# Stochastic Analysis of Active Hydrodynamics

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**Stochastic Analysis of Active Hydrodynamics**

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We study the hydrodynamics of nematic liquid crystal flow perturbed by a multiplicative noise under the Beris-Edwards framework. For the stochastic active liquid crystal system, we built the existence of the weak global martingale solution in a 3-D smooth bounded domain through a four-level approximation scheme. The existence of the limit of approximate solutions in the presence of random variables is guaranteed by the classical Skorokhod representation theorem. For the three-dimensional compressible Navier-Stokes equations coupled with the Q-tensor equation, we first constructed the local existence and uniqueness of strong pathwise solution up to a positive stopping time to the system, then we proved that the local stopping time could be extended to maximal. Note that the construction of the local solution is built upon a cutting-off argument. We also studied the connection between the compressible Navier-Stokes equations coupled by the Q-tensor equation for liquid crystals with the incompressible system in the periodic case. As the Mach number approaches zero, we proved that, both in the deterministic and stochastic case, that the weak solutions of the compressible nematic liquid crystal model would converge to the solution of the incompressible one.

**Keywords:** Navier-Stokes equations, stochastic active liquid crystal system, stochastic compressible liquid crystal system, weak solution, global martingale solution, local strong pathwise solution, uniqueness, stochastic compactness, Mach number, incompressible limit.

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## Preface

This dissertation is about the stochastic analysis of the active hydrodynamics, which is quite a challenging and interesting problem for me. I have been working on the problem for several years, from day to night, with joy and sweat. Look back on the road that has led me here, I realize that I can not go so far without the involvement of numerous people, to whom I want to express my gratitude for making my graduate study successful.

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# 1.0 Introduction

Starting from the 1880's, a new material that shares both the property of conventional liquids and those of solid crystals was found and named as liquid crystals [37]. Instead of showing a single transition of solid to liquid, liquid crystals are more similar to a cascade of transitions involving new phases. The forms of liquid crystals can be divided into nematics, smectics, and columnar phases, and these forms can be observed by the structure of the constituent molecules or groups of molecules. Among all the types of liquid crystals, nematic liquid crystals are one of the most common liquid crystalline phases. In fact, the word "nematic" was invented by G. Friedel to refer to certain thread-like defects that are commonly observed in certain materials. Nematics are often made of elongated or rod-like objects, these elongated molecules flow about freely as in a conventional liquid, but tend to align along certain distinguished directions.

Due to the great importance of liquid crystals, a large amount of research concerning this has appeared starting from the early 1950's concerning its mathematical model. The liquid crystal model is considered as a vector model in the beginning, including the Oseen-Frank theory proposed in 1933 [72], the Ericksen-Leslie theory proposed in 1961 [29]. In these models, the alignment of liquid crystal molecules at a certain point were described by a direction field $\mathbf{n}$. The setting is widely applicable due to its simplicity, but they can not reflect the symmetry of these rod-like molecules. As a result, Doi [27] in 1986 and Onsager [71] in 1949 proposed the Doi-Onsager theory, which describes a molecular model. In this model, the alignment of liquid crystal molecules is described by an orientational distribution function, which contains more information than a direction vector. Later, Gennes [37] proposed the Landau-de Gennes theory in 1995. A traceless symmetric $3\times3$ matrix $Q$ was used to describe the alignment of liquid crystal molecules. In the viewpoint of physics, the order-parameter $Q$ can be considered as a special form of orientational distribution function. The vector theory and $Q$-tensor theory are also called as macroscopic theories, the construction of the corresponding models are based on continuum mechanics. The molecular theory, however, is called the microscopic theory, and the molecular model is built by statistical mechanics.

There are also some studies that can show these theories, under certain assumptions, are related to each other. See [34, 61, 89, 92, 94] as examples.

Active hydrodynamics describes the collective motion of active constituent particles, each particle is driven by internal energy source that drives the system out of equilibrium. Typical example including swarms of bacteria, vibrated granular rods, bird flocks and more [23, 83]. If the particles have elongated shapes, the collective motion would make them undergo orientational ordering at high concentration. Therefore, the active system can be referred to as liquid crystals, forming liquid crystalline phases. Active hydrodynamics are widely applicable in many fields. For examples, see [52, 78, 90] and the references within.

The PDEs perturbed randomly are considered as a primary tool in the modeling of uncertainty, they are especially useful while describing fundamental phenomenon in physics, climate dynamics, communication systems as well as gene regulation systems. In recent years, the study of the well-posedness and dynamical behaviour of PDEs perturbed by the noise, which is largely applied to the theoretical and practical areas, has drawn a lot of attention. Among all the results, the studies concerning the Navier-Stokes equations, or the dynamical system of fluid mechanics, has seen a drastic rise during all these years. For example, there are abundant results about the weak solution of compressible Navier-Stokes equations. At the beginning, [67, 68, 69] established the global existence of weak solution to compressible viscous and heat-conductive fluids with restriction on the initial data. Next, when adiabatic exponent $\gamma > \frac{9}{5}$, [59] gave the global existence of weak solution with large initial data and the appearance of vacuum by introducing the re-normalized solution to surmount the difficulty of large oscillations. Then, [32] extended the result to the case that adiabatic exponent $\gamma > \frac{3}{2}$, which by now is the result that allows the maximum range of $\gamma$. The existence results of the deterministic case had been extended to the stochastic case. In [30], authors obtained the existence of global weak pathwise solution to the equation forced by additive noise, where the special form of noise allows us to transform the stochastic system into the random equation, enabling the deterministic result to be exploited. As for the existence result of the equation driven by multiplicative noise, there are also some pioneering works, check [84] for global weak martingale solution with finite-dimensional Brownian motion, check [43, 81] for global weak martingale solution with cylindrical Wiener process, check [13] for stationary solution, check [11] for local strong pathwise solution, check [82] for weak martingale solution to non-isentropic, compressible Navier-Stokes equation, check [9] for weak martingale solution to non-isentropic, compressible Navier-Stokes-Fourier equation where energy balance equation is also forced by a random heat source.

There are also a lot of results for the $Q$-tensor framework. Regarding the incompressible $Q$-tensor liquid crystal model, Paicu-Zarnescu [74] have proved the existence of a global weak solution to a system describing the evolution of a nematic liquid crystal flow in both 2D and 3D, they also proved higher global regularity as well as the weak-strong uniqueness in two dimensions. Then, Paicu-Zarnescu continued his study and got the same results in [73] for the full system, in which the presence of a certain item would allow quadruply exponential increase of the high norms. De Anna in [2] propagated the result of [74] to the low regularity space $W^s$ for $0 < s < 1$ and proved the uniqueness of weak solutions in 2D, which filled the gap in [74]. Wilkinson [91] obtained the existence and the regularity property for weak solution in the general d-dimensional case in the presence of a singular potential. The existence of a global in time weak solution for system with thermal effects is proved in [33], where the natural physical constraints are enforced by a singular free energy bulk potential. The existence and uniqueness of global strong solution for the density-dependent system is established by Li-Wang in [56]. When it comes to the compressible model, there are fewer results due to its complexity. In [85], Wang-Xu-Yu established the existence as well as long time dynamics of global weak solutions. In fact, there are more results on the hydrodynamic system for the three-dimensional flow of nematic liquid crystals. For example, Jiang-Jiang-Wang [47] has proved the existence of a global weak solution to a two-dimensional simplified Ericksen-Leslie system of compressible flow of nematic liquid crystals, and the existence of a weak solution in a bounded domain for both 2D and 3D can be seen in [46] and [87]. For more studies related to the topic, check [22, 21] and the references within. For the stochastic liquid crystal hydrodynamics system, we refer the readers to [17, 16, 18, 88] for the well-posedness result of incompressible case,

The connection between the compressible flow and the incompressible flow has also been well studied. Lions-Masmoudi had justified the limit of the global weak solutions in [60], the convergence are proven to be global in time, and the result had no restrictions to the size of the initial conditions. Next, in [25], Desjardins-Grenier worked on the same problem, but they used a different approach. They had taken into account the presence of acoustic waves, and used Strichartz's estimates for the linear wave equation, the convergence result were thus improved. Later, Desjardins-Grenier-Lions studied the case of a viscous flow with the Dirichlet boundary in [24], showed that the acoustic waves can be damped due to the thin boundary layer, thus the strong convergence were obtained. The results were extended by others, see [1, 14, 15, 66, 79] for example. In particular, for the compressible magnetohydrodynamic equations, Hu-Wang studied the convergence of the weak solutions. In [44], they considered all the cases, including the periodic domain, the whole space and the bounded domain. Jiang-Ju-Li [48] also studied the convergence of the weak solutions to the compressible magnetohydrodynamic equations in the periodic case, they proved that the weak solutions would converge to the strong solution of the viscous incompressible magnetohydrodynamic equations or the inviscid one, given that the strong solution exists. Later, the incompressible limit of the nematic liquid crystal model was studied in Wang-Yu [87] where the authors justified the weak convergence in a bounded domain. For more related results, see [49, 42, 57, 31, 54, 65, 28, 26, 93].

In Chapter 2, we are devoted to establishing the existence of global weak martingale solution to the active hydradynamics system, that is system (2.0.1)-(2.0.3). It's worth noting that in the stochastic case, unlike the deterministic case, there is no compactness in random element $\omega$ since sample space has no topology structure. As a result, the usual compactness criteria, such as the Aubin or Arzelà-Ascoli type theorems, can not be applied directly. A common method to overcome this difficulty is to invoke the Skorokhod theorem. Using the Skorokhod argument, we can obtain that there exists a sequence of new random variables on a new probability space that converges almost surely to a certain random variable, and its distribution is same as the original one, consequently, the new random variables also satisfy the system on the new probability space. Our proof mainly relies on the four level approximation developed by [32] and [43] which also consists of the Galerkin approximation, the artificial viscosity and the artificial pressure. Each level approximation contains the argument of compactness and identify the limit. Here, we point out that the boundedness of concentration $c$ acquired via the maximum principle and the special construction of

$Q$-tensor(symmetric and traceless) play a key role in obtaining the a priori estimates (cancel certain high-order nonlinear term) and establishing the weak continuity of the effective viscous flow. Without this remarkable property of $Q$-tensor, we are not able to handle the higher order nonlinear term $\nabla\cdot(Q\triangle Q - \triangle QQ)$. In addition, the coupled constitution of four equations makes the analysis much more complicated and more delicate arguments including identifying the stochastic integral and showing the tightness of probability measures set are necessary. The results in this chapter has been published in [77].

In Chapter 3, we are going to prove the existence and uniqueness of strong pathwise solution to the stochastic system (3.0.2), where the "strong" means the strong existence in both PDE and probability sense. That is, the solution has sufficient space regularity and satisfies the system in the pointwise sense when the probability space is given. It's worth noting that even if in the deterministic case, there is no related result for the existence and uniqueness result of strong solution for which the state space lies in $(H^s)^2 \times H^{s+1}$ for integer $s > \frac{9}{2}$. We would introduce the symmetric system considering the energy estimate of the strong solution to compressible fluid. Therefore, for the convenience of the symmetrization, we require the density $\rho > 0$, which means the vacuum state shall not appear. The main difficulty in obtaining the strong solution is the high-order energy estimate of the approximate solution. Therefore, when we apply Moser-type estimate, we could get the form $(\|\mathbf{u}\|_{2,\infty} + \|Q\|_{3,\infty}) \cdot (\|\mathbf{u}\|_{s,2}^2 + \|Q\|_{s+1,2}^2)$ and $(\|\mathbf{u}\|_{2,\infty} + \|\rho\|_{1,\infty}) \cdot \|\rho, \mathbf{u}\|_{s,2}^2$, making it difficult to get the estimate. Inspired by [53], we could deal with the nonlinear terms by adding a cut-off function. We could get that $\|\rho\|_{1,\infty}$ would be bounded if $\|\rho_0\|_{1,\infty}, \|\mathbf{u}\|_{1,\infty}$ are bounded, then the cut-off function only depends on $\|\mathbf{u}\|_{2,\infty}$ and $\|Q\|_{3,\infty}$ under the assumption that $\|\rho_0\|_{1,\infty}$ is bounded. The benefit is, while building Galerkin approximation system, for every fixed $\mathbf{u}$ we could first solve the mass equation directly which actually is a linear transport equation and solve the "parabolic-type" $Q$-tensor equation. In turn, we obtain the existence of approximate solution $\mathbf{u}$ in a finite dimensional space. Different from the deterministic case, we will develop a new extra layer approximation to deal with the difficulty arising from the stochastic integral, constructing the Galerkin approximate solution with the spirit of [43]. Also, the cut-off function brings downside in proving the uniqueness. We have to restrict our regularity index to integer $s > \frac{9}{2}$ comparing with the martingale solution result which only requires $s > \frac{7}{2}$. The results in this chapter has been published in [76].

In Chapter 4, we just consider the convergence in periodic case, and establish the incompressible limit of (4.0.2). The main difficulty lies in the possible oscillation of the density, the coupling effect between the $Q$-tensor and the motion, and the $Q$-tensor should be considered with new estimates. We overcome all the difficulties by using the weak convergence method, then establish the new compactness criteria using the estimates. More precisely, when the density goes to a constant, also when $\varepsilon$ goes to zero, we will apply the Helmholtz decomposition and prove that the divergence-free part of the velocity converges strongly to a divergence-free vector, the curl-free part will converge weakly to zero at the same time. Next, we consider the case when the equation is driven by a stochastic force. Our approach in this part mainly relies on the method of finite energy weak martingale solution, and it is motivated by [10]. Note that the existence of the weak martingale solution was obtained in [77]. To get the uniform estimate, we follow the basic properties of Itô's formula. Compared with the deterministic case, the classical compactness criteria is not applicable when we bring in the stochastic argument, since we do not have any compactness property posed on the sample space $\Omega$. As a result, we are motivated by the Yamada-Watanabe argument and apply the Jakubowski-Skorokhod representation theorem, a modification of the Skorokhod representation theorem to pass to a weakly convergent subsequence. By using this theorem, we can get the existence of a new probability space, together with a sequence of random variables that has the same law as the original variable.

In summary, the rest of the dissertation is organized as follows. In Chapter 2, we established the existence of global weak martingale solution to the compressible active liquid crystal system. The proof is mainly relied on four level approximations, including the Galerkin approximation, the artificial viscosity and the artificial pressure. In Chapter 3, we first established the existence of global strong martingale solution and strong pathwise solution to the truncated symmetric system, then we built a series of local strong pathwise solution. After generalizing the initial data, we proved that the solutions has a maximal stopping time. In Chapter 4, we prove the convergence of the incompressible limit and verify the limit in the deterministic case, then we present and prove similar results in the stochastic case, where we note that the proof of compactness and the verification of the limit is more complex than the deterministic case.

## 2.0 Weak Martingale Solution to Active Liquid Crystals

In this chapter, we consider the following hydrodynamic partial differential equations that model compressible active nematic systems:

$$\begin{cases} \partial_t c + (u\cdot\nabla)c = \triangle c, \\ \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla p = \mu_1\Delta u + (\mu_1+\mu_2)\nabla(\mathrm{div} u) + \sigma^*\nabla\cdot(c^2 Q) \\ \quad +\nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q) + \nabla\cdot(Q\triangle Q - \triangle Q Q) + \rho f(\rho,\rho u, c, Q)\frac{d\mathcal{W}}{dt}, \\ \partial_t Q + (u\cdot\nabla)Q + Q\Psi - \Psi Q = \Gamma \mathrm{H}(Q,c), \end{cases} \tag{2.0.1}$$

where $c, \rho, u$ denote the concentration of active particles, the density, and the flow velocity, $p(\rho) = \rho^\gamma$ stands for the pressure with the adiabatic exponent $\gamma > \frac{3}{2}$, the nematic tensor order parameter $Q$ is a traceless and $3\times 3$ symmetric matrix, $\mathrm{I}_3$ is the $3\times 3$ identity matrix, $\mu_1, \mu_2$ are the viscosity coefficients satisfying the physical assumptions $\mu_1 \geq 0$ and $2\mu_1 + 3\mu_2 \geq 0$, $\Gamma^{-1} > 0$ is the rotational viscosity, $\sigma^* \in \mathbb{R}$ is the stress generated by the active particles along the director field. $\Psi = \frac{1}{2}(\nabla u - \nabla u^\perp)$ is the skew-symmetric part of the rate of strain tensor. $\mathcal{W}$ is a cylindrical Wiener process which will be introduced later. Furthermore,

$$\mathrm{F}(Q) = \frac{1}{2}|\nabla Q|^2 + \frac{1}{2}\mathrm{tr}(Q^2) + \frac{c_*}{4}\mathrm{tr}^2(Q^2),$$

and

$$\mathrm{H}(Q,c) = \triangle Q - \frac{c-c_*}{2}Q + b\left(Q^2 - \frac{\mathrm{tr(Q^2)}}{3}\mathrm{I}_3\right) - c_* Q\mathrm{tr}(Q^2),$$

where the constant $c_*$ is the critical concentration for the isotropic-nematic transition and $b$ is material-dependent constant. And the term $\nabla Q\odot\nabla Q$ stands for a $3\times 3$ matrix, its $(i,j)$-th entry is defined $(\nabla Q\odot\nabla Q)_{ij} = \sum_{k,l=1}^3 \partial_i Q_{kl}\partial_j Q_{kl}$.

The system is supplied with the following initial data,

$$\rho(0,x) = \rho_0(x),\ \rho u(0,x) = m_0(x),\ c(0,x) = c_0(x),\ Q(0,x) = Q_0(x), \tag{2.0.2}$$

and the boundary conditions,

$$\left.\frac{\partial c}{\partial n}\right|_{\partial\mathcal{D}} = \left.\frac{\partial Q}{\partial n}\right|_{\partial\mathcal{D}} = 0, \ u|_{\partial\mathcal{D}} = 0, \tag{2.0.3}$$

here we omit the random element $\omega$.

### 2.1 Preliminaries and Main Result

In this section, we begin by reviewing some deterministic and stochastic preliminaries associated with system (2.0.1)-(2.0.3), followed by main result.

Define the inner product between two $3 \times 3$ matrices $A$ and $B$

$$(A, B) = \int_{\mathcal{D}} A : B dx = \int_{\mathcal{D}} \mathrm{tr}(AB) dx,$$

and $S_0^3 \subset \mathbb{M}^{3\times 3}$ the space of $Q$-tensor

$$S_0^3 = \left\{ Q \in \mathbb{M}^{3\times 3} : \ Q_{ij} = Q_{ji}, \ \mathrm{tr}(Q) = 0, \ i, j = 1, 2, 3 \right\},$$

and the norm of a matrix using the Frobenius norm

$$|Q|^2 = \mathrm{tr}(Q^2) = Q_{ij}Q_{ij}.$$

The Sobolev space of $Q$-tensor is defined by

$$H^1(\mathcal{D}; S_0^3) = \left\{ Q : \mathcal{D} \to S_0^3, \text{ and } \int_{\mathcal{D}} |\nabla Q|^2 + |Q|^2 dx < \infty \right\}.$$

Set $|\nabla Q|^2 = \partial_k Q_{ij} \partial_k Q_{ij}$ and $|\triangle Q|^2 = \triangle Q_{ij} \triangle Q_{ij}$.

The space $C_w([0, T]; X)$ consists of all weakly continuous functions $u : [0, T] \to X$ and $u_n \to u$ in $C_w([0, T]; X)$ if and only if $\langle u_n(t), \phi\rangle \to \langle u(t), \phi\rangle$ uniformly in $t$, $\phi \in X^*$, where $X^*$ is the dual space of $X$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ be a fixed stochastic basis and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. $\mathcal{W}$ is a cylindrical Wiener process defined on the Hilbert space $\mathcal{H}$, which is adapted to the complete, right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Namely, $\mathcal{W} = \sum_{k\geq 1} e_k \beta_k$ with

$\{e_k\}_{k\geq 1}$ being the complete orthonormal basis of $\mathcal{H}$ and $\{\beta_k\}_{k\geq 1}$ being a sequence of independent standard one-dimensional Brownian motions. In addition, $L_2(\mathcal{H}, X)$ denotes the collection of Hilbert-Schmidt operators, the set of all linear operators $G$ from $\mathcal{H}$ to $X$, with the norm $\|G\|^2_{L_2(\mathcal{H},X)} = \sum_{k\geq 1} \|Ge_k\|^2_X$.

Consider an auxiliary space $\mathcal{H}_0 \supset \mathcal{H}$, define by

$$\mathcal{H}_0 = \left\{ h = \sum_{k\geq 1} \alpha_k e_k : \sum_{k\geq 1} \alpha_k^2 k^{-2} < \infty \right\},$$

with the norm $\|h\|^2_{\mathcal{H}_0} = \sum_{k\geq 1} \alpha_k^2 k^{-2}$. Observe that the mapping $\Phi : \mathcal{H} \to \mathcal{H}_0$ is Hilbert-Schmidt. We also have that $\mathcal{W} \in C([0,\infty), \mathcal{H}_0)$ almost surely, see [75].

For an $X$-valued predictable process $f \in L^2(\Omega; L^2_{loc}([0,\infty), L_2(\mathcal{H}, X)))$ by taking $f_k = fe_k$, the Burkholder-Davis-Gundy inequality holds

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t f d\mathcal{W} \right\|_X^p \right] &\leq c_p \mathbb{E}\left( \int_0^T \|f\|^2_{L_2(\mathcal{H},X)} dt \right)^{\frac{p}{2}} \\
&= c_p \mathbb{E}\left( \int_0^T \sum_{k\geq 1} \|f_k\|^2_X dt \right)^{\frac{p}{2}},
\end{aligned}$$

for any $1 \leq p < \infty$.

Next, we define the global weak martingale solution of system (2.0.1)-(2.0.3).

**Definition 2.1.1.** Let $\mathcal{P}$ be a Borel probability measure on $L^\gamma(\mathcal{D}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{D}) \times (H^1(\mathcal{D}))^2$ with $\gamma > \frac{3}{2}$. $\{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}), \rho, u, c, Q, \mathcal{W}\}$ is a global weak martingale solution to system (2.0.1)-(2.0.3) if the following conditions hold:

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a stochastic basis and $\mathcal{W}$ is an $\mathcal{F}_t$ cylindrical Wiener process,

(ii) the processes $\rho \in C_w([0,T]; L^\gamma(\mathcal{D})), \rho u \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathcal{D}))$,
$c \in C_w([0,T]; L^2(\mathcal{D})), Q \in C_w([0,T]; H^1(\mathcal{D}))$ are $\mathcal{F}_t$ progressively measurable, satisfying

$$\begin{aligned}
&\rho \in L^p(\Omega; L^\infty(0,T; L^\gamma(\mathcal{D}))),\\
&\rho u \in L^p(\Omega; L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\mathcal{D}))), \sqrt{\rho}u \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{D}))),\\
&c \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{D})) \cap L^2(0,T; H^1(\mathcal{D}))),\\
&Q \in L^p(\Omega; L^\infty(0,T; H^1(\mathcal{D})) \cap L^2(0,T; H^2(\mathcal{D}))),
\end{aligned}$$

for any $1 \le p < \infty, 0 < T < \infty$,

(iii) the velocity $u$ is a random distribution adapted to $\mathcal{F}_t$, for the definition see [12, Definition 2.2.13], satisfying

$$u \in L^p(\Omega; L^2(0,T; H^1(\mathcal{D}))),$$

for any $1 \le p < \infty, 0 < T < \infty$,

(iv) $\mathcal{P} = \mathbb{P} \circ (\rho_0, m_0, c_0, Q_0)^{-1}$,

(v) for $\ell \in C^\infty(\mathcal{D}), \phi \in C^\infty(\mathcal{D}), \varphi \in C^\infty(\mathcal{D}), \psi \in C^\infty(\mathcal{D})$ and $t \in [0,T]$, $\mathbb{P}$ a.s.

$$\begin{aligned}
&\int_{\mathcal{D}} c(t)\ell dx = \int_{\mathcal{D}} c(0)\ell dx - \int_0^t \int_{\mathcal{D}} (u \cdot \nabla) c \cdot \ell dx ds - \int_0^t \int_{\mathcal{D}} \nabla c \cdot \nabla \ell dx ds, \\
&\int_{\mathcal{D}} \rho(t)\psi dx = \int_{\mathcal{D}} \rho(0)\psi dx + \int_0^t \int_{\mathcal{D}} \rho u \cdot \nabla \psi dx ds, \\
&\int_{\mathcal{D}} \rho u(t)\phi dx = \int_{\mathcal{D}} m(0)\phi dx + \int_0^t \int_{\mathcal{D}} \rho u \otimes u \cdot \nabla \phi dx ds - \mu_1 \int_0^t \int_{\mathcal{D}} \nabla u \cdot \nabla \phi dx ds \\
&\qquad - (\mu_1 + \mu_2) \int_0^t \int_{\mathcal{D}} \mathrm{div} u \cdot \mathrm{div} \phi dx ds + \int_0^t \int_{\mathcal{D}} \rho^\gamma \cdot \mathrm{div} \phi dx ds \\
&\qquad - \int_0^t \int_{\mathcal{D}} ((\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q) + (Q \triangle Q - \triangle Q Q) + \sigma^* c^2 Q) \cdot \nabla \phi dx ds \\
&\qquad + \int_0^t \int_{\mathcal{D}} \rho f(\rho, \rho u, c, Q) \phi dx d\mathcal{W}, \\
&\int_{\mathcal{D}} Q(t)\varphi dx = \int_{\mathcal{D}} Q(0)\varphi dx - \int_0^t \int_{\mathcal{D}} \varphi (u \cdot \nabla) Q + \varphi Q \Psi - \varphi \Psi Q dx ds \\
&\qquad\quad + \int_0^t \int_{\mathcal{D}} \Gamma \varphi \mathrm{H}(Q, c) dx ds,
\end{aligned}$$

(vi) for all $\psi \in C^\infty(\mathcal{D})$ and $t \in [0,T]$, $\rho$ satisfies the following re-normalized equation

$$\begin{aligned}
\int_{\mathcal{D}} b(\rho)\psi dx &= \int_{\mathcal{D}} b(\rho(0))\psi dx + \int_0^t \int_{\mathcal{D}} b(\rho) u \cdot \nabla \psi dx ds \\
&\quad + \int_0^t \int_{\mathcal{D}} (b'(\rho)\rho - b(\rho)) \mathrm{div} u \cdot \psi dx ds,
\end{aligned}$$

where the function $b \in C^1(\mathbb{R})$ satisfies $b'(z) = 0$ for all $z \in \mathbb{R}$ large enough.

Throughout the paper, we assume that the operator $f$ satisfies the following conditions: there exists a constant $C$ such that

$$\sum_{k\geq 1}|f(\rho,\rho u,c,Q)e_k|^2\leq C\left(|\rho|^{\gamma-1}+|c,\nabla Q|^{\frac{2(\gamma-1)}{\gamma}}+|u|^2\right), \tag{2.1.1}$$

and

$$\begin{aligned}&\sum_{k\geq 1}|(\rho_1 f(\rho_1,\rho_1 u_1,c_1,Q_1)-\rho_2 f(\rho_2,\rho_2 u_2,c_2,Q_2))e_k|^2\\ &\leq C|\rho_1-\rho_2,\rho_1 u_1-\rho_2 u_2,c_1-c_2,Q_1-Q_2|^{\frac{\gamma+1}{2\gamma}},\end{aligned} \tag{2.1.2}$$

where $|u,v|:=|u|+|v|$ and $|\cdot|$ stands for the absolute value. Condition (2.1.1) will be used for obtaining the a priori estimate, while Condition (2.1.2) will be applied to identify the limit.

In addition, we assume that initial data satisfy the following conditions for all $1\leq p<\infty$

$$\rho_0\in L^p(\Omega;L^\gamma(\mathcal{D})),\ \rho_0\geq 0 \text{ and } m_0=0 \text{ if } \rho_0=0, \tag{2.1.3}$$

$$\frac{|m_0|^2}{\rho_0}\in L^p(\Omega;L^1(\mathcal{D})), \tag{2.1.4}$$

$$c_0\in L^p(\Omega;H^1(\mathcal{D})) \text{ and } 0<\underline{c}\leq c_0\leq\bar{c}<\infty, \tag{2.1.5}$$

$$Q_0\in L^p(\Omega;H^1(\mathcal{D};S_0^3)), \tag{2.1.6}$$

where the lower and upper bounds $\underline{c},\overline{c}$ are two fixed constants.

Now, we state the main result.

**Theorem 2.1.2.** *Let $\gamma>\frac{3}{2}$. Suppose that the initial data $(\rho_0,m_0,c_0,Q_0)$ satisfy the assumptions (2.1.3)-(2.1.6), and the operator $f$ satisfies the conditions (2.1.1), (2.1.2). Then, there exists a global martingale weak solution to system (2.0.1)-(2.0.3) in the sense of Definition 2.1.1.*

The proof of Theorem 2.1.2 mainly relies on the four-level approximation. The first level approximation actually contains two-step approximation. Because of the difficulty technically, we need to cut-off the approximate solution such that in a sense the bound is uniform in random element corresponding to the truncated parameter $K$ inspired by [43]. Then, for any fixed $n, K$, the existence and uniqueness of the Galerkin approximate solution in small time is established using the Banach fixed point argument. The uniform bound of $c$ obtained via maximum principle and the property of symmetric and traceless of Q-tensor can be used for cancelling the high order nonlinear term, allowing us to get the uniform a priori estimates, see Lemma 2.2.3. With the a priori estimates established, we can extend the existence time to global. Next, we let $K \to \infty$ to establish the Galerkin approximate solution for fixed $n$. Define a stopping time $\tau_K$, on interval $[0, \tau_K)$, we can define the approximate solution using the uniqueness of solution and the monotonicity of the sequence of stopping time $\tau_K$. Using the a priori estimates, it holds $\lim_{K\to\infty} \tau_K = T$, $\mathbb{P}$ a.s. This means no blow-up arises in finite time. Except for this level approximation, all other three level approximations contain the compactness argument. Owing to the complex structure of the system, we have to work with weak compactness, the classical Skorokhod theorem is replaced by the Skorokhod-Jakubowski theorem applying to quasi-Polish space. To employ the theorem, it is necessary to show the tightness of the set of probability measures generated by the distribution of approximate solution, which can be achieved using the Aubin-Lions Lemma, the a priori estimates and the delicate analysis. After obtaining the compactness of the new processes on the new probability space, we identify all the nonlinear term by passing limit as $n \to \infty$.

In the second level approximation, the solution obtained in first level approximation will be used as approximate solution. Here the boundedness of $\sqrt{\epsilon}\rho_{\epsilon,\delta} \in L^p(\Omega; L^2(0,T; H^1(\mathcal{D})))$ is not helpful in getting the strong convergence of density, which makes it difficult to identify the nonlinear term with respect to $\rho$ (the pressure term and stochastic term). Following the ideas from [32, 59], we shall show the strong convergence of density by improving the integrability of density, establishing the weak continuity of the effective viscous flow (here the symmetric of $Q$-tensor plays a crucial role, see Step 1 in Section 4) and using the Minty trick.

Following the same line as the second level approximation, improving the integrability of density, establishing the weak continuity of the effective viscous flow, introducing the renormalized solution to control the large oscillations and the truncated technique are required to get the strong convergence of density. Here, the proof is standard, we only give the necessary tightness argument and improve the integrability of density, for further details, we refer the reader to [43, 81, 84].

## 2.2 The Existence of Martingale Solution When n Tends to Infinity

In this section, we are devoted to building the existence of global weak martingale solution to the following modified viscous system

$$\begin{cases} \partial_t c + (u\cdot\nabla)c = \triangle c, \\ \partial_t\rho + \mathrm{div}(\rho u) = \epsilon\triangle\rho, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla(\rho^\gamma + \delta\rho^\beta) + \epsilon\nabla\rho\cdot\nabla u = \mu_1\Delta u + (\mu_1+\mu_2)\nabla(\mathrm{div} u) \\ \qquad\qquad +\sigma^*\nabla\cdot(c^2Q) + \nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q) + \nabla\cdot(Q\triangle Q - \triangle QQ) \\ \qquad\qquad +\rho f(\rho,\rho u,c,Q)\frac{d\mathcal{W}}{dt}, \\ \partial_t Q + (u\cdot\nabla)Q + Q\Psi - \Psi Q = \Gamma\mathrm{H}(Q,c), \end{cases} \tag{2.2.1}$$

where $\epsilon,\delta>0$ and $\beta>\{6,\gamma\}$, with the boundary conditions

$$\left.\frac{\partial\rho}{\partial n}\right|_{\partial\mathcal{D}} = \left.\frac{\partial c}{\partial n}\right|_{\partial\mathcal{D}} = \left.\frac{\partial Q}{\partial n}\right|_{\partial\mathcal{D}} = 0,\ \ u|_{\partial\mathcal{D}} = 0, \tag{2.2.2}$$

and the modified initial data

$$\rho(0) = \rho_{0,\delta} \in L^p(\Omega; C^{2+\alpha}(\mathcal{D})), \tag{2.2.3}$$

$$\rho u(0) = m_{0,\delta} \in L^p(\Omega; C^2(\mathcal{D})), \tag{2.2.4}$$

$$Q(0) = Q_0 \in L^p(\Omega; H^1(\mathcal{D}; S_0^3)), \tag{2.2.5}$$

$$c(0) = c_0 \in L^p(\Omega; H^1(\mathcal{D})) \text{ and } 0 < \underline{c} \le c_0 \le \bar{c} < \infty. \tag{2.2.6}$$

Moreover, assume that the initial data $\rho_{0,\delta}$ satisfies the following conditions

$$0<\delta\le\rho_{0,\delta}\le\delta^{-\frac{1}{\beta}}<\infty,\quad (\rho_{0,\delta})_m\le M,\quad \rho_{0,\delta}\to\rho_0 \text{ in } L^\gamma(\mathcal{D}) \text{ as } \delta\to 0, \tag{2.2.7}$$

where the $(\rho_{0,\delta})_m$ denotes the mean value of $\rho_{0,\delta}$ in domain $\mathcal{D}$, and

$$m_{0,\delta}=h_\delta\sqrt{\rho_{0,\delta}},$$

$h_\delta$ is defined as follows. Let

$$\tilde{m}_{0,\delta}=\begin{cases} m_0\sqrt{\frac{\rho_{0,\delta}}{\rho_0}}, & \text{if} \quad \rho_0>0,\\ 0, \quad \text{if} \quad \rho_0=0. \end{cases}$$

According to the assumption (2.1.4), we have $\frac{|\tilde{m}_{0,\delta}|^2}{\rho_{0,\delta}}\in L^p(\Omega;L^1(\mathcal{D}))$ uniformly in $\delta$ for $1\le p<\infty$. Therefore, we can find $C^2(\overline{\mathcal{D}})$-valued random variables $h_\delta$ such that

$$\left\|\frac{\tilde{m}_{0,\delta}}{\sqrt{\rho_{0,\delta}}}-h_\delta\right\|_{L^2(\mathcal{D})}\le\delta.$$

Then, we have

$$\begin{aligned} &\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}}\in L^p(\Omega;L^1(\mathcal{D})) \text{ uniformly in } \delta,\\ &\frac{m_{0,\delta}}{\sqrt{\rho_{0,\delta}}}\to\frac{m_0}{\sqrt{\rho_0}} \text{ in } L^p(\Omega;L^2(\mathcal{D})) \text{ for } 1\le p<\infty. \end{aligned}$$

Let $\mathcal{P}_\delta=\mathbb{P}\circ(\rho_{0,\delta},m_{0,\delta},c_0,Q_0)^{-1}$. According to the construction, we have $\mathcal{P}_\delta$ is Borel probability measure on $C^{2+\alpha}(\mathcal{D})\times C^2(\mathcal{D})\times(H^1(\mathcal{D}))^2$, satisfying

$$\begin{aligned} &\mathcal{P}_\delta\big\{(\rho,\rho u,c,Q)\in C^{2+\alpha}(\mathcal{D})\times C^2(\mathcal{D})\times(H^1(\mathcal{D}))^2;\\ &\quad \text{and } 0<\delta\le\rho\le\delta^{-\frac{1}{\beta}}<\infty,\ (\rho)_m\le M,\ 0<\underline{c}\le c_0\le\bar{c}<\infty,\ Q\in S_0^3\big\}=1, \end{aligned}$$

and

$$\begin{aligned} &\int_{C^{2+\alpha}(\mathcal{D})\times C^2(\mathcal{D})\times(H^1(\mathcal{D}))^2}\left\|\frac{|\rho u|^2}{2\rho}+\frac{1}{\gamma-1}\rho^\gamma+|\nabla c,\nabla Q|^2\right\|_{L^1(\mathcal{D})}^p d\mathcal{P}_\delta\\ &\to\int_{L^\gamma(\mathcal{D})\times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{D})\times(H^1(\mathcal{D}))^2}\left\|\frac{|\rho u|^2}{2\rho}+\frac{1}{\gamma-1}\rho^\gamma+|\nabla c,\nabla Q|^2\right\|_{L^1(\mathcal{D})}^p d\mathcal{P}, \end{aligned}\tag{2.2.8}$$

where $\mathcal{P}$ is a Borel probability measure on $L^\gamma(\mathcal{D})\times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{D})\times(H^1(\mathcal{D}))^2$.

The proof will be divided into three subsections. For the first subsection, we establish the Galerkin approximate solution and the a priori estimates. Then, the compactness result is obtained in second subsection. In the third subsection, we get the existence of global weak martingale solution by taking the limit as $n\to\infty$.

### 2.2.1 The approximate solution and a priori estimates

First of all, we build the approximate solution to system (2.2.1)-(2.2.6) for fixed $\varepsilon > 0$ and $\delta > 0$, we would need an extra approximation layer compared to the deterministic case following the ideas of [43]. At the beginning, we introduce the following well-posedness results taken from [22, 62].

**Lemma 2.2.1.** *Suppose that the initial data $\rho_0$ satisfies (2.2.3). If $u \in C([0,T];C^2(\mathcal{D}))$ with $u|_{\partial\mathcal{D}} = 0$, then there exists a mapping $\mathcal{S} = \mathcal{S}(u)$*

$$\mathcal{S} : C([0,T];C^2(\overline{\mathcal{D}})) \to C([0,T];C^{2+\alpha}(\overline{\mathcal{D}})),$$

*with the following properties:*

*(1) $\rho = \mathcal{S}(u)$ is a unique classical solution of system $(2.2.1)_2, (2.2.3), (2.2.7)$ with the mapping $\mathcal{S}$ continuous on bounded subset of $C([0,T];C^2(\mathcal{D}))$.*

*(2) It holds*

$$\delta \exp\left(-\int_0^t \|\mathrm{div} u\|_{L^\infty} dr\right) \leq \rho \leq \delta^{-\frac{1}{\beta}} \exp\left(\int_0^t \|\mathrm{div} u\|_{L^\infty} dr\right),$$

*for all $t \in [0,T]$.*

**Lemma 2.2.2.** *For each $u \in C([0,T];C^2(\mathcal{D}))$ with $u|_{\partial\mathcal{D}} = 0$, then there exists a unique strong solution $(c,Q) \in [L^\infty(0,T;H^1(\mathcal{D})) \cap L^2(0,T;H^2(\mathcal{D}))]^2$ to the system*

$$\begin{cases} \partial_t c + (u\cdot\nabla)c = \triangle c, \\ \partial_t Q + (u\cdot\nabla)Q + Q\Psi - \Psi Q = \Gamma \mathrm{H}(Q,c), \\ Q(0) = Q_0 \in H^1(\mathcal{D}; S_0^3) \ a.e. \text{ and } \left.\frac{\partial Q}{\partial n}\right|_{\partial\mathcal{D}} = 0, \\ c(0) = c_0 \in H^1(\mathcal{D}) \text{ and } 0 < \underline{c} \leq c_0 \leq \bar{c} < \infty,\ \left.\frac{\partial c}{\partial n}\right|_{\partial\mathcal{D}} = 0. \end{cases} \tag{2.2.9}$$

*Moreover, we have $0 < \underline{c} \leq c \leq \bar{c} < \infty$. Furthermore, the mapping*

$$\widetilde{\mathcal{S}} : C([0,T];C^2(\mathcal{D})) \to \left[L^\infty(0,T;H^1(\mathcal{D})) \cap L^2(0,T;H^2(\mathcal{D}))\right]^2,$$

*is continuous on set $B_R := \left\{u \in C([0,T];C^2(\mathcal{D})); \|u\|_{C([0,T];C^2(\mathcal{D}))} \leq R\right\}$ and $Q \in S_0^3$ a.e.*

**Remark**. The proof of $Q \in S_0^3$ a.e. depends on the uniqueness of solution to system (2.2.9), therefore, we have to lift the regularity of initial data $c$ to $H^1(\mathcal{D})$ rather than $L^2(\mathcal{D})$ in establishing the existence of strong solutions, making further effort to achieve the uniqueness. It will also be used for defining the Galerkin approximate solution for fixed truncation parameter $K$ introduced later.

With these results in hand, we now find the approximate velocity field $u_n$ satisfying the integral equation

$$\begin{aligned}
&\int_{\mathcal{D}} \rho u_n \phi dx - \int_{\mathcal{D}} m_0 \phi dx \\
= &- \int_0^t \int_{\mathcal{D}} (\mathrm{div}(\rho u_n \otimes u_n) - \mu_1 \triangle u_n \\
&\qquad -(\mu_1 + \mu_2)\nabla(\mathrm{div} u_n) + \nabla \rho^\gamma + \delta \nabla \rho^\beta)\phi dxds \\
&+ \int_0^t \int_{\mathcal{D}} \sigma^* \nabla \cdot (c^2 Q)\phi dxds - \int_0^t \int_{\mathcal{D}} \epsilon \phi \nabla \rho \cdot \nabla u_n dxds \\
&+ \int_0^t \int_{\mathcal{D}} \nabla \cdot (\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q)\phi + \nabla \cdot (Q \triangle Q - \triangle Q Q)\phi dxds \\
&+ \int_0^t \int_{\mathcal{D}} \sqrt{\rho} P_n(\sqrt{\rho} f(\rho, \rho u_n, c, Q))\phi dx d\mathcal{W},
\end{aligned} \tag{2.2.10}$$

for $t \in [0, T]$, and $\phi$ belongs to the finite dimensional space $X_n$, which is defined by $\mathrm{span}\{h_i\}_{i=1}^n$. The family of smooth functions $\{h_i\}_{i=1}^n$ is an orthonormal basis of $H^1(\mathcal{D})$, and $P_n$ be the orthogonal projection from $L^2(\mathcal{D})$ into $X_n$.

Define by the operator $\mathcal{M}[\rho] : X_n \to X_n^*$

$$\langle \mathcal{M}[\rho]u, v \rangle = \int_{\mathcal{D}} \rho u \cdot v dx \text{ for } u, v \in X_n.$$

From the definition, we know $\mathcal{M}[\rho]u = P_n(\rho u)$. Moreover, $\mathcal{M}[\rho]$ is a positive symmetric operator with following properties,

$$\|\mathcal{M}[\rho]^{-1}\|_{\mathcal{L}(X_n^*, X_n)} \le \|\rho^{-1}\|_{C(\mathcal{D})}, \tag{2.2.11}$$

and

$$\|\mathcal{M}[\rho_1]^{-1} - \mathcal{M}[\rho_2]^{-1}\|_{\mathcal{L}(X_n^*, X_n)} \le \|\rho_1^{-1}\|_{C(\mathcal{D})} \|\rho_2^{-1}\|_{C(\mathcal{D})} \|\rho_1 - \rho_2\|_{L^1(\mathcal{D})}. \tag{2.2.12}$$

Introduce the functional $\mathcal{N}[\rho,u,c,Q](\varphi)$ by

$$\begin{aligned}
&\mathcal{N}[\rho,u,c,Q](\varphi)\\
=&\int_{\mathcal{D}}(-\mathrm{div}(\rho u\otimes u)-\nabla(\rho^{\gamma}+\delta\rho^{\beta})+\mu_1\Delta u+(\mu_1+\mu_2)\nabla(\mathrm{div}u)-\epsilon\nabla\rho\cdot\nabla u\\
&\quad+\nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3-\nabla Q\odot\nabla Q)+\nabla\cdot(Q\triangle Q-\triangle QQ)+\sigma^*\nabla\cdot(c^2Q))\cdot\varphi dx,
\end{aligned}$$

for all $\varphi\in X_n$. Due to the technical difficulty, we need further truncation to $u_n$. Following the idea of [43], define the $C^\infty$-smooth cut-off function

$$\xi_K(z)=\begin{cases}1, & |z|\leq K,\\ 0, & |z|>2K.\end{cases}$$

Let $u^K=\sum_{i=1}^n\xi_K(\alpha_i)\alpha_ih_i$, then we have $\|u^K\|_{C([0,T];C^2(\mathcal{D}))}\leq 2K$ and the truncation operator $\mathrm{Tr}:u\to u^K$ satisfies

$$\mathrm{Tr}:X_n\to X_n,\quad \text{and}\ \ \|\mathrm{Tr}(u)-\mathrm{Tr}(v)\|_{X_n}\leq C(n)\|u-v\|_{X_n}. \tag{2.2.13}$$

Then, we rewrite (2.2.10) as

$$\begin{aligned}
u_n(t)=&\mathcal{M}^{-1}\left[\mathcal{S}(u_n^K)\right]\Bigg(m_0^*+\int_0^t\mathcal{N}[\mathcal{S}(u_n^K),u_n^K,\widetilde{\mathcal{S}}(u_n^K)]ds\\
&+\mathrm{Tr}\left(\int_0^t\mathcal{M}^{\frac12}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K)))d\mathcal{W}\right)\Bigg).
\end{aligned}\tag{2.2.14}$$

Here, the stochastic integral should be understood evolving on space $X_n^*$.

The mapping $\mathcal{Y}$ from $L^2(\Omega;C([0,T];C^2(\mathcal{D})))$ into itself is defined by the right hand side of (2.2.14). For fixed $n,K$, we can show the mapping $\mathcal{Y}$ is the contraction for $T^*$ small enough, for further details see [85, 21] for the deterministic part. Next, we give the estimate of stochastic term. Using (2.2.13) and the triangle inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T^*}\Bigg\|&\mathcal{M}^{-1}(\mathcal{S}(u_n^K))\\
&\times\mathrm{Tr}\left(\int_0^t\mathcal{M}^{\frac12}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K)))d\mathcal{W}\right)\\
&-\mathcal{M}^{-1}(\mathcal{S}(v_n^K))\\
&\times\mathrm{Tr}\left(\int_0^t\mathcal{M}^{\frac12}(\mathcal{S}(v_n^K))P_n(\sqrt{\mathcal{S}(v_n^K)}f(\mathcal{S}(v_n^K),\mathcal{S}(v_n^K)v_n^K,\widetilde{\mathcal{S}}(v_n^K)))d\mathcal{W}\right)\Bigg\|_{X_n}^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\sup_{0\leq t\leq T^*}\left\|\mathcal{M}^{-1}(\mathcal{S}(u_n^K))-\mathcal{M}^{-1}(\mathcal{S}(v_n^K))\right\|^2_{\mathcal{L}(X_n^*,X_n)}\\
&\qquad\times\left\|\mathrm{Tr}\left(\int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K)))d\mathcal{W}\right)\right\|^2_{X_n}\\
&\quad + C(n)\mathbb{E}\sup_{0\leq t\leq T^*}\left\|\mathcal{M}^{-1}(\mathcal{S}(v_n^K))\right\|^2_{\mathcal{L}(X_n^*,X_n)}\\
&\qquad\times\left\|\int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K)))d\mathcal{W}\right.\\
&\qquad\quad\left.-\int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(v_n^K))P_n(\sqrt{\mathcal{S}(v_n^K)}f(\mathcal{S}(v_n^K),\mathcal{S}(v_n^K)v_n^K,\widetilde{\mathcal{S}}(v_n^K)))d\mathcal{W}\right\|^2_{X_n}\\
&=: L_1+L_2.
\end{aligned}\tag{2.2.15}
$$

For $L_1$, using (2.2.12), Lemma 2.2.1(2) and the property of operator Tr, we have

$$
L_1 \leq T^*C(n,K,T^*)\mathbb{E}\sup_{0\leq t\leq T^*}\|u_n^K-v_n^K\|^2_{X_n}. \tag{2.2.16}
$$

For $L_2$, using (2.2.11), the boundedness of $\mathcal{S}(u_n^K)$, we have from the Burkholder-Davis-Gundy inequality

$$
\begin{aligned}
L_2 &\leq C(K,\delta,n)\\
&\quad\times\mathbb{E}\sup_{0\leq t\leq T^*}\left\|\int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K)))d\mathcal{W}\right.\\
&\qquad\qquad\left.-\int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(v_n^K))P_n(\sqrt{\mathcal{S}(v_n^K)}f(\mathcal{S}(v_n^K),\mathcal{S}(v_n^K)v_n^K,\widetilde{\mathcal{S}}(v_n^K)))d\mathcal{W}\right\|^2_{X_n}\\
&\leq C(K,\delta,n)\mathbb{E}\int_0^{T^*}\sum_{k\geq 1}\left\|\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k)\right.\\
&\qquad\qquad\left.-\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(v_n^K))P_n(\sqrt{\mathcal{S}(v_n^K)}f(\mathcal{S}(v_n^K),\mathcal{S}(v_n^K)v_n^K,\widetilde{\mathcal{S}}(v_n^K))e_k)\right\|^2_{X_n}ds\\
&\leq C(K,\delta,n)\mathbb{E}\Bigg(\int_0^{T^*}\sum_{k\geq 1}\left\|(\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K))-\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(v_n^K)))\right.\\
&\qquad\qquad\left.\times P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k)\right\|^2_{X_n}ds\\
&\qquad+\int_0^{T^*}\sum_{k\geq 1}\left\|\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(v_n^K))P_n(\sqrt{\mathcal{S}(v_n^K)}f(\mathcal{S}(v_n^K),\mathcal{S}(v_n^K)v_n^K,\widetilde{\mathcal{S}}(v_n^K))e_k\right.\\
&\qquad\qquad\left.-\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k)\right\|^2_{X_n}ds\Bigg)
\end{aligned}
$$

$$=: L_{21} + L_{22}.$$

For $L_{21}$, using the continuity of $\mathcal{S}(u_n^K)$, the equivalence of norms on finite dimensional space, condition (2.1.1), the boundedness of $u_n^K, \mathcal{S}(u_n^K)$ and Lemma 2.2.2, we have

$$\begin{aligned}
L_{21} \le C(K,\delta,n)\mathbb{E}\int_0^{T^*} & \left\|\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K)) - \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(v_n^K))\right\|^2_{\mathcal{L}(X_n,X_n^*)} \\
& \times \sum_{k\ge 1}\left\|P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k)\right\|^2_{L^2} ds\Bigg) \\
\le T^*C(n,K,T^*,\delta)\mathbb{E}\Bigg( & \sup_{0\le t\le T^*}\|u_n^K - v_n^K\|^2_{X_n} \\
& \times \int_0^{T^*}\sum_{k\ge 1}\left\|\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k\right\|^2_{L^2} ds\Bigg) \\
\le T^*C(n,K,T^*,\delta)\mathbb{E}\Bigg( & \sup_{0\le t\le T^*}\|u_n^K - v_n^K\|^2_{X_n} \\
& \times \int_0^{T^*}\int_{\mathcal{D}}|\rho_n^K|^\gamma + \rho_n^K|c_n^K,\nabla Q_n^K|^{\frac{2(\gamma-1)}{\gamma}} + \rho_n^K|u_n^K|^2 dxds\Bigg) \\
\le T^*C(n,K,T^*,\delta)\mathbb{E}\Bigg( & \sup_{0\le t\le T^*}\|u_n^K - v_n^K\|^2_{X_n} \\
& \times \int_0^{T^*}\int_{\mathcal{D}}|\rho_n^K|^\gamma + |c_n^K,\nabla Q_n^K|^2 + \rho_n^K|u_n^K|^2 dxds\Bigg) \\
\le (T^*)^2C(n,K,T^*,\delta)\mathbb{E} & \sup_{0\le t\le T^*}\|u_n^K - v_n^K\|^2_{X_n}. \qquad (2.2.17)
\end{aligned}$$

For $L_{22}$, using the continuity of $\mathcal{S}(u_n^K), \widetilde{\mathcal{S}}(u_n^K)$(see Lemma 2.2.2), the boundedness of $\mathcal{S}(u^K)$, condition (2.1.2) and the equivalence of norms on finite dimensional space, we also have

$$L_{22} \le T^*C(n,K,T^*)\mathbb{E}\sup_{0\le t\le T^*}\|u_n^K - v_n^K\|^2_{X_n}. \qquad (2.2.18)$$

Then, taking into account of (2.2.15)-(2.2.18), we infer that there exists a sequence of approximate solutions $u_n^K \in L^2(\Omega; C([0,T_*]; X_n))$ to equation (2.2.14) for small time $T^*$ by the Banach fixed point theorem. Here we first assume that the a priori estimates (2.2.20) hold uniformly in $n, K$ which allows us to extend the existence time $T^*$ to $T$ for any $T > 0$. Namely, we proved the existence and uniqueness of solution $u_n^K \in L^2(\Omega; C([0,T]; C^2(\mathcal{D})))$ to equation (2.2.14) for fixed $n, K$.

Next, we build the global existence of Galerkin approximate solution to system (2.2.1) for any fixed $n$ by letting $K \to \infty$. Define the stopping time,

$$
\begin{aligned}
\tau_K = \inf \Bigg\{ & t \geq 0; \sup_{s\in[0,t]} \|u_n^K(s)\|_{L^2} \\
& + \sup_{s\in[0,t]} \left\| \int_0^s \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K)) P_n(\sqrt{\mathcal{S}(u_n^K)} f(\mathcal{S}(u_n^K), \mathcal{S}(u_n^K)u_n^K, \widetilde{\mathcal{S}}(u_n^K))) d\mathcal{W} \right\|_{L^2} \geq K \Bigg\}.
\end{aligned}
$$

Observe that the sequence of the stopping time $\tau_K$ is increasing. Define $\rho_n^K = \mathcal{S}(u_n^K)$, $(c_n^K, Q_n^K) = \widetilde{\mathcal{S}}(u_n^K)$, then $(\rho_n^K, u_n^K, c_n^K, Q_n^K)$ is the unique solution to system (2.2.1). Using the monotonicity of the stopping time and the uniqueness of solution, for $K_1 \leq K_2$, we have $(\rho_n^{K_1}, u_n^{K_1}, c_n^{K_1}, Q_n^{K_1}) = (\rho_n^{K_2}, u_n^{K_2}, c_n^{K_2}, Q_n^{K_2})$ on $[0, \tau_{K_1})$. Therefore, we could define the solution $(\rho_n, u_n, c_n, Q_n) = (\rho_n^K, u_n^K, c_n^K, Q_n^K)$ on interval $[0, \tau_K)$. In order to extend the existence time to $[0, T]$, we show that

$$
\mathbb{P}\left\{ \sup_{K\in\mathbb{N}^+} \tau_K = T \right\} = 1.
$$

Since the stopping time $\tau_K$ is increasing, we have

$$
\begin{aligned}
& \mathbb{P}\left\{ \sup_{K\in\mathbb{N}^+} \tau_K < T \right\} < \mathbb{P}\{\tau_K < T\} \\
& \leq \mathbb{P}\left\{ \sup_{t\in[0,T]} \|u_n^K\|_{L^2} > \frac{K}{2} \right\} \\
& + \mathbb{P}\left\{ \sup_{t\in[0,T]} \left\| \int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K)) P_n(\sqrt{\mathcal{S}(u_n^K)} f(\mathcal{S}(u_n^K), \mathcal{S}(u_n^K)u_n^K, \widetilde{\mathcal{S}}(u_n^K))) d\mathcal{W} \right\| > \frac{K}{2} \right\} \\
& =: J_1 + J_2.
\end{aligned}
$$

Using the Burkholder-Davis-Gundy inequality, the Chebyshev inequality and the equivalence of norms on finite-dimensional space, and the embedding $H^{-l}(\mathcal{D}) \hookrightarrow L^1(\mathcal{D})$ for $l > \frac{3}{2}$, and the condition (2.1.1), the bound (2.2.20), we have

$$
\begin{aligned}
J_2 & \leq \frac{2}{K} \mathbb{E} \left( \sup_{t\in[0,T]} \left\| \int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K)) P_n \sqrt{\mathcal{S}(u_n^K)} f(\mathcal{S}(u_n^K), \mathcal{S}(u_n^K)u_n^K, \widetilde{\mathcal{S}}(u_n^K)) d\mathcal{W} \right\|_{L^2} \right) \\
& \leq \frac{C}{K} \mathbb{E} \left( \sup_{t\in[0,T]} \left\| \int_0^t \mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K)) P_n \sqrt{\mathcal{S}(u_n^K)} f(\mathcal{S}(u_n^K), \mathcal{S}(u_n^K)u_n^K, \widetilde{\mathcal{S}}(u_n^K)) d\mathcal{W} \right\|_{H^{-l}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C}{K}\mathbb{E}\int_0^T \sum_{k\geq 1}\left\|\mathcal{M}^{\frac{1}{2}}(\mathcal{S}(u_n^K))P_n(\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k)\right\|_{L^1}^2 dt\\
&\leq \frac{C}{K}\mathbb{E}\int_0^T \int_{\mathcal{D}}\sum_{k\geq 1}\left|\sqrt{\mathcal{S}(u_n^K)}f(\mathcal{S}(u_n^K),\mathcal{S}(u_n^K)u_n^K,\widetilde{\mathcal{S}}(u_n^K))e_k\right|^2 dx\int_{\mathcal{D}}\mathcal{S}(u_n^K)dxdt\\
&\leq \frac{C}{K}\left(\rho_n^K(0)\right)_m \mathbb{E}\int_0^T\int_{\mathcal{D}}|\rho_n|^\gamma+\rho_n|u_n|^2+|c_n,\nabla Q_n|^2dxdt<\frac{C}{K},
\end{aligned}
\tag{2.2.19}
$$

where $C$ is independent of $K$, leading to $J_2 \to 0$ as $K \to \infty$. Corollary 3.2 in [43] given $J_1 \to 0$ as $K \to \infty$. Therefore, passing $K \to \infty$, we have

$$
\mathbb{P}\left\{\sup_{K\in\mathbb{N}^+}\tau_K<T\right\}=0.
$$

This means that no blow up appears in a finite time, we could extend the existence time to $[0,T]$ for any $T>0$.

We next establish the necessary a priori estimates of approximate solution. To simplify the notation, we replace $(\rho_n^K,\rho_n^K u_n^K,c_n^K,Q_n^K)$ by $(\rho,\rho u,c,Q)$.

**Lemma 2.2.3.** *Suppose that $(\rho,\rho u,c,Q)$ is the Galerkin approximate solution to system (2.2.1)-(2.2.6), and $f$ satisfies the condition (2.1.1). Then, there exists a constant $C$ which is independent of $n,K$ but depends on $(\mu_1,\mu_2,\sigma^*,c_*,b,T,p,\Gamma)$ and initial data such that for all $1\leq p<\infty$*

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|c,\nabla Q,\sqrt{\rho}u\|_{L^2}^2+\|Q\|_{L^4}^4+\frac{1}{\gamma-1}\|\rho\|_{L^\gamma}^\gamma+\frac{\delta}{\beta-1}\|\rho\|_{L^\beta}^\beta\right)\right]^p\\
&+\mathbb{E}\left(\int_0^T\|\nabla c,\nabla u,\triangle Q,\operatorname{div}u\|_{L^2}^2+\|Q\|_{L^6}^6dt\right)^p\\
&+\mathbb{E}\left(\int_0^T\int_{\mathcal{D}}\epsilon(\gamma\rho^{\gamma-2}+\delta\beta\rho^{\beta-2})|\nabla\rho|^2dxdt\right)^p\leq C.
\end{aligned}
\tag{2.2.20}
$$

*Moreover, we have*

$$
\sqrt{\epsilon}\rho\in L^p(\Omega;L^2(0,T;H^1(\mathcal{D}))), \tag{2.2.21}
$$

$$
\rho u\in L^p(\Omega;L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))). \tag{2.2.22}
$$

*Proof.* Denote $\Phi(\rho, m) = (m, \mathcal{M}^{-1}(\rho)m)$, we obtain

$$\nabla_m \Phi(\rho, m) = 2\mathcal{M}^{-1}(\rho)m, \quad \nabla_m^2 \Phi(\rho, m) = 2\mathcal{M}^{-1}(\rho),$$
$$\nabla_\rho \Phi(\rho, m) = -(m, \mathcal{M}^{-1}(\rho)\mathcal{M}^{-1}(\rho)m).$$

Applying the Itô formula to function $\Phi(\rho, \rho u)$, integrating with respect to time, taking the supremum on interval $[0, t \wedge \tau_K]$, then

$$\begin{aligned}
&\sup_{s\in[t\wedge\tau_K]} \int_{\mathcal{D}} \rho|u|^2 dx \\
\leq& \int_{\mathcal{D}} \rho_0|u_0|^2 dx - \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla|u|^2 \cdot \rho u dxds + \epsilon \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla|u|^2 \cdot \nabla\rho dxds \\
&- 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla(\rho^\gamma + \delta\rho^\gamma) \cdot udxds \\
&+ 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (\mu_1 \triangle u + (\mu_1 + \mu_2)\nabla \mathrm{div} u) \cdot udxds \\
&+ 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \rho u \otimes u : \nabla udxds - 2\epsilon \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla u \nabla\rho \cdot udxds \\
&- 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (\nabla Q \odot \nabla Q - \mathrm{F}(Q)\mathrm{I}_3) : \nabla udxds \\
&+ 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (Q\triangle Q - \triangle QQ) : \nabla udxds + 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \sigma^*(c^2 Q) : \nabla udxds \\
&+ \int_0^{t\wedge\tau_K} \sum_{k\geq 1} (\mathcal{M}^{-1}(\rho)\mathcal{M}^{\frac{1}{2}}(\rho)P_n(\sqrt{\rho} f(\rho, \rho u, c, Q)e_k), \\
&\qquad\qquad\qquad\qquad \mathcal{M}^{\frac{1}{2}}(\rho)P_n(\sqrt{\rho} f(\rho, \rho u, c, Q)e_k))ds \\
&+ 2\sup_{s\in[0,t\wedge\tau_K]} \left| \int_0^s \sum_{k\geq 1} \int_{\mathcal{D}} \mathcal{M}^{\frac{1}{2}}(\rho)uP_n(\sqrt{\rho} f(\rho, \rho u, c, Q)e_k)dxd\beta_k \right|. \qquad (2.2.23)
\end{aligned}$$

By equation $(2.2.1)_2$, we have

$$\begin{aligned}
-2\int_{\mathcal{D}} \nabla(\rho^\gamma + \delta\rho^\beta) \cdot udx &= -2\int_{\mathcal{D}} \left( \frac{\gamma}{\gamma - 1}\nabla\rho^{\gamma-1} + \frac{\delta\beta}{\beta - 1}\nabla\rho^{\beta-1} \right) \rho udx \\
&= 2\int_{\mathcal{D}} \left( \frac{\gamma}{\gamma - 1}\rho^{\gamma-1} + \frac{\delta\beta}{\beta - 1}\rho^{\beta-1} \right) \mathrm{div}(\rho u)dx \\
&= 2\int_{\mathcal{D}} \left( \frac{\gamma}{\gamma - 1}\rho^{\gamma-1} + \frac{\delta\beta}{\beta - 1}\rho^{\beta-1} \right) (-\partial_t\rho + \epsilon\triangle\rho)dx \\
&= -2\int_{\mathcal{D}} \frac{1}{\gamma - 1}d\rho^\gamma + \frac{\delta}{\beta - 1}d\rho^\beta dx
\end{aligned}$$

$$- 2\epsilon \int_{\mathcal{D}} (\gamma \rho^{\gamma-2} + \delta\beta\rho^{\beta-2})|\nabla \rho|^2 dxdt.$$

Multiplying equation $(2.2.1)_1$ with c, integrating over $\mathcal{D}$, then

$$d\left(\frac{1}{2}\int_{\mathcal{D}} c^2 dx\right) + \int_{\mathcal{D}} |\nabla c|^2 dxdt = -\int_{\mathcal{D}} cu \cdot \nabla c dxdt. \tag{2.2.24}$$

Also, multiplying equation $(2.2.1)_4$ with $-(\triangle Q - Q - c_* Q\mathrm{tr}(Q^2))$, taking the trace and integrating over $\mathcal{D}$, adding (2.2.23) and (2.2.24), then we get

$$\begin{aligned}
&\sup_{s\in[t\wedge\tau_K]} \int_{\mathcal{D}} \left(\rho|u|^2 + c^2 + \frac{2\delta}{\beta-1}\rho^\beta + \frac{2}{\gamma-1}\rho^\gamma + |Q,\nabla Q|^2 + \frac{c_*}{2}|Q|^4\right) dx \\
&\quad + 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \mu_1|\nabla u|^2 + (\mu_1+\mu_2)|\mathrm{div} u|^2 + |\nabla c|^2 \\
&\qquad\qquad + \Gamma(|\nabla Q|^2 + |\triangle Q|^2) + \Gamma(c_*|Q|^4 + c_*^2|Q|^6) dxds \\
&\quad + 2\epsilon \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (\gamma\rho^{\gamma-2} + \delta\beta\rho^{\beta-2})|\nabla\rho|^2 dxds \\
&= \int_{\mathcal{D}} \left(\rho_0|u_0|^2 + c_0^2 + \frac{2\delta}{\beta-1}\rho_0^\beta + \frac{2}{\gamma-1}\rho_0^\gamma + |Q_0,\nabla Q_0|^2 + \frac{c_*}{2}|Q_0|^4\right) dx \\
&\quad - \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla|u|^2 \cdot \rho u dxds + \epsilon \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla|u|^2 \cdot \nabla\rho dxds \\
&\quad +2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \rho u \otimes u : \nabla u dxds - 2\epsilon \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \nabla u \nabla\rho \cdot u dxds \\
&\quad + 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (\nabla Q \odot \nabla Q - \mathrm{F}(Q)\mathrm{I}_3) : \nabla u dxds \\
&\quad - 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (Q\triangle Q - \triangle Q Q) : \nabla u dxds - 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \sigma^*(c^2 Q) : \nabla u dxds \\
&\quad - 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} cu \cdot \nabla c dxds + 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} u \cdot \nabla Q : (\triangle Q - Q - c_* Q\mathrm{tr}(Q^2)) dxds \\
&\quad - 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} (\Psi Q - Q\Psi) : (\triangle Q - Q - c_* Q\mathrm{tr}(Q^2)) dxds \\
&\quad + \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} \Gamma(c - c_*)Q : (\triangle Q - Q - c_* Q\mathrm{tr}(Q^2)) dxds \\
&\quad - 2\int_0^{t\wedge\tau_K} \int_{\mathcal{D}} b\Gamma Q^2 : (\triangle Q - Q - c_* Q\mathrm{tr}(Q^2)) dxds \\
&\quad + \int_0^{t\wedge\tau_K} \int_{\mathcal{D}} 2c_*\Gamma Q|Q|^2 : \triangle Q dxds \\
&\quad + \int_0^{t\wedge\tau_K} \sum_{k\geq 1} (\mathcal{M}^{-1}(\rho)\mathcal{M}^{\frac{1}{2}}(\rho) P_n(\sqrt{\rho} f(\rho,\rho u, c, Q)e_k),
\end{aligned}$$

$$\mathcal{M}^{\frac{1}{2}}(\rho)P_n(\sqrt{\rho}f(\rho,\rho u,c,Q)e_k))ds$$
$$+2\sup_{s\in[0,t\wedge\tau_K]}\left|\int_0^s\sum_{k\geq 1}\int_{\mathcal{D}}\mathcal{M}^{\frac{1}{2}}(\rho)uP_n(\sqrt{\rho}f(\rho,\rho u,c,Q)e_k)dxd\beta_k\right|$$
$$=:\int_0^{t\wedge\tau_K}\sum_{i=1}^{14}J_ids+2\sup_{s\in[0,t\wedge\tau_K]}\left|\int_0^s\sum_{k\geq 1}J_{15}d\beta_k\right|. \tag{2.2.25}$$

Next, we control all the right hand side terms of (2.2.25). Note that $J_1+J_3=0$, $J_2+J_4=0$ and

$$\begin{aligned}
J_6+J_{10}=&-2\int_{\mathcal{D}}(Q\triangle Q-\triangle QQ):\nabla udx\\
&+2\int_{\mathcal{D}}(\Psi Q-Q\Psi):(\triangle Q-Q-c_*Q\mathrm{tr}(Q^2))dx\\
=&-2\int_{\mathcal{D}}\mathrm{tr}(Q\triangle Q\nabla u-\triangle QQ\nabla u)dx+2\int_{\mathcal{D}}\mathrm{tr}(\Psi Q\triangle Q-Q\Psi\triangle Q)dx\\
&+2\int_{\mathcal{D}}\mathrm{tr}((\Psi Q-Q\Psi)(-Q-c_*Q\mathrm{tr}(Q^2)))dx=:\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.
\end{aligned}$$

Due to the fact that $Q$ is symmetric and traceless and $\Psi$ is skew-symmetric we have $\mathcal{J}_3=0$, $\mathcal{J}_1+\mathcal{J}_2=0$, also $J_5+J_9=0$, see [21].

Applying Young's inequality and the boundedness of $c$, we have

$$\begin{aligned}
|J_7|=&\left|2\int_{\mathcal{D}}\sigma^*c^2Q:\nabla udx\right|\\
\leq&\, C\|c\|^2_{L^\infty([0,T]\times\mathcal{D})}\|\nabla u\|_{L^2}\|Q\|_{L^2}\leq\|\nabla u\|^2_{L^2}+C\|Q\|^2_{L^2},\\
|J_8|=&\left|2\int_{\mathcal{D}}cu\cdot\nabla cdx\right|\\
\leq&\, C\|c\|_{L^\infty([0,T]\times\mathcal{D})}\|\nabla c\|_{L^2}\|u\|_{L^2}\leq\|\nabla c\|^2_{L^2}+C\|u\|^2_{L^2}\\
\leq&\,\|\nabla c\|^2_{L^2}+\mu_1\|\nabla u\|^2_{L^2}+C.
\end{aligned}$$

In addition

$$\begin{aligned}
|J_{11}|=&\left|\int_{\mathcal{D}}\Gamma(c-c_*)Q:(\triangle Q-Q-c_*Q\mathrm{tr}(Q^2))dx\right|\\
\leq&\,\frac{\Gamma}{2}\|\triangle Q\|^2_{L^2}+C\|Q\|^2_{L^2}+C\|Q\|^4_{L^4},\\
|J_{12}|=&\left|\int_{\mathcal{D}}2b\Gamma Q^2:(\triangle Q-Q-c_*Q\mathrm{tr}(Q^2))dx\right|
\end{aligned}$$

$$
\begin{aligned}
&\le b\Gamma\|\triangle Q\|_{L^2}\|Q^2\|_{L^2} + \frac{c_*\Gamma}{2}\|Q\|_{L^6}^6 + C\|Q\|_{L^4}^4 \\
&\le \frac{\Gamma}{2}\|\triangle Q\|^2 + C\|Q\|_{L^4}^4 + \frac{c_*\Gamma}{2}\|Q\|_{L^6}^6 + C\|Q\|^2, \\
J_{13} = &\int_{\mathcal{D}} 2c_*\Gamma Q|Q|^2 : \triangle Q dx = -2c_*\Gamma \int_{\mathcal{D}} |\nabla Q|^2|Q|^2 dx - c_*\Gamma \int_{\mathcal{D}} |\nabla \mathrm{tr}(Q)|^2 dx \\
&\le 0.
\end{aligned}
$$

Using the condition (2.1.1), $J_{14}$ can be treated as

$$
\begin{aligned}
J_{14} &\le \|\sqrt{\rho} f(\rho, \rho u, c, Q)\|_{L_2(\mathcal{H};L^2(\mathcal{D}))}^2 \\
&\le \int_{\mathcal{D}} \sum_{k\ge 1} |\sqrt{\rho} f(\rho, \rho u, c, Q) e_k|^2 dx \\
&\le C \int_{\mathcal{D}} \rho^\gamma + |\sqrt{\rho} u|^2 + \rho |c|^{\frac{2(\gamma-1)}{\gamma}} + \rho |\nabla Q|^{\frac{2(\gamma-1)}{\gamma}} dx \\
&\le C \int_{\mathcal{D}} \rho^\gamma + |\sqrt{\rho} u|^2 + |c|^2 + |\nabla Q|^2 dx.
\end{aligned}
$$

Define the stopping time $\tau_R$

$$
\tau_R = \inf \left\{ t \ge 0; \sup_{s\in[0,t]} \|\sqrt{\rho} u\|_{L^2}^2 \ge R \right\} \wedge \tau_K, \tag{2.2.26}
$$

if the set is empty, taking $\tau_R = T$. Note that, $\tau_R$ is an increasing sequence with

$$
\lim_{R\wedge K\to\infty} \tau_R = T.
$$

Regarding the stochastic term, by the Burkholder-Davis-Gundy inequality and condition (2.1.1) for all $1 \le p < \infty$

$$
\begin{aligned}
&\mathbb{E}\left[ \sup_{s\in[0,t\wedge\tau_R]} \left| \int_0^s \sum_{k\ge 1} J_{15} d\beta_k \right| \right]^p \\
&\le C\mathbb{E} \left( \int_0^{t\wedge\tau_R} \sum_{k\ge 1} \left( \int_{\mathcal{D}} \mathcal{M}^{\frac{1}{2}}(\rho) u P_n(\sqrt{\rho} f(\rho, \rho u, c, Q) e_k) dx \right)^2 ds \right)^{\frac{p}{2}} \\
&\le C\mathbb{E} \left( \int_0^{t\wedge\tau_R} \|\mathcal{M}^{\frac{1}{2}}(\rho) u\|_{L^2}^2 \int_{\mathcal{D}} \sum_{k\ge 1} |\rho f^2(\rho, \rho u, c, Q) e_k^2| dx ds \right)^{\frac{p}{2}} \\
&\le C\mathbb{E}\left[ \sup_{s\in[0,t\wedge\tau_R]} \|\sqrt{\rho} u\|_{L^2}^p \right] \times
\end{aligned}
$$

$$\left(\int_0^{t\wedge\tau_R}\int_{\mathcal{D}}\rho^\gamma+|\sqrt{\rho}u|^2+\rho|c|^{\frac{2(\gamma-1)}{\gamma}}+\rho|\nabla Q|^{\frac{2(\gamma-1)}{\gamma}}dxds\right)^{\frac{p}{2}}$$
$$\le\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_R]}\|\sqrt{\rho}u\|_{L^2}^{2p}\right]$$
$$+C\mathbb{E}\left(\int_0^{t\wedge\tau_R}\int_{\mathcal{D}}\rho^\gamma+|\sqrt{\rho}u|^2+|c|^2+|\nabla Q|^2dxds\right)^p.$$

Considering all these estimates, taking the integral with respect to time, taking the supremum on interval $[0,t\wedge\tau_R]$, then power $p$ and taking expectation on both sides, the Gronwall lemma yields

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_R]}\left(\|c,Q,\nabla Q,\sqrt{\rho}u\|_{L^2}^2+\|Q\|_{L^4}^4+\frac{1}{\gamma-1}\|\rho\|_{L^\gamma}^\gamma+\frac{\delta}{\beta-1}\|\rho\|_{L^\beta}^\beta\right)\right]^p$$
$$+\mathbb{E}\Bigg(\int_0^{t\wedge\tau_R}\mu_1\|\nabla u\|_{L^2}^2+(\mu_1+\mu_2)\|\mathrm{div}u\|_{L^2}^2+\Gamma\|\triangle Q\|_{L^2}^2$$
$$+\|\nabla c\|_{L^2}^2+c_*^2\Gamma\|Q\|_{L^6}^6ds\Bigg)^p$$
$$+\mathbb{E}\left(\int_0^{t\wedge\tau_R}\int_{\mathcal{D}}\epsilon(\gamma\rho^{\gamma-2}+\delta\beta\rho^{\beta-2})|\nabla\rho|^2dxds\right)^p\le C,$$

where $C$ is constant independent of $n$. Finally, we get the bound (2.2.20) by the monotone convergence theorem.

Using the bound (2.2.20), we have

$$\mathbb{E}\int_{\mathcal{D}}|\rho|^2dx+2\epsilon\mathbb{E}\int_0^t\int_{\mathcal{D}}|\nabla\rho|^2dxds=\mathbb{E}\int_{\mathcal{D}}|\rho_0|^2dx-\mathbb{E}\int_0^t\int_{\mathcal{D}}\mathrm{div}u|\rho|^2dxds$$
$$\le\mathbb{E}\int_{\mathcal{D}}|\rho_0|^2dx+\mathbb{E}\left(\int_0^t\int_{\mathcal{D}}|\mathrm{div}u|^2+|\rho|^4dxds\right)^4\le C,$$

consequently, (2.2.21) holds. We can obtain the bound (2.2.22) using the fact that $\sqrt{\rho}u\in L^p(\Omega;L^\infty(0,T;L^2(\mathcal{D})))$ and $\rho\in L^p(\Omega;L^\infty(0,T;L^\beta(\mathcal{D})))$. This completes the proof. □

**Remark**. By taking inner product with $-(\triangle Q-Q-c_*Q\mathrm{tr}(Q^2))$ in equation $(2.2.1)_4$ in place of $\mathrm{H}(Q,c)$, we can prevent the interaction term of $c$ and $Q$-tensor arising, making the estimates concise.

### 2.2.2 The compactness of approximate solution

Unlike the deterministic case, it may not be the case that the embedding $L^2(\Omega; X)$ into $L^2(\Omega; Y)$ is compact, even if $X \hookrightarrow Y$ is compact. Therefore, in order to obtain the compactness of approximate solution, the key point is to obtain the compactness of the set of probability measures generated by the approximate solution sequences. Define the path space

$$\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_\rho \times \mathcal{X}_{\rho u} \times \mathcal{X}_c \times \mathcal{X}_Q \times \mathcal{X}_{\mathcal{W}},$$

where

$$\begin{aligned}
&\mathcal{X}_u := L^2_w(0,T; H^1(\mathcal{D})), \mathcal{X}_{\rho u} := C([0,T]; H^{-1}(\mathcal{D})),\\
&\mathcal{X}_\rho := L^\infty(0,T; H^{-\frac{1}{2}}(\mathcal{D})) \cap L^2(0,T; L^2(\mathcal{D})) \cap L^2_w(0,T; H^1(\mathcal{D})),\\
&\mathcal{X}_c := L^2_w(0,T; H^1(\mathcal{D})) \cap L^2(0,T; L^2(\mathcal{D})),\\
&\mathcal{X}_Q := L^2_w(0,T; H^2(\mathcal{D})) \cap L^2(0,T; H^1(\mathcal{D})),\ \mathcal{X}_{\mathcal{W}} := C([0,T]; \mathcal{H}_0).
\end{aligned}$$

Define the probability measures

$$\nu^n = \nu^n_u \otimes \nu^n_\rho \otimes \nu^n_{\rho u} \otimes \nu^n_c \otimes \nu^n_Q \otimes \nu_{\mathcal{W}}, \tag{2.2.27}$$

where $\nu^n_{(\cdot)}(B) = \mathbb{P}\{\cdot \in B\}$ for any $B \in \mathcal{B}(\mathcal{X}_{(\cdot)})$, $\mathcal{X}_{(\cdot)}$ is the path space defined above, respectively.

Next, we establish the following compactness result.

**Proposition 2.2.4.** *There exists a subsequence of probability measures $\{\nu^n\}_{n\geq 1}$ still denoted by $\{\nu^n\}_{n\geq 1}$, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with $\mathcal{X}$-valued measurable random variables*

$$(\tilde{u}_n, \tilde{\rho}_n, P_n(\tilde{\rho}_n \tilde{u}_n), \tilde{c}_n, \widetilde{Q}_n, \widetilde{\mathcal{W}}_n) \text{ and } (\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}),$$

*such that*

$$(\tilde{u}_n, \tilde{\rho}_n, P_n(\tilde{\rho}_n \tilde{u}_n), \tilde{c}_n, \widetilde{Q}_n, \widetilde{\mathcal{W}}_n) \to (\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}),\ \widetilde{\mathbb{P}} \ a.s. \tag{2.2.28}$$

*in the topology of* $\mathcal{X}$ *and*

$$\widetilde{\mathbb{P}}\{(\tilde{u}_n, \tilde{\rho}_n, P_n(\tilde{\rho}_n\tilde{u}_n), \tilde{c}_n, \widetilde{Q}_n, \widetilde{\mathcal{W}}_n) \in \cdot\} = \nu^n(\cdot), \tag{2.2.29}$$

$$\widetilde{\mathbb{P}}\{(\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}) \in \cdot\} = \nu(\cdot), \tag{2.2.30}$$

*where* $\nu$ *is a Radon measure and* $\widetilde{\mathcal{W}}_n$ *is cylindrical Wiener process, relative to the filtration* $\widetilde{\mathcal{F}}_t^n$ *generated by the completion of* $\sigma(\tilde{u}_n(s), \tilde{\rho}_n(s), \tilde{c}_n(s), \widetilde{Q}_n(s), \widetilde{\mathcal{W}}_n(s); s \leq t)$. *Moreover, the process* $(\tilde{u}_n, \tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n, \widetilde{\mathcal{W}}_n)$ *also satisfies the system (2.2.1) and shares the following uniform a priori estimates*

$$\tilde{\rho}_n \in L^p(\widetilde{\Omega}; L^\infty(0,T; L^\beta(\mathcal{D})) \cap L^2(0,T; H^1(\mathcal{D}))), \tag{2.2.31}$$

$$\tilde{u}_n \in L^p(\widetilde{\Omega}; L^2(0,T; H^1(\mathcal{D}))), \tag{2.2.32}$$

$$\sqrt{\tilde{\rho}_n}\tilde{u}_n \in L^p(\widetilde{\Omega}; L^\infty(0,T; L^2(\mathcal{D}))), \tag{2.2.33}$$

$$\tilde{\rho}_n\tilde{u}_n \in L^p(\widetilde{\Omega}; L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))), \tag{2.2.34}$$

$$\tilde{c}_n \in L^p(\widetilde{\Omega}; L^\infty(0,T; L^2(\mathcal{D})) \cap L^2(0,T; H^1(\mathcal{D}))), \tag{2.2.35}$$

$$\widetilde{Q}_n \in L^p(\widetilde{\Omega}; L^\infty(0,T; H^1(\mathcal{D})) \cap L^2(0,T; H^2(\mathcal{D}))). \tag{2.2.36}$$

Combining the bound (2.2.31) and strong convergence $\tilde{\rho}_n$ in $L^2(0,T; L^2(\mathcal{D}))$, the Vitali convergence theorem A.0.3 implies that $\widetilde{\mathbb{P}}$ a.s.

$$\tilde{\rho}_n \to \tilde{\rho} \text{ in } L^4(0,T; L^4(\mathcal{D})).$$

In order to employ the Skorokhod-Jakubowski theorem, we next show the tightness of set $\{\nu^n\}_{n\geq 1}$.

**Lemma 2.2.5.** *The set of probability measures* $\{\nu^n\}_{n\geq 1}$ *is tight on path space* $\mathcal{X}$.

*Proof.* It is enough to show that each set of probability measures $\{\nu^n_{(\cdot)}\}_{n\geq 1}$ is tight on the corresponding path space $\mathcal{X}_{(\cdot)}$.

**Claim 1**. The set $\{\nu^n_{P_n(\rho u)}\}_{n\geq 1}$ is tight on path spaces $\mathcal{X}_{\rho u}$.

Decompose $P_n(\rho_n u_n) = X_n + Y_n$, where

$$\begin{aligned}X_n = m_{0,n} + P_n \int_0^t & -\mathrm{div}(\rho_n u_n \otimes u_n) - \nabla(\rho_n^\gamma + \delta\rho_n^\beta) + \mu_1 \Delta u_n \\ & +(\mu_1+\mu_2)\nabla(\mathrm{div} u_n) + \nabla\cdot(\mathrm{F}(Q_n)\mathrm{I}_3 - \nabla Q_n \odot \nabla Q_n) \\ & +\nabla\cdot(Q_n \triangle Q_n - \triangle Q_n Q_n) + \sigma^* \nabla\cdot(c_n^2 Q_n) ds \\ & + \int_0^t \mathcal{M}^{\frac{1}{2}}(\rho_n) P_n \sqrt{\rho_n} f(\rho_n, \rho_n u_n, c_n, Q_n) d\mathcal{W},\end{aligned}$$

and

$$Y_n = \epsilon \int_0^t P_n(\nabla\rho_n \cdot \nabla u_n) ds.$$

The main goal is to get

$$\mathbb{E}\|P_n(\rho_n u_n)\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} \leq C, \tag{2.2.37}$$

where $C$ is independent of $n$ for $\alpha \in [0, \frac{1}{2})$ and $k \geq \frac{5}{2}$.

Regarding the stochastic term, similar to (2.2.19), using the Burkholder-Davis-Gundy inequality and condition (2.1.1), we get for all $\alpha \in [0, \frac{1}{2})$

$$\begin{aligned}& \mathbb{E}\left\|\int_0^t \mathcal{M}^{\frac{1}{2}}(\rho_n) P_n \sqrt{\rho_n} f(\rho_n, \rho_n u_n, c_n, Q_n) d\mathcal{W}\right\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} \\ \leq & \mathbb{E}\left[\sup_{t,t'\in[0,T]} \frac{\left\|\int_{t'}^t \mathcal{M}^{\frac{1}{2}}(\rho_n) P_n \sqrt{\rho_n} f(\rho_n, \rho_n u_n, c_n, Q_n) d\mathcal{W}\right\|_{H^{-k}}}{|t-t'|^\alpha}\right] \\ \leq & \frac{\mathbb{E}\left\|\int_{t'}^t \mathcal{M}^{\frac{1}{2}}(\rho_n) P_n \sqrt{\rho_n} f(\rho_n, \rho_n u_n, c_n, Q_n) d\mathcal{W}\right\|_{H^{-k}}}{|t-t'|^\alpha} + \delta' \\ \leq & \frac{\mathbb{E}\left(\int_{t'}^t \|\mathcal{M}^{\frac{1}{2}}(\rho_n)\sum_{k\geq 1} P_n \sqrt{\rho_n} f(\rho_n, \rho_n u_n, c_n, Q_n) e_k\|_{L^1}^2 dr\right)^{\frac{1}{2}}}{|t-t'|^\alpha} + \delta' \\ \leq & C|t-t'|^{\frac{1}{2}-\alpha} \mathbb{E}\left[\sup_{t\in[0,T]} (\|\rho_n\|_{L^\gamma} + \|\sqrt{\rho_n} u_n\|_{L^2}^2 + \|c_n\|_{L^2}^2 + \|\nabla Q_n\|_{L^2}^2)\right] \\ & +\delta' \leq C.\end{aligned}$$

Using (2.2.20) and the Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}\|\nabla \cdot (Q_n \triangle Q_n - \triangle Q_n Q_n)\|^p_{L^2(0,T;H^{-k}(\mathcal{D}))} \\
&\leq \mathbb{E}\left(\int_0^T \|Q_n \triangle Q_n - \triangle Q_n Q_n\|^2_{L^1} dt\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\left(\int_0^T \|Q_n\|^2_{L^2}\|\triangle Q_n\|^2_{L^2} dt\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\left[\sup_{t\in[0,T]} \|Q_n\|^{2p}_{L^2}\right] \mathbb{E}\left(\int_0^T \|\triangle Q_n\|^2_{L^2} dt\right)^p \\
&\leq C,
\end{aligned}
$$

and

$$
\mathbb{E}\|\sigma^* \nabla \cdot (c_n^2 Q_n)\|^p_{L^\infty(0,T;H^{-1}(\mathcal{D}))} \leq C\|c_n\|^{2p}_{L^\infty([0,T]\times\mathcal{D})} \mathbb{E}\|Q_n\|^p_{L^\infty(0,T;L^2(\mathcal{D}))} \leq C,
$$

moreover

$$
\mathbb{E}\|\nabla \cdot (\mathrm{F}(Q_n)\mathrm{I}_3 - \nabla Q_n \odot \nabla Q_n)\|^p_{L^\infty(0,T;H^{-k}(\mathcal{D}))} \leq \mathbb{E}\left[\sup_{t\in[0,T]} \|\nabla Q_n\|^p_{L^2}\right] \leq C,
$$

where the constant $C$ is independent of $n$.

Furthermore, the bound (2.2.20) together with the Hölder inequality yields,

$$
\mathrm{div}(\rho_n u_n \otimes u_n) \in L^p(\Omega; L^2(0,T; H^{-1,\frac{6\beta}{4\beta+3}}(\mathcal{D}))),
$$

then the Sobolev embedding $H^{-1,\frac{6\beta}{4\beta+3}}(\mathcal{D}) \hookrightarrow H^{-k}(\mathcal{D})$ for $k \geq \frac{5}{2}$ implies that

$$
\mathrm{div}(\rho_n u_n \otimes u_n) \in L^p(\Omega; L^2(0,T; H^{-k}(\mathcal{D}))).
$$

Also, the bound (2.2.20) implies $\nabla(\rho_n^\gamma + \delta\rho_n^\beta) \in L^p(\Omega; L^{\frac{\beta+1}{\beta}}(0,T; H^{-k}(\mathcal{D})))$ using the Sobolev embedding $H^{-1,\frac{\beta+1}{\beta}}(\mathcal{D}) \hookrightarrow H^{-k}(\mathcal{D})$.

To find the boundedness of $Y_n$, we need to improve the time integrability of $\rho_n$ following Lemma 2.4 in [30]. By (2.2.20) and the Hölder inequality, we have

$$
\rho_n u_n \in L^p(\Omega; L^2(0,T; L^{\frac{6\beta}{\beta+6}}(\mathcal{D}))) \cap L^p(\Omega; L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))).
$$

The interpolation lemma A.0.2 implies that there exists $q > 2$ such that

$$\rho_n u_n \in L^p(\Omega; L^q(0,T; L^2(\mathcal{D}))),$$

this estimate together with the bound $\rho_n \in L^p(\Omega; L^\infty(0,T; L^\beta(\mathcal{D})))$ and equation $(2.2.1)_2$ yields

$$\rho_n \in L^p(\Omega; L^q(0,T; H^1(\mathcal{D}))), \text{ for } q > 2. \tag{2.2.38}$$

Using (2.2.38) and (2.2.20), we have

$$\begin{aligned}
&\mathbb{E}\left\|\epsilon \int_0^t P_n(\nabla\rho_n \cdot \nabla u_n) ds\right\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} \\
&\leq \mathbb{E}\frac{\left\|\epsilon \int_{t'}^t P_n(\nabla\rho_n \cdot \nabla u_n) ds\right\|_{H^{-k}}}{|t-t'|^\alpha} + \delta' \leq \mathbb{E}\frac{\epsilon \int_{t'}^t \|\nabla\rho_n \cdot \nabla u_n\|_{L^1} ds}{|t-t'|^\alpha} + \delta' \\
&\leq \frac{1}{|t-t'|^\alpha}\mathbb{E}\int_0^t \|\nabla u\|_{L^2}^2 ds \mathbb{E}\int_0^t \epsilon \|\nabla\rho\|_{L^2}^2 ds + \delta' \\
&\leq C|t-t'|^{\frac{q-2}{q}-\alpha} \epsilon \mathbb{E}\left(\int_0^t \|\nabla\rho\|_{L^2}^q ds\right)^{\frac{2}{q}} + \delta' \leq C,
\end{aligned} \tag{2.2.39}$$

for any $\alpha \in [0, \frac{q-2}{q}]$ and $k \geq \frac{5}{2}$.

Combining all estimates, we get the desired bound (2.2.37). For any $R > 0$, define the set

$$\begin{aligned}
B_{1,R} = \Big\{ &P_n(\rho_n u_n) \in L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{D})) \cap C^\alpha([0,T]; H^{-k}(\mathcal{D})) : \\
&\|\rho_n u_n\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))} + \|\rho_n u_n\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} \leq R \Big\}.
\end{aligned}$$

By the Aubin-Lions lemma A.0.1, we know

$$L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{D})) \cap C^\alpha([0,T]; H^{-k}(\mathcal{D})) \hookrightarrow L^\infty(0,T; H^{-1}(\mathcal{D})), \tag{2.2.40}$$

is compact, therefore, the set $B_{1,R}$ is relatively compact in $L^\infty(0,T; H^{-1}(\mathcal{D}))$. Considering (2.2.37), (2.2.22) and the Chebyshev inequality, to conclude

$$\begin{aligned}
\nu_{\rho u}^n(B_{1,R}^\complement) \leq\ &\mathbb{P}\left(\|\rho_n u_n\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))} > \frac{R}{2}\right) \\
&+ \mathbb{P}\left(\|\rho_n u_n\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} > \frac{R}{2}\right)
\end{aligned}$$

$$\leq \frac{2}{R}\mathbb{E}\left(\|\rho_n u_n\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))} + \|\rho_n u_n\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))}\right) \leq \frac{C}{R},$$

leading to the tightness of set $\{\nu^n_{\rho u}\}_{n\geq 1}$.

**Claim 2**. The set $\{\nu^n_c\}_{n\geq 1}$ is tight on path space $\mathcal{X}_c$.

Note that, for any $R > 0$, by the Banach-Alaoglu theorem, the set

$$B_{2,R} := \left\{c_n \in L^2(0,T;H^1(\mathcal{D})) : \|c_n\|_{L^2(0,T;H^1(\mathcal{D}))} \leq R\right\},$$

is relatively compact on path space $L^2_w(0,T;H^1(\mathcal{D}))$. On the other hand, we have

$$\partial_t c_n \in L^p(\Omega; L^2(0,T;H^{-1}(\mathcal{D}))). \tag{2.2.41}$$

Define the set

$$\begin{aligned} B_{3,R} = \big\{c_n \in L^2(0,T;H^1(\mathcal{D})) \cap W^{1,2}(0,T;H^{-1}(\mathcal{D})) :& \\ \|c_n\|_{L^2(0,T;H^1(\mathcal{D}))} + \|c_n\|_{W^{1,2}(0,T;H^{-1}(\mathcal{D}))} \leq R\big\},& \end{aligned}$$

which is compact on $L^2(0,T;L^2(\mathcal{D}))$. The bounds (2.2.20), (2.2.41) and the Chebyshev inequality imply

$$\nu^n_c\left((B_{2,R}\cap B_{3,R})^\complement\right) \leq \nu^n_c(B^\complement_{2,R}) + \nu^n_c(B^\complement_{3,R}) \leq \frac{C}{R}.$$

**Claim 3**. The set $\{\nu^n_Q\}_{n\geq 1}$ is tight on path space $\mathcal{X}_Q$.

The proof follows the same line as above, here we only give the necessary estimates. Using (2.2.20) and the Hölder inequality again, we have

$$\begin{aligned} &\mathbb{E}\left\| -(u_n\cdot\nabla)Q_n - (Q_n\Psi_n - \Psi_n Q_n) + \Gamma\mathrm{H}(Q_n,c_n)\right\|_{L^2(0,T;L^{\frac{3}{2}}(\mathcal{D}))} \\ &\leq C\mathbb{E}\left[\int_0^t \|u_n\|^2_{L^6}\|\nabla Q_n\|^2_{L^2} + \|Q_n\|^2_{L^6}\|\nabla u_n\|^2_{L^2} + \|\triangle Q_n\|^2_{L^2} + \|c_n\|^2_{L^2}\|Q_n\|^2_{L^6}ds\right]^{\frac{1}{2}} \\ &\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\|Q_n\|^2_{H^1}\right]\mathbb{E}\int_0^t\|\nabla u_n\|^2_{L^2}ds + \mathbb{E}\int_0^t\|\triangle Q_n\|^2_{L^2}ds \\ &\quad + \mathbb{E}\left[\sup_{t\in[0,T]}\left(\|c_n\|^2_{L^2} + \|Q_n\|^2_{H^1}\right)\right] \leq C, \end{aligned}$$

leading to

$$\begin{aligned}
&\mathbb{E}\|Q_n\|_{C^\alpha([0,T];L^{\frac{3}{2}}(\mathcal{D}))}\\
&\le \mathbb{E}\frac{\int_{t'}^{t}\|-(u_n\cdot\nabla)Q_n-(Q_n\Psi_n-\Psi_nQ_n)+\Gamma\mathrm{H}(Q_n,c_n)\|_{L^{\frac{3}{2}}}ds}{|t-t'|^\alpha}+\delta'\\
&\le |t-t'|^{\frac{1}{2}-\alpha}\cdot\mathbb{E}\|-(u_n\cdot\nabla)Q_n-(Q_n\Psi_n-\Psi_nQ_n)+\Gamma\mathrm{H}(Q_n,c_n)\|_{L^2(0,T;L^{\frac{3}{2}}(\mathcal{D}))}\\
&\quad+\delta'\le C,
\end{aligned}$$

where $C$ is independent of $n$.

**Claim 4**. The sets $\{\nu_u^n\}_{n\ge1}$ and $\{\nu_\rho^n\}_{n\ge1}$ are tight on path spaces $\mathcal{X}_u,\mathcal{X}_\rho$.

Here, we only focus on the tightness of $\{\nu_\rho^n\}_{n\ge1}$ on space $L^2([0,T]\times\mathcal{D})$. Since $\rho_n\in L^2(0,T;H^1(\mathcal{D}))$ and $\partial_t\rho_n\in L^2(0,T;H^{-1}(\mathcal{D}))$, then we can show the tightness using the same argument as Claim 2.

Finally, Lemma 2.2.5 follows the result of Claims 1-4. □

Proof of Proposition 2.2.4. With the tightness established, the Skorokhod-Jakubowski theorem is invoked to get that there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ with $\mathcal{X}$-valued measurable random variables

$$(\tilde{u}_n,\tilde{\rho}_n,\tilde{q}_n,\tilde{c}_n,\widetilde{Q}_n,\widetilde{\mathcal{W}}_n)\text{ and }(\tilde{u},\tilde{\rho},\tilde{q},\tilde{c},\widetilde{Q},\widetilde{\mathcal{W}}),$$

such that

$$(\tilde{u}_n,\tilde{\rho}_n,\tilde{q}_n,\tilde{c}_n,\widetilde{Q}_n,\widetilde{\mathcal{W}}_n)\to(\tilde{u},\tilde{\rho},\tilde{q},\tilde{c},\widetilde{Q},\widetilde{\mathcal{W}}),\ \ \widetilde{\mathbb{P}}\ \text{ a.s.}$$

in the topology of $\mathcal{X}$. Moreover, the joint distribution of $(\tilde{u}_n,\tilde{\rho}_n,\tilde{q}_n,\tilde{c}_n,\widetilde{Q}_n,\widetilde{\mathcal{W}}_n)$ is the same as the law of $(u_n,\rho_n,P_n(\rho_nu_n),c_n,Q_n,\mathcal{W}_n)$, consequently, we have $\tilde{q}_n=P_n(\tilde{\rho}_n\tilde{u}_n)$, $\widetilde{Q}_n\in S_0^3$, a.s. and the energy estimates (2.2.31)-(2.2.36) hold. Moreover, the process $(\tilde{\rho}_n,P_n(\tilde{\rho}_n\tilde{u}_n),\tilde{c}_n,\widetilde{Q}_n,\widetilde{\mathcal{W}}_n)$ also satisfies the system (2.2.1) using the same argument as [84].

It remains to identify $\tilde{q}=\tilde{\rho}\tilde{u}$. On the one hand, we have

$$P_n(\tilde{\rho}_n\tilde{u}_n)\to\tilde{q}\text{ in }C(0,T;H^{-1}(\mathcal{D})),\ \ \widetilde{\mathbb{P}}\ \text{ a.s.}$$

On the other hand, from

$$\tilde{\rho}_n \to \tilde{\rho} \ \text{ in } \ L^\infty(0,T;H^{-\frac{1}{2}}(\mathcal{D})).$$

and

$$\tilde{u}_n \rightharpoonup \tilde{u} \ \text{ in } \ L^2(0,T;H^1(\mathcal{D})).$$

it follows that $P_n(\tilde{\rho}_n\tilde{u}_n) \rightharpoonup \tilde{\rho}\tilde{u}$ in $L^2(0,T;H^{-1}(\mathcal{D}))$, $\widetilde{\mathbb{P}}$ a.s. Then, we infer $\tilde{q} = \tilde{\rho}\tilde{u}$. □

### 2.2.3 Taking the limit for n tends to infinity

Based on the Proposition 2.2.4, we identify the limit of the nonlinear term.

**Lemma 2.2.6.** *For any $\phi \in L^\infty(0,T;H^{1,6}(\mathcal{D}))$ and $t \in [0,T]$, the following convergence holds $\widetilde{\mathbb{P}}$ a.s.*

$$\begin{aligned}
&\int_0^t \langle \mathrm{F}(\widetilde{Q}_n)\mathrm{I}_3 - \nabla\widetilde{Q}_n \odot \nabla\widetilde{Q}_n + \widetilde{Q}_n\triangle\widetilde{Q}_n - \triangle\widetilde{Q}_n\widetilde{Q}_n + \sigma^*\tilde{c}_n^2\widetilde{Q}_n, \nabla\phi\rangle ds \\
&\to \int_0^t \langle \mathrm{F}(\widetilde{Q})\mathrm{I}_3 - \nabla\widetilde{Q} \odot \nabla\widetilde{Q} + \widetilde{Q}\triangle\widetilde{Q} - \triangle\widetilde{Q}\widetilde{Q} + \sigma^*\tilde{c}^2\widetilde{Q}, \nabla\phi\rangle ds,
\end{aligned}$$

*as $n \to \infty$.*

*Proof.* Decompose

$$\begin{aligned}
&\int_0^t \langle \widetilde{Q}_n\triangle\widetilde{Q}_n - \triangle\widetilde{Q}_n\widetilde{Q}_n - (\widetilde{Q}\triangle\widetilde{Q} - \triangle\widetilde{Q}\widetilde{Q}), \nabla\phi\rangle ds \\
&= \int_0^t \langle (\widetilde{Q}_n - \widetilde{Q})\triangle\widetilde{Q}_n, \nabla\phi\rangle ds + \int_0^t \langle \widetilde{Q}(\triangle\widetilde{Q}_n - \triangle\widetilde{Q}), \nabla\phi\rangle ds \\
&\quad + \int_0^t \langle (\triangle\widetilde{Q} - \triangle\widetilde{Q}_n)\widetilde{Q}, \nabla\phi\rangle ds + \int_0^t \langle \triangle\widetilde{Q}_n(\widetilde{Q} - \widetilde{Q}_n), \nabla\phi\rangle ds \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

For $J_1, J_4$, by Proposition 2.2.4(2.2.28) and (2.2.36), we have $\widetilde{\mathbb{P}}$ a.s.

$$\begin{aligned}
|J_1 + J_4| &\le \int_0^t \|\nabla\varphi\|_{L^3}\|\widetilde{Q}_n - \widetilde{Q}\|_{L^6}\|\triangle\widetilde{Q}_n\|_{L^2} ds \\
&\le \|\nabla\varphi\|_{L^\infty(0,T;L^3(\mathcal{D}))}\left(\int_0^t \|\widetilde{Q}_n - \widetilde{Q}\|_{H^1}^2 ds\right)^{\frac{1}{2}}\left(\int_0^t \|\triangle\widetilde{Q}_n\|_{L^2}^2 ds\right)^{\frac{1}{2}} \to 0.
\end{aligned}$$

Also, we have $J_2, J_3 \to 0$ as $n \to \infty$ using the fact $\triangle\widetilde{Q}_n \rightharpoonup \triangle\widetilde{Q}$ in $L^2([0,T]\times\mathcal{D})$.

On the other hand, by Proposition 2.2.4(2.2.28), (2.2.35) and (2.2.36), the following convergences hold $\widetilde{\mathbb{P}}$ a.s.

$$
\begin{aligned}
&\int_0^t \langle \nabla \widetilde{Q}_n \odot \nabla \widetilde{Q}_n - \nabla \widetilde{Q} \odot \nabla \widetilde{Q}, \nabla \phi \rangle ds \\
&\le \int_0^t \|\nabla \widetilde{Q}_n - \nabla \widetilde{Q}\|_{L^2} \|\nabla \widetilde{Q}, \nabla \widetilde{Q}_n\|_{L^6} \|\nabla \phi\|_{L^3} ds \\
&\le \|\nabla \phi\|_{L^\infty(0,T;L^3(\mathcal{D}))} \left( \int_0^t \|\widetilde{Q}_n - \widetilde{Q}\|_{H^1}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \widetilde{Q}_n, \nabla \widetilde{Q}\|_{H^1}^2 ds \right)^{\frac{1}{2}} \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \langle \tilde{c}_n^2 \widetilde{Q}_n - \tilde{c}^2 \widetilde{Q}, \nabla \phi \rangle ds = \int_0^t \langle (\tilde{c}_n^2 - \tilde{c}^2) \widetilde{Q}_n + \tilde{c}^2 (\widetilde{Q}_n - \widetilde{Q}), \nabla \phi \rangle ds \\
&\le \int_0^t \|\tilde{c}_n - \tilde{c}\|_{L^2} \|\tilde{c}_n, \tilde{c}\|_{L^6} \|\widetilde{Q}_n\|_{L^6} \|\nabla \phi\|_{L^6} + \|\tilde{c}\|_{L^6} \|\tilde{c}\|_{L^2} \|\widetilde{Q}_n - \widetilde{Q}\|_{L^6} \|\nabla \phi\|_{L^6} ds \\
&\le \left( \|\widetilde{Q}_n\|_{L^\infty(0,T;L^6(\mathcal{D}))} + \|\tilde{c}\|_{L^\infty(0,T;L^2(\mathcal{D}))} \right) \|\nabla \phi\|_{L^\infty(0,T;L^6(\mathcal{D}))} \\
&\quad \times \left( \int_0^t \|\tilde{c}_n - \tilde{c}\|_{L^2}^2 + \|\widetilde{Q}_n - \widetilde{Q}\|_{H^1}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t (1 + \|\tilde{c}_n, \tilde{c}\|_{L^6})^2 ds \right)^{\frac{1}{2}} \to 0.
\end{aligned}
$$

Similarly, we have $\widetilde{\mathbb{P}}$ a.s.

$$
\int_0^t \langle \mathrm{F}(\widetilde{Q}_n) \mathrm{I}_3 - \mathrm{F}(\widetilde{Q}) \mathrm{I}_3, \nabla \phi \rangle ds \to 0.
$$

This completes the proof. □

**Lemma 2.2.7.** *For any $\varphi \in L^\infty(0,T;L^3(\mathcal{D}))$ and $t \in [0,T]$, the following convergence holds $\widetilde{\mathbb{P}}$ a.s.*

$$
\begin{aligned}
&\int_0^t \langle (\tilde{u}_n \cdot \nabla) \widetilde{Q}_n + \widetilde{Q}_n \widetilde{\Psi}_n - \widetilde{\Psi}_n \widetilde{Q}_n - \Gamma \mathrm{H}(\widetilde{Q}_n, \tilde{c}_n), \varphi \rangle ds \\
&\to \int_0^t \langle (\tilde{u} \cdot \nabla) \widetilde{Q} + \widetilde{Q} \widetilde{\Psi} - \widetilde{\Psi} \widetilde{Q} - \Gamma \mathrm{H}(\widetilde{Q}, \tilde{c}), \varphi \rangle ds,
\end{aligned}
$$

*as $n \to \infty$.*

*Proof.* Decompose

$$
\begin{aligned}
&\int_0^t \langle (\tilde{u}_n \cdot \nabla)\widetilde{Q}_n - (\tilde{u} \cdot \nabla)\widetilde{Q}, \varphi \rangle ds \\
&= \int_0^t \langle (\tilde{u}_n \cdot \nabla)(\widetilde{Q}_n - \widetilde{Q}), \varphi \rangle ds + \int_0^t \langle ((\tilde{u}_n - \tilde{u}) \cdot \nabla)\widetilde{Q}, \varphi \rangle ds \\
&=: J_1 + J_2.
\end{aligned}
$$

For $J_1$, using the Proposition 2.2.4(2.2.28), (2.2.32) and the Hölder inequality, to get $\widetilde{\mathbb{P}}$ a.s.

$$
\begin{aligned}
|J_1| &\leq \int_0^t \|\tilde{u}_n\|_{L^6} \|\nabla(\widetilde{Q}_n - \widetilde{Q})\|_{L^2} \|\varphi\|_{L^3} ds \\
&\leq \|\varphi\|_{L^\infty(0,T;L^3(\mathcal{D}))} \left( \int_0^t \|\nabla(\widetilde{Q}_n - \widetilde{Q})\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\tilde{u}_n\|_{H^1}^2 ds \right)^{\frac{1}{2}} \to 0.
\end{aligned}
$$

Also, we have $J_2 \to 0$ as $n \to \infty$ which is the result of the convergence of $\tilde{u}_n$ in $\mathcal{X}_u$, $\widetilde{\mathbb{P}}$ a.s. By the similar argument as the first term, using the strong convergences of $\widetilde{Q}_n$ in $\mathcal{X}_Q$ and $\tilde{c}_n$ in $\mathcal{X}_c$, $\widetilde{\mathbb{P}}$ a.s. and (2.2.32), (2.2.35), (2.2.36), we have $\widetilde{\mathbb{P}}$ a.s.

$$
\int_0^t \langle \widetilde{Q}_n \widetilde{\Psi}_n - \widetilde{\Psi}_n \widetilde{Q}_n - \Gamma \mathrm{H}(\widetilde{Q}_n, \tilde{c}_n), \varphi \rangle ds \to \int_0^t \langle \widetilde{Q} \widetilde{\Psi} - \widetilde{\Psi} \widetilde{Q} - \Gamma \mathrm{H}(\widetilde{Q}, \tilde{c}), \varphi \rangle ds.
$$

as $n \to \infty$. □

For the sake of elaborating the convergence of term $\epsilon \nabla \tilde{\rho}_n \cdot \nabla \tilde{u}_n$, in [32], Feireisl-Novotný-Petzeltová showed $\tilde{\rho}_n \to \tilde{\rho}$ in $L^2(0,T;H^1(\mathcal{D}))$, $\widetilde{\mathbb{P}}$ a.s. Then, we have

$$
\nabla \tilde{\rho}_n \cdot \nabla \tilde{u}_n \to \nabla \tilde{\rho} \cdot \nabla \tilde{u} \text{ in } L^\infty([0,T] \times \mathcal{D})', \ \widetilde{\mathbb{P}} \text{ a.s.} \tag{2.2.42}
$$

Moreover, [43, 81, 84] give

$$
\tilde{\rho}_n \tilde{u}_n \otimes \tilde{u}_n \to \tilde{\rho} \tilde{u} \otimes \tilde{u} \text{ in } L^\infty([0,T] \times \mathcal{D})', \ \widetilde{\mathbb{P}} \text{ a.s.} \tag{2.2.43}
$$

and for $q \in [1, \frac{2\beta}{\beta+1})$

$$
\tilde{\rho}_n \tilde{u}_n \to \tilde{\rho} \tilde{u} \text{ in } L^q([0,T] \times \mathcal{D}), \ \widetilde{\mathbb{P}} \text{ a.s.} \tag{2.2.44}
$$

Furthermore, using the Proposition 2.2.4(2.2.28), (2.2.32), (2.2.35), we have

$$
\tilde{u}_n \cdot \nabla \tilde{c}_n \to \tilde{u} \cdot \nabla \tilde{c} \text{ in } L^\infty([0,T] \times \mathcal{D})', \ \widetilde{\mathbb{P}} \text{ a.s.} \tag{2.2.45}
$$

Define the functional for any $\phi \in \cup X_n$

$$
\begin{aligned}
\mathcal{N}(\rho, u, c, Q) =& \int_{\mathcal{D}} \rho u \phi dx - \int_{\mathcal{D}} m(0)\phi dx \\
&- \int_0^t \int_{\mathcal{D}} (\rho u \otimes u) - \mu_1 \nabla u)\nabla\phi - ((\mu_1+\mu_2)\mathrm{div} u + \rho^\gamma + \delta\rho^\beta)\mathrm{div}\phi dx ds \\
&+ \int_0^t \int_{\mathcal{D}} \sigma^*(c^2 Q)\nabla\phi dx ds - \int_0^t \int_{\mathcal{D}} \epsilon\phi\nabla\rho\cdot\nabla u dx ds \\
&+ \int_0^t \int_{\mathcal{D}} (\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q + Q\triangle Q - \triangle Q Q)\nabla\phi dx ds.
\end{aligned}
$$

Following ideas of [43, 20], we are able to obtain the limit $(\tilde{c}, \tilde{\rho}, \tilde{u}, \widetilde{Q}, \widetilde{\mathcal{W}})$ satisfies the momentum equation once we show that the process $\mathcal{N}(\tilde{c}, \tilde{\rho}, \tilde{u}, \widetilde{Q})_t$ is a square integral martingale and its quadratic and cross variations satisfy

$$
\ll \mathcal{N}(\tilde{c}, \tilde{\rho}, \tilde{u}, \widetilde{Q})_t \gg = \sum_{k\geq 1} \int_0^t \langle \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q})\beta_k, \phi\rangle ds, \tag{2.2.46}
$$

$$
\ll \mathcal{N}(\tilde{c}, \tilde{\rho}, \tilde{u}, \widetilde{Q})_t, \tilde{\beta}_k \gg = \int_0^t \langle \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q})\beta_k, \phi\rangle ds. \tag{2.2.47}
$$

Here, we only focus on the noise term. It is enough to show that $\widetilde{\mathbb{P}} \otimes \mathcal{L}$ a.e.

$$
\langle \mathcal{M}^{\frac{1}{2}}(\tilde{\rho}_n) P_n(\sqrt{\tilde{\rho}_n} f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n)\cdot), \phi\rangle \to \langle \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q})\cdot, \phi\rangle \text{ in } L_2(\mathcal{H}; \mathbb{R}). \tag{2.2.48}
$$

Toward proving the convergence, we estimate by the Minkowski inequality

$$
\begin{aligned}
&\left\| \langle \mathcal{M}^{\frac{1}{2}}(\tilde{\rho}_n) P_n(\sqrt{\tilde{\rho}_n} f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n)\cdot, \phi\rangle - \langle \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q})\cdot, \phi\rangle \right\|_{L_2(\mathcal{H};\mathbb{R})} \\
&\leq C \left\| \mathcal{M}^{\frac{1}{2}}(\tilde{\rho}_n) P_n(\sqrt{\tilde{\rho}_n} f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n) - \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}) \right\|_{L_2(\mathcal{H};H^{-k})} \\
&\leq C \left( \sum_{k\geq 1} \|\tilde{\rho}_n f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n)e_k - \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q})e_k\|_{L^1}^2 \right)^{\frac{1}{2}} \\
&\quad + \left\| \mathcal{M}^{\frac{1}{2}}(\tilde{\rho}_n) P_n(\sqrt{\tilde{\rho}_n} f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n) - \tilde{\rho}_n f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n) \right\|_{L_2(\mathcal{H};H^{-k})} \\
&\leq C \int_{\mathcal{D}} \left( \sum_{k\geq 1} |\tilde{\rho}_n f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n)e_k - \tilde{\rho} f(\tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q})e_k|^2 \right)^{\frac{1}{2}} dx \\
&\quad + \left\| \mathcal{M}^{\frac{1}{2}}(\tilde{\rho}_n) P_n(\sqrt{\tilde{\rho}_n} f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n)) - \tilde{\rho}_n f(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{c}_n, \widetilde{Q}_n) \right\|_{L_2(\mathcal{H};H^{-k})} \\
&=: \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}
$$

Next, we show that $\mathcal{J}_1, \mathcal{J}_2 \to 0$, as $n \to \infty$, $\widetilde{\mathbb{P}} \otimes \mathcal{L}$ a.e. Indeed, by condition (2.1.2) as well as Proposition 2.2.4, we have

$$|\mathcal{J}_1| \leq C\|\tilde{\rho}_n - \tilde{\rho}, \tilde{\rho}_n\tilde{u}_n - \tilde{\rho}\tilde{u}, \tilde{c}_n - \tilde{c}, \widetilde{Q}_n - \widetilde{Q}\|_{L^{\frac{2\gamma}{\gamma+1}}} \to 0.$$

Also, using the Hölder inequality, condition (2.1.1), the bound (2.2.20) and Proposition 2.2.4, we have $\mathcal{J}_2 \to 0$, as $n \to \infty$, see also [43, Proposition 4.11]. Then, (2.2.48) follows. We could obtain equalities (2.2.46), (2.2.47) by combining (2.2.42)-(2.2.44), (2.2.48), Proposition 2.2.4 and the Vitali convergence theorem A.0.3.

Using the same argument as above, we infer that it holds $\widetilde{\mathbb{P}}$ a.s.

$$\begin{aligned}
\int_{\mathcal{D}} \tilde{c}(t)\ell dx &= \int_{\mathcal{D}} \tilde{c}(0)\ell dx - \int_0^t \int_{\mathcal{D}} (\tilde{u}\cdot\nabla)\tilde{c}\cdot\ell dxds - \int_0^t\int_{\mathcal{D}} \nabla\tilde{c}\cdot\nabla\ell dxds,\\
\int_{\mathcal{D}} \widetilde{Q}(t)\varphi dx &= \int_{\mathcal{D}} \widetilde{Q}(0)\varphi dx - \int_0^t \int_{\mathcal{D}} ((\tilde{u}\cdot\nabla)\widetilde{Q} + \widetilde{Q}\widetilde{\Psi} - \widetilde{\Psi}\widetilde{Q})\varphi dxds\\
&\quad + \int_0^t\int_{\mathcal{D}} \Gamma\varphi \mathrm{H}(\widetilde{Q},\tilde{c})dxds,
\end{aligned}$$

for $\ell \in C^\infty(\mathcal{D}), \varphi \in C^\infty(\mathcal{D})$, $t \in [0,T]$. We summarize the result for this section,

**Proposition 2.2.8.** *For $\beta > \max\{6,\gamma\}$, fixed $\delta > 0$. If conditions (2.1.1), (2.1.2) hold. There exists a global weak martingale solution to modified system (2.2.1)-(2.2.6).*

## 2.3 The Existence of Martingale Solution for Vanishing Artificial Viscosity

In this section, we let $\epsilon \to 0$ to build the existence of global weak martingale solution to the following system

$$\begin{cases}
\partial_t c + (u\cdot\nabla)c = \triangle c,\\
\partial_t\rho + \mathrm{div}(\rho u) = 0,\\
\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla(\rho^\gamma + \delta\rho^\beta) = \mu_1\Delta u + (\mu_1+\mu_2)\nabla(\mathrm{div} u)\\
\qquad +\sigma^*\nabla\cdot(c^2Q) + \nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q) + \nabla\cdot(Q\triangle Q - \triangle QQ)\\
\qquad +\rho f(\rho,\rho u,c,Q)\frac{d\mathcal{W}}{dt},\\
\partial_t Q + (u\cdot\nabla)Q + Q\Psi - \Psi Q = \Gamma\mathrm{H}(Q,c).
\end{cases} \tag{2.3.1}$$

The solutions $(\rho_{\epsilon,\delta}, u_{\epsilon,\delta}, c_{\epsilon,\delta}, Q_{\epsilon,\delta})$ obtained in the first level approximation will be used for the approximate solution in this section, which shares the same energy bounds with (2.2.31)-(2.2.36). Namely,

$$\rho_{\epsilon,\delta}u_{\epsilon,\delta} \in L^p(\Omega; L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))), \tag{2.3.2}$$

$$\rho_{\epsilon,\delta} \in L^p(\Omega; L^\infty(0,T; L^\beta(\mathcal{D}))), \tag{2.3.3}$$

$$u_{\epsilon,\delta} \in L^p(\Omega; L^2(0,T; H^1(\mathcal{D}))), \tag{2.3.4}$$

$$\sqrt{\rho_{\epsilon,\delta}}u_{\epsilon,\delta} \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{D}))), \tag{2.3.5}$$

$$c_{\epsilon,\delta} \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{D})) \cap L^2(0,T; H^1(\mathcal{D}))), \tag{2.3.6}$$

$$Q_{\epsilon,\delta} \in L^p(\Omega; L^\infty(0,T; H^1(\mathcal{D})) \cap L^2(0,T; H^2(\mathcal{D}))). \tag{2.3.7}$$

The proof also consists of the argument of tightness and identifying the limit. Note that, here we are not allowed to make use of the a priori bound $\sqrt{\epsilon}\rho_{\epsilon,\delta} \in L^p(\Omega; L^2(0,T; H^1(\mathcal{D})))$ to gain the tightness of the distribution of density on path space $L^2(0,T; L^2(\mathcal{D}))$. Therefore, we are not able to identify the pressure and stochastic term. To overcome this difficulty, we first improve the integrability of density.

We replace $(\rho_{\epsilon,\delta}, u_{\epsilon,\delta}, c_{\epsilon,\delta}, Q_{\epsilon,\delta})$ by $(\rho_\epsilon, u_\epsilon, c_\epsilon, Q_\epsilon)$ to simplify the notation.

Recall the operator $\mathcal{T}$ constructed by Bogovskii [7] related to the problem

$$\mathrm{div} v = f, \quad v|_{\partial\mathcal{D}} = 0,$$

with the following properties:

1. $\mathcal{T} : \left\{ f \in L^p : \int_{\mathcal{D}} f dx = 0 \right\} \to H_0^{1,p}(\mathcal{D})$ is a bounded linear operator such that for all $p > 1$

$$\|\mathcal{T}(f)\|_{H_0^{1,p}(\mathcal{D})} \leq C\|f\|_{L^p(\mathcal{D})}. \tag{2.3.8}$$

2. $v = \mathcal{T}(f)$ is a solution to above equation.

3. For any function $g \in L^r(\mathcal{D})$ with $g \cdot \vec{n}|_{\partial\mathcal{D}} = 0$, it holds

$$\|\mathcal{T}(\mathrm{div} g)\|_{L^r(\mathcal{D})} \leq C\|g\|_{L^r(\mathcal{D})}. \tag{2.3.9}$$

The proof of above properties, we refer the readers to [8, 36] and the references therein for details.

**Lemma 2.3.1.** *The approximate sequence $\rho_\epsilon$ satisfies the following estimate*

$$\mathbb{E}\int_0^T \int_{\mathcal{D}} \rho_\epsilon^{\gamma+1} + \delta\rho_\epsilon^{\beta+1} dxdt \leq C,$$

*where the constant $C$ is independent of $\epsilon$.*

*Proof.* The proof is similar to that of [32]. Applying the Itô formula to the function

$$\Phi(\rho_\epsilon u_\epsilon, \mathcal{T}[\rho_\epsilon - (\rho_0)_m]) = \int_{\mathcal{D}} \rho_\epsilon u_\epsilon \cdot \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dx.$$

Then

$$\begin{aligned}
&\int_{\mathcal{D}} \rho_\epsilon u_\epsilon \cdot \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dx \\
= &\int_0^t \int_{\mathcal{D}} \rho_\epsilon^{\gamma+1} + \delta\rho_\epsilon^{\beta+1} dxds + \int_{\mathcal{D}} m_0 \cdot \mathcal{T}[\rho_0 - (\rho_0)_m] dx \\
&- \mu_1 \int_0^t \int_{\mathcal{D}} \nabla u_\epsilon : \nabla \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxds \\
&- (\mu_1 + \mu_2) \int_0^t \int_{\mathcal{D}} \mathrm{div} u_\epsilon \cdot [\rho_\epsilon - (\rho_0)_m] dxds \\
&+ \int_0^t \int_{\mathcal{D}} \rho_\epsilon u_\epsilon \otimes u_\epsilon : \nabla \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxds - \int_0^t (\rho_0)_m \int_{\mathcal{D}} \rho_\epsilon^\gamma + \delta \rho_\epsilon^\beta dxds \\
&- \epsilon \int_0^t \int_{\mathcal{D}} \nabla u_\epsilon \nabla \rho_\epsilon \cdot \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxds - \int_0^t \int_{\mathcal{D}} \rho_\epsilon u_\epsilon \cdot \mathcal{T}[\mathrm{div}(\rho_\epsilon u_\epsilon)] dxds \\
&+ \epsilon \int_0^t \int_{\mathcal{D}} \rho_\epsilon u_\epsilon \cdot \mathcal{T}[\triangle \rho_\epsilon] dxds - \int_0^t \int_{\mathcal{D}} \sigma^* c_\epsilon^2 Q_\epsilon : \nabla \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxds \\
&- \int_0^t \int_{\mathcal{D}} (\nabla Q_\epsilon \odot \nabla Q_\epsilon - \mathrm{F}(\mathrm{Q}_\epsilon)\mathrm{I}_3) : \nabla \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxds \\
&- \int_0^t \int_{\mathcal{D}} (Q_\epsilon \triangle Q_\epsilon - \triangle Q_\epsilon Q_\epsilon) : \nabla \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxds \\
&+ \int_0^t \int_{\mathcal{D}} \rho_\epsilon f(\rho_\epsilon, \rho_\epsilon u_\epsilon, c_\epsilon, Q_\epsilon) \cdot \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dxd\mathcal{W} \\
= &\int_0^t \int_{\mathcal{D}} \rho_\epsilon^{\gamma+1} + \delta\rho_\epsilon^{\beta+1} dxds + J_0 + \int_0^t J_1 + \cdots + J_{10} ds + \int_0^t J_{11} d\mathcal{W}.
\end{aligned} \tag{2.3.10}$$

Taking expectation on both sides of (2.3.10), rearranging and to obtain

$$\begin{aligned}
&\mathbb{E}\int_0^t \int_{\mathcal{D}} \rho_\epsilon^{\gamma+1} + \delta\rho_\epsilon^{\beta+1} dxds \\
&= -\mathbb{E}\left( J_0 + \sum_{i=1}^{10} \int_0^t J_i ds \right) + \mathbb{E}\int_{\mathcal{D}} \rho_\epsilon u_\epsilon \cdot \mathcal{T}[\rho_\epsilon - (\rho_0)_m] dx,
\end{aligned} \tag{2.3.11}$$

in fact $\mathbb{E}\int_0^t J_{11} d\mathcal{W} = 0$, since the process $\int_0^t J_{11} d\mathcal{W}$ is a square integrable martingale. Indeed, using condition (2.1.1)

$$\begin{aligned}
&\mathbb{E}\int_0^t \sum_{k\geq 1}\left(\int_{\mathcal{D}} \rho_\epsilon f(\rho_\epsilon, \rho_\epsilon u_\epsilon, c_\epsilon, Q_\epsilon)\cdot \mathcal{T}[\rho_\epsilon - (\rho_0)_m] e_k dx\right)^2 ds \\
&\leq C\mathbb{E}(\|\mathcal{T}[\rho_\epsilon - (\rho_0)_m]\|^2_{L^\infty([0,T]\times\mathcal{D})}\times \\
&\qquad \int_0^t \int_{\mathcal{D}} \sum_{k\geq 1} \rho_\epsilon |f(\rho_\epsilon, \rho_\epsilon u_\epsilon, c_\epsilon, Q_\epsilon) e_k|^2 dx \int_{\mathcal{D}} \rho_\epsilon dx ds) \\
&\leq C\mathbb{E}\|\mathcal{T}[\rho_\epsilon - (\rho_0)_m]\|^4_{L^\infty([0,T]\times\mathcal{D})} \\
&\quad + C\mathbb{E}\left(\int_{\mathcal{D}} \rho_\epsilon(0) dx\right)^4 \mathbb{E}\left(\int_0^t \int_{\mathcal{D}} \sum_{k\geq 1} \rho_\epsilon |f(\rho_\epsilon, \rho_\epsilon u_\epsilon, c_\epsilon, Q_\epsilon) e_k|^2 dx ds\right)^4 \\
&\leq C\mathbb{E}\|\mathcal{T}[\rho_\epsilon - (\rho_0)_m]\|^4_{L^\infty([0,T]\times\mathcal{D})} \\
&\quad + C\delta^{-\frac{1}{\beta}}\mathbb{E}\left(\int_0^t \int_{\mathcal{D}} \rho_\epsilon^\gamma + |\sqrt{\rho_\epsilon} u_\epsilon|^2 + c_\epsilon^2 + |Q_\epsilon|^2 dx ds\right)^4 \leq C(\delta).
\end{aligned}$$

The desired result follows once each term on the right hand side of (2.3.11) can be controlled. By the Hölder inequality, (2.3.2) and (2.3.9), we obtain

$$\begin{aligned}
&\left|-\mathbb{E}\int_0^t J_3 + J_6 ds\right| \\
&\leq C\mathbb{E}\int_0^T \|\rho_\epsilon u_\epsilon\|_{L^{\frac{2\beta}{\beta+1}}} \|u_\epsilon\|_{L^6} \|\rho_\epsilon - (\rho_0)_m\|_{L^\beta} \\
&\qquad + \|\rho_\epsilon\|_{L^\beta} \|u_\epsilon\|_{L^6} \|\mathcal{T}[\operatorname{div}(\rho_\epsilon u_\epsilon)]\|_{L^{\frac{2\beta}{\beta+1}}} dt \\
&\leq C\mathbb{E}\int_0^T \|\rho_\epsilon u_\epsilon\|_{L^{\frac{2\beta}{\beta+1}}} \|\nabla u_\epsilon\|_{L^2} \|\rho_\epsilon - (\rho_0)_m\|_{L^\beta} dt \\
&\leq C\mathbb{E}\left(\sup_{t\in[0,T]} \|\rho_\epsilon u_\epsilon\|_{L^{\frac{2\beta}{\beta+1}}} \|\rho_\epsilon - (\rho_0)_m\|_{L^\beta} \int_0^T \|\nabla u_\epsilon\|_{L^2} dt\right) \leq C.
\end{aligned}$$

Again, using the Hölder inequality, the Sobolev embedding $H^{1,p}(\mathcal{D}) \hookrightarrow L^\infty(\mathcal{D})$ for $p > 3$ and (2.3.8)

$$\begin{aligned}
\left|\mathbb{E}\int_0^t -J_5 ds\right| &\leq \epsilon C\mathbb{E}\int_0^T \|\nabla u_\epsilon\|_{L^2} \|\nabla \rho_\epsilon\|_{L^2} \|\mathcal{T}[\rho_\epsilon - (\rho_0)_m]\|_{L^\infty} dt \\
&\leq \epsilon C\mathbb{E}\int_0^T \|\nabla u_\epsilon\|_{L^2} \|\nabla \rho_\epsilon\|_{L^2} \|\mathcal{T}(\rho_\epsilon)\|_{H^{1,\beta}} dt \\
&\leq \epsilon C\mathbb{E}\left[\sup_{t\in[0,T]} \|\rho_\epsilon\|_{L^\beta}\right] \mathbb{E}\int_0^T \|\nabla u_\epsilon\|^2_{L^2} dt \mathbb{E}\int_0^T \|\nabla \rho_\epsilon\|^2_{L^2} dt \leq C.
\end{aligned}$$

Moreover, by the bounds (2.3.3), (2.3.4), (2.3.6), (2.3.7), we have

$$
\begin{aligned}
\left|\mathbb{E}\int_0^t -J_7 ds\right| &\le \epsilon C\mathbb{E}\int_0^T \|\nabla u_\epsilon\|_{L^2}\|\rho_\epsilon\|_{L^\beta}\|\nabla\rho_\epsilon\|_{L^2}dt \le C,\\
\left|\mathbb{E}\int_0^t -J_8 ds\right| &\le C\mathbb{E}\int_0^T \|c_\epsilon\|_{L^6}^2\|Q_\epsilon\|_{L^6}\|\nabla\mathcal{T}[\rho_\epsilon-(\rho_0)_m]\|_{L^2}dt\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}(\|Q_\epsilon\|_{L^6}^4+\|\rho_\epsilon-(\rho_0)_m\|_{L^2}^4)\right]\mathbb{E}\left(\int_0^T\|c_\epsilon\|_{L^6}^2dt\right)^2\le C,\\
\left|\mathbb{E}\int_0^t -J_9 ds\right| &\le C\mathbb{E}\int_0^T \|\nabla Q_\epsilon\|_{L^{\frac{10}{3}}}^2\|\nabla\mathcal{T}[\rho_\epsilon-(\rho_0)_m]\|_{L^{\frac{5}{2}}}+(\|Q_\epsilon\|_{L^3}^2\\
&\quad+\|Q_\epsilon\|_{L^6}^4)\|\rho_\epsilon-(\rho_0)_m\|_{L^3}dt\\
&\le C\mathbb{E}\int_0^T\|Q_\epsilon\|_{H^2}^2\|\rho_\epsilon-(\rho_0)_m\|_{L^\beta}+(1+\|Q_\epsilon\|_{L^6}^4)\|\rho_\epsilon-(\rho_0)_m\|_{L^\beta}dt\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\|\rho_\epsilon-(\rho_0)_m\|_{L^\beta}^2\right]\mathbb{E}\left(\int_0^T\|\triangle Q_\epsilon\|_{L^2}^2+(1+\|Q_\epsilon\|_{L^6}^4)dt\right)^2\\
&\le C,\\
\left|\mathbb{E}\int_0^t -J_{10} ds\right| &\le C\mathbb{E}\int_0^T\|Q_\epsilon\|_{L^4}\|\triangle Q_\epsilon\|_{L^2}\|\nabla\mathcal{T}[\rho_\epsilon-(\rho_0)_m]\|_{L^\beta}dt\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\|\rho_\epsilon-(\rho_0)_m\|_{L^\beta}^4\right]\times\\
&\qquad\qquad \mathbb{E}\left[\sup_{t\in[0,T]}\|Q_\epsilon\|_{H^1}^4\right]\mathbb{E}\left(\int_0^T\|\triangle Q_\epsilon\|_{L^2}^2dt\right)^2\\
&\le C,
\end{aligned}
$$

where $C$ is independent of $\epsilon$. The proof is complete. □

### 2.3.1 Compactness argument

In order to acquire the compactness of the approximate sequence, we implement the same procedures as the first level approximation. Define the path space

$$\mathcal{X}=\mathcal{X}_u\times\mathcal{X}_\rho\times\mathcal{X}_{\rho u}\times\mathcal{X}_c\times\mathcal{X}_Q\times\mathcal{X}_{\mathcal{W}},$$

where

$$\mathcal{X}_\rho := L^\infty(0,T;H^{-\frac{1}{2}}(\mathcal{D}))\cap L_w^{\beta+1}(0,T;L^{\beta+1}(\mathcal{D})),$$

and $\mathcal{X}_u, \mathcal{X}_{\rho u}, \mathcal{X}_c, \mathcal{X}_Q, \mathcal{X}_{\mathcal{W}}$ are the same as the definition in subsection 2.2.2. Let $\widetilde{\mathcal{X}} = \mathcal{X} \times L_w^{\frac{\beta+1}{\beta}}((0,T)\times\mathcal{D})$. The set of probability measures $\{\nu^\epsilon\}_{\epsilon>0}$ is constructed similarly to (2.2.27). We also have the following result.

**Proposition 2.3.2.** *There exists new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, subsequence $\{\nu^{\epsilon_k}\}_{k\geq 1}$* (still denote by $\epsilon$) *and $\widetilde{\mathcal{X}}$-valued measurable random variables*

$$(\tilde{u}_\epsilon, \tilde{\rho}_\epsilon, \tilde{\rho}_\epsilon\tilde{u}_\epsilon, \tilde{\rho}_\epsilon^\gamma + \delta\tilde{\rho}_\epsilon^\beta, \tilde{c}_\epsilon, \widetilde{Q}_\epsilon, \widetilde{\mathcal{W}}_\epsilon) \text{ and } (\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \overline{\tilde{\rho}^\gamma} + \delta\overline{\tilde{\rho}^\beta}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}),$$

*such that*

$$(\tilde{u}_\epsilon, \tilde{\rho}_\epsilon, \tilde{\rho}_\epsilon\tilde{u}_\epsilon, \tilde{c}_\epsilon, \widetilde{Q}_\epsilon, \widetilde{\mathcal{W}}_\epsilon) \to (\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}),\ \widetilde{\mathbb{P}} \text{ a.s.} \tag{2.3.12}$$

*in the topology of $\mathcal{X}$, moreover, we have $\widetilde{\mathbb{P}}$ a.s.*

$$\tilde{\rho}_\epsilon^\gamma + \delta\tilde{\rho}_\epsilon^\beta \to \overline{\tilde{\rho}^\gamma} + \delta\overline{\tilde{\rho}^\beta} \text{ in } L_w^{\frac{\beta+1}{\beta}}((0,T)\times\mathcal{D}), \tag{2.3.13}$$

*and*

$$\begin{aligned}
&\widetilde{\mathbb{P}}\left\{(\tilde{u}_\epsilon, \tilde{\rho}_\epsilon, \tilde{\rho}_\epsilon\tilde{u}_\epsilon, \tilde{\rho}_\epsilon^\gamma + \delta\tilde{\rho}_\epsilon^\beta, \tilde{c}_\epsilon, \widetilde{Q}_\epsilon, \widetilde{\mathcal{W}}_\epsilon) \in \cdot\right\} = \nu^\epsilon(\cdot),\\
&\widetilde{\mathbb{P}}\left\{(\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \overline{\tilde{\rho}^\gamma} + \delta\overline{\tilde{\rho}^\beta}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}) \in \cdot\right\} = \nu(\cdot),
\end{aligned}$$

*where $\nu$ is a Radon measure and $\widetilde{\mathcal{W}}_\epsilon$ is cylindrical Wiener process, relative to the filtration $\widetilde{\mathcal{F}}_t^\epsilon$ generated by the completion of $\sigma(\tilde{u}_\epsilon(s), \tilde{\rho}_\epsilon(s), \tilde{c}_\epsilon(s), \widetilde{Q}_\epsilon(s), \widetilde{\mathcal{W}}_\epsilon(s); s\leq t)$. In addition,*

$$\tilde{\rho}_\epsilon \mathrm{div}\tilde{u}_\epsilon \to \overline{\tilde{\rho}_\epsilon \mathrm{div}\tilde{u}_\epsilon} \text{ weakly in } L^p(\widetilde{\Omega}; L^2(0,T; L^{\frac{2\beta}{\beta+2}})), \tag{2.3.14}$$

$$\tilde{\rho}_\epsilon \ln\tilde{\rho}_\epsilon \to \overline{\tilde{\rho}_\epsilon \ln\tilde{\rho}_\epsilon} \text{ weakly star in } L^p(\widetilde{\Omega}; L^2(0,T; L^{\frac{2\beta}{\beta+2}})). \tag{2.3.15}$$

*Furthermore, the process $(\tilde{u}_\epsilon, \tilde{\rho}_\epsilon, \tilde{\rho}_\epsilon\tilde{u}_\epsilon, \tilde{c}_\epsilon, \widetilde{Q}_\epsilon, \widetilde{\mathcal{W}}_\epsilon)$ also satisfies the system (2.3.1) and shares the uniform bounds with (2.3.2)-(2.3.7).*

**Lemma 2.3.3.** *The sequence of probability measures $\{\nu^\epsilon\}_{\epsilon>0}$ is tight on path space $\widetilde{\mathcal{X}}$.*

*Proof.* In order not to repeat the trivial procedures, we mainly show limited parts that are different from the Lemma 2.2.5. Decompose $\rho_\epsilon u_\epsilon = X^\epsilon + Y^\epsilon$, where

$$\begin{aligned} X_\epsilon = m_0 + \int_0^t & -\mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \nabla(\rho_\epsilon^\gamma + \delta\rho_\epsilon^\beta) + \mu_1 \Delta u_\epsilon + (\mu_1+\mu_2)\nabla(\mathrm{div} u_\epsilon) \\ & + \nabla\cdot(\mathrm{F}(Q_\epsilon)\mathrm{I}_3 - \nabla Q_\epsilon \odot \nabla Q_\epsilon) + \nabla\cdot(Q_\epsilon \triangle Q_\epsilon - \triangle Q_\epsilon Q_\epsilon) \\ & + \sigma^* \nabla\cdot(c_\epsilon^2 Q_\epsilon) ds + \int_0^t \rho_\epsilon f(\rho_\epsilon, \rho_\epsilon u_\epsilon, c_\epsilon, Q_\epsilon) d\mathcal{W}, \end{aligned}$$

and

$$Y_\epsilon = \epsilon \int_0^t \nabla\rho_\epsilon \cdot \nabla u_\epsilon ds.$$

For the process $X_\epsilon$, one can treat it by the same argument as Lemma 2.2.5 Claim 1, obtaining $\mathbb{E}\|X_\epsilon(t)\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} \le C$ for any $\alpha \in [0, \frac{1}{2}), k \ge \frac{5}{2}$.

For process $Y_\epsilon$, Using the bound (2.3.4), we have $\mathbb{P}$ a.s.

$$\begin{aligned} \int_0^t \int_{\mathcal{D}} |\epsilon \nabla\rho_\epsilon \cdot \nabla u_\epsilon| dx ds &\le \sqrt{\epsilon} \left( \int_0^t \|\nabla u_\epsilon\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\sqrt{\epsilon}\nabla\rho_\epsilon\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\ &\le C\sqrt{\epsilon} \to 0, \text{ as } \epsilon \to 0, \end{aligned}$$

which leads to $Y_\epsilon \to 0$ in $C([0,T]; L^1(\mathcal{D}))$, $\mathbb{P}$ a.s. Note that, the convergence a.s. implies the convergence in distribution, therefore, we have

$$Y_\epsilon \to 0, \text{ in } C([0,T]; L^1(\mathcal{D})),$$

in the sense of distribution. The Sobolev compactness embedding $L^1(\mathcal{D}) \hookrightarrow H^{-k}(\mathcal{D})$ for $k > \frac{3}{2}$, implies that there exists a compact set $\mathcal{K} \subset C([0,T]; H^{-k}(\mathcal{D}))$ such that $\mathbb{P}(Y_\epsilon \subset \mathcal{K}^\mathfrak{c}) < \epsilon$. We obtain the law of set $\{\mathbb{P}\circ(Y_\epsilon)^{-1}\}$ is tight on space $C([0,T]; H^{-k}(\mathcal{D}))$.

Define the set $\widetilde{\mathcal{K}} = \mathcal{K}_1 \cap (\mathcal{K}_2 + \mathcal{K})$, where

$$\begin{aligned} \mathcal{K}_1 &:= \left\{ \varphi \in L^\infty(0,T; L^{\frac{2\beta}{\beta+1}}(\mathcal{D})) : \|\varphi\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))} \le R \right\}, \\ \mathcal{K}_2 &:= \left\{ \varphi \in C^\alpha([0,T]; H^{-k}(\mathcal{D})) : \|\varphi\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} \le R \right\}. \end{aligned}$$

Then, by the Aubin-Lions lemma A.0.1, we can get that the set $\mathcal{K}$ is relatively compact in $C([0,T];H^{-1}(\mathcal{D}))$. Using the bound (2.3.2) and the Chebyshev inequality, to conclude

$$\mu^{\epsilon}_{\rho u}(\widetilde{\mathcal{K}}^{\mathsf{c}}) \leq \mathbb{P}\left(\|\rho_\epsilon u_\epsilon\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\mathcal{D}))} > R\right) + \mathbb{P}\left(\|X_\epsilon\|_{C^\alpha([0,T];H^{-k}(\mathcal{D}))} > R\right)$$
$$+ \mathbb{P}(Y_\epsilon \subset \mathcal{K}^{\mathsf{c}}) \leq \frac{C}{R} + \epsilon.$$

Thus, we obtain the tightness of $\{\mu^{\epsilon}_{\rho u}\}_{\epsilon>0}$ on path space $\mathcal{X}_{\rho u}$.

The tightness of $\{\nu^{\epsilon}_{\rho}\}_{\epsilon>0}$ on path space $L^{\beta+1}_w(0,T;L^{\beta+1}(\mathcal{D}))$ and $\mathbb{P}\circ(\tilde{\rho}^{\gamma}_{\epsilon}+\delta\tilde{\rho}^{\beta}_{\epsilon})^{-1}$ on path space $L_w^{\frac{\beta+1}{\beta}}([0,T]\times\mathcal{D})$ is a result of the bound (2.3.3) and the Banach-Alagolu theorem using the same argument as Lemma 2.2.5 Claim 2. □

*Proof of Proposition 2.3.2.* The proofs follow the same manner as Proposition 2.2.4. The convergence results (2.3.14), (2.3.15) follows from the bounds (2.3.2)-(2.3.5) after using the Banach-Alagolu theorem, see also [81, Proposition 6.3].

### 2.3.2 Taking the limit for the artificial viscosity coefficient goes to zero

Now, we can pass to the limit $\epsilon \to 0$ for fixed $\delta$, to obtain:

**Proposition 2.3.4.** *For all $\ell \in C^\infty(\mathcal{D}), \phi \in C^\infty(\mathcal{D}), \varphi \in C^\infty(\mathcal{D}), \psi \in C^\infty(\mathcal{D})$, $t \in [0,T]$, there exist pressure $\overline{\tilde{\rho}^{\gamma}} + \delta\overline{\tilde{\rho}^{\beta}}$ and an $L_2(\mathcal{H};H^{-l})$-valued martingale $\widetilde{W}$ such that the process $(\tilde{\rho},\tilde{u},\tilde{c},\widetilde{Q},\widetilde{W})$ satisfies equations, $\widetilde{\mathbb{P}}$ a.s.*

$$\int_{\mathcal{D}} \tilde{c}(t)\ell dx = \int_{\mathcal{D}} \tilde{c}(0)\ell dx - \int_0^t\int_{\mathcal{D}} (\tilde{u}\cdot\nabla)\tilde{c}\cdot\ell dxds - \int_0^t\int_{\mathcal{D}} \nabla\tilde{c}\cdot\nabla\ell dxds,$$
$$\int_{\mathcal{D}} \tilde{\rho}(t)\psi dx = \int_{\mathcal{D}} \tilde{\rho}(0)\psi dx + \int_0^t\int_{\mathcal{D}} \tilde{\rho}\tilde{u}\cdot\nabla\psi dxds,$$
$$\begin{aligned}\int_{\mathcal{D}} \tilde{\rho}\tilde{u}(t)\phi dx = &\int_{\mathcal{D}} \tilde{m}(0)\phi dx + \int_0^t\int_{\mathcal{D}} (\tilde{\rho}\tilde{u}\otimes\tilde{u})\cdot\nabla\phi dxds - \int_0^t\int_{\mathcal{D}} \mu_1\nabla\tilde{u}\nabla\phi dxds\\
&- \int_0^t\int_{\mathcal{D}} (\mu_1+\mu_2)\mathrm{div}\tilde{u}\mathrm{div}\phi dxds + \int_0^t\int_{\mathcal{D}} \left(\overline{\tilde{\rho}^{\gamma}} + \delta\overline{\tilde{\rho}^{\beta}}\right)\mathrm{div}\phi dxds\\
&- \int_0^t\int_{\mathcal{D}} \Big((\mathrm{F}(\widetilde{Q})\mathrm{I}_3 - \nabla\widetilde{Q}\odot\nabla\widetilde{Q}) + (\widetilde{Q}\triangle\widetilde{Q} - \triangle\widetilde{Q}\widetilde{Q})\\
&+\sigma^*(\tilde{c}^2\widetilde{Q})\Big)\nabla\phi dxds + \int_{\mathcal{D}} \phi\widetilde{W}dx,\end{aligned}$$
$$\int_{\mathcal{D}} \widetilde{Q}(t)\varphi dx = \int_{\mathcal{D}} \widetilde{Q}(0)\varphi dx - \int_0^t\int_{\mathcal{D}} ((\tilde{u}\cdot\nabla)\widetilde{Q} + \widetilde{Q}\widetilde{\Psi} - \widetilde{\Psi}\widetilde{Q})\varphi dxds$$

$$+\int_0^t\int_{\mathcal{D}}\Gamma\varphi \mathrm{H}(\widetilde{Q},\tilde{c})dxds.$$

*Proof.* The argument is similar to that one in subsection 3.3. Note that due to the lack of strong convergence of density, we can not identify the specific form of stochastic term, but we could verify that $\widetilde{W}$ is still a martingale process, see [43]. □

In order to identify the nonlinear term of $\rho$ (the pressure term and the stochastic term), the strong convergence of density is necessary, which can be acquired by two steps following the idea of [32, 59].

**Step 1**. Weak convergence of the effective viscous flux.

The quantity $\rho^\gamma+\delta\rho^\beta-(\mu_2+2\mu_1)\mathrm{div}u$ usually called the effective viscous flux which enjoys many remarkable properties, see [41, 59]. Introduce the operator $\mathcal{A}$

$$\mathcal{A}[f]=\nabla\triangle^{-1}f,\ \ \mathcal{A}_j[f]=\partial_j\triangle^{-1}f,$$

with the following properties:

$$\mathrm{div}\mathcal{A}[f]=f,\ \ \triangle\mathcal{A}[f]=\nabla f, \tag{2.3.16}$$

$$\|\mathcal{A}[f]\|_{H^{1,p}(\mathcal{D})}\leq C\|f\|_{L^p(\mathcal{D})},\ \text{for all } p\geq 1, \tag{2.3.17}$$

$$\|\mathcal{A}[f]\|_{L^\infty(\mathcal{D})}\leq C\|f\|_{L^p(\mathcal{D})},\ \text{for all } p>3. \tag{2.3.18}$$

Note that $\tilde{\rho}_\epsilon,\tilde{u}_\epsilon$ could be extended to zero outside $\mathcal{D}$ satisfying

$$\partial_t\tilde{\rho}_\epsilon+\mathrm{div}(\tilde{\rho}_\epsilon\tilde{u}_\epsilon)=\epsilon\mathrm{div}(1_{\mathcal{D}}\nabla\tilde{\rho}_\epsilon),$$

in the weak sense, where $1_{\mathcal{D}}$ stands for the indicator function. We could also do the zero extension to limit function $\tilde{\rho},\tilde{u}$ to $\mathbb{R}^3$ which satisfies the equation $(2.3.1)_2$ in the weak sense, for further detail see [32, 22, 84].

Using the Itô formula to functions

$$f_1(\tilde{\rho}_\epsilon,\tilde{\rho}_\epsilon\tilde{u}_\epsilon)=\int_{\mathcal{D}}\tilde{\rho}_\epsilon\tilde{u}_\epsilon\cdot\tilde{\psi}\tilde{\phi}\mathcal{A}[\tilde{\rho}_\epsilon]dx.$$

and

$$f_2(\tilde{\rho},\tilde{\rho}\tilde{u})=\int_{\mathcal{D}}\tilde{\rho}\tilde{u}\cdot\tilde{\psi}\tilde{\phi}\mathcal{A}[\tilde{\rho}]dx.$$

where the functions $\tilde{\psi} \in C_c^\infty(0,T), \tilde{\phi} \in C_c^\infty(\mathcal{D})$., taking expectation, to obtain

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}(\tilde{\rho}_\epsilon^{\gamma+1} + \delta\tilde{\rho}_\epsilon^{\beta+1} - (\mu_2+2\mu_1)\mathrm{div}\tilde{u}_\epsilon)\cdot\tilde{\rho}_\epsilon dxds \\
=& -\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}(\tilde{\rho}_\epsilon^{\gamma} + \delta\tilde{\rho}_\epsilon^{\beta})\partial_i\tilde{\phi}\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds \\
&+(\mu_1+\mu_2)\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \mathrm{div}\tilde{u}_\epsilon\partial_i\tilde{\phi}\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds \\
&+\mu_1\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \partial_j\tilde{u}_\epsilon^i\partial_j\tilde{\phi}\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds \\
&-\mu_1\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{u}_\epsilon^i\partial_j\tilde{\phi}\partial_j\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds + \mu_1\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{u}_\epsilon^i\partial_i\tilde{\phi}\tilde{\rho}_\epsilon dxds \\
&-\epsilon\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}\tilde{\rho}_\epsilon\tilde{u}_\epsilon^i\mathcal{A}_i[\mathrm{div}(1_{\mathcal{D}}\nabla\tilde{\rho}_\epsilon)]dxds \\
&+\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}\tilde{u}_\epsilon^i(\tilde{\rho}_\epsilon\partial_i\mathcal{A}_j[\tilde{\rho}_\epsilon\tilde{u}_\epsilon^j] - \tilde{\rho}_\epsilon\tilde{u}_\epsilon^i\partial_i\mathcal{A}_j[\tilde{\rho}_\epsilon])dxds \\
&-\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}_s(s)\int_{\mathcal{D}} \tilde{\phi}\tilde{\rho}_\epsilon\tilde{u}_\epsilon^i\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds - \widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\rho}_\epsilon\tilde{u}_\epsilon^i\tilde{u}_\epsilon^j\partial_j\tilde{\phi}\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds \\
&+\epsilon\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}\partial_j\tilde{u}_\epsilon^i\partial_j\tilde{\rho}_\epsilon\mathcal{A}_i[\tilde{\rho}_\epsilon]dxds \\
&+\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \sigma^*(\tilde{c}_\epsilon^2\widetilde{Q}_\epsilon) : (\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon] + \nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}_\epsilon])dxds \\
&+\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} (\mathrm{F}(\widetilde{Q}_\epsilon)\mathrm{I}_3 - \nabla\widetilde{Q}_\epsilon\odot\nabla\widetilde{Q}_\epsilon) : (\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon] + \nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}_\epsilon])dxds \\
&+\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} (\widetilde{Q}_\epsilon\triangle\widetilde{Q}_\epsilon - \triangle\widetilde{Q}_\epsilon\widetilde{Q}_\epsilon) : (\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon] + \nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}_\epsilon])dxds \\
=:& J_1^\epsilon + \cdots + J_{13}^\epsilon,
\end{aligned} \tag{2.3.19}
$$

and

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}(\overline{\tilde{\rho}^\gamma} + \delta\overline{\tilde{\rho}^\beta})\tilde{\rho} - \tilde{\phi}(\mu_2+2\mu_1)\mathrm{div}\tilde{u}\cdot\tilde{\rho}dxds \\
=& -\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{\phi}(\overline{\tilde{\rho}^\gamma} + \delta\overline{\tilde{\rho}^\beta})\partial_i\tilde{\phi}\mathcal{A}_i[\tilde{\rho}]dxds \\
&+(\mu_1+\mu_2)\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \mathrm{div}\tilde{u}\partial_i\tilde{\phi}\mathcal{A}_i[\tilde{\rho}]dxds \\
&+\mu_1\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \partial_j\tilde{u}^i\partial_j\tilde{\phi}\mathcal{A}_i[\tilde{\rho}]dxds \\
&-\mu_1\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{u}^i\partial_j\tilde{\phi}\partial_j\mathcal{A}_i[\tilde{\rho}]dxds + \mu_1\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}} \tilde{u}^i\partial_i\tilde{\phi}\tilde{\rho}dxds
\end{aligned}
$$

$$
\begin{aligned}
&+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\tilde{\phi}\tilde{u}^i(\tilde{\rho}\partial_i\mathcal{A}_j[\tilde{\rho}\tilde{u}^j]-\tilde{\rho}_\epsilon\tilde{u}^i\partial_i\mathcal{A}_j[\tilde{\rho}])dxds\\
&-\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}_s(s)\int_{\mathcal{D}}\tilde{\phi}\tilde{\rho}\tilde{u}^i\mathcal{A}_i[\tilde{\rho}]dxds-\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\tilde{\rho}\tilde{u}^i\tilde{u}^j\partial_j\tilde{\phi}\mathcal{A}_i[\tilde{\rho}]dxds\\
&+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}^2\widetilde{Q}):(\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}]+\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}])dxds\\
&+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\mathrm{F}(\widetilde{Q})\mathrm{I}_3-\nabla\widetilde{Q}\odot\nabla\widetilde{Q}):(\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}]+\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}])dxds\\
&+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\widetilde{Q}\triangle\widetilde{Q}-\triangle\widetilde{Q}\widetilde{Q}):(\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}]+\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}])dxds\\
=:&\ J_1+\cdots+J_{11},
\end{aligned}
\tag{2.3.20}
$$

here the stochastic integral is also cancelled resulting from the property of martingale.

Our main goal is to get for all $t\in[0,T]$

$$
\begin{aligned}
&\lim_{\epsilon\to0}\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\tilde{\phi}(\tilde{\rho}_\epsilon^{\gamma+1}+\delta\tilde{\rho}_\epsilon^{\beta+1}-(\mu_2+2\mu_1)\mathrm{div}\tilde{u}_\epsilon\cdot\tilde{\rho}_\epsilon)dxds\\
&=\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\tilde{\phi}(\overline{\tilde{\rho}^\gamma}+\delta\overline{\tilde{\rho}^\beta})\tilde{\rho}-\tilde{\phi}(\mu_2+2\mu_1)\mathrm{div}\tilde{u}\cdot\tilde{\rho}dxds,
\end{aligned}
\tag{2.3.21}
$$

it suffices to show that all right hand side terms of (2.3.19) converges to the right hand side terms of (2.3.20). Denote $J_{11}^\epsilon=J_{11,a}^\epsilon+J_{11,b}^\epsilon$, $J_9=J_{9,a}+J_{9,b}$, decompose

$$
\begin{aligned}
J_{11,a}^\epsilon-J_{9,a}=&\ \widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}_\epsilon^2\widetilde{Q}_\epsilon-\tilde{c}^2\widetilde{Q}):\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon]dxds\\
&+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}^2\widetilde{Q}):\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon-\tilde{\rho}]dxds=\mathcal{J}_{1,a}+\mathcal{J}_{2,a},\\
J_{11,b}^\epsilon-J_{9,b}=&\ \widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}_\epsilon^2\widetilde{Q}_\epsilon-\tilde{c}^2\widetilde{Q}):\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}_\epsilon]dxds\\
&+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}^2\widetilde{Q}):\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}_\epsilon-\tilde{\rho}]dxds=\mathcal{J}_{1,b}+\mathcal{J}_{2,b}.
\end{aligned}
$$

By Proposition 2.3.2(2.3.13) and the bounds (2.3.6), (2.3.7), we have

$$
\mathcal{J}_{2,a}\to0,\ \text{ as }\epsilon\to0.
\tag{2.3.22}
$$

$\mathcal{J}_{1,a}$ can be handled as follows, by Proposition 2.3.2(2.3.12), the bounds (2.3.3), (2.3.6), (2.3.7) and the assumption $\beta>6$

$$
|\mathcal{J}_{1,a}|\leq\left|\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}_\epsilon^2-\tilde{c}^2)\widetilde{Q}_\epsilon:\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon]dxds\right|
$$

$$
\begin{aligned}
&+\left|\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}}\sigma^*(\tilde{c}^2(\widetilde{Q}_\epsilon-\widetilde{Q}):\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon]dxds\right|\\
&\le \sigma^* C\widetilde{\mathbb{E}}\int_0^t \|\tilde{c}_\epsilon-\tilde{c}\|_{L^2}\|\tilde{c}_\epsilon,\tilde{c}\|_{L^6}\|\widetilde{Q}_\epsilon\|_{L^6}\|\tilde{\rho}_\epsilon\|_{L^\beta}ds\\
&\quad+\sigma^* C\widetilde{\mathbb{E}}\int_0^t \|\tilde{c}\|_{L^2}\|\tilde{c}\|_{L^6}\|\widetilde{Q}_\epsilon-\widetilde{Q}\|_{L^6}\|\tilde{\rho}_\epsilon\|_{L^\beta}ds\\
&\le \sigma^* C\widetilde{\mathbb{E}}\int_0^t \|\tilde{c}_\epsilon-\tilde{c}\|_{L^2}^2+\|\widetilde{Q}_\epsilon-\widetilde{Q}\|_{H^1}^2ds\widetilde{\mathbb{E}}\left[\sup_{t\in[0,T]}(\|\tilde{\rho}_\epsilon\|_{L^\beta}^2+\|\tilde{c},\widetilde{Q}_\epsilon,\nabla\widetilde{Q}_\epsilon\|_{L^2}^2)\right]\\
&\quad\times\widetilde{\mathbb{E}}\int_0^t\|\tilde{c}_\epsilon,\tilde{c}\|_{L^6}^2ds\to 0,\quad\text{as }\epsilon\to 0. \qquad (2.3.23)
\end{aligned}
$$

In addition, we could get $\lim_{\epsilon\to 0}J_{11,b}^\epsilon=J_{9,b}$, then it follows $\lim_{\epsilon\to0}J_{11}^\epsilon=J_9$ using (2.3.22) and (2.3.23). Similarly, decompose

$$
\begin{aligned}
&J_{12,a}^\epsilon-J_{10,a}\\
&=\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\mathrm{F}(\widetilde{Q}_\epsilon)\mathrm{I}_3-\mathrm{F}(\widetilde{Q})\mathrm{I}_3-(\nabla\widetilde{Q}_\epsilon\odot\nabla\widetilde{Q}_\epsilon-\nabla\widetilde{Q}\odot\nabla\widetilde{Q})):\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon]dxds\\
&\quad+\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\mathrm{F}(\widetilde{Q})\mathrm{I}_3-\nabla\widetilde{Q}\odot\nabla\widetilde{Q}):\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon-\tilde{\rho}]dxds=\mathcal{I}_{1,a}+\mathcal{I}_{2,a}.
\end{aligned}
$$

Due to the same arguments as (2.3.22) and (2.3.23), as $\epsilon\to0$

$$
\begin{aligned}
&\mathcal{I}_{2,a}\to 0, \qquad (2.3.24)\\
&|\mathcal{I}_{1,a}|\le\left|\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\nabla(\widetilde{Q}_\epsilon-\widetilde{Q})\odot\nabla\widetilde{Q}_\epsilon-\nabla\widetilde{Q}\odot\nabla(\widetilde{Q}_\epsilon-\widetilde{Q})):\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon]dxds\right|\\
&\qquad\le C\widetilde{\mathbb{E}}\int_0^t\|\nabla\widetilde{Q}_\epsilon,\nabla\widetilde{Q}\|_{L^6}\|\widetilde{Q}_\epsilon-\widetilde{Q}\|_{L^2}\|\nabla\mathcal{A}[\tilde{\rho}_\epsilon]\|_{L^\beta}ds\\
&\qquad\le C\widetilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\tilde{\rho}_\epsilon\|_{L^\beta}\right]\widetilde{\mathbb{E}}\int_0^t\|\widetilde{Q}_\epsilon-\widetilde{Q}\|_{L^2}^2ds\widetilde{\mathbb{E}}\int_0^t\|\nabla\widetilde{Q}_\epsilon,\nabla\widetilde{Q}\|_{L^6}^2ds\to0, \qquad (2.3.25)\\
&\quad\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\mathrm{F}(\widetilde{Q}_\epsilon)-\mathrm{F}(\widetilde{Q}))\mathrm{I}_3:\tilde{\phi}\nabla\mathcal{A}[\tilde{\rho}_\epsilon]dxds\to0. \qquad (2.3.26)
\end{aligned}
$$

Combining the convergence (2.3.24)-(2.3.26), we get $\lim_{\epsilon\to0}J_{12,a}^\epsilon=J_{10,a}$. Using similar estimate, we could get $\lim_{\epsilon\to0}J_{12,b}^\epsilon=J_{10,b}$, it follows $\lim_{\epsilon\to0}J_{12}^\epsilon=J_{10}$.

Denote $J_{13}^\epsilon=J_{13,a}^\epsilon+J_{13,b}^\epsilon$, $J_{11}=J_{11,a}+J_{11,b}$, decompose

$$
J_{13,b}^\epsilon-J_{11,b}=\widetilde{\mathbb{E}}\int_0^t\tilde{\psi}(s)\int_{\mathcal{D}}(\widetilde{Q}_\epsilon\triangle\widetilde{Q}_\epsilon-\triangle\widetilde{Q}_\epsilon\widetilde{Q}_\epsilon):\nabla\tilde{\phi}\otimes(\mathcal{A}[\tilde{\rho}_\epsilon]-\mathcal{A}[\tilde{\rho}])dxds
$$

$$
\begin{aligned}
&+\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}}((\widetilde{Q}_\epsilon-\widetilde{Q})\triangle\widetilde{Q}_\epsilon-\triangle\widetilde{Q}_\epsilon(\widetilde{Q}_\epsilon-\widetilde{Q})):\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}]dxds\\
&+\widetilde{\mathbb{E}}\int_0^t \tilde{\psi}(s)\int_{\mathcal{D}}(\widetilde{Q}_\epsilon\triangle(\widetilde{Q}_\epsilon-\widetilde{Q})-\triangle(\widetilde{Q}_\epsilon-\widetilde{Q})\widetilde{Q}_\epsilon):\nabla\tilde{\phi}\otimes\mathcal{A}[\tilde{\rho}]dxds. \quad (2.3.27)
\end{aligned}
$$

Using Proposition 2.3.2(2.3.12), the bound (2.3.7), the estimate (2.3.18) and the Hölder inequality, we deduce that the right hand terms of (2.3.27) go to 0, (similar to (2.3.25)).

Since the matrices $\widetilde{Q}_\epsilon$, $\widetilde{Q}$ are symmetric, hence $\widetilde{Q}_\epsilon\triangle\widetilde{Q}_\epsilon-\triangle\widetilde{Q}_\epsilon\widetilde{Q}_\epsilon$, $\widetilde{Q}\triangle\widetilde{Q}-\triangle\widetilde{Q}\widetilde{Q}$ are skew-symmetric, and note that $\nabla\mathcal{A}[\tilde{\rho}_\epsilon], \nabla\mathcal{A}[\tilde{\rho}]$ are symmetric, to conclude that $J^\epsilon_{13,a}, J_{11,a}=0$. (The special structure of $Q$-tensor makes the weak convergence possible, otherwise we are not able to handle the high-order nonlinear term).

We also have $J^\epsilon_{10}\to 0$ as $\epsilon\to 0$. For $J^\epsilon_6$, as $\epsilon\to 0$

$$
\begin{aligned}
|J^\epsilon_3| &\le \sqrt{\epsilon}C\widetilde{\mathbb{E}}\int_0^t \|\sqrt{\epsilon}\nabla\tilde{\rho}_\epsilon\|_{L^2}\|\tilde{u}_\epsilon\|_{L^6}\|\tilde{\rho}_\epsilon\|_{L^\beta}ds\\
&\le \sqrt{\epsilon}C\widetilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\tilde{\rho}_\epsilon\|_{L^\beta}\right]\widetilde{\mathbb{E}}\int_0^t\|\sqrt{\epsilon}\nabla\tilde{\rho}_\epsilon\|^2_{L^2}ds\widetilde{\mathbb{E}}\int_0^t\|\nabla\tilde{u}_\epsilon\|^2_{L^2}ds\\
&\le C\sqrt{\epsilon}\to 0.
\end{aligned}
$$

The proofs of convergence of rest terms are standard, we refer the readers to [43, 84, 81]. Finally, we obtain the convergence result (2.3.21).

**Step 2**. Strong convergence of density.

In this step, we could show the strong convergence of density using the re-normalized mass equation and the Minty idea, for further details see [32].

Now we can pass the limit to identify the stochastic term and nonlinear pressure term using the same argument as (2.2.48), obtaining the following result,

**Proposition 2.3.5.** *For $\beta>\max\{6,\gamma\}$, $\delta>0$ and if conditions (2.1.1), (2.1.2) hold. There exists a global weak martingale solution to the modified system (2.3.1).*

## 2.4 Vanishing Artificial Pressure

In this section, we shall pass the artificial pressure coefficient $\delta \to 0$ to establish the Theorem 2.1.2. Also, the following uniform bounds hold for process $(\rho_\delta, u_\delta, c_\delta, Q_\delta)$

$$\rho_\delta \in L^p(\Omega; L^\infty(0,T; L^\gamma(\mathcal{D}))), \tag{2.4.1}$$

$$\sqrt[\beta]{\delta}\rho_\delta \in L^p(\Omega; L^\infty(0,T; L^\beta(\mathcal{D}))), \tag{2.4.2}$$

$$u_\delta \in L^p(\Omega; L^2(0,T; H^1(\mathcal{D}))), \tag{2.4.3}$$

$$\sqrt{\rho_\delta} u_\delta \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{D}))), \tag{2.4.4}$$

$$c_\delta \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{D})) \cap L^2(0,T; H^1(\mathcal{D}))), \tag{2.4.5}$$

$$Q_\delta \in L^p(\Omega; L^\infty(0,T; H^1(\mathcal{D})) \cap L^2(0,T; H^2(\mathcal{D}))). \tag{2.4.6}$$

**Lemma 2.4.1.** *The approximate sequence $\rho_\delta$ satisfies the following estimate*

$$\mathbb{E}\int_0^T \int_{\mathcal{D}} \rho_\delta^{\gamma+\theta} + \delta\rho_\delta^{\beta+\theta} dxdt \leq C,$$

*where the constant $C$ is independent of $\delta$ and the constant $\theta \in \left(0, \min\left\{\frac{2\gamma-3}{3}, \frac{\gamma}{3}\right\}\right)$.*

*Proof.* The proof follows the same line as Lemma 2.3.1. By the Di Perna-Lions commutator lemmas, we infer that the following equation holds in the weak sense

$$d\left[\rho_\delta^\theta - (\rho_\delta^\theta)_m\right] + \mathrm{div}(\rho_\delta^\theta u_\delta)dt + \left[(\theta-1)\rho_\delta^\theta \mathrm{div} u_\delta - (\rho_\delta^\theta \mathrm{div} u_\delta)_m\right] dt = 0, \ \mathbb{P} \text{ a.s.} \tag{2.4.7}$$

Then, applying the operator $\mathcal{T}$ on both sides of (2.4.7), to obtain

$$d\mathcal{T}[\rho_\delta^\theta - (\rho_\delta^\theta)_m] + \mathcal{T}[\mathrm{div}(\rho_\delta^\theta u_\delta)]dt + (\theta-1)\mathcal{T}[\rho_\delta^\theta \mathrm{div} u_\delta - (\rho_\delta^\theta \mathrm{div} u_\delta)_m]dt = 0. \tag{2.4.8}$$

Applying the Itô product formula to function $\Phi(\rho_\delta u_\delta, \rho_\delta^\theta) = \int_{\mathcal{D}} \rho_\delta u_\delta \cdot \mathcal{T}[\rho_\delta^\theta - (\rho_\delta^\theta)_m]dx$, and taking expectation, we have

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{D}} \rho_\delta u_\delta \cdot \mathcal{T}[\rho_\delta^\theta - (\rho_\delta^\theta)_m]dx \\
&= \mathbb{E}\int_0^t \int_{\mathcal{D}} \rho_\delta^{\gamma+\theta} + \delta\rho_\delta^{\beta+\theta} dxds - \mathbb{E}\int_0^t (\rho_\delta^\theta)_m \int_{\mathcal{D}} \rho_\delta^\gamma + \delta\rho_\delta^\beta dxds \\
&\quad + \mathbb{E}\int_{\mathcal{D}} m_{0,\delta} \cdot \mathcal{T}[\rho_{0,\delta}^\theta - (\rho_{0,\delta}^\theta)_m]dx - (\mu_1+\mu_2)\mathbb{E}\int_0^t \int_{\mathcal{D}} \mathrm{div} u_\delta \cdot \rho_\delta^\theta dxds
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^t\int_{\mathcal{D}}\rho_\delta u_\delta\otimes u_\delta:\nabla\mathcal{T}[\rho_\delta^\theta-(\rho_\delta^\theta)_m]dxds\\
&-\mu_1\mathbb{E}\int_0^t\int_{\mathcal{D}}\nabla u_\delta:\nabla\mathcal{T}[\rho_\delta^\theta-(\rho_\delta^\theta)_m]dxds\\
&+\mathbb{E}\int_0^t\int_{\mathcal{D}}\rho_\delta^\theta u_\delta\mathcal{T}[\mathrm{div}(\rho_\delta u_\delta)]dxds-\mathbb{E}\int_0^t\int_{\mathcal{D}}\sigma^*c_\delta^2Q_\delta:\nabla\mathcal{T}[\rho_\delta^\theta-(\rho_\delta^\theta)_m]dxds\\
&+(1-\theta)\mathbb{E}\int_0^t\int_{\mathcal{D}}\rho_\delta u_\delta\mathcal{T}[\rho_\delta^\theta\mathrm{div}u_\delta-(\rho_\delta^\theta\mathrm{div}u_\delta)_m]dxds\\
&-\mathbb{E}\int_0^t\int_{\mathcal{D}}(\nabla Q_\delta\odot\nabla Q_\delta-\mathrm{F}(\mathrm{Q}_\delta)\mathrm{I}_3):\nabla\mathcal{T}[\rho_\delta^\theta-(\rho_\delta^\theta)_m]dxds\\
&-\mathbb{E}\int_0^t\int_{\mathcal{D}}(Q_\delta\triangle Q_\delta-\triangle Q_\delta Q_\delta):\nabla\mathcal{T}[\rho_\delta^\theta-(\rho_\delta^\theta)_m]dxds.
\end{aligned}
$$

Our goal is to get the bound of $\mathbb{E}\int_0^t\int_{\mathcal{D}}\rho_\delta^{\gamma+\theta}+\delta\rho_\delta^{\beta+\theta}dxds$, which can be achieved after all other terms get controlled. Here we just give a limit amount of details.

For $\theta\le\frac{2\gamma-3}{3}$, the bounds (2.4.1), (2.4.3), the Hölder inequality and the Sobolev embedding $H^{1,\frac{6\gamma}{7\gamma-6}}(\mathcal{D})\hookrightarrow L^{\frac{6\gamma}{5\gamma-6}}(\mathcal{D})$ imply

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^t\int_{\mathcal{D}}\rho_\delta u_\delta\mathcal{T}[\rho_\delta^\theta\mathrm{div}u_\delta]dxds\right|\\
&\le C\mathbb{E}\int_0^t\|\rho_\delta\|_{L^\gamma}\|u_\delta\|_{L^6}\left\|\mathcal{T}[\rho_\delta^\theta\mathrm{div}u_\delta]\right\|_{L^{\frac{6\gamma}{5\gamma-6}}}ds\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\|\rho_\delta\|_{L^\gamma}\int_0^t\|u_\delta\|_{L^6}\|\rho_\delta^\theta\mathrm{div}u_\delta\|_{L^{\frac{6\gamma}{7\gamma-6}}}ds\right]\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\|\rho_\delta\|_{L^\gamma}\int_0^t\|u_\delta\|_{L^6}\|\mathrm{div}u_\delta\|_{L^2}\|\rho_\delta^\theta\|_{L^{\frac{3\gamma}{2\gamma-3}}}ds\right]\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\|\rho_\delta\|_{L^\gamma}\|\rho_\delta\|_{L^\gamma}^\theta\int_0^t\|u_\delta\|_{L^6}\|\mathrm{div}u_\delta\|_{L^2}ds\right]\le C,
\end{aligned}
$$

where $C$ is independent of $\delta$. For $\theta<\frac{\gamma}{3}$, using the Hölder inequality and the bounds (2.4.1), (2.4.6), to get

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^t\int_{\mathcal{D}}\sigma^*c_\delta^2Q_\delta:\nabla\mathcal{T}[\rho_\delta^\theta]dxds\right|\\
&\le C\mathbb{E}\int_0^t\|c_\delta\|_{L^6}^2\|Q_\delta\|_{L^6}\|\rho_\delta\|_{L^{2\theta}}^\theta ds\\
&\le C\mathbb{E}\left(\int_0^t\|c_\delta\|_{L^6}^2ds\right)^2\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|Q_\delta\|_{L^6}^4+\|\rho_\delta\|_{L^\gamma}^{4\theta}\right)\right]\le C,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^t\int_{\mathcal{D}}(\nabla Q_\delta\odot\nabla Q_\delta-\mathrm{F}(\mathrm{Q}_\delta)\mathrm{I}_3):\nabla\mathcal{T}[\rho_\delta^\theta]dxds\right|\\
&\le C\mathbb{E}\int_0^t(\|\nabla Q_\delta\|_{L^2}\|\nabla Q_\delta\|_{L^6}+\|Q_\delta\|_{L^6})\|\rho_\delta^\theta\|_{L^3}ds\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}(\|Q_\delta\|_{H^1}^4+\|\rho_\delta\|_{L^\gamma}^{4\theta})\right]\mathbb{E}\int_0^t\|\nabla Q_\delta\|_{L^6}^2ds\le C,\\
&\left|\mathbb{E}\int_0^t\int_{\mathcal{D}}(Q_\delta\triangle Q_\delta-\triangle Q_\delta Q_\delta):\nabla\mathcal{T}[\rho_\delta^\theta]dxds\right|\\
&\le C\mathbb{E}\int_0^t\|Q_\delta\|_{L^6}\|\triangle Q_\delta\|_{L^2}\|\rho_\delta^\theta\|_{L^3}ds\\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|Q_\delta\|_{L^6}^4+\|\rho_\delta\|_{L^\gamma}^{4\theta}\right)\right]\mathbb{E}\int_0^t\|\triangle Q_\delta\|_{L^2}^2ds\le C,
\end{aligned}
$$

where $C$ is independent of $\delta$. This completes the proof. □

### 2.4.1 Compactness argument

Define the cut off functions

$$
T_k(z)=kT(\frac{z}{k}),\ k=1,2,3\cdots
$$

where $T(z)$ is a smooth concave function on $\mathbb{R}$ such that $T(z)=z$ if $z\le 1$ and $T(z)=2$ if $z\ge 3$. The definition of $T_k(z)$ implies that

$$
T_k(z)=\begin{cases} z,\ z\le k,\\ 2k, z\ge 3k.\end{cases} \tag{2.4.9}
$$

Here, we define the path space $\mathcal{X}_1=\mathcal{X}_u\times\mathcal{X}_\rho\times\mathcal{X}_{\rho u}\times\mathcal{X}_c\times\mathcal{X}_Q\times\mathcal{X}_{\mathcal{W}}$, where

$$
\mathcal{X}_\rho:=L^\infty(0,T;H^{-\frac{1}{2}}(\mathcal{D}))\cap L_w^{\gamma+\theta}(0,T;L^{\gamma+\theta}(\mathcal{D})),
$$

and $\mathcal{X}_u,\mathcal{X}_{\rho u},\mathcal{X}_c,\mathcal{X}_Q,\mathcal{X}_{\mathcal{W}}$ are same as the definition in subsection 3.2. Let $\mathcal{X}=\mathcal{X}_1\times C_w([0,T];L^p(\mathcal{D}))\times L_w^2(0,T;L^2(\mathcal{D}))\times L_w^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\mathcal{D})$ for all $1\le p<\infty$. Similarly, we can define the set of probability measures $\{\nu^\delta\}_{\delta>0}$ as before. Following the same line as previous section to build the compactness result,

**Proposition 2.4.2.** *There exists a new probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, *a subsequence of* $\{\nu^\delta\}_{\delta>0}$ (still denoted by $\nu^\delta$) *and* $\mathcal{X}$*-valued measurable random variables*

$$(\tilde{u}_\delta, \tilde{\rho}_\delta, \tilde{\rho}_\delta\tilde{u}_\delta, \tilde{\rho}_\delta^\gamma, \tilde{c}_\delta, \widetilde{Q}_\delta, T_k(\tilde{\rho}_\delta), \quad (\tilde{\rho}_\delta T_k^{'}(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta))\mathrm{div}\tilde{u}_\delta, \widetilde{\mathcal{W}}_\delta),$$

*and* $(\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{T}_{1,k}, \widetilde{T}_{2,k}, \widetilde{\mathcal{W}})$ *such that*

$$\begin{aligned}
&\widetilde{\mathbb{P}}\left\{(\tilde{u}_\delta, \tilde{\rho}_\delta, \tilde{\rho}_\delta\tilde{u}_\delta, \tilde{\rho}_\delta^\gamma, \tilde{c}_\delta, \widetilde{Q}_\delta, T_k(\tilde{\rho}_\delta), (\tilde{\rho}_\delta T_k^{'}(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta))\mathrm{div}\tilde{u}_\delta, \widetilde{\mathcal{W}}_\delta) \in \cdot\right\} = \nu^\delta(\cdot),\\
&\widetilde{\mathbb{P}}\left\{(\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \rho^*, \tilde{c}, \widetilde{Q}, \widetilde{T}_{1,k}, \widetilde{T}_{2,k}, \widetilde{\mathcal{W}}) \in \cdot\right\} = \nu(\cdot),
\end{aligned}$$

*where* $\nu$ *is a Radon measure and* $\widetilde{\mathcal{W}}_\delta$ *is cylindrical Wiener process, relative to the filtration* $\widetilde{\mathcal{F}}_t^\delta$ *generated by the completion of* $\sigma(\tilde{u}_\delta(s), \tilde{\rho}_\delta(s), \tilde{c}_\delta(s), \widetilde{Q}_\delta(s), \widetilde{\mathcal{W}}_\delta(s); s \le t)$, *and the following convergence results hold,* $\widetilde{\mathbb{P}}$ *a.s.*

$$(\tilde{u}_\delta, \tilde{\rho}_\delta, \tilde{\rho}_\delta\tilde{u}_\delta, \tilde{c}_\delta, \widetilde{Q}_\delta, \widetilde{\mathcal{W}}_\delta) \to (\tilde{u}, \tilde{\rho}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{\mathcal{W}}), \tag{2.4.10}$$

*in the topology of* $\mathcal{X}_1$, *in addition*

$$\tilde{\rho}_\delta^\gamma \to \rho^* \text{ in } L_w^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\mathcal{D}), \tag{2.4.11}$$

$$T_k(\tilde{\rho}_\delta) \to \widetilde{T}_{1,k} \text{ in } C_w([0,T]; L^p(\mathcal{D})), \text{ for all } 1 \le p < \infty, \tag{2.4.12}$$

$$(\tilde{\rho}_\delta T_k^{'}(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta))\mathrm{div}\tilde{u}_\delta \rightharpoonup \widetilde{T}_{2,k} \text{ in } L^2(0,T; L^2(\mathcal{D})). \tag{2.4.13}$$

*Moreover, the bounds* (2.4.1)-(2.4.6) *still hold for* $(\tilde{u}_\delta, \tilde{\rho}_\delta, \tilde{\rho}_\delta\tilde{u}_\delta, \tilde{c}_\delta, \widetilde{Q}_\delta)$ *uniformly in* $\delta$.

**Lemma 2.4.3.** *The set of induced laws* $\{\nu^\delta\}_{\delta>0}$ *is tight on path space* $\mathcal{X}$.

*Proof.* Observe that the argument used in Lemma 2.3.3 can be adopted. Decompose $\rho_\delta u_\delta = X_\delta + Y_\delta$, where

$$
\begin{aligned}
X_\delta = m_{0,\delta} + \int_0^t & -\mathrm{div}(\rho_\delta u_\delta \otimes u_\delta) - \nabla \rho_\delta^\gamma + \mu_1 \Delta u_\delta + (\mu_1+\mu_2)\nabla(\mathrm{div} u_\delta) \\
& + \nabla \cdot (\mathrm{F}(Q_\delta)\mathrm{I}_3 - \nabla Q_\delta \odot \nabla Q_\delta) + \nabla \cdot (Q_\delta \triangle Q_\delta - \triangle Q_\delta Q_\delta) \\
& + \sigma^* \nabla \cdot (c_\delta^2 Q_\delta) ds + \int_0^t \rho_\delta f(\rho_\delta, \rho_\delta u_\delta, c_\delta, Q_\delta) d\mathcal{W},
\end{aligned}
$$

and

$$
Y_\delta = \delta \int_0^t \nabla \rho_\delta^\beta ds.
$$

Also, the process $X_\delta$ has the bound

$$
\mathbb{E}\|X_\delta(t)\|_{C^\alpha([0,T];H^{-l}(\mathcal{D}))} \leq C \text{ for any } \alpha \in [0, \frac{1}{2}), \ l \geq \frac{5}{2}.
$$

For the process $Y_\delta$, using the bound (2.4.2), we have as $\delta \to 0$

$$
\delta \rho_\delta^\beta \to 0 \text{ in } L^{\frac{\beta+\theta}{\beta}}((0,T)\times\mathcal{D}), \ \mathbb{P} \text{ a.s.}
$$

which implies

$$
\begin{aligned}
\|Y_\delta\|_{C([0,T];H^{-1,\frac{\beta+\theta}{\beta}})} &= \sup_{t\in[0,T]} \left\| \int_0^t \delta \nabla \rho_\delta ds \right\|_{H^{-1,\frac{\beta+\theta}{\beta}}} \\
&\leq \int_0^T \|\delta \nabla \rho_\delta\|_{H^{-1,\frac{\beta+\theta}{\beta}}} dt \to 0, \ \text{as } \delta \to 0, \ \mathbb{P} \text{ a.s.}
\end{aligned}
$$

this convergence gives $Y_\delta \to 0$ in $C([0,T], H^{-1,\frac{\beta+\theta}{\beta}}(\mathcal{D}))$ in the sense of distribution.

On the other hand, using the boundness of $T_k$ and $(\tilde{\rho}_\delta T_k'(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta))\mathrm{div}\tilde{u}_\delta$ on spaces $C([0,T];L^p(\mathcal{D}))$ and $L^2(0,T;L^2(\mathcal{D}))$ respectively, we can show the sequence of probability measures $\mathbb{P}\circ(T_k(\tilde{\rho}_\delta))^{-1}$ and $\mathbb{P}\circ((\tilde{\rho}_\delta T_k'(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta))\mathrm{div}\tilde{u}_\delta)^{-1}$ are tight on path spaces $C_w([0,T];L^p(\mathcal{D}))$ and $L^2_w(0,T;L^2(\mathcal{D}))$, see Claim 2, Lemma 2.2.5. This completes the proof. $\square$

*Proof of Proposition 2.4.2.* The proof follows the same line of the Proposition 2.3.2.

### 2.4.2 Passing limit for the artificial pressure coefficient goes to zero

Note that,

$$
\begin{aligned}
\widetilde{\mathbb{E}}\int_0^t \langle \delta\tilde{\rho}_\delta^\beta, \nabla\phi\rangle ds &\leq \delta^{\frac{\theta}{\theta+\beta}}\|\nabla\phi\|_{L^\infty}\widetilde{\mathbb{E}}\int_0^t\int_{\mathcal{D}}\delta^{\frac{\beta}{\theta+\beta}}\tilde{\rho}_\delta^\beta dxds\\
&\leq C\delta^{\frac{\theta}{\theta+\beta}}\|\nabla\phi\|_{L^\infty}\widetilde{\mathbb{E}}\int_0^t\int_{\mathcal{D}}\delta\tilde{\rho}_\delta^{\beta+\theta}dxds\\
&\leq C\delta^{\frac{\theta}{\theta+\beta}}\|\nabla\phi\|_{L^\infty}\to 0,\ \text{as } \delta\to 0,
\end{aligned}
$$

this convergence result together with Proposition 2.4.2, using the same argument as subsection 3.3, to conclude that there exists an $L_2(\mathcal{H}, H^{-l})$-valued process $\widetilde{W}$ and $L^{\frac{\gamma+\theta}{\gamma}}$-valued pressure $\rho^*$ such that $(\tilde{u}, \tilde{\rho}\tilde{u}, \tilde{c}, \widetilde{Q}, \widetilde{W})$ satisfies the momentum equation $\widetilde{\mathbb{P}}$ a.s.

$$
\begin{aligned}
&\partial_t(\tilde{\rho}\tilde{u}) + \operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + \nabla\rho^*\\
&= \mu_1\Delta\tilde{u} + (\mu_1+\mu_2)\nabla(\operatorname{div}\tilde{u}) + \nabla\cdot(\mathrm{F}(\widetilde{Q})\mathrm{I}_3 - \nabla\widetilde{Q}\odot\nabla\widetilde{Q})\\
&\quad +\nabla\cdot(\widetilde{Q}\triangle\widetilde{Q} - \triangle\widetilde{Q}\widetilde{Q}) + \sigma^*\nabla\cdot(\tilde{c}^2\widetilde{Q}) + \frac{d\widetilde{W}}{dt},
\end{aligned}
\tag{2.4.14}
$$

in the weak sense.

The proof of Theorem 2.1.2 will be completed once we build the strong convergence of density, to identify the pressure term and the stochastic term in equation (2.4.14). Following the idea of [32, 59], the proof of strong convergence of density shall be obtained by three steps.

**Step 1**. Weak continuity of the effective viscous flow.

Choosing $b = T_k(\tilde{\rho}_\delta)$ in the re-normalized continuity equation, it holds $\widetilde{\mathbb{P}}$ a.s. in the weak sense

$$
dT_k(\tilde{\rho}_\delta) + \operatorname{div}(T_k(\tilde{\rho}_\delta)\tilde{u}_\delta)dt + (T_k'(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta))\operatorname{div}\tilde{u}_\delta dt = 0. \tag{2.4.15}
$$

In addition, (2.4.12) implies

$$
T_k(\tilde{\rho}_\delta) \to \widetilde{T}_{1,k} \text{ in } C([0,T]; H^{-1}(\mathcal{D})). \tag{2.4.16}
$$

Then, combining (2.4.10), (2.4.16) and (2.4.13), letting $\delta \to 0$ in (2.4.15), to obtain that

$$d\widetilde{T}_{1,k} + \mathrm{div}(\widetilde{T}_{1,k}\tilde{u})dt + \widetilde{T}_{2,k}dt = 0, \tag{2.4.17}$$

holds $\widetilde{\mathbb{P}}$ a.s. in the weak sense. We aim to get

$$\begin{aligned}
&\lim_{\delta\to 0}\widetilde{\mathbb{E}}\int_0^T \tilde{\psi}(t)\int_{\mathcal{D}} \tilde{\phi}(\tilde{\rho}_\delta^\gamma - (\mu_2+2\mu_1)\mathrm{div}\tilde{u}_\delta)T_k(\tilde{\rho}_\delta)dxdt \\
&= \widetilde{\mathbb{E}}\int_0^T \tilde{\psi}(t)\int_{\mathcal{D}} \tilde{\phi}(\rho^* - (\mu_2+2\mu_1)\mathrm{div}\tilde{u})\widetilde{T}_{1,k}dxdt,
\end{aligned} \tag{2.4.18}$$

where the functions $\tilde{\psi}, \tilde{\phi}$ are the same as in (2.3.21). The proof of (2.4.18) is very similar to that of the argument (2.3.21). Here, we skip it.

**Step 2**. Re-normalized solution.

Define the oscillations defect measure related to the family $\{\tilde{\rho}_\delta\}$ by

$$\mathcal{O}_{\gamma+1}[\tilde{\rho}_\delta \to \tilde{\rho}](\mathcal{D}) = \sup_{k\geq 1}\left(\limsup_{\delta\to 0^+}\widetilde{\mathbb{E}}\int_0^T\int_{\mathcal{D}} |T_k(\tilde{\rho}_\delta) - T_k(\tilde{\rho})|^{\gamma+1}dxdt\right).$$

**Lemma 2.4.4.**[84, Lemma 5.3] *There exists a constant $C$ independent of $k$ such that*

$$\mathcal{O}_{\gamma+1}[\tilde{\rho}_\delta \to \tilde{\rho}](\mathcal{D}) \leq C.$$

With the help of the Lemma 2.4.4, we may show that the limit $(\tilde{\rho}, \tilde{u})$ satisfies the renormalized continuity equation using the same argument as Lemma 5.4 in [84]

$$\partial_t b(\tilde{\rho}) + \mathrm{div}(b(\tilde{\rho})\tilde{u}) + (b'(\tilde{\rho})\tilde{\rho} - b(\tilde{\rho}))\mathrm{div}\tilde{u} = 0, \tag{2.4.19}$$

$\widetilde{\mathbb{P}}$ a.s. in the weak sense.

**Step 3**. The strong convergence of density.

The proof is also standard, we refer the reader to [32, 84, 81, 43] for the deterministic and stochastic case. The proof of Theorem 2.1.2 is completed.

## 3.0 Local Strong Pathwise Solution to Liquid Crystals

Liquid crystal is a kind of material whose mechanical properties and symmetry properties are intermediate between those of a liquid and those of a crystal. The complex structure of liquid crystals made it the ideal material for the study of topological defects. As a result, several mathematical models have been brought out to describe the dynamics of a liquid crystal. For example, in [37], the Ericksen-Leslie-Parodi system has been used to model the flow of liquid crystals, based on the fact that a nematic flow is very similar to a conventional liquid with molecules of similar size. The challenge is, the flow would disturb the alignment, thus a new flow in the nematic is induced. In order to analyze the coupling between orientation and flow, a macroscopic approach has been used, and a direction field $\mathbf{d}$ with unit length is adopted to describe the local state of alignment. However, the model is restricted to an uniaxial order parameter field of constant magnitude.

In an effort to describe the motion of biaxial liquid crystals, a tensor order parameter $Q$ replacing the unit vector $\mathbf{d}$ was brought up in [6, 37] to describe the primary and secondary directions of nematic alignments along with variations in the degree of nematic order, which reflects better the properties of nematic liquid crystals and can be modeled by the Navier-Stokes equations governing the fluid velocity coupled with a parabolic equation of $Q$-tensor; see [3, 4, 64] for further background discussions. The compressible case we focus on reads as

$$\begin{cases} d\rho + \mathrm{div}_x(\rho \mathbf{u})dt = 0, \\ d(\rho\mathbf{u}) + \mathrm{div}_x(\rho\mathbf{u}\otimes\mathbf{u})dt + \nabla_x p dt \\ \quad = \mathcal{L}\mathbf{u}dt - \mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3)dt + \mathrm{div}_x(Q\mathcal{H}(Q) - \mathcal{H}(Q)Q)dt, \\ dQ + \mathbf{u}\cdot\nabla_x Q dt - (\Theta Q - Q\Theta)dt = \Gamma\mathcal{H}(Q)dt, \end{cases} \tag{3.0.1}$$

where $\rho, \mathbf{u}$ denote the density, and the flow velocity, respectively; $p(\rho) = A\rho^\gamma$ stands for the pressure with the adiabatic exponent $\gamma > 1$, $A > 0$ is the squared reciprocal of the Mach number. The nematic tensor order parameter $Q$ is a traceless and $3\times 3$ symmetric matrix. Furthermore, $\mathcal{L}$ stands for the Lamé operator

$$\mathcal{L}\mathbf{u} = \upsilon\triangle\mathbf{u} + (\upsilon+\lambda)\nabla\mathrm{div}_x\mathbf{u},$$

where $\upsilon > 0, \lambda \geq 0$ are shear viscosity and bulk viscosity coefficient of the fluid, respectively. The term $\nabla_x Q \odot \nabla_x Q$ stands for the $3\times 3$ matrix with its $(i,j)$-th entry defined by

$$(\nabla_x Q \odot \nabla_x Q)_{ij} = \sum_{k,l=1}^{3} \partial_i Q_{kl}\partial_j Q_{kl},$$

and $\mathrm{I}_3$ stands for the $3\times 3$ identity matrix. Define the free energy density of the director field $\mathcal{F}(Q)$

$$\mathcal{F}(Q) = \frac{L}{2}|\nabla_x Q|^2 + \frac{a}{2}\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\mathrm{tr}^2(Q^2),$$

and denote

$$\Gamma\mathcal{H}(Q) = \Gamma L\triangle Q + \Gamma\left(-aQ + b\left[Q^2 - \frac{\mathrm{I}_3}{3}\mathrm{tr}(Q^2)\right] - cQ\mathrm{tr}(Q^2)\right) =: \Gamma L\triangle Q + \mathcal{K}(Q).$$

The coefficients in the formula are elastic constants: $L > 0$, $\Gamma > 0$, $a \in \mathbb{R}$, $b > 0$ and $c > 0$, which are dependent on the material. Finally $\Theta = \frac{\nabla_x\mathbf{u} - \nabla_x\mathbf{u}^\top}{2}$ is the skew-symmetric part of the rate of strain tensor. From the specific form $\mathcal{K}(Q)$, we remark that

$$Q\mathcal{H}(Q) - \mathcal{H}(Q)Q = L(Q\triangle Q - \triangle QQ).$$

The PDEs perturbed randomly are considered as a primary tool in the modeling of uncertainty, especially while describing fundamental phenomenon in physics, climate dynamics, communication systems, nanocomposites and gene regulation systems. Hence, the study of the well-posedness and dynamical behaviour of PDEs subject to the noise which is largely applied to the theoretical and practical areas has drawn a lot of attention. Here, we consider the system (3.0.1) driven by a multiplicative noise:

$$\begin{cases} d\rho + \mathrm{div}_x(\rho\mathbf{u})dt = 0, \\ d(\rho\mathbf{u}) + \mathrm{div}_x(\rho\mathbf{u}\otimes\mathbf{u})dt + A\nabla_x\rho^\gamma dt \\ \quad = \mathcal{L}\mathbf{u}dt - \mathrm{div}_x(L\nabla_x Q\odot\nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3)dt + L\mathrm{div}_x(Q\triangle Q - \triangle QQ)dt \\ \qquad +\mathbb{G}(\rho,\rho\mathbf{u})dW, \\ dQ + \mathbf{u}\cdot\nabla_x Qdt - (\Theta Q - Q\Theta)dt = \Gamma\mathcal{H}(Q)dt, \end{cases} \tag{3.0.2}$$

where $W$ is a cylindrical Wiener process which will be introduced later. The system is equipped with the initial data

$$\rho(0,x) = \rho_0(x), \quad \mathbf{u}(0,x) = \mathbf{u}_0(x), \quad Q(0,x) = Q_0(x), \tag{3.0.3}$$

and the periodic boundary, where each period is a cube $\mathbb{T}\subset\mathbb{R}^3$ defined as follows

$$\mathbb{T} = (-\pi,\pi)|_{\{-\pi,\pi\}^3}. \tag{3.0.4}$$

## 3.1 Preliminary and Main Result

First, we present some deterministic as well as stochastic preliminaries associated with system (3.0.2). For each integer $s\in\mathbb{N}^+$, denote $W^{s,2}(\mathbb{T})$ as the Sobolev space containing all the functions having distributional derivatives up to order $s$, and the derivatives are integrable in $L^2(\mathbb{T})$, endowed with the norm

$$\|u\|^2_{W^{s,2}} = \sum_{k\in\mathbb{Z}^3}(1+k^2)^s|\hat{u}_k|^2,$$

where $\hat{u}_k$ is the Fourier coefficients of $u$. $W^{s,2}(\mathbb{T})$ is an Hilbert space, and for any $u$, $v\in W^{s,2}$, the inner product can be denoted as

$$(u,v)_{s,2} = \sum_{|\alpha|\le s}\int_{\mathbb{T}}\partial_x^\alpha u\cdot\partial_x^\alpha v dx.$$

For simplicity, we denote the notations $\|\cdot\|$ as the $L^2$-norm, $\|\cdot\|_\infty$ as the $L^\infty$-norm, and$\|\cdot\|_{s,p}$ as the $W^{s,p}$-norm for all $1 \le s < \infty, 1 \le p \le \infty$.

Define the inner product between two $3\times 3$ matrices $\mathrm{M}_1$ and $\mathrm{M}_2$

$$(\mathrm{M}_1, \mathrm{M}_2) = \int_{\mathbb{T}} \mathrm{M}_1 : \mathrm{M}_2 dx = \int_{\mathbb{T}} \mathrm{tr}(\mathrm{M}_1\mathrm{M}_2) dx,$$

and $S_0^3 \subset \mathbb{M}^{3\times 3}$ the space of $Q$-tensor

$$S_0^3 = \left\{Q \in \mathbb{M}^{3\times 3} : \ Q_{ij} = Q_{ji}, \ \mathrm{tr}(Q) = 0, \ i, j = 1, 2, 3\right\},$$

and the norm of a matrix using the Frobenius norm

$$|Q|^2 = \mathrm{tr}(Q^2) = \sum_{i,j=1}^{3} Q_{ij}Q_{ij}.$$

Set $|\partial_x^\alpha Q|^2 = \sum\limits_{i,j=1}^{3} \partial_x^\alpha Q_{ij}\partial_x^\alpha Q_{ij}$. The Sobolev space of $Q$-tensor is defined by

$$W^{s,2}(\mathbb{T}; S_0^3) = \left\{ Q : \mathbb{T} \to S_0^3, \ \text{and} \ \sum_{|\alpha|\le s} \|\partial_x^\alpha Q\|^2 < \infty \right\},$$

endowed with the norm

$$\|Q\|^2_{W^{s,2}(\mathbb{T};S_0^3)} := \|Q\|^2_{s,2} = \sum_{|\alpha|\le s} \|\partial_x^\alpha Q\|^2.$$

To deal with the estimate of the nonlinear terms in the equations, we present the following lemmas that involves commutator and Moser estimates. The proof of these lemmas can be found in [51, 63].

**Lemma 3.1.1.** *For $u, v \in W^{s,2}(\mathbb{T})$, $s > \frac{d}{2} + 1$, $d = 2, 3$ is the dimension of space, it holds*

$$\sum_{0\le|\alpha|\le s} \|\partial_x^\alpha (u\cdot\nabla_x)v - u\cdot\nabla_x\partial_x^\alpha v\| \le C(\|\nabla_x u\|_\infty \|v\|_{s,2} + \|\nabla_x v\|_\infty \|u\|_{s,2}), \tag{3.1.1}$$

*and*

$$\|uv\|_{s,2} \le C(\|u\|_\infty\|v\|_{s,2} + \|v\|_\infty\|u\|_{s,2}), \tag{3.1.2}$$

*for some positive constant $C = C(s, \mathbb{T})$ independent of $u$ and $v$.*

**Lemma 3.1.2.** *Let $f$ be a $s$-order continuously differentiable function on the neighborhood of compact set $G = \text{range}[u]$ and $u \in W^{s,2}(\mathbb{T}) \cap C(\mathbb{T})$, it holds*

$$\|\partial_x^\alpha f(u)\| \leq C\|\partial_u f\|_{C^{s-1}(G)}\|u\|_\infty^{|\alpha|-1}\|\partial_x^\alpha u\|,$$

*for all $\alpha \in \mathbb{N}^N, 1 < |\alpha| \leq s$.*

The following result is crucial to handle the highest-order derivative terms in the momentum and $Q$-tensor equations.

**Lemma 3.1.3.** *Assume that $Q$ and $Q'$ are two $3 \times 3$ symmetric matrices, and $\Theta = \frac{1}{2}(\nabla_x \mathbf{u} - \nabla_x \mathbf{u}^{\mathrm{T}})$, as $\nabla_x \mathbf{u}$ is also a $3 \times 3$ matrix, and $(\nabla_x \mathbf{u})_{ij} = \partial_i u_j$, $f(r)$ is a scalar function. Then*

$$(f(r)(\Theta Q' - Q'\Theta), \triangle Q) + (f(r)(Q'\triangle Q - \triangle Q Q'), \nabla_x \mathbf{u}^{\mathrm{T}}) = 0.$$

*Proof.* In a similar way to [21, Lemma A.1], using the fact that $\mathrm{tr}(\mathrm{M}_1\mathrm{M}_2) = \mathrm{tr}(\mathrm{M}_1\mathrm{M}_2)$ and $Q', Q, \Theta + \nabla_x \mathbf{u}^{\mathrm{T}}$ are symmetric, $f(r)$ is scalar function, we get

$$\begin{aligned}
&(f(r)(\Theta Q' - Q'\Theta), \triangle Q) + (f(r)(Q'\triangle Q - \triangle Q Q'), \nabla_x \mathbf{u}^{\mathrm{T}}) \\
&= (f(r)(Q'\triangle Q - \triangle Q Q'), \Theta) + (f(r)(Q'\triangle Q - \triangle Q Q'), \nabla_x \mathbf{u}^{\mathrm{T}}) \\
&= (f(r)(Q'\triangle Q - \triangle Q Q'), \Theta + \nabla_x \mathbf{u}^{\mathrm{T}}) = 0,
\end{aligned}$$

we finish the proof. □

Next, we introduce the following fractional-order Sobolev space with respect to time $t$, since noise term is only Hölder's continuous of order strictly less than $\frac{1}{2}$ in time.

For any fixed $p > 1$ and $\alpha \in (0, 1)$ we define

$$W^{\alpha,p}(0, T; X) = \left\{ v \in L^p(0, T; X) : \int_0^T \int_0^T \frac{\|v(t_1) - v(t_2)\|_X^p}{|t_1 - t_2|^{1+\alpha p}} dt_1 dt_2 < \infty \right\},$$

endowed with the norm

$$\|v\|_{W^{\alpha,p}(0,T;X)}^p := \int_0^T \|v(t)\|_X^p dt + \int_0^T \int_0^T \frac{\|v(t_1) - v(t_2)\|_X^p}{|t_1 - t_2|^{1+\alpha p}} dt_1 dt_2,$$

for any separable Hilbert space $X$. If we take $\alpha = 1$, then

$$W^{1,p}(0, T; X) := \left\{ v \in L^p(0, T; X) : \frac{dv}{dt} \in L^p(0, T; X) \right\},$$

we could see that the space returns to the classical Sobolev space endowed with the usual norm

$$\|v\|^p_{W^{1,p}(0,T;X)} := \int_0^T \|v(t)\|^p_X + \left\|\frac{dv}{dt}(t)\right\|^p_X dt.$$

Note that for $\alpha \in (0,1)$, $W^{1,p}(0,T;X)$ is a subspace of $W^{\alpha,p}(0,T;X)$.

For any $\alpha \le \beta - \frac{1}{p}$, it holds

$$W^{\beta,p}(0,T;L^2(\mathbb{T})) \hookrightarrow C^\alpha([0,T];L^2(\mathbb{T})). \tag{3.1.3}$$

Let $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P}, W)$ be a fixed stochastic basis and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $W$ be a Wiener process defined on an Hilbert space $\mathfrak{U}$, which is adapted to the complete, right continuous filtration $\{\mathcal{F}_t\}_{t\ge0}$. If $\{e_k\}_{k\ge1}$ is a complete orthonormal basis of $\mathfrak{U}$, then $W$ can be written formally as the expansion $W(t,\omega) = \sum_{k\ge1} e_k\beta_k(t,\omega)$ where $\{\beta_k\}_{k\ge1}$ is a sequence of independent standard one-dimensional Brownian motions.

Define an auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ by

$$\mathfrak{U}_0 = \left\{ v = \sum_{k\ge1} \alpha_k e_k : \sum_{k\ge1} \frac{\alpha_k^2}{k^2} < \infty \right\},$$

with the norm $\|v\|^2_{\mathfrak{U}_0} = \sum_{k\ge1} \frac{\alpha_k^2}{k^2}$. Note that the embedding of $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt. We also have that $W \in C([0,\infty), \mathfrak{U}_0)$ almost surely, see [75].

Now considering another separable Hilbert space $X$ and let $L_2(\mathfrak{U}, X)$ be the set of all Hilbert-Schmidt operators $S : \mathfrak{U} \to X$ with the norm $\|S\|^2_{L_2(\mathfrak{U},X)} = \sum_{k\ge1}\|Se_k\|^2_X$. For a predictable process $G \in L^2(\Omega; L^2_{loc}([0,\infty), L_2(\mathfrak{U}, X)))$ by taking $G_k = Ge_k$, one can define the stochastic integral

$$\mathcal{M}_t := \int_0^t G dW = \sum_k \int_0^t Ge_k d\beta_k = \sum_k \int_0^t G_k d\beta_k,$$

which is an $X$-valued square integrable martingale, and the Burkholder-Davis-Gundy inequality holds

$$\mathbb{E}\left(\sup_{0\le t\le T}\left\|\int_0^t GdW\right\|^p_X\right) \le c_p \mathbb{E}\left(\int_0^T \|G\|^2_{L_2(\mathfrak{U},X)} dt\right)^{\frac{p}{2}}, \tag{3.1.4}$$

for any $1 \le p < \infty$, for more details see [75]. The notation $\mathbb{E}$ represents the expectation.

We shall present the main result of this paper. First, we define local strong pathwise solution. For this type of solution, "strong" means in PDE and probability sense, "local" means existence in finite time.

**Definition 3.1.4.** (Local strong pathwise solution). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a fixed probability space, W be an $\mathcal{F}_t$-cylindrical Wiener process. Then $(\rho, \mathbf{u}, Q, \mathfrak{t})$ is a local strong pathwise solution to system (3.0.2) if the following conditions hold

1. $\mathfrak{t}$ is a strictly positive a.s. $\mathcal{F}_t$-stopping time;
2. $\rho$, $\mathbf{u}$, $Q$ are $\mathcal{F}_t$-progressively measurable processes, satisfying $\mathbb{P}$ a.s.

$$
\begin{aligned}
&\rho(\cdot \wedge \mathfrak{t}) > 0,\ \rho(\cdot \wedge \mathfrak{t}) \in C([0,T]; W^{s,2}(\mathbb{T})),\\
&\mathbf{u}(\cdot \wedge \mathfrak{t}) \in L^\infty(0,T; W^{s,2}(\mathbb{T},\mathbb{R}^3)) \cap L^2(0,T; W^{s+1,2}(\mathbb{T},\mathbb{R}^3)) \cap C([0,T]; W^{s-1,2}(\mathbb{T},\mathbb{R}^3)),\\
&Q(\cdot \wedge \mathfrak{t}) \in L^\infty(0,T; W^{s+1,2}(\mathbb{T},S_0^3)) \cap L^2(0,T; W^{s+2,2}(\mathbb{T},S_0^3)) \cap C([0,T]; W^{s,2}(\mathbb{T},S_0^3));
\end{aligned}
$$

3. for any $t \in [0,T]$, $\mathbb{P}$ a.s.

$$
\begin{aligned}
\rho(\mathfrak{t}\wedge t) &= \rho_0 - \int_0^{\mathfrak{t}\wedge t} \mathrm{div}_x(\rho\mathbf{u})d\xi,\\
(\rho\mathbf{u})(\mathfrak{t}\wedge t) &= \rho_0\mathbf{u}_0 - \int_0^{\mathfrak{t}\wedge t} \mathrm{div}_x(\rho\mathbf{u}\otimes\mathbf{u})d\xi - \int_0^{\mathfrak{t}\wedge t} \nabla_x(A\rho^\gamma)d\xi + \int_0^{\mathfrak{t}\wedge t} \mathcal{L}\mathbf{u}d\xi\\
&\quad - \int_0^{\mathfrak{t}\wedge t} \mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3)d\xi\\
&\quad + \int_0^{\mathfrak{t}\wedge t} L\mathrm{div}_x(Q\triangle Q - \triangle Q Q)d\xi + \int_0^{\mathfrak{t}\wedge t} \mathbb{G}(\rho,\rho\mathbf{u})dW,\\
Q(\mathfrak{t}\wedge t) &= Q_0 - \int_0^{\mathfrak{t}\wedge t} \mathbf{u}\cdot\nabla_x Q d\xi + \int_0^{\mathfrak{t}\wedge t} (\Theta Q - Q\Theta)d\xi + \int_0^{\mathfrak{t}\wedge t} \Gamma\mathcal{H}(Q)d\xi.
\end{aligned}
$$

We say that the pathwise uniqueness holds: if $(\rho_1, \mathbf{u}_1, Q_1, \mathfrak{t}_1)$ and $(\rho_2, \mathbf{u}_2, Q_2, \mathfrak{t}_2)$ are two local strong pathwise solutions of system (3.0.2) with

$$
\mathbb{P}\{(\rho_1(0), \mathbf{u}_1(0), Q_1(0)) = (\rho_2(0), \mathbf{u}_2(0), Q_2(0))\} = 1,
$$

then

$$
\mathbb{P}\left\{(\rho_1(t,x), \mathbf{u}_1(t,x), Q_1(t,x)) = (\rho_2(t,x), \mathbf{u}_2(t,x), Q_2(t,x)); \forall t \in [0, \mathfrak{t}_1 \wedge \mathfrak{t}_2]\right\} = 1.
$$

**Definition 3.1.5.** (Maximal strong pathwise solution) A maximal pathwise solution is a quintuple $(\rho, \mathbf{u}, Q, \{\tau_n\}_{n\geq 1}, \mathfrak{t})$ such that each $(\rho, \mathbf{u}, Q, \tau_n)$ is a local pathwise solution in the sense of Definition 3.1.4 and $\{\tau_n\}$ is an increasing sequence with $\lim_{n\to\infty} \tau_n = \mathfrak{t}$ and

$$\sup_{t\in[0,\tau_n]} \|\mathbf{u}(t)\|_{2,\infty} \geq n, \quad \sup_{t\in[0,\tau_n]} \|Q(t)\|_{3,\infty} \geq n, \quad \text{on the set } \{\mathfrak{t} < \infty\}.$$

From the Definition 3.1.5, we can see that

$$\sup_{t\in[0,\mathfrak{t})} \|\mathbf{u}(t)\|_{2,\infty} = \infty, \; \sup_{t\in[0,\mathfrak{t})} \|Q(t)\|_{3,\infty} = \infty, \; \text{on the set } \{\mathfrak{t} < \infty\}.$$

This means the existence time for the solution is determined by the explosion time of the $W^{2,\infty}$-norm of the velocity and $W^{3,\infty}$-norm of the $Q$-tensor.

Throughout the paper, we impose the following assumptions on the noise intensity $\mathbb{G}$: there exists a constant $C$ such that for any $s \geq 0, \rho > 0$,

$$\|\rho^{-1}\mathbb{G}(\rho, \rho\mathbf{u})\|^2_{L_2(\mathfrak{U};W^{s,2}(\mathbb{T}))} \leq C(\|\rho\|^2_{1,\infty} + \|\mathbf{u}\|^2_{2,\infty})\|\rho, \mathbf{u}\|^2_{s,2}, \tag{3.1.5}$$

and

$$\begin{aligned} &\|\rho_1^{-1}\mathbb{G}(\rho_1, \rho_1\mathbf{u}_1) - \rho_2^{-1}\mathbb{G}(\rho_2, \rho_2\mathbf{u}_2)\|^2_{L_2(\mathfrak{U};W^{s,2}(\mathbb{T}))} \\ &\leq C(\|\rho_1, \rho_2\|^2_{1,\infty} + \|\mathbf{u}_1, \mathbf{u}_2\|^2_{2,\infty})\|\rho_1 - \rho_2, \mathbf{u}_1 - \mathbf{u}_2\|^2_{s,2}, \end{aligned} \tag{3.1.6}$$

where the norm $\|u, v\|^2_{s,2} := \|u\|^2_{s,2} + \|v\|^2_{s,2}$ for $u, v \in W^{s,2}$. A typical example for $\mathbb{G}$ is

$$\mathbf{G}_k(x, \rho, \mathbf{q}) = \mathbb{A}_k(x)\rho\mathbf{q}.$$

With $\mathbb{A}_k(x)$: $\mathbb{T} \to \mathbb{R}^{3\times 3}$ a smooth matrix function, and $\mathbf{G}_k = \mathbb{G}e_k$. Assumption (3.1.5) will be used for constructing the a priori estimate, while assumption (3.1.6) will be applied to identify the limit and establish the uniqueness.

**Remark 3.1.6.** Set $r = \sqrt{\frac{2A\gamma}{\gamma-1}}\rho^{\frac{\gamma-1}{2}}$. If the initial data $r_0$ satisfies some certain assumption, see Theorem 3.3.2, then the assumptions (3.1.5),(3.1.6) still hold if we replace $\rho$ by $r$ and $\rho^{-1}\mathbb{G}(\rho, \mathbf{u})$ by $\mathbb{F}(r, \mathbf{u}) = \frac{1}{\rho(r)}\mathbb{G}(\rho(r), \rho(r)\mathbf{u})$.

Our main result of this paper is below.

**Theorem 3.1.7.** *Assume $s \in \mathbb{N}$ satisfies $s > \frac{9}{2}$, and the coefficient $\mathbb{G}$ satisfies the assumptions* (3.1.5),(3.1.6)*, and the initial data $(\rho_0, \mathbf{u}_0, Q_0)$ is $\mathcal{F}_0$-measurable random variable, with values in $W^{s,2}(\mathbb{T}) \times W^{s,2}(\mathbb{T};\mathbb{R}^3) \times W^{s+1,2}(\mathbb{T};S_0^3)$, also $\rho_0 > 0$, $\mathbb{P}$ a.s.. Then there exists a unique maximal strong pathwise solution $(\rho, \mathbf{u}, Q, \mathfrak{t})$ to system* (3.0.2)-(3.0.4) *in the sense of Definition 3.1.5.*

## 3.2 Construction of Truncated Symmetric System

Before the construction of the strong solution, we need to assume first that the vacuum state does not appear. By doing so, we are able to rewrite the system (3.0.2) into the symmetric system following the operation in [11]. To begin with, applying equation (3.0.2)(1), then equation (3.0.2)(2) can be written into the following form

$$\begin{aligned}&\rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla_x\mathbf{u} + A\nabla_x\rho^\gamma\\ =&\mathcal{L}\mathbf{u} - \mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3) + L\mathrm{div}_x(Q\triangle Q - \triangle Q Q) + \mathbb{G}(\rho,\rho\mathbf{u})\frac{dW}{dt},\end{aligned}$$

as $\rho > 0$, divide the above equation by $\rho$ on both sides, we could have

$$\begin{aligned}&\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_x\mathbf{u} + \frac{A}{\rho}\nabla_x\rho^\gamma\\ =&\frac{1}{\rho}\mathcal{L}\mathbf{u} - \frac{1}{\rho}\mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3) + L\frac{1}{\rho}\mathrm{div}_x(Q\triangle Q - \triangle Q Q)\\ &+ \frac{1}{\rho}\mathbb{G}(\rho,\rho\mathbf{u})\frac{dW}{dt}.\end{aligned} \tag{3.2.1}$$

The pressure term can be written into a symmetric form:

$$\frac{A}{\rho}\nabla_x\rho^\gamma = \frac{A}{\gamma-1}\nabla_x\rho^{\gamma-1} = \frac{2A\gamma}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\nabla_x\rho^{\frac{\gamma-1}{2}}.$$

Considering this, define

$$r = \sqrt{\frac{2A\gamma}{\gamma-1}}\rho^{\frac{\gamma-1}{2}},$$

and

$$D(r) = \frac{1}{\rho(r)} = \left(\frac{\gamma-1}{2A\gamma}\right)^{-\frac{1}{\gamma-1}} r^{-\frac{2}{\gamma-1}}, \ \mathbb{F}(r,\mathbf{u}) = \frac{1}{\rho(r)}\mathbb{G}(\rho(r),\rho(r)\mathbf{u}).$$

Then, the system (3.0.2) can be transformed into

$$\begin{cases} dr + \left(\mathbf{u}\cdot\nabla_x r + \frac{\gamma-1}{2} r\mathrm{div}_x\mathbf{u}\right)dt = 0, \\ d\mathbf{u} + (\mathbf{u}\cdot\nabla_x\mathbf{u} + r\nabla_x r)dt \\ \quad = D(r)(\mathcal{L}\mathbf{u} - \mathrm{div}_x(L\nabla_x Q\odot\nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3) + L\mathrm{div}_x(Q\triangle Q - \triangle QQ))dt \\ \qquad + \mathbb{F}(r,\mathbf{u})dW, \\ dQ + (\mathbf{u}\cdot\nabla_x Q - \Theta Q + Q\Theta)dt = \Gamma\mathcal{H}(Q)dt. \end{cases} \tag{3.2.2}$$

As mentioned in the introduction, we add a cut-off function to render the nonlinear terms, where the cut-off function depends only on $\|\mathbf{u}\|_{2,\infty}, \|Q\|_{3,\infty}$.

Let $\Phi_R : [0,\infty) \to [0,1]$ be a $C^\infty$-smooth function defined as follows

$$\Phi_R(x) = \begin{cases} 1, & \text{if } 0 < x < R, \\ 0, & \text{if } x > 2R. \end{cases}$$

Then, define $\Phi_R^{\mathbf{u},Q} = \Phi_R^{\mathbf{u}}\cdot\Phi_R^Q$, where $\Phi_R^{\mathbf{u}} = \Phi_R(\|\mathbf{u}\|_{2,\infty})$, $\Phi_R^Q = \Phi_R(\|Q\|_{3,\infty})$ and add the cut-off function in front of nonlinear terms of system (3.2.2), we have

$$\begin{cases} dr + \Phi_R^{\mathbf{u},Q}\left(\mathbf{u}\cdot\nabla_x r + \frac{\gamma-1}{2} r\mathrm{div}_x\mathbf{u}\right)dt = 0, \\ d\mathbf{u} + \Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x\mathbf{u} + r\nabla_x r)dt \\ \quad = \Phi_R^{\mathbf{u},Q}D(r)(\mathcal{L}\mathbf{u} - \mathrm{div}_x(L\nabla_x Q\odot\nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3) + L\mathrm{div}_x(Q\triangle Q - \triangle QQ))dt \\ \qquad + \Phi_R^{\mathbf{u},Q}\mathbb{F}(r,\mathbf{u})dW, \\ dQ + \Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x Q - \Theta Q + Q\Theta)dt = \Gamma L\triangle Q dt + \Phi_R^{\mathbf{u},Q}\mathcal{K}(Q)dt. \end{cases} \tag{3.2.3}$$

**Remark 3.2.1.** In system (3.2.3), we use the same cut-off function $\Phi_R^{\mathbf{u},Q}$ in front of the all nonlinear terms to simplify the notation. Actually, we can replace $\Phi_R^{\mathbf{u},Q}$ by $\Phi_R^{\mathbf{u}}$ on the left hand side of equations (3.2.3)(1)(2) and in front of the stochastic term, replace $\Phi_R^{\mathbf{u},Q}$ by $\Phi_R^Q$ on the right hand side of equation (3.2.3)(3).

In the following, we mainly discuss the truncated system (3.2.3).

### 3.3 Existence of Strong Martingale Solution

In this section, the main aim is that, proving the existence of a strong martingale solution to system (3.2.3) which is strong in PDE sense and weak in probability sense if the initial condition is good enough. To start, we bring in the concept of strong martingale solution.

**Definition 3.3.1.** (Strong martingale solution) Assume that $\Lambda$ is a Borel probability measure on the space $W^{s,2}(\mathbb{T})\times W^{s,2}(\mathbb{T},\mathbb{R}^3)\times W^{s+1,2}(\mathbb{T},S_0^3)$ for integer $s>\frac{7}{2}$, then the quintuple

$$((\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}),r,\mathbf{u},Q,W)$$

is a strong martingale solution to the truncated system (3.2.3) equipped with the initial law $\Lambda$ if the following conditions hold

1. $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ is a stochastic basis with a complete right-continuous filtration, $W$ is a Wiener process relative to the filtration $\mathcal{F}_t$;
2. $r$, $\mathbf{u},Q$ are $\mathcal{F}_t$-progressively measurable processes with values in $W^{s,2}(\mathbb{T}),W^{s,2}(\mathbb{T},\mathbb{R}^3)$, $W^{s+1,2}(\mathbb{T},S_0^3)$, satisfying

$$\begin{aligned}
&r\in L^2(\Omega;C([0,T];W^{s,2}(\mathbb{T}))),\ \ r(t)>0,\ \ \mathbb{P}\text{ a.s., for all } t\in[0,T],\\
&\mathbf{u}\in L^2(\Omega;L^\infty(0,T;W^{s,2}(\mathbb{T};\mathbb{R}^3))\cap C([0,T];W^{s-1,2}(\mathbb{T},\mathbb{R}^3))),\\
&Q\in L^2(\Omega;L^\infty(0,T;W^{s+1,2}(\mathbb{T};S_0^3))\cap L^2(0,T;W^{s+2,2}(\mathbb{T};S_0^3))\cap C([0,T];W^{s,2}(\mathbb{T};S_0^3)));
\end{aligned}$$

3. the initial law $\Lambda=\mathbb{P}\circ(r_0,\mathbf{u}_0,Q_0)^{-1}$;
4. for all $t\in[0,T]$, $\mathbb{P}$ a.s.

$$\begin{aligned}
r(t)&=r(0)-\int_0^t\Phi_R^{\mathbf{u},Q}\left(\mathbf{u}\cdot\nabla_x r+\frac{\gamma-1}{2}r\mathrm{div}_x\mathbf{u}\right)d\xi,\\
\mathbf{u}(t)&=\mathbf{u}(0)-\int_0^t\Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x\mathbf{u}+r\nabla_x r)d\xi\\
&\quad+\int_0^t\Phi_R^{\mathbf{u},Q}D(r)(\mathcal{L}\mathbf{u}-\mathrm{div}_x(L\nabla_x Q\odot\nabla_x Q-\mathcal{F}(Q)\mathrm{I}_3)\\
&\quad+L\mathrm{div}_x(Q\triangle Q-\triangle QQ))d\xi+\int_0^t\Phi_R^{\mathbf{u},Q}\mathbb{F}(r,\mathbf{u})dW,\\
Q(t)&=Q(0)-\int_0^t\Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x Q-\Theta Q+Q\Theta)d\xi+\int_0^t\Gamma L\triangle Q+\Phi_R^{\mathbf{u},Q}\mathcal{K}(Q)d\xi.
\end{aligned}$$

We state our main result for this section.

**Theorem 3.3.2.** *Assume the initial data* $(r_0, \mathbf{u}_0, Q_0)$ *satisfies*

$$(r_0, \mathbf{u}_0, Q_0) \in L^p(\Omega; W^{s,2}(\mathbb{T}) \times W^{s,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s+1,2}(\mathbb{T}, S_0^3)),$$

*for any* $1 \leq p < \infty$, $s > \frac{7}{2}$ *be the integer, and in addition*

$$\|Q_0\|_{1,2} < R,\ \|r_0\|_{1,\infty} < R,\ r_0 > \frac{1}{R},\quad \mathbb{P}\quad a.s.$$

*for constant* $R > 0$, *the coefficient* $\mathbb{G}$ *satisfies assumptions* (3.1.5),(3.1.6), *then there exists a strong martingale solution to the system* (3.2.3) *with the initial law* $\Lambda = \mathbb{P} \circ (r_0, \mathbf{u}_0, Q_0)^{-1}$ *in the sense of Definition 3.3.1 and we also have*

$$r(t, \cdot) \geq \mathcal{C}(R) > 0,\ \mathbb{P}\ a.s.,\ \text{for all } t \in [0, T],$$

*where* $\mathcal{C}(R)$ *is a constant depending on* $R$, *and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} (\|r(t), \mathbf{u}(t)\|_{s,2}^2 + \|Q(t)\|_{s+1,2}^2) + \int_0^T \Phi_R^{\mathbf{u},Q} \|\mathbf{u}\|_{s+1,2}^2 + \|Q\|_{s+2,2}^2 dt\right]^p \leq C,$$

*for any* $T > 0$, *where* $C = C(p, s, R, \mathbb{T}, T, L, \Gamma)$ *is a constant.*

**Remark 3.3.3.** Here, we assume that $\|Q_0\|_{1,2} < R$, $\mathbb{P}$ a.s. for establishing the Galerkin approximate solution, which could also be relaxed to general case, see Section 6.

The following part is devoted to proving Theorem 3.3.2 which is divided into three steps. First, we construct the approximate solution in the finite-dimensional space. Then we get the uniform estimate of the approximate solution, and show the stochastic compactness. Next, the existence of the strong martingale solution can be derived from taking the limit of the approximate system.

### 3.3.1 Galerkin approximate system

In this subsection, we construct the Galerkin approximate solution of system (3.2.3). First, for any smooth functions $\mathbf{u}, Q$, the transport equation (3.2.3)(1) would admit a classical solution $r = r[\mathbf{u}]$, and the solution is unique if the initial data $r_0$ is given. The solution $r[\mathbf{u}]$ shares the same regularity with the initial data $r_0$. In addition, for certain constant c, we have, see also [11, Section 3.1]

$$\begin{aligned} &\frac{1}{R}\exp(-cRt) \le \exp(-cRt)\inf_{x\in\mathbb{T}} r_0 \le r(t,\cdot) \le \exp(cRt)\sup_{x\in\mathbb{T}} r_0 \le R\exp(cRt), \\ &|\nabla_x r(t,\cdot)| \le \exp(cRt)|\nabla_x r_0| \le R\exp(cRt), \quad \text{for any} \quad t\in[0,T]. \end{aligned} \tag{3.3.1}$$

Using the bound (3.3.1), after a simple calculation, yields

$$\|D(r)^{-1}\|_{1,\infty} + \|D(r)\|_{1,\infty} \le C(R)\exp(cRt). \tag{3.3.2}$$

In addition, by the mean value theorem, the bound (3.3.1) and Lemmas 3.1.1, 3.1.2, we have for any $s > \frac{d}{2}$

$$\|D(r)\|_{s,2} \le C(R,T)\|r\|_{s,2}, \tag{3.3.3}$$

and

$$\|D(r_1) - D(r_2)\|_{s,2} \le C(R,T)\|r_1, r_2\|_{s,2}\|r_1 - r_2\|_{s,2}. \tag{3.3.4}$$

Indeed, due to the mean value theorem, there exists some $\theta \in (0,1)$ such that

$$\begin{aligned} &\|D(r_1) - D(r_2)\|_{s,2} \\ &= \left\|\frac{dD}{dr}(\theta r_1 + (1-\theta)r_2)\cdot(r_1 - r_2)\right\|_{s,2} \\ &\le C\left\|\frac{dD}{dr}(\theta r_1 + (1-\theta)r_2)\right\|_{\infty}\|r_1 - r_2\|_{s,2} + C\|r_1 - r_2\|_{\infty}\left\|\frac{dD}{dr}(\theta r_1 + (1-\theta)r_2)\right\|_{s,2} \\ &\le C(R,T)\|r_1, r_2\|_{s,2}\|r_1 - r_2\|_{s,2}. \end{aligned}$$

**Lemma 3.3.4.** *For any smooth function* $\mathbf{u} \in C([0,T]; X_N(\mathbb{T}))$ *and integer* $s > \frac{7}{2}$*, there exists a unique solution*

$$Q \in C([0,T]; W^{s+1,2}(\mathbb{T}, S_0^3)) \cap L^2(0,T; W^{s+2,2}(\mathbb{T}, S_0^3))$$

*to the initial value problem*

$$\begin{cases} Q_t + \Phi_R^{\mathbf{u},Q}(\mathbf{u} \cdot \nabla_x Q - \Theta Q + Q\Theta) = \Gamma L \triangle Q + \Phi_R^{\mathbf{u},Q} \mathcal{K}(Q), \textit{ in } \mathbb{T} \times (0,T) \\ Q|_{t=0} = Q_0(x) \in W^{s+1,2}(\mathbb{T}, S_0^3). \end{cases} \tag{3.3.5}$$

*Moreover, the mapping*

$$\mathbf{u} \to Q[\mathbf{u}] : C([0,T]; X_N(\mathbb{T})) \to C([0,T]; W^{s+1,2}(\mathbb{T}, S_0^3)) \cap L^2(0,T; W^{s+2,2}(\mathbb{T}, S_0^3)) \tag{3.3.6}$$

*is continuous on a bounded set* $B \in C([0,T]; X_N(\mathbb{T}))$*, where* $X_N$ *is a finite dimensional space spanned by* $\{\psi_m\}_{m=1}^N$*, see (3.3.17).*

*Proof.* Existence: Step 1. Since the system (3.3.5) is a type of parabolic evolution system, we are able to establish the existence and uniqueness of finite-dimensional local approximate solutions $Q_m$ using the Galerkin method and the fixed point theorem, for further details, see [22, 86]. Then, we could extend the local solution to global in time using the following uniform a priori estimate.

Step 2. Let $\alpha$ be a multi index such that $|\alpha| \leq s$. Taking $\alpha$-order derivative on both sides of the $m$-th order finite-dimensional approximate system of (3.3.5), multiplying by $-\triangle \partial_x^\alpha Q_m$, then the trace and integrating over $\mathbb{T}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_x^{\alpha+1} Q_m\|^2 + \Gamma L \|\triangle \partial_x^\alpha Q_m\|^2 \\ &= \Phi_R^{\mathbf{u},Q_m}(\partial_x^{\alpha+1}(\mathbf{u} \cdot \nabla_x Q_m - \Theta Q_m + Q_m \Theta), \partial_x^{\alpha+1} Q_m) \\ &\quad + \Phi_R^{\mathbf{u},Q_m}(\partial_x^{\alpha+1} \mathcal{K}(Q_m), \partial_x^{\alpha+1} Q_m). \end{aligned} \tag{3.3.7}$$

For the first term on the right hand side of (3.3.7), using the Hölder inequality and Lemma 3.1.1, we obtain

$$|\Phi_R^{\mathbf{u},Q_m}(\partial_x^{\alpha+1}(\mathbf{u} \cdot \nabla_x Q_m - \Theta Q_m + Q_m \Theta), \partial_x^{\alpha+1} Q_m)|$$

$$
\begin{aligned}
&\leq C\Phi_R^{\mathbf{u},Q_m}\|\partial_x^{\alpha+1}Q_m\|\|\partial_x^{\alpha+1}(\mathbf{u}\cdot\nabla_x Q_m-\Theta Q_m+Q_m\Theta)\|\\
&\leq C\Phi_R^{\mathbf{u},Q_m}\|\partial_x^{\alpha+1}Q_m\|(\|\mathbf{u}\|_\infty\|\partial_x^{\alpha+1}\nabla_x Q_m\|+C\|\nabla_x Q_m\|_\infty\|\partial_x^{\alpha+1}\mathbf{u}\|\\
&\qquad\qquad\qquad+\|\nabla_x\mathbf{u}\|_\infty\|\partial_x^{\alpha+1}Q_m\|+\|Q_m\|_\infty\|\partial_x^{\alpha+1}\Theta\|)\\
&\leq C\|\partial_x^{\alpha+1}Q_m\|^2+\frac{\Gamma L}{4}\|\partial_x^{\alpha+2}Q_m\|^2.
\end{aligned}
\tag{3.3.8}
$$

We only deal with the high-order term $Q_m\mathrm{tr}(Q_m^2)$ in $\mathcal{K}(Q_m)$, the rest of terms are trivial, using the Hölder inequality and Lemma 3.1.1, to get

$$
\begin{aligned}
&|\Phi_R^{\mathbf{u},Q_m}(\partial_x^{\alpha+1}(Q_m\mathrm{tr}(Q_m^2)),\partial_x^{\alpha+1}Q_m)|\\
&\leq\Phi_R^{\mathbf{u},Q_m}\|\partial_x^{\alpha+1}Q_m\|\|\partial_x^{\alpha+1}(Q_m\mathrm{tr}(Q_m^2))\|\\
&\leq\Phi_R^{\mathbf{u},Q_m}\|\partial_x^{\alpha+1}Q_m\|(\|\partial_x^{\alpha+1}Q_m\|\|Q_m\|_\infty^2+\|\mathrm{tr}(Q_m^2)\|_\infty\|\partial_x^{\alpha+1}Q_m\|)\\
&\leq C\|\partial_x^{\alpha+1}Q_m\|^2.
\end{aligned}
\tag{3.3.9}
$$

Taking into account of (3.3.7)-(3.3.9), taking sum of $|\alpha|\leq s$ and using the Gronwall lemma, we have

$$
\sup_{t\in[0,T]}\|Q_m\|_{s+1,2}^2+\int_0^T\Gamma L\|Q_m\|_{s+2,2}^2dt\leq C.
\tag{3.3.10}
$$

Then, using the estimate (3.3.10), it is also easy to show that

$$
\|Q_m\|_{W^{1,2}(0,T;L^2(\mathbb{T},S_0^3))}\leq C,
\tag{3.3.11}
$$

where the constant $C$ is independent of $m$.

Step 3. Using the a priori estimates (3.3.10), (3.3.11) and the Aubin-Lions lemma A.0.5, we could show the compactness of the sequence of approximate solutions $Q_m$, actually the proof is easier than the argument of Lemma 3.3.7. Then, we could pass $m\to\infty$ to identify the limit, the proof is also easier than the argument in Subsection 4.4, here we omit it. This completes the proof of existence.

Uniqueness: The proof of uniqueness is similar to the following continuity argument.

Next, we focus on showing the continuity of the mapping $\mathbf{u}\to Q[\mathbf{u}]$. Taking $\{\mathbf{u}_n\}_{n\geq1}$ is a bounded sequence in $C([0,T];X_N(\mathbb{T}))$ with

$$
\lim_{n\to\infty}\|\mathbf{u}_n-\mathbf{u}\|_{C([0,T];X_N(\mathbb{T}))}=0.
\tag{3.3.12}
$$

Denote $Q_n = Q[\mathbf{u}_n]$, $Q = Q[\mathbf{u}]$, and $\bar{Q}_n = Q_n - Q$, then the continuity result (3.3.6) would follow if we could prove

$$\|\bar{Q}_n\|^2_{C([0,T];W^{s+1,2}(\mathbb{T},S_0^3))} + \|\bar{Q}_n\|^2_{L^2(0,T;W^{s+2,2}(\mathbb{T},S_0^3))} \leq C \sup_{t\in[0,T]} \|\mathbf{u}_n - \mathbf{u}\|^2_{X_N}. \tag{3.3.13}$$

From (3.3.5), we can get that $\bar{Q}_n$ satisfies the following system

$$\begin{cases} \frac{d}{dt}\bar{Q}_n - \Gamma L\triangle\bar{Q}_n = \Phi_R^{\mathbf{u},Q}\big[(\mathbf{u}-\mathbf{u}_n)\cdot\nabla_x Q - \mathbf{u}_n\cdot\nabla_x\bar{Q}_n \\ \qquad\qquad +\Theta_n\bar{Q}_n - \bar{Q}_n\Theta_n + (\Theta_n-\Theta)Q - Q(\Theta_n-\Theta)\big] \\ \qquad\qquad + \left(\Phi_R^{\mathbf{u}_n,Q_n} - \Phi_R^{\mathbf{u},Q}\right)(\mathbf{u}_n\cdot\nabla_x Q_n - \Theta_n Q_n + Q_n\Theta_n) \\ \qquad\qquad +\Phi_R^{\mathbf{u},Q}(\mathcal{K}(Q_n)-\mathcal{K}(Q)) \\ \qquad\qquad + \left(\Phi_R^{\mathbf{u}_n,Q_n} - \Phi_R^{\mathbf{u},Q}\right)\mathcal{K}(Q_n), \quad \text{in } \mathbb{T}\times(0,T), \\ \bar{Q}_n(0) = 0. \end{cases} \tag{3.3.14}$$

Taking $\alpha$-order derivative on both sides of (3.3.14) for $|\alpha| \leq s$, multiplying by $-\triangle\partial_x^\alpha\bar{Q}_n$, then the trace and integrating over $\mathbb{T}$, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_x^{\alpha+1}\bar{Q}_n\|^2 + \Gamma L\|\triangle\partial_x^\alpha\bar{Q}_n\|^2 \\ =&\int_{\mathbb{T}} \Phi_R^{\mathbf{u},Q}\partial_x^\alpha\bigg(\big[(\mathbf{u}-\mathbf{u}_n)\cdot\nabla_x Q - \mathbf{u}_n\cdot\nabla_x\bar{Q}_n \\ &+\Theta_n\bar{Q}_n - \bar{Q}_n\Theta_n + (\Theta_n-\Theta)Q - Q(\Theta_n-\Theta)\big] \\ &+ \left(\Phi_R^{\mathbf{u}_n,Q_n} - \Phi_R^{\mathbf{u},Q}\right)(\mathbf{u}_n\cdot\nabla_x Q_n - \Theta_n Q_n + Q_n\Theta_n) \\ &+ \Phi_R^{\mathbf{u},Q}(\mathcal{K}(Q_n)-\mathcal{K}(Q)) \\ &+ \left(\Phi_R^{\mathbf{u}_n,Q_n} - \Phi_R^{\mathbf{u},Q}\right)\mathcal{K}(Q_n)\bigg) : (-\triangle\partial_x^\alpha\bar{Q}_n)dx \\ =&: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.3.15}$$

As $\{Q_n\}_{n\geq 1}$ and $Q$ are uniform bounded in

$$C([0,T];W^{s+1,2}(\mathbb{T},S_0^3)) \cap L^2(0,T;W^{s+2,2}(\mathbb{T},S_0^3)).$$

We can estimate $I_1$ by Lemma 3.1.1 and the Hölder inequality

$$|I_1| \leq \left\|\partial_x^\alpha\big[(\mathbf{u}-\mathbf{u}_n)\cdot\nabla_x Q - \mathbf{u}_n\cdot\nabla_x\bar{Q}_n + \Theta_n\bar{Q}_n - \bar{Q}_n\Theta_n + (\Theta_n-\Theta)Q - Q(\Theta_n-\Theta)\big]\right\|$$

$$
\begin{aligned}
&\times \|\triangle\partial_x^\alpha \bar{Q}_n\| \\
&\le C(\|\mathbf{u}-\mathbf{u}_n\|_{s,2}\|Q\|_{s+1,2}+\|\mathbf{u}_n\|_{s+1,2}\|\partial_x^{\alpha+1}\bar{Q}_n\|+\|\mathbf{u}-\mathbf{u}_n\|_{s+1,2}\|Q\|_{s,2})\|\triangle\partial_x^\alpha\bar{Q}_n\| \\
&\le \frac{\Gamma L}{4}\|\triangle\partial_x^\alpha\bar{Q}_n\|^2+C\|\partial_x^{\alpha+1}\bar{Q}_n\|^2+C\|\mathbf{u}-\mathbf{u}_n\|_{s+1,2}^2.
\end{aligned}
$$

For $I_2$, by Lemma 3.1.1 and the Hölder inequality again, we have

$$
\begin{aligned}
|I_2| &\le C(\|\mathbf{u}-\mathbf{u}_n\|_{2,\infty}+\|\bar{Q}_n\|_{3,\infty})\|\partial_x^\alpha(\mathbf{u}_n\cdot\nabla_x Q_n-\Theta_n Q_n+Q_n\Theta_n)\|\|\triangle\partial_x^\alpha\bar{Q}_n\| \\
&\le \frac{\Gamma L}{4}\|\triangle\partial_x^\alpha\bar{Q}_n\|^2+C\|\partial_x^{\alpha+1}\bar{Q}_n\|^2+C\|\mathbf{u}-\mathbf{u}_n\|_{s+1,2}^2.
\end{aligned}
$$

Similarly, for terms $I_3, I_4$

$$
|I_3+I_4| \le \frac{\Gamma L}{4}\|\triangle\partial_x^\alpha\bar{Q}_n\|^2+C\|\partial_x^{\alpha+1}\bar{Q}_n\|^2+C\|\mathbf{u}-\mathbf{u}_n\|_{s+1,2}^2.
$$

Summing all the estimates up and taking sum for $|\alpha|\le s$, we get

$$
\frac{d}{dt}\|\bar{Q}_n\|_{s+1,2}^2+\frac{\Gamma L}{2}\|\bar{Q}_n\|_{s+2,2}^2 \le C\|\bar{Q}_n\|_{s+1,2}^2+C\|\mathbf{u}_n-\mathbf{u}\|_{X_N}^2.
$$

Applying the Gronwall lemma, then

$$
\|\bar{Q}_n(t)\|_{s+1,2}^2+\frac{\Gamma L}{2}\int_0^t\|\bar{Q}_n\|_{s+2,2}^2 d\xi \le Ce^{CT}\sup_{t\in[0,T]}\|\mathbf{u}_n-\mathbf{u}\|_{X_N}^2,
$$

for any $t\in[0,T]$. So let $n\to\infty$, since $\sup_{t\in[0,T]}\|\mathbf{u}_n-\mathbf{u}\|_{X_N}^2\to 0$, then (3.3.13) follows.

Finally, we prove $Q\in S_0^3$, namely $\mathrm{tr}(Q)=0$ and $Q=Q^{\mathrm{T}}$ a.e in $\mathbb{T}\times[0,T]$. If we apply the transpose to the equation (3.3.5)(1), using the fact that $\|Q\|_{3,\infty}=\|Q^{\mathrm{T}}\|_{3,\infty}$, we have

$$
(Q^{\mathrm{T}})_t+\Phi_R^{\mathbf{u},Q^{\mathrm{T}}}(\mathbf{u}\cdot\nabla_x Q^{\mathrm{T}}-\Theta Q^{\mathrm{T}}+Q^{\mathrm{T}}\Theta)=\Gamma L\triangle Q^{\mathrm{T}}+\Phi_R^{\mathbf{u},Q^{\mathrm{T}}}\mathcal{K}(Q^{\mathrm{T}}).
$$

So $Q^{\mathrm{T}}$ also satisfies the equation. The uniqueness result leads to $Q=Q^{\mathrm{T}}$. The proof of $\mathrm{tr}(Q)=0$, we refer the reader to [22]. $\square$

For $r_1 = r[\mathbf{v}_1]$, $r_2 = r[\mathbf{v}_2]$, $r_1 - r_2$ satisfies

$$\begin{aligned} d(r_1 - r_2) + \mathbf{v}_1 \cdot \nabla_x (r_1 - r_2) dt - \frac{\gamma - 1}{2} \mathrm{div}_x \mathbf{v}_1 \cdot (r_1 - r_2) dt \\ = -\nabla_x r_2 \cdot (\mathbf{v}_1 - \mathbf{v}_2) dt - \frac{\gamma - 1}{2} r_2 \cdot \mathrm{div}_x (\mathbf{v}_1 - \mathbf{v}_2) dt, \end{aligned}$$

where $\mathbf{v}_1 = \Phi_R^{\mathbf{u}_1, Q[\mathbf{u}_1]} \mathbf{u}_1$, $\mathbf{v}_2 = \Phi_R^{\mathbf{u}_2, Q[\mathbf{u}_2]} \mathbf{u}_2$. Using the same argument as Breit-Feireisl-Hofmanová [11, Section 3.1] and the continuity of $Q[\mathbf{u}]$, see (3.3.6), we are able to obtain the continuity of $r[\mathbf{u}]$ with respect to $\mathbf{u} \in C([0,T]; X_N(\mathbb{T}))$, that is,

$$\sup_{0 \le t \le T} \| r[\mathbf{u}_1] - r[\mathbf{u}_2] \|^2 \le T C(N, R, T) \sup_{0 \le t \le T} \| \mathbf{u}_1 - \mathbf{u}_2 \|_{X_N}^2. \tag{3.3.16}$$

We proceed to construct the approximate solution to the momentum equation. Let $\{\psi_m\}_{m=1}^\infty$ be an orthonormal basis of the space $H^1(\mathbb{T}, \mathbb{R}^3)$. Set the space

$$X_n = \mathrm{span}\{\psi_1, \ldots, \psi_n\}. \tag{3.3.17}$$

Let $P_n$ be an orthogonal projection from $H^1(\mathbb{T}, \mathbb{R}^3)$ into $X_n$.

We now find the approximate velocity field $\mathbf{u}_n \in L^2(\Omega, C([0,T]; X_n))$ to the following momentum equation

$$\begin{cases} d\langle \mathbf{u}_n, \psi_i \rangle + \Phi_R^{\mathbf{u}_n, Q_n} \langle \mathbf{u}_n \nabla_x \mathbf{u}_n + r[\mathbf{u}_n] \nabla_x r[\mathbf{u}_n], \psi_i \rangle dt \\ = \Phi_R^{\mathbf{u}_n, Q_n} \langle D(r[\mathbf{u}_n]) (\mathcal{L} \mathbf{u}_n - \mathrm{div}_x (L \nabla_x Q[\mathbf{u}_n] \odot \nabla_x Q[\mathbf{u}_n] - \mathcal{F}(Q[\mathbf{u}_n]) \mathrm{I}_3) \\ \qquad\qquad + L \mathrm{div}_x (Q[\mathbf{u}_n] \triangle Q[\mathbf{u}_n] - \triangle Q[\mathbf{u}_n] Q[\mathbf{u}_n])), \psi_i \rangle dt \\ \quad + \Phi_R^{\mathbf{u}_n, Q_n} \langle \mathbb{F}(r[\mathbf{u}_n], \mathbf{u}_n), \psi_i \rangle dW, i = 1, \ldots, n \\ \mathbf{u}_n(0) = P_n \mathbf{u}_0. \end{cases}$$

To handle the nonlinear $Q$-tensor terms and the noise term above, with the spirit of [43], we define another $C^\infty$-smooth cut-off function

$$\Psi_K(z) = \begin{cases} 1, \ |z| \le K, \\ 0, \ |z| > 2K. \end{cases}$$

For any $\mathbf{v} = \sum_{i=1}^n \mathbf{v}_i \psi_i \in X_n$, define $\mathbf{v}^K = \sum_{i=1}^n \Psi_K(\mathbf{v}_i) \mathbf{v}_i \psi_i$, then we have $\|\mathbf{v}^K\|_{C([0,T]; X_n)} \le 2K$.

Define the mapping

$$
\begin{aligned}
\langle \mathcal{T}[\mathbf{u}]; \psi_i \rangle = \langle \mathbf{u}_0^K; \psi_i \rangle - \int_0^{\cdot} \Phi_R^{\mathbf{u}^K, Q[\mathbf{u}^K]} \langle \mathbf{u}^K \nabla_x \mathbf{u}^K + r[\mathbf{u}^K] \nabla_x r[\mathbf{u}^K]; \psi_i \rangle dt \\
+ \int_0^{\cdot} \Phi_R^{\mathbf{u}^K, Q[\mathbf{u}^K]} \langle D(r[\mathbf{u}^K])(\mathcal{L}\mathbf{u}^K - \mathrm{div}_x(L \nabla_x Q[\mathbf{u}^K] \odot \nabla_x Q[\mathbf{u}^K] - \mathcal{F}(Q[\mathbf{u}^K])\mathrm{I}_3) \\
+ L \mathrm{div}_x(Q[\mathbf{u}^K] \triangle Q[\mathbf{u}^K] - \triangle Q[\mathbf{u}^K] Q[\mathbf{u}^K])); \psi_i \rangle dt \\
+ \int_0^{\cdot} \Phi_R^{\mathbf{u}^K, Q[\mathbf{u}^K]} \langle \mathbb{F}(r[\mathbf{u}^K], \mathbf{u}^K); \psi_i \rangle dW, i = 1, \dots, n.
\end{aligned}
\tag{3.3.18}
$$

Next, we show that the mapping $\mathcal{T}$ is a contraction on $\mathcal{B} = L^2(\Omega; C([0, T^*]; X_n))$ with fixed $K, n$ for $T^*$ small enough. Denote the right side of (3.3.18) $\mathcal{T}_{det}$ as the deterministic part, and $\mathcal{T}_{sto}$ as the component $\int_0^{\cdot} \Phi_R^{\mathbf{u}, Q} \langle \mathbb{F}(r[\mathbf{u}], \mathbf{u}); \psi_i \rangle dW$ respectively.

Combining the assumption on initial data $Q_0$ and the definition of $\mathbf{u}^K$, we have after a easily calculation

$$
\|Q[\mathbf{u}^K]\|_{C([0,T]; W^{1,2})} \le C(K, R), \ \mathbb{P} \text{ a.s.} \tag{3.3.19}
$$

Together estimates (3.3.1), (3.3.4), (3.3.19), the continuity results (3.3.16), (3.3.6) with the equivalence of norms on finite dimensional space $X_n$, we can show that the mapping $\mathcal{T}_{det}$ satisfies the estimate

$$
\|\mathcal{T}_{det}(\mathbf{u}_1) - \mathcal{T}_{det}(\mathbf{u}_2)\|_{\mathcal{B}}^2 \le T^* C(n, R, T, K) \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{B}}^2, \tag{3.3.20}
$$

see also [22, 86] and using the Burkholder-Davis-Gundy inequality (3.1.5), the mapping $\mathcal{T}_{sto}$ satisfies the estimate

$$
\begin{aligned}
&\|\mathcal{T}_{sto}(\mathbf{u}_1) - \mathcal{T}_{sto}(\mathbf{u}_2)\|_{\mathcal{B}}^2 \\
&= \mathbb{E} \sup_{t \in [0, T^*]} \left\| \int_0^t \Phi_R^{\mathbf{u}_1^K, Q[\mathbf{u}_1^K]} \mathbb{F}(r[\mathbf{u}_1^K], \mathbf{u}_1^K) - \Phi_R^{\mathbf{u}_2^K, Q[\mathbf{u}_2^K]} \mathbb{F}(r[\mathbf{u}_2^K], \mathbf{u}_2^K) dW \right\|_{X_n}^2 \\
&\le C\mathbb{E} \int_0^{T^*} \left\| \Phi_R^{\mathbf{u}_1^K, Q[\mathbf{u}_1^K]} \mathbb{F}(r[\mathbf{u}_1^K], \mathbf{u}_1^K) - \Phi_R^{\mathbf{u}_2^K, Q[\mathbf{u}_2^K]} \mathbb{F}(r[\mathbf{u}_2^K], \mathbf{u}_2^K) \right\|_{L_2(\mathfrak{U}; X_n)}^2 dt \\
&\le C\mathbb{E} \int_0^{T^*} \left| \Phi_R^{\mathbf{u}_1^K, Q[\mathbf{u}_1^K]} - \Phi_R^{\mathbf{u}_2^K, Q[\mathbf{u}_2^K]} \right|^2 \left\| \mathbb{F}(r[\mathbf{u}_1^K], \mathbf{u}_1^K) \right\|_{L_2(\mathfrak{U}; X_n)}^2 dt \\
&\quad + C\mathbb{E} \int_0^{T^*} (\Phi_R^{\mathbf{u}_2^K, Q[\mathbf{u}_2^K]})^2 \left\| \mathbb{F}(r[\mathbf{u}_1^K], \mathbf{u}_1^K) - \mathbb{F}(r[\mathbf{u}_2^K], \mathbf{u}_2^K) \right\|_{L_2(\mathfrak{U}; X_n)}^2 dt
\end{aligned}
$$

$$=: J_1 + J_2. \tag{3.3.21}$$

Using the equivalence of norms on finite-dimensional space, assumption (3.1.6) and the continuity result (3.3.16), the bound (3.3.1), we have

$$\begin{aligned}
J_2 &\leq C\mathbb{E}\int_0^{T^*} (\Phi_R^{\mathbf{u}_2^K, Q[\mathbf{u}_2^K]})^2 \left\|\mathbb{F}(r[\mathbf{u}_1^K], \mathbf{u}_1^K) - \mathbb{F}(r[\mathbf{u}_2^K], \mathbf{u}_2^K)\right\|_{L_2(\mathfrak{U};L^2)}^2 dt \\
&\leq C\mathbb{E}\int_0^{T^*} (\Phi_R^{\mathbf{u}_2^K, Q[\mathbf{u}_2^K]})^2 (\|r[\mathbf{u}_1^K], r[\mathbf{u}_2^K]\|_{1,\infty}^2 + \|\mathbf{u}_1^K, \mathbf{u}_2^K\|_{2,\infty}^2) \|r[\mathbf{u}_1^K] - r[\mathbf{u}_2^K], \mathbf{u}_1^K - \mathbf{u}_2^K\|^2 dt \\
&\leq T^* C(n, R, K, T)\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{B}}^2.
\end{aligned} \tag{3.3.22}$$

By the mean value theorem, the equivalence of norms on finite-dimensional space, assumption (3.1.5) and continuity result (3.3.6), we also have

$$J_1 \leq T^* C(n, K, T)\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{B}}^2. \tag{3.3.23}$$

Combining (3.3.19)-(3.3.23), we infer that there exists approximate solution sequence belonging to $L^2(\Omega; C([0, T_*]; X_n))$ to momentum equation for small time $T^*$ by the Banach fixed point theorem. Here we first assume that the estimates (3.3.51),(3.3.52) hold. Then, we could extend the existence time $T^*$ to any $T > 0$ for any fixed $n, K$.

Next, we pass $K \to \infty$ to construct the approximate solution $(r_n, \mathbf{u}_n, Q_n)$ for any fixed $n$. Define the stopping time $\tau_K$

$$\tau_K = \inf\left\{t \in [0, T];\ \sup_{\xi\in[0,t]} \left\|\mathbf{u}_n^K(\xi)\right\|_{X_n} \geq K\right\},$$

with the convention $\inf\emptyset = T$. Note that $\tau_{K_1} \geq \tau_{K_2}$ if $K_1 \geq K_2$, due to the uniqueness, we have $(r_n^{K_1}, \mathbf{u}_n^{K_1}, Q_n^{K_1}) = (r_n^{K_2}, \mathbf{u}_n^{K_2}, Q_n^{K_2})$ on the interval $[0, \tau_{K_2})$. Therefore, we can define $(r_n, \mathbf{u}_n, Q_n) = (r_n^K, \mathbf{u}_n^K, Q_n^K)$ on interval $[0, \tau_K)$. Note that

$$\begin{aligned}
\mathbb{P}\left\{\sup_{K\in\mathbb{N}^+} \tau_K = T\right\} &= 1 - \mathbb{P}\left\{\left(\sup_{K\in\mathbb{N}^+} \tau_K = T\right)^c\right\} = 1 - \mathbb{P}\left\{\sup_{K\in\mathbb{N}^+} \tau_K < T\right\} \\
&\geq 1 - \mathbb{P}\left\{\tau_K < T\right\} = 1 - \mathbb{P}\left\{\sup_{t\in[0,T]} \|\mathbf{u}_n^K\|_{X_n} \geq K\right\}.
\end{aligned}$$

From the Chebyshev inequality, estimate (3.3.51) and the equivalence of norms on finite dimensional space, we know

$$\lim_{K\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,T]} \|\mathbf{u}_n^K\|_{X_n} \geq K \right\} = 0,$$

which leads to

$$\mathbb{P}\left\{ \sup_{K\in\mathbb{N}^+} \tau_K = T \right\} = 1.$$

As a result, we could extend the existence time interval $[0, \tau_K)$ to $[0, T]$ for any $T > 0$, obtaining the global existence of approximate solution sequence $(r_n, \mathbf{u}_n, Q_n)$.

### 3.3.2 Uniform estimates

In this subsection, we derive the a priori estimates that hold uniformly for $n \geq 1$, which allow us to extend the existence interval to any $T > 0$ and provide a preliminary for our stochastic compactness argument.

Taking $\alpha$-order derivative on both sides of system (3.2.3) in the x-variable for $|\alpha| \leq s$, then taking inner product with $\partial_x^\alpha r_n$ on both sides of equation (3.2.3)(1) and applying the Itô formula to function $\|\partial_x^\alpha \mathbf{u}_n\|^2$, we obtain

$$\begin{aligned}
&\frac{1}{2} d\|\partial_x^\alpha r_n\|^2 + \Phi_R^{\mathbf{u}_n,Q_n}\left(\mathbf{u}_n \cdot \nabla_x \partial_x^\alpha r_n + \frac{\gamma-1}{2} r_n \mathrm{div}_x \partial_x^\alpha \mathbf{u}_n, \partial_x^\alpha r_n\right) dt\\
=&\Phi_R^{\mathbf{u}_n,Q_n}\left(\mathbf{u}_n \cdot \partial_x^\alpha \nabla_x r_n - \partial_x^\alpha(\mathbf{u}_n \cdot \nabla_x r_n), \partial_x^\alpha r_n\right) dt\\
&+ \frac{\gamma-1}{2}\Phi_R^{\mathbf{u}_n,Q_n}\left(r_n \partial_x^\alpha \mathrm{div}_x \mathbf{u}_n - \partial_x^\alpha(r_n \mathrm{div}_x \mathbf{u}_n), \partial_x^\alpha r_n\right) dt\\
=:& \left(T_1^n dt + T_2^n dt, \partial_x^\alpha r_n\right),
\end{aligned} \tag{3.3.24}$$

and

$$\begin{aligned}
&\frac{1}{2} d\|\partial_x^\alpha \mathbf{u}_n\|^2 + \Phi_R^{\mathbf{u}_n,Q_n}(\mathbf{u}_n \nabla_x \partial_x^\alpha \mathbf{u}_n + r_n \nabla_x \partial_x^\alpha r_n, \partial_x^\alpha \mathbf{u}_n) dt\\
&- \Phi_R^{\mathbf{u}_n,Q_n}(D(r_n)\mathcal{L}(\partial_x^\alpha \mathbf{u}_n), \partial_x^\alpha \mathbf{u}_n) dt\\
&+ \Phi_R^{\mathbf{u}_n,Q_n}\left(D(r_n)\mathrm{div}_x \partial_x^\alpha \left(L\nabla_x Q_n \odot \nabla_x Q_n - \frac{L}{2}|\nabla_x Q_n|^2 \mathrm{I}_3\right), \partial_x^\alpha \mathbf{u}_n\right) dt
\end{aligned}$$

$$
\begin{aligned}
&-\Phi_R^{\mathbf{u}_n,Q_n}\left(D(r_n)\mathrm{div}_x\partial_x^\alpha\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2)-\frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3)+\frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right),\partial_x^\alpha\mathbf{u}_n\right)dt\\
&-\Phi_R^{\mathbf{u}_n,Q_n}(D(r_n)\mathrm{div}_x\left(L\partial_x^\alpha(Q_n\triangle Q_n-\triangle Q_nQ_n)\right),\partial_x^\alpha\mathbf{u}_n)dt\\
=&\Phi_R^{\mathbf{u}_n,Q_n}\left(\mathbf{u}_n\partial_x^\alpha\nabla_x\mathbf{u}_n-\partial_x^\alpha(\mathbf{u}_n\nabla_x\mathbf{u}_n),\partial_x^\alpha\mathbf{u}_n\right)dt\\
&+\Phi_R^{\mathbf{u}_n,Q_n}(r_n\partial_x^\alpha\nabla_x r_n-\partial_x^\alpha(r_n\nabla_x r_n),\partial_x^\alpha\mathbf{u}_n)dt\\
&-\Phi_R^{\mathbf{u}_n,Q_n}(D(r_n)\partial_x^\alpha\mathcal{L}\mathbf{u}_n-\partial_x^\alpha(D(r_n)\mathcal{L}\mathbf{u}_n),\partial_x^\alpha\mathbf{u}_n)dt\\
&+\Phi_R^{\mathbf{u}_n,Q_n}\Bigg(D(r_n)\partial_x^\alpha\mathrm{div}_x\left(L\nabla_xQ_n\odot\nabla_xQ_n-\frac{L}{2}|\nabla_xQ_n|^2\mathrm{I}_3\right)\\
&\qquad-\partial_x^\alpha\left(D(r_n)\mathrm{div}_x\left(L\nabla_xQ_n\odot\nabla_xQ_n-\frac{L}{2}|\nabla_xQ_n|^2\mathrm{I}_3\right)\right),\partial_x^\alpha\mathbf{u}_n\Bigg)dt\\
&-\Phi_R^{\mathbf{u}_n,Q_n}\Bigg(D(r_n)\partial_x^\alpha\mathrm{div}_x\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2)-\frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3)+\frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right)\\
&\qquad-\partial_x^\alpha\left(D(r_n)\mathrm{div}_x\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2)-\frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3)+\frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right)\right),\partial_x^\alpha\mathbf{u}_n\Bigg)dt\\
&-\Phi_R^{\mathbf{u}_n,Q_n}\Bigg(D(r_n)\partial_x^\alpha\mathrm{div}_xL(Q_n\triangle Q_n-\triangle Q_nQ_n)\\
&\qquad-\partial_x^\alpha(D(r_n)\mathrm{div}_xL(Q_n\triangle Q_n-\triangle Q_nQ_n)),\partial_x^\alpha\mathbf{u}_n\Bigg)dt\\
&+\Phi_R^{\mathbf{u}_n,Q_n}(\partial_x^\alpha\mathbb{F}(r_n,\mathbf{u}_n),\partial_x^\alpha\mathbf{u}_n)dW+\frac{1}{2}(\Phi_R^{\mathbf{u}_n,Q_n})^2\sum_{k\geq1}\int_{\mathbb{T}}|\partial_x^\alpha\mathbb{F}(r_n,\mathbf{u}_n)e_k|^2dxdt\\
=:&\sum_{i=3}^{8}(T_i^n,\partial_x^\alpha\mathbf{u}_n)dt+\Phi_R^{\mathbf{u}_n,Q_n}(\partial_x^\alpha\mathbb{F}(r_n,\mathbf{u}_n),\partial_x^\alpha\mathbf{u}_n)dW\\
&+\frac{1}{2}(\Phi_R^{\mathbf{u}_n,Q_n})^2\sum_{k\geq1}\int_{\mathbb{T}}|\partial_x^\alpha\mathbb{F}(r_n,\mathbf{u}_n)e_k|^2dxdt.
\end{aligned}\tag{3.3.25}
$$

To handle the highest order term $\mathrm{div}_x(Q_n\triangle Q_n-\triangle Q_nQ_n)$, we multiply $-D(r_n)\triangle\partial_x^\alpha Q_n$ in equation 3.2.3(3) instead of $-\triangle\partial_x^\alpha Q_n$, then take the trace and integrate over $\mathbb{T}$, to get

$$
\begin{aligned}
&\frac{1}{2}d\|\sqrt{D(r_n)}\nabla_x\partial_x^\alpha Q_n\|^2-\frac{1}{2}\int_{\mathbb{T}}D(r_n)_t|\nabla_x\partial_x^\alpha Q_n|^2dxdt\\
&-\int_{\mathbb{T}}\nabla_xD(r_n)(\partial_x^\alpha Q_n)_t:\nabla_x\partial_x^\alpha Q_ndxdt+\Gamma L\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2dt\\
&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\mathbf{u}_n\cdot\nabla_x\partial_x^\alpha Q_n):\triangle\partial_x^\alpha Q_ndxdt\\
&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)((\partial_x^\alpha\Theta_n)Q_n-Q_n(\partial_x^\alpha\Theta_n)):\triangle\partial_x^\alpha Q_ndxdt\\
&-\Gamma\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)\partial_x^\alpha\left(aQ_n-b\left(Q_n^2-\frac{\mathrm{I}_3}{3}\mathrm{tr}(Q_n^2)+cQ_n\mathrm{tr}(Q_n^2)\right)\right):\triangle\partial_x^\alpha Q_ndxdt
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\mathbb{T}} D(r_n)\Phi_R^{\mathbf{u}_n,Q_n}(\mathbf{u}_n\cdot\partial_x^\alpha\nabla_x Q_n - \partial_x^\alpha(\mathbf{u}_n\cdot\nabla_x Q_n)) : \triangle\partial_x^\alpha Q_n dxdt\\
&\quad -\int_{\mathbb{T}} D(r_n)\Phi_R^{\mathbf{u}_n,Q_n}((\partial_x^\alpha\Theta_n)Q_n - Q_n(\partial_x^\alpha\Theta_n) - \partial_x^\alpha(\Theta_n Q_n - Q_n\Theta_n)) : \triangle\partial_x^\alpha Q_n dxdt\\
&=: \int_{\mathbb{T}} (T_9+T_{10}) : \triangle\partial_x^\alpha Q_n dxdt. \qquad (3.3.26)
\end{aligned}
$$

We next estimate all the right hand side terms. Using Lemma 3.1.1 and the Hölder inequality,

$$
\begin{aligned}
|\langle T_1^n,\partial_x^\alpha r_n\rangle| &\le C\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x\mathbf{u}_n\|_\infty\|\partial_x^\alpha r_n\| + \|\nabla_x r_n\|_\infty\|\partial_x^\alpha\mathbf{u}_n\|)\|\partial_x^\alpha r_n\|\\
&\le C(R)(\|\partial_x^\alpha r_n\|^2 + \|\partial_x^\alpha\mathbf{u}_n\|^2),\\
|\langle T_2^n,\partial_x^\alpha r_n\rangle| &\le C\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x r_n\|_\infty\|\partial_x^\alpha\mathbf{u}_n\| + \|\mathrm{div}_x\mathbf{u}_n\|_\infty\|\partial_x^\alpha r_n\|)\|\partial_x^\alpha r_n\|\\
&\le C(R)(\|\partial_x^\alpha\mathbf{u}_n\|^2 + \|\partial_x^\alpha r_n\|^2).
\end{aligned} \qquad (3.3.27)
$$

Also using Lemma 3.1.1, estimates (3.3.2), (3.3.3) and the Hölder inequality, we have the following estimates for $T_3^n$ to $T_{10}^n$

$$
|(T_3^n,\partial_x^\alpha\mathbf{u}_n)| \le C\Phi_R^{\mathbf{u}_n,Q_n}\|\nabla_x\mathbf{u}_n\|_\infty\|\partial_x^\alpha\mathbf{u}_n\|^2 \le C(R)\|\partial_x^\alpha\mathbf{u}_n\|^2, \qquad (3.3.28)
$$

$$
|(T_4^n,\partial_x^\alpha\mathbf{u}_n)| \le C\Phi_R^{\mathbf{u}_n,Q_n}\|\nabla_x r_n\|_\infty\|\partial_x^\alpha r_n\|\|\partial_x^\alpha\mathbf{u}_n\| \le C(R)(\|\partial_x^\alpha r_n\|^2 + \|\partial_x^\alpha\mathbf{u}_n\|^2), \qquad (3.3.29)
$$

$$
\begin{aligned}
|(T_5^n,\partial_x^\alpha\mathbf{u}_n)| &\le C\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x D(r_n)\|_\infty\|\partial_x^{\alpha-1}\mathcal{L}\mathbf{u}_n\| + \|\mathcal{L}\mathbf{u}_n\|_\infty\|\partial_x^\alpha D(r_n)\|)\|\partial_x^\alpha\mathbf{u}_n\|\\
&\le C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|\partial_x^{\alpha+1}\mathbf{u}_n\| + \|\partial_x^\alpha r_n\|)\|\partial_x^\alpha\mathbf{u}_n\|\\
&\le \frac{\nu}{8}\Phi_R^{\mathbf{u}_n,Q_n}\|\sqrt{D(r_n)}\partial_x^{\alpha+1}\mathbf{u}_n\|^2 + C(R)(\|\partial_x^\alpha r_n\|^2 + \|\partial_x^\alpha\mathbf{u}_n\|^2), \qquad (3.3.30)
\end{aligned}
$$

$$
\begin{aligned}
|(T_6^n,\partial_x^\alpha\mathbf{u}_n)| &\le C\Phi_R^{\mathbf{u}_n,Q_n}\left(\|\nabla_x D(r_n)\|_\infty\left\|\partial_x^\alpha\left(L\nabla_x Q_n\odot\nabla_x Q_n - \frac{L}{2}|\nabla_x Q_n|^2\mathrm{I}_3\right)\right\|\right)\|\partial_x^\alpha\mathbf{u}_n\|\\
&\quad + \left\|\mathrm{div}_x\left(L\nabla_x Q_n\odot\nabla_x Q_n - \frac{L}{2}|\nabla_x Q_n|^2\mathrm{I}_3\right)\right\|_\infty\|\partial_x^\alpha r_n\|\|\partial_x^\alpha\mathbf{u}_n\|\\
&\le C\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x D(r_n)\|_\infty\|\nabla_x Q_n\|_\infty\|\partial_x^{\alpha+1}Q_n\|\\
&\qquad + \|\nabla_x Q_n\|_\infty\|Q_n\|_{2,\infty}\|\partial_x^\alpha r_n\|)\|\partial_x^\alpha\mathbf{u}_n\|\\
&\le C(R)(\|\partial_x^{\alpha+1}Q_n\|^2 + \|\partial_x^\alpha r_n\|^2 + \|\partial_x^\alpha\mathbf{u}_n\|^2), \qquad (3.3.31)
\end{aligned}
$$

$$
\begin{aligned}
|(T_7^n,\partial_x^\alpha\mathbf{u}_n)| &\le C\Phi_R^{\mathbf{u}_n,Q_n}\Bigg(\|\nabla_x D(r_n)\|_\infty\left\|\partial_x^{\alpha-1}\mathrm{div}_x\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2) - \frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3) + \frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right)\right\|\\
&\quad + \left\|\mathrm{div}_x\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2) - \frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3) + \frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right)\right\|_\infty\|\partial_x^\alpha r_n\|\Bigg)\|\partial_x^\alpha\mathbf{u}_n\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\Phi_R^{\mathbf{u}_n,Q_n}(\|Q_n\|_\infty^3+\|Q_n\|_{1,\infty})(\|\partial_x^\alpha Q_n\|^2+\|\partial_x^\alpha r_n\|^2+\|\partial_x^\alpha \mathbf{u}_n\|^2)\\
&\leq C(R)(\|\partial_x^\alpha Q_n\|^2+\|\partial_x^\alpha r_n\|^2+\|\partial_x^\alpha \mathbf{u}_n\|^2), \qquad (3.3.32)\\
|(T_8^n,\partial_x^\alpha \mathbf{u}_n)| &\leq C\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x D(r_n)\|_\infty\|\partial_x^\alpha(Q_n\triangle Q_n-\triangle Q_nQ_n)\|\\
&\qquad +\|\mathrm{div}_x(Q_n\triangle Q_n-\triangle Q_nQ_n)\|_\infty\|\partial_x^\alpha D(r_n)\|)\|\partial_x^\alpha \mathbf{u}_n\|\\
&\leq C\Phi_R^{\mathbf{u}_n,Q_n}(\|Q_n\|_\infty\|\triangle\partial_x^\alpha Q_n\|+\|Q_n\|_{2,\infty}\|\partial_x^\alpha Q_n\|)\|\partial_x^\alpha \mathbf{u}_n\|\\
&\quad +C\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x Q_n\|_\infty\|Q_n\|_{2,\infty}+\|Q_n\|_\infty\|Q_n\|_{3,\infty})\|\partial_x^\alpha r_n\|\|\partial_x^\alpha \mathbf{u}_n\|\\
&\leq C(R)(\|\partial_x^\alpha Q_n\|^2+\|\partial_x^\alpha r_n\|^2+\|\partial_x^\alpha \mathbf{u}_n\|^2)+\frac{\Gamma L}{8}\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2, \qquad (3.3.33)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{\mathbb{T}}(T_9+T_{10}):\triangle\partial_x^\alpha Q_n dx\right|\\
&\leq C(R)\left\|\Phi_R^{\mathbf{u}_n,Q_n}(\mathbf{u}_n\cdot\partial_x^\alpha\nabla_x Q_n-\partial_x^\alpha(\mathbf{u}_n\cdot\nabla_x Q_n))\right\|\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|\\
&\quad +C(R)\left\|\Phi_R^{\mathbf{u}_n,Q_n}((\partial_x^\alpha\Theta_n)Q_n-Q_n(\partial_x^\alpha\Theta_n)-\partial_x^\alpha(\Theta_nQ_n-Q_n\Theta_n))\right\|\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|\\
&\leq C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x \mathbf{u}_n\|_\infty\|\partial_x^\alpha Q_n\|+\|\nabla_x Q_n\|_\infty\|\partial_x^\alpha \mathbf{u}_n\|)\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|\\
&\quad +C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x Q_n\|_\infty\|\partial_x^\alpha \mathbf{u}_n\|+\|\nabla_x \mathbf{u}_n\|_\infty\|\partial_x^\alpha Q_n\|)\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|\\
&\leq C(R)(\|\partial_x^\alpha Q_n\|^2+\|\partial_x^\alpha \mathbf{u}_n\|^2)+\frac{\Gamma L}{8}\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2. \qquad (3.3.34)
\end{aligned}
$$

According to the assumption (3.1.5) on $\mathbb{G}$ and the Remark 3.1.6, we could have the estimate

$$
\begin{aligned}
&\sum_{k\geq 1}\int_0^t(\Phi_R^{\mathbf{u}_n,Q_n})^2\int_{\mathbb{T}}|\partial_x^\alpha\mathbb{F}(r_n,\mathbf{u}_n)e_k|^2dxd\xi\\
&\leq C\int_0^t(\Phi_R^{\mathbf{u}_n,Q_n})^2\int_{\mathbb{T}}\|r_n,\nabla_x\mathbf{u}_n\|_{1,\infty}^2(\|\partial_x^\alpha r_n\|^2+\|\partial_x^\alpha \mathbf{u}_n\|^2)dxd\xi\\
&\leq C(R)\int_0^t\int_{\mathbb{T}}(\|\partial_x^\alpha r_n\|^2+\|\partial_x^\alpha \mathbf{u}_n\|^2)dxd\xi. \qquad (3.3.35)
\end{aligned}
$$

Next, we proceed to estimate the terms on the left hand side of (3.3.24)-(3.3.26). Integration by parts, we get

$$
\begin{aligned}
&\left|\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\mathbf{u}_n\cdot\nabla_x\partial_x^\alpha r_n\partial_x^\alpha r_n dx\right|=\left|\frac{1}{2}\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\mathbf{u}_n\cdot\nabla_x(\partial_x^\alpha r_n)^2dx\right|\\
&=\left|\frac{1}{2}\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\mathrm{div}_x\mathbf{u}_n|\partial_x^\alpha r_n|^2dx\right|\leq C(R)\|\partial_x^\alpha r_n\|^2, \qquad (3.3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}(\mathbf{u}_n\cdot\nabla_x\partial_x^\alpha\mathbf{u}_n+r_n\cdot\nabla_x\partial_x^\alpha r_n)\cdot\partial_x^\alpha\mathbf{u}_n dx\\
=&-\frac{1}{2}\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}|\partial_x^\alpha\mathbf{u}_n|^2\mathrm{div}_x\mathbf{u}_n dx-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}r_n\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n\partial_x^\alpha r_n dx\\
&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\nabla_x r_n\cdot\partial_x^\alpha\mathbf{u}_n\partial_x^\alpha r_n dx\\
\leq& C(R)(\|\partial_x^\alpha\mathbf{u}_n\|^2+\|\partial_x^\alpha r_n\|^2)-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}r_n\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n\partial_x^\alpha r_n dx,
\end{aligned}
\tag{3.3.37}
$$

as well as we have by estimate (3.3.2)

$$
\begin{aligned}
&\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)\mathcal{L}(\partial_x^\alpha\mathbf{u}_n)\cdot\partial_x^\alpha\mathbf{u}_n dx\\
=&\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\upsilon\triangle\partial_x^\alpha\mathbf{u}_n+(\upsilon+\lambda)\nabla_x\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n)\cdot\partial_x^\alpha\mathbf{u}_n dx\\
=&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\upsilon|\nabla_x\partial_x^\alpha\mathbf{u}_n|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n|^2)dx\\
&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\nabla_x D(r_n)(\upsilon\nabla_x\partial_x^\alpha\mathbf{u}_n+(\upsilon+\lambda)\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n)\partial_x^\alpha\mathbf{u}_n dx\\
\leq&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\upsilon|\nabla_x\partial_x^\alpha\mathbf{u}_n|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n|^2)dx\\
&+\Phi_R^{\mathbf{u}_n,Q_n}\|\nabla_x D(r_n)\|_\infty(\upsilon\|\nabla_x\partial_x^\alpha\mathbf{u}_n\|+(\upsilon+\lambda)\|\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n\|)\|\partial_x^\alpha\mathbf{u}_n\|\\
\leq&-\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\upsilon|\nabla_x\partial_x^\alpha\mathbf{u}_n|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n|^2)dx+C(R)\|\partial_x^\alpha\mathbf{u}_n\|^2\\
&+\frac{1}{8}\Phi_R^{\mathbf{u}_n,Q_n}(\upsilon\|\sqrt{D(r_n)}\nabla_x\partial_x^\alpha\mathbf{u}_n\|^2+(\upsilon+\lambda)\|\sqrt{D(r_n)}\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n\|^2).
\end{aligned}
\tag{3.3.38}
$$

Also integration by parts, estimate (3.3.2), Lemmas 3.1.1,3.1.3 and the Hölder inequality give

$$
\begin{aligned}
&\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)\mathrm{div}_x(L\partial_x^\alpha(Q_n\triangle Q_n-\triangle Q_nQ_n))\cdot\partial_x^\alpha\mathbf{u}_n dx\\
=&-L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)\partial_x^\alpha(Q_n\triangle Q_n-\triangle Q_nQ_n):\partial_x^\alpha\nabla_x\mathbf{u}_n^{\mathrm{T}}dx\\
&-L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\nabla_x D(r_n)\partial_x^\alpha(Q_n\triangle Q_n-\triangle Q_nQ_n)\cdot\partial_x^\alpha\mathbf{u}_n dx\\
=&-L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(Q_n\triangle\partial_x^\alpha Q_n-\triangle\partial_x^\alpha Q_nQ_n):\partial_x^\alpha\nabla_x\mathbf{u}_n^{\mathrm{T}}dx\\
&-L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\nabla_x D(r_n)\partial_x^\alpha(Q_n\triangle Q_n-\triangle Q_nQ_n)\cdot\partial_x^\alpha\mathbf{u}_n dx
\end{aligned}
$$

$$
\begin{aligned}
&- L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)(\partial_x^\alpha(Q_n\triangle Q_n - \triangle Q_nQ_n) - (Q_n\triangle\partial_x^\alpha Q_n - \triangle\partial_x^\alpha Q_nQ_n)) : \partial_x^\alpha\nabla_x\mathbf{u}_n^{\mathrm{T}}dx \\
\leq &- L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)((\partial_x^\alpha\Theta_n)Q_n - Q_n(\partial_x^\alpha\Theta_n)) : \triangle\partial_x^\alpha Q_n dx \\
&+ C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|\partial_x^\alpha(Q_n\triangle Q_n - \triangle Q_nQ_n)\|\|\partial_x^\alpha\mathbf{u}_n\| \\
&\qquad + \|\partial_x^\alpha(Q_n\triangle Q_n - \triangle Q_nQ_n) - (Q_n\triangle\partial_x^\alpha Q_n - \triangle\partial_x^\alpha Q_nQ_n)\|\|\partial_x^{\alpha+1}\mathbf{u}_n\|) \\
\leq &- L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)((\partial_x^\alpha\Theta_n)Q_n - Q_n(\partial_x^\alpha\Theta_n)) : \triangle\partial_x^\alpha Q_n dx \\
&+ C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|Q_n\|_\infty\|\triangle\partial_x^\alpha Q_n\| + \|Q_n\|_{2,\infty}\|\partial_x^\alpha Q_n\|)\|\partial_x^\alpha\mathbf{u}_n\| \\
&+ C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|\nabla_x Q_n\|_\infty\|\partial_x^{\alpha+1}Q_n\| + \|Q_n\|_{2,\infty}\|\partial_x^\alpha Q_n\|)\|\partial_x^{\alpha+1}\mathbf{u}_n\| \\
\leq &- L\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)((\partial_x^\alpha\Theta_n)Q_n - Q_n(\partial_x^\alpha\Theta_n)) : \triangle\partial_x^\alpha Q_n dx \\
&+ C(R)(\|\triangle\partial_x^\alpha Q_n\| + \|\partial_x^\alpha Q_n\|)\|\partial_x^\alpha\mathbf{u}_n\| + C(R)(\|\partial_x^{\alpha+1}Q_n\| + \|\partial_x^\alpha Q_n\|)\|\partial_x^{\alpha+1}\mathbf{u}_n\|. \qquad (3.3.39)
\end{aligned}
$$

**Remark 3.3.5.** Actually, Lemma 3.1.3 requires that the symmetric matrices are $3\times 3$, here, the $\triangle\partial_x^\alpha Q, \partial_x^\alpha\nabla\mathbf{u}^{\mathrm{T}}$ can be seen as the vector with each component is a $3\times 3$ matrix, therefore, we could apply Lemma 3.1.3 to each component in above argument, adding them together then get the result.

Also, using the estimate (3.3.2), we have

$$
\begin{aligned}
&\left|\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)\mathrm{div}_x\partial_x^\alpha\left(L\nabla_x Q_n\odot\nabla_x Q_n - \frac{L}{2}|\nabla_x Q_n|^2\mathrm{I}_3\right)\cdot\partial_x^\alpha\mathbf{u}_n dx\right| \\
=&\left|\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\nabla_x D(r_n)\partial_x^\alpha\left(L\nabla_x Q_n\odot\nabla_x Q_n - \frac{L}{2}|\nabla_x Q_n|^2\mathrm{I}_3\right)\cdot\partial_x^\alpha\mathbf{u}_n dx\right. \\
&\left.+ \Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)\partial_x^\alpha\left(L\nabla_x Q_n\odot\nabla_x Q_n - \frac{L}{2}|\nabla_x Q_n|^2\mathrm{I}_3\right) : \partial_x^\alpha\nabla_x\mathbf{u}_n^{\mathrm{T}}dx\right| \\
\leq& C(R)\Phi_R^{\mathbf{u}_n,Q_n}\|\partial_x^{\alpha+1}Q_n\|(\|\partial_x^\alpha\mathbf{u}_n\| + \|\partial_x^{\alpha+1}\mathbf{u}_n\|) \\
\leq& C(R)(\|\partial_x^{\alpha+1}Q_n\|^2 + \|\partial_x^\alpha\mathbf{u}_n\|^2) + \frac{\upsilon}{8}\Phi_R^{\mathbf{u}_n,Q_n}\|\sqrt{D(r_n)}\partial_x^{\alpha+1}\mathbf{u}_n\|^2, \qquad (3.3.40)
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\left|\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)\mathrm{div}_x\partial_x^\alpha\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2) - \frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3) + \frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right)\cdot\partial_x^\alpha\mathbf{u}_n dx\right| \\
=&\left|\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\nabla_x D(r_n)\partial_x^\alpha\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2) - \frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3) + \frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right)\cdot\partial_x^\alpha\mathbf{u}_n dx\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)\partial_x^\alpha\left(\frac{a}{2}\mathrm{I}_3\mathrm{tr}(Q_n^2)-\frac{b}{3}\mathrm{I}_3\mathrm{tr}(Q_n^3)+\frac{c}{4}\mathrm{I}_3\mathrm{tr}^2(Q_n^2)\right):\partial_x^\alpha\nabla_x\mathbf{u}_n^{\mathrm{T}}dx\Bigg|\\
&\le C\Phi_R^{\mathbf{u}_n,Q_n}(\|D(r_n)\|_{1,\infty}+\|D(r_n)\|_\infty)(1+\|Q_n\|_\infty^3)\|\partial_x^\alpha Q_n\|(\|\partial_x^\alpha\mathbf{u}_n\|+\|\partial_x^{\alpha+1}\mathbf{u}_n\|)\\
&\le C(R)(\|\partial_x^\alpha Q_n\|^2+\|\partial_x^\alpha\mathbf{u}_n\|^2)+\frac{\upsilon}{8}\Phi_R^{\mathbf{u}_n,Q_n}\|\sqrt{D(r_n)}\partial_x^{\alpha+1}\mathbf{u}_n\|^2. \qquad (3.3.41)
\end{aligned}
$$

According to equation (3.2.3)(1), we have the estimate of $D(r_n)_t$

$$
\begin{aligned}
\|D(r_n)_t\|_\infty &= \left\|\frac{\rho[r_n]_t}{\rho[r_n]^2}\right\|_\infty \le C(R)\Phi_R^{\mathbf{u}_n,Q_n}\|\mathrm{div}_x(\rho[r_n]\mathbf{u}_n)\|_\infty\\
&\le C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|r_n\|_\infty\|\mathrm{div}_x\mathbf{u}_n\|_\infty+\|\nabla_x r_n\|_\infty\|\mathbf{u}_n\|_\infty)\le C(R). \qquad (3.3.42)
\end{aligned}
$$

Considering equation (3.3.5), by Lemma 3.1.1, we can get the estimate of $(\partial_x^\alpha Q_n)_t$ as follows

$$
\begin{aligned}
\|(\partial_x^\alpha Q_n)_t\| = \Bigg\|&\Gamma L\triangle\partial_x^\alpha Q_n+\Phi_R^{\mathbf{u}_n,Q_n}\Big[-\partial_x^\alpha(\mathbf{u}_n\cdot\nabla_x Q_n)+\partial_x^\alpha(\Theta_n Q_n-Q_n\Theta_n)\\
&-\Gamma\partial_x^\alpha\left(aQ_n-b\left(Q_n^2-\frac{\mathrm{I}_3}{3}\mathrm{tr}(Q_n^2)\right)+cQ_n\mathrm{tr}(Q_n^2)\right)\Big]\Bigg\|\\
&\le C(R)(\|\partial_x^\alpha Q_n\|+\|\partial_x^{\alpha+1}Q_n\|+\|\partial_x^\alpha\mathbf{u}_n\|+\|\triangle\partial_x^\alpha Q_n\|). \qquad (3.3.43)
\end{aligned}
$$

The above two estimates combine with (3.3.2), yielding

$$
\begin{aligned}
&\left|\frac{1}{2}\int_{\mathbb{T}} D(r_n)_t|\nabla_x\partial_x^\alpha Q_n|^2dx+\int_{\mathbb{T}}\nabla_x D(r_n)(\partial_x^\alpha Q_n)_t:\nabla_x\partial_x^\alpha Q_n dx\right|\\
&\le C\|D(r_n)_t\|_\infty\|\partial_x^{\alpha+1}Q_n\|^2+C\|\nabla_x D(r_n)\|_\infty\|(\partial_x^\alpha Q_n)_t\|\|\partial_x^{\alpha+1}Q_n\|\\
&\le C(R)\|\partial_x^{\alpha+1}Q_n\|^2+C(R)(\|\partial_x^{\alpha+1}Q_n\|+\|\partial_x^\alpha\mathbf{u}_n\|+\|\triangle\partial_x^\alpha Q_n\|)\|\partial_x^{\alpha+1}Q_n\|\\
&\le C(R)(\|\partial_x^{\alpha+1}Q_n\|^2+\|\partial_x^\alpha\mathbf{u}_n\|^2)+\frac{\Gamma L}{8}\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2. \qquad (3.3.44)
\end{aligned}
$$

In addition, we also have

$$
\begin{aligned}
&\Phi_R^{\mathbf{u}_n,Q_n}\left|\int_{\mathbb{T}} D(r_n)(\mathbf{u}_n\cdot\nabla_x\partial_x^\alpha Q_n):\triangle\partial_x^\alpha Q_n dx\right|\\
&\le\Phi_R^{\mathbf{u}_n,Q_n}\|D(r_n)\|_\infty\|\mathbf{u}_n\|_\infty\|\partial_x^{\alpha+1}Q_n\|\|\triangle\partial_x^\alpha Q_n\|\\
&\le C(R)\|\partial_x^{\alpha+1}Q_n\|^2+\frac{\Gamma L}{8}\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2, \qquad (3.3.45)
\end{aligned}
$$

and

$$
\Phi_R^{\mathbf{u}_n,Q_n}\left|\int_{\mathbb{T}} D(r_n)\partial_x^\alpha\left(aQ_n-b\left(Q_n^2-\frac{\mathrm{I}_3}{3}\mathrm{tr}(Q_n^2)\right)+cQ_n\mathrm{tr}(Q_n^2)\right):\triangle\partial_x^\alpha Q_n dx\right|
$$

$$\leq \Phi_R^{\mathbf{u}_n,Q_n}\|D(r_n)\|_\infty\|\triangle\partial_x^\alpha Q_n\|\left\|\partial_x^\alpha\left(aQ_n - b\left(Q_n^2 - \frac{\mathrm{I}_3}{3}\mathrm{tr}(Q_n^2)\right) + cQ_n\mathrm{tr}(Q_n^2)\right)\right\|$$
$$\leq C(R)\Phi_R^{\mathbf{u}_n,Q_n}(\|Q_n\|_\infty + \|Q_n\|_\infty^2)\|\partial_x^\alpha Q_n\|\|\triangle\partial_x^\alpha Q_n\|$$
$$\leq C(R)\|\partial_x^{\alpha+1}Q_n\|^2 + \frac{\Gamma L}{8}\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2, \tag{3.3.46}$$

in the last step, we also use the estimate (3.3.2). Summing all the estimates (3.3.27)-(3.3.46), note that the first term in (3.3.39) was cancelled with the forth integral on the left hand side of (3.3.26), also the second term in (3.3.37) was cancelled with the second term on the left hand side of (3.3.24) after matching the constant, we conclude

$$\begin{aligned}
&d(\|\partial_x^\alpha r_n(t)\|^2 + \|\partial_x^\alpha \mathbf{u}_n(t)\|^2 + \|\sqrt{D(r_n)}\nabla_x\partial_x^\alpha Q_n(t)\|^2)\\
&\quad + \Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}} D(r_n)(\upsilon|\nabla_x\partial_x^\alpha\mathbf{u}_n|^2 + (\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha\mathbf{u}_n|^2)dxdt\\
&\quad + \Gamma L\|\sqrt{D(r_n)}\triangle\partial_x^\alpha Q_n\|^2 dt\\
&\leq C(R)(\|\partial_x^\alpha r_n\|^2 + \|\partial_x^\alpha\mathbf{u}_n\|^2 + \|\sqrt{D(r_n)}\partial_x^{\alpha+1}Q_n\|^2)dt\\
&\quad + \Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\partial_x^\alpha\mathbb{F}(r_n,\mathbf{u}_n)\cdot\partial_x^\alpha\mathbf{u}_n dxdW.
\end{aligned}\tag{3.3.47}$$

Define the stopping time $\tau_M$

$$\tau_M = \inf\left\{t\geq 0; \sup_{\xi\in[0,t]}\|r_n(\xi),\mathbf{u}_n(\xi)\|_{s,2}^2 \geq M\right\},$$

if the set is empty, choosing $\tau_M = T$. Then, taking sum for $|\alpha|\leq s$, taking integral with respect to time and sumpremum on interval $[0, t\wedge\tau_M]$, power $p$, finally expectation on both sides of (3.3.47), we arrive at

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\xi\in[0,t\wedge\tau_M]}(\|\partial_x^s r_n\|^2 + \|\partial_x^s\mathbf{u}_n\|^2 + \|\sqrt{D(r_n)}\nabla_x\partial_x^s Q_n\|^2)\right]^p\\
&\quad + \mathbb{E}\left(\int_0^{t\wedge\tau_M}\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\upsilon|\nabla_x\partial_x^s\mathbf{u}_n|^2 + (\upsilon+\lambda)|\mathrm{div}_x\partial_x^s\mathbf{u}_n|^2)dxd\xi\right)^p\\
&\quad + \mathbb{E}\left(\int_0^{t\wedge\tau_M}\Gamma L\|\sqrt{D(r_n)}\triangle\partial_x^s Q_n\|^2 d\xi\right)^p\\
&\leq C\mathbb{E}(\|r_0,\mathbf{u}_0\|_{s,2}^2 + \|Q_0\|_{s+1,2}^2)^p\\
&\quad + C\mathbb{E}\left(\int_0^{t\wedge\tau_M}\|\partial_x^s r_n\|^2 + \|\partial_x^s\mathbf{u}_n\|^2 + \|\sqrt{D(r_n)}\partial_x^{s+1}Q_n\|^2 d\xi\right)^p
\end{aligned}$$

$$+ C\mathbb{E}\left[\sup_{\xi\in[0,t\wedge\tau_M]}\left|\int_0^\xi \Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\partial_x^s\mathbb{F}(r_n,\mathbf{u}_n)\cdot\partial_x^s\mathbf{u}_n dxdW\right|\right]^p. \tag{3.3.48}$$

Regarding the stochastic integral term, we could apply the Burkholder-Davis-Gundy inequality (3.1.4) and assumption (3.1.5)(Remark 3.1.6), for any $1 \le p < \infty$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\xi\in[0,t\wedge\tau_M]}\left|\int_0^\xi \Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}\partial_x^s\mathbb{F}(r_n,\mathbf{u}_n)\cdot\partial_x^s\mathbf{u}_n dxdW\right|\right]^p\\
&\le C(p)\mathbb{E}\left[\int_0^{t\wedge\tau_M}(\Phi_R^{\mathbf{u}_n,Q_n})^2\|\mathbb{F}(r_n,\mathbf{u}_n)\|^2_{L_2(\mathfrak{U};W^{s,2}(\mathbb{T}))}\|\mathbf{u}_n\|^2_{s,2}d\xi\right]^{\frac{p}{2}}\\
&\le C(p)\mathbb{E}\left[\int_0^{t\wedge\tau_M}(\Phi_R^{\mathbf{u}_n,Q_n})^2\|r_n,\nabla\mathbf{u}_n\|^2_{1,\infty}\|r_n,\mathbf{u}_n\|^4_{s,2}d\xi\right]^{\frac{p}{2}}\\
&\le C(p,R)\mathbb{E}\left[\sup_{\xi\in[0,t\wedge\tau_M]}\|r_n,\mathbf{u}_n\|^2_{s,2}\int_0^{t\wedge\tau_M}\|r_n,\mathbf{u}_n\|^2_{s,2}d\xi\right]^{\frac{p}{2}}\\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{\xi\in[0,t\wedge\tau_M]}\|r_n,\mathbf{u}_n\|^{2p}_{s,2}\right]+C(p,R)\mathbb{E}\left[\int_0^{t\wedge\tau_M}\|r_n,\mathbf{u}_n\|^2_{s,2}d\xi\right]^p.
\end{aligned} \tag{3.3.49}$$

Combining (3.3.48)-(3.3.49), the Gronwall lemma gives

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\xi\in[0,t\wedge\tau_M]}(\|\partial_x^s r_n\|^2+\|\partial_x^s\mathbf{u}_n\|^2+\|\sqrt{D(r_n)}\nabla_x\partial_x^s Q_n\|^2)\right]^p\\
&+\mathbb{E}\left(\int_0^{t\wedge\tau_M}\Phi_R^{\mathbf{u}_n,Q_n}\int_{\mathbb{T}}D(r_n)(\upsilon|\nabla_x\partial_x^s\mathbf{u}_n|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^s\mathbf{u}_n|^2)dxd\xi\right)^p\\
&+\mathbb{E}\left(\int_0^{t\wedge\tau_M}\Gamma L\|\sqrt{D(r_n)}\triangle\partial_x^s Q_n\|^2d\xi\right)^p\le C,
\end{aligned} \tag{3.3.50}$$

where the constant $C$ is independent of $n$, but depends on $(s,p,R,\mathbb{T},T)$ and the initial data. Taking $M\to\infty$ in (3.3.50), using the fact that $\frac{1}{C(R)}\le D(r_n)\le C(R)$ and the monotone convergence theorem, we establish the a priori estimates

$$r_n\in L^p(\Omega;L^\infty(0,T;W^{s,2}(\mathbb{T}))),\quad \mathbf{u}_n\in L^p(\Omega;L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3))), \tag{3.3.51}$$

$$Q_n\in L^p(\Omega;L^\infty(0,T;W^{s+1,2}(\mathbb{T},S_0^3))\cap L^2(0,T;W^{s+2,2}(\mathbb{T},S_0^3))), \tag{3.3.52}$$

for all $1\le p<\infty$, integer $s>\frac{7}{2}$.

### 3.3.3 Compactness argument

Let $\{r_n, \mathbf{u}_n, Q_n\}_{n\geq 1}$ be the sequence of approximate solution to system (3.2.3) relative to the fixed stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ and $\mathcal{F}_0$-measurable random variable $(r_0, \mathbf{u}_0, Q_0)$. We define the path space

$$\mathcal{X} = \mathcal{X}_r \times \mathcal{X}_{\mathbf{u}} \times \mathcal{X}_Q \times \mathcal{X}_W,$$

where

$$\begin{aligned}
&\mathcal{X}_r = C([0,T]; W^{s-1,2}(\mathbb{T})),\ \mathcal{X}_{\mathbf{u}} = L^\infty(0,T; W^{s-\varepsilon,2}(\mathbb{T},\mathbb{R}^3)),\\
&\mathcal{X}_Q = C([0,T]; W^{s,2}(\mathbb{T}, S_0^3)) \cap L^2(0,T; W^{s+1,2}(\mathbb{T}, S_0^3)),\quad \mathcal{X}_W = C([0,T]; \mathfrak{U}_0),
\end{aligned}$$

where $\varepsilon$ is small enough such that integer $s-\varepsilon > \frac{3}{2}+2$.

Define the sequence of probability measures

$$\mu^n = \mu_r^n \otimes \mu_{\mathbf{u}}^n \otimes \mu_Q^n \otimes \mu_W, \tag{3.3.53}$$

where $\mu_r^n(\cdot) = \mathbb{P}\{r_n \in \cdot\}$, $\mu_{\mathbf{u}}^n(\cdot) = \mathbb{P}\{\mathbf{u}_n \in \cdot\}$, $\mu_Q^n(\cdot) = \mathbb{P}\{Q_n \in \cdot\}$, $\mu_W(\cdot) = \mathbb{P}\{W \in \cdot\}$. We show that the set $\{\mu^n\}_{n\geq 1}$ is in fact weakly compact. According to the Prokhorov theorem, it suffices to show that each set $\{\mu_{(\cdot)}^n\}_{n\geq 1}$ is tight on the corresponding path space $\mathcal{X}_{(\cdot)}$.

**Lemma 3.3.6.** *The set of the sequence of measures $\{\mu_{\mathbf{u}}^n\}_{n\geq 1}$ is tight on path space $\mathcal{X}_{\mathbf{u}}$.*

*Proof.* First, we show that for any $\alpha \in [0, \frac{1}{2})$

$$\mathbb{E}\|\mathbf{u}_n\|_{C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3))} \leq C, \tag{3.3.54}$$

where $C$ is independent of $n$.

Decompose $\mathbf{u}_n = X_n + Y_n$, where

$$\begin{aligned}
X_n = X_n(0) + \int_0^t &-\Phi_R^{\mathbf{u}_n,Q_n} P_n(\mathbf{u}_n \nabla_x \mathbf{u}_n + r_n \nabla_x r_n) + \Phi_R^{\mathbf{u}_n,Q_n} P_n(D(r_n)(\mathcal{L}\mathbf{u}_n\\
&- \mathrm{div}_x(L\nabla_x Q_n \odot \nabla_x Q_n - \mathcal{F}(Q_n)\mathrm{I}_3) + L\mathrm{div}_x(Q_n \triangle Q_n - \triangle Q_n Q_n)))d\xi,\\
Y_n = \int_0^t &\Phi_R^{\mathbf{u}_n,Q_n} P_n \mathbb{F}(r_n, \mathbf{u}_n) dW.
\end{aligned}$$

Using the a priori estimates (3.3.51), (3.3.52) and the Hölder inequality, we have

$$\mathbb{E}\|X_n\|_{W^{1,2}(0,T;L^2(\mathbb{T},\mathbb{R}^3))} \leq C,$$

where $C$ is independent of $n$. By the embedding (3.1.3), we obtain the estimate

$$\mathbb{E}\|X_n\|_{C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3))} \leq C.$$

Note that, for a.s. $\omega$, and for any $\delta' > 0$, there exists $t_1, t_2 \in [0,T]$ such that

$$\sup_{t,t' \in [0,T], t \neq t'} \frac{\left\| \int_t^{t'} f dW \right\|_{L^2}}{|t'-t|^\alpha} \leq \frac{\left\| \int_{t_1}^{t_2} f dW \right\|_{L^2}}{|t_2-t_1|^\alpha} + \delta'.$$

Regarding the stochastic term $Y_n$, using the Burkholder-Davis-Gundy inequality (3.1.4) and assumption (3.1.5), we get

$$\begin{aligned}
&\mathbb{E}\left\| \int_0^t \Phi_R^{\mathbf{u}_n,Q_n} P_n \mathbb{F}(r_n,\mathbf{u}_n) dW \right\|_{C^\alpha([0,T];L^2(\mathbb{T}))} \\
&\leq \mathbb{E}\left[ \sup_{t,t' \in [0,T], t \neq t'} \frac{\left\| \int_{t'}^{t} \Phi_R^{\mathbf{u}_n,Q_n} P_n \mathbb{F}(r_n,\mathbf{u}_n) dW \right\|_{L^2}}{|t-t'|^\alpha} \right] \\
&\leq \frac{\mathbb{E}\left\| \int_{t_1}^{t_2} \Phi_R^{\mathbf{u}_n,Q_n} P_n \mathbb{F}(r_n,\mathbf{u}_n) dW \right\|_{L^2}}{|t_2-t_1|^\alpha} + \delta' \\
&\leq \frac{C\mathbb{E}\left( \int_{t_1}^{t_2} \left\| \Phi_R^{\mathbf{u}_n,Q_n} \mathbb{F}(r_n,\mathbf{u}_n) \right\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}))} d\xi \right)^{\frac{1}{2}}}{|t_2-t_1|^\alpha} + \delta' \\
&\leq C(R)|t_2-t_1|^{\frac{1}{2}-\alpha} + \delta' \leq C.
\end{aligned}$$

Thus, we get the estimate (3.3.54). Fix any $\alpha \in (0, \frac{1}{2})$, by the Aubin-Lions lemma A.0.5, we have

$$C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3)) \cap L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3)) \hookrightarrow\hookrightarrow L^\infty(0,T;W^{s-\varepsilon,2}(\mathbb{T},\mathbb{R}^3)).$$

Therefore, for any fixed $K > 0$, the set

$$\begin{aligned}
B_K := \Big\{ &\mathbf{u} \in C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3)) \cap L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3)) : \\
&\|\mathbf{u}\|_{C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3))} + \|\mathbf{u}\|_{L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3))} \leq K \Big\}
\end{aligned}$$

is compact in $L^\infty(0,T;W^{s-\varepsilon,2}(\mathbb{T},\mathbb{R}^3))$. Applying the Chebyshev inequality and the estimates $(3.3.51)_2$, (3.3.54), we have

$$\begin{aligned}\mu^n_{\mathbf{u}}(B^c_K) &= \mathbb{P}\left(\|\mathbf{u}_n\|_{L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3))} + \|\mathbf{u}_n\|_{C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3))} > K\right)\\ &\le \mathbb{P}\left(\|\mathbf{u}_n\|_{L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3))} > \frac{K}{2}\right) + \mathbb{P}\left(\|\mathbf{u}_n\|_{C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3))} > \frac{K}{2}\right)\\ &\le \frac{2}{K}\left(\mathbb{E}\|\mathbf{u}_n\|_{L^\infty(0,T;W^{s,2}(\mathbb{T},\mathbb{R}^3))} + \mathbb{E}\|\mathbf{u}_n\|_{C^\alpha([0,T];L^2(\mathbb{T},\mathbb{R}^3))}\right) \le \frac{C}{K},\end{aligned}$$

where the constant $C$ is independent of $n, K$. Thus, we obtain the tightness of the sequence of measures $\{\mu^n_{\mathbf{u}}\}_{n\ge1}$. □

**Lemma 3.3.7.** *The set of the sequence of measures $\{\mu^n_Q\}_{n\ge1}$ is tight on path space $\mathcal{X}_Q$.*

*Proof.* We only need to show that the set $\{\mu^n_Q\}_{n\ge1}$ is tight on space $L^2(0,T;W^{s+1,2}(\mathbb{T},S^3_0))$, the proof of tightness on space $C([0,T];W^{s,2}(\mathbb{T},S^3_0))$ is the same as the proof of the set $\{\mu^n_{\mathbf{u}}\}_{n\ge1}$.

From the equation (3.2.3)(3), we can easily show that

$$\mathbb{E}\|Q_n\|_{W^{1,2}(0,T;L^2(\mathbb{T},S^3_0))} \le C, \tag{3.3.55}$$

where $C$ is a constant independence of $n$. For any fixed $K>0$, define the set

$$\begin{aligned}\overline{B}_K := \Big\{&Q \in L^2(0,T;W^{s+2,2}(\mathbb{T},S^3_0)) \cap W^{1,2}(0,T;L^2(\mathbb{T},S^3_0)) :\\ &\|Q\|_{L^2(0,T;W^{s+2,2}(\mathbb{T},S^3_0))} + \|Q\|_{W^{1,2}(0,T;L^2(\mathbb{T},S^3_0))} \le K\Big\},\end{aligned}$$

which is thus compact in $L^2(0,T;W^{s+1,2}(\mathbb{T},S^3_0))$ as a result of the compactness embedding

$$L^2(0,T;W^{s+2,2}(\mathbb{T},S^3_0)) \cap W^{1,2}(0,T;L^2(\mathbb{T},S^3_0)) \hookrightarrow L^2(0,T;W^{s+1,2}(\mathbb{T},S^3_0)).$$

Applying the Chebyshev inequality and the estimates (3.3.52), (3.3.55), we get

$$\mu^n_u\left(\overline{B}^c_K\right) \le \frac{C}{K},$$

where the constant $C$ is independent of $n, K$. □

Using the same argument as above, we can show the tightness of the sequences of set $\{\mu_r^n\}_{n\geq 1}$. Since the sequence $W$ is only one element and thus, the set $\{\mu_W^n\}_{n\geq 1}$ is weakly compact. Then, the tightness of measure set $\{\mu^n\}_{n\geq 1}$ follows.

With the weakly compact of set $\{\mu^n\}_{n\geq 1}$ in hand, using the Skorokhod representation theorem A.0.7, we have:

**Proposition 3.3.8.** *There exists a subsequence of $\{\mu^n\}_{n\geq 1}$, also denoted as $\{\mu^n\}_{n\geq 1}$, and a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ as well as a sequence of random variables $(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n, \tilde{W}_n)$, $(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$ such that*
*(a) the joint law of $(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n, \tilde{W}_n)$ is $\mu^n$, and the joint law of $(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$ is $\mu$, where $\mu$ is the weak limit of the sequence $\{\mu^n\}_{n\geq 1}$;*
*(b) $(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n, \tilde{W}_n)$ converges to $(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$, $\tilde{\mathbb{P}}$ a.s. in the topology of $\mathcal{X}$;*
*(c) the sequence of $\tilde{Q}_n$ and $\tilde{Q}$ belong to $S_0^3$, almost everywhere.*
*(d) $\tilde{W}_n$ is a cylindrical Wiener process, relative to the filtration $\tilde{\mathcal{F}}_t^n$ given below.*

*Proof.* The results (a), (b), (d) are a direct consequence of the Skorokhod representation theorem. The result (c) is a consequence of result (a). $\square$

**Proposition 3.3.9.** *The sequence $(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n, \tilde{W}_n)$ still satisfies the n-th order Galerkin approximate system relative to the stochastic basis $\widetilde{S}^n := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t^n\}_{t\geq 0}, \tilde{W}_n)$, where $\tilde{\mathcal{F}}_t^n$ is a canonical filtration defined by*

$$\sigma\left(\sigma\left(\tilde{r}_n(s), \tilde{\mathbf{u}}_n(s), \tilde{Q}_n(s), \tilde{W}_n(s) : s \leq t\right) \cup \left\{\Sigma \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(\Sigma) = 0\right\}\right).$$

*Proof.* The proof is similar to the one in [39, 84], here we omit the details. $\square$

### 3.3.4 Identification of limit

We verify that $(\widetilde{\mathcal{S}}, \tilde{r}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$ is a strong martingale solution to system (3.2.3), where $\widetilde{\mathcal{S}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0})$ and the canonical filtration $\tilde{\mathcal{F}}_t$ was given by

$$\tilde{\mathcal{F}}_t = \sigma\left(\sigma\left(\tilde{r}(s), \tilde{\mathbf{u}}(s), \tilde{Q}(s), \tilde{W}(s) : s \leq t\right) \cup \left\{\Sigma \in \tilde{\mathcal{F}}; \tilde{\mathbb{P}}(\Sigma) = 0\right\}\right).$$

Define the following functionals

$$\begin{aligned}
&\mathcal{P}(r,\mathbf{u})_t = r(t) - r(0) + \int_0^t \Phi_R^{\mathbf{u},Q}\left(\mathbf{u}\cdot\nabla_x r + \frac{\gamma-1}{2} r\mathrm{div}_x\mathbf{u}\right)d\xi,\\
&\mathcal{N}(Q,\mathbf{u})_t = Q(t) - Q(0) - \int_0^t \Gamma L\triangle Q d\xi + \int_0^t \Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x Q - \Theta Q + Q\Theta - \mathcal{K}(Q))d\xi,\\
&\mathcal{M}(r,\mathbf{u},Q)_t = \mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x\mathbf{u} + r\nabla_x r)d\xi\\
&\quad - \int_0^t \Phi_R^{\mathbf{u},Q}(D(r)(\mathcal{L}\mathbf{u} - \mathrm{div}_x(L\nabla_x Q\odot\nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3) + L\mathrm{div}_x(Q\triangle Q - \triangle QQ))d\xi.
\end{aligned}$$

First, we show that for any function $\mathbf{h}\in L^2(\mathbb{T})$, almost every $(\omega,t)\in\tilde{\Omega}\times(0,T]$

$$\langle\mathcal{P}(\tilde{r}_n,\tilde{\mathbf{u}}_n)_t,\mathbf{h}\rangle \to \langle\mathcal{P}(\tilde{r},\tilde{\mathbf{u}})_t,\mathbf{h}\rangle,\quad \langle\mathcal{N}(\tilde{Q}_n,\tilde{\mathbf{u}}_n)_t,\mathbf{h}\rangle \to \langle\mathcal{N}(\tilde{Q},\tilde{\mathbf{u}})_t,\mathbf{h}\rangle,$$

as $n\to\infty$. We only give the argument of high-order term $Q\mathrm{tr}(Q^2)$ in $\mathcal{K}(Q)$. Note that

$$\begin{aligned}
&\left|\int_0^t (\Phi_R^{\tilde{\mathbf{u}}_n,\tilde{Q}_n}\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2) - \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}}\tilde{Q}\mathrm{tr}(\tilde{Q}^2),\mathbf{h})d\xi\right|\\
&\le \left|\int_0^t ((\Phi_R^{\tilde{\mathbf{u}}_n,\tilde{Q}_n} - \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}})\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2),\mathbf{h})d\xi\right| + \left|\int_0^t \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}}(\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2) - \tilde{Q}\mathrm{tr}(\tilde{Q}^2),\mathbf{h})d\xi\right|\\
&\le \left|\int_0^t ((\Phi_R^{\tilde{\mathbf{u}}_n,\tilde{Q}_n} - \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}})\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2),\mathbf{h})d\xi\right| + \left|\int_0^t \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}}((\tilde{Q}_n - \tilde{Q})\mathrm{tr}(\tilde{Q}_n^2),\mathbf{h})d\xi\right|\\
&\quad + \left|\int_0^t \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}}(\tilde{Q}(\mathrm{tr}(\tilde{Q}_n^2) - \mathrm{tr}(\tilde{Q}^2)),\mathbf{h})d\xi\right|\\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Using the mean value theorem, the Hölder inequality and Proposition 3.3.8(b), we get

$$\begin{aligned}
J_1 &\le C\|\mathbf{h}\|\int_0^t (\|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}\|_{2,\infty} + \|\tilde{Q}_n - \tilde{Q}\|_{3,\infty})\|\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2)\|_\infty d\xi\\
&\le C\|\mathbf{h}\|\int_0^t (\|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}\|_{s-\varepsilon,2} + \|\tilde{Q}_n - \tilde{Q}\|_{s+1,2})\|\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2)\|_\infty d\xi\\
&\le C\|\mathbf{h}\|\sup_{t\in[0,T]}\|\tilde{Q}_n\mathrm{tr}(\tilde{Q}_n^2)\|_\infty\int_0^t (\|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}\|_{s-\varepsilon,2} + \|\tilde{Q}_n - \tilde{Q}\|_{s+1,2})d\xi\\
&\to 0,\ \text{as } n\to\infty,\ \tilde{\mathbb{P}}\ \text{a.s.}
\end{aligned}$$

We could use the same argument to get $J_2, J_3\to 0$, $n\to\infty$, $\tilde{\mathbb{P}}$ a.s.. Furthermore, by the Vitali convergence theorem A.0.6, we infer that $(\tilde{r},\tilde{\mathbf{u}},\tilde{Q})$ solves equations (3.2.3)(1)(3).

It remains to verify that $(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$ solves equation (3.2.3)(2) by passing $n \to \infty$. With the spirit of [20], we are able to obtain the limit $(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$ satisfies the equation (3.2.3)(2) once we show that the process $\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t$ is a square integral martingale and its quadratic and cross variations satisfy,

$$\langle\langle \mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t \rangle\rangle = \int_0^t (\Phi_R^{\tilde{\mathbf{u}}, \tilde{Q}})^2 \|\mathbb{F}(\tilde{r}, \tilde{\mathbf{u}})\|^2_{L_2(\mathfrak{U}; L^2(\mathbb{T}, \mathbb{R}^3))} d\xi, \tag{3.3.56}$$

$$\langle\langle \mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t, \tilde{\beta}_k \rangle\rangle = \int_0^t \Phi_R^{\tilde{\mathbf{u}}, \tilde{Q}} \|\mathbb{F}(\tilde{r}, \tilde{\mathbf{u}}) e_k\| d\xi. \tag{3.3.57}$$

We clarify that the $\tilde{\mathcal{F}}_t$-Wiener process $\tilde{W}$ can be written in the form of $\tilde{W} = \sum_{k\geq 1} \tilde{\beta}_k e_k$. Since $\tilde{W}_n$ has the same distribution as $W_n$, then clearly its distribution is the same to $W$. That is, for any $n \in \mathbb{N}$, there exists a collection of mutually independent real-valued $\tilde{\mathcal{F}}_t^n$-Wiener processes $\{\tilde{\beta}_k^n\}_{k\geq 1}$, such that $\tilde{W}_n = \sum_{k\geq 1} \tilde{\beta}_k^n e_k$. Due to the convergence property of $\tilde{W}_n$, therefore the same thing holds for $\tilde{W}$.

For any function $\mathbf{h} \in L^2(\mathbb{T}, \mathbb{R}^3)$, by Proposition 3.3.9, we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[h(\mathbf{r}_s\tilde{r}_n, \mathbf{r}_s\tilde{\mathbf{u}}_n, \mathbf{r}_s\tilde{Q}_n, \mathbf{r}_s\tilde{W}_n)\langle \mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_t - \mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_s, \mathbf{h}\rangle\right] = 0,\\
&\tilde{\mathbb{E}}\Bigg[h(\mathbf{r}_s\tilde{r}_n, \mathbf{r}_s\tilde{\mathbf{u}}_n, \mathbf{r}_s\tilde{Q}_n, \mathbf{r}_s\tilde{W}_n)\Bigg(\langle \mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_t, \mathbf{h}\rangle^2 - \langle \mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_s, \mathbf{h}\rangle^2\\
&\quad - \int_s^t (\Phi_R^{\tilde{\mathbf{u}}_n, \tilde{Q}_n})^2 \|(P_n\mathbb{F}(\tilde{r}_n, \tilde{\mathbf{u}}_n))^*\mathbf{h}\|^2_{\mathfrak{U}} d\xi\Bigg)\Bigg] = 0,\\
&\tilde{\mathbb{E}}\Bigg[h(\mathbf{r}_s\tilde{r}_n, \mathbf{r}_s\tilde{\mathbf{u}}_n, \mathbf{r}_s\tilde{Q}_n, \mathbf{r}_s\tilde{W}_n)\Bigg(\tilde{\beta}_k^n(t)\langle \mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_t, \mathbf{h}\rangle - \tilde{\beta}_k^n(s)\langle \mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_s, \mathbf{h}\rangle\\
&\quad - \int_s^t \Phi_R^{\tilde{\mathbf{u}}_n, \tilde{Q}_n} \langle P_n\mathbb{F}(\tilde{r}_n, \tilde{\mathbf{u}}_n)e_k, \mathbf{h}\rangle d\xi\Bigg)\Bigg] = 0,
\end{aligned}$$

where $h$ is a continuous function defined by

$$h : \mathcal{X}_r|_{[0,s]} \times \mathcal{X}_{\mathbf{u}}|_{[0,s]} \times \mathcal{X}_Q|_{[0,s]} \times \mathcal{X}_W|_{[0,s]} \to [0, 1]$$

and $\mathbf{r}_t$ is an operator as the restriction of the path spaces $\mathcal{X}_r$, $\mathcal{X}_{\mathbf{u}}$, $\mathcal{X}_Q$ and $\mathcal{X}_W$ to the interval $[0, t]$ for any $t \in [0, T]$.

In order to pass the limit in above equality, we show that for almost every $(\omega, t) \in \tilde{\Omega} \times (0, T]$

$$(\mathcal{M}(\tilde{r}_n, \tilde{\mathbf{u}}_n, \tilde{Q}_n)_t, \mathbf{h}) \to (\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t, \mathbf{h}). \tag{3.3.58}$$

We only consider the nontrivial term $\mathrm{div}_x(Q\triangle Q - \triangle QQ)$. Note that

$$
\begin{aligned}
&\int_0^t \Big(\Phi_R^{\tilde{\mathbf{u}}_n,\tilde{Q}_n} D(\tilde{r}_n)L\mathrm{div}_x(\tilde{Q}_n\triangle\tilde{Q}_n - \triangle\tilde{Q}_n\tilde{Q}_n) - \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}} D(\tilde{r})L\mathrm{div}_x(\tilde{Q}\triangle\tilde{Q} - \triangle\tilde{Q}\tilde{Q}), \mathbf{h}\Big) d\xi \\
&\le \int_0^t \Big((\Phi_R^{\tilde{\mathbf{u}}_n,\tilde{Q}_n} - \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}})D(\tilde{r}_n)L\mathrm{div}_x(\tilde{Q}_n\triangle\tilde{Q}_n - \triangle\tilde{Q}_n\tilde{Q}_n), \mathbf{h}\Big)\, d\xi \\
&\quad + \int_0^t \Big(\Phi_R^{\tilde{\mathbf{u}},\tilde{Q}}(D(\tilde{r}_n) - D(\tilde{r}))L\mathrm{div}_x(\tilde{Q}_n\triangle\tilde{Q}_n - \triangle\tilde{Q}_n\tilde{Q}_n), \mathbf{h}\Big)\, d\xi \\
&\quad + \int_0^t \Big(\Phi_R^{\tilde{\mathbf{u}},\tilde{Q}} D(\tilde{r})L\mathrm{div}_x(\tilde{Q}_n\triangle\tilde{Q}_n - \triangle\tilde{Q}_n\tilde{Q}_n - \triangle\tilde{Q}\tilde{Q} + \tilde{Q}\triangle\tilde{Q}), \mathbf{h}\Big)\, d\xi \\
&=: K_1 + K_2 + K_3.
\end{aligned}
$$

Using the mean value theorem, the Hölder inequality, (3.3.2) and Proposition 3.3.8(b), we get

$$
\begin{aligned}
K_1 &\le C\|\mathbf{h}\| \int_0^t (\|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}\|_{2,\infty} + \|\tilde{Q}_n - \tilde{Q}\|_{3,\infty})\|D(\tilde{r}_n)\|_\infty \|\tilde{Q}_n\|_{1,\infty}\|\tilde{Q}_n\|_{3,\infty} d\xi \\
&\le C\|\mathbf{h}\| \sup_{t\in[0,T]} \|\tilde{Q}_n\|_{s+1,2}^2 \int_0^t (\|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}\|_{2,\infty} + \|\tilde{Q}_n - \tilde{Q}\|_{3,\infty}) d\xi \\
&\to 0, \text{ as } n \to \infty, \ \tilde{\mathbb{P}} \text{ a.s.}
\end{aligned}
$$

Similarly, using (3.3.4), the Hölder inequality and Proposition 3.3.8(b), we get $K_2 \to 0$, $\tilde{\mathbb{P}}$ a.s.. Using the Hölder inequality, (3.3.2) and Proposition 3.3.8(b), we also get $K_3 \to 0$, $\tilde{\mathbb{P}}$ a.s..

Last, let $n \to \infty$, by (3.3.58) and the Vitali convergence theorem A.0.6, we could find

$$
\begin{aligned}
&\tilde{\mathbb{E}}\left[h(\mathbf{r}_s\tilde{r}, \mathbf{r}_s\tilde{\mathbf{u}}, \mathbf{r}_s\tilde{Q}, \mathbf{r}_s\tilde{W})\langle\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t - \mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_s, \mathbf{h}\rangle\right] = 0, \\
&\tilde{\mathbb{E}}\Bigg[h(\mathbf{r}_s\tilde{r}, \mathbf{r}_s\tilde{\mathbf{u}}, \mathbf{r}_s\tilde{Q}, \mathbf{r}_s\tilde{W})\Bigg(\langle\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t, \mathbf{h}\rangle^2 - \langle\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_s, \mathbf{h}\rangle^2 \\
&\quad - \int_s^t (\Phi_R^{\tilde{\mathbf{u}},\tilde{Q}})^2 \|(\mathbb{F}(\tilde{r}, \tilde{\mathbf{u}}))^*\mathbf{h}\|_{\mathfrak{U}}^2 d\xi\Bigg)\Bigg] = 0, \\
&\tilde{\mathbb{E}}\Bigg[h(\mathbf{r}_s\tilde{r}, \mathbf{r}_s\tilde{\mathbf{u}}, \mathbf{r}_s\tilde{Q}, \mathbf{r}_s\tilde{W})\Bigg(\tilde{\beta}_k(t)\langle\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_t, \mathbf{h}\rangle - \tilde{\beta}_k(s)\langle\mathcal{M}(\tilde{r}, \tilde{\mathbf{u}}, \tilde{Q})_s, \mathbf{h}\rangle \\
&\quad - \int_s^t \Phi_R^{\tilde{\mathbf{u}},\tilde{Q}} \langle\mathbb{F}(\tilde{r}, \tilde{\mathbf{u}})e_k, \mathbf{h}\rangle d\xi\Bigg)\Bigg] = 0.
\end{aligned}
$$

Thus, we obtain the desired equalities (3.3.56) and (3.3.57), the Definition 3.3.1(4) follows.

From the estimate $(3.3.51)_1$ and the mass equation itself, we are able to deduce that the process $r$ is continuous with respect to time $t$ in $W^{s,2}(\mathbb{T})$ using the [55, Theorem 3.1], see also [11] for the compressible Navier-Stokes equations. Moreover, by the initial data condition and estimate (3.3.1), we infer the process $r$ has the uniform lower bound which depends on $R$, $\tilde{\mathbb{P}}$ a.s.. Since the high-order terms $\operatorname{div}_x(Q\triangle Q - \triangle QQ)$ and $\Theta Q - Q\Theta$ arise in momentum and $Q$-tensor equations, again by [55, Theorem 3.1] and the estimates $(3.3.51), (3.3.52)$ and the equations itself, we could only infer that $(\mathbf{u}, Q)$ is continuous with respect to time $t$ in $W^{s-1,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s,2}(\mathbb{T}, S_0^3)$. This completes the proof of Theorem 3.3.2.

## 3.4 Existence and Uniqueness of Strong Pathwise Solution to Truncated System

In this section, we establish the existence and uniqueness of strong pathwise solution to system (3.2.3) and start with the definition and result.

**Definition 3.4.1.** (Strong pathwise solution) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a fixed stochastic basis and $W$ be a given cylindrical Wiener process. The triple $(r, \mathbf{u}, Q)$ is called a global strong pathwise solution to system (3.2.3) with initial data $(r_0, \mathbf{u}_0, Q_0)$ if the following conditions hold

1. $r$, $\mathbf{u}$ are $\mathcal{F}_t$-progressively measurable processes with values in $W^{s,2}(\mathbb{T}), W^{s,2}(\mathbb{T}, \mathbb{R}^3)$, $Q$ is $\mathcal{F}_t$-progressively measurable process with value in $W^{s+1,2}(\mathbb{T}, S_0^3)$, satisfying

$$
\begin{aligned}
&r \in L^2(\Omega; C([0,T]; W^{s,2}(\mathbb{T}))), \ r > 0, \ \mathbb{P} \text{ a.s.} \\
&\mathbf{u} \in L^2(\Omega; L^\infty(0,T; W^{s,2}(\mathbb{T}; \mathbb{R}^3)) \cap C([0,T]; W^{s-1,2}(\mathbb{T}; \mathbb{R}^3))), \\
&Q \in L^2(\Omega; L^\infty(0,T; W^{s+1,2}(\mathbb{T}; S_0^3)) \cap L^2(0,T; W^{s+2,2}(\mathbb{T}; S_0^3)) \cap C([0,T]; W^{s,2}(\mathbb{T}; S_0^3)));
\end{aligned}
$$

2. for all $t \in [0,T]$, $\mathbb{P}$ a.s.

$$
\begin{aligned}
r(t) &= r_0 - \int_0^t \Phi_R^{\mathbf{u},Q} \left( \mathbf{u} \cdot \nabla_x r + \frac{\gamma - 1}{2} r \operatorname{div}_x \mathbf{u} \right) d\xi, \\
\mathbf{u}(t) &= \mathbf{u}_0 - \int_0^t \Phi_R^{\mathbf{u},Q} (\mathbf{u} \cdot \nabla_x \mathbf{u} + r \nabla_x r) d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \Phi_R^{\mathbf{u},Q} D(r)(\mathcal{L}\mathbf{u} - \mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)\mathrm{I}_3) \\
&+ L\mathrm{div}_x(Q\triangle Q - \triangle Q Q))d\xi + \int_0^t \Phi_R^{\mathbf{u},Q}\mathbb{F}(r,\mathbf{u})dW, \\
Q(t) = Q_0 - &\int_0^t \Phi_R^{\mathbf{u},Q}(\mathbf{u}\cdot\nabla_x Q - \Theta Q + Q\Theta)d\xi + \int_0^t \Gamma L\triangle Q + \Phi_R^{\mathbf{u},Q}\mathcal{K}(Q)d\xi.
\end{aligned}
$$

In this section, we shall obtain the following result.

**Theorem 3.4.2.** *Assume the initial data* $(r_0, \mathbf{u}_0, Q_0)$ *satisfies the same conditions with Theorem 3.3.2 and the coefficient* $\mathbb{G}$ *satisfies the assumptions* (3.1.5),(3.1.6). *For any integer* $s > \frac{9}{2}$*, the system (3.2.3) has a unique global strong pathwise solution in the sense of Definition 3.4.1.*

Following the Yamada-Watanabe argument, the pathwise uniqueness in probability "1" in turn reveals that the solution is also strong in probability sense, this means the solution is constructed with respect to the fixed probability space in advance. Therefore, we next establish the pathwise uniqueness.

**Proposition 3.4.3.** (Uniqueness) *Fix any integer* $s > \frac{9}{2}$. *Suppose that* $\mathbb{G}$ *satisfies assumption* (3.1.6)*, and* $((\mathcal{S}, r_1, \mathbf{u}_1, Q_1), (\mathcal{S}, r_2, \mathbf{u}_2, Q_2))$ *are two martingale solutions of system (3.2.3) with the same stochastic basis* $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$. *Then if*

$$
\mathbb{P}\{(r_1(0), \mathbf{u}_1(0), Q_1(0)) = (r_2(0), \mathbf{u}_2(0), Q_2(0))\} = 1,
$$

*then pathwise uniqueness holds in the sense of Definition 3.1.4.*

*Proof.* Owing to the complexity of constitution and the similarity of argument with the a priori estimate, here we only focus on the estimate of high-order nonlinearity term. Let $\alpha$ be any vector such that $|\alpha| \leq s-1$, taking the difference of $r_1$ and $r_2$, then $\alpha$-order derivative, we have

$$
\begin{aligned}
&d\partial_x^\alpha(r_1 - r_2) \\
=& -\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\left(\mathbf{u}_1\cdot\nabla_x r_1 + \frac{\gamma-1}{2}r_1\mathrm{div}_x\mathbf{u}_1\right)dt \\
&+\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\left(\mathbf{u}_2\cdot\nabla_x r_2 + \frac{\gamma-1}{2}r_2\mathrm{div}_x\mathbf{u}_2\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\Phi_R^{\mathbf{u}_1,Q_1} - \Phi_R^{\mathbf{u}_2,Q_2}\right)\partial_x^\alpha\left(\mathbf{u}_1\cdot\nabla_x r_1 + \frac{\gamma-1}{2} r_1 \mathrm{div}_x \mathbf{u}_1\right)dt\\
&\quad - \Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\Bigg(\mathbf{u}_2\cdot\nabla_x(r_1-r_2) + (\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla_x r_1\\
&\qquad\qquad + \frac{\gamma-1}{2}(r_1-r_2)\mathrm{div}_x\mathbf{u}_1 + \frac{\gamma-1}{2} r_2 \mathrm{div}_x(\mathbf{u}_1-\mathbf{u}_2)\Bigg)dt. \qquad (3.4.1)
\end{aligned}
$$

Multiplying (3.4.1) by $\partial_x^\alpha(r_1-r_2)$ and integrating over $\mathbb{T}$, then the highest order term can be treated as follows

$$
\begin{aligned}
&- \Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}\left(\mathbf{u}_2\cdot\nabla_x\partial_x^\alpha(r_1-r_2) + \frac{\gamma-1}{2} r_2 \mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\right)\cdot\partial_x^\alpha(r_1-r_2)dx\\
=&\ \frac{1}{2}\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}\mathrm{div}_x\mathbf{u}_2|\partial_x^\alpha(r_1-r_2)|^2dx - \frac{\gamma-1}{2}\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}} r_2\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(r_1-r_2)dx.
\end{aligned}
$$

From the smoothness of $\Phi$, the mean value theorem and the Sobolev embedding, we have for $s > \frac{3}{2}+3$

$$
\begin{aligned}
\left|\Phi_R^{\mathbf{u}_1,Q_1} - \Phi_R^{\mathbf{u}_2,Q_2}\right| &\le C(\|\mathbf{u}_1-\mathbf{u}_2\|_{2,\infty} + \|Q_1-Q_2\|_{3,\infty})\\
&\le C(\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2} + \|Q_1-Q_2\|_{s,2}). \qquad (3.4.2)
\end{aligned}
$$

Thus we get from above estimates

$$
\begin{aligned}
&\frac{1}{2}d\|\partial_x^\alpha(r_1-r_2)\|^2\\
&\le C(R)\left(1+\sum_{j=1}^2\|r_j,\mathbf{u}_j\|_{s,2}^2\right)(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2 + \|Q_1-Q_2\|_{s,2}^2)dt\\
&\quad - \frac{\gamma-1}{2}\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}} r_2\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(r_1-r_2)dxdt. \qquad (3.4.3)
\end{aligned}
$$

Similarly, for $\mathbf{u}_1$ and $\mathbf{u}_2$, we have the equation

$$
\begin{aligned}
&d\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\\
&= -\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\left(\mathbf{u}_1\cdot\nabla_x\mathbf{u}_1 + r_1\nabla_x r_1 - D(r_1)\mathcal{L}\mathbf{u}_1\right)dt\\
&\quad + \Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\left(\mathbf{u}_2\cdot\nabla_x\mathbf{u}_2 + r_2\nabla_x r_2 - D(r_2)\mathcal{L}\mathbf{u}_2\right)dt\\
&\quad - \Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\left(D(r_1)\mathrm{div}_x(L\nabla_x Q_1\odot\nabla_x Q_1 - \mathcal{F}(Q_1)\mathrm{I}_3 - L(Q_1\triangle Q_1 - \triangle Q_1 Q_1))\right)dt\\
&\quad + \Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\left(D(r_2)\mathrm{div}_x(L\nabla_x Q_2\odot\nabla_x Q_2 - \mathcal{F}(Q_2)\mathrm{I}_3 - L(Q_2\triangle Q_2 - \triangle Q_2 Q_2))\right)dt\\
&\quad + \Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)dW - \Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2)dW
\end{aligned}
$$

$$
\begin{aligned}
=&-\left(\Phi_R^{\mathbf{u}_1,Q_1}-\Phi_R^{\mathbf{u}_2,Q_2}\right)\partial_x^\alpha\big(\mathbf{u}_1\cdot\nabla_x\mathbf{u}_1+r_1\nabla_x r_1-D(r_1)\mathcal{L}\mathbf{u}_1\big)dt\\
&-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\big((\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla_x\mathbf{u}_1+\mathbf{u}_1\cdot\nabla_x(\mathbf{u}_1-\mathbf{u}_2)+(r_1-r_2)\nabla_x r_1+r_2\nabla_x(r_1-r_2)\\
&\qquad\qquad-(D(r_1)-D(r_2))\mathcal{L}\mathbf{u}_1-D(r_2)\mathcal{L}(\mathbf{u}_1-\mathbf{u}_2)\big)dt\\
&-\left(\Phi_R^{\mathbf{u}_1,Q_1}-\Phi_R^{\mathbf{u}_2,Q_2}\right)\partial_x^\alpha\Bigg(D(r_1)\mathrm{div}_x(L\nabla_x Q_1\odot\nabla_x Q_1-\mathcal{F}(Q_1)\mathrm{I}_3\\
&\qquad\qquad\qquad\qquad\qquad-L(Q_1\triangle Q_1-\triangle Q_1Q_1))\Bigg)dt\\
&-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\Bigg((D(r_1)-D(r_2))\,\mathrm{div}_x(L\nabla_x Q_1\odot\nabla_x Q_1-\mathcal{F}(Q_1)\mathrm{I}_3\\
&\qquad\qquad-L(Q_1\triangle Q_1-\triangle Q_1Q_1))\Bigg)dt\\
&-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\Bigg(D(r_2)\mathrm{div}_x(L\nabla_x(Q_1-Q_2)\odot\nabla_x Q_1+L\nabla_x Q_2\odot\nabla_x(Q_1-Q_2)\\
&\qquad\qquad-(\mathcal{F}(Q_1)-\mathcal{F}(Q_2))\mathrm{I}_3)\Bigg)dt\\
&+\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\Bigg(D(r_2)\mathrm{div}_x L(Q_1\triangle(Q_1-Q_2)-\triangle(Q_1-Q_2)Q_1+(Q_1-Q_2)\triangle Q_2\\
&\qquad\qquad+\triangle Q_2(Q_1-Q_2))\Bigg)dt\\
&+\left(\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2)\right)dW. \qquad (3.4.4)
\end{aligned}
$$

Applying the Itô formula to function $\frac{1}{2}\|\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\|^2$, then the high-order term in the formula reads

$$
\begin{aligned}
&-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}r_2\nabla_x\partial_x^\alpha(r_1-r_2)\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
=&\,\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}r_2\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(r_1-r_2)dx+\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}\nabla_x r_2\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\partial_x^\alpha(r_1-r_2)dx\\
\le&\, C(R)\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}r_2\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(r_1-r_2)dx.
\end{aligned}
$$

The last integral in above could be cancelled with the last term in (3.4.3) after matching the constant. Furthermore, integration by parts and the Hölder inequality give

$$
\begin{aligned}
&\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)\mathcal{L}(\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2))\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
=&-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)(\upsilon|\nabla_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2)dx
\end{aligned}
$$

$$
\begin{aligned}
&- \upsilon\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}\nabla_x D(r_2)\nabla_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
&- (\upsilon+\lambda)\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}\nabla_x D(r_2)\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
\le{}& C(R)\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2\\
&- \Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)(\upsilon|\nabla_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2)dx\\
&+ \frac{1}{4}\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)(\upsilon|\nabla_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2)dx,
\end{aligned}
$$

as well as using Lemma 3.1.3

$$
\begin{aligned}
&\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)L\mathrm{div}_x(Q_1\triangle\partial_x^\alpha(Q_1-Q_2)-\triangle\partial_x^\alpha(Q_1-Q_2)Q_1\\
&\qquad+(Q_1-Q_2)\triangle\partial_x^\alpha Q_2-\triangle\partial_x^\alpha Q_2(Q_1-Q_2))\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
={}&-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)L(Q_1\triangle\partial_x^\alpha(Q_1-Q_2)-\triangle\partial_x^\alpha(Q_1-Q_2)Q_1):\partial_x^\alpha\nabla_x(\mathbf{u}_1-\mathbf{u}_2)^{\mathrm{T}}dx\\
&-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}\nabla_x D(r_2)L(Q_1\triangle\partial_x^\alpha(Q_1-Q_2)-\triangle\partial_x^\alpha(Q_1-Q_2)Q_1)\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
&+\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)L\mathrm{div}_x((Q_1-Q_2)\triangle\partial_x^\alpha Q_2-\triangle\partial_x^\alpha Q_2(Q_1-Q_2))\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
\le{}&-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)L(\partial_x^\alpha(\Theta_1-\Theta_2)Q_1-Q_1\partial_x^\alpha(\Theta_1-\Theta_2)):\triangle\partial_x^\alpha(Q_1-Q_2)dx\\
&+C(R)\left(\sum_{j=1}^2\|Q_j\|_{s+2,2}^2\right)(\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)\\
&+\frac{\Gamma L}{8}\int_{\mathbb{T}}D(r_2)|\triangle\partial_x^\alpha(Q_1-Q_2)|^2dx.
\end{aligned}
$$

By Lemma 3.1.1, estimate (3.3.4), we have

$$
\begin{aligned}
&\int_{\mathbb{T}}\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\big((D(r_1)-D(r_2))\,L\mathrm{div}_x(Q_1\triangle Q_1-\triangle Q_1Q_1)\big)\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
\le{}&\Phi_R^{\mathbf{u}_2,Q_2}\left\|\partial_x^\alpha\big((D(r_1)-D(r_2))\,L\mathrm{div}_x(Q_1\triangle Q_1-\triangle Q_1Q_1)\big)\right\|\,\|\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\|\\
\le{}&C\Phi_R^{\mathbf{u}_2,Q_2}(\|r_1,r_2\|_{s,2}\|Q_1\|_{s,2}^2\|r_1-r_2\|_{s-1,2}+\|Q_1\|_{s,2}\|Q_1\|_{s+2,2}\|r_1,r_2\|_{s,2}\|r_1-r_2\|_{s-1,2})\\
&\qquad\times\|\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\|\\
\le{}&C(\|r_1,r_2\|_{s,2}\|Q_1\|_{s,2}^2+\|Q_1\|_{s,2}\|Q_1\|_{s+2,2}\|r_1,r_2\|_{s,2})\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2.
\end{aligned}
$$

Finally, by Lemma 3.1.1 and the Hölder inequality

$$
\begin{aligned}
&\int_{\mathbb{T}} \Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\big(D(r_2)\mathrm{div}_x(\mathrm{tr}^2(Q_1^2)\mathrm{I}_3-\mathrm{tr}^2(Q_2^2)\mathrm{I}_3)\big)\cdot\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)dx\\
&\le \Phi_R^{\mathbf{u}_2,Q_2}\|\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\|\left\|\partial_x^\alpha\big(D(r_2)\mathrm{div}_x(\mathrm{tr}^2(Q_1^2)\mathrm{I}_3-\mathrm{tr}^2(Q_2^2)\mathrm{I}_3)\big)\right\|\\
&\le \Phi_R^{\mathbf{u}_2,Q_2}\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}\\
&\quad\times(\|D(r_2)\|_\infty\|\mathrm{tr}^2(Q_1^2)-\mathrm{tr}^2(Q_2^2)\|_{s,2}+\|D(r_2)\|_{s,2}\|\mathrm{tr}^2(Q_1^2)-\mathrm{tr}^2(Q_2^2)\|_{1,\infty})\\
&\le C(R)(1+\|Q_1\|_{1,\infty}^3)\|r_2\|_{s,2}\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}\|Q_1-Q_2\|_{s,2}\\
&\quad+C(R)(1+\|Q_1,Q_2\|_{s,2}^3)\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}\|Q_1-Q_2\|_{s,2}.
\end{aligned}
$$

Since the order of the rest of nonlinearity terms is lower than above, these terms can be handled using the same way, so we skip the details. In summary, we could get

$$
\begin{aligned}
&\frac{1}{2}d\|\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\|^2\\
&\quad+\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}} D(r_2)(\upsilon|\nabla_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2+(\upsilon+\lambda)|\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)|^2)dxdt\\
&\le C(R)\sum_{j=1}^2(1+\|r_j,u_j\|_{s,2}^2+\|Q_j\|_{s+1,2}^3)(1+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2)\\
&\quad\times(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)dt\\
&\quad+\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}} r_2\mathrm{div}_x\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\cdot\partial_x^\alpha(r_1-r_2)dxdt+\frac{\Gamma L}{2}\int_{\mathbb{T}} D(r_2)|\triangle\partial_x^\alpha(Q_1-Q_2)|^2dxdt\\
&\quad-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}} D(r_2)L(\partial_x^\alpha(\Theta_1-\Theta_2)Q_1-Q_1\partial_x^\alpha(\Theta_1-\Theta_2)):\triangle\partial_x^\alpha(Q_1-Q_2)dxdt\\
&\quad+\left(\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2),\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\right)dW\\
&\quad+\frac{1}{2}\left\|\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2)\right\|_{L_2(\mathfrak{U};L^2(\mathbb{T},\mathbb{R}^3))}^2dt. \qquad (3.4.5)
\end{aligned}
$$

The second and forth terms on the right side of (3.4.5) could be cancelled later. By assumptions (3.1.5),(3.1.6), we could handle

$$
\begin{aligned}
&\left\|\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2)\right\|_{L_2(\mathfrak{U};L^2(\mathbb{T},\mathbb{R}^3))}^2\\
&\le\left\|\left(\Phi_R^{\mathbf{u}_1,Q_1}-\Phi_R^{\mathbf{u}_2,Q_2}\right)\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)\right\|_{L_2(\mathfrak{U};L^2(\mathbb{T},\mathbb{R}^3))}^2\\
&\quad+\left\|\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha(\mathbb{F}(r_1,\mathbf{u}_1)-\mathbb{F}(r_2,\mathbf{u}_2))\right\|_{L_2(\mathfrak{U};L^2(\mathbb{T},\mathbb{R}^3))}^2
\end{aligned}
$$

$$\leq C(R)\sum_{i=1}^{2}(1+\|r_i,\mathbf{u}_i\|_s^2)(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2).$$

For the $Q$-tensor equation, we also get

$$\begin{aligned}
&d\partial_x^\alpha(Q_1-Q_2)-\Gamma L\triangle\partial_x^\alpha(Q_1-Q_2)dt\\
=&-\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha(\mathbf{u}_1\cdot\nabla_x Q_1-\Theta_1Q_1+Q_1\Theta_1-\mathcal{K}(Q_1))dt\\
&+\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha(\mathbf{u}_2\cdot\nabla_x Q_2-\Theta_2Q_2+Q_2\Theta_2-\mathcal{K}(Q_2))dt\\
=&-\left(\Phi_R^{\mathbf{u}_1,Q_1}-\Phi_R^{\mathbf{u}_2,Q_2}\right)\partial_x^\alpha(\mathbf{u}_1\cdot\nabla_x Q_1-\Theta_1Q_1+Q_1\Theta_1-\mathcal{K}(Q_1))dt\\
&-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha((\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla_x Q_1+\mathbf{u}_2\cdot\nabla_x(Q_1-Q_2))dt\\
&-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha((\Theta_1-\Theta_2)Q_1-Q_1(\Theta_1-\Theta_2)+\Theta_2(Q_1-Q_2)-(Q_1-Q_2)\Theta_2\\
&\qquad+\mathcal{K}(Q_1)-\mathcal{K}(Q_2))dt.
\end{aligned}\tag{3.4.6}$$

Multiplying (3.4.6) by $-D(r_2)\partial_x^\alpha\triangle(Q_1-Q_2)$ on both sides, taking the trace and integrating over $\mathbb{T}$, as the a priori estimates, we consider the first term

$$\begin{aligned}
&-\int_{\mathbb{T}}\partial_x^\alpha(Q_1-Q_2)_t:D(r_2)\partial_x^\alpha\triangle(Q_1-Q_2)dx\\
=&\frac{1}{2}\partial_t\int_{\mathbb{T}}D(r_2)|\partial_x^\alpha\nabla_x(Q_1-Q_2)|^2dx-\frac{1}{2}\int_{\mathbb{T}}D(r_2)_t|\partial_x^\alpha\nabla_x(Q_1-Q_2)|^2dx\\
&+\int_{\mathbb{T}}\nabla_x D(r_2)\partial_x^\alpha\nabla_x(Q_1-Q_2):\partial_x^\alpha(Q_1-Q_2)_tdx.
\end{aligned}$$

Using (3.3.42) once more, similar estimate as (3.3.43) , estimate (3.3.1) and Lemma 3.1.1, the Hölder inequality

$$\begin{aligned}
&\left|\frac{1}{2}\int_{\mathbb{T}}D(r_2)_t|\partial_x^\alpha\nabla_x(Q_1-Q_2)|^2dx\right|\leq C(R)\|Q_1-Q_2\|_{s,2}^2,\\
&\left|\int_{\mathbb{T}}\nabla_x D(r_2)\partial_x^\alpha\nabla_x(Q_1-Q_2):\partial_x^\alpha(Q_1-Q_2)_tdx\right|\\
\leq&\, C(R)\|Q_1-Q_2\|_{s,2}\Bigg(\|Q_1-Q_2\|_{s+1,2}+\Phi_R^{\mathbf{u}_2,Q_2}\|Q_1\|_{s,2}\|\mathbf{u}_1-\mathbf{u}_2\|_{s,2}\\
&\qquad+\sum_{j=1}^{2}(\|\mathbf{u}_j\|_{s+1,2}+\|Q_j\|_{s+2,2})(\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}+\|Q_1-Q_2\|_{s,2})\Bigg)\\
\leq&\,\frac{\Gamma L}{8}\int_{\mathbb{T}}D(r_2)|\triangle\partial_x^\alpha(Q_1-Q_2)|^2dx+\frac{\upsilon}{8}\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)|\partial_x^s(\mathbf{u}_1-\mathbf{u}_2)|^2dx
\end{aligned}$$

$$+ C(R)\sum_{j=1}^{2}(1+\|Q_j\|_{s,2}^2+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2)(\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2).$$

We rewrite the highest-order term in (3.4.6) as

$$\begin{aligned}
&-\int_{\mathbb{T}}\Phi_R^{\mathbf{u}_2,Q_2}(\partial_x^\alpha(\Theta_1-\Theta_2)Q_1-Q_1\partial_x^\alpha(\Theta_1-\Theta_2)):(-D(r_2)\partial_x^\alpha\triangle(Q_1-Q_2))dx\\
&=\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)(\partial_x^\alpha(\Theta_1-\Theta_2)Q_1-Q_1\partial_x^\alpha(\Theta_1-\Theta_2)):\triangle\partial_x^\alpha(Q_1-Q_2)dx,
\end{aligned}$$

which can be cancelled with the forth term on the right hand side of (3.4.5). Again, by Lemma 3.1.1, (3.4.2) and the Hölder inequality

$$\begin{aligned}
&\left(\Phi_R^{\mathbf{u}_1,Q_1}-\Phi_R^{\mathbf{u}_2,Q_2}\right)\int_{\mathbb{T}}\partial_x^\alpha(\mathbf{u}_1\cdot\nabla_xQ_1-\Theta_1Q_1+Q_1\Theta_1-\mathcal{K}(Q_1)):D(r_2)\partial_x^\alpha\triangle(Q_1-Q_2)dx\\
&\le\frac{\Gamma L}{8}\int_{\mathbb{T}}D(r_2)|\triangle\partial_x^\alpha(Q_1-Q_2)|^2dx+C\|\mathbf{u}_1\|_{s,2}^2\|Q_1\|_{s,2}^2(\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2).
\end{aligned}$$

After all the estimates we could have

$$\begin{aligned}
&\frac{1}{2}d\|\sqrt{D(r_2)}\partial_x^{\alpha+1}(Q_1-Q_2)\|^2+\Gamma L\int_{\mathbb{T}}D(r_2)|\triangle\partial_x^\alpha(Q_1-Q_2)|^2dxdt\\
&\le C\sum_{j=1}^{2}(1+\|\mathbf{u}_j\|_{s,2}^2+\|Q_j\|_{s+1,2}^2)(1+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2)\\
&\quad\times(\|\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)dt\\
&\quad-\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)(\partial_x^\alpha(\Theta_1-\Theta_2)Q_1-Q_1\partial_x^\alpha(\Theta_1-\Theta_2)):\triangle\partial_x^\alpha(Q_1-Q_2)dxdt\\
&\quad+\frac{\Gamma L}{4}\int_{\mathbb{T}}D(r_2)|\triangle\partial_x^\alpha(Q_1-Q_2)|^2dxdt+\frac{\upsilon}{4}\Phi_R^{\mathbf{u}_2,Q_2}\int_{\mathbb{T}}D(r_2)|\partial_x^s(\mathbf{u}_1-\mathbf{u}_2)|^2dxdt. \qquad (3.4.7)
\end{aligned}$$

Adding (3.4.3), (3.4.5) and (3.4.7), taking sum for $|\alpha|\le s-1$, also using the fact that $\frac{1}{C(R)}\le D(r_2)\le C(R)$, then the following holds

$$\begin{aligned}
&d(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)\\
&\le C(R)\sum_{j=1}^{2}(1+\|r_j,\mathbf{u}_j\|_{s,2}^2+\|Q_j\|_{s+1,2}^3)(1+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2)\\
&\quad\times(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)dt\\
&\quad+C\sum_{|\alpha|\le s-1}\left(\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2),\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\right)dW.
\end{aligned}$$

Denote

$$G(t)=C(R)\sum_{j=1}^{2}(1+\|r_j,\mathbf{u}_j\|_{s,2}^2+\|Q_j\|_{s+1,2}^3)(1+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2).$$

Then we could apply the Itô product formula to function

$$\exp\left(-\int_0^t G(\tau)d\tau\right)(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2),$$

obtaining

$$\begin{aligned}
&d\left[\exp\left(-\int_0^t G(\tau)d\tau\right)(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)\right]\\
&=\left[-G(t)\exp\left(-\int_0^t G(\tau)d\tau\right)(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)\right]dt\\
&\quad+\exp\left(-\int_0^t G(\tau)d\tau\right)d(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)\\
&\le C(R)\sum_{|\alpha|\le s-1}\left(\Phi_R^{\mathbf{u}_1,Q_1}\partial_x^\alpha\mathbb{F}(r_1,\mathbf{u}_1)-\Phi_R^{\mathbf{u}_2,Q_2}\partial_x^\alpha\mathbb{F}(r_2,\mathbf{u}_2),\partial_x^\alpha(\mathbf{u}_1-\mathbf{u}_2)\right)dW\\
&\quad\times\exp\left(-\int_0^t G(\tau)d\tau\right).
\end{aligned}$$

Integrating on $[0,t]$ and then expectation, we have by the Gronwall lemma

$$\mathbb{E}\left[\exp\left(-\int_0^t G(\tau)d\tau\right)(\|r_1-r_2,\mathbf{u}_1-\mathbf{u}_2\|_{s-1,2}^2+\|Q_1-Q_2\|_{s,2}^2)\right]=0.$$

Here, we use the fact that the stochastic integral term is a square integral martingale which its expectation vanishes. As

$$\exp\left(-\int_0^t G(\tau)d\tau\right)>0,\quad \mathbb{P}\text{ a.s.}$$

since

$$\begin{aligned}
&\int_0^t G(\tau)d\tau\\
&\le\sum_{j=1}^{2}\sup_{t\in[0,T]}(1+\|r_j,\mathbf{u}_j\|_{s,2}^2+\|Q_j\|_{s+1,2}^3)\int_0^T 1+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2dt\\
&\le C\sum_{j=1}^{2}\left[\sup_{t\in[0,T]}(1+\|r_j,\mathbf{u}_j\|_{s,2}^2+\|Q_j\|_{s+1,2}^3)^2+\left(\int_0^T 1+\|\mathbf{u}_j\|_{s+1,2}^2+\|Q_j\|_{s+2,2}^2dt\right)^2\right]\\
&<\infty,\quad \mathbb{P}\text{ a.s..}
\end{aligned}$$

We conclude that for any $t \in [0, T]$

$$\mathbb{E}\left(\|r_1 - r_2, \mathbf{u}_1 - \mathbf{u}_2\|_{s-1,2}^2 + \|Q_1 - Q_2\|_{s,2}^2\right) = 0,$$

then the pathwise uniqueness holds. □

From the uniqueness, we shall use the following Gyöngy-Krylov characterization which can be found in [40] to recover the convergence a.s. of the approximate solution on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Lemma 3.4.4.** *Let $X$ be a complete separable metric space and suppose that $\{Y_n\}_{n\geq 0}$ is a sequence of $X$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mu_{m,n}\}_{m,n\geq 1}$ be the set of joint laws of $\{Y_n\}_{n\geq 1}$, that is*

$$\mu_{m,n}(E) := \mathbb{P}\{(Y_n, Y_m) \in E\}, \quad E \in \mathcal{B}(X \times X).$$

*Then $\{Y_n\}_{n\geq 1}$ converges in probability if and only if for every subsequence of the joint probability laws $\{\mu_{m_k,n_k}\}_{k\geq 1}$, there exists a further subsequence that converges weakly to a probability measure $\mu$ such that*

$$\mu\{(u, v) \in X \times X : u = v\} = 1.$$

Next, we verify the condition for the above lemma is valid. Denote by $\mu_{n,m}$ the joint law of

$$(r_n, \mathbf{u}_n, Q_n; r_m, \mathbf{u}_m, Q_m) \quad \text{on the path space } \mathcal{X} = \mathcal{X}_r \times \mathcal{X}_{\mathbf{u}} \times \mathcal{X}_Q \times \mathcal{X}_r \times \mathcal{X}_{\mathbf{u}} \times \mathcal{X}_Q,$$

where $\{r_n, \mathbf{u}_n, Q_n; r_m, \mathbf{u}_m, Q_m\}_{n,m\geq 1}$ are two sequences of approximate solutions to system (3.2.3) relative to the given stochastic basis $\mathcal{S}$, and denote by $\mu_W$ the law of $W$ on $\mathcal{X}_W$. We introduce the extended phase space

$$\mathcal{X}^J = \mathcal{X} \times \mathcal{X}_W,$$

and denote by $\nu_{n,m}$ the joint law of $(r_n, \mathbf{u}_n, Q_n; r_m, \mathbf{u}_m, Q_m; W)$ on $\mathcal{X}^J$. Using a similar argument as the proof of the tightness in subsection 4.3, we obtain the following result.

**Proposition 3.4.5.** *The collection of joint laws $\{\nu_{m,n}\}_{n,m\geq 1}$ is tight on $\mathcal{X}^J$.*

For any subsequence $\{\nu_{n_k,m_k}\}_{k\geq 1}$, there exists a measure $\nu$ such that $\{\nu_{n_k,m_k}\}_{k\geq 1}$ converges to $\nu$. Applying the Skorokhod representation theorem A.0.7, we have a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{X}^J$-valued random variables

$$(\tilde{r}_{n_k}, \tilde{\mathbf{u}}_{n_k}, \tilde{Q}_{n_k}; \tilde{r}_{m_k}, \tilde{\mathbf{u}}_{m_k}, \tilde{Q}_{m_k}; \tilde{W}_k) \text{ and } (\tilde{r}_1, \tilde{\mathbf{u}}_1, \tilde{Q}_1; \tilde{r}_2, \tilde{\mathbf{u}}_2, \tilde{Q}_2; \tilde{W})$$

such that

$$\begin{aligned}
&\tilde{\mathbb{P}}\{(\tilde{r}_{n_k}, \tilde{\mathbf{u}}_{n_k}, \tilde{Q}_{n_k}; \tilde{r}_{m_k}, \tilde{\mathbf{u}}_{m_k}, \tilde{Q}_{m_k}; \tilde{W}_k) \in \cdot\} = \nu_{n_k,m_k}(\cdot), \\
&\tilde{\mathbb{P}}\{(\tilde{r}_1, \tilde{\mathbf{u}}_1, \tilde{Q}_1; \tilde{r}_2, \tilde{\mathbf{u}}_2, \tilde{Q}_2; \tilde{W}) \in \cdot\} = \nu(\cdot)
\end{aligned}$$

and

$$(\tilde{r}_{n_k}, \tilde{\mathbf{u}}_{n_k}, \tilde{Q}_{n_k}; \tilde{r}_{m_k}, \tilde{\mathbf{u}}_{m_k}, \tilde{Q}_{m_k}; \tilde{W}_k) \to (\tilde{r}_1, \tilde{\mathbf{u}}_1, \tilde{Q}_1; \tilde{r}_2, \tilde{\mathbf{u}}_2, \tilde{Q}_2; \tilde{W}), \quad \tilde{\mathbb{P}} \text{ a.s.}$$

in the topology of $\mathcal{X}^J$. Analogously, this argument can be applied to both

$$(\tilde{r}_{n_k}, \tilde{\mathbf{u}}_{n_k}, \tilde{Q}_{n_k}, \tilde{W}_k), \quad (\tilde{r}_1, \tilde{\mathbf{u}}_1, \tilde{Q}_1, \tilde{W}) \quad \text{and} \quad (\tilde{r}_{m_k}, \tilde{\mathbf{u}}_{m_k}, \tilde{Q}_{m_k}, \tilde{W}_k), \quad (\tilde{r}_2, \tilde{\mathbf{u}}_2, \tilde{Q}_2, \tilde{W})$$

to show that $(\tilde{r}_1, \tilde{\mathbf{u}}_1, \tilde{Q}_1, \tilde{W})$ and $(\tilde{r}_2, \tilde{\mathbf{u}}_2, \tilde{Q}_2, \tilde{W})$ are two martingale solutions relative to the same stochastic basis $\widetilde{\mathcal{S}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{W})$.

In addition, we have $\mu_{n,m} \rightharpoonup \mu$ where $\mu$ is defined by

$$\mu(\cdot) = \tilde{\mathbb{P}}\{(\tilde{r}_1, \tilde{\mathbf{u}}_1, \tilde{Q}_1; \tilde{r}_2, \tilde{\mathbf{u}}_2, \tilde{Q}_2) \in \cdot\}.$$

Proposition 3.4.3 implies that $\mu\{(r_1, \mathbf{u}_1, Q_1; r_2, \mathbf{u}_2, Q_2) \in \mathcal{X} : (r_1, \mathbf{u}_1, Q_1) = (r_2, \mathbf{u}_2, Q_2)\} = 1$. Also since $W^{s,2} \subset W^{s-1,2}$, uniqueness in $W^{s-1,2}$ implies uniqueness in $W^{s,2}$. Therefore, Lemma 3.4.4 can be used to deduce that the sequence $(r_n, \mathbf{u}_n, Q_n)$ defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ converges a.s. in the topology of $\mathcal{X}_r \times \mathcal{X}_{\mathbf{u}} \times \mathcal{X}_Q$ to random variable $(r, \mathbf{u}, Q)$.

Again by the same argument as in subsection 4.4, we get the Theorem 3.4.2 in the sense of Definition 3.4.1.

## 3.5 Proof of Theorem 3.1.7.

In the process of proving the Theorem 3.4.2, it is worth noting that due to technical reason, we assume that the initial data is integrable with respect to the random element $\omega$, and that the density is uniformly bounded from below. Next, based on the Theorem 3.4.2, we are able to remove these restrictions on the initial data and discuss the general case, thus the proof of the main Theorem 3.1.7 will be completed.

We start with the proof of the existence of the strong pathwise solution, which is divided into three steps. For the first step, we show the existence of the strong pathwise solution under the assumption that the initial data satisfies

$$\rho_0 > \underline{\rho} > 0,\ \|\rho_0\|_{s,2} \leq M,\ \|\mathbf{u}_0\|_{s,2} \leq M,\ \|Q_0\|_{s+1,2} \leq M,\ Q_0 \in S_0^3, \tag{3.5.1}$$

for a fixed constant $M > 0$ such that $R > \mathcal{C}M$, where $\mathcal{C}$ is a constant satisfying

$$\|\mathbf{u}\|_{2,\infty} < \mathcal{C}\|\mathbf{u}\|_{s-1,2}, \|Q\|_{3,\infty} < \mathcal{C}\|Q\|_{s,2}.$$

Introduce a stopping time $\tau_R = \tau_R^1 \wedge \tau_R^2$, where

$$\tau_R^1 = \inf\left\{t \in [0,T];\ \sup_{\gamma\in[0,t]} \|\mathbf{u}_R\|_{2,\infty} \geq R\right\},\ \tau_R^2 = \inf\left\{t \in [0,T];\ \sup_{\gamma\in[0,t]} \|Q_R\|_{3,\infty} \geq R\right\}.$$

If two sets are empty, choosing $\tau_R^i = T, i = 1,2$. The fact that $\mathbf{u}, Q$ having continuous trajectories in $W^{s-1,2}(\mathbb{T},\mathbb{R}^3)$ and in $W^{s,2}(\mathbb{T}, S_0^3)$ for integer $s > \frac{9}{2}$ respectively and the Sobolev embedding $W^{s,2} \hookrightarrow W^{\alpha,\infty}$ for $s > \frac{3}{2} + \alpha$, $\mathbb{P}$ a.s. guarantee the well-defined of $\tau_R$ and strictly positive $\mathbb{P}$ a.s..

Since $r_R(t,\cdot) \geq \mathcal{C}(R) > 0$, $\mathbb{P}$ a.s. for all $t \in [0,T]$, we could construct a local strong pathwise solution $(\rho_R, \mathbf{u}_R, Q_R, \tau_R)$ of system (3.0.2), based on the existence of unique pathwise solution $(r_R, \mathbf{u}_R, Q_R)$ of the truncated system (3.2.3) with initial data conditions (3.5.1), where $\rho_R = \left(\frac{\gamma-1}{2A\gamma}\right)^{\frac{1}{\gamma-1}} r_R^{\frac{2}{\gamma-1}}$.

For the second step, we drop the auxiliary boundedness assumption of the initial data following the ideas of [39]. For the solution $(r_R, \mathbf{u}_R, Q_R)$ of the system (3.2.3), define the following stopping time

$$
\begin{aligned}
\tau_M^1 &= \inf\left\{ t \in [0,T];\ \sup_{\gamma\in[0,t]} \|\mathbf{u}_R\|_{s,2} \geq M \right\},\\
\tau_M^2 &= \inf\left\{ t \in [0,T];\ \sup_{\gamma\in[0,t]} \|Q_R\|_{s+1,2} \geq M \right\},\\
\tau_M^3 &= \inf\left\{ t \in [0,T];\ \sup_{\gamma\in[0,t]} \|r_R\|_{s,2} \geq M \right\},\\
\tau_M^4 &= \inf\left\{ t \in [0,T]; \inf_{x\in\mathbb{T}} r_R(t) \leq \frac{1}{M} \right\},
\end{aligned}
$$

where $M$ relies on $R$ such that $M \to \infty$ as $R \to \infty$ and $M \leq \min\left(\frac{R}{C}, R\right)$. Then we could define $\tau_M = \tau_M^1 \wedge \tau_M^2 \wedge \tau_M^3 \wedge \tau_M^4$, such that in $[0, \tau_M]$, again using the Sobolev embedding $W^{s,2} \hookrightarrow W^{\alpha,\infty}$ for $s > \frac{3}{2} + \alpha$, $\mathbb{P}$ a.s., we have

$$
\begin{aligned}
&\sup_{t\in[0,\tau_M]} \|r_R(t)\|_{1,\infty} < R, \quad \sup_{t\in[0,\tau_M]} \|\mathbf{u}_R(t)\|_{2,\infty} < R,\\
&\sup_{t\in[0,\tau_M]} \|Q_R(t)\|_{3,\infty} < R, \quad \inf_{t\in[0,\tau_M]} \inf_{\mathbb{T}} r_R(t) > \frac{1}{R}.
\end{aligned}
$$

According to the Theorem 3.4.2, we could construct the solution with respect to the stopping time $\tau_M$ for the general data. Indeed, define

$$
\begin{aligned}
\Sigma_M = \Big\{ &(r, \mathbf{u}, Q) \in W^{s,2}(\mathbb{T}) \times W^{s,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s+1,2}(\mathbb{T}, S_0^3) :\\
&\|r(t)\|_{s,2} < M, \|\mathbf{u}(t)\|_{s,2} < M, \|Q(t)\|_{s+1,2} < M, r(t) > \frac{1}{M} \Big\},
\end{aligned}
$$

then, we have there exists a unique solution $(r_M, \mathbf{u}_M, Q_M)$ to system (3.2.2) with the initial data $(r_0, \mathbf{u}_0, Q_0)\mathbf{1}_{(r_0,\mathbf{u}_0,Q_0)\in\Sigma_M\setminus\cup_{j=1}^{M-1}\Sigma_j}$, which is also a solution to the original system (3.0.2) with the stopping time $\tau_M$.

Define

$$
\begin{aligned}
\tau &= \sum_{M=1}^{\infty} \tau_M \mathbf{1}_{(r_0,\mathbf{u}_0,Q_0)\in\Sigma_M\setminus\cup_{j=1}^{M-1}\Sigma_j},\\
(r, \mathbf{u}, Q) &= \sum_{M=1}^{\infty} (r_M, \mathbf{u}_M, Q_M)\mathbf{1}_{(r_0,\mathbf{u}_0,Q_0)\in\Sigma_M\setminus\cup_{j=1}^{M-1}\Sigma_j}.
\end{aligned}
$$

Using the same argument as [38, Proposition 4.2], we infer that the $(r, \mathbf{u}, Q, \tau)$ is a solution to system (3.2.2) with the initial condition $(r_0, \mathbf{u}_0, Q_0)$ being $\mathcal{F}_0$-measurable random variable, with values in $W^{s,2}(\mathbb{T}) \times W^{s,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s+1,2}(\mathbb{T}, S_0^3)$ and $r_0 > 0$, $\mathbb{P}$ a.s..

Next, we show $(r, \mathbf{u}, Q)$ has continuous trajectory in the space $W^{s,2}(\mathbb{T}) \times W^{s-1,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s,2}(\mathbb{T}, S_0^3)$, $\mathbb{P}$ a.s.. Define

$$\Omega_M = \left\{ \omega \in \Omega : \|r_0(\omega)\|_{s,2} < M, \|\mathbf{u}_0(\omega)\|_{s,2} < M, \|Q_0(\omega)\|_{s+1,2} < M, r_0(\omega) > \frac{1}{M} \right\}.$$

Observe that $\bigcup_{M=1}^{\infty} \Omega_M = \Omega$. Therefore, for any $\omega \in \Omega$, there exists a set $\Omega_M$ such that $\omega \in \Omega_M$, and by the construction, we have $(r, \mathbf{u}, Q)(\omega) = (r_M, \mathbf{u}_M, Q_M)(\omega)$. Since $(r_M, \mathbf{u}_M, Q_M)$ has continuous trajectories in $W^{s,2}(\mathbb{T}) \times W^{s-1,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s,2}(\mathbb{T}, S_0^3)$ and $r_M(t \wedge \tau_M, \cdot) > \mathcal{C}(M)$, $\mathbb{P}$ a.s. for all $t \in [0, T]$, then we deduce that $(r, \mathbf{u}, Q)$ has continuous trajectories in $W^{s,2}(\mathbb{T}) \times W^{s-1,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s,2}(\mathbb{T}, S_0^3)$, $\mathbb{P}$ a.s. and $r(t \wedge \tau, \cdot) > 0$, $\mathbb{P}$ a.s. for all $t \in [0, T]$. In addition, for the fixed $\omega$, we have $\Phi_R^{\mathbf{u},Q} = 1$ on $[0, \tau_M(\omega)]$, thus $\mathbf{u}_M \mathbf{1}_{t \le \tau_M} \in L^2(0, T; W^{s+1,2}(\mathbb{T}, \mathbb{R}^3))$, then by the construction, we deduce that $\mathbf{u}\mathbf{1}_{t \le \tau} \in L^2(0, T; W^{s+1,2}(\mathbb{T}, \mathbb{R}^3))$, $\mathbb{P}$ a.s..

Finally, since $r(t \wedge \tau, \cdot) > 0$, $\mathbb{P}$ a.s. for all $t \in [0, T]$, after a transformation, we summarize that if $(\rho_0, \mathbf{u}_0, Q_0)$ just lies in $W^{s,2}(\mathbb{T}) \times W^{s,2}(\mathbb{T}, \mathbb{R}^3) \times W^{s+1,2}(\mathbb{T}, S_0^3)$ and $\rho_0 > 0$, $\mathbb{P}$ a.s. this means dropping the integrability with respect to $\omega$ and the positive lower bound of $\rho_0$, we establish the existence of a local strong pathwise solution $(\rho, \mathbf{u}, Q)$ to system (3.0.2) in the sense of Definition 3.1.4, up to a stopping time $\tau$ which is strictly positive, $\mathbb{P}$ a.s..

The final step would be constructing the maximal strong solutions. That is, extending the strong solution $(\rho, \mathbf{u}, Q)$ to a maximal existence time $\mathfrak{t}$. The proof is standard, so we refer the reader to [19, 39, 70] for details.

Regarding the proof of uniqueness to Theorem 3.1.7, first, under the assumption (3.5.1), we could prove the uniqueness result by introducing a stopping time and applying the pathwise uniqueness result derived before. Then, we can remove the extra assumption on the initial data by a same cutting argument as above. This completes the proof of Theorem 3.1.7.

## 4.0 Incompressible Limit of the Compressible Q-tensor System of Liquid Crystals

In this paper, we are focused on the incompressible limit of the three-dimensional Navier-Stokes equations coupled with the Q-tensor equation in this paper. By [37, 27], the equations of the compressible nematic liquid crystal model have the following form:

$$\begin{cases} \tilde{\rho}_t + \mathrm{div}_x(\tilde{\rho}\tilde{\mathbf{u}}) = 0, \\ (\tilde{\rho}\tilde{\mathbf{u}})_t + \mathrm{div}_x(\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \nabla_x A\tilde{\rho}^\gamma = \tilde{\nu}\triangle\tilde{\mathbf{u}} - \mathrm{div}_x(L\nabla_x\tilde{Q} \odot \nabla_x\tilde{Q} - \mathcal{F}(\tilde{Q})I_3) \\ \qquad\qquad\qquad\qquad + L\mathrm{div}_x(\tilde{Q}\mathcal{H}(\tilde{Q}) - \mathcal{H}(\tilde{Q})\tilde{Q}), \\ \tilde{Q}_t + \tilde{\mathbf{u}} \cdot \nabla_x\tilde{Q} - \tilde{\Theta}\tilde{Q} + \tilde{Q}\tilde{\Theta} = \tilde{\Gamma}\mathcal{H}(\tilde{Q}), \end{cases} \tag{4.0.1}$$

where $\tilde{\rho} > 0$ denotes the density, $\tilde{\mathbf{u}}$ denotes the velocity, and $\tilde{Q}$ is a symmetric traceless $3\times3$ matrix denoting the nematic tensor order. For the pressure $A\tilde{\rho}^\gamma$, we require that $A > 0$ and $\gamma > 1$. The constant $\tilde{\nu} > 0$ denotes the viscosity. We remark that the term $\nabla\mathrm{div}_x\tilde{\mathbf{u}}$ in the Lamé operator is dropped in the momentum equation for the simplicity of presentation since it can be treated easily. The term $\nabla_x\tilde{Q} \odot \nabla_x\tilde{Q}$ is a $3\times3$ matrix, and its value on the $(i,j)$-th entry is

$$[\nabla_x\tilde{Q} \odot \nabla_x\tilde{Q}]_{ij} = \sum_{k,l=1}^{3} \partial_i\tilde{Q}_{kl}\partial_j\tilde{Q}_{kl}.$$

The term $\mathrm{I}_3$ stands for the $3 \times 3$ identity matrix. In (4.0.1), the free energy density of the director field $\mathcal{F}(\tilde{Q})$ is

$$\mathcal{F}(\tilde{Q}) = \frac{L}{2}|\nabla_x\tilde{Q}|^2 + \frac{a}{2}\mathrm{tr}(\tilde{Q}^2) - \frac{b}{3}\mathrm{tr}(\tilde{Q}^3) + \frac{c}{4}\mathrm{tr}^2(\tilde{Q}^2),$$

and we denote

$$\mathcal{H}(\tilde{Q}) = L\triangle\tilde{Q} - a\tilde{Q} + b\left[\tilde{Q}^2 - \frac{\mathrm{I}_3}{3}\mathrm{tr}(\tilde{Q}^2)\right] - c\tilde{Q}\mathrm{tr}(\tilde{Q}^2).$$

The numbers $L$, $\tilde{\Gamma}$, $a$, $b$ and $c$ are often called as elastic constants with: $L > 0$, $\tilde{\Gamma} > 0$, $a \in \mathbb{R}$, $b > 0$ and $c > 0$, and these coefficients are dependent on the material. The term

$\tilde{\Theta} = \frac{\nabla_x \tilde{\mathbf{u}} - \nabla_x^{\mathsf{T}} \tilde{\mathbf{u}}}{2}$ is the skew-symmetric part of the rate of strain tensor, note that the notation "$\mathsf{T}$" represents the transpose. From the structure of $\mathcal{H}(\tilde{Q})$, we remark that

$$\tilde{Q}\mathcal{H}(\tilde{Q}) - \mathcal{H}(\tilde{Q})\tilde{Q} = \tilde{Q}\triangle\tilde{Q} - \triangle\tilde{Q}\tilde{Q}.$$

From the pointview of physics, when the density approaches a constant, the compressible flow behaves asymptotically like the incompressible flow. We can describe this phenomenon in the following way. For the compressible flow, we can define the Mach number as:

$$M = \frac{|\tilde{\mathbf{u}}|}{\sqrt{a\gamma\tilde{\rho}^{\gamma-1}}}.$$

In the case when the Mach number approaches zero, we expect that $\tilde{\rho}$ keeps the scale, without loss of generality we can assume the scale to be 1. So $\tilde{\mathbf{u}}$ and $\tilde{Q}$ are of the order $\varepsilon$, with $\varepsilon > 0$ and $\varepsilon$ could be infinitely small. The scaling of $\tilde{\rho}$, $\tilde{\mathbf{u}}$ and $\tilde{Q}$ is described as follows:

$$\tilde{\rho}(t,x) = \rho_\varepsilon(\varepsilon t, x), \quad \tilde{\mathbf{u}}(t,x) = \varepsilon \mathbf{u}_\varepsilon(\varepsilon t, x), \quad \tilde{Q}(t,x) = \varepsilon Q_\varepsilon(\varepsilon t, x).$$

The viscosity coefficient and the elastic coefficient are scaled as

$$\tilde{\nu} = \varepsilon \nu_\varepsilon, \quad \tilde{\Gamma} = \varepsilon \Gamma_\varepsilon, \quad \nu_\varepsilon \to \nu \text{ as } \varepsilon \to 0^+, \quad \Gamma_\varepsilon \to \Gamma \text{ as } \varepsilon \to 0^+.$$

Then the corresponding free energy is

$$\mathcal{F}_\varepsilon(Q_\varepsilon) = \frac{L}{2}|\nabla_x Q_\varepsilon|^2 + \frac{a}{2}\mathrm{tr}(Q_\varepsilon^2) - \frac{\varepsilon b}{3}\mathrm{tr}(Q_\varepsilon^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2(Q_\varepsilon^2).$$

We also have

$$\mathcal{H}_\varepsilon(Q_\varepsilon) = L\triangle Q_\varepsilon - aQ_\varepsilon + \varepsilon b\left[Q_\varepsilon^2 - \frac{\mathrm{I}_3}{3}\mathrm{tr}(Q_\varepsilon^2)\right] - \varepsilon^2 c Q_\varepsilon \mathrm{tr}(Q_\varepsilon^2).$$

After the above scaling, the system (4.0.1) yields

$$\begin{cases} (\rho_\varepsilon)_t + \mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon) = 0, \\ (\rho_\varepsilon \mathbf{u}_\varepsilon)_t + \mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x A\rho_\varepsilon^\gamma = \nu_\varepsilon \triangle \mathbf{u}_\varepsilon - \mathrm{div}_x(L\nabla_x Q_\varepsilon \odot \nabla_x Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon) I_3) \\ \qquad + L\mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon), \\ (Q_\varepsilon)_t + \mathbf{u}_\varepsilon \cdot \nabla_x Q_\varepsilon - \Theta_\varepsilon Q_\varepsilon + Q_\varepsilon \Theta_\varepsilon = \Gamma_\varepsilon \mathcal{H}_\varepsilon(Q_\varepsilon), \end{cases} \tag{4.0.2}$$

with $\Theta_\varepsilon = \frac{\nabla_x \mathbf{u}_\varepsilon - \nabla_x^\intercal \mathbf{u}_\varepsilon}{2}$. Considering our prior assumption, we assume that $\rho_\varepsilon^0$ is of the order $1+O(\varepsilon)$, and when $\varepsilon \to 0$, we can expect that $\rho_\varepsilon \to 1$. Furthermore, from $(4.0.2)_1$, it follows that $\mathrm{div}_x \mathbf{u}_\varepsilon \to 0$ in distribution, which implies the fact that the fluid is incompressible. Then the corresponding incompressible system reads

$$\begin{cases} \mathbf{u}_t + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \pi = \nu \triangle \mathbf{u} - \mathrm{div}_x(L \nabla_x Q \odot \nabla_x Q) + L\mathrm{div}_x(Q\triangle Q - \triangle Q Q), \\ Q_t + \mathbf{u} \cdot \nabla_x Q - \Theta Q + Q\Theta = \Gamma \mathcal{H}(Q), \\ \mathrm{div}_x \mathbf{u} = 0. \end{cases} \tag{4.0.3}$$

The term $\nabla_x \pi$ is the limit of $\nabla_x \frac{1}{\varepsilon^2} A \rho_\varepsilon^\gamma$ when $\varepsilon \to 0$. In this paper, we are devoted to the convergence of the above incompressible limit for the global weak solutions to the compressible equations of nematic liquid crystals in the periodic case. Note that the existence of the global weak solutions to the compressible model was established in [85].

## 4.1 The Deterministic Case

First, we consider the problem in the deterministic setting. To avoid the boundary layer, we only discuss the case of the periodic domain, that is, the case of the equations in the flat torus $\mathbb{T} = (-\pi, \pi)^3$. Recall the "div-curl" decomposition, define the two projectors $\mathcal{P}$ and $\mathcal{Q}$, such that for any $\mathbf{u} \in W^{k,p}(\mathbb{T})$, with $1 < p < \infty$ and $k \geq 0$, $\mathbf{u} = \mathcal{P}\mathbf{u} + \mathcal{Q}\mathbf{u}$, $\mathrm{div}_x \mathcal{P}\mathbf{u} = \mathrm{curl}\mathcal{Q}\mathbf{u} = 0$. Note that $\mathcal{P}$ and $\mathcal{Q}$ are both bounded linear operators in $W^{k,p}(\mathbb{T})$.

Now consider a sequence of weak solutions $\{\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon\}_{\varepsilon>0}$ to the system (4.0.2) in the periodic case, equipped with the following initial condition:

$$\rho_\varepsilon|_{t=0} = \rho_\varepsilon^0, \quad \rho_\varepsilon \mathbf{u}_\varepsilon|_{t=0} = \mathbf{m}_\varepsilon^0, \quad Q_\varepsilon|_{t=0} = Q_\varepsilon^0. \tag{4.1.1}$$

They satisfy the following conditions:

$$\begin{aligned}
&\rho_\varepsilon^0 \geq 0 \text{ a.e in } \mathbb{T}, \quad \frac{1}{\varepsilon^2}(\rho_\varepsilon^0)^\gamma \in L^1(\mathbb{T}),\\
&\mathbf{m}_\varepsilon^0 \in L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}), \quad \mathbf{m}_\varepsilon^0 = 0 \text{ a.e in } \{\rho_\varepsilon^0\},\\
&\frac{|\mathbf{m}_\varepsilon^0|^2}{\rho_\varepsilon^0} \in L^1(\mathbb{T}), \text{ and } \frac{|\mathbf{m}_\varepsilon^0|^2}{\rho_\varepsilon^0} = 0 \text{ a.e in } \{\rho_\varepsilon^0\},\\
&Q_\varepsilon^0 \in H^1(\mathbb{T}).
\end{aligned} \tag{4.1.2}$$

Letting $\varepsilon \to 0^+$, we can assume that

$$\begin{aligned}
&\frac{|\mathbf{m}_\varepsilon^0|}{\sqrt{\rho_\varepsilon^0}} \text{ converges weakly to some } \mathbf{u}_0 \text{ in } L^2(\mathbb{T}) \text{ as } \varepsilon \to 0^+,\\
&Q_\varepsilon^0 \text{ converges weakly to some } Q_0 \text{ in } L^2(\mathbb{T}) \text{ as } \varepsilon \to 0^+.
\end{aligned} \tag{4.1.3}$$

Moreover, we assume that the initial values are uniformly bounded:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}} \frac{|\mathbf{m}_\varepsilon^0|^2}{\rho_\varepsilon^0} dx + \frac{A}{\varepsilon^2(\gamma-1)}\int_{\mathbb{T}}((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + \gamma - 1)dx\\
&+\int_{\mathbb{T}}(\frac{L}{2}|\nabla_x Q_\varepsilon^0|^2 + \frac{a}{2}\mathrm{tr}((Q_\varepsilon^0)^2) - \frac{\varepsilon b}{3}\mathrm{tr}((Q_\varepsilon^0)^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2((Q_\varepsilon^0)^2)) \leq C.
\end{aligned} \tag{4.1.4}$$

For some constant $C > 0$. Note that from the fact that $\int_{\mathbb{T}}((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + \gamma - 1)dx \leq C\varepsilon^2$, and that the function $f(x) = x^\gamma$ is a convex function when $\gamma > 1$, then $\rho_\varepsilon^0$ is of order $1 + O(\varepsilon)$. According to [85], for any $\varepsilon > 0$, there exists a weak solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon)$ to the compressible flow (4.0.2) satisfying

$$\begin{aligned}
&\rho_\varepsilon \in L^\infty([0,T]; L^\gamma(\mathbb{T})),\\
&\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \in L^\infty([0,T]; L^2(\mathbb{T})),\\
&\mathbf{u}_\varepsilon \in L^2([0,T]; H^1(\mathbb{T})),\\
&Q_\varepsilon \in L^\infty([0,T]; H^1(\mathbb{T})) \cap L^2([0,T]; H^2(\mathbb{T})).
\end{aligned}$$

The main result of the deterministic part is presented as the following:

**Theorem 4.1.1.** *Assume that $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon)\}_{\varepsilon>0}$ is a sequence of weak solutions to the compressible flow of liquid crystals* (4.0.2)*, in the domain $\mathbb{T} \subset \mathbb{R}$ with the initial condition* (4.1.2)-(4.1.4)*, and $\gamma > \frac{3}{2}$, $a > \frac{b^2}{2c}$. Then for any $T > 0$, as $\varepsilon \to 0$, $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon)\}$ converges to a weak solution $(\mathbf{u}, Q)$ of the incompressible flow of liquid crystals* (4.0.3)*, with the initial data $\mathbf{u}|_{t=0} = \mathcal{P}\mathbf{u}_0$, $Q|_{t=0} = Q_0$ periodic. More precisely, as $\varepsilon \to 0$*

$$\begin{aligned}
&\rho_\varepsilon \to 1 \quad \text{in } L^\infty([0,T]; L^\kappa(\mathbb{T})),\\
&\mathbf{u}_\varepsilon \ \to \mathbf{u} \quad \text{weakly in } L^2([0,T]; H^1(\mathbb{T})),\\
&Q_\varepsilon \to Q \quad \text{strongly in } L^2([0,T]; H^1(\mathbb{T})) \quad \text{and weakly in } L^2([0,T]; H^2(\mathbb{T})).
\end{aligned}$$

### 4.1.1 Proof of theorem 2.1

To prove Theorem 4.1.1, we have taken the idea in [44], [60] and [87]. So we start from the energy estimate of (4.0.2). First, multiply $(4.0.2)_2$ with $\mathbf{u}_\varepsilon$, integrate over $\mathbb{T}$ and apply $(4.0.2)_1$, then it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}} (\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{A}{\varepsilon^2(\gamma-1)}\rho_\varepsilon^\gamma)dx + \nu_\varepsilon \int_{\mathbb{T}} |\nabla_x \mathbf{u}_\varepsilon|^2 dx\\
&= -\int_{\mathbb{T}} \mathrm{div}_x(L\nabla_x Q_\varepsilon \odot \nabla_x Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon) I_3)\mathbf{u}_\varepsilon dx + L\int_{\mathbb{T}} \mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon)\mathbf{u}_\varepsilon dx.
\end{aligned} \tag{4.1.5}$$

Next, multiply $(4.0.2)_3$ with $-\mathcal{H}_\varepsilon(Q_\varepsilon)$, take the trace then integrate over $\mathbb{T}$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}} (\frac{L}{2}|\nabla_x Q_\varepsilon|^2 + \frac{a}{2}\mathrm{tr}(Q_\varepsilon^2) - \frac{\varepsilon b}{3}\mathrm{tr}(Q_\varepsilon^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2(Q_\varepsilon^2))dx + \Gamma_\varepsilon \int_{\mathbb{T}} \mathrm{tr}(\mathcal{H}_\varepsilon^2(Q_\varepsilon))dx\\
&= \int_{\mathbb{T}} (\mathbf{u}_\varepsilon \cdot \nabla_x Q_\varepsilon) : \mathcal{H}_\varepsilon(Q_\varepsilon)dx - \int_{\mathbb{T}} (\Theta_\varepsilon Q_\varepsilon - Q_\varepsilon \Theta_\varepsilon) : \mathcal{H}_\varepsilon(Q_\varepsilon)dx.
\end{aligned} \tag{4.1.6}$$

Adding (4.1.5) and (4.1.6), we integrate with respect to time, then for any given $t \in [0,T]$, the following holds:

$$\begin{aligned}
&\int_{\mathbb{T}} (\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{A}{\varepsilon^2(\gamma-1)}(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1)\\
&\qquad + \frac{L}{2}|\nabla_x Q_\varepsilon|^2 + \frac{a}{2}\mathrm{tr}(Q_\varepsilon^2) - \frac{\varepsilon b}{3}\mathrm{tr}(Q_\varepsilon^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2(Q_\varepsilon^2))dx\\
&+ \nu_\varepsilon \int_0^t \int_{\mathbb{T}} |\nabla_x \mathbf{u}_\varepsilon|^2 dxdt + \Gamma_\varepsilon \int_0^t \int_{\mathbb{T}} \mathrm{tr}(\mathcal{H}_\varepsilon^2(Q_\varepsilon))dxdt\\
&= \int_{\mathbb{T}} (\frac{1}{2}\frac{|\mathbf{m}_\varepsilon^0|^2}{\rho_\varepsilon^0} + \frac{A}{\varepsilon^2(\gamma-1)}((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + \gamma - 1))dx\\
&+ \int_{\mathbb{T}} (\frac{L}{2}|\nabla_x Q_\varepsilon^0|^2 + \frac{a}{2}\mathrm{tr}((Q_\varepsilon^0)^2) - \frac{\varepsilon b}{3}\mathrm{tr}((Q_\varepsilon^0)^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2((Q_\varepsilon^0)^2)) \le C.
\end{aligned} \tag{4.1.7}$$

Considering the fact that $\mathrm{tr}(Q_\varepsilon) = 0$, assume that the eigenvalues of $Q_\varepsilon$ are $\lambda_\varepsilon^1$, $\lambda_\varepsilon^2$ and $\lambda_\varepsilon^3$, then $\mathrm{tr}(Q_\varepsilon) = \lambda_\varepsilon^1 + \lambda_\varepsilon^2 + \lambda_\varepsilon^3 = 0$. So we can get

$$\begin{aligned}
\mathrm{tr}(Q_\varepsilon^2) &= (\lambda_\varepsilon^1)^2 + (\lambda_\varepsilon^2)^2 + (\lambda_\varepsilon^3)^2 = 2((\lambda_\varepsilon^1)^2 + (\lambda_\varepsilon^2)^2 + \lambda_\varepsilon^1\lambda_\varepsilon^2),\\
\mathrm{tr}(Q_\varepsilon^3) &= (\lambda_\varepsilon^1)^3 + (\lambda_\varepsilon^2)^3 + (\lambda_\varepsilon^3)^3 = -3\lambda_\varepsilon^1\lambda_\varepsilon^2(\lambda_\varepsilon^1 + \lambda_\varepsilon^2),
\end{aligned}$$

as $|\lambda_\varepsilon^1\lambda_\varepsilon^2| \le \frac{1}{2}\mathrm{tr}(Q_\varepsilon^2)$, and $|\lambda_\varepsilon^1 + \lambda_\varepsilon^2| = |\lambda_\varepsilon^3| \le \sqrt{\mathrm{tr}(Q_\varepsilon^2)}$. So $|\mathrm{tr}(Q_\varepsilon^3)| \le (\mathrm{tr}(Q_\varepsilon^2))^{\frac{3}{2}} = |Q_\varepsilon|^3$, the following estimate holds

$$|-\frac{\varepsilon b}{3}\mathrm{tr}(Q_\varepsilon^3)| \le \frac{\varepsilon b}{2}|Q_\varepsilon|^3 \le \frac{\varepsilon^2 c}{4}|Q_\varepsilon|^4 + \frac{b^2}{4c}|Q_\varepsilon|^2.$$

Thus from the estimate (4.1.7), by the assumption $a > \frac{b^2}{2c}$,

$$\int_{\mathbb{T}}(\frac{L}{2}|\nabla_x Q_\varepsilon|^2 + (\frac{a}{2} - \frac{b^2}{4c})|Q_\varepsilon|^2) \le \int_{\mathbb{T}}(\frac{L}{2}|\nabla_x Q_\varepsilon|^2 + \frac{a}{2}\mathrm{tr}(Q_\varepsilon^2) - \frac{\varepsilon b}{3}\mathrm{tr}(Q_\varepsilon^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2(Q_\varepsilon^2))dx \le C.$$

So we can get the estimate that $Q_\varepsilon \in L^\infty([0,T]; H^1(\mathbb{T}))$. From the definition of $\mathcal{H}_\varepsilon(Q_\varepsilon))dx$, there is a further estimate about $Q_\varepsilon$,

$$\begin{aligned}
\int_{\mathbb{T}}|\triangle Q_\varepsilon|^2 dx \le& C(\int_{\mathbb{T}}\mathrm{tr}(\mathcal{H}_\varepsilon^2(Q_\varepsilon))dx + a^2\int_{\mathbb{T}}|Q_\varepsilon|^2 dx + b^2\int_{\mathbb{T}}|Q_\varepsilon|^4 dx + c^2\int_{\mathbb{T}}|Q_\varepsilon|^6 dx)\\
\le& C\int_{\mathbb{T}}\mathrm{tr}(\mathcal{H}_\varepsilon^2(Q_\varepsilon))dx + C(a,b,c)(\|Q_\varepsilon\|_{H^1}^6 + \|Q_\varepsilon\|_{H^1}^2).
\end{aligned}$$

Combing the above calculation and (4.1.7), the following estimate holds

$$\begin{aligned}
&\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \text{ is bounded in } L^\infty([0,T]; L^2(\mathbb{T})),\\
&Q_\varepsilon \text{ is bounded in } L^\infty([0,T]; H^1(\mathbb{T})),\\
&\frac{1}{\varepsilon^2}(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1) \text{ is bounded in } L^\infty([0,T]; L^1(\mathbb{T})),
\end{aligned} \tag{4.1.8}$$

together with

$$\begin{aligned}
&\nabla_x \mathbf{u}_\varepsilon \text{ is bounded in } L^2([0,T]; L^2(\mathbb{T})),\\
&\mathcal{H}_\varepsilon(Q_\varepsilon)) \text{ is bounded in } L^2([0,T]; L^2(\mathbb{T})),\\
&\triangle Q_\varepsilon \text{ is bounded in } L^2([0,T]; L^2(\mathbb{T})).
\end{aligned} \tag{4.1.9}$$

Note that all the bounds above are uniform with respect to $\varepsilon$, and hold for any $T > 0$. Consider the function $f(x) = x^\gamma$, as $\gamma > 1$, then the following fact stands true to any $x \geq 0$, with some $c_0 > 0$ and any positive number $R$,

$$
\begin{aligned}
&x^\gamma - \gamma x + \gamma - 1 \geq c_0|x-1|^2 \quad \text{if } \gamma \geq 2,\\
&x^\gamma - \gamma x + \gamma - 1 \geq c_0|x-1|^2 \quad \text{if } \gamma < 2 \ \text{ and } x \leq R,\\
&x^\gamma - \gamma x + \gamma - 1 \geq c_0|x-1|^\gamma \quad \text{if } \gamma < 2 \ \text{ and } x \geq R.
\end{aligned}
$$

Choose $R = \frac{1}{2}$, then from the uniform bounds of $\rho_\varepsilon$ in (4.1.8), it is easy to see that

$$
\begin{aligned}
&\int_{\mathbb{T}} (\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1)\chi_{|\rho_\varepsilon - 1| \leq \frac{1}{2}} dx + \int_{\mathbb{T}} (\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1)\chi_{|\rho_\varepsilon - 1| > \frac{1}{2}} dx\\
=& \int_{\mathbb{T}} (\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1) \leq C\varepsilon^2.
\end{aligned}
\tag{4.1.10}
$$

If $\gamma \geq 2$, then clearly we can get $\|\rho_\varepsilon\|_{L^2} \leq C\varepsilon$. If $\gamma < 2$, then

$$
\|\rho_\varepsilon\|_{L^\gamma} \leq \|\rho_\varepsilon \chi_{|\rho_\varepsilon - 1| \leq \frac{1}{2}}\|_{L^\gamma} + \|\rho_\varepsilon \chi_{|\rho_\varepsilon - 1| > \frac{1}{2}}\|_{L^\gamma} \leq C\varepsilon^{\frac{2}{\gamma}}.
$$

Denote $\kappa = \min\{2, \gamma\}$, then

$$
\rho_\varepsilon \to 1 \text{ in } C([0,T]; L^\kappa(\mathbb{T})) \qquad\qquad \text{as } \varepsilon \to 0. \tag{4.1.11}
$$

Next we split $\mathbf{u}_\varepsilon$ using the bounds of $\rho_\varepsilon$

$$
\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon \chi_{|\rho_\varepsilon - 1| \leq \frac{1}{2}} + \mathbf{u}_\varepsilon \chi_{|\rho_\varepsilon - 1| > \frac{1}{2}} = \mathbf{u}_\varepsilon^1 + \mathbf{u}_\varepsilon^2,
$$

from the estimate (4.1.8), it is obvious that

$$
\int_{\mathbb{T}} |\mathbf{u}_\varepsilon^1|^2 dx \leq 2\int_{\mathbb{T}} \rho_\varepsilon |\mathbf{u}_\varepsilon|^2 dx,
$$

and

$$
\int_{\mathbb{T}} |\mathbf{u}_\varepsilon^2|^2 dx \leq 2\int_{\mathbb{T}} |\rho_\varepsilon - 1||\mathbf{u}_\varepsilon|^2 dx \leq 2\|\rho_\varepsilon - 1\|_{L^\kappa} \|\mathbf{u}_\varepsilon\|^2_{L^{\frac{2\kappa}{\kappa-1}}} \leq C\varepsilon \|\mathbf{u}_\varepsilon\|^2_{L^{\frac{2\kappa}{\kappa-1}}},
$$

by embedding, as $H^1(\mathbb{T}) \hookrightarrow L^6(\mathbb{T})$, we require that $\frac{2\kappa}{\kappa-1} \leq 6$, namely $\gamma \geq \frac{3}{2}$. Then we have deprived that $\mathbf{u}_\varepsilon^1$ is bounded in $L^\infty([0,T]; L^2(\mathbb{T}))$ and that $\varepsilon^{-\frac{1}{2}}\mathbf{u}_\varepsilon^2$ is bounded in $L^2([0,T]; L^2(\mathbb{T}))$.

Consider the convergence of $\rho_\varepsilon$, and from the equation $(4.0.2)_1$ and the fact that $\mathrm{div}_x \mathbf{u}_\varepsilon$ is bounded in $L^2([0,T];L^2(\mathbb{T}))$, then

$$\mathrm{div}_x \mathbf{u}_\varepsilon \to 0 \quad \text{weakly in } L^2([0,T];L^2(\mathbb{T})),$$

by the elliptic equation theory, as $\nabla_x^2 Q_\varepsilon \in L^2([0,T];L^2(\mathbb{T}))$, thus $Q_\varepsilon \in L^2([0,T];H^2(\mathbb{T}))$. In summary the following convergence results hold

$$\begin{aligned}
&\mathbf{u}_\varepsilon \ \to \mathbf{u} \ \text{ weakly in } L^2([0,T];H^1(\mathbb{T})),\\
&\mathrm{div}_x \mathbf{u}_\varepsilon \to 0 \ \text{ weakly in } L^2([0,T];L^2(\mathbb{T})),\\
&\rho_\varepsilon \to 1 \ \text{ in } L^\infty([0,T];L^\kappa(\mathbb{T})),\\
&\triangle Q_\varepsilon \to \triangle Q \ \text{ weakly in } L^2([0,T];L^2(\mathbb{T})),\\
&Q_\varepsilon \to Q \ \text{ weakly in } L^\infty([0,T];H^1(\mathbb{T})) \cap L^2([0,T];H^2(\mathbb{T})),
\end{aligned}$$

in order to show that the convergence of $Q_\varepsilon$ is strong, we can make use of the Aubin-Lions compactness lemma (see [58]), the lemma reads as follows:

**Lemma 4.1.2.** *For $X_0$, $X$ and $X_1$ three Banach spaces, assume that $X_0$ and $X_1$ are reflexive, $X_0$ embeds in $X$ compactly and $X$ embeds in $X_1$ continuously. If for any constants $p$ and $q$, with $1 < p, q < \infty$, we have*

$$W = \left\{ u \in L^p([0,T];X_0) | \frac{du}{dt} \in L^q([0,T];X_1) \right\}.$$

*Then the embedding from $W$ into the space $L^p([0,T];X)$ is compact.*

From $(4.0.2)_3$, we can get the bounds of $\partial_t Q_\varepsilon$, that is

$$\begin{aligned}
\|\partial_t Q_\varepsilon\|_{L^{\frac{3}{2}}} &\le C(\|\mathbf{u}_\varepsilon \cdot \nabla_x Q_\varepsilon\|_{L^{\frac{3}{2}}} + \|\Theta_\varepsilon Q_\varepsilon\|_{L^{\frac{3}{2}}} + \|Q_\varepsilon \Theta_\varepsilon\|_{L^{\frac{3}{2}}} + \Gamma_\varepsilon \|\mathcal{H}_\varepsilon(Q_\varepsilon)\|_{L^{\frac{3}{2}}})\\
&\le C(\|\mathbf{u}_\varepsilon\|_{L^6} \|\nabla_x Q_\varepsilon\|_{L^2} + \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2} \|Q_\varepsilon\|_{L^6} + \|\mathcal{H}_\varepsilon(Q_\varepsilon)\|_{L^2}),
\end{aligned} \tag{4.1.12}$$

the estimate (4.1.7) yields that $\partial_t Q_\varepsilon$ is uniformly bounded in $L^2([0,T];L^{\frac{3}{2}}(\mathbb{T}))$. Consider the fact that $Q_\varepsilon$ is uniformly bounded in $L^2([0,T];H^2(\mathbb{T}))$, as $H^2(\mathbb{T})$ is compactly embedded in

$H^2(\mathbb{T})$ and $H^1(\mathbb{T})$ is continuously embedded in $L^{\frac{3}{2}}(\mathbb{T})$, by lemma 4.1.2, $Q_\varepsilon$ is precompact in $L^2([0,T];H^1(\mathbb{T}))$. Then by taking subsequence we have

$$\begin{aligned} &Q_\varepsilon \to Q \ \text{ weakly in } L^2([0,T];H^2(\mathbb{T})),\\ &Q_\varepsilon \to Q \ \text{ strongly in } L^2([0,T];H^1(\mathbb{T})). \end{aligned} \tag{4.1.13}$$

Also by (4.1.13), it is easy to deduce that

$$\begin{aligned} &\mathrm{div}_x(L\nabla_x Q_\varepsilon \odot \nabla_x Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon)I_3) - L\mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon)\\ &\qquad \to \mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)I_3) + L\mathrm{div}_x(Q\triangle Q - \triangle Q Q) \quad \text{as } \varepsilon \to 0, \end{aligned}$$

in distribution. Denote

$$\pi_\varepsilon = \frac{A}{\varepsilon^2}\rho_\varepsilon^\gamma - \mathcal{F}_\varepsilon(Q_\varepsilon).$$

So we can rewrite $(4.0.2)_2$ and apply the divergence-free projector on the equation, that is

$$\begin{aligned} &\partial_t \mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon) + \mathcal{P}(\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon))\\ =&\nu_\varepsilon \triangle \mathcal{P}\mathbf{u}_\varepsilon - \mathcal{P}(\mathrm{div}_x(L\nabla_x Q_\varepsilon \odot \nabla_x Q_\varepsilon)) + \mathcal{P}(L\mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon)), \end{aligned} \tag{4.1.14}$$

then from (4.1.14),

$$\partial_t \mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon) \ \text{ is bounded in } L^2([0,T];H^{-1}(\mathbb{T})) + L^1([0,T];W^{-1,1}(\mathbb{T})) + L^\infty([0,T];W^{-1,1}(\mathbb{T})),$$

from above, we know that $\partial_t \mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon)$ is bounded in $L^1([0,T];W^{-1,1}(\mathbb{T}))$. Furthermore, as the operator $\mathcal{P}$ is bounded, we can get the estimate of $\mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon)$ from (4.1.7)

$$\mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon) \ \text{ is bounded in } L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T})) \cap L^2([0,T];L^{\frac{6\gamma}{\gamma+6}}(\mathbb{T})).$$

The convergence of the nonlinear term follows from the following lemma (see Lemma 5.1 in [59]).

**Lemma 4.1.3.** *If the sequence $g_n$ converges weakly to $g$ in $L^{p_1}([0,T];L^{p_2}(\mathbb{T}))$, and the sequence $f_n$ converges weakly to $f$ in $L^{q_1}([0,T];L^{q_2}(\mathbb{T}))$, with $1 \le p_1, p_2 \le \infty$ and*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

*Moreover, assume that*

$$\frac{\partial g_n}{\partial t} \quad \text{is bounded in } L^1([0,T];W^{-m,1}(\mathbb{T})) \quad \text{for some } m \ge 0 \quad \text{independent of } n,$$

*and that*

$$\|h_n(t,\cdot) - h_n(t,\cdot+\zeta)\|_{L^{q_1}([0,T];L^{q_2}(\mathbb{T}))} \to 0 \quad \text{as } |\zeta| \to 0 \quad \text{uniformly in } n.$$

*Then $g_n h_n$ would converge to $gh$ in distribution in $[0,T]\times\mathbb{T}$.*

Apply the Lemma 4.1.3, we get that $\mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathcal{P}(\mathbf{u}_\varepsilon)$ converges to $|\mathbf{u}|^2$ in distribution. Since $\mathbf{u} = \mathcal{P}(\mathbf{u})$ and the convergence of $\mathbf{u}_\varepsilon$, we have $\mathcal{P}(\mathbf{u}_\varepsilon)$ converge to $\mathbf{u}$ in distribution from

$$\int_0^T \int_{\mathbb{T}} (|\mathcal{P}(\mathbf{u}_\varepsilon)|^2 - \mathcal{P}(\rho_\varepsilon \mathbf{u}_\varepsilon)\cdot\mathcal{P}(\mathbf{u}_\varepsilon))dxdt \le C\|\rho_\varepsilon - 1\|_{L^\infty([0,T];L^\gamma(\mathbb{T}))}\|\mathbf{u}_\varepsilon\|_{L^2([0,T];L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}))}.$$

The detail for the convergence of $\mathcal{Q}\mathbf{u}_\varepsilon$ follows the same steps as in [24], [44] and [87], then the proof is complete.

## 4.2 Stochastic Case

In this section, we will discuss the case of the equations driven by the stochastic force. The system is written as

$$\begin{cases} d\tilde{\rho} + \mathrm{div}_x(\tilde{\rho}\tilde{\mathbf{u}})dt = 0, \\ d(\tilde{\rho}\tilde{\mathbf{u}}) + [\mathrm{div}_x(\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \nabla_x A\tilde{\rho}^\gamma]dt = [\tilde{\nu}\triangle\tilde{\mathbf{u}} - \mathrm{div}_x(L\nabla_x\tilde{Q} \odot \nabla_x\tilde{Q} - \mathcal{F}(\tilde{Q})I_3)]dt \\ \qquad\qquad +L\mathrm{div}_x(\tilde{Q}\mathcal{H}(\tilde{Q}) - \mathcal{H}(\tilde{Q})\tilde{Q})dt + \tilde{C}\Phi(\tilde{\rho}, \tilde{\rho}\tilde{\mathbf{u}})dW, \\ d\tilde{Q} + [\tilde{\mathbf{u}} \cdot \nabla_x\tilde{Q} - \tilde{\Theta}\tilde{Q} + \tilde{Q}\tilde{\Theta}]dt = \tilde{\Gamma}\mathcal{H}(\tilde{Q})dt, \end{cases} \tag{4.2.1}$$

where $W$ is a cylindrical Wiener process. We will use the same scaling as the deterministic case, and in addition the scaling of the stochastic coefficient is as follows

$$\tilde{C} = \varepsilon\bar{C}, \quad \text{with } \bar{C} \text{ to be some constant.}$$

Then the system (4.2.1) becomes

$$\begin{cases} (\rho_\varepsilon)_t + \mathrm{div}_x(\rho_\varepsilon\mathbf{u}_\varepsilon) = 0, \\ (\rho_\varepsilon\mathbf{u}_\varepsilon)_t + \mathrm{div}_x(\rho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla_x A\rho_\varepsilon^\gamma = \nu_\varepsilon\triangle\mathbf{u}_\varepsilon - \mathrm{div}_x(L\nabla_x Q_\varepsilon \odot \nabla_x Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon)I_3) \\ \qquad\qquad +L\mathrm{div}_x(Q_\varepsilon\triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon) + \bar{C}\Phi(\rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon)dW, \\ (Q_\varepsilon)_t + \mathbf{u}_\varepsilon \cdot \nabla_x Q_\varepsilon - \Theta_\varepsilon Q_\varepsilon + Q_\varepsilon\Theta_\varepsilon = \Gamma_\varepsilon\mathcal{H}_\varepsilon(Q_\varepsilon). \end{cases} \tag{4.2.2}$$

Also, the corresponding incompressible system reads

$$\begin{cases} \mathbf{u}_t + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x\pi = \nu\triangle\mathbf{u} - \mathrm{div}_x(L\nabla_x Q \odot \nabla_x Q) \\ \qquad\qquad +L\mathrm{div}_x(Q\triangle Q - \triangle QQ) + \Psi(\mathbf{u})dW, \\ Q_t + \mathbf{u} \cdot \nabla_x Q - \Theta Q + Q\Theta = \Gamma\mathcal{H}(Q), \\ \mathrm{div}_x\mathbf{u} = 0. \end{cases} \tag{4.2.3}$$

The stochastic term $\Psi(\mathbf{u}) = \mathcal{P}_H\Phi(1, \mathbf{u})$, where the operator $\mathcal{P}_H$ is the Helmholtz projector onto the divergence-free field and will be specified later. We are going to show that, given an initial law $\Lambda$ for (4.2.1), when the density converges to a constant, the velocity as well as the

Q-tensor will converge in law to a weak martingale solution to the system (4.2.3) equipped with the same initial law.

### 4.2.1 Main result in the stochastic case

We first set up the condition for the stochastic perturbation in (4.2.2). For the stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$,the filtration $(\mathfrak{F}_t)_{t\geq 0}$ is defined to be complete as well as right-continuous. Also, the process $W$ is defined to be a cylindrical Wiener process with the form $W(t) = \sum_{k\geq 1} \beta_k(t) e_k$, where the set $(\beta_k(t))_{k\geq 1}$ are mutually independent real-valued standard Wiener processes relative to $(\mathfrak{F}_t)_{t\geq 0}$, and $(e_k)_{k\geq 1}$ a complete orthonormal basis of a separable Hilbert space $\mathfrak{U}$. When it comes to the diffusion term $\Phi$, given its coefficients $\rho \in L^\gamma(\mathbb{T})$, $\rho \geq 0$; and $\mathbf{v} \in L^2(\mathbb{T})$, combined with $\sqrt{\rho}\mathbf{v} \in L^2(\mathbb{T})$. Denote $\mathbf{q} = \rho\mathbf{v}$, then $\Phi = \Phi(\rho, \mathbf{q})$, as a mapping from $\mathfrak{U}$ to $L^1(\mathbb{T})$ satisfies the following conditions:

$$\Phi(\rho, \mathbf{q})e_k = \mathbf{g}_k(\cdot, \rho(\cdot), \mathbf{q}(\cdot)) = \mathbf{h}_k(\cdot, \rho(\cdot)) + \alpha_k \mathbf{q}(\cdot),$$

the coefficients $\alpha_k$ are real-valued constants, the function $\mathbf{h}_k :\ \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are $C^1$-functions and satisfy

$$\begin{aligned}
&\sum_{k\geq 1} |\alpha_k|^2 < \infty, \\
&\sum_{k\geq 1} |\mathbf{h}_k(x,\rho)|^2 \leq C(\rho^2 + |\rho|^{\gamma+1}), \\
&\sum_{k\geq 1} |\nabla_\rho \mathbf{h}_k(x,\rho)|^2 \leq C(1 + |\rho|^{\gamma-1}).
\end{aligned} \tag{4.2.4}$$

Due to the lack of a priori estimates, we can not simply assume that the value of $\Phi(\rho, \mathbf{q})$ is integrable. As a result, we consider the embedding $L^1(\mathbb{T}) \hookrightarrow W^{-l,2}(\mathbb{T})$, with $l > \frac{3}{2}$, then we can assume that the stochastic integral is a process in $W^{-l,2}(\mathbb{T})$. Then, from (4.2.4)

$$\begin{aligned}
\|\Phi(\rho, \rho\mathbf{v})\|^2_{L_2(\mathfrak{U}; W^{-l,2})} &= \sum_{k\geq 1} \|\mathbf{g}_k(\rho, \rho\mathbf{v})\|^2_{W^{-l,2}} \leq C \sum_{k\geq 1} \|\mathbf{g}_k(\rho, \rho\mathbf{v})\|^2_{L^l} \\
&\leq C \sum_{k\geq 1} (\int_{\mathbb{T}} (|\mathbf{h}_k(x,\rho)| + \rho|\alpha_k \mathbf{v}|) dx)^2 \\
&\leq C(\rho)_{\mathbb{T}} \int_{\mathbb{T}} (\sum_{k\geq 1} \rho^{-1} |\mathbf{h}_k(x,\rho)|^2 + \sum_{k\geq 1} \rho |\alpha_k \mathbf{v}|^2) dx \\
&\leq C(\rho)_{\mathbb{T}} \int_{\mathbb{T}} (\rho + \rho^\gamma + \rho|\mathbf{v}|^2) dx < \infty.
\end{aligned} \tag{4.2.5}$$

Note that the term $(\rho)_{\mathbb{T}} = \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} \rho dx$, the mean value of $\rho$ on $\mathbb{T}$. If we have the additional assumption that

$$\rho \in L^{\gamma}(\Omega \times (0,T), \mathfrak{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{dt}; L^{\gamma}(\mathbb{T})),$$
$$\sqrt{\rho}\mathbf{v} \in L^{2}(\Omega \times (0,T), \mathfrak{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{dt}; L^{2}(\mathbb{T})),$$

with the symbol $\mathfrak{P}$ denoting the progressively measurable $\theta$-algebra associated to $(\mathfrak{F}_t)$, here the mean value $(\rho)_{\mathbb{T}}$ is essentially bounded. As a result, we know that the stochastic integral $\int_0^{\cdot} \Phi(\rho, \rho\mathbf{v})dW$ is a well-defined $(\mathfrak{F}_t)$-martingale, taking values in $W^{-l,2}(\mathbb{T})$. At last, we can define the auxiliary space $\mathfrak{U}_0 \supseteq \mathfrak{U}$ by

$$\mathfrak{U}_0 = \{v = \sum_{k\geq 1} c_k e_k;\ v = \sum_{k\geq 1} \frac{c_k^2}{k^2} < \infty\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{c_k^2}{k^2}, \quad v = \sum_{k\geq 1} c_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt, with the trajectories of $W$ are $\mathbb{P}$-a.s in $C([0,T];\mathfrak{U}_0)$.

In this section, our aim is to build the convergence result for the finite energy weak martingale solution to the stochastic compressible Navier-Stokes system coupled with the Q-tensor equation. We start from the definition of the solution.

**Definition 4.2.1.** The quantity $\{(\Omega, \mathfrak{F}, (\mathfrak{F})_t, \mathbb{P});\ \rho, \mathbf{u}, Q, W\}$ is a weak martingale solution to the equations (4.2.2) equipped with the initial law $\Lambda$, given that the following holds:

1. $(\Omega, \mathfrak{F}, (\mathfrak{F})_t, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
2. $W$ is an $(\mathfrak{F})_t$-cylindrical Wiener process;
3. The density $\rho \geq 0$, and for any $\psi \in C^{\infty}(\mathbb{T})$, the mapping $t \mapsto \langle \rho(t,\cdot), \psi\rangle \in C[0,T]$ $\mathbb{P}$-a.s, and it is progressively measurable. The following holds:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\rho(t,\cdot)\|_{L^{\gamma}(\mathbb{T})}^p\right] < \infty \quad \text{for all } 1 \leq p < \infty;$$

4. The velocity field $\mathbf{u}$ is adapted, $\mathbf{u} \in L^2(\Omega \times (0,T); W^{1,2}(\mathbb{T}))$,

$$\mathbb{E}\left[(\int_0^T \|\mathbf{u}\|_{W^{1,2}(\mathbb{T})}^2 dt)^p\right] < \infty \quad \text{for all } 1 \leq p < \infty;$$

5. The momentum $\rho\mathbf{u}$ satisfies, $t \mapsto \langle \rho\mathbf{u}(t,\cdot), \phi\rangle \in C[0,T]$ $\mathbb{P}$-a.s for any $\phi \in C^\infty(\mathbb{T})^3$, the mapping is also progressively measurable

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\rho\mathbf{u}(t,\cdot)\|^p_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T})}\right] < \infty \quad \text{for all } 1 \le p < \infty;$$

6. The Q-tensor $Q$ is adapted, $Q \in L^2(\Omega \times (0,T); W^{2,2}(\mathbb{T}))$, for any $\varphi \in C^\infty(\mathbb{T})^{3\times 3}$, the mapping $t \mapsto \langle Q(t,\cdot), \varphi\rangle \in C[0,T]$ $\mathbb{P}$-a.s, and it is progressively measurable,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Q(t,\cdot)\|_{W^{1,2}(\mathbb{T})}]^p + \mathbb{E}[(\int_0^T \|\mathbf{u}\|^2_{W^{2,2}(\mathbb{T})} dt)^p\right] < \infty \quad \text{for all } 1 \le p < \infty;$$

7. $\Lambda = \mathbb{P} \circ (\rho(0), \rho\mathbf{u}(0), Q(0))^{-1}$;
8. For all $\psi \in C^\infty(\mathbb{T})$, $\phi \in C^\infty(\mathbb{T})^3$ and $\varphi \in C^\infty(\mathbb{T})^{3\times 3}$, it holds $\mathbb{P}$-a.s

$$\begin{aligned}
\langle \rho(t), \psi\rangle &= \langle \rho(0), \psi\rangle + \int_0^t \langle \rho\mathbf{u}(t), \psi\rangle ds, \\
\langle \rho\mathbf{u}(t), \phi\rangle &= \langle \rho\mathbf{u}(0), \phi\rangle + \int_0^t \langle \rho\mathbf{u} \otimes \mathbf{u}, \nabla_x\phi\rangle ds - \nu\int_0^t \langle \nabla_x\mathbf{u}, \nabla_x\phi\rangle ds \\
&\quad + \frac{A}{\varepsilon^2}\int_0^t \langle \rho^\gamma, \mathrm{div}_x\phi\rangle + \int_0^t \langle (L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q)I_3), \nabla_x\phi\rangle ds \\
&\quad - L\int_0^t \langle (Q\triangle Q - \triangle QQ), \nabla_x\phi\rangle ds + \bar{C}\int_0^t \langle \Phi(\rho, \rho\mathbf{u})dW, \phi\rangle, \\
\langle Q(t), \varphi\rangle &= \langle Q(0), \varphi\rangle - \int_0^t \langle \mathbf{u}\cdot\nabla_x Q, \varphi\rangle ds + \int_0^t \langle (\Theta Q - Q\Theta), \varphi\rangle ds + \Gamma\int_0^t \langle \mathcal{H}(Q), \varphi\rangle ds.
\end{aligned}$$

Next, we present the definition of the martingale solution to the incompressible model, that is, the solution to system (4.2.3).

**Definition 4.2.2.** If $\Lambda$ is a Borel probability measure on $L^2(\mathbb{T})$, then the quantity

$$\{(\Omega, \mathfrak{F}, (\mathfrak{F})_t, \mathbb{P});\ \mathbf{u}, Q, W\}$$

is a weak martingale solution to the equations (4.2.3) with the initial law $\Lambda$, provided the following holds:

1. $(\Omega, \mathfrak{F}, (\mathfrak{F})_t, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
2. $W$ is an $(\mathfrak{F})_t$-cylindrical Wiener process;

3. The velocity field $\mathbf{u}$ is $(\mathfrak{F})_t$-adapted, $\mathbf{u} \in C_\omega([0,T]; L^2_{\mathrm{div}}(\mathbb{T})) \cap L^2([0,T]; W^{1,2}_{\mathrm{div}}(\mathbb{T}))$,

$$\mathbb{E}\left[\sup_{(0,T)} \|\mathbf{u}\|_{L^2(\mathbb{T})}\right]^p + \mathbb{E}\left[(\int_0^T \|\mathbf{u}\|^2_{W^{1,2}(\mathbb{T})} dt)^p\right] < \infty \quad \text{for all } 1 \le p < \infty;$$

4. The Q-tensor $Q$ is adapted, $Q \in L^2(\Omega \times (0,T); W^{2,2}(\mathbb{T}))$, for any $\varphi \in C^\infty(\mathbb{T})^{3\times 3}$, the mapping $t \mapsto \langle Q(t,\cdot), \varphi\rangle \in C[0,T]$ $\mathbb{P}$-a.s, and it is progressively measurable,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Q(t,\cdot)\|_{W^{1,2}(\mathbb{T})}\right]^p + \mathbb{E}\left[(\int_0^T \|\mathbf{u}\|^2_{W^{2,2}(\mathbb{T})} dt)^p\right] < \infty \quad \text{for all } 1 \le p < \infty;$$

5. For all $\phi \in C^\infty_{\mathrm{div}}(\mathbb{T})^3$ and $\varphi \in C^\infty(\mathbb{T})^{3\times 3}$, it holds $\mathbb{P}$-a.s

$$\begin{aligned}
\langle \mathbf{u}(t), \phi\rangle = \langle \mathbf{u}(0), \phi\rangle &+ \int_0^t \langle \mathbf{u}\otimes\mathbf{u}, \nabla_x\phi\rangle ds - \nu\int_0^t \langle \nabla_x \mathbf{u}, \nabla_x\phi\rangle ds \\
&+ \int_0^t \langle (L\nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q) I_3), \nabla_x\phi\rangle ds \\
&- L\int_0^t \langle (Q\triangle Q - \triangle Q Q), \nabla_x\phi\rangle ds + \bar{C}\int_0^t \langle \Psi(\mathbf{u}) dW, \phi\rangle,
\end{aligned}$$

$$\langle Q(t), \varphi\rangle = \langle Q(0), \varphi\rangle - \int_0^t \langle \mathbf{u}\cdot\nabla_x Q, \varphi\rangle ds + \int_0^t \langle (\Theta Q - Q\Theta), \varphi\rangle ds + \Gamma\int_0^t \langle \mathcal{H}(Q), \varphi\rangle ds.$$

The main result in the stochastic case is presented as the following:

**Theorem 4.2.3.** *Assume that $\Lambda$ is a given Borel probability measure on $L^2(\mathbb{T})$, $\Lambda_\varepsilon$ is a Borel probability measure on $L^\gamma(\mathbb{T}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}) \times L^6(\mathbb{T})$, for certain constant $M > 0$, $M$ independent of $\varepsilon$, it holds that*

$$\Lambda_\varepsilon\{(\rho, \mathbf{q}, Q) \in L^\gamma(\mathbb{T}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}) \times L^6(\mathbb{T}); \rho \ge \frac{1}{M},\ (\rho)_{\mathbb{T}} \le M,\ |\frac{\rho-1}{\varepsilon}| \le M\} = 1,$$

*and for all $1 \le p < \infty$,*

$$\int_{L^\gamma_x \times L_x^{\frac{2\gamma}{\gamma+1}} \times L^6_x} (\|\frac{1}{2}\frac{|\mathbf{q}|^2}{\rho}\|^p_{L^1_x} + \|Q\|^p_{L^1_x}) d\Lambda_\varepsilon(\rho, \mathbf{q}, Q) \le C(p).$$

*If the martingale law of $\Lambda_\varepsilon$ corresponding to the second and the third component converges to $\Lambda$ weakly in the sense of measure on $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}) \times L^6(\mathbb{T})$, and $((\Omega^\varepsilon, \mathfrak{F}^\varepsilon, (\mathfrak{F}^\varepsilon)_t, \mathbb{P}^\varepsilon),\ \rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon, W_\varepsilon)$ is a finite energy weak martingale solution to (4.2.2) with the initial law $\Lambda_\varepsilon$, $\varepsilon \in (0,1)$, then*

$$\begin{aligned}
&\rho_\varepsilon \to 1 \quad \textit{in law on } L^\infty(0,T; L^\gamma(\mathbb{T})), \\
&\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{in law on } (L^2(0,T; W^{1,2}(\mathbb{T})), \omega), \\
&Q_\varepsilon \to Q \quad \textit{in law on } (L^2(0,T; W^{2,2}(\mathbb{T})), \omega),
\end{aligned}$$

*where $(\mathbf{u}, Q)$ is a weak martingale solution to (4.2.3) with the initial law $\Lambda$.*

To prove Theorem 4.2.3, we will follow the similar arguments to [10]: first, we will get the estimates of the solutions that are uniform in $\varepsilon$; next, by the compactness criteria, we can prove that the limit of $(\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon)$ exists; last, we will verify that the limit of $(\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon)$ is the solution to (4.2.3).

### 4.2.2 Uniform estimate

In this subsection, we will focus on the study of the limit $\varepsilon \to 0$ for the system (4.2.2). For every $\varepsilon \in (0,1)$, there exists at least one quantity

$$\{(\Omega^\varepsilon, \mathfrak{F}^\varepsilon, (\mathfrak{F}^\varepsilon)_t, \mathbb{P}^\varepsilon);\ \mathbf{u}_\varepsilon, Q_\varepsilon, W_\varepsilon\},$$

which is, by the means of Definition 4.2.1, a weak martingale solution. Without loss of generality, it suffices to consider in just one probability space, that is

$$(\Omega^\varepsilon, \mathfrak{F}^\varepsilon, (\mathfrak{F}^\varepsilon)_t, \mathbb{P}^\varepsilon) = ([0,1], \mathcal{B}([0,1]), \mathcal{L}) \ \ \forall \varepsilon \in (0,1),$$

with $\mathcal{L}$ denotes the Lebesgue measure on $[0,1]$. Moreover, we can assume that there exists one common Wiener process $W$ for all $\varepsilon$.

In the beginning, we get the energy estimate of (4.2.2). From Itô's formula, the following estimate holds.

**Proposition 4.2.4.** *For any $p \in [1,\infty)$, the following estimate holds uniformly in $\varepsilon$*

$$\begin{aligned}
&\mathbb{E}[\sup_{0\le t\le T}(\int_{\mathbb{T}}(\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{A}{\varepsilon^2(\gamma-1)}(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1) \\
&\qquad + \frac{L}{2}|\nabla_x Q_\varepsilon|^2 + \frac{a}{2}\mathrm{tr}(Q_\varepsilon^2) - \frac{\varepsilon b}{3}\mathrm{tr}(Q_\varepsilon^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2(Q_\varepsilon^2))dx)]^p \\
&\quad + \mathbb{E}\left[\nu_\varepsilon \int_0^t \int_{\mathbb{T}} |\nabla_x \mathbf{u}_\varepsilon|^2 dxdt + \Gamma_\varepsilon \int_0^t \int_{\mathbb{T}} \mathrm{tr}(\mathcal{H}_\varepsilon^2(Q_\varepsilon))dxdt\right]^p \\
&\le C_p \mathbb{E}\left[\int_{\mathbb{T}} (\frac{1}{2}\frac{|\mathbf{m}_\varepsilon^0|^2}{\rho_\varepsilon^0} + \frac{A}{\varepsilon^2(\gamma-1)}((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + \gamma - 1))dx\right]^p \\
&\quad + C_p\mathbb{E}\left[\int_{\mathbb{T}} (\frac{L}{2}|\nabla_x Q_\varepsilon^0|^2 + \frac{a}{2}\mathrm{tr}((Q_\varepsilon^0)^2) - \frac{\varepsilon b}{3}\mathrm{tr}((Q_\varepsilon^0)^3) + \frac{\varepsilon^2 c}{4}\mathrm{tr}^2((Q_\varepsilon^0)^2))\right]^p \le C_p.
\end{aligned} \tag{4.2.6}$$

Similar to the calculations in the previous section, the following estimates also hold.

**Proposition 4.2.5.** *For all $p \in [1, \infty)$, we have the following uniform bounds,*

$$\begin{aligned} &\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \text{ is bounded in } L^p(\Omega; L^\infty([0,T]; L^2(\mathbb{T}))), \\ &Q_\varepsilon \text{ is bounded in } L^p(\Omega; L^\infty([0,T]; W^{1,2}(\mathbb{T}))), \\ &\frac{1}{\varepsilon^2}(\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1) \text{ is bounded in } L^p(\Omega; L^\infty([0,T]; L^1(\mathbb{T}))), \end{aligned} \tag{4.2.7}$$

*together with the following uniform bounds*

$$\begin{aligned} &\nabla_x \mathbf{u}_\varepsilon \text{ is bounded in } L^p(\Omega; L^2([0,T]; L^2(\mathbb{T}))), \\ &\mathcal{H}_\varepsilon(Q_\varepsilon)) \text{ is bounded in } L^p(\Omega; L^2([0,T]; L^2(\mathbb{T}))), \\ &\triangle Q_\varepsilon \text{ is bounded in } L^p(\Omega; L^2([0,T]; L^2(\mathbb{T}))). \end{aligned} \tag{4.2.8}$$

*Note that all the bounds above are uniform with respect to $\varepsilon$, and hold for any $T > 0$.*

To deal with the estimate for the pressure term, for any function $h$, we can define the essential and residual component for this function:

$$\begin{aligned} &h = h_{\text{ess}} + h_{\text{res}}, \\ &h_{\text{ess}} = \chi(\rho_\varepsilon)h,\ \chi \in C_c^\infty(0,\infty), \quad 0 \le \chi \le 1, \quad \chi \equiv 1 \text{ in some open interval containing } 1, \\ &h_{\text{res}} = (1 - \chi(\rho_\varepsilon))h. \end{aligned}$$

Consider the function $f(x) = x^\gamma$, as $\gamma > 1$, we have the following lemma.

**Lemma 4.2.6.** *Let $P(\rho) = \rho^\gamma - 1 - \gamma(\rho - 1)$, with $\rho \in [0, \infty)$, then there exist some constants $C_1$, $C_2$, $C_3$, $C_4$ positive such that*

$$\begin{aligned} &C_1|\rho - 1|^2 \le P(\rho) \le C_2|\rho - 1|^2 \quad \textit{if } \rho \in \ \textit{supp } \chi, \\ &P(\rho) \ge C_3\rho^\gamma \quad \textit{if } \rho \notin \ \textit{supp } \chi, \\ &P(\rho) \ge C_4 \quad \textit{if } \rho \notin \ \textit{supp } \chi. \end{aligned}$$

*Proof.* The first conclusion is a natural result of Taylor's theorem. We observe that the function $\frac{P(\rho)}{\rho^\gamma}$ is increasing for $\rho \in [1, \infty)$, and achieves minimum at $\rho = 1$. Moreover, the function $P$ is strictly convex, so the second and third conclusions hold true. $\square$

Therefore, from Lemma 4.2.6, and the uniform bounds (4.2.8), we can get that for all $p \in [1, \infty)$, the following bounds are uniform:

$$[\frac{\rho_\varepsilon}{\varepsilon}]_{\text{ess}} \in L^p(\Omega; L^\infty([0,T]; L^2(\mathbb{T}))),$$
$$[\frac{\rho_\varepsilon}{\varepsilon}]_{\text{res}} \in L^p(\Omega; L^\infty([0,T]; L^\gamma(\mathbb{T}))).$$

That is, if we define $\varphi_\varepsilon := \frac{1}{\varepsilon}(\rho_\varepsilon - 1)$, then uniformly in $\varepsilon$, we have

$$\varphi_\varepsilon \in L^p(\Omega; L^\infty([0,T]; L^{\min(\gamma,2)}(\mathbb{T}))). \tag{4.2.9}$$

In the following, we want to show that

$$\rho_\varepsilon \to 1 \ \text{ in } L^p(\Omega; L^\infty([0,T]; L^\gamma(\mathbb{T}))), \tag{4.2.10}$$

and that leads to

$$\rho_\varepsilon \in L^p(\Omega; L^\infty([0,T]; L^\gamma(\mathbb{T}))).$$

Combined with the estimates (4.2.7) and (4.2.8), we have the following uniform estimate for all $p \in [0, \infty)$

$$\rho_\varepsilon \mathbf{u}_\varepsilon \in L^p(\Omega; L^\infty([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}))),$$
$$\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \in L^p(\Omega; L^2([0,T]; L^{\frac{6\gamma}{4\gamma+3}}(\mathbb{T}))).$$

To verify (4.2.10), we need to use the fact that for all $\delta > 0$, there exists a constant $C_\delta > 0$, such that

$$\rho^\gamma - 1 - \gamma(\rho - 1) \geq C_\delta |\rho - 1|^\gamma$$

if $|\rho - 1| \geq \delta$ and $\rho \geq 0$, thus we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\int_{\mathbb{T}} |\rho_\varepsilon - 1|^\gamma dx\right]^p \\
=&\mathbb{E}\left[\sup_{0\le t\le T}\int_{\mathbb{T}} |\rho_\varepsilon - 1|^\gamma \chi_{|\rho_\varepsilon - 1|\le\delta} dx\right]^p + \mathbb{E}\left[\sup_{0\le t\le T}\int_{\mathbb{T}} |\rho_\varepsilon - 1|^\gamma \chi_{|\rho_\varepsilon - 1|\ge\delta} dx\right]^p \\
\le& C_\delta \mathbb{E}\left[\int_{\mathbb{T}} (\rho_\varepsilon^\gamma - \gamma\rho_\varepsilon + \gamma - 1) dx\right]^p + C\delta^{\gamma p} \\
\le& C_\delta \varepsilon^{2p} + C\delta^{\gamma p}.
\end{aligned} \tag{4.2.11}$$

Let $\varepsilon \to 0$, then $\delta \to 0$ and the claim is proven.

Next, we can apply the Helmholtz projection $\mathcal{P}_H$, which projects $L^2(\mathbb{T})$ onto divergence free vector fields

$$L^2_{\mathrm{div}}(\mathbb{T}) := \overline{C}^{\infty}_{\mathrm{div}}(\mathbb{T})^{\|\cdot\|_2}.$$

Furthermore, the operator $\mathcal{Q} = \mathrm{Id} - \mathcal{P}_H$ is the projection onto the curl free vector fields. It has been proven that both $\mathcal{P}_H$ and $\mathcal{Q}$ are bounded in $W^{l,q}(\mathbb{T})$, with any real number $l$ and $p \in [1, \infty)$. Project $(4.2.2)_2$ onto the divergence-free field, then we get the following equation:

$$\begin{aligned} &d\mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon) + \mathcal{P}_H(\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon))dt \\ =&\nu_\varepsilon \triangle \mathcal{P}_H \mathbf{u}_\varepsilon dt - \mathcal{P}_H(\mathrm{div}_x(L\nabla_x Q_\varepsilon \odot \nabla_x Q_\varepsilon))dt + \mathcal{P}_H(L\mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon))dt \\ &+ \bar{C}\mathcal{P}\Phi(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)dW. \end{aligned} \tag{4.2.12}$$

After we have got all the uniform bounds, we will get the weak convergence using the following compactness criteria.

### 4.2.3 Compactness

First, we define the path space

$$\mathcal{X} = \mathcal{X}_\rho \times \mathcal{X}_{\mathbf{u}} \times \mathcal{X}_{\rho\mathbf{u}} \times \mathcal{X}_Q \times \mathcal{X}_{\mathcal{W}},$$

with

$$\begin{aligned} \mathcal{X}_\rho &= C_w([0,T]; L^\gamma(\mathbb{T})), \\ \mathcal{X}_{\mathbf{u}} &:= (L^2([0,T]; W^{1,2}(\mathbb{T})), \omega), \\ \mathcal{X}_{\rho\mathbf{u}} &:= C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T})), \\ \mathcal{X}_Q &:= C_w([0,T]; W^{1,2}(\mathbb{T})) \cap L^2([0,T]; W^{2,2}_w(\mathbb{T})), \\ \mathcal{X}_{\mathcal{W}} &:= C([0,T]; \mathcal{U}_0). \end{aligned}$$

We denote the corresponding law of $\rho_\varepsilon$, $\mathbf{u}_\varepsilon$, $\mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon)$ and $Q_\varepsilon$ by $\mu_{\rho_\varepsilon}$, $\mu_{\mathbf{u}_\varepsilon}$, $\mu_{\mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon)}$ and $\mu_{Q_\varepsilon}$. Denote by $\mu_W$ the law of $W$ on $\mathcal{X}_W$, and their joint law on $\mathcal{X}$ is denoted by $\mu^\varepsilon$. After that, we establish the tightness of $\{\mu^\varepsilon;\ \varepsilon \in (0,1)\}$.

**Proposition 4.2.7.** *The set* $\{\mu_{\mathbf{u}_\varepsilon};\ \varepsilon \in (0,1)\}$ *is tight on* $\mathcal{X}_{\mathbf{u}}$.

*Proof.* The tightness is a consequence of the estimate (4.2.8). Moreover, for any fixed $R > 0$, the set

$$B_R = \{\mathbf{u} \in L^2([0,T]; W^{1,2}(\mathbb{T})); \|\mathbf{u}\|_{L^2([0,T];W^{1,2}(\mathbb{T}))} \leq R\},$$

is relatively compact in $\mathcal{X}_{\mathbf{u}}$ and

$$\mu_{\mathbf{u}_\varepsilon}(B_R^c) = \mathbb{P}(\|\mathbf{u}_\varepsilon\|_{L^2([0,T];W^{1,2}(\mathbb{T}))} \geq R) \leq \frac{1}{R}\mathbb{E}\|\mathbf{u}_\varepsilon\|_{L^2([0,T];W^{1,2}(\mathbb{T}))} \leq \frac{C}{R}.$$

Then the claim holds true. □

**Proposition 4.2.8.** *The set* $\{\mu_{\rho_\varepsilon};\ \varepsilon \in (0,1)\}$ *is tight on* $\mathcal{X}_\rho$.

*Proof.* From (4.2.7), it's easy for us to get that, the set $\{\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon)\}_{\varepsilon\in(0,1)}$ is bounded in $L^p(\Omega;\ L^\infty([0,T]; W^{-1,\frac{2\gamma}{\gamma+1}}(\mathbb{T})))$. Then from the continuity equation, the following bound is uniform, for all $p \in [1,\infty)$:

$$\rho_\varepsilon \in L^p(\Omega;\ C^{0,1}([0,T]; W^{-1,\frac{2\gamma}{\gamma+1}}(\mathbb{T}))).$$

Then the desired tightness follows from the embedding

$$L^\infty([0,T]; L^\gamma(\mathbb{T})) \cap C^{0,1}([0,T]; W^{-1,\frac{2\gamma}{\gamma+1}}(\mathbb{T})) \hookrightarrow^c C_w([0,T]; L^\gamma(\mathbb{T})).$$

□

**Proposition 4.2.9.** *The set* $\{\mu_{\mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon)};\ \varepsilon \in (0,1)\}$ *is tight on* $\mathcal{X}_{\rho\mathbf{u}}$.

*Proof.* First, we decompose $\mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon)$ into three parts. That is

$$\begin{aligned}
\mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon) &= \mathcal{P}_H(\rho_\varepsilon \mathbf{u}_\varepsilon(0)) - \int_0^t \mathcal{P}_H(\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu \triangle \mathbf{u}_\varepsilon) ds \\
&\quad - \int_0^t \mathcal{P}_H(\mathrm{div}_x(L Q_\varepsilon \odot Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon)\mathrm{I}_3)) ds + \int_0^t L\mathcal{P}_H(\mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon)) ds \\
&\quad + \int_0^t \mathcal{P}_H(\Phi(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)) dW(s) \\
&= Y_1^\varepsilon(t) + Y_2^\varepsilon(t) + Z^\varepsilon(t).
\end{aligned}$$

We will prove the Hölder continuity of all the above terms, starting from $\{Y_1^\varepsilon(t)\}$. It suffices for us to show that there exists $l \in \mathbb{N}$, such that for any $\kappa \in (0, \frac{1}{2})$, it holds true that

$$\mathbb{E}\|Y_1^\varepsilon\|_{C^\kappa([0,T];W^{-l,2}(\mathbb{T}))} \leq C. \tag{4.2.13}$$

Choose $l > \frac{5}{2}$, such that $L^1(\mathbb{T}) \hookrightarrow W^{1-l,2}(\mathbb{T})$, then from the a priori estimate (4.2.6) and (4.2.7), combined with the boundedness of $\mathcal{P}_H$, it yields that

$$\begin{aligned}
&\mathbb{E}\|Y_1^\varepsilon(t) - Y_1^\varepsilon(s)\|_{W^{-l,2}(\mathbb{T})}^\theta \\
=&\mathbb{E}\|\int_s^t \mathcal{P}_H(\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu\triangle \mathbf{u}_\varepsilon)ds\|_{W^{-l,2}(\mathbb{T})}^\theta \\
\leq& C\mathbb{E}\|\int_s^t \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon ds\|_{W^{1-l,2}(\mathbb{T})}^\theta + C\mathbb{E}\|\int_s^t \nabla_x \mathbf{u}_\varepsilon ds\|_{W^{1-l,2}(\mathbb{T})}^\theta \\
\leq& C\mathbb{E}\|\int_s^t \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon ds\|_{L^1(\mathbb{T})}^\theta + C\mathbb{E}\|\int_s^t \nabla_x \mathbf{u}_\varepsilon ds\|_{L^1(\mathbb{T})}^\theta \leq C|t-s|^{\frac{\theta}{2}}.
\end{aligned}$$

Then (4.2.13) follows from the Kolmogorov continuity criterion.

Now we prove the Hölder continuity of $\{Y_2^\varepsilon(t)\}$. Follow the same way as above, we can have the following estimate from (4.2.8),

$$\begin{aligned}
&\mathbb{E}\|Y_2^\varepsilon(t) - Y_2^\varepsilon(s)\|_{W^{-l,2}(\mathbb{T})}^\theta \\
=&\mathbb{E}\| - \int_s^t \mathcal{P}_H(\mathrm{div}_x(LQ_\varepsilon \odot Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon)\mathrm{I}_3))ds \\
&\qquad\qquad\qquad + \int_s^t L\mathcal{P}_H(\mathrm{div}_x(Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon))ds\|_{W^{-l,2}(\mathbb{T})}^\theta \\
\leq& C\mathbb{E}\|\int_s^t (LQ_\varepsilon \odot Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon)\mathrm{I}_3)\|_{W^{1-l,2}(\mathbb{T})}^\theta + C\mathbb{E}\|\int_s^t (Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon))ds\|_{W^{1-l,2}(\mathbb{T})}^\theta \\
\leq& C\mathbb{E}\|\int_s^t (LQ_\varepsilon \odot Q_\varepsilon - \mathcal{F}_\varepsilon(Q_\varepsilon)\mathrm{I}_3)\|_{L^1(\mathbb{T})}^\theta + C\mathbb{E}\|\int_s^t (Q_\varepsilon \triangle Q_\varepsilon - \triangle Q_\varepsilon Q_\varepsilon))ds\|_{L^1(\mathbb{T})}^\theta \leq C|t-s|^{\frac{\theta}{2}}.
\end{aligned}$$

Therefore, we can get

$$\mathbb{E}\|Y_2^\varepsilon\|_{C^\kappa([0,T];W^{-l,2}(\mathbb{T}))} \leq C. \tag{4.2.14}$$

Finally, we prove the Hölder continuity of $\{Z^\varepsilon(t)\}$. We also need to show that

$$\mathbb{E}\|Z^\varepsilon\|_{C^\kappa([0,T];W^{-l,2}(\mathbb{T}))} \leq C. \tag{4.2.15}$$

Following the similar way to the above two proofs, we have

$$\begin{aligned}
&\mathbb{E}\|Z^\varepsilon(t)-Z^\varepsilon(s)\|^\theta_{W^{-l,2}(\mathbb{T})}\\
=&\mathbb{E}\|\int_s^t \mathcal{P}_H(\Phi(\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon))dW\|^\theta_{W^{-l,2}(\mathbb{T})}\\
\leq& C\mathbb{E}\|\int_s^t \Phi(\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon)dW\|^\theta_{W^{-l,2}(\mathbb{T})}\\
\leq& C\mathbb{E}(\int_s^t\sum_{k\geq 1}\|\mathbf{g}_k(\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon)\|_{W^{-l,2}(\mathbb{T})}dr)^{\frac{\theta}{2}}\\
\leq& C\mathbb{E}(\int_s^t\sum_{k\geq 1}\|\mathbf{g}_k(\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon)\|_{L^1(\mathbb{T})}dr)^{\frac{\theta}{2}}\\
\leq& C\mathbb{E}(\int_s^t\int_{\mathbb{T}}(\rho_\varepsilon+\rho_\varepsilon|\mathbf{u}_\varepsilon|^2+\rho_\varepsilon^\gamma)dxdr)^{\frac{\theta}{2}}\\
\leq& C|t-s|^{\frac{\theta}{2}}(1+\mathbb{E}(\sup_{[0,T]}\|\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon\|^\theta_{L^2})+\mathbb{E}(\sup_{[0,T]}\|\rho_\varepsilon\|^{\frac{\theta\gamma}{2}}_{L^\gamma}))\leq C|t-s|^{\frac{\theta}{2}}.
\end{aligned}$$

Then the proof is complete. □

**Proposition 4.2.10.** *The set* $\{Q_\varepsilon\};\ \varepsilon\in(0,1)\}$ *is tight on* $\mathcal{X}_Q$.

*Proof.* From the uniform bounds in (4.2.7) and (4.2.8), the tightness is a natural consequence. That is, from the equation $(4.2.2)_3$, we have

$$(Q_\varepsilon)_t=-\mathbf{u}_\varepsilon\cdot\nabla_x Q_\varepsilon+\Theta_\varepsilon Q_\varepsilon-Q_\varepsilon\Theta_\varepsilon+\Gamma_\varepsilon\mathcal{H}_\varepsilon(Q_\varepsilon).$$

Then, we have that, for any $p\in[1,\infty)$

$$Q_\varepsilon \text{ is uniformly bounded in } L^p(\Omega;\ C^\kappa([0,T];L^{\frac{3}{2}}(\mathbb{T}))).$$

Therefore, by embedding,

$$Q_\varepsilon \text{ is compact in } L^p(\Omega;\ L^2([0,T];W^{2,2}(\mathbb{T})))\cap L^p(\Omega;\ C_w([0,T];W^{1,2}(\mathbb{T}))).$$

□

By all the compactness results, since we also have that the law $\mu_W$ is tight on the Polish space $\mathcal{X}_W$, the tightness of the joint law follows.

**Proposition 4.2.11.** *The set* $\{\mu^\varepsilon\};\ \varepsilon\in(0,1)\}$ *is tight on* $\mathcal{X}$.

As the path space $\mathcal{X}$ is not a Polish space, we use the Jakubowski-Skorokhod representation theorem, a refinement of the Skorokhod representation theorem, see [45] for details.

**Proposition 4.2.12.** *One can find a subsequence $\mu^\varepsilon$, a probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$, and $\mathcal{X}$-valued Borel measurable random variables $(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{q}}_\varepsilon, \tilde{Q}_\varepsilon, \tilde{W}_\varepsilon)$, for any $\varepsilon in(0,1)$, also there exists the set$(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \tilde{Q}, \tilde{W})$, such that the following holds*

1. *the law of $(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{q}}_\varepsilon, \tilde{Q}_\varepsilon, \tilde{W}_\varepsilon)$ is the same as $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{q}_\varepsilon, Q_\varepsilon, W_\varepsilon)$, that is, given by $\mu^\varepsilon$;*
2. *similarly, the law of $(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \tilde{Q}, \tilde{W})$ is the same as $(\rho, \mathbf{u}, \mathbf{q}, Q, W)$, can be denoted by $\mu$;*
3. *$(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{q}}_\varepsilon, \tilde{Q}_\varepsilon, \tilde{W}_\varepsilon)$ converges to $(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \tilde{Q}, \tilde{W})$ $\mathbb{P}$-a.s, in the topology of $\mathcal{X}$.*

Next we fix some notations that will be used later. For any $t \in [0, T]$, denote $\mathbf{r}_t$ as the restriction operator onto the interval $[0, t]$, which restrict the various path spaces. That is, if $X$ is one of the path spaces mentioned above, and $t \in [0, T]$, we define

$$\mathbf{r}_t : \; X \to \; X|_{[0,T]}, \; f \to \; f|_{[0,T]}, \tag{4.2.16}$$

and $\mathbf{r}_t$ is a continuous mapping. Denote by $\{\tilde{\mathfrak{F}}_t^\varepsilon\}_{\varepsilon>0}$ and $\tilde{\mathfrak{F}}_t$ the $\tilde{\mathbb{P}}$–augmented canonical filtration of the process $(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon, \tilde{W}_\varepsilon)$ and $(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$. That is

$$\tilde{\mathfrak{F}}_t^\varepsilon = \sigma(\sigma(\mathbf{r}_t\tilde{\rho}_\varepsilon, \mathbf{r}_t\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_t\tilde{Q}_\varepsilon, \mathbf{r}_t\tilde{W}_\varepsilon) \cup \{N \in \tilde{\mathfrak{F}};\; \tilde{\mathbb{P}}(N) = 0\}), \quad \text{for } t \in [0, T],$$
$$\tilde{\mathfrak{F}}_t = \sigma(\sigma(\mathbf{r}_t\tilde{\mathbf{u}}, \mathbf{r}_t\tilde{Q}, \mathbf{r}_t\tilde{W}) \cup \{N \in \tilde{\mathfrak{F}};\; \tilde{\mathbb{P}}(N) = 0\}), \quad \text{for } t \in [0, T].$$

### 4.2.4 Justification of the limit

In this section, we aim at verifying the limit process we got in Proposition 4.2.11 is a weak martingale solution to (4.2.3). Namely, we prove Theorem 4.2.3 by proving this result.

**Theorem 4.2.13.** *The quadruplet $((\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}}), \tilde{\mathbf{u}}, \tilde{Q}, \tilde{W})$ is a weak martingale solution to equation* (4.2.3) *equipped with the initial law* $\Lambda$*, and the process $\tilde{W}$ is a $(\tilde{\mathfrak{F}}_t)$-cylindrical Wiener process.*

The proof can be divided into several steps. At first, we show that the approximate system $\tilde{\rho}_\varepsilon,\; \tilde{\mathbf{u}}_\varepsilon,\; \tilde{Q}_\varepsilon$ solves the system (4.2.2) in the new probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$.

**Proposition 4.2.14.** *For any $\varepsilon \in (0,1)$, the quadruplet*

$$((\tilde{\Omega}, \tilde{\mathfrak{F}}_t^{\varepsilon}, \tilde{\mathbb{P}}), \tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon, \tilde{W}_\varepsilon)$$

*is a finite energy weak martingale solution to* (4.2.3) *with the initial law $\Lambda_\varepsilon$, and the process $\tilde{W}_\varepsilon$ is a $(\tilde{\mathfrak{F}}_t)$-cylindrical Wiener process.*

*Proof.* As $\tilde{W}_\varepsilon$ has the same law as $W$, it follows that $\tilde{W}_\varepsilon$ is a $(\tilde{\mathfrak{F}}_t)$-cylindrical Wiener process. Therefore, there exists a collection of mutually independent real-valued $(\tilde{\mathfrak{F}}_t)$-Wiener processes $(\tilde{\beta}_k^{\varepsilon})_{k\geq 1}$, such that $\tilde{W}_\varepsilon = \sum_{k\geq 1} \tilde{\beta}_k^{\varepsilon} e_k$.

We show that the continuity equation $(4.2.2)_1$ is satisfied. For all $t \in [0,T]$, and $\psi \in C^\infty(\mathbb{T})$, define the functional

$$\mathcal{L}(\rho, \mathbf{q})_t = \langle \rho(t), \psi \rangle - \langle \rho(0), \psi \rangle - \int_0^t \langle \mathbf{q}, \psi \rangle ds.$$

The mapping $(\rho, \mathbf{q}) \to \mathcal{L}(\rho, \mathbf{q})_t$ is continuous on the path space $\mathcal{X}_\rho \times \mathcal{X}_{\rho\mathbf{u}}$. In the Proposition 4.2.12, we know that the law of $\mathcal{L}(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)_t$ coincide with the law of $\mathcal{L}(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t$. Note that $\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon$ solves $(4.2.2)_1$

$$\tilde{\mathbb{E}} |\mathcal{L}(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t|^2 = \mathbb{E} |\mathcal{L}(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)_t|^2 = 0.$$

This also means that $(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)$ solves $(4.2.2)_1$.

The next step is to verify that the momentum equation $(4.2.2)_2$ holds true. Similarly, for any $t \in [0,T]$, $\varphi \in C^\infty(\mathbb{T})^3$, define the functional

$$\begin{aligned}
\mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_t = {} & \langle \mathbf{q}(t), \varphi \rangle - \langle \mathbf{q}(0), \varphi \rangle + \int_0^t \langle \mathbf{q} \otimes \mathbf{v}, \nabla_x \varphi \rangle ds - \nu \int_0^t \langle \nabla_x \mathbf{v}, \nabla_x \varphi \rangle ds \\
& + \frac{A}{\varepsilon^2} \int_0^t \langle \rho^\gamma, \mathrm{div} \varphi \rangle ds - \int_0^t \langle L \nabla_x Q \odot \nabla_x Q - \mathcal{F}(Q) I_3, \nabla_x \varphi \rangle ds \\
& + L \int_0^t \langle Q \triangle Q - \triangle Q Q, \nabla_x \varphi \rangle ds, \\
\mathcal{N}(\rho, \mathbf{q})_t = {} & \sum_{k\geq 1} \int_0^t \langle \mathbf{g}_k(\rho, \mathbf{q}), \varphi \rangle^2 ds, \\
\mathcal{N}_k(\rho, \mathbf{q})_t = {} & \int_0^t \langle \mathbf{g}_k(\rho, \mathbf{q}), \varphi \rangle ds.
\end{aligned}$$

For any $s,\ t \in [0,T]$, we can define the $\mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_{s,t}$ as the increment of the functional from $s$ to $t$, that is,

$$\mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_{s,t} = \mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_t - \mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_s.$$

The same definition holds for $\mathcal{N}(\rho, \mathbf{q})_{s,t}$ and $\mathcal{N}_k(\rho, \mathbf{q})_{s,t}$. From the uniform estimates (4.2.7) and (4.2.8), we can infer that the following mappings

$$\begin{aligned}
&(\rho, \mathbf{v}, \mathbf{q}, Q) \mapsto \mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_t,\\
&(\rho, \mathbf{v}, \mathbf{q}, Q) \mapsto \mathcal{N}(\rho, \mathbf{q})_t,\\
&(\rho, \mathbf{v}, \mathbf{q}, Q) \mapsto \mathcal{N}_k(\rho, \mathbf{q})_t
\end{aligned}$$

are well defined and measurable on the path space $\mathcal{X}_\rho \times \mathcal{X}_{\mathbf{u}} \times \mathcal{X}_{\rho\mathbf{u}} \times \mathcal{X}_Q$, and all the estimates hold true.

To be specific, for $\mathcal{N}(\rho, \mathbf{v}, \mathbf{q}, Q)_t$

$$\sum_{k\geq 1} \int_0^t \langle \mathbf{g}_k(\rho, \mathbf{q})\varphi\rangle^2 ds \leq C \sum_{k\geq 1} \int_0^t \|\mathbf{g}_k(\rho, \mathbf{q})\|_{L^1}^2 ds \leq C.$$

The estimates of $\mathcal{M}(\rho, \mathbf{v}, \mathbf{q}, Q)_t$ and $\mathcal{N}_k(\rho, \mathbf{v})_t$ follow the same way. As stated before, the following variables have the same laws

$$\begin{aligned}
\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon, Q_\varepsilon) &\backsim^d \mathcal{M}(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon),\\
\mathcal{N}(\rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon) &\backsim^d \mathcal{N}(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon),\\
\mathcal{N}_k(\rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon) &\backsim^d \mathcal{N}_k(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon).
\end{aligned}$$

Now fix time $s,\ t \in [0,T]$ with $s < t$, and define

$$h:\ \mathcal{X}_\rho|_{[0,s]} \times \mathcal{X}_{\mathbf{u}}|_{[0,s]} \times \mathcal{X}_Q|_{[0,s]} \times \mathcal{X}_W|_{[0,s]} \to [0,1]$$

as a continuous function. It is easy to infer that from the equality $(4.2.2)_2$

$$\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon, Q_\varepsilon)_t = \int_0^t \langle \Phi(\rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon), \varphi\rangle ds = \sum_{k\geq 1} \int_0^t \langle \mathbf{g}_k(\rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon), \varphi\rangle^2 d\beta_k$$

is a square integrable $(\mathfrak{F}_t)$-martingale, as a result

$$[\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon, Q_\varepsilon)]^2 - \mathcal{N}(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)$$

and

$$\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon, Q_\varepsilon)\beta_k - \mathcal{N}_k(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)$$

are $(\mathfrak{F}_t)$-martingales as well. By the equality of laws

$$\begin{aligned} &\tilde{\mathbb{E}} h(\mathbf{r}_s \tilde{\rho}_\varepsilon, \mathbf{r}_s \tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s \tilde{Q}_\varepsilon, \mathbf{r}_s \tilde{W}_\varepsilon)[\mathcal{M}(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)_{s,t}] \\ =&\mathbb{E} h(\mathbf{r}_s \rho_\varepsilon, \mathbf{r}_s \mathbf{u}_\varepsilon, \mathbf{r}_s Q_\varepsilon, \mathbf{r}_s W_\varepsilon)[\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon, Q_\varepsilon)_{s,t}] = 0, \end{aligned} \tag{4.2.17}$$

$$\begin{aligned} &\tilde{\mathbb{E}} h(\mathbf{r}_s \tilde{\rho}_\varepsilon, \mathbf{r}_s \tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s \tilde{Q}_\varepsilon, \mathbf{r}_s \tilde{W}_\varepsilon) \left[ [\mathcal{M}(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)^2]_{s,t} - \mathcal{N}(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_{s,t} \right] \\ =&\mathbb{E} h(\mathbf{r}_s \rho_\varepsilon, \mathbf{r}_s \mathbf{u}_\varepsilon, \mathbf{r}_s Q_\varepsilon, \mathbf{r}_s W_\varepsilon) \left[ [\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon, Q_\varepsilon)^2]_{s,t} - \mathcal{N}(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)_{s,t} \right] = 0, \end{aligned} \tag{4.2.18}$$

$$\begin{aligned} &\tilde{\mathbb{E}} h(\mathbf{r}_s \tilde{\rho}_\varepsilon, \mathbf{r}_s \tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s \tilde{Q}_\varepsilon, \mathbf{r}_s \tilde{W}_\varepsilon) \left[ [\mathcal{M}(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)\tilde{\beta}_k^\varepsilon]_{s,t} - \mathcal{N}(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_{s,t} \right] \\ =&\mathbb{E} h(\mathbf{r}_s \rho_\varepsilon, \mathbf{r}_s \mathbf{u}_\varepsilon, \mathbf{r}_s Q_\varepsilon, \mathbf{r}_s W_\varepsilon) \left[ [\mathcal{M}(\rho_\varepsilon, \mathbf{u}_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon, Q_\varepsilon)\beta_k]_{s,t} - \mathcal{N}(\rho_\varepsilon, \rho_\varepsilon \mathbf{u}_\varepsilon)_{s,t} \right] = 0. \end{aligned} \tag{4.2.19}$$

The last step is to verify that the Q-tensor equation, $(4.2.2)_3$ is true. Then for any $t \in [0, T]$, $\phi \in C^\infty(\mathbb{T})^{3\times3}$, define the functional

$$\mathcal{P}(\mathbf{v}, Q)_t = \langle Q(t), \phi \rangle - \langle Q(0), \phi \rangle + \int_0^t \langle \mathbf{v} \cdot \nabla_x Q, \phi \rangle ds - \int_0^t \langle \Theta Q - Q\Theta, \phi \rangle ds - \Gamma \int_0^t \langle \mathcal{H}(Q), \phi \rangle ds.$$

We know that the mapping $(\mathbf{v}, Q) \to \mathcal{P}(\mathbf{v}, Q)_t$ is also continuous on the path space $\mathcal{X}_\mathbf{u} \times \mathcal{X}_Q$. Furthermore, we know that the law of $\mathcal{P}(\mathbf{u}_\varepsilon, Q_\varepsilon)_t$ coincide with the law of $\mathcal{P}(\tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)_t$. As $\mathbf{u}_\varepsilon, Q_\varepsilon$ is the solution to $(4.2.2)_3$

$$\tilde{\mathbb{E}}|\mathcal{P}(\tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)_t|^2 = \mathbb{E}|\mathcal{P}(\mathbf{u}_\varepsilon, Q_\varepsilon)_t|^2 = 0.$$

That is, $(\tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)$ also solves $(4.2.2)_3$, so the proof is complete. $\square$

By proving that $(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)$ solve the equation (4.2.2) as well, all the estimates of $(\rho_\varepsilon, \mathbf{u}_\varepsilon, Q_\varepsilon)$ also apply. In particular, we have from (4.2.10)

$$\tilde{\rho}_\varepsilon \to 1 \ \text{ in } L^p(\Omega; L^\infty([0,T]; L^\gamma(\mathbb{T}))) \ \tilde{\mathbb{P}} - \text{a.s}, \tag{4.2.20}$$

and the following bounds hold.

**Proposition 4.2.15.** *For all $p \in [1, \infty)$, we have the following uniform bounds if $l > \frac{3}{2}$,*

$$\begin{aligned}
&\sqrt{\tilde{\rho}_\varepsilon}\tilde{\mathbf{u}}_\varepsilon \textit{ is bounded in } L^p(\Omega; L^\infty([0,T]; L^2(\mathbb{T}))),\\
&\tilde{Q}_\varepsilon \textit{ is bounded in } L^p(\Omega; L^\infty([0,T]; W^{1,2}(\mathbb{T}))),\\
&\tilde{\varphi}_\varepsilon \textit{ is bounded in } L^p(\Omega; L^\infty([0,T]; L^{\min(2,\gamma)}(\mathbb{T}))),\\
&\tilde{\mathbf{F}}_\varepsilon \textit{ is bounded in } L^p(\Omega; L^\infty([0,T]; W^{-l,2}(\mathbb{T}))).
\end{aligned} \tag{4.2.21}$$

*with $\tilde{\varphi}_\varepsilon = \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon}$ and*

$$\begin{aligned}
\tilde{\mathbf{F}}_\varepsilon = {} & \nu_\varepsilon \triangle \mathcal{Q}\tilde{\mathbf{u}}_\varepsilon - \mathcal{Q}\mathrm{div}_x(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) - \frac{1}{\varepsilon^2}\nabla_x A(\tilde{\rho}_\varepsilon^\gamma - 1 - \gamma(\tilde{\rho}_\varepsilon - 1))\\
& - \mathcal{Q}(\mathrm{div}_x(L\nabla_x \tilde{Q}_\varepsilon \odot \nabla_x \tilde{Q}_\varepsilon - \mathcal{F}_\varepsilon(\tilde{Q}_\varepsilon)I_3)) + L\mathcal{Q}(\mathrm{div}_x(\tilde{Q}_\varepsilon \triangle \tilde{Q}_\varepsilon - \triangle \tilde{Q}_\varepsilon \tilde{Q}_\varepsilon)),
\end{aligned}$$

*Note that all these bounds are uniform with respect to $\varepsilon$, and hold for any $T > 0$.*

After we have got all the uniform bounds, we can easily get that

$$\mathcal{P}_H \tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{u}} \text{ in } L^2(0,T; L^q(\mathbb{T})), \ \tilde{\mathbb{P}} - a.s, \ \text{for any } q < 6, \tag{4.2.22}$$

we denote its dual space by

$$W^{-l,2}_{\mathrm{div}}(\mathbb{T}) = \left[W^{l,2}_{\mathrm{div}}(\mathbb{T})\right]^*.$$

Note that two elements of $W^{-l,2}_{\mathrm{div}}(\mathbb{T})$ are identical if their difference is a gradient. With the definition, we can prove that

$$\mathrm{div}_x(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \mathrm{div}_x(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \text{ in } L^1(0,T; W^{-l,2}_{\mathrm{div}}(\mathbb{T})). \tag{4.2.23}$$

The proof of (4.2.22) and (4.2.23) follows from [10].

Then we proceed to the proof of Theorem 4.2.13. From Proposition 4.2.14, it's easy for us to get that for any $\varepsilon \in (0,1)$, all $\tilde{W}_\varepsilon$ are cylindrical Wiener processes. As a result, there exists a collection of mutually independent real-valued $(\tilde{\mathfrak{F}}_t)$-Wiener processes $(\tilde{\beta}_k^\varepsilon)_{k\geq 1}$, such that $\tilde{W} = \sum_{k\geq 1} \tilde{\beta}_k^\varepsilon e_k$.

It remains to show that the equation (4.2.3) is satisfied. Take any divergence free test function $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T})$, use all the notations in Proposition 4.2.14, when $\varepsilon \to 0$, then we need to show that

$$\tilde{\mathbb{E}} h(\mathbf{r}_s \tilde{\mathbf{u}}, \mathbf{r}_s \tilde{Q}, \mathbf{r}_s \tilde{W}_\varepsilon)[\mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q})_{s,t}] = 0, \tag{4.2.24}$$

$$\tilde{\mathbb{E}} h(\mathbf{r}_s \tilde{\mathbf{u}}, \mathbf{r}_s \tilde{Q}, \mathbf{r}_s \tilde{W}_\varepsilon) \left[ [\mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q})^2_{s,t}] - \mathcal{N}(1, \tilde{\mathbf{u}})_{s,t} \right] = 0, \tag{4.2.25}$$

$$\tilde{\mathbb{E}} h(\mathbf{r}_s \tilde{\mathbf{u}}, \mathbf{r}_s \tilde{Q}, \mathbf{r}_s \tilde{W}_\varepsilon) \left[ [\mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q}) \tilde{\beta}_k]_{s,t} - \mathcal{N}_k(1, \tilde{\mathbf{u}})_{s,t} \right] = 0. \tag{4.2.26}$$

By proving these, the proof is complete. Note that from the equation, we can get that the process $\mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q})$ is a $\tilde{\mathfrak{F}}_t$-martingale, and its quadratic and cross variations satisfy

$$\langle\langle \mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q}) \rangle\rangle = \mathcal{N}(1, \tilde{\mathbf{u}}), \qquad \langle\langle \mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q}), \tilde{\beta}_k \rangle\rangle = \mathcal{N}_k(1, \tilde{\mathbf{u}}),$$

also, we have

$$\left\langle\left\langle \mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q}) - \int_0^{\cdot} \langle \Phi(1, \tilde{\mathbf{u}}) d\tilde{W}, \varphi \rangle \right\rangle\right\rangle = 0.$$

Then the equation $(4.2.3)_1$ is satisfied in the sense of definition 4.2.2.

Now we verify (4.2.24)-(4.2.26). By Proposition 4.2.12 and (4.2.23), we know that

$$\mathcal{M}(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon, \tilde{Q}_\varepsilon)_t \to \mathcal{M}(1, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{Q})_t \text{ a.s.}$$

Considering all the uniform estimates (4.2.6)-(4.2.8), the passage to the limit is thus justified.

Last, we work on the limit of the stochastic integral. We can start by showing that

$$\langle \Phi(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \cdot, \varphi) \rangle \to \langle \Phi(1, \tilde{\mathbf{u}}) \cdot, \varphi) \rangle \text{ in } L^2(\mathcal{U}; \mathbb{R}) \quad \tilde{\mathbb{P}} \otimes \mathcal{L} - a.e. \tag{4.2.27}$$

We can write that

$$\begin{aligned}
&\| \langle \Phi(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \cdot, \varphi) \rangle - \langle \Phi(1, \tilde{\mathbf{u}}) \cdot, \varphi) \rangle \|_{L^2(\mathcal{U};\mathbb{R})} \\
&\leq (\sum_{k\geq 1} |\langle \mathbf{h}_k(\tilde{\rho}_\varepsilon) - \mathbf{h}_k(1), \varphi \rangle|^2)^{\frac{1}{2}} + (\sum_{k\geq 1} |\alpha_k|^2 |\langle \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{u}}, \varphi \rangle|^2)^{\frac{1}{2}} = I_1 + I_2.
\end{aligned}$$

By (4.2.4) and (4.2.22), $I_2 \to 0$ for a.e $(\omega, t)\mathcal{U} \times \mathbb{R}$. We then use the Minkowski integral inequality to deal with $I_1$, that is

$$\begin{aligned} I_1 &\leq C(\sum_{k\geq 1} \|\mathbf{h}_k(\tilde{\rho}_\varepsilon) - \mathbf{h}_k(1)\|_{L^1}^2)^{\frac{1}{2}} \leq C\int_{\mathbb{T}}(\sum_{k\geq 1} \|\mathbf{h}_k(\tilde{\rho}_\varepsilon) - \mathbf{h}_k(1)\|_{L^1}^2)^{\frac{1}{2}}dx \\ &\leq C\int_{\mathbb{T}}(1+\tilde{\rho}_\varepsilon^{\frac{\gamma-1}{2}})|\tilde{\rho}_\varepsilon - 1|dx \leq C[\int_{\mathbb{T}}(1+\tilde{\rho}_\varepsilon^{\frac{\gamma-1}{2}})^p dx]^{\frac{1}{p}}[\int_{\mathbb{T}}|\tilde{\rho}_\varepsilon - 1|^q dx]^{\frac{1}{q}}, \end{aligned}$$

where the constants $p$, $q \in (1, \infty)$ are conjugate and satisfy

$$p\frac{\gamma-1}{2} < \gamma + 1 \quad \text{and } q < r.$$

From the estimate of $\tilde{\rho}_\varepsilon$ and (4.2.20)

$$\tilde{\mathbb{E}}\int_0^T I_1 dt \to 0.$$

So $I_1 \to 0$ for a.e $(\omega, t) \in \mathcal{U} \times \mathbb{R}$, up to a subsequence. Therefore, (4.2.27) follows, and we can get

$$\sum_{k\geq 1}\langle \mathbf{g}_k(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon), \varphi\rangle^2 \to \sum_{k\geq 1}\langle \mathbf{g}_k(1, \tilde{\mathbf{u}}), \varphi\rangle^2, \quad \tilde{\mathbb{P}} \otimes \mathcal{L} - a.e.$$

Moreover, for all $p \geq 2$,

$$\begin{aligned} &\tilde{\mathbb{E}}\int_s^t \|\langle \Phi(\tilde{\rho}_\varepsilon, \tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\cdot, \varphi)\rangle\|_{L^2(\mathcal{U};\mathbb{R}}^p dr \\ \leq& C\tilde{\mathbb{E}}\int_s^t \|\tilde{\rho}_\varepsilon\|_{L^2}^{\frac{p}{2}}(1 + \|\tilde{\rho}_\varepsilon\|_{L^\gamma}^\gamma + \|\sqrt{\tilde{\rho}_\varepsilon}\tilde{\mathbf{u}}_\varepsilon\|_{L^2}^2)^{\frac{p}{2}} dr \\ \leq& C(1 + \tilde{\mathbb{E}}\sup_{0\leq t\leq T}\|\tilde{\rho}_\varepsilon\|_{L^\gamma}^{\gamma p} + \tilde{\mathbb{E}}\sup_{0\leq t\leq T}\|\sqrt{\tilde{\rho}_\varepsilon}\tilde{\mathbf{u}}_\varepsilon\|_{L^2}^{2p}) \leq C. \end{aligned}$$

From the boundedness, the weak convergence yields, then the limit $(\tilde{\mathbf{u}}, \tilde{Q})$ solves (4.2.3). It follows that for all $p \in [1, \infty)$

$$\tilde{\mathbf{u}} \in L^p(\tilde{\Omega}; L^2(0, T; W^{1,2}_{\text{div}}(\mathbb{T}))).$$

As from Proposition 4.2.12 and (4.2.20), we have

$$\sqrt{\tilde{\rho}_\varepsilon}\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde{\mathbf{u}} \text{ in } L^1(\tilde{\Omega}; L^1(0, T; L^1(\mathbb{T}))).$$

By the semi-continuity of the functional

$$\tilde{w} \mapsto \tilde{\mathbb{E}}\left[\sup_{0<t<T} \int_{\mathbb{T}} |\tilde{w}|^2 dx\right]^{\frac{p}{2}},$$

one has that $\tilde{\mathbf{u}} \in L^p(\tilde{\Omega}; L^\infty(0,T; L^2(\mathbb{T})))$ by (4.2.21). Finally, the convergence in distribution implies that

$$\tilde{\mathbf{u}} \in L^p(\tilde{\Omega}; C_w(0,T; L^2_{\mathrm{div}}(\mathbb{T}))).$$

Then the proof is complete.

## Appendix Some Basic Theories and Lemmas

In the appendix, we present some theorems and lemmas that will be used frequently in this paper.

**Lemma A.0.1.** [35, Theorem 2.1] *Suppose that $X_1 \subset X_0 \subset X_2$ are Banach spaces, $X_1$ and $X_2$ are reflexive, and the embedding of $X_1$ into $X_0$ is compact. Then for any $1 < p < \infty$, $0 < \alpha < 1$, the embedding*

$$L^p(0,T;X_1) \cap W^{\alpha,p}(0,T;X_2) \hookrightarrow L^p(0,T;X_0)$$

*is compact.*

**Lemma A.0.2.** [5, Theorem 1.1.1] *Let $\mathcal{L} : L^{p_1}(0,T) \to L^{p_2}(\mathcal{D})$ and $L^{q_1}(0,T) \to L^{q_2}(\mathcal{D})$ be a linear operator with $q_1 > p_1$ and $q_2 < p_2$. Then, for any $s \in (0,1)$, the operator $\mathcal{L} : L^{r_1}(0,T) \to L^{r_2}(\mathcal{D})$, where $r_1 = \frac{1}{s/p_1+(1-s)/q_1}$, $r_2 = \frac{1}{s/p_2+(1-s)/q_2}$.*

**Theorem A.0.3.** [50, Chapter 3] *Let $p \geq 1$, $\{X_n\}_{n\geq 1} \in L^p$ and $X_n \to X$ in probability. Then, the following are equivalent*
(1). *$X_n \to X$ in $L^p$;*
(2). *the sequence $|X_n|^p$ is uniformly integrable;*
(3). *$\mathbb{E}|X_n|^p \to \mathbb{E}|X|^p$.*

**Theorem A.0.4.** [45, Theorem 1] *Let $X$ be a quasi-Polish space. If the set of probability measures $\{\nu_n\}_{n\geq 1}$ on $\mathcal{B}(X)$ is tight, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of random variables $u_n, u$ such that theirs laws are $\nu_n$, $\nu$ and $u_n \to u$, $\mathbb{P}$ a.s. as $n \to \infty$.*

**Lemma A.0.5.** *(The Aubin-Lions Lemma,* [58, Chapter I]*) Suppose that $X_1 \subset X_0 \subset X_2$ are Banach spaces, $X_1$ and $X_2$ are reflexive, and the embedding of $X_1$ into $X_0$ is compact. Then for any $1 < p < \infty$, $0 < \alpha < 1$, the embedding*

$$L^p(0,T;X_1) \cap W^{\alpha,p}(0,T;X_2) \hookrightarrow L^p(0,T;X_0),$$

$$L^\infty(0,T;X_1) \cap C^\alpha([0,T];X_2) \hookrightarrow L^\infty(0,T;X_0)$$

*is compact.*

**Theorem A.0.6.** *(The Vitali convergence theorem,* [50, Chapter 3]*) Let* $p \geq 1$, $\{u_n\}_{n\geq 1} \in L^p$ *and* $u_n \to u$ *in probability. Then, the following are equivalent*

(1). $u_n \to u$ *in* $L^p$*;*

(2). *the sequence* $|u_n|^p$ *is uniformly integrable;*

(3). $\mathbb{E}|u_n|^p \to \mathbb{E}|u|^p$.

**Theorem A.0.7.** *(The Skorokhod representation theorem,* [80, Theorem 1]*) Let* $X$ *be a Polish space. If the set of probability measures* $\{\nu_n\}_{n\geq 1}$ *on* $\mathcal{B}(X)$ *is tight, then there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a sequence of random variables* $u_n, u$ *such that theirs laws are* $\nu_n$, $\nu$ *and* $u_n \to u$, $\mathbb{P}$ *a.s. as* $n \to \infty$.

## Bibliography


[1] T. Alazard. Low mach number limit of the full navier-stokes equations. *Arch. Ration. Mech. Anal.*, 180(1):1–73, 2006.

[2] F. De Anna. A global 2d well-posedness result on the order tensor liquid crystal theory. *J. Differential Equations*, 262(7):3932–3979, 2017.

[3] J. M. Ball and A.Majumdar. Nematic liquid crystals: from maier-saupe to a continuum theory. *Molecular crystals and liquid crystals*, 525(1):1–11, 2010.

[4] J. M. Ball and A. Zarnescu. Orientability and energy minimization in liquid crystal models. *Archive for rational mechanics and analysis*, 202(2):493–535, 2011.

[5] J. Bergh and J. Löfström. *Interpolation spaces: an introduction*, volume 223. Springer Science & Business Media, 2012.

[6] A. N. Beris and B. J. Edwards. *Thermodynamics of flowing systems: with internal microstructure*. Number 36. Oxford University Press on Demand, 1994.

[7] M. E. Bogovskii. Solution of some vector analysis problems connected with operators div and grad (in russian). *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, 80:5–40, 1980.

[8] W. Borchers and H. Sohr. On the equations rot v= g and div u= f with zero boundary conditions. *Hokkaido Math. J.*, 19(1):67–87, 1990.

[9] D. Breit and E. Feireisl. Stochastic navier-stokes-fourier equations. *Indiana Univ. Math. J.*, 69(3):911—975, 2020.

[10] D. Breit, E. Feireisl, and M. Hofmanová. Incompressible limit for compressible fluids with stochastic forcing. *Arch. Ration. Mech. Anal.*, 222(2):895–926, 2016.

[11] D. Breit, E. Feireisl, and M. Hofmanová. Local strong solutions to the stochastic compressible navier–stokes system. *Comm. Partial Differential Equations*, 43(2):313–345, 2018.

[12] D. Breit, E. Feireisl, and M. Hofmanová. *Stochastically Forced Compressible Fluid Flows*, volume 3. Walter de Gruyter GmbH & Co KG, 2018.

[13] D. Breit, E. Feireisl, M. Hofmanová, and B. Maslowski. Stationary solutions to the compressible navier–stokes system driven by stochastic forces. *Probab. Theory Related Fields*, 174(3):981–1032, 2019.

[14] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1):211–223, 2003.

[15] D. Bresch and B. Desjardins. On the existence of global weak solutions to the navier–stokes equations for viscous compressible and heat conducting fluids. *J. Math. Pures Appl.*, 87(1):57–90, 2007.

[16] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby. Some results on the penalised nematic liquid crystals driven by multiplicative noise: weak solution and maximum principle. *Stoch. Partial Differ. Equ. Anal. Comput.*, 7(3):417–475, 2019.

[17] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby. Strong solution to stochastic penalised nematic liquid crystals model driven by multiplicative gaussian noise. *Indiana Univ. Math. J.*, 70(5):2177–2235, 2021.

[18] Z. Brzeźniak, U. Manna, and A. A. Panda. Martingale solutions of nematic liquid crystals driven by pure jump noise in the marcus canonical form. *J. Differential Equations*, 266(10):6204–6283, 2019.

[19] Z. Brzeźniak, B. Maslowski, and J. Seidler. Stochastic nonlinear beam equations. *Probab. Theory Related Fields*, 132(1):119–149, 2005.

[20] Z. Brzeźniak and M. Ondreját. Strong solutions to stochastic wave equations with values in riemannian manifolds. *J. Funct. Anal.*, 253(2):449–481, 2007.

[21] G. Q. Chen, A. Majumdar, D. Wang, and R. Zhang. Global existence and regularity of solutions for active liquid crystals. *J. Differential Equations*, 263(1):202–239, 2017.

[22] G. Q. Chen, A. Majumdar, D. Wang, and R. Zhang. Global weak solutions for the compressible active liquid crystal system. *SIAM J. Math. Anal.*, 50(4):3632–3675, 2018.

[23] A. Czirók, E. Ben-Jacob, I. Cohen, and T. Vicsek. Formation of complex bacterial colonies via self-generated vortices. *Physical Review E*, 54(2):1791, 1996.

[24] B. Desjardins, P-L. Lions E. Grenier, and N. Masmoudi. Incompressible limit for solutions of the isentropic navier–stokes equationswith dirichlet boundary conditions. *Journal de Mathématiques Pures et Appliquées*, 78(5):461–471, 1999.

[25] B. Desjardins and E. Grenier. Low mach number limit of viscous compressible flows in the whole space. (english summary). *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455(1986):2271–2279, 1999.

[26] S. Ding, J. Huang, H. Wen, and R. Zi. Incompressible limit of the compressible nematic liquid crystal flow. *J. Funct. Anal.*, 264(7):1711–1756, 2013.

[27] M. Doi and S. F. Edwards. *The theory of polymer dynamics*, volume 73. oxford university press, 1988.

[28] C. Dou, S. Jiang, and Y. Ou. Low mach number limit of full navier–stokes equations in a 3d bounded domain. *J. Differential Equations*, 258(2):379–398, 2015.

[29] J. L. Ericksen. Conservation laws for liquid crystals. *Trans. Soc. Rheol.*, 5(1):23–34, 1961.

[30] E. Feireisl, B. Maslowski, and A. Novotnỳ. Compressible fluid flows driven by stochastic forcing. *J. Differential Equations*, 254(3):1342–1358, 2013.

[31] E. Feireisl and A. Novotnỳ. The low mach number limit for the full navier–stokes–fourier system. *Arch. Ration. Mech. Anal.*, 186(1):77–107, 2007.

[32] E. Feireisl, A. Novotnỳ, and H. Petzeltová. On the existence of globally defined weak solutions to the navier—stokes equations. *J. Math. Fluid Mech.*, 3(4):358–392, 2001.

[33] E. Feireisl, E. Rocca, G. Schimperna, and A. Zarnescu. Evolution of non-isothermal landau–de gennes nematic liquid crystals flows with singular potential. *Commun. Math. Sci.*, 12(2):317–343, 2014.

[34] J. Feng, C. V. Chaubal, and L. G. Leal. Closure approximations for the doi theory: Which to use in simulating complex flows of liquid-crystalline polymers? *Journal of Rheology*, 42(5):1095–1119, 1998.

[35] F. Flandoli and D. Gatarek. Martingale and stationary solutions for stochastic navier-stokes equations. *Probab. Theory Related Fields*, 102(3):367–391, 1995.

[36] G. P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Volume I: Linearised Steady Problems*, volume 38. Springer Science & Business Media, 1998.

[37] P. G. De Gennes and J. Prost. *The physics of liquid crystals*. Number 83. Oxford university press, 1993.

[38] N. Glatt-Holtz and M. Ziane. Strong pathwise solutions of the stochastic navier-stokes system. *Adv. Differential Equations*, 14(5/6):567–600, 2009.

[39] N. E. Glatt-Holtz and V. C. Vicol. Local and global existence of smooth solutions for the stochastic euler equations with multiplicative noise. *Ann. Probab.*, 42(1):80–145, 2014.

[40] I. Gyöngy and N. Krylov. Existence of strong solutions for itô's stochastic equations via approximations. *Probab. Theory Related Fields*, 105(2):143–158, 1996.

[41] D. Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.*, 132(1):1–14, 1995.

[42] D. Hoff. Dynamics of singularity surfaces for compressible, viscous flows in two space dimensions. *Comm. Pure Appl.*, 55(11):1365–1407, 2002.

[43] M. Hofmanová and D. Breit. Stochastic navier-stokes equations for compressible fluids. *Indiana Univ. Math. J.*, 65(4), 2016.

[44] X. Hu and D. Wang. Low mach number limit of viscous compressible magnetohydrodynamic flows. *SIAM J. Math. Anal.*, 41(3):1272–1294, 2009.

[45] A. Jakubowski. The almost sure skorokhod representation for subsequences in nonmetric spaces. *Theory Probab. Appl.*, 42(1):167–174, 1998.

[46] F. Jiang, S. Jiang, and D. Wang. On multi-dimensional compressible flows of nematic liquid crystals with large initial energy in a bounded domain. *J. Funct. Anal.*, 265(12):3369–3397, 2013.

[47] F. Jiang, S. Jiang, and D. Wang. Global weak solutions to the equations of compressible flow of nematic liquid crystals in two dimensions. *Arch. Ration. Mech. Anal.*, 214(2):403–451, 2014.

[48] S. Jiang, Q. Ju, and F. Li. Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. *Comm. Math. Phys.*, 297(2):371–400, 2010.

[49] S. Jiang, Q. Ju, and F. Li. Low mach number limit for the multi-dimensional full magnetohydrodynamic equations. *Nonlinearity*, 25(5):1351, 2012.

[50] O. Kallenberg. *Foundations of modern probability*, volume 2. Springer, 1997.

[51] T. Kato and G. Ponce. Commutator estimates and the euler and navier-stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[52] J. Kierfeld, K. Frentzel, P. Kraikivski, and R. Lipowsky. Active dynamics of filaments in motility assays. *The European Physical Journal Special Topics*, 157(1):123–133, 2008.

[53] J. U. Kim. Existence of a local smooth solution in probability to the stochastic euler equations in r3. *J. Funct. Anal.*, 256(11):3660–3687, 2009.

[54] S. Klainerman and A. Majda. Compressible and incompressible fluids. *Comm. Pure Appl. Math.*, 35:629–651, 1982.

[55] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. *Stochastic differential equations: theory and applications*, pages 1–69, 2007.

[56] X. Li and D. Wang. Global strong solution to the density-dependent incompressible flow of liquid crystals. *Trans. Amer. Math. Soc.*, 367(4):2301–2338, 2015.

[57] C. K. Lin. On the incompressible limit of the compressible navier-stokes equations. *Comm. Partial Differential Equations*, 20(3-4):677–707, 1995.

[58] J. L. Lions. *Quelques méthodes de résolution des problemes aux limites non linéaires.* Dunod, 1969.

[59] P. L. Lions. *Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Mod-*

*els*, volume 2. Oxford University Press on Demand, 199.

[60] P. L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. (english, french summary). *J. Math. Pures Appl.*, 77(6):585–627, 1998.

[61] Y. Liu and W. Wang. The oseen–frank limit of onsager's molecular theory for liquid crystals. *Arch. Ration. Mech. Anal.*, 227(3):1061–1090, 2018.

[62] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Springer Science & Business Media, 2012.

[63] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53. Springer Science & Business Media, 2012.

[64] A. Majumdar. Equilibrium order parameters of nematic liquid crystals in the landau-de gennes theory. *European J. Appl. Math.*, 21(2):181–203, 2010.

[65] N. Masmoudi. Incompressible, inviscid limit of the compressible navier–stokes system. (english, french summary). *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(2):199–224, 2001.

[66] N. Masmoudi. Examples of singular limits in hydrodynamics. *Handb. Differ. Equ.*, 3:195–275, 2007.

[67] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(9):337–342, 1979.

[68] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1):67–104, 1980.

[69] A. Matsumura and T. Nishida. Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.*, 89(4):445–464, 1983.

[70] R. Mikulevicius and B. L. Rozovskii. Stochastic navier–stokes equations for turbulent flows. *SIAM J. Math. Anal.*, 35(5):1250–1310, 2004.

[71] L. Onsager. The effects of shape on the interaction of colloidal particles. *Annals of the New York Academy of Sciences*, 51(4):627–659, 1949.

[72] C. W. Oseen. The theory of liquid crystals. *Transactions of the Faraday Society*, 29(140):883–899, 1933.

[73] M. Paicu and A. Zarnescu. Global existence and regularity for the full coupled navier–stokes and q-tensor system. *SIAM J. Math. Anal.*, 43(5):2009–2049, 2011.

[74] M. Paicu and A. Zarnescu. Energy dissipation and regularity for a coupled navier–stokes and q-tensor system. *Arch. Ration. Mech. Anal.*, 203(1):45–67, 2012.

[75] G. Da Prato and J. Zabczyk. Stochastic equations in infinite dimensions. *Encyclopedia of Mathematics and its Applications*, 44, 1992.

[76] Z. Qiu and Y. Wang. Strong solution for compressible liquid crystal system with random force. *arXiv preprint arXiv:2003.06074*, 2020.

[77] Z. Qiu and Y. Wang. Martingale solution for stochastic active liquid crystal system. *Discrete Contin. Dyn. Syst.*, 41(5):2227–2268, 2021.

[78] T. Sanchez, D. TN. Chen, S. J. DeCamp, M. Heymann, and Z. Dogic. Spontaneous motion in hierarchically assembled active matter. *Nature*, 491(7424):431–434, 2012.

[79] S. Schochet. The mathematical theory of low mach number flows. *M2AN Math. Model. Numer. Anal.*, 39(3):441–458, 2005.

[80] A. V. Skorokhod. Limit theorems for stochastic processes. (russian). *Teor. Veroyatnost. i Primenen.*, 1(3):289–319, 1956.

[81] S. A. Smith. Random perturbations of viscous, compressible fluids: global existence of weak solutions. *SIAM J. Math. Anal.*, 49(6):4521–4578, 2017.

[82] S. A. Smith and K. Trivisa. The stochastic navier–stokes equations for heat-conducting, compressible fluids: global existence of weak solutions. *J. Evol. Equ.*, 18(2):411–465, 2018.

[83] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical review letters*, 75(6):1226, 1995.

[84] D. Wang and H. Wang. Global existence of martingale solutions to the three-dimensional stochastic compressible navier-stokes equations. *Differential Integral Equations*, 28(11/12):1105–1154, 2015.

[85] D. Wang, X. Xu, and C. Yu. Global weak solution for a coupled compressible navier-stokes and $q$-tensor system. *Commun. Math. Sci.*, 13(1):49–82, 2015.

[86] D. Wang and C. Yu. Global weak solution and large-time behavior for the compressible flow of liquid crystals. *Arch. Ration. Mech. Anal.*, 204(3):881–915, 2012.

[87] D. Wang and C. Yu. Incompressible limit for the compressible flow of liquid crystals. *J. Math. Fluid Mech.*, 16(4):771–786, 2014.

[88] L. Wang, J. L. Wu, and G. Zhou. Global well-posedness of stochastic nematic liquid crystals with random initial and random boundary conditions driven by multiplicative noise. *arXiv preprint arXiv:1906.11472*, 2019.

[89] W. Wang, P. Zhang, and Z. Zhang. Rigorous derivation from landau–de gennes theory to ericksen-leslie theory. (english summary). *SIAM J. Math. Anal.*, 47(1):127–158, 2015.

[90] H. H. Wensink, J. Dunkel, S. Heidenreich, K. Drescher, R. E. Goldstein, H. Löwen, and J. M. Yeomans. Meso-scale turbulence in living fluids. *Proceedings of the national academy of sciences*, 109(36):14308–14313, 2012.

[91] M. Wilkinson. Strictly physical global weak solutions of a navier–stokes q-tensor system with singular potential. *Arch. Ration. Mech. Anal.*, 218(1):487–526, 2015.

[92] Z. Xin and X. Zhang. From the landau-de gennes theory to the ericksen-leslie theory in dimension two. *arXiv preprint arXiv:2105.10652*, 2021.

[93] X. Yang. Uniform well-posedness and low mach number limit to the compressible nematic liquid crystal flows in a bounded domain. (english summary). *Nonlinear Anal.*, 120:118–126, 2015.

[94] P. Zhang et al. A molecular kinetic theory of inhomogeneous liquid crystal flow and the small deborah number limit. *Methods Appl. Anal.*, 13(2):181–198, 2006.