An Unexplored Aspect of Following a Rule

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Though Wittgenstein has been most often identified as opposing Platonism in his writings about mathematics, I argue that Wittgenstein's radical contextualism about mathematics finds its most natural opponent not in Platonism, but in a variety of formalism. One of Wittgenstein's obvious formalist targets is his colleague the mathematician G. H. Hardy. If we discard this—still influential—picture of mathematics and replace it with a more nuanced account of mathematical activity as exemplified in the metamathematical thinking of the nineteenth century mathematician Augustus De Morgan, the example of the wayward pupil takes on a different significance. Against a more complex background, the wayward pupil can be reinterpreted as representing an exemplar of mathematical discovery. I consider the example of the nineteenth century engineer Oliver Heaviside whose unconventional approach in mathematics, driven by a need to efficiently elicit results from complex formulae for the purposes of aiding his research in electrical engineering, resulted in extraordinary mathematical advances. Yet, his approach to algebraic manipulation has the aspect of a wayward pupil. The wayward pupil, who may be making an error according to our ordinary criteria of rule-following, may be initiating new and fruitful paths. This possibility is largely unexamined in the larger discussion of Wittgenstein's remarks on following a rule, and it explains Wittgenstein's hesitation to label the wayward pupil's actions straightforwardly incorrect.

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Abbreviations

BB WITTGENSTEIN, L., Preliminary studies for the Philosophical Investigations : generally known as The blue and brown books, Blackwell, Oxford 1958.

OC WITTGENSTEIN, L., ANSCOMBE, G. E. M. – PAUL, D. (eds.), On Certainty, Harper & Row, New York 1972.

PR WITTGENSTEIN, L., RHEES, R. (ed.), Philosophical Remarks, Blackwell, 1989.

FM WITTGENSTEIN, L., WRIGHT, G. H. VON – RHEES, R. – ANSCOMBE, G. E. M. (eds.), Remarks on the Foundations of Mathematics, ANSCOMBE, G. E. M. (trad.), Basil Blackwell, Oxford 1978[Revised ed].

AWL WITTGENSTEIN, L., AMBROSE, A. (ed.), Wittgenstein's Lectures, Cambridge 1932–1935 From the Notes of Alice Ambrose and Margaret Macdonald, Prometheus Books, Amherst, NY 2001.

PI WITTGENSTEIN, L., HACKER, P. M. S. – SCHULTE, J. (eds.), Philosophical Investigations, ANSCOMBE, G. E. M. — HACKER, P. M. S. — SCHULTE, J. (trads.), Wiley-Blackwell, Oxford 2009[Revised 4t].

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Preface

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I dedicate this work to my brave and lovely boys Rufus, Hugo and Jasper.

1.0 Introduction

My aim in this dissertation is to offer a novel interpretation of Wittgenstein's example of the wayward pupil. My interpretation arises as a result of reconsidering the example of the wayward pupil in the context of what I will argue is a sustained attempt by Wittgenstein to attack a particular formalist picture of mathematics. This formalist picture of mathematics can be found in the writings of Wittgenstein's colleague—the mathematician G. H. Hardy. If we discard this—still influential—picture of mathematics and replace it with a more nuanced account of mathematical activity, the example of the wayward pupil takes on a different significance.

In the first chapter, I will argue that the literature on Wittgenstein's remarks on rule-following and the wayward pupil has been largely unfruitful. The focus has tended to dwell on the sources of correctness in following a rule with the only viable possibilities seeming to commit Wittgenstein to an implausible conventionalism. There is substantial textual evidence to support the claim that this was not Wittgenstein's position. Instead, I argue that Wittgenstein's discussion of the wayward pupil is best understood as situated against his radical contextualism. Against this background, the example of the wayward pupil reveals itself as a "grammatical investigation," the outcome of which is merely to draw our attention to what we might say in various cases, of which the wayward pupil is one striking example. The more general conclusion Wittgenstein wishes to draw is that we can only make judgments about whether someone is competently following a rule when we have access to a huge range of background considerations. That is, whether someone counts as following a rule will depend on all kinds of facts about this individual and the context of her actions. This is part of what I label Wittgenstein's "radical contextualism".

In the second chapter, I develop an account of what I term Wittgenstein's "radical contextualism". I argue that this position is most clearly articulated in Wittgenstein's discussion of Moore's "proof" of the external world in *On Certainty.* According to Wittgenstein's view, an utterance or an action has no "full" sense outside an actual occurrence. Thus, any ruling about whether an individual is successfully following a rule must take place against a rich background of information. In particular, it will not be clear in all cases whether we ought to label the behaviour on some occasions as "following a rule" or not.

Though Wittgenstein has been most often identified as opposing Platonism in his writings about mathematics, I argue that Wittgenstein's radical contextualism about mathematics finds its most natural opponent not in Platonism, but in a variety of formalism. One of Wittgenstein's obvious formalist targets is his colleague the mathematician G. H. Hardy.

In chapter three, I attempt to articulate a clearer account of Wittgenstein's conception of mathematics by making use of alternative pictures of the nature of mathematical certainty. The first of these is G. H. Hardy's formalist picture as articulated in his mathematical writings, which I argue is a central target of Wittgenstein's remarks on mathematics. One reason Hardy's position and Wittgenstein's rejection of it is interesting is because Hardy's picture of mathematics strikes a modern audience as entirely natural. The formalist picture of mathematics that I have in mind as exemplified by Hardy asserts that the characteristic activity of the mathematician is the laying down of definitions and the deriving of results from those definitions. Mathematics gets its certainty, according to this picture, from the definitions and the rules according to which it operates. Under this conception, we can distinguish the hard kernel of "legitimate" mathematics from various decorative and inessential remarks that surround that kernel.

As many have discovered, Wittgenstein's position in the philosophy of mathematics resists simple articulation. There are, however, curious similarities between Wittgenstein's picture of mathematics as it opposes Hardy's formalist position and the complicated attitudes articulated earlier by the nineteenth century British mathematician and logician Augustus De Morgan. I will make use of these similarities and the less cryptic nature of De Morgan's writing in order to give some order to Wittgenstein's remarks.

The surprising feature De Morgan and Wittgenstein have in common is an ambivalence about the "certainty" of a mathematical claim. On the one hand, clear cut rules of proof or calculation leading to a mathematical result seem required to grant this result its "certainty." Hardy's own picture of mathematics accords with this as we can see in his insistence on well-formed "definitions". On the other hand, mathematics gains many of its most surprising utilities by wandering into previously uncharted territories. For this reason, De Morgan wants his guiding rules of calculation to be broader than Hardy.

While there is no reason to suppose that Wittgenstein was influenced by De Morgan, both thinkers respond in a similar way to the question of how we should think about the “certainty” of mathematical claims. Since De Morgan’s view shares some features with Wittgenstein’s, and since Hardy takes aim at De Morgan in presenting his formalist scruples, I will examine De Morgan in order to see how one could have a Wittgensteinian view about certainty in philosophy of mathematics opposed to a formalist position.

In order to make vivid the tension between these two positions, I use the example of an episode in the history of mathematics concerning divergent series. We see how these two attitudes toward mathematical certainty play out by examining the contrasting attitudes expressed by Hardy and De Morgan toward this area of mathematics. While De Morgan is open to the use of not strictly regimented methods in the use of divergent series, Hardy dismisses any progress that does not yield formal definitions as “no real progress” at all. Divergent series is also a fascinating example because it is an area of mathematics that—to this day—resists the kind of formal articulation Hardy demands. To that end, I discuss the case of asymptotic series.

In the fourth chapter, I focus on Wittgenstein’s remarks concerning river-beds in *On Certainty*. Once dislodged, the formalist picture of mathematical activity no longer serves as the background against which to understand the example of the wayward pupil. Against a more complex background, the wayward pupil can be reinterpreted as representing an exemplar of mathematical discovery. To illustrate the kind of situation I am imagining, I consider the example of the nineteenth century engineer Oliver Heaviside. Heaviside's unconventional approach in mathematics, driven by a need to efficiently elicit results from complex formulae for the purposes of aiding his research in electrical engineering, resulted in extraordinary mathematical advances. Yet, his approach to algebraic manipulation has the aspect of a wayward pupil. The wayward pupil, who may be making an error according to our ordinary criteria of rule-following, may be initiating new and fruitful paths. This possibility is largely unexamined in the larger discussion of Wittgenstein's remarks on following a rule and it explains Wittgenstein's hesitation to label the wayward pupil's actions straight-forwardly incorrect.

2.0 Following a Rule

In this chapter, I will argue that the literature on Wittgenstein's remarks on rule-following and the wayward pupil have been largely unfruitful. The focus of the secondary literature has tended to dwell on the sources of correctness in following a rule with the only viable possibilities seeming to commit Wittgenstein to an implausible conventionalism. However, there is substantial textual evidence to support the claim that this was not Wittgenstein's position. Instead, I argue that Wittgenstein's discussion of the wayward pupil is best understood as situated against his radical contextualism. Against this background, the example of the wayward pupil reveals itself as a "grammatical investigation," the outcome of which is merely to draw our attention to what we might say in various cases, of which the wayward pupil is one striking example. The more general conclusion Wittgenstein wishes to draw is that we can only make judgments about whether someone is competently following a rule when we have access to a huge range of background considerations. That is, whether someone counts as following a rule will depend on all kinds of facts about this individual and the context of her actions. This is part of what I label Wittgenstein's "radical contextualism".

In the first part of this chapter, I will survey various interpretations of Wittgenstein's discussion of following a rule. In the second section, I will argue that the example of the wayward pupil has similarities with Wittgenstein's earlier discussion in the *Philosophical Investigations* of the Augustinian language learner. These similarities should invite us to question whether the example of the wayward pupil is supposed to be taken at face value. Instead, I think Wittgenstein intended the case of the wayward pupil to strain our notion of coherence. In the third and final section of this chapter, I propose that the example of the wayward pupil is best understood as an example of a "grammatical investigation."

2.1 The Wayward Pupil

Wittgenstein's remarks on following a rule in the *Philosophical Investigations*[1] have generated about as much attention as any part of that work, and rightly so. Many of the important ideas of the later Wittgenstein emerge in these passages, including those concerning meaning, understanding, criteria and agreement. Much of the discussion has focused on the various kinds of doubts that can arise from the disquieting example of the wayward child of §185 who is able to write down the series of numbers according to the rule +2 up to 1000, but then goes on to write 1000, 1004, 1008, … and resists correction when her error is pointed out, insisting that she is going on in the same way.

These remarks on rule-following in the *Philosophical Investigations* have given rise to much discussion in the secondary literature of the various hyperbolic doubts that can emerge from

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[1] Ludwig Wittgenstein, *Philosophical Investigations*, ed. by P. M. S Hacker and Joachim Schulte, trans. by Gertrude Elizabeth Margaret Anscombe, P. M. S. Hacker, and Joachim Schulte, Revised 4t (Oxford: Wiley-Blackwell, 2009).

the example of the wayward pupil, perhaps at least in part as a result of Kripke's discussion in *Wittgenstein on Rules and Private Language.*[2] On one interpretation, the example of the wayward pupil demonstrates that since we only ever have a finite amount of behaviour as evidence that someone understands how to follow a rule, there is no guarantee that he or she actually does have "full" understanding. The possibility of future behaviour not according with our understanding of how to follow a rule is always open. This possibility leads to the further question of how we know that *we* have the answer about what it is to continue the rule in the same way.

These doubts have encouraged a largely fruitless search for where Wittgenstein locates the true sources of "correctness" for such behaviours and has encouraged an extensive literature on "societal agreement" that does not seem very persuasive.

Dummett, takes Wittgenstein to be committed to a 'full-blooded conventionalism',[3] which requires an explicit decision at every stage to endorse some piece of mathematics as correct. According to Dummett, Wittgenstein takes this view of both 'deep theorems' and elementary computations such as the rule-following in which the wayward child is engaged. So, the source of the correctness of rule-following behaviour according to Dummett's account of Wittgenstein is grounded in an explicit decision to accept one way of going on as the correct way. He asserts that

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[2] Saul A Kripke, *Wittgenstein on Rules and Private Language* (Oxford: Blackwell, 1982).

[3] Michael Dummett, 'Wittgenstein's Philosophy of Mathematics', *Philosophical Review*, 68.3 (1959), 324–48. p.329

Wittgenstein believes it possible that someone could understand and accept all of the relevant axioms and rules of inference, and yet decide to reject any given proof. Dummett himself finds this account problematic, and thinks that the examples Wittgenstein gives are uncharacteristically 'thin and unconvincing.'[4] Barry Stroud, in response, states that Dummett is right to find Wittgenstein's supposed account unconvincing, but argues that Wittgenstein would agree and that the true source of the difficulty in Dummett's account is an implausible interpretation of Wittgenstein.[5]

If one takes seriously the problems that Wittgenstein's discussion of following a rule seems to illuminate, one likely takes the resolution of the problems to lie in some area associated with the agreement of a people. There are many subtleties about exactly how this is spelled out. For example, I take it that John McDowell and Crispin Wright both have views of roughly this kind, yet there remains serious disagreement between the two about the details of this. Crispin Wright, for example, takes Wittgenstein to ground correctness in the authority of communal agreement. He says:

> None of us unilaterally can make sense of the idea of correct employment of language save by reference to the authority of securable communal assent on the matter; and for the community itself there is no authority, so no standard to meet.[6]

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[4] Dummett. p.333

[5] Barry Stroud, 'Wittgenstein and Logical Necessity', *Philosophical Review*, 74.4 (1965), 504–18.

[6] Crispin Wright, *Wittgenstein on the Foundations of Mathematics* (London: Duckworth, 1980). P.220

Wright's account is dissatisfying because it merely pushes the question of the ground of "correctness" up to the level of the community. The individual follows a rule correctly if she is in agreement with her community, but for the community itself, whatever seems correct *is* correct.[7] John McDowell says of Wright's account that 'the notion of right and wrong that we have made room for is at best a thin surrogate for what would be required by the intuitive notion of objectivity.'[8]

It is clear that Wittgenstein himself takes the notion of a "form of life" to play a central role in making sense of the notion of a correctness that is not the kind that Dummett takes it to be, nor Crispin Wright's communal agreement. Some authors have criticized Kripke for prioritizing the notion of "agreement" in his remarks, mistakenly attributing to it the role of a "ground" that is lacking in Wittgenstein's actual picture. Warren Goldfarb, for example, takes it that Kripke's notion of agreement is at odds with the role Wittgenstein takes it to play. He writes: 'for Kripke, agreement among speakers is, in some sense, to ground ascriptions of meaning or adherence to a rule. Wittgenstein, however, puts no such onus on agreement, I believe. *Agreement is exhibited in rule-following, but does not ground it*'[9] (my emphasis). Similarly, Cora Diamond argues that if we think of "agreement" as some claim of the form 'Smith goes "1000, 1002" and so do the rest of

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[7] Wittgenstein, *PI*. See PI §258: 'And that only means that here we can't talk about "correct."'

[8] John McDowell, 'Wittgenstein on Following a Rule', in *Rule-Following and Meaning*, ed. by Alexander Miller and Crispin Wright (Ithaca: McGill-Queen's University Press, 2002), pp. 45–80. p.49

[9] Warren Goldfarb, 'Kripke on Wittgenstein on Rules', in *Rule-Following and Meaning*, ed. by Alexander Miller and Crispin Wright (Ithaca: McGill-Queen's University Press, 2002), pp. 92–107. p.104

us', we will have questions about 'whether our all going '1000, 1002' has anything to do with the possibility of Smith's being *right* in carrying on with the rule 'Add 2' as he does.'[10] And, she thinks, this focus on communal agreement will fail to settle how it can be possible for an individual's response to be right or wrong, in precisely the way that Wright does. Diamond thinks that the problem of those accounts that focus on "agreement" in the context of Wittgenstein's remarks on following a rule is that they ignore the role that rule-following has in our lives:

> What we are ignoring, then, is the place of this procedure in a life in which following rules of all sorts comes in an enormous number of ways. In fact, of course, we are not just trained to go '446, 448, 450' etc. and other similar things; we are brought into a life in which we rest on, depend on, people's following rules of many sorts, and in which people depend on us: rules, and agreement in following them, and criticising or rounding on people who do not do it right – all this is woven into the texture of life; and it is in the context of its having a place in such a form of human life that a 'mistake' is recognisably that.[11]

I take it that both Diamond and Goldfarb are appealing to something to do with a "form of life" as the correct place to look in trying to make sense of Wittgenstein's thoughts about rule-following. It does seem clear that Wittgenstein himself takes the notion of "form of life" to be of fundamental importance in resolving these issues. However, the examples of rule-following that Wittgenstein himself discusses in the *Philosophical Investigations* are such that it can be difficult to see why we would need to appeal to anything so complex in order to understand a situation where a pupil fails to follow a rule according to our ordinary criteria. Goldfarb describes the example of the wayward pupil as 'bizarre', and it is certainly difficult to imagine circumstances in

[10] Cora Diamond, 'Rules: Looking in the Right Place', in *Wittgenstein: Attention to Particulars*, ed. by D. Z Phillips and Peter Winch (New York: St. Martin's Press, 1989), pp. 12–34. p.27

[11] Diamond, 'Rules: Looking in the Right Place'. pp.27-28

which the pupil would insist that she was going on in the same way, or be so resistant to correction.[12] We can be left struggling, as Dummett was, in comprehending what it would be to endorse, or even more importantly perhaps, to *fail to endorse* some portion of mathematics that seems to follow inexorably from the portion that is already accepted.

2.2 The Coherence of the Wayward Pupil

Despite the preoccupation with "correctness" and "societal agreement" in the secondary literature, there is textual evidence to support the that this was not Wittgenstein's purpose in his discussion of the wayward pupil.

Philosophers, perhaps prompted by Kripke's reading of the rule following passages in the *Philosophical Investigations,* have taken the example of the wayward pupil at face value. However, there are clues that suggest that the naïve reading of the wayward pupil fails for the same reasons that the Augustinian picture of language learning presented in the opening remarks of the *Investigations* fails.

[12] Goldfarb. p.104

For the example to work, the wayward pupil must know how to write, be able to count, be able to perform addition (even if perhaps not be able to master the conceptual issue of "+n"). The wayward pupil, in other words, is a fluent language user and has a rudimentary understanding of mathematics. Yet, somehow his teacher is unable to communicate to him what it is to "go on" in this case. The wayward pupil is fully capable of being trained in how to follow a rule up to an arbitrary point, after which all attempts at training him fail. Part of the strangeness of the example lies in the notion that someone could be capable of understanding how to follow a rule up to some point without apparent difficulty, but then be entirely unable to understand the further steps despite no increase in complexity.

The wayward pupil has similarities with Wittgenstein's discussion of the Augustinian language learner at the beginning of the *Investigations*. Wittgenstein begins the *Philosophical Investigations* with a quote from St. Augustine:

> When they (my elders) named some object, and accordingly moved towards something, I saw this and I grasped that the thing was called by the sound they uttered when they meant to point it out. Their intention was shewn by their bodily movements, as it were the natural language of all peoples: the expression of the face, the play of the eyes, the movement of other parts of the body and the tone of voice which expresses our state of mind in seeking, having, rejecting, or avoiding something. Thus, I heard words repeatedly used in their proper places in various sentences, I gradually learnt to understand what objects they signified; and after I had trained my mouth to form these signs, I used them to express my own desires.[13]

[13]PI §1, English translation provided in the footnote to that section.

Wittgenstein remarks that such a description of the process of learning language involves 'thinking primarily of nouns like "table", "chair", "loaf", and of people's names and only secondarily of the names of certain actions and properties; and of the remaining kinds of word as something that will take care of itself.'[14] Wittgenstein also comments that the picture St. Augustine gives us is of a language in which each word names an object. This picture brings with it the idea that '[e]very word has a meaning. This meaning is correlated with the word. It is the object for which the word stands.'[15] Once the meaning of each individual word has been established, we can build sentences, which are 'combinations of such names'.[16]

Wittgenstein does not claim that St. Augustine was trying to outline a detailed philosophical theory of language or of meaning, but he does take the picture seriously in his subsequent discussion of it. He makes use of the picture to illustrate aspects of language and language learning that we may misunderstand and to draw our attention to how unsatisfactory the notion of ostensive definition is in determining meaning.

We are to suppose that the act of pointing to an object and saying its name teaches someone else the meaning of the word. It attaches a label to a thing so that the learner may now use the word

[14]PI §1

[15]PI §1

[16]PI §1

to refer to the thing. Wittgenstein's essential criticism of this view is that it presupposes what it seeks to explain. He does not argue that ostensive definition is an impossible means of conveying meaning, but that it is inadequate as an account of how we come to learn language. I will go through the main remarks Wittgenstein makes about the notion of ostensive definition.

First, Wittgenstein draws our attention to the fact that giving an ostensive definition or asking a thing's name presupposes an understanding of the way a culture and its language functions. I think we may say that even the act of pointing only makes sense within a community which knows how to respond. That is, there is nothing about pointing a finger which explains what one ought to do in response. Later, in the context of rule following, Wittgenstein gives the example of a person who 'naturally reacted to the gesture of pointing with the hand by looking in the direction of the line from finger-tip to wrist, not from wrist to finger-tip'.[17] It is not at all obvious that someone without the appropriate background understanding of a community's practices (and we would assume the child language-learner to be entirely ignorant in this matter) would know what the significance of pointing was; would know to look in the direction of the pointing, instead of, say, (as dogs seem to do) to look at the finger-tip being extended. Wittgenstein describes the two connected acts - of asking a thing's name and giving an ostensive definition - as forming a language-game of its own, which he says means that 'we are brought up, trained, to ask: "What is that called?" - upon which the name is given'.[18] Asking a thing's name and understanding even

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[17]PI §185

[18]PI §27

what is being attempted in giving an ostensive definition requires mastery of some complex ideas. Asking a thing's name will be clearly impossible for the language-learning child who cannot yet speak at all. Understanding that someone is trying to give her an ostensive definition might require the child to understand that "things" have "names" and that she is to direct her attention to the object her teacher's finger points towards. Stanley Cavell writes that learning language involves learning 'not merely what the names of things are, but what a name is'.[19] Wittgenstein is here pointing out that the language-game of asking for names of objects or comprehending someone who is naming an object is not an intuitive process. It requires an understanding of customs and practices in language that a child does not yet have, which leaves the Augustinian picture wanting as an explanation of how a child might come to learn language.

Assuming that the ostension worked, and we were able to direct the attention of the language-learner to a given object and also that she understood that we were trying to teach her the name of the object, Wittgenstein claims that merely setting up an association between a word and an object is insufficient to give a word meaning.[20] He writes that we could imagine establishing an association between a word and an object for the child, so that saying the word makes a picture of the object come before the child's mind, however there is no aspect of creating this association that enables the child to go on to use the word in the future. Say we were able to create an association for the child between a chair and the word "chair". How can she then go on to know

[19] Stanley Cavell, *The Claim of Reason* (Oxford: Oxford University Press, 1999). p.177

[20]PI §6

that “chair” applies not only to this red wooden chair, but also to other red wooden chairs, and even to blue wooden chairs and perhaps also to armchairs, dentist’s chairs and to stools? How is she to know that chairs are for sitting in? She may be able to associate this one chair with its name, but she does not know how the word is used or how to determine what other objects the word applies to. Wittgenstein describes this association between an object and its name as being like attaching a label to a thing. He says: ‘Attaching a label is preparatory to the use of a word. But what is it a preparation for?’[21] Attaching a label does not tell us how the word is used and that seems to suggest that we haven’t learnt the word at all. The thing that it is preparatory for seems to me to be the actual learning of the word. We start with the association, perhaps, and then gradually come to develop criteria for its use through an increased understanding of how the word functions. We see what various kinds of things are called “chairs" and what people do with them. We learn its use. Cavell describes this alternative to the Augustinian picture of teaching children language: ‘We initiate them, into the relevant forms of life held in language and gathered around the objects and persons of our world’.[22] The child slowly learns to use words as she learns and accepts the customs, practices and conventions in her world.

As a third objection, Wittgenstein demonstrates that the act of pointing is itself ambiguous. There is nothing that tells the child which aspect of an object the label is being attached to.

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[21]PI §26

[22]Cavell. p.178

Wittgenstein gives the example of pointing to two nuts and uttering "this is called 'two'".[23] In this case the child may think that you are naming this particular group of nuts. Or perhaps you are giving her the name of the variety of nuts in question (e.g. these are "almonds"), or perhaps you are just telling her that these are "nuts." It becomes clear that 'an ostensive definition can be variously interpreted in every case'.[24] There is nothing in the pointing that fixes which aspect of an object the child is supposed to direct her attention towards. If she is lacking the language ability to clarify what it is you are indicating, how can the ostension succeed?

Having looked at problems Wittgenstein offers for the Augustinian picture of language-learning, it is useful to look at the cases where ostensive definition *does* succeed. Returning to the case of pointing to the two nuts in order to define "two," this could tell someone the meaning of the word if they already understood the concept "number" and, furthermore, knew that it was a number that was being defined. Wittgenstein states that in this case, the word "number" is able to 'shew the post at which we station the word'.[25] Ostensive definition is able to explain the meaning of a word if the learner already understands its role in the language. We could imagine a person who had already developed an understanding of the use of the word "chair" (was able to identify whether something was or wasn't a chair in various cases) but was yet to learn the thing's name. He has a "chair" shaped hole in his language, that needs only the right word attached in order to

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[23]PI §28

[24]PI §28

[25]PI §29

for him to be said to understand what a "chair" is. This is the kind of situation someone is in when he is learning a second language. He may already understand the role that the word "chair" plays in a language and just needs to learn the new word in order to be able to use it. Wittgenstein asserts that this is the idea that the Augustinian picture gives us: children language-learners are like foreigners who already have their own language and thereby learn the new language by attaching the new words to their old understandings.[26] The child in St. Augustine's description already has the background understanding of someone who has mastered a language.

So, Wittgenstein's criticism of the notion of ostensive definition as a basis for learning a first language centres around the lack of background information a child will have about the practices and customs involved in understanding the act of ostensive definition; the inadequacy of creating an association between an object and its name as a means of determining a word's meaning and the ambiguity of what exactly is being fixed upon in the act of pointing. The Augustinian picture of language-learning betrays an over dependence on the explanatory role that ostensive definition could possibly play.

Just like the wayward pupil who knows how to write, etc, but not how to write down a series of numbers after a certain point, the wayward pointee knows that pointing plays a gesturing role without knowing how to react to it. He has a natural reaction. The ostensive definition picture

[26]PI §32

of language learning assumes that the learner is like a foreigner who already has ready-made concepts that merely need linking to the new words. The wayward pupil is in a similar position. He knows how to write numbers, follow a rule, etc., but naturally responds to a rule in slightly different way to a typical learner.

He is like a language speaker who has come into a foreign land. Wittgenstein says:

> In such a case, we might perhaps say: this person finds it natural, once given our explanations, to understand our order as we would understand the order "Add 2 up to 1000, 4 up to 2000, 6 up to 3000, and so on".
>
> This case would have similarities to that in which it comes naturally to a person to react to the gesture of pointing with the hand by looking in the direction from fingertip to wrist, rather than from wrist to fingertip.[27]

Wittgenstein's use of the example of how one might "naturally react" to pointing is extremely striking, given his earlier discussion of ostension. In particular, an early lesson of the *Investigations* is how little use pointing or signposts have without being taught how to respond to them. Here, Wittgenstein considers the possibility that *we might* say of the wayward pupil that "he responds naturally" to the continuation of a series in the same way that someone else might react naturally to pointing by looking from finger to elbow, but if his discussion of ostensive definition has made anything clear for us, it is the extent to which pointing by itself has no natural interpretation.

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[27] PI §185

2.3 The Wayward Pupil as a Grammatical Investigation

If Wittgenstein *is* trying to undermine this picture of learning to write a series of numbers and yet failing after a certain arbitrary point, what could be his goal? The example of the wayward pupil is strange, and I will argue that—in part, at least—the strangeness is the point. I will argue in this section that Wittgenstein's discussion of the wayward pupil is best understood as an example of a "grammatical investigation." Wittgenstein has—I will propose—a radically contextual view of meaning. His example of the wayward pupil is best understood as situated against his radical contextualism. Against this background, the example of the wayward pupil reveals itself as a grammatical investigation, the outcome of which is to draw our attention to what we might say in various cases, of which the wayward pupil is one striking example.

2.3.1 Wittgenstein's Use of Thought Experiments

Wittgenstein's thought experiments—unlike many famous thought experiments in philosophy—do not seem to drive straightforwardly at any definite conclusion. It is certainly the case that Wittgenstein does not use them to explicitly *argue* for some conclusion. Instead, Wittgenstein tends to ask the reader to imagine something, or sometimes to ask what we would say about some case without supplying what he takes to be the "correct" answer. The purpose of a given thought experiment—if there is one—is usually left to the reader to determine for herself. They serve as prompts to contemplate connections.

The open-ended nature of Wittgenstein's thought experiments form part of his conception of a "grammatical investigation." In response to Ernst Mach's discussion of thought experiments,[28] Wittgenstein states:

> What Mach calls a thought experiment is of course not an experiment at all. At bottom it is a grammatical investigation.[29]

A thought experiment, according to Wittgenstein, is a way of examining how we use language in an attempt to get a clear view. This attempt at clarifying our use of language is a fundamental part of the later Wittgenstein's conception of philosophy. The later Wittgenstein conceives of confusion about our language as the root of problems in philosophy, and thus obtaining a clear view of our language is the way out of this confusion. In §109 of the *Philosophical Investigations*, Wittgenstein writes:

> We must do away with all *explanation*, and description alone must take its place. And this description gets its light, that is to say its purpose, from the philosophical problems. These are, of course, not empirical problems; they are solved, rather, by looking into the workings of our language, and that in such a way as to make us recognize those workings: *in despite of* an urge to misunderstand them. The problems are solved, not by reporting new experience, but by

[28] 1905, "Über Gedankenexperimente", in *Erkenntnis und Irrtum*, Leipzig: Verlag von Johann Ambrosius Barth, 181–197, (translated by J. McCormack, in *Knowledge and Error*, Dordrecht: Reidel, 134–147).

[29] PR p.52 Ludwig Wittgenstein, *Philosophical Remarks*, ed. by Rush Rhees (Blackwell, 1989).

> arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of our language.[30]

According to Wittgenstein, the problems of philosophy are solved by 'looking into the workings of our language.' Wittgenstein tells us that: 'A philosophical problem has the form: "I don't know my way about".'[31] To be under the sway of a philosophical problem is to be disoriented about the geography of our concepts.[32] Part of what constitutes looking into the workings of our language is recalling the kinds of statements we make about phenomena in the way Wittgenstein describes.

Quite early on in the *Investigations*, Wittgenstein tells us that 'philosophical problems arise when language *goes on holiday*.'[33] What Wittgenstein has in mind here is that the kind of conceptual disorientation that is characteristic of philosophical puzzlement tends to arise precisely when we remove our language from its everyday context and make it the object of our study independent of any actual use we are accustomed to make of it. Take as an example Wittgenstein's discussion of Augustine's remark about time. Augustine says 'quid est ergo tempus? si nemo ex

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[30]PI §109

[31]PI §123

[32] In The Big Typescript, Wittgenstein explicitly compares philosophy with geography: 'Teaching philosophy involves the same immense difficulty as instruction in geography would have if a pupil brought with him a mass of false and far too simple … ideas about the course and connections of the routes of rivers … and mountain chains.' PO p.185 (TS 213 §90) Ludwig Wittgenstein, *Philosophical Occasions 1912-1952*, ed. by James C. Klagge and Alfred Nordmann (Indianpolis & Cambridge: Hackett Publishing Company, 1993).

[33] PI §38

me quaerat scio; si quaerenti explicarevelim, nescio'.[34] Augustine runs into trouble when he tries to give an explanation or a definition of a concept independently of any account of how we ordinarily talk about it. Wittgenstein responds to Augustine's remark with the following: 'Something that we know when no one asks us, but no longer know when we are supposed to give an account of it, is something that we need to *remind* ourselves of.'[35] This is a case ripe for grammatical investigation. Augustine describes a situation where he is initially confident about his understanding of a concept but becomes disoriented when called upon to give an account of it. In his succeeding remark, Wittgenstein says 'We feel as if we had to *penetrate* phenomena.'[36]

Instead of feeling that we had to 'penetrate phenomena', Wittgenstein says, our investigation is to be directed 'towards the '*possibilities*' of phenomena.'[37] The 'possibilities of phenomena' draw us back to our ordinary employment of language. Wittgenstein glosses his investigation of the possibilities of phenomena in this way:

> What that means is that we call to mind the *kinds of statement* that we make about phenomena. So too, Augustine calls to mind the different statements that are made about the duration of events, about their being past, present or future. (These are, of course, not *philosophical* statements about time, the past, the present and the future.)
>
> Our inquiry is therefore a grammatical one. And this inquiry sheds light on our problem by clearing misunderstandings away. Misunderstandings concerning the use of words, brought about, among other things, by certain analogies between the forms of expression in different

[34] Quoted in PI §89 'What is time? I know well enough what it is, provided that nobody asks me; but if I am asked to explain, I am baffled.' Augustine, *Confessions*, trans. by R. S. Pine-Coffin (Hamondsworth: Penguin books, 1961), bk. XI.14.

[35] PI §89

[36]PI §90

[37]PI §90

> regions of our language. —Some of them can be removed by substituting one form of expression for another; this may be called 'analysing' our forms of expression, for sometimes this procedure resembles taking a thing apart.[38]

This is what, according to Wittgenstein, would constitute undertaking a grammatical investigation. When faced with Augustine's confusion, we remind ourselves of the way we ordinarily employ concepts. We can now understand what Wittgenstein means in the passage I quoted earlier when he writes 'We must do away with all *explanation*, and description alone must take its place.'[39] An *explanation* is the kind of answer Augustine felt obliged to give in response to the question 'what is time?'. Instead of feeling as if we had to penetrate phenomena and give an explanation, Wittgenstein exhorts us to stick to description. Description, which in this case consists in recalling the 'the different statements that are made about the duration, past present or future, of events' is the calling-up of reminders. It draws us back from philosophical confusion to the everyday. And this is the approach we should adopt towards philosophy in general. Thus, Wittgenstein says, concluding with his approach to philosophy as a slogan:

> When philosophers use a word--"knowledge", "being", "object", "I", "proposition", "name"--and try to grasp the *essence* of the thing, one must always ask oneself: is the word ever actually used in this way in the language-game which is its original home?--
>
> What *we* do is to bring words back from their metaphysical to their everyday use.[40]

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[38] PI §90

[39]PI §109

[40]PI §116

In the remark directly following his discussion of Augustine and time, Wittgenstein writes that such an investigation is 'grammatical'. He continues:

> Such an investigation sheds light on our problem by clearing misunderstandings away. Misunderstandings concerning the use of words, caused, among other things, by certain analogies between the forms of expression in different regions of language.[41]

Looking into the workings of our language in a particular case involves undertaking a grammatical investigation. Wittgenstein instructs us at one point to 'Carry out a grammatical investigation as follows:'[42] and then goes on to consider various statements that we may or may not ordinarily make, asks whether we would apply certain adverbs, and asks us to compare sentences that are seemingly analogous. A grammatical investigation involves attending to how we use language: when it makes sense to us to apply a certain adverb and when not, when we would be puzzled if someone said something to us in the ordinary run of things that in the context of doing philosophy appears to be meaningful, noticing when a single word is used in multiple closely-related, but divergent ways, looking at where an analogy between two concepts breaks down, and so on.

Indeed, one source of philosophical puzzlement, according to Wittgenstein in the remark given above, are misleading analogies between forms of expression in different areas of language.

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[41]PI §90

[42]PI p.59

When Wittgenstein mentions Augustine’s lack of an answer to the question “what is time?”, he says of the forgetting: ‘This could not be said about a question of natural science ("What is the specific gravity of hydrogen?" for instance)’.[43] One way of thinking about Augustine’s disorientation is in terms of a false analogy operative in his thought. In the *Blue Book*, Wittgenstein says:

> The questions "What is length?", "What is meaning?", "What is the number one?" etc., produce in us a mental cramp. We feel that we can't point to anything in reply to them and yet ought to point to something. (We are up against one of the great sources of philosophical bewilderment: a substantive makes us look for a thing that corresponds to it.)[44]

Augustine’s question produces that variety of mental cramp Wittgenstein envisages. The form of the expression “What is time?” is analogous to the question “What is the specific gravity of hydrogen?”, however they are unlike in Wittgenstein’s sense of “grammar.” A grammatical investigation here would uncover the ways we speak about duration, past present or future, etc and would also make plain the ways in which the question concerning time diverges from questions in the natural sciences.

Immediately prior to remarking that philosophical problems have the form of not knowing one’s way about, Wittgenstein says:

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[43] PI §89

[44] BB p.1 Ludwig Wittgenstein, Preliminary Studies for the Philosophical Investigations : Generally Known as The Blue and Brown Books (Oxford: Blackwell, 1958).

> A main source of our failure to understand is that we don't have *an overview* of the use of our words. Our grammar is deficient in surveyability. A surveyable representation produces precisely that kind of understanding which consists in 'seeing connections'. Hence the importance of finding and inventing *intermediate links*.[45]

The result of a grammatical investigation is the laying out of the uses of our words, the similarities and differences between analogous cases and the connections between various instances made plain. In other words, the result of a grammatical investigation is intended to provide a clear map of the geography of the concepts that are entangled in our philosophical confusion and thus remove the sense of not knowing one's way about. This perspicuous representation aims at complete clarity, and the result of this, according to Wittgenstein, is that 'philosophical problems should *completely* disappear.'[46]

2.3.2 The Wayward Pupil as an Episode of a Grammatical Investigation

Since Wittgenstein conceives of thought experiments as part of a grammatical investigation, let us entertain the possibility that the example of the wayward pupil should be understood as forming part of a "grammatical investigation" rather than a parable about which we are supposed to deduce a moral. If we think of the example as an episode in a grammatical

[45] PI §122

[46] PI §133

investigation, the outcome is merely to draw our attention to what we might say in various cases, of which the wayward pupil is one striking example.

The wayward pupil, by our usual criteria, has mastered the series of natural numbers. We are then to suppose that we have trained and tested the pupil in the series 2, 4, 6, 8, … up to 1000, and that the pupil again masters this series according to our usual criteria. When the pupil then fails to continue the series beyond 1000, it can seem that our criteria are incomplete or inadequate. But Wittgenstein's point here is not to endorse this worry, but to *affirm* our usual criteria by presenting the strangeness of the case. As an episode in a grammatical investigation, the wayward pupil invites us to imagine what we would say about it, but the strangeness of the case makes for a situation in which our criteria do not make it clear what to say. In the earlier, less strange, cases, Wittgenstein outlines possible responses of the pupil and suggests what we might conclude in each case. We can distinguish between, for example, random errors and systematic errors. The case of the wayward pupil does not fit easily into our usual understanding of learning to follow a rule. It constitutes what Wittgenstein refers to as *an abnormal* case.

Wittgenstein makes a distinction between "normal" and "abnormal" cases and states that:

> It is only in the normal cases that the use of a word is clearly laid out in advance for us; we know, are in no doubt, what we have to say in this or that case. The more abnormal the case, the more doubtful it become what we are to say.[47]

[47] PI §142

Abnormal cases are those where it is unclear in advance what we ought to say. We would ordinarily say that the wayward pupil had in fact mastered the series +2, since she had demonstrated the ability to produce the series accurately up to *1000*! But this accomplishment coupled with the unorthodox method of continuing the series beyond 1000 and the bizarre resistance to attempts at correction make for an "abnormal" situation. We might feel inclined to think that the level of understanding demonstrated in the production of the first several hundred terms of the series suggests that the pupil is making a legitimate interpretation of the series that happens not to coincide with the most common reaction.

The case of the wayward pupil is supposed to invite us to imagine various responses to attempts to teach a pupil to follow a rule in order to clarify the geography of the concepts involved. There are cases where it is clear what we would say and others where it is not. We are not supposed to draw the conclusion that our concept of following a rule and when it does or does not apply is hopelessly inadequate. I do not think that there are any morals of that kind that Wittgenstein hopes us to draw.

I *do*, however, think there is a more general conclusion that Wittgenstein wishes to draw that undergirds his use of thought experiments as a mode of philosophical investigation and that is: that the context in which anything occurs is crucial to our making sense of it and for determining what we would say about any particular case. This statement of Wittgenstein's conclusion is hopelessly vague, so I will attempt to explain it in some detail.

3.0 Radical Contextualism

In this chapter, I develop an account of what I term Wittgenstein's "radical contextualism." Frege's familiar exhortation to never ask the meaning of a word outside its use in a sentence is radically expanded in Wittgenstein's hands to—very roughly—"never ask the meaning of a sentence or action outside the context of a form of life." First, I will examine Wittgenstein's discussion of Moore's "proof of the external world" in *On Certainty*, which offers the clearest articulation of the position. According to Wittgenstein's view, an utterance or an action has no determinate sense outside an actual occurrence. Thus, any ruling about whether an individual is successfully following a rule must take place against a rich background of information. In particular, it will not be clear in all cases whether we ought to label the behaviour on some occasions as "following a rule" or not.

In this chapter, I will make use of Grice's discussion of the distinction between the inherent meaning of a sentence and the meanings that can arise through the *use* of a sentence to clarify Wittgenstein's position. I will argue that Charles Travis is right to claim that Wittgenstein's own view rejects this distinction. Instead, he holds that a sentence only has meaning in certain contexts. He rejects the view that there is a "literal" sense of a sentence that can be identified outside of any particular use of it. Indeed, for Wittgenstein, the understanding of the meaning of an utterance can only be identified in the context of a form of life. This is the position I am calling Wittgenstein's "radical contextualism."

I will then argue that this radical contextualism also applies to the behaviour of the wayward pupil. That is, I argue that—according to the Wittgenstein—*actions* are subject to interpretation against a complex background of individual considerations. Since Wittgenstein believes that the actions of the rule-follower can only be meaningfully interpreted against a rich background that he calls the *form of life*, an understanding of what Wittgenstein intends by this is crucial for understanding his remarks concerning following a rule.

3.1 The Importance of Context

The important point in Wittgenstein's view for my purposes is the relationship he sees as holding between a pupil's behaviour and its context in determining what we will or won't be willing to say on any occasion. To elucidate this position, I will begin with Wittgenstein's understanding of the importance of context in making sense of an utterance with the aim of working my way toward a more complete account of the special case of following a rule.

Wittgenstein takes an understanding of the context of an utterance as essential to an understanding of the utterance itself. There are at least two ways that an insistence on the importance of the context of the use of an expression in making sense of it can be taken. The first, sometimes attributed to Wittgenstein, presupposes a sharp distinction between the semantic component of a sentence which is fixed for every use of it and is determined by the mere words involved, and the incidental implications of these words that arise as a result of the particular circumstances in which they are uttered. I will argue that this view is both untrue of Wittgenstein and implausible. I will look at Wittgenstein's discussion of Moore's proof of the external world as a means of suggesting an alternative account of Wittgenstein's conception of sense and nonsense.

Wittgenstein's view is elucidated in his discussion of Moore's refutation of skepticism about the external world in *On Certainty*. Moore, holding out his hand, uttered "I know that this is a hand". There are at least two ways one might question the meaningfulness of this sentence given the context in which it was uttered. One way is to suppose that there are certain meaningful propositions that are not compatible with certain situations. Trying to use them in such situations renders them nonsensical. Under this conception there is a sharp distinction between what is said and the act of saying it. The utterance is nonsensical because its *sense* does not fit with the situation in which it finds itself. We are able to separate the sense of an individual sentence from the additional sense or nonsense that arises in the act of saying it.

Grice attributes a view of this kind to Wittgenstein. Wittgenstein's complaint, according to Grice, was that Moore 'misused the word "know" when he said that he knew that this was one human hand and that this was another human hand'[48], the reason being that 'an essential part of

[48] H. Paul Grice, *Studies in the Way of Words* (Cambridge, Massachusetts: Harvard University Press, 1989). p.5

the concept "know" is the implication that an enquiry is under way.'[49] Under this picture, Wittgenstein's objection to Moore's claim is that his statement is somehow inappropriate on account of the situation in which Moore tries to apply it. This is because Moore is not undertaking an *enquiry*: there is no context of doubt about whether or not he knows there is a hand. Grice sees this protest about Moore's use of "know" as involving the assertion that Moore's kind of claim can only be made when certain conditions are met.

Grice's discussion of this view takes place in the context of proposing his own account of the relation between the meaning of a sentence, its use and his notion of conversational implicature. Charles Travis points out that much of Grice's rejection of the view he attributes to Wittgenstein and others turns on his understanding of "saying"[50]. Grice distinguishes between 'what the speaker has *said* ... and what he has *implicated* (e.g. implied, indicated, suggested)'.[51] Grice intends, by "say", that 'what someone has said to be closely related to the conventional meaning of the words (the sentence) he has uttered.'[52] So, for example, if someone were to say, "Bill is a philosopher and he is, therefore, brave", this would not be to have *said* 'that Bill's being courageous follows from his being a philosopher.'[53] Instead, Grice says, 'I would wish to maintain that the semantic

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[49] Grice. p.5

[50] Charles Travis, 'Review : Critical Notice : Annals of Analysis', *Mind*, 100.2 (1991), 237–64. p.238

[51] Grice. p.118

[52] Grice. p.25

[53] Grice. p.121

function of the word "therefore" is to enable a speaker to *indicate* though not to *say*, that a certain consequence holds.'[54] The picture we have is that each utterance can be split into two parts: what is "said", and what is otherwise conveyed. The first portion is fixed and determined by the words used, the second is context-dependent and can vary with the situation.

With this distinction in mind, Travis discusses Grice's example sentence, "The table is covered with butter". There are various ways one could understand this sentence. Travis says:

> Suppose I buy masses of foil-wrapped packets of butter. I then arrange them on the table so that no bit of surface is showing. Is the table covered with butter? Intuitively, there is something true to be said in describing it, and also something false. What one *would* say in such words, equally, what the words quite literally *said*, whether a true thing or a false one, depends on such things as why one is saying it.[55]

The point is that it is not clear which precise sense is the *literal* sense of any given sentence which is isolable from the particular acts and contexts of saying it. Grice assumes that it is possible to isolate a literal meaning that is common to all uses of a sentence, but it is not clear that this is correct. When we hear a statement ("The table was covered with butter"), we can see how it might be used. We understand its meaning insofar as we think of what someone who said it *might* mean. We think of possible meanings and think that we therefore know *what is meant*. It does not, however, mean that we have found a full determinate sense.

[54] Grice. p.121

[55] Travis, p. 240.

There is certainly something to be said for the intuitive appeal of Grice's conception of sentence-meaning. That we are able to construct and understand new sentences made up of old words is evidence of the close relationship between the meaning of a sentence and the meanings of its component words. The point to be made is rather that the meanings of the component words *underdetermine* the sense of a sentence in its use in a particular context. The contrast, as Travis sees it, is that the Gricean account involves a commitment to selecting one possible sense of a sentence as the literal one. The question is whether this can be done in a way that does not seem merely arbitrary.

Whether it is possible to determine a literal meaning of a sentence outside of its significant use, or not, this is not the view Wittgenstein takes. He says:

> the words "I am here" have a meaning only in certain contexts and not when I say them to someone who is sitting in front of me and sees me clearly,—and not because they are superfluous, but because their meaning is not *determined* by the situation, yet stands in need of such determination.[56]

The meaning of the sentence "I am here", according to Wittgenstein, 'is not *determined* by the situation'.[57] The contrast between the two views is that in the Gricean interpretation, the

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[56] Ludwig Wittgenstein, *On Certainty*, ed. by G. E. M Anscombe and Denis Paul (New York: Harper & Row, 1972). OC §348

[57]OC §348

sentence has a sense that does not fit with the situation, whereas in Wittgenstein's own description, the sentence does not yet have a sense. It is not a case of the sense clashing with the context; the meaning has not been determined by the situation; the use has failed to confer any determinate sense whatsoever.

James Conant characterizes Wittgenstein's view like this:

> What we are tempted to call 'the meaning of a sentence' is not a property the sentence already has in abstraction from any possibility of use and which it carries with it – like an atmosphere accompanying it – into each specific occasion of use. It is, as Wittgenstein keeps saying, *in* the circumstances in which it is 'actually used' that the sentence has sense.[58]

Conant's remark immediately brings to mind the exhortation in Frege's context principle 'never to ask for the meaning of a word in isolation, but only in the context of a proposition.'[59]

Frege says that if this principle is not observed, 'one is almost forced to take as the meanings of words mental pictures or acts of the individual mind, and so to offend against the first principle as well.'[60] Frege argues that it is a mistake to seek guidance in the ideas that numbers

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[58] James Conant, 'Wittgenstein on Meaning and Use', *Philosophical Investigations*, 21.3 (1998), 222–50 <http://doi.wiley.com/10.1111/1467-9205.00069>. p.241

[59] Gottlob Frege, *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number*, ed. by J L (trans) Austin, 2nd (Revis (Evanston, Illinois: Northwestern University Press, 1980). p.x

[60] Frege. p.x

conjure in us in seeking a definition of number. He argues that a word calling up an idea in us is neither a necessary nor a sufficient condition for determining the meaning of a word. It is not a necessary condition since there are many words that are meaningful that do not call up a suitable idea corresponding to it. Frege says that: 'Even so concrete a thing as the Earth we are unable to imagine it as we know it to be; instead we content ourselves with a ball of moderate size, which serves as a symbol for the Earth, though we know quite well it is very different from it.'[61] Likewise, it is not a sufficient condition since even if every word did call up an idea in us,[62] 'this idea need not correspond to the content of the word; it may be quite different in different men.'[63]

It is due to the considerations mentioned above that Frege introduces his context principle as follows:

> That we can form no idea of its content is no reason for denying all meaning to a word, or for excluding it from our vocabulary. We are indeed only imposed on by the opposite view because we will, when asking for the meaning of a word, consider it in isolation, which leads us to accept an idea as the meaning. Accordingly, any word for which we can find no corresponding mental picture appears to have no content. But we ought always to keep before our eyes a complete proposition. Only in a proposition have the words really a meaning. It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content.[64]

[61] Frege. p.71, §60

[62] We might, for example. doubt that words like "only" *do* call up ideas in us, but suppose nevertheless that they do (see p.70, §59).

[63] Frege. p.70, §59

[64] Frege. p.71, §60

Frege takes it that by focusing on a word's use in context, we turn our attention not to the psychological associations or ideas we have of the concept, but to the logical role it plays. Wittgenstein is committed to a stronger formulation of Frege's principle. Indeed, Wittgenstein's conception of nonsense can be seen as a radicalization of Frege's context principle. He holds that a *whole sentence* can have no determinate sense outside a context in which it has a significant use. He extends Frege's argument to show that analogous difficulties occur in the context of sentences that arose in the context of words. Frege's worry that to ask for the meaning of a word outside the context of a sentence would lead to seeking an answer in the realm of the psychological carries over to this radicalized context principle. This new principle instructs us to only ask for the meaning of a sentence within a language-game. Wittgenstein explicitly draws our attention to the lesson we ought by now to have learnt from Frege as applying also to sentences in the following remark:

> The sentence 'I know that that's a tree' if it were said outside its language-game, might also be a quotation (from an English grammar-book perhaps).—'But suppose I *mean* it while I'm saying it?' The old misunderstanding about the concept of 'mean'.[65]

A feeling that accompanies the saying of a sentence does not determine its sense. An implication of this radicalized context principle is that we cannot determine whether a sentence is sense or nonsense in isolation.

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[65] OC §393

Wittgenstein's talk of whether I can be said to know that I am in pain is only related to whether this series of words may be uttered intelligibly or not. His point is that, as with every other utterance, there is no determinate sense to this sequence of words outside any particular context of saying them. The philosophical confusion at work here relates not to knowledge, but to sense. Just as Wittgenstein says that the words "I am here" might be used meaningfully at any moment 'if suitable occasion presented itself',[66] so we must conclude he would say the same of the sentence "I know that I am in pain" (it might, for example, be uttered in the context of a joke). However, under Wittgenstein's conception of nonsense, this is true of *any* sequence of words. So what is it about the utterance "I know I am in pain" that leads Wittgenstein to isolate this in particular as something one cannot meaningfully say? I think the reason for Wittgenstein's focus on this is that it is precisely the kind of statement philosophers make in the context of doing philosophy, in this case in the context of philosophical reflection on the privacy of sensations, where we are liable to feel that it is meaningful without reflecting too deeply on what precisely is being said. Wittgenstein tells us that 'philosophical problems arise when language *goes on holiday*.'[67] What Wittgenstein has in mind here is that the kind of conceptual disorientation that is characteristic of philosophical puzzlement tends to arise precisely when we remove our language from its everyday context and make it the object of our study independent of any actual use we are accustomed to make of it. But in this sense, the words "I know I am in pain" are akin to his other examples "I know that that's a tree" or "I am here".

[66] OC §10, p.3

[67] PI §38

Having described what I take to be Wittgenstein's attitude toward the role of context in making sense of an utterance, I will now argue that he takes a similar view on the subject of non-verbal behaviour.

3.2 Context and Following a Rule

I have argued that Wittgenstein has a radically contextual view of sentence meaning. So, what is the relevance of this radical contextual view for Wittgenstein's discussion of the wayward pupil? The various examples of rule following instruction Wittgenstein relates do sometimes contain utterances, but we can also imagine a pupil writing down a series of numbers silently. That is, the example of the wayward pupil does not seem obviously relevant, since it does not obviously concern verbal behaviour. However, I will argue that Wittgenstein's attitude toward the importance of context permeates all regions of human behaviour, not merely utterances. In the *Philosophical Investigations* he describes a case of wanting to make a remark:

> Remember this case: if one urgently wants to make some remark, some objection, in a discussion, it often happens that one opens one's mouth, draws a breath, and holds it; if one then decides to let the objection drop, one lets one's breath out. …

> An observer will realize that I wanted to say something and then thought better of it. In *this* situation, that is. In a different one, he would not interpret my behaviour in this way, however characteristic of the intention to speak it may be in the present situation.[68]

According to Wittgenstein, the context of human behaviour determines what one will say about it in just the way that I argued it does in the case of utterances.

Wittgenstein gives a number of thought experiments both in the *Philosophical Investigations* and elsewhere that prompt us to examine situations where there seems to be a "mismatch"[69] between someone's behaviour and the circumstances surrounding it. Sometimes he asks us to imagine someone who appears to have all of the usual background experiences and understanding, but who fails to behave in the way that we would consider normal, as he does in the example of the wayward pupil. Sometimes he does the converse: he asks us to imagine someone who behaves in the way we would expect someone to *without* the usual background.

The example of Two-minute England is a case of the second kind. In *Remarks on the Foundations of Mathematics*, Wittgenstein gives the following thought experiment:

> Let us imagine a god creating a country instantaneously in the middle of the wilderness, which exists for two minutes and is an exact reproduction of a part of England, with everything

[68] PI §591

[69] I have much trepidation in characterizing this as a "mismatch" between context and behaviour. Wittgenstein might want to say that there can't be a "mismatch" between one's behaviour and the circumstances surrounding it since the behaviour derives its sense from the circumstances in which it occurs.

> that is going on there is two minutes. Just like those in England, the people are pursuing a variety of occupations. Children are in school. Some people are doing mathematics. Now let us contemplate the activity of some human being during these two minutes. One of these people is doing exactly what a mathematician in England is doing, who is just doing a calculation.—Ought we to say that this two-minute man is calculating? Could we for example not imagine a past and a continuation of these two minutes, which would make us call the process something different?[70]

Wittgenstein's thought experiment pushes us to ask: "What is it to calculate?", "In what circumstances would we call something calculating?" His example suggests that the outward appearance or behaviour of a person is not sufficient to guarantee that this is the activity we would like to call calculating. We can imagine a past and a future for the two-minute man that might make us doubt that his activity ought to be called calculating. Imagine, for example, that he has been initiated into the rites of a particular religious practice that involves laying down symbols on a page that exactly mimics the patterns we would make were we to be doing some calculations. For him, however, the laying down of symbols is an act of veneration. Without seeing which of the two histories the two-minute man has—training in mathematics, or initiation into the veneration rites—we cannot say which of the two activities he is engaged in. Wittgenstein's point is to demonstrate the importance of the background against which someone acts in making sense of the action. He says: 'What in a complicated surrounding, we call "following a rule" we should certainly not call that if it stood in isolation.'[71]

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[70]RFM VI.34

[71]RFM VI.33

There are, likewise, intermediate cases related to the correctness of someone's calculations. We would likely describe an individual who makes the occasional error as nevertheless "calculating". However, if he makes not just occasional errors, but repeated mistakes we may come to say that he does not know how to calculate properly, and is not engaged in calculating, even if he has apparently been trained in the appropriate way. We can imagine a child who has memorized the patterns of symbols someone makes when performing a certain calculation and who writes them down for fun. The child's behaviour appears to be that of someone who is calculating, but we would be hesitant to call it that if the child's activity does not have the appropriate surroundings—if she has never been taught to calculate, for example, or she does not know how to count, or if we ask her what she is doing she says "drawing", etc.

Thus, for Wittgenstein, it only makes sense to evaluate whether someone is following a rule (or reading, or calculating, or any other distinctly human activity) within a rich context that involves the entire circumstances surrounding the activity including what training the agent has been given, the kinds and frequency of error involved, the language being used and so forth. This gives us a different understanding of the purpose of the example of the wayward pupil. First, we understand that Wittgenstein is undertaking a grammatical investigation into the notion of following a rule. Wittgenstein fills in a background of this example in such a way that we have conflicting information about whether we ought rightly to think that the wayward pupil has mastered a series.

There is textual evidence to suggest that this is precisely what Wittgenstein intended in his discussion of following a rule. He asks: ‘Is what we call “following a rule” something that it would be possible for only *one* person, only *once* in a lifetime to do?—And this is, of course a gloss on the *grammar* of the expression “to follow a rule”.’[72] Investigating whether what we call “following a rule” could be said to occur in certain circumstances is—according to Wittgenstein—an investigation into the logical grammar of the expression. His answer to this particular question about the expression “following a rule” is—unlike many other questions raised in the *Investigations*—unequivocal. He states:

> It is not possible that there should have been only one occasion on which a person followed a rule. It is not possible that there should have been only one occasion on which a report was made, an order given or understood, and so on. –To follow a rule, to make a report, to give an order, to play a game of chess, are *customs* (usages, institutions).[73]

Whatever else we might think it right to say about the case of the wayward pupil, Wittgenstein thinks that we can be sure that “following a rule” is the kind of human activity that derives its sense from shared understanding and practice. He says: ‘Following a rule is analogous to obeying an order.’[74] One cannot count as obeying an order without many other background features occurring such as the existence of a shared language and a custom of giving orders.

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[72] PI §199

[73] PI §199

[74] PI §206

In order to profit from the Wayward Pupil example properly, we should try to understand the sort of context in which (according to Wittgenstein) mathematical activity takes place. However, before I do that, I will clarify what I take to be Hardy's "formalism," which is a view about mathematics that thinks this task can be safely ignored.

4.0 Hardyan Formalism

Wittgenstein has been most often identified as opposing Platonism in his writings about mathematics. However, Wittgenstein's radical contextualism about mathematics finds its most natural opponent not in Platonism, but in a variety of formalism. One of Wittgenstein's obvious formalist targets is his colleague the mathematician G. H. Hardy. In this chapter, I attempt to articulate a clearer account of Wittgenstein's conception of mathematics by making use of Hardy's alternative picture of the nature of mathematical certainty. I argue that Hardy's formalist picture as articulated in his mathematical writings is a central target of Wittgenstein's remarks on mathematics. One reason Hardy's position and Wittgenstein's rejection of it is interesting is because Hardy's picture of mathematics strikes a modern audience as entirely natural. The formalist picture of mathematics that I have in mind is exemplified by Hardy's assertion that the characteristic activity of the mathematician is the laying down of definitions and the deriving of results from those definitions. Mathematics gets its certainty, according to this picture, from the definitions and the rules according to which it operates.

In order to make vivid the tension between Wittgenstein and Hardy's positions, I use the example of an episode in the history of mathematics concerning divergent series.

4.1 Conventionalism vs. Platonism

There have been many and varied attempts to locate Wittgenstein's later thought about mathematics by reference to one or more recognizable positions in the terrain of the philosophy of mathematics. Much of the discussion of Wittgenstein's work in the philosophy of mathematics has conceived of the central issues as adjudicating between positions clustered within either a conventionalist or a Platonist viewpoint.

As I mention above, Dummett's assessment of Wittgenstein's *Remarks on the Foundation of Mathematics*[75] is that Wittgenstein endorses a radical conventionalism. Dummett begins by outlining two positions one might occupy in the philosophy of mathematics. 'In the philosophy of mathematics,' Dummett remarks, 'Platonism stands opposed to various degrees of constructivism.'[76] Dummett describes Platonism about mathematics as the view that 'mathematical objects are there and stand in certain relations to one another, independently of us, and what we do is to discover these objects and their relations to one another.'[77] In contrast, the constructivist presents a 'picture of our making, constructing, the mathematical entities as we go

[75] Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics*, ed. by Georg Henrik Von Wright, R. Rhees, and Gertrude Elizabeth Margaret Anscombe, trans. by Gertrude Elizabeth Margaret Anscombe, Revised ed (Oxford: Basil Blackwell, 1978).

[76] Dummett, p. 324.

[77] Dummett, p. 325.

along.'[78] While Dummett himself does not think we need necessarily view these two positions as rivals, he asserts that Wittgenstein's aim in the *Remarks on the Foundations of Mathematics* is to argue for constructivism against Platonism. Dummett states that the meaning of mathematical statements according to the Platonist is 'to be explained in terms of its truth conditions; for each statement there is something in mathematical reality in virtue of which it is either true of false.'[79] In contrast, for the constructivist—according to Dummett—'the general form of an explanation of meaning must be in terms of the conditions under which we regard ourselves as justified in asserting a statement, that is, the circumstances in which we are in possession of a proof.'[80]

More recent (and sympathetic) readings of Wittgenstein's philosophy of mathematics have viewed him as treading—or attempting to tread—a middle path between Platonism and conventionalism[81] rather than as defending a form of conventionalism. For example, Steve Gerrard argues that Wittgenstein's 'ultimate achievement in the philosophy of mathematics was to stake out a defensible intermediate position between two untenable warring factions.'[82] He takes it that Wittgenstein perceived his 'chief foe' as picturing 'mathematics as transcendent: a mathematical

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[78] Dummett, p. 325.

[79] Dummett, p. 325.

[80] Dummett, p. 325.

[81] Mark Steiner, 'Empirical Regularities in Wittgenstein's Philosophy of Mathematics', *Philosophia Mathematica*, 17.1 (2009), 1–34 <https://doi.org/10.1093/philmat/nkn016>; Steve Gerrard, 'A Philosophy of Mathematics between Two Camps', in *Cambridge Companion to Wittgenstein*, ed. by Hans Sluga and David Stern (Cambridge: Cambridge University Press, 1996), pp. 171–97.

[82] Gerrard, 'A Philosophy of Mathematics between Two Camps', p. 171.

proposition has truth and meaning regardless of human rules or use.'[83] Gerrard states that Wittgenstein sometimes identified this picture with Hardy. The *other* side, according to Gerrard, was instantiated in various ways throughout Wittgenstein's opposition to it but was characterized by a rejection of the metaphysical garb needed to guarantee the objectivity within the first picture. Those in the 'anarchist camp,' as Gerrard labels it, 'deny objectivity in all fields in all ways.'[84] Against this position, Gerrard states, 'Wittgenstein and Hardy were allies; Wittgenstein never rejected his predecessors' antipsychologism, and no matter how much he objected to some characterizations of the *metaphysics* of objectivity (not objectivity itself!), he never recoiled into the anarchist camp.'[85]

While I agree with Gerrard's general characterization of Wittgenstein as resisting both Platonism and some "anarchic" position, there is a third aspect of Wittgenstein's approach that is missing in Gerrard's depiction. While Hardy adhered to some form of Platonism, there is also a distinctly formalist picture of mathematics that held sway over Hardy and to which Wittgenstein also objected.

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[83] Gerrard, 'A Philosophy of Mathematics between Two Camps', p. 172.

[84] Gerrard, 'A Philosophy of Mathematics between Two Camps', p. 171.

[85] Gerrard, 'A Philosophy of Mathematics between Two Camps', p. 173.

Later in this chapter I will set out Hardy's formalist position, but first I will briefly describe his more familiar Platonistic picture.

4.2 Hardy's Platonism

Wittgenstein was, of course, familiar with Russell's philosophy and there is clear evidence that Wittgenstein was also acquainted with Hardy and aware of his work.[86] Indeed, Wittgenstein's *Lectures on the Foundations of Mathematics*[87] reads in many ways as a response to Hardy's paper (which was the text of Hardy's Rouse Ball lecture presented at Cambridge in 1928) 'Mathematical Proof.'[88] Steve Gerrard has argued that Hardy's attitude towards mathematics presents a useful

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[86] According to Raymond Monk's biography, Hardy (along with Moore) had been Wittgenstein's examiner for his fellowship at Trinity (Monk 1990, p.304). John King mentions in passing that Wittgenstein used to visit Hardy, but does not give any other details about the precise nature of these visits John King, 'Recollections of Wittgenstein', in *Recollections of Wittgenstein*, ed. by Rush Rhees, Revised ed (Oxford: Oxford University Press, 1984), pp. 68–75. p.73. Wittgenstein explicitly refers to Hardy's remarks in 'Mathematical Proof' in Lectures XXV and XXVI. (*Wittgenstein's Lectures on the Foundations of Mathematics*, ed. by Cora Diamond (Ithaca, NY: Cornell University Press, 1976). Wittgenstein also mentions Hardy's book *A Mathematician's Apology* in MS 124 p.35: "the sentences that Hardy sets forth as expression of his philosophy of mathematics in his miserable book 'Apology of a Mathematician' are in no way philosophy, but could—like all similar outpourings—be conceived as raw material of philosophizing." [Translation provided by Juliet Floyd in (Juliet Floyd, 'Das Überraschende: Wittgenstein on the Surprising in Mathematics', in *Wittgenstein and the Philosophy of Mind*, ed. by Jonathan Ellis and Daniel Guevara (New York: Oxford University Press, 2012).) Juliet Floyd also describes the notes that Wittgenstein made in his own copy of Hardy's undergraduate text *A Course of Pure Mathematics* (G. H. Hardy, *A Course of Pure Mathematics*, 5th edn (Cambridge: Cambridge University Press, 1928). She writes: These apparent annotations are striking indeed, and remarkably odd at first glance, a kind of midrashic editorial commentary, strewn around the edges of the book, circling, castigating, rewording, crossing out and substituting individual words, especially those connected with generality.' p.255

[87] Diamond, *Wittgenstein's Lectures on the Foundations of Mathematics*.

[88] Floyd; Steve Gerrard, 'Wittgenstein's Philosophies of Mathematics', *Synthese*, 87.1 (1991), 125–42.

foil for understanding Wittgenstein's own ideas.[89] Gerrard argues that Hardy's notion of mathematics corresponding to some kind of "reality" played a role in Wittgenstein's thinking about mathematics that mirrors the role that the Augustinian picture of language acquisition had in his thinking about language.

While I agree with Gerrard that considering Hardy's conception of mathematics can be illuminating in understanding Wittgenstein's thinking about mathematics, I wish to focus less on Hardy's explicitly Platonistic conception and more on his (largely unnoticed) implicitly formalistic conception. I think paying attention to this aspect of Hardy's conception of mathematics will yield a clearer understanding of some of Wittgenstein's puzzling remarks about mathematics. I will first, however, briefly describe Hardy's explicitly Platonistic account of mathematics.

Hardy describes the kind of reality he conceives mathematics as having as akin to a distant mountain range. He likens the experience of the mathematician to that of an observer gazing at it and attempting to pass on his observations to others:

> I have myself always thought of a mathematician as in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B.[90]

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[89] G. H. Hardy, 'Mathematical Proof', *Mind*, 38.149 (1929), 1–25.

[90] Hardy, 'Mathematical Proof'. p.18

This picture of mathematics is clearly a variety of Platonism. Hardy's description of mathematical enterprise is of an experience of gradually discerning mathematical truths already fully formed and inhabiting some kind of mathematical reality. He takes it as a precondition of any acceptable philosophy of mathematics that it takes this view of mathematical truth:

> It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality.[91]

However, this view of mathematics that is most frequently ascribed to Hardy is in some respects in tension with another mathematical picture that can also be found in Hardy and that Wittgenstein was reacting against. While Hardy's explicitly "philosophical" writings are undoubtedly Platonistic, his mathematical writings have a formalistic flavour.

4.3 Hardy's Formalism

There are many varieties of formalism in the philosophy of mathematics. In his *Introduction to the Philosophy of Mathematics*, Mark Colyvan states that formalism 'takes

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[91] Hardy, 'Mathematical Proof'. p.4

mathematical notation and its manipulation to be the core business of mathematics.'[92] In this section, I will argue that the defining feature of the Hardyan formalist picture is a strict division between "legitimate" mathematics and the residual material that has not quite attained the status of mathematics. Hardy's use of this distinction between "legitimate", formally defined mathematics and the residual "nonsense" firmly places him in a formalist camp and invites critical attention from Wittgenstein.

Hardy introduces the notion of mathematical "gas" in his paper 'Mathematical Proof'.[93] In his comments immediately following his likening mathematical enterprise to the observation of a distant mountain range mentioned above, Hardy turns his attention to the process of communicating mathematical knowledge to a pupil. Hardy describes the mathematician-discoverer and his pupil as follows:

> [W]hen he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he points to it, either directly or through the chain of summits which led him to recognise it himself. When his pupil also sees it, the research, the argument, the proof is finished.[94]

According to Hardy, a mathematical demonstration serves to "show" a pupil the mathematical truth in question—here represented as a distant mountain range. He later continues

[92] Mark Colyvan, 'An Introduction to the Philosophy of Mathematics', *Cambridge Introductions to Philosophy*, 2012, ix, 188 pages (p. 5) <https://doi.org/10.1093/aesthj/ayi021>.

[93] Hardy, 'Mathematical Proof'.

[94] Hardy, 'Mathematical Proof', p. 18.

that mathematics is also full of proofs 'whose purpose is not in the least to secure conviction,' but whose interest 'depends on their formal and aesthetic properties.'[95] He writes that if the analogy is pushed to its extreme we would be 'led to a rather paradoxical conclusion; that there is, strictly, no such thing as mathematical proof.'[96] According to Hardy, there would be no such thing as mathematical proof since:

> we can, in the last analysis, do nothing but point; that proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils.[97]

While Hardy asserts that the analogy is imperfect, he believes that it captures something right about the way mathematical instruction works. Hardy and Littlewood's concept of "gas" represents the additional informal instruction a teacher might give a pupil in order to help him or her to comprehend mathematical content. In the preface to his textbook *The Elements of the Theory of Real Functions*, Littlewood describes the importance of students attending lectures due to the possibility of the instructor conveying the real "point" of the mathematics. Littlewood does not use the term "gas," but it is clear that he has in mind something that would fit the description Hardy gives. He emphasises the importance of the instructor's informal attempts to help the pupil see "the point":

> I am one of those who believe that lectures can have great value, and particularly at a certain moderately advanced stage of a mathematical education. The modern standard of conciseness and lucidity in original papers and advanced text-books is on the whole a high one, but the style

[95] Hardy, 'Mathematical Proof', p. 18.

[96] Hardy, 'Mathematical Proof', p. 18.

[97] Hardy, 'Mathematical Proof', p. 18. My emphasis.

> is one for the expert only. We may demand two things of an original paper, a complete and accurate exposition on the one hand, and on the other that it should convey what is the real "point" of the subject-matter. For various reasons, among which a sufficient one nowadays is sheer lack of space, the second demand is inevitably sacrificed to the first. A lecture, however, more particularly when it is supported by a complete exposition in print, is the very place for the provisional nonsense that the second generally calls for. This would appear ridiculous if enshrined permanently in print, and its real function is to disappear when it has served its turn. …The infinitely greater flexibility of speech enables me here to do without a blush what I shrink from doing in print.[98]

Littlewood describes two goals that an original paper in mathematics should attain. The first is 'a complete and accurate exposition', which might be brief and fail to fully explain the "point" of the mathematics within the paper. The second goal is to convey the real "point", but this may call for the use of 'provisional nonsense' in order to successfully convey the point.

This is the background against which what I am calling the Hardyan formalist picture of mathematics arises. First, there is perceived to be a clear distinction between on the one hand what Hardy and Littlewood conceive of as "real mathematics," where this takes the form of rigorous definition and methods of deduction, and on the other hand, the gaseous, provisional nonsense which is used in practice to convey the "point" of the real mathematics. Second, there is a belief that "real mathematics" is monolithic. As Philip Davis describes this view, mathematics is conceived of as a 'set of static, formal, deductive structures, permanent in arrangement and fixed for all time.'[99]

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[98] J.E. Littlewood, *The Elements of the Theory of Real Functions*, 3rd (revis (New York: Dover, 1954).

[99] Philip J. Davis, 'When Mathematics Says No', in *No Way: The Nature of the Impossible*, ed. by Philip J Davis and David Park (New York: W. H. Freeman and Company, 1987), pp. 161–77 (p. ??).

The Hardyan formalist picture is not fully developed in Hardy's writings, but however one might choose to spell out the details there is a sharp distinction between the "real" or "legitimate" mathematical kernel, and extraneous, inessential portion. And that is what makes Hardy part of Wittgenstein's target. While seeming to pursue a certainty that distinguishes respectable mathematics as such, Hardy actually ends up with inflexible criteria of mathematical legitimacy that excludes valuable material from the class of respectable mathematics.

Wittgenstein himself uses this curious phrase "gas" in the context of speaking of the interpretation of mathematical symbols, and seems to have Hardy's formalism particularly in mind:

> Mathematicians tend to think that interpretations of mathematical symbols are a lot of jaw—some kind of gas which surrounds the real process, the essential mathematical kernel. A philosopher provides gas, or decoration-like squiggles on the wall of a room.[100]

Wittgenstein seems to have picked up this phraseology from Hardy's use of it in his paper 'Mathematical Proof.' However, Wittgenstein's attitude toward it is crucially at odds with that of Hardy and Littlewood. In Hardy and Littlewood's descriptions, this "gas" is a slightly embarrassing but necessary part of conveying a proper understanding of mathematics to pupils.

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[100] Lecture I Diamond, *Wittgenstein's Lectures on the Foundations of Mathematics*, pp. 13–14.

Wittgenstein returns to this theme of the essential and inessential parts in discussing Hardy in his 'Philosophy for Mathematicians' lectures:

> [Hardy] conceived philosophy as a decoration, an atmosphere, around the hard realities of mathematics and science. These disciplines, on the one hand, and of philosophy on the other, are thought of as being like the necessities and decoration of a room. Hardy is thinking of philosophical opinions. I conceive of philosophy as an activity of clearing up thought.[101]

Wittgenstein is critical of this attitude in two respects. The first is in thinking of the less formal discussions of mathematics as inessential or decorative. The second is the notion that the "essential" mathematical kernel is "watertight." In his *Lectures on the Foundations of Mathematics*, Wittgenstein asks us to imagine that there are two kinds of proof::

> Suppose I say, "Some proofs are strictly logical—watertight, airtight, foolproof; and some proofs are meant merely to convince." It seems almost as though what is meant is that they should have a certain psychological effect.[102]

There are a few different issues here that Wittgenstein addresses. Littlewood states that a modern original paper in mathematics gives a complete and accurate exposition of a topic, but that the style is such as to leave the "point" unclear. There is an immediate question here about clarity. On the one hand, Hardy and Littlewood seem to conceive of the telegraphic formal writings of the

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[101] Ludwig Wittgenstein, *Wittgenstein's Lectures, Cambridge 1932–1935 From the Notes of Alice Ambrose and Margaret Macdonald*, ed. by Alice Ambrose (Amherst, NY: Prometheus Books, 2001), p. 225.

[102] Diamond, *Wittgenstein's Lectures on the Foundations of Mathematics*, pp. 130–31.

professional mathematician as a benchmark of clarity, and yet on the other hand, the pupil of mathematics cannot be expected to understand the "point" without the addition of some "provisional nonsense" from her instructor. The conception of clarity at work here is an unusual one. This is not "clarity" in the sense of "the quality of being coherent and intelligible" since that is precisely what Littlewood and Hardy take the mathematical prose to *not* be to the pupil. "Clarity" in their sense seems to be some kind of "logical" property of the mathematics. This clarity is supposed to be a guarantee about the status of the mathematics in question.

4.4 Divergent Series

We can see how Hardyan formalism plays itself out in the particular example of Hardy's criticisms of earlier treatments of divergent series. In his mathematical text *Divergent Series*,[103] Hardy gives a brief account and evaluation of the contribution of past mathematicians to the study of divergent series. By attending to Hardy's specific criticism of prior mathematical approaches to divergent series, we can piece together his formalistic picture of mathematics.

The study of divergent series was a controversial area of mathematics until well into the twentieth century. Littlewood, in his preface to Hardy's book states that in the early twentieth

[103] G. H. Hardy, *Divergent Series* (London: Oxford University Press, 1949).

century, divergent series 'while in no way mystical or unrigorous *was* regarded as sensational, and about the present title [*Divergent Series*], now colourless, there hung an aroma of paradox and audacity.'[104] The mathematical historian Morris Kline tells us that the serious consideration of divergent series 'indicates how radically mathematicians have revised their own conception of the nature of mathematics.'[105]

An infinite series is an infinite string of terms arranged together with addition or subtraction signs, such as:

A $1 + 2 + 3 + \cdots$

B $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

C $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

D $1 - 1 + 1 - 1 + \cdots$ "Grandi's Series"

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[104] J. E. LITTLEWOOD, *Preface*, in Hardy, *Divergent Series*, p. p.vii.

[105] Morris Kline, *Mathematical Thought From Ancient To Modern Times* (New York: Oxford University Press, 1972). p.1096

The "…" in the series above present varying degrees of difficulty for determining a solution to the equations. We call the "partial sum" of a series the sum of some finite set of initial terms. So, for example, the partial sum S_n of "D" is the sum of the first n terms in the series:

$$S_n = \underbrace{1 - 1 + 1 - 1 + \cdots + 1} = \begin{cases} 0 \ for\ even\ values\ of\ n \\ 1\ for\ odd\ values\ of\ n \end{cases}$$

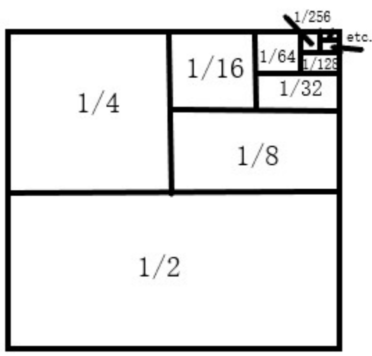

Figure 1 Terms from the series "C" inside a unit square.

An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be *convergent* if the partial sum $s_{n= a_1+a_2+\cdots+a_n}$ tends to a finite limit s when $n \to \infty$. For example, the series "C" $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is convergent since the sum tends to 2. Intuitively, we can see that this is the case by noting that the portion of "C" after the first term fits into a unit square.

A series is *divergent* if it is not convergent, which occurs when the sum tends to infinity, or the partial sums oscillate between values. So the series "A" $1 + 2 + 3 + \cdots$ is clearly divergent since $\lim_{n\to\infty} \sum_{n=1}^{\infty} n \ \to \infty$. And the series "D" $1 - 1 + 1 - 1 + \cdots$ (sometimes known as "Grandi's series") fails to converge since the partial sum s_n of this series is 0 for even n and 1 for odd n. i.e.:

$s_1 = \mathbf{1}$

$s_2 = 1 - 1 = \mathbf{0}$

$s_3 = 1 - 1 + 1 = \mathbf{1}$

$s_4 = 1 - 1 + 1 - 1 = \mathbf{0}$

$\ldots$

Note that series "B" $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is also divergent. Its terms tend to zero, but not "quickly" enough to yield a finite sum, unlike series "C". The partial sums in series "B" tend to infinity.

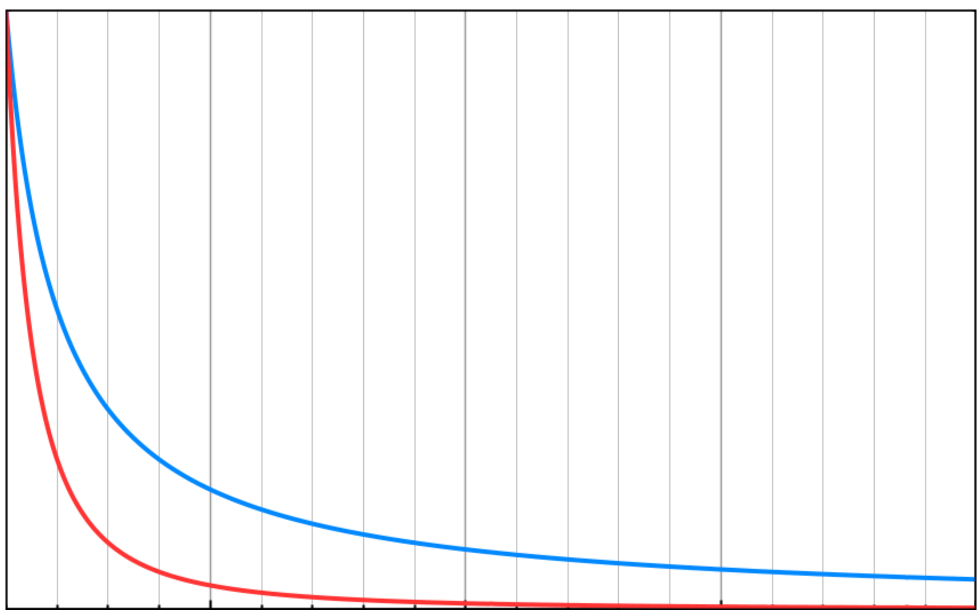

Figure 2 The blue line denotes the function 1/x and the red line 1/x^2. You can see that the red line gets closer to zero faster than the blue line.

Prior to Cauchy's formal definition for convergence in his *Cours D'analyse*[106], mathematicians found it useful to employ divergent series despite the spectre of paradox that hovered over them. Generally, divergent series were used 'with more or less conscious recognition of their divergence.'[107] There was an awareness that something was not quite in order about divergent series, but the combination of mathematical dividends reaped through operations indiscriminately performed with them and important results that were independently verifiable seemed to ensure that any difficulties would be worked out in time. Despite some hesitation (Euler explicitly stated in *Institutiones* 1755 that one should not use the term "sum" when discussing divergent series since this refers to actual addition), many mathematicians made use of the notion of the "sum" of a divergent series, without defining exactly what that should be taken to mean. Instead they were guided by the intuitive "meaning" of a series.

So let's return briefly to Grandi's series "D" in order to see what is puzzling about divergent series:

D $\quad 1-1+1-1+\cdots \quad$ "Grandi's Series"

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[106] A. L. Cauchy, *Cours d'analyse a l'Ecole Royal Polytechnique* (Paris: Debure Frères, 1821).

[107] Kline. p.1096

Our ordinary notion of "sum" does not give us a solution to this series. The partial sums oscillate between 1 and 0. Earlier mathematicians attempted to derive something that could be called its sum using various approaches, but all were in some respect problematic.

We could rearrange the series like this:

$$(1-1)+(1-1)+\cdots=0+0+\cdots=0$$

And therefore, conclude that the series sums to zero.

However, it seems equally reasonable to rearrange the series like this:

$$1-(1-1)-(1-1)-\cdots=1-0-0-\cdots=1$$

And thus, conclude that the series sums to one.

Alternatively, we could get the following result:

Let s = $1-1+1-1+\cdots$, then

$$1 - s = 1 - (1 - 1 + 1 - 1 + \cdots) = 1 - 1 + 1 - 1 + \cdots = s$$

So, $1 - s = s$, thus $s = \frac{1}{2}$.

The latter result is also obtainable by using an appropriate substitution with the binomial expression as follows:[108]

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

We let $x = 1$, which yields:

$$\frac{1}{1+1} = 1 - 1 + (1)^2 - (1)^3 + \cdots$$

Thus,

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

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[108] Euler also used a method akin to this, using the expression $\frac{1}{1-x} = 1 + x^2 + x^3 + \cdots$ and substituting $x = -1$. (*Institutiones)*

Thus, there are at least three reasonable contenders for the value of the "sum" of Grandi's series and an assortment of possible methods for obtaining them. There was a long dispute amongst mathematicians of the eighteenth century concerning how to understand the "sum" of this series, though almost all agreed that the value of the sum should be $\frac{1}{2}$. Guido Grandi, after whom the series is named, provided a geometrical proof that the series summed to $\frac{1}{2}$.[109] Leibniz, in response, endorsed Grandi's solution and, furthermore, gave his own argument in favour of the value $\frac{1}{2}$. Leibniz argued that the partial sums, as we have seen, alternate between 0 and 1, so either value is equally probable. Therefore, neither value can be the whole sum of the series, so one should take the arithmetic mean $\frac{1}{2}$ as the correct solution.[110]

So, the problem with divergent series was that though notion of a "sum" was useful in the context of a divergent series, existing mathematics did not give a clear guide as to how to determine what the "sum" ought to be. There were clear considerations to be taken into account, but no definitive grounds to settle the dispute.

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[109] Guido Grandi, *Quadratura Circula et Hyperbolae per Infinitas Hyperbolas Geometrice Exhibita* (Pisis: ex typographia F. Bindi, 1703). p.29

[110] GMS, 5:386

When Cauchy provided the now familiar definition of convergence, he regarded the rejection of the "sum" of a divergent series a consequence of his adherence to a particular methodology by which he hoped to 'make uncertainty vanish'. However, he thought that this was a difficult pill to swallow in the effort to embrace a desirable formal methodology.[111] More than twenty years after Cauchy's *Cours d'analyse*, De Morgan in his textbook *The Differential and Integral Calculus*, still deplored the rejection of divergent series, warning that to discard a method on the basis of one or more erroneous results would be to reject large swathes of fruitful mathematics.[112]

Though Cauchy's decree that divergent series have no sum was accepted for a time, new methods of summation were later accepted, which allowed mathematicians to again make use of divergent series with a clear conscience. Now, instead of relying on intuitive notions of what the correct "sum" of a divergent series should be, mathematicians developed rigorous definitions of the "summability" of a series, usually relying on properties of the series' partial sums. The new rigorous methods often gave results that were in line with the intuitions of the mathematicians of the past. So, for example, the Cesàro summation of Grandi's series is indeed $\frac{1}{2}$.

[111] Cauchy writes: 'As for methods, I have sought to give them all the rigor that one requires in geometry, so as never to have recourse to the reasons drawn from the generality of algebra. Reasons of this kind, although commonly admitted, particularly in the passage from convergent series to divergent series, and from real quantities to imaginary expressions, can, it seems to me, only sometimes be considered as inductions suitable for presenting the truth, but which are little suited to the exactitude so vaunted in the mathematical sciences…. In order to remain faithful to these principles, I admit that I was forced to accept several propositions which seemed slightly had at first sight. For example … a divergent series has no sum.' Cauchy. p.ii cited in Giovanni Ferraro, *The Rise and Development of the Theory of Series up to the Early 1820s*, 2008. p.347.

[112] Augustus De Morgan, *The Differential and Integral Calculus* (London: R. Baldwin, 1842)., p.566.

4.5 Hardy and Divergent Series

The history of divergent series that I have sketched is typical. It is a characteristic example of mathematical progress achieved through the extension of a rule into an expanded domain. In the case of divergent series, the concept of a "sum" was extended into an infinite domain, and mathematicians were guided by their intuitions about how to understand first, the concept of an infinite sum and later, the "sum" of an infinite series that diverged. The outcome of this process can be understood as weighing on the one hand the requirements of consistency (hence Cauchy's reluctant dispensing of the notion of the "sum" of a divergent series in order to adhere to his new method) and on the other, the desire to preserve those portions of mathematics that have been found to be useful (hence the work on summability that allowed mathematicians to continue to employ divergent series in areas in which they had been found to be fruitful).

Mathematicians of the past were not seriously troubled by contradictions that arose in this process. The mathematicians who made use of divergent series, for example, proceeded despite concerns about the consistent use of these concepts. Israel Kleiner emphasizes that this putting aside of concern about the inconsistent underpinnings of their methods often stemmed from an understanding that the early stages of developing a new system required skirting contradiction while the new mathematical terrain was explored. He writes: 'It was not uncommon for mathematicians of the seventeenth and eighteenth centuries to resort to mathematical techniques which were at best questionable, often inconsistent.'[113] He notes that they usually recognized that their methods were unsatisfactory, but 'were willing to tolerate them because they yielded correct results.'[114] However, he argues, it is not merely successful results that have served as a justification for the use of unsatisfactory methods. It is also justified by the fact that mathematical progress *requires* an exploratory period during which contradiction frequently arises. He writes:

> Textbooks usually present the end product of mathematical activity, but of course before one can *prove* one has to *discover*. And the method of discovery of a given result may differ radically from its method of demonstration.[115]

Mathematicians of the eighteenth century were willing to suspend their concerns about consistency in the interest of exploration. We might think that the provisional nature of the extension of a system is a phenomenon of the history of mathematics and that mathematics *now* is not vulnerable to inconsistency in the same way, due to the program of rigorization that took place in the nineteenth century.

Hardy's diagnosis of why earlier mathematicians were less troubled by contradiction is that they believed themselves to be guided by intuitions about some inherent meaning of the concepts involved. Mathematicians *now*, according to Hardy, have developed the habit of definition which

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[113] Israel Kleiner, *Excursions in the History of Mathematics*, *Springer* (New York, 2013). pp.80-81

[114] Kleiner. p.81

[115] Kleiner. pp.80-81

allows them to avoid contradiction and confusion. Hardy views the "inherent meanings" that he thought played a role in the thinking of earlier mathematicians as a kind of unscientific naïveté.

Hardy claims that there was a fundamental difference in how mathematicians prior to Cauchy conceived of the work of the mathematician. In *Divergent Series*, he writes:

> [I]t does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, 'by *X* we *mean* Y'. There are reservations to be made…, but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1 - 1 + 1 - \cdots$?' but 'What *is* $1 - 1 + 1 - \cdots$?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.[116]

According to Hardy, De Morgan and his mathematical predecessors attempted to solve the problem of finding the "sum" of Grandi's series, by asking themselves "What *is* 1 – 1 + 1 – 1 + …?" This led them, according to Hardy, into 'unnecessary perplexities and controversies which were often verbal.'[117] According to Hardy's interpretation, the English analysts and mathematicians of earlier generations were led into confusion by dwelling on the question of what the "meaning" of a mathematical equation was in attempting to determine an answer about what its sum could be.

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[116] Hardy, *Divergent Series*. pp.5-6

[117] Hardy, *Divergent Series*, p. 6.

Hardy, by contrast, conceived of himself as skirting this kind of confusion by beginning with definitions. The *first* step, as he remarks, in making sense of the "sum" of a divergent series 'must be some definition, or definitions, of the 'sum' of an infinite series, more widely applicable than the classical definition of Cauchy.'[118] Given that he thinks the definition is the first step, it is not surprising that his attitude toward earlier generations of mathematicians who did not take this view is disparaging.

Hardy's picture of modern mathematics as obtaining meanings only through explicit acts of christening omits the reality that, despite Hardy's belief that they shouldn't be, mathematicians *are* guided in their investigations by some implicit promise of "meaning" in the concepts they investigate. The confidence in some meaningful sense of the "sum" of a divergent series in the face of contradiction seems to have stemmed from the fact that they had proved useful, coupled with the experience of similar computations that were eventually satisfactorily resolved. De Morgan's admonition not to discard portions of mathematics on the basis of 'one or more erroneous results' is a rejection of the formalist picture on pragmatic grounds. Since the criterion of legitimacy in the formalist picture of mathematics is consistency, one would expect that a contradiction under such a picture would signal the end of the line for a mathematical system. If mathematicians conceived of their field as primarily concerning the manipulation of symbols

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[118] Hardy, *Divergent Series*, p. 5.

which derive their meanings only through acts of explicit definition, when some such manipulation leads to contradiction we would expect them to discard the system that led to the contradiction, or to set to work to correct the starting principles that yielded the contradiction. Instead, we see the effort to preserve what seems like the "natural" meaning (Hardy himself refers to the value "$\frac{1}{2}$" as the *natural* value of the "sum" of Grandi's series). The actual project Hardy undertakes in *Divergent Series* is to rehabilitate the intuitive understanding of the concepts at play, preserving the meanings that those eighteenth-century mathematicians found latent in their computations. This project is strikingly at odds with the picture of mathematics that he endorses within it. Indeed, Hardy's picture of mathematics is full of tension. His description of mathematical proof as akin to observing a distant mountain range sits strangely with his rejection of the question "What *is X*?"[119]

Wittgenstein directly responds to this Hardy-type picture of the meaning of mathematical concepts. In response to this formalist picture, he claims that the intuitions about meaning that

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[119] Hardy's attempt to maintain both a Platonistic and formalist position was, according to Morris Kline, a fairly common response to the move to rigorization in mathematics. Kline argues that many mathematicians in the period immediately following the move to rigorization in mathematics embraced a Platonistic view of mathematics as a direct response to the acceptance of a formalist picture of mathematics as the study of arbitrary structures:

> The loss of truth and the seeming arbitrariness, the subjective nature of mathematical ideas and results, deeply disturbed many men who considered this a denigration of mathematics. Some therefore adopted a mystical view that sought to grant some reality and objectivity to mathematics. These mathematicians subscribed to the idea that mathematics is a reality in itself, an independent body of truths, and that the objects of mathematics are given to us as are the objects of the real world. Mathematicians merely discover the concepts and their properties. Kline, p. 1035.

guide our investigation of mathematical concepts are connected to, and flow from, meanings in ordinary life. However, Wittgenstein also claims that these meanings are tentative and we can be wrong in believing that they will always turn out to yield a determinate sense. I will discuss each of these two points in turn.

Wittgenstein seems to conceive of our mathematical language games as enveloped by our form of life, and as such, the meanings in mathematics are rooted in meanings outside of mathematics. For example, the concept of following a rule, according to Wittgenstein, is not a special mathematical activity, but forms part of our ordinary life:

> The concept of the rule for the formation of an infinite decimal is--of course--not a specifically mathematical one. It is a concept connected with a rigidly determined *activity* in human life. The concept of this rule is not more mathematical than that of: following the rule. Or again: this latter is not less sharply defined than the concept of such a rule itself.--For the expression of the rule and its sense is only a part of the language-game: following the rule.[120]

Wittgenstein understands the activity of following a rule as importantly part of our ordinary lives. So, he thinks, the criteria by which we judge whether a rule is followed or whether a rule is projected in a way that makes sense in our ordinary lives are related to those same aspects in the context of mathematical enterprise. Wittgenstein repeats this notion of mathematical concepts depending on meanings from our ordinary life in several places. For example, he writes that he

[120] RFM Part VII §42

wants to say that 'it is essential to mathematics that its signs are also employed in *mufti*.'[121] We must pay attention when Wittgenstein says he "wants to say" something and realise that he may be giving voice to a temptation rather than endorsing some particular statement, but in this case, the way he goes on makes it clear that Wittgenstein does endorse this thought: 'It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics.'[122]

A little later in the *Remarks on the Foundations of Mathematics*, Wittgenstein states that:

> Concepts which occur in 'necessary' propositions must also occur and have a meaning in non-necessary ones.[123]

Wittgenstein's insistence here on mathematical signs or activities having meaning outside of a mathematical enterprise could be reformulated in terms of a form of life. Mathematical practices are a part of our form of life and derive some form of meaning from their continuity with meaningful practices in ordinary life.

There is, of course, something quite different in the way in which mathematical terms get their meanings. We do have *the habit of definition* in the context of mathematics. But

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[121] Wittgenstein, *Remarks on the Foundations of Mathematics*. Part V, §2

[122] Wittgenstein, *Remarks on the Foundations of Mathematics*. Part V, §2

[123] Wittgenstein, *Remarks on the Foundations of Mathematics*. Part V, §41

Wittgenstein's point, I think, is that these definitions do not materialize as in a conjuring trick. The clean appearance of the chosen axioms might disguise a long and muddled process of fiddling with definitions until the resulting system proves satisfactory, and this process is itself guided by intuitions about the latent meanings of the concepts at hand. The formal explication is the end product of the investigation and is not possible prior to it. Wittgenstein's interest in pointing out the connections between meanings inside mathematics with meanings in our ordinary life is clear evidence that he wants to question the formalist picture of meaning ascription in mathematics.

5.0 De Morgan's Alternative

As many have discovered, Wittgenstein's position in the philosophy of mathematics resists simple articulation. There are, however, curious similarities between Wittgenstein's picture of mathematics as it opposes Hardy's formalist vision and the complicated attitudes articulated earlier by the nineteenth century British mathematician and logician Augustus De Morgan. I will make use of these similarities and the less cryptic nature of De Morgan's writing in order to give some order to Wittgenstein's remarks.

The surprising feature De Morgan and Wittgenstein have in common is an ambivalence about the "certainty" of a mathematical claim. On the one hand, clear cut rules of proof or calculation leading to a mathematical result seem required to grant this result its "certainty." Hardy's own picture of mathematics accords with this as we can see in his insistence on well-formed "definitions". On the other hand, mathematics gains many of its most surprising utilities by wandering into previously uncharted territories. For this reason, De Morgan wants his guiding rules of calculation to be broader than Hardy's.

We can deepen our understanding not only of formalism but also of De Morgan's alternative by examining the aspects of De Morgan's view against which Hardy gives some spirited formalist objections. While Hardy did not react to the criticism that I allege Wittgenstein lobbed at his formalist picture, he did respond to De Morgan who shares some important views with Wittgenstein.

De Morgan's picture of mathematics is complex, and I will not attempt to give a complete account of it. For our purposes, the two salient characteristics of De Morgan's view are an appreciation of the utility of rigorously formalizing the "rules" one uses in mathematics, combined with an appreciation of the unexpected manner in which the discovery of novel suggestions from calculation can lead to new arenas of mathematics in which one's old "rules' may be overturned in the new setting.

De Morgan's guiding principle—above all—is to preserve as much useful mathematics as possible, even when we do not yet have great confidence in our methods. In his textbook, *The differential and integral calculus*, De Morgan clearly articulates this policy with the warning that doing otherwise would have led to the loss for mathematics of both negative and complex numbers:

> The history of algebra shows us nothing is more unsound than the rejection of any method which naturally arises, on account of one or more apparently valid cases in which such a method leads to erroneous results. Such cases should indeed teach caution, but not rejection; if the latter had been preferred to the former, negative quantities, and still more their square roots, would have been an effectual bar to the progress of algebra… and those immense fields of analysis over which even the rejectors of divergent series now range without fear, would have been not so much as discovered, much less cultivated and settled.[124]

[124] De Morgan, *The Differential and Integral Calculus*, p. 566.

Yet, De Morgan also recognized the need for rigorous definitions, when available. The need to keep these two concerns in counterbalance can be dramatically illustrated by the methodological difficulties that English mathematicians faced in the early part of the nineteenth century. De Morgan inhabited a time of upheaval in mathematics. British mathematicians of De Morgan's period were grappling with questions about algebra that would lead to a fundamental change in our understanding of numbers and abstraction.

5.1 The British Algebraists

By 1800, mathematicians had started to worry about the looseness of some concepts within mathematics. Hardy identifies the "Cambridge Symbolists" as the dominant school of English analysts in the first half of the nineteenth century. He counts amongst these Woodhouse, Peacock and D. F. Gregory. In 1812, a group of undergraduates at Cambridge—including Peacock—founded the Analytical Society, the purported goal of which was to introduce Leibnizian notation into English mathematics.[125] The idea for forming the group was Babbage's and was initially conceived as a kind of joke. Slegg told Babbage about a controversy concerning the establishment of the Cambridge Auxiliary of the British and Foreign Bible Society. The recently established

[125] The original group consisted of Michael Slegg, Charles Babbage, Edward Bromhead, George Peacock, Richard Gwatkin, John Herschel, John Whittaker and Henry Wilkinson.

Society had prompted a debate about whether the Bible should be distributed without commentary or accompanied by the prayer book. In his autobiography, Babbage states:

> Cambridge was agitated by a fierce controversy. Societies had been formed for printing and circulating the Bible. One party proposed to circulate it with notes, in order to make it intelligible; whilst the other scornfully rejected all explanations of the word of God as profane attempts to mend that which was perfect.
>
> The walls of the town were placarded with broadsides, and posters were sent from house to house. One of the latter form of advertisement was lying upon my table (...) Taking up the paper and looking through it, I thought it, from its exaggerated tone, a good subject for parody.
>
> I then drew up the sketch of a society to be instituted for translating the small work of Lacroix on the *Differential and Integral Calculus*. It proposed that we should have periodical meetings for the propagation of D's; and consigned to perdition all who supported the heresy of Dots. It maintained that the work of Lacroix was perfect that any comment was unnecessary.[126]

Babbage's talk of "D's" and "Dots" refers to the conflict between the use of the Leibnizian and the Newtonian calculus notation. One distinguishing feature of the Cambridge Symbolists (or the "'f(D)' school of analysis", as Hardy refers to them) is a focus on mathematical notation. The purpose of translating Lacroix's work was in part intended as a way of bringing Leibnizian notation to Cambridge, which the group viewed as superior.

The purpose of bringing Leibnizian notation to Cambridge was to make available to British mathematicians the fruits of some questionable algebraic manipulations. Richards writes:

> This effort at changing the symbols was seen as important not because it would allow easier transmission of Continental results to England, but because it promised to strengthen British

[126] Charles Babbage, *Passages from the Life of a Philosopher* (London: Longman, 1864), pp. 28–29. as quoted in Menachem Fisch, 'The Making of Peacock's Treatise on Algebra: A Case of Creative Indecision', *Archive for History of Exact Sciences*, 54.2 (1999), 137–79 (p. 139) <https://doi.org/10.1007/s004070050037>.

> mathematical research. Algebraic manipulations of the Continental symbology of dy/dx and its inverse $\int y\, dx$ had long proved suggestive of new results. Although it was widely acknowledged that such practices as reaching results by "multiplying both sides by dx" could not be rigorously justified, they were remarkably effective. Much of the fertility of eighteenth-century Continental mathematics sprang from the suggestive powers of the Leibnizian symbology. A large part of the Promethean program of the Analytical Society involved making this symbolical power accessible to their compatriots.[127]

5.2 Negative and Complex Numbers

In this period, there was perceived to be a problem within mathematics concerning the status of negative and imaginary numbers. Arithmetic was supposed to be the science of quantity and, as such, was concerned only with the real and non-negative. Since "quantity" was taken to be the subject matter of algebra, operations in unknowns such as "$a - b$", must include the proviso that "$a \geq b$". The notion of quantity invoked here is the one we experience in our everyday dealings with the world. If one has three apples, one cannot take four apples away. Mathematicians wished to make use of more abstract operations within algebra, but it was unclear in what sense such operations could be legitimate. The uncritical manipulation of symbols in algebra had proved fruitful, but since the "meaning" of algebra was supposed to derive from its subject matter of quantity, it was felt that there was a lack of ground for the truth or meaning in the mere operations of symbols.

[127] Joan L. Richards, 'Augustus De Morgan, the History of Mathematics, and the Foundations of Algebra', *Isis*, 78.1 (2012), 6–30 (p. 11).

Some British mathematicians of the period endorsed doing away with the problematic numbers altogether. William Frend, a Cambridge mathematician slightly older than Peacock (and De Morgan's father-in-law) was one of the English figures who proposed that the proper response to this perceived difficulty was to banish negative numbers from algebra entirely. William Frend intended his mathematical text *The Principles of Algebra* to be an antidote to what he conceived of as the confused state of algebra instruction at the time.[128] In the preface to his textbook, he states that 'the first error in teaching the principles of algebra is obvious on perusing a few pages only in the first part of Maclaurin's Algebra.'[129] The problem, according to Frend was that Maclaurin included negative and imaginary numbers in his text. Frend objected to the inclusion of these on the grounds that they involve ideas that are not adequately clear and distinct. Frend tells us that '[t]he ideas of number are the clearest and most distinct in the human mind' and that 'the acts of the mind upon them are equally simple and clear.'[130] We can tell, according to Frend, that something is amiss with the very idea of a negative number since in attempting to explain these ideas, Maclaurin resorts to 'allusions to book-debts and other arts.'[131] Frend asserts that 'when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject.'[132]

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[128] William Frend, *The Principles of Algebra* (London: printed by J. Davis, for G. G. and J. Robinson, 1796).

[129] Frend. p.x

[130] Frend. ix-x

[131] p.x

[132] p.x

Frend was a Lockean who thought that the use of metaphor in explaining negative numbers showed that the ideas were not adequately clear and distinct. He believed that since numbers are the most clear and distinct ideas we have, any confusion about them indicates some deficiency in our reasoning. He states that concerning numbers there 'cannot be confusion in them, unless numbers too great for the comprehension of the learner are employed, or some arts are used which are not justifiable'.[133]

The British algebraists of the generations following Frend expanded mathematics in a variety of ways that violated his proto-formalist scruples, and the fruitfulness of this expansion casts doubt on the idea that mathematics should be limited by such scruples in the first place. Most mathematicians of the period were not as willing to discard mathematical concepts that seemed useful. Pycior, in her paper 'George Peacock and the British origins of abstract algebra', argues that the move in favour of keeping negative numbers was largely about their utility:

> Although admitting the absence of a satisfactory definition of the negatives, most argued in favour of their retention, if only because of practical considerations.[134]

As Pycior's title suggests, this attempt to keep the problematic negatives and imaginaries was critical to the genesis of abstract algebra. The English analysts were at once trying to bring rigorous continental methods to Britain and also open to the use of not fully regimented methods.

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[133] Frend. p.ix-x

[134] Helena M. Pycior, 'George Peacock and the British Origins of Symbolical Algebra', *Historia Mathematica*, 8.1 (1981), 23–45 (p. 29) <https://doi.org/10.1016/0315-0860(81)90003-3>.

This is clearly visible in the distinction they sought to draw between "arithmetical" and "symbolical" algebra. Peacock and the other British analysts associated with the Analytic Society attempted to make sense of these concepts through this division of algebra into two separate entities: arithmetical and symbolical algebra. The first—"arithmetical algebra"—was constrained by manifestations of quantity in the real world in just the way that Frend thought that algebra itself was constrained. The second— "symbolical algebra"—was unbound by any restrictions beyond consistency. He writes:

> Algebra may be considered, in its most general form, as *the science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws* : for we may *assume* any laws for the combination and incorporation of such symbols, so long as our assumptions are independent, and therefore not inconsistent with each other.

However, though symbolical algebra *could* be interpreted in any way, Peacock argues that its meanings should be drawn from those in arithmetical algebra since that condition would assure us that it will be useful. Peacock writes:

> [I]n order, however, that such a science may not be one of useless and barren speculations, we choose some subordinate science as the guide merely, and not as the foundation of our assumptions, and frame them in such a manner that Algebra may become the most general form of that science, when the symbols denote the same quantities which are the objects of its operations : and as Arithmetic is the science of calculation, to the dominion of which all other sciences, in their application at least, are in a greater or less degree subject, it is the one which is usually, because most usefully, selected for this purpose.[135]

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[135] George Peacock, *A Treatise on Algebra*, 1830, pp. 71–72.

Peacock's idea is that while the meanings of the terms and operations in arithmetical algebra determine the rules, symbolical algebra gains its meanings *from* the rules. He writes:

> In arithmetical algebra, the definitions of the operations determine the rules; in symbolical algebra, the rules determine the meaning of the operations, or more properly speaking, they furnish the means of interpreting them.[136]

Peacock and the other British analysts trusted that the unconstrained algebraic manipulation of symbolical algebra could yield something sensible. This trust stemmed from the combination of a belief in the meaningfulness of arithmetical algebra given its ties to the empirical world of quantities, and a faith that the manipulation of symbols whose potential meanings rested on these empirical notions would lead us to something fruitful.

Mark Wilson, in his book *Wandering Significance* finds a similar thread in the writings of Boole and Venn, who were active at Cambridge in roughly the same period. He writes:

> John Venn explains that we should regard Boole's inverse combinations (''$A\ /(1-B\ )$'') as ''interrogative hints'': ''Would you, perhaps, care to use us?"[137]

[136] George Peacock, 'Report on the Recent Progress and Present State of Certain Branches of Analysis', in *Report of the Third Meeting of the British Association for the Advancement of Science*, 1833, pp. 185–352 (p. 200).

[137] Mark Wilson, *Wandering Significance: An Essay on Conceptual Behavior* (Oxford: Clarendon Press, 2006), p. 543.

Wilson goes on to quote Venn's description of the way that the symbols of mathematics can offer suggestions to the mathematician:

> We might conceive the symbols conveying the following hint to us: Look out and satisfy yourselves on logical grounds whether or not there be not an inverse operation to the above. If you can ascertain its existence, then there is one of our number at your service to express it. In fact, having chosen one of us to represent your logical analogue to multiplication, there is another which you are bound in consistency to employ as representative of your logical analogue to its inverse, division,—supposing such an operation to exist.[138]

The British algebraists succeeded in pivoting our whole conception of meaning within mathematics. Pycior describes this shift in the conception of meaning in algebra as the revolutionary move that led the way to the development of abstract algebra. She writes:

> In short, in lieu of the definitions of symbols and signs demanded by his predecessors, Peacock used laws of combination to determine the content of symbolical algebra, thus shifting the emphasis in algebra from the meaning of symbols and signs to the laws of operation. It was this aspect of his work, similar in spirit and effect to the introduction into physics during the Scientific Revolution of concern for the "how" rather than the "why", upon which Augustus De Morgan, Duncan F. Gregory, George Boole, and other British pioneers of mathematics constructed modern abstract algebra during the second third of the 19th century.[139]

It is clear that had the British algebraists followed Frend in discarding the negative and complex numbers, this revolutionary advance within algebra would not have taken place the way it did.

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[138] John Venn, *Symbolic Logic* (London: MacMillan and Co., 1894), p. 74. as quoted in Wilson, p. 543.

[139] Pycior, p. 36.

5.3 De Morgan's Conception of Mathematics

This is the background against which De Morgan approaches the study of divergent series. De Morgan, like Venn and other British algebraists, was impressed by the suggestive power of notation within mathematics. According to De Morgan, the *language* of mathematics contributes its own suggestions about possible future paths. In his 1828 speech at the University of London, De Morgan states:

> I might proceed to show that this novel system of writing, this compendious language (for such in fact it is) contains, in its very formation, the germ of the most valuable improvements which the mathematics have ever received, and has been from its peculiar structure, a never failing guide to new discoveries.[140]

This "compendious language", according to De Morgan, contains within it suggestions of future expansions of mathematics. And furthermore, we are right to trust that manipulations of mathematical notation will lead us somewhere interesting. De Morgan continues:

> I might lay before you the opinions of men the most distinguished for acuteness of reasoning as well as fertility of imagination, in support of the assertion that the study of this language, without reference to any of its applications, is instrumental in furnishing the mind with new ideas, and calling into exercise some of the powers which most peculiarly distinguish man from the brute creation.[141]

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[140] Augustus De Morgan, 'An Introductory Lecture Delivered to the Opening of the Mathematical Classes in the University of London Nov. 5th, 1828', *Mathematical Intelligencer*, 28.3 (2006), 19–28 (p. 19) <https://doi.org/10.1007/BF02986880>.

[141] De Morgan, 'An Introductory Lecture Delivered to the Opening of the Mathematical Classes in the University of London Nov. 5th, 1828', p. 19.

De Morgan endorsed Peacock's approach to the difficulty posed by negative and complex numbers which led to the development of abstract algebra. His views also aligned with the other algebraists who sought to make cautious, exploratory use of questionable, but fruitful mathematical methods. However, De Morgan's views were in some respects more radical than his fellow algebraists.

De Morgan's picture of mathematics is of a subject matter that is deeply intertwined with life. De Morgan held to the view that the historical, philosophical, and notational context of mathematical thinking contributed crucially to our understanding of the resulting mathematics.

In his Presidential address to the newly formed London Mathematical Society, De Morgan offered his ideas about which directions he thought ought to be pursued by the mathematicians of that group. He clearly articulated his view concerning the importance of the history of mathematics:

> The History of Mathematics is another thing that is unfairly neglected. I say that no art or science is a liberal art or a liberal science unless it be studied in connection with the mind of man in past times. It is astonishing how strangely mathematicians talk of the Mathematics, because they do not know the history of their subject. By asserting what they conceive to be facts they distort its history in this manner. There is in the idea of every one some particular sequence of propositions, which he has in his own mind, and he imagines that that sequence exists in history; that his own order is the historical order in which the propositions have been successively evolved. The mathematician needs to know what the course of invention has been in the different branches of Mathematics; he wants to see Newton bringing out and evolving the Binomial Theorem by suggestion of the higher theorem which Wallis had already given. If he be to have

his own researches guided in the way which will best lead him to success, he must have seen the curious ways in which the lower proposition has constantly been evolved from the higher.[142]

De Morgan thought that mathematicians wrongly suppose that the historical development of mathematical ideas mirrors their own mathematical education and this misunderstanding can lead to a fundamental misunderstanding of the mathematical ideas themselves since—he thinks—a full understanding requires an extensive understanding of the true historical background.

We can see the radically contextual attitude that De Morgan takes to mathematics in a letter to his friend, the Irish mathematician William Rowan Hamilton. De Morgan, fancifully, said:

> In reading an old mathematician you will not read his riddle unless you plough with his heifer; you must see with his light if you want to know how much he saw.[143]

This remark references a strange biblical story about an impossible riddle Samson gives to his wife's people, the Philistines.[144] In the story, Samson kills a lion with his bare hands and then some time later returns to the lion's body to find that bees have nested inside its carcass. Samson eats some of the honey from the lion. He later sets a riddle for the relatives of his new wife:

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142 Augustus De Morgan, 'Speech of Professor De Morgan', *Proceedings of the London Mathematical Society*, 1, 1–9 (p. 6).

143 Quoted in Richards, p. 9..

144 The story takes place in Judges Chapter 14.

“Out of the eater, something to eat; out of the strong, something sweet.”[145]

The riddle is so obscure (who would think of honey inside a lion’s carcass?), that when the Philistines solve the riddle, Samson realizes that his wife must have given them the solution. Samson says:

“If you had not ploughed with my heifer, you would not have solved my riddle.”[146]

“Ploughing with my heifer” here is an accusation that the Philistines have had access to inside information (via Samson’s wife). De Morgan is claiming that in order to understand another’s mathematical writing, one must immerse oneself in his or her perspective. One must—according to De Morgan—‘see with his light’ if you want to know how much a mathematician understood. And although we may reasonably wonder how De Morgan would prefer us to cash out the details of this metaphor in clarifying his strongly contextual view of mathematics, it is clear that if one must plough with a heifer, mathematics cannot be like a distant mountain range or an unchanging city. If one must see another mathematician’s mathematics ‘with his light,’ then that mathematics varies with the very light it is seen by.

145 *The Bible* (New International Version: Zondervan, 2011) Judges 14:14..

146 *The Bible* Judges 14:18..

De Morgan considered an interest in meta-mathematical reasoning as the essential part of the work of the mathematician. Indeed, he seemed to think that reflection *about* mathematics was appropriate for even the beginner student. De Morgan was an active member of the Society for the Diffusion of Useful Knowledge, which published the *Penny Cyclopaedia*.[147] The *Penny Cyclopaedia* was intended as a cheap and accessible source of information for the general public.[148] De Morgan authored an astonishing *six hundred* individual entries of the encyclopaedia and despite its target audience being amateurs, he did not confine himself to elementary topics. As Joan Richards writes:

> He felt that the problems that posed the major difficulties for the student learning mathematics were just the problems that lay at the foundations of the subject itself. His attempts to clarify mathematics for beginners parallel his struggles to formulate a view of it for himself. A contemporary map of the British mathematical landscape is clearly laid out in De Morgan's popular tracts.[149]

For De Morgan, there was no clear division between mathematics proper and reflection *about* mathematics and both were (or ought to be) open to any thinker.

As I have mentioned, De Morgan's father-in-law William Frend was emphatic that negative numbers should be excluded from algebra. De Morgan himself was sympathetic toward

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[147] *Penny Cyclopaedia and Supplement, 29 Volumes*, ed. by Charles Knight (London: Charles Knight and Co.).

[148] Herman Melville apparently consulted the *Penny Cyclopaedia* extensively in writing *Moby-Dick* see P. McCarthy, 'Forms of Insanity and Insane Characters in Moby Dick.', *The Colby Library Quarterly*, 7.3 (1987), 39–51 (p. 39). for further details.

[149] Richards, p. 10.

this view for a time and even after he had reconciled himself to the use of negative numbers, De Morgan still took the attitude that the question of what ought to be included in the study of mathematics should be examined *within mathematics*. That is, the work of the mathematician includes examining our criteria concerning what constitutes "legitimate mathematics". Thus, Hardy's criticism of De Morgan's paper on divergent series, the main goal of which is to examine the history and main issues in the study of divergent series, would likely have left De Morgan completely unconcerned. His goal was to consider divergent series' place within mathematics and his general attitude was that we ought not to exclude any potentially useful portion of mathematics from use. His goal was not to propose formal definitions. Since De Morgan conceived of the meta-mathematical tasks as a legitimate part of mathematics, focusing on those aspects would not have been to "miss the essential points" from his point of view.

We thus find in De Morgan's work a conception of mathematics in which a concern for meta-mathematical issues about the meaning and application of mathematical language is essential to legitimate mathematics and is not dispensable "gas." De Morgan developed a view that rejected the limitations that Frend's scruples would have imposed on mathematics while at the same time integrating into the core of mathematics Frend's meta-mathematical practice of reflecting on what mathematical expressions mean in a way that is guided by a concern for the historical development of mathematics as a way of addressing certain specific practical purposes. In contrast, we shall see below that Hardy did the opposite: he retained Frend-style scruples about legitimate mathematics but expelled meta-mathematical reflection from the core of mathematics.

5.4 Hardy's Criticism

Hardy states that the British writings in the period 1840-50 'show a singular and often entertaining mixture of occasional shrewdness and fundamental incompetence.'[150] Hardy does, however, describe De Morgan as an 'ingenious writer, both on logic and on mathematics'[151] and states that his book *Differential and integral calculus*[152] is 'the best of the early English text-books on the calculus' and that it contains 'much that is still interesting to read and difficult to find in any other book,'[153] but his overall assessment of De Morgan is that he failed to make any substantial progress with divergent series.

We might suppose that Hardy's criticism of these earlier mathematicians would be directed at some kind of deficit of "rigour" since Hardy was a central figure in the move towards new rigorous methods in mathematics in England in the early part of the twentieth century. His fixation on certain kinds of definition might be understood as a concern to instil rigour in the study of divergent series. While mathematics in Germany and France had long embraced the new rigorous definitions, mathematics in England was still in the thrall of Newton. The modern formal

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[150] Hardy, *Divergent Series*. p.18

[151] Hardy, *Divergent Series*. p.19

[152] De Morgan, *The Differential and Integral Calculus*.

[153] Hardy, *Divergent Series*. p.19

definitions were slow to make inroads. Hardy's textbook *A Course of Pure Mathematics*,[154] which was first published in 1908, was part of a change in how mathematics was taught at Cambridge. In that text, Hardy gives the 'first rigorous English exposition of number, function, limit, and so on, adapted to the undergraduate, and thus it transformed university teaching' [155].

However, the complaints Hardy levels against mathematics of the past is not that it was insufficiently rigorous. Hardy is explicit that in their study of divergent series, earlier mathematicians were not lacking in "rigour", but had rather a deficit of what he calls "technique". Hardy describes it as a 'habit of mind'.[156] Hardy has in mind by the term "technique", the giving of a particular kind of definition.

Hardy's evaluation of De Morgan's work is that he was a talented mathematician, but too susceptible to the poor influence of the mathematics of his time. Considering De Morgan's paper on divergent series, Hardy remarks that in it De Morgan 'attempts a reasoned statement of his attitude toward divergent series.'[157] However, Hardy's evaluation is that De Morgan's attempt is

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[154] Hardy, *A Course of Pure Mathematics*.

[155] J. C. Burkill, 'Hardy, Godfrey Harold', *Complete Dictionary of Scientific Biography* (Charles Scribner's Sons, 2008), pp. 113–14.

[156] Hardy, *Divergent Series*. p.6

[157] Hardy, *Divergent Series*, p. 19.

unsuccessful. He states: 'He talks much excellent sense, but the habits of the time are too strong for him: logician though he is, he cannot, or will not, give definitions.'[158]

Hardy's critical attitude toward De Morgan is shaped by his view of the sharp distinction between "legitimate mathematics" where this *begins* with the laying down of formal definitions, and "gas". De Morgan's discussion of divergent series in this paper would be categorised in its entirety as "gas" by Hardy's lights, and thereby inessential. De Morgan, for his part, does not even set out to give definitions.

De Morgan begins his paper by noting that divergent series is 'the only subject yet remaining, of an elementary character, on which a serious schism exists among mathematicians as to absolute correctness or incorrectness of results.'[159] According to De Morgan, this disagreement amongst mathematician is a sign that error is likely if caution is not used:

> When such a question arises upon a method of pure mathematics, there can be little doubt that it must be one which is likely to lead to error if not cautiously used ; and it is probable that the contending parties have not made any close agreement upon the use of terms. A review of the leading points of the controversy may be useful, accompanied by an examination of the maxims which have been adopted, but I think not very plainly stated, in the rejection of the series called divergent.[160]

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[158] Hardy, *Divergent Series*, p. 19.

[159] Augustus De Morgan, 'On Divergent Series, and Various Points of Analysis Connected with Them', *Transactions of the Cambridge Philosophical Society*, 8.Part II (1849), 182–202.. P.182

[160] De Morgan, 'On Divergent Series, and Various Points of Analysis Connected with Them', p. 182.

De Morgan's solution to this risk of error is to canvas the various positions taken regarding divergent series. His object is to evaluate the various considerations of different approaches one might take to formalisation of concepts in the study of divergent series. However, since he does not give the actual formal definitions, Hardy's conclusion is that De Morgan misses the essential points.

Hardy's criticism demonstrates a schism between his picture of mathematics and De Morgan's own. While Hardy has a simple picture of either 'legitimate mathematics' or 'gas', De Morgan has a complicated picture of mathematics that is deeply entwined with our lives and the world.

5.5 Hardy and William Frend

There is a parallel to Hardy's position that he would doubtlessly find unflattering: William Frend. There is a common thread here between Frend's way of thinking and Hardy's. While Hardy would not have countenanced Frend's rejection of the negative numbers, there is a connection between their notion that what counts as legitimate mathematics has precise and definable criteria. While Frend's criteria invoke the notion of clear and distinct ideas, Hardy's involves a definition of some kind.

Both Hardy and Frend want to lay down scruples for all time limiting what counts as legitimate mathematics. Nevertheless, the British algebraists of the generation following Frend expanded mathematics in a variety of ways that violated his proto-formalist scruples, and the fruitfulness of this expansion casts doubt on the idea that mathematics should be limited by such scruples in the first place. We can use the work and metamathematical thought of Augustus De Morgan as a particularly illustrative example of a position that suggests a potentially plausible alternative to formalism.

5.6 Asymptotic Series

The limitations of Hardy's position are clearly revealed in the particular criticism he levels at De Morgan's discussion of alternating series. According to Hardy, merely canvasing attitudes toward the status of an area of mathematics and describing the various considerations at issue is to fail to contribute to mathematics. For example, De Morgan's demonstration that a useful property of some alternating series is not true of *all* alternating series invites something approaching scorn from Hardy.

However, Hardy omits in his survey of the history of divergent series the progress that had been made through the use of series expansions that did not converge to real values in a Cauchy-like manner. Some mathematicians continued to entertain divergent series despite grave misgivings. For example, Abel who famously said, 'divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever'[161] continued in his 1826 letter to Holmboë to say:

> That most of these things are correct in spite of that is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

Hardy, quoting De Morgan, writes:

> 'When an alternating series is convergent, and a certain number of its terms are taken … the first term neglected is a superior limit to the error of approximation … This very useful property was observed to belong to large classes of alternating series, when finitely or even infinitely divergent: I do not remember that anyone has *denied* that it is universally true …' De Morgan shows by examples that it is not, but without making any substantial contribution to the subject. Indeed these supplementary discussions merely confirm the impression left by the earlier sections of the paper, of astonishment that so acute a reasoner should be able to say so much that is interesting and yet to miss the essential points so completely.[162]

De Morgan's demonstration, according to Hardy, does not constitute a 'substantial contribution to the subject' and is merely further confirmation that De Morgan missed 'the essential points' so completely. However, the kind of divergent series under discussion by De Morgan are just the set that brings the serious problems with Hardy's formalism into stark relief.

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[161] *Œuvres,* 2, 256. Quoted in Kline, p. 973.

[162] Hardy, *Divergent Series*, p. 20.

Hardy's book *Divergent Series* succeeds in giving a formal account of many of the different methods of summation of divergent series. However, the types of series that he examines in that work are merely a subset of the possible divergent series and he neglects to mention another kind of subset that is both useful and resists Hardy-type formalization.

Asymptotic series are a variety of divergent series that have resisted the kind of formalist consolidation we usually expect in mathematics, but they are of fundamental importance in some mathematical contexts such as the explanation of the light distribution in rainbows. Asymptotic series provide useful methods of approximation when other methods are computationally intractable. However, rather than providing mere useful approximations, asymptotic series can sometimes be explanatory and can even be essential for a full understanding of the phenomena under investigation.[163] The example of asymptotic series suggests that the formal consolidation that we envisage as necessary to meet modern standards of mathematical rigour is not always available. In this section, I will discuss an episode from the history of mathematics concerning the optical properties of rainbows as an example that illustrates this point.

In certain conditions, in addition to the usual arc of a rainbow, some additional, fainter bands are visible below the rainbow. These bands alternate between pink and green tones and fade

[163] Robert W Batterman, '`Into a Mist': Asymptotic Theories on a Caustic', *Studies in History and Philosophy of Modern Physics*, 28.3 (1997), 395–413.

in brightness away from the primary bows. These additional bands are known as “supernumerary rainbows.” Unlike the spectrum of colours in the main bow, the supernumerary rainbows can not be explained using classical geometric optics. Supernumerary rainbows are therefore historically significant because they provided evidence in favour of the wave theory of light in the early nineteenth century.

The main arc of a rainbow can be explained using a ray theory of light, in which light travels along rays that are refracted or reflected by water droplets. The supernumerary rainbows, however, are explained using a wave theory of light. The alternating bands of pink and green and the dark bands between them are explained by interference between the waves of light reflected or refracted in raindrops on slightly different paths and with different wavelengths.

Mathematical investigations involving the main arcs of a rainbow had been performed by Newton and Descartes in the eighteenth century. In 1838, Airy derived an equation from the wave theory of light from which one can determine properties of the supernumerary rainbows.[164] The Airy equation is defined by an oscillatory definite integral. Airy succeeded in calculating the integral for various values, but the calculations required were impractically time-consuming.

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[164] George Biddell Airy, ‘On the Intensity of Light in the Neighbourhood of a Caustic’, *Transactions of the Cambridge Philosophical Society*, 6 (1838), 379–403.

The mathematician Stokes supplied a method for approximating the Airy integral, which made use of divergent asymptotic series. Stokes' method involved forming a differential equation of which Airy's integral is a particular solution and solving the differential equation in terms of divergent series that could be used for calculation. Stokes found that by using only the first few terms of the series, he could get an excellent approximation, despite the series being divergent.[165] Michael Berry writes:

> This *Stokes phenomenon*, connecting different exponentials representing the same function, is central to our current understanding of such divergent series, and is the feature that distinguishes them most sharply from convergent ones. In view of this seminal contribution, it is ironic that George ('G H') Hardy makes no mention of the Stokes phenomenon in his textbook 'Divergent series'. Nor does he exempt Stokes from his devastating assessment of 19th century English mathematics: "there [has been] no first-rate subject, except music, in which England has occupied so consistently humiliating a position. And what have been the peculiar characteristics of such English mathematics . . .? . . . for the most part, amateurism, ignorance, incompetence, and triviality."[166]

Stokes' asymptotic representation provided a practical solution to the problem of calculations to do with supernumerary rainbows, as well as other phenomena. Indeed, according to Robert Batterman, the recalcitrance of asymptotic series to modern standards of mathematical rigour is no obstacle to their performing an *informative* role in our understanding of rainbows. Batterman argues that 'the asymptotic representation of the Airy integral is explanatory in a way that the convergent series *definition* of the function is not.'[167] He continues:

[165] Kline, pp. 1102–3.

[166] M. V. Berry and C. J. Howls, 'Divergent Series: Taming the Tails', in *A Half-Century of Physical Asymptotics and Other Diversions*, 2017, pp. 425–31 (p. 428) <https://doi.org/10.1142/9789813221215_0033>.

[167] Batterman, p. 406.

> This is because the asymptotic representation describes, *mathematically,* the dominant structures 'in' the function. By this I mean that such a representation makes explicit the fundamental characteristics of the solutions of the differential equation … by reference to structural features of the solutions in the complex plane-for instance, the locations of the Stokes and anti-Stokes lines… These characteristics in turn help to describe both qualitatively and quantitatively the observable features of the physical situation-in this case the rainbow, or fold caustic.[168]

Asymptotic series thus provide us with a very clear example of a bit of mathematics that—in spite of the difficulties that continue to prevent its rigorization—mathematicians have retained and continued to use, precisely because they continue to be more useful for various calculations than more well-behaved mathematical devices. All of this would be lost to us if we were to stick to Hardy's scruples—as indeed it *is* lost to Hardy, whose work on divergent series does not mention Stokes phenomenon.

[168] Batterman, p. 406.

6.0 Reinterpreting Wittgenstein

In this dissertation, I have argued that Wittgenstein adheres to a radical contextualist view of mathematics and that this position finds its most natural opponent in Hardy's formalist picture of mathematics. Once dislodged, the formalist picture of mathematical activity no longer serves as the background against which to understand the example of the wayward pupil. Against a more complex background, the wayward pupil can be reinterpreted as a potential figure of mathematical discovery. In this chapter, I will focus on Wittgenstein's remarks concerning river-beds in *On Certainty*.

In the first part of this chapter, I will describe Wittgenstein's view of the character of certainty about mathematics as it is revealed primarily in *On Certainty*. My assertion is that Wittgenstein held two apparently conflicting positions about mathematical certitude. On the one hand, he endorses the kind of certitude Hardy seeks in the form of rigorous calculus. One the other, he does not think that this approach solves the foundationalist difficulties we might hope it does. Instead, it is to be regarded as a valuable when available position that will not always be obtainable.

6.1 Wittgenstein's Ambivalence

In his discussions of mathematics, Wittgenstein expresses an ambivalence about the certainty of a mathematical claim. In the *Remarks on the Foundations of Mathematics*, Wittgenstein imagines the following dialogue with an interlocutor:

> Counting (and that means: counting like *this*) is a technique that is employed daily in the most various operations of our lives. And that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is inexorably insisted that we shall all say "two" after "one", "three" after "two" and so on.--But is this counting only a *use*, then; isn't there also some truth corresponding to this sequence?" The *truth* is that counting has proved to pay.--"Then do you want to say that 'being true' means: being usable (or useful)?"--No, not that; but that it can't be said of the series of natural numbers--any more than of our language—that it is true, but: that it is usable, and, above all, *it is used*.
>
> "But doesn't it follow with logical necessity that you get two when you add one to one, and three when you add one to two? and isn't this inexorability the same as that of logical inference?"--Yes! it is the same.--"But isn't there a truth corresponding to logical inference? Isn't it *true* that this follows from that?"--The proposition: "It is true that this follows from that" means simply: this follows from that.[169]

In this passage and the surrounding text, Wittgenstein resists the appeal to label the claim that the number "three" follows "two" *true*. He hesitates to agree that our way of counting is "true," but does agree that addition has the same inexorability as logical inference. His response to direct questions about the truth of a mathematical claim here and in both the *Philosophical Investigations* and *On Certainty* is to shift the focus of the interlocutor to the use or usefulness of a claim.

[169] Wittgenstein, *Remarks on the Foundations of Mathematics*. Part I, §4

So why does Wittgenstein say that a particular claim can be certified as the "right thing to do" given the prevailing rules of some calculus without thereby establishing that the claim fully represents a true proposition of properly descriptive nature? Wittgenstein seems to regard the certainty of a mathematical claim as a feature of the role that we give it in our practices. So, for example, in *On Certainty*, Wittgenstein states:

> *This* is how calculation is done, in such circumstances a calculation is *treated* as absolutely reliable, as certainly correct.[170]

We *treat* our calculation as certain and correct, rather than its being so. Yet, he is also willing to describe the propositions of mathematics as "incontrovertible", "inexorable."[171]

This focus on questions of the "correctness" of the rule-follower has left another aspect of Wittgenstein's discussion of rule-following largely unexplored: namely his interest in what I am inclined to label the rigidity of *the mathematics itself*. In particular, he repeatedly draws our attention to cases where following rules in a particular context leads one to unexpected places, often involving contradictions.[172] Wittgenstein makes several puzzling remarks that suggest that it is a persistent possibility that an established piece of mathematics will collapse into contradiction or will run into some other unforeseen fatal difficulty. He seems to suggest not merely that any

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[170] OC §39

[171] See for example RFM part I, §4 and OC §657

[172] See for example PI §125, RFM part III §§80, 84 & 85 and RFM part VII §§11, 12, 27 & 29 (Wittgenstein, *Remarks on the Foundations of Mathematics*.)

individual rule-follower may run into trouble, but that the rules themselves may end in confusion. Take for example, Wittgenstein's discussion of the movements of a machine in §§193-194 of the *Philosophical Investigations*. There, Wittgenstein draws an explicit comparison between the movements of a machine and the expansion of a series:

> We use a machine, or a picture of a machine, as a symbol of a particular mode of operation. For instance, we give someone such a picture and assume that he will derive the successive movements of the parts from it. (Just as we can give someone a number by telling him that it is the twenty-fifth in the series 1, 4, 9, 16, …)[173]

Yet, earlier in §193, when Wittgenstein discusses the possible movements of a machine, he includes the possibility of collapse:

> We talk as if these parts [of a machine] could only move in this way, as if they could not do anything else. Is this how it is? Do we forget the possibility of their bending, breaking off, melting, and so on? Yes; in many cases we don't think of that at all.[174]

Wittgenstein here draws a comparison between the potential movements of a mechanism, given the details of its construction, and the potential expansion of an algorithm (i.e. the actual calculations of a person). If we take Wittgenstein's analogy here seriously, then the possible collapse of the mechanism should correspond not to the individual failure of any particular rule-follower, but to the algorithm itself. The algorithm, like the machine, holds the promise of outputting a stream of expected results, but Wittgenstein seems to be suggesting that just as the

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[173] PI §193

[174] PI §193

machine gives us no guarantee that its working will be a success, so too the algorithm could fail us.

Given the upheaval in mathematics--in part arising from Russell's paradox--at the time, one might suppose that Wittgenstein's "scepticism" about current mathematics is a result of some loss of confidence in the consistency of mathematics. However, far from being excessively concerned about future contradictions in mathematics, Wittgenstein seems to suggest that contradictions should not preoccupy us in quite the way that they do. In the *Remarks on the Foundations of Mathematics* he remarks: 'Contradiction. Why just this *one* bogy? That is surely very suspicious.'[175]

Wittgenstein seems to conclude that had Russell's paradox not been found, it need not have rendered Frege's a false calculus in the remarks below:

> Let us suppose that the Russellian contradiction had never been found. Now--is it quite clear that in that case we should have possessed a false calculus? For aren't there various possibilities here?
>
> And suppose the contradiction had been discovered but we were not excited about it, and had settled e.g. that no conclusions were to be drawn from it. (As no one does draw conclusions from the 'Liar'.) Would this have been an obvious mistake?

[175] RFM Part IV §56

> "But in that case it isn't a proper calculus! It loses all *strictness*!" Well, not *all*. And it is only lacking in full strictness, if one has a particular ideal of rigour, wants a particular style in mathematics.
>
> 'But a contradiction in mathematics is incompatible with its application.
>
> 'If it is consistently applied, i.e. applied to produce arbitrary results, it makes the application of mathematics into a farce, or some kind of superfluous ceremony. Its effect is e.g. that of non-rigid rulers which permit various results of measuring by being expanded and contracted.' But was measuring by pacing not measuring at all? And if people worked with rulers made of dough, would that of itself have to be called wrong?
>
> Couldn't reasons be easily imagined, on account of which a certain elasticity in rulers might be desirable?
>
> "But isn't it right to manufacture rulers out of ever harder, more unalterable material?" Certainly it is right; if that is what one wants![176]

Yet, Wittgenstein does think that Russell's paradox is disturbing, as he states in an earlier passage:

> The Russellian contradiction is disquieting, not because it is a contradiction, but because the whole growth culminating in it is a cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly.[177]

Wittgenstein's point here is not to deny that Russell's paradox is disturbing, but to emphasize that it is disturbing to us only because of how it thwarts our ambition to use Frege's calculus in the way that we had hoped: as a basis for mathematical expression that conformed to a certain ideal of rigour. But there might be other bits of mathematics—or other ways in which we

[176] RFM Part VII §15

[177] RFM Part VII, §11

might hope to use Frege's calculus—in relation to which paradoxes would simply be less disturbing, perhaps even not disturbing at all, as in the case of the 'Liar'.

So how are we to make sense of this position that Wittgenstein occupies that seems to entertain grave doubts about the reliability of mathematics on the one hand and yet to adopt a nonchalant attitude towards contradiction—or to at least mock what he labels the 'superstitious dread and veneration by mathematicians in face of contradiction'[178] on the other? I propose that making sense of these two apparently conflicting attitudes will further illuminate Wittgenstein's opposition to the formalist picture of mathematics that I have ascribed to Hardy. Hardy, from Wittgenstein's perspective, is both too complacent about the extent to which our problems disappear through rigorous definition and too strict in denying potentially useful explorations the proper status of mathematics. Wittgenstein wants to preserve the possibility that our current conceptions of rigour may no longer seem desirable to us at some point. He also wants to entertain the possibility that adhering to our standards of rigour may not spare us from the possibility of unforeseen disturbances—paradoxes and contradictions—that will require more or less radical revision of the elements of our mathematical framework that currently seem most stable.

[178] RFM Part III, §17

6.2 Shifting River-beds

The key to understanding the ambivalence in Wittgenstein's thinking about the status of mathematical certainty is his radical contextualism. Wittgenstein radical contextualist position in its application to mathematical contexts resembles the position of De Morgan described in the previous chapter, where we saw that De Morgan regarded various features of the context in which mathematicians do their work as essential to mathematics rather than merely dispensable "gas." What we do not yet have is a clear model for how to understand this claim, and that is something we can get from the radical contextualist position developed by Wittgenstein, especially in *On Certainty*.

The basic features of this radical contextualist model are put in place in Wittgenstein's striking passage on river-bed truths:

> The propositions describing this world-picture might be part of a kind of mythology. And their role is like that of rules of a game; and the game can be learned purely practically, without learning any explicit rules.
>
> It might be imagined that some propositions, of the form of empirical propositions, were hardened and functioned as channels for such empirical propositions as were not hardened but fluid; and that this relation altered with time, in that fluid propositions hardened, and hard ones became fluid.
>
> The mythology may change back into a state of flux, the river-bed of thoughts may shift. But I distinguish between the movement of the waters on the river-bed and the shift of the bed itself; though there is not a sharp division of the one from the other.

> But if someone were to say "So logic too is an empirical science" he would be wrong. Yet this is right: the same proposition may get treated at one time as something to test by experience, at another as a rule of testing.[179]

Wittgenstein's puzzling remarks concerning riverbeds that shift reflect many of these same concerns with respect to certainty in mathematics. His description of propositions that harden and function as channels, only to become fluid again provides a potent metaphorical reflection of the description De Morgan gives with respect to progress in mathematics, where individuals, properties and relations solidify in a renewed system of mathematical endeavour, only to eventually themselves become the fluid moving parts of a yet more expanded system. Wittgenstein's radical contextualism, on which the sense of a term, expression or sentence depends on the context in which it is used, meets up here with a potent image for understanding how (at least) one of the things that can vary from one context to another is which propositions function as river-bed truths in that context.

In the *Philosophical Investigations*, Wittgenstein appeals to the notion of a form of life as the central location of human agreement:

> "So you are saying that human agreement decides what is true and what is false?" What is true or false is what human beings *say*; and it is in their *language* that human beings agree. This is agreement not in opinions, but rather in form of life.[180]

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[179] OC §§95-98

[180] Wittgenstein, *PI*. §241

Then, directly prior to his remarks concerning shifting riverbeds in *On Certainty*, Wittgenstein repeats this thought, but now emphasizes that this domain of agreement is not generally brought into question:

> But I did not get my picture of the world by satisfying myself of its correctness; nor do I have it because I am satisfied of its correctness. No: it is the inherited background against which I distinguish between true and false.[181]

The first quote makes it clear that Wittgenstein takes the notion of a form of life to be the location of agreement, and the second quote describes this form of life as a background against which we make a judgment, which is not itself in the ordinary course of things subject to interrogation. However, quickly following this remark, Wittgenstein describes his shifting river-bed metaphor which underlies our background picture of the world. So, according to Wittgenstein, this form of life is that background against which we make judgments, and this background itself is subject to alteration over time, though the shifting of the background is like the shifting of a river-bed and so, presumably, gradual and involving movement only around the edges at any one time. Something—perhaps most things—must be preserved as the propositions we take to be the hardened channels along which the fluid propositions flow slowly shift.

In the context of mathematics, although Wittgenstein does think that the "certainty" attaching to a mathematical conclusion depends on the result being reached via an established set

[181] Wittgenstein, *On Certainty*. §94

of rules ("calculus," "grammar"), he also thinks calculations that deviate from or even violate these procedures can sometimes open the door to new computational territory. In river-bed terms this means that sometimes the river-bed—the bedrock on which our language and our mathematical practice "flows"—can shift in a way that opens up the possibility of saying, knowing, or understanding new things.

The opacity of our future requirements of mathematical conceptions, or in our notions of rigour, is part of what Wittgenstein has in mind in his remarks concerning the shifting riverbed in *On Certainty*. I take this to be part of Wittgenstein's point with the river-bed analogy: that the axioms we endorse in any mathematical system are hardened into rules for the system and treated in a special way, but that what we now take to be the hardened rules might lead to future dissatisfaction and revision, should our endeavours change. Steve Gerrard labels the special status that Wittgenstein attributes to these river-bed propositions in mathematics 'the nonrevisability of mathematical statements.'[182] The "nonrevisability" refers to the way we treat certain statements in the process of doing mathematics. Certain propositions act as the hardened contours of the river-bed once they have been accepted as rules within mathematics. However, these propositions are not "nonrevisable" in the face of mathematical need.[183]

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[182] Gerrard, 'A Philosophy of Mathematics between Two Camps', p. 179.

[183] Gerrard, 'A Philosophy of Mathematics between Two Camps', p. 179.

I have argued that a formalist picture that takes it that mathematical concepts gain meaning through explicit acts of definition conceals the exploratory nature of projecting a rule and that Wittgenstein is interested to dislodge this picture of mathematics. There is a second misleading aspect to the formalist picture that Wittgenstein also challenges. This *habit of definition* can look like it provides a permanent answer to the search for meaning of our mathematical concepts. The formalist may think that we have a final answer in the form of modern standards of mathematical rigour. The new rigorous definition can be taken to be unassailable. While mathematicians of the past used to run into these kinds of difficulties, rigorization has led to an end to these problems, the thinking goes. This line of thought is clearly exhibited in Hardy's discussion of divergent series. However, there is nothing about having established a rigorous footing for a mathematical system guarantees that we have solved that problem for all time. In particular, we cannot know in advance to what purposes we will want to put our mathematical language in the future, just as we cannot know how we will want to project words from our ordinary language in the future. Wittgenstein draws this parallel in the *Philosophical Investigations* in the reverse direction. There, he takes it as clear that our mathematical concepts are not fixed and final, and uses that fact as a way of illuminating our picture of ordinary language:

> But how many kinds of sentence are there? Say assertion, question and command? There are *countless* kinds; countless different kinds of use of all the things we call "signs", "words", "sentences". And this diversity is not something fixed, given once for all; but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten.
>
> (We can get a *rough picture* of this from the changes in mathematics.)[184]

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[184] PI §23

We might have need of other mathematical concepts in the future, and furthermore, we have no guarantee that current forms of rigour will satisfy us in future enterprises.

On Wittgenstein's view, we should object to the notion that there is a final solution available to us. Moreover, even if it were possible to "harden" these propositions for all time, this would be a disastrous outcome from the point of view of progress in mathematics. Just as it would not be desirable to remove the possibility of finding new uses of our words in ordinary language, it would not be desirable to remove the possibility of projecting a rule into new, as yet unimagined, mathematical domains. Progress is made through the extension of rules. It is just this that gives mathematics its power, but it is also this fact that denies the formalist the mechanical guarantee that he is after.

Just as in ordinary life, in mathematics we encounter new needs, new objects, and hope to find new potencies in our existing mathematical concepts, and thus want a mathematics that will allow the new needs to be met and the new potencies to be exploited. Once our mathematics is extended in this sort of way, a new collection of results will be available for "certainty" in accordance with a rigorous application of its rules—but this collection too can become subject to change if another shift in the riverbed takes place.

Wittgenstein's radical contextualism says that the sense of a term, expression or sentence depends on the context in which it is used. That is why he thinks that Russell's Paradox might not have been so devastating if it had turned up in a different context where we were trying to use Frege's calculus in a different way.

6.3 Hardy Revisited

Let us consider again the contrast between Hardy and Wittgenstein's conceptions of mathematics. Whereas Hardy restricts the label "real mathematics" to the results of calculations that strictly conform to established rules ("definitions" he calls them), Wittgenstein and De Morgan recognize that certain achievements of the intellect that it would be dogmatic not to call mathematical—and essentially so—consist instead in finding a new way to consolidate various topics that have eluded mathematical handling by tinkering with established rules or definitions (for example, by interpreting them in a novel and perhaps surprising way).

Hardy describes the breakthrough of the modern mathematician as asking oneself not "What *is* X?", but "How shall we *define* X?" For Hardy, this change in emphasis of the characteristic question of the mathematician cleared away a great deal of confusion and led the way to an efficient, precise and above all "clear" approach to definition in modern mathematics.

Wittgenstein, as always, sceptical of the *appearance* of clarity, gives a more complex version of the characteristic question of the mathematician. He describes the guiding question of the mathematician as follows:

> "Do something which I shall be inclined to accept as a solution, though I do not know now what it will be like."[185]

Wittgenstein gives this description in a 1935 lecture where he considers the difference between the kind of mathematical problem one sets a child and the kinds of problems a professional mathematician tries to solve. The kind one sets a pupil are those 'which it gets an answer according to the rules it has been taught'[186] as one hopes will be the case in the example of the wayward pupil. However, the problems of the professional mathematician are quite different:

> But there are also those to which a mathematician tries to find an answer which are stated without a method of solution. They are like the problem set by the king in the fairy tale who told the princess to come neither naked nor dressed, and she came wearing fish net. That might have been called not naked and yet not dressed either. He didn't really know what he wanted her to do, but when she came thus he was forced to accept it. It was of the form, Do something which I shall be inclined to call neither naked nor dressed. It's the same with the mathematical problem. Do something which I shall be inclined to accept as a solution, though I don't know now what it will be like.[187]

So, we have Hardy's endorsed question:

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[185] Wittgenstein, *Wittgenstein's Lectures, Cambridge 1932–1935 From the Notes of Alice Ambrose and Margaret Macdonald*, p. 186.

[186] Wittgenstein, *Wittgenstein's Lectures, Cambridge 1932–1935 From the Notes of Alice Ambrose and Margaret Macdonald*, p. 185.

[187] Wittgenstein, *Wittgenstein's Lectures, Cambridge 1932–1935 From the Notes of Alice Ambrose and Margaret Macdonald*, pp. 185–86.

How shall we *define* X?

The question Hardy attributes to De Morgan:

What *is* X?

And Wittgenstein's description of the mathematician's characteristic task:

Do something I shall be inclined to call a solution to the problem X, though I don't know now what it will be like.

These three questions are clearly related, but the difference highlights subtle differences in their conceptions of mathematics, certainty, and truth. Hardy's characterisation may indeed seem to equate to Wittgenstein's. "How shall I *define* X?" perhaps implicitly conveys that the mathematician's way of answering this question will involve providing something that one would be inclined to call a solution to the question. However, Hardy's formulation is in fact an attempt to strip this sort of context from the mathematics. Hardy conceives of the question "How shall I define X?” as preparatory to doing mathematics, whereas Wittgenstein conceives of that question as the riddle in which the essential business of mathematics consists.

Their disagreement, I contend, is actually about what constitutes the activity of mathematics. According to Hardy, something is wrong or deficient if you are discussing considerations outside of formal definitions which delineate the mathematical landscape. According to De Morgan and Wittgenstein, we use a variety of considerations, but we should err on the side of keeping as many concepts or techniques as we have found to be useful.

6.4 The Wayward Pupil Revisited

One advantage of this interpretation of Wittgenstein’s understanding of mathematical activity is that it allows us to make sense of Wittgenstein’s reluctance to describe the wayward pupil as making a mistake. Wittgenstein gives a lengthy discussion of the concept of a “mistake” in *On Certainty*. He argues that there are transgressions that we will regard as “mistakes”, but that if the departure from what we expect is great, our response may be to exclude such behaviour or judgements from the realm of rational human behaviour. Wittgenstein gives the following example: ‘If my friend were to imagine one day that he had been living for a long time past in such and such a place, etc. etc., I should not call this a *mistake*, but rather a mental disturbance'.[188] There is a limit to what we are willing to call a mistake. We may, for example, be willing to call the erroneous belief that there is a table here in front of me a mistake, but be less sure of what to say in the case where 'I believe wrongly that I have seen this table, or one like it, every day for several months past, and have regularly used it'.[189] According to Wittgenstein, we presuppose a degree of agreement in our talk with other human beings. Not every mistaken belief is taken as evidence of mental disturbance, but we could say, 'In order to make a mistake, a man must already judge in conformity with mankind'[190].

So what does this agreement consist in? Wittgenstein states that this agreement is in *language*; that it is 'not agreement in opinions, but in form of life'.[191] This agreement must be 'not only in definitions, but also (queer as this may sound) in judgements.'[192] Our agreement is in how we view the world and ourselves. It is the whole edifice of our language, customs, institutions, and practices—our *form of life*. There are certain beliefs that we accord priority based on the way we have been inducted into a language-using society. We derive our picture of the world not through

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[188]OC §71

[189]OC §75

[190]OC §156

[191]PI §241

[192]PI §242

empirical investigation, but through giving our assent to the view in which we are imbued from infancy. So, Wittgenstein says:

> I did not get my picture of the world by satisfying myself of its correctness; nor do I have it because I am satisfied of its correctness. No: it is the inherited background against which I distinguish between true and false.[193]

According to Wittgenstein, when we meet with someone operating radically outside of these judgements, we cannot make sense of them. Wittgenstein conceives of Moore as stating the facts that are unshakeable in his conception of the world, as they are for most of us. Moore appeals to the propositions we do not ordinarily doubt as though this will somehow reinforce our commitment to them. But Wittgenstein's point is that on the one hand, it makes no sense to either assert them or deny them when we cannot conceive of an alternative, and on the other, our commitment to them is not something we debate and change in the way it would seem Moore is attempting. They are part of the system within which we make judgements. The system itself 'is not so much the point of departure, as the element in which arguments have their life.'[194]

Given Wittgenstein's conception of making a mistake coupled with his radical contextualism, the wayward pupil example in abstract does not provide enough information to determine whether or not the wayward pupil should be said to be making a mistake. Wittgenstein's

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[193]OC §94

[194]OC §105

reticence in passing judgment on the wayward pupil is explained by his belief that there is always the possibility that the circumstances surrounding a situation may shine on it a new light. The wayward pupil, who may be making an error according to our ordinary criteria of rule-following, may instead be initiating new and fruitful paths. To illustrate the kind of situation I am imagining, I will consider the example of the nineteenth century engineer Oliver Heaviside.

6.5 Heaviside

Heaviside was a self-taught electrical engineer who wrote extensively on electromagnetism and made extraordinary contributions to mathematics, though mathematicians at the time viewed his work as lacking in rigour. But Heaviside was driven by a practical need to efficiently elicit results from complex formulae for the purposes of aiding his research in electrical engineering, and for him, these practical concerns were always more important than any scruples about mathematical rigour. He introduced his operational calculus to allow the solution of ordinary differential equations which arose from the theory of electrical circuits. Heaviside's unconventional approach to algebraic manipulation was responsible for tremendous mathematical advances, but this was also the very feature of Heaviside's work that scandalized his mathematical contemporaries.

Heaviside viewed adherence to scruples of rigour as an obstacle to mathematical progress. He seems to have agreed with Wittgenstein's articulation of the characteristic activity of the mathematician. Rather than proceeding from definitions that fix the all the possibilities in advance, Heaviside thought mathematics should begin with a process of "experimentation":

> For it is in mathematics just as in the real world; you must observe and experiment to find the go of it . . . All experimentation is deductive work in a sense, only it is done by trial and error, followed by new deductions and changes of direction to fit circumstances. Only afterwards, when the go of it is known, is any formal explication possible. Nothing could be more fatal to progress than to make fixed rules and conventions at the beginning, and then go on by mere deduction. You would be fettered by your own conventions, and be in the same fix as the House of Commons with respect to the dispatch of business, stopped by its own rules.[195]

Interestingly, Wittgenstein expresses something very similar to this quote from Heaviside in his *Lectures on the Foundations of Mathematics*:

> It is like finding the best place to build a road across the moors. We may first send people across, and see which is the most natural way for them to go, and then build the road that way. Before the calculation was invented or the technique fixed, there was no right or wrong result.
>
> …
>
> If you say that you make an experiment to see what result the rules will lead him to, this is only an experiment so long as the rules do not prescribe what it has to lead him to-so long as there is not a right and wrong. We say of the child, not "He has followed the rules in this way" but "He has followed the rules."[196]

Both Heaviside and Wittgenstein claim that mathematicians may be guided by practice and "experimentation" in developing new rules or definitions. According to this picture of

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[195] Oliver Heaviside, *Electromagnetic Theory, Volume II* (London: The Electrician, 1899). p.33 quoted in Wilson. p.223.

[196] Diamond, *Wittgenstein's Lectures on the Foundations of Mathematics*, p. Lecutre X, 95.

mathematical discovery, the mathematician need not be constrained by mathematical convention in "finding the go of it." Just as in Wittgenstein's example of the fish net, Heaviside conceives of the work of the mathematician as finding something we will be inclined to call an answer to our question, without yet knowing precisely what kind of thing it might be.

Mark Wilson discusses Heaviside's unorthodox approach to the solution of differential equations in *Wandering Significance*. He gives an example of the kind of surprising algebraic manipulation Heaviside experimented with as follows:[197]

> If the reader has only taken a standard calculus course and never witnessed such techniques before, the maneuvers of Heaviside's calculus will seem utterly whimsical. Specifically, its general propensity is to manipulate differential *operators* as if they were *numbers*. That is, beginning with the equation:
>
> (1) $\frac{dy}{dt} + y = t^2$
>
> Heaviside will "factor" it
>
> (2) $\left[\left(\frac{d}{dt}\right) + 1\right] y = t^2$,
>
> then "divide" it
>
> (3) $y = \frac{t^2}{\left(\frac{d}{dt}+1\right)}$,
>
> and finally "expand" it (in analogy to $\frac{1}{1+x} = \sum(-1)^{n+1}\frac{1}{x}$, valid if $|x| > 1$)
>
> (4) $y = \left\{\frac{t^2}{\frac{d}{dt}} - \frac{t^2}{\frac{d^2}{dt^2}} + \frac{t^2}{\frac{d^3}{dt^3}} - \cdots\right\}$
>
> …

[197] Wilson, pp. 519–20.

> Heaviside then maneuvers the gobbledegook on the right side of his equation to a form he could interpret by one of his self-styled ‘‘algebratizing rules’’. For example, the rule ‘‘Algebratize $\frac{1}{\frac{d^n}{dt^n}}$ as n-fold integration’’ would convert the nonsense above to:
>
> $$(5) \quad y = \frac{t^3}{3} - \frac{t^4}{3.4} + \frac{t^5}{3.4.5} - \cdots$$
>
> which reduces, by standard results on series to
>
> $$y = 2 - 2t + t^2 - 2e^{-t}$$

Mathematical orthodoxy does not generally allow the manipulation of the separate “d”, “y” and “t” of the initial $\frac{dy}{dt}$, yet Heaviside experiments with treating each part as though it could be meaningfully manipulated like a standard algebraic variable. Furthermore, the outcome of his unconventional approach is great mathematical success. Wilson comments:

> Heaviside invariably obtained correct answers through such apparent lunacy. Moreover, his algorithm generally found the right answer more quickly than orthodox methods (when the latter could be made to work at all).[198]

Heaviside thus provides us with a very clear example of a mathematician whose work—although it frequently appears to descend into what Wilson calls “lunacy” or “gobbledygook”—consistently opened up new possibilities for calculation. Heaviside’s methods—like the wayward pupil’s after he reaches 1000—struck his mathematical community as bizarre. But in spite of their bizarreness, Heaviside’s methods represented a bold and fruitful new way of deploying the mathematical resources available to him by proceeding from a hitherto unenvisaged conception of what is and is not permissible in manipulating them—a shift in the mathematical river-bed. And it

[198] Wilson, p. 520.

is precisely because Wittgenstein is alive to the possibility of cases like this, cases in which someone finds a way to shift the river-bed of mathematical practice and produce useful if initially bizarre results, that he refrains from giving any definite verdict about whether the wayward pupil is making a mistake. The wayward pupil just *might* be a mathematical genius, like Heaviside, who has stumbled upon an important discovery that we do not yet understand.

7.0 Conclusion

This dissertation takes its point of departure from the controversy generated by Wittgenstein's notoriously difficult remarks on rule-following. Wittgenstein's remarks on rule-following have traditionally been interpreted and debated as if they are intended to support some sort of thesis about the criteria of correctness in following a rule. But I have argued that this way in which the debate has traditionally been framed overlooks the significance of Wittgenstein's remarks as understood as an episode of a grammatical investigation. As elements of a grammatical investigation, the primary purpose of Wittgenstein's remarks is not to support any particular thesis about rule-following, but to draw our attention to the variety of ways in which we speak of someone's following a rule, and to the variety of different cases in which we affirm or refrain from affirming that someone has followed a rule correctly. There is no particular thesis about rule-following that is more important to Wittgenstein than the overarching project of clarifying what we might call the grammatical lie of the land in this area.

In order to develop this idea and show its interpretive fruitfulness, I have emphasized how Wittgenstein's rule-following remarks are informed by an aspect of his thinking that opposes a variety of formalism in the philosophy of mathematics. Formalism, as it is articulated in the work of Hardy to which Wittgenstein was reacting, is the view that we can distinguish a hard kernel of legitimate mathematics from various decorative remarks that surround that kernel. What qualifies this hard kernel as "legitimate mathematics," according to Hardyan formalism, is the way in which it conforms to a particular ideal of rigor in drawing out the consequences of definitions and principles whose mathematical significance is supposed to be entirely independent of their potential utility in other realms, either for helping us to understand non-mathematical topics or to address various practical concerns. Everything else is "gas."

Although Wittgenstein does not oppose the ideal of rigour in general, we have seen that he does oppose this idea that the mathematical significance of an insight or result must be independent of its potential utility, and thus must hold that there is a viable alternative for understanding the mathematical significance of an insight or result. I have shown how to appreciate the possibility of viable alternatives by considering the metamathematical thinking of De Morgan, who emphasizes the indispensability for mathematical work of various sorts of context to the possibility of what Hardy would regard as the "hard kernel" of mathematics. If De Morgan's view is right, then even what Hardy would regard as the "hard kernel" of mathematics could not be the mathematical achievement that it is if it were entirely independent of other realms—the various sorts of context whose importance De Morgan emphasizes—in the way that Hardy requires it to be.

De Morgan thus gives us an example of a metamathematical thinker whose appreciation for the importance for mathematics of its surrounding contexts makes for a poignant contrast with Hardy's formalism. And I have argued that we can read Wittgenstein as providing a model for how to understand De Morgan's view in his remarks about river-bed truths in *On Certainty*. According to the picture Wittgenstein develops with the image of a shifting river-bed linguistic activity is made possible by the way in which it rests on a "river-bed" of stable truths or rules that determine the criteria of correctness for the use of language. But this river-bed can shift over time, so that what serves at one time as a river-bed truth can at another time become a bit of language that is to be tested against something else that has usurped its position in the river-bed. With this conception of language in mind, we can see that it is always a live question, when confronted with a bit of *mathematical* language-use, what bits of language are supposed to serve as the river-bed (the "context") in the light of which it has its mathematical significance. Thus, according to the point of this Wittgensteinian image, there can be no such thing as a hard kernel of mathematics whose mathematical significance isolated from broader questions about how it relates to other, non-mathematical realms.

This point underwrites my final suggestion concerning Wittgenstein's remarks on rule-following. Wittgenstein is remarkably hesitant to give a clear verdict as to whether the wayward pupil in his example is making a mathematical mistake when he goes on with 1004, 1008, and so on, and we can now appreciate why. Since the river-bed of our language—including our mathematical language—can shift over time, there is a standing possibility of reconfiguring the river-bed truths so that what it includes and excludes differs from what we have recently been used to. Many important episodes in the progress of mathematics can be regarded as insights into how a more or less radical shift in what we hold fixed in the mathematical riverbed can open up possibilities for saying and understanding things that were previously closed off to us. I have described the work of Heaviside in some detail as an example in order to emphasize that those who first work out insights of this sort can appear to their mathematical peers to be operating with mathematical concepts and resources in a way that is baffling or incoherent, even though they are in fact making important advances. Wittgenstein's meta-linguistic and meta-mathematical perspective, I have claimed, makes him particularly sensitive to the possibility of such cases, and it is for that reason that he refrains from saying definitively that the "wayward pupil" really is making a mistake.

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