction from (3.4). We skip the details here.

For the third component, we set $T^k_{ij} = q_{ijk} = \epsilon_{ijl}\, q^l_k$. From Proposition 3.2, we have

$$T^l_{jk} - T^k_{jl} - T^j_{kl} \;=\; q_{jkl} - q_{jlk} - q_{klj} \;=\; -\,2\, q_{klj}.$$

By (3.4), we have

$$X^H \;=\; X^h - \big(\lambda \times q(X)\big)^v \;=\; X^h - \lambda_k\, \epsilon_{kml}\, \big(q^m_j \xi_j\big) \frac{\partial}{\partial \lambda_l}. \tag{3.5}$$

The almost complex structure $J$ on the horizontal and vertical bundles is given by

$$JX^H \;=\; (\lambda \times X)^H \;=\; (\lambda \times X)^h - \big(\lambda \times q(\lambda \times X)\big)^v \quad \text{and} \quad JX^v \;=\; (\lambda \times X)^v. \tag{3.6}$$

at $(x, \lambda)$ with $X \perp \lambda$. Let $\mathfrak{D}(q)$ be the antiholomorphic bundle regarding (3.5) and (3.6). In general, $\mathfrak{D}(q)$ is different from $\mathfrak{D}$.

**Proposition 3.3.** *The complex distribution $\mathfrak{D}(q)$ is a CR structure on $N$.*

The vector field $X^H$ (3.5) is pulled back to the local chart of $(x, u)$ on $N$. Let $\Phi$ be the identification map in (2.10) from $N$ to the sphere bundle $S$. When $\lambda = \Phi(u)$, we get

$$\lambda_1 = \frac{u + \overline{u}}{1 + |u|^2}, \quad \lambda_2 = \frac{1 - |u|^2}{1 + |u|^2} \quad \text{and} \quad \lambda_3 = \frac{i(u - \overline{u})}{1 + |u|^2}.$$

Recall that $[\mu] = f([u : 1])$ in (2.3), and we have $\mu = \big(u^2 - 1, 2u, i(u^2 + 1)\big)$. We know that

$$d\Phi\big(\frac{\partial}{\partial \overline{u}}\big) = \frac{-1}{(1 + |u|^2)^2}\big(\mu_l \frac{\partial}{\partial \lambda_l}\big) \quad \text{and} \quad d\Phi\big(\overline{X}_1\big) = \mu_j e_j - \mu_m G^k_{ml} \lambda_l \frac{\partial}{\partial \lambda_k},$$

where $\overline{X}_1$ is the vector in $\mathfrak{D}$ from (2.4). From (3.5) and (3.6), the complex distribution $\mathfrak{D}(q)$ at $(x, \lambda)$ is spanned by the vectors $\overline{Y}_1$ and $\overline{Y}_2$.

$$\begin{aligned} \overline{Y}_1 \;&=\; \mu_j e_j - \mu_m G^l_{mk} \lambda_k \frac{\partial}{\partial \lambda_l} - \lambda_k\, \epsilon_{kml}\, \big(q^m_j \mu_j\big) \frac{\partial}{\partial \lambda_l} \\ \overline{Y}_2 \;&=\; \mu_l \frac{\partial}{\partial \lambda_l} \end{aligned}$$

Note that $d\Phi\Big( -(1 + |u|^2)^2 \dfrac{\partial}{\partial \overline{u}}\Big) = \overline{Y}_2$.

Let $\overline{Y}_0$ be the third component of $Y_2$. That is,

$$\begin{aligned}
\overline{Y}_0 &= -\lambda_k\,\epsilon_{kml}\left(q_j^m\mu_j\right)\frac{\partial}{\partial\lambda_l} \\
&= -\mu_j\Big[\left(\lambda_2\,q_3^j-\lambda_3\,q_2^j\right)\frac{\partial}{\partial\lambda_1}+\left(\lambda_3\,q_1^j-\lambda_1\,q_3^j\right)\frac{\partial}{\partial\lambda_2}+\left(\lambda_1\,q_2^j-\lambda_2\,q_1^j\right)\frac{\partial}{\partial\lambda_3}\Big].
\end{aligned}$$

Let $\overline{Y}_0 = c_1\,\mu_l\dfrac{\partial}{\partial\lambda_l}+c_2\ ,\overline{\mu}_l\dfrac{\partial}{\partial\lambda_l}$. So we have,

$$\overline{Y}_0\ \cdot\ \Big(\mu_l\,\frac{\partial}{\partial\lambda_l}\Big)=c_2|\mu|^2=2(1+|u|^2)^2c_2$$

under the Sasaki metric on $TM$.

$$\begin{aligned}
c_2 &= \frac{-\mu_j}{2(1+|u|^2)^2}\Big[\Big(\lambda_2\,q_3^j-\lambda_3\,q_2^j\Big)\mu_1+\Big(\lambda_3\,q_1^j-\lambda_1\,q_3^j\Big)\mu_2+\Big(\lambda_1\,q_2^j-\lambda_2\,q_1^j\Big)\mu_3\Big] \\
&= \frac{1}{2(1+|u|^2)^2}\Big[q_1^1\Big(-\mu_1\,\mu_2\,\lambda_3+\mu_1\,\mu_3\,\lambda_2\Big)+q_2^2\Big(\mu_1\,\mu_2\,\lambda_3-\mu_2\,\mu_3\,\lambda_1\Big) \\
&\qquad +q_3^3\Big(-\mu_1\,\mu_3\,\lambda_2+\mu_2\,\mu_3\,\lambda_1\Big)+q_1^2\Big(\mu_1^2\,\lambda_3-\mu_2^2\,\lambda_3-\mu_1\,\mu_3\,\lambda_1+\mu_2\,\mu_3\,\lambda_2\Big) \\
&\qquad +q_1^3\Big(-\mu_1^2\,\lambda_2+\mu_3^2\,\lambda_2-\mu_2\,\mu_3\,\lambda_3+\mu_1\,\mu_2\,\lambda_1\Big) \\
&\qquad +q_2^3\Big(\mu_2^2\,\lambda_1-\mu_3^2\,\lambda_1+\mu_1\,\mu_3\,\lambda_3-\mu_1\,\mu_2\,\lambda_2\Big)\Big].
\end{aligned}$$

By the fact that $\lambda\times\mu=-i\,\mu$, $\lambda\times\overline{\mu}=i\,\overline{\mu}$, we may simplify $c_2$ to get

$$c_2 = \frac{1}{2(1+|u|^2)^2}\Big[i\,(\mu_3^2-\mu_1^2)\,q_1^1+i\,(\mu_3^2-\mu_2^2)\,q_2^2-2\,i\,\mu_1\,\mu_2\,q_1^2-2\,i\,\mu_1\,\mu_3\,q_1^3-2\,i\,\mu_2\,\mu_3\,q_2^3\Big].$$

Therefore,

$$\begin{aligned}
d\Phi^{-1}(\overline{Y}_1) &= d\Phi^{-1}\Big(\mu_je_j-\mu_mG_{mk}^l\lambda_k\frac{\partial}{\partial\lambda_l}\Big)+d\Phi^{-1}(\overline{Y}_0) \\
&= \overline{X}_1+d\Phi^{-1}\Big(c_1\mu_l\frac{\partial}{\partial\lambda_l}+c_2\overline{\mu}_l\frac{\partial}{\partial\lambda_l}\Big) \\
&= \overline{X}_1-c_1(1+|u|^2)^2\frac{\partial}{\partial\overline{u}}-c_2(1+|u|^2)^2\frac{\partial}{\partial u}
\end{aligned}$$

$$
\begin{aligned}
d\Phi^{-1}(\overline{Y}_1) \;=\;& \mu_j\, e_j \;-\; \frac{i}{2}\mu_m\Big(G^2_{m1}\,\mu_3 + G^1_{m3}\,\mu_2 + G^3_{m2}\,\mu_1\Big)\,\frac{\partial}{\partial u} \\
&+ \Big[\frac{i}{2}\,(\mu_1^2-\mu_3^2)\,q_1^1 + \frac{i}{2}\,(\mu_2^2-\mu_3^2)\,q_2^2 + i\,\mu_1\,\mu_2\,q_1^2 + i\,\mu_1\,\mu_3\,q_1^3 + \mu_2\,\mu_3\,q_2^3\Big]\,\frac{\partial}{\partial u} \\
&- c_1\,(1+|u|^2)^2\,\frac{\partial}{\partial \overline{u}}.
\end{aligned}
$$

As a result, the CR structure $\mathfrak{D}(q)$ at the point $(x,u)$, is spanned by $\overline{X}_2 = \dfrac{\partial}{\partial \overline{u}}$ and

$$
\overline{X}_1 \;=\; \mu_j\, e_j - \frac{i}{2}\,\mu_m\,\Big(G^2_{m1}\,\mu_3 + G^1_{m3}\,\mu_2 + G^3_{m2}\,\mu_1\Big)\,\frac{\partial}{\partial u} + \mathbf{u}^T\cdot\mathbf{C}\cdot\mathbf{q}\,\frac{\partial}{\partial u}. \tag{3.7}
$$

Here we define

$$
\mathbf{u}^T\cdot\mathbf{C}\cdot\mathbf{q} \;=\; \begin{pmatrix} 1 & u & u^2 & u^3 & u^4 \end{pmatrix}
\begin{pmatrix}
i & 0 & 1 & \frac{i}{2} & 0 \\
0 & -2i & 0 & 0 & -2 \\
0 & 0 & 0 & 3i & 0 \\
0 & 2i & 0 & 0 & -2 \\
i & 0 & -1 & \frac{i}{2} & 0
\end{pmatrix}
\begin{pmatrix} q_1^1 \\ q_1^2 \\ q_1^3 \\ q_2^2 \\ q_2^3 \end{pmatrix}
$$

correspondingly.

In general, for every trace-free and symmetric (1,1)-tensor $q$ on $M$, we may define a corresponding CR structure $\mathfrak{D}(q)$ on $N$ by (3.7). We would say that $q$ is the *trace-free torsion tensor* on $M$. The complex function $w = \mathbf{u}^T\cdot\mathbf{C}\cdot\mathbf{q}$ is holomorphic in $u$, and we would call it by the *torsion function* of $\mathfrak{D}(q)$ in the following context.

## 3.3 THE TRACE-FREE SECOND FUNDAMENTAL FORM

When $M$ is embedded to a 4-manifold $\tilde{M}$, the trace-free second fundamental form of $M$ would become the trace-free torsion tensor $q$. Suppose $\tilde{g}$ is the metric on $\tilde{M}$ and $\tilde{\nabla}$ is its Levi-Civita connection. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame on $M$ and let $\mathbf{n}$ be the unit normal vector to $M$. The second fundamental form on $M$ is then defined by

$$\mathrm{II}(u,v) = -\tilde{g}\big(\tilde{\nabla}_u \mathbf{n}, v\big) \quad \text{for } u, v \text{ on } TM.$$

Following from LeBrun's paper [10], we would define a metric connection with torsion ($\nabla'$ ) on $M$ by letting $\nabla'_u v = \tilde{\nabla}_u v - \big(v \times \tilde{\nabla}_u n\big)$ for any $u, v$ on $TM$. Through our discussion in Section 3.2, the trace-free torsion tensor $q$ is given by

$$q(u) \;=\; -\tilde{\nabla}_u \mathbf{n} - \frac{1}{3}\mathrm{tr}(\mathrm{II}) \cdot \mathrm{Id} \quad \text{for } u \in TM. \tag{3.8}$$

When $\tilde{M}$ is a Riemannian manifold, we let

$$\tilde{\nabla}_{e_i} e_j = \mathcal{G}^k_{ij}\, e_k + \mathcal{G}^0_{ij}\, \mathbf{n}, \quad \tilde{\nabla}_{e_i}\mathbf{n} = \mathcal{G}^k_{i0}\, e_k \quad \text{and} \quad \tilde{\nabla}_{\mathbf{n}} e_j = \mathcal{G}^k_{0j}\, e_k + \mathcal{G}^0_{0j}\, \mathbf{n}. \tag{3.9}$$

We have the symmetries $\mathcal{G}^k_{ij} = -\mathcal{G}^j_{ik}$ and $\mathcal{G}^0_{ij} = \mathcal{G}^0_{ji} = -\mathcal{G}^j_{i0}$. Note that $\mathcal{G}^k_{ij}$'s are also the Christoffel symbols of the Riemannian connection on $M$. We then get to $\mathrm{II}(e_m, e_n) = \mathcal{G}^0_{mn}$ and $\mathrm{tr}(\mathrm{II}) = \sum_{k=1}^3 \mathcal{G}^0_{kk}$. The components of $q$ could be found by

$$q^l_k = \mathcal{G}^0_{kl} - \frac{1}{3}\Big(\sum_{m=1}^{3} \mathcal{G}^0_{mm}\Big)\, \delta_{kl}.$$

Putting $q$ to (3.7), the torsion function $w = \mathbf{u}^T \cdot \mathbf{C} \cdot \mathbf{q}$ becomes

$$\begin{aligned} w \;=\; & \Big(\frac{i}{2}\,\mathcal{G}^0_{11} - \frac{i}{2}\,\mathcal{G}^0_{33} + \mathcal{G}^0_{13}\Big) - \big(2i\,\mathcal{G}^0_{12} + 2\,\mathcal{G}^0_{23}\big)\, u + \big(2i\,\mathcal{G}^0_{22} - i\,\mathcal{G}^0_{11} - i\,\mathcal{G}^0_{33}\big)\, u^2 \\ & + \big(2i\,\mathcal{G}^0_{12} - 2\,\mathcal{G}^0_{23}\big)\, u^3 + \Big(\frac{i}{2}\,\mathcal{G}^0_{11} - \frac{i}{2}\,\mathcal{G}^0_{33} - \mathcal{G}^0_{13}\Big)\, u^4. \end{aligned} \tag{3.10}$$

The same definition of $q$ in (3.8) by the trace-free second fundamental form could also be carried out naturally when $\tilde{M}$ is Lorentzian. The torsion function $w$ obtained would then be identical to (3.10) but with a minus sign to every $\mathcal{G}^0_{ij}$ on the right.

When $\tilde{M}$ is a Lorentzian 4-manifold and $M$ is a space-like submanifold of $\tilde{M}$, we could also define a corresponding CR structure $\mathfrak{D}(q)$ on $M$ by the trace-free second fundamental form but in an alternative way. It results in using $i$ times $q$ (3.8) to construct the CR structure by (3.7) on the twistor CR manifold $N$ of $M$.

We keep the orthonormal frame of $\mathbf{n}$, $e_1$, $e_2$ and $e_3$ on $\tilde{M}$ but with $\tilde{g}(\mathbf{n},\mathbf{n}) = -1$. Following the definition in (3.9), we have $\mathrm{II}(e_m, e_n) = -\mathcal{G}^n_{m0} = -\mathcal{G}^0_{mn}$ and $\mathrm{tr}(\mathrm{II}) = -\sum_{k=1}^3 \mathcal{G}^0_{kk}$. Since $\tilde{g}$ is a Lorentzian metric, the null cones on $M$ form a 6-dimensional manifold

$$\tilde{N} = \Big\{(x,v) \in T\tilde{M} \mid x \in M,\ v \in T_x\tilde{M},\ \tilde{g}(v,v) = 0 \ \text{ with } \ v \neq 0\Big\}.$$

The parallel transport of null vectors along tangential vectors on $M$, would define a CR structure on the sphere bundle $S$ of $M$.

For a point $(x,v)$ on $\tilde{N}$, we let $v = v_0\,\mathbf{n} + \sum_{j=1}^3 v_j e_j$. Let $u = u_j e_j$ be a tangent vector on $T_xM$. The horizontal lift of $u$ by $\tilde{\nabla}$ at $(x,v)$ is

$$u^H \ = \ u_i\, e_i - u_j\, \mathcal{G}^l_{jk}\, v_k\, \frac{\partial}{\partial v_l} - u_j\, \mathcal{G}^l_{j0}\, v_0\, \frac{\partial}{\partial v_l} - u_j\, \mathcal{G}^0_{jk}\, v_k\, \frac{\partial}{\partial v_0}.$$

$u^H$ is a vector on $T\tilde{N}$ since

$$\big(-v_0\, dv_0 + v_i\, dv_i\big)(u^H) \ = \ u_j\, \mathcal{G}^0_{jk}\, v_k\, v_0 - u_j\, \mathcal{G}^i_{jk}\, v_k\, v_i - u_j\, \mathcal{G}^i_{j0}\, v_0\, v_i \ = \ 0.$$

Suppose $(x,\lambda)$ are the coordinates on $S$ corresponding to $\big\{e_1, e_2, e_3\big\}$. We would define a projection map $\tilde{\Phi}$ from $\tilde{N}$ to $S$, $\tilde{\Phi}(v) = \big(\frac{v_j}{v_0}\big)e_j$. Through the differential map $d\tilde{\Phi}$ at $(x,v)$,

$$d\tilde{\Phi}(u^H) \ = \ u_j\, e_j - u_j\, \mathcal{G}^l_{jk}\, \lambda_k\, \frac{\partial}{\partial \lambda_l} - u_j\, \mathcal{G}^l_{j0}\, \frac{\partial}{\partial \lambda_l} + u_j\, \mathcal{G}^k_{j0}\, \lambda_k\, \lambda_l\, \frac{\partial}{\partial \lambda_l} \tag{3.11}$$

at $\lambda = \tilde{\Phi}(v)$. From here we denote $d\tilde{\Phi}(u^H)$ by $u^H$ directly. (3.11) is equivalent to

$$u^H = u^h - \Big(\tilde{\nabla}_u \mathbf{n} - \tilde{g}(\tilde{\nabla}_u \mathbf{n}, \lambda)\, \lambda\Big)^v. \tag{3.12}$$

Here $u^h$ is the horizontal lift of $u$ at $(x,\lambda)$ by the Riemannian connection on $M$, defined by the first two terms of (3.11).

The vertical lift of $u$ at $(x,\lambda)$ remains as $u^v = u_j \frac{\partial}{\partial \lambda_j}$. The horizontal and vertical bundles of $S$ at $(x,\lambda)$ consist of all horizontal and vertical lifts of tangent vectors $u$ at $x$ respectively, given that $\tilde{g}(u,\lambda)=0$. The almost complex structure $J$ on $TS$ is defined by $Ju^H = (\lambda\times u)^H$ and $Ju^v = (\lambda\times u)^v$.

Therefore, the corresponding antiholomorphic bundle $\tilde{\mathfrak{D}}$ defines a CR structure on $S$. By Proposition 2.7, it is equivalent to say that $\tilde{\mathfrak{D}}$ is on the twistor CR manifold $N$ of $(M,\tilde{g})$.

**Proposition 3.4.** *$\tilde{\mathfrak{D}}$ coincides with the CR structure $\mathfrak{D}(q)$ on $N$, where the trace-free torsion tensor $q$ is defined by the trace-free second fundamental form of $M$ multiplied by $i$, i.e.,*

$$q \;=\; -\,i\,\tilde{\nabla}\mathbf{n} - \frac{i}{3}\,tr(\mathrm{II})\cdot \mathrm{Id}.$$

*Proof.* By (3.6), the CR distribution $\mathfrak{D}(q)$ at $(x,\lambda)$ is spanned by

$$u^H + i(\lambda\times u)^H \;=\; u^h + i(\lambda\times u)^h - \big(\lambda\times q(u)\big)^v - i\big(\lambda\times q(\lambda\times u)\big)^v$$

and $u^v + i(\lambda\times u)^v$ given that $\tilde{g}(u,\lambda)=0$. Put $q(u) = -i\,\tilde{\nabla}_u\mathbf{n} - \frac{i}{3}\,\mathrm{tr}(\mathrm{II})\,u$. First of all we assume that $u$ is a unit vector orthogonal to $\lambda$. So we get

$$\mathrm{tr}(\mathrm{II}) \;=\; -\,\tilde{g}(\tilde{\nabla}_u\mathbf{n},u) - \tilde{g}(\tilde{\nabla}_{\lambda\times u}\mathbf{n},\lambda\times u) - \tilde{g}(\tilde{\nabla}_\lambda\mathbf{n},\lambda).$$

We compute for

$$\begin{aligned}
\lambda\times q(u) \;&=\; \lambda\times\big(-\,i\,\tilde{\nabla}_u\mathbf{n} - \frac{i}{3}\,\mathrm{tr}(\mathrm{II})\,u\big)\\
&=\; -i\,\Big(\tilde{g}\big(\tilde{\nabla}_u\mathbf{n},u\big)\,\lambda\times u - \tilde{g}\big(\tilde{\nabla}_u\mathbf{n},\lambda\times u\big)\,u\Big) - \frac{i}{3}\,\mathrm{tr}(\mathrm{II})\,\lambda\times u\\
&=\; i\,\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},u\big)\,u - i\,\tilde{g}\big(\tilde{\nabla}_u\mathbf{n},u\big)\,\lambda\times u - \frac{i}{3}\,\mathrm{tr}(\mathrm{II})\,\lambda\times u\\
&=\; i\,\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},u\big)\,u + i\,\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},\lambda\times u\big)\,\lambda\times u\\
&\quad +i\,\tilde{g}\big(\tilde{\nabla}_\lambda\mathbf{n},\lambda\big)\,\lambda\times u + i\,\mathrm{tr}(\mathrm{II})\,\lambda\times u - \frac{i}{3}\,\mathrm{tr}(\mathrm{II})\,\lambda\times u.\\
&=\; i\,\Big(\tilde{\nabla}_{\lambda\times u}\mathbf{n} - \tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},\lambda\big)\,\lambda\Big) + i\,\tilde{g}\big(\tilde{\nabla}_\lambda\mathbf{n},\lambda\big)\,\lambda\times u + \frac{2i}{3}\,\mathrm{tr}(\mathrm{II})\,\lambda\times u.
\end{aligned}$$

Then we compute for

$$
\begin{aligned}
i\,\lambda\times q(\lambda\times u) &= i\,\lambda\times\big(-i\,\tilde{\nabla}_{\lambda\times u}\mathbf{n}-\frac{i}{3}\,\mathrm{tr}(\mathrm{II})\,\lambda\times u\big)\\
&= \lambda\times\tilde{\nabla}_{\lambda\times u}\mathbf{n}-\frac{1}{3}\,\mathrm{tr}(\mathrm{II})\,u\\
&= \lambda\times\Big(\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},u\big)\,u+\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},\lambda\times u\big)\,\lambda\times u\Big)-\frac{1}{3}\,\mathrm{tr}(\mathrm{II})\,u\\
&= \tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},u\big)\,\lambda\times u-\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},\lambda\times u\big)\,u-\frac{1}{3}\mathrm{tr}(\mathrm{II})\,u\\
&= \tilde{g}\big(\tilde{\nabla}_{u}\mathbf{n},\lambda\times u\big)\,\lambda\times u+\tilde{g}\big(\tilde{\nabla}_{u}\mathbf{n},u\big)\,u+\tilde{g}\big(\tilde{\nabla}_{\lambda}\mathbf{n},\lambda\big)\,u+\frac{2}{3}\,\mathrm{tr}(\mathrm{II})\,u\\
&= \Big(\tilde{\nabla}_{u}\mathbf{n}-\tilde{g}\big(\tilde{\nabla}_{u}\mathbf{n},\lambda\big)\,\lambda\Big)+\tilde{g}\big(\tilde{\nabla}_{\lambda}\mathbf{n},\lambda\big)\,u+\frac{2}{3}\,\mathrm{tr}(\mathrm{II})\,u.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&u^H+i\,(\lambda\times u)^H\\
=\;&\big(u^h+i\,(\lambda\times u)^h\big)-\Big[\Big(\tilde{\nabla}_{u}\mathbf{n}-\tilde{g}\big(\tilde{\nabla}_{u}\mathbf{n},\lambda\big)\,\lambda\Big)^v+i\,\Big(\tilde{\nabla}_{\lambda\times u}\mathbf{n}-\tilde{g}\big(\tilde{\nabla}_{\lambda\times u}\mathbf{n},\lambda\big)\,\lambda\Big)^v\Big]\\
&-\tilde{g}\big(\tilde{\nabla}_{\lambda}\mathbf{n},\lambda\big)\,\big(u^v+i\,(\lambda\times u)^v\big)-\frac{2}{3}\,\mathrm{tr}(\mathrm{II})\,\big(u^v+i(\lambda\times u)^v\big).
\end{aligned}
$$

The assumption $\tilde{g}(u,u)=1$ is redundant since $u$ is linear on every component of the sum in $\lambda\times q(u)$ and $i\,\lambda\times q(\lambda\times u)$. Compare the last equation with (3.12), it is clear that $\mathfrak{D}(q)$ coincides with $\tilde{\mathfrak{D}}$ on $N$. □

Proposition 3.4 suggests that the trace-free torsion tensor $q$ could be complex-valued in general. When we go back to (3.7), we may replace every $q^l_k$'s by

$$
q^l_k \;=\; -\,i\,\mathcal{G}^0_{kl}+\frac{i}{3}\big(\sum_{m=1}^{3}\mathcal{G}^0_{mm}\big)\,\delta_{kl}.
$$

Therefore the torsion function $w=\mathbf{u}^T\cdot\mathbf{C}\cdot\mathbf{q}$ is

$$
\begin{aligned}
w \;=\;&\Big(\frac{1}{2}\,\mathcal{G}^0_{11}-\frac{1}{2}\,\mathcal{G}^0_{33}-i\,\mathcal{G}^0_{13}\Big)+\big(-2\mathcal{G}^0_{12}+2\,i\,\mathcal{G}^0_{23}\big)\,u+\big(2\,\mathcal{G}^0_{22}-\mathcal{G}^0_{11}-\mathcal{G}^0_{33}\big)\,u^2\\
&+\big(2\,\mathcal{G}^0_{12}+2i\,\mathcal{G}^0_{23}\big)\,u^3+\Big(\frac{1}{2}\,\mathcal{G}^0_{11}-\frac{1}{2}\,\mathcal{G}^0_{33}+i\,\mathcal{G}^0_{13}\Big)\,u^4.
\end{aligned}
\tag{3.13}
$$

## 4.0 FEFFERMAN METRIC (I)

Given a 3-manifold $M$, we could define CR structures on its twistor CR manifold $N$ by a trace-free torsion tensor $q$. In general we may choose a complex-valued function $w$ on $N$ holomorphic in the vertical parameter to replace $\mathbf{u}^T \cdot \mathbf{C} \cdot \mathbf{q}$ in (3.7). The corresponding CR structure would be named $\mathfrak{D}(w)$. In this chapter, we assume that $M$ is equipped with a flat metric.

### 4.1 THE LOCAL MODEL OF $\mathfrak{D}(w)$

Suppose $M$ is a flat space of coordinates $x = (x_1, x_2, x_3)$ and $N$ is the twistor CR manifold (sphere bundle) of $M$. The local model of the CR structure $\mathfrak{D}(w)$ on $N$ is described by

$$\begin{cases} \overline{X}_1 &= (u^2-1)\dfrac{\partial}{\partial x_1} + 2u\dfrac{\partial}{\partial x_2} + i(u^2+1)\dfrac{\partial}{\partial x_3} + w(x,u)\dfrac{\partial}{\partial u} \\ X_1 &= (\overline{u}^2-1)\dfrac{\partial}{\partial x_1} + 2\overline{u}\dfrac{\partial}{\partial x_2} - i(\overline{u}^2+1)\dfrac{\partial}{\partial x_3} + \overline{w(x,u)}\dfrac{\partial}{\partial \overline{u}} \\ \overline{X}_2 &= \dfrac{\partial}{\partial \overline{u}} \\ X_2 &= \dfrac{\partial}{\partial u} \\ T &= \dfrac{u+\overline{u}}{1+|u|^2}\dfrac{\partial}{\partial x_1} + \dfrac{1-|u|^2}{1+|u|^2}\dfrac{\partial}{\partial x_2} + \dfrac{i(u-\overline{u})}{1+|u|^2}\dfrac{\partial}{\partial x_3} \end{cases} \tag{4.1}$$

at $(x, u)$, where $w$ is an arbitrary complex function holomorphic in $u$. The contact form of $\mathfrak{D}(w)$ is always chosen by (2.6), i.e.

$$\alpha = \frac{u+\overline{u}}{1+|u|^2}dx_1 + \frac{1-|u|^2}{1+|u|^2}dx_2 + \frac{i(u-\overline{u})}{1+|u|^2}dx_3. \tag{4.2}$$

The vector field $T$ in (4.1) is the Reeb vector field of $\alpha$. Also, we let $\{\theta^1, \theta^2, \theta^{\bar{1}}, \theta^{\bar{2}}, \alpha\}$ be the dual coframe of $\{X_1, X_2, \overline{X}_1, \overline{X}_2, T\}$.

Let $\mathcal{L}$ be the Levi form of $\alpha$, then it components are given by $h_{\alpha\bar{\beta}} = \mathcal{L}(X_\alpha, \overline{X}_\beta)$ with

$$h = \begin{bmatrix} h_{1\bar{1}} & h_{1\bar{2}} \\ h_{2\bar{1}} & h_{2\bar{2}} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{4.3}$$

In particular, we have $h^{-1} = h$ and so $h^{\bar{\alpha}\beta} = h_{\alpha\beta}$ for $\alpha, \beta = 1, 2$. Moreover, the eigenvalues of $h$ is $\pm 1$, so $\mathfrak{D}(w)$ is anticlastic.

Let $g$ be the Webster metric associated with $\alpha$, and let $\nabla$ be the Tanaka Webster-connection of $\alpha$. Following our notation in Section 1.4, we let

$$\nabla_{X_m} X_n = \Gamma^k_{mn} X_k, \quad \nabla_{\overline{X}_m} X_n = \Gamma^k_{\bar{m}n} X_k \quad \text{and} \quad \nabla_T X_n = \Gamma^k_{0n} X_k$$

accordingly. The Christoffel symbols $\Gamma^k_{mn}$, $\Gamma^k_{\bar{m}n}$ and $\Gamma^k_{0n}$ could be found by (1.1), replacing $T_n$ by $X_n$ and $T_{\bar{n}}$ by $\overline{X}_n$. In this process, we will have to compute for the Lie brackets between $X_1$, $\overline{X}_1$, $X_2$, $\overline{X}_2$ and $T$.

Let $\mu = \mu_j \frac{\partial}{\partial x_j}$ and $\overline{\mu} = \overline{\mu}_j \frac{\partial}{\partial x_j}$, in which $\mu_1 = u^2 - 1$, $\mu_2 = 2u$ and $\mu_3 = i(u^2 + 1)$. We denote the directional derivative of $w$ along $\mu$ (or $\overline{\mu}$) by $D_\mu w$ (or $D_{\overline{\mu}} w$). So,

$$D_\mu w = (u^2 - 1) w_x + 2u\, w_y + i(u^2 + 1) w_z \quad \text{and} \quad D_{\overline{\mu}} w = (\overline{u}^2 - 1)\overline{w}_x + 2\overline{u}\,\overline{w}_y - i(\overline{u}^2 + 1)\overline{w}_z.$$

The vector $\frac{v}{|v|}$ in (2.10) coincides with $T$ when $M$ is flat but not in general. We also denote the directional derivative of $w$ along $\frac{v}{|v|}$ by $D_{\frac{v}{|v|}} w$. Moreover, the symbol $D_u D_\mu w$ would mean the second derivative of $w$ first by $\mu$ and then by $u$. Other symbols of second derivatives are similarly defined.

**Proposition 4.1.** *The Lie brackets are given as follows.*

$$
\begin{aligned}
[X_1, \overline{X}_2] &= \frac{-2u}{1+|u|^2}\, X_1 + \Big( -\overline{w}_{\overline{u}} + \frac{2u}{1+|u|^2}\overline{w}\Big)\, \overline{X}_2 - 2\,T \\
[X_1, \overline{X}_1] &= D_{\overline{\mu}} w \cdot X_2 - D_{\mu}\overline{w} \cdot \overline{X}_2 \\
[X_1, X_2] &= 0 \\
[\overline{X}_1, T] &= -\frac{w}{(1+|u|^2)^2}\, X_1 - D_{\frac{v}{|v|}} w \cdot X_2 + \frac{|w|^2}{(1+|u|^2)^2}\, \overline{X}_2 \\
[X_2, T] &= -\frac{1}{(1+|u|^2)^2}\, X_1 + \frac{\overline{w}}{(1+|u|^2)^2}\, \overline{X}_2
\end{aligned}
$$

**Proposition 4.2.** *The coefficients of the Tanaka-Webster connection are given by*

$$
\begin{aligned}
\Gamma^1_{11} &= \overline{w}_{\overline{u}} - \frac{2u}{1+|u|^2}\overline{w} \\
\Gamma^2_{11} &= -D_{\mu}\overline{w} \\
\Gamma^2_{22} &= -\frac{2\overline{u}}{1+|u|^2} \\
\Gamma^1_{12} &= \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^1_{22} = 0 \\
\Gamma^2_{\overline{1}1} &= -D_{\overline{\mu}} w \\
\Gamma^2_{\overline{1}2} &= -w_u + \frac{2\overline{u}}{1+|u|^2} w \\
\Gamma^1_{\overline{2}1} &= \frac{2u}{1+|u|^2} \\
\Gamma^1_{\overline{1}1} &= \Gamma^1_{\overline{1}2} = \Gamma^2_{\overline{2}1} = \Gamma^1_{\overline{2}2} = \Gamma^2_{\overline{2}2} = 0 \\
\Gamma^2_{01} &= -\frac{|w|^2}{(1+|u|^2)^2} \\
\Gamma^1_{02} &= \frac{1}{(1+|u|^2)^2} \\
\Gamma^1_{01} &= \Gamma^2_{02} = 0.
\end{aligned}
$$

The curvature tensor of $\nabla$ is denoted by $R$. According to (1.2), we may find out ${R_m}^n{}_{k\bar{l}}$, which represents the coefficient of $X_n$ in the vector $R(X_k, \overline{X}_l)X_m$. The Ricci tensor (ric) and the scalar curvature ($\rho$) of $\nabla$ follow from (1.3).

Many of our equations and statements are found and justified by computer programming. In this simplified model that $M$ is flat, however, we may also justify results by direct argument.

**Proposition 4.3.** *The Ricci tensor of the Tanaka-Webster connection is given by*

$$\begin{aligned}
ric(X_1, \overline{X}_1) &= \frac{4|w|^2}{(1+|u|^2)^2} - D_u D_{\overline{\mu}} w - D_{\overline{u}} D_\mu \overline{w} + \frac{4\overline{u}}{1+|u|^2} D_{\overline{\mu}} w + \frac{4u}{1+|u|^2} D_\mu \overline{w}, \\
ric(X_1, \overline{X}_2) &= -\Big(\overline{w}_{\overline{u}\,\overline{u}} - \frac{6u}{1+|u|^2}\overline{w}_{\overline{u}} + \frac{12u^2}{(1+|u|^2)^2}\overline{w}\Big), \\
ric(X_2, \overline{X}_1) &= -\Big(w_{uu} - \frac{6\overline{u}}{1+|u|^2} w_u + \frac{12\overline{u}^2}{(1+|u|^2)^2} w\Big), \\
ric(X_2, \overline{X}_2) &= \frac{4}{(1+|u|^2)^2}.
\end{aligned}$$

*Proof.* The first coefficient is $R_{1\bar{1}} = {R_1}^1{}_{1\bar{1}} + {R_1}^2{}_{2\bar{1}}$. We have

$$\begin{aligned}
{R_1}^1{}_{1\bar{1}} &= d\Gamma^1_{\bar{1}1}(X_1) - d\Gamma^1_{11}(\overline{X}_1) - \Gamma^p_{11}\Gamma^1_{\bar{1}p} + \Gamma^p_{\bar{1}1}\Gamma^1_{1p} + \Gamma^p_{\bar{1}1}\Gamma^1_{p1} - \Gamma^{\bar{p}}_{1\bar{1}}\Gamma^1_{\bar{p}1} + 2i\Gamma^1_{01}h_{1\bar{1}} \\
&= 0 - d\Gamma^1_{11}(\overline{X}_1) - 0 + 0 + 0 - \Gamma^{\bar{2}}_{1\bar{1}}\Gamma^1_{\bar{2}1} + 0 \\
&= -\Big(D_{\overline{u}} D_\mu \overline{w} - \frac{2u}{1+|u|^2} D_\mu \overline{w} - \frac{2|w|^2}{(1+|u|^2)^2}\Big) + \frac{2u}{1+|u|^2} D_\mu \overline{w} \\
&= -D_{\overline{u}} D_\mu \overline{w} + \frac{4u}{1+|u|^2} D_\mu \overline{w} + \frac{2|w|^2}{(1+|u|^2)^2}, \\
{R_1}^2{}_{2\bar{1}} &= d\Gamma^2_{\bar{1}1}(X_2) - d\Gamma^2_{21}(\overline{X}_1) - \Gamma^p_{21}\Gamma^2_{\bar{1}p} + \Gamma^p_{\bar{1}1}\Gamma^2_{2p} + \Gamma^p_{\bar{1}2}\Gamma^2_{p1} - \Gamma^{\bar{p}}_{2\bar{1}}\Gamma^2_{\bar{p}1} + 2i\Gamma^2_{01}h_{2\bar{1}} \\
&= d\Gamma^2_{\bar{1}1}(X_2) - 0 - 0 + \Gamma^2_{\bar{1}1}\Gamma^2_{22} + 0 - \Gamma^{\bar{1}}_{2\bar{1}}\Gamma^2_{\bar{1}1} - 2\Gamma^2_{01} \\
&= (-D_u D_{\overline{\mu}} w) + \frac{2\overline{u}}{1+|u|^2} D_{\overline{\mu}} w + \frac{2\overline{u}}{1+|u|^2} D_{\overline{\mu}} w + \frac{2|w|^2}{(1+|u|^2)^2} \\
&= -D_u D_{\overline{\mu}} w + \frac{4\overline{u}}{1+|u|^2} D_{\overline{\mu}} w + \frac{2|w|^2}{(1+|u|^2)^2}.
\end{aligned}$$

So $R_{1\bar{1}}$ is obtained. Next we consider $R_{1\bar{2}} = R_1{}^1{}_{1\bar{2}} + R_1{}^2{}_{2\bar{2}}$.

$$
\begin{aligned}
R_1{}^1{}_{1\bar{2}} &= d\Gamma^1_{\bar{2}1}(X_1) - d\Gamma^1_{11}(\overline{X}_2) - \Gamma^p_{11}\Gamma^1_{\bar{2}p} + \Gamma^p_{\bar{2}1}\Gamma^1_{1p} + \Gamma^p_{\bar{2}1}\Gamma^1_{p1} - \Gamma^{\bar{p}}_{1\bar{2}}\Gamma^1_{\bar{p}1} + 2i\Gamma^1_{01}h_{1\bar{2}} \\
&= d\Big(\frac{2u}{1+|u|^2}\Big)(X_1) - d\Big(\overline{w}_{\overline{u}} - \frac{2u}{1+|u|^2}\overline{w}\Big)(\overline{X}_2) + \Gamma^1_{\bar{2}1}\Gamma^1_{11} - \Gamma^{\bar{2}}_{1\bar{2}}\Gamma^1_{\bar{2}1} \\
&= -\frac{2u^2}{(1+|u|^2)^2}\overline{w} - \Big(\overline{w}_{\overline{u}\,\overline{u}} - \frac{2u}{1+|u|^2}\overline{w}_{\overline{u}} + \frac{2u^2}{(1+|u|^2)^2}\overline{w}\Big) \\
&\quad + \frac{2u}{1+|u|^2}\Big(\overline{w}_{\overline{u}} - \frac{2u}{1+|u|^2}\overline{w}\Big) - \frac{2u}{1+|u|^2}\Big(-\overline{w}_{\overline{u}} + \frac{2u}{1+|u|^2}\overline{w}\Big) \\
&= -\overline{w}_{\overline{u}\,\overline{u}} + \frac{6u}{1+|u|^2}\,\overline{w}_{\overline{u}} - \frac{12u^2}{(1+|u|^2)^2}\,\overline{w} \\
R_1{}^2{}_{2\bar{2}} &= d\Gamma^2_{\bar{2}1}(X_2) - d\Gamma^2_{21}(\overline{X}_2) - \Gamma^p_{21}\Gamma^2_{\bar{2}p} + \Gamma^p_{\bar{2}1}\Gamma^2_{2p} + \Gamma^p_{\bar{2}2}\Gamma^2_{p1} - \Gamma^{\bar{p}}_{2\bar{2}}\Gamma^2_{\bar{p}1} + 2i\Gamma^2_{01}h_{2\bar{2}} \\
&= 0
\end{aligned}
$$

Therefore, $R_{1\bar{2}} = -\overline{w}_{\overline{u}\,\overline{u}} + \dfrac{6u}{1+|u|^2}\,\overline{w}_{\overline{u}} - \dfrac{12u^2}{(1+|u|^2)^2}\overline{w}$. Similarly, we get

$$
\begin{aligned}
R_2{}^1{}_{1\bar{1}} &= d\Gamma^1_{\bar{1}2}(X_1) - d\Gamma^1_{12}(\overline{X}_1) - \Gamma^p_{12}\Gamma^1_{\bar{1}p} + \Gamma^p_{\bar{1}2}\Gamma^1_{1p} + \Gamma^p_{\bar{1}1}\Gamma^1_{p2} - \Gamma^{\bar{p}}_{1\bar{1}}\Gamma^1_{\bar{p}2} + 2i\Gamma^1_{02}h_{1\bar{1}} \\
&= 0 \\
R_2{}^2{}_{2\bar{1}} &= d\Gamma^2_{\bar{1}2}(X_2) - d\Gamma^2_{22}(\overline{X}_1) - \Gamma^p_{22}\Gamma^2_{\bar{1}p} + \Gamma^p_{\bar{1}2}\Gamma^2_{2p} + \Gamma^p_{\bar{1}2}\Gamma^2_{p2} - \Gamma^{\bar{p}}_{2\bar{1}}\Gamma^2_{\bar{p}2} + 2i\Gamma^2_{02}h_{2\bar{1}} \\
&= d\Big(-w_u + \frac{2\overline{u}}{1+|u|^2}w\Big)(X_2) - d\Big(-\frac{2\overline{u}}{1+|u|^2}\Big)(\overline{X}_1) + \Gamma^2_{\bar{1}2}\Gamma^2_{22} - \Gamma^{\bar{1}}_{2\bar{1}}\Gamma^2_{\bar{1}2} \\
&= \Big(-w_{uu} + \frac{2\overline{u}}{1+|u|^2}w_u - \frac{2\overline{u}^2}{(1+|u|^2)^2}w\Big) - \frac{2\overline{u}^2}{(1+|u|^2)^2}w \\
&\quad - \frac{2\overline{u}}{1+|u|^2}\Big(-w_u + \frac{2\overline{u}}{1+|u|^2}w\Big) - \frac{2\overline{u}}{1+|u|^2}\Big(-w_u + \frac{2\overline{u}}{1+|u|^2}w\Big) \\
&= -w_{uu} + \frac{6\overline{u}}{1+|u|^2}\,w_u - \frac{12\overline{u}^2}{(1+|u|^2)^2}\,w.
\end{aligned}
$$

So we obtain the identity for $R_{2\bar{1}}$.

Finally, $R_{2\bar{2}} = R_2{}^1{}_{1\bar{2}} + R_2{}^2{}_{2\bar{2}}$.

$$\begin{aligned}
R_2{}^1{}_{1\bar{2}} &= d\Gamma^1_{\bar{2}2}(X_1) - d\Gamma^1_{12}(\overline{X}_2) - \Gamma^p_{12}\Gamma^1_{\bar{2}p} + \Gamma^p_{\bar{2}2}\Gamma^1_{1p} + \Gamma^p_{\bar{2}1}\Gamma^1_{p2} - \Gamma^{\bar{p}}_{1\bar{2}}\Gamma^1_{\bar{p}2} + 2i\Gamma^1_{02}h_{1\bar{2}} \\
&= 2\Gamma^1_{02} \;=\; \frac{2}{(1+|u|^2)^2} \\
R_2{}^2{}_{2\bar{2}} &= d\Gamma^2_{\bar{2}2}(X_2) - d\Gamma^2_{22}(\overline{X}_2) - \Gamma^p_{22}\Gamma^2_{\bar{2}p} + \Gamma^p_{\bar{2}2}\Gamma^2_{2p} + \Gamma^p_{\bar{2}2}\Gamma^2_{p2} - \Gamma^{\bar{p}}_{2\bar{2}}\Gamma^2_{\bar{p}2} + 2i\Gamma^2_{02}h_{2\bar{2}} \\
&= -d\Big(-\frac{2\overline{u}}{1+|u|^2}\Big)(\overline{X}_2) \;=\; \frac{2}{(1+|u|^2)^2}
\end{aligned}$$

So we obtain $R_{2\bar{2}} = \dfrac{4}{(1+|u|^2)^2}$.

□

For any function $w$ on $N$ which is holomorphic in $u$, we let

$$\phi_w \;=\; \Big(\frac{\partial}{\partial u} - \frac{3\overline{u}}{1+|u|^2}\Big)^2(w) \;=\; w_{uu} - \frac{6\overline{u}}{1+|u|^2}\,w_u + \frac{12\overline{u}^2}{(1+|u|^2)^2}\,w. \tag{4.4}$$

By Proposition 4.3 and (1.3), we have the next result.

**Proposition 4.4.** *The scalar curvature is given by* $\rho = i\big(\phi_w - \overline{(\phi_w)}\big)$.

As a remark, Proposition 4.4 and the formula (4.4) depend on the choice of contact form $\alpha$. By [11], given another contact form $\tilde{\alpha} = e^{2f}\alpha$ with $f$ being a real-valued function on $N$, the scalar curvature of $\tilde{\alpha}$ is found by

$$\tilde{\rho} \;=\; e^{-2f}\Big(\rho - 6\,\Delta_b f - 24\,f_\gamma\,f_{\bar{\delta}}\,h^{\bar{\delta}\gamma}\Big). \tag{4.5}$$

Here, $f_\gamma = X_\gamma(f)$ and $f_{\bar{\gamma}} = \overline{X}_\gamma(f)$. The second covariant derivatives of $f$ are defined by

$$f_{\alpha\bar{\beta}} = \overline{X}_\beta X_\alpha(f) - \Gamma^\gamma_{\bar{\beta}\alpha}f_\gamma \quad \text{and} \quad f_{\bar{\alpha}\beta} = X_\beta\overline{X}_\alpha(f) - \Gamma^{\bar{\gamma}}_{\beta\bar{\alpha}}f_{\bar{\gamma}}.$$

Then, the sub-Laplacian operator $\Delta_b$ (of $f$) is defined by $\Delta_b f = f_{\alpha\bar{\beta}}\,h^{\bar{\beta}\alpha} + f_{\bar{\alpha}\beta}\,h^{\bar{\alpha}\beta}$.

Back to our model of $\mathfrak{D}(w)$, we obtain that

$$\begin{aligned} f_{1\bar{2}} &= D_{\overline{\mu}} D_{\overline{u}} f + 2D_{\frac{v}{|v|}} f + \overline{w} f_{\overline{u}\,\overline{u}} + \Big(\overline{w}_{\overline{u}} - \frac{2u\,\overline{w}}{1+|u|^2}\Big) f_{\overline{u}}, \\ f_{\bar{2}1} &= D_{\overline{\mu}} D_{\overline{u}} f + \overline{w} f_{\overline{u}\,\overline{u}} + \Big(\overline{w}_{\overline{u}} - \frac{2u\,\overline{w}}{1+|u|^2}\Big) f_{\overline{u}}. \end{aligned}$$

Also, $f_{\bar{1}2} = \overline{f_{1\bar{2}}}$ and $f_{2\bar{1}} = \overline{f_{\bar{2}1}}$ since $f$ is real-valued. Therefore,

$$\Delta_b f = 2i\Big(D_{\overline{\mu}} D_{\overline{u}} f - D_\mu D_u f + \overline{w} f_{\overline{u}\,\overline{u}} - w f_{uu} + \big(\overline{w}_{\overline{u}} - \frac{2u\,\overline{w}}{1+|u|^2}\big) f_{\overline{u}} - \Big(w_u - \frac{2\overline{u}\,w}{1+|u|^2}\Big) f_u\Big). \quad (4.6)$$

By (4.5) and (4.6), if $f$ is independent of $u$, then the scalar curvature of $\tilde{\alpha}$ could be found by $\tilde{\rho} = e^{-2f}\rho$. However, this result doesn't hold for a general function $f$ on $N$.

Back to Proposition 4.3, we have found 8 out of 16 components of the curvature tensor $R$. The other 8 coefficients are listed below.

$$\begin{aligned} R_1{}^1{}_{2\bar{1}} &= 0 \\ R_1{}^2{}_{1\bar{1}} &= D_\mu D_\mu \overline{w} - D_{\overline{\mu}} D_{\overline{\mu}} w - w_u\, D_\mu \overline{w} + \overline{w}_{\overline{u}}\, D_{\overline{\mu}} w - 2\,\overline{w}\, D_{\frac{v}{|v|}} w + 2\, w\, D_{\frac{v}{|v|}} \overline{w} \\ &\quad - \frac{4u}{1+|u|^2}\,\overline{w}\, D_{\overline{\mu}} w + \frac{4\overline{u}}{1+|u|^2}\, w\, D_\mu \overline{w} \\ R_1{}^1{}_{2\bar{2}} &= \frac{2}{(1+|u|^2)^2} \\ R_1{}^2{}_{1\bar{2}} &= D_\mu D_{\overline{u}} \overline{w} - \frac{4u}{1+|u|^2}\, D_\mu \overline{w} - \frac{2|w|^2}{(1+|u|^2)^2} \\ R_2{}^1{}_{2\bar{1}} &= -\frac{2}{(1+|u|^2)^2} \\ R_2{}^2{}_{1\bar{1}} &= -D_{\overline{\mu}} D_u w + \frac{4\overline{u}}{1+|u|^2}\, D_{\overline{\mu}} w + \frac{2|w|^2}{(1+|u|^2)^2} \\ R_2{}^1{}_{2\bar{2}} &= 0 \\ R_2{}^2{}_{1\bar{2}} &= 0 \end{aligned}$$

Let $C$ be the Chern-Moser curvature tensor of $\nabla$ defined in (1.4) and (1.5). That means

$$C_m{}^n{}_{k\bar{l}} = \theta^n\big(C(X_k, \overline{X}_l)X_m\big) \quad \text{and} \quad C_{m\bar{n}k\bar{l}} = g\big(C(X_k, \overline{X}_l)X_m, \overline{X}_n\big). \tag{4.7}$$

The coefficients of the (1,3)-Chern tensor of $\mathfrak{D}(w)$ are given by

$$\begin{array}{llll}
C_1{}^1{}_{1\bar{1}} = R_1{}^1{}_{1\bar{1}} - \frac{1}{2}R_{1\bar{1}}, & C_1{}^2{}_{1\bar{1}} = R_1{}^2{}_{1\bar{1}}, & C_1{}^1{}_{2\bar{1}} = -\frac{i}{6}\rho, & C_1{}^2{}_{2\bar{1}} = R_1{}^2{}_{2\bar{1}} - \frac{1}{2}R_{1\bar{1}}, \\
C_2{}^1{}_{1\bar{1}} = -\frac{i}{6}\rho, & C_2{}^2{}_{1\bar{1}} = R_2{}^2{}_{1\bar{1}} - \frac{1}{2}R_{1\bar{1}}, & C_2{}^1{}_{2\bar{1}} = 0, & C_2{}^2{}_{2\bar{1}} = \frac{i}{6}\rho, \\
C_1{}^1{}_{1\bar{2}} = -\frac{i}{6}\rho, & C_1{}^2{}_{1\bar{2}} = R_1{}^2{}_{1\bar{2}} + \frac{1}{2}R_{1\bar{1}} & C_1{}^1{}_{2\bar{2}} = 0, & C_1{}^2{}_{2\bar{2}} = \frac{i}{6}\rho, \\
C_2{}^1{}_{1\bar{2}} = 0, & C_2{}^2{}_{1\bar{2}} = \frac{i}{6}\rho, & C_2{}^1{}_{2\bar{2}} = 0 & C_2{}^2{}_{2\bar{2}} = 0.
\end{array}$$

It leads to the following results about $C_{m\bar{n}k\bar{l}}$'s.

**Proposition 4.5.**

*(1)* $C_{m\bar{n}k\bar{l}} = -\frac{\rho}{6} = -\frac{i}{6}(\phi_w - \overline{\phi_w})$ *for* $C_{1\bar{1}2\bar{2}}$, $C_{1\bar{2}1\bar{2}}$, $C_{1\bar{2}2\bar{1}}$, $C_{2\bar{1}1\bar{2}}$, $C_{2\bar{1}2\bar{1}}$ *and* $C_{2\bar{2}1\bar{1}}$.

*(2)* $C_{m\bar{n}k\bar{l}} = 0$ *for* $C_{1\bar{2}2\bar{2}}$, $C_{2\bar{2}2\bar{1}}$, $C_{2\bar{1}2\bar{2}}$, $C_{2\bar{2}1\bar{2}}$ *and* $C_{2\bar{2}2\bar{2}}$.

*(3)*
$$\begin{aligned}
C_{1\bar{1}1\bar{1}} = & -i\Big(D_{\bar{\mu}}D_{\bar{\mu}}w + w_u\, D_\mu\overline{w} + 2\overline{w}\, D_{\frac{v}{|v|}}w + \frac{4u\overline{w}}{1+|u|^2}\, D_{\bar{\mu}}w\Big) \\
& + i\Big(D_\mu D_\mu\overline{w} + \overline{w}_{\overline{u}}\, D_{\bar{\mu}}w + 2w\, D_{\frac{v}{|v|}}\overline{w} + \frac{4\overline{u}w}{1+|u|^2}\, D_\mu\overline{w}\Big).
\end{aligned}$$

*(4)*
$$C_{m\bar{n}k\bar{l}} = -\frac{i}{2}D_{\bar{\mu}}D_u w + \frac{i}{2}D_\mu D_{\bar{u}}\overline{w} + \frac{2i\overline{u}}{1+|u|^2}D_{\bar{\mu}}w - \frac{2iu}{1+|u|^2}D_\mu\overline{w}$$

*for* $C_{1\bar{1}1\bar{2}}$, $C_{1\bar{1}2\bar{1}}$, $C_{1\bar{2}1\bar{1}}$ *and* $C_{2\bar{1}1\bar{1}}$.

## 4.2 FEFFERMAN BUNDLE AND FEFFERMAN METRIC

Let $C(N)$ be the Fefferman bundle of $N$ . The construction of the Fefferman metric $F$ of $\mathfrak{D}(w)$ involves the coframe $\{\theta^1, \theta^2, \theta^{\bar{1}}, \theta^{\bar{2}}, \alpha\}$ as well as the connection forms $\omega^n_m$ of $\nabla$ defined in (1.6). Let $\mu = \mu_i\, dx_i$ and $\overline{\mu} = \overline{\mu}_i\, dx_i$. We have

$$\begin{aligned}
\theta^1 &= \frac{1}{2(1+|u|^2)^2}\,\mu \\
\theta^{\bar{1}} &= \frac{1}{2(1+|u|^2)^2}\,\overline{\mu} \\
\theta^2 &= du - w\,\theta^{\bar{1}} \;=\; du - \frac{w}{2(1+|u|^2)^2}\,\overline{\mu} \\
\theta^{\bar{2}} &= d\overline{u} - \overline{w}\,\theta^1 \;=\; d\overline{u} - \frac{\overline{w}}{2(1+|u|^2)^2}\,\mu.
\end{aligned}$$

The connection forms are given by

$$\begin{aligned}
\omega^1_1 &= \Gamma^1_{11}\,\theta^1 + \Gamma^1_{\bar{2}1}\,\theta^{\bar{2}} \;=\; \Big(\overline{w}_{\overline{u}} - \frac{2u}{1+|u|^2}\overline{w}\Big)\,\theta^1 + \frac{2u}{1+|u|^2}\,\theta^{\bar{2}}, \\
\omega^1_2 &= \Gamma^1_{02}\,\alpha \;=\; \frac{1}{(1+|u|^2)^2}\,\alpha, \\
\omega^2_1 &= \Gamma^2_{11}\,\theta^1 + \Gamma^2_{\bar{1}1}\,\theta^{\bar{1}} + \Gamma^2_{01}\,\alpha \;=\; -\,(D_\mu\overline{w})\,\theta^1 - (D_{\overline{\mu}}w)\,\theta^{\bar{1}} - \frac{|w|^2}{(1+|u|^2)^2}\,\alpha, \\
\omega^2_2 &= \Gamma^2_{22}\,\theta^2 + \Gamma^2_{\bar{1}2}\,\theta^{\bar{1}} \;=\; \Big(-\,w_u + \frac{2\overline{u}}{1+|u|^2}w\Big)\,\theta^{\bar{1}} - \frac{2\overline{u}}{1+|u|^2}\,\theta^2.
\end{aligned}$$

Note that the restriction of Webster metric $g$to the Levi distribution $\bar{\mathfrak{D}}(w) \oplus \mathfrak{D}(w)$ equals $2i\big(\theta^2 \odot \theta^{\bar{1}} - \theta^1 \odot \theta^{\bar{2}}\big)$. The 1-form $\sigma$ on $C(N)$ in (1.7), is given by

$$\sigma = \frac{1}{4}\big(i\omega^1_1 + i\omega^2_2\big) - \frac{\rho}{48}\,\alpha + \frac{1}{4}\,d\gamma.$$

Therefore, the Fefferman metric $F$ of $\alpha$, is given by

$$\begin{aligned}
F \;=\;& 2i\;\theta^2 \odot \theta^{\bar{1}} - 2i\;\theta^1 \odot \theta^{\bar{2}} - \frac{\rho}{24}\;\alpha \odot \alpha + \frac{1}{2}\;\alpha \odot d\gamma \\
&+ \frac{i}{2}\Gamma^1_{11}\;\alpha \odot \theta^1 + \frac{i}{2}\Gamma^2_{\bar{1}2}\;\alpha \odot \theta^{\bar{1}} + \frac{i}{2}\Gamma^2_{22}\;\alpha \odot \theta^2 + \frac{i}{2}\Gamma^1_{\bar{2}1}\;\alpha \odot \theta^{\bar{2}}.
\end{aligned} \tag{4.8}$$

Note that $\Gamma^2_{\bar{1}2} = -\overline{(\Gamma^1_{11})}$ and $\Gamma^2_{22} = -\overline{(\Gamma^1_{\bar{2}1})}$. The matrix representation of $F$ corresponding to the basis $\{\theta^1, \theta^{\bar{1}}, \theta^2, \theta^{\bar{2}}, \alpha, d\gamma\}$ is

$$[F] = \begin{bmatrix} 0 & 0 & 0 & -i & \frac{i}{4}\Gamma^1_{11} & 0 \\ 0 & 0 & i & 0 & -\frac{i}{4}\overline{(\Gamma^1_{11})} & 0 \\ 0 & i & 0 & 0 & -\frac{i}{4}\overline{(\Gamma^1_{\bar{2}1})} & 0 \\ -i & 0 & 0 & 0 & \frac{i}{4}\Gamma^1_{\bar{2}1} & 0 \\ \frac{i}{4}\Gamma^1_{11} & -\frac{i}{4}\overline{(\Gamma^1_{11})} & -\frac{i}{4}\overline{(\Gamma^1_{\bar{2}1})} & \frac{i}{4}\Gamma^1_{\bar{2}1} & -\frac{1}{24}\rho & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \end{bmatrix}.$$

The entry $F_{ij}$ equals $F(u_i, u_j)$, where the vectors $u_j$'s are:

$$u_1 = X_1, \quad u_2 = \overline{X}_1, \quad u_3 = X_2, \quad u_4 = \overline{X}_2, \quad u_5 = T \text{ and } u_6 = \frac{\partial}{\partial\gamma}. \tag{4.9}$$

To be more specific, we have

$$\begin{aligned} F(X_1, T) &= \frac{i}{4}\Gamma^1_{11} = \frac{i}{4}\Big(\overline{w}_{\overline{u}} - \frac{2u}{1+|u|^2}\overline{w}\Big), \\ F(\overline{X}_2, T) &= \frac{i}{4}\Gamma^1_{\bar{2}1} = \frac{iu}{2(1+|u|^2)}, \\ F(T,T) = -\frac{\rho}{24} &\quad \text{and} \quad F\big(T, \frac{\partial}{\partial\gamma}\big) = \frac{1}{4}. \end{aligned}$$

Moreover, the inverse of $[F]$ is:

$$[F^{-1}] = \begin{bmatrix} 0 & 0 & 0 & i & 0 & \Gamma^1_{\bar{2}1} \\ 0 & 0 & -i & 0 & 0 & \overline{(\Gamma^1_{\bar{2}1})} \\ 0 & -i & 0 & 0 & 0 & \overline{(\Gamma^1_{11})} \\ i & 0 & 0 & 0 & 0 & \Gamma^1_{11} \\ 0 & 0 & 0 & 0 & 0 & 4 \\ \Gamma^1_{\bar{2}1} & \overline{(\Gamma^1_{\bar{2}1})} & \overline{(\Gamma^1_{11})} & \Gamma^1_{11} & 4 & \begin{matrix}\frac{2}{3}\rho - 2i\Gamma^1_{11}\Gamma^1_{\bar{2}1} \\ +2i\overline{(\Gamma^1_{11})}\,\overline{(\Gamma^1_{\bar{2}1})}\end{matrix} \end{bmatrix}.$$

Since the signature of $g$ on $N$ is $(+++--)$, the signature of $F$ is $(+++---)$.

We proceed to consider the Levi-Civita connection $\hat{\nabla}$ of $F$ on $C(N)$. Let $\hat{\nabla}_{u_i} u_j = \hat{\Gamma}^k_{ij}\, u_k$. The Christoffel symbol $\hat{\Gamma}^k_{ij}$ is obtained by (1.8). In terms of the connection forms $\hat{\omega}^j{}_i$,

$$\hat{\nabla} u_i = \hat{\omega}^j_i \otimes u_j \quad \text{with} \quad \hat{\omega}^j_i(u_k) = \hat{\Gamma}^j_{ki}.$$

To obtain the Riemann tensor of $\hat{\nabla}$, we have to find most of the connection forms defined by (1.13). In the following, we set

$$y = 1 + |u|^2, \quad \Gamma^1_{11} = \overline{w}_{\overline{u}} - \frac{2u}{y}\overline{w}, \quad \Gamma^2_{11} = -D_\mu \overline{w}, \quad \Gamma^2_{01} = -\frac{|w|^2}{y^2}, \tag{4.10}$$

Moreover, $\Gamma^1_{11}$ and $\Gamma^2_{11}$ satisfy the identities

$$D_{\overline{X}_1}\Gamma^1_{11} \;=\; D_\mu \Gamma^1_{11} + 2\Gamma^2_{01} \quad \text{and} \quad D_u \Gamma^2_{11} \;=\; \frac{2\overline{u}}{y}\Gamma^2_{11} - 2D_{\frac{v}{|v|}}\overline{w} \tag{4.11}$$

respectively. The exterior derivatives, $d\theta^1$, $d\theta^{\bar{1}}$, $d\theta^2$, $d\theta^{\bar{2}}$ and $d\alpha$ are given by

$$\begin{aligned}
d\theta^1 &= \frac{2u}{y}\,\theta^1 \wedge \theta^{\bar{2}} + \frac{1}{y^2}\,\theta^2 \wedge \alpha + \frac{w}{y^2}\,\theta^{\bar{1}} \wedge \alpha, \\
d\theta^{\bar{1}} &= \frac{2\overline{u}}{y}\,\theta^{\bar{1}} \wedge \theta^2 + \frac{1}{y^2}\,\theta^{\bar{2}} \wedge \alpha + \frac{\overline{w}}{y^2}\,\theta^1 \wedge \alpha, \\
d\theta^2 &= \Gamma^2_{\bar{1}1}\,\theta^1 \wedge \theta^{\bar{1}} - \Gamma^2_{\bar{1}2}\,\theta^{\bar{1}} \wedge \theta^2 + \Gamma^2_{01}\,\theta^1 \wedge \alpha + (D_{\frac{v}{|v|}} w)\,\theta^{\bar{1}} \wedge \alpha - \frac{w}{y^2}\,\theta^{\bar{2}} \wedge \alpha, \\
d\theta^{\bar{2}} &= -\Gamma^2_{11}\,\theta^1 \wedge \theta^{\bar{1}} + \Gamma^1_{11}\,\theta^1 \wedge \theta^{\bar{2}} + (D_{\frac{v}{|v|}} \overline{w})\,\theta^1 \wedge \alpha + \Gamma^2_{01}\,\theta^{\bar{1}} \wedge \alpha - \frac{\overline{w}}{y^2}\,\theta^2 \wedge \alpha, \\
d\alpha &= 2\,\theta^{\bar{1}} \wedge \theta^2 + 2\,\theta^1 \wedge \theta^{\bar{2}}.
\end{aligned}$$

**List of the connection forms $\hat{\omega}^n_m$ of $\hat{\nabla}$:**

$$\begin{aligned}
\hat{\omega}^1_1 &= \frac{1}{2}\Gamma^1_{11}\,\theta^1 - \frac{1}{4}\Gamma^2_{\bar{1}2}\,\theta^{\bar{1}} + \frac{\overline{u}}{2y}\,\theta^2 + \frac{3u}{2y}\,\theta^{\bar{2}} + \Big(\frac{\phi_w}{8} + \frac{i\rho}{12}\Big)\alpha + \frac{i}{4}\,d\gamma \\
\hat{\omega}^2_1 &= \frac{1}{4}\Gamma^1_{11}\,\theta^{\bar{1}} \\
\hat{\omega}^3_1 &= \Gamma^2_{11}\,\theta^1 + \Gamma^2_{\bar{1}1}\,\theta^{\bar{1}} - \frac{1}{4}\Gamma^1_{11}\,\theta^2 - \Big(\frac{1}{8}D_{\overline{X}_1}\Gamma^1_{11} + \frac{1}{8}D_{X_1}\overline{\Gamma^1_{11}} + \frac{u}{4y^2}\Gamma^2_{11} + \frac{\overline{u}}{4y^2}\Gamma^2_{\bar{1}1}\Big)\alpha \\
\hat{\omega}^4_1 &= \frac{1}{4}\Gamma^1_{11}\,\theta^{\bar{2}} \\
\hat{\omega}^5_1 &= \theta^{\bar{2}}
\end{aligned}$$

$$
\begin{aligned}
\hat{\omega}_1^6 &= \Big(iD_{X_1}\Gamma_{11}^1 - 4iD_{\frac{v}{|v|}}\overline{w} - \frac{i}{2}(\Gamma_{11}^1)^2 + \frac{2i\overline{u}}{y}\Gamma_{11}^2\Big)\theta^1 \\
&\quad + \Big(\frac{i}{2}D_{\overline{X}_1}\Gamma_{11}^1 - \frac{i}{2}D_{X_1}\overline{\Gamma_{11}^1} + \frac{i\overline{u}}{y}\Gamma_{\bar{1}1}^2 - \frac{iu}{y}\Gamma_{11}^2\Big)\theta^{\bar{1}} + \Big(\frac{2i\overline{w}}{y^2} - \frac{i\overline{u}}{y}\Gamma_{11}^1\Big)\theta^2 \\
&\quad + \Big(\frac{i}{2}\overline{w}_{\overline{u}\,\overline{u}} - \frac{iu}{y}\overline{w}_{\overline{u}}\Big)\theta^{\bar{2}} + \hat{\Gamma}_{51}^6\ \alpha + \frac{1}{4}\Gamma_{11}^1\ d\gamma \\
\hat{\omega}_3^1 &= \frac{\overline{u}}{2y}\ \theta^1 + \frac{1}{2y^2}\ \alpha \\
\hat{\omega}_3^2 &= -\frac{\overline{u}}{2y}\ \theta^{\bar{1}} \\
\hat{\omega}_3^3 &= -\frac{1}{4}\Gamma_{11}^1\ \theta^1 + \frac{3}{4}\Gamma_{\bar{1}2}^2\ \theta^{\bar{1}} - \frac{\overline{u}}{y}\ \theta^2 - \frac{u}{2y}\ \theta^{\bar{2}} - \Big(\frac{i\rho}{24} + \frac{\phi_w}{8}\Big)\alpha + \frac{i}{4}\ d\gamma \\
\hat{\omega}_3^4 &= -\frac{\overline{u}}{2y}\ \theta^{\bar{2}} \\
\hat{\omega}_3^5 &= -\theta^{\bar{1}} \\
\hat{\omega}_3^6 &= \Big(\frac{2i\overline{w}}{y^2} - \frac{i\overline{u}}{y}\Gamma_{11}^1\Big)\theta^1 + \Big(-\frac{i}{2}w_{uu} + \frac{i\overline{u}}{y}w_u\Big)\theta^{\bar{1}} + \hat{\Gamma}_{53}^6\ \alpha - \frac{\overline{u}}{2y}\ d\gamma \\
\hat{\omega}_5^1 &= \Big(\frac{i\rho}{12} + \frac{\phi_w}{8}\Big)\theta^1 - \frac{w}{y^2}\ \theta^{\bar{1}} - \frac{1}{2y^2}\ \theta^2 + \Big(\frac{i}{48}D_{\overline{u}}\rho + \frac{1}{4y^2}\Gamma_{\bar{1}2}^2 + \frac{\overline{u}w}{2y^3}\Big)\alpha \\
\hat{\omega}_5^3 &= -\Big(\frac{1}{8}D_{X_1}\overline{\Gamma_{11}^1} + \frac{1}{8}D_{\overline{X}_1}\Gamma_{11}^1 + \frac{u}{4y}\Gamma_{11}^2 + \frac{\overline{u}}{4y}\Gamma_{\bar{1}1}^2 - \frac{1}{4}\Gamma_{01}^2\Big)\theta^1 - (D_{\frac{v}{|v|}}w)\ \theta^{\bar{1}} \\
&\quad - \Big(\frac{i\rho}{24} + \frac{\phi_w}{8}\Big)\theta^2 + \frac{w}{y^2}\theta^{\bar{2}} - \Big(\frac{i}{48}D_{\overline{X}_1}\rho + \frac{1}{4}D_{\frac{v}{|v|}}D_u w + \frac{w}{4y^2}\Gamma_{11}^1 + \frac{u}{2y}\Gamma_{01}^2\Big)\alpha \\
\hat{\omega}_5^5 &= 0 \\
\hat{\omega}_5^6 &= \hat{\Gamma}_{15}^6\ \theta^1 + \hat{\Gamma}_{25}^6\ \theta^{\bar{1}} + \hat{\Gamma}_{35}^6\ \theta^2 + \hat{\Gamma}_{45}^6\ \theta^{\bar{2}} + \hat{\Gamma}_{55}^6\ \alpha \\
\hat{\omega}_6^1 &= \frac{i}{4}\ \theta^1 \\
\hat{\omega}_6^2 &= -\frac{i}{4}\ \theta^{\bar{1}} \\
\hat{\omega}_6^3 &= \frac{i}{4}\ \theta^2 \\
\hat{\omega}_6^4 &= -\frac{i}{4}\ \theta^{\bar{2}} \\
\hat{\omega}_6^5 &= 0 \\
\hat{\omega}_6^6 &= \frac{1}{4}\Gamma_{11}^1\ \theta^1 - \frac{1}{4}\Gamma_{\bar{1}2}^2\ \theta^{\bar{1}} - \frac{\overline{u}}{2y}\ \theta^2 - \frac{u}{2y}\ \theta^{\bar{2}}
\end{aligned}
$$

We skip some of the most complicated Christoffel symbols here, e.g. $\hat{\Gamma}^6_{51}$. Other unlisted connection 1-forms can be obtained by the complex conjugation of a 1-form above.

Let $\hat{R}$ be the curvature tensor of $\hat{\nabla}$ according to (1.9) and (1.10). It means

$$\hat{R}(u_i, u_j)u_k = \hat{R}^l_{ijk}u_l \quad \text{and} \quad \hat{R}_{ijkl} = F\big(\hat{R}(u_i, u_j)u_k, u_l\big).$$

Also, we let *Ric* be the Ricci tensor and $S$ be the scalar curvature of $\hat{\nabla}$ as in (1.11) and (1.12). The raise-index of Ric is denoted by $\mathrm{Ric}^\sharp$ with $\mathrm{Ric}^\sharp(u_i) = \hat{R}^j_i\, u_j$. From Theorem 1.4, we immediately have

$$S \;=\; \frac{5}{3}\rho \;=\; \frac{5i}{3}\big(\phi_w - \overline{(\phi_w)}\big),$$

where $\phi_w$ is defined in (4.4). The curvature forms of $\hat{R}$ are denoted by $\hat{\Omega}^n_m$, which are given in (1.14) with $\Omega^n_m \;=\; d\hat{\omega}^n_m - \hat{\omega}^k_m \wedge \hat{\omega}^l_k$.

Following (1.14) and (1.15), we let $\mathcal{W}$ be the Weyl curvature tensor of $\hat{\nabla}$. In particular

$$\mathcal{W}(u_i, u_j)u_k \;=\; \mathcal{W}^l_{ijk}\, u_l \quad \text{and} \quad \mathcal{W}_{ijkl} \;=\; F\big(\mathcal{W}(u_i, u_j)u_k, u_l\big).$$

## 4.3 FURTHER RESULTS WHEN $M$ IS FLAT

**Theorem 4.6.** *Suppose $\mathfrak{D}(w)$ is a CR structure on the twistor CR manifold $N$ over a flat space $M$. Associated with the contact form $\alpha$ (4.2), let $F$ be the Fefferman metric of $\mathfrak{D}(w)$ on $C(N)$ and $S$ the scalar curvature of $F$.*

*Suppose $w$ is holomorphic in $u$ at the point $u = 0$. Then, $S = 0$ if and only if*

$$w(x, u) \;=\; \lambda_0(x) + \lambda_1(x)u + K(x)u^2 - \overline{\lambda_1(x)}u^3 + \overline{\lambda_0(x)}u^4 \tag{4.12}$$

*for $\lambda_0$, $\lambda_1 \in C^\infty(M, \mathbb{C})$ and $K \in C^\infty(M, \mathbb{R})$.*

*Proof.* By (4.4), $S = 0$ if and only if $\phi(w) = \Big(\dfrac{\partial}{\partial u} - \dfrac{3\overline{u}}{1+|u|^2}\Big)^2(w)$ is real-valued. By fixing $x = (x_1, x_2, x_3)$ on $M$, we assume $w$ depends on $u$ only. Let $w = \sum_{n=0}^{\infty} a_n u^n$ near the point $u = 0$.

$$\begin{aligned}(1+|u|^2)^2\phi(w) &= (1+u\overline{u})^2 w_{uu} - 6\overline{u}(1+u\overline{u})w_u + 12\overline{u}^2 w \\ &= w_{uu} + \big(2u\, w_{uu} - 6w_u\big)\overline{u} + \big(u^2\, w_{uu} - 6u\, w_u + 12w\big)\overline{u}^2\end{aligned} \tag{4.13}$$

Let the right hand side of (4.13) be $f(u) + g(u)\overline{u} + h(u)\overline{u}^2$. In power series, we have

$$f(u) \;=\; \sum_{n=0}^{\infty} a_n (u^n)_{uu} \;=\; \sum_{n\geq 2} n(n-1)a_n u^{n-2},$$

$$g(u) \;=\; 2u\sum_{n=0}^{\infty}(u^n)_{uu} - 6\sum_{n=0}^{\infty} a_n (u^n)_u \;=\; -6a_1 + \sum_{n\geq 2}(2n^2-8n)a_n u^{n-1},$$

$$h(u) \;=\; \sum_{n\geq 2} a_n\, n(n-1)u^n - \sum_{n\geq 1} 6a_n\, nu^n + \sum_{n=0}^{\infty} 12a_n u^n \;=\; 12a_0 + 6a_1 u + \sum_{n\geq 2}(n-3)(n-4)a_n u^n.$$

Since $\phi(w)$ is real-valued, $(1+|u|^2)^2\phi(w)$ is also real-valued. Moreover,

$$(1+|u|^2)^2\phi(w) = \overline{(1+|u|^2)^2}\;\overline{\phi(w)} = \overline{f}(\overline{u}) + \overline{g}(\overline{u})u + \overline{h}(\overline{u})u^2.$$

Therefore, all $f(u)$, $g(u)$ and $h(u)$ are polynomials in $u$ up to degree 2. Previously we got

$$f(u) \;=\; \sum_{n\geq 2} n(n-1)a_n u^{n-2} = 2a_2 + 6a_3 u + 12a_4 u^2 + 20a_5 u^3 + \cdots.$$

Hence $a_n = 0$ whenever $n \geq 5$. We could get to the same conclusion if we further examine the power series expansion of $g(u)$ and $h(u)$.

Therefore, $w(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4$ for some constants $a_j$'s. By (4.13),

$$(1+|u|^2)^2\phi(w) = \big(2a_2 + 6a_3 u + 12a_4 u^2\big) + \big(-6a_1 - 8a_2 u - 6a_3 u^2\big)\overline{u} + \big(12a_0 + 6a_1 u + 2a_2 u^2\big).$$

This term is real-valued if and only if $a_2 \in \mathbb{R}$, $a_3 = -\overline{a_1}$ and $a_4 = \overline{a_0}$. We must have

$$w(u) = \lambda_0 + \lambda_1 u + Ku^2 - \overline{\lambda_1}u^3 + \overline{\lambda_0}u^4$$

for $\lambda_0, \lambda_1 \in \mathbb{C}$ and $K \in \mathbb{R}$. $\square$

As a remark, the torsion function $w$ in (3.13) satisfies the condition (4.11). In fact, if both $\lambda_0$, $\lambda_1$ and $K$ are constant functions in (4.11), we obtain a further result that the Weyl tensor $\mathcal{W}$ vanishes. (Theorem 7.3)

In our exploration for the properties of $\mathcal{D}(w)$, computational work shows that the Chern tensor (4.7) coincides with the Weyl tensor on $\bar{\mathfrak{D}} \oplus \mathfrak{D}$, when $M$ is a flat space or a general 3-manifold. However, the statement that '$C = \mathcal{W}$' is a rather general result in CR geometry. We include a direct proof of this statement when $M$ is flat for an extra reference.

As a preliminary, we find out the components of $\mathrm{Ric}^\sharp$. We follow the notation in (4.10).

$$\hat{R}^1_1 = \frac{7\rho}{12} - \frac{i\phi_w}{2}, \quad \hat{R}^2_1 = -\frac{4i\overline{w}}{y^2}, \quad \hat{R}^3_1 = \frac{i}{2}D_{\overline{X}_1}\Gamma^1_{11} + \frac{i}{2}D_{X_1}(\overline{\Gamma^1_{11}}) + \frac{iu}{y}\Gamma^2_{11} + \frac{i\overline{u}}{y}\Gamma^2_{\bar{1}1},$$

$$\hat{R}^4_1 = -4iD_{\frac{v}{|v|}}\overline{w}, \quad \hat{R}^5_1 = i\Gamma^1_{11}, \quad \hat{R}^1_3 = \frac{2i}{y^2}, \quad \hat{R}^2_3 = 0, \quad \hat{R}^3_3 = \frac{i}{2}\phi_w + \frac{\rho}{12},$$

$$\hat{R}^4_3 = \frac{4i\overline{w}}{y^2}, \quad \hat{R}^5_3 = -\frac{2i\overline{u}}{y}, \quad \hat{R}^6_3 = \frac{6}{y^2}\Gamma^1_{11} + \frac{i}{3}D_u\rho - \frac{\overline{u}}{y}\phi_w - \frac{i\overline{u}}{3y}\rho, \quad \hat{R}^1_5 = \frac{1}{12}D_{\overline{u}}\rho + \frac{i}{y^2}(\overline{\Gamma^1_{11}}) - \frac{2i\overline{u}}{y^3}w,$$

$$\hat{R}^5_5 = \frac{\rho}{12}, \quad \hat{R}^1_6 = \hat{R}^2_6 = \hat{R}^3_6 = \hat{R}^4_6 = 0, \quad \hat{R}^5_6 = 1, \quad \hat{R}^6_6 = \frac{\rho}{4}.$$

The terms $\hat{R}^6_1$, $\hat{R}^3_5$ and $\hat{R}^6_5$ are skipped here because of their complexity. Other components of $\hat{R}^j_i$ could be found by complex conjugation, e.g. $\hat{R}^1_2 = \overline{\hat{R}^2_1}$.

**Theorem 4.7.** *Let $\mathcal{W}(X,Y,Z,W) = F\big(\mathcal{W}(X,Y)Z,W\big)$ for any tangent vectors $X,Y,Z,W$ on $C(N)$. Let $C$ is the (0,4)-Chern tensor of $\mathfrak{D}(w)$ given by (4.7). Then,*

$$C_{m\bar{n}k\bar{l}} \;=\; \mathcal{W}\big(X_m, \overline{X}_n, X_k, \overline{X}_l\big)$$

*for every $m,n,k,l = 1,2$.*

Under the indexing of $\mathcal{W}_{ijkl}$'s with respect to the basis of $u_j$'s (4.9), we have

$$\mathcal{W}_{1212} = \mathcal{W}(X_1, \overline{X}_1, X_1, \overline{X}_1), \quad \mathcal{W}_{1234} = \mathcal{W}(X_1, \overline{X}_1, X_2, \overline{X}_2), \quad \mathcal{W}_{1515} = \mathcal{W}(X_1, T, X_1, T)$$

and so on. We would follow the items in Proposition 4.5 to prove Theorem 4.7.

*Proof.* **Step 1: Show that $\mathcal{W}(\mathbf{X_1}, \mathbf{X_2}, \mathbf{Z_1}, \mathbf{Z_2}) = \mathbf{0}$ for any $\mathbf{Z_1}, \mathbf{Z_2} \in \overline{\mathfrak{D}} \oplus \mathfrak{D}$.**

It suffices to show that $\mathcal{W}_{1313}$, $\mathcal{W}_{1312}$, $\mathcal{W}_{1314}$, $\mathcal{W}_{1323}$, $\mathcal{W}_{1324}$ and $\mathcal{W}_{1334}$ are zero. We quote the Sparling condition [6] [18] that $\iota_{\frac{\partial}{\partial\gamma}}\mathcal{W} = 0$, which implies $\mathcal{W}^5_{ijk} = \mathcal{W}_{ijk6}\, F^{65} = 0$.

$$\begin{aligned}
\hat{R}^2_{131} &= 2\big(d\hat{\omega}^2_1 - \hat{\omega}^k_1 \wedge \hat{\omega}^2_k\big)(X_1, X_2) \;=\; 0 \\
\hat{R}^3_{131} &= 2\big(d\hat{\omega}^3_1 - \hat{\omega}^k_1 \wedge \hat{\omega}^3_k\big)(X_1, X_2) \\
&= -D_u\Gamma^2_{11} + \frac{2\overline{u}}{y}\Gamma^2_{11} + \frac{1}{16}(\Gamma^1_{11})^2 - D_{\frac{v}{|v|}}\overline{w} \\
&= D_{\frac{v}{|v|}}\overline{w} + \frac{1}{16}(\Gamma^1_{11})^2 \\
\hat{R}^1_{131} &= 2\big(d\hat{\omega}^1_1 - \hat{\omega}^k_1 \wedge \hat{\omega}^1_k\big)(X_1, X_2) \;=\; \frac{\overline{w}}{y^2} + \frac{\overline{u}}{8y}\Gamma^1_{11} \\
\hat{R}^4_{132} &= 2\big(d\hat{\omega}^4_2 - \hat{\omega}^k_2 \wedge \hat{\omega}^4_k\big)(X_1, X_2) \;=\; -D_u\Gamma^2_{11} + \frac{2\overline{u}}{y}\Gamma^2_{11} - D_{\frac{v}{|v|}}\overline{w} \;=\; D_{\frac{v}{|v|}}\overline{w} \\
\hat{R}^3_{132} &= 2\big(d\hat{\omega}^3_2 - \hat{\omega}^k_2 \wedge \hat{\omega}^3_k\big)(X_1, X_2) \\
&= \frac{1}{4}D_{X_1}\overline{(\Gamma^1_{11})} - \frac{1}{16}|\Gamma^1_{11}|^2 - \frac{1}{2}\Gamma^2_{01} - \frac{1}{4}D_{\overline{\mu}}\overline{(\Gamma^1_{11})} \\
&= -\frac{1}{16}|\Gamma^1_{11}|^2 \\
\hat{R}^3_{133} &= 2\big(d\hat{\omega}^3_3 - \hat{\omega}^k_3 \wedge \hat{\omega}^3_k\big) \;=\; -\frac{\overline{w}}{y^2} - \frac{\overline{u}}{8y}\Gamma^1_{11}
\end{aligned}$$

Therefore, we have the following $\mathcal{W}^l_{ijk}$'s:

$$\begin{aligned}
\mathcal{W}^2_{131} &= \hat{R}^2_{131} = 0 \\
\mathcal{W}^3_{131} &= \hat{R}^3_{131} + \frac{1}{4}\hat{R}_{11} = D_{\frac{v}{|v|}}\overline{w} + \frac{1}{16}(\Gamma^1_{11})^2 + \frac{1}{4}\Big( -4D_{\frac{v}{|v|}}\overline{w} - \frac{1}{4}(\Gamma^1_{11})^2\Big) = 0 \\
\mathcal{W}^1_{131} &= \hat{R}^1_{131} - \frac{1}{4}\hat{R}_{13} = \frac{\overline{w}}{y^2} + \frac{\overline{u}}{8y}\Gamma^1_{11} - \frac{1}{4}\Big(\frac{\overline{u}}{2y}\overline{w}_{\overline{u}} - \frac{\overline{w}}{y} + \frac{5\overline{w}}{y^2}\Big) \\
&= \frac{\overline{w}}{y^2} + \frac{\overline{w}}{4y^2} - \frac{5\overline{w}}{4y^2} = 0 \\
\mathcal{W}^4_{132} &= \hat{R}^4_{132} - \frac{i}{4}\hat{R}^4_1 = D_{\frac{v}{|v|}}\overline{w} - \frac{i}{4}\big( -4iD_{\frac{v}{|v|}}\overline{w}\big) = 0 \\
\mathcal{W}^3_{132} &= \hat{R}^3_{132} - \frac{i}{4}\hat{R}^3_1 + \frac{1}{4}\hat{R}_{12} \\
&= -\frac{1}{16}|\Gamma^1_{11}|^2 - \frac{i}{4}\Big(\frac{i}{2}D_{\overline{X}_1}\Gamma^1_{11} + \frac{i}{2}D_{X_1}\overline{(\Gamma^1_{11})} + \frac{iu}{y}\Gamma^2_{11} + \frac{i\overline{u}}{y}\Gamma^2_{\overline{1}1}\Big) \\
&\quad + \frac{1}{4}\Big( -\frac{1}{2}D_{\overline{X}_1}\Gamma^1_{11} - \frac{1}{2}D_{X_1}\overline{(\Gamma^1_{11})} - \frac{u}{y}\Gamma^2_{11} - \frac{\overline{u}}{y}\Gamma^2_{\overline{1}1} + \frac{1}{4}|\Gamma^1_{11}|^2\Big) \\
&= 0 \\
\mathcal{W}^3_{133} &= \hat{R}^3_{133} + \frac{1}{4}\hat{R}_{13} = -\frac{\overline{w}}{y^2} - \frac{\overline{u}}{8y}\Gamma^1_{11} + \frac{1}{4}\Big(\frac{\overline{u}}{2y}\Gamma^1_{11} + \frac{4\overline{w}}{y^2}\Big) = 0
\end{aligned}$$

As a result, $\mathcal{W}_{1313} = \mathcal{W}_{1312} = \mathcal{W}_{1314} = \mathcal{W}_{1323} = \mathcal{W}_{1324} = \mathcal{W}_{1334} = 0$. It implies that $\mathcal{W}(X_1, X_2, Z_1, Z_2)$ or $\mathcal{W}(\overline{X}_1, \overline{X}_2, Z_1, Z_2)$ is zero whenever $Z_1$, $Z_2$ are on $\overline{\mathfrak{D}} \oplus \mathfrak{D}$.

**Step 2: Show that $\mathcal{W}_{1434}$ and $\mathcal{W}_{3434}$ are zero.**

$$\begin{aligned}
\hat{R}^1_{143} &= 2\big(d\hat{\omega}^1_3 - \hat{\omega}^k_3 \wedge \hat{\omega}^1_k\big)(X_1, \overline{X}_2) = \frac{1}{2y^2} + \frac{|u|^2}{4y^2} \\
\hat{R}^1_{343} &= 2\big(d\hat{\omega}^1_3 - \hat{\omega}^k_3 \wedge \hat{\omega}^1_k\big)(X_2, \overline{X}_2) = 0
\end{aligned}$$

So we have

$$\begin{aligned}
\mathcal{W}^1_{143} &= \hat{R}^1_{143} - \frac{1}{4}\hat{R}_{34} = \frac{1}{2y^2} + \frac{|u|^2}{4y^2} - \frac{1}{4}\Big(\frac{2+|u|^2}{y^2}\Big) = 0, \\
\mathcal{W}^1_{343} &= \hat{R}^1_{343} = 0.
\end{aligned}$$

Therefore, $\mathcal{W}_{1434} = \mathcal{W}_{3434} = 0$ are zero. Note that $\mathcal{W}_{2334} = -\overline{\mathcal{W}_{1434}} = 0$. All of $\mathcal{W}_{1434}$, $\mathcal{W}_{3234}$, $\mathcal{W}_{3414}$, $\mathcal{W}_{3432}$ and $\mathcal{W}_{3434}$ become zero, and they correspond to $C_{1\bar{2}2\bar{2}}$, $C_{2\bar{1}2\bar{2}}$, $C_{2\bar{2}1\bar{2}}$, $C_{2\bar{2}2\bar{1}}$ and $C_{2\bar{2}2\bar{2}}$ respectively.

**Step 3: Show that $\mathcal{W}_{1234} = -\frac{\rho}{6}$.**

We first use the fact that $\mathrm{tr}(\mathcal{W}) = 0$, so $\sum_{j,k} \mathcal{W}_{1jk4} F^{jk} = 0$. Since $\mathcal{W}_{1144} = \mathcal{W}_{1324} = 0$, we have $\mathcal{W}_{1234} = \mathcal{W}_{1414}$. Moreover, we may consider the Bianchi identity

$$\mathcal{W}_{1234} + \mathcal{W}_{2314} + \mathcal{W}_{3124} = 0.$$

It gives $\mathcal{W}_{1432} = \mathcal{W}_{1234}$. Once we establish that $\mathcal{W}_{1234} = -\frac{\rho}{6}$, the coefficients $\mathcal{W}_{1414}$, $\mathcal{W}_{1432}$, $\mathcal{W}_{3214}$, $\mathcal{W}_{3232}$ and $\mathcal{W}_{3412}$ are of the same value, which correspond to $C_{1\bar{1}2\bar{2}}$, $C_{1\bar{2}1\bar{2}}$, $C_{1\bar{2}2\bar{1}}$, $C_{2\bar{1}1\bar{2}}$, $C_{2\bar{1}2\bar{1}}$ and $C_{2\bar{2}1\bar{1}}$ respectively.

$$\begin{aligned}
\hat{R}^1_{123} &= 2\big(d\hat{\omega}^1_3 - \hat{\omega}^k_3 \wedge \hat{\omega}^1_k\big)(X_1, \overline{X}_1) \\
&= -D_{\overline{X}_1}\big(\frac{\overline{u}}{2y}\big) - \frac{\overline{u}}{8y}\overline{(\Gamma^1_{11})} - \frac{\overline{u}}{8y}\overline{(\Gamma^1_{11})} - \frac{3\overline{u}}{8y}\overline{(\Gamma^1_{11})} - \frac{1}{8}\phi_w \\
&\quad -\frac{i\rho}{12} - \frac{\overline{u}^2 w}{2y^2} + \frac{1}{8}\phi_w + \frac{\overline{u}}{2y}\overline{(\Gamma^1_{11})} \\
&= \Big(\frac{\overline{u}^2 w}{2y^2}\Big) - \frac{5\overline{u}}{8y}\overline{(\Gamma^1_{11})} - \frac{i\rho}{12} - \frac{\overline{u}^2 w}{2y^2} + \frac{\overline{u}}{2y}\overline{(\Gamma^1_{11})} \;=\; -\frac{i\rho}{12} - \frac{\overline{u}}{8y}\overline{(\Gamma^1_{11})}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{W}^1_{123} &= \hat{R}^1_{123} - \frac{i}{4}\hat{R}^1_1 - \frac{1}{4}\hat{R}_{23} + \frac{i}{20}S \\
&= \Big(-\frac{i\rho}{12} - \frac{\overline{u}}{8y}\overline{\Gamma^1_{11}}\Big) - \frac{i}{4}\Big(\frac{7\rho}{12} - \frac{i}{2}\phi_w\Big) - \frac{1}{4}\Big(-\frac{1}{12}w_{uu} - \frac{1}{2}\phi_w + \frac{1}{12}\overline{\phi_w}\Big) + \frac{i\rho}{12} \\
&= -\frac{7i}{48}\rho - \frac{1}{48}\overline{\phi_w} + \frac{1}{48}w_{uu} - \frac{\overline{u}}{8y}\Big(w_u - \frac{2\overline{u}}{y}w\Big) \\
&= \frac{7}{48}\big(\phi_w - \overline{\phi_w}\big) - \frac{1}{48}\overline{\phi_w} + \frac{1}{48}\phi_w \;=\; \frac{1}{6}\big(\phi_w - \overline{\phi_w}\big).
\end{aligned}$$

It implies that $\mathcal{W}_{1234} = \mathcal{W}^1_{123} F_{14} = -\frac{\rho}{6}$.

**Step 4: Show that $\mathcal{W}_{1214}$ equals $C_{1\bar{1}1\bar{2}}$.**

Note that for any vectors $Z_1$, $Z_2$ on $\overline{\mathfrak{D}} \oplus \mathfrak{D}$, $\overline{\mathcal{W}\big(X_1, \overline{X}_1, Z_1, Z_2\big)} = -\mathcal{W}\big(X_1, \overline{X}_1, \overline{Z}_1, \overline{Z}_2\big)$. It implies that $\overline{\mathcal{W}_{1214}} = -\mathcal{W}_{1223}$. Hence, $\mathcal{W}_{1214}$, $\mathcal{W}_{1232}$, $\mathcal{W}_{1412}$ and $\mathcal{W}_{3212}$ equal $C_{1\bar{1}1\bar{2}}$, $C_{1\bar{1}2\bar{1}}$, $C_{1\bar{2}1\bar{1}}$ and $C_{2\bar{1}1\bar{1}}$ respectively. From Proposition 4.5, these coefficients of the Chern tensor equal $C_{1\bar{1}1\bar{2}}$, which could be written as

$$C_{1\bar{1}1\bar{2}} \;=\; \frac{i}{2}\Big(D_{\overline{X}_1}\Gamma^1_{11} + \frac{2u}{y}\Gamma^2_{11} - D_{X_1}\overline{(\Gamma^1_{11})} - \frac{2\overline{u}}{y}\Gamma^2_{\bar{1}1}\Big).$$

$$\begin{aligned}
\hat{R}^1_{121} &= 2\big(d\hat{\omega}^1_1 - \hat{\omega}^k_1 \wedge \hat{\omega}^1_k\big)(X_1, \overline{X}_1) \\
&= \Big(-\frac{1}{2}D_{\overline{X}_1}\Gamma^1_{11} + \frac{1}{4}D_{X_1}\overline{(\Gamma^1_{11})} + \frac{\overline{u}}{2y}\Gamma^2_{\bar{1}1} - \frac{3u}{2y}\Gamma^2_{11}\Big) + \frac{1}{16}|\Gamma^1_{11}|^2 + \frac{\overline{u}}{2y}\Gamma^2_{\bar{1}1} \\
&\quad + \Big(-\frac{1}{8}D_{\overline{X}_1}\Gamma^1_{11} + \frac{1}{8}D_{X_1}\overline{(\Gamma^1_{11})} - \frac{\overline{u}}{4y}\Gamma^2_{\bar{1}1} + \frac{u}{4y}\Gamma^2_{11}\Big) \\
&= -\frac{5}{8}D_{\overline{X}_1}\Gamma^1_{11} + \frac{3}{8}D_{X_1}\overline{(\Gamma^1_{11})} + \frac{1}{16}|\Gamma^1_{11}|^2 + \frac{3\overline{u}}{4y}\Gamma^2_{\bar{1}1} - \frac{5u}{4y}\Gamma^2_{11}
\end{aligned}$$

So we have

$$\begin{aligned}
\mathcal{W}^1_{121} &= \hat{R}^1_{121} - \frac{1}{4}\hat{R}_{12} \\
&= -\frac{5}{8}D_{\overline{X}_1}\Gamma^1_{11} + \frac{3}{8}D_{X_1}\overline{(\Gamma^1_{11})} + \frac{1}{16}|\Gamma^1_{11}|^2 + \frac{3\overline{u}}{4y}\Gamma^2_{\bar{1}1} - \frac{5u}{4y}\Gamma^2_{11} \\
&\quad -\frac{1}{4}\Big(-\frac{1}{2}D_{\overline{X}_1}\Gamma^1_{11} - \frac{1}{2}D_{X_1}\overline{(\Gamma^1_{11})} - \frac{u}{y}\Gamma^2_{11} - \frac{\overline{u}}{y}\Gamma^2_{\bar{1}1} + \frac{1}{4}|\Gamma^1_{11}|^2\Big) \\
&= -\frac{1}{2}D_{\overline{X}_1}\Gamma^1_{11} + \frac{1}{2}D_{X_1}\overline{(\Gamma^1_{11})} - \frac{u}{y}\Gamma^2_{11} + \frac{\overline{u}}{y}\Gamma^2_{\bar{1}1}.
\end{aligned}$$

Therefore,

$$\mathcal{W}_{1214} \;=\; \frac{i}{2}\Big(D_{\overline{X}_1}\Gamma^1_{11} - D_{X_1}\overline{(\Gamma^1_{11})} + \frac{2u}{y}\Gamma^2_{11} - \frac{2\overline{u}}{y}\Gamma^2_{\bar{1}1}\Big) \;=\; C_{1\bar{1}1\bar{2}}.$$

**Step 5: Show that $\mathcal{W}_{1212} = \mathbf{C}_{1\bar{1}1\bar{1}}$.**

Recall that $C_{1\bar{1}1\bar{1}} = i\, D_{X_1}\Gamma^2_{\bar{1}1} - i\, D_{\overline{X}_1}\Gamma^2_{11} + i\,\Gamma^2_{11}\,\overline{(\Gamma^1_{11})} - i\,\Gamma^2_{\bar{1}1}\Gamma^1_{11}$.

$$
\begin{aligned}
\hat{R}^3_{121} &= 2\big(d\hat{\omega}^3_1 - \hat{\omega}^k_1 \wedge \hat{\omega}^3_l\big)(X_1, \overline{X}_1) \\
&= -D_{\overline{X}_1}\Gamma^2_{11} + D_{X_1}\Gamma^2_{\bar{1}1} - \frac{1}{4}\Gamma^1_{11}\,\Gamma^2_{\bar{1}1} + \Big(-\frac{1}{2}\Gamma^1_{11}\,\Gamma^2_{\bar{1}1} + \frac{1}{4}\overline{(\Gamma^1_{11})}\,\Gamma^2_{11}\Big) \\
&\quad + \Big(\frac{3}{4}\Gamma^2_{11}\,\overline{(\Gamma^1_{11})} - \frac{1}{4}\Gamma^2_{\bar{1}1}\,\Gamma^1_{11}\Big) \\
&= -D_{\overline{X}_1}\Gamma^2_{11} + D_{X_1}\Gamma^2_{\bar{1}1} - \Gamma^1_{11}\,\Gamma^2_{\bar{1}1} + \overline{(\Gamma^1_{11})}\,\Gamma^2_{11}
\end{aligned}
$$

Therefore, $\mathcal{W}^3_{121} = \hat{R}^3_{121}$. Also,

$$
W_{1212} = -\,i\, D_{\overline{X}_1}\Gamma^2_{11} + i\, D_{X_1}\Gamma^2_{\bar{1}1} - i\,\Gamma^1_{11}\,\Gamma^2_{\bar{1}1} + i\,\overline{(\Gamma^1_{11})}\,\Gamma^2_{11} = C_{1\bar{1}1\bar{1}}.
$$

$\square$

## 5.0 FEFFERMAN METRIC (II)

We continue our discussion about the CR structure $\mathfrak{D}(w)$ on the sphere bundle $N$, assuming that $M$ is a any Riemannian 3-manifold. The general model of $\mathfrak{D}(w)$ requires us to introduce more variables regarding the geometry of $M$. Comparing to the flat case in Chapter 4, we could obtain similar results regarding the Tanaka-Webster connection and the Fefferman metric of $\mathfrak{D}(w)$.

### 5.1 CHANGE OF COORDINATES ON $N$

Following the notation in Chapter 2, let $x = (x_1, x_2, x_3)$ be local coordinates on $M$ and $\mathcal{B} = \{e_1, e_2, e_3\}$ be an orthonormal frame under the metric $\mathsf{g}$ on $M$. Also, if $\nabla^M$ is the Riemannian connection of $\mathsf{g}$, then we set $\nabla^M_{e_i} e_j = G^k_{ij} e_k$. The twistor CR manifold of $(M, \mathsf{g})$ is then denoted by $N$.

On $\mathbb{C}T^*M$, we impose the coordinate system $(x, \mu)$ with respect to $\mathcal{B}$ such that $\mu$ represents the covector $\mu_j e^j$. This allows us to parametrize $N$ by $(x, u)$, where $u$ is mapped to the equivalence class of $\mu = \mu_j e^j$ with $\mu_1 = u^2 - 1$, $\mu_2 = 2u$ and $\mu_3 = i(u^2 + 1)$. According to (2.4), $\mathfrak{D}$ is spanned by $\overline{X}_2 = \dfrac{\partial}{\partial \overline{u}}$ and

$$\overline{X}_1 = \mu_j e_j - \frac{i}{2}\mu_m\Big(G^2_{m1}\mu_3 + G^3_{m2}\mu_1 + G^1_{m3}\mu_2\Big)\frac{\partial}{\partial u} + \frac{i}{2}\mu_m\Big(G^2_{m1}\overline{\mu}_3 + G^3_{m2}\overline{\mu}_1 + G^1_{m3}\overline{\mu}_2\Big)\frac{\partial}{\partial \overline{u}}.$$

Since the construction of $\mathfrak{D}$ involves the choice of an orthonormal frame on $M$, it is essential to discuss how the CR structure $\mathfrak{D}$ (or $\mathfrak{D}(w)$) transforms under coordinate change on $N$.

Let $\mathcal{B}' = \{f_1, f_2, f_3\}$ be another orthonormal frame on $M$. We have $f_j = a_{kj}(x)\, e_k$ with $[a_{kj}(x)]$ in $SO(3,\mathbb{R})$ for every $x$. It also means that $e_k = a^{jk}\, f_j = a_{kj}\, f_j$, and for the coframe of $\mathcal{B}'$, $f^m = a_{lm}\, e^l$.

Let $(x,\gamma)$ be the coordinate system on $T^*M$ with respect to the frame $\mathcal{B}'$. Define $\Phi$ to be the transition map from $(x,\mu)$ to $(x,\gamma)$. If we assume $\gamma_m\, f^m = \mu_l\, e^l = \gamma_m\, a_{lm}\, e_l$, then $\mu_l = a_{lm}\, \gamma_m$ and $\gamma_m = a_{lm}\, \mu_l$.

Suppose $T^k_{ij} = \mathsf{g}\big(\nabla^M_{f_i} f_j, f_k\big)$. The horizontal vector field $\mu^h$ (2.1) is defined by

$$\mu^h \;=\; \mu_j\, e_j - \mu_m\, G^l_{mk}\, \mu_k\, \frac{\partial}{\partial \mu_l} - \mu_m\, G^l_{mk}\, \overline{\mu}_k\, \frac{\partial}{\partial \overline{\mu}_l}$$

at $(x,\mu)$. If $\gamma = \Phi(\mu)$, then $\gamma^h = d\Phi(\mu^h)$ with

$$\gamma^h \;=\; \gamma_j\, f_j - \gamma_m\, T^l_{mk}\, \gamma_k\, \frac{\partial}{\partial \gamma_l} - \gamma_m\, T^l_{mk}\, \overline{\gamma}_k\, \frac{\partial}{\partial \overline{\gamma}_l}$$

at the point $(x,\gamma)$ under the coordinate system with respect to $\mathcal{B}'$.

The space of null covectors $\hat{N}$ can be parametrized by the rational parametrization $\hat{f}$,

$$\hat{f} : \mathbb{C}^2 \backslash \mathbf{0} \to \hat{N}_x, \quad \big(\mu_1,\ \mu_2,\ \mu_3\big) = \hat{f}(s,t) = \big(s^2 - t^2,\ 2st,\ i(s^2+t^2)\big)$$

at the point $x$ in $M$. Let $\mathcal{S}_x$ be the complex plane of $(s,t)$ defined above at $x$. $\mathcal{S} = \bigcup_{x\in M} \mathcal{S}_x$ is a complex vector bundle of rank 2 over $M$, called the spinor bundle of $M$ [5]. With respect to $\mathcal{B}'$, we construct another rational parametrization

$$\hat{h} : \mathbb{C}^2 \backslash \mathbf{0} \to \hat{N}_x, \quad \big(\gamma_1,\ \gamma_2,\ \gamma_3\big) = \hat{h}(s',t') = \big(s'^2 - t'^2,\ 2s't',\ i(s'^2+t'^2)\big).$$

The transition map $\hat{\phi}$ from $(s,t)$ to $(s',t')$ is defined in a way that this diagram commutes.

$$\begin{array}{ccc}
\Big(\mathcal{S}_x, (s,t)\Big) & \overset{\hat{\phi}}{\to} & \Big(\mathcal{S}_x, (s',t')\Big) \\
\hat{f} \downarrow & & \downarrow \hat{h} \\
\Big(\hat{N}_x, (\mu^1,\mu^2,\mu^3)\Big) & \overset{\Phi}{\to} & \Big(\hat{N}_x, (\gamma^1,\gamma^2,\gamma^3)\Big)
\end{array}$$

For any matrix $P$ in $SU(2)$, $P$ is in the form of

$$P = \begin{bmatrix} b & -\bar{c} \\ c & \bar{b} \end{bmatrix}$$

for complex numbers $b$ and $c$ such that $|b|^2 + |c|^2 = 1$. We may consider the 2:1 covering map $q$ from $SU(2)$ to $SO(3, \mathbb{R})$ defined by

$$q(P) = \begin{bmatrix} \frac{1}{2}(b^2 - c^2 + \bar{b}^2 - \bar{c}^2) & -b\bar{c} - \bar{b}c & -\frac{i}{2}(b^2 - c^2 - \bar{b}^2 + \bar{c}^2) \\ bc + \bar{b}\bar{c} & |b|^2 - |c|^2 & -i(bc - \bar{b}\bar{c}) \\ \frac{i}{2}(b^2 + c^2 - \bar{b}^2 - \bar{c}^2) & i(-b\bar{c} + \bar{b}c) & \frac{1}{2}(b^2 + c^2 + \bar{b}^2 + \bar{c}^2) \end{bmatrix}.$$

**Proposition 5.1.** *Let $(s', t')$ and $(s, t)$ be two coordinate systems on $\mathcal{S}$ such that on $\mathcal{S}_x$,*

$$\begin{bmatrix} s' \\ t' \end{bmatrix} = \begin{bmatrix} b & -\bar{c} \\ c & \bar{b} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}.$$

*Let $P = \begin{bmatrix} b & -\bar{c} \\ c & \bar{b} \end{bmatrix}$ in $SU(2)$. Then, on the fibre $\hat{N}_x$, we have*

$$\begin{bmatrix} s'^2 - t'^2 \\ 2s't' \\ i(s'^2 + t'^2) \end{bmatrix} = \begin{bmatrix} q(P) \end{bmatrix} \begin{bmatrix} s^2 - t^2 \\ 2st \\ i(s^2 + t^2) \end{bmatrix}.$$

*for every $(s, t)$ on $\mathcal{S}_x$.*

Under the coordinate transformation $\gamma_j = a_{kj}(x)\,\mu_k$ on $T^*M$, we let $A = [a_{ij}(x)]$ and then find a matrix function $P$,

$$P(x) = \begin{bmatrix} b(x) & -\bar{c}(x) \\ c(x) & \bar{b}(x) \end{bmatrix}$$

such that $|b|^2 + |c|^2 = 1$ and $q(P) = A^T$ at every $x$. If $\hat{\phi}$ is defined by $s' = bs - \bar{c}t$ and $t' = cs + \bar{b}t$ on $S_x$, then we obtain $(\hat{h} \circ \hat{\phi})(s, t) = \Phi \circ \hat{f}(s, t)$ for every $(s, t)$.

Back to the manifold $N$, with respect to the rational parametrization $\hat{h}$ given by $\mathcal{B}'$, we define a complex parameter $v$ on $N$ such that $s' = v$ and $t' = 1$. The parameter $v$ represents the equivalence class of the covector $(v^2-1)f^1 + 2vf^2 + i(v^2+1)f^3$.

Let $v = \phi(u)$ be the transition map from $(x,u)$ to $(x,v)$. Indeed,

$$v \;=\; \frac{s'}{t'} \;=\; \frac{bu-\bar{c}}{cu+\bar{b}}.$$

Using Proposition 5.1, we get to the equation

$$t'^2 \begin{bmatrix} v^2-1 \\ 2v \\ i(v^2+1) \end{bmatrix} = \begin{bmatrix} s'^2-t'^2 \\ 2s't' \\ i(s'^2+t'^2) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} u^2-1 \\ 2u \\ i(u^2+1) \end{bmatrix}.$$

Let $\mathbf{v}_1 = v^2-1$, $\mathbf{v}_2 = 2v$ and $\mathbf{v}_3 = i(v^2+1)$. The differential map $d\phi$ at $(x,u)$is given by

$$\begin{aligned} d\phi\Big(\frac{\partial}{\partial u}\Big) &= \frac{1}{(cu+\bar{b})^2}\frac{\partial}{\partial v}, \\ d\phi\big(\overline{X}_1\big) &= t'^2\Big[\mathbf{v}_j f_j - \frac{i\mathbf{v}_m}{2}\Big(T^2_{m1}\mathbf{v}_3 + T^3_{m2}\mathbf{v}_1 + T^1_{m3}\mathbf{v}_2\Big)\frac{\partial}{\partial v} + \frac{i\mathbf{v}_m}{2}\Big(T^2_{m1}\overline{\mathbf{v}}_3 + T^3_{m2}\overline{\mathbf{v}}_1 + T^1_{m3}\overline{\mathbf{v}}_2\Big)\frac{\partial}{\partial \overline{v}}\Big]. \end{aligned}$$

Therefore, in the coordinates $(x,v)$, the CR structure $\mathfrak{D}$ is spanned by $\overline{Y}_2 = \dfrac{\partial}{\partial \overline{v}}$ and

$$\overline{Y}_1 \;=\; \mathbf{v}_j f_j - \frac{i\mathbf{v}_m}{2}\Big(T^2_{m1}\mathbf{v}_3 + T^3_{m2}\mathbf{v}_1 + T^1_{m3}\mathbf{v}_2\Big)\frac{\partial}{\partial v}.$$

In the next step, we introduce the torsion function $w(x,u)$ to the model and consider the CR structure $\mathfrak{D}(w)$. At the point $(x,u)$, $\mathfrak{D}(w)$ is spanned by $\dfrac{\partial}{\partial \overline{u}}$ and

$$\overline{X} \;=\; \mu_j e_j - \frac{i}{2}\mu_m\big(G^2_{m1}\mu_3 + G^1_{m3}\mu_2 + G^3_{m2}\mu_1\big)\frac{\partial}{\partial u} + w\frac{\partial}{\partial u} + \overline{w}\frac{\partial}{\partial \overline{u}}.$$

The image of $\overline{X}$ by $d\phi$ is the vector

$$d\phi\Big(\overline{X}_1 + w(x,u)\frac{\partial}{\partial u}\Big) \;=\; Y_1 + w\big(x,\phi^{-1}(v)\big)\,\frac{1}{t'^2}\frac{\partial}{\partial v}.$$

Note that $u = \phi^{-1}(v) = \dfrac{-\bar{b}v - \bar{c}}{cv - b}$. In the coordinate system $(x, v)$ with respect to $\mathcal{B}'$, $\mathfrak{D}(w)$ is spanned by $\dfrac{\partial}{\partial \overline{v}}$ and

$$\overline{Y}_1 \;=\; \mathbf{v}_j f_j - \frac{i\mathbf{v}_m}{2}\Big(T^2_{m1}\mathbf{v}_3 + T^3_{m2}\mathbf{v}_1 + T^1_{m3}\mathbf{v}_2\Big)\frac{\partial}{\partial v} + (cv-b)^4\, w\big(x, \frac{-\bar{b}v - \bar{c}}{cv - b}\big)\,\frac{\partial}{\partial v}. \tag{5.1}$$

The equation (5.1) allows us to keep the rational parametrization on $N$ in the same style while using different orthonormal frames on $M$ to analyze $\mathfrak{D}(w)$.

For example, the complex parameter $u$ could not cover every point on $N_x$, so we need to set $v = \dfrac{1}{u}$ to cover the missing point. By Proposition 5.1, we may let $b = 0$ and $c = i$, which also means to consider the orthonormal frame $f_1 = e_1$, $f_2 = -e_2$ and $f_3 = -e_3$. In the coordinates $(x, v)$, $v = 0$ refers to the missing covector $-e^1 - i\, e^3$. The transition map from $(x, u)$ to $(x, v)$ sends $\mathfrak{D}(w)$ under $\mathcal{B}$ to the CR structure $\mathfrak{D}(w')$ under $\mathcal{B}'$ with

$$w'(x, v) \;=\; v^4\; w\Big(x, \frac{1}{v}\Big).$$

In this way, our results from the local model can be applied to the coordinate chart of $(x, v)$.

In addition to transitions between orthonormal frames, change of coordinates on $N$ can be realized by a transition between conformal moving frames. Suppose $\tilde{\mathsf{g}} = \lambda^2\, \mathsf{g}$ for a positive function $\lambda$ on $M$. Since $[\mathsf{g}] = [\tilde{\mathsf{g}}]$, $N$ (equipped with $\mathfrak{D}$) is also the twistor CR manifold of $(M, \tilde{\mathsf{g}})$.

Let $\tilde{e}_j = \frac{1}{\lambda} e_j$ ($\tilde{e}^j = \lambda\, e^j$) and so $\mathcal{B}'' = \big\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\big\}$ is an orthonormal frame on $M$ under $\tilde{\mathsf{g}}$. We also let $\tilde{\nabla}$ be the Riemannian connection of $\tilde{\mathsf{g}}$ with $\tilde{\nabla}_{\tilde{e}_i}\tilde{e}_j = \tilde{G}^k_{ij}\, \tilde{e}_k$.

$\mathcal{B}''$ defines a coordinate system $(\gamma_1, \gamma_2, \gamma_2)$ on $\mathbb{C}T^*M$. The change of coordinates from $\gamma$ to $\mu$ is given by $\mu_j = \lambda\, \gamma_j$. The rational parametrization on $N$ with respect to $\mathcal{B}''$ maps the complex parameter $v$ to $[\gamma_j \tilde{e}^j] = [\mu_j e^j]$. It means that $v = u$.

Suppose we begin with the CR structure $\mathfrak{D}(w,\tilde{\mathfrak{g}})$ on $N$ which is described by

$$\begin{cases} \overline{Y}_1 &= \mu_j\,\tilde{e}_j - \dfrac{i}{2}\,\mu_m\Big(\tilde{G}^2_{m1}\,\mu_3 + \tilde{G}^3_{m2}\,\mu_1 + \tilde{G}^1_{m3}\,\mu_2\Big)\dfrac{\partial}{\partial u} + w\,\dfrac{\partial}{\partial u} \\ \overline{Y}_2 &= \dfrac{\partial}{\partial \overline{u}} \\ \tilde{\alpha} &= \dfrac{u+\overline{u}}{1+|u|^2}\,\tilde{e}^1 + \dfrac{1-|u|^2}{1+|u|^2}\,\tilde{e}^2 + \dfrac{i(u-\overline{u})}{1+|u|^2}\,\tilde{e}^3 \end{cases} \tag{5.2}$$

in the coordinates $(x,u)$. Here $\mu_1 = u^2-1$, $\mu_2 = 2u$, $\mu_3 = i(u^2+1)$.

If we let $[e_i,e_j] = A^k_{ij}\,e_k$ and $[\tilde{e}_i,\tilde{e}_j] = B^k_{ij}\,\tilde{e}_k$, then

$$[\tilde{e}_i,\tilde{e}_j] = \frac{\lambda_j}{\lambda^2}\,\tilde{e}_i - \frac{\lambda_i}{\lambda^2}\,\tilde{e}_j + \frac{1}{\lambda^2}\,[e_i,e_j] \qquad \Longrightarrow \qquad B^k_{ij} = \frac{\delta_{ik}\,\lambda_j - \delta_{jk}\,\lambda_i}{\lambda^2} + \frac{1}{\lambda}\,A^k_{ij}.$$

We denote $e_j(\lambda)$ by $\lambda_j$ above. By Koszul formula,

$$\tilde{G}^k_{ij} \;=\; \frac{1}{2}\big(-B^j_{ik} - B^i_{jk} + B^k_{ij}\big) \;=\; \frac{1}{\lambda^2}\big(\delta_{ik}\,\lambda_j - \delta_{ij}\,\lambda_k\big) + \frac{1}{\lambda}G^k_{ij}.$$

Therefore, in the local model (5.2), we get

$$\tilde{f} \;=\; -\frac{i}{2}\Big(\tilde{G}^2_{m1}\,\mu_3 + \tilde{G}^3_{m2}\,\mu_1 + \tilde{G}^1_{m3}\,\mu_2\Big) \;=\; -\frac{i}{2\lambda}\Big(G^2_{m1}\,\mu_3 + G^3_{m2}\,\mu_1 + G^1_{m3}\,\mu_2\Big) \;=\; \frac{1}{\lambda}f.$$

It allows us to rewrite $\overline{Y}_1$ (5.2) by $\overline{Y}_1 = \dfrac{1}{\lambda}\Big(\mu_j\,e_j + (f+\lambda w)\,\dfrac{\partial}{\partial u}\Big)$. Therefore, the model of $\mathfrak{D}(w,\tilde{\mathfrak{g}})$ is equivalent to the model of $\mathfrak{D}(\lambda w,\mathfrak{g})$, described by

$$\begin{cases} \overline{X}_1 &= \mu_j\,e_j - \dfrac{i}{2}\,\mu_m\Big(G^2_{m1}\,\mu_3 + G^3_{m2}\,\mu_1 + G^1_{m3}\,\mu_2\Big)\dfrac{\partial}{\partial u} + \lambda\,w\,\dfrac{\partial}{\partial u} \\ \overline{X}_2 &= \dfrac{\partial}{\partial \overline{u}} \\ \alpha &= \dfrac{u+\overline{u}}{1+|u|^2}\,e^1 + \dfrac{1-|u|^2}{1+|u|^2}\,e^2 + \dfrac{i(u-\overline{u})}{1+|u|^2}\,e^3 \end{cases} \tag{5.3}$$

Note that the contact forms in (5.2) and (5.3) differ by a factor of $\lambda$.

## 5.2 THE LOCAL MODEL OF $\mathfrak{D}(w)$: GENERAL CASE

On the sphere bundle $N$ of $M$, we define the CR structure $\mathfrak{D}(w)$ by substituting the torsion function $w$ for the term $\mathbf{u}^T \cdot \mathbf{C} \cdot \mathbf{q}$ in (3.7). $\mathfrak{D}(w)$ is then spanned by $\overline{X}_2 = \dfrac{\partial}{\partial \overline{u}}$ and

$$\overline{X}_1 = \mu_j\, e_j - \frac{i}{2}\,\mu_m\Big(G^2_{m1}\,\mu_3 + G^3_{m2}\,\mu_1 + G^1_{m3}\,\mu_2\Big)\frac{\partial}{\partial u} + w\,\frac{\partial}{\partial u}$$

at $(x,u)$ with $\mu_1 = u^2 - 1$, $\mu_2 = 2u$ and $\mu_3 = i(u^2+1)$. We let

$$f(x,u) = -\frac{i}{2}\,\mu_m\Big(G^2_{m1}\,\mu_3 + G^3_{m2}\,\mu_1 + G^1_{m3}\,\mu_2\Big). \tag{5.4}$$

Since $f$ is holomorphic in $u$, we can combine the coefficients of $\dfrac{\partial}{\partial u}$ in $\overline{X}_1$.

On $N$, the local model of $\mathfrak{D}(w)$ is described by

$$\begin{cases} \overline{X}_1 &= (u^2-1)\,e_1 + 2u\,e_2 + i(u^2+1)\,e_3 + w(x,u)\,\dfrac{\partial}{\partial u} \\ X_1 &= (\overline{u}^2-1)\,e_1 + 2\overline{u}\,e_2 - i(\overline{u}^2+1)\,e_3 + \overline{w(x,u)}\,\dfrac{\partial}{\partial \overline{u}} \\ \overline{X}_2 &= \dfrac{\partial}{\partial \overline{u}} \\ X_2 &= \dfrac{\partial}{\partial u} \\ T &= \dfrac{u+\overline{u}}{1+|u|^2}\,e_1 + \dfrac{1-|u|^2}{1+|u|^2}\,e_2 + \dfrac{i(u-\overline{u})}{1+|u|^2}\,e_3 + T_4\,\dfrac{\partial}{\partial u} + \overline{T}_4\,\dfrac{\partial}{\partial \overline{u}}. \end{cases} \tag{5.5}$$

Note that the actual torsion function is $(w-f)$ under our setting. By the notation $v = i\,\mu\times\overline{\mu}$ and $|v| = 2(1+|u|^2)^2$, we have

$$\frac{v_1}{|v|} = \frac{u+\overline{u}}{1+|u|^2}, \quad \frac{v_2}{|v|} = \frac{1-|u|^2}{1+|u|^2} \quad \text{and} \quad \frac{v_3}{|v|} = \frac{i(u-\overline{u})}{1+|u|^2}.$$

So $T = \dfrac{v}{|v|} + T_4\dfrac{\partial}{\partial u} + \overline{T}_4\dfrac{\partial}{\partial \overline{u}}$. The contact form of $\mathfrak{D}(w)$ is chosen to be

$$\alpha = \frac{v_1}{|v|}\,e^1 + \frac{v_2}{|v|}\,e^2 + \frac{v_3}{|v|}\,e^3. \tag{5.6}$$

The exterior derivative of $\alpha$ is given by,

$$\begin{aligned} d\alpha &= -\frac{1}{(1+|u|^2)^2}\Big((\overline{\mu}_1\,du+\mu_1\,d\overline{u})\wedge e^1+(\overline{\mu}_2\,du+\mu_2\,d\overline{u})\wedge e^2+(\overline{\mu}_3\,du+\mu_3\,d\overline{u})\wedge e^3\Big) \\ &\quad +\Big(\frac{v_1}{|v|}\,G^2_{11}-\frac{v_2}{|v|}\,G^1_{22}-\frac{v_3}{|v|}\,(G^1_{23}+G^3_{12})\Big)\;e^1\wedge e^2 \\ &\quad +\Big(-\frac{v_1}{|v|}\,(G^1_{23}+G^2_{31})+\frac{v_2}{|v|}\,G^3_{22}-\frac{v_3}{|v|}\,G^2_{33}\Big)\;e^2\wedge e^3 \\ &\quad +\Big(-\frac{v_1}{|v|}\,G^3_{11}-\frac{v_2}{|v|}\,(G^2_{31}+G^3_{12})+\frac{v_3}{|v|}\,G^1_{33}\Big)\;e^3\wedge e^1. \end{aligned}$$

Being the Reeb vector field corresponding to $\alpha$, $T$ is characterized by $d\alpha(T,\cdot)=0$, where

$$T_4 \;=\; -\frac{1}{2}\frac{v_m}{|v|}\Big(G^l_{mk}\,\mu_k\,\frac{v_l}{|v|}\Big) \;=\; -\frac{i}{2}\frac{v_m}{|v|}\Big(G^2_{m1}\,\mu_3+G^1_{m3}\,\mu_2+G^3_{m2}\,\mu_1\Big). \tag{5.7}$$

The Levi-form of $\alpha$ is defined by $\mathcal{L}=-i\,d\alpha$. Its components are $h_{\alpha\bar\beta}=\mathcal{L}(X_\alpha,\overline{X}_\beta)$. We have $h_{1\bar2}=-i$, $h_{2\bar1}=i$ and $h_{2\bar2}=0$. The remaining coefficient is,

$$\begin{aligned} h_{1\bar1} &= -|\mu_1|^2\,G^3_{12}-|\mu_2|^2\,G^1_{23}-|\mu_3|^2\,G^2_{31} \\ &\quad +\frac{1}{2|v|}\Big(G^2_{11}\,v_1\,v_3-G^3_{11}\,v_1\,v_2-G^1_{22}\,v_3\,v_2+G^3_{22}\,v_2\,v_1+G^1_{33}\,v_2\,v_3-G^2_{33}\,v_1\,v_3\Big). \end{aligned} \tag{5.8}$$

The inverse of $h$ is given by $h^{\bar11}=0$, $h^{\bar12}=-i$, $h^{\bar21}=i$ and $h^{\bar22}=-h_{1\bar1}$.

The coframe to $\big\{X_1,\overline{X}_1,X_2,\overline{X}_2,T\big\}$ consists of

$$\theta^1 \;=\; \frac{1}{2(1+|u|^2)^2}\,\big(\mu_j\,e^j\big), \qquad \theta^{\bar1} \;=\; \frac{1}{2(1+|u|^2)^2}\,\big(\overline{\mu}_j\,e^j\big),$$

$$\theta^2 \;=\; du-w\,\theta^{\bar1}-T_4\,\alpha, \qquad \theta^{\bar2} \;=\; d\overline{u}-\overline{w}\,\theta^1-\overline{T}_4\,\alpha,$$

together with the contact form $\alpha$. Let $g$ be the Webster metric associated with $\alpha$, where

$$g \;=\; 2h_{11}\,\theta^1\odot\theta^{\bar1}+2i\,\theta^2\odot\theta^{\bar1}-2i\,\theta^1\odot\theta^{\bar2}+\alpha\odot\alpha.$$

In the following context, we simply write $\mu = \mu_j\, e_j$ and $\overline{\mu} = \overline{\mu}_j\, e_j$. Define the scalar variables $a_M$, $a_V$ and $b_V$ on $N$ by the Lie brackets between vectors $\mu$, $\overline{\mu}$ and $\dfrac{v}{|v|}$.

$$\begin{aligned}
\big[\mu, \overline{\mu}\big] &= a_M\, \mu - \overline{a}_M\, \overline{\mu} + 2ih_{1\bar{1}}\, \frac{v}{|v|} \\
\Big[\mu, \frac{v}{|v|}\Big] &= a_V\, \mu + b_V\, \overline{\mu} + 2T_4\, \frac{v}{|v|} \\
\Big[\mu, \frac{\partial}{\partial u}\Big] &= -\frac{\partial \mu}{\partial u} \;=\; -\frac{2\overline{u}}{1+|u|^2}\, \mu - 2\frac{v}{|v|} \\
\Big[\frac{v}{|v|}, \frac{\partial}{\partial u}\Big] &= -\frac{\partial}{\partial u}\Big(\frac{v}{|v|}\Big) \;=\; \frac{1}{(1+|u|^2)^2}\, \overline{\mu}
\end{aligned}$$

In particular, we have

$$\frac{\partial \mu}{\partial u} = \frac{2\overline{u}}{1+|u|^2}\, \mu + 2\frac{v}{|v|} \quad \text{and} \quad \frac{\partial}{\partial u}\Big(\frac{v}{|v|}\Big) = -\frac{1}{(1+|u|^2)^2}\, \overline{\mu}.$$

Using the identities $v = i\, \mu \times \overline{\mu}$ and $\dfrac{v}{|v|} \times \mu = -i\, \mu$, we have

$$\big[\mu, \overline{\mu}\big] = \big[\mu_j e_j, \overline{\mu}_k e_k\big] = -iv_3\big[e_1, e_2\big] - iv_1\big[e_2, e_3\big] - iv_2\big[e_3, e_1\big],$$

$$\big[\mu, \frac{v}{|v|}\big] = \big[\mu_j e_j, \frac{v_k}{|v|} e_k\big] = i\mu_3\big[e_1, e_2\big] + i\mu_1\big[e_2, e_3\big] + i\mu_2\big[e_3, e_1\big],$$

The variables $a_M$, $a_V$ and $b_V$ could then be obtained by

$$a_M = \frac{1}{|\mu|^2}\Big([\mu, \overline{\mu}] \cdot \overline{\mu}\Big), \quad a_V = \frac{1}{|\mu|^2}\Big(\big[\mu, \frac{v}{|v|}\big] \cdot \overline{\mu}\Big) \quad \text{and} \quad b_V = \frac{1}{|\mu|^2}\Big(\big[\mu, \frac{v}{|v|}\big] \cdot \mu\Big)$$

in terms of the Christoffel symbols on $M$. For example,

$$a_V = -\frac{i\, \overline{\mu}_m}{|\mu|^2}\Big(G^2_{m1}\, \mu_3 + G^3_{m2}\, \mu_1 + G^1_{m3}\, \mu_2\Big) + i\big(G^3_{12} + G^1_{23} + G^2_{31}\big). \tag{5.9}$$

These variables $a_M$, $a_V$, $b_V$, together with $f$, $T_4$ and $h_{1\bar{1}}$, would get involved in the geometry of $N$ and $C(N)$. Both of them are independent of $w$. When $M$ is equipped with a flat metric, all these variables vanish. Moreover, we have the following statement relating them to each other.

**Proposition 5.2.**

$$
\begin{aligned}
&(1) \quad a_M &&= -(1+|u|^2)^2\, a_{V,u} \\
&(2) \quad T_4 &&= \frac{(1+|u|^2)^2}{2}\, a_{V,\overline{u}} \\
&(3) \quad b_V &&= u\,(1+|u|^2)\,\overline{a}_{V,\overline{u}} + \frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,\overline{u}\,\overline{u}} \\
&(4) \quad h_{1\bar{1}} &&= i(1+|u|^2)^2\, a_V + \frac{i(1+|u|^2)^4}{2}\, a_{V,u\overline{u}} \\
&(5) \quad f &&= (1+|u|^2)^2\, b_V
\end{aligned}
$$

*Proof.* We first look at the identity,

$$\frac{\partial}{\partial u}\Big[\mu, \frac{v}{|v|}\Big] = \Big[\frac{\partial \mu}{\partial u}, \frac{v}{|v|}\Big] + \Big[\mu, \frac{\partial}{\partial u}\big(\frac{v}{|v|}\big)\Big]. \tag{5.10}$$

The left hand side is given by

$$\Big(\frac{\partial a_V}{\partial u} \;+\; \frac{2\,\overline{u}\,a_V}{1+|u|^2}\Big)\mu \;+\; \Big(\frac{\partial b_V}{\partial u} - \frac{2T_4}{1+|u|^2}\Big)\overline{\mu} + \Big(2\frac{\partial T_4}{\partial u} + 2a_V\Big)\frac{v}{|v|},$$

and the right hand side is

$$\Big(\frac{2\,\overline{u}\,a_V}{1+|u|^2} - \frac{a_M}{(1+|u|^2)^2}\Big)\mu \;+\; \Big(\frac{2\,\overline{u}\,b_V}{1+|u|^2} + \frac{\overline{a}_M}{(1+|u|^2)^2}\Big)\overline{\mu} \;+\; \Big(\frac{4\,\overline{u}\,T_4}{1+|u|^2} - \frac{2\,i\,h_{1\bar{1}}}{(1+|u|^2)^2}\Big)\frac{v}{|v|}.$$

Next, we differentiate $\big[\mu, \frac{v}{|v|}\big]$ by $\overline{u}$ instead of $u$, so we obtain

$$\frac{\partial}{\partial \overline{u}}\Big[\mu, \frac{v}{|v|}\Big] \;=\; \Big[\mu, \frac{\partial}{\partial \overline{u}}\big(\frac{v}{|v|}\big)\Big] \;=\; \Big[\mu, \frac{-1}{(1+|u|^2)^2}\mu\Big] \;=\; 0. \tag{5.11}$$

The left hand side is given by

$$\frac{\partial}{\partial \overline{u}}\Big[\mu, \frac{v}{|v|}\Big] \;=\; \Big(\frac{\partial a_V}{\partial \overline{u}} - \frac{2T_4}{(1+|u|^2)^2}\Big)\mu + \Big(\frac{\partial b_V}{\partial \overline{u}} + \frac{2\,u\,b_V}{1+|u|^2}\Big)\overline{\mu} + \Big(2b_V + 2\frac{\partial T_4}{\partial \overline{u}}\Big)\frac{v}{|v|}.$$

The third formula will be obtained by differentiating the Lie bracket $[\mu, \overline{\mu}]$.

$$\frac{\partial}{\partial u}\Big[\mu, \overline{\mu}\Big] \;=\; \Big[\frac{\partial \mu}{\partial u}, \overline{\mu}\Big] \tag{5.12}$$

By expansion, the left hand side is

$$\Big(\frac{\partial a_M}{\partial u} + \frac{2\,\overline{u}\,a_M}{1+|u|^2}\Big)\mu \;-\; \Big(\frac{\partial \overline{a}_M}{\partial u} + \frac{2\,i\,h_{1\bar{1}}}{(1+|u|^2)^2}\Big)\overline{\mu} \;+\; \Big(2a_M + 2i\frac{\partial h_{1\bar{1}}}{\partial u}\Big)\frac{v}{|v|}.$$

And the right hand side is

$$\Big(\frac{2\,\overline{u}\,a_M}{1+|u|^2} - 2\bar{b}_V\Big)\mu \;-\; \Big(\frac{2\,\overline{u}\,\overline{a}_M}{1+|u|^2} + 2\overline{a}_V\Big)\overline{\mu} \;+\; \Big(\frac{4\,i\,\overline{u}\,h_{1\bar{1}}}{1+|u|^2} - 4\overline{T}_4\Big)\frac{v}{|v|}.$$

We would apply equations (5.10), (5.11) and (5.12) to justify Proposition 5.2. For example, the $\mu$-coefficients of (5.10) give

$$a_M = -(1+|u|^2)^2\, a_{V,u},$$

and the $\mu$-coefficients of (5.9) implies that

$$T_4 = \frac{(1+|u|^2)^2}{2}\, a_{V,\overline{u}}.$$

Considering the $\mu$-coefficients of (5.12), we get

$$\begin{aligned} b_V &= -\frac{1}{2}\,\overline{a}_{M,\overline{u}} \;=\; -\frac{1}{2}\frac{\partial}{\partial \overline{u}}\Big(-(1+|u|^2)^2\,\overline{a}_{V,\overline{u}}\Big) \\ &= u\,(1+|u|^2)\,\overline{a}_{V,\overline{u}} + \frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,\overline{u}\,\overline{u}}. \end{aligned}$$

For the last item, the $\frac{v}{|v|}$-components of (5.10) gives

$$\begin{aligned} T_{4,u} + 2a_V &= \frac{4\,\overline{u}\,T_4}{1+|u|^2} - \frac{2\,i\,h_{1\bar{1}}}{(1+|u|^2)^2} \\ (1+|u|^2)^2\,a_{V,u\overline{u}} + 2a_V &= -\frac{2\,i\,h_{1\bar{1}}}{(1+|u|^2)^2} \\ h_{1\bar{1}} &= i\,(1+|u|^2)^2\,a_V + \frac{i}{2}\,(1+|u|^2)^4\,a_{V,u\overline{u}}. \end{aligned}$$

□

As a remark, $h_{1\bar{1}} = \mathcal{L}(X_1, \overline{X}_1)$ is real-valued, so $h_{1\bar{1}} = \overline{h_{1\bar{1}}}$. By item (4), we have

$$a_V + \overline{a}_V + \frac{1}{2}(1+|u|^2)^2\big(a_{V,u\overline{u}} + \overline{a}_{V,u\overline{u}}\big) \;=\; 0. \tag{5.13}$$

We also derive relations between the $u$- or $\overline{u}$-derivatives of $a_V$ by (5.10), (5.11) and (5.12).

**Proposition 5.3.**

$$
\begin{aligned}
(1)\quad a_{V,u\overline{u}} &= \frac{2}{(1+|u|^2)^2}\Big(2i(G^3_{12}+G^1_{23}+G^2_{31})+\overline{a}_V-2a_V\Big)\\
(2)\quad a_{V,\overline{u}\,\overline{u}} &= \frac{-2u}{1+|u|^2}\big(a_{V,\overline{u}}+\overline{a}_{V,\overline{u}}\big)-\overline{a}_{V,\overline{u}\,\overline{u}}\\
(3)\quad a_{V,uuu} &= -\frac{6\overline{u}^2}{(1+|u|^2)^2}\,a_{V,u}-\frac{6\overline{u}}{1+|u|^2}\,a_{V,uu}
\end{aligned}
$$

*Proof.* For the item(2), the $\dfrac{v}{|v|}$-component of (5.11) gives $b_V=-T_{4,\overline{u}}$. By expansion,

$$
\begin{aligned}
u\,(1+|u|^2)\,\overline{a}_{V,\overline{u}}+\frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,\overline{u}\,\overline{u}} &= -\,u\,(1+|u|^2)\,a_{V,\overline{u}}-\frac{(1+|u|^2)^2}{2}\,a_{V,\overline{u}\,\overline{u}}\\
a_{V,\overline{u}\,\overline{u}} &= \frac{-2u}{1+|u|^2}\big(a_{V,\overline{u}}+\overline{a}_{V,\overline{u}}\big)-\overline{a}_{V,\overline{u}\,\overline{u}}.
\end{aligned}
$$

To prove the item (3), we look at the $\overline{\mu}$-coefficients of (5.11).

$$
\begin{aligned}
b_{V,\overline{u}}+\frac{2u}{1+|u|^2}\,b_V &= 3u^2\,\overline{a}_{V,\overline{u}}+3u\,(1+|u|^2)\,\overline{a}_{V,\overline{u}\,\overline{u}}+\frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,\overline{u}\,\overline{u}\,\overline{u}} = 0\\
\Longrightarrow\qquad a_{V,uuu} &= -\frac{6\overline{u}^2}{(1+|u|^2)^2}\,a_{V,u}-\frac{6\overline{u}}{1+|u|^2}\,a_{V,uu}
\end{aligned}
$$

Finally, We compare the $\overline{\mu}$-terms on both sides of (5.10).

$$
b_{V,u}-\frac{2T_4}{(1+|u|^2)^2} = \frac{2\overline{u}b_V}{1+|u|^2}+\frac{\overline{a}_M}{(1+|u|^2)^2}.
$$

In terms of $a_V$ and its derivatives,

$$
\begin{aligned}
b_{V,u}-\frac{2T_4}{(1+|u|^2)^2} &= \big(1+2|u|^2\big)\,\overline{a}_{V,\overline{u}}+u\,(1+|u|^2)\,\overline{a}_{V,u\overline{u}}+\overline{u}\,(1+|u|^2)\,\overline{a}_{V,\overline{u}\,\overline{u}}\\
&\quad+\frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,u\overline{u}\,\overline{u}}-a_{V,\overline{u}}\\
\frac{2\,\overline{u}\,b_V}{1+|u|^2}+\frac{\overline{a}_M}{(1+|u|^2)^2} &= \big(2|u|^2-1\big)\,\overline{a}_{V,\overline{u}}+\overline{u}\,(1+|u|^2)\,\overline{a}_{V,\overline{u}\,\overline{u}}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,u\overline{u}\,\overline{u}} + u\,(1+|u|^2)\,\overline{a}_{V,u\overline{u}} + 2\overline{a}_{V,\overline{u}} - a_{V,\overline{u}} &= 0\\
\frac{\partial}{\partial \overline{u}}\Big(2\overline{a}_V - a_V + \frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,u\overline{u}}\Big) &= 0\\
\Longrightarrow \qquad \frac{\partial}{\partial u}\Big(2a_V - \overline{a}_V + \frac{(1+|u|^2)^2}{2}\,a_{V,u\overline{u}}\Big) &= 0.
\end{aligned}$$

By (5.13), we may substitute $a_{V,u\overline{u}}$ with $-\overline{a}_{V,u\overline{u}} - \dfrac{2}{(1+|u|^2)^2}(a_V + \overline{a}_V)$. So,

$$\frac{\partial}{\partial u}\Big(2\overline{a}_V - a_V + \frac{(1+|u|^2)^2}{2}\,\overline{a}_{V,u\overline{u}}\Big) = 0.$$

Hence we know that the expression $2a_V - \overline{a}_V + \dfrac{(1+|u|^2)^2}{2}\,a_{V,u\overline{u}}$ is independent of $u$ or $\overline{u}$. At the point $u=0$, which refers to $\mu_1=-1$, $\mu_2=0$ and $\mu_3=i$, we obtain

$$\begin{aligned}
a_V &= \frac{i}{2}\big(G^3_{12}+G^2_{31}\big) + i\,G^1_{23} - \frac{1}{2}\big(G^2_{11}+G^2_{33}\big),\\
\overline{a}_V &= -\frac{i}{2}\big(G^3_{12}+G^2_{31}\big) - i\,G^1_{23} - \frac{1}{2}\big(G^2_{11}+G^2_{33}\big),\\
a_{V,u\overline{u}} &= -2i\,G^1_{23} + i\,G^3_{12} + i\,G^2_{31} + G^2_{11} + G^2_{33}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& 2a_V - \overline{a}_V + \frac{(1+|u|^2)^2}{2}\,a_{V,u\overline{u}}\\
= \;& \Big(2a_V - \overline{a}_V + \frac{(1+|u|^2)^2}{2}\,a_{V,u\overline{u}}\Big)\Big|_{u=0}\\
= \;& \Big(i\,G^3_{12} + i\,G^2_{31} + 2i\,G^1_{23} - G^2_{11} - G^2_{33}\Big) + \Big(\frac{i}{2}\,G^3_{12} + \frac{i}{2}\,G^2_{31} + i\,G^1_{23} + \frac{1}{2}\,G^2_{11} + \frac{1}{2}\,G^2_{33}\Big)\\
& + \Big(-i\,G^1_{23} + \frac{i}{2}\,G^3_{12} + \frac{i}{2}\,G^2_{31} + \frac{1}{2}\,G^2_{11} + \frac{1}{2}\,G^2_{33}\Big)\\
= \;& 2\,i\,\big(G^3_{12} + G^1_{23} + G^2_{31}\big),
\end{aligned}$$

which justifies the item (1) of Proposition 5.3. □

The function $f$ in (5.4) could be rewritten as

$$f \;=\; (1+|u|^2)^2\, b_V \;=\; -\,u\,(1+|u|^2)^3\, a_{V,\overline{u}} - \frac{(1+|u|^2)^4}{2}\, a_{V,\overline{u}\,\overline{u}}. \tag{5.14}$$

By Proposition 5.2 and 5.3, we would make use of the variables $a_M$, $a_V$, $b_V$, $h_{1\bar{1}}$, $T_4$ to express important variables on $N$. Moreover, we could replace these variables by $a_V$ and its derivatives if need be.

In the following context, we follow the same notation (right before Proposition 4.1) about directional derivatives of functions. If $q = q(x, u, \overline{u})$ is a function on $N$, then $D_\mu q$, $D_{\overline{\mu}} q$ and $D_{\frac{v}{|v|}} q$ are the directional derivatives of $q$ by $\mu$, $\overline{\mu}$ and $\frac{v}{|v|}$ respectively.

We are ready the find the Lie brackets between $X_1$, $X_2$, $\overline{X}_1$, $\overline{X}_2$ and $T$ of $\mathfrak{D}(w)$ in (5.5).

**Proposition 5.4.**

$$\begin{aligned}
[X_1, \overline{X}_2] \;&=\; -\frac{2u}{1+|u|^2}\, X_1 + 2T_4\, X_2 + \Big(\frac{2\,u\,\overline{w}}{1+|u|^2} + 2\overline{T}_4 - \overline{w}_{\overline{u}}\Big)\, \overline{X}_2 - 2\,T \\
[X_1, \overline{X}_1] \;&=\; \overline{a}_M\, X_1 - a_M\, \overline{X}_1 + \Big(D_{\overline{\mu}} w + a_M\, w + 2i\, h_{1\bar{1}}\, T_4\Big)\, X_2 \\
&\quad + \Big(-\,D_\mu \overline{w} - \overline{a}_M\, \overline{w} + 2i\, h_{1\bar{1}}\, \overline{T}_4\Big)\, \overline{X}_2 - 2i\, h_{1\bar{1}}\, T \\
[\overline{X}_1, T] \;&=\; \Big(b_V - \frac{w}{(1+|u|^2)^2}\Big)\, X_1 + \Big(a_V - \frac{2\,\overline{u}\,T_4}{1+|u|^2}\Big)\, \overline{X}_1 \\
&\quad + \Big(-\,D_{\frac{v}{|v|}} w - T_4\, w_u + \Big(T_{4,u} + \frac{2\,\overline{u}\,T_4}{1+|u|^2} - a_V\Big)\, w + D_\mu T_4\Big)\, X_2 \\
&\quad + \Big(\overline{T}_{4,u}\, w - b_V\, \overline{w} + \frac{|w|^2}{(1+|u|^2)^2} + D_\mu \overline{T}_4\Big)\, \overline{X}_2 \\
[X_2, T] \;&=\; -\frac{1}{(1+|u|^2)^2}\, X_1 + T_{4,u}\, X_2 + \Big(\frac{\overline{w}}{(1+|u|^2)^2} + \overline{T}_{4,u}\Big)\, \overline{X}_2
\end{aligned}$$

*Also,* $[X_1, X_2] = [X_2, \overline{X}_2] = 0$.

## 5.3 THE SCALAR CURVATURE FORMULA

We denote the Tanaka-Webster connection of $g$ by $\nabla$, and define its Christoffel symbols in the same style: $\nabla_{X_m} X_n = \Gamma^k_{mn}\, X_k$, $\nabla_{\overline{X}_m} X_n = \Gamma^k_{\bar{m}n}\, X_k$ and $\nabla_T X_n = \Gamma^k_{0n}\, X_k$. See equation (1.1) for their formulas.

**Proposition 5.5.**

$$
\begin{aligned}
\Gamma^1_{11} &= \overline{w}_{\overline{u}} - \frac{2\,u\,\overline{w}}{1+|u|^2} - 2\overline{T}_4 \\
\Gamma^2_{11} &= -D_\mu \overline{w} + i\,h_{1\bar{1}}\,\overline{w}_{\overline{u}} - \Big(i\,\frac{\partial h_{1\bar{1}}}{\partial \overline{u}} + \frac{2\,i\,u\,h_{1\bar{1}}}{1+|u|^2} + \overline{a}_M\Big)\overline{w} - i\,D_{\overline{\mu}} h_{1\bar{1}} - i\,a_M\,h_{1\bar{1}} \\
\Gamma^2_{12} &= a_M \\
\Gamma^2_{21} &= -i\,\frac{\partial h_{1\bar{1}}}{\partial u} + \frac{2\,i\,\overline{u}\,h_{1\bar{1}}}{1+|u|^2} - 2\overline{T}_4 \\
\Gamma^2_{22} &= -\frac{2\overline{u}}{1+|u|^2} \\
\Gamma^1_{12} &= \Gamma^1_{21} \;=\; \Gamma^1_{22} \;=\; 0 \\
\Gamma^1_{\bar{1}1} &= -\overline{a}_M \\
\Gamma^2_{\bar{1}1} &= -D_{\overline{\mu}} w - a_M\,w - 2\,i\,h_{1\bar{1}}\,T_4 \\
\Gamma^2_{\bar{1}2} &= -w_u + \frac{2\,\overline{u}\,w}{1+|u|^2} + 2T_4 \\
\Gamma^1_{\bar{2}1} &= \frac{2u}{1+|u|^2} \\
\Gamma^2_{\bar{2}1} &= -2T_4 \\
\Gamma^1_{\bar{1}2} &= \Gamma^1_{\bar{2}2} \;=\; \Gamma^2_{\bar{2}2} \;=\; 0 \\
\Gamma^1_{01} &= -\overline{a}_V + \frac{2\,u\,\overline{T}_4}{1+|u|^2} \\
\Gamma^2_{01} &= -\frac{|w|^2}{(1+|u|^2)^2} + \overline{b}_V\,w - D_{\overline{\mu}} T_4 - \overline{w}\,\frac{\partial T_4}{\partial \overline{u}} \\
\Gamma^1_{02} &= \frac{1}{(1+|u|^2)^2} \\
\Gamma^2_{02} &= -\frac{\partial T_4}{\partial u}
\end{aligned}
$$

Note that we could express $a_M$, $T_4$ and $a_V$ in terms of the Christoffel symbols,

$$a_M = \Gamma^2_{12}, \quad T_4 = -\frac{1}{2}\Gamma^2_{\bar{2}1} \quad \text{and} \quad a_V = \frac{1}{2}\,\Gamma^2_{\bar{2}1}\,\Gamma^2_{22} - \Gamma^{\bar{1}}_{0\bar{1}}.$$

Let $R$ be the Riemann tensor of the Tanaka-Webster connection on $N$. Let

$$R_m{}^n{}_{k\bar{l}} \;=\; \theta^n\big(R(X_k, \overline{X}_l)X_m\big)$$

according to (1.2). Also, we let $ric$ be the Ricci tensor and $\rho$ be the scalar curvature of $\nabla$ defined by (1.3). Extending the result in Proposition 4.4, we may obtain a general formula for the scalar curvature $\rho$.

**Proposition 5.6.** *For $R_{m\bar{n}} = ric(X_m, \overline{X}_n)$, we have the following.*

$$\begin{aligned}
R_{1\bar{1}} \;=\;& -D_{\overline{\mu}}D_u w - D_\mu D_{\overline{u}}\overline{w} + \frac{4\overline{u}}{1+|u|^2}D_{\overline{\mu}}w + \frac{4u}{1+|u|^2}D_\mu\overline{w} + \frac{4|w|^2}{(1+|u|^2)^2} \\
&+(1+|u|^2)^2\big(w_u\, a_{V,u} + \overline{w}_{\overline{u}}\,\overline{a}_{V,\overline{u}}\big) - 4(1+|u|^2)\big(\overline{u}\,w\,a_{V,u} - u\,\overline{w}\,\overline{a}_{V,\overline{u}}\big) \\
&+12i(1+|u|^2)^2(\overline{a}_V - a_V)\big(G^3_{12} + G^1_{23} + G^2_{31}\big) - 8(1+|u|^2)^2\big(G^3_{12} + G^1_{23} + G^2_{31}\big)^2 \\
&+(1+|u|^2)^2\Big(4(a_V - \overline{a}_V)^2 + D_\mu D_u a_V + D_{\overline{\mu}}D_{\overline{u}}a_V + D_\mu D_u\overline{a}_V + D_{\overline{\mu}}D_{\overline{u}}\overline{a}_V\Big) \\
&-(1+|u|^2)^4\Big(a_{V,u}\,a_{V,\overline{u}} + \overline{a}_{V,u}\,\overline{a}_{V,\overline{u}} - 2|a_{V,u}|^2\Big) \\
R_{1\bar{2}} \;=\;& -\overline{\phi_w} - 4i(G^3_{12} + G^1_{23} + G^2_{31}) + 2a_V - 6\overline{a}_V \\
R_{2\bar{1}} \;=\;& -\phi_w + 4i(G^3_{12} + G^1_{23} + G^2_{31}) - 6a_V + 2\overline{a}_V \\
R_{2\bar{2}} \;=\;& \frac{4}{(1+|u|^2)^2}
\end{aligned}$$

The term $\phi_w$ above is defined by (4.4). The scalar curvature $\rho$ can be found by (1.3).

$$\rho \;=\; h^{\bar{1}1}R_{1\bar{1}} + h^{\bar{1}2}R_{2\bar{1}} + h^{\bar{2}1}R_{1\bar{2}} + h^{\bar{2}2}R_{2\bar{2}} \;=\; i\big(R_{1\bar{2}} - R_{2\bar{1}}\big) - h_{1\bar{1}}R_{2\bar{2}}$$

In terms of our notation,

$$\begin{aligned}\rho &= i(\phi_w - \overline{\phi_w}) + \Big(4(G^3_{12} + G^1_{23} + G^2_{31}) + 2i\, a_V - 6i\, \overline{a}_V\Big) \\ &\quad -\Big(-4(G^3_{12} + G^1_{23} + G^2_{31}) - 6i\, a_V + 2i\, \overline{a}_V\Big) - h_{1\bar{1}} R_{2\bar{2}} \\ &= i(\phi_w - \overline{\phi_w}) + 8\big(G^3_{12} + G^1_{23} + G^2_{31}\big) + 8i(a_V - \overline{a}_V) - \frac{4h_{1\bar{1}}}{(1+|u|^2)^2}.\end{aligned}$$

For the last term above, we can substitute $a_V$ and its derivatives for $h_{1\bar{1}}$.

$$\begin{aligned}h_{1\bar{1}} &= i(1+|u|^2)^2\, a_V + \frac{i(1+|u|^2)^4}{2}\, a_{V,u\overline{u}} \\ &= i(1+|u|^2)^2\, a_V + \frac{i(1+|u|^2)^4}{2}\Big[\frac{2}{(1+|u|^2)^2}\Big(2i(G^3_{12} + G^1_{23} + G^2_{31}) + \overline{a}_V - 2a_V\Big)\Big]\end{aligned}$$

Therefore,

$$-\frac{4h_{1\bar{1}}}{(1+|u|^2)^2} = 8\big(G^3_{12} + G^1_{23} + G^2_{31}\big) - 4i(\overline{a}_V - a_V).$$

It gives

$$\rho = i(\phi_w - \overline{\phi_w}) + 16\big(G^3_{12} + G^1_{23} + G^2_{31}\big) + 12i(a_V - \overline{a}_V). \tag{5.15}$$

**Theorem 5.7.** *In the local model of $\mathfrak{D}(w)$ on $N$ (5.5), of which the antiholomorphic bundle is spanned by $\overline{X}_2 = \dfrac{\partial}{\partial \overline{u}}$ and*

$$\overline{X}_1 = (u^2-1)e_1 + 2ue_2 + i(u^2+1)e_3 + w(x,u)\frac{\partial}{\partial u},$$

*the scalar curvature of the Tanaka-Webster connection associated with $\alpha$ (5.3) is given by*

$$\rho = i\big(\phi_w - \overline{(\phi_w)}\big) - i\big(\phi_f - \overline{(\phi_f)}\big). \tag{5.16}$$

The function $f$ above is defined by (5.4). The *actual* torsion function of $\mathfrak{D}(w)$ is $(w-f)$. We could rewrite $\overline{X}_1$ as

$$\overline{X}_1 = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 - \frac{i}{2}\mu_m\Big(G^2_{m1}\mu_3 + G^3_{m2}\mu_1 + G^1_{m3}\mu_2\Big)\frac{\partial}{\partial u} + (w-f)\frac{\partial}{\partial u}.$$

Note that $\phi_{w-f} = \phi_w - \phi_f$ Theorem 5.7 states that $\rho$ depends on the $\phi$-value of the torsion function $(w-f)$ and is regardless of the geometry of $M$.

*Proof.* We have to show that (5.15) implies (5.16). By comparison, it suffices to show

$$\phi_f \;=\; 8i\big(G^3_{12}+G^1_{23}+G^2_{31}\big)-6a_V+6\overline{a}_V.$$

We try to express $f$, $f_u$ and $f_{uu}$ in terms of lower derivatives of $a_V$, which are $a_{V,u}$, $a_{V,\overline{u}}$, $a_{V,uu}$ and their complex conjugates. On the left hand side,

$$\phi_f \;=\; f_{uu}-\frac{6\,\overline{u}\,f_u}{1+|u|^2}+\frac{12\,\overline{u}^2\,f}{(1+|u|^2)^2}.$$

We start from simplifying $f$.

$$\begin{aligned}
f \;&=\; -u\,(1+|u|^2)^3\,a_{V,\overline{u}}-\frac{(1+|u|^2)^4}{2}\,a_{V,\overline{u}\,\overline{u}}\\
&=\; -u\,(1+|u|^2)^3\,a_{V,\overline{u}}-\frac{(1+|u|^2)^4}{2}\Big[\frac{-2u}{1+|u|^2}\big(a_{V,\overline{u}}+\overline{a}_{V,\overline{u}}\big)-\overline{a}_{V,\overline{u}\,\overline{u}}\Big]\\
&=\; u(1+|u|^2)^3\,\overline{a}_{V,\overline{u}}+\frac{(1+|u|^2)^4}{2}\,\overline{a}_{V,\overline{u}\,\overline{u}}
\end{aligned}$$

The $u$-derivative of $f$ is

$$\begin{aligned}
\frac{\partial f}{\partial u} \;&=\; \Big((1+|u|^2)^3+3u\,\overline{u}\,(1+|u|^2)^2\Big)\,\overline{a}_{V,\overline{u}}+u\,(1+|u|^2)^3\,\overline{a}_{V,u\overline{u}}\\
&\quad +2\overline{u}\,(1+|u|^2)^3\,\overline{a}_{V,\overline{u}\,\overline{u}}+\frac{(1+|u|^2)^4}{2}\,\overline{a}_{V,u\overline{u}\,\overline{u}}\\
&=\; (1+|u|^2)^2(1+4|u|^2)\,\overline{a}_{V,\overline{u}}+2\overline{u}\,(1+|u|^2)^3\,\overline{a}_{V,\overline{u}\,\overline{u}}\\
&\quad +u\,(1+|u|^2)^3\cdot\frac{2}{(1+|u|^2)^2}\Big(-2i\big(G^3_{12}+G^1_{23}+G^2_{31}\big)+a_V-2\overline{a}_V\Big)\\
&\quad +\frac{(1+|u|^2)^4}{2}\cdot\frac{\partial}{\partial\overline{u}}\Big[\frac{2}{(1+|u|^2)^2}\big(-2i(G^3_{12}+G^1_{23}+G^2_{31})+a_V-2\overline{a}_V\big)\Big]\\
&=\; (1+|u|^2)^2\,(1+4|u|^2)\,\overline{a}_{V,\overline{u}}+2\overline{u}\,(1+|u|^2)^3\,\overline{a}_{V,\overline{u}\,\overline{u}}\\
&\quad +2u\,(1+|u|^2)\,\big(-2i(G^3_{12}+G^1_{23}+G^2_{31})+a_V-2\overline{a}_V\big)\\
&\quad +\frac{(1+|u|^2)^4}{2}\left[\begin{array}{l}\dfrac{-4u}{(1+|u|^2)^3}\big(-2i(G^3_{12}+G^1_{23}+G^2_{31})+a_V-2\overline{a}_V\big)\\ +\dfrac{2}{(1+|u|^2)^2}\Big(a_{V,\overline{u}}-2\overline{a}_{V,\overline{u}}\Big)\end{array}\right]\\
&=\; (1+|u|^2)^2(1+4|u|^2)\overline{a}_{V,\overline{u}}+2\overline{u}(1+|u|^2)^3\overline{a}_{V,\overline{u}\,\overline{u}}+(1+|u|^2)^2a_{V,\overline{u}}-2(1+|u|^2)^2\overline{a}_{V,\overline{u}}.
\end{aligned}$$

Therefore,

$$\frac{\partial f}{\partial u} = (1+|u|^2)^2\,(4|u|^2-1)\,\overline{a}_{V,\overline{u}} + (1+|u|^2)^2\,a_{V,\overline{u}} + 2\overline{u}\,(1+|u|^2)^3\,\overline{a}_{V,\overline{u}\,\overline{u}}.$$

The second derivative of $f$ by $u$ is

$$\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \Big(2\overline{u}(1+|u|^2)(4|u|^2-1) + 4\overline{u}(1+|u|^2)^2\Big)\overline{a}_{V,\overline{u}} + (1+|u|^2)^2(4|u|^2-1)\overline{a}_{V,u\overline{u}} \\
&\quad +2\overline{u}(1+|u|^2)a_{V,\overline{u}} + (1+|u|^2)^2 a_{V,u\overline{u}} + 6\overline{u}^2(1+|u|^2)^2\overline{a}_{V,\overline{u}\,\overline{u}} + 2\overline{u}(1+|u|^2)^3\overline{a}_{V,u\overline{u}\,\overline{u}} \\
&= 2\overline{u}(1+|u|^2)(6|u|^2+1)\overline{a}_{V,\overline{u}} + 2\overline{u}(1+|u|^2)a_{V,\overline{u}} + 6\overline{u}^2(1+|u|^2)^2\overline{a}_{V,\overline{u}\,\overline{u}} \\
&\quad +(1+|u|^2)^2(4|u|^2-1)\cdot\frac{2}{(1+|u|^2)^2}\Big[-2i(G^3_{12}+G^1_{23}+G^2_{31}) + a_V - 2\overline{a}_V\Big] \\
&\quad +(1+|u|^2)^2\cdot\frac{2}{(1+|u|^2)^2}\Big[2i(G^3_{12}+G^1_{23}+G^2_{31}) + \overline{a}_V - 2a_V\Big] \\
&\quad +2\overline{u}(1+|u|^2)^3\cdot\frac{\partial}{\partial\overline{u}}\Big[\frac{2}{(1+|u|^2)^2}\Big(-2i(G^3_{12}+G^1_{23}+G^2_{31}) + a_V - 2\overline{a}_V\Big)\Big] \\
&= 2\overline{u}(1+|u|^2)(6|u|^2+1)\overline{a}_{V,\overline{u}} + 2\overline{u}(1+|u|^2)a_{V,\overline{u}} + 6\overline{u}^2(1+|u|^2)^2\overline{a}_{V,\overline{u}\,\overline{u}} \\
&\quad +2(4|u|^2-1)\Big(-2i(G^3_{12}+G^1_{23}+G^2_{31}) + a_V - 2\overline{a}_V\Big) \\
&\quad +2\Big(2i(G^3_{12}+G^1_{23}+G^2_{31}) + \overline{a}_V - 2a_V\Big) + 2\overline{u}(1+|u|^2)^3\cdot\frac{2}{(1+|u|^2)^2}\Big(a_{V,\overline{u}} - 2\overline{a}_{V,\overline{u}}\Big) \\
&\quad +2\overline{u}(1+|u|^2)^3\cdot\frac{-4u}{(1+|u|^2)^3}\Big(-2i(G^3_{12}+G^1_{23}+G^2_{31}) + a_V - 2\overline{a}_V\Big) \\
&= (1+|u|^2)\big(12u\overline{u}^2 + 2\overline{u} - 8\overline{u}\big)\overline{a}_{V,\overline{u}} + (1+|u|^2)\big(2\overline{u} + 4\overline{u}\big)a_{V,\overline{u}} + 6\overline{u}^2(1+|u|^2)^2\overline{a}_{V,\overline{u}\,\overline{u}} \\
&\quad +\Big(-4i(4|u|^2-1) + 4i + 16i|u|^2\Big)\big(G^3_{12}+G^1_{23}+G^2_{31}\big) \\
&\quad +\Big((8|u|^2-2) - 4 - 8|u|^2\Big)a_V + \Big(-4(4|u|^2-1) + 2 + 16|u|^2\Big)\overline{a}_V \\
&= (1+|u|^2)(12u\overline{u}^2 - 6\overline{u})\overline{a}_{V,\overline{u}} + 6\overline{u}(1+|u|^2)a_{V,\overline{u}} + 6\overline{u}^2(1+|u|^2)^2\overline{a}_{V,\overline{u}\,\overline{u}} \\
&\quad +8i\big(G^3_{12}+G^1_{23}+G^2_{31}\big) - 6a_V + 6\overline{a}_V.
\end{aligned}$$

Combining what we get on $f$, $f_u$ and $f_{uu}$,

$$\begin{aligned}
&\frac{\partial^2 f}{\partial u^2} - \frac{6\overline{u}}{1+|u|^2}\,\frac{\partial f}{\partial u} + \frac{12\overline{u}^2}{(1+|u|^2)^2}\,f \\
=\;& 8i\big(G^3_{12}+G^1_{23}+G^2_{31}\big) - 6a_V + 6\overline{a}_V \\
&+6\overline{u}\,(1+|u|^2)\,(2|u|^2-1)\,\overline{a}_{V,\overline{u}} + 6\overline{u}\,(1+|u|^2)\,a_{V,\overline{u}} + 6\overline{u}^2\,(1+|u|^2)^2\,\overline{a}_{V,\overline{u}\,\overline{u}} \\
&-\frac{6\overline{u}}{1+|u|^2}\Big[(1+|u|^2)^2\,(4|u|^2-1)\,\overline{a}_{V,\overline{u}} + (1+|u|^2)^2\,a_{V,\overline{u}} + 2\overline{u}\,(1+|u|^2)^3\,\overline{a}_{V,\overline{u}\,\overline{u}}\Big] \\
&+\frac{12\overline{u}^2}{(1+|u|^2)^2}\Big[u\,(1+|u|^2)^3\,\overline{a}_{V,\overline{u}} + \frac{(1+|u|^2)^4}{2}\,\overline{a}_{V,\overline{u}\,\overline{u}}\Big] \\
=\;& 8i\big(G^3_{12}+G^1_{23}+G^2_{31}\big) - 6a_V + 6\overline{a}_V \\
&+6\overline{u}\,(1+|u|^2)\Big((2|u|^2-1)-(4|u|^2-1)+2|u|^2\Big)\,\overline{a}_{V,\overline{u}} \\
&+\Big(6\overline{u}\,(1+|u|^2)-6\overline{u}\,(1+|u|^2)\Big)\,a_{V,\overline{u}} + (1+|u|^2)^2\,\Big(6\overline{u}^2-12\overline{u}^2+6\overline{u}^2\Big)\,\overline{a}_{V,\overline{u}\,\overline{u}} \\
=\;& 8i\big(G^3_{12}+G^1_{23}+G^2_{31}\big) - 6a_V + 6\overline{a}_V.
\end{aligned}$$

Therefore, $\phi_f = 8i\big(G^3_{12}+G^1_{23}+G^2_{31}\big) - 6a_V + 6\overline{a}_V$. □

As a remark, if the contact form $\alpha$ is replaced by $\tilde{\alpha} = e^{2\lambda}\alpha$ for $\lambda \in C^\infty(N,\mathbb{R})$, then the scalar curvature associated with $\tilde{\alpha}$ could be found by (4.5):

$$\tilde{\rho} \;=\; e^{-2\lambda}\Big(\rho - 6\,\Delta_b\lambda - 24\,\lambda_\gamma\,\lambda_{\bar{\delta}}\,h^{\bar{\delta}\gamma}\Big).$$

We assume $\lambda$ is independent of $u$ or $\overline{u}$. So $\lambda_2 = D_{X_2}\lambda = 0$ and $\lambda_{\bar{2}} = D_{\overline{X}_2}\lambda = 0$. Also,

$$\lambda_{1\bar{2}} = 2D_{\frac{v}{|v|}}\lambda, \quad \lambda_{2\bar{1}} = 0 \quad \text{and} \quad \lambda_{2\bar{2}} = 0.$$

$$\Longrightarrow\;\; \Delta_b\lambda \;=\; \lambda_{\alpha\bar{\beta}}\,h^{\bar{\beta}\alpha} + \lambda_{\bar{\alpha}\beta}\,h^{\bar{\alpha}\beta} \;=\; i\,\lambda_{1\bar{2}} + \overline{(i\,\lambda_{1\bar{2}})} \;=\; 0.$$

Therefore, $\tilde{\rho} = e^{-2\lambda}\rho$ when $\lambda$ is in fact a function on $M$.

## 5.4 THE CHERN-MOSER CURVATURE TENSOR

Let $C$ be the Chern-Moser curvature tensor of the CR structure $\mathfrak{D}(w)$ on $N$. According to (1.4) and (1.5), we define

$$C_m{}^n{}_{k\bar{l}} = \theta^n\big(C(X_k, \overline{X}_l)X_m\big) \quad \text{and} \quad C_{m\bar{n}k\bar{l}} = g\big(C(X_k, \overline{X}_l)X_m, \overline{X}_n\big). \tag{5.17}$$

Right before Theorem 4.7, we state that the Chern tensor field is part of the Weyl tensor of the Fefferman metric in general. It means that $C$ would follow the same symmetries with the Weyl tensor. We summarize them in Proposition 5.8.

**Proposition 5.8.**

*(1)* $C_{m\bar{n}k\bar{l}} = -\dfrac{\rho}{6}$ *for* $C_{1\bar{1}2\bar{2}}$, $C_{1\bar{2}1\bar{2}}$, $C_{1\bar{2}2\bar{1}}$, $C_{2\bar{1}1\bar{2}}$, $C_{2\bar{1}2\bar{1}}$ *and* $C_{2\bar{2}1\bar{1}}$.

*(2)* $C_{m\bar{n}k\bar{l}} = 0$ *for* $C_{1\bar{2}2\bar{2}}$, $C_{2\bar{1}2\bar{2}}$, $C_{2\bar{2}1\bar{2}}$, $C_{2\bar{2}2\bar{1}}$ *and* $C_{2\bar{2}2\bar{2}}$.

*(3)* $C_{1\bar{1}1\bar{2}} = C_{1\bar{2}1\bar{1}} = \overline{C_{1\bar{1}2\bar{1}}} = \overline{C_{2\bar{1}1\bar{1}}}$.

*(4) The coefficient* $C_{1\bar{1}1\bar{1}}$ *is real-valued.*

In the item (3), the term $C_{1\bar{1}2\bar{1}}$ is given by

$$\begin{aligned}
C_{1\bar{1}2\bar{1}} \;=\; & -\frac{i}{2}\, D_{\bar{\mu}} D_u w + \frac{i}{2}\, D_\mu D_{\bar{u}} \overline{w} + \frac{2\, i\, \overline{u}}{1+|u|^2}\, D_{\bar{\mu}} w - \frac{2\, i\, u}{1+|u|^2}\, D_\mu \overline{w} \\
& + (1+|u|^2)^2 \left(2(G^3_{12} + G^1_{23} + G^2_{31}) + i a_V - i\overline{a}_V\right)\Big(-\frac{1}{3}\phi_w - \frac{1}{6}\overline{(\phi_w)}\Big) \\
& + \frac{i}{2}(1+|u|^2)^2 \Big(w_u\, a_{V,u} - \overline{w}_{\overline{u}}\, \overline{a}_{V,\overline{u}} + 2\, w\, a_{V,uu} - 2\, \overline{w}\, \overline{a}_{V,\overline{u}\,\overline{u}}\Big) \\
& + \frac{i}{2}(1+|u|^2)^2 \Big(D_\mu D_u\, a_V + D_{\overline{\mu}} D_{\overline{u}} a_V - D_{\overline{\mu}} D_{\overline{u}} \overline{a}_V - D_\mu D_u \overline{a}_V\Big) \\
& - \frac{i}{2}(1+|u|^2)^4 \Big(a_{V,u}\, a_{V,\overline{u}} - \overline{a}_{V,u}\, \overline{a}_{V,\overline{u}}\Big) + \frac{8i}{3}(1+|u|^2)^2 \left(G^3_{12} + G^1_{23} + G^2_{31}\right)^2 \\
& + \frac{10}{3}(1+|u|^2)^2\, (\overline{a}_V - a_V)\big(G^3_{12} + G^1_{23} + G^2_{31}\big) - i(1+|u|^2)^2\, (a_V - \overline{a}_V)^2.
\end{aligned}$$

In the item (4), before the result '$C = \mathcal{W}$' is proven, it is not obvious that $C_{1\bar{1}1\bar{1}}$ is real-valued. We may justify this fact directly. In the following, we let $\theta = G^3_{12} + G^1_{23} + G^2_{31}$ and $y = 1 + |u|^2$. $C_{1\bar{1}1\bar{1}}$ can be broken down to two terms:

$$C_{1\bar{1}1\bar{1}} = \mathrm{Term}_1 + \mathrm{Term}_2.$$

$\mathrm{Term}_1$ contains all terms involving $w$ and its derivatives, and $\mathrm{Term}_2$ contains the rest.

$$\begin{aligned}
\mathrm{Term}_1 = \; & -i\Big(D_{\bar{\mu}}D_{\bar{\mu}}w + w_u\, D_\mu\overline{w} + 2\,\overline{w}\, D_{\frac{v}{|v|}}w + \frac{4\,u\,\overline{w}}{y}D_{\bar{\mu}}w\Big) \\
& +i\Big(D_\mu D_\mu\overline{w} + \overline{w}_{\overline{u}}\, D_{\bar{\mu}}w + 2\,w\, D_{\frac{v}{|v|}}\overline{w} + \frac{4\,\overline{u}\,w}{y}D_\mu\overline{w}\Big) \\
& +\frac{iy^2}{6}\big(2\theta + ia_V - i\overline{a}_V\big)^2\Big(\phi_w - \overline{(\phi_w)}\Big) \\
& -y^2\big(2\theta + ia_V - i\overline{a}_V\big)\big(D_{\bar{\mu}}D_u\, w + D_\mu D_{\overline{u}}\overline{w}\big) \\
& +iy^2\Big[D_{\bar{\mu}}w\Big(4a_{V,u} - \overline{a}_{V,u}\Big) - D_\mu\overline{w}\Big(4\overline{a}_{V,\overline{u}} - a_{V,\overline{u}}\Big)\Big] \\
& +4y\big(2\theta + ia_V - i\overline{a}_V\big)\Big(\overline{u}\, D_{\bar{\mu}}w + u\, D_\mu\overline{w} - u\,\overline{w}\, w_u - \overline{u}\, w\, \overline{w}_{\overline{u}} + 4|w|^2\Big) \\
& +y^2\big(2\theta + ia_V - i\overline{a}_V\big)|w_u|^2 \\
& +iy^2\Big[-\overline{w}\, w_u\Big(a_{V,\overline{u}} - 2\overline{a}_{V,\overline{u}}\Big) + w\,\overline{w}_{\overline{u}}\Big(\overline{a}_{V,u} - 2a_{V,u}\Big)\Big] \\
& -y^2\Big[w_u\Big(2D_{\bar{\mu}}\theta + iD_{\bar{\mu}}a_V - iD_{\bar{\mu}}\overline{a}_V\Big) + \overline{w}_{\overline{u}}\Big(2D_\mu\theta + iD_\mu a_V - iD_\mu\overline{a}_V\Big)\Big] \\
& +y^4\big(2\theta + ia_V - i\overline{a}_V\big)\Big(2\,w_u\, a_{V,u} + w\, a_{V,uu} + 2\,\overline{w}_{\overline{u}}\,\overline{a}_{V,\overline{u}} + \overline{w}\,\overline{a}_{V,\overline{u}\,\overline{u}}\Big) \\
& +y\,|w|^2\Big(8i\,u\, a_{V,u} - 8i\,\overline{u}\,\overline{a}_{V,\overline{u}} + 4i\,\overline{u}\, a_{V,\overline{u}} - 4i\,u\,\overline{a}_{V,u}\Big) \\
& +iy^2\Big[w\big(2D_{\bar{\mu}}D_u a_V - D_{\bar{\mu}}D_u\overline{a}_V\big) - \overline{w}\big(2D_\mu D_{\overline{u}}\overline{a}_V - D_\mu D_{\overline{u}}a_V\big)\Big] \\
& +y\Big[4\,\overline{u}\, w\big(2D_{\bar{\mu}}\theta + iD_{\bar{\mu}}a_V - iD_{\bar{\mu}}\overline{a}_V\big) + 4\,u\,\overline{w}\big(2D_\mu\theta + iD_\mu a_V - iD_\mu\overline{a}_V\big)\Big] \\
& -y^3\big(2\theta + ia_V - i\overline{a}_V\big)\Big(6\,\overline{u}\, w\, a_{V,u} + 6\,u\,\overline{w}\,\overline{a}_{V,\overline{u}}\Big) \\
& +2iy^4\Big[w\Big(a_{V,u}\,\overline{a}_{V,u} - 2\big(a_{V,u}\big)^2\Big) - \overline{w}\Big(a_{V,\overline{u}}\,\overline{a}_{V,\overline{u}} - 2\big(\overline{a}_{V,\overline{u}}\big)^2\Big)\Big] \\
& -|w|^2\Big(40\theta + 24ia_V - 24i\overline{a}_V\Big)
\end{aligned}$$

$\text{Term}_1$ is real-valued by observation. Moreover, we have

$$\begin{aligned}
\text{Term}_2 \;=\;& (1+|u|^2)^2\Big(2D_{\overline{\mu}}D_\mu\theta + iD_{\overline{\mu}}D_\mu a_V - iD_\mu D_{\overline{\mu}}\overline{a}_V\Big)\\
&-(1+|u|^2)^4\big(2\theta + ia_V - i\overline{a}_V\big)\Big(4iD_{\frac{v}{|v|}}\theta + D_{\overline{\mu}}D_{\overline{u}}a_V - 2D_{\frac{v}{|v|}}a_V\Big)\\
&-(1+|u|^2)^4\Big(4(D_\mu\theta)\,a_{V,u} + 2(D_{\overline{\mu}}\theta)\,\overline{a}_{V,\overline{u}}\Big)\\
&-i(1+|u|^2)^4\Big[\big(2D_\mu a_V - D_\mu\overline{a}_V\big)a_{V,u} - \big(2D_{\overline{\mu}}\overline{a}_V - D_{\overline{\mu}}a_V\big)\overline{a}_{V,\overline{u}}\Big]\\
&+(1+|u|^2)^6\big(2\theta + ia_V - i\overline{a}_V\big)\Big(a_{V,u}\,a_{V,\overline{u}} + a_{V,u}\,\overline{a}_{V,\overline{u}}\Big)\\
&+(1+|u|^2)^4\left[\begin{array}{l} -\dfrac{16}{3}\theta^3 - \dfrac{8i}{3}\theta^2\big(5a_V - 2\overline{a}_V\big)\\ +\dfrac{4}{3}\theta\big(7a_V - \overline{a}_V\big)\big(a_V - \overline{a}_V\big) + 2ia_V\big(a_V - \overline{a}_V\big)^2\end{array}\right].
\end{aligned}$$

$\text{Term}_2$ can be divided into three parts, each of them being a real component. The first component is called $\sigma_1$.

$$\begin{aligned}
\sigma_1 =\;& y^2\big(iD_{\overline{\mu}}D_\mu a_V - iD_\mu D_{\overline{\mu}}\overline{a}_V\big) - y^4\Big(2\,(D_\mu\theta)\,a_{V,u} + 2\,(D_{\overline{\mu}}\theta)\,\overline{a}_{V,\overline{u}}\Big)\\
&-iy^4\Big[\big(2D_\mu a_V - D_\mu\overline{a}_V\big)a_{V,u} - \big(2D_{\overline{\mu}}\overline{a}_V - D_{\overline{\mu}}a_V\big)\overline{a}_{V,\overline{u}}\Big]\\
&+y^6\big(2\theta + ia_V - i\overline{a}_V\big)a_{V,u}\,\overline{a}_{V,\overline{u}}\\
&+y^4\Big(-\frac{16}{3}\theta^3 - \frac{16i}{3}\theta^2\big(a_V - \overline{a}_V\big) + \frac{4}{3}\theta\big(a_V - \overline{a}_V\big)^2\Big).
\end{aligned}$$

$\sigma_1$ is naturally a real-valued term. Next, we let

$$\begin{aligned}
\sigma_2 =\;& \big(2\theta + ia_V - i\overline{a}_V\big)\Big(y^6\,a_{V,u}\,a_{V,\overline{u}} - y^4\big(2iD_{\frac{v}{|v|}}\theta + D_{\overline{\mu}}D_{\overline{u}}a_V - 2D_{\frac{v}{|v|}}a_V\big)\Big)\\
&+y^4\Big(-8i\,\theta^2\,a_V + 8\,\theta\,a_V\,(a_V - \overline{a}_V) + 2i\,a_V\,\big(a_V - \overline{a}_V\big)^2\Big), \qquad .\\
\sigma_3 =\;& 2y^2\,(D_{\overline{\mu}}D_\mu\theta) - y^4\big(2\theta + ia_V - i\overline{a}_V\big)\Big(2iD_{\frac{v}{|v|}}\theta\Big) - 2y^4\,(D_\mu\theta)\,a_{V,u}.
\end{aligned}$$

The last part of $\sigma_2$ can be simplified as

$$\begin{aligned}
&-8i\,\theta^2\,a_V + 8\,\theta\,a_V\,(a_V - \overline{a}_V) + 2i\,a_V\,\big(a_V - \overline{a}_V\big)^2\\
=\;& (2\theta + ia_V - i\overline{a}_V)\big(-2\overline{a}_V^2 - 4i\,\theta\,\overline{a}_V\big) - 8i\,\theta^2\,(a_V - \overline{a}_V) + 8\,\theta\,(a_V^2 + \overline{a}_V^2) - 12\,\theta\,|a_V|^2\\
&+2i\,(a_V - \overline{a}_V)\,\big(a_V^2 + \overline{a}_V^2 - |a_V|^2\big).
\end{aligned}$$

So, we have

$$\sigma_2 = y^4(2\theta + ia_V - i\overline{a}_V)\Big(y^2\, a_{V,u}\, a_{V,\overline{u}} - D_{\overline{\mu}}D_{\overline{u}}a_V + 2D_{\frac{v}{|v|}}a_V - 2iD_{\frac{v}{|v|}}\theta - 2\overline{a}_V^2 - 4i\theta\,\overline{a}_V\Big)$$

$$+y^4\Big(-8i\theta^2(a_V - \overline{a}_V) + 8\theta(a_V^2 + \overline{a}_V^2) - 12\theta\,|a_V|^2 + 2i(a_V - \overline{a}_V)\big(a_V^2 + \overline{a}_V^2 - |a_V|^2\big)\Big).$$

We may show that the expression

$$D_{\overline{\mu}}D_{\overline{u}}a_V - y^2\, a_{V,u}\, a_{V,\overline{u}} - 2D_{\frac{v}{|v|}}a_V + 2i\, D_{\frac{v}{|v|}}\theta + 4i\,\theta\,\overline{a}_V + 2\overline{a}_V^2 \tag{5.18}$$

is of real values later in this chapter. So $\sigma_2$ is real-valued. We then consider the term $\sigma_3$ in the way that

$$\sigma_3 \;=\; \mathrm{Re}\big(2y^2\, D_{\overline{\mu}}D_\mu\theta\big) + i\;\mathrm{Im}\big(2y^2\, D_{\overline{\mu}}D_\mu\theta\big) - 2i\, y^4\,\big(2\theta + ia_V - i\overline{a}_V\big)\, D_{\frac{v}{|v|}}\theta - 2\, y^4\,(D_\mu\theta)\, a_{V,u}.$$

The imaginary part of $2y^2 D_{\overline{\mu}}D_\mu\theta$ multiplied by $i$ is:

$$\begin{aligned} i\,\mathrm{Im}\big(2y^2\, D_{\overline{\mu}}D_\mu\theta\big) &= y^2\big(D_{\overline{\mu}}D_\mu\theta - D_\mu D_{\overline{\mu}}\theta\big) \;=\; y^2\, d\theta\big([\overline{\mu},\mu]\big) \\ &= y^2\, d\theta\Big(-a_M\,\mu + \overline{a}_M\,\overline{\mu} - 2i\, h_{1\bar{1}}\,\frac{v}{|v|}\Big) \\ &= -y^2\, a_M\, D_\mu\theta + y^2\,\overline{a}_M\, D_{\overline{\mu}}\theta - 2i\, y^2\, h_{1\bar{1}}\, D_{\frac{v}{|v|}}\theta \\ &= y^4\, a_{V,u}\, D_\mu\theta - y^4\,\overline{a}_{V,\overline{u}}\, D_{\overline{\mu}}\theta + 2i\, y^4\,\big(2\theta + ia_V - i\overline{a}_V\big)\, D_{\frac{v}{|v|}}\theta. \end{aligned}$$

Therefore, $\sigma_3 = \mathrm{Re}\big(2y^2\, D_{\overline{\mu}}D_\mu\theta\big) - y^4\Big(a_{V,u}\, D_\mu\theta + \overline{a}_{V,\overline{u}}\, D_{\overline{\mu}}\theta\Big)$, which is also of real values.

Our discussion above justifies that $C_{1\bar{1}1\bar{1}}$ is real-valued.

## 5.5 FEFFERMAN METRIC IN THE GENERAL SETTING

Let $F$ be the Fefferman metric of the CR structure $\mathcal{D}(w)$ on $C(N)$. The connection forms $\omega_1^1$ and $\omega_2^2$ of the Tanaka-Webster connection are given by:

$$\begin{aligned}
\omega_1^1 &= \Gamma_{11}^1\,\theta^1 + \Gamma_{\bar{1}1}^1\,\theta^{\bar{1}} + \Gamma_{\bar{2}1}^1\,\theta^{\bar{2}} + \Gamma_{01}^1\,\alpha \\
&= \Big(\overline{w}_{\overline{u}} - \frac{2\,u\,\overline{w}}{1+|u|^2} - 2\overline{T}_4\Big)\,\theta^1 - \overline{a}_M\,\theta^{\bar{1}} + \Big(\frac{2u}{1+|u|^2}\Big)\,\theta^{\bar{2}} + \Big(-\overline{a}_V + \frac{2\,u\,\overline{T}_4}{1+|u|^2}\Big)\,\alpha;
\end{aligned}$$

$$\begin{aligned}
\omega_2^2 &= \Gamma_{12}^2\,\theta^1 + \Gamma_{22}^2\,\theta^2 + \Gamma_{\bar{1}2}^2\,\theta^{\bar{1}} + \Gamma_{02}^2\,\alpha \\
&= a_M\,\theta^1 - \frac{2\overline{u}}{1+|u|^2}\,\theta^2 + \Big(\frac{2\,\overline{u}\,w}{1+|u|^2} + 2T_4 - w_u\Big)\,\theta^{\bar{1}} - \frac{\partial T_4}{\partial u}\,\alpha.
\end{aligned}$$

By (1.7), we have

$$\begin{aligned}
F \;=\; & 2\,h_{1\bar{1}}\,\theta^1\odot\theta^{\bar{1}} + 2\,i\,\theta^2\odot\theta^{\bar{1}} - 2\,i\,\theta^1\odot\theta^{\bar{2}} + \frac{1}{2}\,\alpha\odot d\gamma \\
& + \Big[-\frac{\rho}{24} + \frac{i}{2}\Big(-\overline{a}_V + \frac{2\,u\,\overline{T}_4}{1+|u|^2} - \frac{\partial T_4}{\partial u}\Big)\Big]\,\alpha\odot\alpha - \frac{i\,\overline{u}}{1+|u|^2}\,\alpha\odot\theta^2 + \frac{i\,u}{1+|u|^2}\,\alpha\odot\theta^{\bar{2}} \\
& + \frac{i}{2}\Big(\overline{w}_{\overline{u}} - \frac{2\,u\,\overline{w}}{1+|u|^2} - 2\overline{T}_4 + a_M\Big)\,\alpha\odot\theta^1 - \frac{i}{2}\Big(w_u - \frac{2\,\overline{u}\,w}{1+|u|^2} - 2T_4 + \overline{a}_M\Big)\,\alpha\odot\theta^{\bar{1}}.
\end{aligned}$$

The metric coefficients of $F$ are then found by:

$$\begin{aligned}
F(X_1,\overline{X}_1) &= h_{1\bar{1}} \\
F(X_1,\overline{X}_2) &= -i \\
F(X_1,T) &= \frac{i}{4}\big(\Gamma_{11}^1 + \Gamma_{12}^2\big) \;=\; \frac{i}{4}\Big(\overline{w}_{\overline{u}} - \frac{2\,u\,\overline{w}}{1+|u|^2} - 2\overline{T}_4 + a_M\Big) \\
F(X_2,T) &= \frac{i}{4}\Gamma_{22}^2 \;=\; -\frac{i\,\overline{u}}{2(1+|u|^2)} \\
F(T,T) &= -\frac{\rho}{24} + \frac{i}{2}\big(\Gamma_{01}^1 + \Gamma_{02}^2\big) \;=\; -\frac{\rho}{24} + \frac{i}{2}\Big(-\overline{a}_V + \frac{2\,u\,\overline{T}_4}{1+|u|^2} - \frac{\partial T_4}{\partial u}\Big) \\
F(T,\tfrac{\partial}{\partial\gamma}) &= \frac{1}{4}
\end{aligned}$$

Similar to (4.9), we let $u_1 = X_1,\;\; u_2 = \overline{X}_1,\;\; u_3 = X_2,\;\; u_4 = \overline{X}_2,\;\; u_5 = T$ and $u_6 = \dfrac{\partial}{\partial\gamma}$.

Note that $F$ is of signature $(+++---)$ since the Webster metric $g$ is of signature $(+++--)$. Corresponding to the frame $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ on $C(N)$, we let $F_{ij} = F(u_i, u_j)$ or any $i, j = 1, \cdots, 6$. The matrix representation of $F$ is

$$[F] \;=\; \begin{bmatrix} 0 & h_{1\bar{1}} & 0 & -i & F_{15} & 0 \\ h_{1\bar{1}} & 0 & i & 0 & \overline{F}_{15} & 0 \\ 0 & i & 0 & 0 & F_{35} & 0 \\ -i & 0 & 0 & 0 & \overline{F}_{35} & 0 \\ F_{15} & \overline{F}_{15} & F_{35} & \overline{F}_{35} & F_{55} & 1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \end{bmatrix}.$$

The inverse $F^{-1}$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & i & 0 & -4i\overline{F}_{35} \\ 0 & 0 & -i & 0 & 0 & 4iF_{35} \\ 0 & -i & 0 & -h_{1\bar{1}} & 0 & 4(h_{1\bar{1}}\overline{F}_{35} + i\overline{F}_{15}) \\ i & 0 & -h_{1\bar{1}} & 0 & 0 & 4(h_{1\bar{1}}F_{35} - iF_{15}) \\ 0 & 0 & 0 & 0 & 0 & 4 \\ -4i\overline{F}_{35} & 4iF_{35} & 4(h_{1\bar{1}}\overline{F}_{35} + i\overline{F}_{15}) & 4(h_{1\bar{1}}F_{35} - iF_{15}) & 4 & \begin{pmatrix} -16F_{55} - 32|F_{35}|^2 h_{1\bar{1}} \\ +32i(F_{15}\overline{F}_{35} - F_{35}\overline{F}_{15}) \end{pmatrix} \end{bmatrix}.$$

Explicitly, we have

$$\begin{aligned} 4(h_{1\bar{1}}\overline{F}_{35} + i\overline{F}_{15}) \;&=\; w_u - \frac{2\overline{u}}{1+|u|^2} w - (1+|u|^2)^2\big(a_{V,\overline{u}} + \overline{a}_{V,\overline{u}}\big) \\ &\quad -2iu(1+|u|^2)\Big(2\big(G^3_{12} + G^1_{23} + G^2_{31}\big) + ia_V - i\overline{a}_V\Big), \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} -16F_{55} - 32|F_{35}|^2 h_{1\bar{1}} \\ +32i(F_{15}\overline{F}_{35} - F_{35}\overline{F}_{15}) \end{bmatrix} \;&=\; \frac{4\,i\,\overline{u}\,w_u}{1+|u|^2} - \frac{4\,i\,u\,\overline{w}_{\overline{u}}}{1+|u|^2} - \frac{8\,i\,\overline{u}^2\,w}{(1+|u|^2)^2} + \frac{8\,i\,u^2\,\overline{w}}{(1+|u|^2)^2} + \frac{2}{3}\rho \\ &\quad +4iu(1+|u|^2)\Big(a_{V,u} - \overline{a}_{V,u}\Big) + 4i\overline{u}(1+|u|^2)\big(a_{V,\overline{u}} - \overline{a}_{V,\overline{u}}\big) \\ &\quad +16(|u|^2-1)\big(G^3_{12} + G^1_{23} + G^2_{31}\big) + 8i\big(|u|^2-2\big)(a_V - \overline{a}_V). \end{aligned}$$

Let $\hat{\nabla}$ be the Levi-Civita connection of $F$ and define $\hat{\nabla}_{u_i} u_j = \hat{\Gamma}^k_{ij} u_k$. The Christoffel symbols $\hat{\Gamma}^k_{ij}$ could be found by the Koszul formula (1.8). We also define $\hat{R}$ to be the Riemann tensor of $\hat{\nabla}$ with

$$\hat{R}(u_i, u_j)u_k = \hat{R}^l_{ijk}\, u_l \quad \text{and} \quad \hat{R}_{ijkl} = F\big(\hat{R}(u_i, u_j)u_k, u_l\big).$$

See (1.9) and (1.10). Moreover, *Ric* denotes the Ricci tensor of $\hat{R}$ with $\hat{R}_{ij}$ (1.11), and $S$ denotes the scalar curvature of $\hat{\nabla}$ (1.12). The raise-index of Ric is Ric$^\sharp$ ($\hat{R}^j_i = \hat{R}_{ik}F^{kj}$).

Let $\mathcal{W}$ be the Weyl curvature tensor of $\hat{\nabla}$ on $C(N)$. From (1.15) and (1.16), we define

$$\mathcal{W}(u_i, u_j)u_k \;=\; \mathcal{W}^l_{ijk}\, u_l \quad \text{and} \quad \mathcal{W}_{ijkl} \;=\; \mathcal{W}\big(u_i, u_j, u_k, u_l\big) \;=\; F\big(\mathcal{W}(u_i, u_j)u_k, u_l\big). \tag{5.19}$$

The discussion of the Weyl tensor will be delayed to the next chapter. By Theorem 1.4 (J.M.Lee), the scalar curvature $S$ can be found by

$$S \;=\; \frac{5}{3}\rho \;=\; \frac{5i}{3}\Big(\phi_w - \overline{(\phi_w)}\Big) - \frac{5i}{3}\Big(\phi_f - \overline{(\phi_f)}\Big). \tag{5.20}$$

See Theorem 5.7. We mentioned before that the expression (5.18) is real-valued. Observe that $\hat{R}_{12} = \mathrm{Ric}(X_1, \overline{X}_1)$, is real-valued by default. Using the notation $\theta = G^3_{12} + G^1_{23} + G^2_{31}$ and $y = 1 + |u|^2$, we have

$$\begin{aligned}
\hat{R}_{12} = &-\frac{1}{2}D_{\overline{\mu}}D_u w - \frac{1}{2}D_\mu D_{\overline{u}}\overline{w} + \frac{2\overline{u}}{y}D_{\overline{\mu}}w + \frac{2u}{y}D_\mu\overline{w} - \frac{iy^2}{12}\big(2\theta + ia_V - i\overline{a}_V\big)\big(w_{uu} - \overline{w}_{\overline{u}\,\overline{u}}\big)\\
&+\frac{1}{4}|w_u|^2 - \frac{1}{2y}\big(\overline{u}\,w\,\overline{w}_{\overline{u}} + u\,\overline{w}\,w_u\big) + \frac{y^2}{4}\big(a_{V,u} - \overline{a}_{V,u}\big)w_u + \frac{y^2}{4}\big(\overline{a}_{V,\overline{u}} - a_{V,\overline{u}}\big)\overline{w}_{\overline{u}}\\
&+\frac{iy}{2}\big(2\theta + ia_V - i\overline{a}_V\big)\big(\overline{u}\,w_u - u\,\overline{w}_{\overline{u}}\big) - i\big(2\theta + ia_V - i\overline{a}_V\big)\big(\overline{u}^2\,w - u^2\,\overline{w}\big)\\
&+\overline{u}\,y\Big(-\frac{3}{2}a_{V,u} + \frac{1}{2}\overline{a}_{V,u}\Big)w + u\,y\Big(-\frac{3}{2}\overline{a}_{V,\overline{u}} + \frac{1}{2}a_{V,\overline{u}}\Big)\overline{w} + \frac{y+1}{y^2}|w|^2\\
&+\frac{y^4}{4}\Big(\big|a_{V,\overline{u}}\big|^2 - 3\big|a_{V,u}\big|^2 - 3\,a_{V,u}\,a_{V,\overline{u}} + \overline{a}_{V,\overline{u}}\,\overline{a}_{V,u}\Big)\\
&+y^2\Big(\frac{1}{2}D_\mu D_u a_V + \frac{1}{2}D_{\overline{\mu}}D_{\overline{u}}\overline{a}_V + D_{\overline{\mu}}D_{\overline{u}}a_V - D_{\frac{v}{|v|}}a_V + D_{\frac{v}{|v|}}\overline{a}_V\Big)\\
&+y^2\Big(2iD_{\frac{v}{|v|}}\theta - \frac{20}{3}\theta^2 - \frac{22i}{3}\,\theta\,a_V + \frac{34i}{3}\,\theta\,\overline{a}_V - 6|a_V|^2 + 2a_V^2 + 4\overline{a}_V^2\Big)
\end{aligned}$$

$\hat{R}_{12}$ consists of a real term added by

$$
\begin{aligned}
& -y^4\, a_{V,u}\, a_{V,\overline{u}} + y^2\, D_{\overline{\mu}} D_{\overline{u}} a_V - 2\, y^2\, D_{\frac{v}{|v|}} a_V + y^2 \Big( 2i\, D_{\frac{v}{|v|}} \theta + 4i\, \theta\, \overline{a}_V + 2\overline{a}_V^2 \Big) \\
= \;\; & y^2 \Big( D_{\overline{\mu}} D_{\overline{u}} a_V - y^2\, a_{V,u}\, a_{V,\overline{u}} - 2\, D_{\frac{v}{|v|}} a_V + 2i\, D_{\frac{v}{|v|}} \theta + 4i\, \theta\, \overline{a}_V + 2\overline{a}_V^2 \Big),
\end{aligned}
$$

which must be also a real term. Therefore, the expression (5.18) is real.

## 6.0 WEYL CURVATURE TENSOR (I)

Based on the local model of $\mathfrak{D}(w)$ in (5.5), we are going to analyze the Weyl curvature tensor of the Fefferman metric on $C(N)$. As a remark, the CR distribution $\mathfrak{D}(w)$ is spanned by $\overline{X}_2 = \dfrac{\partial}{\partial \overline{u}}$ and

$$\overline{X}_1 = (u^2 - 1)\, e_1 + 2u\, e_2 + i(u^2 + 1)\, e_3 + w(x, u)\, \frac{\partial}{\partial u}$$

at $(x, u)$ on $N$. The associated contact form of $\mathfrak{D}(w)$ is chosen as $\alpha$ in (5.6).

## 6.1 PROPERTIES OF THE WEYL TENSOR

The Weyl tensor (5.19) obeys the symmetries that

$$\mathcal{W}_{[ij]kl} = \mathcal{W}_{ij[kl]} = 0, \quad \mathcal{W}_{ijkl} = \mathcal{W}_{klij}, \quad \mathcal{W}_{[ijk]l} = \frac{1}{3}\big(\mathcal{W}_{ijkl} + \mathcal{W}_{jkil} + \mathcal{W}_{kijl}\big) = 0.$$

By first two symmetries, we have 120 components of $\mathcal{W}_{ijkl}$'s in the collection

$$\Big\{\mathcal{W}_{ijkl} \mid i < j,\ k < l,\ i \leq k \text{ and } j \leq l\Big\}.$$

These coefficients are divided into different categories according to their properties.

**Class 1: $\mathcal{W}_{ijkl}$'s with at least one of $i$ ,$j$, $k$, $l$ being '6'**

The is a number of 65 $\mathcal{W}_{ijkl}$'s with one or more of the indices being '6'. The Sparling condition [6] [18] implies that $\iota_{\frac{\partial}{\partial \overline{\gamma}}} \mathcal{W} = 0$ and so these coefficients are zero.

**Theorem 6.1.** *$\mathcal{W}\big(\frac{\partial}{\partial\gamma}, Z_1, Z_2, Z_3\big) = 0$ for any tangent vectors $Z_1$, $Z_2$ and $Z_3$ on $C(N)$.*

**Class 2: $\mathcal{W}_{ijkl}$'s evaluated on the Levi distribution**

We consider the values of $\mathcal{W}$ on $\bar{\mathfrak{D}}(w) \oplus \mathfrak{D}(w)$ in Class 2. In other words, all $i$, $j$, $k$ and $l$ are from $\{1, 2, 3, 4\}$. There are 21 components of $\mathcal{W}_{ijkl}$'s in this class:

$$\mathcal{W}_{1212},\ \mathcal{W}_{1213},\ \mathcal{W}_{1214},\ \mathcal{W}_{1223},\ \mathcal{W}_{1224},\ \mathcal{W}_{1234},\ \mathcal{W}_{1313},\ \mathcal{W}_{1314},\ \mathcal{W}_{1323},\ \mathcal{W}_{1324},\ \mathcal{W}_{1334},$$

$$\mathcal{W}_{1414},\ \mathcal{W}_{1423},\ \mathcal{W}_{1424},\ \mathcal{W}_{1434},\ \mathcal{W}_{2323},\ \mathcal{W}_{2324},\ \mathcal{W}_{2334},\ \mathcal{W}_{2424},\ \mathcal{W}_{2434},\ \mathcal{W}_{3434}.$$

As a general fact, the coefficients in Class 2 coincide with that of the Chern curvature tensor in equation (5.17).

**Theorem 6.2.** *For the CR structure $\mathfrak{D}(w)$ on $N$, let $C$ be the Chern curvature tensor (5.17) associated with the contact form $\alpha$. Also let $F$ be the Fefferman metric associated with $\alpha$ on $C(N)$ and $\mathcal{W}$ (5.19) be its Weyl tensor. Then,*

*(1) : $C_{m\bar{n}k\bar{l}} = \mathcal{W}(X_m, \overline{X}_n, X_k, \overline{X}_l)$ for every $m$, $n$, $k$, $l$.*

*(2) : $\mathcal{W}(X_m, X_n, Z_1, Z_2) = 0$ for any $Z_1$, $Z_2$ are tangent vectors on $C(N)$.*

We include a proof of Theorem 6.2 in the Appendix B. The assertion (1) implies

$$\mathcal{W}_{1212} = C_{1\bar{1}1\bar{1}}, \quad \mathcal{W}_{1214} = C_{1\bar{1}1\bar{2}}, \quad \mathcal{W}_{1223} = -C_{1\bar{1}2\bar{1}}, \quad \mathcal{W}_{1234} = C_{1\bar{1}2\bar{2}}, \quad \mathcal{W}_{1414} = C_{1\bar{2}1\bar{2}}.$$

$$\mathcal{W}_{1423} = -C_{1\bar{2}2\bar{1}}, \quad \mathcal{W}_{1434} = C_{1\bar{2}2\bar{2}}, \quad \mathcal{W}_{2323} = C_{2\bar{1}2\bar{1}}, \quad \mathcal{W}_{2334} = -C_{2\bar{1}2\bar{2}}, \quad \mathcal{W}_{3434} = C_{2\bar{2}2\bar{2}}.$$

The above terms can be found by Proposition 5.8. Moreover, we have

$$\mathcal{W}_{1213} = \mathcal{W}_{1224} = \mathcal{W}_{1313} = \mathcal{W}_{1314} = \mathcal{W}_{1323} = \mathcal{W}_{1324} = 0,$$

$$\mathcal{W}_{1334} = \mathcal{W}_{1424} = \mathcal{W}_{2324} = \mathcal{W}_{2424} = \mathcal{W}_{2434} = 0.$$

**Class 3: Other coefficients of the form $\mathcal{W}_{13kl}$, $\mathcal{W}_{24kl}$, $\mathcal{W}_{ij13}$ or $\mathcal{W}_{ij24}$.**

We may use the assertion (2) of Theorem 6.2 to show that these coefficients are zero. They include: $\mathcal{W}_{1315}$, $\mathcal{W}_{1325}$, $\mathcal{W}_{1335}$, $\mathcal{W}_{1345}$, $\mathcal{W}_{1524}$, $\mathcal{W}_{2425}$, $\mathcal{W}_{2435}$, $\mathcal{W}_{2445}$.

## 6.2 COMPONENTS OF THE WEYL TENSOR INVOLVING $T$

**Class 4: $\mathcal{W}_{ijkl}$'s related to the scalar curvature $\rho$**

The components of $\mathcal{W}_{ijkl}$'s in Class 4 contain two indices selected from '3' or '4', and the other two from '1', '2', or '5'. They include

$$\mathcal{W}_{1435},\ \mathcal{W}_{1445},\ \mathcal{W}_{1534},\ \mathcal{W}_{2335},\ \mathcal{W}_{2345},\ \mathcal{W}_{2534},\ \mathcal{W}_{3535},\ \mathcal{W}_{3545},\ \mathcal{W}_{4545}.$$

All of these terms could be expressed by the scalar curvature $\rho$ and its $u$- or $\overline{u}$-derivatives. If $\rho$ vanishes, then every one of them vanishes.

**Proposition 6.3.** *Referring to the model (5.5) of $\mathfrak{D}(w)$, we have the following.*

*(1)* $\mathcal{W}_{1435} = \mathcal{W}_{1534} = -\mathcal{W}_{2335} = -\dfrac{1}{24}\dfrac{\partial\rho}{\partial u}$

*(2)* $\mathcal{W}_{1445} = -\mathcal{W}_{2345} = \mathcal{W}_{2534} = \dfrac{1}{24}\dfrac{\partial\rho}{\partial\overline{u}}$

*(3)* $\mathcal{W}_{3535} = -\dfrac{1}{48}\dfrac{\partial^2\rho}{\partial u^2} - \dfrac{\overline{u}}{24(1+|u|^2)}\dfrac{\partial\rho}{\partial u}$

*(4)* $\mathcal{W}_{3545} = -\dfrac{\rho}{24(1+|u|^2)^2}$

*(5)* $\mathcal{W}_{4545} = -\dfrac{1}{48}\dfrac{\partial^2\rho}{\partial\overline{u}^2} - \dfrac{u}{24(1+|u|^2)}\dfrac{\partial\rho}{\partial\overline{u}}$

We include formulas for $\rho$, $\rho_u$ and $\rho_{uu}$ here. Write $\theta = G^3_{12} + G^1_{23} + G^2_{31}$ and $y = 1 + |u|^2$.

$$
\begin{aligned}
\rho &= i\left(\phi_w - \overline{(\phi_w)}\right) + 16\,\theta + 12i\,a_V - 12i\,\overline{a}_V \\
\rho_u &= i\,w_{uuu} - \frac{6i\overline{u}}{y}\,w_{uu} + \frac{18i\overline{u}^2}{y^2}\,w_u - \frac{24\,i\,\overline{u}^3}{y^3}\,w + \frac{6i}{y^2}\,\overline{w}_{\overline{u}} - \frac{24\,i\,u}{y^3}\,\overline{w} + 12i\,a_{V,u} - 12i\,\overline{a}_{V,u} \\
\rho_{uu} &= i\,w_{uuuu} - \frac{6\,i\,\overline{u}}{y}\,w_{uuu} + \frac{24\,i\,\overline{u}^2}{y^2}\,w_{uu} - \frac{60\,i\,\overline{u}^3}{y^3}\,w_u + \frac{72\,i\,\overline{u}^4}{y^4}\,w \\
&\quad - \frac{12\,i\,\overline{u}}{y^3}\,\overline{w}_{\overline{u}} + \frac{72\,i\,|u|^2}{y^4}\,\overline{w} - \frac{24i}{y^3}\,\overline{w} + 24i\,a_{V,uu} + \frac{24\,i\,\overline{u}}{y}\left(\overline{a}_{V,u} + a_{V,u}\right)
\end{aligned}
$$

As a remark, in our computation, we find that

$$
\mathcal{W}_{3435} \;=\; \mathcal{W}_{3445} \;=\; 0.
$$

We would put these two terms to Class 4 for convenience.

**Class 5: The coefficients of $\mathcal{W}_{ijkl}$'s almost linear in $w$**

There are 15 (out of 120) coefficients of the Weyl tensor left on our list. If we collect the values of

$$
\mathcal{W}\big(X_1, \overline{X}_1, \eta_1, \eta_2\big), \quad \mathcal{W}\big(X_1, T, \eta_1, \eta_2\big) \quad \text{and} \quad \mathcal{W}\big(\overline{X}_1, T, \eta_1, \eta_2\big),
$$

where the pair of vectors $(\eta_1, \eta_2)$ is either $(X_1, \overline{X}_2)$, $(\overline{X}_1, X_2)$, $(X_2, T)$ or $(\overline{X}_2, T)$, then we obtain the Class 5 of 10 Weyl tensor coefficients:

$$
\mathcal{W}_{1235},\ \mathcal{W}_{1245},\ \mathcal{W}_{1415},\ \mathcal{W}_{1425},\ \mathcal{W}_{1523},\ \mathcal{W}_{1535},\ \mathcal{W}_{1545},\ \mathcal{W}_{2325},\ \mathcal{W}_{2535},\ \mathcal{W}_{2545}.
$$

We may apply the symmetries of $\mathcal{W}$ to these 10 terms. First of all, $\mathcal{W}_{1235} = -\overline{\mathcal{W}_{1245}}$. By the identity $\mathcal{W}_{1jk5}F^{jk} = 0$,

$$
\mathcal{W}_{1145}F^{14} + \mathcal{W}_{1235}F^{23} + \mathcal{W}_{1325}F^{32} + \mathcal{W}_{1415}F^{41} = 0.
$$

It leads to $\mathcal{W}_{1415} = \mathcal{W}_{1235}$. Using the Bianchi identity to $\mathcal{W}_{1245}$, we also have $\mathcal{W}_{1425} = \mathcal{W}_{1245}$. In summary, we have

$$
\mathcal{W}_{1235} \;=\; -\,\overline{\mathcal{W}_{1245}} \;=\; \mathcal{W}_{1415} \;=\; \overline{\mathcal{W}_{2325}} \;=\; -\,\overline{\mathcal{W}_{1425}} \;=\; -\,\mathcal{W}_{1523}.
$$

Moreover, $\mathcal{W}_{1535} = \overline{\mathcal{W}_{2545}}$ and $\mathcal{W}_{1545} = \overline{\mathcal{W}_{2535}}$. Therefore, it suffices to study $\mathcal{W}_{1235}$, $\mathcal{W}_{1535}$ and $\mathcal{W}_{1545}$. We may also put $\mathcal{W}_{1214}$ (or $\mathcal{W}_{1223}$) to this category of $\mathcal{W}_{ijkl}$'s since it shares the main properties with the above 10 coefficients.

**Proposition 6.4.**

*Assume $w = f$, where $f$ is defined by (5.4), and so the CR structure $\mathfrak{D}(w)$ is of zero torsion. Then, $\mathcal{W}_{1214} = \mathcal{W}_{1235} = \mathcal{W}_{1535} = \mathcal{W}_{1545} = 0$.*

In Proposition 5.8, we include an explicit formula for $C_{1\bar{1}2\bar{1}}$, which is as same as $\mathcal{W}_{1214}$. Proposition 6.4 helps us rewrite $\mathcal{W}_{1214}$ as

$$\begin{aligned}
\mathcal{W}_{1214} \;=\; & -\frac{i}{2}\, D_{\bar{\mu}} D_u (w-f) + \frac{i}{2}\, D_{\mu} D_{\bar{u}} (\overline{w} - \overline{f}) + \frac{2\, i\, \overline{u}}{y}\, D_{\bar{\mu}} (w-f) - \frac{2\, i\, u}{y}\, D_{\mu} (\overline{w} - \overline{f}) \\
& + y^2 \left(2\,\theta + i\, a_V - i\, \overline{a}_V\right) \Big( -\frac{1}{3}\, \phi_w - \frac{1}{6}\, \overline{(\phi_w)} + \frac{1}{3}\, \phi_f + \frac{1}{6}\, \overline{(\phi_f)} \Big) \\
& + \frac{i y^2}{2} \Big( (w_u - f_u)\, a_{V,u} - (\overline{w}_{\overline{u}} - \overline{f}_{\overline{u}})\, \overline{a}_{V,\overline{u}} + 2(w-f)\, a_{V,uu} - 2(\overline{w} - \overline{f})\, \overline{a}_{V,\overline{u}\,\overline{u}} \Big)
\end{aligned}$$

We include the results for $\mathcal{W}_{1235}$, $\mathcal{W}_{1535}$ and $\mathcal{W}_{1545}$ here.

$$\begin{aligned}
\mathcal{W}_{1235} = \; & \frac{1}{24}\, \overline{w}\, \rho_{\overline{u}} - \frac{i}{12}\, D_{\bar{\mu}} D_u D_u (w-f) - \frac{i}{24}\, D_{\bar{\mu}} D_{\overline{u}} D_{\overline{u}} (\overline{w} - \overline{f}) + \frac{i}{4}\, D_{\frac{v}{|v|}} D_{\overline{u}} (\overline{w} - \overline{f}) \\
& - \frac{y^2}{24} \big(2\,\theta + i\, a_V - i\, \overline{a}_V\big)(w_{uuu} - f_{uuu}) + \frac{iu}{4y}\, D_{\bar{\mu}} D_{\overline{u}} (\overline{w} - \overline{f}) + \frac{i\overline{u}}{2y}\, D_{\bar{\mu}} D_u (w-f) \\
& - \frac{iu}{y}\, D_{\frac{v}{|v|}} (\overline{w} - \overline{f}) + \frac{i y^2}{8}\, (\overline{w}_{\overline{u}\,\overline{u}} - \overline{f}_{\overline{u}\,\overline{u}})\, \overline{a}_{V,u} + \frac{\overline{u}\, y}{4} \big(2\,\theta + i\, a_V - i\, \overline{a}_V\big)(w_{uu} - f_{uu}) \\
& - \frac{i}{2y^2}\, D_{\mu} (\overline{w} - \overline{f}) - \frac{iu^2}{2y^2}\, D_{\bar{\mu}} (\overline{w} - \overline{f}) - \frac{i\overline{u}^2}{y^2}\, D_{\bar{\mu}} (w-f) \\
& + \Big( \frac{3\, i\, y^2}{8}\, a_{V,uu} + \frac{3\, i\, \overline{u}\, y}{4}\, a_{V,u} - \frac{3\overline{u}^2}{4} \big(2\,\theta + i\, a_V - i\, \overline{a}_V\big) \Big)(w_u - f_u) \\
& + \Big( \frac{1}{2}\, \theta + \frac{i}{4}\, a_V - \frac{3\, i\, u\, y}{4}\, \overline{a}_{V,u} \Big)(\overline{w}_{\overline{u}} - \overline{f}_{\overline{u}}) \\
& + \Big( \frac{\overline{u}^3}{y} \big(2\,\theta + i\, a_V - i\, \overline{a}_V\big) - \frac{3\, i\, \overline{u}\, y}{2}\, a_{V,uu} - 3i\, \overline{u}^2\, a_{V,u} \Big)(w-f) \\
& + \Big( \frac{3\, i\, u^2}{2}\, \overline{a}_{V,u} + i\, \overline{a}_{V,\overline{u}} - \frac{i}{2}\, a_{V,\overline{u}} - \frac{u}{y} \big(2\,\theta + i\, a_V\big) \Big)(\overline{w} - \overline{f})
\end{aligned}$$

$$
\begin{aligned}
\mathcal{W}_{1535} = & -\frac{1}{12y^2}\overline{w}\,\rho - \frac{i}{48}D_{\overline{\mu}}D_uD_uD_u(w-f) + \frac{i\overline{u}}{8y}D_{\overline{\mu}}D_uD_u(w-f) - \frac{iy^2}{48}(w_{uuu}-f_{uuu})a_{V,u} \\
& -\frac{3i\,\overline{u}^2}{8y^2}\,D_{\overline{\mu}}D_u(w-f) + \frac{i}{8y^2}\,D_{\overline{\mu}}D_{\overline{u}}(\overline{w}-\overline{f}) + \Big(\frac{5i\,y^2}{48}\,a_{V,uu} + \frac{i\,\overline{u}\,y}{3}\,a_{V,u}\Big)(w_{uu}-f_{uu}) \\
& +\Big(\frac{iy^2}{48}\,a_{V,uu} + \frac{i\,\overline{u}\,y}{24}\,a_{V,u}\Big)(\overline{w}_{\overline{u}\,\overline{u}}-\overline{f}_{\overline{u}\,\overline{u}}) + \frac{i\overline{u}^3}{2y^3}\,D_{\overline{\mu}}(w-f) - \frac{iu}{2y^3}\,D_{\overline{\mu}}(\overline{w}-\overline{f}) \\
& -\frac{i}{y^2}\,D_{\frac{v}{|v|}}(\overline{w}-\overline{f}) - \Big(\frac{5i\,\overline{u}\,y}{8}\,a_{V,uu} + \frac{13i\,\overline{u}^2}{8}\,a_{V,u}\Big)(w_u-f_u) \\
& +\Big(\frac{3i}{8}\,a_{V,u} - \frac{i}{2}\,\overline{a}_{V,u} - \frac{i\,u\,y}{8}\,a_{V,uu} - \frac{iy}{4}\,a_{V,u}\Big)(\overline{w}_{\overline{u}}-\overline{f}_{\overline{u}}) \\
& +\Big(\frac{5i\,\overline{u}^2}{4}\,a_{V,uu} + \frac{3i\,\overline{u}^3}{y}\,a_{V,u}\Big)(w-f) \\
& +\Big(\frac{iu^2}{4}\,a_{V,uu} + \frac{iu}{2}\,a_{V,u} - \frac{iu}{y}\,a_{V,u} + \frac{2i\,u}{y}\,\overline{a}_{V,u} + \frac{1}{y^2}\big(\theta + i\,a_V - 2i\,\overline{a}_V\big)\Big)(\overline{w}-\overline{f})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{1545} = & -\frac{1}{96}\Big(w\,\rho_{uu}+\overline{w}\,\rho_{\overline{u}\,\overline{u}}\Big) - \frac{1}{96}\Big(w_u-\frac{2\overline{u}}{y}w\Big)\rho_u - \frac{1}{96}\Big(\overline{w}_{\overline{u}}-\frac{2u}{y}\overline{w}\Big)\rho_{\overline{u}} + \frac{\rho}{192}\Big(\phi_w+\overline{(\phi_w)}\Big) \\
& -\frac{i}{96}D_{\mu}D_uD_uD_u(w-f) + \frac{i}{96}D_{\overline{\mu}}D_{\overline{u}}D_{\overline{u}}D_{\overline{u}}(\overline{w}-\overline{f}) + \frac{i\overline{u}}{16y}D_{\mu}D_uD_u(w-f) \\
& -\frac{iu}{16y}D_{\overline{\mu}}D_{\overline{u}}D_{\overline{u}}(\overline{w}-\overline{f}) + \frac{i}{24}D_{\frac{v}{|v|}}D_uD_u(w-f) - \frac{i}{24}D_{\frac{v}{|v|}}D_{\overline{u}}D_{\overline{u}}(\overline{w}-\overline{f}) \\
& +\frac{iy^2}{32}a_{V,\overline{u}}(w_{uuu}-f_{uuu}) - \frac{iy^2}{32}\overline{a}_{V,u}(\overline{w}_{\overline{u}\,\overline{u}\,\overline{u}}-\overline{f}_{\overline{u}\,\overline{u}\,\overline{u}}) \\
& -\Big(\frac{\theta}{12} + \frac{i}{16}a_V - \frac{i}{16}\overline{a}_V\Big)\Big(\phi_w + \overline{(\phi_w)} - \phi_f - \overline{(\phi_f)}\Big) \\
& +\frac{i}{4y}\Big(u\,D_{\frac{v}{|v|}}D_{\overline{u}}(\overline{w}-\overline{f}) - \overline{u}\,D_{\frac{v}{|v|}}D_u(w-f)\Big) + \frac{i}{16y^2}\Big(D_{\mu}D_{\overline{u}}(\overline{w}-\overline{f}) - D_{\overline{\mu}}D_u(w-f)\Big) \\
& +\frac{3i}{16y^2}\Big(u^2\,D_{\overline{\mu}}D_{\overline{u}}(\overline{w}-\overline{f}) - \overline{u}^2\,D_{\mu}D_u(w-f)\Big) + \Big(\frac{ia_V}{24} - \frac{3i\,\overline{u}\,y}{16}a_{V,\overline{u}}\Big)(w_{uu}-f_{uu}) \\
& +\Big(\frac{3i\,u\,y}{16}\overline{a}_{V,u} - \frac{ia_V}{24} - \frac{\theta}{12}\Big)(\overline{w}_{\overline{u}\,\overline{u}}-\overline{f}_{\overline{u}\,\overline{u}}) + \frac{i\overline{u}^2}{2y^2}D_{\frac{v}{|v|}}(w-f) - \frac{iu^2}{2y^2}D_{\frac{v}{|v|}}(\overline{w}-\overline{f}) \\
& +\frac{i}{4y^3}\Big(\overline{u}\,D_{\overline{\mu}}(w-f) - u\,D_{\mu}(\overline{w}-\overline{f}) + \overline{u}^3\,D_{\mu}(w-f) - u^3\,D_{\overline{\mu}}(\overline{w}-\overline{f})\Big) \\
& +\Big(\frac{i}{8}a_{V,u} - \frac{3i}{16}\overline{a}_{V,u} + \frac{9i\,\overline{u}^2}{16}a_{V,\overline{u}} - \frac{i\,\overline{u}\,a_V}{4y}\Big)(w_u-f_u) \\
& +\Big(\frac{3i}{16}a_{V,\overline{u}} - \frac{i}{8}\overline{a}_{V,\overline{u}} - \frac{9i\,u^2}{16}\overline{a}_{V,u} + \frac{u}{4y}\big(2\theta + ia_V\big)\Big)(\overline{w}_{\overline{u}}-\overline{f}_{\overline{u}}) \\
& +\Big(\frac{i}{8}a_{V,uu} - \frac{i}{8}\overline{a}_{V,uu} - \frac{i\overline{u}}{4y}a_{V,u} + \frac{i\overline{u}}{2y}\overline{a}_{V,u} - \frac{3i\,\overline{u}^3}{4y}a_{V,\overline{u}} + \frac{i\,\overline{u}^2\,a_V}{2y^2}\Big)(w-f) \\
& +\Big(\frac{i}{8}a_{V,\overline{u}\,\overline{u}} - \frac{i}{8}\overline{a}_{V,\overline{u}\,\overline{u}} + \frac{3i\,u^3}{4y}\overline{a}_{V,u} - \frac{iu}{2y}a_{V,\overline{u}} + \frac{iu}{4y}\overline{a}_{V,\overline{u}} - \frac{u^2}{2y^2}\big(2\theta + ia_V\big)\Big)(\overline{w}-\overline{f})
\end{aligned}
$$

By observation, as long as $\rho = 0$, $\mathcal{W}_{1214}$, $\mathcal{W}_{1235}$, $\mathcal{W}_{1535}$ and $\mathcal{W}_{1545}$ would be linear in $w$, its derivatives and their respective complex conjugations over the ring of complex-valued functions in $u$, $G^k_{ij}$'s and $G^k_{ij,m}$'s. In other words, should $\rho = 0$, these four terms would be in the form of

$$\begin{aligned}\sum_k \Big( & a_k\, D^k_u w \,+\, b_k\, D_\mu(D^k_u w) \,+\, c_k\, D_{\overline{\mu}}(D^k_u w) \,+\, d_k\, D_{\frac{v}{|v|}}(D^k_u w) \\ & + \tilde{a}_k\, D^k_{\overline{u}} \overline{w} \,+\, \tilde{b}_k\, D_\mu(D^k_{\overline{u}} \overline{w}) \,+\, \tilde{c}_k\, D_{\overline{\mu}}(D^k_{\overline{u}} \overline{w}) \,+\, \tilde{d}_k\, D_{\frac{v}{|v|}}(D^k_{\overline{u}} \overline{w}) \Big)\end{aligned} \tag{6.1}$$

Here $D^k_u w$ is the $k$-th order $u$-derivative of $w$. $a_k$, $b_k$, $c_k$, $d_k$ and other coefficients are complex-valued functions in $u$, $G^k_{ij}$'s and $G^k_{ij,m}$'s. As a remark, we say that the coefficients in Class 5 are *almost linear in $w$* because they all satisfy the property (6.1).

**Proposition 6.5.**

*Under the assumption that $\rho = 0$, the following coefficients of the Weyl tensor,*

$$\mathcal{W}_{1214},\ \mathcal{W}_{1223},\ \mathcal{W}_{1235},\ \mathcal{W}_{1245},\ \mathcal{W}_{1415},\ \mathcal{W}_{1425},\ \mathcal{W}_{1523},\ \mathcal{W}_{1535},\ \mathcal{W}_{1545},\ \mathcal{W}_{2325},\ \mathcal{W}_{2535},\ \mathcal{W}_{2545},$$

*are linear in $w$, its derivatives and their respective complex conjugations over the ring of complex-valued functions in $u$, $G^k_{ij}$'s and $G^k_{ij,m}$'s.*

## 6.3 THE TWISTOR CR MANIFOLD OF ZERO TORSION

The **Class 6** of coefficients of the Weyl tensor consists of the remaining terms. Combining $\mathcal{W}_{1212}$ to them, we have the terms

$$\mathcal{W}_{1212},\ \mathcal{W}_{1215},\ \mathcal{W}_{1225},\ \mathcal{W}_{1515},\ \mathcal{W}_{1525} \text{ and } \mathcal{W}_{2525}.$$

By complex conjugation, it suffices to consider $\mathcal{W}_{1212}$, $\mathcal{W}_{1215}$, $\mathcal{W}_{1515}$ and $\mathcal{W}_{1525}$.

They are the most complicated components in the Weyl tensor since they involve the second derivatives of $G^k_{ij}$'s on the 3-manifold $M$. Much work is still required to study these terms in the general case. However, when the CR structure is of zero torsion ($w = f$), we would understand them fully by the Schouten tensor $\mathsf{P}$ (and hence the Cotton tensor) on $M$.

From our notation in Chapter 2, $\{e_1, e_2, e_3\}$ is an orthonormal frame on $M$. $\nabla^M$ represents the Riemannian connection of the metric $\mathsf{g}$ on $M$. Let $G^k_{ij} = \mathsf{g}\big(\nabla^M_{e_i} e_j, e_k\big)$ be the Christoffel symbols. We then denote the Riemann tensor on $M$ by $R_M$.

$$\mathsf{R}_{ijkl} \ = \ \mathsf{g}\big(R_M(e_i, e_j)e_k, e_l\big).$$

The Ricci tensor is named by $\mathrm{Ric}_M$ with $\mathsf{R}_{ij} = \mathrm{Ric}_M(e_i, e_j)$. Also we set the scalar curvature to be $\mathsf{R}$. The Schouten tensor $\mathsf{P}$ in 3-dimension is a symmetric (0,2) tensor such that

$$\mathsf{P} = \mathrm{Ric}_M - \frac{\mathsf{R}}{4}\,\mathsf{g} \quad \text{with} \quad \mathsf{P}_{ij} = \mathsf{P}(e_i, e_j) = \mathsf{R}_{ij} - \frac{\mathsf{R}}{4}\,\mathsf{g}_{ij}. \tag{6.2}$$

If we write $\nabla_i \mathsf{P}_{jk} = (\nabla^M_{e_i}\mathsf{P})(e_j, e_k)$, then

$$\nabla_i \mathsf{P}_{jk} \ = \ e_i(\mathsf{R}_{jk}) - \frac{e_i(\mathsf{R})}{4}\,\delta_{jk} - G^m_{ij}\,\mathsf{P}_{mk} - G^m_{ik}\,\mathsf{P}_{mj}. \tag{6.3}$$

It is a fact that in three dimension, the metric $\mathsf{g}$ is conformally flat if and only if the Schouten tensor is a Codazzi tensor: $(\nabla^M_{e_i}\mathsf{P})(e_j, e_k) = (\nabla^M_{e_j}\mathsf{P})(e_i, e_k)$ for every $i, j, k = 1, 2, 3$.

Back to the twistor CR manifold of zero torsion, the CR structure $(\mathfrak{D})$ of zero torsion on $N$ is spanned by $\overline{X}_2 = \dfrac{\partial}{\partial \overline{u}}$ and

$$\overline{X}_1 \;=\; \mu_1\, e_1 + \mu_2\, e_2 + \mu_3\, e_3 - \frac{i}{2}\,\mu_m \left( G^2_{m1}\,\mu_3 + G^3_{m2}\,\mu_1 + G^1_{m3}\,\mu_2 \right) \frac{\partial}{\partial u}$$

with $\mu_1 = u^2 - 1$, $\mu_2 = 2u$ and $\mu_3 = i(u^2+1)$. From our discussion in Section 6.1 and 6.2, all coefficients of the Weyl tensor vanish except those from the Class 6. Indeed these terms could be related to the Schouten tensor $\mathsf{P}$ on $M$.

**Proposition 6.6.** *Let $F$ be the Fefferman metric of $\mathfrak{D}$ associated with $\alpha$ (5.6) and let $\mathcal{W}$ be the Weyl tensor of $F$. Let $\mathsf{P}$ be the Schouten tensor on $M$ defined in (6.2) and (6.3). Then, under the local model (5.5), we have the following results.*

*(1)*

$$\begin{aligned}
\mathcal{W}_{1212} \;=\;& 4i\,(1+|u|^2)^2\,(\overline{u}^2 - u^2)\left(\nabla_2 \mathsf{P}_{11} - \nabla_1 \mathsf{P}_{12}\right) \\
&+ 4\,(1+|u|^2)^2\,(u+\overline{u})\,(1-|u|^2)\left(\nabla_3 \mathsf{P}_{11} - \nabla_1 \mathsf{P}_{13}\right) \\
&+ 4i\,(1+|u|^2)^2\,(u-\overline{u})\,(1-|u|^2)\left(\nabla_1 \mathsf{P}_{22} - \nabla_2 \mathsf{P}_{12}\right) \\
&+ 4\,(1+|u|^2)^2\,(u^2+\overline{u}^2)\,\nabla_2 \mathsf{P}_{13} \\
&- 2\,(1+|u|^2)^2\left((u-\overline{u})^2 + (1-|u|^2)^2\right)\nabla_1 \mathsf{P}_{23} \\
&- 2\,(1+|u|^2)^2\left((u+\overline{u})^2 - (1-|u|^2)^2\right)\nabla_3 \mathsf{P}_{12}
\end{aligned}$$

*(2)*

$$\begin{aligned}
\mathcal{W}_{1515} \;=\;& i\,(1-\overline{u}^4)\left(\nabla_1 \mathsf{P}_{12} - \nabla_2 \mathsf{P}_{11}\right) \;+\; 2\,(\overline{u}-\overline{u}^3)\left(\nabla_1 \mathsf{P}_{13} - \nabla_3 \mathsf{P}_{11}\right) \\
&+ 2i\,(\overline{u}+\overline{u}^3)\left(\nabla_2 \mathsf{P}_{12} - \nabla_1 \mathsf{P}_{22}\right) \;-\; \left(3\overline{u}^2 + \frac{\overline{u}^4}{2} + \frac{1}{2}\right)\nabla_1 \mathsf{P}_{23} \\
&+ (1+\overline{u}^4)\,\nabla_2 \mathsf{P}_{13} \;+\; \left(3\overline{u}^2 - \frac{\overline{u}^4}{2} - \frac{1}{2}\right)\nabla_3 \mathsf{P}_{12}
\end{aligned}$$

$$
\begin{aligned}
(3) \qquad \mathcal{W}_{1215} \;=\;& 2i\,(1+|u|^2)\,(u+\overline{u}^3)\,\big(\nabla_1\mathsf{P}_{12}-\nabla_2\mathsf{P}_{11}\big)\\
&+\, i\,(1+|u|^2)\,\Big(u\,\overline{u}^3-3\overline{u}^2+3|u|^2-1\Big)\,\big(\nabla_2\mathsf{P}_{12}-\nabla_1\mathsf{P}_{22}\big)\\
&+\,(1+|u|^2)\,\Big(u\,\overline{u}^3-3\overline{u}^2-3|u|^2+1\Big)\,\big(\nabla_3\mathsf{P}_{11}-\nabla_1\mathsf{P}_{13}\big)\\
&-\,(1+|u|^2)\,\Big(u-3\overline{u}+3u\,\overline{u}^2-\overline{u}^3\Big)\,\nabla_1\mathsf{P}_{23}\\
&+\,2\,(1+|u|^2)\,(u-\overline{u}^3)\,\nabla_2\mathsf{P}_{13}\\
&-\,(1+|u|^2)\,\Big(u+3\overline{u}-3u\,\overline{u}^2-\overline{u}^3\Big)\,\nabla_3\mathsf{P}_{12}
\end{aligned}
$$

$$
\begin{aligned}
(4) \qquad \mathcal{W}_{1525} \;=\;& i\,(u^2-\overline{u}^2)\,\big(\nabla_1\mathsf{P}_{12}-\nabla_2\mathsf{P}_{11}\big)\\
&+\,\Big(u\,\overline{u}^2+u^2\,\overline{u}-u-\overline{u}\Big)\,\big(\nabla_1\mathsf{P}_{13}-\nabla_3\mathsf{P}_{11}\big)\\
&+\,i\,\Big(u\,\overline{u}^2-u^2\,\overline{u}+u-\overline{u}\Big)\,\big(\nabla_1\mathsf{P}_{22}-\nabla_2\mathsf{P}_{12}\big)\\
&+\,\frac{1}{2}\Big(4|u|^2-|u|^4-u^2-\overline{u}^2-1\Big)\,\nabla_1\mathsf{P}_{23}\;+\;(u^2+\overline{u}^2)\,\nabla_2\mathsf{P}_{13}\\
&+\,\frac{1}{2}\Big(|u|^4-4|u|^2-u^2-\overline{u}^2+1\Big)\,\nabla_3\mathsf{P}_{12}
\end{aligned}
$$

Proposition 6.6 is justified by computer programming in MATLAB. We replace the variable $a_V$ and its derivatives by the Christoffel symbols $G^k_{ij}$ and their derivatives. In the process, the second derivatives, $G^k_{ij,pl}=e_l\,e_p\left(G^k_{ij}\right)$, can be substituted with $\nabla_i\mathsf{P}_{jk}$'s. So we get the formulas regarding $\mathcal{W}_{1212}$, $\mathcal{W}_{1515}$, $\mathcal{W}_{1215}$ and $\mathcal{W}_{1525}$. A detailed explanation regarding the computational model is included in Appendix C.

**Theorem 6.7.** *Let $M$ be a 3-manifold equipped with the metric $\mathsf{g}$, and let $(N,\mathfrak{D})$ be the twistor CR manifold of $(M,\mathsf{g})$. Let $F$ be the Fefferman metric of $\mathfrak{D}$ associated with any contact form $\theta$ on $N$. Then, $\mathsf{g}$ is conformally flat if and only if $F$ is conformally flat (which is equivalent to the vanishing of the Weyl tensor).*

*Proof.* If $\mathsf{g}$ is conformally flat, then Theorem 7.3 implies that $w = 0$ is a sufficient condition for the Weyl tensor to vanish. Therefore, $F$ is conformally flat. On the other hand, if we assume that the Weyl tensor is zero, then it suffices to prove that the Cotton tensor of $(M, \mathsf{g})$ vanishes.

Let $\mathsf{C}_{ijk}$ be the component of the Cotton tensor with $\mathsf{C}_{ijk} = \nabla_k \mathsf{P}_{ij} - \nabla_j \mathsf{P}_{ik}$. Note that $\mathsf{C}_{ijk} = -\mathsf{C}_{ikj}$ and $\mathsf{C}_{ijk} + \mathsf{C}_{jki} + \mathsf{C}_{kij} = 0$. Moreover, in 3 dimension, we have the result:

$$\mathsf{C}_{121} = \mathsf{C}_{332}, \quad \mathsf{C}_{221} = \mathsf{C}_{313}, \quad \mathsf{C}_{131} = \mathsf{C}_{223}. \tag{6.4}$$

Therefore, we obtain the following symmetries.

$$\begin{cases} \mathsf{C}_{112} = -\mathsf{C}_{121} = -\mathsf{C}_{332} = \mathsf{C}_{323} \\ \mathsf{C}_{113} = -\mathsf{C}_{131} = -\mathsf{C}_{223} = \mathsf{C}_{232} \\ \mathsf{C}_{212} = -\mathsf{C}_{221} = -\mathsf{C}_{313} = \mathsf{C}_{331} \\ \mathsf{C}_{123} = -\mathsf{C}_{132}, \ \mathsf{C}_{231} = -\mathsf{C}_{213}, \ \mathsf{C}_{312} = -\mathsf{C}_{321} \\ \mathsf{C}_{123} + \mathsf{C}_{231} + \mathsf{C}_{312} = 0 \end{cases} \tag{6.5}$$

When $\mathcal{W} = 0$, in particular, $\mathcal{W}_{1212} = 0$. We could write

$$\begin{aligned} \frac{1}{4(1+|u|^2)^2}\mathcal{W}_{1212} \;=\; & \big(i\,\overline{u}^2 - i\,u^2\big)\mathsf{C}_{112} + \Big(u + \overline{u} - u\,\overline{u}^2 - u^2\,\overline{u}\Big)\mathsf{C}_{113} \\ & + \Big(i\,u - i\,\overline{u} + i\,u\,\overline{u}^2 - i\,u^2\,\overline{u}\Big)\mathsf{C}_{221} \\ & + \frac{1}{2}\Big(4|u|^2 - |u|^4 - 1 - u^2 - \overline{u}^2\Big)\mathsf{C}_{321} \\ & + \frac{1}{2}\Big(|u|^4 - 4|u|^2 + 1 - u^2 - \overline{u}^2\Big)\mathsf{C}_{123}. \end{aligned}$$

When $\mathcal{W}_{1212} = 0$, by comparing the coefficients of $|u|^4$, we get $\mathsf{C}_{321} = \mathsf{C}_{123}$. Also, considering the terms of $u^2\,\overline{u}$, $-\mathsf{C}_{113} - i\,\mathsf{C}_{221} = 0$. So $\mathsf{C}_{221} = i\,\mathsf{C}_{113}$. As a result,

$$\big(i\overline{u}^2 - iu^2\big)\mathsf{C}_{112} + \big(2\overline{u} - 2u\,\overline{u}^2\big)\mathsf{C}_{113} - \big(u^2 + \overline{u}^2\big)\mathsf{C}_{123} = 0$$

Immediately, we have $\mathsf{C}_{113} = 0$, $i\,\mathsf{C}_{112} - \mathsf{C}_{123} = 0$ and $-i\,\mathsf{C}_{112} - \mathsf{C}_{123} = 0$. Therefore, $\mathsf{C}_{113} = \mathsf{C}_{112} = \mathsf{C}_{123} = 0$.

□

## 7.0 WEYL CURVATURE TENSOR (II)

When $M$ is equipped with the Euclidean metric, the model of the twistor CR structure (4.1) becomes much simpler. So we can obtain specific results for the Weyl tensor of the Fefferman metric. Especially, we may solve for non-trivial torsion function $w$ such that the Weyl tensor vanishes completely for $\mathfrak{D}(w)$.

### 7.1 THE ALMOST $w$-LINEAR COMPONENTS

Suppose $M$ is a flat 3-manifold and $x = (x_1, x_2, x_3)$ is the coordinate system on $M$. The CR structure $\mathfrak{D}(w)$ on the sphere bundle $N$ (of $M$) is described by

$$\begin{cases} \overline{X}_1 &= (u^2-1)\dfrac{\partial}{\partial x_1} + 2u\dfrac{\partial}{\partial x_2} + i(u^2+1)\dfrac{\partial}{\partial x_3} + w(x,u)\dfrac{\partial}{\partial u} \\ \overline{X}_2 &= \dfrac{\partial}{\partial \overline{u}} \end{cases}$$

See (4.1). Moreover, by Theorem 4.6, the scalar curvature $S$ (or $\rho$) of the Fefferman metric $F$ is zero if and only if $w$ is in the form of (4.12):

$$w = \lambda_0 + \lambda_1\, u + K\, u^2 - \overline{\lambda}_1\, u^3 + \overline{\lambda}_0\, u^4.$$

$\lambda_0$, $\lambda_1$ are complex-valued functions on $M$, and $K$ is a real-valued function on $M$.

From our discussion in Chapter 6, if $w$ satisfies the condition (4.12), then all the coefficients of the Weyl tensor would vanish except those from Class 5 and Class 6.

The remaining terms could be reduced to 8 terms:

$$\mathcal{W}_{1214},\ \mathcal{W}_{1235},\ \mathcal{W}_{1535},\ \mathcal{W}_{1545}, \quad \text{(from Class 5)}$$

$$\mathcal{W}_{1212},\ \mathcal{W}_{1215},\ \mathcal{W}_{1515},\ \mathcal{W}_{1525}. \quad \text{(from Class 6)}$$

We first compute for the Class 5 coefficients, which are *linear in* $w$ when $\rho = 0$. Let

$$\begin{aligned}
A &= \frac{\partial \lambda_1}{\partial x_3} - i\frac{\partial \lambda_1}{\partial x_1} + \frac{\partial \overline{\lambda}_1}{\partial x_3} + i\frac{\partial \overline{\lambda}_1}{\partial x_1}, \\
B &= \frac{\partial K}{\partial x_3} + i\frac{\partial K}{\partial x_1} + 2i\frac{\partial \lambda_0}{\partial x_1} - 2\frac{\partial \lambda_0}{\partial x_3} + i\frac{\partial \lambda_1}{\partial x_2}, \\
C &= \frac{\partial \lambda_1}{\partial x_3} + i\frac{\partial \lambda_1}{\partial x_1} + 4i\frac{\partial \lambda_0}{\partial x_2}.
\end{aligned}$$

We then have the following results.

$$\begin{aligned}
\mathcal{W}_{1214} &= -\frac{A}{2}\big(1 - 4|u|^2 + |u|^4\big) - \big(B\overline{u} + \overline{B}u\big)(1 - |u|^2) + \big(C\overline{u}^2 + \overline{C}u^2\big) \\
\mathcal{W}_{1235} &= -\frac{1}{1+|u|^2}\Big(\frac{\overline{u}^2}{4}\big(|u|^2 - 3\big)B + \frac{1}{4}\big(1 - 3|u|^2\big)\overline{B} + \frac{3\overline{u}}{4}\big(|u|^2 - 1\big)A + \frac{\overline{u}^3}{2}C - \frac{u}{2}\overline{C}\Big) \\
\mathcal{W}_{1535} &= \frac{1}{(1+|u|^2)^2}\Big(-\frac{3\overline{u}^2}{4}A - \frac{\overline{u}^3}{2}B + \frac{\overline{u}}{2}\overline{B} + \frac{\overline{u}^4}{4}C + \frac{1}{4}\overline{C}\Big) \\
\mathcal{W}_{1545} &= -\frac{1}{(1+|u|^2)^2}\Big(\frac{1}{8}\big(|u|^4 - 4|u|^2 + 1\big)A + (1 - |u|^2)\big(\frac{\overline{u}}{4}B + \frac{u}{4}\overline{B}\big) - \frac{1}{4}\big(\overline{u}^2C + u^2\overline{C}\big)\Big)
\end{aligned}$$

**Proposition 7.1.**

*$\mathcal{W}_{1214}$, $\mathcal{W}_{1235}$, $\mathcal{W}_{1535}$ and $\mathcal{W}_{1545}$ are zero if and only if $\lambda_0$, $\lambda_1$ and $K$ satisfy both of the following equalities.*

*(1)* $\dfrac{\partial \lambda_1}{\partial x_3} - i\dfrac{\partial \lambda_1}{\partial x_1} + \dfrac{\partial \overline{\lambda}_1}{\partial x_3} + i\dfrac{\partial \overline{\lambda}_1}{\partial x_1} = 0$

*(2)* $\dfrac{\partial K}{\partial x_3} + i\dfrac{\partial K}{\partial x_1} + 2i\dfrac{\partial \lambda_0}{\partial x_1} - 2\dfrac{\partial \lambda_0}{\partial x_3} + i\dfrac{\partial \lambda_1}{\partial x_2} = 0$

*(3)* $\dfrac{\partial \lambda_1}{\partial x_3} + i\dfrac{\partial \lambda_1}{\partial x_1} + 4i\dfrac{\partial \lambda_0}{\partial x_2} = 0$

Define the complex variable $p = x_1 + ix_3$. The above equations could be translated to

$$(1) \quad \frac{\partial \lambda_1}{\partial \overline{p}} = \frac{\partial \overline{\lambda}_1}{\partial p}, \quad (2) \quad \frac{\partial K}{\partial p} + 2\frac{\partial \lambda_0}{\partial \overline{p}} + \frac{1}{2}\frac{\partial \lambda_1}{\partial x_2} = 0, \quad (3) \quad \frac{\partial \lambda_1}{\partial p} + 2\frac{\partial \lambda_0}{\partial x_2} = 0. \tag{7.1}$$

We let $\lambda_1 = \alpha_{px_2} + i\,\beta_{px_2}$ for two real-valued functions $\alpha(p, \overline{p}, x_2)$ and $\beta(p, \overline{p}, x_2)$. The equation (1) holds whenever $\beta_{p\overline{p}x_2} = 0$. It means that $\beta$ is decomposed to $\beta = f + \bar{f} + g$, where $f = f(p, \overline{p}, x_2)$ is a complex-valued function holomorphic in $p$, and $g = g(p, \overline{p})$ is real-valued. As a result, we have

$$\lambda_1 \;=\; \alpha_{px_2} + i\,\beta_{px_2} \;=\; \alpha_{px_2} + i\,f_{px_2}. \tag{7.2}$$

The equation (3) means that $\dfrac{\partial \lambda_0}{\partial x_2} = -\dfrac{1}{2}\dfrac{\partial \lambda_1}{\partial p} = -\dfrac{1}{2}\alpha_{ppx_2} + i\,\beta_{ppx_2}$. Integrating by $x_2$,

$$\lambda_0 \;=\; -\frac{1}{2}(\alpha_{pp} + i\,\beta_{pp}) + \tau_{pp}(p, \overline{p}), \tag{7.3}$$

where $\tau$ is a complex-valued function depending on $p$ and $\overline{p}$ only. Finally, (2) implies that

$$\frac{\partial K}{\partial p} \;=\; -2\frac{\partial \lambda_0}{\partial \overline{p}} - \frac{1}{2}\frac{\partial \lambda_1}{\partial x_2} \;=\; \big(\alpha_{pp\overline{p}} + i\,\beta_{pp\overline{p}} - 2\,\tau_{pp\overline{p}}\big) - \frac{1}{2}\big(\alpha_{px_2x_2} + i\,\beta_{px_2x_2}\big).$$

Integrating by $p$, we have

$$K \;=\; \big(\alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2}\big) + i\big(\beta_{p\overline{p}} - \frac{1}{2}\beta_{x_2x_2}\big) - 2\,\tau_{p\overline{p}} + h(p, \overline{p}, x_2). \tag{7.4}$$

Here $h(p, \overline{p}, x_2)$ is a complex-valued function such that $h_p = 0$. Moreover, since $K$ is real-valued, we obtain the identities

$$\begin{cases} K \;=\; \mathrm{Re}(K) \;=\; \alpha_{p\overline{p}} - \dfrac{1}{2}\alpha_{x_2x_2} - (\tau_{p\overline{p}} + \overline{\tau}_{p\overline{p}}) + \dfrac{1}{2}(h + \overline{h}) \\ \mathrm{Im}(K) \;=\; \beta_{p\overline{p}} - \dfrac{1}{2}\beta_{x_2x_2} + i(\tau_{p\overline{p}} - \overline{\tau}_{p\overline{p}}) - \dfrac{i}{2}(h - \overline{h}) \;=\; 0. \end{cases} \tag{7.5}$$

from (7.4). Substituting $f + \overline{f} + g$ for $\beta$ in the second equation of (7.5), we get

$$g_{p\overline{p}} - \frac{1}{2}\big(f_{x_2x_2} + \overline{f}_{x_2x_2}\big) + i\big(\tau_{p\overline{p}} - \overline{\tau}_{p\overline{p}}\big) - \frac{i}{2}(h - \overline{h}) = 0 \tag{7.6}$$

When we differentiate (7.6) by $x_2$,

$$\begin{aligned} -\frac{1}{2}\big(f_{x_2x_2x_2}+\overline{f}_{x_2x_2x_2}\big)-\frac{i}{2}(h_{x_2}-\overline{h}_{x_2}) &= 0 \\ \big(f_{x_2x_2x_2}-i\,\overline{h}_{x_2}\big)+\big(\overline{f}_{x_2x_2x_2}+i\,h_{x_2}\big) &= 0. \end{aligned}$$

Since $f_{x_2x_2x_2}-i\,\overline{h}_{x_2}$ is holomorphic in $p$ and $\overline{f}_{x_2x_2x_2}+i\,h_{x_2}$ is antiholomorphic in $p$, $f_{x_2x_2x_2}-i\,\overline{h}_{x_2}$ is then a function depending only on $x_2$. Integrating it by $x_2$,

$$f_{x_2x_2}-i\,\overline{h}=r(x_2)+s(p,\overline{p})$$

for two complex-valued functions $r(x_2)$ and $s(p,\overline{p})$ with $s_{\overline{p}}=0$. Moreover,

$$\begin{aligned} h(p,\overline{p},x_2) &= i\Big(\overline{f}_{x_2x_2}-\overline{r}(x_2)-\overline{s}(p,\overline{p})\Big), \\ \overline{h}(p,\overline{p},x_2) &= i\Big(-f_{x_2x_2}+r(x_2)+s(p,\overline{p})\Big). \end{aligned} \tag{7.7}$$

To solve for $r$ and $s$, note that from (7.6)

$$\begin{aligned} g_{p\overline{p}}+i\big(\tau_{p\overline{p}}-\overline{\tau}_{p\overline{p}}\big)-\tfrac{1}{2}\big(f_{x_2x_2}-i\,\overline{h}\big)-\tfrac{1}{2}\big(\overline{f}_{x_2x_2}+i\,h\big) &= 0 \\ g_{p\overline{p}}+i\big(\tau_{p\overline{p}}-\overline{\tau}_{p\overline{p}}\big)-\tfrac{1}{2}\big(r+\overline{r}\big)-\tfrac{1}{2}\big(s+\overline{s}\big) &= 0 \\ g_{p\overline{p}}-2\,\mathrm{Im}(\tau)_{p\overline{p}}-\tfrac{1}{2}\big(s+\overline{s}\big) &= \tfrac{1}{2}\big(r+\overline{r}\big). \end{aligned} \tag{7.8}$$

The LHS of (7.8) depends on $p$ and $\overline{p}$ only, while the RHS depends on $x_2$ only. Therefore, there exists a constant $C_0$ such that

$$\frac{1}{2}(r+\overline{r}) = C_0 = g_{p\overline{p}}-2\,\mathrm{Im}(\tau)_{p\overline{p}}-\frac{1}{2}\big(s+\overline{s}\big). \tag{7.9}$$

Accordingly, we may write

$$r(x_2)=C_0+i\rho(x_2) \tag{7.10}$$

for a real-valued function $\rho$. In addition, by (7.8),

$$\big(g_{p\overline{p}}-2\,\mathrm{Im}(\tau)_{p\overline{p}}\big)_{p\overline{p}} = \big(C_0+\frac{1}{2}(s+\overline{s})\big)_{p\overline{p}} = 0.$$

So $g_{p\overline{p}}-2\,\mathrm{Im}(\tau)_{p\overline{p}}=\theta_{p\overline{p}}$ for a real-valued function $\theta(p,\overline{p})$ such that $\theta_{pp\overline{p}\,\overline{p}}=0$.

By this condition $\theta_{pp\overline{p}\,\overline{p}} = 0$, the general form of $\theta$ is

$$\theta \;=\; \theta_1(p,\overline{p})\cdot\overline{p} \;+\; \theta_2(p,\overline{p}) \;+\; \overline{\theta}_1(p,\overline{p})\cdot p \;+\; \overline{\theta}_2(p,\overline{p})$$

where $\theta_1$, $\theta_2$ are complex-valued functions such that $\theta_{1,\overline{p}} = 0$ and $\theta_{2,\overline{p}} = 0$.

If we integrate the equation $g_{p\overline{p}} - 2\,\mathrm{Im}(\tau)_{p\overline{p}} = \theta_{p\overline{p}}$ by $p$ and then by $\overline{p}$, we get

$$\begin{aligned} g - 2\,\mathrm{Im}(\tau) &= \theta + \delta + \overline{\delta} \\ \mathrm{Im}(\tau) &= \frac{1}{2}\big(g - \theta - \delta - \overline{\delta}\big) \end{aligned} \tag{7.11}$$

for a complex-valued function $\delta(p,\overline{p})$ holomorphic in $p$. In order to solve for $s$, we let $\mathcal{H}\big(g_{p\overline{p}} - 2\,\mathrm{Im}(\tau)_{p\overline{p}}\big)$ represent the harmonic conjugate of $g_{p\overline{p}} - 2\,\mathrm{Im}(\tau)_{p\overline{p}}$. By (7.9), we have

$$\mathrm{Re}(s) = g_{p\overline{p}} - 2\,\mathrm{Im}(\tau)_{p\overline{p}} - C_0 \quad\Rightarrow\quad s = \Big(g_{p\overline{p}} - 2\,\mathrm{Im}(\tau)_{p\overline{p}}\Big) + i\,\mathcal{H}\big(g_{p\overline{p}} - 2\,\mathrm{Im}(\tau)_{p\overline{p}}\big) + C_1$$

where $C_1$ is a complex number. Therefore,

$$s \;=\; \theta_{p\overline{p}} + i\;\mathcal{H}\big(\theta_{p\overline{p}}\big) + C_1 \quad \text{and} \quad s_p = 2\theta_{pp\overline{p}}. \tag{7.12}$$

We are now ready to substitute the above variables for $\lambda_0$, $\lambda_1$ and $K$.

$$\begin{aligned}
(7.2) \implies \quad \lambda_1 &= \alpha_{px_2} + i f_{px_2} \\
(7.3) \implies \quad \lambda_0 &= -\frac{1}{2}\big(\alpha_{pp} + i\,\beta_{pp}\big) + \tau_{pp} \\
&= -\frac{1}{2}\alpha_{pp} - \frac{i}{2}(f_{pp} + g_{pp}) + \tau_{pp} \\
&= -\frac{1}{2}\big(\alpha + i\,f\big)_{pp} - \frac{i}{2}g_{pp} + \tau_{pp} \\
&\overset{(7.11)}{=} -\frac{1}{2}\big(\alpha + i\,f\big)_{pp} - \frac{i}{2}\Big(\theta_{pp} + \delta_{pp} + 2\,\mathrm{Im}(\tau)_{pp}\Big) + \tau_{pp} \\
&= -\frac{1}{2}\big(\alpha + i\,f + i\,\theta + i\,\delta\big)_{pp} + \mathrm{Re}(\tau)_{pp}
\end{aligned}$$

$$
\begin{aligned}
(7.4) \implies \quad K \quad &= \quad \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - \big(\tau_{p\overline{p}} + \overline{\tau}_{p\overline{p}}\big) + \frac{1}{2}(h + \overline{h}) \\
&\overset{(7.7)}{=} \quad \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - 2\,\mathrm{Re}(\tau)_{p\overline{p}} + \frac{i}{2}\Big(\overline{f}_{x_2x_2} - \overline{r} - \overline{s} - f_{x_2x_2} + r + s\Big) \\
&= \quad \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - \frac{i}{2}\big(f_{x_2x_2} - \overline{f}_{x_2x_2}\big) - 2\,\mathrm{Re}(\tau)_{p\overline{p}} + \frac{i}{2}\big(r - \overline{r} + s - \overline{s}\big) \\
&\overset{(7.10),\,(7.12)}{=} \quad \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - \frac{i}{2}\big(f_{x_2x_2} - \overline{f}_{x_2x_2}\big) - 2\,\mathrm{Re}(\tau)_{p\overline{p}} - \rho(x_2) - \mathcal{H}(\theta_{p\overline{p}})
\end{aligned}
$$

We could eliminate $f$, $\mathrm{Re}(\tau)$ and $\delta$ by letting

$$
\alpha' \;=\; \alpha - 2\,\mathrm{Re}(\tau) + i(f - \overline{f}) \quad \text{and} \quad \theta' \;=\; \theta + \delta + \overline{\delta}.
$$

First of all, we have $\alpha'_{px_2} = \alpha_{px_2} + i\, f_{px_2}$, so $\lambda_1 = \alpha'_{px_2}$. Then, we note that

$$
\alpha'_{pp} = \alpha_{pp} - 2\,\mathrm{Re}(\tau)_{pp} + i\, f_{pp} \quad \text{and} \quad \theta'_{pp} = \theta_{pp} + \delta_{pp}.
$$

Therefore, $\lambda_0 = -\frac{1}{2}\big(\alpha' + i\,\theta'\big)_{pp}$. Lastly, we have

$$
\begin{aligned}
\alpha'_{p\overline{p}} - \frac{1}{2}\alpha'_{x_2x_2} \quad &= \quad \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - 2\,\mathrm{Re}(\tau)_{p\overline{p}} + i(f - \overline{f})_{p\overline{p}} - \frac{i}{2}\big(f_{x_2x_2} - \overline{f}_{x_2x_2}\big) \\
&= \quad \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - \frac{i}{2}\big(f_{x_2x_2} - \overline{f}_{x_2x_2}\big) - 2\,\mathrm{Re}(\tau)_{p\overline{p}}, \\
\theta'_{p\overline{p}} \quad &= \quad \theta_{p\overline{p}} + \delta_{p\overline{p}} + \overline{\delta}_{p\overline{p}} \quad = \quad \theta_{p\overline{p}}.
\end{aligned}
$$

Therefore, $K = \alpha'_{p\overline{p}} - \frac{1}{2}\alpha'_{x_2x_2} - \rho(x_2) - \mathcal{H}(\theta'_{p\overline{p}})$.

We may also put $\theta' = \theta_1\,\overline{p} + \theta_2 + \overline{\theta}_1\,p + \overline{\theta}_2$, then we get

$$
\theta'_{pp} = \theta_{1,pp}\,\overline{p} + \theta_{2,pp} \quad \text{and} \quad \mathcal{H}(\theta'_{p\overline{p}}) = \mathcal{H}(2\,\mathrm{Re}(\theta_{1,p}) = 2\,\mathrm{Im}(\theta_{1,p}) = -i(\theta_{1,p} - \overline{\theta}_{1,\overline{p}}).
$$

By substitution,

$$
\begin{aligned}
\lambda_0 \quad &= \quad -\frac{1}{2}\big(\alpha' + i\,\theta'\big)_{pp} \quad = \quad -\frac{1}{2}\alpha' - \frac{i}{2}\big(\theta_{1,pp}\,\overline{p} + \theta_{2,pp}\big), \\
K \quad &= \quad \alpha'_{p\overline{p}} - \frac{1}{2}\alpha'_{x_2x_2} - \rho(x_2) - \mathcal{H}(\theta'_{p\overline{p}}) \quad = \quad \alpha'_{p\overline{p}} - \frac{1}{2}\alpha'_{x_2x_2} - \rho(x_2) + i\big(\theta_{1,p} - \overline{\theta}_{1,\overline{p}}\big).
\end{aligned}
$$

**Proposition 7.2.** *The general solution to the linear system of differential equations in Proposition 7.1 (or (7.1)) is given by*

$$\begin{cases} \lambda_1 &= \alpha_{px_2} \\ \lambda_0 &= -\frac{1}{2}\alpha_{pp} - \frac{i}{2}\big(\theta_{1,pp}\,\overline{p} + \theta_{2,pp}\big) \\ K &= \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - \rho(x_2) + i\big(\theta_{1,p} - \overline{\theta}_{1,\overline{p}}\big) \end{cases} \tag{7.13}$$

*for arbitrary real-valued functions* $\alpha(p,\overline{p},x_2)$, $\rho(x_2)$, *and complex-valued functions* $\theta_1(p,\overline{p})$, $\theta_2(p,\overline{p})$ *which are holomorphic in* $p$.

It is possible to further reduce the system (7.13)} if we let

$$\phi \;=\; \alpha + i\big(\theta_1\,\overline{p} + \theta_2 - \overline{\theta}_1\,p - \overline{\theta}_2\big) + R(x_2).$$

Here $R(x_2)$ is a real-valued function such that $R_{x_2x_2} = 2\rho$. (7.13) is then equivalent to,

$$\begin{cases} \lambda_1 &= \alpha_{px_2} \;=\; \phi_{px_2} \\ \lambda_0 &= -\frac{1}{2}\alpha_{pp} - \frac{i}{2}\big(\theta_{1,pp}\,\overline{p} + \theta_{2,pp}\big) \;=\; -\frac{1}{2}\phi_{pp} \\ K &= \alpha_{p\overline{p}} - \frac{1}{2}\alpha_{x_2x_2} - \rho(x_2) + i\big(\theta_{1,p} - \overline{\theta}_{1,\overline{p}}\big) \;=\; \phi_{p\overline{p}} - \frac{1}{2}\phi_{x_2x_2} \end{cases} \tag{7.14}$$

for an arbitrary real-valued functions $\phi(p,\overline{p},x_2)$.

## 7.2 THE VANISHING OF THE WEYL TENSOR

In this section we will investigate under what kind of conditions on $w$, the Weyl tensor vanishes completely. Suppose $w = \lambda_0 + \lambda_1 u + K u^2 - \overline{\lambda}_1 u^3 + \overline{\lambda}_0 u^4$ satisfies (7.13). We would discuss a very special case that the real-valued function $\alpha$ in (7.13) is in the form

$$\alpha = x_2(\Psi + \overline{\Psi}). \tag{7.15}$$

for a holomorphic function $\Psi = \Psi(p)$. We also let $\psi(p) = \Psi'(p)$.

(7.15) is a strict condition imposed on (7.13). It makes $\lambda_1 = \psi(p)$ be holomorphic in $p$ and eliminate the term $\alpha_{p\overline{p}} - (1/2)\alpha_{x_2x_2}$ in the third equation. In the following, we also let $K_0(x_2) = -\rho(x_2)$, $\theta_{1,p} = \zeta_1$ and $\theta_{2,pp} = \zeta_2$. Therefore, (7.13) becomes

$$\begin{cases} \lambda_1 &= \psi(p) \\ \lambda_0 &= -\dfrac{x_2}{2}\,\psi'(p) - \dfrac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) \\ K &= K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1) \end{cases} \tag{7.16}$$

where both $\psi$, $\zeta_1$ and $\zeta_2$ are holomorphic in $p$ and independent of $x_2$.

The Weyl tensor of $\mathfrak{D}(w)$ vanishes if and only if $\mathcal{W}_{1212}$, $\mathcal{W}_{1215}$, $\mathcal{W}_{1515}$ and $\mathcal{W}_{1525}$ are all zero. Both of these four coefficients of $\mathcal{W}$ are polynomials in $u$ and $\overline{u}$. More precisely,

$$\begin{aligned} \frac{\mathcal{W}_{1212}}{(1+|u|^2)^2} \;=\; & -F_1(p,\overline{p})\,\big(u^2\,\overline{u}^2 - 4u\,\overline{u} + 1\big) \\ & -\big(F_{21}\,x_2^2 + F_{22}\,x_2 + F_{23}(p,\overline{p},x_2)\big)\,(u^2\,\overline{u} - u) \\ & -\big(\overline{F}_{21}\,x_2^2 + \overline{F}_{22}\,x_2 + \overline{F}_{23}(p,\overline{p},x_2)\big)\,(\overline{u}^2\,u - \overline{u}) \\ & -\big(F_{31}(p,\overline{p},x_2)\,x_2 + F_{32}(p,\overline{p},x_2)\big)\,u^2 \\ & -\big(\overline{F}_{31}(p,\overline{p},x_2)\,x_2 + \overline{F}_{32}(p,\overline{p},x_2)\big)\,\overline{u}^2. \end{aligned}$$

The coefficient functions of $\mathcal{W}_{1212}$ are defined as follows.

$$
\begin{aligned}
F_1 &= \psi\,\overline{\zeta}_1' + \zeta_1'\,\overline{\psi} + \frac{1}{2}\,\psi'\big(\overline{\zeta}_1'\,p + \overline{\zeta}_2\big) + \frac{1}{2}\,\overline{\psi}'\big(\zeta_1'\,\overline{p} + \zeta_2\big) \\
F_{21} &= i\,\psi'\,\overline{\psi}'' \\
F_{22} &= \psi'\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \overline{\psi}''\big(\zeta_1'\,\overline{p} + \zeta_2\big) - \zeta_1'\,\overline{\psi}' \\
F_{23} &= -2\,\overline{\zeta}_1'\big(K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1)\big) + i\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big)\big(\zeta_1'\,\overline{p} + \zeta_2\big) + i\zeta_1'\big(\overline{\zeta}_1'\,p + \overline{\zeta}_2\big) \\
F_{31} &= i\big(\psi\,\overline{\psi}'' - K_0'(x_2)\,\overline{\psi}'\big) \\
F_{32} &= -K_0'(x_2)\big(\overline{\zeta}_1'\,p + \overline{\zeta}_2\big) - i\,\overline{\psi}'\big(K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1)\big) + \psi\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \overline{\psi}\,\overline{\zeta}_1'
\end{aligned}
$$

In terms of $F_1$, $F_{21}$, $F_{22}$, $F_{23}$, $F_{31}$ and $F_{32}$, we also get other coefficients.

$$
\begin{aligned}
\frac{\mathcal{W}_{1215}}{1+|u|^2} &= -\frac{3}{2}\,F_1(u\,\overline{u}^2 - \overline{u}) - \frac{1}{4}\big(F_{21}\,x_2^2 + F_{22}\,x_2 + F_{23}\big)(3u\,\overline{u} - 1) \\
&\quad -\frac{1}{4}\big(\overline{F}_{21}\,x_2^2 + \overline{F}_{22}\,x_2 + \overline{F}_{23}\big)(3\overline{u}^2 - u\,\overline{u}^3) - \frac{1}{2}\big(F_{31}\,x_2 + F_{32}\big)u \\
&\quad +\frac{1}{2}\big(\overline{F}_{31}\,x_2 + \overline{F}_{32}\big)\overline{u}^3 \\
\\
\mathcal{W}_{1515} &= -\frac{3}{2}\,F_1\,\overline{u}^2 - \frac{1}{2}\big(F_{21}\,x_2^2 + F_{22}\,x_2 + F_{23}\big)\overline{u} + \frac{1}{2}\big(\overline{F}_{21}\,x_2^2 + \overline{F}_{22}\,x_2 + \overline{F}_{23}\big)\overline{u}^3 \\
&\quad -\frac{1}{4}\big(F_{31}\,x_2 + F_{32}\big) - \frac{1}{4}\big(\overline{F}_{31}\,x_2 + \overline{F}_{32}\big)\overline{u}^4 \\
\\
\mathcal{W}_{1525} &= -\frac{1}{4}\,F_1\big(u^2\,\overline{u}^2 - 4u\,\overline{u} + 1\big) - \frac{1}{4}\big(F_{21}\,x_2^2 + F_{22}\,x_2 + F_{23}\big)(u^2\,\overline{u} - u) \\
&\quad -\frac{1}{4}\big(\overline{F}_{21}\,x_2^2 + \overline{F}_{22}\,x_2 + \overline{F}_{23}\big)(u\,\overline{u}^2 - \overline{u}) - \frac{1}{4}\big(F_{31}\,x_2 + F_{32}\big)u^2 \\
&\quad -\frac{1}{4}\big(\overline{F}_{31}\,x_2 + \overline{F}_{32}\big)\overline{u}^2
\end{aligned}
$$

Therefore, the Weyl tensor vanishes if and only if

$$
\begin{cases}
F_1(p,\overline{p}) &= 0, \\
F_{21}\,x_2^2 + F_{22}\,x_2 + F_{23}(p,\overline{p},x_2) &= 0, \\
F_{31}(p,\overline{p},x_2)\,x_2 + F_{32}(p,\overline{p},x_2) &= 0.
\end{cases}
$$

The first equation $F_1(p,\overline{p}) = 0$ is

$$\psi\,\overline{\zeta}_1' + \zeta_1'\,\overline{\psi} + \frac{1}{2}\,\psi'\big(\overline{\zeta}_1'\,p + \overline{\zeta}_2\big) + \frac{1}{2}\,\overline{\psi}'\big(\zeta_1'\,\overline{p} + \zeta_2\big) = 0. \tag{7.17}$$

The second equation $F_{21}\,x_2^2 + F_{22}\,x_2 + F_{23} = 0$ is

$$\begin{aligned} & i\,\psi'\,\overline{\psi}''\,x_2^2 \;+\; \Big[\psi'\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \overline{\psi}''\big(\zeta_1'\,\overline{p} + \zeta_2\big) - \zeta_1'\,\overline{\psi}'\Big]\,x_2 \\ & + \Big[-2\,\overline{\zeta}_1'\big(K_0 + i(\zeta_1 - \overline{\zeta}_1)\big) + i\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big)\big(\zeta_1'\,\overline{p} + \zeta_2\big) + i\,\zeta_1'\big(\overline{\zeta}_1'\,p + \overline{\zeta}_2\big)\Big] \end{aligned} = 0. \tag{7.18}$$

The third equation $F_{31}\,x_2 + F_{32} = 0$ is

$$i\big(\psi\,\overline{\psi}'' - K_0'\,\overline{\psi}'\big)x_2 + \Big[-K_0'\big(\overline{\zeta}_1'\,p + \overline{\zeta}_2\big) - i\overline{\psi}'\big(K_0 + i(\zeta_1 - \overline{\zeta}_1)\big) + \psi\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \overline{\psi}\,\overline{\zeta}_1'\Big] = 0. \tag{7.19}$$

(7.17) doesn't depend on $x_2$. Moreover, while $\zeta_1' = 0$, (7.18) becomes a quadratic function in $x_2$. We would make use of this fact and divide (7.16) into two separate cases: (1) $\zeta_1' = 0$ and (2) $\zeta_1' \neq 0$.

### Condition (1): $\boldsymbol{\zeta_1' = 0}$.

Given that $\zeta_1' = 0$, $\zeta_1$ is a complex constant so that it can be absorbed to $K_0(x_2)$ in (7.16). We assume that $\zeta_1 = 0$. Moreover, (7.18) becomes

$$i\,\psi'\,\overline{\psi}''\,x_2^2 + \big(\psi'\,\overline{\zeta}_2' - \overline{\psi}''\,\zeta_2\big)\,x_2 + i\,\overline{\zeta}_2'\,\zeta_2 \;=\; 0. \tag{7.20}$$

On the last component of (7.20), we obtain $\overline{\zeta}_2'\,\zeta_2 = 0$, which implies that $\zeta_2' = 0$. We write $\zeta_2(p) = z_2$ for a complex number $z_2$. The first component gives $\psi'\,\overline{\psi}'' = 0$, leading us to:

$$\text{Subcase (1.1)} \;\; \psi' = 0 \quad \text{and} \quad \text{subcase (1.2)} \;\; \psi'' = 0 \text{ but } \psi' \neq 0.$$

Note that the second component of (7.20) vanishes once $\zeta_2'$ and $\psi''$ are zero.

When (1.1) $\psi' = 0$ happens, we have $\psi(p) = \psi_0$ for $\psi_0 \in \mathbb{C}$. (7.17) is satisfied since both $\zeta_1'$ and $\psi'$ are zero. Also, the equation (7.19) becomes

$$-\,\overline{\zeta}_2\,K_0' \;=\; -\,\overline{z}_2\,K_0' \;=\; 0. \tag{7.21}$$

If $z_2 = 0$ in (7.21), we don't have any restriction on $K_0'(x_2)$, so the Weyl tensor is zero when

$$(1.11)\quad \begin{cases} \lambda_1 &= \psi(p) &= \psi_0 \\ \lambda_0 &= -\dfrac{x_2}{2}\,\psi'(p) - \dfrac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) &= 0 \\ K &= K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1) &= K_0(x_2) \end{cases}$$

for an arbitrary real-valued function $K_0(x_2)$ and a complex constant $\psi_0$.

On the other hand, if $z_2 \neq 0$ in (7.21), we require that $K_0'(x_2) = 0$. Therefore, $\mathcal{W}$ is zero when

$$(1.12)\quad \begin{cases} \lambda_1 &= \psi(p) &= \psi_0 \\ \lambda_0 &= -\dfrac{x_2}{2}\,\psi'(p) - \dfrac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) &= -\dfrac{i\,z_2}{2} \\ K &= K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1) &= K_0 \end{cases}$$

for arbitrary complex constants $\psi_0$, $z_2$ and a real constant $K_0$.

When (1.2) $\psi'' = 0$ but $\psi' \neq 0$ happens, we let $\psi(p) = \alpha_1\,p + \alpha_2$ with $\alpha_1 \neq 0$. (7.17) becomes

$$\frac{1}{2}\,\psi'\,\overline{\zeta}_2 + \frac{1}{2}\,\overline{\psi}'\,\zeta_2 \;=\; 0 \qquad \Longrightarrow \qquad \alpha_1\,\overline{z}_2 + \overline{\alpha}_1\,z_2 \;=\; 0. \tag{7.22}$$

Moreover, (7.19) implies that:

$$\begin{aligned} \Big(-\,i\,K_0'\,\overline{\psi}'\Big)x_2 + \Big(-\,K_0'\,\overline{\zeta}_2 - i\,\overline{\psi}'\,K_0\Big) &= 0 \\ -i\,\overline{\alpha}_1\,K_0'\,x_2 - \overline{z}_2\,K_0' - i\,\overline{\alpha}_1\,K_0 &= 0. \end{aligned} \tag{7.23}$$

If $z_2 = 0$ in (7.22), then we don't have any restriction on $\alpha_1$. In addition, $x_2\,K_0' + K_0 = 0$ from (7.23). So $K_0(x) = \dfrac{b}{x_2}$ for a real constant $b$. Hence, the Weyl tensor vanishes when

$$(1.21)\quad \begin{cases} \lambda_1 &= \psi(p) &= \alpha_1\,p + \alpha_2 \\ \lambda_0 &= -\dfrac{x_2}{2}\,\psi'(p) - \dfrac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) &= -\dfrac{\alpha_1}{2}x_2 \\ K &= K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1) &= \dfrac{b}{x_2} \end{cases}$$

for arbitrary complex numbers $\alpha_1(\neq 0)$, $\alpha_2$ and a real number $b$.

Suppose $z_2 \neq 0$ in (7.22). (7.22) becomes $z_2 = r \cdot i\,\alpha_1$ for some $r \in \mathbb{R}$ and $r \neq 0$.

$$\begin{aligned}
-i\,\overline{\alpha}_1\,K_0'\,x_2 - \overline{z}_2\,K_0' - i\,\overline{\alpha}_1\,K_0 &= 0 \\
-i\,\overline{\alpha}_1\,K_0'\,x_2 + i\,r\,\overline{\alpha}_1\,K_0' - i\,\overline{\alpha}_1\,K_0 &= 0 \\
K_0'\,x_2 - r\,K_0' + K_0 &= 0 \\
(x_2 - r)K_0' + K_0 &= 0 \\
\frac{K_0'}{K_0} &= -\frac{1}{x_2 - r} \\
K_0(x_2) &= \frac{b}{x_2 - r}
\end{aligned}$$

for two real numbers $b$ and $r$. Therefore, the Weyl tensor is zero when

$$(1.22)\quad \begin{cases}
\lambda_1 = \psi(p) = \alpha_1\,p + \alpha_2 \\
\lambda_0 = -\dfrac{x_2}{2}\,\psi'(p) - \dfrac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) = -\dfrac{\alpha_1}{2}\,(x_2 - r) \\
K = K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1) = \dfrac{b}{x_2 - r}
\end{cases}$$

whenever $\alpha_1(\neq 0), \alpha_2 \in \mathbb{C}$ and $r, b \in \mathbb{R}$ with $r \neq 0$.

Note that the solution (1.21) is a special case ($r = 0$) for the solution (1.22). The solution sets in (1.11), (1.12) and (1.22) provide us with all choices of $\lambda_1$, $\lambda_0$ and $K$ such that $\mathcal{W}$ is zero when $\zeta_1' = 0$.

**Condition (2): $\zeta_1' \neq 0$.**

We differentiate (7.18) and (7.19) by $x_2$. The first derivative of (7.18) by $x_2$ is

$$2i\,\psi'\,\overline{\psi}''\,x_2 + \Big[\psi'\,\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \overline{\psi}''\big(\zeta_1'\,\overline{p} + \zeta_2\big) - \zeta_1'\,\overline{\psi}'\Big] - 2\,\overline{\zeta}_1'\,K_0' \;=\; 0. \tag{7.24}$$

The second derivative of (7.18) by $x_2$ is

$$2i\,\psi'\,\overline{\psi}'' - 2\,\overline{\zeta}_1'\,K_0'' \;=\; 0. \tag{7.25}$$

The first derivative of (7.19) by $x_2$ is

$$\begin{aligned} i\,\psi\,\overline{\psi}'' - i\,\overline{\psi}'\,K_0' - i\,\overline{\psi}'\,K_0''\,x_2 - (\overline{\zeta}_1'\,p + \overline{\zeta}_2)K_0'' - i\,\overline{\psi}'\,K_0' &= 0 \\ -\big(i\,\overline{\psi}'\,x_2 + \overline{\zeta}_1'\,p + \overline{\zeta}_2\big)K_0'' - 2i\,\overline{\psi}'\,K_0' + i\,\psi\,\overline{\psi}'' &= 0. \end{aligned} \tag{7.26}$$

The second derivative of (7.19) by $x_2$ is

$$\begin{aligned} -i\,\overline{\psi}'\,K_0'' - \big(i\,\overline{\psi}'\,x_2 + \overline{\zeta}_1'\,p + \overline{\zeta}_2\big)K_0''' - 2i\,\overline{\psi}'\,K_0'' &= 0 \\ \big(i\,\overline{\psi}'\,x_2 + \overline{\zeta}_1'\,p + \overline{\zeta}_2\big)K_0''' + 3i\,\overline{\psi}'\,K_0'' &= 0. \end{aligned} \tag{7.27}$$

Since $\zeta_1' \neq 0$, (7.25) becomes

$$K_0'' = \frac{i\,\psi'\,\overline{\psi}''}{\overline{\zeta}_1'}.$$

$K_0$ depends on $x_2$ only, but the RHS above depends exclusively on $p$ and $\overline{p}$. Therefore,

$$K_0'' \;=\; C \;=\; \frac{i\,\psi'\,\overline{\psi}''}{\overline{\zeta}_1'} \tag{7.28}$$

for a constant $C$. On the right of (7.28), we have $i\,\psi'\,\overline{\psi}'' = C\,\overline{\zeta}_1'$. Differentiating it by $p$,

$$i\,\psi''\,\overline{\psi}'' \;=\; 0 \quad \text{which implies} \quad \psi'' \;=\; 0.$$

So $C = 0$. Since $\psi'' = 0$, we are going to consider three subcases separately:

(2.1): $\psi(p) = 0$ for any $p$, (2.2): $\psi' = 0$ but $\psi \neq 0$, and (2.3): $\psi'' = 0$ but $\psi' \neq 0$.

*Condition (2.1): $\zeta_1' \neq 0$ and $\psi = 0$.*

Assuming $\psi = 0$, (7.17) is simultaneously satisfied. (7.19) becomes

$$-K_0'\,(\overline{\zeta}_1'\,p + \overline{\zeta}_2) \;=\; 0. \tag{7.29}$$

(7.29) leads to $K_0' = 0$. We may further assume that $K_0 = 0$ since $K_0$ could be absorbed to the term $i(\zeta_1 - \overline{\zeta}_1)$. Under the condition (2.1), (7.18) becomes

$$-2i\,\overline{\zeta}_1'(\zeta_1 - \overline{\zeta}_1) + i(\overline{\zeta}_1''\,p + \overline{\zeta}_2')(\zeta_1'\,\overline{p} + \zeta_2) + i\,\zeta_1'(\overline{\zeta}_1'\,p + \overline{\zeta}_2) \;=\; 0. \tag{7.30}$$

We may put $\lambda_1 = 0$, $\lambda_0 = -\dfrac{i}{2}(\zeta_1'\,\overline{p} + \zeta_2)$ and $K = i(\zeta_1 - \overline{\zeta}_1)$ to (7.30).

$$\begin{aligned}
-2\Big(i\,\frac{\partial K}{\partial \overline{p}}\Big)K + i\Big(-2i\,\frac{\partial \overline{\lambda}_0}{\partial \overline{p}}\Big)(2i\,\lambda_0) + i\Big(2i\,\frac{\partial \lambda_0}{\partial \overline{p}}\Big)(-2i\,\overline{\lambda}_0) \;&=\; 0\\
-2i\,K\,\frac{\partial K}{\partial \overline{p}} + 4i\,\lambda_0\,\frac{\partial \overline{\lambda}_0}{\partial \overline{p}} + 4i\,\overline{\lambda}_0\,\frac{\partial \lambda_0}{\partial \overline{p}} \;&=\; 0\\
\frac{\partial K^2}{\partial \overline{p}} \;&=\; 4\,\frac{\partial(\lambda_0\,\overline{\lambda}_0)}{\partial \overline{p}}
\end{aligned}$$

Therefore, $K^2 = 4\lambda_0\overline{\lambda}_0 + C_0$ for some real number $C_0$. In terms of $\zeta_1$ and $\zeta_2$,

$$-\;(\zeta_1 - \overline{\zeta}_1)^2 \;=\; (\zeta_1'\,\overline{p} + \zeta_2)(\overline{\zeta}_1'\,p + \overline{\zeta}_2) \;+\; C_0. \tag{7.31}$$

The general solution to (7.31) is included in the appendix of this article. The result is

$$\zeta_1(p) = \alpha_1\,p + \alpha_2 \quad \text{and} \quad \zeta_2(p) = -\frac{\alpha_1^2}{\overline{\alpha}_1}\,p - 2\,\frac{\alpha_1\,\alpha_2}{\overline{\alpha}_1} + r\,\alpha_1^2 \tag{7.32}$$

for any complex numbers $\alpha_1 \neq 0$, $\alpha_2$ and any real number $r$. Let $b = -2\,\dfrac{\alpha_1\,\alpha_2}{\overline{\alpha}_1} + r\,\alpha_1^2$.

Put (7.32) back to (7.16). We have

$$\begin{aligned}
\lambda_0 \;&=\; -\frac{x_2}{2}\,\psi'(p) - \frac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) \;=\; -\,\frac{i}{2}\Big(\alpha_1\,\overline{p} - \frac{\alpha_1^2}{\overline{\alpha}_1}\,p + b\Big)\\
&=\; \frac{i\,\alpha_1}{2\overline{\alpha}_1}\big(-\,\overline{\alpha}_1\,\overline{p} + \alpha_1\,p\big) - \frac{i}{2}\,b.
\end{aligned}$$

Note that

$$\alpha_2 \;=\; -\frac{\overline{\alpha}_1}{2\alpha_1}\,(b - r\alpha_1^2) \;=\; -\frac{\overline{\alpha}_1}{2\alpha_1}\,b + \frac{r}{2}\,|\alpha_1|^2$$

$$\Longrightarrow\;\; i(\alpha_2 - \overline{\alpha}_2) \;=\; i\Big(-\frac{\overline{\alpha}_1}{2\alpha_1}\,b + \frac{\alpha_1}{2\overline{\alpha}_1}\,\overline{b}\Big) \;=\; i\Big(\frac{\alpha_1^2\,\overline{b} - \overline{\alpha}_1^2\,b}{2\,|\alpha_1|^2}\Big).$$

Therefore,

$$K \;=\; i(\zeta_1 - \overline{\zeta}_1) \;=\; i\Big(\alpha_1\,p - \overline{\alpha}_1\,\overline{p} + \alpha_2 - \overline{\alpha}_2\Big) \;=\; i\big(\alpha_1\,p - \overline{\alpha}_1\,\overline{p}\big) + \frac{i(\alpha_1^2\,\overline{b} - \overline{\alpha}_1^2\,b)}{2\,|\alpha_1|^2}.$$

In summary, given $\zeta_1' \neq 0$ and $\psi = 0$, the Weyl tensor vanishes when

$$(2.1)\begin{cases} \lambda_1 \;=\; 0 \\ \lambda_0 \;=\; \dfrac{i\,\alpha_1^2}{2\overline{\alpha}_1}\,p - \dfrac{i}{2}\,\alpha_1\,\overline{p} - \dfrac{i}{2}\,b \\ K \;=\; i\,\alpha_1\,p - i\,\overline{\alpha}_1\,\overline{p} + \dfrac{i\,(\alpha_1^2\,\overline{b} - \overline{\alpha}_1^2\,b)}{2|\alpha_1|^2} \end{cases}$$

for complex numbers $\alpha_1(\neq 0)$ and $b$.

*Condition (2.2): $\zeta_1' \neq 0$, $\psi' = 0$ and $\psi \neq 0$.*

Let $\psi = \psi_0 \neq 0$. By the equation (7.24),

$$-2\,\overline{\zeta}_1'\,K_0' = 0 \qquad \Longrightarrow\; K_0' = 0.$$

We may then put $K_0 = 0$. Under the assumption that $\psi' = 0$, (7.17) becomes

$$\psi_0\,\overline{\zeta}_1' + \zeta_1'\,\overline{\psi}_0 = 0 \qquad \Longrightarrow \qquad \mathrm{Re}(\zeta_1'\,\overline{\psi}_0) = 0 \qquad \Longrightarrow \qquad \zeta_1 = \big(i\,r\,\psi_0\big)\,p + \alpha_2 \tag{7.33}$$

for a non-zero real number $r$ and $\alpha_2 \in \mathbb{C}$. On the other hand, (7.19) gives

$$\begin{aligned} \psi_0\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \overline{\psi}_0\,\overline{\zeta}_1' \;&=\; 0 \\ \psi_0\,\overline{\zeta}_2' \;&=\; \overline{\psi}_0\,\overline{\zeta}_1' \\ \overline{\psi}_0\,\zeta_2' \;&=\; \psi_0\,\zeta_1' \;=\; i\,r\,\psi_0^2. \end{aligned}$$

It implies that $\zeta_2' = \dfrac{i\,r\,\psi_0^2}{\overline{\psi}_0}$. Hence, $\zeta_2 = \Big(\dfrac{i\,r\,\psi_0^2}{\overline{\psi}_0}\Big)\,p + b$ for a complex constant $b$.

Since both $\psi$ and $K_0$ are constant functions, (7.18) becomes

$$-\,2i\,\overline{\zeta}_1'(\zeta_1 - \overline{\zeta}_1) + i(\overline{\zeta}_1''\,p + \overline{\zeta}_2')(\zeta_1'\,\overline{p} + \zeta_2) + i\,\zeta_1'(\overline{\zeta}_1'\,p + \overline{\zeta}_2) = 0. \tag{7.34}$$

By the formulas of $\zeta_1$ and $\zeta_2$, the left hand side of (7.34) becomes

$$\begin{aligned}
&-2i\,\overline{\zeta}_1'(\zeta_1 - \overline{\zeta}_1) + i(\overline{\zeta}_1''\,p + \overline{\zeta}_2')(\zeta_1'\,\overline{p} + \zeta_2) + i\,\zeta_1'(\overline{\zeta}_1'\,p + \overline{\zeta}_2) \\
= \;& -2i(-i\,r\,\overline{\psi}_0)\big(i\,r\,\psi_0\,p + \alpha_2 + i\,r\,\overline{\psi}_0\,\overline{p} - \overline{\alpha}_2\big) + i\Big(\frac{-i\,r\,\overline{\psi}_0^2}{\psi_0}\Big)\big(i\,r\,\psi_0\,\overline{p} + \frac{i\,r\,\psi_0^2}{\overline{\psi}_0}\,p + b\big) \\
& +i(i\,r\,\psi_0)\Big[-\,i\,r\,\overline{\psi}_0\,p + \Big(\frac{-i\,r\,\overline{\psi}_0^2}{\psi_0}\Big)\,\overline{p} + \overline{b}\Big] \\
= \;& -2\,r\,\overline{\psi}_0\Big(i\,r\,\psi_0\,p + i\,r\,\overline{\psi}_0\,\overline{p} + \alpha_2 - \overline{\alpha}_2\Big) + \Big[i\,r^2\,\overline{\psi}_0^2\,\overline{p} + i\,r^2\,\psi_0\,\overline{\psi}_0\,p + \frac{r\,\overline{\psi}_0^2}{\psi_0}\,b\Big] \\
& -\Big[-\,i\,r^2\,\psi_0\,\overline{\psi}_0\,p - i\,r^2\,\overline{\psi}_0^2\,\overline{p} + r\,\psi_0\,\overline{b}\Big] \\
= \;& \Big(-\,2i\,r^2\,|\psi_0|^2 + i\,r^2\,|\psi_0|^2 + i\,r^2\,|\psi_0|^2\Big)\,p + \Big(-\,2i\,r^2\,\overline{\psi}_0^2 + i\,r^2\,\overline{\psi}_0^2 + i\,r^2\,\overline{\psi}_0^2\Big)\,\overline{p} \\
& -2\,r\,\overline{\psi}_0(\alpha_2 - \overline{\alpha}_2) + \frac{r\,\overline{\psi}_0^2}{\psi_0}\,b - r\,\psi_0\,\overline{b} \\
= \;& \frac{r}{\psi_0}\Big(-\,2\,|\psi_0|^2(\alpha_2 - \overline{\alpha}_2) + \overline{\psi}_0^2\,b - \psi_0^2\,\overline{b}\Big).
\end{aligned}$$

Therefore, (7.18) is satisfied when

$$-\,2\,|\psi_0|^2\,(\alpha_2 - \overline{\alpha}_2) + \overline{\psi}_0^2\,b - \psi_0^2\,\overline{b} \;=\; 0. \tag{7.35}$$

When $\lambda_1 = \psi_0$, we have

$$\lambda_0 \;=\; -\,\frac{x_2}{2}\,\psi'(p) - \frac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) \;=\; -\,\frac{i}{2}\Big(\big(i\,r\,\psi_0\big)\,\overline{p} + \big(\frac{i\,r\,\psi_0^2}{\overline{\psi}_0}\big)\,p + b\Big).$$

Let $\alpha_0 = i\,r\,\psi_0$. Note $\dfrac{\psi_0}{\overline{\psi}_0} = -\dfrac{\alpha_0}{\overline{\alpha}_0}$. The term $\lambda_0$ becomes

$$\lambda_0 \;=\; -\,\frac{i}{2}\Big(\alpha_0\,\overline{p} - \frac{\alpha_0^2}{\overline{\alpha}_0}\,p\Big)\,p - \frac{i}{2}\,b \;=\; \frac{i\,\alpha_0}{2\overline{\alpha}_0}\big(\alpha_0\,p - \overline{\alpha}_0\,\overline{p}\big) - \frac{i}{2}\,b. \tag{7.36}$$

On the other hand, the function $K$ is given by

$$\begin{aligned} K &= i(\zeta_1 - \overline{\zeta}_1) = i\Big(\big(i\,r\,\psi_0\big)\,p + \alpha_2 - \big(-i\,r\,\overline{\psi}_0\big)\,\overline{p} + \overline{\alpha}_2\Big) \\ &= i\big(\alpha_0\,p - \overline{\alpha}_0\,\overline{p}\big) + i(\alpha_2 - \overline{\alpha}_2). \end{aligned} \tag{7.37}$$

Note that (7.35) is equivalent to

$$\alpha_2 - \overline{\alpha}_2 = \frac{\overline{\psi}_0^2\,b - \psi_0^2\,\overline{b}}{2\,|\psi_0|^2} = -\frac{\overline{\alpha}_0}{2\alpha_0}\,b + \frac{\alpha_0}{2\overline{\alpha}_0}\,\overline{b} = \frac{\alpha_0^2\,\overline{b} - \overline{\alpha}_0^2\,b}{2|\alpha_0|^2}. \tag{7.38}$$

Therefore, under the condition (2.2), the Weyl tensor vanishes when

$$(2.2)\begin{cases} \lambda_1 = \dfrac{-i\,\alpha_0}{r} \\ \lambda_0 = \dfrac{i\,\alpha_0^2}{2\overline{\alpha}_0}\,p - \dfrac{i}{2}\,\alpha_0\,\overline{p} - \dfrac{i}{2}\,b \\ K = i\,\alpha_0\,p - i\,\overline{\alpha}_0\,\overline{p} + \dfrac{i\,(\alpha_0^2\,\overline{b} - \overline{\alpha}_0^2\,b)}{2|\alpha_0|^2} \end{cases}$$

for complex numbers $\alpha_0, b$, and a real number $r$ with $\alpha_0 \neq 0$ and $r \neq 0$. Note that system (2.1) is a special case to system (2.2) if we put $r = \infty$ ($\lambda_1 = 0$).

*Condition (2.3): $\zeta_1' \neq 0$, $\psi'' = 0$ and $\psi' \neq 0$.*

Let $\psi(p) = \alpha_1\,p + \alpha_2$ with $\alpha_1 \neq 0$. First of all, since $K_0'' = C = 0$, (7.26) becomes

$$-2i\,\overline{\psi}'\,K_0' = -2i\,\overline{\alpha}_1\,K_0' = 0.$$

It implies that $K_0' = 0$ and $K_0$ is a constant function. Then, (7.24) gives

$$\psi'\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) - \zeta_1'\,\overline{\psi}' = 0 \quad\Longrightarrow\quad \alpha_1\big(\overline{\zeta}_1''\,p + \overline{\zeta}_2'\big) = \overline{\alpha}_1\,\zeta_1'. \tag{7.39}$$

Differentiate (7.39) by $p$, and we have

$$\alpha_1\,\overline{\zeta}_1'' = \overline{\alpha}_1\,\zeta_1' \quad\Longrightarrow\quad \mathrm{Im}(\overline{\alpha}_1\,\zeta_1'') = 0 \quad\Longrightarrow\quad \zeta_1'' = r\,\alpha_1$$

for some $r \in \mathbb{R}$. We may write $\zeta_1'(p) = r\,\alpha_1\,p + b$ for $b \in \mathbb{C}$.

Back to (7.39),

$$\alpha_1\big(r\,\overline{\alpha}_1\,p+\overline{\zeta}_2'\big)=\overline{\alpha}_1\big(r\,\alpha_1\,p+b\big)\qquad\Longrightarrow\qquad\alpha_1\,\overline{\zeta}_2'=\overline{\alpha}_1\,b\qquad\Longrightarrow\qquad\zeta_2'=\frac{\alpha_1\,\overline{b}}{\overline{\alpha}_1}$$

Therefore, we may write $\zeta_2(p)=\Big(\dfrac{\alpha_1\,\overline{b}}{\overline{\alpha}_1}\Big)\,p+c$ for $c\in\mathbb{C}$.

From our analysis on $\psi$, $\zeta_1$ and $\zeta_2$, (7.17) implies that

$$\psi\,\overline{\zeta}_1'+\zeta_1'\,\overline{\psi}+\frac{1}{2}\,\psi'\big(\overline{\zeta}_1'\,p+\overline{\zeta}_2\big)+\frac{1}{2}\,\overline{\psi}'\big(\zeta_1'\,\overline{p}+\zeta_2\big)\;=\;0$$

$$\left[\begin{array}{l}(\alpha_1\,p+\alpha_2)(r\,\overline{\alpha}_1\,\overline{p}+\overline{b})+(r\,\alpha_1\,p+b)(\overline{\alpha}_1\,\overline{p}+\overline{\alpha}_2)\\ +\dfrac{1}{2}\,\alpha_1\Big((r\,\alpha_1\,\overline{p}+\overline{b})\,p+\dfrac{\overline{\alpha}_1\,b}{\alpha_1}\,\overline{p}+\overline{c}\Big)+\dfrac{1}{2}\,\overline{\alpha}_1\Big((r\,\alpha_1\,p+b)\,\overline{p}+\dfrac{\alpha_1\,\overline{b}}{\overline{\alpha}_1}\,p+c\Big)\end{array}\right]\;=\;0. \tag{7.40}$$

The $p\overline{p}$-terms of the left hand side of (7.40) gives

$$\big(r\,\alpha_1\,\overline{\alpha}_1\big)\,p\,\overline{p}+\big(r\,\alpha_1\,\overline{\alpha}_1\big)\,p\,\overline{p}+\big(\frac{1}{2}\,r\,\alpha_1\,\overline{\alpha}_1\big)\,\overline{p}+\big(\frac{1}{2}\,r\,\alpha_1\,\overline{\alpha}_1\big)\,p\,\overline{p}\;=\;3\,|\alpha_1|^2\,r\;=\;0.$$

Therefore, $r=0$ and so $\zeta_1'=b\neq 0$. We put $r=0$ to (7.40).

$$(\alpha_1\,p+\alpha_2)\,\overline{b}+(\overline{\alpha}_1\,\overline{p}+\overline{\alpha}_2)\,b+\frac{1}{2}\,\alpha_1\Big(\overline{b}\,p+\frac{\overline{\alpha}_1\,b}{\alpha_1}\,\overline{p}+\overline{c}\Big)+\frac{1}{2}\,\overline{\alpha}_1\Big(b\,\overline{p}+\frac{\alpha_1\,\overline{b}}{\overline{\alpha}_1}\,p+c\Big)\;=\;0$$

$$2\,\alpha_1\,\overline{b}\,p\;+\;2\,\overline{\alpha}_1\,b\,\overline{p}\;+\;\alpha_2\,\overline{b}\;+\;\overline{\alpha}_2\,b\;+\;\frac{1}{2}\,\alpha_1\,\overline{c}\;+\;\frac{1}{2}\,\overline{\alpha}_1 c\;=\;0.$$

So $\alpha_1\,\overline{b}=0$. Under the condition (2.3), $\psi'=\alpha_1\neq 0$, so we must have $b=0$, which leads to a contradiction. Hence, when $\zeta_1'\neq 0$, $\psi''=0$ and $\psi\neq 0$ occur, there is no solution set of $\lambda_1$, $\lambda_0$ and $K$ so that the Weyl tensor vanishes.

**Theorem 7.3.** *Suppose in the local model (4.1) of $\mathfrak{D}(w)$, the torsion function is defined by*

$$w = \lambda_0 + \lambda_1\, u + K\, u^2 - \overline{\lambda}_1\, u^3 + \overline{\lambda}_0\, u^4,$$

*where $\lambda_0$, $\lambda_1$ and $K$ satisfies the system (7.16). Then, the Weyl tensor $\mathcal{W}$ of $\mathfrak{D}(w)$ vanishes if and only if one of the following cases occurs.*

*Case(1.11)* : $\lambda_1 \in \mathbb{C},\ \ \lambda_0 = 0$ *and* $K = K(x_2)$ *for an arbitrary real-valued function.*

*Case(1.12)* : $\lambda_1 \in \mathbb{C},\ \ \lambda_0 \in \mathbb{C}$ *and* $K \in \mathbb{R}$ *for arbitrary constants.*

*Case(1.2)* : $\lambda_1 = \alpha_1\, p + \alpha_2,\ \ \lambda_0 = -\dfrac{\alpha_1}{2}\,(x_2 - r)$ *and* $K = \dfrac{b}{x_2 - r}$

*for arbitrary constants* $\alpha_1\,(\neq 0)$, $\alpha_2 \in \mathbb{C}$ *and* $b, r \in \mathbb{R}$.

*Case(2)* : $\lambda_1 \;=\; \dfrac{i\,\alpha}{r}$ *or* $0$, $\quad \lambda_0 \;=\; \dfrac{i\,\alpha^2}{2\overline{\alpha}}\,p - \dfrac{i}{2}\alpha\,\overline{p} - \dfrac{i}{2}\,\beta$

*and* $\quad K \;=\; i\,\alpha\,p - i\,\overline{\alpha}\,\overline{p} + \dfrac{i\,(\alpha^2\,\overline{\beta} - \overline{\alpha}^2\,\beta)}{2|\alpha|^2}$

*for arbitrary constants* $\alpha\,(\neq 0)$, $\beta \in \mathbb{C}$ *and* $r\,(\neq 0)\ \in \mathbb{R}$

As a remark to Theorem 7.3, if we express the solutions of $\lambda_1$, $\lambda_0$ and $K$ above in terms of (7.14), then we could obtain the respective $\phi$ functions, which are unique up to a $\mathbb{R}$-linear combination between $|p|^2 + x_2$, $c_1\,p + \overline{c}_1\,\overline{p}$ $(c_1 \in \mathbb{C})$, $x_2$ and $1$.

$$(1.11): \quad \phi \;=\; x_2\big(\lambda_1\,p + \overline{\lambda}_1\,\overline{p}\big) - 2\int\!\!\int K\; dx_2\, dx_2$$

$$(1.12): \quad \phi \;=\; x_2\big(\overline{\lambda}_1\,p + \overline{\lambda}_1\,\overline{p}\big) - \lambda_0\,p^2 - \overline{\lambda}_0\,\overline{p}^2 - K_0\,x_2^2$$

$$(1.2): \quad \phi \;=\; x_2\Big(\frac{\alpha_1}{2}\,p^2 + \alpha_2\,p + \frac{\overline{\alpha}_1}{2}\,\overline{p}^2 + \overline{\alpha}_2\,\overline{p}\Big) - \frac{r\,\alpha_1}{2}\,p^2 - \frac{r\,\overline{\alpha}_1}{2}\,\overline{p}^2 - 2\,b\int \ln(x_2 - r)\,dx_2$$

$$(2): \quad \phi \;=\; \frac{i\,x_2}{r}\big(\alpha\,p - \overline{\alpha}\,\overline{p}\big) + \frac{i}{2}\,|p|^2\big(\alpha\,p - \overline{\alpha}\,\overline{p}\big) - \frac{i}{6|\alpha|^2}\big(\alpha^3\,p^3 - \overline{\alpha}^3\,\overline{p}^3\big)$$
$$+\frac{i\,(\alpha^2\,\overline{\beta} - \overline{\alpha}^2\,\beta)}{2|\alpha|^2}\,|p|^2 + \frac{i}{2}\big(\beta\,p^2 - \overline{\beta}\,\overline{p}^2\big)$$

## 7.3 THE GENERAL SOLUTION

In addition to (7.13), the system (7.14) also gives a general form of $\lambda_0$, $\lambda_1$ and $K$ solving (7.1) in terms of an arbitrary real-valued function $\phi(p, \overline{p}, x_2)$:

$$\lambda_1 = \phi_{px_2}, \quad \lambda_0 = -\frac{1}{2}\,\phi_{pp} \quad \text{and} \quad K = \phi_{p\overline{p}} - \frac{1}{2}\,\phi_{x_2x_2}.$$

We imposed an extra condition (7.15) to system (7.13) to solve for coefficients $\lambda_0$, $\lambda_1$ and $K$ so that $\mathcal{W} = 0$. If we refer to the system (7.14), then the assumption on (7.15) is that both $\phi_{p\overline{p}y}$ and $\phi_{pp\overline{p}\,\overline{p}}0$ are zero. If $\lambda_0$, $\lambda_1$ and $K$ satisfies the format in (7.16),

$$\lambda_1 = \psi, \quad \lambda_0 = -\frac{x_2}{2}\,\psi' - \frac{i}{2}\big(\zeta_1'\,\overline{p} + \zeta_2\big) \quad \text{and} \quad K = K_0(x_2) + i(\zeta_1 - \overline{\zeta}_1),$$

then we have $\lambda_{1,x_2} = \lambda_{0,\overline{p}\,\overline{p}} = 0$. It implies that $\phi_{p\overline{p}y} = \phi_{pp\overline{p}\,\overline{p}} = 0$.

On the other hand, if $\phi_{p\overline{p}y} = 0$, then we get $\alpha_{p\overline{p}y} = 0$ by the definition of $\phi$ right before (7.14). So there is a real-valued function $\sigma$ of $p$ and $\overline{p}$ such that $\alpha = x_2(\Psi + \overline{\Psi}) + \sigma$. The other condition $\phi_{pp\overline{p}\,\overline{p}} = 0$ is equivalent to $\sigma_{pp\overline{p}\,\overline{p}} = 0$. It means that

$$\sigma_{pp} \;=\; \gamma(p)\,\overline{p} + \gamma_1(p)$$

for two holomorphic functions $\gamma$ and $\gamma_1$. If $\Gamma$ is a holomorphic antiderivative of $\gamma$, then we get $\sigma_{p\overline{p}} = \Gamma + \overline{\Gamma}$ up to a real constant. Therefore,

$$\begin{aligned}
\lambda_1 &= \psi, \\
\lambda_0 &= -\frac{x_2}{2}\,\psi' - \frac{1}{2}\big(\gamma\,\overline{p} + \gamma_1\big) - \frac{i}{2}\big(\theta_1''\,\overline{p} + \theta_2''\big) \;=\; -\frac{x_2}{2}\,\psi' - \frac{i}{2}\Big((\theta_1'' - i\,\gamma)\overline{p} + (\theta_2'' - i\,\gamma_1)\Big), \\
K &= (\Gamma + \overline{\Gamma}) + i(\theta_1' - \overline{\theta}_1) - \rho(x_2) \;=\; i\Big((\theta_1' - i\,\Gamma) - (\overline{\theta}_1' + i\,\overline{\Gamma})\Big) - \rho(x_2).
\end{aligned}$$

We can now set $\zeta_1 = \theta_1'' - i\,\Gamma$ and $\zeta_2 = \theta_2'' - i\,\gamma_1$ to express $\lambda_0$, $\lambda_1$ and $K$ in the format of system (7.16).

Back to our main concern, we are going to extract a general condition on $\phi$ such that the remaining terms in the Weyl tensor, $\mathcal{W}_{1212}$, $\mathcal{W}_{1215}$, $\mathcal{W}_{1515}$ and $\mathcal{W}_{1525}$ vanish.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be the following terns.

$$\begin{aligned}
\mathcal{F}_1 &= \frac{i}{2}\big(\phi_{pp}\,\phi_{\overline{p}x_2}\big)_{\overline{p}} - \frac{i}{2}\big(\phi_{px_2}\,\phi_{p\overline{p}}\big)_{\overline{p}} + \frac{i}{2}\big(\phi_{p\overline{p}}\,\phi_{\overline{p}x_2}\big)_{p} - \frac{i}{2}\big(\phi_{\overline{p}\,\overline{p}}\,\phi_{px_2}\big)_{p} \\
\mathcal{F}_2 &= i\big(\phi^2_{\overline{p}x_2}\big)_{p} - i\big(\phi_{x_2x_2}\,\phi_{\overline{p}\,\overline{p}}\big)_{p} + i\big(\phi^2_{p\overline{p}}\big)_{\overline{p}} - i\big(|\phi_{pp}|^2\big)_{\overline{p}} \qquad (7.41) \\
\mathcal{F}_3 &= i\big(\phi_{p\overline{p}}\,\phi_{\overline{p}x_2}\big)_{\overline{p}} - i\big(\phi_{px_2}\,\phi_{\overline{p}\,\overline{p}}\big)_{\overline{p}} + \frac{i}{2}\big(\phi^2_{\overline{p}x_2}\big)_{x_2} - \frac{i}{2}\big(\phi_{x_2x_2}\,\phi_{\overline{p}\,\overline{p}}\big)_{x_2}
\end{aligned}$$

Here, $\big(\phi^2_{p\overline{p}}\big)_{\overline{p}}$ means $\big((\phi_{p\overline{p}})^2\big)_{\overline{p}}$, and $\big(\phi^2_{\overline{p}x_2}\big)_{p}$ means $\big((\phi_{\overline{p}x_2})^2\big)_{p}$.

The terms $\mathcal{W}_{1212}$, $\mathcal{W}_{1215}$, $\mathcal{W}_{1515}$ and $\mathcal{W}_{1525}$ are polynomials of $u$ and $\overline{u}$ with coefficients being $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ or their complex conjugates.

**Proposition 7.4.**

$$\begin{aligned}
\mathcal{W}_{1212} &= \mathcal{F}_1\left(|u|^8 - 2|u|^6 - 6|u|^4 - 2|u|^2 + 1\right) \\
&\quad + \mathcal{F}_2\left(u^4\,\overline{u}^3 + u^3\,\overline{u}^2 - u^2\,\overline{u} - u\right) + \overline{\mathcal{F}}_2\left(u^3\,\overline{u}^4 + u^2\,\overline{u}^3 - u\,\overline{u}^2 - \overline{u}\right) \\
&\quad + \mathcal{F}_3\left(u^4\,\overline{u}^2 + 2u^3\,\overline{u} + u^2\right) + \overline{\mathcal{F}}_3\left(u^2\,\overline{u}^4 + 2u\,\overline{u}^3 + \overline{u}^2\right) \\
\mathcal{W}_{1215} &= \frac{3}{2}\,\mathcal{F}_1\left(u^2\,\overline{u}^3 - \overline{u}\right) + \frac{1}{4}\,\mathcal{F}_2\left(3|u|^4 + 2|u|^2 - 1\right) + \frac{1}{4}\,\overline{\mathcal{F}}_2\left(3\overline{u}^2 + 2u\,\overline{u}^3 - u^2\,\overline{u}^4\right) \\
&\quad + \frac{1}{2}\,\mathcal{F}_3\left(u^2\,\overline{u} + u\right) - \frac{1}{2}\,\overline{\mathcal{F}}_3\left(u\,\overline{u}^4 + \overline{u}^3\right) \\
\mathcal{W}_{1515} &= \frac{3}{2}\,\mathcal{F}_1\,\overline{u}^2 + \frac{1}{2}\,\mathcal{F}_2\,\overline{u} - \frac{1}{2}\,\overline{\mathcal{F}}_2\,\overline{u}^3 + \frac{1}{4}\,\mathcal{F}_3 + \frac{1}{4}\,\overline{\mathcal{F}}_3\,\overline{u}^4 \\
\mathcal{W}_{1525} &= \frac{1}{4}\,\mathcal{F}_1\left(|u|^4 - 4|u|^2 + 1\right) + \frac{1}{4}\,\mathcal{F}_2\left(u^2\,\overline{u} - u\right) + \frac{1}{4}\,\overline{\mathcal{F}}_2\left(u\,\overline{u}^2 - \overline{u}\right) \\
&\quad + \frac{1}{4}\,\mathcal{F}_3\,u^2 + \frac{1}{4}\,\overline{\mathcal{F}}_3\,\overline{u}^2
\end{aligned}$$

Therefore, these terms are zero when both of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are zero. By Theorem 7.3, we know that the system of $(\mathcal{F}_j = 0,\ j = 1, 2, 3)$ is consistent with non-trivial solution. The formulas of $\mathcal{F}_j$'s in (7.41) can be simplified using the Hessian of $\phi$. We let

$$d^2\phi \;=\; \begin{bmatrix} \phi_{pp} & \phi_{p\overline{p}} & \phi_{px_2} \\ \phi_{p\overline{p}} & \phi_{\overline{p}\,\overline{p}} & \phi_{\overline{p}x_2} \\ \phi_{px_2} & \phi_{\overline{p}x_2} & \phi_{x_2x_2} \end{bmatrix}.$$

Let $\mathsf{C}$ be the cofactor matrix of $d^2\phi$. We have

$$\mathsf{C}_{11} \;=\; \phi_{\overline{p}\,\overline{p}}\,\phi_{x_2x_2} - \phi^2_{\overline{p}x_2}, \quad \mathsf{C}_{12} \;=\; |\phi_{px_2}|^2 - \phi_{p\overline{p}}\,\phi_{x_2x_2}, \quad \mathsf{C}_{22} \;=\; \phi_{pp}\,\phi_{x_2x_2} - \phi^2_{px_2},$$

$$\mathsf{C}_{13} \;=\; \phi_{p\overline{p}}\,\phi_{\overline{p}x_2} - \phi_{\overline{p}\,\overline{p}}\,\phi_{px_2}, \quad \mathsf{C}_{23} \;=\; \phi_{p\overline{p}}\,\phi_{px_2} - \phi_{pp}\,\phi_{\overline{p}x_2}, \quad \mathsf{C}_{33} \;=\; |\phi_{pp}|^2 - \phi^2_{p\overline{p}}.$$

Under this notation, we get to the following expression.

$$\begin{cases} \mathcal{F}_1 &=\; \dfrac{i}{2}\Big(\big(\mathsf{C}_{13}\big)_p - \big(\mathsf{C}_{23}\big)_{\overline{p}}\Big), \\ \mathcal{F}_2 &=\; -i\Big(\big(\mathsf{C}_{11}\big)_p + \big(\mathsf{C}_{33}\big)_{\overline{p}}\Big) \\ \mathcal{F}_3 &=\; i\big(\mathsf{C}_{13}\big)_{\overline{p}} - \dfrac{i}{2}\big(\mathsf{C}_{11}\big)_{x_2} \end{cases} \tag{7.42}$$

As an application of (7.41) and (7.42), we may consider an embedding of a 3-manifold to a 4-manifold. The torsion function $w$ is determined by the trace-free second fundamental form. In particular, let $M$ be $\mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$ and let $f : \mathbb{R}^3 \to \mathbb{R}$. We could define an embedding from $\mathbb{R}^3$ to $\mathbb{R}^4$ by the graph of $f$:

$$F : \mathbb{R}^3 \to \mathbb{R}^4, \quad F(x_1, x_2, x_3) = \big(x_1, x_2, x_3, f(x_1, x_2, x_3)\big).$$

The fourth coordinate of $\mathbb{R}^4$ is labeled by $t$. For simplicity we assume that $f$ depends on $x_1$ only. An orthonormal frame for the image of $F$ is given by $\{e_1, e_2, e_3\}$ with

$$e_1 = \frac{1}{\sqrt{1 + (f')^2}}\Big(\frac{\partial}{\partial x_1} + f'\,\frac{\partial}{\partial t}\Big), \quad e_2 = \frac{\partial}{\partial x_2} \quad \text{and} \quad e_3 = \frac{\partial}{\partial x_3}.$$

Moreover, let $\mathbf{n}$ be the unit normal to the image of $F$ in the outward direction.

$$\mathbf{n} \;=\; \frac{1}{\sqrt{1+(f')^2}}\Big(\frac{\partial}{\partial t} - f'\,\frac{\partial}{\partial x_1}\Big)$$

The second fundamental form is given by

$$\mathcal{G}^0_{ij} \;=\; \mathrm{II}(e_i, e_j) \;=\; -\,g_{\mathbb{E}}\big(\nabla_{e_i}\mathbf{n},\, e_j\big),$$

where $\nabla$ is the Riemannian connection of the Euclidean metric $g_{\mathbb{E}}$ on $\mathbb{R}^4$. Explicitly, all $\mathcal{G}^0_{ij}$'s vanish except $\mathcal{G}^0_{11}$, where

$$\mathcal{G}^0_{11} \;=\; \frac{f''}{(1+(f')^2)^{3/2}}.$$

Let $q$ be the trace-free second fundamental form. We call the twistor CR structure of the torsion tensor $i\,q$ by $\mathfrak{D}_f$. By (3.7) and (3.10), the torsion function $w$ is obtained by

$$w \;=\; i\,\mathbf{u}^T\cdot\mathbf{C}\cdot\mathbf{q} \;=\; -\,\frac{1}{2}\,\mathcal{G}^0_{11}\,\big(1-u^2\big)^2.$$

We have already know that $w$ is in the form of (4.12), where

$$\lambda_1 = 0, \quad \lambda_0 = -\frac{f''}{2(1+(f')^2)^{3/2}} \quad \text{and} \quad K = \frac{f''}{(1+(f')^2)^{3/2}}.$$

Moreover, $\lambda_0$, $\lambda_1$ and $K$ solve the system (7.1) because we could find out a real-valued function $\phi$ so that (7.14) is satisfied. Indeed, we can take

$$\phi(x_1) \;=\; 4\int \frac{f'}{\sqrt{1+(f')^2}}\,dx_1.$$

It is then obvious that both $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are zero since $\phi_{pp} = \phi_{p\bar{p}} = (1/4)\,\phi''$. Therefore, the Weyl tensor of $\mathfrak{D}(q)$ vanishes everywhere on the Fefferman bundle whenever $f$ is determined by $x_1$ only.

In general, we have the following results when $\phi$ depends on a single variable only.

**Theorem 7.5.** *Under the following choices of $\phi$ from (7.14), which defines the function*

$$w \;=\; \lambda_0 + \lambda_1\, u + K\, u^2 - \overline{\lambda}_1\, u^3 + \overline{\lambda}_0\, u^4,$$

*the Weyl tensor of the respective CR structure $\mathfrak{D}(w)$ on a flat space $M$ vanishes.*

$$(\phi = \phi(x_1)) : \quad \lambda_1 = 0, \;\; \lambda_0 = -\frac{1}{8}\,\phi_{x_1x_1} \quad \textit{and} \quad K = \frac{1}{4}\,\phi_{x_1x_1}$$

$$(\phi = \phi(x_2)) : \quad \lambda_1 = 0, \;\; \lambda_0 = 0 \quad \textit{and} \quad K = -\frac{1}{2}\,\phi_{x_2x_2}$$

$$(\phi = \phi(x_3)) : \quad \lambda_1 = 0, \;\; \lambda_0 = \frac{1}{8}\,\phi_{x_3x_3} \quad \textit{and} \quad K = \frac{1}{4}\,\phi_{x_3x_3}$$

As a remark, the case that $\phi = \phi(x_2)$ has been included in the case (1.11) of Theorem 7.3. However, for $\phi = \phi(x_1)$ or $\phi = \phi(x_3)$, the conclusion in Theorem 7.5 is not covered since we don't necessarily have $\phi_{pp\overline{p}\,\overline{p}} = 0$ under the new hypothesis.

## APPENDIX A. SOLUTION TO EQUATION (7.31)

This appendix chapter serves as a supplement to the Chapter 7, regarding (7.31):

$$-(\zeta_1 - \overline{\zeta}_1)^2 \;=\; (\zeta_1' \,\overline{p} + \zeta_2)(\overline{\zeta}_1' \, p + \overline{\zeta}_2) + C_0. \tag{A1}$$

Here $p$ is a complex variable, $\zeta_1$, $\zeta_2$ are two holomorphic functions in $p$ and $C_0$ a is real constant. Assuming that both $\zeta_1$ and $\zeta_2$ are defined on a neighborhood of $p = 0$, we are going to find all possible solutions of $\zeta_1$, $\zeta_2$ and $C_0$ such that (7.31) is satisfied.

Let $z_1 = \zeta_1(0)$ and $z_2 = \zeta_2(0)$. The $p$-power series of $\zeta_1$ and $\zeta_2$ are

$$\zeta_1 \;=\; z_1 + \sum_{n=1}^{\infty} a_n \, p^n \quad \text{and} \quad \zeta_2 \;=\; z_2 + \sum_{n=1}^{\infty} b_n \, p^n. \tag{A2}$$

In particular $a_1 = \zeta_1'(0)$. We also let $a_0 = z_1$ and $b_0 = z_2$.

Put the power series back to (A1). The left hand side becomes

$$\begin{aligned}
& -\Big( \sum_{n=1}^{\infty} a_n \, p^n - \sum_{n=1}^{\infty} \overline{a}_n \, \overline{p}^n + (z_1 - \overline{z}_1) \Big)^2 \\
= \;& 2\Big( \sum_{n=1}^{\infty} a_n \, p^n \Big) \Big( \sum_{k=1}^{\infty} \overline{a}_k \, \overline{p}^k \Big) - \Big( \sum_{n=1}^{\infty} a_n \, p^n \Big)^2 - \Big( \sum_{n=1}^{\infty} \overline{a}_n \, \overline{p}^n \Big)^2 \\
& -2(z_1 - \overline{z}_1) \Big( \sum_{n=1}^{\infty} a_n \, p^n - \sum_{n=1}^{\infty} \overline{a}_n \, \overline{p}^n \Big) - (z_1 - \overline{z}_1)^2.
\end{aligned}$$

And the right hand side becomes

$$\begin{aligned}
&\Big[\big(\sum_{n=1}^{\infty} n\, a_n\, p^{n-1}\big)\overline{p} + \sum_{n=1}^{\infty} b_n\, p^n + z_2\Big]\Big[\big(\sum_{n=1}^{\infty} n\, \overline{a}_n\, \overline{p}^{n-1}\big)p + \sum_{n=1}^{\infty} \overline{b}_n\, \overline{p}^n + \overline{z}_2\Big] + C_1 \\
= \;& p\,\overline{p}\Big(\sum_{n=1}^{\infty} n\, a_n\, p^{n-1}\Big)\Big(\sum_{k=1}^{\infty} k\, \overline{a}_k\, \overline{p}^{k-1}\Big) + \overline{p}\Big(\sum_{n=1}^{\infty} n\, a_n\, p^{n-1}\Big)\Big(\sum_{k=1}^{\infty} \overline{b}_k\, \overline{p}^k + \overline{z}_2\Big) \\
&+ p\Big(\sum_{n=1}^{\infty} n\, \overline{a}_n\, \overline{p}^{n-1}\Big)\Big(\sum_{k=1}^{\infty} b_k\, p^k + z_2\Big) + \Big(\sum_{k=1}^{\infty} b_k\, p^k + z_2\Big)\Big(\sum_{k=1}^{\infty} \overline{b}_k\, \overline{p}^k + \overline{z}_2\Big) + C_0
\end{aligned}$$

In genera, for $k \geq 1$ and $n \geq 1$, the $p^k\, \overline{p}^n$-terms follow the formula:

$$2a_k\, \overline{a}_n = k\, a_k \cdot n\, \overline{a}_n + (k+1)\, a_{k+1}\, \overline{b}_{n-1} + (n+1)\, \overline{a}_{n+1}\, b_{k-1} + b_k\, \overline{b}_n. \tag{A3}$$

Multiply both sides of (A3) by $\overline{p}^n$ for $n \geq 1$ and then take the sum over $n$ from 1 to infinity.

$$\begin{aligned}
2a_k \sum_{n=1}^{\infty} \overline{a}_n\, \overline{p}^n \;&=\; k\, a_k\Big(\sum_{n=1}^{\infty} n\, \overline{a}_n\, \overline{p}^n\Big) + (k+1)\, a_{k+1}\Big(\sum_{n=1}^{\infty} \overline{b}_{n-1}\, \overline{p}^n\Big) \\
&\quad + b_{k-1}\Big(\sum_{n=1}^{\infty} (n+1)\, \overline{a}_{n+1}\, \overline{p}^n\Big) + b_k\Big(\sum_{n=1}^{\infty} \overline{b}_n\, \overline{p}^n\Big) \\
2a_k(\overline{\zeta}_1 - \overline{z}_1) \;&=\; k\, a_k \cdot \overline{p}\, \overline{\zeta}_1' + (k+1)\, a_{k+1} \cdot \overline{p}\, \overline{\zeta}_2 + b_{k-1}(\overline{\zeta}_1' - \overline{a}_1) + b_k(\overline{\zeta}_2 - \overline{z}_2)
\end{aligned}$$

Take complex conjugate over the equation.

$$2(\zeta_1 - z_1)\overline{a}_k = p\, \zeta_1' \cdot k\, \overline{a}_k + p\, \zeta_2 \cdot (k+1)\, \overline{a}_{k+1} + (\zeta_1' - a_1)\overline{b}_{k-1} + (\zeta_2 - z_2)\overline{b}_k$$

Let $q$ be another complex variable. For $q \geq 1$, multiply $\overline{q}^k$ and sum over $k$ from 1 to infinity.

$$\begin{aligned}
2\big(\zeta_1(p) - z_1\big)\Big(\sum_{k=1}^{\infty} \overline{a}_k\, \overline{q}^k\Big) \;&=\; p\, \zeta_1'(p)\Big(\sum_{k=1}^{\infty} k\, \overline{a}_k\, \overline{q}^k\Big) + \zeta_1'(p)\Big(\sum_{k=1}^{\infty} \overline{b}_{k-1}\, \overline{q}^k\Big) \\
&\quad + p\, \zeta_2(p)\Big(\sum_{k=1}^{\infty} (k+1)\, \overline{a}_{k+1}\, \overline{q}^k\Big) \\
&\quad + \zeta_2(p)\Big(\sum_{k=1}^{\infty} \overline{b}_k\, \overline{q}^k\Big) - a_1\Big(\sum_{k=1}^{\infty} \overline{b}_{k-1}\, \overline{q}^k\Big) - z_2\Big(\sum_{k=1}^{\infty} \overline{b}_k\, \overline{q}^k\Big)
\end{aligned}$$

$$\begin{aligned}
2\big(\zeta_1(p) - z_1\big)\big(\overline{\zeta}_1(\overline{q}) - \overline{z}_1\big) = \;& p\, \zeta_1'(p) \cdot \overline{q}\, \overline{\zeta}_1'(\overline{q}) \;+\; \zeta_1'(p) \cdot \overline{q}\, \overline{\zeta}_2(\overline{q}) \;+\; p\, \zeta_2(p)\big(\overline{\zeta}_1'(\overline{q}) - \overline{a}_1\big) \\
&+ \zeta_2(p)\big(\overline{\zeta}_2(q) - \overline{z}_2\big) \;-\; a_1\, \overline{q}\, \overline{\zeta}_2(\overline{q}) - z_2\big(\overline{\zeta}_2(\overline{q}) - \overline{z}_2\big)
\end{aligned}$$

Therefore, for any $p$ and $q$ in $\mathbb{C}$,

$$\begin{aligned} 2\big(\zeta_1(p)-z_1\big)\big(\overline{\zeta}_1(\overline{q})-\overline{z}_1\big) \;&=\; p\,\overline{q}\,\zeta_1'(p)\,\overline{\zeta}_1'(\overline{q}) \;+\; \big(\zeta_1'(p)-a_1\big)\,\overline{q}\,\overline{\zeta}_2(\overline{q}) \\ &\quad + p\,\zeta_2(p)\big(\overline{\zeta}_1'(\overline{q})-\overline{a}_1\big) + \big(\zeta_2(p)-z_2\big)\big(\overline{\zeta}_2(\overline{q})-\overline{z}_2\big). \end{aligned} \tag{A4}$$

Let $\mathcal{B}=\Big\{\zeta_1-z_1,\ p\,\zeta_1',\ \zeta_1'-a_1,\ p\,\zeta_2,\ \zeta_2-z_2\Big\}$. From (A4), these four holomorphic functions (in $p$) are linearly dependent. First of all, in (A4), we let

$$g_1(q) \;=\; 2(\zeta_1(q)-z_1), \quad g_2(q) \;=\; -q\,\zeta_1'(q), \qquad g_3(q) \;=\; -q\,\zeta_2(q),$$

$$g_4(q) \;=\; -(\zeta_1'(q)-a_1), \quad g_5(q) \;=\; -(\zeta_2(q)-z_2).$$

The equation (A4) is then translated to

$$\overline{g}_1(\overline{q})(\zeta_1-z_1) \;+\; \overline{g}_2(\overline{q})\,p\,\zeta_1' \;+\; \overline{g}_3(\overline{q})(\zeta_1'-a_1) \;+\; \overline{g}_4(\overline{q})\,p\,\zeta_2 \;+\; \overline{g}_5(\overline{q})(\zeta_2-z_2) \;=\; 0. \tag{A5}$$

The function $g_4(q)\equiv 0$ if and only if $\zeta_1'=a_1$ for all $q$, in which case $\zeta_1$ is a linear function (so we are done). *Wlog*, $g_4(q)\neq 0$. Take $q_1$ such that $g_4(q_1)\neq 0$. As a result,

$$p\zeta_2 \;=\; -\frac{\overline{g}_1(\overline{q}_1)}{\overline{g}_4(\overline{q}_1)}\big(\zeta_1-z_1\big) - \frac{\overline{g}_2(\overline{q}_1)}{\overline{g}_4(\overline{q}_1)}\,p\,\zeta_1' - \frac{\overline{g}_3(\overline{q}_1)}{\overline{g}_4(\overline{q}_1)}\big(\zeta_1'-a_1\big) - \frac{\overline{g}_5(\overline{q}_1)}{\overline{g}_4(\overline{q}_1)}\big(\zeta_2-z_2\big).$$

Let $\lambda_i=\dfrac{g_i(q_1)}{g_4(q_1)}$ for $i=1,2,3,5$. Hence, we have

$$\begin{aligned} &\Big(\overline{g}_1(\overline{q})-\overline{g}_4(\overline{q})\,\overline{\lambda}_1\Big)(\zeta_1-z_1)+\Big(\overline{g}_2(\overline{q})-\overline{g}_4(\overline{q})\,\overline{\lambda}_2\Big)\,p\,\zeta_1' \\ &+\Big(\overline{g}_3(\overline{q})-\overline{g}_4(\overline{q})\,\overline{\lambda}_3\Big)(\zeta_1'-a_1)+\Big(\overline{g}_5(\overline{q})-\overline{g}_4(\overline{q})\,\overline{\lambda}_5\Big)(\zeta_2-z_2) \end{aligned} \;=\; 0. \tag{A6}$$

Next, we consider the term $\Big(g_5(q)-\lambda_5\,g_4(q)\Big)$. If it happens to vanish, then

$$\begin{aligned} g_5(q)-\lambda_5\,g_4(q) \;&=\; 0 \\ -\big(\zeta_2(q)-z_2\big)+\lambda_5\big(\zeta_1'(q)-a_1\big) \;&=\; 0 \\ -\zeta_2(q)+z_2+\lambda_5\,\zeta_1'(q)-\lambda_5\,a_1 \;&=\; 0 \\ \zeta_2 \;&=\; \lambda_5\,\zeta_1'+\big(z_2-\lambda_5\,a_1\big). \end{aligned}$$

When $g_5(q) - \lambda_5 g_4(q) \equiv 0$, we are in the special case that

$$\zeta_2 \,=\, \overline{\mu}_1\, \zeta_1' \,+\, \mu_2, \qquad \text{for} \;\; \mu_1,\, \mu_2 \in \mathbb{C}. \tag{A7}$$

We will go back to (A7) later. Assume there exists $q_2$ such that $g_5(q_2) - \lambda_5\, g_4(q_2) \neq 0$. From (A6), we have

$$\begin{aligned} \zeta_2 - z_2 \;\; = \;\; & -\frac{\overline{g}_1(\overline{q}_2) - \overline{g}_4(\overline{q}_2)\,\overline{\lambda}_1}{\overline{g}_5(\overline{q}_2) - \overline{g}_4(\overline{q}_2)\,\overline{\lambda}_5}\,(\zeta_1 - z_1) - \frac{\overline{g}_2(\overline{q}_2) - \overline{g}_4(\overline{q}_2)\,\overline{\lambda}_2}{\overline{g}_5(\overline{q}_2) - \overline{g}_4(\overline{q}_2)\,\overline{\lambda}_5}\,p\,\zeta_1' \\ & -\frac{\overline{g}_3(\overline{q}_2) - \overline{g}_4(\overline{q}_2)\,\overline{\lambda}_3}{\overline{g}_5(\overline{q}_2) - \overline{g}_4(\overline{q}_2)\,\overline{\lambda}_5}\,(\zeta_1' - a_1). \end{aligned}$$

Let $\mu_i = \dfrac{g_i(q_2) - g_4(q_2)\,\lambda_i}{g_5(q_2) - g_4(q_2)\,\lambda_5}$ for $i = 1, 2, 3$. We then get to a linear relation between $\zeta_1 - z_1$, $p\zeta_1'$ and $\zeta_1' - a_1$.

$$\begin{aligned} & \Big[\big(\overline{g}_1(\overline{q}) - \overline{g}_4(\overline{q})\,\overline{\lambda}_1\big) - \big(\overline{g}_5(\overline{q}) - \overline{g}_4(\overline{q})\,\overline{\lambda}_5\big)\,\overline{\mu}_1\Big](\zeta_1 - z_1) \\ & + \Big[\big(\overline{g}_2(\overline{q}) - \overline{g}_4(\overline{q})\,\overline{\lambda}_2\big) - \big(\overline{g}_5(\overline{q}) - \overline{g}_4(\overline{q})\,\overline{\lambda}_5\big)\,\overline{\mu}_2\Big]\,p\,\zeta_1' \qquad = \; 0 \\ & + \Big[\big(\overline{g}_3(\overline{q}) - \overline{g}_4(\overline{q})\,\overline{\lambda}_3\big) - \big(\overline{g}_5(\overline{q}) - \overline{g}_4(\overline{q})\,\overline{\lambda}_5\big)\,\overline{\mu}_3\Big](\zeta_1' - a_1) \end{aligned}$$

If coefficients of $\zeta_1 - z_1$, $p\,\zeta_1'$ and $\zeta_1' - a_1$ are not all zero, then $\zeta_1$ satisfies the differential relation

$$\alpha_1\, p\,\zeta_1' + \alpha_2\,\zeta_1' + \alpha_3\,\zeta_1 + \alpha_4 \;=\; 0 \qquad \text{for} \;\; \alpha_i \in \mathbb{C}. \tag{A8}$$

Moreover, it can be shown that even if these three coefficients are all zero, $\zeta_1$ still satisfies a differential relation in the form of (A8). In summary, in order to find holomorphic solutions $\zeta_1$ and $\zeta_2$ to (A1), it suffices to solve for $\zeta_1$ in either (A7) or (A8).

We shall consider solutions to (A7) in Part (1), and solutions to (A8) in Part (2).

## Part (1): Solution to (A7)

From the power series of $\zeta_1$ and $\zeta_2$ (A2), $\zeta_2 = \overline{\mu}_1\,\zeta_1' + \mu_2$ implies $z_2 = \overline{\mu}_1\,a_1 + \mu_2$. For any $n \geq 1$,

$$b_n \;=\; \frac{(\zeta_2)^{(n)}(0)}{n!} \;=\; \frac{1}{n!}\,\overline{\mu}_1\,\zeta_1^{(n+1)}(0) \;=\; \frac{(n+1)!}{n!}\cdot\overline{\mu}_1\,\frac{\zeta_1^{(n+1)}(0)}{(n+1)!} \;=\; (n+1)\,\overline{\mu}_1\,a_{n+1}.$$

Next, we apply the $p^k\,\overline{p}^n$-formula (A3). For any $k \geq 2$ and $n \geq 2$, we get

$$\begin{aligned}
2a_k\,\overline{a}_n \;&=\; k\,a_k\cdot n\,\overline{a}_n + (k+1)\,a_{k+1}\,\overline{b}_{n-1} + (n+1)\,\overline{a}_{n+1}\,b_{k-1} + b_k\,\overline{b}_n \\
&=\; n\,k\,a_k\,\overline{a}_n + (k+1)\,a_{k+1}\cdot\mu_1\,n\,\overline{a}_n + (n+1)\,\overline{a}_{n+1}\cdot\overline{\mu}_1\,k\,a_k \\
&\quad +(k+1)\,\overline{\mu}_1\,a_{k+1}\cdot(n+1)\,\mu_1\,\overline{a}_{n+1} \\
&=\; n\,\overline{a}_n\big(k\,a_k + (k+1)\,\mu_1\,a_{k+1}\big) + (n+1)\,\overline{\mu}_1\cdot\overline{a}_{n+1}\big(k\,a_k + (k+1)\,\mu_1\,a_{k+1}\big) \\
&=\; \big(k\,a_k + (k+1)\,\mu_1\,a_{k+1}\big)\big(n\,\overline{a}_n + (n+1)\,\overline{\mu}_1\,\overline{a}_{n+1}\big).
\end{aligned}$$

In particular, when $n = k$, $2|a_k|^2 = |k\,a_k + (k+1)\,\mu_1\,a_{k+1}|^2$ for any $k \geq 2$. It implies that if any $a_{k_0} = 0$ for an integer $k_0 \geq 2$, then $a_k = 0$ for all $k \geq 2$ and so $\zeta_1$ is linear in $p$. *Wlog*, we assume that $a_n \neq 0$ for all $n \geq 2$.

Rearranging the terms of the above equation, we obtain

$$\frac{k\,a_k + (k+1)\,\mu_1\,a_{k+1}}{a_k} = \frac{2\,\overline{a}_n}{n\,\overline{a}_n + (n+1)\,\overline{\mu}_1\,\overline{a}_{n+1}} \tag{A9}$$

for any $n, k \geq 2$. Since $n$ and $k$ are arbitrary in (A9), there exists a constant $\gamma_0$ such that

$$\gamma_0 \;=\; \frac{k\,a_k + (k+1)\,\mu_1\,a_{k+1}}{a_k} \quad \text{for } k \geq 2,$$

Note that $\gamma_0 = \dfrac{2}{\overline{\gamma}_0}$ so $|\gamma_0|^2 = 2$. Back to (A9), we have

$$a_{k+1} \;=\; \frac{\gamma_0 - k}{(k+1)\,\mu_1}\,a_k \quad \text{for } k \geq 2.$$

Recursively,

$$\begin{aligned} a_{k+1} &= \frac{\gamma_0 - k}{(k+1)\,\mu_1} a_k = \frac{\big(\gamma_0 - k\big)\big(\gamma_0 - (k-1)\big)}{(k+1)\,k\,\mu_1^2} a_{k-1} \\ &= \frac{(\gamma_0 - k)(\gamma_0 - k + 1)\cdots(\gamma_0 - 2)}{(k+1)\,k\cdots 3\,\mu_1^{k-1}} a_2 \\ &= \frac{(\gamma_0 - k)(\gamma_0 - k + 1)\cdots(\gamma_0 - 2)(\gamma_0 - 1)\gamma_0}{(k+1)!\,\mu_1^{k+1}} \cdot \frac{2\,a_2\,\mu_1^2}{\gamma_0(\gamma_0 - 1)}. \end{aligned}$$

We are to find $f(p) = c_0 + \sum_{n=1}^{\infty} c_n\, p^n$ such that $c_n = a_n$ for all $n \geq 2$. Indeed,

$$f(p) = \frac{2\,a_2\,\mu_1^2}{\gamma_0(\gamma_0 - 1)} \Big(\frac{p}{\mu_1} + 1\Big)^{\gamma_0}$$

implies that $f^{(n)}(0)/(n!) = a_n$ for all $n \geq 2$. Therefore, for some $d_1$, $d_2 \in \mathbb{C}$,

$$\zeta_1 = \frac{2a_2\mu_1^2}{\gamma_0(\gamma_0 - 1)} \Big(\frac{p}{\mu_1} + 1\Big)^{\gamma_0} + d_1\, p + d_2. \tag{A10}$$

Differentiate (A10) by $p$ to get

$$\zeta_1' = \frac{2\,a_2\,\mu_1}{\gamma_0 - 1} \Big(\frac{p}{\mu_1} + 1\Big)^{\gamma_0 - 1} + d_1.$$

By the relation $\zeta_2 = \overline{\mu}_1\, \zeta_1' + \mu_2$,

$$\zeta_2 = \frac{2a_2\,|\mu_1|^2}{\gamma_0 - 1} \Big(\frac{p}{\mu_1} + 1\Big)^{\gamma_0 - 1} + \overline{\mu}_1\, d_1 + \mu_2. \tag{A11}$$

For the moment, we go back to the main equation (A1). The holomorphic part of (A1) gives

$$\begin{aligned} 2\zeta_1\,\overline{\zeta}_1 - \zeta_1^2 - \overline{\zeta}_1^2 &= p\,\overline{p}\,\zeta_1'\,\overline{\zeta}_1' + p\,\overline{\zeta}_1'\,\zeta_2 + \overline{p}\,\zeta_1'\,\overline{\zeta}_2 + \zeta_2\,\overline{\zeta}_2 + C_0 \\ \Longrightarrow\ 2\zeta_1\,\overline{\zeta}_1(0) - \zeta_1^2 - \overline{\zeta}_1^2(0) &= p\,\overline{\zeta}_1'(0)\,\zeta_2 + \zeta_2\,\overline{\zeta}_2(0) + C_0 \\ -\big(\zeta_1 - \overline{z}_1\big)^2 &= \big(\overline{a}_1\,p + \overline{z}_2\big)\zeta_2 + C_0. \end{aligned}$$

Therefore,

$$\zeta_2 = \frac{-(\zeta_1 - \overline{z}_1)^2 + (z_1 - \overline{z}_1)^2 + z_2\,\overline{z}_2}{\overline{a}_1\,p + \overline{z}_2}. \tag{A12}$$

We can equalize (A11) and (A12).

$$\frac{2a_2\,|\mu_1|^2}{\gamma_0-1}\Big(\frac{p}{\mu_1}+1\Big)^{\gamma_0-1}+\overline{\mu}_1\,d_1+\mu_2=\frac{-\Big(\frac{2a_2\,\mu_1^2}{\gamma_0(\gamma_0-1)}\big(\frac{p}{\mu_1}+1\big)^{\gamma_0}+d_1\,p+d_2-\overline{z}_1\Big)^2-C_0}{\overline{a}_1\,p+\overline{z}_2}$$

$$\big(\overline{a}_1\,p+\overline{z}_2\big)\Big[\frac{2a_2\,|\mu_1|^2}{\gamma_0-1}\Big(\frac{p}{\mu_1}+1\Big)^{\gamma_0-1}+\overline{\mu}_1\,d_1+\mu_2\Big]=-\Big(\frac{2a_2\,\mu_1^2}{\gamma_0(\gamma_0-1)}\big(\frac{p}{\mu_1}+1\big)^{\gamma_0}+d_1\,p+d_2-\overline{z}_1\Big)^2-C_0.$$

Let $y=\dfrac{p}{\mu_1}+1$. The above equation becomes

$$\Big(\overline{a}_1\mu_1(y-1)+\overline{z}_2\Big)\Big[\frac{2a_2|\mu_1|^2}{\gamma_0-1}y^{\gamma_0-1}+\overline{\mu}_1d_1+\mu_2\Big]=-\Big(\frac{2a_2\mu_1^2}{\gamma_0(\gamma_0-1)}y^{\gamma_0}+d_1\mu_1(y-1)+d_2-\overline{z}_1\Big)^2-C_0.$$

On left hand side, powers of $y$ are: $\quad y^{\gamma_0},\ y^{\gamma_0-1},\ y,\ 1.$

On right hand side, powers of $y$ are: $\quad y^{2\gamma_0},\ y^{\gamma_0+1},\ y^{\gamma_0},\ y^2,\ y,\ 1.$

The coefficient of $y^{2\gamma_0}$ on the right hand side is $-\Big(\frac{2a_2\mu_1^2}{\gamma_0(\gamma_0-1)}\Big)^2\neq 0$. So we may have

$$2\gamma_0=\gamma_0,\quad 2\gamma_0=\gamma_0-1,\quad 2\gamma_0=1,\quad \text{or}\quad 2\gamma_0=0.$$

Hence, $\gamma_0=0,\ -1$ or $1/2$. None of them gives $|\gamma_0|^2=2$. We arrive at a contradiction to such a constant $\gamma_0$ exists. The only possible case in (A7) is that $a_n=0$ for all $n\geq 2$. So $\zeta_1$ is linear in $p$.

## Part (2): Solution to (A8)

The equation (A8): $\alpha_1\, p\, \zeta_1' + \alpha_2\, \zeta_1' + \alpha_3\, \zeta_1 + \alpha_4 = 0$ for $\alpha_i \in \mathbb{C}$, is an ordinary differential equation which could be solved to different solutions.

(I) Both $\alpha_1 \neq 0$ and $\alpha_3 \neq 0$ : $\zeta_1(p) = c_1\big(p+\alpha\big)^{\gamma} + c_2$ for $c_1, c_2, \alpha, \gamma \in \mathbb{C}$.

(II) $\alpha_1 = 0$ and $\alpha_3 \neq 0$ $(\alpha_2 \neq 0)$ : $\zeta_1(p) = c_1\, e^{\gamma p} + c_2$ for $c_1, c_2, \gamma \in \mathbb{C}$.

(III) $\alpha_3 = 0$ and $\alpha_1 \neq 0$ $(\alpha_4 \neq 0)$ : $\zeta_1(p) = c_1 \log\big(\alpha_1\, p + \alpha_2\big) + c_2$ for $c_j,\ \alpha_j \in \mathbb{C}$

(IV) Both $\alpha_1 = 0$ and $\alpha_3 = 0$ : $\zeta_1$ is a linear function in $p$.

We would work on the case (I) while a similar method is applied to show that there is no solution from case (II) or case (III) satisfying (A1).

Let $\zeta_1(p) = c_1(p+\alpha)^{\gamma} + c_2$. We have

$$\zeta_1' = c_1\, \gamma (p+\alpha)^{\gamma-1}, \quad z_1 = \zeta_1(0) = c_1\, \alpha^{\gamma} + c_2, \quad \text{and} \quad a_1 = \zeta_1'(0) = c_1\, \gamma\, \alpha^{\gamma-1}. \tag{A13}$$

Denote $y = p + \alpha$. The LHS of (A1) becomes

$$\begin{aligned}
-(\zeta_1 - \overline{\zeta}_1)^2 &= -\Big(c_1(p+\alpha)^{\gamma} - \overline{c}_1(\overline{p+\alpha})^{\overline{\gamma}} + c_2 - \overline{c}_2\Big)^2 \\
&= -\big(c_1(p+\alpha)^{\gamma} - \overline{c}_1(\overline{p+\alpha})^{\overline{\gamma}}\big)^2 \\
&\quad -2(c_2 - \overline{c}_2)\big(c_1(p+\alpha)^{\gamma} - \overline{c}_1(\overline{p+\alpha})^{\overline{\gamma}}\big) - (c_2 - \overline{c}_2)^2 \\
&= -c_1^2\,(p+\alpha)^{2\gamma} - \overline{c}_1^2\,(\overline{p+\alpha})^{2\overline{\gamma}} + 2|c_1|^2\,(p+\alpha)^{\gamma}(\overline{p+\alpha})^{\overline{\gamma}} \\
&\quad -2c_1(c_2 - \overline{c}_2)(p+\alpha)^{\gamma} + 2\overline{c}_1(c_2 - \overline{c}_2)(\overline{p+\alpha})^{\overline{\gamma}} - (c_2 - \overline{c}_2)^2 \\
-(\zeta_1 - \overline{\zeta}_1)^2 &= -c_1^2\, y^{2\gamma} - \overline{c}_1^2\, \overline{y}^{2\overline{\gamma}} + 2|c_1|^2\, y^{\gamma}\, \overline{y}^{\overline{\gamma}} - 2c_1(c_2 - \overline{c}_2)\, y^{\gamma} \\
&\quad +2\overline{c}_1(c_2 - \overline{c}_2)\, \overline{y}^{\overline{\gamma}} - (c_2 - \overline{c}_2)^2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
C_0 &= -(z_1 - \overline{z}_1)^2 - |z_2|^2 = -\big(c_1\, \alpha^{\gamma} - \overline{c}_1\, \overline{\alpha}^{\overline{\gamma}} + c_2 - \overline{c}_2\big)^2 - |z_2|^2 \\
&= 2|c_1|^2\, \alpha^{\gamma}\, \overline{\alpha}^{\overline{\gamma}} - c_1^2\, \alpha^{2\gamma} - \overline{c}_1^2\, \overline{\alpha}^{2\overline{\gamma}} - (c_2 - \overline{c}_2)^2 - 2c_1(c_2 - \overline{c}_2)\alpha^{\gamma} + 2\overline{c}_1(c_2 - \overline{c}_2)\overline{\alpha}^{\overline{\gamma}} - |z_2|^2.
\end{aligned}$$

From the formula in (A12),

$$\zeta_2 = \frac{-(\zeta_1(p)-\overline{z}_1)^2 - C_0}{\overline{w}_1 p + \overline{z}_2} = \frac{-\big(c_1(p+\alpha)^\gamma + c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma}\big)^2 - C_0}{\overline{a}_1 p + \overline{z}2}$$

$$= \frac{\begin{pmatrix} -c_1^2\,(p+\alpha)^{2\gamma} - 2c_1\big(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma}\big)(p+\alpha)^\gamma + c_1^2\,\alpha^{2\gamma} \\ +2c_1\alpha^\gamma(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma}) + |z_2|^2 \end{pmatrix}}{\overline{a}_1 p + \overline{z}_2}.$$

Note that

$$\overline{a}_1 p + \overline{z}_2 = \overline{a}_1(p+\alpha) + (\overline{z}_2 - \overline{a}_1\alpha) = \overline{a}_1 y + (\overline{z}_2 - \overline{a}_1\alpha),$$

$$a_1\overline{p} + z_2 = a_1(\overline{p+\alpha}) + (z_2 - a_1\overline{\alpha}) = a_1\overline{y} + (z_2 - a_1\overline{\alpha}).$$

As a result,

$$\zeta_2 = \frac{\Big(-c_1^2\,y^{2\gamma} - 2c_1(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma})y^\gamma + c_1^2\,\alpha^{2\gamma} + 2c_1\alpha^\gamma(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma}) + |z_2|^2\Big)}{\overline{a}_1 y + (\overline{z}_2 - \overline{a}_1\alpha)}.$$

We then rewrite (A1) in the form of

$$-|\overline{a}_1 p + \overline{z}_2|^2(\zeta_1 - \overline{\zeta}_1)^2 = \Big((\overline{a}_1 p + \overline{z}_2)(\zeta_1'\overline{p} + \zeta_2)\Big)\Big((a_1\overline{p} + z_2)(\overline{\zeta}_1' p + \overline{\zeta}_2)\Big) + C_0|\overline{a}_1 p + \overline{z}_2|^2. \quad \text{(A14)}$$

On the left of (A14), we get

$$-|\overline{a}_1 p + \overline{z}_2|^2\big(\zeta_1 - \overline{\zeta}_1\big)^2 = \Big(|a_1|^2 y\overline{y} + (z_2 - a_1\overline{\alpha})\overline{a}_1 y + (\overline{z}_2 - \overline{a}_1\alpha)a_1\overline{y} + |z_2 - a_1\overline{\alpha}|^2\Big)$$
$$\cdot \begin{pmatrix} -c_1^2 y^{2\gamma} - \overline{c}_1^2\overline{y}^{2\overline{\gamma}} + 2|c_1|^2 y^\gamma\overline{y}^{\overline{\gamma}} \\ -2c_1(c_2 - \overline{c}_2)y^\gamma + 2\overline{c}_1(c_2 - \overline{c}_2)\overline{y}^{\overline{\gamma}} - (c_2 - \overline{c}_2)^2 \end{pmatrix}.$$

Regarding the right hand side of (A14), we first find out

$$\zeta_1'\overline{p} + \zeta_2 = \zeta_1'(\overline{p+\alpha}) - \overline{\alpha}\zeta_1' + \zeta_2 = c_1\gamma(p+\alpha)^{\gamma-1}(\overline{p+\alpha}) - \overline{\alpha}c_1\gamma(p+\alpha)^{\gamma-1}$$
$$+ \frac{1}{\overline{w}_1 p + \overline{z}_2}\begin{pmatrix} -c_1^2(p+\alpha)^{2\gamma} - 2c_1(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma})(p+\alpha)^\gamma \\ +c_1^2\alpha^{2\gamma} + 2c_1\alpha^\gamma(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha^\gamma}) + |z_2|^2 \end{pmatrix}.$$

It leads to the expression of $(\overline{a}_1 p + \overline{z}_2)(\zeta_1' \overline{p} + \zeta_2)$.

$$
\begin{aligned}
& (\overline{a}_1 p + \overline{z}_2)(\zeta_1' \overline{p} + \zeta_2) \\
= \; & \big(\overline{a}_1(p+\alpha) + (\overline{z}_2 - \overline{a}_1\alpha)\big) \cdot \big(c_1\gamma(p+\alpha)^{\gamma-1}(\overline{p+\alpha}) - \overline{\alpha} c_1 \gamma (p+\alpha)^{\gamma-1}\big) \\
& + \Big( - c_1^2 (p+\alpha)^{2\gamma} - 2c_1(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}})(p+\alpha)^{\gamma} + c_1^2 \alpha^{2\gamma} \\
& \qquad + 2c_1\alpha^{\gamma}(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}}) + |z_2|^2 \Big) \\
= \; & \overline{a}_1 c_1 \gamma (p+\alpha)^{\gamma}(\overline{p+\alpha}) \; + \; (\overline{z}_2 - \overline{a}_1\alpha) c_1\gamma(p+\alpha)^{\gamma-1}(\overline{p+\alpha}) \; - \; \overline{a}_1\overline{\alpha} c_1\gamma(p+\alpha)^{\gamma} \\
& - (\overline{z}_2 - \overline{a}_1\alpha)\overline{\alpha} c_1\gamma(p+\alpha)^{\gamma-1} \; - \; c_1^2(p+\alpha)^{2\gamma} \; - \; 2c_1(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}})(p+\alpha)^{\gamma} \\
& + c_1^2\alpha^{2\gamma} \; + \; 2c_1\alpha^{\gamma}(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}}) \; + \; |z_2|^2 \\
= \; & \overline{a}_1 c_1\gamma y^{\gamma}\overline{y} \; + \; (\overline{z}_2 - \overline{a}_1\alpha)c_1\gamma y^{\gamma-1}\overline{y} \; - \; \overline{a}_1\overline{\alpha}c_1\gamma y^{\gamma} \; - \; (\overline{z}_2 - \overline{a}_1\alpha)\overline{\alpha}c_1\gamma y^{\gamma-1} \; - \; c_1^2 y^{2\gamma} \\
& - 2c_1(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}})y^{\gamma} \; + \; c_1^2\alpha^{2\gamma} \; + \; 2c_1\alpha^{\gamma}(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}}) \; + \; |z_2|^2
\end{aligned}
$$

Overall, the right hand side of (A14) is as follows.

$$
\begin{aligned}
& \Big((\overline{a}_1 p + \overline{z}_2)(\zeta_1'\overline{p} + \zeta_2)\Big)\Big((a_1\overline{p} + z_2)(\overline{\zeta}_1' p + \overline{\zeta}_2)\Big) + C_0|\overline{a}_1 p + \overline{z}_2|^2 \\
= & \begin{pmatrix} \overline{a}_1 c_1\gamma y^{\gamma}\overline{y} \; + \; (\overline{z}_2 - \overline{a}_1\alpha)c_1\gamma y^{\gamma-1}\overline{y} \; - \; \overline{a}_1\overline{\alpha}c_1\gamma y^{\gamma} \; - \; (\overline{z}_2 - \overline{a}_1\alpha)\overline{\alpha}c_1\gamma y^{\gamma-1} \; - \; c_1^2 y^{2\gamma} \\ - 2c_1(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}})y^{\gamma} \; + \; c_1^2\alpha^{2\gamma} \; + \; 2c_1\alpha^{\gamma}(c_2 - \overline{c}_2 - \overline{c}_1\overline{\alpha}^{\overline{\gamma}}) \; + \; |z_2|^2 \end{pmatrix} \\
& \cdot \begin{pmatrix} a_1\overline{c}_1\overline{\gamma}y\overline{y}^{\overline{\gamma}} \; + \; (z_2 - a_1\overline{\alpha})\overline{c}_1\overline{\gamma}y\overline{y}^{\overline{\gamma}-1} \; - \; a_1\alpha\overline{c}_1\overline{\gamma}\overline{y}^{\overline{\gamma}} \; - \; (z_2 - a_1\overline{\alpha})\alpha\overline{c}_1\overline{\gamma}\overline{y}^{\overline{\gamma}-1} \; - \; \overline{c}_1^2\overline{y}^{2\overline{\gamma}} \\ - 2\overline{c}_1(\overline{c}_2 - c_2 - c_1\alpha^{\gamma})\overline{y}^{\overline{\gamma}} \; + \; \overline{c}_1^2\overline{\alpha}^{2\overline{\gamma}} \; + \; 2\overline{c}_1\overline{\alpha}^{\overline{\gamma}}(\overline{c}_2 - c_2 - c_1\alpha^{\gamma}) \; + \; |z_2|^2 \end{pmatrix} \\
& - \Big((c_1\alpha^{\gamma} - \overline{c}_1\overline{\alpha}^{\overline{\gamma}} + c_2 - \overline{c}_2)^2 + |z_2|^2\Big) \\
& \quad \cdot \Big(|a_1|^2|y|^2 + (z_2 - a_1\overline{\alpha})\overline{a}_1 y + (\overline{z}_2 - \overline{a}_1\alpha)a_1\overline{y} + |z_2 - a_1\overline{\alpha}|^2\Big)
\end{aligned}
$$

We are ready to compare the exponents of $y$ and $\overline{y}$ on two sides of (A14). On the left, there are 24 different exponents of $y$ or $\overline{y}$:

$$
\begin{aligned}
& y^{2\gamma+1}\overline{y}, \quad y\overline{y}^{2\overline{\gamma}+1}, \quad y^{2\gamma+1}, \quad \overline{y}^{2\overline{\gamma}+1}, \quad y^{2\gamma}\overline{y}, \quad y\overline{y}^{2\overline{\gamma}}, \quad y^{2\gamma}, \quad \overline{y}^{2\overline{\gamma}}, \\
& y^{\gamma+1}\overline{y}^{\overline{\gamma}+1}, \quad y^{\gamma+1}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}^{\overline{\gamma}+1}, \quad y^{\gamma+1}\overline{y}, \quad y\overline{y}^{\overline{\gamma}+1}, \quad y^{\gamma+1}, \quad \overline{y}^{\overline{\gamma}+1}, \\
& y^{\gamma}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}, \quad y\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}, \quad \overline{y}^{\overline{\gamma}}, \quad y\overline{y}, \quad y, \quad \overline{y}, \quad 1.
\end{aligned}
$$

On the right hand side of (A14), there 32 different kind of exponents of $y$ or $\overline{y}$:

$$y^{2\gamma+1}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}^{2\overline{\gamma}+1}, \quad y^{2\gamma+1}\overline{y}^{\overline{\gamma}-1}, \quad y^{\gamma-1}\overline{y}^{2\overline{\gamma}+1},$$

$$y^{2\gamma}\overline{y}^{2\overline{\gamma}}, \quad y^{2\gamma}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}^{2\overline{\gamma}}, \quad y^{2\gamma}\overline{y}^{\overline{\gamma}-1}, \quad y^{\gamma-1}\overline{y}^{2\overline{\gamma}}, \quad y^{2\gamma}, \quad \overline{y}^{2\overline{\gamma}},$$

$$y^{\gamma+1}\overline{y}^{\overline{\gamma}+1}, \quad y^{\gamma+1}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}^{\overline{\gamma}+1}, \quad y^{\gamma+1}\overline{y}^{\overline{\gamma}-1}, \quad y^{\gamma-1}\overline{y}^{\overline{\gamma}+1},$$

$$y^{\gamma}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}^{\overline{\gamma}-1}, \quad y^{\gamma-1}\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}\overline{y}, \quad y\overline{y}^{\overline{\gamma}}, \quad y^{\gamma}, \quad \overline{y}^{\overline{\gamma}},$$

$$y^{\gamma-1}\overline{y}^{\overline{\gamma}-1}, \quad y^{\gamma-1}\overline{y}, \quad y\overline{y}^{\overline{\gamma}-1}, \quad y^{\gamma-1}, \quad \overline{y}^{\overline{\gamma}-1}, \quad y\overline{y}, \quad y, \quad \overline{y}, \quad 1.$$

From the right of (A14), the coefficient of $y^{2\gamma}\,\overline{y}^{2\overline{\gamma}}$ is $|c_1|^4 \neq 0$. This term must belong to an exponent of $y$ and $\overline{y}$ on the left hand side. If it is represented by a *real* monomial $y^{\delta}\,\overline{y}^{\overline{\delta}}$ on the left, then we have

$$2\gamma = \gamma + 1, \quad 2\gamma = \gamma, \quad 2\gamma = 1 \quad \text{or} \quad 2\gamma = 0.$$

So $\gamma = 0, 1/2$ or $1$ in this situation.

On the other hand, it may happen that $y^{2\gamma}\,\overline{y}^{2\overline{\gamma}}$ is represented by a *complex* monomial $y^{\delta_1}\,\overline{y}^{\overline{\delta}_2}$ on the left, which happens to be real-valued with a particular $\gamma$. We may have

$$\begin{array}{llll} 2\gamma = 2\gamma + 1 = 1, & 2\gamma = 2\gamma + 1 = 0, & 2\gamma = 2\gamma = 1, & 2\gamma = 2\gamma = 0, \\ 2\gamma = 2\gamma + 1 = \gamma, & 2\gamma = \gamma + 1 = 1, & 2\gamma = \gamma + 1 = 0, & 2\gamma = \gamma = 1 \\ 2\gamma = \gamma = 0, & 2\gamma = 1 = 0, & \text{or} \;\; 2\gamma = 0. & \end{array}$$

From any one of the above equalities, we must have $\gamma = 0$ or $1/2$. When $\gamma = 1/2$, note that the coefficient of $y^{2\gamma+1}\,\overline{y}^{\overline{\gamma}}$ on the right is $-|c_1|^2 c_1 a_1 \overline{\gamma}$ with

$$a_1 = c_1\,\gamma\,\alpha^{\gamma-1} = \frac{c_1}{2}\,\alpha^{-1/2} \neq 0.$$

So $y^{2\gamma+1}\,\overline{y}^{\overline{\gamma}}$ is a non-zero term when $\gamma = 1/2$. On the left, we have exponents of $y$ or $\overline{y}$ from:

$$y^2\overline{y}, \quad y\overline{y}^2, \quad y^2, \quad \overline{y}^2, \quad y\overline{y}, \quad y\overline{y}, \quad y, \quad \overline{y}, \quad y^{3/2}\overline{y}^{3/2}, \quad y^{3/2}\overline{y}^{1/2}, \quad y^{1/2}\overline{y}^{3/2}, \quad y^{3/2}\overline{y},$$

$$y\overline{y}^{3/2}, \quad y^{3/2}, \quad \overline{y}^{3/2}, \quad y^{1/2}\overline{y}^{1/2}, \quad y^{1/2}\overline{y}, \quad y\overline{y}^{1/2}, \quad y^{1/2}, \quad \overline{y}^{1/2}, \quad y\overline{y}, \quad y, \quad \overline{y}, \quad 1.$$

There is no $y^2\,\overline{y}^{1/2}$ term on the left hand side, so we arrive at a contradiction. $\gamma \neq 1/2$. It means that $\gamma = 0$ or 1. $\gamma = 0$ is a trivial case, and when $\gamma = 1$, $\zeta_1$ is linear in $p$. We have shown that the possible solutions of $\zeta_1$ to (A8), is either a constant or a linear function in $p$.

At the end of our analysis, we explore what happens when $\zeta_1$ is exactly a linear function in $p$. Let $\zeta_1(p) = a_0 + a_1\,p$ with $a_1 \neq 0$. Consider (A3) with $n = 1$ and $k \geq 1$, we get

$$2\,\overline{a}_1\,a_k \;=\; k\,\overline{a}_1\,a_k + (k+1)\,a_{k+1}\,\overline{b}_0 + 2\,\overline{a}_2\,b_{k-1} + \overline{b}_1\,b_k \;=\; k\,\overline{a}_1\,a_k + \overline{b}_1\,b_k.$$

It means $(2-k)\,\overline{a}_1\,a_k = \overline{b}_1\,b_k$ for any $k \geq 1$. Put $k = 1$, we have $|a_1|^2 = |b_1|^2 \neq 0$. So, $b_1 \neq 0$. Whenever $k \geq 2$, the left hand side is zero, so $b_k = 0$ for all $k \geq 2$. In other words, we may let $\zeta_2 = b_0 + b_1\,p$ with $b_0, b_1 \in \mathbb{C}$.

Here we may apply the formula for $\zeta_2$ in (A12). We have

$$\zeta_2 \;=\; \frac{-(\zeta_1 - \overline{a}_0)^2 + (a_0 - \overline{a}_0)^2 + |b_0|^2}{\overline{a}_1\,p + \overline{b}_0} \;=\; \frac{-a_1^2\,p^2 - 2a_1(a_0 - \overline{a}_0)\,p + |b_0|^2}{\overline{a}_1\,p + \overline{b}_0}.$$

It implies that $(\overline{a}_1\,p + \overline{b}_0)(b_1\,p + b_0) = -a_1^2\,p^2 - 2a_1(a_0 - \overline{a}_0)\,p + |b_0|^2$. By comparison, we get

$$\overline{a}_1\,b_1 = -a_1^2 \quad \text{and} \quad \overline{b}_0\,b_1 + \overline{a}_1\,b_0 = -2a_1(a_0 - \overline{a}_0).$$

On solving, the first equation gives $b_1 = -\dfrac{a_1^2}{\overline{a}_1}$. The second equation is equivalent to

$$i\big(\overline{a}_1^2\,b_0 - a_1^2\,\overline{b}_0\big) \;=\; -\,2i\,|a_1|^2\big(a_0 - \overline{a}_0\big).$$

A general solution to the above equation is

$$b_0 \;=\; -\,\frac{2\,a_1\,a_0}{\overline{a}_1} + k\,a_1^2 \quad \text{for any} \quad k \in \mathbb{R}.$$

Therefore, $\zeta_2 \;=\; -\,\dfrac{a_1^2}{\overline{a}_1}\,p - \dfrac{2\,a_1\,a_0}{\overline{a}_1} + k\,a_1^2$ for any real number $k$.

## APPENDIX B. PROOF OF THEOREM 6.2

This appendix is to justify Theorem 6.2 in order to complete our logical argument. We will mainly follow the notations from J. M. Lee's paper [11] on Fefferman metric (but with the Levi form multiplied by a factor of 2) and also refer to [4]. Moreover, since we are proving a general result in CR geometry, notations and variables defined in the following will be used exclusively to the current chapter.

The statement of Theorem 6.2 is as follows. The definition of the Chern tensor follows from (1.5), and that of the Weyl tensor from (1.15) and (1.16).

**Theorem.** *Let $N$ be a CR manifold of hypersurface type on which the CR structure $\mathbb{L}$ is non-degenerate. Let $\nu$ be the CR dimension of $\mathbb{L}$. Associated with the contact form $\theta$, let $C$ be the Chern curvature tensor and $F$ be the Fefferman metric on $C(N)$. Suppose $\mathcal{W}$ is the Weyl tensor of $F$. Then,*

*(1)* : $C_{m\bar{n}k\bar{l}} = \mathcal{W}(T_m, T_{\bar{n}}, T_k, T_{\bar{l}})$ *for every $m$, $n$, $k$, $l$.*

*(2)* : $\mathcal{W}(T_m, T_n, Z_1, Z_2) = 0$ *for any $m, n$. $Z_1$, $Z_2$ are tangent vectors on $C(N)$.*

To begin with, let $\mathcal{B} = \{T_1, T_2, \cdots, T_\nu\}$ be the basis for the holomorphic bundle $T^{1,0}N$ and $\{T_{\bar{1}}, T_{\bar{2}}, \cdots, T_{\bar{\nu}}\}$ be the basis for the antiholomorphic bundle $T^{0,1}N$. With respect to the contact form $\theta$, let $\mathcal{L}_\theta$ be the Levi form and $T_0$ $(= T)$ be the Reeb vector field.

The derivative of $\theta$ is

$$d\theta = 2i\, h_{\alpha\bar{\beta}}\, \theta^{\alpha} \wedge \theta^{\bar{\beta}} \quad \text{for} \quad h_{\alpha\bar{\beta}} = \mathcal{L}_{\theta}(T_{\alpha}, T_{\bar{\beta}}). \tag{B1}$$

Define $g$ to be the Webster metric associated with $\theta$. $\nabla$ is the Tanaka-Webster connection on $N$ such that its Christoffel symbols are given by

$$\nabla_{T_{\alpha}} T_{\beta} = \Gamma^{\gamma}_{\alpha\beta}\, T_{\gamma}, \quad \nabla_{T_{\bar{\alpha}}} T_{\beta} = \Gamma^{\gamma}_{\bar{\alpha}\beta}\, T_{\gamma} \quad \text{and} \quad \nabla_{T_0} T_{\beta} = \Gamma^{\gamma}_{0\beta}\, T_{\gamma}. \tag{B2}$$

Let $\left\{\theta^1, \theta^2, \cdots, \theta^{\nu}\right\}$ be the dual coframe of $\mathcal{B}$ and so $\theta^{\bar{\alpha}} = \overline{\theta^{\alpha}}$ is the dual 1-form of $T_{\bar{\alpha}}$. The connection forms of $\nabla$ are then defined by

$$\omega^{\gamma}_{\beta} = \Gamma^{\gamma}_{\alpha\beta}\, \theta^{\alpha} + \Gamma^{\gamma}_{\bar{\alpha}\beta}\, \theta^{\bar{\alpha}} + \Gamma^{\gamma}_{0\beta}\, \theta. \tag{B3}$$

Here we write $\Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} = \overline{(\Gamma^{\gamma}_{\alpha\beta})}$, $\Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}} = \overline{(\Gamma^{\gamma}_{\bar{\alpha}\beta})}$ and $\Gamma^{\bar{\gamma}}_{0\bar{\beta}} = \overline{(\Gamma^{\gamma}_{0\beta})}$. Also, we have

$$dh_{\alpha\bar{\beta}} = h_{\alpha\bar{\gamma}}\, \omega^{\bar{\gamma}}_{\bar{\beta}} + h_{\gamma\bar{\beta}}\, \omega^{\alpha}_{\gamma}, \tag{B4}$$

with $\omega^{\bar{\gamma}}_{\bar{\beta}} = \overline{\omega^{\gamma}_{\beta}}$. Similar to the Cartan first structural equation, we have

$$d\theta^{\alpha} = \theta^{\beta} \wedge \omega^{\alpha}_{\beta} + \theta \wedge \tau^{\alpha} \quad \text{and} \quad d\theta^{\bar{\alpha}} = \theta^{\bar{\beta}} \wedge \omega^{\bar{\alpha}}_{\bar{\beta}} + \theta \wedge \tau^{\bar{\alpha}}. \tag{B5}$$

In this way, the Lie brackets between $T_{\alpha}$, $T_{\bar{\beta}}$ and $T_0$ are found by

$$\begin{cases} [T_{\alpha}, T_{\bar{\beta}}] &= -2i\, h_{\alpha\bar{\beta}}\, T_0 - \Gamma^{\gamma}_{\bar{\beta}\alpha}\, T_{\gamma} + \Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}}\, T_{\bar{\gamma}} \\ [T_{\alpha}, T_{\beta}] &= \left(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}\right) T_{\gamma} \\ [T_{\alpha}, T_0] &= -\Gamma^{\gamma}_{0\alpha}\, T_{\gamma} + A^{\bar{\gamma}}_{\alpha}\, T_{\bar{\gamma}} \end{cases} \tag{B6}$$

The $\tau$-operator in (B5) is defined by: for any $X$ on $TN$,

$$\tau(X) = \mathrm{tor}(T_0, X) = \nabla_{T_0} X - [T_0, X].$$

It allows us to write

$$\tau = \tau^{\alpha} \otimes T_{\alpha} + \tau^{\bar{\alpha}} \otimes T_{\bar{\alpha}} \quad \text{with} \quad \tau^{\alpha} = A^{\alpha}_{\bar{\beta}}\, \theta^{\bar{\beta}} \ \text{ and } \ \tau^{\bar{\alpha}} = A^{\bar{\alpha}}_{\beta}\, \theta^{\beta}. \tag{B7}$$

The coefficients $A_\alpha^{\bar\beta}$'s appear both in (B6) and (B7). We can find them by the formula

$$A_\alpha^{\bar\beta} = -h^{\bar\beta\gamma}\, g\big([T_0, T_\alpha], T_\gamma\big).$$

The lower-index of $A_\alpha^{\bar\beta}$ by $g$ is defined by $A_{\alpha\gamma} = A_\alpha^{\bar\beta}\, h_{\gamma\bar\beta}$. Taking the complex conjugation,

$$A_{\bar\alpha}^{\beta} = \overline{(A_\alpha^{\bar\beta})} \quad \text{and} \quad A_{\bar\alpha\bar\gamma} = \overline{(A_{\alpha\gamma})}.$$

Correspondingly, the lower-index of the (1,1)-tensor $\tau$ is denoted by $A$, which is a (0,2)-tensor given by $A(u,v) = g\big(\tau(u), v\big)$ for any $u$, $v$ on $TN$. Note that $\tau$ is self-adjoint with respect to $g$, so $A_{\alpha\gamma} = A_{\gamma\alpha}$. It implies that

$$A_\alpha^{\bar\gamma} = h^{\bar\gamma\beta}\, A_\beta^{\bar\delta}\, h_{\alpha\bar\delta} \quad \text{and} \quad A_{\bar\alpha}^{\gamma} = h^{\bar\beta\gamma}\, A_{\bar\beta}^{\delta}\, h_{\delta\bar\alpha}. \tag{B8}$$

We would also compute for the covariant derivative of $A$ by $\nabla$. Let

$$\nabla_{\bar\sigma} A_{\alpha\lambda} \;=\; \big(\nabla_{T_{\bar\sigma}} A\big)\big(T_\alpha, T_\lambda\big) \;=\; T_{\bar\sigma}(A_{\alpha\lambda}) - A_{\delta\lambda}\,\Gamma_{\bar\sigma\alpha}^{\delta} - A_{\alpha\delta}\,\Gamma_{\sigma\lambda}^{\delta}$$

Let $V_{\alpha\lambda}^{\beta} = h^{\bar\sigma\beta}\,\nabla_{\bar\sigma} A_{\alpha\lambda}$ and $V_{\alpha\bar\mu}^{\beta} = h^{\bar\sigma\beta}\,\nabla_\alpha A_{\bar\mu\bar\sigma}$. We also have

$$V_\lambda \;\triangleq\; \sum_{\alpha=1}^{\nu} V_{\alpha\lambda}^{\alpha} \;=\; \sum_{\bar\alpha=1}^{\nu} V_{\bar\alpha\lambda}^{\bar\alpha}. \tag{B9}$$

Following by $\tau$, we look into the Ricci tensor of $\nabla$. Let $R$ be the Riemann tensor.

$$R(u,v)T_\alpha = 2\big(d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta\big)(u,v) \otimes T_\alpha \tag{B10}$$

The curvature form on the RHS of (B10) is found by

$$\begin{aligned} d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta \;&=\; R_\alpha{}^\beta{}_{\lambda\bar\mu}\;\theta^\lambda \wedge \theta^{\bar\mu} + V_{\alpha\lambda}^{\beta}\;\theta^\lambda \wedge \theta - V_{\alpha\bar\lambda}^{\beta}\;\theta^{\bar\lambda} \wedge \theta \\ &\quad +2i\, h_{\alpha\bar\gamma}\;\theta^{\bar\gamma} \wedge \tau^\beta + 2i\, h_{\alpha\bar\gamma}\;\theta^\beta \wedge \tau^{\bar\gamma} \\ &=\; R_\alpha{}^\beta{}_{\lambda\bar\mu}\;\theta^\lambda \wedge \theta^{\bar\mu} + V_{\alpha\lambda}^{\beta}\;\theta^\lambda \wedge \theta - V_{\alpha\bar\lambda}^{\beta}\;\theta^{\bar\lambda} \wedge \theta \\ &\quad +2i\, A_{\delta\alpha}\;\theta^\beta \wedge \theta^\delta + 2i\, h_{\alpha\bar\gamma} A_{\bar\delta}^{\beta}\;\theta^{\bar\gamma} \wedge \theta^{\bar\delta}. \end{aligned} \tag{B11}$$

Contracting $\alpha$ and $\beta$ in (B11), $R_{\lambda\bar{\mu}} = R_{\alpha}{}^{\alpha}{}_{\lambda\bar{\mu}} = 2(d\omega^{\alpha}_{\alpha})\big(T_{\lambda}, T_{\bar{\mu}}\big)$ with

$$\sum_{\alpha=1}^{\nu} d\omega^{\alpha}_{\alpha} = R_{\lambda\bar{\mu}}\ \theta^{\lambda}\wedge\theta^{\bar{\mu}} + V_{\gamma}\ \theta^{\gamma}\wedge\theta - V_{\bar{\delta}}\ \theta^{\bar{\delta}}\wedge\theta. \tag{B12}$$

As a crucial component to the Chern tensor, we let $D$ be the (0,2)-tensor,

$$D_{\alpha\bar{\beta}} \ = \ \frac{i}{\nu+2}R_{\alpha\bar{\beta}} - \frac{i}{2(\nu+1)(\nu+2)}\rho\, h_{\alpha\bar{\beta}}. \tag{B13}$$

Here $\rho$ is the scalar curvature of $\nabla$. The raise-index of $D$ is correspondingly given by

$$D^{\gamma}_{\alpha} \ = \ D_{\alpha\bar{\beta}}\, h^{\bar{\beta}\gamma} \ = \ \frac{i}{\nu+2}\, R^{\gamma}_{\alpha} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,\delta_{\alpha\gamma}. \tag{B14}$$

Because both $R_{\alpha\bar{\beta}}$ and $h_{\alpha\bar{\beta}}$ are hermitian on their indices, we have

$$D_{\beta\bar{\alpha}} = -\overline{(D_{\alpha\bar{\beta}})} \quad \text{and} \quad D^{\bar{\gamma}}_{\bar{\alpha}} = D_{\beta\bar{\alpha}}\, h^{\bar{\gamma}\beta} = -\overline{(D^{\gamma}_{\alpha})}.$$

The trace of $D$ can be obtained by

$$\sum_{\gamma=1}^{\nu}(D^{\gamma}_{\gamma}) \ = \ \frac{i}{\nu+2}\,\rho - \frac{i\,\nu}{2(\nu+1)(\nu+2)}\rho \ = \ \frac{i}{2(\nu+1)}\rho.$$

It also means that the sum $(D^{\bar{\gamma}}_{\bar{\gamma}}) = -\overline{(D^{\gamma}_{\gamma})} = (D^{\gamma}_{\gamma})$.

Let $C(N)$ be the Fefferman bundle of $(N, \mathbb{L})$ and $\pi : C(N) \to N$ be the projection map. In addition to the local coordinates $x$ on $N$, we let $\gamma$ be the fibre coordinate on $C(N)$ which represents the point $e^{i\gamma}\zeta$ on $C_x(N)$. $\zeta$ is a prescribed $(\nu+1, 0)$-form.

The Fefferman metric $F$ is defined by

$$F = 2h_{\alpha\bar{\beta}}\,\theta^{\alpha}\odot\theta^{\bar{\beta}} + 2\,\theta\odot\sigma.$$

We will denote $\pi^*\omega$ by $\omega$ for any smooth forms $\omega$ on $N$. The $\sigma$-tensor is given by

$$\sigma = \frac{1}{\nu+2}\Big(d\gamma + i\,\omega^{\alpha}_{\alpha} - \frac{i}{2}\,h^{\bar{\beta}\alpha}\,dh_{\alpha\bar{\beta}} - \frac{1}{4(\nu+1)}\,\rho\,\theta\Big) \tag{B15}$$

Let $\mathcal{Z}$ be a moving frame on $C(N)$ which consists of vectors $Z_\alpha$, $Z_{\bar\alpha}$, $Z_0$ and $Z_c$. $(1 \le \alpha \le \nu)$.

$$\begin{cases} Z_\alpha &= T_\alpha - \dfrac{i}{2}\big(\Gamma^\gamma_{\alpha\gamma} - \Gamma^{\bar\gamma}_{\alpha\bar\gamma}\big)\dfrac{\partial}{\partial\gamma} \\ Z_{\bar\alpha} &= T_{\bar\alpha} - \dfrac{i}{2}\big(\Gamma^\gamma_{\bar\alpha\gamma} - \Gamma^{\bar\gamma}_{\bar\alpha\bar\gamma}\big)\dfrac{\partial}{\partial\gamma} \\ Z_0 &= T_0 - \Big(\dfrac{i}{2}\big(\Gamma^\alpha_{0\alpha} - \Gamma^{\bar\alpha}_{0\bar\alpha}\big) - \dfrac{\rho}{4(\nu+1)}\Big)\dfrac{\partial}{\partial\gamma} \\ Z_c &= (\nu+2)\,\dfrac{\partial}{\partial\gamma} \end{cases}$$

The matrix representation of $F$ corresponding to $\mathcal{Z} = \Big\{Z_0, Z_1, \cdots, Z_\nu, Z_{\bar 1}, \cdots Z_{\bar\nu}, Z_c\Big\}$, is

$$[F] = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0}_\nu & \big[h_{\alpha\bar\beta}\big] & \mathbf{0} \\ \mathbf{0} & \big[h_{\beta\bar\alpha}\big] & \mathbf{0}_\nu & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} & 0 \end{bmatrix} \quad \text{and} \quad [F^{-1}] = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0}_\nu & \big[h^{\bar\beta\alpha}\big] & \mathbf{0} \\ \mathbf{0} & \big[h^{\bar\alpha\beta}\big] & \mathbf{0}_\nu & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}.$$

Here $\alpha$ and $\beta$ are the respective row and column indices.

The Lie brackets between vectors in $\mathcal{Z}$ are described by

$$\begin{cases} [Z_\alpha, Z_\beta] &= \big(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\big)\, Z_\gamma, \\ [Z_\alpha, Z_{\bar\beta}] &= -2i\, h_{\alpha\bar\beta}\, Z_0 - \Gamma^\gamma_{\bar\beta\alpha}\, Z_\gamma + \Gamma^{\bar\gamma}_{\alpha\bar\beta}\, Z_{\bar\gamma} - D_{\alpha\bar\beta}\, Z_c, \\ [Z_\alpha, Z_0] &= -\Gamma^\gamma_{0\alpha}\, Z_\gamma + A^{\bar\gamma}_\alpha\, Z_{\bar\gamma} + \Big(\dfrac{1}{4(\nu+1)(\nu+2)} T_\alpha(\rho) - \dfrac{i}{(\nu+2)} V_\alpha\Big)\, Z_c. \end{cases} \tag{B16}$$

In the following context, an index of capital letters runs within the set $\big\{0, 1, \cdots, \nu, \bar 1, \cdots \bar\nu, c\big\}$. For example, if $X = \xi_P\, Z_P$ is a vector field on $C(N)$. then it means

$$X = \xi_0\, Z_0 + \xi_1\, Z_1 + \cdots + \xi_\nu\, Z_\nu + \xi_{\bar 1}\, Z_{\bar 1} + \cdots + \xi_{\bar\nu}\, Z_{\bar\nu} + \xi_c\, Z_c.$$

Meanwhile, an index of Greek or small letters runs from 1 to $\nu$ only, or from $\bar 1$ to $\bar\nu$.

Let $\hat{\nabla}$ be the Levi-Civita connection of $F$. The connection forms are defined by

$$\hat{\nabla}_X Z_Q \;=\; \hat{\omega}^P_Q(X) \otimes Z_P \;=\; \xi_B\, \hat{\Gamma}^P_{BQ}\, Z_P. \tag{B17}$$

They are explicitly given by

$$\begin{aligned}
\hat{\omega}^\gamma_\beta &= \omega^\gamma_\beta + \frac{1}{2} D^\gamma_\beta\, \theta + i\, \delta_{\beta\gamma}\, \sigma \\
\hat{\omega}^{\bar{\gamma}}_{\bar{\beta}} &= \omega^{\bar{\gamma}}_{\bar{\beta}} - \frac{1}{2} D^{\bar{\gamma}}_{\bar{\beta}}\, \theta - i\, \delta_{\beta\gamma}\, \sigma \\
\hat{\omega}^\gamma_{\bar{\beta}} &= 0 \\
\hat{\omega}^{\bar{\gamma}}_\beta &= 0 \\
\hat{\omega}^0_\beta &= i\, h_{\beta\bar{\alpha}}\, \theta^{\bar{\alpha}} \\
\hat{\omega}^0_{\bar{\beta}} &= -i\, h_{\alpha\bar{\beta}}\, \theta^\alpha \\
\hat{\omega}^c_\beta &= -A_{\beta\alpha}\, \theta^\alpha + \frac{1}{2} D_{\beta\bar{\alpha}}\, \theta^{\bar{\alpha}} + \frac{i}{2(\nu+2)} \Big( 2V_\beta + \frac{i}{2(\nu+1)} T_\beta(\rho) \Big) \theta \\
\hat{\omega}^c_{\bar{\beta}} &= -A_{\bar{\beta}\bar{\alpha}}\, \theta^{\bar{\alpha}} - \frac{1}{2} D_{\alpha\bar{\beta}}\, \theta^\alpha - \frac{i}{2(\nu+2)} \Big( 2V_{\bar{\beta}} - \frac{i}{2(\nu+1)} T_{\bar{\beta}}(\rho) \Big) \theta \\
\hat{\omega}^\gamma_c &= i\, \theta^\gamma \\
\hat{\omega}^{\bar{\gamma}}_c &= -i\, \theta^{\bar{\gamma}} \\
\hat{\omega}^\gamma_0 &= \frac{1}{2} D^\gamma_\alpha\, \theta^\alpha + A^\gamma_{\bar{\alpha}}\, \theta^{\bar{\alpha}} + \frac{i}{2(\nu+2)} h^{\bar{\delta}\gamma} \Big( 2V_{\bar{\delta}} - \frac{i}{2(\nu+1)} T_{\bar{\delta}}(\rho) \Big) \theta \\
\hat{\omega}^{\bar{\gamma}}_0 &= -\frac{1}{2} D^{\bar{\gamma}}_{\bar{\alpha}}\, \theta^{\bar{\alpha}} + A^{\bar{\gamma}}_\alpha\, \theta^\alpha - \frac{i}{2(\nu+2)} h^{\bar{\gamma}\delta} \Big( 2V_\delta + \frac{i}{2(\nu+1)} T_\delta(\rho) \Big) \theta \\
\hat{\omega}^0_0 &= \hat{\omega}^c_0 \;=\; \hat{\omega}^0_c \;=\; \hat{\omega}^c_c \;=\; 0
\end{aligned}$$

## Part (1): Proof of $\mathcal{W}_{m\bar{n}k\bar{l}} = C_{m\bar{n}k\bar{l}}$

We assume $\hat{R}$ to be the Riemann tensor and Ric to be the Ricci tensor of $\hat{\nabla}$. We set up the notation that

$$\begin{aligned} \hat{R}(Z_P, Z_Q)Z_J &= \hat{R}^K_{PQJ}\, Z_K = 2\,\Omega^K_J(Z_P, Z_Q) \otimes Z_K, \\ \hat{R}_{PQJK} &= \hat{R}^A_{PQJ}\, f_{AK} \quad \text{and} \quad \hat{R}_{PQ} = \mathrm{Ric}(Z_P, Z_Q). \end{aligned} \tag{B18}$$

The symbol $f_{AK}$ means the metric coefficient $F(Z_A, Z_K)$. Let $S$ be the scalar curvature.

The Weyl tensor (1.15) of $\hat{\nabla}$, restricted to $T^{1,0}N \oplus T^{0,1}N$, is represented by

$$\begin{aligned} \mathcal{W}_{m\bar{n}k\bar{l}} &= \mathcal{W}(Z_m, Z_{\bar{n}}, Z_k, Z_{\bar{l}}) \\ &= \hat{R}_{m\bar{n}k\bar{l}} - \frac{1}{2\nu}\Big(\hat{R}_{m\bar{l}}\, f_{k\bar{n}} + \hat{R}_{k\bar{n}}\, f_{m\bar{l}} - \hat{R}_{mk}\, f_{\bar{n}\bar{l}} - \hat{R}_{\bar{n}\bar{l}}\, f_{mk}\Big) \\ &\quad + \frac{S}{(2\nu)(2\nu+1)}\big(f_{m\bar{l}}\, f_{k\bar{n}} - f_{mk}\, f_{\bar{n}\bar{l}}\big) \\ &= \hat{R}_{m\bar{n}k\bar{l}} - \frac{1}{2\nu}\Big(\hat{R}_{m\bar{l}}\, f_{k\bar{n}} + \hat{R}_{k\bar{n}}\, f_{m\bar{l}}\Big) + \frac{S}{(2\nu)(2\nu+1)} f_{m\bar{l}}\, f_{k\bar{n}} \end{aligned} \tag{B19}$$

and so on. Note that $\iota_{\frac{\partial}{\partial\gamma}}\mathcal{W} = 0$. So

$$\mathcal{W}_{m\bar{n}k\bar{l}} = \mathcal{W}(Z_m, Z_{\bar{n}}, Z_k, Z_{\bar{l}}) = \mathcal{W}(T_m, T_{\bar{n}}, T_k, T_{\bar{l}}).$$

Together with the symmetry that $\mathcal{W}_{m\bar{n}k\bar{l}} = \mathcal{W}_{k\bar{l}m\bar{n}}$, we are going to show that

$$\mathcal{W}_{m\bar{n}k\bar{l}} = C_{k\bar{l}m\bar{n}}$$

for any $m$, $n$, $k$ and $l$. To begin with, we find out $\hat{R}_{m\bar{n}k\bar{l}}$ by

$$\hat{R}^p_{m\bar{n}k} = 2\,\Omega^p_k(Z_m, Z_{\bar{n}}) = 2\big(d\hat{\omega}^p_k - \hat{\omega}^Q_k \wedge \hat{\omega}^p_Q\big)(Z_m, Z_{\bar{n}}) \tag{B20}$$

The first term of (B20) is

$$\begin{aligned} d\hat{\omega}^p_k(Z_m, Z_{\bar{n}}) &= d\big(\omega^p_k + \frac{1}{2}\, D^p_k\, \theta + i\, \delta_{kp}\, \sigma\big)(Z_m, Z_{\bar{n}}) \\ &= d\omega^p_k(T_m, T_{\bar{n}}) + \frac{i}{2}\, D^p_k\, h_{m\bar{n}} + i\, \delta_{kp}\; d\sigma(T_m, T_{\bar{n}}) \\ &= d\omega^p_k(T_m, T_{\bar{n}}) + \frac{i}{2}\, D^p_k\, h_{m\bar{n}} + \frac{i}{2}\delta_{kp} D_{m\bar{n}}. \end{aligned}$$

Note that

$$d\sigma \;=\; \frac{1}{\nu+2}\; d\Big(d\gamma + i\, \omega^\alpha_\alpha - \frac{i}{2}\, h^{\bar{\beta}\alpha}\, dh_{\alpha\bar{\beta}} - \frac{1}{4(\nu+1)}\, \rho\, \theta\Big).$$

For an invertible hermitian matrix function $\mathsf{H} = [\mathsf{h}_{mn}]$, we have

$$\mathsf{h}^{nm}\, d\mathsf{h}_{mn} \;=\; d\Big(\log\big(\det(\mathsf{H})\big)\Big) \tag{B21}$$

We diagonalize $\mathsf{H}$ by $\mathsf{H} = \mathsf{U}^*\, \mathsf{D}\, \mathsf{U}$ such that $\mathsf{U}$ is a unitary matrix. Suppose $\mathsf{u}_{km} = \mathsf{U}_{km}$ and $\lambda_{km} = \mathsf{D}_{km} = \lambda_k\, \delta_{km}$.

$$\begin{aligned} \mathsf{h}_{mn} &= \overline{\mathsf{u}_{km}}\, \lambda_{kp}\, \mathsf{u}_{pn} \\ \Longrightarrow\quad d\mathsf{h}_{mn} &= \lambda_{kp}\, \mathsf{u}_{pn}\, d\big(\overline{\mathsf{u}_{km}}\big) + \overline{\mathsf{u}_{km}}\, \lambda_{kp}\, d(\mathsf{u}_{pn}) + \overline{\mathsf{u}_{km}}\, \mathsf{u}_{pn}\, d\lambda_{kp} \end{aligned}$$

When $\overline{\mathsf{u}_{km}}\, \mathsf{u}_{kq} = \delta_{mq}$, we get $\mathsf{u}_{kq}\, d(\overline{\mathsf{u}_{km}}) + \overline{\mathsf{u}}_{pm}\, d\mathsf{u}_{pq} = 0$. It makes $d(\overline{\mathsf{u}_{km}}) = -\overline{\mathsf{u}}_{pm}\, \overline{\mathsf{u}}_{kq}\, d\mathsf{u}_{pq}$.

$$\begin{aligned} \Longrightarrow\quad d\mathsf{h}_{mn} &= -\lambda_{kp}\, \mathsf{u}_{pn}\, \big(\overline{\mathsf{u}_{am}}\, \overline{\mathsf{u}_{kb}}\, d\mathsf{u}_{ab}\big) + \overline{\mathsf{u}_{km}}\, \lambda_{kp}\, d\mathsf{u}_{pn} + \overline{\mathsf{u}_{km}}\, \mathsf{u}_{pn}\, d\lambda_{kp} \\ &= -\mathsf{h}_{bn}\, \overline{\mathsf{u}_{am}}\, d\mathsf{u}_{ab} + \mathsf{h}_{mb}\, \overline{\mathsf{u}_{pb}}\, d\mathsf{u}_{pn} + \overline{\mathsf{u}_{km}}\, \mathsf{u}_{pn}\, d\lambda_{kp} \\ \Longrightarrow\quad \mathsf{h}^{nm}\, d\mathsf{h}_{mn} &= -\delta_{mb}\, \overline{\mathsf{u}_{am}}\, d\mathsf{u}_{ab} + \delta_{nb}\, \overline{\mathsf{u}_{pb}}\, d\mathsf{u}_{pn} + \big(\overline{\mathsf{u}_{km}}\, \mathsf{h}^{nm}\, \mathsf{u}_{pn}\big)\, d\lambda_{kp} \\ &= -\overline{\mathsf{u}_{ab}}\, d\mathsf{u}_{ab} + \overline{\mathsf{u}_{pn}}\, d\mathsf{u}_{pn} + \frac{1}{\lambda_{pk}}\, d\lambda_{kp} \;=\; \sum_k \frac{1}{\lambda_k}\, d\lambda_k \end{aligned}$$

Since $\det(\mathsf{H}) = \lambda_1\lambda_2\cdots\lambda_\nu$, (B21) is justified. As an application,

$$d\big(h^{\bar{\beta}\alpha}\; dh_{\alpha\bar{\beta}}\big) = d^2\Big(\log\big(\det([h_{\alpha\bar{\beta}}])\big)\Big) \;=\; 0$$

$$\Longrightarrow\quad d\sigma \;=\; \frac{1}{\nu+2}\Big(i\, d\omega^\alpha_\alpha - \frac{1}{4(\nu+1)}\, d\rho \wedge \theta - \frac{1}{4(\nu+1)}\, \rho\, d\theta\Big).$$

Therefore, we get

$$d\sigma(T_m, T_{\bar{n}}) \;=\; \frac{i}{2(\nu+2)}\, R_{m\bar{n}} - \frac{1}{4(\nu+1)(\nu+2)}\, \rho\,(i\, h_{m\bar{n}}) \;=\; \frac{1}{2}\, D_{m\bar{n}}$$

in the first term of (B20). Other terms of (B20) are given by

$$\begin{aligned}
\big(\hat{\omega}^{\gamma}_{k} \wedge \hat{\omega}^{p}_{\gamma}\big)(Z_m, Z_{\bar{n}}) &= \big(\omega^{\gamma}_{k} \wedge \omega^{p}_{\gamma}\big)(T_m, T_{\bar{n}}) \\
\big(\hat{\omega}^{0}_{k} \wedge \hat{\omega}^{p}_{0}\big)(Z_m, Z_{\bar{n}}) &= -\frac{i}{4}\, h_{k\bar{n}}\, D^{p}_{m} \\
\big(\hat{\omega}^{c}_{k} \wedge \hat{\omega}^{p}_{c}\big)(Z_m, Z_{\bar{n}}) &= -\frac{i}{4}\, \delta_{mp}\, D_{k\bar{n}}
\end{aligned}$$

Therefore,

$$\hat{R}^{p}_{m\bar{n}k} \;=\; R_k{}^{p}{}_{m\bar{n}} + i\, D^{p}_{k}\, h_{m\bar{n}} + i\, \delta_{kp}\, D_{m\bar{n}} + \frac{i}{2}\, h_{k\bar{n}}\, D^{p}_{m} + \frac{i}{2}\, \delta_{mp}\, D_{k\bar{n}}.$$

By the formula $\hat{R}_{m\bar{n}k\bar{l}} = \hat{R}^{p}_{m\bar{n}k}\, h_{p\bar{l}}$,

$$\hat{R}_{m\bar{n}k\bar{l}} \;=\; R_{m\bar{n}k\bar{l}} + i\, D_{k\bar{l}}\, h_{m\bar{n}} + i\, D_{m\bar{n}}\, h_{k\bar{l}} + \frac{i}{2}\, D_{m\bar{l}}\, h_{k\bar{n}} + \frac{i}{2}\, D_{k\bar{n}}\, h_{m\bar{l}}. \tag{B22}$$

On the other hand, the Ricci terms in (B19) can be found by

$$\hat{R}_{m\bar{n}} \;=\; \hat{R}^{Q}_{Qm\bar{n}} \;=\; 2\big(d\hat{\omega}^{Q}_{\bar{n}} - \hat{\omega}^{P}_{\bar{n}} \wedge \hat{\omega}^{Q}_{P}\big)(Z_Q, Z_m) \tag{B23}$$

**Q** $= \gamma$ **in** (B23):

$$\begin{aligned}
d\hat{\omega}^{\gamma}_{\bar{n}}(Z_\gamma, Z_m) &= 0 \\
\big(\hat{\omega}^{k}_{\bar{n}} \wedge \hat{\omega}^{\gamma}_{k}\big)(Z_\gamma, Z_m) &= \big(\hat{\omega}^{\bar{k}}_{\bar{n}} \wedge \hat{\omega}^{\gamma}_{\bar{k}}\big)(Z_\gamma, Z_m) \;=\; 0 \\
\big(\hat{\omega}^{0}_{\bar{n}} \wedge \hat{\omega}^{\gamma}_{0}\big)(Z_\gamma, Z_m) &= -\frac{i}{4}\, h_{\gamma\bar{n}}\, D^{\gamma}_{m} + \frac{i}{4}\, h_{m\bar{n}}\, (D^{\gamma}_{\gamma}) \\
\big(\hat{\omega}^{c}_{\bar{n}} \wedge \hat{\omega}^{\gamma}_{c}\big)(Z_\gamma, Z_m) &= -\frac{i}{4}\, D_{\gamma\bar{n}}\, \delta_{m\gamma} + \frac{i}{4}\, D_{m\bar{n}}\, (\delta_{\gamma\gamma})
\end{aligned}$$

Therefore,

$$\big(d\hat{\omega}^{\gamma}_{\bar{n}} - \hat{\omega}^{P}_{\bar{n}} \wedge \hat{\omega}^{\gamma}_{P}\big)(Z_{\gamma}, Z_m) \;=\; \frac{i}{4}\, h_{\gamma\bar{n}}\, D^{\gamma}_{m} - \frac{i}{4}\, h_{m\bar{n}}\, (D^{\gamma}_{\gamma}) + \frac{i}{4}\, D_{m\bar{n}} - \frac{i}{4}\, \nu\, D_{m\bar{n}}.$$

**$\mathbf{Q} = \bar{\gamma}$ in** (B23):

$$\begin{aligned} d\hat{\omega}^{\bar{\gamma}}_{\bar{n}}(Z_{\bar{\gamma}}, Z_m) &= d\omega^{\bar{\gamma}}_{\bar{n}}(T_{\bar{\gamma}}, T_m) + \frac{i}{2} h_{m\bar{\gamma}} D^{\bar{\gamma}}_{\bar{n}} + i\delta_{n\gamma} \cdot \frac{1}{(\nu+2)}\Big(\frac{i}{2} R_{m\bar{\gamma}} - \frac{i}{4(\nu+1)} \rho h_{m\bar{\gamma}}\Big) \\ &= d\omega^{\bar{\gamma}}_{\bar{n}}(T_{\bar{\gamma}}, T_m) + \frac{i}{2}\, h_{m\bar{\gamma}}\, D^{\bar{\gamma}}_{\bar{n}} + \frac{i}{2}\, D_{m\bar{n}} \end{aligned}$$

$$\begin{aligned} \big(\hat{\omega}^{\bar{k}}_{\bar{n}} \wedge \hat{\omega}^{\bar{\gamma}}_{\bar{k}}\big)(Z_{\bar{\gamma}}, Z_m) &= \big(\omega^{\bar{k}}_{\bar{n}} \wedge \omega^{\bar{\gamma}}_{\bar{k}}\big)(T_{\bar{\gamma}}, T_m) \\ \big(\hat{\omega}^{0}_{\bar{n}} \wedge \hat{\omega}^{\bar{\gamma}}_{0}\big)(Z_{\bar{\gamma}}, Z_m) &= -\frac{i}{4}\, h_{m\bar{n}}\, (D^{\bar{\gamma}}_{\bar{\gamma}}) \\ \big(\hat{\omega}^{c}_{\bar{n}} \wedge \hat{\omega}^{\bar{\gamma}}_{c}\big)(Z_{\bar{\gamma}}, Z_m) &= -\frac{i}{4}\, D_{m\bar{n}}\, (\delta_{\gamma\gamma}) \end{aligned}$$

Therefore,

$$\big(d\hat{\omega}^{\bar{\gamma}}_{\bar{n}} - \hat{\omega}^{P}_{\bar{n}} \wedge \hat{\omega}^{\bar{\gamma}}_{P}\big)(Z_{\bar{\gamma}}, Z_m) \;=\; \frac{1}{2}\, R_{m\bar{n}} + \frac{i}{2}\, h_{m\bar{\gamma}}\, D^{\bar{\gamma}}_{\bar{n}} + \frac{i}{2}\, D_{m\bar{n}} + \frac{i}{4}\, h_{m\bar{n}}\, (D^{\bar{\gamma}}_{\bar{\gamma}}) + \frac{i}{4}\, \nu\, D_{m\bar{n}}.$$

**$\mathbf{Q} = \mathbf{0}$ in** (B23):

$$d\hat{\omega}^{0}_{\beta} \;=\; d\big(i\, h_{\beta\bar{\alpha}}\, \theta^{\bar{\alpha}}\big) \;=\; i\, h_{\beta\bar{\gamma}}\, \omega^{\bar{\gamma}}_{\bar{\alpha}} \wedge \theta^{\bar{\alpha}} + i\, h_{\gamma\bar{\alpha}}\, \omega^{\gamma}_{\beta} \wedge \theta^{\bar{\alpha}} + i\, h_{\beta\bar{\alpha}}\, \theta^{\bar{\gamma}} \wedge \omega^{\bar{\alpha}}_{\bar{\gamma}} + i\, h_{\beta\bar{\alpha}}\, \theta \wedge \tau^{\bar{\alpha}}$$

$$d\hat{\omega}^{0}_{\bar{\beta}} \;=\; d\big(-i\, h_{\alpha\bar{\beta}}\, \theta^{\alpha}\big) \;=\; -i\, h_{\gamma\bar{\beta}}\, \omega^{\gamma}_{\alpha} \wedge \theta^{\alpha} - i\, h_{\alpha\bar{\gamma}}\, \omega^{\bar{\gamma}}_{\bar{\beta}} \wedge \theta^{\alpha} - i\, h_{\alpha\bar{\beta}}\, \theta^{\gamma} \wedge \omega^{\alpha}_{\gamma} - i\, h_{\alpha\bar{\beta}}\, \theta \wedge \tau^{\alpha}$$

It implies that

$$d\hat{\omega}^{0}_{\bar{n}}(Z_0, Z_m) \;=\; -\frac{i}{2}\, \Gamma^{\gamma}_{0m}\, h_{\gamma\bar{n}} - \frac{i}{2}\, \Gamma^{\bar{\gamma}}_{0\bar{n}}\, h_{m\bar{\gamma}} + \frac{i}{2}\, \Gamma^{\alpha}_{0m}\, h_{\alpha\bar{n}} \;=\; -\frac{i}{2}\, \Gamma^{\bar{\gamma}}_{0\bar{n}}\, h_{m\bar{\gamma}}.$$

Moreover, we have

$$\big(\hat{\omega}^{P}_{\bar{n}} \wedge \hat{\omega}^{0}_{P}\big)(Z_0, Z_m) \;=\; \big(\hat{\omega}^{\bar{\gamma}}_{\bar{n}} \wedge \hat{\omega}^{0}_{\bar{\gamma}}\big)(Z_0, Z_m) \;=\; -\frac{i}{2}\, \Gamma^{\bar{\gamma}}_{0\bar{n}}\, h_{m\bar{\gamma}} + \frac{i}{4}\, D^{\bar{\gamma}}_{\bar{n}}\, h_{m\bar{\gamma}}.$$

Therefore, $\quad \big(d\hat{\omega}^{0}_{\beta} - \hat{\omega}^{P}_{\bar{n}} \wedge \hat{\omega}^{0}_{P}\big)(Z_0, Z_m) \;=\; -\frac{i}{4}\, D^{\bar{\gamma}}_{\bar{n}}\, h_{m\bar{\gamma}}.$

$\mathbf{Q} = \mathbf{c}$ **in** (B23):

$$
\begin{aligned}
d\hat{\omega}^c_{\bar{n}}(Z_c, Z_m) &= 0 \\
\big(\hat{\omega}^P_{\bar{n}} \wedge \hat{\omega}^c_P\big)(Z_c, Z_m) &= \big(\hat{\omega}^{\bar{\gamma}}_{\bar{n}} \wedge \hat{\omega}^c_{\bar{\gamma}}\big)(Z_c, Z_m) \;=\; \frac{i}{4}\, D_{m\bar{n}}
\end{aligned}
$$

Therefore, $\quad \big(d\hat{\omega}^c_\beta - \hat{\omega}^P_{\bar{n}} \wedge \hat{\omega}^c_P\big)(Z_c, Z_m) = -\frac{i}{4}\, D_{m\bar{n}}.$

Combining everything, we get

$$
\begin{aligned}
\hat{R}_{m\bar{n}} &= 2\Big(\frac{i}{4}\, D_{m\bar{n}} - \frac{i}{4}\, h_{m\bar{n}}\, (D^\gamma_\gamma) + \frac{i}{4}\, D_{m\bar{n}} - \frac{i}{4}\, \nu\, D_{m\bar{n}}\Big) \\
&\quad +2\Big(\frac{1}{2}\, R_{m\bar{n}} + \frac{i}{2}\, D_{m\bar{n}} + \frac{i}{2}\, D_{m\bar{n}} + \frac{i}{4}\, h_{m\bar{n}}\, (D^{\bar{\gamma}}_{\bar{\gamma}}) + \frac{i}{4}\, \nu\, D_{m\bar{n}}\Big) - \frac{i}{2}\, D_{m\bar{n}} - \frac{i}{2}\, D_{m\bar{n}} \\
&= R_{m\bar{n}} + 2i\, D_{m\bar{n}}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\hat{R}_{m\bar{n}} &= R_{m\bar{n}} + 2i\, D_{m\bar{n}} \;=\; R_{m\bar{n}} + 2i\, \Big(\frac{i}{\nu+2}\, R_{m\bar{n}} - \frac{i}{2(\nu+1)(\nu+2)}\, \rho\, h_{m\bar{n}}\Big) \\
\Longrightarrow \quad \hat{R}_{m\bar{n}} &= \frac{\nu}{\nu+2}\, R_{m\bar{n}} + \frac{1}{(\nu+1)(\nu+2)}\, \rho\, h_{m\bar{n}}
\end{aligned}
\tag{B24}
$$

The Chern tensor (assoicated with $\theta$ on $N$) is defined by $C(T_m, T_{\bar{n}})T_k = C_k{}^p{}_{m\bar{n}}\, T_p$, where

$$
\begin{aligned}
C_k{}^p{}_{m\bar{n}} &= R_k{}^p{}_{m\bar{n}} - \frac{1}{\nu+2}\Big(R^p_k\; h_{m\bar{n}} + R^p_m\; h_{k\bar{n}} + \delta_{kp}\; R_{m\bar{n}} + \delta_{mp}\; R_{k\bar{n}}\Big) \\
&\quad + \frac{\rho}{(\nu+1)(\nu+2)}\big(\delta_{kp}\; h_{m\bar{n}} + \delta_{mp}\; h_{k\bar{n}}\big) \\
\Longrightarrow\; C_{k\bar{l}m\bar{n}} &= R_{k\bar{l}m\bar{n}} - \frac{1}{\nu+2}\Big(R_{k\bar{l}}\; h_{m\bar{n}} + R_{m\bar{l}}\; h_{k\bar{n}} + R_{m\bar{n}}\; h_{k\bar{l}} + R_{k\bar{n}}\; h_{m\bar{l}}\Big) \\
&\quad + \frac{\rho}{(\nu+1)(\nu+2)}\big(h_{k\bar{l}}\; h_{m\bar{n}} + h_{m\bar{l}}\; h_{k\bar{n}}\big).
\end{aligned}
$$

The $D$-tensor defined above plays a similar role to the Schouten tensor in Riemannian geometry. We can express $C_{k\bar{l}m\bar{n}}$ in terms of the coefficients of the $D$-tensor.

$$
\begin{aligned}
C_{k\bar{l}m\bar{n}} &= R_{k\bar{l}m\bar{n}} + i\left(\frac{i}{\nu+2}\,R_{k\bar{l}} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,h_{k\bar{l}}\right)h_{m\bar{n}} \\
&\quad + i\left(\frac{i}{\nu+2}\,R_{m\bar{l}} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,h_{m\bar{l}}\right)h_{k\bar{n}} \\
&\quad + i\left(\frac{i}{\nu+2}\,R_{m\bar{n}} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,h_{m\bar{n}}\right)h_{k\bar{l}} \\
&\quad + i\left(\frac{i}{\nu+2}\,R_{k\bar{n}} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,h_{k\bar{n}}\right)h_{m\bar{l}}
\end{aligned}
$$

$$
\Longrightarrow \quad C_{k\bar{l}m\bar{n}} = R_{k\bar{l}m\bar{n}} + i\Big(D_{k\bar{l}}\,h_{m\bar{n}} + D_{m\bar{l}}\,h_{k\bar{n}} + D_{m\bar{n}}\,h_{k\bar{l}} + D_{k\bar{n}}\,h_{m\bar{l}}\Big)
$$

**Show that $\mathcal{W}_{\mathbf{m\bar{n}k\bar{l}}} = \mathbf{C}_{\mathbf{k\bar{l}m\bar{n}}}$**

We will apply the statement of Theorem 1.4, $S = \dfrac{2\nu+1}{\nu+1}\,\rho$, in the work-out.

$$
\begin{aligned}
\mathcal{W}_{m\bar{n}k\bar{l}} &= R_{k\bar{l}m\bar{n}} + i\,D_{k\bar{l}}\,h_{m\bar{n}} + i\,D_{m\bar{n}}\,h_{k\bar{l}} + \frac{i}{2}\,D_{m\bar{l}}\,h_{k\bar{n}} + \frac{i}{2}\,D_{k\bar{n}}\,h_{m\bar{l}} && (\hat{R}_{m\bar{n}k\bar{l}}) \\
&\quad - \frac{h_{k\bar{n}}}{2\nu}\left(\frac{\nu}{\nu+2}\,R_{m\bar{l}} + \frac{1}{(\nu+1)(\nu+2)}\,\rho\,h_{m\bar{l}}\right) && (\hat{R}_{m\bar{l}}) \\
&\quad - \frac{h_{m\bar{l}}}{2\nu}\left(\frac{\nu}{\nu+2}\,R_{k\bar{n}} + \frac{1}{(\nu+1)(\nu+2)}\,\rho\,h_{k\bar{n}}\right) && (\hat{R}_{k\bar{n}}) \\
&\quad + \frac{1}{(2\nu)(2\nu+1)}\left(\frac{2\nu+1}{\nu+1}\right)\rho\,h_{m\bar{l}}\,h_{k\bar{n}} && (S) \\
&= R_{k\bar{l}m\bar{n}} + i\,D_{k\bar{l}}\,h_{m\bar{n}} + i\,D_{m\bar{n}}\,h_{k\bar{l}} + \frac{i}{2}\,D_{m\bar{l}}\,h_{k\bar{n}} + \frac{i}{2}\,D_{k\bar{n}}\,h_{m\bar{l}} \\
&\quad + \frac{i\,h_{k\bar{n}}}{2}\Big(\frac{i}{\nu+2}\,R_{m\bar{l}} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,h_{m\bar{l}}\Big) \\
&\quad + \frac{i\,h_{m\bar{l}}}{2}\Big(\frac{i}{\nu+2}\,R_{k\bar{n}} - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,h_{k\bar{n}}\Big) \\
&\quad - \frac{h_{k\bar{n}}\,h_{m\bar{l}}}{2\nu(\nu+1)(\nu+2)}\,\rho - \frac{h_{k\bar{n}}\,h_{m\bar{l}}}{2\nu(\nu+1)(\nu+2)}\,\rho \\
&\quad - \frac{h_{k\bar{n}}\,h_{m\bar{l}}}{4(\nu+1)(\nu+2)}\,\rho - \frac{h_{k\bar{n}}\,h_{m\bar{l}}}{4(\nu+1)(\nu+2)}\,\rho + \frac{1}{2\nu(\nu+1)}\,\rho\,h_{m\bar{l}}\,h_{k\bar{n}} \\
&= R_{k\bar{l}m\bar{n}} + i\,D_{k\bar{l}}\,h_{m\bar{n}} + i\,D_{m\bar{n}}\,h_{k\bar{l}} + \frac{i}{2}\,D_{m\bar{l}}\,h_{k\bar{n}} + \frac{i}{2}\,D_{k\bar{n}}\,h_{m\bar{l}} \\
&\quad + \frac{i}{2}\,h_{k\bar{n}}\,D_{m\bar{l}} + \frac{i}{2}\,h_{m\bar{l}}\,D_{k\bar{n}} \\
&= R_{k\bar{l}m\bar{n}} + i\,D_{k\bar{l}}\,h_{m\bar{n}} + i\,D_{m\bar{n}}\,h_{k\bar{l}} + i\,D_{m\bar{l}}\,h_{k\bar{n}} + i\,D_{k\bar{n}}\,h_{m\bar{l}} \\
&= C_{k\bar{l}m\bar{n}}
\end{aligned}
$$

Therefore, we justify that $\mathcal{W}_{k\bar{l}m\bar{n}} = \mathcal{W}_{m\bar{n}k\bar{l}} = C_{m\bar{n}k\bar{l}}$ for every $m$, $n$, $k$, $l$.

**Part (2): $\mathcal{W}\big(T_m, T_n, u_1, u_2\big) = 0$ when $u_1$, $u_2$ are on $T^{1,0}N \oplus T^{0,1}N$**

**(2.1) Show $\mathcal{W}_{\mathbf{mnkl}} = \mathbf{0}$.** Note that $\mathcal{W}_{mnkl} = \hat{R}_{mnkl} = \hat{R}^{\bar{p}}_{mnk}\, h_{l\bar{p}}$ and

$$\hat{R}^{\bar{p}}_{mnk} = 2\big(d\hat{\omega}^{\bar{p}}_k - \hat{\omega}^Q_k \wedge \hat{\omega}^{\bar{p}}_Q\big)(Z_m, Z_n).$$

The first term $d\hat{\omega}^{\bar{p}}_k = 0$ and

$$\big(\hat{\omega}^Q_k \wedge \hat{\omega}^{\bar{p}}_Q\big)(Z_m, Z_n) = \big(\hat{\omega}^0_k \wedge \hat{\omega}^{\bar{p}}_0 + \hat{\omega}^c_k \wedge \hat{\omega}^{\bar{p}}_c\big)(Z_m, Z_n) = 0.$$

Therefore, $W_{mnkl} = \hat{R}_{mnkl} = 0$.

**(2.2) Show $\mathcal{W}_{\mathbf{mnk\bar{l}}} = \mathbf{0}$.** By definition,

$$\mathcal{W}_{mnk\bar{l}} = \hat{R}_{mnk\bar{l}} - \frac{1}{2\nu}\,\hat{R}_{nk}\, h_{m\bar{l}} + \frac{1}{2\nu}\,\hat{R}_{mk}\, h_{n\bar{l}}. \tag{B25}$$

The first term of (B25) is $\hat{R}_{mnk\bar{l}} = \hat{R}^p_{mnk}\, h_{p\bar{l}}$ with $\hat{R}^p_{mnk} = 2\big(d\hat{\omega}^p_k - \hat{\omega}^Q_k \wedge \hat{\omega}^p_Q\big)(Z_m, Z_n)$.

$$d\hat{\omega}^p_k(Z_m, Z_n) \;=\; d\omega^p_k(T_m, T_n) + i\delta_{kp}\; d\sigma(Z_m, Z_n) \;=\; d\omega^p_k(T_m, T_n)$$

$$\begin{aligned}
\big(\hat{\omega}^Q_k \wedge \hat{\omega}^p_Q\big)(Z_m, Z_n) &= (\hat{\omega}^\gamma_k \wedge \hat{\omega}^p_\gamma)(Z_m, Z_n) + (\hat{\omega}^c_k \wedge \hat{\omega}^p_c)(Z_m, Z_n) \\
&= (\omega^\gamma_k \wedge \omega^p_\gamma)(T_m, T_n) + \frac{i}{2}\big(A_{kn}\,\delta_{mp} - A_{km}\,\delta_{np}\big)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{R}^p_{mnk} &= 2\big(d\omega^p_k - \omega^\gamma_k \wedge \omega^p_\gamma\big)(T_m, T_n) - i\big(A_{kn}\,\delta_{mp} - A_{km}\,\delta_{np}\big) \\
&= R_k{}^p{}_{mn} - i\big(A_{kn}\,\delta_{mp} - A_{km}\,\delta_{np}\big) \\
&= 2i\big(A_{nk}\,\delta_{mp} - A_{mk}\,\delta_{np}\big) - i\big(A_{kn}\,\delta_{mp} - A_{km}\;,\delta_{np}\big) \\
&= i\big(A_{nk}\,\delta_{mp} - A_{mk}\,\delta_{np}\big).
\end{aligned}$$

In the third step above, we apply the formula regarding ${R_k}^p{}_{mn}$:

$$ {R_k}^p{}_{mn} \;=\; 2\big(d\omega_k^p - \omega_k^\gamma \wedge \omega_\gamma^p\big)(T_m, T_n) \;=\; \Big(2i\,\delta_{mp}\,A_{kn} - 2i\,\delta_{np}\,A_{km}\Big). \tag{B26} $$

(B26) will be justify later in Part (3).

On the other hand, $\hat{R}_{mk} = 2\sum_{P,Q}\big(d\hat{\omega}_k^Q - \hat{\omega}_k^P \wedge \hat{\omega}_P^Q\big)(Z_Q, Z_m)$.

$$ \begin{aligned} (Q=\gamma) \qquad \big(d\hat{\omega}_k^\gamma - \hat{\omega}_k^P \wedge \hat{\omega}_P^\gamma\big)(Z_\gamma, Z_m) &= \frac{1}{2}\,\hat{R}^\gamma_{\gamma mk} \\ &= \frac{1}{2}\sum_{\gamma=1}^{\nu}\Big(i\,A_{mk}\,\delta_{\gamma\gamma} - i\,A_{\gamma k}\,\delta_{m\gamma}\Big) \\ &= \frac{i\,\nu}{2}\,A_{mk} - \frac{i}{2}\,A_{mk} \end{aligned} $$

$$ \begin{aligned} (Q=\bar{\gamma}) \qquad \big(d\hat{\omega}_k^{\bar{\gamma}} - \hat{\omega}_k^P \wedge \hat{\omega}_P^{\bar{\gamma}}\big)(Z_{\bar{\gamma}}, Z_m) &= -\big(\hat{\omega}_k^0 \wedge \hat{\omega}_0^{\bar{\gamma}}\big)(Z_{\bar{\gamma}}, Z_m) - \big(\hat{\omega}_k^c \wedge \hat{\omega}_c^{\bar{\gamma}}\big)(Z_{\bar{\gamma}}, Z_m) \\ &= -\frac{1}{2}(i\,h_{k\bar{l}})(A_m^{\bar{\gamma}}) + \frac{1}{2}(-A_{km})(-i\,\delta_{\gamma\gamma}) \\ &= -\frac{i}{2}\,A_{km} + \frac{i\,\nu}{2}\,A_{km} \end{aligned} $$

$$ \begin{aligned} (Q=0) \qquad \big(d\hat{\omega}_k^0 - \hat{\omega}_k^P \wedge \hat{\omega}_P^0\big)(Z_0, Z_m) &= i\,h_{k\bar{\alpha}}\;d\theta^{\bar{\alpha}}(T_0, T_m) - (\hat{\omega}_k^\gamma \wedge \hat{\omega}_\gamma^0)(Z_0, Z_m) \\ &= i\,h_{k\bar{\alpha}}\Big(\frac{1}{2}\,A_m^{\bar{\alpha}}\Big) \;=\; \frac{i}{2}\,A_{km} \end{aligned} $$

$$ \begin{aligned} (Q=c) \qquad \big(d\hat{\omega}_k^c - \hat{\omega}_k^P \wedge \hat{\omega}_P^c\big)(Z_c, Z_m) &= -(\hat{\omega}_k^\gamma \wedge \hat{\omega}_\gamma^c)(Z_c, Z_m) \\ &= -\frac{1}{2}(i\,\delta_{k\gamma})(-A_{\gamma m}) \;=\; \frac{i}{2}\,A_{km} \end{aligned} $$

As a result, $\hat{R}_{mk} \;=\; 2i\,(\nu-1)\,A_{mk} + 2i\,A_{mk} \;=\; 2i\,\nu\,A_{mk}$. Hence,

$$ \begin{aligned} \mathcal{W}_{mnk\bar{l}} &= i\big(A_{nk}\,\delta_{mp} - A_{mk}\,\delta_{np}\big)h_{p\bar{l}} - \frac{1}{2\nu}\,h_{m\bar{l}}\big(2i\,\nu\,A_{nk}\big) + \frac{1}{2\nu}\,h_{n\bar{l}}\big(2i\,\nu A_{mk}\big) \\ &= i\,A_{nk}\,h_{m\bar{l}} - i\,A_{mk}\,h_{n\bar{l}} - i\,A_{nk}\,h_{m\bar{l}} + i\,A_{mk}\,h_{n\bar{l}} \\ &= 0. \end{aligned} $$

**(2.3) Show $\mathcal{W}_{\mathbf{mn\bar{k}\bar{l}}} = \mathbf{0}$.** We would need the formula :

$$\begin{aligned}\mathcal{W}_{mn\bar{k}\bar{l}} &= \hat{R}_{mn\bar{k}\bar{l}} - \frac{1}{2\nu}\Big(\hat{R}_{m\bar{l}}\,h_{n\bar{k}} + \hat{R}_{n\bar{k}}\,h_{m\bar{l}} - \hat{R}_{m\bar{k}}\,h_{n\bar{l}} - \hat{R}_{n\bar{l}}\,h_{m\bar{k}}\Big)\\ &\quad + \frac{S}{(2\nu)(2\nu+1)}\big(h_{m\bar{l}}\,h_{n\bar{k}} - h_{m\bar{k}}\,h_{n\bar{l}}\big)\end{aligned}$$

Note that $\hat{R}_{mn\bar{k}\bar{l}} = \hat{R}^{p}_{mn\bar{k}}\,h_{p\bar{l}}$.

$$\begin{aligned}R^{p}_{mn\bar{k}} &= 2\big(d\hat{\omega}^{p}_{\bar{k}} - \hat{\omega}^{P}_{\bar{k}} \wedge \hat{\omega}^{p}_{P}\big)(Z_m, Z_n) \;=\; -2\big(\hat{\omega}^{0}_{\bar{k}} \wedge \hat{\omega}^{p}_{0}\big)(Z_m, Z_n) - 2\big(\hat{\omega}^{c}_{\bar{k}} \wedge \hat{\omega}^{p}_{c}\big)(Z_m, Z_n)\\ &= -2\Big(-\frac{i}{4}\,h_{m\bar{k}}\,D^{p}_{n} + \frac{i}{4}\,h_{n\bar{k}}\,D^{p}_{m}\Big) - 2\Big(-\frac{i}{4}\,D_{m\bar{k}}\,\delta_{np} + \frac{i}{4}\,D_{n\bar{k}}\,\delta_{mp}\Big)\\ &= \frac{i}{2}\Big(h_{m\bar{k}}\,D^{p}_{n} - h_{n\bar{k}}\,D^{p}_{m} + D_{m\bar{k}}\,\delta_{np} - D_{n\bar{k}}\,\delta_{mp}\Big)\end{aligned}$$

Therefore, $\hat{R}_{mn\bar{k}\bar{l}} \;=\; \frac{i}{2}\Big(h_{m\bar{k}}\,D_{n\bar{l}} - h_{n\bar{k}}\,D_{m\bar{l}} + D_{m\bar{k}}\,h_{n\bar{l}} - D_{n\bar{k}}\,h_{m\bar{l}}\Big)$.

$$\begin{aligned}&\mathcal{W}_{mn\bar{k}\bar{l}}\\ &= \frac{i}{2}\Big(h_{m\bar{k}}\,D_{n\bar{l}} - h_{n\bar{k}}\,D_{m\bar{l}} + D_{m\bar{k}}\,h_{n\bar{l}} - D_{n\bar{k}}\,h_{m\bar{l}}\Big)\\ &\quad -\frac{1}{2\nu}\Big(h_{n\bar{k}}(R_{m\bar{l}} + 2iD_{m\bar{l}}) + h_{m\bar{l}}(R_{n\bar{k}} + 2iD_{n\bar{k}}) - h_{n\bar{l}}(R_{m\bar{k}} + 2iD_{m\bar{k}}) - h_{m\bar{k}}(R_{n\bar{l}} + 2iD_{n\bar{l}})\Big)\\ &\quad + \frac{S}{(2\nu)(2\nu+1)}\big(h_{m\bar{l}}\,h_{n\bar{k}} - h_{m\bar{k}}\,h_{n\bar{l}}\big)\\ &= \frac{i}{2}\cdot\Big(\frac{i}{\nu+2}\Big)\Big(h_{m\bar{k}}\,R_{n\bar{l}} - h_{n\bar{k}}\,R_{m\bar{l}} + h_{n\bar{l}}\,R_{m\bar{k}} - h_{m\bar{l}}\,R_{n\bar{k}}\Big) && \text{[first line]}\\ &\quad + \frac{\rho}{4(\nu+1)(\nu+2)}\Big(h_{m\bar{k}}\,h_{n\bar{l}} - h_{n\bar{k}}\,h_{m\bar{l}} + h_{m\bar{k}}\,h_{n\bar{l}} - h_{n\bar{k}}\,h_{m\bar{l}}\Big) && \text{[first line]}\\ &\quad -\frac{1}{2\nu}\cdot\Big(1 - \frac{2}{\nu+2}\Big)\Big(h_{n\bar{k}}\,R_{m\bar{l}} + h_{m\bar{l}}\,R_{n\bar{k}} - h_{n\bar{l}}\,R_{m\bar{k}} - h_{m\bar{k}}\,R_{n\bar{l}}\Big) && \text{[2nd line]}\\ &\quad -\frac{1}{2\nu}\cdot\Big(\frac{\rho}{(\nu+1)(\nu+2)}\Big)\Big(2h_{n\bar{k}}\,h_{m\bar{l}} - 2h_{n\bar{l}}\,h_{m\bar{k}}\Big) && \text{[2nd line]}\\ &\quad + \frac{\rho}{(2\nu)(\nu+1)}\big(h_{m\bar{l}}\,h_{n\bar{k}} - h_{m\bar{k}}\,h_{n\bar{l}}\big) && \text{[3rd line]}\\ &= h_{m\bar{k}}\,R_{n\bar{l}}\Big(-\frac{1}{2(\nu+2)} + \frac{1}{2\nu}\Big(\frac{\nu}{\nu+2}\Big)\Big) + h_{n\bar{k}}\,R_{m\bar{l}}\Big(\frac{1}{2(\nu+2)} - \frac{1}{2\nu}\Big(\frac{\nu}{\nu+2}\Big)\Big)\\ &\quad + h_{n\bar{l}}\,R_{m\bar{k}}\Big(-\frac{1}{2(\nu+2)} + \frac{1}{2\nu}\Big(\frac{\nu}{\nu+2}\Big)\Big) + h_{m\bar{l}}\,R_{n\bar{k}}\Big(\frac{1}{2(\nu+2)} - \frac{1}{2\nu}\Big(\frac{\nu}{\nu+2}\Big)\Big)\\ &\quad + h_{m\bar{k}}\,h_{n\bar{l}}\,\rho\Big(\frac{2}{4(\nu+1)(\nu+2)} + \frac{2}{2\nu(\nu+1)(\nu+2)} - \frac{1}{2\nu(\nu+1)}\Big)\\ &\quad + h_{n\bar{k}}\,h_{m\bar{l}}\,\rho\Big(-\frac{2}{4(\nu+1)(\nu+2)} - \frac{2}{2\nu(\nu+1)(\nu+2)} + \frac{1}{2\nu(\nu+1)}\Big)\\ &= 0\end{aligned}$$

**Part (3): $\mathcal{W}\big(T_m, T_n, u_1, u_2\big) = 0$ for any $u_1$, $u_2$ on $C(N)$**

Recall that $g$ is the Webster metric associated with $\theta$. Let $\tilde{\nabla}$ be the Levi-Civita connection of $g$. Define Let $\tilde{\nabla}$ be the Levi-Civita connection of $g$. Define ($T = T_0$)

$$\begin{cases} \tilde{\nabla}_{T_\alpha} T_\beta &= \tilde{\Gamma}^{\gamma}_{\alpha\beta}\, T_\gamma + \tilde{\Gamma}^{\bar{\gamma}}_{\alpha\beta}\, T_{\bar{\gamma}} + \tilde{\Gamma}^{0}_{\alpha\beta}\, T, \\ \tilde{\nabla}_{T_{\bar{\alpha}}} T_\beta &= \tilde{\Gamma}^{\gamma}_{\bar{\alpha}\beta}\, T_\gamma + \tilde{\Gamma}^{\bar{\gamma}}_{\bar{\alpha}\beta}\, T_{\bar{\gamma}} + \tilde{\Gamma}^{0}_{\bar{\alpha}\beta}\, T, \\ \tilde{\nabla}_{T} T_\beta &= \tilde{\Gamma}^{\gamma}_{0\beta}\, T_\gamma + \tilde{\Gamma}^{\bar{\gamma}}_{0\beta}\, T_{\bar{\gamma}} + \Gamma^{0}_{0\beta}\, T \end{cases} \tag{B27}$$

We could obtain the Christoffel symbols in (B27) from the identities below.

$$\tilde{\nabla}_{T_m} T_n = \nabla_{T_m} T_n - A_{mn}\, T \qquad \tilde{\nabla}_{T_{\bar{m}}} T_{\bar{n}} = \nabla_{T_{\bar{m}}} T_{\bar{n}} - A_{\bar{m}\bar{n}}\, T$$

$$\tilde{\nabla}_{T_{\bar{m}}} T_n = \nabla_{T_{\bar{m}}} T_n + i\, h_{n\bar{m}}\, T \qquad \tilde{\nabla}_{T_m} T_{\bar{n}} = \nabla_{T_m} T_{\bar{n}} - i\, h_{m\bar{n}}\, T$$

$$\tilde{\nabla}_{T} T_m = \nabla_{T} T_m + i\, T_m \qquad \tilde{\nabla}_{T} T_{\bar{m}} = \nabla_{T} T_{\bar{m}} - i\, T_{\bar{m}}$$

$$\tilde{\nabla}_{T_m} T = i\, T_m + A^{\bar{k}}_{m}\, T_{\bar{k}} \qquad \tilde{\nabla}_{T_{\bar{m}}} T = -i\, T_{\bar{m}} + A^{k}_{\bar{m}}\, T_k \qquad \tilde{\nabla}_{T} T = 0$$

Let $\tilde{R}$ be the Riemann tensor of $\tilde{\nabla}$. We will find $\tilde{R}(T_m, T_n)T_k$.

$$\tilde{R}(T_m, T_n)T_k = \tilde{\nabla}_{T_m}\tilde{\nabla}_{T_n} T_k - \tilde{\nabla}_{T_n}\tilde{\nabla}_{T_m} T_k - \tilde{\nabla}_{[T_m, T_n]} T_k$$

$$\begin{aligned} \tilde{\nabla}_{T_m}\tilde{\nabla}_{T_n} T_k &= \tilde{\nabla}_{T_m}\big(\nabla_{T_n} T_k - A_{kn}\, T\big) \\ &= T_m(\Gamma^{\gamma}_{nk}) + \Gamma^{\gamma}_{nk}\, \tilde{\nabla}_{T_m} T_\gamma - T_m(A_{kn})T - A_{kn}\, \tilde{\nabla}_{T_m} T \\ &= T_m(\Gamma^{\gamma}_{nk}) + \Gamma^{\gamma}_{nk}\, \nabla_{T_m} T_\gamma - \Gamma^{\gamma}_{nk}\, A_{m\gamma}\, T - T_m(A_{kn})\, T - A_{kn}\big(i\, T_m + A^{\bar{\gamma}}_{m}\, T_{\bar{\gamma}}\big) \\ &= \nabla_{T_m}\nabla_{T_n} T_k - i\, A_{kn}\, T_m - A_{kn}\, A^{\gamma}_{m}\, T_{\bar{\gamma}} - \big(T_m(A_{kn}) + \Gamma^{\gamma}_{nk}\, A_{m\gamma}\big)T \end{aligned}$$

$$\tilde{\nabla}_{T_n}\tilde{\nabla}_{T_m} T_k = \nabla_{T_n}\nabla_{T_m} T_k - i\, A_{km}\, T_n - A_{km}\, A^{\gamma}_{n}\, T_{\bar{\gamma}} - \big(T_n(A_{km}) + \Gamma^{\gamma}_{mk}\, A_{n\gamma}\big)T$$

$$\tilde{\nabla}_{[T_m, T_n]} T_k = (\Gamma^{\gamma}_{mn} - \Gamma^{\gamma}_{nm})\big(\nabla_{T_\gamma} T_k - A_{\gamma k}\, T\big) = \nabla_{[T_m, T_n]} T_k - A_{\gamma k}\big(\Gamma^{\gamma}_{mn} - \Gamma^{\gamma}_{nm}\big)T$$

Therefore,

$$\begin{aligned}\tilde{R}(T_m,T_n)T_k \;=\; & R(T_m,T_n)T_k + i\,A_{km}\,T_n - i\,A_{kn}\,T_m + \big(A_{km}\,A_n^{\bar\gamma} - A_{kn}\,A_m^{\bar\gamma}\big)T_{\bar\gamma} \\ & + \Big(T_n(A_{km}) - T_m(A_{kn}) + \Gamma^\gamma_{mk}\,A_{n\gamma} - \Gamma^\gamma_{nk}\,A_{m\gamma} + A_{\gamma k}\big(\Gamma^\gamma_{mn} - \Gamma^\gamma_{nm}\big)\Big)T.\end{aligned} \tag{B28}$$

Next we find $\tilde{R}(T_k,T)T_m$.

$$\tilde{R}(T_k,T)T_m = \tilde{\nabla}_{T_k}\tilde{\nabla}_T T_m - \tilde{\nabla}_T\tilde{\nabla}_{T_k}T_m - \tilde{\nabla}_{[T_k,T]}T_m$$

$$\begin{aligned}\tilde{\nabla}_{T_k}\tilde{\nabla}_T T_m &= \tilde{\nabla}_{T_k}\big(\nabla_T T_m + i\,T_m\big) \\ &= T_k(\Gamma^\gamma_{0m})\,T_\gamma + \Gamma^\gamma_{0m}\big(\nabla_{T_k}T_\gamma - A_{k\gamma}\,T\big) + i\,\Gamma^\gamma_{km}\,T_\gamma - i\,A_{mk}\,T \\ &= \nabla_{T_k}\nabla_T T_m + i\,\Gamma^\gamma_{km}\,T_\gamma - \big(i\,A_{mk} + A_{k\gamma}\,\Gamma^\gamma_{0m}\big)T\end{aligned}$$

$$\begin{aligned}\tilde{\nabla}_T\tilde{\nabla}_{T_k}T_m &= \tilde{\nabla}_T\big(\nabla_{T_k}T_m - A_{mk}\,T\big) \\ &= T(\Gamma^\gamma_{km})\,T_\gamma + \Gamma^\gamma_{km}\big(\nabla_T T_\gamma + i\,T_\gamma\big) - T(A_{mk})\,T \\ &= \nabla_T\nabla_{T_k}T_m + i\,\Gamma^\gamma_{km}\,T_\gamma - T(A_{mk})\,T\end{aligned}$$

$$\begin{aligned}\tilde{\nabla}_{[T_k,T]}T_m &= -\Gamma^\gamma_{0k}\,\tilde{\nabla}_{T_\gamma}T_m + A_k^{\bar\gamma}\,\tilde{\nabla}_{T_{\bar\gamma}}T_m \\ &= -\Gamma^\gamma_{0k}\big(\nabla_{T_\gamma}T_m - A_{m\gamma}\,T\big) + A_k^{\bar\gamma}\big(\nabla_{T_{\bar\gamma}}T_m + i\,h_{m\bar\gamma}\,T\big) \\ &= \nabla_{[T_k,T]}T_m + \big(\Gamma^\gamma_{0k}\,A_{m\gamma} + i\,A_{mk}\big)T\end{aligned}$$

Therefore,

$$\tilde{R}(T_k,T)T_m \;=\; R(T_k,T)T_m + \Big(T(A_{mk}) - A_{k\gamma}\,\Gamma^\gamma_{0m} - A_{m\gamma}\,\Gamma^\gamma_{0k} - 2i\,A_{mk}\Big)T. \tag{B29}$$

(B28), (B29): **Show that $\mathcal{W}_{\mathbf{mnk0}} = \mathbf{0}$.**

$$\mathcal{W}_{mnk0} \;=\; \hat{R}_{mnk0} \;=\; \hat{R}^c_{mnk} \;=\; 2\big(d\hat{\omega}^c_k - \hat{\omega}^P_k \wedge \hat{\omega}^c_P\big)(Z_m, Z_n)$$

$$\begin{aligned}
d\hat{\omega}^c_k(Z_m, Z_n) &= -\big(dA_{k\alpha} \wedge \theta^\alpha + A_{k\alpha}\, d\theta^\alpha\big)(Z_m, Z_n) \\
&= -\big(dA_{k\alpha} \wedge \theta^\alpha + A_{k\alpha}\, \theta^\beta \wedge \omega^\alpha_\beta\big)(Z_m, Z_n) \\
&= -\frac{1}{2}\big(T_m(A_{kn}) - T_n(A_{km})\big) - \frac{1}{2}\, A_{k\alpha}\big(\Gamma^\alpha_{nm} - \Gamma^\alpha_{mn}\big) \\
&= \frac{1}{2}\Big(T_n(A_{km}) - T_m(A_{kn}) + A_{k\alpha}\,\Gamma^\alpha_{mn} - A_{k\alpha}\,\Gamma^\alpha_{nm}\Big)
\end{aligned}$$

$$\begin{aligned}
(\hat{\omega}^P_k \wedge \hat{\omega}^c_P)(Z_m, Z_n) &= (\hat{\omega}^\gamma_k \wedge \hat{\omega}^c_\gamma)(Z_m, Z_n) \\
&= \frac{1}{2}\Big((\Gamma^\gamma_{mk})(-A_{\gamma n}) - (\Gamma^\gamma_{nk})(-A_{\gamma m})\Big) \\
&= \frac{1}{2}\big(A_{\gamma m}\,\Gamma^\gamma_{nk} - A_{\gamma n}\,\Gamma^\gamma_{mk}\big)
\end{aligned}$$

Therefore,

$$\hat{R}^c_{mnk} \;=\; \frac{1}{2}\Big(T_n(A_{km}) - T_m(A_{kn}) + A_{k\alpha}\,\Gamma^\alpha_{mn} - A_{k\alpha}\,\Gamma^\alpha_{nm} + A_{\gamma n}\,\Gamma^\gamma_{mk} - A_{\gamma m}\,\Gamma^\gamma_{nk}\Big).$$

We know that $R_{k0mn} = g\big(R(T_m, T_n)T_k, T\big) = 0$. From (B28),

$$\hat{R}^c_{mnk} = 0 \quad \Longleftrightarrow \quad \tilde{R}(T_m, T_n, T_k, T) = g\big(\tilde{R}(T_m, T_n)T_k, T\big) = 0.$$

From (B29), we have $\tilde{R}(T_m, T_n, T_k, T) = \tilde{R}(T_k, T, T_m, T_n) = 0$. It justifies that $\mathcal{W}_{mnk0} = 0$.

As another application of the Levi-Civita connection of $g$, we compute for $\tilde{R}(T_k, T_{\bar{l}})T_m$.

$$\tilde{R}(T_k, T_{\bar{l}})T_m \;=\; \tilde{\nabla}_{T_k}\tilde{\nabla}_{T_{\bar{l}}}T_m - \tilde{\nabla}_{T_{\bar{l}}}\tilde{\nabla}_{T_k}T_m - \tilde{\nabla}_{[T_k, T_{\bar{l}}]}T_m$$

$$\begin{aligned}
\tilde{\nabla}_{T_k}\tilde{\nabla}_{T_{\bar{l}}}T_m &= \tilde{\nabla}_{T_k}\big(\tilde{\nabla}_{T_{\bar{l}}}T_m + i\, h_{m\bar{l}}\, T\big) \\
&= T_k(\Gamma^\gamma_{\bar{l}m})\, T_\gamma + \Gamma^\gamma_{\bar{l}m}\big(\nabla_{T_k}T_\gamma - A_{k\gamma}\, T\big) + i\, T_k(h_{m\bar{l}})\, T + i\, h_{m\bar{l}}\big(i\, T_k + A^{\bar{\gamma}}_k\, T_{\bar{\gamma}}\big) \\
&= \nabla_{T_k}\nabla_{T_{\bar{l}}}T_m - h_{m\bar{l}}\, T_k + i\, h_{m\bar{l}}\, A^{\bar{\gamma}}_k\, T_{\bar{\gamma}} + \big(i\, h_{m\bar{\delta}}\, \Gamma^{\bar{\delta}}_{k\bar{l}} + i\, h_{\delta\bar{l}}\, \Gamma^\delta_{km} - \Gamma^\gamma_{\bar{l}m}\, A_{k\gamma}\big)T
\end{aligned}$$

$$
\begin{aligned}
\tilde{\nabla}_{T_{\bar{l}}}\tilde{\nabla}_{T_k}T_m &= \tilde{\nabla}_{T_{\bar{l}}}\big(\nabla_{T_k}T_m - A_{km}\,T\big) \\
&= T_{\bar{l}}(\Gamma^{\gamma}_{km})\,T_\gamma + \Gamma^{\gamma}_{km}\big(\nabla_{T_{\bar{l}}}T_\gamma + i\,h_{\gamma\bar{l}}\,T\big) - T_{\bar{l}}(A_{km})\,T - A_{km}\big(-i\,T_{\bar{l}} + A^{\gamma}_{\bar{l}}\,T_\gamma\big) \\
&= \nabla_{T_{\bar{l}}}\nabla_{T_k}T_m - A_{km}\,A^{\gamma}_{\bar{l}}\,T_\gamma + i\,A_{km}\,T_{\bar{l}} + \big(-T_{\bar{l}}(A_{km}) + i\,h_{\gamma\bar{l}}\,\Gamma^{\gamma}_{km}\big)T \\
\tilde{\nabla}_{[T_k,T_{\bar{l}}]}T_m &= -2i\,h_{k\bar{l}}\big(\nabla_T T_m + i\,T_m\big) - \Gamma^{\gamma}_{\bar{l}k}\big(\nabla_{T_\gamma}T_m - A_{m\gamma}\,T\big) + \Gamma^{\bar{\gamma}}_{k\bar{l}}\big(\nabla_{T_{\bar{\gamma}}}T_m + i\,h_{m\bar{\gamma}}\,T\big) \\
&= \nabla_{[T_k,T_{\bar{l}}]}T_m + 2\,h_{k\bar{l}}\,T_m + \big(i\,h_{m\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{k\bar{l}} + \Gamma^{\gamma}_{\bar{l}k}\,A_{m\gamma}\big)T
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{R}(T_k,T_{\bar{l}})T_m &= R(T_k,T_{\bar{l}})T_m - h_{m\bar{l}}\,T_k + i\,h_{m\bar{l}}\,A^{\bar{\gamma}}_k\,T_{\bar{\gamma}} \\
&\quad + \big(i\,h_{m\bar{\delta}}\,\Gamma^{\bar{\delta}}_{k\bar{l}} + i\,h_{\delta\bar{l}}\,\Gamma^{\delta}_{km} - \Gamma^{\gamma}_{\bar{l}m}\,A_{k\gamma}\big)T \\
&\quad + A_{km}\,A^{\gamma}_{\bar{l}}\,T_\gamma - i\,A_{km}\,T_{\bar{l}} + \big(T_{\bar{l}}(A_{km}) - i\,h_{\gamma\bar{l}}\,\Gamma^{\gamma}_{km}\big)T \\
&\quad - 2h_{k\bar{l}}\,T_m - \big(i\,h_{m\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{k\bar{l}} + \Gamma^{\gamma}_{\bar{l}k}\,A_{m\gamma}\big)T \\
&= R(T_k,T_{\bar{l}})T_m + A_{km}\,A^{\gamma}_{\bar{l}}\,T_\gamma - h_{m\bar{l}}\,T_k - 2h_{k\bar{l}}\,T_m \\
&\quad + i\,h_{m\bar{l}}\,A^{\bar{\gamma}}_k\,T_{\bar{\gamma}} - i\,A_{km}\,T_{\bar{l}} + \Big(T_{\bar{l}}(A_{km}) - \Gamma^{\gamma}_{\bar{l}m}\,A_{k\gamma} - \Gamma^{\gamma}_{\bar{l}k}\,A_{m\gamma}\Big)T
\end{aligned}
\tag{B30}
$$

**(B28), (B30): Find $\mathbf{R_k{}^p{}_{mn}}$ in (B26).**

$$
\begin{aligned}
R_{k\bar{l}mn} &= g\big(R(T_m,T_n)T_k, T_{\bar{l}}\big) \\
&= g\Big(\tilde{R}(T_m,T_n)T_k - i\,A_{km}\,T_n + i\,A_{kn}\,T_m\,,\,T_{\bar{l}}\Big) && \text{(B28)} \\
&= \tilde{R}_{mnk\bar{l}} - i\,A_{km}\,h_{n\bar{l}} + i\,A_{kn}\,h_{m\bar{l}} \\
&= \tilde{R}_{k\bar{l}mn} - i\,A_{km}\,h_{n\bar{l}} + i\,A_{kn}\,h_{m\bar{l}} \\
&= \big(i\,h_{m\bar{l}}\,A^{\bar{\gamma}}_k\,h_{n\bar{\gamma}} - i\,A_{km}\,h_{n\bar{l}}\big) - i\,A_{km}\,h_{n\bar{l}} + i\,A_{kn}\,h_{m\bar{l}} && \text{(B30)} \\
&= 2i\,h_{m\bar{l}}\,A_{kn} - 2i\,A_{km}\,h_{n\bar{l}}
\end{aligned}
$$

Therefore, $R_k{}^p{}_{mn} = R_{k\bar{l}mn}\,h^{\bar{l}p} = 2i\,\delta_{mp}\,A_{kn} - 2i\,\delta_{np}\,A_{km}$.

**Show that $\mathcal{W}_{\mathbf{mn\bar{k}0}} = \mathbf{0}$.**

It is the final part to prove Theorem 6.2. We first recall the equation (B12):

$$\sum_{m=1}^{\nu} d\omega_m^m \;=\; R_{\lambda\bar{\mu}}\;\theta^\lambda\wedge\theta^{\bar{\mu}} + V_\lambda\;\theta^\lambda\wedge\theta - V_{\bar{\mu}}\;\theta^{\bar{\mu}}\wedge\theta.$$

Differentiate both sides of (B12).

$$\begin{aligned} 0 \;=\; d^2\omega_m^m \;&=\; dR_{\lambda\bar{\mu}}\wedge\theta^\lambda\wedge\theta^{\bar{\mu}} + R_{\lambda\bar{\mu}}\big(d\theta^\lambda\wedge\theta^{\bar{\mu}} - \theta^\lambda\wedge d\theta^{\bar{\mu}}\big) \\ &\quad + d(V_\lambda)\wedge\theta^\lambda\wedge\theta + V_\lambda\big(d\theta^\lambda\wedge\theta - \theta^\lambda\wedge d\theta\big) \\ &\quad - d(V_{\bar{\mu}})\wedge\theta^{\bar{\mu}}\wedge\theta - V_{\bar{\mu}}\big(d\theta^{\bar{\mu}}\wedge\theta - \theta^{\bar{\mu}}\wedge d\theta\big) \end{aligned}$$

Acting on $(T_m, T_n, T_{\bar{k}})$, we have

$$\begin{aligned} \big(dR_{\lambda\bar{\mu}}\wedge\theta^\lambda\wedge\theta^{\bar{\mu}}\big)(T_m,T_n,T_{\bar{k}}) &= \frac{1}{6}\big(T_m(R_{n\bar{k}}) - T_n(R_{m\bar{k}})\big) \\ \big(R_{\lambda\bar{\mu}}\;d\theta^\lambda\wedge\theta^{\bar{\mu}}\big)(T_m,T_n,T_{\bar{k}}) &= \frac{1}{6}\,R_{\lambda\bar{k}}\big(\Gamma^\lambda_{nm} - \Gamma^\lambda_{mn}\big) \\ \big(R_{\lambda\bar{\mu}}\;\theta^\lambda\wedge d\theta^{\bar{\mu}}\big)(T_m,T_n,T_{\bar{k}}) &= \frac{1}{6}\big(-R_{m\bar{\mu}}\,\Gamma^{\bar{\mu}}_{n\bar{k}} + R_{n\bar{\mu}}\,\Gamma^{\bar{\mu}}_{m\bar{k}}\big) \\ \big(V_\lambda\;\theta^\lambda\wedge d\theta\big) &= \frac{1}{3}\big(V_m\cdot ih_{n\bar{k}} - V_n\cdot ih_{m\bar{k}}\big). \end{aligned}$$

Putting all together, for fixed $m$, $n$ and $\bar{k}$,

$$T_m(R_{n\bar{k}}) - T_n(R_{m\bar{k}}) + R_{\lambda\bar{k}}(\Gamma^\lambda_{nm} - \Gamma^\lambda_{mn}) + R_{m\bar{\mu}}\Gamma^{\bar{\mu}}_{n\bar{k}} - R_{n\bar{\mu}}\Gamma^{\bar{\mu}}_{m\bar{k}} + 2iV_n\,h_{m\bar{k}} - 2iV_m\,h_{n\bar{k}} = 0.$$

Note that

$$\begin{aligned} T_m(R_{n\bar{k}}) \;&=\; T_m(R^p_n\,h_{p\bar{k}}) \;=\; T_m(R^p_n)\,h_{p\bar{k}} + R^p_n\,T_m(h_{p\bar{k}}) \\ \Longrightarrow\quad T_m(R^p_n)\,h_{p\bar{k}} \;&=\; T_m(R_{n\bar{k}}) - R^q_n\,T_m(h_{q\bar{k}}). \end{aligned}$$

As a result,

$$T_m(R^p_n) \;=\; T_m(R_{n\bar{k}})\,h^{\bar{k}p} - R^q_n\,\Gamma^{\bar{\delta}}_{m\bar{k}}\,h_{q\bar{\delta}}\,h^{\bar{k}p} - R^q_n\,\Gamma^p_{mq} \quad \text{for fixed } m,\, n,\, p. \tag{B31}$$

(1) Set $p = m$ in (B31):

$$T_m(R^m_n) \;=\; T_m(R_{n\bar{k}})\,h^{\bar{k}m} - R_{n\bar{\delta}}\,\Gamma^{\bar{\delta}}_{m\bar{k}}\,h^{\bar{k}m} - R^q_n\,\Gamma^m_{mq}.$$

(2) Set $p = m$ and switch $n$ and $m$ in (B31):

$$T_n(R^m_m) \;=\; T_n(R_{m\bar{k}})\, h^{\bar{k}m} - R_{m\bar{\delta}}\, \Gamma^{\bar{\delta}}_{n\bar{k}}\, h^{\bar{k}m} - R^q_m\, \Gamma^m_{nq}$$

Therefore, for particular $m$ and $n$,

$$\begin{aligned}
& T_m(R^m_n) - T_n(R^m_m) \\
= \;& h^{\bar{k}m}\big(T_m(R_{n\bar{k}}) - T_n(R_{m\bar{k}})\big) + R_{m\bar{\gamma}}\, \Gamma^{\bar{\gamma}}_{n\bar{k}}\, h^{\bar{k}m} - R_{n\bar{\gamma}}\, \Gamma^{\bar{\gamma}}_{m\bar{k}}\, h^{\bar{k}m} + R^q_m\, \Gamma^m_{nq} - R^q_n\, (\Gamma^m_{mq}) \\
= \;& h^{\bar{k}m}\Big[R_{\lambda\bar{k}}(\Gamma^\lambda_{mn} - \Gamma^\lambda_{nm}) + R_{n\bar{\mu}}\, \Gamma^{\bar{\mu}}_{m\bar{k}} - R_{m\bar{\mu}}\, \Gamma^{\bar{\mu}}_{n\bar{k}} + 2i\, V_m\, h_{n\bar{k}} - 2i\, V_n\, h_{m\bar{k}}\Big] \\
& + R_{m\bar{\gamma}}\, \Gamma^{\bar{\gamma}}_{n\bar{k}}\, h^{\bar{k}m} - R_{n\bar{\gamma}}\, \Gamma^{\bar{\gamma}}_{m\bar{k}}\, h^{\bar{k}m} + R^q_m\, \Gamma^m_{nq} - R^q_n\, (\Gamma^m_{mq}).
\end{aligned}$$

Summing on $m$, we get to

$$\Big(\sum_{m=1}^{\nu} T_m(R^m_n)\Big) - T_n(\rho) \;=\; -\,2i\,(\nu-1)\,V_n + R^m_\lambda\, \Gamma^\lambda_{mn} - R^q_n\, (\Gamma^m_{mq}).$$

We could now obtain the divergence formula of the $D$-tensor.

$$\begin{aligned}
& T_\gamma(D^\gamma_m) - D^\gamma_\alpha\, \Gamma^\alpha_{\gamma m} + D^p_m\, (\Gamma^\gamma_{\gamma p}) \\
= \;& \frac{i}{\nu+2} T_\gamma(R^\gamma_m) - \frac{i}{2(\nu+1)(\nu+2)}\big(T_\gamma(\rho)\,\delta_{\gamma m}\big) \\
& - \Big(\frac{i}{\nu+2} R^\gamma_\alpha - \frac{i}{2(\nu+1)(\nu+2)}\rho\,\delta_{\alpha\gamma}\Big)\Gamma^\alpha_{\gamma m} + \Big(\frac{i}{\nu+2} R^p_m - \frac{i}{2(\nu+1)(\nu+2)}\rho\,\delta_{pm}\Big)\Gamma^\gamma_{\gamma p} \\
= \;& \frac{i}{\nu+2}\Big[T_m(\rho) - 2i(\nu-1)V_m + R^p_\lambda\, \Gamma^\lambda_{pm} - R^q_m\, (\Gamma^p_{pq})\Big] \\
& - \frac{i}{2(\nu+1)(\nu+2)}\, T_m(\rho) - \frac{i}{\nu+2}\, R^\gamma_\alpha\, \Gamma^\alpha_{\gamma m} + \frac{i}{\nu+2}\, R^p_m\, (\Gamma^\gamma_{\gamma p}) \\
= \;& \frac{i\,(2\nu+1)}{2(\nu+1)(\nu+2)}\, T_m(\rho) + \frac{2(\nu-1)}{(\nu+2)}\, V_m
\end{aligned}$$

We conclude that

$$T_\gamma(D^\gamma_m) - D^\gamma_\alpha\, \Gamma^\alpha_{\gamma m} + D^p_m\, (\Gamma^\gamma_{\gamma p}) \;=\; \frac{i\,(2\nu+1)}{2(\nu+1)(\nu+2)}\, T_m(\rho) + \frac{2(\nu-1)}{(\nu+2)}\, V_m \tag{B32}$$

As an important step, we have to find out $\hat{R}_{m0}$.

$$\hat{R}_{m0} = 2\big(d\hat{\omega}_0^Q - \hat{\omega}_0^P \wedge \hat{\omega}_P^Q\big)(Z_Q, Z_m)$$

(1) $(Q = \gamma)$

$$\begin{aligned} d\hat{\omega}_0^\gamma(Z_\gamma, Z_m) &= \Big(\frac{1}{2}\, d(D_\alpha^\gamma) \wedge \theta^\alpha + \frac{1}{2}\, D_\alpha^\gamma\ d\theta^\alpha\Big)(Z_\gamma, Z_m) \\ &= \frac{1}{4}\big(T_\gamma(D_m^\gamma) - T_m(D_\gamma^\gamma)\big) + \frac{1}{4}\, D_\alpha^\gamma\big(\Gamma_{m\gamma}^\alpha - \Gamma_{\gamma m}^\alpha\big) \end{aligned}$$

$$(\hat{\omega}_0^P \wedge \hat{\omega}_P^\gamma)(Z_\gamma, Z_m) = (\hat{\omega}_0^p \wedge \hat{\omega}_p^\gamma)(Z_\gamma, Z_m) = \frac{1}{4}\, D_\gamma^p\, \Gamma_{mp}^\gamma - \frac{1}{4}\, D_m^p\, (\Gamma_{\gamma p}^\gamma)$$

Therefore,

$$\big(d\hat{\omega}_0^\gamma - \hat{\omega}_0^P \wedge \hat{\omega}_P^\gamma\big)(Z_\gamma, Z_m) = \frac{1}{4}\Big(T_\gamma(D_m^\gamma) - T_m(D_\gamma^\gamma) + D_\alpha^\gamma(\Gamma_{m\gamma}^\alpha - \Gamma_{\gamma m}^\alpha) - D_\gamma^p\, \Gamma_{mp}^\gamma + D_m^p(\Gamma_{\gamma p}^\gamma)\Big).$$

(2) $(Q = \bar{\gamma})$

$$\begin{aligned} & d\hat{\omega}_0^{\bar{\gamma}}(Z_{\bar{\gamma}}, Z_m) \\ = \ & \Big(-\frac{1}{2} d(D_{\bar{\alpha}}^{\bar{\gamma}}) \wedge \theta^{\bar{\alpha}} - \frac{1}{2} D_{\bar{\alpha}}^{\bar{\gamma}}\ d\theta^{\bar{\alpha}} + dA_\alpha^{\bar{\gamma}} \wedge \theta^\alpha + A_\alpha^{\bar{\gamma}}\ d\theta^\alpha\Big)(Z_{\bar{\gamma}}, Z_m) \\ & -\frac{i}{2(\nu+2)} h^{\bar{\gamma}\delta}\Big(2V_\delta + \frac{i}{2(\nu+1)} T_\delta(\rho)\Big)\, d\theta(Z_{\bar{\gamma}}, Z_m) \\ = \ & \frac{1}{4}\, T_m(D_{\bar{\gamma}}^{\bar{\gamma}}) - \frac{1}{4}\, D_{\bar{\alpha}}^{\bar{\gamma}}(\Gamma_{m\bar{\gamma}}^{\bar{\alpha}}) + \frac{1}{2}\, T_{\bar{\gamma}}(A_m^{\bar{\gamma}}) + \frac{1}{2}\, A_\alpha^{\bar{\gamma}}(-\Gamma_{\bar{\gamma}m}^\alpha) \\ & +\frac{i}{2(\nu+2)}\, h^{\bar{\gamma}\delta}\Big(2V_\delta + \frac{i}{2(\nu+1)}\, T_\delta(\rho)\Big)\, i\, h_{m\bar{\gamma}} \end{aligned}$$

$$(\hat{\omega}_0^P \wedge \hat{\omega}_P^{\bar{\gamma}})(Z_{\bar{\gamma}}, Z_m) = (\hat{\omega}_0^{\bar{p}} \wedge \hat{\omega}_{\bar{p}}^{\bar{\gamma}})(Z_{\bar{\gamma}}, Z_m) = -\frac{1}{4}\, D_{\bar{\gamma}}^{\bar{p}}\, \Gamma_{m\bar{p}}^{\bar{\gamma}} - \frac{1}{2}\, A_m^{\bar{p}}\, (\Gamma_{\bar{\gamma}\bar{p}}^{\bar{\gamma}})$$

Therefore,

$$\begin{aligned} & \big(d\hat{\omega}_0^{\bar{\gamma}} - \hat{\omega}_0^P \wedge \hat{\omega}_P^{\bar{\gamma}}\big)(Z_{\bar{\gamma}}, Z_m) \\ = \ & \frac{1}{4} T_m(D_{\bar{\gamma}}^{\bar{\gamma}}) + \frac{1}{2}\Big(T_{\bar{\gamma}}(A_m^{\bar{\gamma}}) - A_\alpha^{\bar{\gamma}}\, \Gamma_{\bar{\gamma}m}^\alpha + A_m^{\bar{p}}\, \Gamma_{\bar{\gamma}\bar{p}}^{\bar{\gamma}}\Big) - \frac{1}{2(\nu+2)}\Big(2V_m + \frac{i}{2(\nu+1)} T_m(\rho)\Big). \end{aligned}$$

(3) $(Q=0)$

$$\begin{aligned}\big(d\hat{\omega}_0^0 - \hat{\omega}_0^P \wedge \hat{\omega}_P^0\big)(Z_0, Z_m) &= -(\hat{\omega}_0^p \wedge \hat{\omega}_p^0)(Z_0, Z_m) - (\hat{\omega}_0^{\bar{p}} \wedge \hat{\omega}_{\bar{p}}^0)(Z_0, Z_m) \\ &= -(\hat{\omega}_0^{\bar{p}} \wedge \hat{\omega}_{\bar{p}}^0)(Z_0, Z_m) \\ &= \frac{1}{4(\nu+2)}\Big(V_m + \frac{i}{2(\nu+1)} T_m(\rho)\Big)\end{aligned}$$

(4) $(Q=c)$

$$\big(d\hat{\omega}_0^c - \hat{\omega}_0^P \wedge \hat{\omega}_P^c\big)(Z_c, Z_m) = -(\hat{\omega}_0^p \wedge \hat{\omega}_p^c)(Z_c, Z_m) - (\hat{\omega}_0^{\bar{p}} \wedge \hat{\omega}_{\bar{p}}^c)(Z_c, Z_m) = 0$$

Combining all results from (1)-(4), we compute for $\hat{R}_{m0}$.

$$\begin{aligned}\hat{R}_{m0} &= \frac{1}{2}\Big(T_\gamma(D_m^\gamma) - T_m(D_\gamma^\gamma) + D_\alpha^\gamma(\Gamma_{m\gamma}^\alpha - \Gamma_{\gamma m}^\alpha) - D_\gamma^p\,\Gamma_{mp}^\gamma + D_m^p(\Gamma_{\gamma p}^\gamma)\Big) && (Q=\gamma) \\ &\quad + \Big(T_{\bar{\gamma}}(A_m^{\bar{\gamma}}) - A_\alpha^{\bar{\gamma}}\,\Gamma_{\bar{\gamma}m}^\alpha + A_m^{\bar{p}}\,\Gamma_{\bar{\gamma}\bar{p}}^{\bar{\gamma}}\Big) && (Q=\bar{\gamma}) \\ &\quad + \frac{1}{2}\,T_m(D_{\bar{\gamma}}^{\bar{\gamma}}) - \frac{1}{(\nu+2)}\Big(2V_m + \frac{i}{2(\nu+1)}\,T_m(\rho)\Big) && (Q=\bar{\gamma}) \\ &\quad + \frac{1}{2(\nu+2)}\Big(V_m + \frac{i}{2(\nu+1)}\,T_m(\rho)\Big) && (Q=0) \\ &= \frac{1}{2}\Big(T_\gamma(D_m^\gamma) - D_\alpha^\gamma\,\Gamma_{\gamma m}^\alpha + D_m^p\,(\Gamma_{\gamma p}^\gamma)\Big) + \Big(T_{\bar{\gamma}}(A_m^{\bar{\gamma}}) - A_\alpha^{\bar{\gamma}}\,\Gamma_{\bar{\gamma}m}^\alpha + A_m^{\bar{p}}\,\Gamma_{\bar{\gamma}\bar{p}}^{\bar{\gamma}}\Big) \\ &\quad - \frac{1}{2(\nu+2)}\Big(V_m + \frac{i}{2(\nu+1)}\,T_m(\rho)\Big)\end{aligned}$$

According to (B7), note that

$$\begin{aligned}\big(\nabla_{T_{\bar{\gamma}}}\tau\big)(T_m) &= \nabla_{T_{\bar{\gamma}}}\big(\tau(T_m)\big) - \tau\big(\nabla_{T_{\bar{\gamma}}} T_m\big) \\ &= T_{\bar{\gamma}}(A_m^{\bar{k}})\,T_{\bar{k}} + A_m^{\bar{k}}\,\Gamma_{\bar{\gamma}\bar{k}}^{\bar{\delta}}\,T_{\bar{\delta}} - A_\delta^{\bar{k}}\,\Gamma_{\bar{\gamma}m}^\delta\,T_{\bar{k}} \\ &= \Big(T_{\bar{\gamma}}(A_m^{\bar{k}}) + A_m^{\bar{p}}\,\Gamma_{\bar{\gamma}\bar{p}}^{\bar{k}} - A_p^{\bar{k}}\,\Gamma_{\bar{\gamma}m}^p\Big)T_{\bar{k}}.\end{aligned}$$

Since $\big(\nabla_{T_{\bar\gamma}} A\big)(T_m, T_k) = g\big((\nabla_{T_{\bar\gamma}}\tau)(T_m), T_k\big)$,

$$\nabla_{\bar\gamma} A_{mk} \;=\; \Big(T_{\bar\gamma}(A^{\bar l}_m) + A^{\bar p}_m\,\Gamma^{\bar l}_{\bar\gamma\bar p} - A^{\bar l}_p\,\Gamma^{p}_{\bar\gamma m}\Big) h_{k\bar l}$$

$$\Longrightarrow \sum_{\bar\gamma=1}^{\nu}\big(T_{\bar\gamma}(A^{\bar\gamma}_m) + A^{\bar p}_m\,\Gamma^{\bar\gamma}_{\bar\gamma\bar p} - A^{\bar\gamma}_p\,\Gamma^{p}_{\bar\gamma m}\big) \;=\; \sum_{\bar\gamma,k}\big(h^{\bar\gamma k}\,\nabla_{\bar\gamma} A_{mk}\big) \;=\; (V^k_{km}) \;=\; V_m.$$

Therefore, with the help of (B32),

$$\begin{aligned}
\hat R_{m0} \;&=\; \frac{1}{2}\Big(\frac{i\,(2\nu+1)}{2(\nu+1)(\nu+2)}\,T_m(\rho) + \frac{2(\nu-1)}{(\nu+2)}\,V_m\Big)\\
&\quad + V_m - \frac{1}{(\nu+2)}\,V_m - \frac{i}{4(\nu+1)(\nu+2)}\,T_m(\rho)\\
&=\; \frac{i\,\nu}{2(\nu+1)(\nu+2)}\,T_m(\rho) + \frac{2\nu}{(\nu+2)}\,V_m.
\end{aligned}$$

Our goal is to show that $\mathcal{W}_{mn\bar k 0} = 0$.

$$\mathcal{W}_{mn\bar k 0} \;=\; \hat R_{mn\bar k 0} - \frac{1}{2\nu}\,\hat R_{m0}\,h_{n\bar k} + \frac{1}{2\nu}\,\hat R_{n0}\,h_{m\bar k} \tag{B33}$$

We have $\hat R_{mn\bar k 0} = \hat R^c_{mn\bar k} = 2\big(d\hat\omega^c_{\bar k} - \hat\omega^P_{\bar k}\wedge\hat\omega^c_P\big)(Z_m, Z_n)$

$$\begin{aligned}
d\hat\omega^c_{\bar k}(Z_m, Z_n) \;&=\; \Big(-\frac{1}{2}\,d(D_{\alpha\bar k})\wedge\theta^\alpha - \frac{1}{2}D_{\alpha\bar k}\,d\theta^\alpha\Big)(Z_m, Z_n)\\
&=\; -\frac{1}{4}\big(T_m(D_{n\bar k}) - T_n(D_{m\bar k})\big) - \frac{1}{4}\,D_{\alpha\bar k}(\Gamma^\alpha_{nm} - \Gamma^\alpha_{mn})\\
&=\; \frac{1}{4}\Big(T_n(D_{m\bar k}) - T_m(D_{n\bar k}) + D_{p\bar k}\,\Gamma^p_{mn} - D_{p\bar k}\,\Gamma^p_{nm}\Big)
\end{aligned}$$

$$\big(\hat\omega^P_{\bar k}\wedge\hat\omega^c_P\big)(Z_m, Z_n) \;=\; \big(\hat\omega^{\bar\gamma}_{\bar k}\wedge\hat\omega^c_{\bar\gamma}\big)(Z_m, Z_n) \;=\; \frac{1}{4}\Big(D_{m\bar\gamma}\,\Gamma^{\bar\gamma}_{n\bar k} - D_{n\bar\gamma}\,\Gamma^{\bar\gamma}_{m\bar k}\Big)$$

Therefore,

$$\hat R_{mn\bar k 0} \;=\; \frac{1}{2}\Big(T_n(D_{m\bar k}) - T_m(D_{n\bar k}) + D_{p\bar k}\,\Gamma^p_{mn} - D_{p\bar k}\,\Gamma^p_{nm} + D_{n\bar\gamma}\,\Gamma^{\bar\gamma}_{m\bar k} - D_{m\bar\gamma}\,\Gamma^{\bar\gamma}_{n\bar k}\Big).$$

From the definition of the $D$-tensor (B13),

$$
\begin{aligned}
& T_n(D_{m\bar{k}}) - T_m(D_{n\bar{k}}) \\
= & \Big[\frac{i}{(\nu+2)}T_n(R_{m\bar{k}}) - \frac{i}{2(\nu+1)(\nu+2)}T_n(\rho)h_{m\bar{k}} - \frac{i}{2(\nu+1)(\nu+2)}\rho\big(h_{m\bar{\gamma}}\Gamma^{\bar{\gamma}}_{n\bar{k}} + h_{\gamma\bar{k}}\Gamma^{k}_{nm}\big)\Big] \\
& -\Big[\frac{i}{(\nu+2)}T_m(R_{n\bar{k}}) - \frac{i}{2(\nu+1)(\nu+2)}T_m(\rho)h_{n\bar{k}} - \frac{i}{2(\nu+1)(\nu+2)}\rho\big(h_{n\bar{\gamma}}\Gamma^{\bar{\gamma}}_{m\bar{k}} + h_{\gamma\bar{k}}\Gamma^{k}_{mn}\big)\Big] \\
= & \frac{i}{(\nu+2)}\Big(R_{\lambda\bar{k}}(\Gamma^{\lambda}_{nm} - \Gamma^{\lambda}_{mn}) + R_{m\bar{\mu}}\Gamma^{\bar{\mu}}_{n\bar{k}} - R_{n\bar{\mu}}\Gamma^{\bar{\mu}}_{m\bar{k}} + 2iV_n h_{m\bar{k}} - 2iV_m h_{n\bar{k}}\Big) \\
& + \frac{i}{2(\nu+1)(\nu+2)}\Big[T_m(\rho)h_{n\bar{k}} - T_n(\rho)h_{m\bar{k}} + \rho\Big(h_{n\bar{\gamma}}\Gamma^{\bar{\gamma}}_{m\bar{k}} + h_{\gamma\bar{k}}\Gamma^{\gamma}_{mn} - h_{m\bar{\gamma}}\Gamma^{\bar{\gamma}}_{n\bar{k}} - h_{\gamma\bar{k}}\Gamma^{\gamma}_{nm}\Big)\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& D_{p\bar{k}}\,\Gamma^{p}_{mn} - D_{p\bar{k}}\,\Gamma^{p}_{nm} + D_{n\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{m\bar{k}} - D_{m\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{n\bar{k}} \\
= & \frac{i}{(\nu+2)}\Big(R_{p\bar{k}}\,\Gamma^{p}_{mn} - R_{p\bar{k}}\,\Gamma^{p}_{nm} + R_{n\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{m\bar{k}} - R_{m\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{n\bar{k}}\Big) \\
& - \frac{i}{2(\nu+1)(\nu+2)}\,\rho\,\Big(h_{p\bar{k}}\,\Gamma^{p}_{mn} - h_{p\bar{k}}\,\Gamma^{p}_{nm} + h_{n\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{m\bar{k}} - h_{m\bar{\gamma}}\,\Gamma^{\bar{\gamma}}_{n\bar{k}}\Big).
\end{aligned}
$$

Therefore,

$$
\hat{R}_{mn\bar{k}0} = \frac{1}{(\nu+2)}\big(V_m\,h_{n\bar{k}} - V_n\,h_{m\bar{k}}\big) + \frac{i}{4(\nu+1)(\nu+2)}\big(T_m(\rho)\,h_{n\bar{k}} - T_n(\rho)\,h_{m\bar{k}}\big).
$$

Finally, we are ready to find $\mathcal{W}_{mn\bar{k}0}$.

$$
\begin{aligned}
\mathcal{W}_{mn\bar{k}0} &= \hat{R}_{mn\bar{k}0} - \frac{1}{2\nu}\,\hat{R}_{m0}\,h_{n\bar{k}} + \frac{1}{2\nu}\,\hat{R}_{n0}\,h_{m\bar{k}} \\
&= \frac{1}{(\nu+2)}\Big(V_m\,h_{n\bar{k}} - V_n\,h_{m\bar{k}}\Big) + \frac{i}{4(\nu+1)(\nu+2)}\Big(T_m(\rho)\,h_{n\bar{k}} - T_n(\rho)\,h_{m\bar{k}}\Big) \\
&\quad - \frac{1}{2\nu}\Big(\frac{i\,\nu}{2(\nu+1)(\nu+2)}\,T_m(\rho) + \frac{2\nu}{(\nu+2)}\,V_m\Big)\,h_{n\bar{k}} \\
&\quad + \frac{1}{2\nu}\Big(\frac{i\nu}{2(\nu+1)(\nu+2)}\,T_n(\rho) + \frac{2\nu}{(\nu+2)}\,V_n\Big)\,h_{m\bar{k}} \\
&= 0 \qquad \square
\end{aligned}
$$

## APPENDIX C. COMPUTATIONAL MODEL IN MATLAB

We would introduce the use of software MATLAB [12] to computer for formulas and justify equalities. This results in justifying Proposition 5.8, Theorem 6.2, Proposition 6.6, and other formulas. The main objective in our programming work is to define an effective model of (5.5) and compute for components of various curvature tensors so that they could be compared with each other.

### Part (1): The main model of $\mathfrak{D}(w)$

In the model (5.5), $e_1$, $e_2$ and $e_3$ form an orthonormal frame on the 3-manifold $M$. Any tangent vector on the sphere bundle $N$ of $M$, belongs to the span of $e_1$, $e_2$, $e_3$, $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial \overline{u}}$. We define the variables

```
conj_X1  =  [u^2-1; 2*u; i*(u^2+1); w; 0];
     X1  =  [conj(u)^2-1; 2*conj(u); -i*(conj(u)^2+1); 0; conj(w)];
     X2  =  [0; 0; 0; 1; 0];
conj_X2  =  [0; 0; 0; 0; 1];
      T  =  [v1normv; v2normv; v3normv; T4; conj(T4)];
```

to represent $\overline{X}_1$, $X_1$, $X_2$, $\overline{X}_2$ and $T$. Here `v1normv` is a variable in `u` defined by $\dfrac{v_1}{|v|}$.

In addition to `T4`, we also define the *complex* variables: `h11`, `aV`, `bV` and `rho` for the Lie brackets between the above vectors and the Fefferman metric of $\mathfrak{D}(w)$. From Proposition 5.4, the Lie brackets also include the first derivative of $T_4$ by $u$, $T_{4,u}$. To describe any first derivatives of $T_4$, we use the symbols

`dT4_Mu`, `dT4_conjMu`, `dT4_vnormv`, `dT4_u`, `dT4_conju`

to represent $D_\mu T_4$, $D_{\overline{\mu}} T_4$, $D_{\frac{v}{|v|}} T_4$, $T_{4,u}$ and $T_{4,\overline{u}}$ respectively. Similar definition holds for the first derivatives of `h11`, `aV`, `bV` and `rho`. Regarding the second derivatives of $T_4$, we create the following representation:

$D_\mu D_\mu T_4$ : `d2T4_MuMu`, $D_{\overline{\mu}} D_\mu T_4$ : `d2T4_MuconjMu`, $D_{\frac{v}{|v|}} D_\mu T_4$ : `d2T4_Muvnormv`,

$D_{\overline{\mu}} D_{\overline{\mu}} T_4$ : `d2T4_conjMuconjMu`, $D_{\frac{v}{|v|}} D_{\overline{\mu}} T_4$ : `d2T4_conjMuvnormv`,

$D_{\frac{v}{|v|}} D_{\frac{v}{|v|}} T_4$ : `d2T4_vnormvvnormv`, $D_\mu D_u T_4$ : `d2T4_uMu`,

$D_{\overline{\mu}} D_u T_4$ : `d2T4_uconjMu`, $D_{\frac{v}{|v|}} D_u T_4$ : `d2T4_uvnormv`,

$D_\mu D_{\overline{u}} T_4$ : `d2T4_uMu`, $D_{\overline{\mu}} D_{\overline{u}} T_4$ : `d2T4_uconjMu`, $D_{\frac{v}{|v|}} D_{\overline{u}} T_4$ : `d2T4_uvnormv`,

$D_u D_u T_4$ : `d2T4_uu`, $D_{\overline{u}} D_u T_4$ : `d2T4_uconju`, $D_{\overline{u}} D_{\overline{u}} T_4$ : `d2T4_conjuconju`.

Note that $D_Y D_X(f) = D_X D_Y(f) + [Y, X](f)$, so we have

$$D_\mu D_{\overline{\mu}} T_4 \;=\; D_{\overline{\mu}} D_\mu T_4 + \big[\mu, \overline{\mu}\big](T_4) \;=\; D_{\overline{\mu}} D_\mu T_4 + a_M\, D_\mu T_4 - \overline{a}_M D_{\overline{\mu}} T_4 + 2i\, h_{1\overline{1}}\, D_{\frac{v}{|v|}} T_4.$$

In order to obtain the rest of the second derivatives of $T_4$, we describe the vectors

$$\big[\mu, \overline{\mu}\big], \quad \Big[\mu, \frac{v}{|v|}\Big], \quad \Big[\mu, \frac{\partial}{\partial u}\Big] \quad \text{and} \quad \Big[\frac{v}{|v|}, \frac{\partial}{\partial u}\Big]$$

(Chapter 5) under the basis $\Big\{\mu, \overline{\mu}, \dfrac{v}{|v|}\Big\}$.

```
lie_Mu_conjMu = [aM; -conj(aM); 2*i*h11];
lie_Mu_vnormv = [aV; bV; 2*T4];
lie_Mu_ddu = [-2*conj(u)/(1+u*conj(u)); 0; -2];
lie_conjMu_vnormv = [conj(bV); conj(aV); 2*conj(T4)];
lie_conjMu_ddconju = [0; -2*u/(1+u*conj(u)); -2];
lie_vnormv_ddu = [0; 1/(1+u*conj(u))∧2; 0];
lie_vnormv_ddconju = [1/(1+u*conj(u))∧2; 0; 0];
```

Moreover, we let `dT4row = [dT4_Mu, dT4_conjMu, dT4_vnormv]`. Therefore, we get to

```
d2T4_conjMuMu = d2T4_MuconjMu + dT4row*lie_Mu_conjMu;
d2T4_vnormvMu = d2T4_Muvnormv + dT4row*lie_Mu_vnormv;
d2T4_vnormvconjMu = d2T4_conjMuvnormv + dT4row*lie_conjMu_vnormv;
d2T4_Muu = d2T4_uMu - dT4row*lie_Mu_ddu;
d2T4_conjMuu = d2T4_uconjMu;
d2T4_vnormvu = d2T4_uvnormv - dT4row*lie_vnormv_ddu;
d2T4_Muconju = d2T4_conjuMu;
d2T4_conjMuconju = d2T4_conjuconjMu - dT4row*lie_conjMu_ddconju;
d2T4_vnormvconju = d2T4_conjuvnormv - dT4row*lie_vnormv_ddconju;
d2T4_conjuu = d2T4_uconju;
```

The Fefferman metric $F$ on $C(N)$ (Section 5.5) is represented by the matrix `F0`, which corresponds to the basis $\{X_1, \overline{X}_1, X_2, \overline{X}_2, T, \frac{\partial}{\partial\gamma}\}$. We set:

```
 F_T_T  =  -1/24*rho + i/2*(-conj(aV)+2*u*conj(T4)/(1+u*conj(u))-dT4_u);
F_X1_T  =  i/4*(conj(dw_u)- 2*u/(1+u*conj(u))*conj(w)-2*conj(T4)+aM);
F_X2_T  =  -i*conj(u)/(2*(1+u*conj(u))); ,

F0 = [ 0,      h11,        0,         -i,          F_X1_T,        0;
     h11,       0,         i,          0,       conj(F_X1_T),     0;
       0,       i,         0,          0,          F_X2_T,        0;
      -i,       0,         0,          0,       conj(F_X2_T),     0;
   F_X1_T,  conj(F_X1_T),  F_X2_T,  conj(F_X2_T),     F_T_T,      1/4;
       0,       0,         0,          0,           1/4,       0 ]; .
```

Under our convention that $u_1 = X_1$, $u_2 = \overline{X}_1$, $u_3 = X_2$, $u_4 = \overline{X}_2$, $u_5 = T$ and $u_6 = \frac{\partial}{\partial\gamma}$, the Lie brackets between $u_j$'s are expressed in the same basis of `F0`. We let

```
lieTwo = cell(6,6);
```

such that `lieTwo{j,k}` represents the vector $[u_j, u_k]$. For example, $[u_1, u_2]$ is denoted by

```
lieTwo{1,2} =  [conj(aM); -aM; aM*w + 2*i*h11*T4 + dw_conjMu;
               -conj(aM)*conj(w) + 2*i*h11*conj(T4) - conj(dw_conjMu);
                -2*i*h11; 0];
```

according to Proposition 5.4.

The Christoffel symbols $\hat{\Gamma}^k_{ij}$, defined by $\hat{\nabla}_{u_i} u_j = \hat{\Gamma}^k_{ij}\, u_k$ and found by (1.8), are the very first things to compute for. In particular, we differentiate every $F_{ij} = F(u_i, u_j)$ by some $u_k$. The matrix `F0` consists of variables in

`CVarMain2 = [u, w, h11, T4, aM, aV, rho, dw_u, dT4_u].`

A particular function in MATLAB is created to help finding $D_\mu F_{ij}$, $D_{\overline{\mu}} F_{ij}$, $D_{\frac{v}{|v|}} F_{ij}$, $D_u F_{ij}$ and $D_{\overline{u}} F_{ij}$. The main tool is to use the chain rule, for example

$$D_\mu F = \sum_{i=j}^{n_1} \Big( \frac{\partial F}{\partial z_j} D_\mu z_j \Big) + \sum_{i=j}^{n_1} \Big( \frac{\partial F}{\partial \overline{z}_j} \overline{(D_{\overline{\mu}} z_j)} \Big), \tag{C1}$$

where $z_1, z_2, \cdots, z_{n_1}$ are elements in `CVarMain2`. For the first term, $F_{z_j}$ or $F_{\overline{z}_j}$, we modify the default function `diff` so that complex differentiation is executed properly.

```
function   g = complexdiff3(f, z, s)
if isreal(z)==1
    g = diff(f,z);
end
if isreal(z)==0
    syms a
    f1 = subs(f, abs(z), (a*z)∧(0.5));
    f1 = subs(f1, conj(z), a);
    g1 = diff(f1, z); g1 = subs(g1, a, conj(z));
    g2 = diff(f1, a); g2 = subs(g2, a, conj(z));
    if s==0
        g = g1;
    else
        g = g2;
    end
end
```

The terms $D_\mu z_j$ and $D_{\overline{\mu}} z_j$ are defined earlier, e.g. `dT4_Mu` and `dT4_conjMu`. It is then to recall the first derivatives of $z_j$ from a structure array when we run to the $j$-th step finding $D_\mu F$ or others. The structure array `derivativeDict` consists of fields named by elements in `CVarMain2`.

```
derivativeDict.u = [0; 0; 0; 1; 0];
derivativeDict.w = [dw_Mu; dw_conjMu; dw_vnormv; dw_u; 0];
derivativeDict.h11 = [dh11_Mu; dh11_conjMu; dh11_vnormv;
   dh11_u; dh11_conju];
derivativeDict.aM = [daM_Mu; daM_conjMu; daM_vnormv; daM_u; daM_conju];
derivativeDict.aV = [daV_Mu; daV_conjMu; daV_vnormv; daV_u; daV_conju];
derivativeDict.rho = [drho_Mu; drho_conjMu; drho_vnormv;
   drho_u; drho_conju];
derivativeDict.dw_u = [d2w_uMu; d2w_uconjMu; d2w_uvnormv; d2w_uu; 0];
derivativeDict.dT4_u = [d2T4_uMu; d2T4_uconjMu; d2T4_uvnormv;
   d2T4_uu; d2T4_uconju];
```

In the function to find $D_\mu f$ etc. by (C1), the output is a row array `df` in

$$\left[D_\mu f,\ D_{\overline{\mu}} f,\ D_{\frac{v}{|v|}} f,\ D_u f,\ D_{\overline{u}} f,\ D_\gamma f\right]. \tag{C2}$$

The derivative of $f$ by $\gamma$ is found by `diff(f, gamma)` directly.

```
function  df = df_main_MuGamma(f, CVar, derivativeDict, gamma)
length_of_CVar = length(CVar);
df_by_CVar = sym('df_by_CVar', [2, length_of_CVar]);
for j=1:length_of_CVar
    df_by_CVar(1,j) = complexdiff3(f, CVar(j), 0);
    df_by_CVar(2,j) = complexdiff3(f, CVar(j), 1);
end
df = sym([0, 0, 0, 0, 0, 0]);
for j=1:length_of_CVar
    column = derivativeDict.(char(CVar(j)));
    if isreal(CVar(j))==1
        df(1) = df(1) + df_by_CVar(1,j)*column(1);
        df(2) = df(2) + df_by_CVar(1,j)*column(2);
        df(3) = df(3) + df_by_CVar(1,j)*column(3);
```

```
        df(4) = df(4) + df_by_CVar(1,j)*column(4);
        df(5) = df(5) + df_by_CVar(1,j)*column(5);
    else
        df(1) = df(1) + df_by_CVar(1,j)*column(1)...
            + df_by_CVar(2,j)*conj(column(2));
        df(2) = df(2) + df_by_CVar(1,j)*column(2)...
            + df_by_CVar(2,j)*conj(column(1));
        df(1) = df(3) + df_by_CVar(1,j)*column(3)...
            + df_by_CVar(2,j)*conj(column(3));
        df(1) = df(4) + df_by_CVar(1,j)*column(4)...
            + df_by_CVar(2,j)*conj(column(5));
        df(1) = df(5) + df_by_CVar(1,j)*column(5)...
            + df_by_CVar(2,j)*conj(column(4));
    end
end
df(6) = diff(f, gamma);
```

To find the values of $df(u_j)$, we also define a matrix `Uvector` of column vectors being $u_j$'s.

```
Uvector = [0,  1,  0,  0,         0,   0;
           1,  0,  0,  0,         0,   0;
           0,  0,  0,  0,         1,   0;
           0,  w,  1,  0,        T4,   0;
     conj(w),  0,  0,  1,  conj(T4),   0;
           0,  0,  0,  0,         0,  1];
```

Let `RGamma` be a symbolic array such that `RGamma(m,n,k)` represents $\hat{\Gamma}^k_{mn}$. We first find out $\hat{\Gamma}_{mn,k}$ denoted by `RGamma0(m,n,k)`. Then, we apply $\hat{\Gamma}^k_{mn} = \hat{\Gamma}_{mn,l}\, F^{kl}$.

```
for m=1:6
   for n=1:6
      for k=1:6
         part1 = dF0{n,k}*Uvector(:,m) - dF0{m,n}*Uvector(:,k) ...
            + dF0{m,k}*Uvector(:,n);
         part2 = 0;
```

```
            for ll=1:6
               part2 = part2 - F0(m,ll)*lieTwo{n,k}(ll) ...
                  + F0(k,ll)*lieTwo{m,n}(ll) + F0(n,ll)*lieTwo{k,m}(ll);
            end
            RGamma0(m,n,k) = 1/2*(part1+ part2);
         end
      end
   end
```

The next step will be to compute for the Riemann tensor of $\hat{\nabla}$. We will follow the same approach but now it involves more complex variables in the `CVar`-array. By (1.9) and (1.10), we need to differentiate $\hat{\Gamma}^k_{ij}$'s. The Christoffel symbols contain the following variables in `CVarMain3`.

```
CVarMain3 = [u, w, h11, T4, aM, aV, bV, rho, ...
      dw_u, dw_Mu, dw_conjMu, dw_vnormv, dh11_u, dh11_conju, dh11_Mu, ...
      dh11_conjMu, dh11_vnormv, dT4_u, dT4_conju, dT4_Mu, dT4_conjMu, ...
      dT4_vnormv, daM_u, daM_conju, daM_Mu, daM_conjMu, daM_vnormv, ...
      daV_u, daV_conju, daV_Mu, daV_conjMu, daV_vnormv, ...
      drho_u, drho_conju, drho_Mu, drho_conjMu, drho_vnormv, ...
      d2w_uu, d2w_uMu, d2w_uconjMu, d2w_uvnormv, ...
      d2T4_uu, d2T4_uconju, d2T4_uMu, d2T4_uconjMu, d2T4_uvnormv];
```

Therefore, we have to define the second and third derivatives of variables, and expand the structure array `derivativeDict`. For example, the second derivatives of $T_{4,u}$ include

```
d3T4_uMuMu,   d3T4_uMuconjMu,   d3T4_uMuvnormv,   d3T4_uconjMuconjMu,
d3T4_uconjMuvnormv,   d3T4_uvnormvvnormv,   d3T4_uuMu,   d3T4_uuconjMu,
d3T4_uuvnormv,   d3T4_uconjuMu,   d3T4_uconjuconjMu,   d3T4_uconjuvnormv,
d3T4_uuu,   d3T4_uuconju,   d3T4_uconjuconju.
```

Other third derivatives of $T_4$ in the form of $T_{4,u\mathrm{XX}}$ can be found by the identity

$$D_Y D_X(T_{4,u}) \;=\; D_X D_Y(T_{4,u}) + [Y,X](T_{4,u}).$$

We denote $\hat{R}^l_{mnk}$ (1.9) by `RiemCurv0(m,n,k,ll)` and $\hat{R}_{mnkl}$ (1.10) by `RiemCurv(m,n,k,ll)`. The main code to find `RiemCurv0(m,n,k,ll)` is:

```
...
part1 = dRGamma{n,k,ll}*Uvector(:,m) - dRGamma{m,k,ll}*Uvector(:,n);
part2 = 0;
for p = 1:6
   part2 = part2 + RGamma(n,k,p)*RGamma(m,p,ll) ...
      -RGamma(m,k,p)*RGamma(n,p,ll) -lieTwo{m,n}(p)*RGamma(p,k,ll);
end
RiemCurv0(m,n,k,ll) = part1 + part2;
...
```

Here, `dRGamma{n,k,ll}` consists of the directional derivatives of $\hat{\Gamma}^l_{nk}$ in the format of (C2). Both the Ricci tensor, the scalar curvature and the Weyl tensor can be found directly from `RiemCurv`, according to (1.11), (1.12) and (1.16) respectively.

$$\begin{aligned} \hat{R}_{ij} \;&=\; \mathrm{Ric}(u_i,u_j) &&:\; \texttt{RicCurv(ii,j)} \\ S \;&=\; \hat{R}_{ij}\,F^{ij} &&:\; \texttt{S} \\ \mathcal{W}_{ijkl} \;&=\; \mathcal{W}(u_i,u_j,u_k,u_l) &&:\; \texttt{Weyl(ii,j,k,ll)} \end{aligned}$$

At the end of our main program, we try simplifying the components of curvature tensors based on the MATLAB function `simplify`. We would use an original function in order to get rid of any '`abs(VARIABLE)`' in the simplified expression.

```
function   g = complex_simple3(f, MVar)
MVar1=[ ];
for j=1:length(MVar)
   count = has(f, MVar(j)) + has( f, conj(MVar(j)) );
   if count>0
      MVar1=cat(2,MVar1,MVar(j));
   end
end
```

```
if length(MVar1) == 0
   g = simplify(f);
end
if length(MVar1) > 0
   h = f;
   A = sym(‘a’, [1, length(MVar1)]);
   B = sym(‘b’, [2, length(MVar1)], ‘real’);
   for j = 1:length(MVar1)
      absVar = (A(j)*MVar1(j))∧0.5;
      h = subs(h, abs(MVar1(j)), absVar);
      h = subs(h, real(MVar1(j)), 0.5*( MVar1(j)+A(j)) );
      h = subs(h, imag(MVar1(j)), -0.5*i*( MVar1(j)-A(j) ));
      h = subs(h, conj(MVar1(j)), A(j));
      h = subs(h, MVar1(j), B(1,j));
      h = subs(h, A(j), B(2,j));
   end
   g = simplify(h);
   for j = 1:length(MVar1)
      g = subs(g, B(1,j), MVar1(j));
      g = subs( g, B(2,j), conj(MVar1(j)) );
   end
end
```

The input array `MVar` contains all complex variables that the variable `f` depends on, which could be found by `symvar(f)`.

## Part (2): Reduction of variables

The second stage of our programming work, focuses on the components of the Weyl tensor. In particular, by putting other variables in terms of $a_V$ and its derivatives, we have a more effective way to check whether two expressions equal each other or not.

To begin with, we apply symmetries within $\mathcal{W}_{ijkl}$'s so that only 120 components need to be analyzed. See Section 6.1. We define the index array `countIndex120` by the following codes.

```
countIndex120 = [ ];
for m = 1:6
   for n = 1:6
      for k = 1:6
         for ll = 1:6
            if (m<n) && (k<ll) && ((10*m + n)<=(10*k + ll))
               countIndex120 = [countIndex120; m, n, k, ll];
            end
         end
      end
   end
end
```

These 120 terms of the Weyl tensor are contained in the array `WeylTwo` of the size $120 \times 5$.

```
m = countIndex120(j, 1);   n = countIndex120(j, 2);
k = countIndex120(j, 3);   ll = countIndex120(j, 4);
WeylTwo(j,:)  = [m, n, k, ll, Weyl(m,n,k,ll)];
```

These coefficients in `WeylTwo` contain the following symbolic variables that have to be replaced by $a_V$ and its derivatives.

```
variableSet1 = [aM, bV, T4, h11, rho];
variableSet2 = [dT4_u, dT4_Mu, dT4_conju, dT4_conjMu, dT4_vnormv, ...
   daM_u, daM_Mu, daM_conju, daM_conjMu, daM_vnormv, ...
   dbV_u, dbV_Mu, dbV_conju, dbV_conjMu, dbV_vnormv, ...
   dh11_u, dh11_Mu, dh11_conju, dh11_conjMu, dh11_vnormv, ...
   drho_u, drho_Mu, drho_conju, drho_conjMu, drho_vnormv];
```

```
variableSet3 = [d2T4_uu, d2T4_uMu, d2T4_MuMu, d2T4_uconju, ...
   d2T4_conjuMu, d2T4_uconjMu, d2T4_uvnormv, d2T4_MuconjMu, ...
   d2T4_Muvnormv, d2T4_conjuconju, d2T4_conjuconjMu, d2T4_conjuvnormv, ...
   d2T4_conjMuconjMu, d2T4_conjMuvnormv, d2aM_uu, d2aM_uMu, d2aM_MuMu, ...
   d2aM_uconju, d2aM_conjuMu, d2aM_uconjMu, d2aM_uvnormv, ...
   d2aM_MuconjMu, d2aM_Muvnormv, d2aM_conjuconju, ...
   d2aM_conjuconjMu, d2aM_conjuvnormv, d2aM_conjMuvnormv, ...
   d2h11_uu, d2h11_uMu, d2h11_uconju, d2h11_conjuMu, d2h11_uconjMu, ...
   d2h11_uvnormv, d2h11_MuconjMu, d2h11_Muvnormv, d2h11_conjuconju, ...
   d2h11_conjuconjMu, d2h11_conjuvnormv, d2h11_conjMuvnormv, ...
   d2h11_vnormvvnormv, d2rho_uu, d2rho_uMu, d2rho_MuMu, d2rho_uconju, ...
   d2rho_conjuMu, d2rho_uconjMu, d2rho_MuconjMu, d2rho_conjuconju, ...
   d2rho_conjuconjMu, d2rho_conjMuconjMu];

variableSet4 = [d3T4_uuu, d3T4_uuMu, d3T4_uMuMu, d3T4_uuconju, ...
   d3T4_uconjuMu, d3T4_uuconjMu, d3T4_uMuconjMu, d3T4_uconjuconju, ...
   d3T4_uconjuconjMu, d3T4_uconjMuconjMu];
```

We would first replace the simplest variables $a_M$, $T_4$, $b_V$, $h_{1\bar{1}}$, $\rho$ by $a_V$ and its derivatives.

```
 aMSub  =  -(1+u*conj(u))∧2*daV_u;
 T4Sub  =  ((1+u*conj(u))∧2/2)*daV_conju;
 bVSub  =  u*(1+u*conj(u))*conj(daV_u) + (1+u*conj(u))∧2/2*conj(d2aV_uu);
h11Sub  =  i*(1+u*conj(u))∧2*aV + (i/2)*(1+u*conj(u))∧4*d2aV_uconju;
  phiW  =  d2w_uu - 6*conj(u)/(1+u*conj(u))*dw_u
           + 12*conj(u)∧2/(1+u*conj(u))∧2*w;
rhoSub  =  i*(phiW-conj(phiW)) + 16*theta + 12*i*(aV-conj(aV));
```

For the substitution of derivatives of $a_M$, $T_4$, $b_V$, $h_{1\bar{1}}$ and $\rho$, we find their values in terms of $a_V$ and its derivatives by the function `df_main_MuGamma` . The `CVar`-set is taken to be:

```
CVarWeyl = [u, aV, daV_u, daV_conju, d2aV_uu, d2aV_uconju, ...
   daV_Mu, daV_conjMu, daV_vnormv, d2aV_uMu, d2aV_uconjMu, ...
   d2aV_conjuMu, d2aV_conjuconjMu, d2aV_conjuconju, ...
   d3aV_uconjuMu, d3aV_uconjuconjMu, d3aV_uconjuvnormv, d3aV_uuconju, ...
   d3aV_uconjuconju, theta, dtheta_Mu, w, dw_u, d2w_uu, ...
   dw_Mu, dw_conjMu, d2w_uMu, d2w_uconjMu, d3w_uuMu, ...
   d3w_uuconjMu, d3w_uuu] .
```

Here, `theta` represents the value $\theta = G^3_{12} + G^1_{23} + G^2_{31}$, which appears in $\rho$. For example, in order to find the substitution for `d2T4_uMu`, we implement the code:

```
dT4Vec = df_main_MuGamma(T4, CVarWeyl, derivativeDict, gamma);
dT4_uSub = dT4Vec(4);
d2T4_uVec = df_main_MuGamma(dT4_uSub, CVarWeyl, derivativeDict, gamma);
d2T4_uMuSub = d2T4_uVec(1); .
```

Repeating the same procedure for every variable in concern, every coefficient in `WeylTwo` is written by $a_V$ and its derivatives, along with terms in $u$, $w$ and $\theta$. Here comes a list of symbolic variables found in `WeylTwo`.

$a_V$ : `aV, daV_u, daV_conju, daV_Mu, daV_conjMu, daV_vnormv, d2aV_uu, d2aV_uconju, d2aV_conjuconju, d2aV_uMu, d2aV_uconjMu, d2aV_uvnormv, d2aV_conjuMu, d2aV_conjuconjMu, d2aV_conjuvnormv, d2aV_MuMu, d2aV_MuconjMu, d2aV_Muvnormv, d2aV_conjMuconjMu, d2aV_conjMuvnormv, d2aV_vnormvvnormv, d3aV_uuu, d3aV_uuconju, d3aV_uconjuconju, d3aV_conjuconjuconju, d3aV_uuMu, d3aV_uuconjMu, d3aV_uuvnormv, d3aV_conjuconjuMu, d3aV_conjuconjuconjMu, d3aV_conjuconjuvnormv, d3aV_uMuMu, d3aV_uMuconjMu, d3aV_uMuvnormv, d3aV_uconjMuvnormv, d3aV_conjuMuMu, d3aV_conjuMuconjMu, d3aV_conjuMuvnormv, d3aV_conjuconjMuconjMu, d3aV_conjuconjMuvnormv, d3aV_uconjuMu, d3aV_uconjuconjMu, d3aV_uconjuvnormv, d4aV_uuuconju, d4aV_uuconjuconju, d4aV_uconjuconjuconju, d4aV_uuconjuMu, d4aV_uuconjuconjMu, d4aV_uuconjuvnormv, d4aV_uconjuconjuMu, d4aV_uconjuconjuconjMu, d4aV_uconjuconjuvnormv, d4aV_uconjuMuMu, d4aV_uconjuMuconjMu, d4aV_uconjuMuvnormv, d4aV_uconjuconjMuconjMu, d4aV_uconjuconjMuvnormv, d4aV_uconjuvnormvvnormv.`

$w$: u, w, dw_u, dw_Mu, dw_conjMu, dw_vnormv, d2w_uu, d2w_uMu, d2w_uconjMu, d2w_uvnormv, d2w_MuMu, d2w_MuconjMu, d2w_Muvnormv, d2w_conjMuconjMu, d2w_conjMuvnormv, d2w_vnormvvnormv, d3w_uuu, d3w_uuMu, d3w_uuconjMu, d3w_uuvnormv, d3w_uMuMu, d3w_uMuconjMu, d3w_uMuvnormv, d3w_uconjMuconjMu, d3w_uconjMuvnormv, d4w_uuuu, d4w_uuuMu, d4w_uuuconjMu, d4w_uuMuMu, d4w_uuMuconjMu, d4w_uuconjMuconjMu.

$\theta$ : theta, dtheta_Mu, dtheta_vnormv, d2theta_MuMu, d2theta_MuconjMu.

It is possible to reduce the amount of variables in WeylTwo further using differential relations among $a_V$ and its $u$- or $\overline{u}$-derivatives. Proposition 5.3 provides us with the reduction formulas of $a_{V,u\overline{u}}$, $a_{V,\overline{u}\,\overline{u}}$ and $a_{V,uuu}$. Therefore, we set

```
d2aV_conjuconjuR = - (2*u/(1+u*conj(u)))*(daV_conju +conj(daV_u)) ...
   - conj(d2aV_uu);
d2aV_uconjuR = (2/(1+u*conj(u))∧2)*(2*i*theta+conj(aV)-2*aV);
d3aV_uuuR = (-6*conj(u)∧2/(1+u*conj(u))∧2)*daV_u ...
   - (6*conj(u)/(1+u*conj(u)))*d2aV_uu;
d3aV_uuconjuR = - 4*conj(u)/(1+u*conj(u))∧3*(2*i*theta+conj(aV)-2*aV) ...
   + 2/(1+u*conj(u))∧2*(-2*daV_u + conj(daV_conju)); .
```

All derivatives of $a_V$ with a $u$-$\overline{u}$-derivative, two $\overline{u}$-derivatives, or three $u$-derivatives, can be substituted with other $a_V$-terms. They are:

```
d2aV_conjuconju,  d2aV_uconju,  d3aV_uuu,  d3aV_uuconju,
d3aV_conjuconjuMu,  d3aV_conjuconjuconjMu,  d3aV_conjuconjuvnormv,
d3aV_conjuconjuconju, d3aV_uconjuMu, d3aV_uconjuconjMu, d3aV_uconjuvnormv,
d3aV_uconjuconju,  d4aV_uuconjuMu,  d4aV_uuconjuconjMu,
d4aV_uuconjuvnormv,  d4aV_uuconjuconju,  d4aV_uuuconju,  d4aV_uconjuMuMu,
d4aV_uconjuMuconjMu,  d4aV_uconjuMuvnormv,  d4aV_uconjuconjMuconjMu,
d4aV_uconjuconjMuvnormv,  d4aV_uconjuconjuMu,  d4aV_uconjuconjuconjMu,
d4aV_uconjuconjuconju,  d4aV_uconjuconjuvnormv,  d4aV_uconjuvnormvvnormv.
```

We would make use of the function `df_main_MuGamma` again to replace the above variables. The `CVar`-array in use is defined by

```
CVarWeylTwo = [u, theta, aV, daV_u, daV_conju, d2aV_uu, daV_Mu, ...
   daV_conjMu, daV_vnormv, dtheta_Mu, dtheta_vnormv];
```

`derivativeDict` has already contained the fields of `daV_u` and `daV_conju`. We have to modify these existing fields as well as to add fields to `derivativeDict`. For example,

```
derivativeDict.daV_u = [d2aV_uMu; d2aV_uconjMu; d2aV_uvnormv;
   d2aV_uu; d2aV_uconjuR];
derivativeDict.daV_conju = [d2aV_conjuMu; d2aV_conjuconjMu;
   d2aV_conjuvnormv; d2aV_uconjuR; d2aV_conjuconjuR];
derivariveDict.d2aV_uu = [d3aV_uuMu; d3aV_uuconjMu; d3aV_uuvnormv;
   d3aV_uuuR; d3aV_uuconjuR];
```

If we want to replace `d4aV_uconjuMuMu` by the new variable `d4aV_uconjuMuMuR`, we may implement the code:

```
d3aV_uconjuVec = df_main_MuGamma(d2aV_uconjuR, ...
   CVarWeylTwo, derivativeDict, gamma);
d3aV_uconjuMuR = d3aV_uconjuVec(1);
d4aV_uconjuMuVec = df_main_MuGamma(d3aV_uconjuMuR, ...
   CVarWeylTwo, derivativeDict, gamma);
d4aV_uconjuMuMuR = d4aV_uconjuMuVec(1); .
```

We mentioned in (5.18) that the following expression is real-valued:

$$D_{\overline{\mu}} D_{\overline{u}} a_V - (1+|u|^2)^2\, a_{V,u}\, a_{V,\overline{u}} - 2\, D_{\frac{v}{|v|}} a_V + 2i\, D_{\frac{v}{|v|}} \theta + 4i\, \theta\, \overline{a}_V + 2\overline{a}_V^2.$$

In some cases, we may also need to substitute $\overline{(D_{\overline{\mu}} D_{\overline{u}} a_V)}$ accordingly to reduce the running time of the MATLAB program.

```
conj_d2aV_conjuconjMuSub = d2aV_conjuconjMu ...
   - (1+u*conj(u))^2*(daV_u*daV_conju) ...
   + (1+u*conj(u))^2*conj(daV_u)*conj(daV_conju) - 2*daV_vnormv ...
   + 2*conj(daV_vnormv) + 4*i*dtheta_vnormv + 4*i*theta*(aV+conj(aV)) ...
   + 2*conj(aV)^2 - 2*aV^2;
```

```
VARIABLE = subs(VARIABLE, conj(d2aV_conjuconjMu), ...
   conj_d2aV_conjuconjMuSub);
```

The computational model constructed in Part (2), allows us to justify $S = \frac{5}{3}\rho$ (5.20) and provide an explicit formula for $\hat{R}_{12} = \mathrm{Ric}(X_1, \overline{X}_1)$. The statements in Proposition 6.3 concerning the Class 4 components of the Weyl tensor, which are terms in $\rho$, $\rho_u$ and $\rho_{uu}$.

Moreover, under the same methodology in Part (1) and Part (2), we could compute for the Christoffel symbols of the Tanaka-Webster connection (1.1) on $N$, and hence the coefficients of the Chern curvature tensor in Proposition 5.8.

### Part (3): Expressing variables in Christoffel symbols

In Part (3), we are expressing every variable of $a_V$ in terms of $u$, $G^k_{ij}$'s or their derivatives. Here $\nabla$ is the Riemannian connection of $\mathtt{g}$, and $G^k_{ij} = \mathtt{g}\big(\nabla_{e_i} e_j, e_k\big)$. We have 9 terms of $G^k_{ij}$:

`G11_2, G11_3, G12_3, G22_1, G22_3, G23_1, G31_2, G33_1, G33_2.`

For instance, $G^2_{11} =$ `G11_2`. We also denote the first and second derivatives of $G^k_{ij}$ by

$$G^k_{ij,m} = e_m(G^k_{ij}) \quad : \quad \texttt{dG11\_2\_by1},\ \texttt{dG11\_2\_by2},\ \texttt{dG11\_2\_by3},\ \cdots$$

$$G^k_{ij,ml} = e_l\, e_m(G^k_{ij}) \quad : \quad \texttt{d2G11\_2\_by11},\ \texttt{d2G11\_2\_by12},\ \texttt{d2G11\_2\_by13},$$
$$\texttt{d2G11\_2\_by22},\ \texttt{d2G11\_2\_by23},\ \texttt{d2G11\_2\_by33},\ \cdots$$

The second derivatives are related by

$$G^k_{ij,lm} = G^k_{ij,ml} + [e_m, e_l](G^k_{ij}) = G^k_{ij,ml} + G^k_{ij,s}\left(G^s_{ml} - G^s_{lm}\right)$$

For example, concerning $G^2_{11,ml}$, we have

```
d2G11_2_by21 = d2G11_2_by12 - dG11_2_by1*G11_2 +dG11_2_by2*G22_1...
   + dG11_2_by3*(G12_3+G23_1);
d2G11_2_by31 = d2G11_2_by13 - dG11_2_by1*G11_3 - dG11_2_by2*(G12_3+G31_2) ...
   + dG11_2_by3*G33_1;
d2G11_2_by32 = d2G11_2_by23 + dG11_2_by1*(G23_1+G31_2) - dG11_2_by2*G22_3 ...
   + dG11_2_by3*G33_2;
```

Let $R_M$ be the Riemann tensor of $\nabla$ with $\mathsf{R}_{ijkl} = \mathsf{g}\big(R_M(e_i, e_j)e_k, e_l\big)$. So we obtain six terms of $\mathsf{R}_{ijkl}$'s: $\mathsf{R}_{1212}$, $\mathsf{R}_{1213}$, $\mathsf{R}_{1223}$, $\mathsf{R}_{1313}$, $\mathsf{R}_{1323}$ and $\mathsf{R}_{2323}$. By definition,

$$
\begin{aligned}
\mathsf{R}_{1212} &= -G^1_{22,1} - G^2_{11,2} + G^3_{12}G^1_{23} + G^3_{11}G^3_{22} - G^1_{23}G^2_{31} - G^3_{12}G^2_{31} + (G^1_{22})^2 + (G^2_{11})^2,\\
\mathsf{R}_{1213} &= -G^1_{23,1} - G^3_{11,2} - G^1_{22}G^3_{12} - G^2_{11}G^3_{22} + G^1_{22}G^1_{23} + G^1_{23}G^1_{33} + G^2_{11}G^3_{11} + G^3_{12}G^1_{33},\\
\mathsf{R}_{1223} &= G^3_{22,1} - G^3_{12,2} + G^1_{22}G^3_{11} + G^1_{23}G^2_{33} + G^2_{11}G^3_{12} + G^3_{12}G^2_{33} - G^2_{11}G^1_{23} - G^1_{22}G^3_{22},\\
\mathsf{R}_{1313} &= -G^1_{33,1} - G^3_{11,3} + G^2_{31}G^3_{12} + G^2_{11}G^2_{33} - G^2_{31}G^1_{23} - G^3_{12}G^1_{23} + (G^3_{11})^2 + (G^1_{33})^2,\\
\mathsf{R}_{1323} &= -G^2_{33,1} - G^3_{12,3} + G^2_{31}G^3_{22} + G^1_{33}G^2_{33} + G^3_{11}G^3_{12} + G^3_{12}G^3_{22} - G^2_{31}G^3_{11} - G^2_{11}G^1_{33},\\
\mathsf{R}_{2323} &= -G^2_{33,2} - G^3_{22,3} + G^2_{31}G^1_{23} + G^1_{22}G^1_{33} - G^2_{31}G^3_{12} - G^1_{23}G^3_{12} + (G^3_{22})^2 + (G^2_{33})^2.
\end{aligned}
$$

In three dimension, the first Bianchi identity of $R_M$ is equivalent to the symmetries

$$\mathsf{R}_{ijkl} = -\mathsf{R}_{jikl} \quad \text{and} \quad \mathsf{R}_{ijkl} = \mathsf{R}_{klij},$$

while the second Bianchi identity is trivial. As a result, we obtain a set of identities.

$$
\begin{aligned}
\mathsf{R}_{1213} &= \mathsf{R}_{1312} = G^2_{31,1} - G^2_{11,3} + G^1_{33}G^3_{12} + G^3_{11}G^2_{11} - G^3_{11}G^2_{33} - G^2_{31}G^1_{22} - G^1_{33}G^2_{31} - G^3_{12}G^1_{22}\\
\mathsf{R}_{1223} &= \mathsf{R}_{2312} = G^2_{31,2} + G^1_{22,3} + G^1_{33}G^3_{22} + G^1_{23}G^2_{33} - G^2_{31}G^2_{11} - G^2_{33}G^2_{31} - G^1_{23}G^2_{11} - G^3_{22}G^1_{22}\\
\mathsf{R}_{1323} &= \mathsf{R}_{2313} = G^1_{23,3} - G^1_{33,2} + G^2_{31}G^3_{22} + G^2_{33}G^1_{33} - G^1_{22}G^2_{33} - G^2_{31}G^3_{11} - G^1_{23}G^3_{11} - G^3_{22}G^1_{23}
\end{aligned}
$$

We may then replace some $G^k_{ij,l}$'s by others as follows.

$$
\left\{
\begin{aligned}
G^2_{11,3} &= G^1_{23,1} + G^3_{11,2} + G^2_{31,1} - G^3_{11}G^2_{33} + G^2_{11}G^3_{22} - \big(G^1_{22} + G^1_{33}\big)\big(G^2_{31} + G^1_{23}\big)\\
G^3_{22,1} &= G^3_{12,2} + G^2_{31,2} + G^1_{22,3} + G^1_{33}G^3_{22} - G^1_{22}G^3_{11} - \big(G^2_{11} + G^2_{33}\big)\big(G^2_{31} + G^3_{12}\big)\\
G^1_{23,3} &= G^1_{33,2} - G^2_{33,1} - G^3_{12,3} - G^2_{11}G^1_{33} + G^1_{22}G^2_{33} + \big(G^3_{11} + G^3_{22}\big)\big(G^3_{12} + G^1_{23}\big)
\end{aligned}
\right. \tag{C3}
$$

By (C3), we substitute $G^2_{11,3}$, $G^3_{22,1}$, $G^1_{23,3}$ and the corresponding second derivatives with other values. The following lines of code are implemented for the substitution.

```
dG11_2_by3Sub = dG23_1_by1 + dG11_3_by2 + dG31_2_by1 ...
   - G11_3*G33_2 + G11_2*G22_3 - (G22_1+G33_1)*(G31_2+G23_1);
```

```
d2G11_2_by31Sub = d2G23_1_by11 + d2G11_3_by21 + d2G31_2_by11 ...
   - dG11_3_by1*G33_2 - G11_3*dG33_2_by1 + dG11_2_by1*G22_3 ...
   + G11_2*dG22_3_by1 - (dG22_1_by1+dG33_1_by1)*(G31_2+G23_1) ...
   - (G22_1+G33_1)*(dG31_2_by1+dG23_1_by1);

d2G11_2_by13Sub = d2G11_2_by31Sub + G11_3*dG11_2_by1 - G33_1*dG11_2_by3 ...
   + dG11_2_by2*(G12_3 + G31_2);

d2G11_2_by32Sub = d2G23_1_by12 + d2G11_3_by22 + d2G31_2_by12 ...
   - dG11_3_by2*G33_2 - G11_3*dG33_2_by2 + dG11_2_by2*G22_3 ...
   + G11_2*dG22_3_by2 - (dG22_1_by2+dG33_1_by2)*(G31_2+G23_1) ...
   - (G22_1+G33_1)*(dG31_2_by2+dG23_1_by2);

d2G11_2_by23Sub = d2G11_2_by32Sub + G22_3*dG11_2_by2 - G33_2*dG11_2_by3 ...
   - dG11_2_by1*(G23_1 + G31_2);

d2G11_2_by33Sub = d2G23_1_by13 + d2G11_3_by23 + d2G31_2_by13 ...
   - dG11_3_by3*G33_2 - G11_3*dG33_2_by3 + dG11_2_by3*G22_3 ...
   + G11_2*dG22_3_by3 - (dG22_1_by3+dG33_1_by3)*(G31_2+G23_1)
   - (G22_1+G33_1)*(dG31_2_by3+dG23_1_by3);

dG22_3_by1Sub = dG12_3_by2 + dG31_2_by2 + dG22_1_by3 ...
   + G33_1*G22_3 - G22_1*G11_3 - (G11_2+G33_2)*(G31_2+G12_3);

d2G22_3_by11Sub = d2G12_3_by21 + d2G31_2_by21 + d2G22_1_by31 ...
   + dG33_1_by1*G22_3 + G33_1*dG22_3_by1 - dG22_1_by1*G11_3 ...
   - G22_1*dG11_3_by1 - (dG11_2_by1+dG33_2_by1)*(G31_2+G12_3) ...
   - (G11_2+G33_2)*(dG31_2_by1+dG12_3_by1);

d2G22_3_by12Sub = d2G12_3_by22 + d2G31_2_by22 + d2G22_1_by32 ...
   + dG33_1_by2*G22_3 + G33_1*dG22_3_by2 - dG22_1_by2*G11_3 ...
   - G22_1*dG11_3_by2 - (dG11_2_by2+dG33_2_by2)*(G31_2+G12_3)
- (G11_2+G33_2)*(dG31_2_by2+dG12_3_by2);

d2G22_3_by13Sub = d2G12_3_by23 + d2G31_2_by23 + d2G22_1_by33 ...
   + dG33_1_by3*G22_3 + G33_1*dG22_3_by3 - dG22_1_by3*G11_3 ...
   - G22_1*dG11_3_by3 - (dG11_2_by3+dG33_2_by3)*(G31_2+G12_3)
   - (G11_2+G33_2)*(dG31_2_by3+dG12_3_by3);
```

```
dG23_1_by3Sub = dG33_1_by2 - dG33_2_by1 - dG12_3_by3 ...
   - G11_2*G33_1 + G22_1*G33_2 + (G11_3+G22_3)*(G12_3+G23_1);

d2G23_1_by31Sub = d2G33_1_by21 - d2G33_2_by11 - d2G12_3_by31 ...
   - dG11_2_by1*G33_1 - G11_2*dG33_1_by1 + dG22_1_by1*G33_2 ...
   + G22_1*dG33_2_by1 + (dG11_3_by1+dG22_3_by1)*(G12_3+G23_1) ...
   + (G11_3+G22_3)*(dG12_3_by1+dG23_1_by1);

d2G23_1_by13Sub = d2G23_1_by31Sub + G11_3*dG23_1_by1 - G33_1*dG23_1_by3 ...
   + dG23_1_by2*(G12_3 + G31_2);

d2G23_1_by32Sub = d2G33_1_by22 - d2G33_2_by12 - d2G12_3_by32 ...
   - dG11_2_by2*G33_1 - G11_2*dG33_1_by2 + dG22_1_by2*G33_2 ...
   + G22_1*dG33_2_by2 + (dG11_3_by2+dG22_3_by2)*(G12_3+G23_1) ...
   + (G11_3+G22_3)*(dG12_3_by2+dG23_1_by2);

d2G23_1_by23Sub = d2G23_1_by32Sub + G22_3*dG23_1_by2 - G33_2*dG23_1_by3 ...
   - dG23_1_by1*(G23_1 + G31_2);

d2G23_1_by33Sub = d2G33_1_by23 - d2G33_2_by13 - d2G12_3_by33 ...
   - dG11_2_by3*G33_1 - G11_2*dG33_1_by3 + dG22_1_by3*G33_2 ...
   + G22_1*dG33_2_by3 + (dG11_3_by3+dG22_3_by3)*(G12_3+G23_1) ...
   + (G11_3+G22_3)*(dG12_3_by3+dG23_1_by3);
```

In practice, we substitute `dG11_2_by3Sub` for `dG11_2_by3` and so on for the first derivative terms, and `d2G11_2_by13Sub` for `d2G11_2_by13` etc. for the second derivative terms. In total, we are left with 24 free coefficients of $G^k_{ij,m}$'s and 45 of $G^k_{ij,ml}$'s. Back to $a_V$ and its derivatives, we begin with the following substitution for $a_V$, $\theta$ and $f$.

```
aVSub = 1/(2*(1+u*conj(u))^2)*( i*(mu1*conj(mu1)+mu3*conj(mu3))*G23_1 ...
   + i*(mu1*conj(mu1)+mu2*conj(mu2))*G31_2 ...
   + i*(mu2*conj(mu2)+mu3*conj(mu3))*G12_3 ...
   - i*conj(mu1)*mu3*G11_2 + i*conj(mu1)*mu2*G11_3 ...
   + i*conj(mu2)*mu3*G22_1 - i*conj(mu2)*mu1*G22_3 ...
   + i*conj(mu3)*mu1*G33_2 - i*conj(mu3)*mu2*G33_1 );

thetaSub = G12_3 + G23_1 + G31_2;
```

Here, `mu1` equals $u^2 - 1$, `mu2` equals $2u$ and `mu3` equals $i(u^2 + 1)$.

Assume complex variables `dG11_2_Mu`, `dG11_2_conjMu`, `dG11_2_vnormv` to denote $D_\mu G^2_{11}$, $D_{\overline{\mu}} G^2_{11}$, $D_{\frac{v}{|v|}} G^2_{11}$ respectively. Note that

$$D_\mu G^2_{11} \;=\; (u^2-1)\, G^2_{11,1} \;+\; 2u\, G^2_{11,2} \;+\; i(u^2+1)\, G^2_{11,3},$$

$$D_{\overline{\mu}} G^2_{11} \;=\; (\overline{u}^2-1)\, G^2_{11,1} \;+\; 2\overline{u}\, G^2_{11,2} \;-\; i(\overline{u}^2+1)\, G^2_{11,3},$$

$$D_{\frac{v}{|v|}} G^2_{11} \;=\; \frac{u+\overline{u}}{1+|u|^2}\, G^2_{11,1} \;+\; \frac{1-|u|^2}{1+|u|^2}\, G^2_{11,2} \;-\; \frac{i(u-\overline{u})}{1+|u|^2}\, G^2_{11,3}.$$

Similar variables are set up for $G^3_{11}$, $G^3_{12}$, $G^1_{22}$, $G^3_{22}$, $G^1_{23}$, $G^2_{31}$, $G^1_{33}$ and $G^2_{33}$. For the second derivatives by $\mu$, $\overline{\mu}$ or $\dfrac{v}{|v|}$, we define

`d2G11_2_MuMu`, `d2G11_2_MuconjMu`, `d2G11_2_Muvnormv`, `d2G11_2_conjMuconjMu`, `d2G11_2_conjMuvnormv`, `d2G11_2_vnormvvnormv`.

Note that $D_{\overline{\mu}} D_\mu G^k_{ij} \;=\; \overline{\mu}_l\, \mu_m\, G^k_{ij,ml}$ and so on for $D_{\frac{v}{|v|}} D_\mu G^k_{ij}$ and $D_{\frac{v}{|v|}} D_{\overline{\mu}} G^k_{ij}$. We may first construct the 3 x 3-matrix $d^2 G^k_{ij}$ as well as $\mu_{\mathbf{vec}}$ and $\frac{\mathbf{v}}{|\mathbf{v}|}_{\mathbf{vec}}$.

$$d^2 G^k_{ij} \;=\; \begin{bmatrix} G^k_{ij,11} & G^k_{ij,12} & G^k_{ij,13} \\ G^k_{ij,21} & G^k_{ij,22} & G^k_{ij,23} \\ G^k_{ij,31} & G^k_{ij,32} & G^k_{ij,33} \end{bmatrix}, \quad \mu_{\mathbf{vec}} \;=\; \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad \frac{\mathbf{v}}{|\mathbf{v}|}_{\mathbf{vec}} \;=\; \begin{bmatrix} \dfrac{v_1}{|v|} \\ \dfrac{v_2}{|v|} \\ \dfrac{v_3}{|v|} \end{bmatrix}.$$

Therefore, we have

$$\left\{ \begin{aligned} D_\mu D_\mu G^k_{ij} &= (\mu_{\mathbf{vec}})^T \left[d^2 G^k_{ij}\right] (\mu_{\mathbf{vec}}), \\ D_{\overline{\mu}} D_\mu G^k_{ij} &= (\mu_{\mathbf{vec}})^T \left[d^2 G^k_{ij}\right] \overline{(\mu_{\mathbf{vec}})}, \\ D_{\frac{v}{|v|}} D_\mu G^k_{ij} &= \left(\mu_{\mathbf{vec}}\right)^T \left[d^2 G^k_{ij}\right] \left(\frac{\mathbf{v}}{|\mathbf{v}|}_{\mathbf{vec}}\right), \\ D_{\overline{\mu}} D_{\overline{\mu}} G^k_{ij} &= \overline{(\mu_{\mathbf{vec}})}^T \left[d^2 G^k_{ij}\right] \overline{(\mu_{\mathbf{vec}})}, \\ D_{\frac{v}{|v|}} D_{\overline{\mu}} G^k_{ij} &= \overline{\left(\mu_{\mathbf{vec}}\right)}^T \left[d^2 G^k_{ij}\right] \left(\frac{\mathbf{v}}{|\mathbf{v}|}_{\mathbf{vec}}\right), \\ D_{\frac{v}{|v|}} D_{\frac{v}{|v|}} G^k_{ij} &= \overline{\left(\frac{\mathbf{v}}{|\mathbf{v}|}_{\mathbf{vec}}\right)}^T \left[d^2 G^k_{ij}\right] \left(\frac{\mathbf{v}}{|\mathbf{v}|}_{\mathbf{vec}}\right). \end{aligned} \right. \tag{C4}$$

To replace derivatives of $a_V$ or $\theta$, we compute their values through the self-created MATLAB function `df_main_MuGamma`. The procedures involved include two steps: replace these terms by directional derivatives of $G^k_{ij}$ along $\mu$, $\overline{\mu}$ and $\frac{v}{|v|}$. Then, we replace every $D_\mu G^k_{ij}$, $D_{\overline{\mu}} G^k_{ij}$, $D_{\frac{v}{|v|}} G^k_{ij}$ and so on, by $G^k_{ij,m}$'s (`dGij_k_byM`) and $G^k_{ij,ml}$'s (`d2Gij_k_byML`).

In the first step of Part (3), he `CVar`-array is chosen as

```
CVarW72 = [ u, G12_3, G23_1, G31_2, G11_2, G11_3, G22_1, G22_3, ...
   G33_1, G33_2, dG11_2_Mu, dG11_3_Mu, dG12_3_Mu, dG22_1_Mu, dG22_3_Mu, ...
   dG23_1_Mu, dG31_2_Mu, dG33_1_Mu, dG33_2_Mu, ...
   dG11_2_conjMu, dG11_3_conjMu, dG12_3_conjMu, dG22_1_conjMu, ...
   dG22_3_conjMu, dG23_1_conjMu, dG31_2_conjMu, dG33_1_conjMu, ...
   dG33_2_conjMu, dG11_2_vnormv, dG11_3_vnormv, dG12_3_vnormv, ...
   dG22_1_vnormv, dG22_3_vnormv, dG23_1_vnormv, dG31_2_vnormv, ...
   dG33_1_vnormv, dG33_2_vnormv];
```

Inside `derivativeDict`, for example, we create the fields:

```
derivativeDict.G11_2 = [dG11_2_Mu; dG11_2_conjMu; dG11_2_vnormv; 0; 0];
derivativeDict.dG11_2_Mu = [d2G11_2_MuMu; d2G11_2_MuconjMu;
   d2G11_2_Muvnormv; d2G11_2_Muu; 0];
derivativeDict.dG11_2_conjMu = [d2G11_2_conjMuMu; d2G11_2_conjMuconjMu;
   d2G11_2_conjMuvnormv; 0; d2G11_2_conjMuconju];
derivativeDict.dG11_2_vnormv = [d2G11_2_vnormvMu; d2G11_2_vnormvconjMu;
   d2G11_2_vnormvvnormv; d2G11_2_vnormvu; d2G11_2_vnormvconju]; .
```

We have to replace elements of the arrays below by $G^k_{ij}$'s and their $\mu$, $\overline{\mu}$ or $\frac{v}{|v|}$-derivatives.

```
variableSetG1 = [theta, aV, daV_u, daV_conju, d2aV_uu];

variableSetG2 = [daV_Mu, daV_conjMu, daV_vnormv, d2aV_uMu, ...
   d2aV_uconjMu, d2aV_uvnormv, d2aV_conjuMu, d2aV_conjuconjMu, ...
   d2aV_conjuvnormv, d3aV_uuMu, d3aV_uuconjMu, d3aV_uuvnormv, ...
   dtheta_Mu, dtheta_vnormv];
```

```
variableSetG3 = [d2aV_MuMu, d2aV_MuconjMu, d2aV_Muvnormv, ...
   d2aV_conjMuconjMu, d2aV_conjMuvnormv, d2aV_vnormvvnormv, ...
   d3aV_uMuMu, d3aV_uMuconjMu, d3aV_uMuvnormv, d3aV_uconjMuconjMu, ...
   d3aV_uconjMuvnormv, d3aV_uvnormvvnormv, d3aV_conjuMuMu, ...
   d3aV_conjuMuconjMu, d3aV_conjuMuvnormv, d3aV_conjuconjMuconjMu, ...
   d3aV_conjuconjMuvnormv];

variableSetG4 = [d4aV_uuMuMu, d4aV_uuMuconjMu, d4aV_uuMuvnormv, ...
   d4aV_uuvnormvvnormv, d4aV_uuconjMuconjMu, d4aV_uuconjMuvnormv, ...
   d2theta_MuMu, d2theta_MuconjMu, d2theta_Muvnormv, d2theta_vnormvvnormv];
```

For instance, to substitute $D_\mu D_\mu D_u D_u a_V$, we can implement these lines of code:

```
daVVec = df_main_MuGamma(aVSub, CVarW72, derivativeDict, gamma);
daV_uSub = daVVec(4);
d2aV_uVec = df_main_MuGamma(daV_uSub, CVarW72, derivativeDict, gamma);
d2aV_uuSub = d2aV_uVec(4);
d3aV_uuVec = df_main_MuGamma(d2aV_uuSub, CVarW72, derivativeDict, gamma);
d3aV_uuMuSub = d3aV_uuVec(1);
d4aV_uuMuVec = df_main_MuGamma(d3aV_uuMuSub, CVarW72, ...
   derivativeDict, gamma);
d4aV_uuMuMuSub = d4aV_uuMuVec(1);
```

The second step of Part (3) is a rather straightforward replacement following (C3) and (C4).

(1) Replace $D_\mu D_\mu G^k_{ij}$'s and other 2nd derivatives by $G^k_{ij,ml}$'s and $G^k_{ij,m}$'s.

(2) Replace $D_\mu G^k_{ij}$, $D_{\overline{\mu}} G^k_{ij}$ and $D_{\frac{v}{|v|}} G^k_{ij}$ by $G^k_{ij,m}$'s.

(3) Replace $G^2_{11,13}$, $G^2_{11,23}$, $G^2_{11,33}$, $G^3_{22,11}$, $G^3_{22,12}$, $G^3_{22,13}$, $G^1_{23,13}$, $G^1_{23,23}$ and $G^1_{23,33}$.

(4) Replace $G^2_{11,3}$, $G^3_{22,1}$ and $G^1_{23,3}$.

As a final remark to Part (3), it is reasonable for us to define $G^k_{ij,m}$ and $G^k_{ij,ml}$ to be real variables. Alternatively, we may have to insert extra code, for example

```
VARIABLE = subs(VARIABLE, conj(dG11_2_by2), dG11_2_by2); .
```

As a primary application, we would justify the first item of Theorem 6.2 in the case of twistor CR manifolds. The crucial step here is to show $\mathcal{W}_{1212} = C_{1\bar{1}1\bar{1}}$ and $\mathcal{W}_{1214} = C_{1\bar{1}1\bar{2}}$. We denote $\mathcal{W}_{1212}$, $\mathcal{W}_{1214}$ by `W1212`, `W1214`, and $C_{1\bar{1}1\bar{1}}$, $C_{1\bar{1}1\bar{2}}$ by `C1111`, `C1112`.

```
temp = W1212 - C1111;
temp = subs(temp, conj(d2aV_conjuconjMu), conj_d2aV_conjuconjMuSub);
temp = complex_simple3(temp, MVAR);
...
for j = 1:length(variableSetG3)
   char01 = char(variableSetG3(j));
   char02 = [char01, 'Sub'];
   eval( ['temp=subs(temp,' , char01, ',' , char02 , ');' ]);
end
% ( also for variableSetG2 and then variableSetG1.  )
...
temp = subs(temp, d2G11_2_MuMu, transpose(muVec)*d2G11_2*muVec);
temp = subs(temp, d2G11_2_MuconjMu, ...
   transpose(muVec)*d2G11_2*conj(muVec));
temp = subs(temp, d2G11_2_Muvnormv, transpose(muVec)*d2G11_2*vnormvVec);
temp = subs(temp, d2G11_2_conjMuconjMu, ...
   transpose(conj(muVec))*d2G11_2*conj(muVec));
temp = subs(temp, d2G11_2_conjMuvnormv, ...
   transpose(conj(muVec))*d2G11_2*vnormvVec);
temp = subs(temp, d2G11_2_vnormvvnormv, ...
   transpose(vnormvVec)*d2G11_2*vnormvVec);
% ( also for other d2Gij_k's.  )
...
temp = subs(temp, dG11_2_Mu, ...
   mu1*dG11_2_by1 + mu2*dG11_2_by2 + mu3*dG11_2_by3);
temp = subs(temp, dG11_2_conjMu, conj(mu1)*dG11_2_by1 ...
   + conj(mu2)*dG11_2_by2 + conj(mu3)*dG11_2_by3);
temp = subs(temp, dG11_2_vnormv, ...
   v1normv*dG11_2_by1 + v2normv*dG11_2_by2 + v3normv*dG11_2_by3);
% ( also for other dGij_k's.  )
...
```

```
...
variableSetBianchi1 = [dG11_2_by3, dG22_3_by1, dG23_1_by3];
variableSetBianchi2 = [d2G11_2_by13, d2G11_2_by23, d2G11_2_by33, ...
   d2G22_3_by11, d2G22_3_by12, d2G22_3_by13, ...
   d2G23_1_by13, d2G23_1_by23, d2G23_1_by33];
for j=1:length(variableSetBianchi2)
   char03 = char(variableSetBianchi2);
   char04 = [char03, 'Sub'];
   eval( ['temp=subs(temp,' , char03, ',' , char04 , ');' ]);
end
% ( also for variableSetBianchi1.  )
...
```

Finally, we apply the function `complex_simple3` with `MVar` containing $u$ and $w$-terms.

### Part (4): Application to verify Proposition 6.6

As an important application of our computer model of the twistor CR structure $\mathfrak{D}$ (of zero torsion), we will describe how to justify Proposition 6.6 in details. At the beginning, we have to put $w = f$, i.e. substitute $f$ for $w$.

```
f = u*(1+u*conj(u))∧3*conj(daV_u) + (1+u*conj(u))∧4/2*conj(d2aV_uu);
```

Every derivative of $w$ will be replaced by the corresponding derivative of $f$, including

```
wSet = [w, dw_Mu, dw_conjMu, dw_vnormv, dw_u, d2w_uMu, d2w_uconjMu, ...
   d2w_uvnormv, d2w_uu, d2w_MuMu, d2w_MuconjMu, d2w_Muvnormv, ...
   d2w_conjMuconjMu, d2w_conjMuvnormv, d2w_vnormvvnormv, d3w_uMuMu.  ...
   d3w_uMuconjMu, d3w_uMuvnormv, d3w_uconjMuconjMu, d3w_uconjMuvnormv, ...
   d3w_uuMu, d3w_uuconjMu, d3w_uuvnormv, d3w_uuu, d4w_uuMuMu, ...
   d4w_uuMuconjMu, d4w_uuconjMuconjMu, d4w_uuuMu, d4w_uuuconjMu, d4w_uuuu];
```

The derivatives of $f$ are found by the function `df_main_MuGamma`. The `CVar` array is set as

```
CVarW7 = [u, aV, daV_u, daV_conju, d2aV_uu, theta, d2aV_uMu, ...
   d2aV_uconjMu, d2aV_uvnormv, d3aV_uuMu, d3aV_uuconjMu, d3aV_uuvnormv, ...
   d2aV_conjuMu, d2aV_conjuconjMu, daV_Mu, daV_conjMu, dtheta_Mu]; .
```

The next step is to define variables to represent components of the Schouten tensor ($\mathsf{P}$), which are found by (6.2) and (6.3). We let `P` and `covP` be the Schouten tensor and its covariant derivative respectively.

$$\mathsf{P}_{ij} : \texttt{P(ii,j)} \quad \text{and} \quad \nabla_i \mathsf{P}_{jk} : \texttt{covP(ii,j,k)}$$

Both `P(ii,j)` and `covP(ii,j,k)` are in terms of $G^k_{ij}$'s , $G^k_{ij,m}$'s and $G^k_{ij,ml}$'s. On the other hand, we define another set of real variables:

```
covP112,  covP113,  covP122,  covP123,  covP133,
covP211,  covP212,  covP213,  covP223,  covP233,
covP311,  covP312,  covP313,  covP322,  covP323.
```

Let `dGset` be the array of every `Gij_k` and `dGij_k_byM`, and `d2Gset` be that of all variables in the form of '`d2Gij_k_byML`' ($G^k_{ij,ml}$). Moreover, we let

```
covPset = [covP112, covP113, covP122, covP123, covP133, ...
   covP211, covP212, covP213, covP223, covP233, ...
   covP311, covP312, covP313, covP322, covP323]; .
```

We would replace some $G^k_{ij,ml}$'s by $\nabla_i \mathsf{P}_{jk}$, following the lines of code below.

```
...
indexArray = [‘‘112’’, ‘‘113’’, ‘‘122’’, ‘‘123’’, ‘‘133’’, ...
   ‘‘211’’, ‘‘212’’, ‘‘213’’, ‘‘223’’, ‘‘233’’, ‘‘311’’, ...
   ‘‘312’’, ‘‘313’’, ‘‘322’’, ‘‘323’’];
for j=1:15
   char01 = char(indexArray(j));
   m = str2num(char01(1));
   n = str2num(char01(2));
   k = str2num(char01(3));
   [termVec, gVec] = coeffs(covP(m,n,k),d2Gset);
   eval([‘term’ , char01 , ‘=termVec(end);’]);
end
...
```

```
d2G33_2_by11Sub  =  covP112 - d2G12_3_by13 - term112;
d2G33_2_by22Sub  =  d2G11_2_by22 + d2G11_3_by23 + d2G22_1_by12 ...
                    - d2G22_3_by23 + d2G33_1_by12 - 2*covP211 + 2*term211;
d2G31_2_by12Sub  =  covP113 - d2G22_1_by13 - term113;
d2G33_2_by23Sub  =  d2G11_2_by23Sub + d2G11_3_by33 + d2G22_1_by13 ...
                    - d2G22_3_by33 + d2G33_1_by13 - 2*covP311 + 2*term311;
d2G33_2_by12Sub  =  2*covP122 - d2G11_2_by12 + d2G11_3_by13 ...
                    - d2G22_1_by11 - d2G22_3_by13Sub ...
                    + d2G33_1_by11 - 2*term122;
d2G12_3_by23Sub  =  covP212 - d2G33_2_by12 - term212;
d2G33_1_by11Sub  =  covP133 + d2G11_2_by12 - d2G11_3_by13 + d2G22_1_by11 ...
                    -covP122 - term133 + term122;
d2G31_2_by23Sub  =  covP313 - d2G22_1_by33 - term313;
d2G23_1_by12Sub  =  covP223 - d2G11_3_by22 - term223;
d2G22_1_by13Sub  =  covP322 + covP311 - term311- term322- d2G11_2_by23Sub;
d2G11_3_by23Sub  =  covP323 - d2G23_1_by13Sub - term323;
d2G33_1_by12Sub  =  covP233 + covP211 - term233 - term211 - d2G11_3_by23;
d2G23_1_by11Sub  =  covP123 - d2G11_3_by12 - term123;
d2G31_2_by22Sub  =  covP213 - d2G22_1_by23 - term213;
d2G33_2_by13Sub  =  covP312 -d2G12_3_by33 - term312;
...
```

In Proposition 6.6, we interpret $\mathcal{W}_{1212}$, $\mathcal{W}_{1215}$, $\mathcal{W}_{1515}$ and $\mathcal{W}_{1525}$ as a polynomial in $u$ and $\overline{u}$ with coefficients being components of the Cotton tensor:

$$\mathsf{C}_{ijk} \;=\; \nabla_k \mathsf{P}_{ij} - \nabla_j \mathsf{P}_{ik}.$$

Take $\mathcal{W}_{1212}$ as an example. We first find an approximation $\mathcal{A}_{1212}$ such that $(\mathcal{W}_{1212} - \mathcal{A}_{1212})$ is without any second derivatives of $G^k_{ij}$'s.

```
covPVariableSet1 = [d2G33_1_by11, d2G31_2_by23, d2G23_1_by12, ...
   d2G22_1_by13, d2G11_3_by23, d2G33_2_by11, d2G23_1_by11, ...
   d2G31_2_by22, d2G33_2_by13];
covPVariableSet2 = [d2G31_2_by12, d2G33_2_by23, d2G33_2_by12, d2G33_1_by12];
covPVariableSet3 = [d2G12_3_by23, d2G33_2_by22];
...
for j = 1:length(covPVariableSet3)
   char01 = char(covPVariableSet3(j));
   char02 = [char01, 'Sub'] ;
   eval([ 'W1212=subs(W1212' , char01 , ',' , char02 , ');' ]);
end
% (also for covPVariableSet2 and then covPVariableSet1);
...
W1212 = subs(W1212, d2G23_1_by12, covP223 -d2G11_3_by22 - term223);
W1212 = subs( W1212, Gset, zeros(1,length(Gset)) );
[term1212, gVec1212] = coeffs(W1212, [d2Gset, covPset]);
...
```

`gVec1212` doesn't have any entry of '`d2Gij_k_byML`' or 1. The approximation to $\mathcal{W}_{1212}$ is denoted by `A1212` in the program, where every `covPijk` above is replaced by `covP(ii,j,k)` in `A1212`. The difference is `diff1212 = W1212 - A1212`. We apply the following line of codes to show that `diff1212` is indeed zero.

```
for j = 1:length(variableSetBianchi2)
   char01 = variableSetBianchi2(j);
   char02 = ['char01', 'Sub'];
   eval([ 'diff1212=subs(diff1212,' , char01 , ',' , char02 , ');' ]);
end
% (also for variableSetBianchi1)
...
```

```
diff1212 = complex_simple3(diff1212, u);
[termVec, gVec] = coeffs(diff1212, d2Gset);
remainder = termVec(end);
[termVec2, gVec2] = coeffs(remainder, Gset);
for j = 1:length(gVec2)
   termVec2(j) = complex_simple3(termVec2(j), u);
   disp(j); disp(gVec2(j)); disp(termVec2(j));
end
```

Similar procedures are carried out to obtain results for $\mathcal{W}_{1215}$, $\mathcal{W}_{1515}$ and $\mathcal{W}_{1525}$.

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