Analysis of solitary waves for some long-wave water wave models

by

Jie Jin

B.S. in Mathematics, Tongji University, 2012

M.S. in Applied Mathematics, University of Houston, 2014

Submitted to the Graduate Faculty of

the Kenneth P. Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2021

UNIVERSITY OF PITTSBURGH

KENNETH P. DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Jie Jin

It was defended on

May 4th 2021

and approved by

Prof. Ming Chen, Dept. of Mathematics, University of Pittsburgh

Prof. Huiqiang Jiang, Dept. of Mathematics, University of Pittsburgh

Prof. Dehua Wang, Dept. of Mathematics, University of Pittsburgh

Prof. Robert Pego, Dept. of Mathematical Sciences, Carnegie Mellon University

Dissertation Director: Prof. Ming Chen, Dept. of Mathematics, University of Pittsburgh

Analysis of solitary waves for some long-wave water wave models

Jie Jin, PhD

University of Pittsburgh, 2021

The analysis of solitary waves is an important topic in studying the dynamic of water wave models. The thesis will be divided into two parts:

In the first part, we focus on the solitary waves solutions to the Boussinesq *abcd* system. The Boussinesq *abcd* system arises in the modeling of long wave small amplitude water waves in a channel, where the four parameters (a, b, c, d) satisfy one constraint. In particular we work in two parameter regimes where the system does not admit a Hamiltonian structure (corresponding to $b \neq d$). We prove via analytic global bifurcation techniques the existence of solitary waves in such parameter regimes. Some qualitative properties of the solutions are also derived, from which sharp results can be obtained for the global solution curves. Specifically, we first construct solutions bifurcating from the stationary waves, and obtain a global continuous curve of solutions that exhibits a loss of ellipticity in the limit. The second family of solutions bifurcate from the classical Boussinesq supercritical waves. We show that the curve associated to the second class either undergoes a loss of ellipticity in the limit or becomes arbitrarily close to having a stagnation point.

In the second part, we consider the Camassa-Holm-Kadomtsev-Petviashvili-I equation (CH-KP-I), which is a two dimensional generalization of the Camassa-Holm equation (CH). We prove transverse instability of the line solitary waves under periodic transverse perturbations. The proof is based on the framework of [56]. Due to the high nonlinearity, our proof requires necessary modification. In more detail, we first establish the linear instability of the line solitary waves. Then through an approximation procedure, we prove that the linear effect actually dominates the nonlinear behavior.

Keywords: Boussinesq *abcd* system, Solitary waves, Analytic global bifurcation, CH-KP-I equation, Transverse instability.

Table of contents

Preface

Pursuing the Ph.D. degree is really a bittersweet experience and I would like to express my gratitude to all the people who made my graduate study and dissertation successful.

First and foremost, I would like to thank my advisor Prof. Ming Chen for his help and guidance throughout my Ph.D. life. It is extremely inspiring to discuss problems with him. Indeed, his lucid explanation and deep insight gave me lots of confidence to tackle the problem. In addition, when I suffered illness both mentally and physically, his encouragement really guided me to the "lighthouse".

Next, I would also like to thank my committee members Prof. Huiqiang Jiang, Prof. Robert Pego and Prof. Dehua Wang for their valuable time and effort for serving in the dissertation committee with many valuable comments.

Moreover, I would also like to extend my appreciation to my friends Xing Wang, Guanhao Li, Wenlong Pei and Dongyu Wu for sharing happiness with me.

Last but not least, I express my gratitude to all my family members for their constant support. Specially thanks to my girlfriend Peiji Zhong, for her love and care.

1.0 Introduction

The phenomenon of solitary wave was first observed by John Scott Russel [60] almost two centuries ago, which is later used to characterize a wave that does not disperse and retains its original identity as time evolves. Exact existence theory for solitary water waves, however, first appeared more than a century later in the work of Lavrentiev [47], Friedrichs-Hyers [34], and Ter-Krikorov [66] for small-ampliltude irrotational waves. Construction for large-amplitude irrotational waves was achieved by Amick–Toland [4, 5] and Benjemin–Bona–Bose [9].

Russel's experiment also motivated the studies on the mathematical modeling of water waves. The first works can be dated back to Boussinesq [15], Rayleigh [55], Korteweg and de Vries [42], where simpler sets of equations were derived as asymptotic models from the free surface Euler equations in some specific physical regimes. To be more precise, let h and λ denote respectively the mean elevation of the water over the bottom and the typical wavelength, and let a be a typical wave amplitude. The parameter regime considered in the above works corresponds to

$$\varepsilon = \frac{a}{h} \ll 1, \quad \delta = \frac{h}{\lambda} \ll 1, \quad \varepsilon = O(\delta^2).$$

In Chapter 2, we will study a model that lies exactly in the above papameter regime, while in Chapter 3, we will work on a model with the parameter regime corresponding to $\varepsilon = O(\delta)$. These two parameter regimes are both called the small amplitude, shallow water regimes. Physically, ε measures the strength of nonlinearity while δ characterizes the effect of dispersion. Thus solitary waves can be viewed as generated from a perfect balance between nonlinear and dispersive effects. In this thesis, we investigate two topics related to the solitary waves: existence and stability. More specifically, in Chapter 2, we study the existence of solitary waves to the Boussinesq *abcd* system; in Chapter 3, we establish the transverse instability of the Camassa-Holm-Kadomtsev-Petviashvili-I equation (CH-KP-I).

1.1 Global bifurcation of solitary waves to the Boussinesq *abcd* system

In Chapter 2, we will consider the existence of solitary waves to an asymptotic water wave model derived by Bona–Chen–Saut [12] (generalized to include the surface tension in [31] and in higher dimensions Bona–Colin–Lannes [13]) as an extended system of the classical Boussinesq equation. Specifically, it is a three-parameter family of Boussinesq systems for one dimensional surfaces that takes the following form

$$\begin{cases} \eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + \frac{1}{2}(u^2)_x + c\eta_{xxx} - du_{xxt} = 0, \end{cases} \tag{1.1.1}$$

all of which are formally equivalent models of solutions of the Euler equations. In the above system η is proportional to the deviation of the free surface from its rest position, u is proportional to the horizontal velocity taken at the scaled height $0 \leq \theta \leq 1$ ($\theta = 1$ at the free surface and $\theta = 0$ at the bottom). The parameters have the following explicit form

$$a = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\nu, \quad b = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\nu), \quad c = \frac{(1-\theta^2)}{2}\mu - \tau, \quad d = \frac{(1-\theta^2)}{2}(1-\mu)$$

with ν and μ arbitrary real numbers, and $\tau \geq 0$ is the normalized surface tension. These three degrees of freedom arise from the height at which the horizontal velocity is taken and from a double use of the *BBM trick* [10]. The justification of their hydrodynamic relevance was established in [13, 18, 62].

A solitary wave solution to system (1.1.1) is of the type

$$\eta(x,t) = \eta(\xi) = \eta(x - \lambda t) \in H^1(\mathbb{R}), \quad u(x,t) = u(\xi) = u(x - \lambda t) \in H^1(\mathbb{R}), \tag{1.1.2}$$

where λ denotes the traveling speed and $\xi = x - \lambda t$ is the moving coordinate with speed $\lambda \in \mathbb{R}$. We are thus looking in the class of "localized" solutions to the system

$$\begin{cases} c\eta'' + \eta - \lambda u + d\lambda u'' + \frac{1}{2}u^2 = 0, \\ au'' + u - \lambda\eta + b\lambda\eta'' + \eta u = 0, \end{cases} \tag{1.1.3}$$

where $'$ denotes the derivative with respect to ξ. The regularity of the solutions indicates the asymptotic behavior

$$\lim_{|x|\to\infty} (\eta, u) = (0,0). \tag{1.1.4}$$

Note that when $b = d$, the system possesses a Hamiltonian structure with Hamiltonian

$$\mathcal{H}(\eta, u) = \frac{1}{2} \int \left[-c\eta_x^2 - au_x^2 + \eta^2 + (1+\eta)u^2 \right] dx. \tag{1.1.5}$$

The solitary waves correspond to the critical points of the action functional $S_\lambda = \mathcal{H} - \lambda \mathcal{I}$, where

$$\mathcal{I}(\eta, v) = \int (\eta u + b\eta_x u_x)\, dx$$

is called the impulse functional, and the Lagrange multiplier λ gives the speed of the wave.

From (1.1.5) we see that the Hamiltonian $\mathcal{H}(\eta, u)$ is coercive in H^1 provided that $a, c < 0$. In this parameter regime, the existence of solitary waves can be inferred from the existence of minimizers to a constraint minimization problem [20] under an assumption on large surface tension $\tau > 1/3$ and a smallness on $\|\eta\|_{H^2}$. Later in [21] another variational formulation was adapted in the same parameter regime to establish the existence of solitary waves for any $\tau \geq 0$, but with a smallness restriction on the traveling speed λ. Using a Nehari manifold technique, the existence of ground state solutions (nontrivial solitary waves carrying minimum action energy S_λ) was established in [7]. In the case of large surface tension $\tau > 1/3$, these ground states are shown to be depression waves which are symmetric and increasing from its unique trough, consistent with the results in the context of two-dimensional full gravity-capillary water waves [1, 41, 61].

All the above analytical results are crucially based upon the Hamiltonian structure of the system, i.e., $b = d$. In Chapter 2, we extend the existence result to the cases when the parameters fall out of this regime. In particular, we will focus on pure gravity waves, corresponding to $\tau = 0$, and allow either (i) $b \neq d$, so that the Hamiltonian structure is no longer available; (ii) $a, c > 0$, so that the quadratic part of the Hamiltonian (1.1.5) is not positive definite; or (iii) the wave speed λ is large $|\lambda| > 1$, so that the action functional fails to be bounded from below. In all cases, the standard variational method seems hard to apply. The sketch of the proof will be given at the beginning of Chapter 2.

1.2 Transverse instability of the CH-KP-I equation

Stability is a large category of topics in water wave models.

For unidirectional approximation models, like the KdV equation [42], the Camassa-Holm equation (CH) [17, 28], etc., one problem in the above category is called the orbital stability around solitary waves. Roughly speaking, we want to know if the solution consistently stays in the neighborhood of a solitary wave and its translation when its initial data does. A naive reason why it is true is that the solitary wave holds the least Lagrangian action energy, so the object around it is "willing" to evolve like that. One of the universal treatments is by center manifold theory. The center manifold theory is an equivalent but more algebraic form of the original problem (e.g. under Fourier transform), based on spectral decomposition. The "finite dimension" version of the spectral decomposition is purely algebraic in taste, while its corresponding "infinite" counterpart has topology coming into play as a role of approximation to mimic the world of "finite". This thought works well for some class of operators (e.g. normal operators), but not some others. For equations preserving the Hamiltonian structure, the linearized operator around a solitary wave has essential spectrum on the imaginary axis, which corresponds to center manifold part that is hard to deal with. Another treatment is by the Lyaponov method, which is by Benjamin [8] and Bona [11], and later generalized to handle a class of Hamiltonian models by Weinstein [69] and Grillakis-Shatah-Strauss (GSS) [36]. They claim that knowing the information from the Lagrangian action energy allows one to determine the orbital stability and instability. The gain of their method is that instead of working with the original linearized operator, one just needs to study the spectrum of a rather transparent self-adjoint operator. The trade-off is that it is required to carefully weave the domain of the energy functional to balance between the complexity and solvability (due to loss of information from the original problem).

Besides the unidirectional models like KdV and CH, one can also allow transverse effect into modeling, leading to two-dimensional generalizations of the scalar models. Since the transverse perturbation is weak, it is natural to ask whether these models retain transverse stability, i.e. the unidirectional solitary waves remain stable under the two-dimensional flow. However, the answer to this question is much more involved. The first result is by

Alexander-Pego-Sachs [6] on the Kadomtsev-Petviashvili (KP) equation

$$(u_t + uu_x + u_{xxx})_x - \sigma u_{yy} = 0$$

which is a two-dimensional version of the KdV equation. The coefficient σ takes values in $\{-1, 1\}$ representing the strength of capillarity relative to the gravitational forces. The weak surface tension case corresponds to $\sigma = 1$ and is referred to as the KP-I equation; and the strong surface tension leads to the so-called KP-II equation with $\sigma = -1$. In [6], the authors state that the KP-I model is linearly unstable, while the KP-II model is linearly stable. The transition from linear instability to nonlinear instability for the KP-I equation is achieved by Rousset-Tzvetkov [56]. Later on, they employed the same idea to a large class of equations [57]. Transverse stability of the KP-II equation is proved by Mizumachi-Tzvetkov [49] and Mizumachi [50].

In Chapter 3, we will study the Camassa-Holm-Kadomtsev-Petviashvili-I equation (CH-KP-I), which is a two-dimensional generalization of the Camassa-Holm equation (CH):

$$\left[\left(1 - \partial_x^2\right) u_t + 3uu_x + 2\kappa u_x - 2u_x u_{xx} - uu_{xxx}\right]_x - u_{yy} = 0 \tag{1.2.1}$$

with $\kappa > 0$. In [22], Chen derived a generalized version of (4.0.1) in the context of nonlinear elasticity theory. Also in [38], the CH-KP-II model is derived in the context of water wave. Note that in (4.0.1), if we disregard the transverse effect, the CH-KP-I equation is reduced to the CH equation. The CH equation exhibits the wave-breaking phenomenon that is not shown in the KdV equation. From the point of view of modeling, this is because that these two models arise from different physical parameter regimes. More specifically, for the parameters ε and δ mentioned at the beginning of this chapter, the parameter regime considered in the CH equation corresponds to $\varepsilon = O(\delta)$, while the parameter regime for the KdV equation is $\varepsilon = O(\delta^2)$. Thus the CH equation possesses stronger nonlinearity than the KdV equation, which allows for the breaking wave. Like the KdV equation, solitary waves also exist for the CH equation, which are symmetric, monotone decreasing on positive x-axis and decay exponentially as $|x| \to \infty$. Furthermore, the CH solitary waves are also orbitally stable like the KdV solitons, as is proved by Constantin-Strauss [29] using the GSS method. For the CH-KP-I equation, since it could be treated as the CH counterpart of the two-dimensional KdV equation (KP-I), it is reasonable to expect that the CH line solitary waves are also transversely unstable. Here a line solitary wave ϕ is defined such that it is uniform in the transverse direction, and for each cross section, it is exactly the solitary wave of the CH equation. The theorem we prove is as follows:

Theorem 1.2.1 (Transverse instability of line solitary waves)**.** *The CH line solitary wave* ϕ *of the CH-KP-I equation* (4.0.1) *is transversely unstable in the following sense: There exists* $k_0 > 0$ *such that for every* $s \geq 0$*, there exists an* $\eta > 0$ *such that for each* $\delta > 0$*, there exists a solution* u^δ *emanating from an initial data* $u_0^\delta \in H^\infty(\mathbb{R} \times \mathbb{T}_a)$ *with* $\left\|u_0^\delta - \phi\right\|_{H^s(\mathbb{R}\times\mathbb{T}_a)} \leq \delta$*, and a time* $T^\delta \sim |\log \delta|$*, so that* u^δ *satisfies*

$$\inf_{l\in\mathbb{R}} \left\|u^\delta(T^\delta, \cdot) - \phi(\cdot - l)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_a)} \geq \eta, \tag{1.2.2}$$

where $a = \frac{2\pi}{k_0}$*,* $\mathbb{T}_a$ *is the torus* $\mathbb{R}/a\mathbb{Z}$*.*

The sketch of the proof will be given at the beginning of Chapter 3.

2.0 Global bifurcation of solitary waves to the Boussinesq *abcd* system

At the beginning of this chapter, we provide a sketch of the proof.

The main tool we are using is the bifurcation theory. For this to work we need to first choose a good parameter $s \in \mathbb{R}$ with which the problem (1.1.3) can be formulated as an abstract one-parameter problem

$$\mathscr{F}(U, s) = 0$$

where $U := (u, \eta)$. The perturbative construction of solutions relies on a good understanding of the linearized operator $\mathscr{F}_U$ at some special solution (U_0, s_0). It turns out that the translation invariance of the problem naturally generates a nontrivial kernel of the linearized operator $\mathscr{F}_U$ at any solution. With some appropriate choices of the "base point solution" (U_0, s_0), standard ODE techniques can be applied to ensure that the kernel is exactly one dimensional and hence can be removed by suitable choice of the function spaces, allowing us to invoke the Implicit Function Theorem to obtain a local curve of solutions.

As is common for the solitary wave problem, continuing the local curve globally by standard global bifurcation techniques faces a serious obstruction due to the unboundedness of the domain. One classical approach is to approximate the solitary waves by periodic ones as the period tends to infinity. Such a method is used by Toland [67] to treat (1.1.3) with $(a, b, c, d) = (0, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$. He first obtains a global bifurcation theory for the periodic problems, and then proves a uniform estimate. Together with an application of the Whyburn lemma, this leads to the convergence of the global sets of periodic solutions to a global connected set of solitary wave solutions as the period goes to infinity.

We will adapt a recently developed analytic global implicit function theorem in [25] for the global theory, cf. Theorem 2.1.5. As is pointed out in [25], the global curve may not be locally pre-compact, nor can Fredholmness persists. Thus the loss of compactness emerges as an alternative. The ODE nature of the problem easily rules out the failure of Fredholmness. Therefore the theory will become useful in practice if we can rule out the loss of properness or classify how it manifests.

More specifically, we will consider global branches of solutions emanating from two base point solutions: the first one being the stationary solution (corresponding to $\lambda = 0$), and the second being the supercritical ($\lambda > 1$) waves to the classical Boussinesq system (corresponding to $(a, b, c, d) = (0, 0, 0, \frac{1}{3})$). We will also assign different parameters when studying these two types of waves. When bifurcating from the stationary waves, we use the wave speed λ as the bifurcation parameter while fixing the *abcd* system as in (2.1.5), and obtain a continuous curve of solutions all the way into the regime where solutions are traveling with an $O(1)$ speed. For the other case we will fix an arbitrary supercritical speed $\lambda > 1$ and design a family of *abcd* systems (as in (2.2.3)) that can accommodate solitary waves with such a speed λ. In both cases we prove a collection of qualitative properties of the solutions that are crucial for the final global result. In particular, using maximum principle arguments and the symmetry result for weakly coupled cooperative elliptic systems [16] we are able to obtain local uniqueness, local monotonicity, and nodal pattern of the solutions. The fact that we are always considering a *system* makes the maximum arguments more delicate, and possibly more restrictive; see Section 2.1.2–2.1.3 and Section 2.2.1–2.2.2.

Regarding the ruling-out/realization of the loss of compactness alternative in the global theory, as was studied in [24, 25], the established monotonicity property is strong enough to assert a "compactness or front" result stating that this possibility must manifest as a broadening phenomenon, leading to a *monotone front* type of solution at the end of the bifurcation curve. When the underlying system possesses a Hamiltonian structure, a so-called conjugate flow analysis can be casted utilizing the conserved quantities to rule out the broadening alternative [2, 24, 25, 26, 37, 64]. Moreover, for some particular problems such a Hamiltonian structure may also allow one to obtain uniform bounds on solutions that can account for the realization of broadening [37]. In the cases we consider, however, the system is not Hamiltonian, and we do not have any obvious conserved quantities that can be of much use to control the solutions. Taking advantage of the monotonicity and together with delicate algebra we are able to prove the nonexistence of monotone front solutions, cf. Lemma 2.1.7 and Lemma 2.2.2. Using this idea we can also prevent the blowup of solutions (u, η) in the case of bifurcation from stationary waves, which leads to a sharp result ensuring the loss of ellipticity as the only remaining alternative cf. Theorem 2.1.6. For the other case of solutions bifurcating from the classical Boussinesq waves, we are able to winnow the alternatives down to the possibilities of either the loss of ellipticity or that the curve continues up to the appearance of an "extreme wave" that has a stagnation point, cf. Theorem 2.2.4.

2.1 Bifurcation from stationary waves with $a = c < 0$

We start by constructing solutions near the stationary waves corresponding to $\lambda = 0$. To ensure ellipticity we will impose the sign condition $a, c < 0$.

2.1.1 Stationary solutions

Note that in the case when $\lambda = 0$ the terms in system (1.1.3) containing b and d disappear. and becomes

$$\begin{cases} -c\eta'' = \eta + \frac{u^2}{2}, \\ -au'' = u(1+\eta), \end{cases} \tag{2.1.1}$$

By elliptic regularity we know that the solution to (2.1.1) is smooth and $\lim_{|x|\to\infty}(\eta', u') = (0,0)$. Hence solitary wave solutions satisfy the 'first integral' property

$$-a(u')^2 - c(\eta')^2 = u^2(1+\eta) + \eta^2. \tag{2.1.2}$$

The solution theory for (2.1.1) has been carried out in [23]. Here we collect some results which will be important for the later bifurcation argument. For the reader's convenience we provide their proofs in Appendix A.

Lemma 2.1.1. *Any solitary wave solution of* (2.1.1) *satisfies*

$$\eta(x) < 0 \quad on\ \mathbb{R}.$$

Proposition 2.1.1 (Existence and uniqueness of stationary waves [23])**.** *When* $a = c = -\beta^2 < 0$ *we have*

(i) *there is a solitary wave solution such that* $u_0^-(x) < 0$ *on* $\mathbb{R}$. *Up to translation,*

$$u_0^-(x) = -\frac{3\sqrt{2}}{2}\operatorname{sech}^2\left(\frac{x}{2\beta}\right), \quad \text{and} \quad \eta_0(x) = -\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2\beta}\right). \tag{2.1.3}$$

This solution is unique among the class of functions (u, η) *where* $u < \sqrt{2}$*;*

(ii) *there is a solitary wave solution such that* $u_0^+(x) > 0$ *on* $\mathbb{R}$. *Up to translation,*

$$u_0^+(x) = \frac{3\sqrt{2}}{2}\operatorname{sech}^2\left(\frac{x}{2\beta}\right), \quad \text{and} \quad \eta_0(x) = -\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2\beta}\right). \tag{2.1.4}$$

This solution is unique among the class of functions (u, η) *where* $u > -\sqrt{2}$.

2.1.2 Local theory

Now we will construct a local curve of solutions nearby the stationary solution (u_0, η_0). The parameters we are taking satisfy

$$a = c = -d = -\beta^2 < 0, \quad b = \frac{1}{3} + \beta^2. \tag{2.1.5}$$

Obviously we see that $b \neq d$, and hence we are outside the Hamiltonian regime when the surface tension is small. For simplicity we will take $\tau = 0$ in the following discussion.

Writing $U = (u, \eta)$, the system for solitary waves takes the following form

$$\mathscr{F}(U, \lambda) := \begin{pmatrix} \mathcal{L}\left(u - \lambda\left(1 + \frac{1}{3\beta^2}\right)\eta\right) + \left(\frac{\lambda}{3\beta^2} + u\right)\eta, \\ \\ \mathcal{L}\left(\eta - \lambda u\right) + \frac{1}{2}u^2 \end{pmatrix} = 0, \tag{2.1.6}$$

where

$$\mathscr{F} : H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathbb{R} \to L^2(\mathbb{R})$$

and $\mathcal{L} := 1 - \beta^2\partial_x^2$ is an invertible operator from $H^2(\mathbb{R}) \to L^2(\mathbb{R})$.

The discussion in Section 2.1.1 indicates that $\mathscr{F}(U_0^\pm, 0) = 0$ where $U_0^\pm = (u_0^\pm, \eta_0)$. The linearized operator at the solution $(U_0^\pm, 0)$ is

$$\mathscr{F}_U(U_0^\pm, 0)[V] = \mathcal{L}V + \begin{pmatrix} \eta_0 & u_0^\pm \\ u_0^\pm & 0 \end{pmatrix} V, \tag{2.1.7}$$

where $V := (v,\zeta) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. It is clear to see that $\mathscr{F}_u(u_0, 0)$ is self-adjoint on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. The following lemma states that the kernel of $\mathscr{F}_U(U_0^{\pm}, 0)$ is only generated by the translation symmetry.

Lemma 2.1.2. *For any given $\beta > 0$, $\ker \mathscr{F}_U(U_0^{\pm}, 0) = \operatorname{span}\{(U_0^{\pm})'\}$ where $(U_0^{\pm}) := \big(u_0^{\pm}, \eta_0\big)$.*

Proof. Let $V = (v,\zeta) \in \ker \mathscr{F}_U(U_0^{\pm}, 0)$. The V satisfies

$$\mathcal{L}V + \begin{pmatrix} \eta_0 & u_0^{\pm} \\ u_0^{\pm} & 0 \end{pmatrix} V = 0. \tag{2.1.8}$$

The regularity condition implies that V is bounded. Further notice that the Green's function for $\mathcal{L}^{-1}$ is $G(x) = \frac{1}{2\beta} e^{-|x|/\beta}$. Therefore

$$\begin{aligned} V(x) &= -G(x) * \left[\begin{pmatrix} \eta_0 & u_0^{\pm} \\ u_0^{\pm} & 0 \end{pmatrix} V \right](x) \\ &= -\int_{\mathbb{R}} G(x-y) \begin{pmatrix} \eta_0(y) & u_0^{\pm}(y) \\ u_0^{\pm}(y) & 0 \end{pmatrix} V(y)\, dy \\ &= -\frac{1}{G(x)} \int_{\mathbb{R}} \frac{G(x-y)G(y)}{G(x)} \frac{1}{G(y)} \begin{pmatrix} \eta_0(y) & u_0^{\pm}(y) \\ u_0^{\pm}(y) & 0 \end{pmatrix} V(y)\, dy. \end{aligned}$$

Since

$$\left| \frac{G(x-y)G(y)}{G(x)} \right| \lesssim 1, \qquad \left| \frac{\eta_0(y)}{G(y)} \right| + \left| \frac{u_0^{\pm}(y)}{G(y)} \right| \lesssim 1,$$

we conclude that V decay exponentially

$$|G(x)V(x)| \lesssim 1. \tag{2.1.9}$$

Expanding (2.1.8) into a 4×4 first order ODE system and checking the asymptotics we find that there are only two L^2 solutions having asymptotic behavior as

$$e^{-|x|/\beta} \qquad \text{and} \qquad |x| e^{-|x|/\beta}.$$

Together with (2.1.9) we know that $\dim\ker \mathscr{F}_U(U_0^{\pm}, 0) \le 1$. Recalling the fact that

$$\mathscr{F}_U(U_0^{\pm}, 0)[(U_0^{\pm})'] = 0$$

yields the desired result. □

The spectral property of $\mathscr{F}_U(U_0^{\pm}, 0)$ given by Lemma 2.1.2 allows a use of Implicit Function Theorem on the space

$$H^2_{\mathrm{e}}(\mathbb{R}) := \{f \in H^2(\mathbb{R}) : \ f \text{ is even}\}, \qquad L^2_{\mathrm{e}}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \ f \text{ is even}\}. \tag{2.1.10}$$

Notice that for if (u, η, λ) is a solution to (2.1.6) with $\lambda > 0$, then so is $(-u, \eta, -\lambda)$. In fact this corresponds to the same wave propagating in the opposite direction. Therefore in the following analysis we will only consider the case $\lambda > 0$.

Theorem 2.1.1 (Nearly stationary waves)**.** *For any $\beta \in \mathbb{R}$ there exists some positive $\lambda_0 > 0$ and a C^0 solution curve*

$$\mathscr{C}^{\mathrm{slow}}_{\mathrm{loc}} = \{(u^{\pm}(\lambda), \eta(\lambda), \lambda) : \ 0 \le \lambda < \lambda_0\} \subset H^2_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{e}}(\mathbb{R}) \times \mathbb{R}$$

to problem (2.1.6) *with the property that*

$$u(\lambda) = u_0^+ + O(\lambda), \quad \eta(\lambda) = \eta_0 + O(\lambda) \quad \textit{in} \ \ H^2_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{e}}(\mathbb{R}), \tag{2.1.11}$$

$$u(\lambda) > 0 \quad \textit{and} \quad \eta(\lambda) < 0, \tag{2.1.12}$$

where (u_0^+, η_0) is given in (2.1.4)*.*

Proof. The proof of the existence and uniqueness of the solution curves and (2.1.11) follows from Lemma 2.1.2 and a direct application of the Implicit Function Theorem.

Applying the maximum principle to the second equation of (2.1.6) we see that $\lambda u \ge \eta$. From (2.1.6) we also have

$$-\beta^2 \left[1 - \lambda^2 \left(1 + \frac{1}{3\beta^2}\right)\right] \eta'' + (1 - \lambda^2 - \lambda u)\eta + \frac{1}{2}u^2 = 0. \tag{2.1.13}$$

From (2.1.11) we know that for λ sufficiently small $1 - \lambda^2 - \lambda u > 0$. Therefore from maximum principle we conclude that $\eta \le 0$. If there is an x_1 such that $\eta(x_1) = 0 = \max_{x \in \mathbb{R}} \eta$, then we have $\eta'(x_0) = 0$. Substituting this into the above equation leads to $\eta''(x_0) = u(x_0) = 0$. Hence $(\eta - \lambda u)(x_0) = 0$. Since $\eta - \lambda u \le 0$, we see that $(\eta - \lambda u)(x_0) = \max_{x \in \mathbb{R}}(\eta - \lambda u)$, and thus $u'(x_0) = 0$. The uniqueness of ODE then implies that $(\eta, u) \equiv 0$, a contradiction. Therefore we must have

$$\eta < 0.$$

Direct calculation yields the equation for u as

$$-\beta^2\left[1-\lambda^2\left(1+\frac{1}{3\beta^2}\right)\right]u''+\left[1-\lambda^2\left(1+\frac{1}{3\beta^2}\right)+\frac{\lambda}{2}\left(1+\frac{1}{3\beta^2}\right)u\right]u \\ +\left(\frac{\lambda}{3\beta^2}+u\right)\eta=0. \tag{2.1.14}$$

From (2.1.11) and (2.1.4) we know that for any $\varepsilon>0$ there exist $\lambda>0$ sufficient small and $R_0>0$ sufficiently large such that

$$\begin{aligned}&\left\|u-u_0^+\right\|_{H^2(\mathbb{R})}+\|\eta-\eta_0\|_{H^2(\mathbb{R})}+\left\|u_0^+\right\|_{L^\infty(|x|\geq R_0)}+\left\|\eta_0\right\|_{L^\infty(|x|\geq R_0)}<\varepsilon,\\ &u>0 \qquad \text{for} \quad |x|<R_0.\end{aligned} \tag{2.1.15}$$

If $\inf_{x\in\mathbb{R}} u<-\frac{\lambda}{3\beta^2}<0$, then from the above we know that there exists some $x_0>R_0$ such that $u(x_0)=\inf_{x\in\mathbb{R}} u$. Continuity then yields the existence of x_1 with $x_1>R_0$ and $u(x_1)=0$ such that

$$x_1=\min\{x>0:\ u(x)=0\}.$$

Rewriting (2.1.14) as

$$-\beta^2\left[1-\lambda^2\left(1+\frac{1}{3\beta^2}\right)\right]u''+\left[1-\lambda^2\left(1+\frac{1}{3\beta^2}\right)+\eta+\frac{\lambda}{2}\left(1+\frac{1}{3\beta^2}\right)u\right]u+\frac{\lambda}{3\beta^2}\eta=0, \tag{2.1.16}$$

we see from (2.1.15) that $|\eta|<2\varepsilon$ on $[x_1,+\infty)$. Thus for λ and ε sufficiently small, applying the maximum principle on $[x_1,+\infty)$ yields that

$$u\geq 0 \qquad \text{on} \quad [x_1,+\infty),$$

which is a contradiction.

Therefore

$$\inf_{x\in\mathbb{R}} u\geq -\frac{\lambda}{3\beta^2}.$$

Substituting this into (2.1.14), maximum principle infers that $u>0$, which is (2.1.12). □

To investigate further the qualitative properties of the solutions, let us first recall the following result of [16, Theorem 2] on weakly coupled elliptic systems.

Theorem 2.1.2 ([16])**.** *Assume (u,v) is a classical solution to the following elliptic system*

$$\begin{cases} \Delta u + g(u,v) = 0 & \text{in} \quad \mathbb{R}^n, \\ \Delta v + f(u,v) = 0 & \text{in} \quad \mathbb{R}^n, \\ u, \ v > 0 & \text{in} \quad \mathbb{R}^n, \\ u(x), \ v(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases}$$

where $f, g \in C^1([0,\infty) \times [0,\infty), \mathbb{R})$. Suppose further that

(i) $\dfrac{\partial g}{\partial v}, \dfrac{\partial f}{\partial u}$ *are non-negative on* $[0,\infty) \times [0,\infty)$*;* (quasi-monotonicity)

(ii) $\dfrac{\partial g}{\partial u}(0,0) < 0$ *and* $\dfrac{\partial f}{\partial v}(0,0) < 0$*;*

(iii) $\det A > 0$, *where*

$$A := \begin{pmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \\ \\ \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} (0,0).$$

Then there exist points $x_0, x_1 \in \mathbb{R}^n$ such that $u(x) = u(|x - x_0|)$ and $v(x) = v(|x - x_1|)$. Moreover

$$\frac{du}{dr_0} < 0 \qquad \text{and} \qquad \frac{dv}{dr_1} < 0,$$

where $r_0 := |x - x_0|$ and $r_1 := |x - x_1|$.

From the above theorem we immediately obtain

Lemma 2.1.3 (Local monotonicity)**.** *Fix $\beta \in \mathbb{R}$. There exists $\lambda_0 > 0$ such that every solution $(u, \eta, \lambda) \in \mathscr{C}^{\text{slow}}_{\text{loc}}$ with $0 \le \lambda < \lambda_0$ is strictly monotone in that for $x > 0$,*

$$u' < 0 \qquad \text{and} \qquad \eta' > 0. \tag{2.1.17}$$

Proof. We see that (u, η) satisfies equations (2.1.14) and (2.1.13). Setting $v := -\eta$ and putting it into the form as in Theorem 2.1.2 we find that

$$g(u,v) = -\frac{1}{\beta^2}u + \frac{\lambda}{3\beta^4 B}v + \frac{1}{\beta^2 B}uv - \frac{\lambda}{2\beta^2 B}\left(1 + \frac{1}{3\beta^2}\right)u^2,$$
$$f(u,v) = -\frac{1-\lambda^2}{\beta^2 B}v + \frac{\lambda}{\beta^2 B}uv + \frac{1}{2\beta^2 B}u^2,$$

where $B := \left[1-\lambda^2\left(1+\frac{1}{3\beta^2}\right)\right] > 0$ for small λ. Direct computation shows that

$$\begin{pmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \\ \\ \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\beta^2}+\dfrac{1}{\beta^2 B}v-\dfrac{\lambda}{\beta^2 B}\left(1+\dfrac{1}{3\beta^2}\right)u & \dfrac{\lambda}{3\beta^4 B}+\dfrac{1}{\beta^2 B}u \\ \\ \dfrac{\lambda}{\beta^2 B}v+\dfrac{1}{\beta^2 B}u & -\dfrac{1-\lambda^2}{\beta^2 B}+\dfrac{\lambda}{\beta^2 B}u \end{pmatrix}.$$

From Theorem 2.1.1 we know that $u, v > 0$ when λ is small, which implies that (i)–(iii) of Theorem 2.1.2 are satisfied. Therefore (2.1.17) holds. □

Another application of Theorem 2.1.2 to the local solution near the bifurcation point $(u_0^+, \eta_0, 0)$ is the following result on the local uniqueness of the solution curve $\mathscr{C}_{\text{loc}}^{\text{slow}}$. In particular this result shows that all H^2 solutions near $(u_0^+, \eta_0, 0)$ with $\lambda > 0$ must be even and monotone on the positive axis.

Corollary 2.1.3 (Local uniqueness)**.** *Denote $\mathcal{B}_r$ the ball of radius $r > 0$ in $H^2(\mathbb{R})\times H^2(\mathbb{R})\times\mathbb{R}$ centered at $(u_0^+, \eta_0, 0)$. There exists $\varepsilon > 0$ such that for $\lambda > 0$,*

$$\mathscr{F}^{-1}(0)\cap\mathcal{B}_\varepsilon = \mathscr{C}_{\text{loc}}^{\text{slow}}\cap\mathcal{B}_\varepsilon. \tag{2.1.18}$$

Proof. Consider a solution (u, η, λ) to equations (2.1.13)–(2.1.14) with

$$\left\|u-u_0^+\right\|_{H^2(\mathbb{R})}+\|\eta-\eta_0\|_{H^2(\mathbb{R})}+|\lambda|<\varepsilon.$$

There exists an $R_0 > 0$ large enough such that

$$\begin{aligned} &\left\|u-u_0^+\right\|_{H^2(\mathbb{R})}+\|\eta-\eta_0\|_{H^2(\mathbb{R})}+|\lambda|+\left\|u_0^+\right\|_{L^\infty(|x|\geq R_0)}+\left\|\eta_0\right\|_{L^\infty(|x|\geq R_0)}<\varepsilon, \\ &u>0, \quad \eta<0, \quad u'<0, \quad \eta'>0 \qquad \text{for} \quad |x|<R_0. \end{aligned} \tag{2.1.19}$$

Hence if $\sup_{x\in\mathbb{R}}\eta > 0$, then from continuity there exists $x_0 := \min\{x > 0 :\ \eta(x) = 0\}$ such that $\eta(x_0) = 0$ and $x_0 > R_0$. From (2.1.19) we see that

$$1-\lambda^2-\lambda u>0 \qquad \text{on} \quad [x_0,+\infty).$$

Applying maximum principle to (2.1.13) on $[x_0, +\infty)$ yields that $\eta \leq 0$ on $[x_0, +\infty)$. Together with (2.1.19) it contradicts with $\sup_{x\in\mathbb{R}}\eta > 0$. Therefore we must have $\eta \leq 0$.

In a similar way if $\inf_{x\in\mathbb{R}} u < 0$, then we may find $x_1 := \min\{x > 0 : \ u(x) = 0\}$ such that $u(x_0) = 0$ and $x_0 > R_0$. Maximum principle applied to (2.1.16) on $[x_1, +\infty)$ leads to $u \geq 0$, contradicting to the assumption that $\inf_{x\in\mathbb{R}} u < 0$. Thus $u \geq 0$.

If there exists some $x_0 \geq 0$ such that $\eta(x_0) = 0$, then $\eta(x_0) = \sup_{x\in\mathbb{R}} \eta$, and hence $\eta'(x_0) = 0$ and $\eta''(x_0) \leq 0$. From (2.1.13) we find that $u(x_0) = 0$. This also means that $u(x_0) = \inf_{x\in\mathbb{R}} u$, and so $u'(x_0) = 0$. Uniqueness of the ODE then implies that $\eta = u \equiv 0$, which contradicts (2.1.19). The same argument applies to the situation if u touches zero at some finite point.

The above argument indicates that for any small $(u, \eta, \lambda) \in \mathscr{F}^{-1}(0) \cap \mathcal{B}_\varepsilon$,

$$u > 0 \qquad \text{and} \qquad \eta < 0.$$

Then for $\lambda > 0$ one may apply Theorem 2.1.2 to conclude that u and η are both even. Therefore the uniqueness of $\mathscr{C}^{\text{slow}}_{\text{loc}}$ within $\mathscr{F}^{-1}(0) \cap (H^2_{\text{e}}(\mathbb{R}) \times H^2_{\text{e}}(\mathbb{R}) \times \mathbb{R}^+)$ gives (2.1.18). $\square$

2.1.3 Nodal pattern

Now for each fixed $\beta \in \mathbb{R}$ we introduce the set

$$\mathcal{O} := \left\{ (u, \eta, \lambda) \in H^2_{\text{e}}(\mathbb{R}) \times H^2_{\text{e}}(\mathbb{R}) \times \mathbb{R}^+ : \ 1 - \lambda^2 \left(1 + \frac{1}{3\beta^2} \right) > 0 \right\}. \tag{2.1.20}$$

The results of Theorem 2.1.1 and Lemma 2.1.3 naturally suggest us to consider the following "nodal properties"

$$u > 0 \qquad \text{in } \ \mathbb{R}, \tag{2.1.21a}$$

$$\eta < 0 \qquad \text{in } \ \mathbb{R}, \tag{2.1.21b}$$

$$u' < 0 \qquad \text{in } \ \mathbb{R}^+, \tag{2.1.21c}$$

$$\eta' > 0 \qquad \text{in } \ \mathbb{R}^+. \tag{2.1.21d}$$

Lemma 2.1.4 (Open property)**.** *Let $(u_*, \eta_*, \lambda_*) \in \mathcal{O} \cap \mathscr{F}^{-1}(0)$ be given and suppose that it satisfies* (2.1.21)*. There exists $\varepsilon = \varepsilon(u_*, \eta_*, \lambda_*) > 0$ such that, if $(u, \eta, \lambda) \in \mathcal{O} \cap \mathscr{F}^{-1}(0)$ and*

$$\|u - u_*\|_{H^2(\mathbb{R})} + \|\eta - \eta_*\|_{H^2(\mathbb{R})} + |\lambda - \lambda_*| < \varepsilon, \tag{2.1.22}$$

then (u, η, λ) also satisfies (2.1.21).

Proof. The proof of (2.1.21a) and (2.1.21b) follows the same argument as in the proof of Corollary 2.1.3 by replacing $(u_0^+, \eta_0, 0)$ with (u_*, η_*, λ_*). The proof for (2.1.21c)–(2.1.21d) then follows directly from the application of Theorem 2.1.2. □

Lemma 2.1.5 (Closed property). *Let* $\{(u_n, \eta_n, \lambda_n)\} \subset \mathcal{O} \cap \mathscr{F}^{-1}(0)$ *be given and suppose that* $(u_n, \eta_n, \lambda_n) \to (u, \eta, \lambda) \in \mathcal{O} \cap \mathscr{F}^{-1}(0)$ *in* $H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathbb{R}$. *If each* (u_n, η_n, λ_n) *satisfies* (2.1.21), *then* (u, η, λ) *also satisfies* (2.1.21) *unless* $u = \eta \equiv 0$.

Proof. First we see that

$$\begin{aligned} &u \geq 0, \qquad \eta \leq 0, \qquad \lambda \geq 0, \qquad \text{and} \\ &u' \leq 0, \qquad \eta' \geq 0 \qquad \text{in } \mathbb{R}^+. \end{aligned}$$

If there exists x_0 such that $u(x_0) = 0$, then $u(x_0) = \inf_{x\in\mathbb{R}} u$, and hence $u'(x_0) = 0$. From the equation (2.1.14) and maximum principle we see that $\eta(x_0) = 0$. Therefore $\eta(x_0) = \sup_{x\in\mathbb{R}} \eta$. So $\eta'(x_0) = 0$. Thus from the uniqueness of ODE we know that $u = \eta \equiv 0$. □

Lemma 2.1.6 (Nodal property). *If* $\mathcal{K}$ *is any connected subset of* $\mathcal{O} \cap \mathscr{F}^{-1}(0)$ *that contains* $\mathscr{C}_{\mathrm{loc}}^{\mathrm{slow}}$, *then every* $(u, \eta, \lambda) \in \mathcal{K}$ *exhibits* (2.1.21).

Proof. First note that each $(u(\lambda), \eta(\lambda), \lambda) \in \mathscr{C}_{\mathrm{loc}}^{\mathrm{slow}}$ satisfies (2.1.21). Recall the definition of $\mathcal{B}_r$ in Corollary 2.1.3. Fix $0 < \lambda < \lambda_0$ and take ε to be sufficiently small, the local uniqueness of $\mathscr{C}_{\mathrm{loc}}^{\mathrm{slow}}$ implies that

$$\mathcal{K} \cap \mathcal{B}_\varepsilon = \mathscr{C}_{\mathrm{loc}}^{\mathrm{slow}} \cap \mathcal{B}_\varepsilon,$$

and $\mathcal{K} \backslash \mathcal{B}_\varepsilon$ is the connected component containing $(u(\lambda), \eta(\lambda), \lambda)$. Applying Lemmas 2.1.4 and 2.1.5 completes the proof. □

2.1.4 Monotone fronts

Next we define the concept of *monotone fronts*.

Definition 2.1.4. *For $\lambda > 0$ and $\lambda^2\left(1+\frac{1}{3\beta^2}\right) < 1$, we say (u, η, λ) is a monotone front solution of* (2.1.6) *if $(u,\eta) \in C^2_{\mathrm{b}}(\mathbb{R}) \times C^2_{\mathrm{b}}(\mathbb{R})$, and*

$$\lim_{x\to+\infty}(u(x),\eta(x)) = (0,0), \quad \text{and} \quad u > 0, \quad \eta < 0, \quad u' \le 0, \quad \eta' \ge 0 \quad \text{in } \mathbb{R}, \tag{2.1.23}$$

where $C^2_{\mathrm{b}}(\mathbb{R})$ is the set of C^2 functions with bounded norms.

Lemma 2.1.7 (Nonexistence of monotone fronts)**.** *If $\lambda > 0$, and β satisfies $\beta^2 < 0.26$, then system* (2.1.6) *does not admit any monotone front solution in the sense of* (2.1.23)*.*

Proof. Suppose (u, η) is a monotone front solution to (2.1.6). Then since u, η are bounded and monotone,

$$(\bar{u}, \bar{\eta}) := \lim_{x\to-\infty}(u(x), \eta(x))$$

exists, and $\bar{u} > 0$, $\bar{\eta} < 0$. Evaluating (2.1.13) at $-\infty$ leads to

$$\bar{u} < \frac{1-\lambda^2}{\lambda}, \qquad \text{and} \quad \bar{\eta} = -\frac{\bar{u}^2}{2(1-\lambda^2-\lambda\bar{u})}. \tag{2.1.24}$$

Substituting the above into (2.1.14) and evaluating the equation at $-\infty$ yields

$$-(2-B)\,\bar{u}^2 - \lambda B\bar{u} + 2(1-\lambda^2)B = 0,$$

where $B = 1 - \lambda^2\left(1+\frac{1}{3\beta^2}\right) \in (0,1)$. Solving this quadratic equation together with the constraint that $\bar{u} > 0$ yields

$$\bar{u} = \frac{\sqrt{\lambda^2B^2 + 8B(2-B)(1-\lambda^2)} - \lambda B}{2(2-B)}. \tag{2.1.25}$$

On the other hand, multiplying (2.1.13) by η' and multiplying (2.1.14) by u' and summing up, it follows that

$$\begin{aligned}\left[-\frac{\beta^2B}{2}\left((u')^2+(\eta')^2\right) + \frac{1-\lambda^2}{2}\eta^2 + \frac{B}{2}u^2 + \frac{\lambda}{6}\left(1+\frac{1}{3\beta^2}\right)u^3 + \frac{1}{2}u^2\eta\right]' \\ -\lambda u\eta\eta' + \frac{\lambda}{3\beta^2}\eta u' = 0.\end{aligned} \tag{2.1.26}$$

Rewriting the last two terms as

$$-\lambda u\eta\eta' + \frac{\lambda}{3\beta^2}\eta u' = \left(-\frac{\lambda}{2}u\eta^2 + \frac{\lambda}{3\beta^2}\eta u\right)' + \frac{\lambda}{2}u'\eta^2 - \frac{\lambda}{3\beta^2}\eta' u$$

The definition of monotone front implies that $\frac{\lambda}{2}u'\eta^2 - \frac{\lambda}{3\beta^2}\eta' u \le 0$, and hence we have

$$\frac{1-\lambda^2}{2}\bar{\eta}^2 + \frac{B}{2}\bar{u}^2 + \frac{\lambda}{6}\left(1+\frac{1}{3\beta^2}\right)\bar{u}^3 + \frac{1}{2}\bar{u}^2\bar{\eta} - \frac{\lambda}{2}\bar{u}\bar{\eta}^2 + \frac{\lambda}{3\beta^2}\bar{u}\bar{\eta} \le 0.$$

Recalling (2.1.24) and the definition of B the above can be reduced to

$$\bar{u}\left(\frac{1}{4}\bar{\eta} + \frac{B}{2} + \frac{1-B}{6\lambda}\bar{u}\right) + \frac{\lambda}{3\beta^2}\bar{\eta} \le 0,$$

which further leads to

$$\bar{u}\left(\frac{B}{2} + \frac{1-B}{6\lambda}\bar{u}\right) - \left(\frac{1}{4}\bar{u} + \frac{1-B-\lambda^2}{\lambda}\right)\frac{\bar{u}^2}{2(1-\lambda^2-\lambda\bar{u})} \le 0.$$

Solving the above yields

$$\bar{u} \ge \frac{2\sqrt{4(1-B)^2(1-\lambda^2)^2 + 3(7-4B)B\lambda^2(1-\lambda^2)} - 4(1-B)(1-\lambda^2)}{\lambda(7-4B)}.$$

Combining this with (2.1.25) and explicitly solving the resulting inequality leads to

$$G(\lambda^2, t) \ge 0, \tag{2.1.27}$$

where $t = 1 + \frac{1}{3\beta^2} > 1$ and

$$G(z,t) := \left(-20 + \frac{13}{t}\right)z^3 + \left(-60 + \frac{33}{t} + 32t\right)z^2 + \left(-39 + \frac{18}{t} + 32t\right)z - 9.$$

Note that

$$G_z(0,t),\ G_{zz}(0,t) > 0.$$

For a fixed $t > 1$, $G_{zz}(z,t)$ is linear in z. Solving a quartic inequality we find that

$$G_{zz}\left(\frac{1}{t^2}, t\right) > 0 \quad \text{when} \quad t > 1.68.$$

Therefore when $t > 1.68$, $G_{zz}(z,t) > 0$ for $0 < z < \frac{1}{t^2}$, which implies that

$$G_z(z,t) > G_z(0,t) > 0 \quad \text{for} \quad 0 < z < \frac{1}{t^2}.$$

Looking at $G(z,t)$, we find that

$$G\left(\frac{1}{t^2}, t\right) < 0 \quad \text{when} \quad t > 2.264,$$

which leads to

$$G(z,t) < 0 \quad \text{for} \quad 0 < z < \frac{1}{t^2}.$$

Recalling the definition of t we immediately find that (2.1.27) fails when $\beta^2 < 0.26$. □

2.1.5 Global continuation

Having had the local theory carried out, we will extend the local solution curves constructed in Section 2.1.2 to the non-perturbative regime using a global implicit function theorem developed in [25].

In order to quote the result of [25], we define for $\alpha \in (0,1)$ the following Hölder space

$$\mathscr{X} := C^{2+\alpha}_{\mathrm{b,e}}(\mathbb{R}) \times C^{2+\alpha}_{\mathrm{b,e}}(\mathbb{R}), \qquad \mathscr{Y} := C^{\alpha}_{\mathrm{b,e}}(\mathbb{R}) \times C^{\alpha}_{\mathrm{b,e}}(\mathbb{R}),$$

where the subscript 'e' denotes the restriction to even functions. Also we will modify $\mathcal{O}$ as

$$\mathcal{O}_H := \left\{ (u,\eta,\lambda) \in \mathscr{X} \times \mathbb{R}^+ : \ 1 - \lambda^2 \left(1 + \frac{1}{3\beta^2} \right) > 0 \right\}.$$

We will consider our problem in the above spaces instead. Note that by elliptic regularity, the $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ solutions are indeed smooth, and hence $\mathcal{O} \cap \mathscr{F}^{-1}(0) = \mathcal{O}_H \cap \mathscr{F}^{-1}(0)$.

Theorem 2.1.5. *There exists a curve $\mathscr{C}^{\mathrm{slow}}$ containing $\mathscr{C}^{\mathrm{slow}}_{\mathrm{loc}}$, which admits a global C^0 parametrization*

$$\mathscr{C}^{\mathrm{slow}} := \{ (u(s), \eta(s), \lambda(s)) : \ s \in (0,\infty) \} \subset \mathcal{O} \cap \mathscr{F}^{-1}(0)$$

with $\lim_{s \searrow 0} (u(s), \eta(s), \lambda(s)) = \left(u_0^+, \eta_0, 0\right)$ and satisfies the following.

(a) *At each $s \in (0,\infty)$, the linearized operator $\mathscr{F}_{(u,\eta)}(u(s), \eta(s), \lambda(s)) \colon \mathscr{X} \times \mathbb{R}^+ \to \mathscr{Y}$ is Fredholm index 0.*

(b) *One of the following alternatives holds as $s \to \infty$.*

(A1) (Blowup) *The quantity*

$$N(s) := \|(u(s), \eta(s))\|_{\mathscr{X}} + \lambda(s) + \frac{1}{\operatorname{dist}((u(s), \eta(s), \lambda(s)),\, \partial\mathcal{O})} \to \infty. \qquad (2.1.28)$$

(A2) (Loss of compactness) *There exists a sequence $s_n \to \infty$ with $\sup_n N(s_n) < \infty$, but $(u^+(s_n), \eta(s_n), \lambda(s_n))$ has no convergent subsequence in $\mathscr{X} \times \mathbb{R}^+$.*

(A3) (Loss of Fredholmness) *There exists a sequence $s_n \to \infty$ with $\sup_n N(s_n) < \infty$ and so that $(u(s_n), \eta(s_n), \lambda(s_n)) \to (u_*, \eta_*, \lambda_*)$ in $\mathscr{X} \times \mathbb{R}^+$, however $\mathscr{F}_{(u,\eta)}(u_*, \eta_*, \lambda_*)$ is not Fredholm index 0.*

(A4) (Closed loop) *There exists* $T > 0$ *such that* $(u(s+T), \eta(s+T), \lambda(s+T)) = (u(s), \eta(s), \lambda(s))$ *for all* $s \in (0, \infty)$.

(c) *Near each point* $(u(s_0), \eta(s_0), \lambda(s_0)) \in \mathscr{C}^{\text{slow}}$*, we can locally reparameterize* $\mathscr{C}^{\text{slow}}$ *so that* $s \mapsto (u(s), \eta(s), \lambda(s))$ *is real analytic.*

Proof. The proof follows from [25, Theorem B.1] and [24, Theorem 6.1], since from Lemma 2.1.2 we know that $\mathscr{F}_{(u,\eta)}(u_0, \eta_0, \lambda_0) \colon \mathscr{X} \times \mathbb{R}^+ \to \mathscr{Y}$ is an isomorphism. □

To further winnow down the alternatives, notice that for any $(u(s), \eta(s), \lambda(s)) \in \mathscr{C}^{\text{slow}}$, $\mathscr{F}_{(u,\eta)}(u(s), \eta(s), \lambda(s))$ is Fredholm since it can be identified as a fourth order ODE operator. Since $\mathscr{F}_{(u,\eta)}(u_0^+, \eta_0, \lambda_0)$ is an isomorphism $\mathscr{X} \times \mathbb{R}^+ \to \mathscr{Y}$, a standard homotopy argument indicates that $\mathscr{F}_{(u,\eta)}(u(s), \eta(s), \lambda(s)) \colon \mathscr{X} \times \mathbb{R}^+ \to \mathscr{Y}$ is Fredholm index 0. Thus we know that (A3) does not occur.

The loop alternative (A4) can also be ruled out by the nodal property Lemma 2.1.6 combined with the uniqueness results Proposition 2.1.1 and Corollary 2.1.3.

As for (A2), we may invoke [24, Lemma 6.3] in our current setting to give the following

Lemma 2.1.8 (Compactness or front)**.** *Suppose that* $\{(u_n, \eta_n, \lambda_n)\} \subset \mathscr{F}^{-1}(0) \cap \mathcal{O}$ *satisfies*

$$\sup_{n \geq 1} \left(\|(u_n, \eta_n)\|_{\mathscr{X}} + \frac{1}{\operatorname{dist}((u_n, \eta_n, \lambda_n),\, \partial\mathcal{O})} \right) < \infty,$$

and each (u_n, η_n) *is strictly monotone in that* $\partial_x u_n < 0$, $\partial_x \eta_n > 0$ *for* $x > 0$*. Then, either*

(i) (Compactness) $\{(u_n, \eta_n, \lambda_n)\}$ *has a convergent subsequence in* $\mathscr{X} \times \mathbb{R}$*; or*

(ii) (Monotone front) *there exists a sequence of translations* $x_n \to +\infty$ *so that we can extract a convergent subsequence*

$$(u_n, \eta_n)(\,\cdot\, + x_n) \longrightarrow (u, \eta) \in C_{\mathrm{b}}^{2+\alpha}(\mathbb{R}) \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}), \qquad \lambda_n \longrightarrow \lambda,$$

with $(u, \eta, \lambda) \in \mathcal{O}_H$*. The limit is a* monotone front solution *of* (2.1.6) *in the sense of Definition 2.1.4.*

Proof. Given the assumptions of the lemma, we know that up to a subsequence $\lambda_n \to \lambda$ with $1 - \lambda^2 \left(1 + \frac{1}{3\beta^2}\right) > 0$. If (u_n, η_n) is equi-decaying in the sense that for any $\varepsilon > 0$ there exists some $R > 0$ such that

$$\sup_n \|(u_n, \eta_n)\|_{C^2((R,\infty))} < \varepsilon,$$

then obviously (u_n, η_n) has a convergent subsequence in $\mathscr{X}$, and hence leads to (i).

If (u_n, η_n) is not equi-decaying, then there exists some $\varepsilon_0 > 0$ and a sequence $\{x_n\}$ with $x_n \to +\infty$ such that for all $n \geq 1$,

$$\sup_{0 \leq i \leq 2} \left|\partial_x^i (u_n, \eta_n)(x_n)\right| \geq \varepsilon_0.$$

Set $(v_n, \zeta_n) := (u_n, \eta_n)(\,\cdot\, + x_n)$. Since (v_n, ζ_n) is uniformly bounded in $\mathscr{X}$, there is a subsequence, still denoted by the same labeling, $(v_n, \zeta_n) \to (u, \eta) \in \mathscr{X}$ in $C^2_{\text{loc}}(\mathbb{R})$. Local convergence is enough to ensure that (u^+, η) solves (2.1.6). The monotonicity of (u_n, η_n) confirms that

$$\partial_x u \leq 0, \qquad \partial_x \eta \geq 0.$$

By definition of (v_n, ζ_n) we see that

$$\left|\partial_x^i (u, \eta)(0)\right| \geq \varepsilon_0 \qquad \text{for some} \;\; i \leq 2.$$

Thus $(u, \eta) \not\equiv (0, 0)$. Maximum principle then implies that $u > 0$ and $\eta < 0$. □

Putting all of the above, we can finally arrive at our main result of this section.

Theorem 2.1.6 (Slow waves)**.** *For any β with $\beta^2 < 0.26$, the global curve $\mathscr{C}^{\text{slow}}$ constructed in Theorem 2.1.5 enjoys the following properties.*

(a) (Symmetry and monotonicity) *Each solution on $\mathscr{C}^{\text{slow}}$ is even and*

$$\begin{aligned} \eta(s) &< 0 \qquad u(s) > 0 \qquad &&\text{on } \mathbb{R}, \\ \partial_x \eta(s) &> 0 \qquad \partial_x u(s) < 0 \qquad &&\text{on } \mathbb{R}^+. \end{aligned} \tag{2.1.29}$$

(b) (Loss of ellipticity) *Following $\mathscr{C}^{\text{slow}}$ to its extreme, the system loses ellipticity in that*

$$\lim_{s \to \infty} \lambda(s) = \left(1 + \frac{1}{3\beta^2}\right)^{-1/2}. \tag{2.1.30}$$

Proof. Note that property (a) follows from the nodal properties Lemma 2.1.6. From the previous discussion, at the extreme of the solution curve, (A3) and (A4) cannot occur. Lemma 2.1.8 together with Lemma 2.1.7 rules out (A2). Therefore we are only left with alternative blowup alternative. Since λ is always bounded in $\mathcal{O}$, one can remove $\lambda(s)$ from the blowup quantity in (2.1.28).

From the local uniqueness and the nodal properties we know that $\lim_{s\to\infty}\lambda(s) > 0$. So if (2.1.30) is false, then there exists a sequence $\{s_n\}$, $s_n \to \infty$ with the corresponding solutions $(u_n, \eta_n, \lambda_n) := (u, \eta, \lambda)(s_n) \in \mathcal{O} \cap \mathscr{F}^{-1}(0)$ such that

$$\lambda_n \to \lambda_* < \left(1 + \frac{1}{3\beta^2}\right)^{-1/2}, \qquad \|(u_n, \eta_n)\|_{\mathscr{X}} \to \infty. \tag{2.1.31}$$

Moreover $\lambda_* > 0$. Elliptic regularity implies that $\|(u_n, \eta_n)\|_{C^0} \to \infty$. From (a), this is equivalent to

$$u_n(0) - \eta_n(0) \to \infty.$$

From the second equation in (2.1.6) and the fact that $(\eta_n - \lambda_n u_n)(0) = \min_{x\in\mathbb{R}}(\eta_n - \lambda_n u_n)$, it follows that

$$\frac{1}{2}u_n^2(0) + (\eta_n - \lambda_n u_n)(0) = \beta^2(\eta_n - \lambda_n u_n)''(0) \geq 0.$$

From this it must hold that

$$u_n(0) \to \infty.$$

Similarly, evaluating (2.1.14) at $x = 0$ and using that $u_n''(0) \leq 0$ we find that

$$\eta_n(0) \leq -\frac{1 - \lambda_n^2\left(1 + \frac{1}{3\beta^2}\right) + \frac{\lambda_n}{2}\left(1 + \frac{1}{3\beta^2}\right)u_n(0)}{\frac{\lambda_n}{3\beta^2} + u_n(0)}u_n(0).$$

For n sufficiently large, from the above we have

$$\eta_n(0) \leq -\frac{2\lambda_n}{5}\left(1 + \frac{1}{3\beta^2}\right)u_n(0). \tag{2.1.32}$$

Recall (2.1.26). Integrating the equation over $(0, \infty)$ we find that

$$\frac{1 - \lambda_n^2}{2}\eta_n^2(0) + \frac{1}{2}u_n^2(0)\left[B + \frac{\lambda_n}{3}\left(1 + \frac{1}{3\beta^2}\right)u_n(0) + \eta_n(0)\right] > 0.$$

From (2.1.32) we obtain

$$\frac{1-\lambda_n^2}{2}\eta_n^2(0) + \frac{1}{2}u_n^2(0)\left[B - \frac{\lambda_n}{15}\left(1+\frac{1}{3\beta^2}\right)u_n(0)\right] > 0. \tag{2.1.33}$$

Further using (2.1.6) we have that for any $\delta > 0$,

$$\begin{aligned}\mathcal{L}\left\{\delta\lambda_n\left(1+\frac{1}{3\beta^2}\right)\eta_n + \left[1-(1+\delta)\lambda_n^2\left(1+\frac{1}{3\beta^2}\right)\right]u_n\right\}\\ + \left(\frac{\lambda_n}{3\beta^2}+u_n\right)\eta_n + \frac{1}{2}(1+\delta)\lambda_n\left(1+\frac{1}{3\beta^2}\right)u_n^2 = 0.\end{aligned} \tag{2.1.34}$$

From (2.1.31) there exists $\delta_0 > 0$ such that for n sufficiently large,

$$1-(1+\delta)\lambda_n^2\left(1+\frac{1}{3\beta^2}\right) > 0 \qquad \text{for all } \ 0<\delta<\delta_0.$$

From (2.1.33) we see that

$$|\eta_n(0)| = O(|u_n(0)|^{3/2}) \qquad \text{as } \ n\to\infty.$$

Therefore for any $0<\delta<\delta_0$ there exists some n_0 large enough such that for $n \geq n_0$

$$\delta\lambda_n\left(1+\frac{1}{3\beta^2}\right)\eta_n(0) + \left[1-(1+\delta)\lambda_n^2\left(1+\frac{1}{3\beta^2}\right)\right]u_n(0) < 0.$$

Denote $x_n \in [0,\infty)$ the point where $\delta\lambda_n\left(1+\frac{1}{3\beta^2}\right)\eta_n + \left[1-(1+\delta)\lambda_n^2\left(1+\frac{1}{3\beta^2}\right)\right]u_n$ achieves its minimum. Then it holds that $\eta_n(0) \leq \eta_n(x_n) < 0$, and

$$\delta\lambda_n\left(1+\frac{1}{3\beta^2}\right)\eta_n(x_n) \leq \delta\lambda_n\left(1+\frac{1}{3\beta^2}\right)\eta_n(0) + \left[1-(1+\delta)\lambda_n^2\left(1+\frac{1}{3\beta^2}\right)\right][u_n(0)-u_n(x_n)].$$

From this we conclude that

$$|\eta_n(x_n)| = O(|u_n(0)|^{3/2}) \qquad \text{as } \ n\to\infty. \tag{2.1.35}$$

Evaluating equation (2.1.34) at x_n indicates that

$$\left(\frac{\lambda_n}{3\beta^2}+u_n(x_n)\right)\eta_n(x_n) + \frac{1}{2}(1+\delta)\lambda_n\left(1+\frac{1}{3\beta^2}\right)u_n^2(x_n) > 0,$$

which contradicts the asymptotics (2.1.35). □

2.2 Bifurcation from classical Boussinesq supercritical waves

In this section we focus on fast traveling solitary waves with wave speed $\lambda > 1$. Different from the previous section, here we will consider the wave speed as given, and restrict the four parameters (a, b, c, d) on a one-parameter curve to perform the bifurcation. The base point of the bifurcation corresponds to the solution to the classical Boussinesq system which has $a = b = c = 0$ adn $d = \frac{1}{3}$ in (1.1.1) (see, for example, [3, 14, 54, 63]). As is discussed in [19], the solitary waves (u_f, η_f) satisfy

$$\begin{cases} (u_f')^2 = \dfrac{1}{\lambda}\left(-u_f^3 + 3\lambda u_f^2 + 6u_f + 6\lambda \log\left|\dfrac{\lambda - u_f}{\lambda}\right|\right), \\ \eta_f = \dfrac{u_f}{\lambda - u_f}. \end{cases} \tag{2.2.1}$$

From classical ODE technique one obtains that for any $\lambda > 1$ there exists a unique solution $(u_f, \eta_f) \in H^2_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{e}}(\mathbb{R})$ such that

$$\begin{cases} u_f, \eta_f \text{ are both monotonically decreasing from their crests at } x = 0, \text{ and} \\ \qquad\qquad \dfrac{1}{2}\left(3\lambda - \sqrt{\lambda^2 + 8}\right) < \max\limits_{x\in\mathbb{R}} |u_f| < \lambda. \end{cases} \tag{2.2.2}$$

2.2.1 Local solutions

Now for any fixed $k > 0$ with $k < \lambda$, consider the parameter curve

$$a = c = ks, \qquad b = s, \qquad d = \frac{1}{3} - (2k+1)s. \tag{2.2.3}$$

Thus $b = d$ only when $2(k+1)s = \frac{1}{3}$. So in particular $b \neq d$ when s is small. Moreover we also allow a, c to be negative.

Similar as before, in this parameter regime we can rewrite (1.1.3) as

$$\mathscr{F}(U, s) := \begin{pmatrix} ksu'' + \lambda s\eta'' + u - \lambda\eta + \eta u, \\ \\ \left[\dfrac{1}{3} - (2k+1)s\right]\lambda u'' + ks\eta'' - \lambda u + \eta + \dfrac{1}{2}u^2 \end{pmatrix} = 0, \tag{2.2.4}$$

with $\mathscr{F} : H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathbb{R} \to L^2(\mathbb{R})$.

The existence of solitary waves in the parameter regime (2.2.3) is stated as follows

Theorem 2.2.1 (Fast waves near the Boussinesq solutions)**.** *For any $\lambda > 1$, let k be such that $0 < k < \lambda$. Suppose that the parameters of* (1.1.3) *satisfy* (2.2.3). *Then there exist some positive $\delta > 0$ and a unique C^0 solution curve*

$$\mathscr{C}_{\mathrm{loc}}^{\mathrm{fast}} = \{(u_s, \eta_s, s) : \ |s| < \delta\} \subset H_{\mathrm{e}}^2(\mathbb{R}) \times H_{\mathrm{e}}^2(\mathbb{R}) \times \mathbb{R}$$

to problem (2.2.4) *with the property that*

$$(u_s, \eta_s) = (u_f, \eta_f) + O(s) \qquad \textit{in} \quad H_{\mathrm{e}}^2(\mathbb{R}) \times H_{\mathrm{e}}^2(\mathbb{R}), \tag{2.2.5}$$

$$u_s, \ \eta_s > 0 \qquad \textit{for} \ \ s \geq 0, \tag{2.2.6}$$

where (u_f, η_f) is the unique solution to (2.2.1) *satisfying* (2.2.2).

Proof. Denote $U_f := (u_f, \eta_f)$. Working with even functions, direct computation yields that

$$\mathscr{F}_U(U_f, 0) = \begin{pmatrix} 1 + \eta_f & u_f - \lambda \\ \dfrac{\lambda}{3}\partial_x^2 + u_f - \lambda & 1 \end{pmatrix} : H_{\mathrm{e}}^2(\mathbb{R}) \times H_{\mathrm{e}}^2(\mathbb{R}) \to L_{\mathrm{e}}^2(\mathbb{R}) \times L_{\mathrm{e}}^2(\mathbb{R}).$$

Suppose that $\mathscr{F}_U(U_f, 0)[V] = 0$ for some $V = (v, \zeta) \in H_{\mathrm{e}}^2(\mathbb{R}) \times H_{\mathrm{e}}^2(\mathbb{R})$. Writing out the equations we have

$$(1 + \eta_f)v + (u_f - \lambda)\zeta = 0,$$
$$\frac{\lambda}{3}v'' + (u_f - \lambda)v + \zeta = 0,$$

from which we can solve ζ from the first equation and obtain a single ODE for v as

$$\frac{\lambda}{3}v'' + \left[\frac{\lambda}{(\lambda - u_f)^2} - (\lambda - u_f)\right] v = 0. \tag{2.2.7}$$

Since $\lambda > 1$ and u_f satisfies (2.2.2), from the classical ODE theory we know that there is only one bounded nontrivial solution to the above equation. On the other hand from the translation invariance of (2.2.4) we see that u_f' solves (2.2.7). From the fact that $u_f \in H_{\mathrm{e}}^2(\mathbb{R})$, it follows that $\ker \mathscr{F}_U(U_f, 0)$ is trivial in $H_{\mathrm{e}}^2(\mathbb{R}) \times H_{\mathrm{e}}^2(\mathbb{R})$.

Since $\mathscr{F}_U(U_f, 0)$ is self-adjoint on $L_{\mathrm{e}}^2(\mathbb{R}) \times L_{\mathrm{e}}^2(\mathbb{R})$, we further conclude that it is invertible. Therefore the existence of the local solution curve and (2.2.5) follows from the Implicit Function Theorem.

Next let's turn to the sign property. For any $\varepsilon > 0$ we can find an $R_0 > 0$ such that

$$|u_f(x)|,\ |\eta_f(x)| < \varepsilon \qquad \text{for} \quad |x| > R_0.$$

From (2.2.5) we know that by choosing s sufficiently small,

$$\begin{cases} u_s(x),\ \eta_s(x) > 0 & \text{for} \quad |x| \le R_0, \\ |u_s(x)|,\ |\eta_s(x)| < \varepsilon + O(s) & \text{for} \quad |x| > R_0. \end{cases} \tag{2.2.8}$$

From (2.2.4) we have

$$\left(\left[\frac{1}{3} - (2k+1)s\right]\lambda^2 - k^2 s\right) u_s'' - \left[(k+\lambda^2) - \frac{\lambda}{2}u_s\right] u_s + \left[\lambda + k(\lambda - u_s)\right]\eta_s = 0. \tag{2.2.9}$$

So if $\inf_{x\in\mathbb{R}} u_s = u_s(x_0) < 0$, then from (2.2.8) $|x_0| > R_0$. For small s, maximum principle implies that $\eta_s(x_0) < 0$ and

$$u_s(x_0) \ge \frac{\lambda + k(\lambda - u_s(x_0))}{k + \lambda^2 - \lambda u_s(x_0)/2}\eta_s(x_0) > \eta_s(x_0),$$

since

$$0 < \frac{\lambda + k(\lambda - u_s(x_0))}{k + \lambda^2 - \lambda u_s(x_0)/2} < 1$$

for sufficiently small ε and s.

Since $\eta_s(x_0) < 0$, we know that $\inf_{x\in\mathbb{R}} \eta_s = \eta_s(x_1) \le \eta_s(x_0) < 0$ for $|x_1| > R_0$. Looking at the equation for η_s

$$\begin{aligned}\left(\left[\frac{1}{3} - (2k+1)s\right]\lambda^2 s - k^2 s^2\right)\eta_s'' - \left(\left[\frac{1}{3} - (2k+1)s\right]\lambda^2 + ks - \left[\frac{1}{3} - (2k+1)s\right]\lambda u_s\right)\eta_s \\ + \left(\left[\frac{1}{3} - (2k+1)s\right]\lambda + \lambda ks - \frac{ks}{2}u_s\right)u_s = 0,\end{aligned} \tag{2.2.10}$$

it follows that for $s > 0$ small, at x_1 we have $u_s(x_1) < 0$, and

$$\eta_s(x_1) \ge \frac{\left[\frac{1}{3} - (2k+1)s\right]\lambda + \lambda ks - \frac{ks}{2}u_s(x_1)}{\left[\frac{1}{3} - (2k+1)s\right]\lambda^2 + ks - \left[\frac{1}{3} - (2k+1)s\right]\lambda u_s(x_1)} u_s(x_1) > u_s(x_1).$$

The last estimate holds because for ε, s sufficiently small the fraction can be made between 0 and 1. However this would lead to a contradiction since

$$\inf_{x\in\mathbb{R}} u_s = u_s(x_0) \ge \eta_s(x_0) \ge \inf_{x\in\mathbb{R}} \eta_s = \eta_s(x_1) > u_s(x_1).$$

Therefore we have proved that for $s > 0$ sufficiently small, $u_s \geq 0$. A similar argument yields that $\eta_s \geq 0$ as well.

If there is a point x_* where $u_s(x_*) = 0$, then the above argument shows that $\eta_s(x_*) = 0$, and hence x_* is a minimum point for u_s and η_s, indicating that $u_s'(x_*) = \eta_s'(x_*) = 0$. Thus from uniqueness of ODE it must hold that $u_s = \eta_s \equiv 0$, which is a contradiction. This proves (2.2.6). □

Similarly as in Section 2.1, we have the following argument about the local monotonicity and local uniqueness.

Corollary 2.2.2 (Local monotonicity and local uniqueness)**.** *Denote $\mathcal{B}_r$ the ball of radius $r > 0$ in $H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathbb{R}$ centered at $(u_f, \eta_f, 0)$. There exists $\varepsilon > 0$ such that for $s > 0$,*

$$\mathcal{F}^{-1}(0) \cap \mathcal{B}_\varepsilon = \mathscr{C}_{\text{loc}}^{\text{fast}} \cap \mathcal{B}_\varepsilon \tag{2.2.11}$$

In addition, every solution $(u, \eta, s) \in \mathcal{F}^{-1}(0) \cap \mathcal{B}_\varepsilon$ is strictly monotone in that for $x > 0$,

$$u' < 0 \qquad \text{and} \qquad \eta' < 0. \tag{2.2.12}$$

Proof. Similarly as the proof of the sign property in Theorem 2.2.1, for $(u, \eta, s) \in \mathcal{F}^{-1}(0) \cap \mathcal{B}_\varepsilon$, we have $u, \eta > 0$. Thus it suffices to check conditions (i)–(iii) of Theorem 2.1.2. Writing (2.2.9) and (2.2.10) as

$$\begin{cases} \left(\left[\dfrac{1}{3} - (2k+1)s \right] \lambda^2 s - k^2 s^2 \right) u'' + g(u, \eta) = 0, \\ \\ \left(\left[\dfrac{1}{3} - (2k+1)s \right] \lambda^2 s - k^2 s^2 \right) \eta'' + f(u, \eta) = 0, \end{cases}$$

direct computation yields that:

$$\begin{aligned} &\frac{\partial g}{\partial \eta} = \lambda + \lambda k - ku, &&\frac{\partial f}{\partial u} = \left[\frac{1}{3} - (2k+1)s \right] (\lambda + \eta) + ks(\lambda - u), \\ &\frac{\partial g}{\partial u}(0,0) = -(\lambda^2 + k), &&\frac{\partial f}{\partial \eta}(0,0) = -\left[\frac{1}{3} - (2k+1)s \right] \lambda^2 - ks. \end{aligned} \tag{2.2.13}$$

When ε is chosen sufficiently small, conditions (i)–(iii) of Theorem 2.1.2 are satisfied. □

Remark 2.2.1. In the proof above we used a relaxed version of condition (i) which only requires that $\dfrac{\partial g}{\partial v}(u_\alpha, v), \dfrac{\partial f}{\partial u}(u, v_\alpha)$ are non-negative for $(u, v) \in [0, \infty) \times [0, \infty)$ where (u_α, v_α), where (u_α, v_α) are reflection of the solution of the elliptic system with repect to the line $x = \alpha$.

2.2.2 Nodal pattern and monotone fronts

Now for each fixed $\lambda, k \in \mathbb{R}^+$ we introduce the set

$$\mathcal{O} := \left\{(u, \eta, s) \in H^2_{\mathrm{e}}(\mathbb{R}) \times H^2_{\mathrm{e}}(\mathbb{R}) \times \mathbb{R}^+ : s \in \Gamma_1, u \in \Gamma_2\right\}. \tag{2.2.14}$$

where

$$\Gamma_1 := \left\{ s \in \mathbb{R}^+ : \frac{\lambda^2}{3} - \left[(2k+1)\lambda^2 + k^2\right] s > 0 \right\}, \quad \Gamma_2 := \left\{ u : \|u\|_{L^\infty(\mathbb{R})} < \lambda \right\}.$$

The intuition for the choice of $\mathcal{O}$ is that Γ_1 is needed for the ellipticity, and Γ_2 provides a sufficient condition to ensure conditions (i)–(iii) in Theorem 2.1.2, in particular the condition (i) for (strict) quasi-monotonicity. Indeed from (2.2.13) we see that $\partial_\eta g, \partial_u f > 0$ when

$$u < \lambda + \frac{\lambda}{k} \qquad \text{and} \qquad u < \lambda + \frac{\frac{1}{3} - (2k+1)s}{ks}(\lambda + \eta).$$

Moreover, this constraint also allows one to deduce from (2.2.10) an upper bound for η

$$\eta_s(0) \leq \frac{\left[\frac{1}{3} - (2k+1)s\right]\lambda + \left(\lambda - \frac{u_s(0)}{2}\right) ks}{\left[\frac{1}{3} - (2k+1)s\right]\lambda\big(\lambda - u_s(0)\big) + ks} u_s(0). \tag{2.2.15}$$

Constraint Γ_2 can also be understood as a "no stagnation" condition and indicates that the particles travel behind the wave.

From Theorem 2.2.1 and Corollary 2.2.2 we are led to consider the following nodal property:

$$u > 0, \quad \eta > 0 \qquad \text{in } \mathbb{R}, \tag{2.2.16a}$$

$$u' < 0, \quad \eta' < 0 \qquad \text{in } \mathbb{R}^+, \tag{2.2.16b}$$

Similarly as in the previous section, we can prove that the above nodal property persists on the solution curve. The proof follows along the same line as the one in Lemma 2.1.6, and hence we omit it.

Lemma 2.2.1 (Nodal property). *If $\mathcal{K}$ is any connected subset of $\mathcal{O} \cap \mathscr{F}^{-1}(0)$ that contains $\mathscr{C}^{\text{fast}}_{\text{loc}}$, then every $(u, \eta, \lambda) \in \mathcal{K}$ exhibits* (2.2.16).

The next step regards the nonexistence of monotone fronts, which will provide useful information for the global theory. As in Section 2.1.4, we define the concept of monotone fronts as follows.

Definition 2.2.3. *Let* $s \in \Gamma_1, u \in \Gamma_2$*. we say* (u, η, λ) *is a monotone front solution of* (2.2.4) *if* $(u, \eta) \in C^2_{\mathrm{b}}(\mathbb{R}) \times C^2_{\mathrm{b}}(\mathbb{R})$*, and*

$$\lim_{x \to +\infty} (u(x), \eta(x)) = (0, 0), \quad \textit{and} \quad u > 0, \quad \eta > 0, \quad u' \leq 0, \quad \eta' \leq 0 \quad \textit{in} \ \mathbb{R}. \tag{2.2.17}$$

Lemma 2.2.2 (Nonexistence of monotone fronts)**.** *If* $s \in \Gamma_1, u \in \Gamma_2$ *then system* (2.2.4) *does not admit any monotone front solution in the sense of* (2.2.17)*.*

Proof. The proof is similar to Lemma 2.1.7 but the algebra is simpler. Suppose (u, η) is a monotone front solution to (2.2.4). Let

$$(\bar{u}, \bar{\eta}) := \lim_{x \to -\infty} (u(x), \eta(x)).$$

So $\bar{u}, \bar{\eta} > 0$. Evaluating (2.2.4) at $x \to -\infty$ implies that

$$\bar{u} = \frac{3\lambda - \sqrt{\lambda^2 + 8}}{2} \tag{2.2.18}$$

and

$$\bar{\eta} = \frac{\bar{u}}{\lambda - \bar{u}}. \tag{2.2.19}$$

Multiplying the first equation of (2.2.4) by η' and second by u', and summing them up, it follows that

$$\begin{aligned} &\left[\frac{\left(\frac{1}{3} - (2k+1)s\right)\lambda}{2}(u')^2 + \frac{\lambda s}{2}(\eta')^2 + ksu'\eta' + u\eta - \frac{\lambda}{2}u^2 - \frac{\lambda}{2}\eta^2 + \frac{1}{6}u^3 \right]' \\ &+ u\eta\eta' = 0 \end{aligned} \tag{2.2.20}$$

Write $u\eta\eta' = \left(\frac{1}{2}u\eta^2\right)' - \frac{1}{2}u'\eta^2$, by definition of monotone front, we have $u'\eta^2 \leq 0$ and thus

$$\bar{u}\bar{\eta} - \frac{\lambda}{2}\bar{u}^2 - \frac{\lambda}{2}\bar{\eta}^2 + \frac{1}{6}\bar{u}^3 + \frac{1}{2}\bar{u}\bar{\eta}^2 \geq 0 \tag{2.2.21}$$

From (2.2.19) it can be reduced to

$$(\lambda - \bar{u})\left(\frac{\bar{u}}{3} - \lambda\right) + 1 \geq 0$$

and from (2.2.18) we finally have

$$\frac{1}{12}\left(-\lambda+\sqrt{\lambda^2+8}\right)\left(-3\lambda-\sqrt{\lambda^2+8}\right)+1\geq 0$$

the above inequality holds only when $\lambda \leq 1$, which contradicts the fact that $\lambda > 1$. $\square$

2.2.3 Global continuation

As in Section 2.1.5, with Lemma 2.2.1 and 2.2.2, we obtain the following global solution curve:

Theorem 2.2.4. *There exists a curve $\mathscr{C}^{\text{fast}}$ containing $\mathscr{C}^{\text{fast}}_{\text{loc}}$, which admits a global C^0 parametrization*

$$\mathscr{C}^{\text{fast}} := \{(u(t),\eta(t),s(t)):\ t\in(0,\infty)\}\subset \mathcal{O}\cap\mathscr{F}^{-1}(0)$$

with $\lim_{t\searrow 0}(u(t),\eta(t),s(t)) = (u_f,\eta_f,0)$ *and satisfies the following property:*

(a) (Symmetry and monotonicity) *Each solution on $\mathscr{C}^{\text{fast}}$ is even and*

$$\begin{aligned} u(t) &> 0 \qquad \eta(t) > 0 \qquad &&\text{on } \mathbb{R},\\ \partial_x u(t) &< 0 \qquad \partial_x\eta(t) < 0 \qquad &&\text{on } \mathbb{R}^+. \end{aligned}$$

(b) (Loss of ellipticity or stagnation limit) *Following $\mathscr{C}^{\text{fast}}$ to its extreme, either the system loses ellipticity in that*

$$\lim_{t\to\infty} s(t) = \frac{\lambda^2}{3\left((2k+1)\lambda^2+k^2\right)}, \tag{2.2.22}$$

or we encounter waves that are arbitrarily close to having a stagnation point

$$\lim_{t\to\infty}\inf_{x\in\mathbb{R}}(\lambda-u(t)) = 0. \tag{2.2.23}$$

Proof. We will only focus on proving (b). Since $\lim_{t\to\infty} s(t) > 0$, so if (2.2.22) is false, we can find a sequence $\{t_n\} \to \infty$ with the corresponding solutions denoted by $(u_n, \eta_n, s_n) \in \mathcal{O} \cap \mathscr{F}^{-1}(0)$ such that as $n \to \infty$,

$$s_n \to s_* < \frac{\lambda^2}{3\left((2k+1)\lambda^2 + k^2\right)}, \qquad \text{either } \ \|(u_n, \eta_n)\|_{\mathscr{X}} \to \infty \text{ or } \ \|u_n\|_{L^\infty} \to \lambda.$$

Recall from (2.2.15) the upper bound for η_n

$$\bar{\eta}_n \le \frac{\bar{s}_n\lambda + \left(\lambda - \frac{\bar{u}_n}{2}\right) ks_n}{\bar{s}_n\lambda(\lambda - \bar{u}_n) + ks_n}\bar{u}_n \le \frac{2\bar{s}_n\lambda + \lambda ks_n}{2ks_n}\lambda,$$

where $\bar{s}_n := \frac{1}{3} - (2k+1)s_n$. Elliptic regularity then asserts that the latter alternative can be replaced by

$$\|u_n\|_{L^\infty} \to \lambda,$$

proving (2.2.23).

□

3.0 Transverse instability of the CH-KP-I equation

Our proof is based on the pioneering work of Rousset-Tzvetkov [56, 57]. Their main idea is to first construct a most unstable eigenmode, and then prove that the nonlinear effect can be dominated by the linear effect, in the spirit of center manifold theory. The method works perfectly well for semilinear equations. However due to the nature of quasilinearity in our equation, we need to make necessary changes. The strategy is as follows: as in [56, 57], the first step is to prove the linear instability by finding one unstable eigenvalue. Our method relies on [58]. By taking Fourier transform with respect to y, the problem is transformed to finding a positive eigenvalue σ corresponding to one frequency k. To handle this problem, it suffices to know the distribution of spectrum as k evolves. The key issue is that for each k, the spectrum of the corresponding operator is hard to investigate compared with that of the KdV equation. Thus we have turned the problem to a generalized eigenvalue problem for a self-adjoint operator, and the spectrum of self-adjoint operator has much better property.

The second step is to prove the nonlinear instability based on the linear result. First, we choose the most unstable eigenmode v^0. Then we will prove that the solution $u^\delta = \phi + v^\delta$ with initial data $\phi + \delta v^0(0, \cdot)$ could lead to (1.2.2). The estimate is based on the approximation procedure first constructed by Grenier [35]. In details, the approximation of v^δ can be written as $v^{ap} = \delta \left(v^0 + \sum_{k=1}^{M} \delta^k v^k \right)$. Since the nonlinearity of (4.0.1) is power-like, by matching the orders of δ, it turns out that this approximate scheme is iterative. Unlike Picard iteration for the center-manifold theory, each v^k in this scheme solves a differential equation. The main reason why we choose this approximation scheme instead of the semi-group estimate is due to the high nonlinearity. For the semi-group estimates, since we couldn't have an explicit form of the semi-group, it is hard to conduct delicate analysis to close the energy estimates because of the loss of derivative. While for Grenier's approach, since for the jth iteration, v^j is just a finite combination of the Fourier modes, it allows us to use energy estimates to overcome this difficulty. The rest of the proof consists of two parts. We first estimate v^k and show that it can be controlled by v^0. Then an error estimate will follow. For the first part, by the Laplace transform, the original estimate for v^k could be transformed to a resolvent estimate. The difficulty comes from higher order estimates. Compared with the KP-I equation in [56], (4.0.1) has stronger nonlinearity, and the corresponding linearized operator is weakly dispersive and nonlocal, making the energy estimates more challenging. What we do is to utilize the strong "smoothing" property together with a new cancellation mechanism resulting from the special structure of the nonlinearity. In this way, we are able to close the estimate at each iteration step. Finally the roughness of the energy estimates can be compensated by going to sufficiently high order approximation.

The rest of the chapter is organized as follows. In Section 3.1, we present some notation, the Hamiltonian formulation and some preliminary results. In Section 3.2, we will prove the linear instability. In Section 3.3, we will prove the nonlinear instability based on the linear instability. Several existence results will be given in the appendix.

3.1 Preliminary

3.1.1 Notation

We denote $|\cdot|_s$ for $\|\cdot\|_{H^s(\mathbb{R})}$ and $\|\cdot\|_s$ for $\|\cdot\|_{H^s(\mathbb{R}\times\mathbb{T}_a)}$, where $a=\frac{2\pi}{k_0}$ and k_0 will be given later. We also denote $\langle\cdot,\cdot\rangle$ the inner product of $L^2(\mathbb{R})$. Finally, denote ϕ for ϕ_c for simplicity, where $\phi_c(x,y)$ is the line solitary wave of (4.0.1) with $\phi_c(x,y)=Q_c(x)$, and Q_c represents the solitary wave of the CH equation with speed $c>2\kappa$. In the following, we will abuse using the notation of ϕ and Q_c for convenience.

3.1.2 Hamiltonian formulation

The CH-KP-I equation (4.0.1) can be written in the Hamiltonian form:

$$\begin{aligned} u_t &= \mathcal{J}\frac{\delta\mathcal{H}}{\delta u} \\ &= \left(1-\partial_x^2\right)^{-1}\partial_x\left(\frac{1}{2}u_x^2+uu_{xx}-2\kappa u-\frac{3}{2}u^2+\partial_x^{-2}\partial_y^2 u\right), \end{aligned} \tag{3.1.1}$$

where

$$\begin{aligned} \mathcal{J} &= \left(1-\partial_x^2\right)^{-1}\partial_x, \\ \mathcal{H} &= -\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_a}\left[u^3+uu_x^2+2\kappa u^2-\left(\partial_x^{-1}\partial_y u\right)^2\right]dxdy, \end{aligned} \tag{3.1.2}$$

and $\mathcal{H}$ is a conserved energy. A change of variable from x to $x-ct$ yields that

$$\begin{aligned} u_t &= \mathcal{J}\frac{\delta(\mathcal{H}+c\mathcal{Q})}{\delta u} \\ &= \left(1-\partial_x^2\right)^{-1}\partial_x\left(\frac{1}{2}u_x^2+uu_{xx}-2\kappa u-\frac{3}{2}u^2+\partial_x^{-2}\partial_y^2 u+c(u-u_{xx})\right), \end{aligned} \tag{3.1.3}$$

where $\mathcal{Q}=\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_a}\left(u^2+u_x^2\right)dxdy$ is called the impulse which is another conserved quantity. A line solitary wave ϕ with speed c can be regarded as a critical point of $\mathcal{H}+c\mathcal{Q}$:

$$\frac{\delta(\mathcal{H}+c\mathcal{Q})}{\delta u}[\phi]=0. \tag{3.1.4}$$

3.1.3 Preliminary results

We will collect some results that will be used later.

Proposition 3.1.1 ([29]). *The line solitary wave ϕ with speed $c>2\kappa$ satisfies the following property:*

1. *It is smooth and positive with an even profile decreasing from its peak of height $c-2\kappa$.*
2. *It is concave when $\phi\in\left(c-\frac{\kappa}{2}-\sqrt{c\kappa+\frac{\kappa^2}{4}},c-2\kappa\right)$ and convex elsewhere.*
3. *$\phi\sim\exp\left(-\sqrt{1-\frac{2\kappa}{c}}|x|\right)$ for $|x|\to\infty$.*

Theorem 3.1.1 ([29]). *For the linearized operator H_c of the CH equation about the solitary wave ϕ: $H^1(\mathbb{R})\to H^{-1}(\mathbb{R})$,*

$$H_c=-\partial_x\left((c-\phi)\partial_x\right)+\phi''-3\phi-2\kappa+c, \tag{3.1.5}$$

it has exactly one simple negative eigenvalue, one simple zero eigenvalue and the rest of the spectrum is positive and bounded away from zero.

3.2 Linear instability

In this section, we will first prove the linear instability, from which we will construct a most unstable eigenmode in the next section. Denote $v = u - \phi$, the linearized equation of (3.1.3) about ϕ is

$$\partial_t v = \mathcal{J}\mathcal{L}v, \tag{3.2.1}$$

where

$$\mathcal{L} = -\partial_x\left((c-\phi)\partial_x\right) + (\phi'' - 3\phi - 2\kappa + c) + \partial_x^{-2}\partial_y^2. \tag{3.2.2}$$

Let $v = e^{\sigma t}e^{iky}U$, then

$$\sigma U = \mathcal{J}(k) \circ \mathcal{L}(k)U,$$

where

$$\mathcal{L}(k) = e^{-iky}\mathcal{L}e^{iky} = -\partial_x((c-\phi)\partial_x) + (\phi'' - 3\phi - 2\kappa + c) - k^2\partial_x^{-2}, \quad \mathcal{J}(k) = \mathcal{J}.$$

Let $U = \mathcal{J}^*(k)W$. Then

$$\sigma\mathcal{J}(k)^*W = \tilde{\mathcal{L}}(k)W, \tag{3.2.3}$$

where $\tilde{\mathcal{L}}(k) = \mathcal{J}(k)\mathcal{L}(k)\mathcal{J}^*(k)$. The proof of the linear instability is based on the following theorem:

Theorem 3.2.1 ([58]). *Assume the following conditions:*

1. *There exist $K > 0$ and $\alpha > 0$ such that $\tilde{\mathcal{L}}(k) \geq \alpha Id$ for $|k| \geq K$;*
2. *The essential spectrum of $\tilde{\mathcal{L}}(k)$ is included in $[c_k, +\infty)$ with $c_k > 0$ for $k \neq 0$;*
3. *For every $k_1 \geq k_2 \geq 0$, we have $\tilde{\mathcal{L}}(k_1) \geq \tilde{\mathcal{L}}(k_2)$. In addition, if for some $k > 0$ and $U \neq 0$, we have $\tilde{\mathcal{L}}(k)U = 0$, then $\left\langle \tilde{\mathcal{L}}'(k)U, U\right\rangle > 0$;*
4. *The spectrum of $\tilde{\mathcal{L}}(0)$ is under the form $\{-\lambda\} \cup I$ where $-\lambda < 0$ is an isolated simple eigenvalue and I is included in $[0, +\infty)$.*

Then there exist $\sigma > 0$, $k \neq 0$ and U solving (3.2.3).

Proposition 3.2.1. *(Existence of an unstable eigenmode) If $c > 2\kappa > 0$, there exists one unstable eigenmode for* (3.2.1).

Proof. According to Theorem 3.2.1, it suffices to verify conditions (1)-(4) for $\tilde{\mathcal{L}}(k): H^2(\mathbb{R}) \to L^2(\mathbb{R})$.

(1) It is easy to see that

$$\begin{aligned}\left\langle \tilde{\mathcal{L}}(k)u, u \right\rangle =& \left\langle (c-\phi)\left(1-\partial_x^2\right)^{-1} u_{xx}, \left(1-\partial_x^2\right)^{-1} u_{xx} \right\rangle \\ &+ \left\langle (\phi'' - 3\phi - 2\kappa + c)\left(1-\partial_x^2\right)^{-1} u_x, \left(1-\partial_x^2\right)^{-1} u_x \right\rangle \\ &+ k^2 \left\langle \left(1-\partial_x^2\right)^{-1} u, \left(1-\partial_x^2\right)^{-1} u \right\rangle \\ \geq& \kappa \left| \left(1-\partial_x^2\right)^{-1} u_{xx} \right|_0^2 - \alpha \left| \left(1-\partial_x^2\right)^{-1} u_x \right|_0^2 + k^2 \left| \left(1-\partial_x^2\right)^{-1} u \right|_0^2 .\end{aligned}$$

Here we require $\kappa > 0$ and have used Proposition 3.1.1. Then since $\left|(1-\partial_x^2)^{-1} u_x\right|_0$ can be controlled by $\left|(1-\partial_x^2)^{-1} u_{xx}\right|_0$ and $\left|(1-\partial_x^2)^{-1} u\right|_0$, (1) is verified.

(2) Consider $n(k): \;\; H^4(\mathbb{R}) \to L^2(\mathbb{R})$ as

$$\tilde{\mathcal{L}}(k) = \left(1-\partial_x^2\right)^{-1} n(k) \left(1-\partial_x^2\right)^{-1},$$

thus

$$n(k) = -\partial_x \left(-\partial_x \left((c-\phi)\partial_x\right) + (\phi'' - 3\phi - 2\kappa + c)\right)\partial_x + k^2.$$

It can be seen that $\tilde{\mathcal{L}}(k)$ and $n(k)$ have the same Fredholm index since $(1-\partial_x^2)^{-1} : H^{s-2}(\mathbb{R}) \to H^s(\mathbb{R})$ has Fredholm index 0. Thus the essential spectrum of $\tilde{\mathcal{L}}(k)$ and $n(k)$ are the same. By Weyl's lemma, we just need to study the essential spectrum of the limiting operator

$$n_\infty(k) = c\partial_x^4 - (c-2\kappa)\partial_x^2 + k^2.$$

Consider $(n_\infty(k) - \beta)u = f$, since $c > 2\kappa$. Using Fourier transform, we have the essential spectrum lying in $[c_k, \infty)$ for some $c_k > 0$.

(3) Direct computation yields

$$\left\langle \tilde{\mathcal{L}}'(k)u, u \right\rangle = 2k \left\langle \left(1-\partial_x^2\right)^{-1} u, \left(1-\partial_x^2\right)^{-1} u \right\rangle > 0 \;\; \text{for } k > 0.$$

(4) The proof follows from the discussion on KP-I in [58]. Observe from (3.1.5) that we can write $\tilde{\mathcal{L}}(0)$ as $-\mathcal{J}H_c\mathcal{J}$. By Theorem 3.1.1, it has a unique simple negative eigenvalue with the associated eigenvector ψ. By the approximation argument, we have $\mathcal{J}u_n$ tending to ψ, and

$$\left\langle \tilde{\mathcal{L}}(0)u_n, u_n \right\rangle = \langle H_c\mathcal{J}u_n, \mathcal{J}u_n\rangle < 0$$

for n sufficiently large. Thus $\tilde{\mathcal{L}}(0)$ has at least one negative eigenvalue. On the other hand, for u with $\langle \mathcal{J}u, \psi\rangle = 0$, we have $\left\langle \tilde{\mathcal{L}}(0)u, u \right\rangle = \langle H_c\mathcal{J}u, \mathcal{J}u\rangle \geq 0$. Thus we conclude that $\tilde{\mathcal{L}}(0)$ just has one negative eigenvalue which is simple. □

3.3 Nonlinear instability

3.3.1 Construction of a most unstable eigenmode

As discussed in the previous section, there exists an unstable mode $k_0 \neq 0$ which indicates the linear instability. In the rest of the chapter, we consider the period with respect to y to be $a = \frac{2\pi}{k_0}$. Let

$$v = e^{\sigma t}e^{imk_0 y}U(m, x), \quad m \in \mathbb{Z},$$

which solves $\partial_t v = \mathcal{J}\mathcal{L}v$. By Fourier transforming with respect to y, we have

$$\sigma U = \mathcal{J}\mathcal{L}(mk_0)U. \tag{3.3.1}$$

The construction of a most unstable eigenmode is based on the following lemma:

Lemma 3.3.1 ([57]). *Consider the problem* (3.3.1). *There exists $K > 0$ such that for $|mk_0| \geq K$, there is no nontrivial solution with $Re(\sigma) \neq 0$. In addition, for every $k \neq 0$, there is at most one unstable mode with corresponding transverse frequency k.*

Remark 3.3.1. The proof of Lemma 3.3.1 is based on the fact that $\mathcal{L}(mk_0)$ is positive definite, which is easy to check.

According to Lemma 3.3.1, σ_0, U_0 can be chosen corresponding to the maximal m_0, and the most unstable eigenmode v^0 can be written as

$$v^0 = 2\mathrm{Re}(e^{\sigma_0 t} e^{im_0 k_0 y} U_0).$$

We now construct the unstable solution u^δ with initial data $\phi + \delta v^0(0,x,y)$. Let $v = u^\delta - \phi$, then it satisfies

$$\partial_t v = \mathcal{J}\mathcal{L}v + \mathcal{J}\left(\frac{1}{2}v_x^2 + v v_{xx} - \frac{3}{2}v^2\right), \quad v(0,x,y) = \delta v^0(0,x,y). \tag{3.3.2}$$

Thus in order to prove the nonlinear instability of (4.0.1), it suffices to study the behavior of v.

3.3.2 Construction of a high order unstable approximate solution

Define V_K^s as the following truncated space:

$$V_K^s = \left\{ u : \ u = \sum_{j=-K}^{j=K} u_j e^{ijm_0 k_0 y}, \ u_j \in H^s(\mathbb{R}) \right\},$$

where the norm on V_K^s is defined by $|u|_{V_K^s} = \sup_j |u_j|_s$. It can be seen that $v^0 \in V_1^s$ for all $s \in \mathbb{N}$. We look for a high order approximate solution

$$v^{ap} = \delta\left(v^0 + \sum_{k=1}^{M} \delta^k v^k\right), \quad v^k \in V_{k+1}^{s-k}.$$

By matching the orders of δ, it yields that

$$\begin{cases} \partial_t v^k = \mathcal{J}\mathcal{L}v^k + \mathcal{J}\left[\frac{1}{2}\left(\sum_{j+l=k-1} v_x^j v_x^l\right) + \sum_{j+l=k-1} v^j v_{xx}^l - \frac{3}{2}\sum_{j+l=k-1} v^j v^l\right], \\ v^k|_{t=0} = 0 \end{cases} \tag{3.3.3}$$

for $1 \le k \le M$.

Proposition 3.3.1. *Let v^k be the solution of (3.3.3), if $s - k \ge 0$ then*

$$\left|v^k\right|_{V_{k+1}^{s-k}} \le C_{M,s} e^{(k+1)\sigma_0 t}, \tag{3.3.4}$$

where $C_{M,s} > 0$ depends on the approximation order M and regularity s.

Remark 3.3.2. This proposition implies that the effect of v^k can be controlled by v^0.

Indeed, the above proposition can be easily derived by induction from the following theorem:

Theorem 3.3.1. *Consider the solution u of the linear problem*

$$\partial_t u = \mathcal{J}\mathcal{L}u + \mathcal{J}F \tag{3.3.5}$$

with $F \in V_K^{s-1}$ and

$$|F|_{V_K^{s-1}} \leq C_{K,s}e^{\gamma t},\ \gamma \geq 2\sigma_0,$$

then $u \in V_K^s$ and satisfies the estimate

$$|u|_{V_K^s} \leq C_{K,s}e^{\gamma t}.$$

By Fourier transforming with respect to y, we have

$$\partial_t u_j = \mathcal{J}\mathcal{L}(jm_0k_0)u_j + \mathcal{J}F_j, \quad u_j|_{t=0} = 0. \quad \text{for } j = 1\cdots, K, \tag{3.3.6}$$

where u_j, F_j are the jth Fourier modes in y of u and F respectively. Thus the problem is equivalent to proving that if

$$|F_j|_{s-1} \leq C_{K,s}e^{\gamma t},\ \gamma \geq 2\sigma_0, \quad \text{for } j = 1\cdots, K, \tag{3.3.7}$$

then

$$|u_j|_s \leq C_{K,s}e^{\gamma t}, \quad \text{for } j = 1\cdots, K. \tag{3.3.8}$$

Lemma 3.3.2 (Existence of u_j)**.** *For any $s \in \mathbb{R}$, there exists a unique $u_j \in H^s(\mathbb{R})$ solving* (3.3.6)*.*

Proof. The proof of Lemma 3.3.2 is postponed in Appendix B.1. □

To prove (3.3.8), we first give a resolvent estimate. Take γ_0 such that $\sigma_0 < \gamma_0 < \gamma$. For $T > 0$, we introduce

$$G = 0,\ t < 0; \qquad G = 0,\ t > T; \qquad G = F_j,\ t \in [0, T], \tag{3.3.9}$$

then (3.3.6) can be written as

$$\partial_t \tilde{u}_j = \mathcal{J}\mathcal{L}(jm_0k_0)\tilde{u}_j + \mathcal{J}G, \quad \tilde{u}_j|_{t=0} = 0.$$

where $\tilde{u}_j$ is the extension of u_j such that

$$\tilde{u}_j|_{0\le t\le T} = u_j, \quad \tilde{u}_j|_{t<0} = 0.$$

Then the Laplace transform yields that

$$(\gamma_0 + i\tau)w = \mathcal{J}\mathcal{L}(jm_0k_0)w + \mathcal{J}H, \tag{3.3.10}$$

where

$$w = \int_{t\ge 0} e^{-(\gamma_0+i\tau)}\tilde{u}_j dt, \qquad H = \int_{t\ge 0} e^{-(\gamma_0+i\tau)}G dt.$$

Here for simplicity, we denote w as the Laplace transform of $\tilde{u}_j$ for each given j.

Theorem 3.3.2 (Resolvent estimate). *Let $s \ge 1$. Let w be the solution of* (3.3.10)*, then there exists a constant $C(s, \gamma_0, K)$ such that for every τ, we have the estimate*

$$|w|_s^2 \le C(s, \gamma_0, K)|H|_{s-1}^2. \tag{3.3.11}$$

We will split the proof of the above theorem into Lemma 3.3.3 and Lemma 3.3.4.

Lemma 3.3.3. *There exist $M > 0$ and $C(s, \gamma_0, K)$ such that for $|\tau| \ge M$, we have*

$$|w|_s^2 \le C(s, \gamma_0, K)|H|_{s-1}^2. \tag{3.3.12}$$

Proof. First prove the case when $s = 1$. Write

$$\mathcal{L}(jm_0k_0) = L - (jm_0k_0)^2\partial_x^{-2}$$

where

$$L = -\partial_x((c-\phi)\partial_x) + (\phi'' - 3\phi - 2\kappa + c)\,.$$

Then we decompose

$$w = \alpha\phi_{-1} + \beta\phi_0 + w_\perp \tag{3.3.13}$$

such as

$$L\phi_{-1} = \mu\phi_{-1},\ \mu < 0; \quad L\phi_0 = 0; \quad \langle Lw_\perp, w_\perp\rangle \geq c_\perp w_\perp^2,\ c_\perp > 0. \tag{3.3.14}$$

By Theorem 3.1.1, such a decomposition is available. Taking the inner product of (3.3.10) with $\mathcal{L}(jm_0k_0)$ yields that

$$\gamma_0\left(\langle w, Lw\rangle + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2\right) = \operatorname{Re}\left(\langle\mathcal{J}H, Lw\rangle + \left\langle\mathcal{J}H, (jm_0k_0)^2\partial_x^{-2}w\right\rangle\right). \tag{3.3.15}$$

By (3.3.14) and (3.3.15), we have

$$\gamma_0\left(\mu\alpha|\phi_{-1}|_0^2 + c_\perp|w_\perp|_0^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2\right) \leq C\left(|H|_0|w|_1 + (jm_0k_0)^2|H|_{-2}\left|\partial_x^{-1}w\right|_0\right),$$

then

$$|w_\perp|_0^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2 \leq C\left(|\alpha|^2 + |H|_{-2}^2 + |H|_0|w|_1\right). \tag{3.3.16}$$

Taking the inner product of (3.3.10) with ϕ_{-1} and ϕ_0 respectively, we have

$$\begin{aligned}(\gamma_0 + i\tau)\alpha &= -\langle w, L\mathcal{J}\phi_{-1}\rangle - (jm_0k_0)^2\left\langle\mathcal{J}\partial_x^{-2}w, \phi_{-1}\right\rangle + \langle\mathcal{J}H, \phi_{-1}\rangle\,,\\ (\gamma_0 + i\tau)\beta &= -\langle w, L\mathcal{J}\phi_0\rangle - (jm_0k_0)^2\left\langle\mathcal{J}\partial_x^{-2}w, \phi_0\right\rangle + \langle\mathcal{J}H, \phi_0\rangle\,.\end{aligned}$$

Rewriting w as (3.3.13) for the first term on the right-hand side and combining the above two equations, we have

$$(\gamma_0 + |\tau|)(|\alpha| + |\beta|) \leq C\left(|\alpha| + |\beta| + |w_\perp|_0 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0 + |H|_{-2}\right). \tag{3.3.17}$$

Multiplying $|\alpha| + |\beta|$ to (3.3.17) and by Cauchy-Schwarz inequality it follows that

$$(\gamma_0 + |\tau| - C)\left(|\alpha|^2 + |\beta|^2\right) \le C\left(|w_\perp|_0^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2 + |H|_{-2}^2\right),$$

which is a good estimate when $|\tau|$ is large. For a sufficiently large constant B, consider B(3.3.16)+(3.3.17):

$$\begin{aligned}(B - C)&\left(|w_\perp|_0^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2\right) + (\gamma_0 + |\tau| - BC - C)\left(|\alpha|^2 + |\beta|^2\right)\\ \le\ & BC\left(|H|_{-2}^2 + |H|_0|w|_1\right).\end{aligned}$$

When $|\tau| > C + BC$ we have

$$|w|_0^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2 \le C\left(|H|_0|w|_1 + |H|_{-2}^2\right). \tag{3.3.18}$$

On the other hand,

$$\begin{aligned}\langle w, Lw\rangle &= \langle w, -\partial_x\left((c-\phi)\partial_x w\right) + \left(\phi'' - 3\phi - 2\kappa + c\right)w\rangle\\ &\ge a_1|w|_1^2 + \langle\left(\phi'' - 3\phi - 2\kappa + c\right)w, w\rangle\end{aligned} \tag{3.3.19}$$

for $a_1 > 0$. Replacing $\langle w, Lw\rangle$ in (3.3.15) with (3.3.19), we have

$$|w|_1^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2 \le C\left(|w|_0^2 + |H|_{-2}^2 + |H|_0|w|_1\right). \tag{3.3.20}$$

Combining (3.3.18) and (3.3.20) yields

$$|w|_1^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2 \le C\left(|H|_{-2}^2 + |H|_0|w|_1\right).$$

Consequently,

$$|w|_1^2 + (jm_0k_0)^2\left|\partial_x^{-1}w\right|_0^2 \le C|H|_0^2,$$

which proves the case $s = 1$.

For higher order estimates, (3.3.10) can be written as

$$\begin{aligned}(\gamma_0 + i\tau)w = &\left(1 - \partial_x^2\right)^{-1}\left(-\partial_x^2\right)\left[(c-\phi)w_x\right]\\ &+ \left(1 - \partial_x^2\right)^{-1}\partial_x\left[\left(\phi'' - 3\phi - 2\kappa + c\right)w\right]\\ &- (jm_0k_0)^2\left(1 - \partial_x^2\right)^{-1}\partial_x^{-1}w + \left(1 - \partial_x^2\right)^{-1}\partial_x H.\end{aligned} \tag{3.3.21}$$

For the first term on the right-hand side, we can rewrite it as

$$\left(1-\partial_x^2\right)^{-1}\left(-\partial_x^2\right)\left((c-\phi)w_x\right)=(c-\phi)w_x-\left(1-\partial_x^2\right)^{-1}\left((c-\phi)w_x\right). \tag{3.3.22}$$

By induction, assume (3.3.12) is true for s. We prove that it is true for $s+1$. In the rest of this proof, we denote $O(\partial_x^k w)$ as generic polynomial differential operator on w with highest degree k.

Take the inner product of (3.3.21) with

$$(-1)^{s+1}\partial_x^{2s+2}w+(-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right),$$

where $r_{s+1}(x)$ is bounded which will be determined later. We have

$$\begin{aligned}
&\operatorname{Re}\left\langle(\gamma_0+i\tau)w,(-1)^{s+1}\partial_x^{2s+2}w\right\rangle && (3.3.23)\\
&\quad=\ \gamma_0|w|_{s+1}^2,\\
&\operatorname{Re}\left\langle(c-\phi)w_x,(-1)^{s+1}\partial_x^{2s+2}w\right\rangle && (3.3.24)\\
&\quad=\ \operatorname{Re}\left\langle\left(\partial^{s+1}(c-\phi)w_x\right),\partial_x^{s+1}w\right\rangle\\
&\quad=\ \operatorname{Re}\left\langle(c-\phi)\partial_x^{s+2}w-(s+1)\phi'\partial_x^{s+1}w+O(\partial_x^s w),\partial_x^{s+1}w\right\rangle\\
&\quad=\ \left\langle\left(\frac{1}{2}-(s+1)\right)\phi'\partial_x^{s+1}w+O(\partial_x^s w),\partial_x^{s+1}w\right\rangle\\
&\quad=\ -\left\langle\left(s+\frac{1}{2}\right)\phi'\partial_x^{s+1}w+O(\partial_x^s w),\partial_x^{s+1}w\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\left\langle(\gamma_0+i\tau)w,(-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right)\right\rangle && (3.3.25)\\
&\quad=\ \gamma_0\left\langle r_{s+1}(x)\partial_x^{s+1}w,\partial_x^{s+1}w\right\rangle,\\
&\operatorname{Re}\left\langle(c-\phi)w_x,(-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right)\right\rangle && (3.3.26)\\
&\quad=\ \operatorname{Re}\left\langle r_{s+1}(x)\partial_x^{s+1}\left((c-\phi)w_x\right),\partial_x^{s+1}w\right\rangle\\
&\quad=\ \operatorname{Re}\Big\langle r_{s+1}(x)\big((c-\phi)\partial_x^{s+2}w-(s+1)\phi' r_{s+1}(x)\partial_x^{s+1}w\\
&\qquad+O(\partial_x^s w)\big),\partial_x^{s+1}w\Big\rangle\\
&\quad=\ \operatorname{Re}\left\langle\left(\frac{1}{2}\left(\phi' r_{s+1}(x)-r_{s+1}'(x)(c-\phi)\right)-(s+1)\phi' r_{s+1}(x)\right)\partial_x^{s+1}w\right.\\
&\qquad\left.+O(\partial_x^s w),\partial_x^{s+1}w\right\rangle,
\end{aligned}$$

and

$$\left\langle \left(1-\partial_x^2\right)^{-1}((c-\phi)w_x), (-1)^{s+1}\partial_x^{2s+2}w + (-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right)\right\rangle \tag{3.3.27}$$

$$= \left\langle O(\partial_x^s w), \partial_x^{s+1}w\right\rangle,$$

$$\left\langle \left(1-\partial_x^2\right)^{-1}\partial_x\left(\left(\phi''-3\phi-2\kappa+c\right)w\right), (-1)^{s+1}\partial_x^{2s+2}w + (-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right)\right\rangle$$

$$= \left\langle O(\partial_x^s w), \partial_x^{s+1}w\right\rangle,$$

$$\left\langle (jm_0k_0)^2\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}w, (-1)^{s+1}\partial_x^{2s+2}w + (-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right)\right\rangle$$

$$= \left\langle O(\partial_x^{s-1} w), \partial_x^{s+1}w\right\rangle,$$

$$\left\langle \left(1-\partial_x^2\right)^{-1}\partial_x H, (-1)^{s+1}\partial_x^{2s+2}w + (-1)^{s+1}\partial_x^{s+1}\left(r_{s+1}(x)\partial_x^{s+1}w\right)\right\rangle$$

$$= \left\langle O(\partial_x^s H), \partial_x^{s+1}w\right\rangle.$$

We want to use $r_{s+1}(x)$ to eliminate $-(s+\frac{1}{2})\phi'$ in (3.3.24) with (3.3.25), (3.3.26). On the other hand, since $\phi' \to 0$ when $|x| \to \infty$, by (3.3.24), there exists $A > 0$ such that $\gamma_0 + (s+\frac{1}{2})\phi' > 0$ when $|x| > A$. Then we want $r_{s+1}(x)$ to satisfy when $x > -A$,

$$\begin{aligned} -\left(s+\frac{1}{2}\right)\phi' - \gamma_0 r_{s+1}(x) + \frac{1}{2}\left(\phi' r_{s+1}(x) - r'_{s+1}(x)(c-\phi)\right) \\ - (s+1)\phi' r_{s+1}(x) = 0, \end{aligned} \tag{3.3.28}$$

and when $x < -A$

$$\begin{aligned} -\gamma_0 r_{s+1}(x) + \frac{1}{2}(\phi' r_{s+1}(x) - r'_{s+1}(x)(c-\phi)) \\ - (s+1)\phi' r_{s+1}(x) \le \gamma_0 + (s+\frac{1}{2})\phi'. \end{aligned} \tag{3.3.29}$$

One choice could be $r_{s+1}(x) = 0$ when $x \le -A$ and $r_{s+1}(x)$ satisfies (3.3.28) when $x > -A$. Note that (3.3.28) can be written in a form of Bernoulli equation:

$$r'_{s+1}(x) + \frac{2\gamma_0 + (2s+1)\phi'}{c-\phi} r_{s+1}(x) = -(2s+1)\frac{\phi'}{c-\phi}.$$

So when $x > -A$

$$r_{s+1}(x) = -(2s+1)e^{-\int_{-A}^{x}\frac{2\gamma_0+(2s+1)\phi'}{c-\phi}}\int_{-A}^{x} e^{\int_{-A}^{t}\frac{2\gamma_0+(2s+1)\phi'}{c-\phi}}\frac{\phi'}{c-\phi}dt$$

and $r_{s+1}(s)$ is bounded. Indeed, when $x \to +\infty$, $\frac{2\gamma_0+(2s+1)\phi'}{c-\phi}$ is positive, and the forcing term $-(2s+1)\frac{\phi'}{c-\phi} \to 0$, which will prevent $|r_{s+1}(x)| \to \infty$.

So by (3.3.23)-(3.3.27) and (3.3.28)

$$\gamma_0|w|_{s+1}^2 = \mathrm{Re}\left(\left\langle O(\partial_x^s w), \partial_x^{s+1} w\right\rangle + \left\langle O(\partial_x^s H), \partial_x^{s+1} w\right\rangle\right).$$

Since $|w|_k$ is bounded by $|w|_{s+1}$ and $|w|_1$ for $1 < k \leq s$, by Cauchy-Schwartz inequality

$$|w|_{s+1}^2 \leq C|H|_s^2,$$

which proves the lemma. □

Lemma 3.3.4. *For $|\tau| \leq M$, we have*

$$|w|_s^2 \leq C(s, \gamma_0, K, M)|H|_{s-1}^2.$$

Write $\sigma = \gamma_0 + i\tau$ and impose $\left(1-\partial_x^2\right)\partial_x$ on (3.3.10)

$$\begin{aligned}\sigma\left(1-\partial_x^2\right)w_x = &-\partial_x^3\left((c-\phi)w_x\right) + \partial_x^2\left((\phi''-3\phi-2\kappa+c)w\right)\\ &-(jm_0k_0)^2 w + \partial_x^2 H.\end{aligned}$$

Then

$$\begin{aligned}(c-\phi)\partial_x^4 w = &\left(3\phi'+\sigma\right)\partial_x^3 w + \left(3\phi''+(\phi''-3\phi-2\kappa+c)\right)\partial_x^2 w\\ &+\left(\phi'''-\sigma+2\left(\phi''-3\phi-2\kappa+c\right)'\right)\partial_x w\\ &+\left((\phi''-3\phi-2\kappa+c)''-(jm_0k_0)^2\right)w + \partial_x^2 H.\end{aligned}$$

Let $V = (w, \partial_x w, \partial_x^2 w, \partial_x^3 w)^T$. We have

$$\frac{dV}{dx} = A(x,\sigma,j)V + \partial_x^2 H.$$

Here

$$A(x,\sigma,j) = \frac{1}{c-\phi}\begin{pmatrix} 0 & c-\phi & 0 & 0\\ 0 & 0 & c-\phi & 0\\ 0 & 0 & 0 & c-\phi\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix},$$

where

$$A_{41} = (\phi'' - 3\phi - 2\kappa + c)'' - (jm_0k_0)^2,$$

$$A_{42} = \phi''' - \sigma + 2\left(\phi'' - 3\phi - 2\kappa + c\right)',$$

$$A_{43} = 3\phi'' + \left(\phi'' - 3\phi - 2\kappa + c\right),$$

$$A_{44} = 3\phi' + \sigma,$$

and the limiting matrix

$$A_\infty(\sigma, j) = \frac{1}{c}\begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ -(jm_0k_0)^2 & -\sigma & -2\kappa + c & \sigma \end{pmatrix}.$$

The proof of Lemma 3.3.4 is based on the following lemma:

Lemma 3.3.5 ([57])**.** *Assume* $|A(x, \sigma, j) - A_\infty(\sigma, j)| \le Ce^{-\alpha|x|}$ *and the spectrum of* $A_\infty(\sigma, j)$ *doesn't meet the imaginary axis for Re*$(\sigma) > 0$. *Then*

$$|w|_s \le C_{j,K,s}|H|_{s-1}.$$

Remark 3.3.3. The statement of the lemma is slightly different from [57, Lemma 4.2], but it is essentially the same.

Based on the above statement, to prove Lemma 3.3.4, it suffices to show that the spectrum of $A_\infty(\sigma, j)$ doesn't intersect the imaginary axis for $\mathrm{Re}(\sigma) > 0$.

Proof of Lemma 3.3.4. The characteristic polynomial of $A_\infty(\sigma, j)$ can be written as

$$c\lambda^4 - \sigma\lambda^3 - (c - 2\kappa)\lambda^2 + \sigma\lambda + (jm_0k_0)^2,$$

which doesn't have imaginary root for all j. □

Now we are ready to show (3.3.8) and thus Theorem 3.3.1.

Proof of Theorem 3.3.1. By Theorem 3.3.2 and Bessel-Parseval identity, for $T > 0$

$$\int_0^T e^{-2\gamma_0 t}|u_j(t)|_s^2 dt \le \int_0^\infty e^{-2\gamma_0 t}|\tilde{u}_j(t)|_s^2 dt = \int_{\mathbb{R}} |w(\tau)|_s^2 d\tau$$
$$\le C \int_{\mathbb{R}} |H(\tau)|_{s-1}^2 d\tau = \int_0^T e^{-2\gamma_0 t} |F_j(t)|_{s-1}^2 \, dt.$$

From (3.3.7) we have

$$\int_0^T e^{-2\gamma_0 t}|u_j(t)|_s^2 dt \le C \int_0^T e^{2(\gamma-\gamma_0)t} dt \le C e^{2(\gamma-\gamma_0)T}. \tag{3.3.30}$$

From (3.3.6), by the similar argument as in (3.3.24)-(3.3.26), we can obtain the following H^s estimate

$$\frac{d}{dt}|u_j(t)|_s^2 \le C \left(|u_j|_s^2 + |F_j(t)|_{s-1}^2\right) \le C|u_j(t)|_s^2 + Ce^{2\gamma t},$$

and then

$$\frac{d}{dt}\left(e^{-2\gamma_0 t}|u_j(t)|_s^2\right) \le C\left(e^{-2\gamma_0 t}|u_j(t)|_s^2 + e^{2(\gamma-\gamma_0)t}\right).$$

Integrating the above in time and by (3.3.30), it follows that

$$|u_j(t)|_s^2 \le Ce^{2\gamma t},$$

which proves (3.3.8). □

3.3.3 Error estimate and final result

In this subsection, we will first give an error estimate and then prove Theorem 1.2.1.

Let u^δ be decomposed as

$$u^\delta = Q + v^{ap} + w.$$

From (3.3.2), w satisfies

$$\begin{cases} \partial_t w = \mathcal{J}\mathcal{L}w + \mathcal{J}\left(\dfrac{1}{2}w_x^2 + v_x^{ap}w_x + v_{xx}^{ap}w + (w+v^{ap})w_{xx} - \dfrac{3}{2}w^2 - 3v^{ap}w\right) \\ \qquad\quad + G, \\ w|_{t=0} \; = 0, \end{cases} \tag{3.3.31}$$

where

$$G = -\partial_t v^{ap} + \mathcal{J}\mathcal{L}v^{ap} + \mathcal{J}\left(\frac{1}{2}(v_x^{ap})^2 + v^{ap}v_{xx}^{ap} - \frac{3}{2}(v^{ap})^2\right).$$

The existence of w in (3.3.31) will be proved in the Appendix B.2. By Proposition 3.3.1, we have

$$\|G\|_s \le C_{M,s}\delta^{M+2}e^{(M+2)\mathrm{Re}(\sigma_0)t}. \tag{3.3.32}$$

The following priori estimate will be crucial for the proof of the instability result.

Lemma 3.3.6. *Let $w \in H^s(\mathbb{R}\times\mathbb{T}_a)$ satisfy* (3.3.31)*, then*

$$\frac{d}{dt}\|w\|_s^2 \le C_1\left(C_{2,M,s} + \|w\|_s\right)\|w\|_s^2 + C_{3,M,s}\delta^{2(M+2)}e^{2(M+2)Re(\sigma_0)t}. \tag{3.3.33}$$

Proof. In this proof, we denote ∂^k as the derivative with order k, $O(\partial^k w)$ as the polynomial differential operator on w with highest order k, and $\langle \cdot, \cdot \rangle$ as the inner product in $L^2(\mathbb{R} \times \mathbb{T}_a)$. It suffices to consider the estimate for the highest order s. Apply $\partial_x^\alpha \partial_y^\beta$ on (3.3.31) where $\alpha + \beta = s$ and take inner product with $\partial_x^\alpha \partial_y^\beta w$. Here we choose $s \geq 2$.

For the first term on the right-hand side of (3.3.31), by (3.3.22)

$$
\begin{aligned}
&\left\langle \partial_x^\alpha \partial_y^\beta \mathcal{J} \mathcal{L} w, \partial_x^\alpha \partial_y^\beta w \right\rangle \\
&= \left\langle \partial_x^\alpha \partial_y^\beta \left((c-\phi) w_x\right) + \mathcal{J} O(\partial_x^\alpha \partial_y^\beta w), \partial_x^\alpha \partial_y^\beta w \right\rangle \\
&= \left\langle (c-\phi) \partial_x^{\alpha+1} \partial_y^\beta w + O(\partial_x^\alpha \partial_y^\beta w) + \mathcal{J} O(\partial_x^\alpha \partial_y^\beta w), \partial_x^\alpha \partial_y^\beta w \right\rangle \\
&= \left\langle \frac{1}{2} \phi' \partial_x^\alpha \partial_y^\beta w + O(\partial_x^\alpha \partial_y^\beta w) + \mathcal{J} O(\partial_x^\alpha \partial_y^\beta w), \partial_x^\alpha \partial_y^\beta w \right\rangle \\
&\leq C \|w\|_s^2
\end{aligned}
\tag{3.3.34}
$$

since $\mathcal{J}$ is bounded on $H^s(\mathbb{R} \times \mathbb{T}_a)$.

For the second term on the right-hand side of (3.3.31), rewrite it as

$$
\begin{aligned}
&\mathcal{J} \left(\frac{1}{2} w_x^2 + v_x^{ap} w_x + v_{xx}^{ap} w + (w + v^{ap}) w_{xx} - \frac{3}{2} w^2 - 3 v^{ap} w \right) \\
&= \left(1 - \partial_x^2\right)^{-1} \partial_x \left(-\frac{1}{2} w_x^2 + v_{xx}^{ap} w + ((w + v^{ap}) w_x)_x - \frac{3}{2} w^2 - 3 v^{ap} w \right) \\
&= -\left(w + v^{ap}\right) w_x + \left(1 - \partial_x^2\right)^{-1} \partial_x \left(-\frac{1}{2} w_x^2 + v_{xx}^{ap} w - \frac{3}{2} w^2 - 3 v^{ap} w \right),
\end{aligned}
\tag{3.3.35}
$$

then

$$
\begin{aligned}
&\left\langle \partial_x^\alpha \partial_y^\beta \left((w + v^{ap}) w_x\right), \partial_x^\alpha \partial_y^\beta w \right\rangle \\
&\leq \Big\langle (w + v^{ap}) \partial_x^{\alpha+1} \partial_y^\beta w + w_x \partial_x^\alpha \partial_y^\beta (w + v^{ap}) \\
&\quad + s \sum_{j+k=s-1} \partial^1 (w + v^{ap}) \partial_x^{j+1} \partial_y^k w + O(\partial^{[\frac{s}{2}]+1}(w + v^{ap})) O(\partial^{s-1} w), \partial_x^\alpha \partial_y^\beta w \Big\rangle \\
&\leq \Big\langle -\frac{1}{2} (w_x + v_x^{ap}) \partial_x^\alpha \partial_y^\beta w + w_x \partial_x^\alpha \partial_y^\beta (w + v^{ap}) \\
&\quad + s \sum_{j+k=s-1} \partial^1 (w + v^{ap}) \partial_x^{j+1} \partial_y^k w + O(\partial^{[\frac{s}{2}]+1}(w + v^{ap})) O(\partial^{s-1} w), \partial_x^\alpha \partial_y^\beta w \Big\rangle \\
&\leq C \left\| O(\partial^{[\frac{s}{2}]+1}(w + v^{ap})) \right\|_{L^\infty} \|w\|_s^2 \leq C_1 (C_{2,M,s} + \|w\|_s) \|w\|_s^2,
\end{aligned}
\tag{3.3.36}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\left\langle\partial_x^\alpha\partial_y^\beta\left(\left(1-\partial_x^2\right)^{-1}\partial_x w_x^2\right),\partial_x^\alpha\partial_y^\beta w\right\rangle &= \frac{1}{2}\left\langle\partial_x^\alpha\partial_y^\beta w_x^2,\left(1-\partial_x^2\right)^{-1}\partial_x\partial_x^\alpha\partial_y^\beta w\right\rangle \qquad (3.3.37)\\
\leq\ & \Big\langle w_x\partial_x^{\alpha+1}\partial_y^\beta w+2s\sum_{i+j=s-1}\partial^2 w\partial_x^{j+1}\partial_y^k w\\
&+O(\partial^{[\frac{s}{2}]+1}w)O(\partial^{s-1}w)),\left(1-\partial_x^2\right)^{-1}\partial_x\partial_x^\alpha\partial_y^\beta w\Big\rangle\\
\leq\ & \left\langle w_x\partial_x^{\alpha+1}\partial_y^\beta w,\left(1-\partial_x^2\right)^{-1}\partial_x\partial_x^\alpha\partial_y^\beta w\right\rangle+C\left\|O(\partial^{[\frac{s}{2}]+1}w)\right\|_{L^\infty}\|w\|_s^2\\
=\ & -\left\langle\partial_x^\alpha\partial_y^\beta w,w_x\left(1-\partial_x^2\right)^{-1}\partial_x^2\partial_x^\alpha\partial_y^\beta w+w_{xx}\left(1-\partial_x^2\right)^{-1}\partial_x\partial_x^\alpha\partial_y^\beta w\right\rangle\\
&+C\left\|O(\partial^{[\frac{s}{2}]+1}w)\right\|_{L^\infty}\|w\|_s^2\\
\leq\ & -\left\langle\partial_x^\alpha\partial_y^\beta w,w_x\left(1-\partial_x^2\right)^{-1}\left(\partial_x^2-1+1\right)\partial_x^\alpha\partial_y^\beta w\right\rangle+C\left\|O(\partial^{[\frac{s}{2}]+1}w)\right\|_{L^\infty}\|w\|_s^2\\
\leq\ & C\left\|O(\partial^{[\frac{s}{2}]+1}w)\right\|_{L^\infty}\|w\|_s^2\ \leq\ C_1\left(C_{2,M,s}+\|w\|_s\right)\|w\|_s^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\langle\partial_x^\alpha\partial_y^\beta\left(\left(1-\partial_x^2\right)^{-1}\partial_x\left(v_{xx}^{ap}w-\frac{3}{2}w^2-3v^{ap}w\right)\right),\partial_x^\alpha\partial_y^\beta w\right\rangle \qquad (3.3.38)\\
&=\ \left\langle\left(1-\partial_x^2\right)^{-1}\partial_x\partial_x^\alpha\partial_y^\beta\left(v_{xx}^{ap}w-\frac{3}{2}w^2-3v^{ap}w\right),\partial_x^\alpha\partial_y^\beta w\right\rangle\\
&\leq\ C\left\|\partial_x^\alpha\partial_y^\beta\left(v_{xx}^{ap}w-\frac{3}{2}w^2-3v^{ap}w\right)\right\|_0\|w\|_s\\
&\leq\ C\left\|O(\partial^{[\frac{s}{2}]+1}w)\right\|_{L^\infty}\|w\|_s^2\ \leq\ C_1\left(C_{2,M,s}+\|w\|_s\right)\|w\|_s^2.
\end{aligned}
$$

So by (3.3.32), (3.3.34)-(3.3.38), the estimate (3.3.33) is obtained. □

Now we give an error estimate. Let

$$T^\delta=\frac{\log(\theta/\delta)}{\sigma_0},$$

where θ will be chosen later. Define T^* such that

$$T^*=\sup\{T: T\leq T^\delta \text{ such that for any } t\in[0,T], \|w\|_s\leq 1\}.$$

Then by Lemma 3.3.6, when $0\leq t\leq T^*$,

$$\frac{d}{dt}\|w\|_s^2\leq C_1C_{2,M,s}\|w\|_s^2+C_{3,M,s}\delta^{2(M+2)}e^{2(M+2)\mathrm{Re}(\sigma_0)t}.$$

Note that $C_{2,M,s}$ is only related to v^{ap}. We can rewrite $C_{2,M,s}$ as $\theta C_{2,M,s}$ such that the new $C_{2,M,s}$ is depends on s and M but independent of θ and t. Then we have

$$\frac{d}{dt}\|w\|_s^2 \leq (C_1 + \theta C_{2,M,s})\|w\|_s^2 + C_{3,M,s}\delta^{2(M+2)}e^{2(M+2)\mathrm{Re}(\sigma_0)t}.$$

We can choose M large enough and θ small enough such that

$$2(M+2) - C_1 - \theta C_{2,M,s} > 1.$$

From now on, we fix M. Then by Gronwall's inequality we have

$$\sup_{0\leq t\leq T^*} \|w\|_s \leq C_{M,s}\theta^{M+2}.$$

When θ is sufficiently small, by the definition of T^*, we actually have $T^* = T^\delta$, i.e.

$$\sup_{0\leq t\leq T^\delta} \|w\|_s \leq C_{M,s}\theta^{M+2}$$

for $s \geq 2$. In particular, we have

$$\left\|w(T^\delta, \cdot)\right\|_0 \leq C_{M,s}\theta^{M+2}. \tag{3.3.39}$$

Now we are in the position to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Denote Π the projection onto the zero mode in y, i.e.

$$\Pi\left(u(x,y)\right) = u(x,y) - \int_0^{\frac{2\pi}{k_0}} u(x,y)dy,$$

then

$$\begin{aligned}\|\Pi(v^{ap})\|_0 &\geq \delta\left\|\Pi(v_0)\right\|_0 - \sum_{k=1}^{M}\delta^{k+1}\left\|\Pi(v^k)\right\|_0 \\ &= \delta\|v_0\|_0 - \sum_{k=1}^{M}\delta^{k+1}\left\|\Pi(v^k)\right\|_0 \\ &\geq c_s\delta e^{\sigma_0 t} - \sum_{k=1}^{M} C_{k,s}\delta^{k+1}e^{(k+1)\sigma_0 t}.\end{aligned}$$

When θ is sufficiently small, we have

$$\left\|\Pi(v^{ap}(T^\delta,\cdot))\right\|_0 \geq \frac{c_s\theta}{2}. \tag{3.3.40}$$

Then by (3.3.39) and (3.3.40), for any $l \in \mathbb{R}$,

$$\begin{aligned}
\left\| u^{\delta}(T^{\delta}, \cdot) - \phi(\cdot - l) \right\|_0 &\geq \left\| \Pi \left(u^{\delta}(T^{\delta}, \cdot) - \phi(\cdot - l) \right) \right\|_0 \\
&= \left\| \Pi \left(u^{\delta}(T^{\delta}, \cdot) - \phi(\cdot) \right) \right\|_0 \\
&= \left\| \Pi \left(v^{ap}(T^{\delta}, \cdot) + w(T^{\delta}, \cdot) \right) \right\|_0 \\
&\geq \frac{c_s \theta}{2} - \left\| \Pi \left(w(T^{\delta}, \cdot) \right) \right\|_0 \\
&\geq \frac{c_s \theta}{2} - \left\| w(T^{\delta}, \cdot) \right\|_0 \\
&\geq \frac{c_s \theta}{2} - C_{M,s} \theta^{M+2},
\end{aligned}$$

when θ chosen appropriately, the estimate will be bounded below by a fixed η depending only on s, which proves the theorem . □

4.0 Open question

1. Transverse stability of the CH-KP-II equation

We could ask a natural question, is the CH-KP-II equation, like the KP-II equation, transversely stable? Here the CH-KP-II equation is written as

$$\left[\left(1-\partial_x^2\right) u_t + 3uu_x + 2\kappa u_x - 2u_x u_{xx} - uu_{xxx}\right]_x - u_{yy} = 0. \tag{4.0.1}$$

To prove spectral stability, it suffices to show there is no unstable eigenmode, while for nonlinear stability, a promising track is to follow the methods in [49, 50], which are mentioned in Section 1.2.

2. Asymptotic stability of the CH equation

Based on the orbital stability, we can ask a further question: For dispersive equations, when the initial data stays close to family of solitary waves, will it eventually evolve into a number of solitons (related to the energy for the initial data) plus a radiation term? This problem is called the asymptotic stability of solitary waves. In fact, there is an even stronger argument which is called the soliton resolution conjecture: Is the above argument true for "generic" initial data ?

Now let's back to the problem of the asymptotic stability. The pioneering work is by Pego-Weinstein [53], they proved that for subcritical generalized KdV equations:

$$u_t + u_{xxx} + u^p u_x = 0, \tag{4.0.2}$$

family of solitary waves are asymptotically stable when $p = 1$ (KdV) and $p = 2$ (modified KdV) or $3 \leq p < 4$ and the linearized operator around solitary waves has no eigenvalue in L^2 other than 0. They require that the initial data should exponentially decay. Later, Mizumachi [48] refined the result, his argument only required the initial data to be polynomially decay. A breakthrough in this problem is made by Martel-Merle [43, 44]. The initial data in their paper only requires to be in the natural energy space H^1. In their work, to characterize the long-time behavior, they constructed an asymptotic object, which is exactly the solitary waves mentioned above when time goes to infinity. In [45], they gave a more direct method without constructing the asymptotic object. Furthermore, in [46], they extended the result to the gKdV equation with general nonlinearity. For much more detailed account, we can turn to [65].

For the asymptotic stability of the CH equation, Molinet [52] proved the case when $\kappa = 0$. Since under this situation, the solitary wave becomes a peakon which is no longer smooth, he devised a completely new method to overcome the difficulty. For the case when $\kappa \neq 0$, the most recent result is by Dika-Molinet [32], they proved that any global solution that is H^1-localized and moves fast enough to the right decays exponentially in space uniformly with respect to time. The main difficulty for the case that $\kappa \neq 0$ is the spectral analysis. To prove the orbital stability, as mentioned in Section 1.2, we can transform the problem to analyzing a self-adjoint operator. However, for the problem of the asymptotic stability, if we want to employ the method of Martel Merle in [43, 44], we need to analyze the original linearized operator about a modulated solitary wave, which is quite complicated.

Appendix A Proofs from Section 2.1.1

In this appendix we provide the proofs of the properties for the stationary wave problem (2.1.1) stated in Section 2.1.1.

Proof of Lemma 2.1.1. If there is an x_1 such that $\eta(x_1) \geq 0$. This property, and the fact that $\eta(x) \to 0$ as $x \to \pm\infty$, imply that η has a non-negative maximum on $\mathbb{R}$. Suppose that this maximum is at x_0. Then from (2.1.1), either $\eta(x_0) = u(x_0) = 0$ or $\eta''(x_0) > 0$. The latter being impossible at a maximum, we conclude the former two equalities. Also, $\eta'(x_0) = 0$, and from (2.1.2), $u'(x_0) = 0$. But then, since η, η', u, and u' all vanish at x_0, the uniqueness theorem for ODE's implies that $\eta(x) = u(x) = 0$ for all x, contradicting the definition of a solitary wave, which must be non-constant. This completes the proof of Lemma 2.1.1. □

A quick application of the maximum principle also yields

Lemma A.0.1. *Let (u, η) be a solitary wave solution. Then*

(a) *$(-u, \eta)$ is also a solitary wave solution.*

(b) *If $u \geq 0$ then $u(x) > 0$.*

(c) *If $u \leq 0$ then $u(x) < 0$.*

Proof. Part (a) follows directly from (2.1.1). For (b), suppose that $u(x) \geq 0$ and that $u(x_1) = 0$ for some x_1. Since u is non-negative it must be the case that $u'(x_1) = 0$. Thus, by uniqueness of the constant solution $(u, u') = (0, 0)$ of

$$-au'' = u(1 + \eta),$$

we conclude that $u(x) \equiv 0$. In this case, $\eta'' = \eta$ on $\mathbb{R}$. But since the orbit is homoclinic, η must be bounded, which implies that $\eta(x) = 0$ for all x as well, and this contradicts the definition of solitary waves and Lemma 2.1.1. Part (c) follows by a similar argument. □

Proof of Proposition 2.1.1. The proof part (i) makes use of function

$$h = u - \sqrt{2}\eta,$$

which satisfies

$$\beta^2 h'' = \left(1 - \frac{u}{\sqrt{2}}\right) h. \tag{A.0.1}$$

If (u, η) is a solitary wave solution with $u < \sqrt{2}$, then $1 - u/\sqrt{2} > 0$. Then from maximum principle we know that $h \equiv 0$.

Substituting $h = u - \sqrt{2}\eta = 0$ into (2.1.2) we get

$$\beta^2(\eta')^2 = \eta^2 + \frac{2}{3}\eta^3.$$

which is the classical steady KdV equation. The solution is given by

$$\eta_0(x) = -\frac{3}{2}\text{sech}^2\left(\frac{x}{2\beta}\right), \quad \text{and hence} \quad u_0^-(x) = -\frac{3\sqrt{2}}{2}\text{sech}^2\left(\frac{x}{2\beta}\right).$$

The proof of (ii) makes use of the functional $w = u + \sqrt{2}\eta$, which satisfies

$$\beta^2 w'' = \left(1 + \frac{u}{\sqrt{2}}\right) w.$$

The remainder of the proof follows the same argument as before, with w replacing h. □

Appendix B Proofs from Section 3.3

B.1 Proof of lemma 3.3.2

Proof of Lemma 3.3.2. By Duhamel's principle, it suffices to prove the existence of solution for the homogeneous equation:

$$\partial_t u = \mathcal{J}\mathcal{L}(jm_0k_0)u, \quad u|_{t=0} = \tilde{u} \quad \text{for } j = 1\cdots, K.$$

Since

$$\begin{aligned} \mathcal{J}\mathcal{L}(jm_0k_0) = & \left(1-\partial_x^2\right)^{-1}\left(-\partial_x^2\right)((c-\phi)u_x) \\ & + \left(1-\partial_x^2\right)^{-1}\partial_x\left(\left(\phi''-3\phi-2\kappa+c\right)u\right) - (jm_0k_0)^2\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}u \\ = & (c-\phi)u_x - \left(1-\partial_x^2\right)^{-1}((c-\phi)u_x) \\ & + \left(1-\partial_x^2\right)^{-1}\partial_x\left(\left(\phi''-3\phi-2\kappa+c\right)u\right) - (jm_0k_0)^2\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}u, \end{aligned} \tag{B.1.1}$$

it suffices to study the operator

$$\mathcal{A} = (c-\phi)\partial_x - (jm_0k_0)^2\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}$$

since other terms are just bounded perturbation.

Consider $\mathcal{A} : H^{s+1}(\mathbb{R}) \cap \mathcal{D}\left(\partial_x^{-1}(\mathbb{R})\right) \to H^s(\mathbb{R})$, where $\mathcal{D}(\partial_x^{-1}(\mathbb{R})) = \mathcal{F}^{-1}\left\{u : \hat{u}(0) = 0\right\}$ and $\mathcal{F}$ is the Fourier transform with respect to x. We first prove that

$$\langle \mathcal{A}u, u\rangle_{H^s} \leq \omega\, \langle u, u\rangle_{H^s} \tag{B.1.2}$$

for some $\omega > 0$. For $(c-\phi)\partial_x$,

$$\begin{aligned} \langle (c-\phi)u_x, u\rangle_{H^s} &= \langle \partial_x^s((c-\phi)u_x), \partial_x^s u\rangle + \sum_{\alpha=0}^{s-1}\langle \partial_x^\alpha((c-\phi)u_x), \partial_x^\alpha u\rangle \\ &\leq \langle \partial_x^s((c-\phi)u_x), \partial_x^s u\rangle + \omega_1\,\langle u, u\rangle_{H^s}. \end{aligned}$$

It reduces to control order s term, and we have

$$\begin{aligned}\langle\partial_x^s((c-\phi)u_x),\partial_x^s u\rangle &= \left\langle(c-\phi)\partial_x^{s+1}u+O(\partial_x^s u),\partial_x^s u\right\rangle \\ &= \left\langle\frac{1}{2}\phi'\partial_x^s u+O(\partial_x^s u),\partial_x^s u\right\rangle \\ &\leq \omega_2\left\langle u,u\right\rangle_{H^s}.\end{aligned}$$

For $\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}$,

$$\begin{aligned}\left\langle\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}u,u\right\rangle_{H^s} &= \left\langle\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}u,u\right\rangle+\sum_{\alpha=1}^{s}\left\langle\partial_x^\alpha\left(\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}u\right),\partial_x^\alpha u\right\rangle \\ &\leq \left\langle\left(1-\partial_x^2\right)^{-1}\partial_x^{-1}u,u\right\rangle+\omega_3\left\langle u,u\right\rangle_{H^s} \\ &= 0+\omega_3\left\langle u,u\right\rangle_{H^s}.\end{aligned}$$

Next we prove that $\lambda-(\mathcal{A}-\omega)$ is surjective for $\lambda>0$. Since by (B.1.2), there is no point spectrum larger than 0. It suffices to prove that $\lambda>0$ is not in the essential spectrum of $\mathcal{A}-\omega$. It is enough just to consider the essential spectrum of its limiting operator

$$c\partial_x-(jm_0k_0)^2(1-\partial_x^2)^{-1}\partial_x^{-1}-\omega.$$

By using Fourier transform it is clear that $\lambda>0$ is not in the essential spectrum of the above operator. Based on all the above, by Lumer-Phillips theorem [33], the lemma is concluded. □

B.2 Existence of solution in (3.3.31)

Proof. The proof follows the idea of [27, 51]. Consider the regularized problem

$$\begin{cases} \partial_t w^\varepsilon = \mathcal{J}^\varepsilon \left(\frac{1}{2}(w_x^\varepsilon)^2 + v_x^{ap} w_x^\varepsilon + v_{xx}^{ap} w^\varepsilon + (w^\varepsilon + v^{ap}) w_{xx}^\varepsilon - \frac{3}{2}(w^\varepsilon)^2 - 3v^{ap} w^\varepsilon \right) \\ \qquad\qquad + \mathcal{J}^\varepsilon \mathcal{L} w^\varepsilon + G^\varepsilon, \\ w|_{t=0} \ = 0, \end{cases} \tag{B.2.1}$$

where

$$\mathcal{J}^\varepsilon = \left(1 - \partial_x^2 + \varepsilon \Delta^2\right)^{-1} \partial_x, \ \Delta = \partial_x^2 + \partial_y^2.$$

It can be derived from fixed point argument that the solution w_ε exists. Indeed, since $\mathcal{J}^\varepsilon$ maps $H^s \to H^{s+3}$, it suffices to choose a unit ball in $C([0, t^\varepsilon])$ for the contraction mapping. Next we state the approximating procedure. We could have the same estimate as (3.3.33) for w^ε:

$$\frac{d}{dt}\|w^\varepsilon\|_s^2 \leq C_1 \left(C_{2,M,s} + \|w^\varepsilon\|_s\right) \|w^\varepsilon\|_s^2 + C_{3,M,s}\delta^{2(M+2)} e^{2(M+2)\mathrm{Re}(\sigma_0)t}.$$

Then for each ε, we define T^ε

$$T^\varepsilon = \sup\{T : \|w\|_s \leq C_{2,M,s} \text{ for } 0 \leq t \leq T\}.$$

Then for each ε such that $T^\varepsilon < 1$, we have

$$\frac{d}{dt}\|w^\varepsilon\|_s^2 \leq 2C_1 \|w^\varepsilon\|_s \|w^\varepsilon\|_s^2 + C_{3,M,s}\delta^{2(M+2)} e^{2(M+2)\mathrm{Re}(\sigma_0)t},$$

which yields

$$\begin{aligned} \|w^\varepsilon\|_s^2 \leq & \left(\frac{1}{\sqrt{C_{2,M,s}}} - 2C_1(t - T^\varepsilon) \right)^{-2} \\ & + \int_{T^\varepsilon}^t \left(\frac{1}{\sqrt{C_{3,M,s}\delta^{2(M+2)} e^{2(M+2)\mathrm{Re}(\sigma_0)s}}} - 2C_1(t-s) \right)^{-2} ds. \end{aligned}$$

Since $T^\varepsilon < 1$, for t sufficiently close to T^ε,

$$\frac{1}{\sqrt{C_{3,M,s}\delta^{2(M+2)}e^{2(M+2)\mathrm{Re}(\sigma_0)s}}} - 2C_1(t-s) > c > 0$$

for all ε such that $T^\varepsilon < 1$. So there exists T such that $\|w^\varepsilon\|_s^2$ is uniformly bounded on $[0,T]$ for all ε when $T^\varepsilon < 1$. Then for all ε, $\|w^\varepsilon\|_s^2$ is uniformly bounded on $[0,\tilde{T}]$ where $\tilde{T} = \min(T,1)$. And from (B.2.1), $\{\partial_t w^\varepsilon\}$ is uniformly bounded in $L^\infty([0,\tilde{T}]; L^2_{\mathbb{R}\times\mathbb{T}_a})$. Then by Aubin-Lions lemma, we obtain a solution $u \in L^\infty([0,\tilde{T}], H^s_{\mathbb{R}\times\mathbb{T}_a})$ for (3.3.31). □

Bibliography

[1] M. J. Ablowitz, T. S. Haut, Asymptotic expansions for solitary gravity-capillary waves in two and three dimensions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **465** (2009), 2725–2749.

[2] A. Adelaide, S. Walsh, Solitary water waves with discontinuous vorticity, *J. Math. Pures Appl.*, **124** (2019), 220–272.

[3] C. J. Amick, Regularity and uniqueness of solutions to the Boussinesq system of equations, *J. Differential Equaitons*, **54** (1984), 231–247.

[4] C. J. Amick, J. F. Toland, On periodic water-waves and their convergence to solitary waves in the long-wave limit, *Philos. Trans. Roy. Soc. London Ser. A*, **303** (1981), 633–669.

[5] C. J. Amick, J. F. Toland, On sollitary water-waves of finite amplitude, *Arch. Ration. Mech. Anal.*, **76** (1981), 9–95.

[6] J. C Alexander, R.L. Pego, R.L. Sachs, On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation, *Phys. Lett. A*, **226** (1997), 187-192.

[7] E.S. Bao, R.M. Chen, Q. Liu, Existence and symmetry of ground states to a Boussinesq *abcd* system, *Arch. Ration. Mech. Anal.*, **216** (2015) 569–591.

[8] T. B. Benjamin, The stability of solitary waves, *Proc. London Math. Soc.*, **328** (1972), 153-183.

[9] T. B. Benjamin, J. L. Bona, D. K. Bose, Solitary-wave solutions of nonlinear problems, *Philos. Trans. Roy. Soc. London Ser. A*, **331** (1990), 195–244.

[10] T. B. Benjamin, J. L. Bona, J. J. Mahoney, Model equations for long waves in non-linear dispersive systems, *Phil. Trans. Roy. Soc. London Ser. A*, **227** (1972), 47–78.

[11] J. L. Bona, On the stability theory of solitary waves, *Proc. London Math. Soc.*, **344** (1975), 363-374.

[12] J. L. Bona, M. Chen, J. C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory, *J. Nonlinear Sci.*, **12** (2002), 283–318.

[13] J. L. Bona, T. Colin, D. Lannes, Long wave approximations for water waves, *Arch. Ration. Mech. Anal.*, **178** (2005), 373–410.

[14] J. Boussinesq, Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire, *Comptes Rendus Acad. Sci. Paris*, **72** (1871), 755–759.

[15] J. Boussinesq, Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal, *Comptes Rendus Acad. Sci. Paris*, **73** (1871), 256–260.

[16] J. Busca, B. Sirakov, Symmetry results for semilinear elliptic systems in the whole space, *J. Differential Equations*, **163** (2000), 41–56.

[17] R. Camassa, D.D. Holm, An integrable shallow water equation with peake dsolitons, *Phys. Rev. Lett.* **71** (11)(1993), 1661–1664.

[18] F. Chazel, Influence of topography on water waves, *ESAIM: M2AN*, **41** (2007), 771–799.

[19] M. Chen, Solitary-wave and multi-pulsed traveling-wave solutions of Boussinesq systems, *Appl. Anal.*, **75** (2000), 213–240.

[20] M. Chen, N. Nguyen, S. M. Sun, Solitary-wave solutions to Boussinesq systems with large surface tension, *Discrete Contin. Dyn. Syst. Ser. S.*, **26** (2009), 1153–1184.

[21] M. Chen, N. Nguyen, S. M. Sun, Existence of Traveling-wave Solutions to Boussinesq Systems, *Diff. & Integral Eq.*, **24** (2011), 895–908.

[22] R. M. Chen, Some nonlinear dispersive waves arising in compressible hyperelastic plates, *International Journal of Engineering Science*, **44** (2006), 1188-1204.

[23] R. M. Chen, S. Hasting, W. C. Troy, Solitary waves of the Boussinesq *abcd* system, preprint.

[24] R. M. Chen, S. Walsh, M. Wheeler, Existence and qualitative theory for stratified solitary water waves, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **35** (2018), 517–576.

[25] R. M. Chen, S. Walsh, M. Wheeler, Global bifurcation for monotone fronts of elliptic equations, *arXiv preprint* arXiv:2005.00651 (2020).

[26] R. M. Chen, S. Walsh, M. Wheeler, Global bifurcation of anti-plane shear fronts, *J. Nonlinear Sci.*, **31** (2021), no. 2, 28.

[27] A. Constantin, J. Escher, Global existence and blow-up for a shallow water equation, *Annali Sc. Norm. Sup. Pisa*, **26** (1998), 303-328.

[28] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, *Arch. Ration. Mech. Anal.*, **192** (2009), 165-186.

[29] A. Constantin, W. Strauss, Stability of the Camassa-Holm solitons, *J. Nonlinear Sci.*, **12** (2002), 415-422.

[30] W. Craig, An existence theory for water waves, and Boussinesq and Korteweg-de Vries scaling limits, *Commun. PDEs*, **10** (1985), 787–1003.

[31] P. Daripa, R. K. Dash, A class of model equations for bi-directional propagation of capillary-gravity waves, *Internat. J. Engrg. Sci.*, **41** (2003), 201–218.

[32] K. EL Dika, L. Molinet, Exponential decay of H^1-localized solutions and stability of the train of N solitary waves for the Camassa–Holm equation, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **365** (2007), 2313–2331.

[33] K.J. Engel, R. Nagel, A short course on operator semigroups. *Springer Science and Business Media*, (2006).

[34] K. O. Friedrichs, D.H. Hyers, The existence of solitary waves, *Comm. Pure Appl. Math.*, **7** (1954), 517–550.

[35] E. Grenier, On the nonlinear instability of Euler and Prandtl equations, *Comm. Pure Appl. Math*, **53** (2000), 1067-1091

[36] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry, *J. Funct. Anal*, **74** (1987), 160-197.

[37] T. Hogancamp, Broadening global families of anti-plane shear equilibria, *arXiv preprint* arXiv:2101.07909 (2020).

[38] R. S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water waves, *Journal of Fluid Mechanics*, **455** (2002), 63-82.

[39] B. B. Kadomtsev, V. I. Petviashvii, On the stability of solitary waves in weakly dispersive media, *Sov. Phys. Dokl.*, **15** (1970), 539-541.

[40] K. Kano, T. Nishida, A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves, *Osaka J. Math.*, **23** (1986), 389–413.

[41] K. Kirchgassner,, Wave solutions of reversible systems and applications, *J. Diff. Eq.*, **45** (1982), 113–127.

[42] D. J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, *Phil. Mag.*, **39** (1895), 422–443.

[43] Y. Martel, F. Merle, A Liouville theorem for the critical generalized Korteweg-de Vries equation, *J. Math. Pures Appl.*, **79** (2000), 339-425.

[44] Y. Martel, F. Merle, Asymptotic stability of solitons for subcritical generalized KdV equations, *Arch. Ration. Mech. Anal.*, **157** (2001), no.3, 219-254.

[45] Y. Martel, F. Merle, Asymptotic stability of solitons of the subcritical gKdV equations revisited, *Nonlinearity*, **18** (2005), no.1, 55-80.

[46] Y. Martel, F. Merle, Asymptotic stability of solitons of the gKdV equations with general nonlinearity, *Math. Ann.*, **341** (2008), no.2, 391-427.

[47] M. A. Lavrentiev, I. On the theory of long waves. II. A contribution to the theory of long waves, *Amer. Math. Soc. Transl.*, (1954), 53.

[48] T. Mizumachi, Large time asymptotics of solutions around solitary waves to the generalized Kortewegde-De Vries equations, *SIAM J. Math. Anal.*, **32** (2001), 1050–1080.

[49] T. Mizumachi, N. Tzvetkov, Stability of the line soliton of the KP-II equation under periodic transverse perturbations, *Math. Ann.*, **352** (2012), 659-690.

[50] T. Mizumachi, Stability of line solitons for the KP-II equation in $\mathbb{R}^2$, *Mem. Amer. Math. Soc.*, **238** (2015), no.1125, vii+95 pp.

[51] L. Molinet, On well-posedness results for the Camassa-Holm equation on the line: a survey, *J. Nonlin. Math. Phys.*, **11** (2004), 521-533.

[52] L. Molinet, A Liouville property with application to asymptotic stability for the CamassaHolm equation, *Arch. Ration. Mech. Anal.*, **230** (2018), no.1, 185-230.

[53] R. L. Pego, M. I. Weinstein, Asymptotic stability of solitary waves, *Commun. Math. Phys.*, **164** (1994), 305-349

[54] D. H. Peregrine, Equations for water waves and the approximation behind them, in Waves on beaches and resulting sediment transport; proceedings of an advanced seminar conducted by the Mathematics Research Center, New York, Academic Press, (1972), 95–121.

[55] L. Rayleigh, , On Waves, *Phil. Mag.*, **1** (1876), 257–279.

[56] F. Rousset, N. Tzvetkov, Transverse nonlinear instability for two-dimensional dispersive models, *Ann. IHP, Analyse Non Linéaire*, **26** (2009), 477-496.

[57] F. Rousset, N. Tzvetkov, Transverse nonlinear instability for some Hamiltonian PDE's, *J. Math. Pures Appl.* **90** (2008), 550-590.

[58] F. Rousset, N. Tzvetkov, A simple criterion of transverse linear instability for solitary waves, *Math. Res. Lett.* **17** (2010), no.1, 157-169.

[59] F. Rousset, N. Tzvetkov, Transverse instability of the line solitary water-waves, *Inventiones mathematicae* **184** (2011), 257-388.

[60] J. S. Russell, Report on waves, in 14th meeting of British Association for Advancement of Science, vol. 311, 1844, p. 390.

[61] R. Sachs, On the existence of small amplitude waves with strong surface tension, *J. Differential. Equ.*, **90** (1991), 31–51.

[62] J. C. Saut, L. Xu, The Cauchy problem on large time for surface waves Boussinesq systems, *J. Math. Pures Appl.*, **97** (2012), 635–662.

[63] M. E. Schonbek, Existence of solutions for the Boussinesq system of equations, *J. Differential Equations*, **42** (1981), 325–352.

[64] D. Sinambela, Large-amplitude solitary waves in two-layer density stratified water, *arXiv preprint* arXiv:2012.00142 (2020).

[65] T. Tao, Why are solitons stable?, *Bull. Amer. Math. Soc. (N.S.) 46*, **1** (2009), 1-33.

[66] A. M. Ter-Krikorov, The existence of periodic waves which degenerate into a solitary wave, *J. Appl. Math. Mech.*, **24** (1960), 930–949.

[67] J. F. Toland, Solitary wave solutions for a model of the two-way propagation of water waves in a channel, *Math. Proc. Camb. Phil. Soc.* **90** (1981), 343–360.

[68] F. Ursell, The long-wave paradox in the theory of gravity waves, *Proc. Cambridge Phil. Soc.*, **49** (1953), 685–694.

[69] M. I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, *Comm. Pure Appl. Math.* **39** (1986), 51-67.