Relationships Between Spaces and Their Functional Generators

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A subset G of the set $C(X)$ of all continuous real valued functions on a Tychonoff space X, is a *generator* if whenever x is a point of X not in a closed set C then there is a g in G such that $g(x) \notin \overline{g(C)}$. The set $C(X)$ admits some natural topologies, including the topology of pointwise convergence and the compact open topology. Generators, then, are subspaces of these function spaces. In this work, we examine discrete, compact and first and second countable generators.

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2.0 Introduction

To know the basis of a space is to understand how to generate its topology, to be given the blueprint for what open sets look like within the space. With a base, it's possible to prove a space has certain properties, like metrizability, compactness, and more. But the topology of a space can be encoded in other ways, for instance via continuous real-valued maps – at least in Tychonoff spaces. (We assume all spaces are Tychonoff moving forward.)

Recall that a space is *Tychonoff* if it is T_1 and the continuous real-valued functions, $C(X)$, 'generate the topology' of X, in the sense that the collection $\{g^{-1}U : U$ open in $\mathbb{R}$, $g \in C(X)\}$ is a base for X. It is natural, then, to call any subset, G say, of $C(X)$, a *generator* if $\{g^{-1}U : U$ open in $\mathbb{R}$, $g \in G\}$ is a base for X. Equivalently, G is a generator provided whenever x is a point of X not in a closed set C then there is a g in G such that $g(x) \notin \overline{g(C)}$. As all spaces here are Tychonoff, all have a generator.

Spaces with bases with a host of different combinatorial and topological properties have been intensively studied by topologists. Similar questions can be raised about generators. But for generators there is an entirely different class of questions, because $C(X)$ carries a number of natural topologies, which any generator then inherits. This thesis is dedicated to understanding which spaces have a compact generator, a discrete generator, or a first or second countable generator.

The most important topology on $C(X)$ for us is the topology of pointwise convergence, for which the basic open sets have the form $B(f, F, \epsilon)$ for finite subsets F of X, but we also consider the compact-open topology, whose basic open sets are those $B(f, K, \epsilon)$ for compact subsets K of X. Here, for any f in $C(X)$, subset S of X and $\epsilon > 0$ we define $B(f, S, \epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon$ for all $x \in S\}$. Write $C_p(X)$ for $C(X)$ with the pointwise topology, and $C_k(X)$ for $C(X)$ with the compact-open topology.

The study, 'C_p-theory', of the function space $C_p(X)$ has developed into a deep and active area, lying between, and connecting, topology and analysis. See Arkhangel'skii's classic text [3], or the more recent book series of Tkachuk [21]. Generators feature in these books and many other articles in the C_p-theory literature, but always in the background, as tools. Our aim here is to bring generators into the limelight.

There are a variety of reasons for investigating generators. First, by doing so we better understand and sharpen a commonly used tool. Second, by examining the properties required of a generator, we may reveal in detail the reasons why classic C_p-theory results hold. A third, and perhaps most compelling, reason for studying generators is that they can have significantly better topological properties than the full function space, making them more convenient to use and allowing for more elegant proofs.

As mentioned above, in this research work, we focus on four properties for our generators, namely compactness, discreteness, and first and second countability. Compactness is an important and useful property in general, and we can think of a compact generator as being one which is small and efficient. The topological polar opposite of compactness is discreteness, and discrete generators are spread out, and in some sense have no topology. Since every discrete space is first countable, while second countable spaces are small in a different way to compact spaces, it is a natural further step to consider spaces with first and second countable generators.

Working with generators one quickly realizes that it is often convenient, and sometimes necessary, for a generator to have additional properties. Specifically, that it is more precise in its separation of points and closed sets. A subset G of $C(X)$ is a $(0, \neq 0)$*–generator* (respectively, $(0,1)$*–generator*) for X if whenever x is a point of X not in a closed set C then there is a g in G such that $g(x) \neq 0$ (respectively, $g(x) = 1$) while $g(C) \subseteq \{0\}$.

We begin the thesis with some preliminaries in Chapter 3. The most important results here - used in almost every theorem and example - are those allowing us to 'upgrade' generators (from 'vanilla' to $(0, \neq 0)$ or $(0,1)$) or otherwise manipulate generators. Also important is the construction of a generator for $C_p(Y)$ in terms of Y (Theorem 13).

Then we move into compact generators in Chapter 4. After, we explore discrete generators in Chapter 5 and then end with first and second countable generators in Chapter 6. Ideas for future work and open questions follow. The results of these chapters are now sketched.

2.1 Compact Generators

We begin by asking what kinds of spaces lend themselves to compact generators. Clearly $C_p(X)$ itself is never compact, and Velichko showed $C_p(X)$ is σ-compact if and only if X is finite, see [22] for the extension to σ-countably compact. The problem of when $C_p(X)$ is Lindelöf is an important and challenging one, which we examine further in Section 4.3. It is known for example, see [1, 12], that $C_p(X)$ is Lindelöf for every Corson (hence, every Eberlein) compact, X.

In Section 4.1 we show that a k-space has a compact generator in $C_k(X)$ if and only if X is metrizable (Propositions 19 and 20). While a space has a compact generator in $C_p(X)$ if and only if X is Eberlein-Grothendieck (Proposition 21). Specifically, $C_p(Y)$ has a compact generator if and only if Y is σ-compact (Theorem 22).

In Section 4.2 we investigate which spaces have compact generator in $C_p(X)$ of very specific topological types (supersequences and convergent sequences of finite powers of supersequences). This is motivated by our understanding of Eberlein compacta.

Finally in Section 4.3 we look at the properties of spaces with a Lindelöf generator.

2.2 Discrete Generators

In Section 5.1, we develop and motivate the conjecture that a space X has a discrete generator if and only if $hc^*(X) = w(X)$. We show the conjecture holds for zero-dimensional spaces, Theorem 40, and in some other cases, as well. In Section 5.2, we formulate an analogous conjecture for spaces with a discrete $(0, \neq 0)$–generator, and deduce that there are spaces with a discrete generator but no discrete $(0, \neq 0)$–generator. Initially, we thought that discrete $(0, 1)$–generators might only arise in very specific circumstances and would be much harder to obtain than $(0, \neq 0)$–generators, but our work in Section 5.3 revealed that every metrizable space has one. The converse is not true (all ordinals and several other non-metrizable examples have a $(0, 1)$–generator). We show that the Michael line is a space with a discrete $(0, \neq 0)$–generator that does not have a discrete $(0, 1)$–generator.

2.3 First and Second Countable Generators

In Section 6, we investigate when spaces have first or second countable generators of various types. A point x in a space X is a *point of first countability* if it has a countable local base. A space X is *first countable*, abbreviated 1°, if every point is a point of first countability, and is *second countable*, abbreviated 2°, if it has a countable base.

When considering first and second countable generators the type of generator - 'vanilla', $(0, \neq 0)$ or $(0, 1)$ - becomes a critical factor. By untangling these dependencies we are led to the central questions.

As an example, it is known that if $C_p(X)$ is *cosmic* – that is, it has a countable network – then X is also cosmic (and conversely). However, the standard proof requires only the existence of a cosmic 'vanilla' generator.

For a second example, recall that if $C_p(X)$ has a coarser second countable topology, then every point of $C_p(X)$ is a G_δ point, and this implies X is separable (moreover, both implications reverse). In this case, having a generator with a coarser second countable topology is not sufficient to conclude separability. For the standard proof to work, the requirement for a space X to be separable is the existence of a $(0, \neq 0)$–generator G of $C_p(X)$ that contains the zero function, $\mathbf{0}$, that has all points G_δ.

As a third example, recall that $C_p(X)$ is first or second countable only in very limited circumstances, namely when X is countable. Here, the standard proof actually only requires the existence of a first (or second) countable $(0, 1)$–generator G of $C_p(X)$ that contains $\mathbf{0}$.

These three results, in light of the types of generator needed in each case, raise two pairs of questions: Which spaces have a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$? Does separability suffice? And which spaces have a second countable $(0, \neq 0)$–generator containing $\mathbf{0}$? Does cosmicity suffice?

In Sections 6.1 and 6.3 we establish the claims above about cosmic generators, and first and second countable $(0, 1)$–generators containing $\mathbf{0}$. We show that separable spaces have a $(0, \neq 0)$–generator containing $\mathbf{0}$ as a point of first countability. We show that the requirement that the generator contain the zero function, despite appearances, can be dropped. We also investigate which spaces have a compact, second countable generator and show that they are precisely the subspaces of some $C_p(K)$ where K is compact and second countable.

However, our questions remain unanswered. While we think it likely that there are separable spaces without a first countable $(0, \neq 0)$–generator, and cosmic spaces without a second countable $(0, \neq 0)$–generator, in Sections 6.2 and 6.4 we show that many 'classical' separable and cosmic spaces do have first and second countable generators, respectively.

Section 6.5 explores first and second countable generators for the space $X = C_p(Y)$.

3.0 Background Material

3.1 Definitions and Notation

Our topological definitions and notation are largely standard. See, for example, Engelking [10] or Willard [26]. For completeness we recap some key concepts. The various types of generator are less well-known, and do not have a standard nomenclature. We introduce here our definitions and notation for generators.

3.1.1 Topological Properties

All spaces are Tychonoff. It is convenient to review, and generalize, the topological properties of interest in this thesis via cardinal invariants. Let Y be a space. A *cardinal function* or *cardinal invariant* is an assignment, f, of a cardinal, $f(Y)$, to any space Y so that if Y and Z are homeomorphic then $f(Y) = f(Z)$.

The following cardinal invariants capture well-known global topological properties. Define $L(Y)$, the *Lindelöf degree* of Y, to be the minimal κ such that every open cover of Y has a subcover of size $\leq \kappa$. Define $d(Y)$, the *density* of Y, to be the minimal size of a dense set. Define $c(Y)$, the *cellularity* of Y to be the suprema of sizes of pairwise disjoint family of open sets. Define $e(Y)$, the *extent*, to be the supremum of sizes of closed discrete subsets. Define $w(Y)$, the *weight* of Y, to be the minimal size of a base for Y. Define $nw(Y)$, the *netweight* of Y, to be the minimal size of a network for Y. Observe that a space is *Lindelöf* if it has countable Lindelöf degree, *separable* if its density is countable, *ccc* if its cellularity is countable, *second countable* if it has countable weight, and *cosmic* if it has countable netweight.

To isolate local properties, define for a point y in Y: $\chi(y, Y)$, the *character* of y in Y, to be the minimal size of a local base at y; $\psi(y, Y)$, the *pseudocharacter* of y, $\psi(y, Y)$, to be the minimal size of a family of open sets whose intersection is $\{y\}$; and the *tightness* of y in Y, $t(y, Y)$, to be the minimal κ such that whenever y is in $\overline{A}$ then there is a subset A_0 of A

with $|A_0| \leq \kappa$ such that $y \in \overline{A_0}$. Further set $\chi(Y)$ (the *character* of Y) to be the supremum over y in Y of all $\chi(y, Y)$, $\psi(y, Y) = \sup\{\psi(y, Y) : y \in Y\}$ (the *pseudocharacter* of Y) and $t(Y) = \sup\{t(y, Y) : y \in Y\}$ (the *tightness* of Y). Now a space is *first countable* if it has countable character, and *countably tight* if its tightness is countable. (Spaces with countable pseudocharacter do not have a traditional name, typically one just says, 'all points G_δ'.)

Given any cardinal function f we can define two additional cardinal invariants as follows. First, define $hf(Y) = \sup\{f(A) : A \subseteq Y\}$, the hereditary version of f. In particular we have, $hc(Y)$, the *hereditary cellularity* of Y, and note that this is also equal to the supremum of sizes of discrete subspaces of Y. Clearly we always have $f(Y) \leq hf(Y)$. Second, define $f^*(Y) = \sup\{f(Y^n) : n \in \mathbb{N}\}$. Clearly we always have $f(Y) \leq f^*(Y)$.

The following relationships between the cardinal invariants are well-known, and used frequently, without further comment, below. For any space Y: $hc(Y) = he(Y)$, $h(c) \leq \max(hd(Y), hL(Y)) \leq nw(Y) \leq w(Y)$, $c(Y) \leq d(Y)$, $e(Y) \leq L(Y)$ and $\max(c(Y), e(Y) \leq hc(Y) = he(Y)$. Also, $t(Y) \leq hd(Y)$, $\psi(Y) \leq hL(Y)$ and $\chi(Y) \leq w(Y)$.

Two additional topological properties arising in this thesis, but not naturally related to a cardinal invariant are: *metrizability* and being a *k-space*. A space is metrizable if it admits a compatible metric. While a space Y is a k-space if its compact subsets determine its topology (a subset U of Y is open if $U \cap K$ is open in K for every compact subset K of Y). Every discrete space is metrizable, every metrizable space is first countable, and every first countable space is a k-space. For a family $\{X_\lambda : \lambda \in \Lambda\}$ we write $\bigoplus_\lambda X_\lambda$ for the sum of the spaces X_λ.

3.1.2 Types of Generator

Generator: Let X be a space. A subset G of $C(X)$ is a *generator* if whenever x is a point of X not in a closed set C then there is a g in G such that $g(x) \notin \overline{g(C)}$.

Some proofs using generators require that the generator have properties beyond those of the definition. It is often desirable, for example, that we can separate the values of $g(x)$ and $g(C)$ more precisely.

Zero, non-zero generator: A subset G of $C(X)$ is a $(0, \neq 0)$*–generator* for X if whenever

x is a point of X not in a closed set C then there is a g in G such that $g(x) \neq 0$ while $g(C) \subseteq \{0\}$.

Zero, one generator: A subset G of $C(X)$ is a $(0,1)$*–generator* for X if whenever x is a point of X not in a closed set C then there is a g in G such that $g(x) = 1$ while $g(C) \subseteq \{0\}$.

We show below that we can often 'upgrade' a plain generator to one which is $(0, \neq 0)$ or even $(0,1)$. Less frequently we may need to functionally separate a finite set and a closed set.

n-Generator: For a fixed n we say G is an *n-generator* if whenever $x_1, \ldots, x_n$ are points not in a closed set C then there is a g in G such that $g(x_i) \notin \overline{g(C)}$ for all i. We further say G is a $(0,1)$ n-generator (respectively, $(0, \neq 0)$ n-generator) if whenever $x_1, \ldots, x_n$ are points not in a closed set C then there is a g in G such that $g(x_i) = 1$ (respectively, $g(x_i) \neq 0$) for all i, while $g(C) \subseteq \{0\}$.

3.2 Fundamental Results on Generators

Here we gather together some key results on constructing and upgrading generators. But we start by showing that nice generators exist, and prove some simple topological properties of generators. We conclude by constructing a generator for $X = C_p(Y)$, which plays an important role throughout this thesis. Some additional properties of this generator are then explained.

3.2.1 Basic Examples and Facts

Our first lemma is intended to simplify the task of showing that a subset of $C(X)$ is a $(0, \neq 0)$- (etc) generator. It also highlights the connection between generators and bases. For the $(0,1)$ case, we introduce netbases. A collection of pairs of subsets of a space X is a *pair netbase* if whenever a point x is in an open U, there is a pair (N, B) in the collection such that $x \in N \subseteq B \subseteq U$ where B is open.

Lemma 1. *Let X be a space and G a subset of $C(X)$. Then*

(1) G is a generator if and only if $\mathcal{B} = \{g^{-1}U : g \in G$ and U open in $\mathbb{R}\}$ is a base for X,

(2) G is a $(0,\neq 0)$–generator if and only if $\mathcal{B} = \{g^{-1}(\mathbb{R} \setminus \{0\}) : g \in G\}$ is a base for X, and

(3) G is a $(0,1)$–generator if and only if $\mathcal{P} = \{(N,B) : g \in G, N = g^{-1}\{1\}$ and $B = g^{-1}(\mathbb{R} \setminus \{0\})\}$ is a pair netbase.

Proof. These claims are almost immediate. We prove (2), and omit the others. Suppose G is a $(0,\neq 0)$–generator. Let $\mathcal{B} = \{g^{-1}(\mathbb{R} \setminus \{0\}) : g \in G\}$. We show $\mathcal{B}$ is a base for X. Indeed if x is in some open U, then $x \notin X \setminus U$, which is closed, so there is a g in G such that $g(x) \neq 0$ but $g(X \setminus U) \subseteq \{0\}$. Now we see, $B = g^{-1}(\mathbb{R} \setminus \{0\})$ is open and in $\mathcal{B}$, and $x \in B \subseteq U$, as required for $\mathcal{B}$ to be a base. Conversely, suppose G is a subset of $C(X)$ such that $\mathcal{B}$, as defined in (2), is a base for X. Take any x not in a closed set C. Then x is in $X \setminus C$, which is open, so there is a $B = g^{-1}(\mathbb{R} \setminus \{0\})$ in $\mathcal{B}$ such that $x \in B \subseteq X \setminus C$. But this means that $g(x) \neq 0$ while $g(C)$ is zero, as required for G to be a $(0,\neq 0)$–generator. $\square$

Like bases, generators (plain or $(0,\neq 0)$) for a space can be 'shrunk' to have minimal size, the weight of X.

Lemma 2. *If G is a generator (respectively, $(0,\neq 0)$–generator) for X then there is a subset G' of G of size $w(X)$ which is also a generator (rep., $(0,\neq 0)$–generator).*

Proof. Suppose, to start, that G is a generator for X. Fix a countable basis, $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$, for $\mathbb{R}$. For each g in G and n in $\mathbb{N}$, set $U(g,n) = g^{-1}B_n$. Then $\mathcal{U} = \{U(g,n) : g \in G, n \in \mathbb{N}\}$ is a base for X. It contains a subcollection $\mathcal{U}'$ which has size $w(X)$ and is a base for X. Let $G' = \{g \in G : U(g,n) \in \mathcal{U}'$ for some $n\}$. Then G' has size $w(X)$ and is a generator.

If G is a $(0,\neq 0)$–generator then for each g in G set $U(g) = g^{-1}(\mathbb{R} \setminus \{0\})$, and $\mathcal{U} = \{U(g) : g \in G\}$. Now proceed as above, and note that the G' obtained is a $(0,\neq 0)$–generator. $\square$

Note that the analogous result for $(0,1)$–generators does not hold. Let X be the reals with usual topology. For each x and n let $g_{x,n}$ be the function in $C_p(X)$ which is piecewise linear, has value 0 on $(-\infty, x - 1/n] \cup [x + 1/n, +\infty)$, and value 1 at x. Then $G = \{g_{x,n} : x \in \mathbb{R}, n \in \mathbb{N}\}$ is a $(0,1)$–generator for X. Since, for a fixed x, the $g_{x,n}$'s are the only members of G which take value 1 at x, it is clear that if G' is a subset of G which is a $(0,1)$–generator then $|G'| = |G| = \mathfrak{c}$, while X has countable weight.

However every space does have a $(0,1)$–generator of minimal size, and that generator can additionally be assumed to be an n-generator for all n.

Lemma 3. *Let X be a space. Then there is a subset G of $C(X,[0,1])$ of size $w(X)$ which is a $(0,1)$ n-generator for all n in $\mathbb{N}$.*

Thus every separable metrizable space X has (1) a countable $(0,1)$ n-generator, for all n, and (2) a $(0,\neq 0)$ n-generator, for all n, which is homeomorphic to the convergent sequence, with limit point the zero function, $\mathbf{0}$.

Proof. Fix a base, $\mathcal{B}$ say, for X which has size $w(X)$, is closed under finite unions, and every member is a co-zero set. Fix, for the moment, B in $\mathcal{B}$ and pick $g_B \in C(X)$ such that $B = g_B^{-1}(\mathbb{R} \setminus \{0\})$. Define $g_{B,n}$ in $C(X,[0,1])$ by $g_{B,n}(x) = n \cdot \min(|g(x)|, 1/n)$. Note that if $|g_B(x)| \geq 1/n$ then $g_{B,n}(x) = 1$ while if $g_B(x) = 0$ then $g_{B,n}(x) = 0$.

Set $G = \{g_{B,n} : g \in G \text{ and } n \in \mathbb{N}\}$. Then $|G| = w(X)$. We check G is a $(0,1)$ n-generator for all n. To this end take any $x_1, \ldots, x_n$ not in a closed set C. As $\mathcal{B}$ is closed under finite unions, there is a B in $\mathcal{B}$ such that $x_1, \ldots, x_n \in B$ and $B \cap C = \emptyset$. Pick n so that $1/n \leq \min_i |g_B(x_i)|$. Then $g_{B,n}$ is in G and $g_{B,n}(x_i) = 1$ for all i but $g_{B,n}(C) = \{0\}$.

Now suppose X is separable metrizable, or equivalently, second countable. Clearly G from above shows (1) holds. And $G' = \{f_m/m : f_m \in G\} \cup \{\mathbf{0}\}$ is easily seen to be as required for (2). □

Corollary 4. *A space X is separable and metrizable if and only if it has a countable generator.*

Proof. From Lemma 3 we know if X is separable, metrizable then it has a (nice) countable generator. While from Lemma 2, since separable, metrizable spaces are those with countable weight, they have a countable generator. □

Lemma 5. *Let G be a $(0,\neq 0)$–generator for X. If $\mathbf{0}$ is not in $\overline{G \setminus \{\mathbf{0}\}}$ - here the closure is in the pointwise topology - then X is finite.*

Proof. Towards the contrapositive, suppose that is X is infinite. Take any $B = B(\mathbf{0}, F, \epsilon)$ a basic neighborhood of $\mathbf{0}$. Pick $x \in X \setminus F$. As G is a $(0, \neq 0)$–generator there is a g in G such that $g(x) \neq 0$ but $g(F) = \{0\}$. Then $\mathbf{0} \neq g \in B \cap G$. Thus $\mathbf{0}$ is in $\overline{G \setminus \{\mathbf{0}\}}$, as required. $\square$

We record the following well known result, see Arkhangel'skii [3, Proposition 0.5.4] for example, which we use without further comment.

Lemma 6. *If G is a generator for X then the evaluation map, $e : X \to C_p(G)$ defined by $e(x)(g) = g(x)$, is an embedding of X in $C_p(G)$.*

Okunev [16] proved the following informative technical theorem on embeddings of a $C_p(Y)$ in a $C_p(X)$. Let X be a space. Define $\mathcal{K}_X$ to be the smallest class of spaces containing X, all compacta, and which is closed under taking finite products, disjoint countable sums, closed subspaces, and continuous images.

Theorem 7 (Okunev)**.** *If $C_p(Y)$ embeds in $C_p(X)$ then Y is in $\mathcal{K}_X$.*

For future reference, let us note: if Y is in $\mathcal{K}_X$ and (1) X is σ-compact or (2) X has all finite powers Lindelöf, then Y has the same property.

3.2.2 Creating and Upgrading Generators

Lemma 8. *Let X be a space, and A a subspace. Let G be a subset of $C(X)$, and define $G_A = \pi_A(G) = \{g \restriction A : g \in G\}$. If G has any of the following properties then so does G_A: (i) generator, (ii) $(0, \neq 0)$–generator, (iii) $(0,1)$–generator; (iv) n-generator, (v) $(0, \neq 0)$ n-generator, (vi) $(0,1)$ n-generator.*

Proof. We show (v). The other cases are similar. Let G be an n-generator for X. Take any $a_1, \ldots, a_n$ from A and any C_A a closed subset of A not containing any a_i. Write $C_A = C \cap A$ where C is closed in X, and note no a_i is in C. Hence there is a g in G such that $g(a_i) \neq 0$ for all i but $g(C) \subseteq \{0\}$. Let $g_A = g \restriction A$. Then g_A is in G_A and $g_A(a_i) = g(a_i) \neq 0$ for all i while $g_A(C_A) \subseteq g(C) \subseteq \{0\}$. $\square$

Lemma 9. *(a) Let G be a generator in $C_k(X)$ (respectively, $C_p(X)$) for X. Then $G' = \{\mathbf{1} - \min(\lambda|g - \mu\mathbf{1}|, \mathbf{1}) : g \in G, \lambda, \mu \in \mathbb{R}\}$ is a $(0,1)$-generator for X in $C_k(X)$ (respectively, $C_p(X)$), and G' is the continuous image of $\mathbb{R}^2 \times G$.*

(b) Let G be an n-generator in $C_k(X)$ (respectively, $C_p(X)$) for X. Then there is a subset G'' of $C_k(X)$ (respectively, $C_p(X)$), which is the continuous image of $(\bigoplus_n \mathbb{R}^{2n}) \times G$, and a $(0,1)$ n-generator.

Proof. For (a), first note that min – as a function from $C(X) \times C(X)$ to $C(X)$, where either each $C(X)$ has the pointwise topology or each has the compact-open topology – is continuous, and similarly for absolute value and for linear combinations. Hence the map $((\lambda, \mu), g) \mapsto g_{\lambda,\mu}$, where $g_{\lambda,\mu} = \mathbf{1} - \min(\lambda|g - \mu\mathbf{1}|, \mathbf{1})$, is continuous, and G' is indeed the continuous image of $\mathbb{R}^2 \times G$.

Now take any $x \in X$ and a closed $C \subseteq X$ where $x \notin C$. Since G is a generator for X, then there is some function $g \in G$ such that $g(x) \notin \overline{g(C)}$. Let $\mu = g(x)$ and $g_1 = |g - \mu\mathbf{1}|$ and note that $g_1(x) = 0$, $g_1(x) \notin \overline{g_1(C)}$, and g_1 maps onto $[0, \infty)$. Pick $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \overline{g_1(C)} = \emptyset$. Let $\lambda = 1/\epsilon$ and define $g_2 = \lambda g_1$. Note that $g_2(x) = 0$ as before and that $g_2(C) \subseteq [1, \infty)$. Next, define $g_3 = \min(g_2, \mathbf{1})$ and observe $g_3(x) = 0$ and $g_3(C) \subseteq \{1\}$. Define $g' = 1 - g_3 = g_{\lambda,\mu}$. Then $g'(x) = 1$ and $g'(C) \subseteq \{0\}$, as desired.

For (b), G'' is the image of $(\bigoplus_n \mathbb{R}^{2n}) \times G$ under the map taking $((\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n), g)$ to $g_{\lambda_1,\mu_1} + \cdots + g_{\lambda_n,\mu_n}$. From part (a) we see this map is continuous.

Now take any $x_1, \ldots, x_n$ not in a closed set C. As G is an n-generator there is some g in G such that $g(x_i) \notin \overline{g(C)}$ for $i = 1, \ldots, n$. Pick pairwise disjoint open sets, $U_1, \ldots, U_n$ such that $x_i \in U_i$ and $U_i \cap C = \emptyset$. Applying part (a) to g, x_i and $X \setminus U_i$, we get λ_i, μ_i such that $g_i = g_{\lambda_i,\mu_i}$ maps x_i to 1 and $X \setminus U_i$ to 0. Since the U_i are pairwise disjoint and $\bigcup_i U_i$ is disjoint from C, $g'' = g_{\lambda_1,\mu_1} + \cdots + g_{\lambda_n,\mu_n}$ is as required – $g'' \in G''$, $g''(x_i) = 0$ for all i, and $g''(C) \subseteq g''(X \setminus \bigcup_i U_i) \subseteq \{0\}$. □

Lemma 10. *Let J be an open interval and $h : \mathbb{R} \to J$ a homeomorphism. Let G be a subset of $C(X)$. Let $h \circ G = \{h \circ g : g \in G\}$.*

(1) $h \circ G$ is a subset of $C(X, J)$ and, G and $h \circ G$ are homeomorphic,

(2) if G is generator then $h \circ G$ is also generator,

(3) if G is a $(0, \neq 0)$–generator and $h(0) = 0 \in J$ then $h \circ G$ is also a $(0, \neq 0)$–generator, and

(4) if G is a $(0,1)$–generator, $h(0) = 0$ and $h(1) = 1$ then $h \circ G$ is also a $(0,1)$–generator.

Proof. For (1), recall the sets in $C_p(X,Y)$ of the form $O((x_1,\ldots,x_n), U_1 \times \cdots \times U_n) = \{f \in C(X,Y) : \forall i\, f(x_i) \in U_i\}$, form a basis. Define $\phi : C_p(X) \to C_p(X,J)$ by $\phi(f) = h \circ f$. Then ϕ is a homeomorphism. Indeed $\phi(O((x_1,\ldots,x_n), U_1 \times \cdots \times U_n)) = \{h \circ f : \forall i\, f(x_i) \in U_i\} = \{h \circ f : \forall i\, (h \circ f)(x_i) \in h(U_i)\} = O((x_1,\ldots,x_n), h(U_1) \times \cdots \times h(U_n))$. Now ϕ induces a homeomorphism between G and $h \circ G$.

For (2), suppose x is in open U. As G is a generator there is a g in G and an open subset V' of $\mathbb{R}$ such that $x \in g^{-1}V' \subseteq U$. Let $V = h(V')$, and note this is an open subset of both J and $\mathbb{R}$. Further, $g^{-1}V' = g^{-1}h^{-1}V = (h \circ g)^{-1}V$. Hence $x \in (h \circ g)^{-1}V \subseteq U$, as required for $h \circ G$ to be a generator.

For (3), suppose x is in open U. As G is a $(0, \neq 0)$–generator there is a g in G such that $g(x) \neq 0$ but g is zero outside U. Now $h \circ g$ is in $h \circ G$, and - recalling $h(0) = 0$ - we see $(h \circ g)(x) = h(g(x)) \neq 0$ and $(h \circ g)(X \setminus U) = h(g(X \setminus U)) \subseteq h(\{0\}) = \{0\}$, as required for $h \circ G$ to be a $(0, \neq 0)$–generator. The argument for (4) is similar, and the details are omitted. $\square$

Lemma 11. *Let X be a space, $(F_n)_n$ an increasing sequence of non-empty finite subsets of X, and $(t_n)_n$ a sequence of reals strictly decreasing to 0. For each $n \geq 2$, set $\epsilon_n = \min(t_n - t_{n+1}, t_{n-1} - t_n)/2$, and set $\epsilon_1 = (t_1 - t_2)/2$.*

Suppose G_n is a subset of $C(X, (-t_n - \epsilon_n/2, t_n + \epsilon_n/2))$ such that for every g in G_n we have $|g(x)| \geq t_n$ for some x in F_n. Let $G = \{\mathbf{0}\} \cup \bigcup_n G_n$.

Then (1) for each n the set G_n is clopen in G, and (2) the G_n's are pairwise disjoint and converge to $\mathbf{0}$ in G.

Proof. Observe, first, that the ϵ_n's are defined so that the open intervals, $(t_n - \epsilon_n, t_n + \epsilon_n)$ are all pairwise disjoint. Fix n. Take any g in G_n. Then for some z in F_n, $|g(z)| \geq t_n$. Since $|g|$ has value at least t_n at z, but has value no more than $t_n + \epsilon_n/2$ over all of X, we see that g can not be in any G_m for $m \neq n$. Thus the G_n's are pairwise disjoint.

Let $U_g = B(g, F_n, \epsilon_n) \cap G$, a basic neighborhood of g in G. Any element of U_g is non-zero at z, so $\mathbf{0} \notin U_g$. Suppose $g' \in U_g \cap G_m$. Then we can not have $m > n$ because $z \in F_n$, $|g(z)| \geq t_n$ and $g \in B(g, F_n, \epsilon_n/2)$ imply $|g'(z)| \geq t_n - \epsilon_n/2$, which means $|g'|$ is too large at z to be in G_m. But also we can not have $m < n$ because as $g' \in G_m$, for some $z' \in F_m \subseteq F_n$ we have $|g'(z')| \geq t_m \geq t_{n-1}$, and now $g' \in B(g, F_n, \epsilon_n/2)$ forces $|g(z')| \geq t_{n-1} - \epsilon_{n-1}/2$, which means $|g|$ is too large at z' to be in G_n. Thus the basic neighborhood U_g of g is contained in G_n. This means G_n is open in G.

Define, for each n, the basic neighborhood in G of the zero function, $B_n = B(\mathbf{0}, F_n, t_n + \epsilon_n/2) \cap G$, and also set $T_n = \{\mathbf{0}\} \cup \bigcup_{m \geq n} G_m$. We check that in fact $B_n = T_n$. Indeed, if $g \in B_n$, then for every $m < n$ and every x in $F_m \subseteq F_n$ we have $|g(x)| < t_n < t_m$, so g is not in G_m. So $B_n \subseteq T_n$. Clearly $\mathbf{0}$ is in B_n, and for any g in G_m where $m \geq n$, the function g is in $C(X, (-t_n - \epsilon_n/2, t_n + \epsilon_n/2))$ which is contained in $B(\mathbf{0}, F_n, t_n + \epsilon_n/2)$. So $T_n \subseteq B_n$. It follows that the complement of each G_n in G is open, and so G_n is clopen.

It remains to show that the G_n's converge to $\mathbf{0}$. Take any basic neighborhood, $B(\mathbf{0}, F, \epsilon)$, of $\mathbf{0}$, where F is any finite subset of X, and $\epsilon > 0$. Pick n so that $t_n + \epsilon_n/2 < \epsilon$. Then $B_n = T_n$ is contained in $C(X, (-t_n - \epsilon_n/2, t_n + \epsilon_n/2))$, which is a subset of $B(\mathbf{0}, F, \epsilon)$. □

This result is used in two ways. The first is to help us explicitly construct interesting first and second countable $(0, \neq 0)$–generators containing $\mathbf{0}$. In this case we typically take $t_n = 1/n$ and have elements of G_n mapping into $[0, 1/n]$ and taking value $1/n$ at some point of F_n. We also use it to 'tidy' a given $(0, \neq 0)$–generator for a separable space, so that the new $(0, \neq 0)$–generator is made from pieces of the old and has $\mathbf{0}$ is a point of first countability. It is for this application, presented next, that we need some added flexibility with the values of members of each G_n.

Proposition 12. *If a separable space X has a $(0, \neq 0)$–generator H, then X has another $(0, \neq 0)$–generator G containing $\mathbf{0}$ such that (1) $G \setminus \{\mathbf{0}\}$ is a countable union of pairwise disjoint subspaces that are each clopen in G and homeomorphic to a subspace of H, and (2) $\mathbf{0}$ is a point of first countability in G.*

Proof. Fix a countable dense subset $D = \{x_n : n \in \mathbb{N}\}$ of X, enumerated so that each element is listed infinitely many times. For each n, let $F_n = \{x_1, \ldots, x_n\}$, let $t_n = 1/n$, let ϵ_n

be as in the preceding lemma, and fix a homeomorphism $h_n : \mathbb{R} \to (-1/n - \epsilon_n/2, 1/n + \epsilon_n/2)$ that is the identity on $[-1/n, 1/n]$.

Define, $G'_n = \{g \in H : \text{for some } x \in F_n,\ |g(x)| \geq 1/n\}$ and $G_n = h_n \circ G'_n$ for each n. Observe that each G_n is homeomorphic to a subspace of H. Let $G = \{\mathbf{0}\} \cup \bigcup_n G_n$.

By the preceding lemma, claims (1) and (2) hold for G. It remains to show that G is a $(0, \neq 0)$–generator for X. To see this, take any x in an open subset U of X. As H is a $(0, \neq 0)$–generator there is a g' in H such that $g'(x) \neq 0$ and $g'(X \setminus U) \subseteq \{0\}$. Pick m so that $|g'(x)| > 1/m$. The dense set D meets U, say at z. As z appears in the enumeration infinitely often, there is an $n > m$ so that $z = x_n \in F_n$. Now g' is in G'_n, so $g = h_n \circ g'$ is in G_n, and by choice of h_n we see $|g(x)| \geq 1/n > 0$ and $g(X \setminus U) = g'(X \setminus U) \subseteq \{0\}$, as required. $\square$

3.2.3 A Generator for $C_p(X)$

Let X be any space. We define $s(X)$ to be the space with the underlying set $(\bigoplus_n (X^n \times \mathbb{N})) \cup \{\star\}$ and whose basic open sets are of the form: $U = (U_1 \times \cdots \times U_n) \times \{m\}$ for $n, m \in \mathbb{N}$ and $U_1, \ldots, U_n$ open in X, and $U = s(X) \setminus \bigoplus_{n,m \leq N} (X^n \times \{m\})$ for some $N \in \mathbb{N}$. Define $X^{(n)}$ as the n-th symmetric power of X, that is, $(x_1, \ldots, x_n)$ is in $X^{(n)}$ if and only if $x_i \neq x_j$ if $i \neq j$, considered as a subspace of X^n. Then define $ss(X)$ to be the subspace of $s(X)$ where $ss(X) = \left(\bigoplus_n (X^{(n)} \times \mathbb{N})\right) \cup \{\star\}$. We now show that certain continuous images of $s(X)$ and $ss(X)$ are generators for $C_p(X)$.

The maps come from generators for $\mathbb{R}^n$, as follows. Let $\mathcal{F} = \bigcup_n \mathcal{F}_n$ be a family of functions so that $\mathcal{F}_n = \{f_{n,m} : m \in \mathbb{N}\}$ is contained in $C_p(\mathbb{R}^n, [0,1])$. Let k be in $\mathbb{N}$. We say $\mathcal{F}$ is *k-generating* if, for every n, $\mathcal{F}_n$ is a $(0,1)$ k-generator for $\mathbb{R}^n$. Given an f in $C_p(X)$ and $\underline{x} = (x_1, \ldots, x_n)$ in X^n, we write $f(\underline{x}) = (f(x_1), \ldots, f(x_n))$. Note that this function is continuous. For any k-generating family of $f_{n,m}$'s, define $\phi : s(X) \to C_p(C_p(X))$ by $\phi(\star) = \underline{0}$ and $\phi(\underline{x}, m)(f) = f_{n,m}(f(\underline{x}))/(2^n 3^m)$. Let $G' = \phi(s(X))$ and $G = \phi(ss(X))$.

By Lemma 3, applied to $\mathbb{R}^n$, for each n, we can find a family $\mathcal{F}$ which is k-generating for all k. In the sequel we always use such a family, but by selecting the family in other ways we can adjust the properties of ϕ (see below).

Theorem 13. *For any space X and k-generating family $\mathcal{F}$:*

(1) ϕ is continuous and

(2) G is a $(0, \neq 0)$ k-generator for $C_p(X)$ containing $\mathbf{0}$.

Hence G' is also a $(0, \neq 0)$ k-generator for $C_p(X)$ containing $\mathbf{0}$.

Proof. We first show that ϕ is continuous. Take any $s \in s(X)$.

Suppose $s \neq \star$. Then $s = ((x_1, \ldots, x_n), m)$. Now take any subbasic neighborhood $B(\phi(s), \{f\}, \epsilon)$ of $\phi(s)$ in $C_p(C_p(X))$ where f is in $C_p(X)$ and $\epsilon > 0$. By continuity of the composition of the continuous functions $f_{n,m}$ and f at $(x_1, \ldots, x_n)$ there exist open sets $U_1, \ldots, U_n$ with $x_i \in U_i$ such that if $x'_i \in U_i$ for $i = 1, \ldots, n$ then $|f_{n,m}(f(\underline{x}))/(2^n 3^m) - f_{n,m}(f(\underline{x}'))/(2^n 3^m)| < \epsilon$. Let $U = (U_1 \times \cdots \times U_n) \times \{m\}$ and note that $s \in U$ and that U is open in $s(X)$. Take any $s' = ((x'_1, \ldots, x'_n), m) \in U$. Then $|\phi(s)(f) - \phi(s')(f)| = |f_{n,m}(f(\underline{x}))/(2^n 3^m) - f_{n,m}(f(\underline{x}'))/(2^n 3^m)| < \epsilon$. Thus, if $s' \in U$ then $\phi(s') \in B(\phi(s), \{f\}, \epsilon)$, as required for continuity of ϕ at s.

Now suppose $s = \star$. Take any basic neighborhood around $\phi(\star) = \mathbf{0}$, say $B(\mathbf{0}, F, \epsilon)$. Pick N so that $1/(2^n 3^m) < \epsilon$ if either n and m is strictly greater than N. Set $U = s(X) \setminus \bigoplus_{n,m \leq N}(X^n \times \{m\})$. This is a basic neighborhood of $*$. Take any $s' = (\underline{x}', m) \in X^n \times \{m\} \subseteq U$, and note at least one of n, m is strictly larger than N. Then $\phi(s') \in B(\mathbf{0}, F, \epsilon)$ because $f_{n,m}$ maps into $[0, 1]$ so for every f in F, $|0 - \phi(s')(f)| = |\phi(s')(f)| = |f_{n,m}(f(\underline{x}'))/(2^n 3^m)| \leq 1/(2^n 3^m) < \epsilon$, as required.

It remains to show (2). Evidently, $\phi(\star) = \mathbf{0} \in G$. We show G is a $(0, \neq 0)$ k-generator. To this end, take any $f_1, \ldots, f_k$ from $C_p(X)$, and disjoint closed set C. We can pick $F = \{x_1, \ldots, x_n\}$ a finite subset of X and $\epsilon > 0$ so that $\bigcup_i B(f_i, F, \epsilon)$ is disjoint from C. Set $\underline{x} = (x_1, \ldots, x_n)$. Note that $\underline{x}$ is in $X^{(n)}$. For each i, let $\underline{y}_i = f_i(\underline{x}) = (y_{i,1}, y_{i,2}, \ldots, y_{i,n})$ and $U_i = (y_{i,1} - \epsilon, y_{i,1} + \epsilon) \times \cdots \times (y_{i,n} - \epsilon, y_{i,n} + \epsilon)$. Then, there is an m so that $f_{n,m}$ has value one at each $\underline{y}_i$ and $f_{n,m}(\mathbb{R}^n \setminus \bigcup_{i=1}^n U_i) = \{0\}$. Now $g = \phi(\underline{x}, m)$ is in G. On the one hand, we have $g(f_i) = f_{n,m}(\underline{y}_i)/(2^n 3^m) \neq 0$, as $f_{n,m}(\underline{y}_i) = 1$. While on the other hand, take any f' not in $\bigcup_i B(f_i, F, \epsilon)$. Then, for each i, for some j we have $|f_i(x_j) - f'(x_j)| \geq \epsilon$, and so, $f'(\underline{x})$ is not in U_i. So $f'(\underline{x}) \notin \bigcup_i U_i$, $f_{n,m}(f'(\underline{x})) = 0$ and $g(f') = 0$. $\square$

We present some additional results relating to $ss(X)$, specifically we place constraints on the map ϕ, restricted to $ss(X)$, to G. Passing to a subset of the k-generating family $\mathcal{F}$ going into the definition of ϕ we may, and do, assume that every member $f_{n,m}$ of $\mathcal{F}$ attains the value 1 somewhere, and has bounded support.

We start by showing that the map ϕ, when restricted to $ss(X)$ has finite fibres. Choosing the k-generating family, $\mathcal{F} = \bigcup_n \mathcal{F}_n$, carefully we can ensure that ϕ is injective. In this case, $C_p(X)$ has a generator which is a condensation of $ss(X)$.

We unimaginatively call this carefully chosen family 'special'. Let q be a quadruple of increasing rationals, $(q(1), q(2), q(3), q(4))$. Write $U_q = (q(2), q(3))$ and $V_q = (q(1), q(4))$. Enumerate all such quadruples q as $q_1, q_2, \ldots, q_m, \ldots$. Define $f_{1,m} : \mathbb{R} \to [0,1]$ to be zero outside V_{q_m}, one on U_{q_m}, and to linearly interpolate between 0 and 1 on $(q_m(1), q_m(2))$ and $(q_m(3), q_m(4))$. Observe that the family $f_{1,m}$ is a $(0,1)$–generator for the reals.

For $n \geq 2$, let $f_{n,m}$ enumerate all functions f of the form $f = \prod_{i=1}^n f_{1,k_i}$, so $f(x_1, \ldots, x_n)$ equals $f_{1,k_1}(x_1).f_{1,k_2}(x_2) \ldots f_{1,k_n}(x_n)$, and all the rationals associated with $f_{1,k_1}, \ldots, f_{1,k_n}$ are distinct (i.e. for all $i \neq j$, $\{q_{k_i}(p) : p = 1,2,3,4\} \cap \{q_{k_j}(p) : p = 1,2,3,4\} = \emptyset$). Observe that the support of $f = \prod_{i=1}^n f_{1,k_i}$ is $\prod_{i=1}^n V_{k_i}$ and $f^{-1}\{1\}$ is the closure of $\prod_{i=1}^n U_{k_i}$. Hence the family of all $f_{n,m}$ is 1-generating.

As above, define $\phi : s(X) \to C_p(C_p(X))$ by $\phi(\underline{x}, m)(f) = f_{n,m}(f(\underline{x}))/(2^n 3^m)$ and $\phi(\star) = \mathbf{0}$. Let $G' = \phi(s(X))$ and $G = \phi(ss(X))$. If the family is the special one then call ϕ 'special'.

Theorem 14. *For any space X, ϕ restricted to $ss(X)$ has finite fibers. More precisely, $\phi^{-1}\{\underline{0}\} = \{*\}$ and $\phi((x_1, \ldots, x_n), m) = \phi((x'_1, \ldots, x'_{n'}), m')$ if and only if $n = n'$, $m = m'$ and $\{x_1, \ldots, x_n\} = \{x'_1, \ldots, x'_{n'}\}$. Further, if ϕ is special then it is an injection.*

Proof. Fix $s, s' \in ss(X)$ such that $\phi(s) = \phi(s')$.

Suppose, to start, $s = \star$, but, for a contradiction, that $s' \neq \star$, so say $s' = (\underline{x}, m) = ((x_1, \ldots, x_n), m)$. Pick $\underline{y} = (y_1, \ldots, y_n)$, such that $f_{n,m}(\underline{y}) = 1$. Noting that, as $\underline{x}$ is in $X^{(n)}$ (so $x_1, \ldots, x_n$ are distinct), there is an $f \in C_p(X)$ such that $f(x_i) = y_i$ for each $i \in \{1, \ldots, n\}$. Now we see that $\phi(s')(f) = f_{n,m}(f(\underline{x}))/(2^n 3^m) = f_{n,m}(\underline{y}) = 1/(2^n 3^m) \neq 0 = \underline{0}(f) = \phi(\star)$, so $\phi(s) \neq \phi(s')$.

Now, suppose that neither s nor s' is $\star$. Say $s = (\underline{x}, m) = ((x_1, \ldots, x_n), m)$ and $s' = (\underline{x}', m') = ((x'_1, \ldots, x'_{n'}), m')$. We first show that $n = n'$ and $m = m'$. Recall we have $\underline{t}_{n,m} =$

$(t_1, \ldots, t_n)$ so that $f_{n,m}(\underline{t}_{n,m}) = 1$. As above, pick f in $C_p(X)$ such that $f(x_i) = t_i$ for $1 \leq i \leq n$. Since $\phi(s) = \phi(s')$ we have $1/(2^n 3^m) = \phi(s)(f) = \phi(s')(f) = f_{n',m'}(f(\underline{x}'))/(2^{n'} 3^{m'}) \in [0, 1/(2^{n'} 3^{m'})]$. So $2^n 3^m \geq 2^{n'} 3^{m'}$. Symmetrically, interchanging s and s', from $\phi(s') = \phi(s)$ we see that $2^{n'} 3^{m'} \geq 2^n 3^m$. Combining the two inequalities, $2^n 3^m = 2^{n'} 3^{m'}$, and hence $n = n'$ and $m = m'$.

Thus $s = ((x_1, \ldots, x_n), m)$ and $s' = ((x'_1, \ldots, x'_n), m)$. It remains to show that the two sets $\{x_1, \ldots, x_n\}$ and $\{x'_1, \ldots, x'_n\}$ are equal. If not then there is at least one x'_j where $x'_j \neq x_i$ for all $i \in \{1, \ldots, n\}$. Pick $\underline{y}_{n,m} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $f_{n,m}(\underline{y}_{n,m}) = 1$, and recall that the support of $f_{n',m'}$ is bounded (in the max metric on $\mathbb{R}^n$), say by M'. Choose $f \in C_p(X)$ such that $f(x_i) = y_i$ for $i \in \{1, \ldots, n\}$, and $f(x'_j) = M' + 1$. Then $f_{n,m}(f(\underline{x})) = f_{n,m}(\underline{t}_{n,m}) = 1$. However, as $f(x'_j) = M' + 1$, $f(\underline{x}')$ is not in the support of $f_{n',m'}$ (whatever values f takes on x'_i for $i \neq j$), so $f_{n',m'}(f(\underline{x}')) = 0$. Hence $\phi(s) \neq \phi(s')$ as $\phi(s)(f) = 1/(2^n 3^m) \neq 0 = \phi(s')(f)$.

Now let us assume that ϕ is special. We show ϕ is injective. Suppose, for a contradiction, that $\phi(s) = \phi(s')$ but $s \neq s'$. From the above we know $s = ((x_1, \ldots, x_n), m)$, $s' = ((x'_1, \ldots, x'_n), m)$, and $\{x_1, \ldots, x_n\} = \{x'_1, \ldots, x'_n\}$. An alternative way of stating the latter condition is that there is a non-trivial permutation, π, of $\{1, \ldots, n\}$ such that $\underline{x}' = \pi(\underline{x})$, where we have abused notation and written $\pi(\underline{x})$ for $(x_{\pi(i)}, \ldots, x_{\pi(n)})$.

Take any $\underline{y}$ in $\mathbb{R}^n$. Pick f in $C_p(X)$ such that $f(\underline{x}) = \underline{y}$. Note that $f(\underline{x}') = f(\pi(\underline{x})) = \pi(\underline{y})$. Then $f_{n,m}(\underline{y})/(2^n 3^m) = f_{n,m}(f(\underline{x}))/(2^n 3^m) = \phi(s)(f) = \phi(s')(f)$ and $\phi(s')(f) = f_{n,m}(\pi(\underline{y}))/(2^n 3^m)$. Thus $f_{n,m}(\underline{y}) = f_{n,m}(\pi(\underline{y}))$. In particular, $f_{n,m}$ has value one if and only if $f_{n,m} \circ \pi$ has value one.

However, the construction of the 'special' family of $f_{n,m}$ means we know that $f_{n,m}$ has value one precisely on a product of n closed intervals with rational endpoints *all of which are different*. So $f_{n,m} \circ \pi$ has value one precisely on a product of n closed intervals with rational endpoints, but – because π is non-trivial – a *different* product. This is the desired contradiction to ϕ not injective. $\square$

Thus the map ϕ from $ss(X)$ to the generator G for $C_p(X)$ is continuous and finite-to-one (even, one-to-one for special k-generators). It would have been pleasant if ϕ were an open map. Open maps preserve first and second countability, for example. However we now show that ϕ is never open.

For any set X let $\Delta = \{(x, x) : x \in X\}$, the *diagonal* of X. A straightforward induction on the number of rectangles, establishes the following.

Lemma 15. *Let X be an infinite set. Then $X^2 \setminus \Delta$ is not a finite union of proper rectangles (if $R_1 = S_1 \times T_1, \ldots, R_n = S_n \times T_n$ are subsets of $X^2 \setminus \Delta$ then $\bigcup_{i=1}^n R_i \neq (X^2 \setminus \Delta)$).*

Theorem 16. *Let X be an infinite space. Then ϕ is not an open mapping from $ss(X)$ onto G.*

Proof. Let $n = 2$. Pick m so that the support of $f_{n,m}$ is contained in a rectangle $R = P \times Q$ which is disjoint from the diagonal, Δ, of $\mathbb{R}^2$. Set $N = \max(2, m)$. Note that $n, m \leq N$. Set $U = U_N = ss(X) \setminus \bigoplus_{n',m' \leq N}(X^{(n')} \times \{m'\})$. This is a basic neighborhood of $\star$. We show $\phi(U)$ is not open in G because $\underline{0} = \phi(\star)$ is in $\phi(U)$, but no basic neighborhood, $B(\underline{0}, F, \epsilon) \cap G$, of it is contained in $\phi(U)$.

To see this, fix the basic neighborhood, $B(\underline{0}, \{f_1, \ldots, f_n\}, \epsilon) \cap G$. For $i = 1, \ldots, n$ let A_i be the set of all points (x_1, x_2) in $X^{(n)} = X^2 \setminus \Delta$ such that $f_{n,m}(f_i(x_1), f_i(x_2))/(2^n 3^m) \geq \epsilon$. Note that if (x_1, x_2) is in A_i then $(f_i(x_1), f_i(x_2))$ must be in the support of $f_{n,m}$, and so must be in R. In other words, $A_i \subseteq R_i$ where $R_i = f_i^{-1}P \times f_i^{-1}Q$, which is a rectangle in X^2 not meeting the diagonal.

By the preceding lemma, the rectangles $R_1, \ldots, R_n$ do not cover $X^2 \setminus \Delta$. So there is an (x_1, x_2) in $X^{(2)}$ which is not in any A_i. From the latter property, as $s = ((x_1, x_2), m)$ is in $X^{(n)} \times \{m\}$, we see that $\phi(s)$ is in $B(\underline{0}, \{f_1, \ldots, f_n\}, \epsilon) \cap G$. However, according to Theorem 14, $\phi^{-1}\phi(s)$ is contained in $X^{(n)} \times \{m\}$, which is disjoint from U, so $\phi(s) \notin \phi(U)$, as required. $\square$

4.0 Compact Generators

Which spaces have a compact generator? Or for that matter a σ-compact generator? The answer potentially depends on the generator's topological property (compact or σ-compact), the type of generator ($(0,1)$, $(0,\neq 0)$, or plain) and the topology of the function space (compact-open or pointwise). Fortunately, these twelve potential combinations form just three equivalent groups.

The first case is when the generator is compact and $(0,1)$. Then, by Lemma 17, independently of the function space topology, the space must be discrete (and conversely). Next, we see (Theorem 18) that - for a fixed function space topology - all the remaining combinations of type and topological property are equivalent. In subsection 4.1.2 we focus on the second case, when the function space topology is the compact-open topology, showing in this scenario a space is metrizable if and only if it is a k-space and has a compact generator (Propositions 19 and 20). In subsection 4.1.3 we turn to the last case, the pointwise topology, establishing that a space has a compact generator if and only if it is Eberlein-Grothendieck (Proposition 21).

Next, we examine compact generators of highly specific types in Section 4.2. Motivated by our understanding of compact Eberlein-Grothendieck spaces, or Eberlein compacta, we focus on spaces with a generator homeomorphic to some supersequence, $A(\kappa)$ (Section 4.2.1), or with a generator homeomorphic to a continuous image of some $s(A(\kappa))$ (Section 4.2.2).

We end with results for Lindelöf generators in Section 4.3. It's well known that a finite power of σ-compact spaces remains σ-compact, while even the square of a Lindelöf space may not be Lindelöf. We give an example of a Lindelöf generator whose square is not Lindelöf. (The corresponding result for $C_p(X)$ is a famous open problem.) We show that any space with a Lindelöf generator is countably tight. However a space with a Lindelöf generator need not have a countably tight square (unlike the situation with $C_p(X)$). And a space with all finite powers countably tight need not have a Lindelöf generator.

4.1 General Compact Generators

4.1.1 Grouping Generators

Lemma 17. *A space X has a compact $(0,1)$–generator in $C_k(X)$ if and only if it has a compact $(0,1)$–generator in $C_p(X)$ if and only if X is discrete.*

Proof. First observe that every discrete space, X say, has a compact $(0,1)$-generator in $C_k(X)$. To see this, let $G = \{\chi_{\{x\}} : x \in X\} \cup \{\mathbf{0}\}$, and observe that any neighborhood of $\mathbf{0}$ contains all but finitely many of the $\chi_{\{x\}}$.

It remains to show that if a space has a non-isolated point, x_0 say, then any $(0,1)$-generator, G say, in $C_p(X)$ for X contains the discontinuous function $f = \chi_{\{x_0\}}$ in its closure, and so G can not be compact. Take any basic neighborhood, $B(f, F, \epsilon)$ say, of f in $\mathbb{R}^X$. Let $F' = F \setminus \{x_0\}$. As G is a $(0,1)$-generator and F' is closed, there is a $g = g_{F,\epsilon}$ in G such that $g(F') = 0$ but $g(x_0) = 1$. Then f and g coincide on F, so $g \in G \cap B(f, F, \epsilon)$, as required. $\square$

Theorem 18. *Let X be a space then for generators in $C_k(X)$ (respectively, $C_p(X)$) the following are equivalent:*

(1) X has a σ-compact $(0,1)$–generator, (2) X has a σ-compact $(0,\neq 0)$–generator, (3) X has a σ-compact generator, (4) X has a compact $(0,\neq 0)$–generator, and (5) X has a compact generator.

Proof. Evidently, (1) implies (2) and (2) implies (3). As finite products and continuous images of σ-compact space are again σ-compact, from Lemma 9 we immediately deduce: if a space X has a σ-compact generator in $C_k(X)$ (respectively, $C_p(X)$) then it has a σ-compact $(0,1)$-generator. Thus (1)-(3) are equivalent.

Evidently, (4) implies (5) and (5) implies (3). We show: if a space X has a σ-compact $(0,\neq 0)$–generator in $C_k(X)$ (respectively, $C_p(X)$) then it has a compact $(0,\neq 0)$–generator in $C_k(X)$ (respectively, $C_p(X)$). In other words, (3) implies (4), from which equivalence of all (1)-(5) follows.

Let $G = \bigcup_n G_n$ be a $(0, \neq 0)$–generator for X, where each G_n is compact. Fix n and define $m_n : C(X) \to C(X)$ by $m_n(f)(x) = \mathrm{mid}(-1/n, f(x), 1/n)$. Then m_n is continuous (with both copies of $C(X)$ either both taking the pointwise or compact-open topologies). Let $G'_n = m_n(G_n)$, and note G'_n is a compact subset of $C(X, [1/n, -1/n])$ (with the appropriate topology). Let $G' = \bigcup_n G'_n \cup \{\mathbf{0}\}$. As G is a $(0, \neq 0)$–generator, and if g is G_n and $g(x) \neq 0$, for some point x, then $m_n(g)(x) \neq 0$, we see G' is also a $(0, \neq 0)$–generator for X. Further G' is compact. To see this observe that any basic open neighborhood of $\mathbf{0}$, say $B(\mathbf{0}, F, 1/n)$, contains all G'_m where $m > n$, and each of $G'_1, \ldots, G'_n$ is compact. □

4.1.2 The Compact Open Topology

Proposition 19. *Every metrizable space X has a compact $(0, \neq 0)$–generator in $C_k(X)$ (and hence in $C_p(X)$).*

Proof. Let (X, d) be a metric space with d bounded by 1. Let $\mathcal{B}$ be a basis where $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ with each $\mathcal{B}_n$ locally finite. Let $\mathcal{B}' \subseteq \mathcal{B}$ and define $g_{\mathcal{B}'} = \sum_{n=1}^{\infty} g_{\mathcal{B}',n}/2^n$ where $g_{\mathcal{B}',n}(x) = \sup\left(\{0\} \cup D_{\mathcal{B}',n,x}\right)$ with $D_{\mathcal{B}',n,x} = \{d(x, X \setminus B) : B \in (\mathcal{B}' \cap \mathcal{B}_n)\}$.

Since $\mathcal{B}_n$ is locally finite, $x \mapsto d(x, X \setminus B)$ is continuous and $d(x, X \setminus B) \neq 0$ means $x \in B$, we see $g_{\mathcal{B}',n}$ is continuous for each $n \in \mathbb{N}$. Also, $\mathrm{range}(g_{\mathcal{B}',n}) \subseteq [0, 1]$, as d is bounded by 1. It follows that $g_{\mathcal{B}'}$ is continuous and $\mathrm{range}(g_{\mathcal{B}'}) \subseteq [0, 1]$. Define $G = \{g_{\mathcal{B}'} \in C_k(X) : \mathcal{B}' \subseteq \mathcal{B}\}$.

To show that G is a $(0, \neq 0)$–generator, consider any $x \in X$ and open set $U \ni x$. Since $\mathcal{B}$ is a basis for X, we may find $B \in \mathcal{B}_N \subseteq \mathcal{B}$ for some $N \in \mathbb{N}$ where $x \in B \subseteq U$. Let $\mathcal{B}' = \{B\} \subseteq \mathcal{B}$ and note that $g_{\mathcal{B}'} \in G$. Since $x \in B \in \mathcal{B}' \cap \mathcal{B}_N$, then $d(x, X \setminus B) \neq 0$ and $g_{\mathcal{B}'}(x) \neq 0$. Now take any $z \in X \setminus U$. Then $d(z, X \setminus B) = 0$. For any $n \in \mathbb{N}$, as $\mathcal{B}' = \{B\}$, we see $\mathcal{B}' \cap \mathcal{B}_n \subseteq \{B\}$, so $D_{\mathcal{B}',n,z} \subseteq \{0\}$ and $\sup\left(\{0\} \cup D_{\mathcal{B}',n,z}\right) = 0$. Hence $g_{\mathcal{B}'}(z) = \sum_{n=1}^{\infty} g_{\mathcal{B}',n}(z)/2^n = 0$. Thus $g_{\mathcal{B}'}(X \setminus U) \subseteq \{0\}$, as required.

We now show G is compact in $C_k(X)$. Toward this end, define $\Gamma : \{0,1\}^{\mathcal{B}} \to C_k(X, [0, 1])$ by $\Gamma(\chi_{\mathcal{B}'}) = g_{\mathcal{B}'}$. Observe $G = \Gamma(\{0,1\}^{\mathcal{B}})$, the product space, $\{0,1\}^{\mathcal{B}}$, is compact, and so G is compact provided that Γ is continuous.

Take an arbitrary $\mathcal{B}' \subseteq \mathcal{B}$ and a basic open set $B(g_{\mathcal{B}'}, K, \epsilon)$ where K is compact, $\epsilon > 0$. We will find a finite collection $\mathcal{F} \subseteq \mathcal{B}$ such that $B(\chi_{\mathcal{B}'}, \mathcal{F})$ maps into the basic open set

$B(g_{\mathcal{B}'}, K, \epsilon)$.

Pick $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} 1/2^n < \epsilon$. For each $n \leq N$, then let $\mathcal{F}_n = \{B \in \mathcal{B}_n : B \cap K \neq \emptyset\}$. Note that $\mathcal{F}_n$ is finite because each $\mathcal{B}_n$ is locally finite and K is compact. Last, let $\mathcal{F} = \bigcup_{n \leq N} \mathcal{F}_n$. It is finite.

Now take any $\mathcal{B}'' \in \mathcal{B}(\chi_{\mathcal{B}'}, \mathcal{F})$. In other words, $\mathcal{B}'' \cap \mathcal{F} = \mathcal{B}' \cap \mathcal{F}$. We want to show $\Gamma(\chi_{\mathcal{B}''})$, which is $g_{\mathcal{B}''}$, satisfies the following: for every $x \in K$, $|g_{\mathcal{B}'}(x) - g_{\mathcal{B}''}(x)| < \epsilon$. Take any $x \in K$.

Take any $n \leq N$. Consider $D^-_{\mathcal{B}',n,x} = \{d(x, X \setminus B) : B \in \mathcal{B}' \cap \mathcal{B}_n \setminus \mathcal{F}\}$. When $B \in \mathcal{B}_n \setminus \mathcal{F}$, $x \notin B$, so $d(x, X \setminus B) = 0$ and we see that $D^-_{\mathcal{B}',n,x} \subseteq \{0\}$. Similarly, set $D^-_{\mathcal{B}'',n,x} = \{d(x, X \setminus B) : B \in \mathcal{B}'' \cap \mathcal{B}_n \setminus \mathcal{F}\}$ and note $D^-_{\mathcal{B}'',n,x} \subseteq \{0\}$. On the other hand, as $\mathcal{B}' \cap \mathcal{F} = \mathcal{B}'' \cap \mathcal{F}$, $\mathcal{B}' \cap \mathcal{B}_n \cap \mathcal{F} = \mathcal{B}'' \cap \mathcal{B}_n \cap \mathcal{F}$. So $D^+_{\mathcal{B}',n,x} = \{d(x, X \setminus B) : B \in \mathcal{B}' \cap \mathcal{B}_n \cap \mathcal{F}\} = \{d(x, X \setminus B) : B \in \mathcal{B}'' \cap \mathcal{B}_n \cap \mathcal{F}\} = D^+_{\mathcal{B}'',n,x}$. Thus, $g_{\mathcal{B}'}(x) = \sup\left(\{0\} \cup D_{\mathcal{B}',n,x}\right) = \sup\left(\{0\} \cup D^-_{\mathcal{B}',n,x} \cup D^+_{\mathcal{B}',n,x}\right) = \sup\left(\{0\} \cup D^-_{\mathcal{B}'',n,x} \cup D^+_{\mathcal{B}'',n,x}\right) = \sup\left(\{0\} \cup D_{\mathcal{B}'',n,x}\right) = g_{\mathcal{B}''}(x)$.

Hence, $|g_{\mathcal{B}'}(x) - g_{\mathcal{B}''(x)}| < \sum_{n=1}^{N} |(g_{\mathcal{B}',n}(x) - g_{\mathcal{B}'',n}(x)|/2^n + \epsilon = 0 + \epsilon$, as required. □

We now prove the converse with the assistance of the Collins-Roscoe Metrization Theorem [8]: a T_1-space X is metrizable if (and only if) for each x in X there is a decreasing sequence $\{W(n,x)\}_n$ of neighborhoods of x such that: (A) if $x \in U$, where U is open, then there is an $s = s(x, U)$ and an open $V = V(x, U)$ containing x such that $x \in W(s, y) \subseteq U$ whenever y is in V. (Note that as x is in V, it immediately follows from (A) that $\{W(n,x)\}_n$ is a base at x.)

Proposition 20. *Let X be a k-space with a σ-compact generator G in $C_k(X)$. Then X is metrizable.*

Proof. We show X has a family $\{W(n,x)\}_n$ satisfying (A) of the Collins-Roscoe theorem.

According to Theorem 18 we can assume X has a compact $(0, \neq 0)$–generator G in $C_k(X)$. As G is compact in the compact-open topology and X is a k-space the evaluation map $e : G \times X \to \mathbb{R}$, $e(g,x) = g(x)$, is continuous. For any n set $G_n = \{(g,x) \in G \times X : |g(x)| \geq 1/n\}$, and note it is closed. Further, for any x set $G_{n,x} = \{g \in G : |g(x)| \geq 1/n\}$, and note it is closed in G, and hence compact. Now define $W(n,x) = \{x' : \forall g \in G_{n,x}\, |g(x')| > 0\}$, and note it is a neighborhood of x. Clearly $W(n+1,x) \subseteq W(n,x)$. This gives the family $\{W(n,x)\}_n$.

To show the family satisfies (A), fix x and any open U containing x. As G is a $(0,\neq 0)$–generator there is a $g_{x,U}$ in G such that $g_{x,U}(X \setminus U) = 0$ but $|g_{x,U}(x)| > 0$. Pick $s = s(x,U) \geq 2$ such that $|g_{x,U}(x)| \geq 1/(s(x,U)-1)$. Let $V = V(x,U) = V_1 \cap V_2$, where V_1 and V_2 are defined as follows.

Pick $V_1 = V_1(x,U)$ an open set containing x and contained in U such that if $y \in V_1$ and $g \in G_{s-1,x}$ then $|g(y)| \geq 1/s$. (Here we use compactness of $G_{s-1,x}$ and continuity of e.) Take any $y \in V_1$. Take any $z \in W(s,y)$. Then for all g in $G_{s,y}$ we have $|g(y)| \geq 1/s$. But, as $y \in V_1$ and $g_{x,U} \in G_{s-1,x}$, we see $|g_{x,U}(y)| \geq 1/s$. Now, as z is in $W(s,y)$, we have $|g_{x,U}(z)| > 0$, and hence z is in U. Thus we have: (A1) if $y \in V_1$ then $W(s,y) \subseteq U$.

Pick $V_2 = V_2(x,U)$ an open subset of X containing x and open subset T of G such that: (a) $e(\cdot,x)^{-1}\{0\} \subseteq T$, and (b) $(T \times V_2) \cap G_s = \emptyset$. (This is possible because G_s is closed and disjoint from $e(\cdot,x)^{-1}\{0\} \times \{x\}$, which is compact.) We show: (A2) if $y \in V_2$ then $x \in W(s,y)$. To this end, take any y in V_2. Take any $g \in G_{s,y}$. We require $|g(x)| > 0$. Suppose, for a contradiction, $g(x) = 0$. Then by (a) we have $g \in T$. Thus $(g,y) \in T \times V_2$, so by (b) we see (g,y) is not in G_s, in other words $|g(y)| < 1/s$. This contradicts g in $G_{s,y}$.

Now (A) clearly holds for $V = V_1 \cap V_2$ by (A1) and (A2), and we are done □

4.1.3 The Pointwise Topology

A space is *Eberlein-Grothendieck* if it embeds in a $C_p(K)$ where K is compact.

Proposition 21. *Let X be a space. Then X has a compact generator if and only if X is Eberlein-Grothendieck.*

Proof. Suppose X has a compact generator G. Then X embeds in $C_p(G)$ via the evaluation map, and so X is Eberlein-Grothendieck.

Now suppose, X is Eberlein-Grothendieck, and specifically that X embeds in $C_p(K)$, where K is compact. By Theorem 13 $C_p(K)$ has a generator, G say, homeomorphic to a continuous image of $s(K)$. And since K is compact, $s(K)$ is clearly compact, and so the generator G is compact. By Lemma 8 as $C_p(K)$ has a compact generator then so does X. □

The class of Eberlein-Grothendieck spaces is not well understood. We do know that Eberlein-Grothendieck spaces have all finite powers countably tight (see the discussion of Asanov's theorem in Section 4.3), and Eberlein-Grothendieck spaces are *monolithic*: for every subspace A we have $nw(\overline{A}) \leq |A|$, in particular every separable subspace has a countable network ([4]). Nevertheless, there is no known internal characterization of Eberlein-Grothendieck spaces, and finding such a characterization is a major open problem, see Arkhangel'skii [3, Problem III.1.9] for example.

When the space X is a $C_p(Y)$ we have a complete answer to the question of when X is Eberlein-Grothendieck (equivalently, has a compact generator).

Theorem 22. *A space $X = C_p(Y)$ has a compact generator if and only if Y is σ-compact.*

Proof. Suppose $C_p(Y)$ has a compact generator G. Then, $C_p(Y)$ embeds in $C_p(G)$, and by Okunev's Theorem 7 it follows that Y is σ-compact. Conversely, if Y is σ-compact then a continuous image of $s(Y)$ is a generator for $C_p(Y)$, which is σ-compact. Hence, by Theorem 18, $C_p(Y)$ has a compact $(0, \neq 0)$–generator. $\square$

4.2 Special Compact Generators

In this section, we investigate when spaces have compact generators of highly specific types. This is motivated by our understanding of Eberlein compacta. A space is *Eberlein compact* if it is a compact Eberlein-Grothendieck space. Unlike general Eberlein-Grothendieck spaces, Eberlein compacta are extremely well understood, and have a variety of descriptions, including effective internal characterizations.

Each of the following four conditions is necessary and sufficient for a compact space X to be Eberlein compact: (0) X has a separator homeomorphic to some $A(\kappa)$, (1) X embeds in some $C_p(A(\kappa))$, (2) X has a generator homeomorphic to a continuous image of some $s(A(\kappa))$, and (3) X has a σ-point finite almost subbase.

Here $A(\kappa)$ is the one point compactification of the discrete space of size κ. A *supersequence* is any space homeomorphic to an $A(\kappa)$. Since supersequences are compact, any compact space satisfying (1) above is evidently Eberlein compact. Amir & Lindenstrauss [2] proved the foundational result that the converse is true.

Now if X is a subspace of some $C_p(A(\kappa))$, then it is easy to check that the evaluation map $e : A(\kappa) \to C_p(X)$ has image some supersequence which is a *separator*: a subspace S of $C_p(X)$ such that whenever distinct x and x' are in X then $s(x) \neq s(x')$ for some s in S. Thus (1) implies (0). For the converse suppose S is a separator for X and S is a supersequence. Then the evaluation map $e : X \to C_p(S)$ is continuous, injective (because S is a separator) and so an embedding (because X is compact). This means (0) implies (1).

Next suppose X is a subspace of some $C_p(A(\kappa))$, then (Theorem 13) some continuous image of $s(A(\kappa))$ is a generator for $C_p(A(\kappa))$, and so (Lemma 8) its subspace also has a generator which is the continuous image of $s(A(\kappa))$. So (1) implies (2). Clearly, $s(A(\kappa))$ is compact, so any space with a generator as in (2) is Eberlein-Grothendieck (Proposition 21), and any compact such space is Eberlein compact.

Dimov [9] introduced almost subbases and showed a compact space is Eberlein compact if and only if it has a σ-point finite almost subbase. A family $\mathcal{U}$ of open subsets of a space X is called an *almost subbase* if there are sets $V_m(U) \subseteq U$ for each $m \in \mathbb{N}$ and $U \in \mathcal{U}$ satisfying: (i) $U = \bigcup_{m \in \mathbb{N}} V_m(U)$, (ii) $V_{m+1}(U) \subseteq V_m(U)$, (iii) $V_m(U)$ is a cozero set if m is even, and (iv) $V_m(U)$ is a zero set if m is odd, such that $\mathcal{U} \cup \{X \setminus V_{2m-1}(U) : m \in \mathbb{N} \text{ and } U \in \mathcal{U}\}$ is a subbase for X. An almost subbase $\mathcal{U}$ is σ-point finite if and only if we can write $\mathcal{U} = \bigcup_n \mathcal{U}_n$ where each $\mathcal{U}_n$ is point finite (for each x in X the set $\{U \in \mathcal{U}_n : x \in U\}$ is finite).

Our focus is on spaces with a generator (rather than separator) homeomorphic to some supersequence, $A(\kappa)$, or with a generator homeomorphic to a continuous image of some $s(A(\kappa))$. Specifically, a space has generator homeomorphic to some supersequence if and only if it has a σ-point finite base of cozero sets (Theorem 23); while the equivalence of (1), (2) and (3) above holds for *all* spaces, not just compact ones (Theorem 27).

We remark that not every space with a compact generator (equivalently, Eberlein-Grothendieck space) has a generator as in (2) above, homeomorphic to a continuous image of some $s(A(\kappa))$. A specific example is $C_p(I)$, see Example 29.

4.2.1 Supersequences as Generators

Theorem 23. *A space X has a compact $(0,\neq 0)$–generator homeomorphic to a supersequence if and only if it has a σ-point finite base of cozero sets.*

Proposition 24. *If X has a σ-point finite base of cozero sets, then X has a compact $(0,\neq 0)$-generator in $C_p(X)$ which is homeomorphic to a supersequence, with $\mathbf{0}$ as the limit point.*

Proof. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a base of cozero sets where each $\mathcal{B}_n$ is point finite. Then, for any $B \in \mathcal{B}_n$, fix some x_B in B and $f_B \in C_p(X, [0, 1/n])$ such that $B = f_B^{-1}(\mathbb{R} \setminus \{0\})$ and $f_B(x_B) = 1/n$. For each $n \in \mathbb{N}$, let $K_n = \{f_B \mid B \in \mathcal{B}_n\}$, and define $K = \bigcup_{n\in\mathbb{N}} K_n \cup \{\mathbf{0}\}$. It is clear that K is a $(0,\neq 0)$–generator for X.

To see that K is compact let $\mathcal{U}$ be an open cover of K. Then there is a $U_0 \in \mathcal{U}$ such that $\mathbf{0} \in U_0$, and so there is a basic open set $V_0 = B(\mathbf{0}, F, \epsilon) \subseteq U_0$, where $F \subseteq X$ is finite and $\epsilon > 0$. Pick N such that $1/N < \epsilon$. Observe that, for all $n \geq N$, since each member of K_n maps into $[0, 1/n]$, the set K_n is a subset of V_0, and so U_0. Take any $n < N$. The set $A_n = \{f_B \in K_n \mid B \cap F \neq \emptyset\}$ is finite since F is finite and $\mathcal{B}_n$ is point finite. Now for every $f_B \in K_n \setminus A_n$, we have $B \cap F = \emptyset$, so $f_B(F) \subseteq f_B(X \setminus B) = \{0\}$, which means $f_B \in V_0 \subseteq U_0$. And since A_n is finite, then there is a finite $\mathcal{U}_n \subseteq \mathcal{U}$ covering A_n. Hence, $\{U_0\} \cup \mathcal{U}_n$ is a finite subset of $\mathcal{U}$ covering K_n. Now $\{U_0\} \cup \bigcup_{n<N} \mathcal{U}_n$ is a finite subcover of K from $\mathcal{U}$, as desired for compactness of K.

It remains to show that each f in K' is isolated in K, and for this it suffices (as $C_p(X)$ is T_1) to find an open neighborhood of f meeting only finitely many elements of K'. Fix f in K'. Then f is in some K_n and is associated with some $B \in \mathcal{B}_n$, so $f = f_B$. Let $U_B = B(f_B, \{x_B\}, 1/n - 1/(n + 1))$. This is an open neighborhood of f_B.

For any g in U_B we have $g(x_B) > 1/(n+1)$. As $K_m \subseteq C_p(X, [0, 1/m])$ we have $U_B \cap K_m = \emptyset$ for $m > n$. Now fix $m \leq n$. If $f_{B'} \in U_B \cap K_m$ then $f_{B'}(x_B) > 1/(n + 1) > 0$, so $x_B \in B' \in \mathcal{B}_m$. As $\mathcal{B}_m$ is point finite we see $U_B \cap K_m$ is finite. Hence U_B does indeed only meet finitely many members of K. $\square$

Proposition 25. *If X has a compact $(0,\neq 0)$–generator K which is a supersequence, then X has a σ-point finite base of cozero sets.*

Proof. If X is finite, then the claim is immediate. Hence, assume X is infinite. By Lemma 5 the unique limit point of K must be the zero function, $\mathbf{0}$. Let $K' = K \setminus \{\mathbf{0}\}$. Let $B_{n,f} = f^{-1}(\mathbb{R} \setminus [-1/n, 1/n])$ for each $n \in \mathbb{N}$ and $f \in K'$. Note that each $B_{n,f}$ is a cozero set. Then set $\mathcal{B}_n = \{B_{n,f} : f \in K'\}$ and $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. If $x \in U \subseteq X$ and U is open, then is $f \in K$ such that $f_\alpha(X \setminus U) = \{0\}$ and $f_\alpha(x) \neq 0$. Note $f \neq \mathbf{0}$, so $f \in K'$. Choose $n \in \mathbb{N}$ such that $1/n < |f(x)|$, so $x \in B_{n,f} \in \mathcal{B}$. And $X \setminus U \subseteq f^{-1}(0) \subseteq f^{-1}([-1/n, 1/n])$ implies $B_{n,f} = X \setminus f^{-1}([-1/n, 1/n]) \subseteq U$. So $\mathcal{B}$ is a base for X.

Fix $n \in \mathbb{N}$ and $x \in X$. Since $V = B(\mathbf{0}, \{x\}, 1/n)$ is an open neighborhood of $\mathbf{0}$, which is the limit of the supersequence K, then we have $f \in V$ for all but finitely many $f \in K'$. So there are only finitely many f such that $f(x) \notin [-1/n, 1/n]$, and so x is in only finitely many $B_{n,f}$. Thus each $\mathcal{B}_n$ is point finite, and $\mathcal{B}$ is σ-point finite. □

Uspenskii, see [25], has shown that pseudocompact spaces with a σ-point finite base are compact and metrizable.

Corollary 26. *A pseudocompact space has a supersequence generator if and only if it is compact and metrizable.*

4.2.2 Compact Generators Related to $s(A(\kappa))$

Theorem 27. *Let X be a space. Then the following are equivalent:*

(1) X embeds in some $C_p(A(\kappa))$,

(2) X has a generator homeomorphic to a continuous image of some $s(A(\kappa))$, and

(3) X has a σ-point finite almost subbase.

Proof. [11, Theorem 1.3], and also see [11, Theorem 1.17], says that (3) holds if and only if X embeds in some $C_p(A(\kappa) \times \mathbb{N})$, and since this latter space embeds in $C_p(A(\kappa))$, we have the equivalence of (1) and (3).

Next suppose (1) holds and X is a subspace of some $C_p(A(\kappa))$, then (Theorem 13) some continuous image of $s(A(\kappa))$ is a generator for $C_p(A(\kappa))$, and so (Lemma 8) its subspace X also has a generator which is the continuous image of $s(A(\kappa))$. So (1) implies (2).

Now suppose (2) holds. We will show X embeds in $C_p(A(\kappa))^{\mathbb{N}} = C_p(A(\kappa) \times \mathbb{N})$, and so done as in the first paragraph. Fix, then, a generator, G say, of X homeomorphic to a continuous image of some $s(A(\kappa))$, say via ϕ. Then the dual map, $\phi^\sharp : C_p(G) \to s(A(\kappa))$ where $\phi^\sharp(f) = f \circ \phi$ embeds $C_p(G)$ in $C_p(s(A(\kappa)))$. Thus, as X embeds in $C_p(G)$, X also embeds in $C_p(s(A(\kappa)))$. As $s(A(\kappa))$ is the countable union of copies of $A(\kappa)^n$, and $*$, $C_p(s(A(\kappa)))$ embeds in $\mathbb{R} \times \prod_n C_p(A(\kappa)^n)$ (apply the dual map). The factor $\mathbb{R}$ is absorbed, so we are done once we know $C_p(A(\kappa)^n)$ embeds in $C_p(A(\kappa))$. However, in Proposition 30 below we show that $C_p(A(\kappa)^n)$ has a σ-point finite almost subbase, and so we can apply the equivalence of (1) and (3) to finish the proof. □

From [11] we know (see paper for definitions) that a space with a σ-point finite almost subbase has a nice base (σ-additively Noetherian), network (σ-point finite expandable network) and point-network (σ-finite). We also note here a restriction on the cardinal invariants.

Lemma 28. *Let X have a generator homeomorphic to a continuous image of some $s(A(\kappa))$. Then (1) $hc(X) = w(X)$ and (2) if X is Baire then $c(X) = w(X)$.*

Proof. From above we know X has a σ-point finite almost subbase. It is well known that if $hc(X) \leq \kappa$ then any point-finite family of open sets has size no more than κ. So the almost subbase has size no more than κ. That almost subbase naturally generates a base of the same size, which implies $w(X) \leq \kappa$. If X is Baire then Dimov [9] has shown this, combined with the σ-point finite almost subbase, means X has a dense metrizable subspace, D. Hence, $c(X) = c(D) = d(D)$ and so $c(X) = d(X)$. Again from above we know X is Eberlein-Grothendieck and so monolithic. Thus $nw(X) \leq nw(\overline{D}) \leq d(D) = c(X)$. Since $hc(X) \leq nw(X)$ we deduce from part (1) that indeed $c(X) = d(X) = hc(X) = w(X)$. □

By Theorem 13, $C_p(I)$ has a generator homeomorphic to a continuous image of $s(I)$, which is compact. But $C_p(I)$ is hereditarily ccc while not second countable, so by the preceding lemma it does not have a generator homeomorphic to a continuous image of any $s(A(\kappa))$.

Example 29. The space $C_p(I)$ has a compact generator but does not have a generator homeomorphic to a continuous image of a $s(A(\kappa))$.

Jeremiah Morgan [15] has shown the following.

Proposition 30. *For every n and κ, the space $C_p(A(\kappa)^n)$ has a σ-point finite almost subbase.*

4.3 Lindelöf Generators

Now we turn to the problem of which spaces have a Lindelöf generator in the pointwise topology. Note that if a space has a Lindelöf generator in the compact-open topology then that generator is also Lindelöf in the pointwise topology. Since $\mathbb{R}^2$ is σ-compact and the product of σ-compact space with a Lindelöf space is Lindelöf, by Lemma 9, we can always upgrade a given Lindelöf generator to a Lindelöf $(0,1)$–generator, and thus all the types of generator are equivalent.

A critical difference between Lindelöf spaces and the σ-compact spaces of the previous sections, is that a finite power of σ-compact spaces is again σ-compact, but the square of a Lindelöf space need not be Lindelöf. One might hope that the additional structure of generators would lead to better productivity properties. Indeed it is a famous open problem whether $C_p(X)^2$ is Lindelöf when $C_p(X)$ is Lindelöf. But Example 32 dispels any such hope for generators. However, having a Lindelöf generator does have consequences. Asanov [6] proved that if $C_p(X)$ is Lindelöf then all finite powers of X are countably tight. (A space X is *countably tight* if whenever a point x is in $\overline{A}$ then there countable subset A_0 of A such that $x \in \overline{A_0}$.) It follows from Proposition 33 (with $n = 1$) that any space, X, with a Lindelöf generator is countably tight. To deduce that X^n is countably tight it suffices that X has a Lindelöf n-generator. In Example 34 we see the restriction to n-generators is necessary.

It is well known that Asanov's theorem does not have a converse. The double arrow space, DA, is a well-behaved first countable space (hence countably tight in all finite powers) such that $C_p(DA)$ is not Lindelöf. We show in Proposition 35 that DA does not have a Lindelöf generator.

4.3.1 Finite Powers Lindelöf

Theorem 31. *Let Y be a space. Then $X = C_p(Y)$ has a generator all of whose finite powers are Lindelöf if and only if all finite powers of Y are Lindelöf.*

Proof. If all finite powers of Y are Lindelöf then, by Theorem 13, $C_p(Y)$ has a generator which is the continuous image of $s(Y)$, and has all finite powers Lindelöf. For the converse, suppose G is a generator for $C_p(Y)$ all of whose finite powers are Lindelöf. Then $C_p(Y)$ embeds in $C_p(G)$. Since all finite powers of G are Lindelöf, Okunev's Theorem 7 guarantees that all finite powers of Y are Lindelöf. □

Let X be space and A a subspace. Denote by $X_{(A)}$ the space obtained from X by isolating all points of $X \setminus A$. In [19] Przymusiński showed that, for any $m \geq 1$, any uncountable compact metric space X can be partitioned into subsets $A_0, \ldots, A_m$ such that each $X_{(A_i)}$ has all finite powers Lindelöf, but $\prod_{i=0}^{m} X_{(A_i)}$ is not Lindelöf (or even normal). This construction is the key ingredient in the next two examples.

Example 32. There is a space X such that X has a Lindelöf generator G with G^2 not Lindelöf.

Proof. Partition the Cantor set, $\mathbb{C}$, into A_0, A_1 so that all finite powers of each $Y_k = \mathbb{C}_{(A_k)}$ is Lindelöf, but $Y_0 \times Y_1$ is not Lindelöf. For $k = 0, 1$, let $X_k = C_p(Y_k)$, let H_k be the generator for X_k given by Theorem 13, and observe that H_k is Lindelöf. For $k = 0, 1$, the evaluation map embeds a closed copy, Z_k say, of Y_k in $C_p(C_p(Y_k))$, and note the zero function is not in Z_k. For $k = 0, 1$, set $G_k = H_k \cup Z_k$, note that each G_k is a Lindelöf generator for X_k which contains a closed copy of Y_k not containing the zero function. Define $X = X_0 \oplus X_1$. For $k = 0, 1$ let G'_k be the subset of $C_p(X)$ obtained from G_k by extending each g in G_k so that it is constantly equal to zero on X_{1-k}. Note each G'_k is naturally homeomorphic to G_k, and so is Lindelöf, and $G'_0 \cap G'_1 = \{\mathbf{0}\}$. Finally define $G = G'_0 \cup G'_1$. Now we see that G is a Lindelöf generator for X such that G^2 contains a closed copy of $Y_0 \times Y_1$, and so is not Lindelöf. □

4.3.2 Finite Powers Countably Tight

Recall from Section 3.1.1 the two cardinal invariants of a space Y, *Lindelöf degree*, $L(Y)$, and *tightness*, $t(Y)$. Note $L(Y) \leq \aleph_0$ if and only if Y is Lindelöf and $t(Y) \leq \aleph_0$ if and only if Y is countably tight. Now the following result generalizes Asanov's theorem: for any space X we have, for all n, $t(X^n) \leq L(C_p(X))$.

Proposition 33. *If X has an n-generator G with $L(G) \leq \kappa$, then $t(X^n) \leq \kappa$.*

Proof. By Lemma 9 we can suppose that G is a Lindelöf $(0,1)$–generator.

Take any subset A of X^n and any $x = (x_1, \ldots, x_n) \in \overline{A}$. Let $G_x = \{g \in G : g(x_i) = 1$ for $i = 1, \ldots, n\}$, and note - as it is closed in G - it has Lindelöf degree $\leq \kappa$. For each $a = (a_1, \ldots, a_n)$ in A, let $U_a = B(\mathbf{1}, \{a_1, \ldots, a_n\}, 1/2) \cap G_x$. For any $g \in G_x$, by continuity of g at each x_i, where it has value 1, and recalling that $x \in \overline{A}$, we have that there is an $b^g = (b^g_1, \ldots, b^g_n)$ in A such that $|g(b^g_i) - 1| < 1/2$, for $i = 1, \ldots, n$, in other words, $g \in U_{b^g}$. Thus $\mathcal{U} = \{U_a : a \in A\}$ is an open cover of G_x, and so contains a subcover of size $\leq \kappa$, say $\{U_b : b \in B\}$ where $B \subseteq A$, $|B| \leq \kappa$.

We verify that $x \in \overline{B}$. Take any basic open neighborhood, $V = V_1 \times \cdots \times V_n$, of x. Recalling that G is an n-generator, there is a g in G such that $g(X \setminus \bigcup_i V_i) = 0$ and $g(x_i) = 1$ for $i = 1, \ldots, n$. Note g is in G_x. For some $b = (b_1, \ldots, b_n)$ in B, we must have $g \in U_b$. But then $g(b_i) \neq 0$ for $i = 1, \ldots, n$, and b is forced to be in V. Thus $V \cap B \neq \emptyset$, as desired. □

Example 34. For each n there is a space X with a Lindelöf n-generator so that (X^n is countably tight, but) X^{n+1} is not countably tight.

Proof. Partition the Cantor set, $\mathbb{C}$, into $A_0, \ldots, A_n$ so that all finite powers of each $\mathbb{C}_{(A_k)}$ is Lindelöf. Define, for each k, Y_k to be $\mathbb{C}$ with the points of A_k isolated. Let $Y = Y_0 \oplus \cdots Y_n$. For each k, let $X_k = C_p(Y_k)$. Let $X = X_0 \oplus \cdots \oplus X_n$.

Fix in this paragraph, $k \in \{0, \ldots, n\}$. Since for every $j \neq k$ the identity map from $\mathbb{C}_{(A_k)}$ to Y_j is continuous, we see that any finite product with factors from $\{Y_j : j \neq k\}$ is Lindelöf. Hence $\prod_{j \neq k} s(Y_j)$ is Lindelöf. Further, for each j, a continuous image, say H_j, of $s(Y_j)$ is an n-generator for $C_p(Y_j) = X_j$. Thus $G_k = \prod_{j \neq k} H_j$ is a Lindelöf n-generator for

$\bigoplus_{j\neq k} X_j$. Let G'_k be the subset of $C_p(X)$ obtained from G_k by extending each g in G_k so that it is constantly equal to zero on X_k. Then G'_k is naturally homeomorphic to G_k, and so is Lindelöf.

Define $G = \bigcup\{G'_k : 0 \leq k \leq n\}$. Then G is Lindelöf. And it is an n-generator for X. (To see this observe that any n-element subset of X is contained in some $\bigoplus_{j\neq k} X_j$, and so can be separated from any closed set by an element of G'_k.)

It remains to show that X^{n+1} is not countably tight. Since X^{n+1} contains $C_p(Y_0) \times C_p(Y_1) \times \cdots \times C_p(Y_n) = C_p(Y)$, it suffices to show this latter space is not countably tight. For each k, let $i_k : \mathbb{C} \to Y_k$ be the identity map. For any $F = (F_0, \ldots, F_n)$, where F_k is a finite subset of A_k, and any $U = (U_0, , \ldots, U_n)$, where the U_k's form a pairwise disjoint partition of $\mathbb{C}$ by clopen sets, define a function $f_{F,U} : Y \to \{0, 1\}$ as follows: $f_{F,U}(i_k(y)) = 0$ if $y \in F_k$ or $y \in U_j$ for some $j \neq k$ (and 1 otherwise). Noting that each $i_k(F_k)$ is clopen in Y we see that $f_{F,U}$ is continuous. Let S be the set of all $f_{F,U}$. Note that S is a subset of $C_p(Y)$, the zero function, 0, is not in S, and $|S| = \mathfrak{c}$.

To verify that $C_p(Y)$ is not countably tight (indeed has tightness $\mathfrak{c}$), we show (1) if $S' \subseteq S$ and $|S'| < |S|$ then $0 \notin \overline{S'}$, but (2) $0 \in \overline{S}$.

Towards (1), fix $S' \subseteq S$ with $|S'| < |S|$. Let $T = \bigcup\{\bigcup_{k=0}^{n} F_k : f_{F,U} \in S'\}$. Then $|T| < |S| = \mathfrak{c}$, so we can pick some $y' \in \mathbb{C} \setminus T$. We will show that the basic neighborhood of 0, $B = B(0, \{i_0(y'), \ldots, i_n(y')\}, 1/2)$ is disjoint frm S'. Suppose, for a contradiction, that $f = f_{F,U} \in B \cap S'$. Then $f_{F,U}(i_k(y')) = 0$ for all k. But now, for each k: as $y' \notin T$, $y' \notin F_k$ so y' must be in some U_j where $j \neq k$ – and, in particular, $y' \notin U_k$. This is impossible because the $(U_0, \ldots, U_n)$ partition $\mathbb{C}$.

Now for (2). Take any basic neighborhood, $B(0, K_Y, \epsilon)$, of 0 in $C_p(Y)$. We can suppose $K_Y = \bigcup_{k=0}^{n} i_k(K)$ for some finite subset K of $\mathbb{C}$. We will find $f = f_{F,U}$ in S such that f is zero on K_Y, and so $f \in B \cap S$, as required. For each k, let $F_k = K \cap A_k$. As $F_0, \ldots, F_n$ are pairwise disjoint and finite, we can find a clopen partition, $(U_0, \ldots, U_n)$ of $\mathbb{C}$ such that $F_k \subseteq U_k$ for each k. Then $f = f_{F,U}$ is in S where $F = (F_0, \ldots, F_n)$ and $U = (U_0, \ldots, U_n)$. Take any $i_k(y)$ in K_Y. Then $y \in A_j$ for some j, and so $y \in K \cap A_j = F_j$. If $j = k$ then $y \in F_j = F_k$ and so $f(i_k(y)) = 0$ in this case. While if $j \neq k$ then $x \in F_j \subseteq U_j$ and so $f(i_k(y)) = 0$ in this case, as well. $\square$

4.3.3 No Lindelöf Generator

The double arrow space, DA, has underlying set $I \times \{0,1\}$ and topology coming from the lexicographic order. For notational convenience, we write t^a for $(t,a) \in DA$. Then a basic neighborhood of t^1 has the form $\{t^1\} \cup ((t, t+\epsilon) \times \{0,1\})$, while basic neighborhoods of t^0 are $\{t^0\} \cup ((t-\epsilon, t) \times \{0,1\})$, where $\epsilon > 0$. Taking the basic neighborhoods with $\epsilon = 1/n$ for n in $\mathbb{N}$, we see directly that DA is first countable. The double arrow space is compact, hereditarily Lindelöf, hereditarily separable but not metrizable. (Note that a space is *hereditarily ccc* if every discrete subset is countable.)

Proposition 35. *The double arrow space, which is first countable and so countably tight in all finite powers, has no generator which is Lindelöf or one that is hereditarily ccc.*

Proof. Suppose, for a contradiction, that the double arrow space had a Lindelöf generator. Then by Lemma 9 it would also have a Lindelöf $(0,1)$–generator $G \subseteq C_p(DA, [0,\infty))$. Similarly, if the double arrow space had a hereditarily ccc generator, then by Lemma 9 it would also have a hereditarily ccc $(0,1)$–generator $G \subseteq C_p(DA, [0,\infty))$. We show that every $(0, \neq 0)$–generator G for DA with $G \subseteq C_p(DA, [0,\infty))$ contains an uncountable closed discrete subspace, and so is neither Lindelöf or hereditarily ccc.

For each $t \in I$, define the basic open neighborhoods $B_{t,\epsilon} = ((t, t+\epsilon) \times \{0,1\}) \cup \{t^1\}$ of t^1. In particular, let $B_t = B_{t,1/3}$ for each $t \in [1/3, 2/3]$. For each $t \in [1/3, 2/3]$, there is some $h_t \in G$ such that $h_t(DA \setminus B_t) = \{0\}$ and $\alpha_t = h_t(t^1) > 0$. Since $[1/3, 2/3] = \bigcup_{n\in\mathbb{N}} A_n$, where $A_n = \{t \in [1/3, 2/3] : \alpha_t > 1/n\}$, then there is some $m \in \mathbb{N}$ such that A_m is uncountable. Let $\alpha = 1/m$. For each $t \in A_m$, since $h_t(t^1) = \alpha_t > \alpha$, then by continuity, there is a $\delta_t > 0$ such that $h_t(B_{t,\delta_t}) \subseteq (\alpha, \infty)$. Now $A_m = \bigcup_{n\in\mathbb{N}} D_n$, where $D_n = \{t \in A_m : \delta_t > 1/n\}$, so there is a $k \in \mathbb{N}$ such that D_k is uncountable. Let $\delta = 1/k$.

Let $H = \{h_t : t \in D_k\}$. We will show that H is an uncountable closed discrete subspace of G. First, H is uncountable since $t \mapsto h_t$ is one-to-one. To see that H is discrete, fix $t \in D_k$, define the open neighborhood $U_t = B(h_t, \{t^0, t^1\} \cup (P \times \{1\}), \alpha/2)$, of h_t, where $P = \{(i\delta)/2 : i = 0, \ldots, 2k\}$, and we show $U_t \cap H = \{h_t\}$. Take any other $s \in D_k$. Three cases arise. If $s > t$, then $h_s(t^1) = 0$ while $h_t(t^1) = \alpha_t > \alpha$, so $h_s \notin U_t$. Next if $s < t < s + \delta$, then $h_s(t^0) > \alpha$ while $h_t(t^0) = 0$, so $h_s \notin U_t$. Finally in the case when $(*)$ $s + \delta \leq t$, then pick $r \in P \cap (s, s+\delta)$. Now, $h_s(r^1) > \alpha$, but $h_t(r^1) = 0$ since $r < s+\delta \leq t$. Hence, $h_s \notin U_t$.

It remains to show H is closed. We establish this in two steps. We say that $f \in G$ *jumps* at $t \in I$ if $f([0,t) \times \{0,1\}) = \{0\}$ and $f(B_{t,\delta}) \subseteq [\alpha, \infty)$. Define $F = \{f \in G : f \text{ jumps at some } t_f \in [1/3, 2/3]\}$. Note that for each f, t_f is unique. Also, note that $H \subseteq F$. Indeed, for any $t \in D_k \subseteq [1/3, 2/3]$, h_t jumps at t. In fact, H is closed in F. To see this, fix $f \in F \setminus H$, and note that $f(t_f^0) = 0$ by continuity, so consider $U_f = B(f, \{t_f^0, t_f^1\} \cup (P \times \{1\}), \alpha/2)$. By essentially the same 3-case argument as above, one can show that $h_s \notin U_f$ for any $s \in D_k \setminus \{t_f\}$. If $t_f \in D_k$, then there is an open nbhd V_f of f such that $h_{t_f} \notin V_f$. Otherwise, let $V_f = C_p(DA)$, so in any case, $f \in U_f \cap V_f \cap F \subseteq F \setminus H$, and H is closed in F.

Hence, it suffices to show F is closed in G. But F is contained in the closed subspace $G' = \{g \in G : g([0,1/3) \times \{0,1\}) = \{0\}\}$, so we only need to show F is closed in G'. Fix $g \in G' \setminus F$ and let $t_g = \sup\{t \in I : g([0,t) \times \{0,1\}) = \{0\}\} \geq 1/3$.

If $t_g > 2/3$, let $U_g = B(g, (P \cup \{2/3\}) \times \{1\}, \alpha/2)$. Then for any $f \in F$, there is an $x = s^1 \in ((P \cup \{2/3\}) \times \{1\}) \cap B_{t_f,\delta}$ such that $s < t_g$. Hence, $g(x) = 0$ while $f(x) > \alpha$, so $f \notin U_g$, which means $g \in U_g \cap G' \subseteq G' \setminus F$.

If $t_g \leq 2/3$, then since $g \notin F$, g does not jump at t_g. Hence, there is an $x_1 = t_1^a \in B_{t_g,\delta}$ such that $g(x_1) < \alpha$. By continuity, we may assume $t_1 \neq t_g$, so $t_g < t_1 < t_g + \delta$. Then by definition of t_g, there is an $x_2 = t_2^b$ with $t_g < t_2 < t_1$ such that $g(x_2) > 0$. Let $U_g = B(g, \{x_1, x_2, t_g^0\} \cup (P \times \{1\}), \epsilon)$, where $\epsilon = \min\{g(x_2), \alpha - g(x_1)\}/2$. Note that $\epsilon \leq \alpha/2$ since $g(x_1) \geq 0$. We show $U_g \cap F = \emptyset$. To this end, fix any $f \in F$. Four cases arise. If $t_2 < t_f$, then $f(x_2) = 0$, so $f \notin U_g$. Next, if $t_f < t_g \leq t_f + \delta$, then $f(t_g^0) \geq \alpha$, but $g(t_g^0) = 0$ by continuity, so $f \notin U_g$. While if $t_f + \delta \leq t_g$, then as with $(*)$ above, $f \notin U_g$. Otherwise $t_g \leq t_f \leq t_2$, so $t_f \leq t_2 < t_1 < t_g + \delta \leq t_f + \delta$. In particular, $t_f < t_1 < t_f + \delta$ gives $f(x_1) \geq \alpha$, so $f \notin U_g$. $\square$

5.0 Discrete Generators

What spaces have a discrete generator? In this chapter we only consider the pointwise topology, but note that a set discrete in the compact-open topology is discrete in the pointwise topology. We start in Section 5.1 by looking at spaces with a discrete 'vanilla' generator. We show that a necessary condition for a space X to have a discrete generator is that $w(X) = hc^*(X)$, and we show that in a wide range of situations, for example when X is zero-dimensional, that this condition is sufficient. In Section 5.2 we move from plain generators to $(0, \neq 0)$–generators. We establish a necessary condition – for all open subsets U we have $hc^*(U) = w(U)$ - for a space to have a discrete $(0, \neq 0)$–generator. We apply the condition to exhibit a variety of spaces with a discrete generator but no discrete $(0, \neq 0)$–generator. We show that the condition is sufficient in some limited cases. Generally, constructing discrete $(0, \neq 0)$–generators is more taxing than the task of making discrete generators. With this in mind we give results showing that large classes of spaces do indeed have discrete $(0, \neq 0)$–generators.

In the final section, Section 5.3, we explore which spaces have a discrete $(0, 1)$–generator. We show that metrizable space, all ordinals and the Alexandrov duplicate of a space with a discrete $(0, 1)$–generator all have discrete $(0, 1)$–generators. But we also show that the Michael line is a space with a discrete $(0, \neq 0)$–generator but no discrete $(0, 1)$–generator.

5.1 Discrete Generators

Applying Lemma 2 we see:

Lemma 36. *If X has a discrete generator then $C_p(X)$ contains a discrete subset of size $w(X)$.*

Question 1. Is the converse true: if $C_p(X)$ contains a discrete subset of size $w(X)$ then X has a discrete generator?

Question 2. Can we characterize in terms of X when $C_p(X)$ contains a discrete subset of size $w(X)$?

Lemma 37. *If $C_p(X)$ has a discrete set of infinite size κ then in $\bigoplus_n X^n$ there is a σ-discrete subset S of size κ.*

Proof. Fix a countable basis, $\mathcal{B}$, for $\mathbb{R}$. For any $x = (x_1, \ldots, x_n) \in X^n$ and $B = (B_1, \ldots, B_n)$ in $\mathcal{B}^n$ set $O(x, B) = \{f \in C_p(X) : f(x_i) \in B_i \text{ for } i = 1, \ldots, n\}$. Then the collection of all $O(x, B)$ is a basis for $C_p(X)$. Let $T = \{f_\alpha : \alpha < \kappa\}$ be discrete, say witnessed by $O(x_\alpha, B_\alpha)$. For each $B = (B_1, \ldots, B_n) \in \mathcal{B}^n$ let $S_B = \{x_\alpha : B_\alpha = B\} \subseteq X^n$. Let S be the countable union of all S_B. Note that S has size κ. We complete the proof by showing that each S_B is discrete.

Fix $B = (B_1, \ldots, B_n) \in \mathcal{B}^n$. For each $x_\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,x_n}) \in S_B$, let $U_\alpha = g_\alpha^{-1} B_1 \times \cdots \times g_\alpha^{-1} B_n$. Then x_α is in the open set U_α. But if $x_\beta = (x_{\beta,1}, \ldots, x_{\beta,x_n})$ is in S_B where $\beta \neq \alpha$, then as $O(x_\beta, B)$ witnesses discreteness of T for g_β, for some i we have $g_\alpha(x_{\beta,i})$ not in B_i, so x_β is not in U_α. Thus the U_α's witness discreteness of the x_α's in S_B. $\square$

Observing that for any space X and infinite cardinal κ, there is a σ-discrete subset of $\bigoplus_n X^n$ of size κ if and only if $hc^*(X) \geq \kappa$, we can combine Lemmas 36 and 37, and derive a necessary condition on a space to have a discrete generator.

Theorem 38. *If X has a discrete generator then $w(X) = hc^*(X)$.*

Question 3. Is the converse true: if $w(X) = hc^*(X)$ then must X have a discrete generator?

When the space is zero-dimensional or when the space (and not one of its higher powers) contains a suitably large discrete set, then the answer is positive.

Theorem 39. *If X contains a discrete subset S with $|S| = w(X)$ then X has a discrete generator.*

Proof. For any point x in an open U fix $b_{x,U} \in C_p(X, [0,1])$ such that $b_{x,U}(x) = 1$ and $b_{x,U}$ is zero outside U. Let $\kappa = w(X)$. Enumerate $S = \{x_\alpha : \alpha < \kappa\}$ and fix U_α such that $x_\alpha \in U_\alpha$ open and ($\beta \neq \alpha \implies x_\beta \notin U_\alpha$). Next, fix $\{f_\alpha : \alpha < \kappa\} \subseteq C_p(X, [0,1])$ such that whenever x is not in a closed set C there is an f_α such that $f_\alpha(x) = 1$ but f_α is zero on C.

Fix α. Shrink U_α so that if $f_\alpha(x_\alpha) > 0$ then $f_\alpha(U_\alpha) \subseteq (0, 1]$ (using continuity of f_α at x_α) while if $f_\alpha(x_\alpha) = 0$ then $U_\alpha \cap f_\alpha^{-1}\{1\} = \emptyset$. If $f_\alpha(x_\alpha) > 0$ then set $\lambda_\alpha = 2 - f_\alpha(x_\alpha)$ and note $1 \le \lambda_\alpha < 2$. While if $f_\alpha(x_\alpha) = 0$ then let $\lambda_\alpha = -1$. Define $g_\alpha = f_\alpha + \lambda_\alpha b_{x_\alpha, U_\alpha}$. Note that $g_\alpha = f_\alpha$ outside U_α. Define B_α, a basic open neighborhood of g_α, by $B_\alpha = B(g_\alpha, \{x_\alpha\}, 1)$.

Let $G = \{g_\alpha : \alpha < \kappa\}$. Then G is a generator for X. To see this take any point x not in closed C. For some α we know $f_\alpha(x) = 1$ but f_α is zero on C. Now, by construction, $g_\alpha(x) \ge f_\alpha(x) = 1$, while for z in C we have $g_\alpha(z) \le f_\alpha(z) = 0$ - and $g_\alpha(x)$ is not in $\overline{g_\alpha(C)}$, as required.

It remains to show that the B_α's witness discreteness of G. Suppose, then, some g_β is in B_α, so $|g_\beta(x_\alpha) - g_\alpha(x_\alpha)| < 1$. By the choice of λ_α there are only two possibilities for the value of $g_\alpha(x_\alpha)$, namely 2 or -1. If $g_\alpha(x_\alpha) = 2$ then $g_\beta(x_\alpha) > 1$, and since (i) $g_\beta = f_\beta$ outside U_β but (ii) f_β maps into $[0, 1]$, we see x_α must be in U_β, forcing $\beta = \alpha$ in this case. In the other case $g_\alpha(x_\alpha) = -1$, then $g_\beta(x_\alpha) < 0$, and again because (i) $g_\beta = f_\beta$ outside U_β but (ii) f_β maps into $[0, 1]$, we see x_α must be in U_β, forcing $\beta = \alpha$ in this case, as well. □

Theorem 40. *Let X be zero-dimensional. Then the following are equivalent:*

(1) $\bigoplus_n X^{(n)}$ contains a σ-discrete subset S of size $w(X)$,

(2) $C_p(X)$ contains a discrete subset of size $w(X)$,

(3) X has a discrete generator, and

(4) X has a σ-discrete generator.

Proof. Lemma 41 below says (4) implies (3) (without restriction on the dimension), while the results above say (3) implies (2) and (2) implies (1). We show (1) implies (4).

Let $\kappa = w(X)$. Fix a clopen base, $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ for X. Enumerate $S = \{x_\alpha : \alpha < \kappa\}$, write $S = \bigcup_{m,n} S_{m,n}$ where $S_{m,n}$ is a discrete subset of $X^{(n)}$, and let $\Lambda_{m,n} = \{\alpha : x_\alpha \in S_{m,n}\}$.

Fix m and n. For each α in $\Lambda_{m,n}$, write $x_\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,n})$, fix $U_\alpha = U_{\alpha,1} \times \cdots \times U_{\alpha,n}$ a product of pairwise disjoint, clopen sets such that if $\beta \in \Lambda_{m,n}$ and $x_\beta \in U_\alpha$ then $\beta = \alpha$. Shrink each $U_{\alpha,i}$, if necessary, so that it is either contained in C_α or in $X \setminus C_\alpha$. Define $g_\alpha \in C(X, \{0, \ldots, n+1\})$ by g_α is i on $U_{\alpha,i}$, $n+1$ on $C_\alpha \setminus \bigcup_i U_i$ and 0 everywhere else. Set B_α, a basic neighborhood of g_α, to be $B(g_\alpha, \{x_{\alpha,1}, \ldots, x_{\alpha,n}\}, 1)$. Let $G_{m,n} = \{g_\alpha : \alpha \in \Lambda_{m,n}\}$.

We show the B_α's witness that $G_{m,n}$ is discrete. But if $\beta \in \Lambda_{m,n}$ and $g_\beta \in B_\alpha$, then $g_\beta(x_{\alpha,i}) = g_\alpha(x_{\alpha,i}) = i$, and so $x_{\alpha,i} \in U_{\beta,i}$ for $i = 1, \ldots, n$, which forces $\beta = \alpha$.

Let $G = \bigcup_{m,n} G_{m,n}$. This is σ-discrete. To complete the proof we show G is a generator. Towards this end suppose a point x is not in closed C. Pick C_α from the clopen base so that $x \in C_\alpha$ and C_α is disjoint from C. Then g_α is in some $G_{m,n} \subseteq G$, and $g_\alpha(C_\alpha)$ and $g_\alpha(C) \subseteq g_\alpha(X \setminus C_\alpha)$ are finite and disjoint (because each $U_{\alpha,i}$ is either contained in C_α or $X \setminus C_\alpha$), as required. $\square$

Lemma 41. *A space X has a discrete generator if and only if it has a σ-discrete generator.*

Proof. Let G be a generator for X, where $G = \bigcup_n G_n$ and each G_n is discrete. Fix homeomorphisms $h_n : \mathbb{R} \to (2n, 2n+1)$, and set $H_n = h_n \circ G_n$. Then, by Lemma 10, H_n is homeomorphic to G_n, and hence discrete. Let $H = \bigcup_n H_n$. Since $H_n \subseteq C(X, (2n, 2n+1))$ it is clear that H is discrete, and since $G = \bigcup_n G_n$ is a generator then evidently so is $H = \bigcup_n H_n$. $\square$

The results above easily give examples of spaces with and without discrete generators. For instance, the Stone-Cech compactification of the natural numbers, and its remainder, $\beta\mathbb{N}$ and $\beta\mathbb{N} \setminus \mathbb{N}$, are compact spaces with weight $\mathfrak{c}$ which contain a discrete space of size $\mathfrak{c}$. Hence, perhaps unexpectedly, by Theorem 5.1, they have discrete generators.

In the other direction, any cosmic not metrizable space X has $hc^*(X) = \aleph_0 < w(X)$, and so by Theorem 38 does not have a discrete generator. Specific examples include the bowtie space and $C_p(I)$, and - as we have shown elsewhere - these spaces do have compact, second countable $(0, \neq 0)$–generators.

We now give a stronger example of a space without a discrete generator. This answers Question 2.7 of [7]. A subspace S of $C_p(X)$ is a *separator* if for any two distinct elements x and y of X there is an s in S such that $s(x) \neq s(y)$. Evidently generators are separators. Buzyakova and Okunev showed in [7] that if a space X has a discrete separator then $iw(X) \leq hc^*(X)$, where $iw(X)$ is the minimal weight of a Tychonoff topology on X coarser than the given topology. Their Question 2.7 asks for a ZFC example of a space without a discrete separator.

Example 42. (ZFC) There is a space X which has no discrete separator.

Proof. In ZFC, Todorcevic [23] and independently Shelah [20], have constructed spaces Y such that $hL^*(Y) < hd(Y)$. Passing to a left-separated subset we get $hL^*(Y) < d(Y)$. Let $X = C_p(Y)$. Then $hd^*(X) = hL^*(Y)$ (Zenor, [27]) and $d(Y) = iw(C_p(Y)) = iw(X)$ (Arkhangel'skii, [5]). So we have $hc^*(X) \leq hd^*(X) < iw(X)$. But if X had a discrete separator then $iw(X) \leq hc^*(X)$. $\square$

5.2 Discrete $(0, \neq 0)$-Generators s

Since the cardinal invariant restriction of Theorem 38 was so useful in the context of spaces with a discrete generator we start by formulating an analogous restriction for spaces with a discrete $(0, \neq 0)$–generator.

Theorem 43. *If X has a discrete $(0, \neq 0)$–generator then for all open subsets U of X we have $w(U) = hc^*(U)$.*

Proof. Let G be a discrete $(0, \neq 0)$–generator for X. Take any non-empty subset U of X. Let $G'_U = \{g \in G : g \text{ is zero outside } U\}$. Let $G_U = \pi_U(G'_U) = \{g \restriction U : g \in G'_U\}$. Then it is easy to see that G_U and G'_U are homeomorphic, and so G_U is discrete. Further, it is clear G_U is a $(0, \neq 0)$-generator for U. Thus $w(U) = hc^*(U)$ by Theorem 5.1. $\square$

Question 4. If for all open subsets U of X, we have $w(U) = hc^*(U)$, then does X have a discrete $(0, \neq 0)$–generator?

We give now one (limited) situation where the question has a positive answer, and the cardinal invariant restriction of Theorem 43 is sufficient. This in turn allows us to find a variety of examples related to spaces with or without discrete $(0, \neq 0)$–generators.

Let X be a set, $\mathcal{F}$ a free filter on X. Define $X(\mathcal{F})$ to be the space with underlying set $X \cup \{*\}$, and topology where each x in X is isolated and neighborhoods of $*$ have the form $\{*\} \cup F$ for $F \in \mathcal{F}$.

Theorem 44. *The following are equivalent: (1) $X(\mathcal{F})$ has a discrete $(0,1)$–generator, (2) $X(\mathcal{F})$ has a discrete $(0,\neq 0)$–generator, (3) for all open subsets U of $X(\mathcal{F})$ we have $w(U) = hc^*(U)$, and (4) for all $F \in \mathcal{F}$ we have $|F| \geq \chi(*, X(\mathcal{F}))$.*

Proof. We know (1) implies (2), and (2) implies (3). It is also easy to check that (3) and (4) are equivalent. So it remains to show (4) implies (1).

Assume (4) holds, let $\kappa = \chi(*, X(\mathcal{F}))$, and $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ a cofinal family in $\mathcal{F}$. By assumption, for each α we have $|B_\alpha| \geq \kappa$, so we can pick x_α in B_α so that if $\beta \neq \alpha$ then $x_\beta \neq x_\alpha$.

For each α define $g_\alpha \in C(X(\mathcal{F}))$ by g_α is 1 at $*$ and on $B\alpha \setminus \{x_\alpha\}$, $1/2$ at x_α, and 0 everywhere else. For each x in X define $h_x \in C(X(\mathcal{F}))$ to be the characteristic function of $\{x\}$. Then it is easy to check that $G = \{g_\alpha : \alpha < \kappa\} \cup \{h_x : x \in X\}$ is a $(0,1)$–generator for $X(\mathcal{F})$.

Further, G is discrete. For h_x this is witnessed by $B(h_x, \{*, x\}, 1/2)$. Any g in this set has $g(*) < 1/2$ and $g(x) > 1/2$. The first fact eliminates g and g_α. The second eliminates all h_y except $y = x$. For g_α this is witnessed by $B(g_\alpha, \{x_\alpha\}, 1/2)$. Any g in this set has $g(x_\alpha) \in (0, 1/2)$. But, recalling that $x_\beta \neq x_\alpha$ if $\beta \neq \alpha$, g_α is the only member of G with value at x_α in this range. □

Now for the promised examples.

Example 45. There is a space X with a discrete generator, but no discrete $(0,\neq 0)$–generator.

Proof. Let $\mathfrak{d}$ be the cofinality of $\mathbb{N}^\mathbb{N}$ with the pointwise order, $f \leq g$ if and only if $f(n) \leq g(n)$ for all n in $\mathbb{N}$. Let $X = \mathbb{N}^2 \cup \mathfrak{d}$. Let $\mathcal{F}$ be the filter on X with base all $B_f = \{(m,n) : m \geq f(n)\}$.

Then $|X(\mathcal{F})| = \mathfrak{d} = \chi(*, X(\mathcal{F}))$, so $X(\mathcal{F})$ has a discrete generator. But $F = \mathbb{N}^2$ is in $\mathcal{F}$, while $|F| = \aleph_0 < \mathfrak{d} = \chi(*, X(\mathcal{F}))$, so $X(\mathcal{F})$ does not have a discrete $(0,\neq 0)$–generator. □

Example 46. (Consistently) There is a compact space X with a discrete generator, but no discrete $(0,\neq 0)$–generator.

Proof. Under (CH) Kunen in [13] constructed a space, whose one compactification Y is a 'compact strong S-space': Y is compact, $hd^*(Y) = \aleph_0 < w(Y)$. Todorcevic, [23], has weakened the Continuum Hypothesis to $\mathfrak{b} = \aleph_1$.

Let X be the one point compactification of the disjoint sum of $w(Y)$-many copies of Y. Then X is compact. As $w(X) = w(Y) = hc(X)$ (pick one point from each copy of Y) we see from Theorem 39 that X has a discrete generator. But X has open subspaces U homeomorphic to Y, and so $hc^*(U) \leq hd^*(U) < w(U)$, which (Theorem 43) tells us that X does not have a discrete $(0, \neq 0)$–generator. $\square$

Further examples of spaces with a discrete $(0, \neq 0)$–generator can be constructed from existing ones by isolating points of any given subset. Let X be a space and A a subset. Define $X_{(A)}$, the *Michael line space of X over A*, to be X with the points of $X \setminus A$ isolated.

Theorem 47. *If X has discrete $(0, \neq 0)$–generator then $X_{(A)}$ also has a discrete $(0, \neq 0)$–generator.*

Proof. Let G be a discrete $(0, \neq 0)$–generator for X. We can suppose (Lemma 10) that $G \subseteq C_p(X, (-\infty, 1/2)) \setminus \{\mathbf{0}\}$. As G is a discrete subset of $C_p(X)$, for each g in G, there is a basic neighborhood, say $B(g, F_g, \epsilon_g)$, witnessing discreteness.

For each x in X define g_x to be 1 at x and 0 elsewhere. Let $H_1 = \{g_x : x \in X\}$ and $H = H_1 \cup G$. Clearly H is a $(0, \neq 0)$–generator for $X_{(A)}$. We show H is discrete.

Take any g_x in H_1. Then $B_x = B(g_x, \{x\}, 1/2)$ is an open neighborhood of g_x. Clearly $B_x \cap H_1 = \{g_x\}$. And if $g \in B_x$ then $g(x) > g_x(x) - 1/2 = 1/2$ and so g can not be in $G \subseteq C_p(X, (-\infty, 1/2))$.

Now take any g in G. Since g is not identically zero, pick x_g in X so that $g(x_g) \neq 0$. Let $B_g = B(g, F_g \cup \{x_g\}, \delta_g)$ where $\delta_g = \min(\epsilon_g, 1/2, |g(x_g)|)$. This is a basic neighborhood of g in $C_p(X_{(A)})$. By choice of F_g and $\delta_g \leq \epsilon_g$ we have $B_g \cap G = \{g\}$. Further, $B_g \cap H_1 = \emptyset$. For, if $g_x \in B_g \cap H_1$ then $|g(x_g) - g_x(x_g)| < |g(x_g)|$ so $g_x(x_g) \neq 0$, while $|g(x_g) - g_x(x_g)| < 1/2$ combined with $g(x_g) < 1/2$ forces $g_x(x_g) \neq 1$ - but g_x only takes on values 0 and 1. $\square$

An interesting open question is when spaces of the type $X = C_p(Y)$, for some Y, have discrete generators or discrete $(0, \neq 0)$–generators. (Recall we used a space of this form in

Example 42.) We know: $hc^*(C_p(Y)) = hc^*(Y)$ ([27]) and $w(C_p(Y)) = |Y|$. Further, every basic open $B(0, F, \epsilon)$ clearly contains $C_p(Y, (-\epsilon, \epsilon))$, which is homeomorphic to $C_p(Y)$. Hence, for every open subset U of $C_p(Y)$ we know $hc^*(U) = hc^*(C_p(Y))$ and $w(U) = w(C_p(Y))$. So we are led to ask the following question.

Question 5. Are the following equivalent: (1) $C_p(Y)$ has a discrete $(0, \neq 0)$–generator, (2) $C_p(Y)$ has a discrete generator, (3) $C_p(Y)$ contains a discrete subset of size $|Y|$, and (4) $hc^*(Y) = |Y|$?

Here we know, of course, that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). But also, from the discussion above, that (4) $\implies$ (3), and if (3) implies (2) then they are all equivalent.

5.3 Discrete (0,1)-Generators s

We started by believing that spaces would have a discrete $(0, 1)$–generator only in rare cases. Indeed, our proof that every metrizable space has a discrete $(0, 1)$–generator took some effort. But then we added all ordinals to the list of spaces with a discrete $(0, 1)$–generator, followed by all Alexandrov duplicates of spaces with a discrete $(0, 1)$–generator.

In particular, as the unit inteval, I, is metrizable it has a discrete $(0, 1)$–generator, and so its Alexandrov duplicate, $AD(I)$, has a discrete $(0, 1)$–generator. However we show its subspace the Michael line does *not* have a discrete $(0, 1)$–generator, although the Michael line does have a discrete $(0, \neq 0)$–generator. It appears that the dividing line between spaces with a discrete $(0, \neq 0)$–generator and a discrete $(0, 1)$–generator is delicately placed.

Theorem 48. *Let X be a metrizable space. Then X has a discrete $(0, 1)$–generator.*

Proof. Fix a compatible metric d for X. Recursively we build sequences of open covers $\mathcal{U}_n$, $\mathcal{B}_n$, $\mathcal{B}_n^-$, $\mathcal{V}_n$ and $\mathcal{V}_n'$, a sequence of closed covers $\mathcal{C}_n$, and for each B in $\mathcal{B}_k$ a map g_B, possibly along with a point c_B, as follows. Only the definition of $\mathcal{U}_k$ differs between the base case $(n = 1)$ and the inductive step $(n + 1)$, so we deal with this first and separately.

Now take any $n \geq 1$. Let $\mathcal{B}_n$ be a locally finite open refinement of $\mathcal{U}_n$. Let $\mathcal{V}_n = \{V_x : x \in X,\, x \in V_x \subseteq \overline{V_x} \subseteq \text{some member of } \mathcal{B}_n\}$. Let $\mathcal{V}_n'$ be a locally finite open refinement of

$\mathcal{V}_n$, consisting of non-empty sets.

For each B in $\mathcal{B}_n$ define $C(B) = \bigcup\{\overline{V} : V \in \mathcal{V}'_k \text{ and } \overline{V} \subseteq B\}$. Since $\mathcal{B}_n$ is locally finite, we have $C(B) = \overline{\bigcup\{V : V \in \mathcal{V}'_n \text{ and } \overline{V} \subseteq B\}}$. Then $C(B)$ is a closed subset of B. If $C(B) \subsetneq B$, then we can pick $c_B \in B \setminus C(B)$ and, g_B in $C(X, [0,1])$ so that $g_B^{-1}\{1\} = C(B)$, $g_B^{-1}\{0\} = X \setminus B$ and $g_B(c_B) = 1/4$. If $C(B) = B$, then let $g_B = \chi_B$. Let $B_- = g_B^{-1}(3/4, \infty)$. Note B_- is open, $C(B) \subseteq B_- \subseteq \overline{B_-} \subseteq B$, and - if defined - c_B is in $B \setminus \overline{B_-}$.

Let $\mathcal{C}_n = \{C(B) : B \in \mathcal{B}_n\}$. Since $\mathcal{C}_n$ is a shrinking of $\mathcal{B}_n$, which is locally finite, we see $\mathcal{C}_n$ is locally finite. Let $\mathcal{B}_n^- = \{B_- : B \in \mathcal{B}_n\}$, and note it is locally finite.

Let $\mathcal{U}_1 = \{B_{1/1}(x) : x \in X\}$. At stage $n+1$ we define $\mathcal{U}_{n+1}$ inductively. Take any x in X. As $\mathcal{C}_k$ is locally finite for each $k < n+1$, there is an open set containing x and meeting only finitely many elements of $\bigcup_{k<n+1} \mathcal{C}_k$. So we can pick $\epsilon_x > 0$ such that $\epsilon_x < 1/(n+1)$ and

if $k < n+1$, $B \in \mathcal{B}_k$ and $B_{\epsilon_x}(x) \cap C(B) \neq \emptyset$ then

$(*)$ $B_{\epsilon_x}(x) \subseteq B_-$, and

$(**)$ if B_- is not a singleton then $B_{\epsilon_x}(x) \subsetneq B_-$.

Let $\mathcal{U}_{n+1} = \{B_{\epsilon_x}(x) : x \in X\}$.

With the recursive construction complete, define $G_n = \{g_B : B \in \mathcal{B}_n\}$ and $G = \bigcup_n G_n$. We verify G is a discrete $(0,1)$–generator for X.

To see this take any point x in X and basic open neighborhood, $B_\epsilon(x)$, and find g in G taking x to 1 and $X \setminus B_\epsilon(x)$ to 0. Pick n such that $2/n < \epsilon$. Since $\mathcal{V}'_n$ covers X, by definition of $\mathcal{V}_n, \mathcal{V}'_n$ and $\mathcal{C}_n$ we see that $\mathcal{C}_n$ covers X, so there is a B in $\mathcal{B}_n$ such that $x \in C(B) \subseteq B$. Since the diameter of B is strictly less than $2/n$ and $x \in B$ we see $B \subseteq B_{2/n}(x) \subseteq B_\epsilon(x)$. Of course, $g_B \in G_n \subseteq G$. By definition of g_B, as $x \in C(B)$, we have $g_B(x) = 1$, and as $B \subseteq B_\epsilon(x)$, we have $g_B(X \setminus B_\epsilon(x)) \subseteq \{1\}$. Thus g_B is as required.

Take any $g = g_B$ in G, where $B \in \mathcal{B}_n$ (so g_B is in G_n). As $C_p(X)$ is T_1, for discreteness it suffices to find an open neighborhood of g_B meeting G in a finite set.

Pick a_B in $C(B)$, and note $g_B(a_B) = 1$. Fix, for the moment, m. If $B' \in \mathcal{B}_m$ and $g_{B'} \in G_m \cap B(g_B, \{a_B\}, 1)$ then $g_B(a_B) = 1$ forces $g_{B'}(a_B) > 0$, so $a_B \in B'$. But $\mathcal{B}_m$ is locally finite, so a_B is in only finitely many B' from $\mathcal{B}_m$, and we see that $G_m \cap B(g_B, \{a_B\}, 1)$ is finite. Hence it is sufficient to show that there is a neighborhood $U_B = B(g_B, F_B, 1/4)$ disjoint from $\bigcup_{m>n} G_m$.

Towards this, fix $m > n$. Suppose, for a contradiction, that $g_{B'}$ is in $U_B \cap G_m$ (and $B' \in \mathcal{B}_m$). *Our F_B will contain a_B.* Then, since $g(a_B) = 1$, $a_B \in F_B$ and $g_{B'} \in U_B$, we have that $|1 - g_{B'}(a_B)| < 1/4$, so $a_B \in (B')_-$, and in particular, $(B')_- \cap C(B) \neq \emptyset$. By $(*)$ in the definition of $\mathcal{U}_m$, and as $\mathcal{B}_m$ is a refinement of $\mathcal{U}_m$, we have $B' \subseteq B_-$.

To complete the proof we consider three cases depending on the relationship between B and $C(B)$.

Case: $C(B)$ is a proper subset of B. Let $F_B = \{a_B, c_B\}$. We know $B' \subseteq B_-$. By definition of c_B and g_B, we have $g_B(c_B) = 1/4$. But, as $B' \subseteq B_-$, and $c_B \notin B_-$ (by definition), we have $c_B \notin B'$ and so $g_{B'}(c_B) = 0$. Hence $|g_B(c_B) - g_{B'}(c_B)| = 1/4$, and since c_B is in F_B we have $g_{B'} \notin U_B$, contradiction.

Case: $C(B) = B = \{a_B\}$. Let $F_B = \{a_B\}$. We know $B' \subseteq B_-$. But $B_- = B = \{a_B\}$. So $g_{B'} = \chi_{\{a_B\}} = g_B$, and done.

Case: $C(B) = B$ but B is not a singleton. By local finiteness of $\mathcal{B}_{n+1}$ there is a neighborhood V_B of a_B meeting only finitely many B'' in $\mathcal{B}_{n+1}$. For each such B'', applying our case assumption and $(**)$, pick y'' in $B \setminus B''$, and gather these together in the (finite) set F''.

Let $F_B = \{a_B\} \cup F''$. Again we show there is a contradiction to $g_{B'}$ being in U_B. Since $B'' \in \mathcal{B}_m$ and $m > n$, we know $B' \subseteq B''$ for some B'' in $\mathcal{B}_{n+1}$. If $B'' \cap V_B = \emptyset$ then $a_B \notin B''$, so $g_{B''}(a_B) = 0$, while $g_B(a_B) = 1$ - which is impossible because a_B is in F_B. On the other hand, if $B'' \cap V_B \neq \emptyset$ then the y'' corresponding to B'' is in F_B and in $B \setminus B''$ - which is impossible because we have $g_{B'}(y'') = 0$ ($y'' \notin B''$) while $g_B(y'') = 1$ ($y'' \in B = C(B)$). □

Proposition 49. *Every ordinal has a discrete $(0,1)$–generator.*

Proof. Let ϵ be an ordinal.

Take any α in ϵ. If $\alpha = 0$ then define $F_\alpha = \{0, 1\}$, $C_\alpha = \{0\}$, and $g_\alpha = \chi_{C_\alpha}$. If $\alpha = \beta + 1$ then define $F_\alpha = \{\beta, \alpha, \alpha + 1\}$, $C_\alpha = \{\alpha\}$, and $g_\alpha = \chi_{\{\alpha\}}$. If α is a limit then for every $\delta < \alpha$ define $F_{\alpha,\delta} = \{\delta, \delta + 1, \alpha, \alpha + 1\}$, $C_{\alpha,\delta} = [\delta + 1, \alpha]$ and $g_{\alpha,\delta} = \chi_{C_{\alpha,\delta}}$.

Let $G_1 = \{g_\alpha : \alpha \text{ not a limit}\}$, $G_2 = \{g_{\alpha,\delta} : \delta < \alpha \text{ a limit}\}$ and $G = G_1 \cup G_2$. We claim G is a $(0,1)$–generator. This is clear because every element of G only takes on the values 0 or 1, and $\{g^{-1}\{1\} : g \in G\} = \{C_\alpha : \alpha \text{ not a limit}\} \cup \{C_{\alpha,\delta} : \delta < \alpha \text{ a limit}\}$ which is the standard base of clopen sets for ϵ.

We complete the proof by showing that the basic open sets B_α and $B_{\alpha,\delta}$ demonstrate that G is discrete.

First we check $B_\alpha \cap G = \{g_\alpha\}$ for every $\alpha = \beta + 1$ in ϵ. (The proof for $\alpha = 0$ is almost identical, and so omitted.) Suppose g is in $B_\alpha \cap G$. Then g is 0 at β and $\beta + 2$, and 1 at $\alpha = \beta + 1$. Elements of G_1 take value 1 exactly once, so in this case $g = g_\alpha$. Elements of G_2, on the other hand, if they take the value 1 at some successor $\gamma + 1$ then they also have value 1 at either γ or $\gamma + 2$, and so can not be g.

Now let α be a limit and $\delta < \alpha$. We check $B_{\alpha,\delta} \cap G = \{g_{\alpha,\delta}\}$. Suppose g is in $B_{\alpha,\delta} \cap G$. Then $g(\delta + 1) = 1 = g(\alpha)$. Noting that the successor $\delta + 1 \neq \alpha$, a limit, and that elements of G_1 take value 1 exactly once, we see g is not in G_1. Hence $g = g_{\alpha',\delta'}$ for some limit α' and $\delta' < \alpha$. Again because $g(\delta + 1) = 1 = g(\alpha)$, from the definitions, we see $\delta' \leq \delta$ and $\alpha \leq \alpha'$. But we also know that $g(\delta) = 0 = g(\alpha + 1)$. This implies, from the definitions, that $\delta \leq \delta'$ and $\alpha' \leq \alpha$. All together we see $g = g_{\alpha,\delta}$, as required. $\square$

Define $AD(X)$, the *Alexandrov duplicate of* X, to have underlying set $X \times \{0,1\}$, all points of $X \times \{1\}$ isolated and basic open neighborhoods of $(x,0)$ to be $U \times \{0,1\} \setminus \{(x,1)\}$ for any open neighborhood U of x in X. Observe that $AD(I)$ is the usual 'Alexandrov duplicate' of the unit interval.

Theorem 50. *If X has a discrete $(0,1)$–generator then so does $AD(X)$.*

Proof. Let G be a discrete $(0,1)$–generator for X. For each g in G fix a basic open neighborhood $B(g, F_g, \epsilon_g)$ in $C_p(X)$ witnessing discreteness, where F_g is a finite subset of X and $0 < \epsilon_g < 1/2$.

Fix x in X. Define h_x to be 1 at $(x,1)$ and zero elsewhere. For g in G such that $g(x) = 1$ define $g_{(x)} : X \times \{0,1\} \to \mathbb{R}$ by 'copying' g, specifically, $g_{(x)}(z,i) = g(z)$ - and note this is continuous on $AD(X)$ - and then modifying this by specifying, $g_{(x)}(x,1) = 0$ - and note this is still continuous on $AD(X)$, because $(x, 1)$ is isolated. Now set $H_x = \{g_{(x)} : g \in G$ and $g(x) = 1\}$.

Define $H = H_1 \cup \bigcup_x H_x$, where $H_1 = \{h_x : x \in X\}$. Then H is a subset of $C_p(AD(X))$. It is a $(0, 1)$–generator for $AD(X)$. This is realized for any $(x, 1)$ by h_x. While if $(x, 0)$ is in basic open $U \times \{0, 1\} \setminus \{(x, 1)\}$ then pick g in G so that $g(x) = 1$ but g is zero outside U, and now $g_{(x)}$ is as required.

It remains to show that H is discrete. Fix x. Now $B(h_x, \{(x, 0), (x, 1)\}, 1/2) \cap H = \{h_x\}$ because if h is in this set then $h(x, 1) > 1/2$ and $h(x, 0) < 1/2$, and for h equal to some h_z this forces $z = x$, while for h equal to some $g_{(z)}$ this is impossible as $g_{(z)}$ has the same values at $(x, 0)$ and $(x, 1)$ unless, possibly, when $g_{(z)}(x, 1) = 0$. Now take any $g_{(x)}$ in H_x. Consider $B = B(g_{(x)}, \{(x, 0), (x, 1)\} \cup (F_g \times \{0, 1\}), \epsilon_g)$, a basic open neighborhood of $g_{(x)}$ in $C_p(AD(X))$. Since $\epsilon_g < 1/2$, $g_{(x)}(x, 0) = 1$ and $g_{(x)}(x, 1) = 0$, it is clear $H_1 \cap B = \emptyset$. Now take any $g'_{(x')}$ in $H_{x'} \cap B$. Because $g_{(x)}$ and $g'_{(x')}$ restricted to $X \times \{0\}$ are, essentially, g and g', respectively, and since $B(g, F_g, \epsilon_g)$ witnesses discreteness for g in $C_p(X)$, we clearly have that g' must equal g. We complete the proof by showing $x' = x$. As $g_{(x)}(x, 1) = 0$, we know $g_{(x')}(x, 1) < 1/2$ (recall, $\epsilon_g < 1/2$). As $g_{(x)}(x, 0) = 1$, $g_{(x')}(x, 0) > 1/2$. But now if $x \neq x'$ then $g_{(x')}(x, 1) = g_{(x')}(x, 0) > 1/2$, contradiction. $\square$

Suppose X is metrizable. Then $X_{(A)}$ (the space obtained by isolating all points not in A) is metrizable if and only if A is a G_δ subset of X. (This is well known, and easily follows from Bing's version of the Bing-Nagata-Smirnov metrization theorem: a T_3 space is metrizable if it has a σ-discrete base.)

Theorem 51. *Let X be a metrizable space and A a subset. Then:*

$X_{(A)}$ has a discrete $(0, 1)$–generator if and only if $X_{(A)}$ is metrizable.

Proof. Fix d a compatible metric for X. As metrizable spaces have discrete $(0, 1)$–generators, by Theorem 48, the reverse direction is clear.

Suppose, for a contradiction,$X_{(A)}$ has a discrete $(0, 1)$–generator G, but is not metrizable. Let $B = X \setminus A$. For each $x \in B$, $\{x\}$ is an open set, so there is a g_x in the $(0, 1)$–generator G such that $g_x(x) = 1$ but $g_x(X \setminus \{x\}) \subseteq \{0\}$. By discreteness of G there is a basic open neighborhood of g_x witnessing this, say $B_x = B(g_x, F_x, 1/m_x)$ where F_x is a finite subset of X which, we may assume, contains x. Let $F'_x = F_x \setminus \{x\}$ and pick n_x so that $d(x, F'_x) > 1/n_x$.

Let $S_{m,n} = \{x \in B : m_x = x \text{ and } n_x = n\}$. Note $B = \bigcup_{m,n} S_{m,n}$. As A is not a G_δ set in X, B is not an F_σ, and not every $S_{m,n}$ can be closed. So there is a pair m, n such that $\overline{S_{m,n}} \cap A \neq \emptyset$, say y is in this intersection.

As G is a $(0,1)$–generator there is a g_y in G such that $g_y(y) = 1$ and g_y maps $X \setminus B_d(y, 1/(2n))$ to 0. As g_y is continuous at y there is an $0 < \epsilon < 1/(2n)$ such that $g_y(B_d(y, \epsilon)) \subseteq (1 - 1/m, 1 + 1/m)$. As $y \in \overline{S_{m,n}}$ there is an x in $S_{m,n}$ such that $x \in B_d(y, \epsilon)$. Clearly $g_y \neq g_x$ and we now show that $g_y \in B_x$, contradicting that B_x witnesses discreteness at g_x.

Well, $B_x = B(g_x, F_x, 1/m)$ and for any z in F_x two cases arise. If $z = x$ then as $x \in B_d(y, \epsilon)$ $g_y(x) \in (1 - 1/m, 1 + 1/m)$ and so $|g_x(x) - g_y(x)| = |1 - g_y(x)| < 1/m$. While if $z \neq x$, so $z \in F'_x$, as $d(x, F'_x) > 1/n_x = 1/n$ and $x \in B_d(y, 1/(2n))$, we see $z \notin B_d(y, 1/(2n))$ so $g_y(z) = 0$, from which it follows that $|g_x(z) - g_y(z)| = |0 - 0| = 0$. Thus g_y is indeed in B_x, as required. □

Example 52. There is a space X with a discrete $(0, \neq 0)$–generator but no discrete $(0,1)$–generator.

Proof. Let X be the Michael line. Metrizable spaces have a discrete $(0,1)$–generator (Theorem 48). Hence the Michael line has a discrete $(0, \neq 0)$–generator (Theorem 47). But Theorem 51 tell us that there is no discrete $(0,1)$–generator for the Michael line. □

Question 6. Is there a compact space with a discrete $(0, \neq 0)$–generator but no discrete $(0,1)$–generator?

6.0 First and Second Countable Generators

When does a space have a 1° or 2° generator? More specifically, we examine the two pairs of questions we raised in the introduction: Which spaces have a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$? Is separability sufficient? Likewise, which spaces have a second countable $(0, \neq 0)$–generator containing $\mathbf{0}$? Is cosmicity enough?

In Sections 6.1 and 6.3, we explore the connections between first and second countable generators and separable and cosmic spaces (respectively). As to compact, second countable generators, the spaces which have these are subspaces of some $C_p(K)$ where K is compact and second countable.

In Sections 6.2 and 6.4 we demonstrate that many 'classic' separable spaces do have a first countable $(0, \neq 0)$–generator (containing $\mathbf{0}$) and many 'classic' cosmic spaces have a second countable $(0, \neq 0)$–generator (containing $\mathbf{0}$). Finally, in Section 6.5 we examine the special case when our space X itself is $C_p(Y)$ for some Y. We show that $X = C_p(Y)$ has a compact, second countable generator if and only if Y is cosmic and σ-compact; and we show that if Y is first countable (respectively, second countable) then $X = C_p(Y)$ has a first (respectively, second) countable $(0, \neq 0)$–generator.

6.1 First Countable Generators

Lemma 53. *The following are equivalent: (1) X is countable, (2) $C_p(X)$ is second countable, (3) $C_p(X)$ is first countable, (4) X has a first countable $(0,1)$–generator containing $\mathbf{0}$, and (5) X has a $(0,1)$–generator which contains $\mathbf{0}$ and $\mathbf{0}$ is a point of first countability in G.*

Proof. The implications (1) $\implies$ (2), (2) $\implies$ (3), (3) $\implies$ (4), and (4) $\implies$ (5), are all clear. It remains to show that (5) $\implies$ (1).

Suppose G is a $(0,1)$–generator containing $\mathbf{0}$, and $\mathbf{0}$ has a countable local base in G say, $\mathcal{B}(\mathbf{0}, F_n, \epsilon_n)$ where each F_n is finite and $0 < \epsilon_n < 1$. Let $X' = \bigcup_n F_n$. Then X' is countable. We complete the proof by showing $X' = X$. If not then we can pick $x \in X \setminus X'$. As

$B(\mathbf{0}, \{x\}, 1/2) \cap G$ is a neighborhood of $\mathbf{0}$ in G, there is some m so that $L = B(\mathbf{0}, F_m, \epsilon_m) \cap G$ is contained in $R = B(\mathbf{0}, \{x\}, 1/2) \cap G$. But G is a $(0,1)$–generator, so there is a g in G such that $g(x) = 1$ but $g(F_m) = 0$, and this g is in L but not in R, which yields a contradiction. □

The argument given above is the 'standard' one that if $C_p(X)$ is first countable, then X is countable. Regarding item (4), note that we have a variety of spaces (any metrizable space, the Alexandrov duplicate, etc.) with discrete $(0,1)$–generators *not containing the zero function.* So for the preceding argument, we do need to know that our generator contains $\mathbf{0}$. From item (5), we know that $\mathbf{0}$ being in the generator and being a point of first countability is sufficient to deduce that the space X is countable *provided* the generator is a $(0,1)$–generator.

A related basic C_p-theory result is that if $C_p(X)$ has countable pseudocharacter (all point G_δ), then X is separable, and this implies that $C_p(X)$ has a coarser second countable topology.

Lemma 54. *The following are equivalent: (1) X is separable, (2) $C_p(X)$ has a coarser second countable topology, (3) $C_p(X)$ has countable pseudocharacter, (4) X has a $(0, \neq 0)$–generator containing $\mathbf{0}$ with countable pseudocharacter, and (5) X has a $(0, \neq 0)$–generator containing $\mathbf{0}$ with countable pseudocharacter at $\mathbf{0}$.*

Proof. The only part which requires proof is (5) $\implies$ (1).

Let X have a $(0, \neq 0)$–generator G containing $\mathbf{0}$, and suppose we have basic open sets $B_n = B(\mathbf{0}, F_n, \epsilon_n) \cap G$ in G such that $\{\mathbf{0}\} = \bigcap_n B_n$. Let $D = \bigcup_{n \in \mathbb{N}} F_n$, which is countable. We claim that $X = \overline{D}$. Suppose for the sake of contradiction that $\overline{D} \neq X$. Then pick $x \in X \setminus \overline{D}$. Since G is a $(0, \neq 0)$–generator there is a function $g \in G$ such that $g(x) \neq 0$ and $g(\overline{D}) = \{0\}$. Notice, however, that $g \neq \mathbf{0}$, but it is also in every B_n since it is zero on each F_n. This is a contradiction. □

From the previous two results we might wonder if we can relax '$(0,1)$–generator' to '$(0, \neq 0)$–generator' in Lemma 53. The answer is *no.* From Lemma 54 we know if a space does have a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$, or even just a $(0, \neq 0)$–generator with $\mathbf{0}$ as a point of first countability, then the space must be separable. In the next section we give an array of uncountable, separable spaces – many, indeed, compact and separable – with a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$.

Tacking in the opposite direction, we might consider, instead, whether every separable space has a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$, or at least a $(0, \neq 0)$–generator with $\mathbf{0}$ as a point of first countability. This latter question has a positive answer. Since every space has a $(0, \neq 0)$–generator, namely $C_p(X)$, applying Proposition 12 we deduce the following.

Theorem 55. *A space X is separable if and only if it has a $(0, \neq 0)$–generator containing $\mathbf{0}$ which is first countable at $\mathbf{0}$.*

But the stronger version of our question remains unanswered.

Question 7.

(A) Which spaces have a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$?

(B) Does every separable space have a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$?

Although we don't know answers to these questions, we can show that the requirement that $\mathbf{0}$ be in the generator is *not* necessary, even though the standard proofs do require that.

Theorem 56. *Let X be a separable space.*

(1) If X has a first countable $(0, \neq 0)$–generator, then it has a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$.

(2) If X has a discrete $(0, \neq 0)$–generator, then it has a first countable $(0, \neq 0)$–generator G such that $G \setminus \{\mathbf{0}\}$ is discrete.

Proof. Recalling that subspaces of first countable (respectively, discrete) are first countable (respectively, discrete), and a space is first countable (respectively, discrete) if it is the countable union of open subspaces that are first countable (respectively, discrete), we see this easily follows from Proposition 12. □

6.2 Examples: Spaces with a 1° Generator

We present some illustrative examples of separable spaces with first countable generators. In Section 6.4 we give additional examples of cosmic spaces with second countable generators, so our focus here is on non-cosmic spaces. Our first example is the famous Sorgenfrey line, which is first countable, hereditarily separable, and hereditarily Lindelöf. Our remaining examples are all *compact* and separable: the double arrow space (which, like the Sorgenfrey line, is first countable, hereditarily separable, and hereditarily Lindelöf), the one-point compactification of Ψ-space (which fails to be first countable at just one point), and separable Cantor cubes ($\{0,1\}^\kappa$ for $\kappa \le \mathfrak{c}$, which are nowhere first countable).

The *Sorgenfrey line* is the real line with the 'half-open topology'. We use a homeomorphic copy, and distinguish between the the left and right looking Sorgenfrey topologies. Specifically, we let S^+ and S^- both have underlying set $(0,1)$, and the basic open sets in S^+ have the form $[x, x+\epsilon)$, while in S^- they have the form $(x-\epsilon, x]$, where $x \in (0,1)$ and $\epsilon > 0$. Then S^+, S^-, and the usual Sorgenfrey line are all homeomorphic. Also note that they are all homeomorphic to the convergent sequence of clopen copies of themselves.

Example 57. The Sorgenfrey line has a $(0, \neq 0)$–generator containing $\mathbf{0}$ which is homeomorphic to itself and, hence, first countable.

Proof. We work with S^+. For each x in $(0,1)$ and $n \in \mathbb{N}$, let $B_{x,n} = [x, x+1/n)$ and $g_{x,n} = (1/n)\cdot\chi_{B_{x,n}}$. Note $g_{x,n}^{-1}(\mathbb{R}\setminus\{0\}) = B_{x,n}$. For each n, let $G_{x,n} = \{g_{x,n} : x \in (0,1)\}$, $F_n = \{i/(n+1) : 1 \le i \le n\}$ and $t_n = 1/n$. Clearly $G_n \subseteq C_p(S^+, \{0, 1/n\})$, and for any x in $(0,1)$, $[x, x+1/n)$ meets F_n, so $g_{x,n}$ has value $1/n$ at some member of F_n. Now Lemma 11 tells us that $G = \{\mathbf{0}\} \cup \bigcup_n G_n$ is the convergent sequence of clopen copies of the G_n, converging to $\mathbf{0}$.

Since $\mathcal{B} = \{g^{-1}(\mathbb{R}\setminus\{0\}) : g \in G\} = \{B_{x,n} : x \in (0,1) \text{ and } n \in \mathbb{N}\}$, which is a base for S^+, from Lemma 1(2) we see G is a $(0, \neq 0)$–generator for S^+.

To complete the example we need to show that each G_n is homeomorphic to S^-. Fix n. Take any x and $\epsilon > 0$. Then $B(g_{x,n}, \{x, x-\epsilon\}, 1/2) \cap G_n = \{g_{x',n} : x, x-\epsilon \in B_{x',n}$, which is $\{g_{x',n} : x' \in (x-\epsilon, x]\}$ (a). Further, take any basic neighborhood of $g_{x,n}$ in G_n, say

$B = B(g_{x,n}, F, \delta) \cap G_n$, where we may suppose $0 < \delta < 1$, and F contains x and meets $(0, x)$. Let ϵ be the smaller of $\min\{x - x' : x' \in F$ and $x' < x\}$ and $\min\{(x + 1/n) - x' : x' \in F$ and $x \leq x' < x + 1/n\}$. Then it is straight forward to check that $B = B(g_{x,n}, \{x, x - \epsilon\}, 1/2) \cap G_n$ (b). From (a) and (b) it follows that the map $h : S^- \to G_n$ given by $h(x) = g_{x,n}$ is a homeomorphism. □

The double arrow space, DA, has underlying set $I \times \{0, 1\}$. A basic open neighborhood of $(x, 0)$ is $U_\epsilon(x, 0) = (x - \epsilon, x) \times \{0, 1\} \cup \{(x, 0)\}$, and a basic open neighborhood of and $(x, 1)$ is $U_\epsilon(x, 1) = (x, x + \epsilon) \times \{0, 1\} \cup \{(x, 1)\}$, where $\epsilon > 0$. Note that these basic open sets are clopen.

Example 58. The double arrow space, DA, has a first countable $(0, \neq 0)$–generator G containing $\mathbf{0}$, such that $G \setminus \{\mathbf{0}\}$ is discrete.

Proof. For each (x, i) in DA and $n \in \mathbb{N}$, let $B_{x,i,n} = U_{1/n}(x, i)$ and $g_{x,i,n} = (1/n) \cdot \chi_{B_{x,i,n}}$. Note $g^{-1}_{x,i,n}(\mathbb{R} \setminus \{0\}) = B_{x,i,n}$. For each n, let $G_{i,n} = \{g_{x,i,n} : (x, i) \in DA\}$ (for $i = 0, 1$), $G_n = G_{0,n} \cup G_{1,n}$, $F'_n = \{j/(n + 1) : 0 \leq j \leq n + 1\}$, $F_n = F'_n \times \{0, 1\}$ and $t_n = 1/n$. Clearly $G_n \subseteq C_p(DA, \{0, 1/n\})$, and for any (x, i) in DA, $B_{x,i,n}$ meets F_n, so $g_{x,i,n}$ has value $1/n$ at some member of F_n. Now Lemma 11 tells us that $G = \{\mathbf{0}\} \cup \bigcup_n G_n$ is the convergent sequence of clopen copies of the G_n, converging to $\mathbf{0}$.

Since $\mathcal{B} = \{g^{-1}(\mathbb{R} \setminus \{0\}) : g \in G\} = \{B_{x,i,n} : (x, i) \in DA$ and $n \in \mathbb{N}\}$, which is a base for DA, from Lemma 1(2) we see G is a $(0, \neq 0)$–generator for DA. Summarizing, G is a $(0, \neq 0)$–generator for DA, containing the zero function, which is first countable at $\mathbf{0}$.

As each G_n is an open subset of G, it remains to show that G_n is discrete (in itself). But if $g_{x',i',n}$ is in $B(g_{x,0,n}, \{(x, 0), (x, 1)\}, 1/2)$ then $g_{x',i',n}((x, 0)) = 1$ while $g_{x',i',n}((x, 1)) = 0$, and the only $g_{x',i',n}$ with this combination of values at these two points is $g_{x,i,n}$. Similarly, $B(g_{x,1,n}, \{(x, 0), (x, 1)\}, 1/2)$ witnesses discreteness of $g_{x,1,n}$ in G_n. □

Now we turn to the one-point compactification of Ψ-space. We show it has a first countable $(0, \neq 0)$–generator containing $\mathbf{0}$ which is discrete away from $\mathbf{0}$. Let $\mathcal{A}$ be an almost-disjoint family of subsets of ω (so, for any $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2$ is finite). Let $\Psi(\mathcal{A})$, the Ψ*-space* associated with $\mathcal{A}$, have underlying set $\mathcal{A} \cup \omega$, and topology where points of ω

are isolated and the other basic open sets have the form $A^F = \{A\} \cup (A \setminus F)$ for $A \in \mathcal{A}$ and F a finite subset of ω.

Denote by $\alpha(\Psi(\mathcal{A}))$ the one-point compactification of $\Psi(\mathcal{A})$. In $\alpha(\Psi(\mathcal{A}))$ there are three kinds of basic open set. The first two are the open sets of $\Psi(\mathcal{A})$ mentioned above. The third kind is the open sets around the added point at infinity, $\star$. These sets take the form of $\alpha(\Psi(\mathcal{A})) \setminus K$, where K is a compact subset of $\Psi(\mathcal{A})$. For our convenience we make a couple of remarks about coding compact subsets of $\Psi(\mathcal{A})$ (and hence neighborhoods of $\star$) and basic neighborhoods of A in $\Psi(\mathcal{A})$.

Lemma 59. *If K is a compact subset of $\Psi(\mathcal{A})$ then K is a subset of a set K' where K' is the finite union of sets of the form $A^{\emptyset}$ along with a finite subset of ω, and such a K' is a compact subset of $\Psi(\mathcal{A})$.*

Proof. Let K be a compact subset of $\Psi(\mathcal{A})$. Since $\mathcal{A}$ is a closed discrete subset of $\Psi(\mathcal{A})$, we have $K \cap \mathcal{A}$ finite, say $\{A_1, \ldots, A_n\}$. Let $F = K \setminus (A_1^{\emptyset} \cup \ldots \cup A_n^{\emptyset})$. Then F is a compact subset of ω, and so finite. Now we see that K is a subset of $K' = F \cup \bigcup_{i=1}^{n} A_i^{\emptyset}$. Since K' is a finite union of compact sets it is compact. $\square$

Below, we'll adopt the following notation for compact sets. Let $\mathcal{A}_0 = \{A_1, \ldots, A_n\}$ be a finite subset of $\mathcal{A}$. Let $K_{\mathcal{A}_0} = \bigcup_{i=1}^{n} A_i^{\emptyset} \cup (n+1)$. Observe that from the preceding lemma the collection of all $K_{\mathcal{A}_0}$ is a cofinal family in the compact subsets of $\Psi(\mathcal{A})$.

Lemma 60. *The following is a local base of clopen sets at an $A \in \mathcal{A}$: all sets of the form $B_n(A) = \{A\} \cup (\{n\} \cup A \setminus \{0, \ldots, n-1\})$.*

Proof. Note that the $B_n(A)$ are indeed clopen neighborhoods of A. Take any basic $U = \{A\} \cup (A \setminus F)$. Let $m = \max(F)$ and let $F^\star = m+1 = \{0, \ldots, m\}$. Let $n = \min(A \setminus F^\star)$ and note that $n \in U$. We check $B_n(A)$ is contained in U. Take any $x \in B_n(A)$. If $x = A$, then $x \in U$, done. If $x = n$ then $x \in U$, done. Otherwise, $x \in A \setminus \{0, \ldots, n-1\}$ then $x \in A \setminus F^\star$, as $(A \setminus \{0, \ldots, n-1\}) \subseteq A \setminus F^\star$. So then $x \in U$ in this case, as well. Thus, $B_n(A) \subseteq U$. $\square$

From the above the following is a base for $\alpha(\Psi(\mathcal{A}))$: $\mathcal{B} = \mathcal{B}_\omega \cup \mathcal{B}_{\mathcal{A}} \cup \mathcal{B}_\star$, where $\mathcal{B}_\omega = \{\{n\} : n \in \omega\}$, $\mathcal{B}_{\mathcal{A}} = \{B_n(A) : A \in \mathcal{A}\}$ and $\mathcal{B}_\star = \{\alpha(\Psi(\mathcal{A})) \setminus K_{\mathcal{A}_0} : \text{finite } \mathcal{A}_0 \subseteq \mathcal{A}\}$.

Example 61. The one-point compactification of Ψ-space, $\alpha(\Psi(\mathcal{A}))$, has a first countable $(0,\neq 0)$–generator containing $\mathbf{0}$, such that $G \setminus \{\mathbf{0}\}$ is discrete.

Proof. Let $X = \alpha(\Psi(\mathcal{A}))$. For each n in ω, let $B_n = \{n\}$ and $g_n = (1/(3n+1))\chi_{B_n}$. For each A in $\mathcal{A}$ and $n \in \omega$, let $B_{A,n} = B_n(A)$ and $g_{A,n} = (1/(3n+2))\chi_{B_{A,n}}$. For each finite subset $\mathcal{A}_0$ of $\mathcal{A}$ of size n, let $B_{\mathcal{A}_0} = X \setminus K_{\mathcal{A}_0}$ and $g_{\mathcal{A}_0} = (1/(3n+3))\chi_{B_{\mathcal{A}_0}}$. For each n in ω, let $G_{3n+1} = \{g_n\}$, $G_{3n+2} = \{g_{A,n} : A \in \mathcal{A}\}$ and $G_{3n+3} = \{g_{\mathcal{A}_0} : \text{finite } \mathcal{A}_0 \subseteq \mathcal{A},\ |\mathcal{A}_0| = n\}$. Let $F_n = \{\star\} \cup (n+1)$ and $t_n = 1/(n+1)$. Clearly $G_n \subseteq C(X, \{0, 1/(n+1)\})$ and for each g in G_n the sets F_n and $g^{-1}(\mathbb{R} \setminus \{0\})$ meet (indeed both contain n, or both contain $\star$ – here we use the specific properties of the $B_n(A)$). Now Lemma 11 tells us that $G = \{\mathbf{0}\} \cup \bigcup_n G_n$ is a sequence of clopen copies of the G_n, converging to $\mathbf{0}$.

Since $\mathcal{B} = \{g^{-1}(\mathbb{R} \setminus \{0\}) : g \in G\}$ is precisely the base for X stated above, from Lemma 1(2) we deduce that G is a $(0,\neq 0)$–generator for X containing $\mathbf{0}$, and first countable at $\mathbf{0}$.

It remains to show that each G_m is discrete. This is evident for G_{3n+1}. Now for G_{3n+2}. Take any $g_{A,n} \in G_{3n+2}$. Suppose $g_{A',n}$ is in the basic neighborhood, $B(g_{A,n}, \{A\}, 1/(3n+2))$, of $g_{A,n}$. Then as $g_{A,n}(A) = 1/(3n+2)$ also $g_{A',n}(A) = 1/(3n+2)$. But this forces $A' = A$, as desired. And finally, G_{3n+3}. Take any $g_{\mathcal{A}_0}$ in G_{3n+3}. So $\mathcal{A}_0$ is a subset of $\mathcal{A}$ of size n. Observe, using the specific properties of $K_{\mathcal{A}_0}$, that $\mathcal{A}_0 = K_{\mathcal{A}_0} \cap \mathcal{A}$ and $g_{\mathcal{A}_0}$ is zero precisely on $K_{\mathcal{A}_0}$. Now suppose $g_{\mathcal{A}_0'} \in G_{3n+3}$ is in $B(g_{\mathcal{A}_0}, \mathcal{A}_0, 1/(3n+3))$. Then, for each A in $\mathcal{A}_0$, as $g_{\mathcal{A}_0}(A) = 0$ also $g_{\mathcal{A}_0'}(A) = 0$, and so A is in $\mathcal{A}_0'$. But since $\mathcal{A}_0$ and $\mathcal{A}_0'$ have the same (finite) size, it follows that they are equal, and so $g_{\mathcal{A}_0'} = g_{\mathcal{A}_0}$, as required for discreteness. □

Example 62. For every cardinal κ, the compact space $\{0,1\}^\kappa$ has a discrete $(0,1)$–generator. Further, $\{0,1\}^\kappa$ has a first countable $(0,\neq 0)$–generator containing $\mathbf{0}$ if and only if $\kappa \leq \mathfrak{c}$.

Proof. Since $\{0,1\}^\kappa$ is separable if and only if $\kappa \leq \mathfrak{c}$, the 'further' claim follows from the first combined with Theorem 56(2).

We show $\{0,1\}^\kappa$ has a discrete $(0,1)$–generator. We may suppose κ is infinite. For any f in $\{0,1\}^\kappa$ and pair (α, k) define $f \| (\alpha, k)$ to be the function which is f except at α where it has value k. Fix σ a function whose domain, $dom\,\sigma$, is a finite subset of κ of size at least two, and whose range is contained in $\{0,1\}$. Then $B_\sigma = \{f \in \{0,1\}^\kappa : f \restriction dom\,\sigma = \sigma\}$ is a basic clopen set in $\{0,1\}^\kappa$. Fix some f_σ in B_σ and $\alpha_\sigma \in \kappa \setminus dom\, \sigma$. Define $\sigma^i = \sigma \cup \{(\alpha_\sigma, i)\}$ and $f_{\sigma,i} = f\|(\alpha_\sigma, i)$, for $i = 0, 1$. Define, for $i = 0, 1$, $g_{\sigma,i}$ in $C_p(\{0,1\}^\kappa)$ by $g_{\sigma,i}(f)$ is 1 if $f \in B_{\sigma^{1-i}}$, $(-1)^i \cdot |\sigma|$ if $f \in B_{\sigma^i}$ and 0 otherwise ($f \notin B_\sigma$).

Let $G_{n,i} = \{g_{\sigma,i} : |\sigma| = n\}$ and $G = \bigcup_{n \geq 2, i=0,1} G_{n,i}$. Since $g_{\sigma,0}^{-1}\{1\} \cup g_{\sigma,1}^{-1}\{1\} = B_\sigma$ and $g_{\sigma,0}^{-1}\{0\} = g_{\sigma,1}^{-1}\{0\} = \{0,1\}^\kappa \setminus B_\sigma$, we see G is a $(0,1)$–generator for $\{0,1\}^\kappa$. We show G is discrete.

Fix σ and i. Let $U_0 = B(g_{\sigma,i}, \{f_{\sigma,i}\}, 1/2)$ a neighborhood of $g_{\sigma,i}$. If $g_{\tau,k}$ is in U_0 then $g_{\tau,k}(f_{\sigma,i}) = (-1)^i \cdot |\sigma|$. Since $g_{\tau,k}$ only takes on the values 0, 1 and $(-1)^k \cdot |\tau|$, we see that $k = i$ and $|\tau| = |\sigma|$. Now let $U = U_0 \cap U_1$, where $U_1 = B(g_{\sigma,i}, \{f_{\sigma,1-i}\|(\alpha, k) : \alpha \in dom\, \sigma,\ k \in \{0,1\}\}, 1/2)$. Then U is a neighborhood of $g_{\sigma,i}$. If $g_{\tau,k}$ is in U then from above $k = i$ and $|\tau| = |\sigma|$. Take any $\alpha \in dom\, \sigma$. Then $g_{\sigma,i}(f_{\sigma,1-i}\|(\alpha, \sigma(\alpha))) = 1$ but $g_{\sigma,i}(f_{\sigma,1-i}\|(\alpha, 1 - \sigma(\alpha))) = 0$. Hence $g_{\tau,i}(f_{\sigma,1-i}\|(\alpha, \sigma(\alpha))) = 1$ and $g_{\tau,i}(f_{\sigma,1-i}\|(\alpha, 1 - \sigma(\alpha))) = 0$, equivalently, $f_{\sigma,1-i}\|(\alpha, \sigma(\alpha)) \in B_{\tau^{1-i}} \subseteq B_\tau$ and $f_{\sigma,1-i}\|(\alpha, 1 - \sigma(\alpha)) \notin B_\tau$. This forces α to be in $dom\, \tau$ (so that $B_{\tau^{1-i}}$ can 'see' the difference between $f_{\sigma,1-i}\|(\alpha, \sigma(\alpha))$ and $f_{\sigma,1-i}\|(\alpha, 1 - \sigma(\alpha))$, which differ only at α). Thus $dom\, \sigma \subseteq dom\, \tau$. As $|\sigma| = |\tau|$ we see that $\tau = \sigma$, and so $g_{\tau,k} = g_{\sigma,i}$, as required for discreteness. □

6.3 Second Countable Generators

We know from Lemma 53 that a space has a second countable $(0,1)$–generator containing the zero function if and only if the space is countable. What happens if we weaken '$(0,1)$–generator' to '$(0, \neq 0)$–generator'? Recalling that every second countable space is cosmic, a sufficient condition is given by the next result.

Lemma 63. *Let X be any space. The following are equivalent: (1) X is cosmic, (2) $C_p(X)$ is cosmic and (3) X has a cosmic generator.*

Proof. It is a standard result (see [3, Theorem 1.1.3]) that for any space Y, $C_p(Y)$ is cosmic if and only if Y is cosmic. Hence (1) and (2) are equivalent and imply (3) (since $C_p(X)$ is a generator). Now, suppose X has a cosmic generator G. Then X embeds into $C_p(G)$.

However, since G is cosmic, then $C_p(G)$ must be cosmic, and so X is cosmic as well. $\square$

Further, every second countable space has a compact, second countable $(0,\neq 0)$–generator containing $\mathbf{0}$. In fact, we showed in Lemma 3 that if X is second countable, then X has a $(0,\neq 0)$–generator containing $\mathbf{0}$ homeomorphic to a convergent sequence. But many more spaces have second countable generators. In the next section we present a range of cosmic, non-second countable spaces with a second countable $(0,\neq 0)$–generator containing $\mathbf{0}$, some even compact and second countable. As in the case of first countability, we can drop the requirement that our second countable $(0,\neq 0)$–generator contain $\mathbf{0}$.

Theorem 64. *If a space has a second countable $(0,\neq 0)$–generator, then it has a second countable $(0,\neq 0)$–generator containing $\mathbf{0}$.*

Proof. Let X have a second countable $(0,\neq 0)$–generator. Then X is cosmic by Lemma 63 and, hence, separable. From Proposition 12, we know X has a $(0,\neq 0)$–generator G containing $\mathbf{0}$ as a point of first countability such that $G \setminus \{\mathbf{0}\}$ is a countable union of open subspaces, each of which is second countable (as each is homeomorphic to a subspace of the given second countable generator). The union of the countable bases for each of those open subsets, along with a countable local base at $\mathbf{0}$, gives a countable base for G. $\square$

The above leads us to three questions.

Question 8.

(A) Which spaces have a second countable $(0,\neq 0)$–generator?

(B) Does every cosmic space have a second countable $(0,\neq 0)$–generator?

(C) Which spaces have a compact, second countable $(0,\neq 0)$–generator?

Regarding (C) we can tap into our knowledge of spaces with a compact generator.

Proposition 65. *The following are equivalent: (1) X has a σ-compact, cosmic generator, (2) X has a compact, second countable $(0,\neq 0)$–generator, (3) X embeds in $C_p(K)$ where K is compact and second countable, and (4) X embeds in $C_p(Y)$ where Y is σ-compact and cosmic.*

Proof. We prove this by cycling through (1)-(4).

Suppose (1) holds and G is a σ-compact, cosmic generator for X. Then we can suppose (via Lemma 9) G is a $(0, \neq 0)$–generator and $G = \bigcup_n G_n$ where each G_n is compact and cosmic. Fix n and define $m_n : C_p(X) \to C_p(X)$ by $m_n(f)(x) = \text{mid}(-1/n, f(x), 1/n)$. Then m_n is continuous. Let $G'_n = m_n(G_n)$, and note G'_n is a compact, cosmic subset of $C_p(X, [1/n, -1/n])$. Let $G' = \bigcup_n G'_n \cup \{\mathbf{0}\}$. As G is a $(0, \neq 0)$–generator, and if g is G_n and $g(x) \neq 0$, for some point x, then $m_n(g)(x) \neq 0$, we see G' is also a $(0, \neq 0)$–generator for X. To see this observe that any basic open neighborhood of $\mathbf{0}$, say $B(\mathbf{0}, F, 1/n)$, contains all G'_m where $m > n$, and each of $G'_1, \ldots, G'_n$ is compact. As a countable union of cosmic spaces G' is cosmic. Combining cosmicity with the compactness, we see that G' is compact and second countable. Thus we have (2).

Now if G is a compact, second countable generator for X, then we know X embeds in $C_p(G)$. Thus (2) implies (1). Evidently (2) implies (3).

Now suppose (3) holds and X is a subspace of $C_p(Y)$ where Y is σ-compact, cosmic. We show below in Proposition 69 ((1) $\implies$ (3)) that $C_p(Y)$ has a σ-compact and cosmic generator, say G. Then $G_X = \pi_X(G) = \{g \restriction X : g \in G\}$ is a $(0, \neq 0)$–generator for X, which is the continuous image of G, and is σ-compact and cosmic. Thus (4) holds. $\square$

Since every compact, second countable space K is the continuous image of the Cantor set C, then $C_p(K)$ embeds in $C_p(C)$ via the dual map. Thus, in the preceding result we may assume the compact, second countable space is the Cantor set. We are not aware of an internal characterization of subspaces of $C_p(C)$.

6.4 Examples: Spaces with a 2° Generator

Recall that every second countable space has a compact, second countable $(0, \neq 0)$–generator containing $\mathbf{0}$. Every countable space, X, has a second countable $(0, 1)$–generator containing $\mathbf{0}$, simply because $C_p(X)$ is second countable. However, not every countable space has a compact, second countable generator. Indeed many countable spaces do not embed in any $C_p(K)$ where K is compact (in other words, are not Eberlein-Grothendieck). Specifically, the Frechet-Urysohn fan ($X = Y/A$ where $Y = S \times \mathbb{N}$, S is the convergent sequence, say $S = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, and $A = \{0\} \times \mathbb{N}$) and ultrafilter space $\mathbb{N} \cup \{p\}$ (subspaces of $\beta\mathbb{N}$ for $p \in \beta\mathbb{N} \setminus \mathbb{N}$), are known to be not Eberlein-Grothendieck. Uspenskii [24] has given a characterization of those countable spaces with exactly one non-isolated point which are Eberlein-Grothendieck. But this characterization is not internal, again highlighting the difficulties in characterizing *internally* the spaces with a compact, second countable generator.

Example 66. Every countable space has a second countable $(0,1)$–generator containing the zero function. The spaces $\mathbb{N} \cup \{p\}$ and the Frechet-Urysohn fan do not have a compact, second countable generator.

We now present two examples of nice generators for uncountable cosmic spaces which are not second countable. In the first example, the space has a compact, second countable generator. The second has a second countable $(0, \neq 0)$–generator containing $\mathbf{0}$, but no compact generator.

Towards the first example, in the plane, $\mathbb{R}^2$, let $W_{x,n}$ be the set containing $(x, 0)$ along with all points that lie strictly between the two lines passing through $(x, 0)$ with slope of $\pm 1/n$, and the two vertical lines through $(x - 1/n, 0)$ and $(x + 1/n, 0)$. McAuley's bow tie space [14] is the plane with the usual topology and all $W_{x,n}$, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$, added as basic open sets. For any subset, S, of $\mathbb{R}^2$ we call S with the subspace topology from the bow tie space, the *bow tie space on S*. The bow tie space is a classic example of a first countable, cosmic space which is not second countable.

One can show that the bow tie space has a compact, second countable $(0, \neq 0)$–generator containing $\mathbf{0}$, and that every subspace of the bow tie has a second countable $(0, \neq 0)$–generator containing $\mathbf{0}$. For simplicity we sketch a particular case.

Example 67. The bow tie space on $[-1, 1]^2$ has a compact, second countable $(0, \neq 0)$–generator containing $\mathbf{0}$.

Proof. Let X be the bowtie space on $[-1, 1]^2$. Fix a countable basis, $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$, for $[-1, 1]^2$ with the usual topology, and pick z_n in B_n and $g_n \in C_p(X, [0, 1/(2n - 1)])$ such that $g_n(z_n) = 1/(2n - 1)$ and $g_n^{-1}(\mathbb{R} \setminus \{0\}) = B_n$. Let $G_{2n-1} = \{h_n\}$, and $F'_{2n-1} = \{z_n\}$. For each x in $[-1, 1]$ and n, define $g_{x,n} \in C_p(X)$ such that $g_{x,n}(W_{x,n+1}) = 1/(2n)$, $g_{x,n}(X \setminus W_{x,n}) = 0$, and $g_{x,n}$ goes linearly to 0 on $W_{x,n} \setminus W_{x,n+1}$. Note $g_{x,n}^{-1}(\mathbb{R} \setminus \{0\}) = W_{x,n}$. Let $G_{2n} = \{g_{x,n} : x \in [-1,1]\}$, and $F'_{2n} = \{(i/n, 0) : -n \leq i \leq n\}$. Note that F'_{2n} meets $W_{x,n}$.

Now set $F_n = \bigcup_{i \leq n} F'_i$ and $t_n = 1/n$. Clearly $G_n \subseteq C_p(X, [0, 1/n])$ and for each g in G_n the sets F_n and $g^{-1}(\mathbb{R} \setminus \{0\})$ meet. Since $\mathcal{B} = \{g^{-1}(\mathbb{R} \setminus \{0\}) : g \in G\}$ is $\mathcal{B} \cup \{W_{x,n} : x \in [-1,1]$ and $n \in \mathbb{N}\}$ - a base for X - from Lemma 1(2) we know that G is a $(0, \neq 0)$–generator for X containing $\mathbf{0}$. While from Lemma 11 we deduce that $G = \{\mathbf{0}\} \cup \bigcup_n G_n$ is a sequence of clopen copies of the G_n, converging to $\mathbf{0}$.

It remains to show each G_{2n} is homeomorphic to $[-1,1]$, for then G is homeomorphic to a convergent sequence of compact intervals (odd ones being singletons), and so is compact and second countable. However it is not hard (albeit, a little detailed) to verify that, for each n, the map ψ_n taking x to $g_{x,n}$ is a homeomorphism from $[-1,1]$ to G_{2n}. $\square$

Example 68. Let $Y = I \times \mathbb{N}$, $A = \{1\} \times \mathbb{N}$, and $X = Y/A$. Then X has a second countable $(0,1)$–generator, and a second countable $(0, \neq 0)$–generator containing $\mathbf{0}$. However, X does not have a compact (second countable) generator.

Proof. To see that X does not have a compact generator, note that the Frechet-Urysohn fan embeds in X, and if X had a compact generator then so would the fan, but - as discussed above - this is false. For the positive claims, according to Theorem 64, it suffices to show X has a second countable $(0,1)$–generator.

First, let $\star$ denote the image of A in X, and identify each point of $Y \setminus A$ with its image in X, so that $Y \setminus A = X \setminus \{*\}$. Note that $Y \setminus A = [0,1) \times \mathbb{N}$ has the usual topology, which is separable metrizable. So by Theorem 48 it has a discrete $(0,1)$–generator which, by separability, must be countable. The subspace consisting of those functions, g, whose support, $g^{-1}(\mathbb{R} \setminus \{0\})$, is contained in a *proper* subinterval of some $[0,1) \times \{i\}$, is also discrete and a $(0,1)$–generator for $[0,1) \times \mathbb{N}$. And these functions can be extended continuously over X (by giving them value 0 at $\star$). In summary, there is a countable, discrete subspace, G_0, of $C_p(X)$ such that every g in G_0 is zero at $\star$ and, whenever a point $x \in X \setminus \{\star\}$ is in an open U then there is a g in G_0 such that $g(x) = 1$ but $g(X \setminus U) = \{0\}$.

Now we deal with the neighborhoods of $\star$. For any $n \in \mathbb{N}$, let $f_n \in C_p(I)$ be given by: f_n is zero on $[0, 1 - 1/n]$, one on $[1 - 1/(n+1), 1]$, and linearly interpolates from 0 to 1 between $1 - 1/n$ and $1 - 1/(n+1)$. Then for any $\mathbf{n} = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$, define $f'_{\mathbf{n}} \in C_p(Y)$ by $f'_{\mathbf{n}}((t,k)) = f_{n_k}(t)$. Since $f'_{\mathbf{n}}(a) = 1$ for any $a \in A$, then $f'_{\mathbf{n}}$ induces a well-defined $f_{\mathbf{n}} \in C_p(X)$, namely, $f_{\mathbf{n}}((t,k)) = f_{n_k}(t)$ for each $(t,k) \in X \setminus \{\star\}$ and $f_{\mathbf{n}}(\star) = 1$.

Define $F : \mathbb{N}^{\mathbb{N}} \to C_p(X)$ by $F(\mathbf{n}) = f_{\mathbf{n}}$, which is clearly injective since $f_n \neq f_m$ for any $n \neq m \in \mathbb{N}$. We show F is an embedding.

To see that F is continuous, let $\mathbf{n} = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and fix a basic open neighborhood $V = B(f_{\mathbf{n}}, \{(t_1,k_1), \ldots, (t_l,k_l), \star\}, \epsilon)$ of $f_{\mathbf{n}}$, where $(t_i,k_i) \in Y \setminus A$. Then consider the basic open neighborhood $U = B(\mathbf{n}, \{k_1, \ldots, k_l\}, 1/2)$ of $\mathbf{n}$. For each $\mathbf{m} = (m_k)_k \in U$ and $i = 1, \ldots, l$, we have $m_{k_i} = n_{k_i}$. Hence, $f_{\mathbf{m}}((t_i,k_i)) = f_{m_{k_i}}(t_i) = f_{n_{k_i}}(t_i) = f_{\mathbf{n}}((t_i,k_i))$ for each i, and since $f_{\mathbf{m}}(\star) = 1 = f_{\mathbf{n}}(\star)$, then $f_{\mathbf{m}} \in V$, so $F(U) \subseteq V$.

Let $G_1 = F(\mathbb{N}^{\mathbb{N}})$ and fix $f_{\mathbf{n}} \in G_1$. Let $V = B(\mathbf{n}, \{k_1, \ldots, k_l\}, \epsilon)$ be any basic open neighborhood of $\mathbf{n} = F^{-1}(f_{\mathbf{n}})$. Then consider the basic open neighborhood $U = B(f_{\mathbf{n}}, S, 1/2)$ of $f_{\mathbf{n}}$, where $S = \{x_1, \ldots, x_l, z_1, \ldots, z_l\}$, $x_i = (t_i,k_i)$, $z_i = (s_i,k_i)$, $t_i = 1 - 1/n_{k_i}$, and $s_i = 1 - 1/n_{k_i} + 1$. For each $f_{\mathbf{m}} \in U$ and $i = 1, \ldots, l$, we have $f_{\mathbf{m}}(x_i) = f_{m_{k_i}}(t_i) \in \{0,1\}$ since $f_m(1 - 1/n) \notin (0,1)$ for any $n, m \in \mathbb{N}$. But since $f_{\mathbf{n}}(x_i) = f_{n_{k_i}}(t_i) = 0$ and $|f_{\mathbf{m}}(x_i) - f_{\mathbf{n}}(x_i)| < 1/2$, then $f_{m_{k_i}}(t_i) = 0$. Similarly, $f_{m_{k_i}}(s_i) = 1$. Thus, $1 - 1/n_{k_i} \leq 1 - 1/m_{k_i} < 1 - 1/(m_{k_i}+1) \leq 1 - 1/(n_{k_i}+1)$, so $m_{k_i} = n_{k_i}$ for all i, which means $\mathbf{m} \in V$. As $F^{-1}(U) \subseteq V$, we see $F^{-1} : G_1 \to \mathbb{N}^{\mathbb{N}}$ is continuous.

Hence, G_1, being homeomorphic to $\mathbb{N}^{\mathbb{N}}$, is second countable. Note that G_0 and G_1 are disjoint open sets in $G = G_0 \cup G_1$, since $g(\star) = 0$ for all $g \in G_0$ and $f(\star) = 1$ for all $f \in G_1$. So G is also second countable.

Finally, to check that G is a $(0,1)$–generator, let $x \in U$ with U open in X. If $x \neq \star$, then we are done by construction of G_0. If $x = \star$, then there is some $\mathbf{n} = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ such that $V = \{\star\} \cup (\bigcup_{k \in \mathbb{N}} (1 - 1/n_k, 1) \times \{k\}) \subseteq U$, so $f_{\mathbf{n}}(\star) = 1$ and $f_{\mathbf{n}}(X \setminus U) = \{0\}$, as required. $\square$

6.5 The Case $X = C_p(Y)$

Next, we ask: when does a space of the type $X = C_p(Y)$ have a first or second countable generator? In Section 4.1.3 we introduced a natural generator, $s(Y)$, for $C_p(Y)$. Using this generator, we give a complete solution, see Proposition 69, to the problem of when $C_p(Y)$ has a compact, second countable generator. However neither $s(Y)$, nor its subspace $ss(Y)$, work well with first or second countability, see Theorem 16 and the preceding discussion. Consequently, to give sufficient conditions for $C_p(Y)$ to have a first (Theorem 72) or second countable generator (Theorem 71) we need to develop an entirely different generator for $C_p(Y)$, see Theorem 70.

The details of the definition of $s(Y)$ are not needed here. All that we need to know is that $s(Y) \setminus \{\mathbf{0}\}$ is the continuous image of $\bigoplus_n (Y^n \times \mathbb{N})$. Recall that both σ-compact and cosmic spaces are preserved under finite products, continuous images, and countable unions; and a compact space is second countable if and only if it is cosmic.

Proposition 69. *The following are equivalent: (1) Y is σ-compact and cosmic, (2) $C_p(Y)$ has a compact, second countable $(0, \neq 0)$–generator, and (3) $C_p(Y)$ has a σ-compact and cosmic generator.*

Proof. If (1) holds, and Y is σ-compact and cosmic, then the generator $s(Y)$ for $C_p(Y)$ is also σ-compact and cosmic, so (3) holds. If (3) holds and $X = C_p(Y)$ has a σ-compact and cosmic generator, then by Proposition 65 ((1) $\implies$ (2)), $X = C_p(Y)$ has a compact, second countable $(0, \neq 0)$–generator. Now suppose (2) holds and $C_p(Y)$ has a compact, second countable generator G. Then $C_p(Y)$ embeds in $C_p(G)$. As G is second countable, hence cosmic, $C_p(G)$ is cosmic, and hence so is $C_p(Y)$, and in turn, so must Y be cosmic. Also, by Okunev's theorem, as $C_p(Y)$ embeds in $C_p(G)$ and G is compact, we see Y is σ-compact. Thus (1) holds. $\square$

To introduce our second generator for $C_p(Y)$, we need a little background on hyperspaces. For any space Y, let 2^Y be the space of non-empty, closed subsets of Y with the Vietoris topology. For each $n \in \mathbb{N}$, let $F_n(Y)$ denote the subspace of 2^Y consisting of all n-element subsets of Y. If $F = \{y_1, \ldots, y_n\}$ in $F_n(Y)$, then its basic open neighborhoods have the form $O[F, U_1, \ldots, U_n] = \{A \in F_n(Y) : A \subseteq \bigcup_i U_i, A \cap U_i \neq \emptyset \text{ for each } i\}$, where each U_i is an open neighborhood of y_i, and the U_i's are pairwise disjoint.

Theorem 70. *If $X = C_p(Y)$ is separable, then it has a $(0, \neq 0)$–generator G containing* $\mathbf{0}$ *such that (1)* $\mathbf{0}$ *is a point of first countability in G, and (2) $G \setminus \{0\}$ is a countable union* *of subsets that are clopen in G, each of which is homeomorphic to a subspace of $F_n(Y)$ for some n.*

Proof. For a fixed k in $\mathbb{N}$ define $\beta_k : [0, \infty) \to [0, 1]$ by $\beta_k(t) = \max\{0, 1 - 2^k t\}$. As X is separable, we may fix a countable dense subset $\{x_m : m \in \mathbb{N}\}$ of X. For any $F \in F_n(Y)$ and $m \in \mathbb{N}$, define $d_{m,F} \in C_p(X)$ by $d_{m,F}(x) = \max\{|x(y) - x_m(y)| : y \in F\}$, and let $h_{k,m,F} = \delta_{k,m,n} \cdot \beta_k \circ d_{m,F} \in C_p(X, [0, \delta_{k,m,n}])$, where $\delta_{k,m,n} = 1/(2^k 3^m 5^n)$. Observe that $d_{m,F}(x) = 0$ if and only if $x \restriction F = x_m \restriction F$, and $d_{m,F}(x) \geq 1/2^k$ if and only if $|x(y) - x_m(y)| \geq 1/2^k$ for some $y \in F$. Hence $h_{k,m,F}(x) = 0$ if and only if $x \notin B(x_m, F, 1/2^k)$, and $h_{k,m,F}(x) = \delta_{k,m,n}$ if and only if $x \restriction F = x_m \restriction F$. In particular, $h_{k,m,F}(x_m) = \delta_{k,m,n}$.

For each $k, m, n \in \mathbb{N}$, define $H_{k,m,n} = \{h_{k,m,F} : F \in F_n(Y)\}$ and let $H = \bigcup_{k,m,n} H_{k,m,n}$. We will show that H is homeomorphic to $\bigoplus_n F_n(Y) \times \mathbb{N}$.

Our first objective is to use Lemma 11 to show that the $H_{k,m,n}$ are pairwise disjoint and clopen in H. For each $i \in \mathbb{N}$, there is a unique choice of $k_i, m_i, n_i \in \mathbb{N}$ such that the map $\mathbb{N} \to \mathbb{N}^3$, $i \mapsto (k_i, m_i, n_i)$ is bijective and $(2^{k_i} 3^{m_i} 5^{n_i})_i$ is a strictly increasing sequence. Note that this implies $m_i \leq i$ for each i, since otherwise there would need to be at least i terms in this sequence smaller than $2^{k_i} 3^{m_i} 5^{n_i}$. Let $H_i = H_{k_i, m_i, n_i}$ for each i. It follows that $\{H_i : i \in \mathbb{N}\} = \{H_{k,m,n} : k, m, n \in \mathbb{N}\}$, the sequence $(t_i = \delta_{k_i, m_i, n_i})_i$ is strictly decreasing to 0, and each x_{m_i} is in the corresponding $F_i = \{x_m : m \leq i\}$. Now each $H_i \subseteq C_p(X, [0, t_i])$, and for every $h = h_{k_i, m_i, F} \in H_i$, we have $h(x_{m_i}) = t_i$, so Lemma 11 implies that the H_i are pairwise disjoint and clopen in H, as desired.

Fix $k, m, n \in \mathbb{N}$, we will next show that $\phi = \phi_{k,m,n} : F_n(Y) \to H_{k,m,n}$, $F \mapsto h_{k,m,F}$ is a homeomorphism. To verify that ϕ is bijective, let $F, F' \in F_n(Y)$ with $F \neq F'$. Without loss of generality, we can find $y' \in F' \setminus F$. Then there exists x in X such that $x \restriction F = x_m \restriction F$ but $x(y') \neq x_m(y')$. Then $h_{k,m,F}(x) = \delta_{k,m,n}$ but $h_{k,m,F'}(x) \neq \delta_{k,m,n}$, so $h_{k,m,F} \neq h_{k,m,F'}$. Thus, ϕ is one-to-one, and it is clearly also onto.

To see that ϕ is continuous, fix any $F = \{y_1, \ldots, y_n\}$ in $F_n(Y)$, and take any sub-basic neighborhood of $\phi(F) = h_{k,m,F}$, say $V = B(h_{k,m,F}, \{x\}, \epsilon) \cap H_{k,m,n}$. Since $\delta_{k,m,n} \cdot \beta_k$ is continuous at $d_{m,F}(x)$, then there is an $\eta > 0$ such that $|\delta_{k,m,n} \cdot \beta_k(t) - h_{k,m,F}(x)| < \epsilon$ whenever $|t - d_{m,F}(x)| < \eta$. By the continuity of x_m and x, there exist pairwise disjoint open neighborhoods U_i of the y_i such that, for each $i = 1, \ldots, n$, we have $|(x - x_m)(y) - (x - x_m)(y_i)| < \eta$ for all $y \in U_i$. Consider the basic neighborhood $O = O[F, U_1, \ldots, U_n]$ of F in $F_n(Y)$.

We show $\phi(O) \subseteq V$ and deduce that ϕ is continuous. Take any $F' \in O$. Write $F' = \{y'_1, \ldots, y'_n\}$, where $y'_i \in U_i$ for each i. Now, $|(x - x_m)(y'_i) - (x - x_m)(y_i)| < \eta$ for each i, so $|d_{m,F'}(x) - d_{m,F}(x)| < \eta$ as well. Hence $|h_{k,m,F'}(x) - h_{k,m,F}(x)| < \epsilon$, so $\phi(F') = h_{k,m,F'}$ is in V, as required.

To check continuity of ϕ^{-1}, let $h_{k,m,F} \in H_{k,m,n}$ and write $F = \{y_1, \ldots, y_n\}$. Take any basic neighborhood $O = O[F, U_1, \ldots, U_n]$ of F, where each U_i is an open neighborhood of y_i, and the U_i are pairwise disjoint. For $i = 1, \ldots, n$, choose $w_i \in X$ such that $w_i(Y \setminus U_i) = \{0\}$ and $w_i(y_i) = 1$, and set $z_i = x_m + w_i$. Define $V = H_{k,m,n} \cap B(h_{k,m,F}, \{z_1, \ldots, z_n\}, \delta_{k,m,n})$, which is a neighborhood of $h_{k,m,F}$ in $H_{k,m,n}$.

We will show $\phi^{-1}(V) \subseteq O$, and thus ϕ^{-1} is continuous. Take any $h_{k,m,F'} \in V \subseteq H_{k,m,n}$. We need to show that $\phi^{-1}(h_{k,m,F'}) = F'$ is in O. Since $|F'| = n$ and $U_1, \ldots, U_n$ are pairwise disjoint, it suffices to verify that F' meets every U_i. Take any $i \in \{1, \ldots, n\}$. Note that $w_i(y_i) = 1 > 1/2^k$ implies $z_i \notin B(x_m, F, 1/2^k)$, so $h_{k,m,F}(z_i) = 0$. Now suppose $F' \cap U_i = \emptyset$ for some i. Then w_i is zero on F', so $z_i \restriction F' = x_m \restriction F'$, and hence $h_{k,m,F'}(z_i) = h_{k,m,F'}(x_m) = \delta_{k,m,n}$. Thus $|h_{k,m,F}(z_i) - h_{k,m,F'}(z_i)| = \delta_{k,m,n}$, which contradicts the fact that $h_{k,m,F'} \in V$, so F' must meet each U_i and $\phi^{-1}(V) \subseteq O$, as required.

We have shown that each $H_{k,m,n}$ is homeomorphic to $F_n(Y)$, and since the $H_{k,m,n}$ are pairwise disjoint and clopen in $H = \bigcup_{k,m,n} H_{k,m,n}$, then H is homeomorphic to $\bigoplus_n F_n(Y) \times \mathbb{N}$.

To see that H is a $(0, \neq 0)$–generator, fix x in X and a basic open neighborhood $B(x, F, \epsilon)$, where $F \subseteq Y$ is finite and $\epsilon \in (0, 1)$. Let $n = |F|$, choose $m \in \mathbb{N}$ such that $x_m \in B(x, F, \epsilon/4)$, and pick $k \in \mathbb{N}$ such that $\epsilon/4 \leq 1/2^k < \epsilon/2$. Then $h_{k,m,F} \in H_{k,m,n} \subseteq H$. We have $x \in B(x_m, F, \epsilon/4) \subseteq B(x_m, F, 1/2^k) \subseteq B(x_m, F, \epsilon/2) \subseteq B(x, F, \epsilon)$. But $B(x_m, F, 1/2^k) = h_{k,m,F}^{-1}(\mathbb{R} \setminus \{0\})$, and hence, $h_{k,m,F}(X \setminus B(x, F, \epsilon)) = \{0\}$ while $h_{k,m,F}(x) \neq 0$, as required of a $(0, \neq 0)$–generator.

Finally, Proposition 12 implies that X has another $(0, \neq 0)$–generator G that contains $\mathbf{0}$ as a point of first countability such that $G \setminus \{\mathbf{0}\} = \bigcup_j G_j$, where the G_j are pairwise disjoint and clopen in G, and each G_j is homeomorphic to a subspace of $\bigoplus_n F_n(Y) \times \mathbb{N}$.

Hence, each G_j is itself the union of countably many subsets that are each clopen in G and homeomorphic to a subspace of some $F_n(Y)$. $\square$

Let $Y^{(n)} = \{(y_1, \ldots, y_n) \in Y^n : y_i \neq y_j \text{ if } i \neq j\}$, and note it is an open subset of Y^n. Then it is well known that the map $Y^{(n)} \to F_n(Y), (y_1, \ldots, y_n) \mapsto \{y_1, \ldots, y_n\}$ is continuous and open. It follows that if Y is first countable (respectively, second countable) then so is $F_n(Y)$ (for every n). Recalling that $C_p(Y)$ is separable if and only if Y has a coarser second countable topology, we immediately deduce broad sufficient conditions on Y for $C_p(Y)$ to have a first or second countable generator.

Theorem 71. *If Y is second countable, then $X = C_p(Y)$ has a second countable $(0, \neq 0)$–generator containing* $\mathbf{0}$.

Theorem 72. *If Y is first countable and has a coarser second countable topology, then $X = C_p(Y)$ has a first countable $(0, \neq 0)$–generator containing* $\mathbf{0}$.

7.0 Future Directions

Despite the results presented in this thesis a number of our central questions remain only partially answered. In addition our results raise new questions, problems and directions. In this final chapter we outline what we think are the key areas for future research, and present some specific questions to ponder.

In Section 4, we offered as a characterization of spaces X with a compact generator in $C_p(X)$ that they are those which are Eberlein-Grothendieck. Unfortunately the definition of Eberlein-Grothendieck spaces, they embed in a $C_p(K)$ where K is compact, is not intrinsic, and there is no known internal characterization. It seems natural to start by trying to characterize those spaces which embed in a $C_p(K)$ where K is compact and second countable. This same problem arose in Section 6.3 (see discussion at the end) where we observed that what was a required was an internal characterization of the subspaces of $C_p(C)$, for C the Cantor set. In light of the results on special compact generators, see Section 4.2, perhaps there is a characterization in terms of some kind of 'nice' almost subbase?

At the end of Chapter 3, in Section 4.3, we generalized from (σ-) compact spaces to Lindelöf spaces. The result was a mix of the positive theorems (restrictions on the tightness, Proposition 33) and negative examples (related to behavior in powers). It seems likely that results for spaces with a Lindelöf Σ generator would be better, and proving - or disproving - this would be a natural extension of our results on spaces with a compact generator.

We were hopeful in Section 5.1 that the condition $w(X) = hc^*(X)$ might be both necessary and sufficient for X to have a discrete generator, but we have only been able to show necessity. On the one hand we have sufficiency for all zero-dimensional spaces, Theorem 40, and in set-theoretic topology almost all examples are zero-dimensional (or at the very least, have zero-dimensional versions). This certainly gives strong support to the conjecture. On the other hand the proof of Theorem 5.1 works in dimension 1 (the discrete subset is a subspace of X^1) but runs into (potential) problems in dimension 2 - and these problems arise when X is connected, and the Intermediate Value Theorem undermines our control of values. We see the difference in size of discrete subsets with the Sorgenfrey line, S. All discrete subsets of S are countable, but the anti-diagonal of S^2 is a discrete subset of size $|S|$. Now S is zero-dimensional, so we know it has a discrete generator, but the cone of S ($(S \times [0,1])/A$ where $A = S \times \{1\}$) has similar properties to S and is connected. Does it have a discrete generator? If not then we have a counter-example to our conjecture. If it does then how does it 'evade' the problem of being connected?

Beyond the cone of the Sorgenfrey line, we have a whole family of possible counter-examples, namely $X = C_p(Y)$ (and note these are connected). Specifically, does $C_p(Y)$ containing a discrete subset of size $|Y|$ imply that $C_p(Y)$ has a discrete generator? If there is a counter-example then perhaps there is another condition that needs to be imposed for the converse.

Likewise, we are also interested in a different version of this in terms of open subsets of a space and $(0, \neq 0)$–generators: if, for all open U of a space X, $w(U) = hc^*(U)$, does this mean X has a discrete $(0, \neq 0)$–generator? Are there other conditions that need to be added to assure this? Regrettably, we don't even have a conjectural characterization of spaces with a discrete $(0,1)$–generator. We know, in general, that there are spaces with a discrete $(0, \neq 0)$–generator but no discrete $(0,1)$–generator (the Michael line, Example 52). So the condition given above certainly does not suffice. But is there a compact X with a discrete $(0, \neq 0)$–generator but no discrete $(0,1)$–generator?

Last, Section 6 raised two central pairs of questions. First, what spaces have first countable generators? Is it enough for these spaces to be separable to guarantee the existence of such a generator? Second, what spaces have second countable generators? Is it enough for these spaces to be cosmic? While we were able to answer the first half of each pair of questions (a first countable generator implies a separable space, and likewise for second countable and cosmic), sufficiency eluded us.

We were, however, able to come up with a stable of examples of separable spaces with first countable generators and cosmic spaces with second countable ones. Still, the problem remains thorny: there does not appear to be a pattern in our proofs of these examples, and each space seems to require new techniques to construct a generator. We suspect that counterexamples exist. Continuing our approach of looking at specific spaces, a natural 'next target' would be $\beta\mathbb{N}$, the Stone-Cech compactification of the integers. It is compact and separable. Since it is also zero-dimensional and $w(\beta\mathbb{N}) = \mathbf{c} = hc(\beta\mathbb{N})$ we know from Theorem 40 that it has a discrete generator. Since $w(U) = hc(U)$ for all open subsets of $\beta\mathbb{N}$, our conjecture says that it should have a discrete $(0, \neq 0)$–generator. However, a preliminary investigation has not uncovered such a generator. Indeed we suspect $\beta\mathbb{N}$ does not have a discrete $(0, \neq 0)$–generator, even that it may not have a first countable $(0, \neq 0)$–generator.

However another, likely better, approach to finding a separable space without a first countable $(0, \neq 0)$–generator or a cosmic space without a second countable generator, is to find something additional - beyond 'separable' or 'cosmic' - that having a first or second countable generator implies. (Then the plan, of course, would be to construct a separable or cosmic space without that additional property.) It seems so plausible that we could find and prove that 'extra something'. And yet, to date, no success.

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