Counting parabolic principal G-bundles with nilpotent sections over $\mathbb{P}^1$

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A Higgs bundle over an algebraic curve is a vector bundle with a twisted endomorphism. An important question is to calculate the volume of the groupoid of Higgs bundles over finite fields. In 2014, Olivier Schiffmann succeeded in finding the corresponding generating function and together with Mozvogoy reduced the problem to counting pairs of a vector bundle and a nilpotent endomorphism. It was generalized recently by Anton Mellit to the case of Higgs bundles with regular singularities. An important step in Mellit's calculations is the case of $\mathbb{P}^1$ and two marked points, which allows him to relate the corresponding generating function with the Macdonald polynomials. It is a natural question to generalize Mellit's calculations to arbitrary reductive groups.

We consider the case of $\mathbb{P}^1$ with two marked points and an arbitrary split connected reductive group G over $\mathbb{F}_q$. Firstly, we give an explicit formula for the number of $\mathbb{F}_q$-rational points of generalized Steinberg varieties of G. Secondly, for each principal G-bundle over $\mathbb{P}^1$, we give an explicit formula counting the number of triples consisting of parabolic structures at 0 and ∞ and compatible nilpotent sections of the associated adjoint bundle.

Keywords: Reductive group, principal G-bundle, parabolic structure, generalized Springer variety, generalized Steinberg variety, affine fibration, stratification, coproduct, symmetric function.

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1.0 Introduction.

Let k be a field. Let G be a reductive group over k. Reductive groups include some of the most important groups in mathematics, such as the group of invertible matrices GL_n, the special orthogonal group SO_n and the symplectic group Sp_{2n}.

Let X be an algebraic curve over k. By a principal G-bundle over X, we mean a morphism $\pi : \mathcal{E} \to X$ with a fibrewise action of G on $\mathcal{E}$ such that for any $q \in X$, there is an etale neighbourhood U of q such that there exists an isomorphism $\mathcal{E}_U \cong U \times G$ compatible with the action of G, that is, $\mathcal{E}$ locally looks like $U \times G$ in the etale topology. A typical example of a principal bundle is the frame bundle of a vector bundle, which consists of all ordered bases of the vector space attached to each point. The group GL_n acts on the collection of all ordered basis by changes of basis. From this perspective, principal bundles are a generalization of the vector bundles to more general groups and also include orthogonal and symplectic bundles, which are vector bundles with a non-degenerate symmetric and skew-symmetric bilinear form respectively. Principal bundles are central objects in the geometric Langlands program ([2], [16], [17], [37]).

A Higgs G-bundle over X is a principal G-bundle over X together with a section of the adjoint vector bundle twisted by the canonical bundle (in the case of vector bundles, roughly speaking it can be thought of as a matrix of 1-forms on the curve). Higgs G-bundles have a rich structure and recently received vast attention from researchers ([16], [17], [37]). In this thesis, we are interested in the moduli stack of Higgs G-bundles over an $\mathbb{F}_q$-algebraic curve, whose points parametrize Higgs G-bundles over the curve.

Let us now give more precise definitions (see Section 2.2). Let $\mathbb{F}_q$ denote the finite field of q elements. In this introduction from now on, our base field will be $\mathbb{F}_q$. Let X be a smooth geometrically connected projective curve over $\mathbb{F}_q$ (geometrically connected means that the curve remains connected after base change to the algebraic closure $\overline{\mathbb{F}}_q$). Let $\mathcal{E}$ be a principal G-bundle over X, then we can form a vector bundle $\mathrm{ad}(\mathcal{E})$ over X, which is the vector bundle associated to the adjoint representation of G. It is defined as follows: let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G and Ad: $G \to GL(\mathfrak{g})$ be the adjoint representation, then

$\mathrm{ad}(\mathcal{E})$ is the quotient of $\mathcal{E} \times \mathfrak{g}$ under the right action of G given by $g \cdot (e, f) = (e \cdot g, \mathrm{Ad}_{g^{-1}}(f))$, $e \in \mathcal{E}, f \in \mathfrak{g}, g \in G$.

Definition. Let Ω_X denote the line bundle of algebraic differential 1-forms on X. A *Higgs G-bundle* over X is a pair $(\mathcal{E}, \Theta)$, where $\mathcal{E}$ is a principal G-bundle over X and Θ is a global section of the vector bundle $\mathrm{ad}(\mathcal{E}) \otimes \Omega_X$. The section Θ is called a *Higgs field*.

1.1 Volume of stack of Higgs G-bundles.

It is a natural question to calculate the volume of the groupoid of Higgs G-bundles over a finite field. In addition, we mention two important applications. When $G = GL_n$, counting the number of stable Higgs bundles over X is related to the number of geometrically indecomposable vector bundles (vector bundles that remain indecomposable after base change to the algebraic closure) (see [37, Theorem 1.2]). Now let G be a reductive group. Define a principal G-bundle over X to be *indecomposable* if it does not admit a reduction to a proper Levi subgroup. In this case, we expect that counting the number of stable Higgs G-bundles is related to the number of geometrically indecomposable principal G-bundles, (that is principal G-bundles that remain indecomposable after base change to the algebraic closure).

Another application is related to the E-polynomial [24, Definition 2.1.4], which is an important invariant of an algebraic variety over $\mathbb{C}$. By a theorem of Katz [24, Theorem 2.1.8], in many cases one can compute the E-polynomial of a separated scheme of finite type over $\mathbb{C}$ by counting its number of points over finite fields. Since Higgs G-bundles over a complex algebraic variety form a stack over $\mathbb{C}$, one computes its E-polynomial by calculating the volumes of groupoids of Higgs G-bundles over finite fields.

Let us consider the case $G = GL_n$ as this case is completely solved. One shows that the number of Higgs GL_n-bundles is infinity. In order to overcome this problem, one needs to impose a stability condition on the Higgs bundles that we consider to get a finite answer:

Definition. Let (V, Θ) be a Higgs bundle over X. A subbundle $W \subset V$ is called Θ-invariant if $\Theta(W) \subset W \otimes \Omega_X$. We say that the Higgs bundle (V, Θ) is stable if for every proper Θ-invariant subbundle $W \subset V$, we have

$$\deg(W)/\operatorname{rk}(W) < \deg(V)/\operatorname{rk}(V).$$

In a breakthrough paper [37], O. Schiffmann computed the number of stable Higgs bundles over X of coprime rank and degree when $\mathrm{char}(\mathbb{F}_q)$ is sufficiently large (see [37, Theorem 1.2]). In a later paper [32] with Mozvogoy, the condition on $\mathrm{char}(\mathbb{F}_q)$ was removed. A major step in their calculation is computing the weighted number of vector bundles over X with nilpotent endomorphisms.

It is clear that, while the general strategy of Schiffmann may work for arbitrary reductive group G, there are significant difficulties to overcome in the general case. One of the difficulties comes from the fact that while the conjugacy classes of nilpotent matrices of size n are easily parametrized by partitions of n (thanks to Jordan form theorem), it is complicated to describe conjugacy classes of nilpotent elements of $\mathfrak{g}$ for a general reductive group G.

1.2 Volumes of stacks of parabolic Higgs G-bundles.

A. Mellit in [29] has generalized the result of Mozvogoy and Schiffmann to the parabolic case. In particular, Mellit counts vector bundles over X with nilpotent endomorphisms preserving parabolic structures at marked points. An important part of his calculation is the case of $\mathbb{P}^1$ and two marked points. This case allows him to relate the count with modified Macdonald polynomials. It is a natural question to generalize Mellit's calculation to arbitrary reductive groups. In this thesis, we complete this step, namely, we count the number of principal G-bundles over $\mathbb{P}^1$ with nilpotent sections of adjoint bundles compatible with parabolic structures at 0 and ∞ for any split connected reductive group over $\mathbb{F}_q$ (see Corollary 3.2.2).

1.2.1 $\mathbb{P}^1$ with two marked points.

Fix a set of simple roots Π of G. For $J \subset \Pi$, let P_J denote the standard parabolic $\mathbb{F}_q$-subgroup corresponding to J. We need the following definition.

Definition. Let x be an $\mathbb{F}_q$-rational point of X. A *parabolic structure* on a principal G-bundle $\mathcal{E}$ over X at x of type J is a choice of a $\mathbb{F}_q$-rational point P_x of $\mathcal{E}_x/P_J$ where $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x. For vector bundles, this is equivalent to having a partial flag in the fiber of the corresponding vector bundle at x.

For reductive groups, the case of $\mathbb{P}^1$ with two marked points reduces to the following question.

Problem 1. Fix a principal G-bundle $\mathcal{E}$ over $\mathbb{P}^1$. Count the number of triples (P_0, P_∞, Ψ), where P_0 and P_∞ are parabolic structures on $\mathcal{E}$ at 0 and ∞ and Ψ is a nilpotent section of $\mathrm{ad}(\mathcal{E})$ compatible with P_0 and P_∞ (for vector bundles, this means that the nilpotent endomorphism preserves the corresponding partial flags at 0 and ∞).

As part of this thesis, we have an explicit formula (see Corollary 3.2.2) for Problem 1 when G is a split reductive group. In the case of $\mathbb{P}^1$, Mellit uses Hall algebras, which are not easily accesible for a general reductive group. Instead, we use geometric techniques in our proof. We also derive the Mellit's result in the case of GL_n using our methods.

The counting has two important steps. In the first step, we give an explicit formula for the number of points of generalized Steinberg varieties in Theorem 3.1. To this end we introduce a coproduct (see Section 3.1 for more details) for any reductive group, which might be of independent interest.

In the second step, we reduce the problem to counting the number of points of generalized Steinberg varieties using the Bialynicki–Birula decomposition in Theorem 3.2. We note that the applicability of the Bialynicki–Birula decomposition is not obvious since the schemes that we work with are neither smooth nor projective.

1.2.2 $\mathbb{P}^1$ with an arbitrary number of marked points.

The next step in this project is the case of $\mathbb{P}^1$ with an arbitrary number of marked points.

Problem 2. Given a set D of rational points of $\mathbb{P}^1$, a collection $J_\bullet$ of subsets of Π indexed by D and a nilpotent $n \in \mathrm{Lie}(G)$, calculate the volume of the stack of triples $(\mathcal{E}, P_\bullet, \Psi)$, where $\mathcal{E}$ is a principal G-bundle over $\mathbb{P}^1$, $\mathcal{E}$ satisfies a certain stability condition, P_x is a parabolic structure of type J_x, $x \in D$ and Ψ is a nilpotent section of $\mathrm{ad}(\mathcal{E})$ compatible with parabolic structures such that Ψ is conjugate to n at the generic point.

I am working on this problem with R. Fedorov. The idea is to write the generating functions for the volumes as a product of a global term independent of the points of D and local terms corresponding to the points of D. We plan to follow the strategy of Mellit [29, Thm. 5.6]. If Problem 2 is solved, we are hopeful of calculating the volumes of the stacks of parabolic Higgs G-bundles for $X = \mathbb{P}^1$. A more ambitious goal for the future is to solve Problem 2 for higher genus algebraic curves.

2.0 Preliminaries.

Convention 2.1. *k denotes an arbitrary field. When k is fixed, we denote by $\mathbb{P}^1$ the projective line over k and by $\mathbb{G}_m$ the multiplicative k-group $\mathbb{G}_{m,k}$. We denote by $\mathbb{F}_q$ the finite field with q elements. For any scheme X over $\mathbb{F}_q$, we denote by $|X|$ the number of $\mathbb{F}_q$-rational points of X.*

2.1 Affine algebraic groups.

2.1.1 Split reductive groups and its Lie algebras.

By an affine algebraic group over k, we mean a smooth affine k-group scheme. A torus over k is said to be split if it is isomorphic to $\mathbb{G}_m^r$ for some r. A connected affine algebraic group G over k is said to be reductive ([30, Section 6.46]) if $G_{\overline{k}}$ is reductive. Recall that a connected reductive group over k is called *split* ([30, Definition 19.22]) if it contains a maximal torus that is split.

Let us recall the notion of the Lie algebra of an affine algebraic group over k from [30, Section 10.6]. For an affine algebraic group G over k, the tangent space of G at the identity element e is defined as:

$$T_{e,G} := \ker(G(k[\epsilon]) \to G(k)),$$

where $k[\epsilon]$ is the ring of dual numbers over k. Let I_G be the augmentation ideal, which is defined to be $\ker(\mathcal{O}(G) \xrightarrow{e^*} k)$, where $e^* : \mathcal{O}(G) \to k$ is the co-identity map. One has the following isomorphism

$$T_{e,G} \simeq \mathrm{Hom}_{k-\mathrm{linear}}(I_G/I_G^2, k).$$

We define the Lie algebra of G to be $\mathrm{Hom}_{k-\mathrm{linear}}(I_G/I_G^2, k)$, which we will denote by $\mathfrak{g}$ or sometimes by $\mathrm{Lie}(G)$. For the definition of the Lie bracket on $\mathfrak{g}$, we refer to [30, Section 10.22]

Recall that an element $x \in \mathfrak{g}$ is said to be nilpotent if $r(x)$ is nilpotent for every Lie algebra homomorphism $r : \mathfrak{g} \to \mathfrak{gl}(V)$, where V varies over all finite dimensional vector spaces over k.

2.1.2 Parabolic and Levi k-subgroups.

Recall that a smooth closed k-subgroup $P \subset G$ is parabolic if the coset space G/P is proper over k (see [9, Section 1.3]). Since G/P is quasi-projective over k (see [11, Theorem 18.1.1]), we see that for a parabolic k-subgroup P of G, G/P is projective over k. By a Levi k-subgroup of G we mean a Levi factor of a parabolic k-subgroup.

In the rest of the thesis, G will denote a split connected reductive group over k with a fixed split maximal torus T and a Borel k-subgroup B containing T with unipotent radical U. Denote by W the Weyl group of G relative to T. Further, $X^*(T) := \mathrm{Hom}_k(T, \mathbb{G}_m)$ and $X_*(T) := \mathrm{Hom}_k(\mathbb{G}_m, T)$ will denote the lattices of k-characters of T and k-cocharacters of T respectively. There is a natural perfect pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$, which we denote by $\langle \cdot, \cdot \rangle$. Next, $\Pi \subset \Phi^+ \subset \Phi \subset X^*(T)$ will denote the corresponding simple roots, the positive roots and the root system (see [9, Proposition 11.3.8]).

2.1.3 Parametrization of standard parabolic k-subgroups.

Let us now recall the description of standard parabolic k-subgroups of G and their Levi factors. Pick $J \subset \Pi$ and let L_J be the the scheme-theoretic centralizer of the identity component of $(\bigcap_{\alpha \in J} \mathrm{Ker}\ \alpha)_{\mathrm{red}}$. Then L_J is a split reductive k-group with root system $\Phi_J := \mathbb{Z}J \cap \Phi$ ([30, Proposition 21.90]). Next, let U_J be the k-subgroup of G generated by U_α (root subgroups), $\alpha \in \Phi^+ \setminus \Phi_J$. Then $P_J := L_J U_J$ is a parabolic k-subgroup and U_J is the unipotent radical of P_J ([30, Theorem 21.91]). The subgroups P_J are called standard parabolic k-subgroups and the subgroups L_J are called standard Levi k-subgroups. It is known that every parabolic k-subgroup is $G(k)$-conjugate to P_J for a unique $J \subset \Pi$ (see [30, Theorem 21.91 and Theorem 25.8]). It follows that in the case of $G = GL_n$, parabolic k-subgroups are precisely the stabilizers of flags in k^n and that the Levi k-subgroups are precisely the stabilizers of ordered direct sum decompositions $k^n = V_1 \oplus \ldots \oplus V_m$.

Notation. We denote by $X_+(T)$ the semilattice of dominant k-cocharacters of T, i.e, $\lambda \in X_+(T)$ if and only if $(\alpha, \lambda) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi^+$. We note that every W-orbit of $X_*(T)$ contains exactly one element of $X_+(T)$, so we can identify $X_+(T)$ with $X_*(T)/W$.

2.2 Principal G-bundles.

Convention 2.2. *We make the following convention about fibre products of schemes over k. For any two schemes X and Y over k, we will denote $X \times_k Y$ by $X \times Y$.*

Definition. Let Y be a scheme over k. Let H be a quasi-compact group scheme over Y. Let us review the definition of principal H-bundles. Recall that a Y-scheme $\mathcal{P}$ equipped with a right action

$$\mathcal{P} \times H \to \mathcal{P}$$

of H such that the morphism $\mathcal{P} \to Y$ is H-invariant is called a principal H-bundle over Y, if $\mathcal{P}$ is faithfully flat and quasi-compact over Y and the action is simply transitive, i.e, the natural morphism $\mathcal{P} \times H \to \mathcal{P} \times_Y \mathcal{P}$ is an isomorphism. A morphism of principal H-bundles $\pi_1 : \mathcal{P}_1 \to Y$ and $\pi_2 : \mathcal{P}_2 \to Y$ is a morphism of H-schemes $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ such that $\pi_1 = \pi_2 \circ \phi$.

Remark 2.1. *(a) The above definition is equivalent to requiring the existence of a covering $\mathcal{U} = (U_i \to Y)$ in the fpqc topology such that for any i, $\mathcal{P}_{U_i}$ is H-equivariantly isomorphic to $H \times_Y U_i$ with H acting on $H \times_Y U_i$ by right multiplication on the first factor. ([39, Section 2.2]).*

(b) When the underlying group scheme H is smooth over Y, $\mathcal{P}$ can be trivialized in the etale topology. Indeed, by Lemma 2.2 (to be proved later) it is enough to show that there exists an etale cover $\mathcal{U} = (U_i \to Y)$ such that $\mathcal{P} \times_Y U_i \to U_i$ has a section. Since $H \to Y$ is smooth and smooth morphisms satsify fpqc descent, we have that $\mathcal{P}$ is smooth over Y. Now the claim follows from the fact that every smooth surjective morphism has a section etale-locally ([21, Proposition 17.16.3(ii)]).

Now let H be an affine algebraic group over k. Let us recall the construction of associated bundles. Let Z be a quasi-projective k-scheme equipped with a left H-action and let $\mathcal{E}$ be a principal H-bundle over a k-scheme S. Then we denote by $\mathcal{E} \times^H Z$ (or sometimes $\mathcal{E}(Z)$) the associated bundle with fibre type Z, which is the following scheme (see [18, Proposition 3.1]): $\mathcal{E} \times^H Z = (\mathcal{E} \times Z)/H$ for the right action of H on $\mathcal{E} \times Z$ given by $h \cdot (e, z) = (e \cdot h, h^{-1} \cdot z)$.

Definition. Let H and M be affine algebraic groups over k and let $\mathcal{E}$ be a principal H-bundle over a k-scheme S. If $\rho : H \to M$ is a homomorphism of groups defined over k, then the associated bundle $\mathcal{E} \times^H M$ for the action of H on M by left multiplication through ρ, is naturally a principal M–bundle over S. We denote this principal M–bundle over S often by $\rho_*\mathcal{E}$ and we say this principal M-bundle is obtained from $\mathcal{E}$ by extension of structure group.

Let $\mathcal{F}$ be a principal M-bundle over S. By a reduction of $\mathcal{F}$ to H, we mean a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a principal H-bundle and $\phi : \rho_*\mathcal{E} \to \mathcal{F}$ is an isomorphism of principal M-bundles over S. Two H-reductions $(\mathcal{E}_1, \phi_1)$ and $(\mathcal{E}_2, \phi_2)$ of $\mathcal{F}$ are said to be isomorphic if there exists an isomorphism of principal H-bundles $\omega : \mathcal{E}_1 \to \mathcal{E}_2$ such that the following triangle is commutative:

$$\begin{tikzcd} \rho_*\mathcal{E}_1 \arrow[rr, "\rho_*\omega"] \arrow[dr, "\phi_1"'] & & \rho_*\mathcal{E}_2 \arrow[dl, "\phi_2"] \\ & \mathcal{F} & \end{tikzcd}$$

If $\rho : H \to M$ is a closed subgroup, then we have the following well-known lemma (see [1, Remark 2.5] for the details):

Lemma 2.1. *There is a natural $1-1$ correspondence between sections $S \to \mathcal{F}(M/H)$ and reductions of $\mathcal{F}$ to H up to isomorphism.*

The following lemma tells us when a principal bundle is trivial.

Lemma 2.2. *Let H be an affine algebraic group over k and let $\pi : \mathcal{P} \to Y$ be a principal H-bundle. Then $\mathcal{P}$ is trivial if and only if π has a section.*

We only sketch the proof of Lemma 2.2. If $\mathcal{P}$ is a trivial principal H-bundle, then we compose the identity section of the group scheme $H \times Y \to Y$ with the isomorphism $H \times Y \cong \mathcal{P}$ to get the required section. Conversely, let $\sigma : Y \to \mathcal{P}$ be a section of π. Then the morphism $\phi : H \times Y \to \mathcal{P}$, $(h, y) \mapsto \sigma(y) \cdot h$ gives a morphism of principal H-bundles over Y. Since any morphism of principal H-bundles is an isomorphism (this can be checked etale-locally since isomorphisms satisfy etale descent), ϕ gives the required isomorphism of principal H-bundles.

2.3 The theorem of Grothendieck and Harder.

In this section, we give a sketch of the proof of the existence part of the theorem of Grothendieck and Harder. We follow [35, Lemma 3.3]. The proof serves three purposes: making the exposition more self-contained, making it clear why there is a natural description of principal G-bundles over $\mathbb{P}^1$ and discusses important techniques in the theory of principal G-bundles over curves.

Consider the $\mathbb{G}_m$-bundle $\mathcal{O}(1)^\times$ over $\mathbb{P}^1$, which is $\mathcal{O}(1)$ minus the zero section. Let $\mu \in X_*(T)$, define a principal G-bundle over $\mathbb{P}^1$ as:

$$\mathcal{E}_\mu := \mu_*\mathcal{O}(1)^\times,$$

where we view μ as a morphism $\mu : \mathbb{G}_m \to G$.

2.3.1 Principal T-bundles over $\mathbb{P}^1$.

Recall T from Section 2.1.2. Note that $T \cong \mathbb{G}_m^r$ for some r. Let $\mathcal{E}$ be a T-bundle over $\mathbb{P}^1$. Define a homomorphism $\mu_\mathcal{E} : X^*(T) \to \mathbb{Z}$ by mapping χ to the degree of the line bundle $\mathcal{E} \times^T \mathbb{A}^1_k$, where T acts on $\mathbb{A}^1_k$ via χ. Using the natural duality between $X_*(T)$ and $X^*(T)$, we will view $\mu_\mathcal{E}$ as an element in $X_*(T)$. We have the following classification of principal T-bundles over $\mathbb{P}^1$:

Lemma 2.3. *Let $G = T$ in the above notations. The association $\mu \mapsto \mathcal{E}_\mu$ gives a $1-1$ correspondence between $X_*(T)$ and the isomorphism classes of principal T-bundles over $\mathbb{P}^1$, with the inverse given by $\mathcal{E} \mapsto \mu_\mathcal{E}$.*

Proof. Note that $X_*(T)$ can be identified with $\mathbb{Z}^n$ and a principal T-bundle is just an ordered n-tuple of principal $\mathbb{G}_m$-bundles, which can be identified with line bundles. Now the lemma follows by the well-known isomorphism (see [23, Proposition 6.4]) $\mathbb{Z} \to \mathrm{Pic}\,(\mathbb{P}^1)$, $d \mapsto \mathcal{O}(d)$. $\square$

2.3.2 Principal B-bundles over $\mathbb{P}^1$.

Recall B from Section 2.1.2, which is the Borel k-subgroup containing T with unipotent radical U.

Definition. Let $\mathcal{E}$ be a principal B-bundle over $\mathbb{P}^1$. Let $p : B \to B/U \simeq T$ be the natural projection. By the classification of principal T-bundles over $\mathbb{P}^1$, there exists a unique $\lambda \in X_*(T)$ such that $p_*\mathcal{E} \simeq T_\lambda$. We call λ the T-type of $\mathcal{E}$.

Let $\mathcal{E}$ be a principal B-bundle over $\mathbb{P}^1$ and let B act on U by conjugation. Then the associated bundle $\mathcal{E}(U)$ is a group scheme over $\mathbb{P}^1$, locally isomorphic to U in the etale topology. We have the following lemma.

Lemma 2.4. *Keep notations as above. Let B act on B/T by left multiplication. Then the associated bundle $\mathcal{E}(B/T)$ is a principal $\mathcal{E}(U)$-bundle over $\mathbb{P}^1$.*

Proof. Note that we have a simply transitive action of U on B/T acting by left multiplication. Moreover, the action morphism $U \times B/T \to B/T$ is B-equivariant where B acts on U via conjugation. By functoriality of the construction of associated bundles, we have a morphism of associated bundles

$$\mathcal{E}(U) \times \mathcal{E}(B/T) \to \mathcal{E}(B/T).$$

Now the lemma follows since the above action of $\mathcal{E}(U)$ on $\mathcal{E}(B/T)$ is a simply transitive action as it can be easily checked etale-locally. $\square$

The next lemma guarantees that the B-bundle $\mathcal{E}$ has a reduction to T when its T-type satisfies a certain condition.

Lemma 2.5. *Keep notations as above. Let λ be the T-type of $\mathcal{E}$. If $\langle \alpha, \lambda \rangle \geq -1$ for all $\alpha \in \Phi^+$, then every principal $\mathcal{E}(U)$-bundle over $\mathbb{P}^1$ is trivial and $\mathcal{E} \simeq i_*T_\lambda$, where $i : T \to B$ is the inclusion.*

Proof. Since the first etale cohomology set classifies principal bundles for affine groups (see [31, Chapter III, Corollary 4.7] and Remark 2.1(b)), we show that every principal $\mathcal{E}(U)$-bundle over $\mathbb{P}^1$ is trivial by showing that $H^1(\mathbb{P}^1, \mathcal{E}(U)) = 1$. By ([22, Section 1.1]), U has a filtration by T-invariant normal subgroups such that the successive quotients are isomorphic to $\mathbb{G}_a$ with T acting by positive roots:

$$U_0 = U \supset U_1 \supset \ldots \supset U_i \supset U_{i+1} \supset \ldots \supset U_l = \{e\}.$$

Note that the subgroups U_i, $0 \leq i \leq l$ in the above filtration are B-invariant since they are T-invariant and normal in U. Consider the following exact sequence of affine algebraic groups with action of B:

$$1 \to U_{i+1} \to U_i \to (\mathbb{G}_a)_i \to 1,$$

where $(\mathbb{G}_a)_i \simeq \mathbb{G}_a$ as groups and B is acting by $\alpha_i \in \Phi_+$, $0 \leq i \leq l-1$. We get the exact sequence of "twisted" groups:

$$1 \to \mathcal{E}(U_{i+1}) \to \mathcal{E}(U_i) \to \mathcal{E}((\mathbb{G}_a)_i) \to 1,$$

By [31, Proposition 4.5], we have the associated exact sequence of pointed sets

$$H^1(\mathbb{P}^1, \mathcal{E}(U_{i+1})) \to H^1(\mathbb{P}^1, \mathcal{E}(U_i)) \to H^1(\mathbb{P}^1, \mathcal{E}((\mathbb{G}_a)_i)).$$

Note that $\mathcal{E}((\mathbb{G}_a)_i) \simeq \mathcal{O}(-\langle \alpha_i, \lambda \rangle)$ as group schemes over $\mathbb{P}^1$, which by Serre duality and the assumption gives $H^1(\mathbb{P}^1, \mathcal{E}((\mathbb{G}_a)_i)) = 1$. Therefore, the first map is surjective for all i. Now using induction, we have $H^1(\mathbb{P}^1, \mathcal{E}(U)) = 1$.

Now we can easily prove the second part of the lemma. Since $H^1(\mathbb{P}^1, \mathcal{E}(U)) = 1$, $\mathcal{E}(B/T)$ is a trivial principal $\mathcal{E}(U)$-bundle over $\mathbb{P}^1$ and so there is a section $\mathbb{P}^1 \to \mathcal{E}(B/T)$. Hence by Lemma 2.1, $\mathcal{E}$ has a reduction to T. The lemma follows by noting that this reduction is given by T_λ. $\square$

2.3.3 Sketch of the proof of the existence theorem of Grothendieck-Harder.

The following theorem says that every Zariski locally trivial principal G-bundle $\mathcal{E}$ over $\mathbb{P}^1$ is isomorphic to exactly one $\mathcal{E}_\mu$, $\mu \in X_+(T)$:

Theorem 2.6. *(Grothendieck-Harder) Let $\mathcal{E} \to \mathbb{P}^1$ be a principal G-bundle, which is locally trivial in the Zariski topology. Then $\mathcal{E} \simeq \mathcal{E}_\mu$ for some $\mu \in X_*(T)$. For $\mu_1, \mu_2 \in X_*(T)$, $\mathcal{E}_{\mu_1} \simeq \mathcal{E}_{\mu_2}$ if and only if $\mu_1 = w \cdot \mu_2$ for some $w \in W$. Therefore the Zariski locally trivial principal G-bundles over $\mathbb{P}^1$ are classified by $X_*(T)/W$.*

Before we proceed, we need the following useful consequence of the valuative criterion for properness:

Proposition 2.1. *Let X be a smooth projective curve over k and let $f : Y \to X$ be a proper morphism. Let K be the function field of X. Then any morphism $\eta : \mathrm{Spec}(K) \to Y$ of X-schemes can be uniquely extended to X, that is, there exists a unique morphism $\widetilde{\eta} : X \to Y$ of X-schemes such that $\widetilde{\eta}_{|\mathrm{Spec}(K)} = \eta$.*

Proof. (of Theorem 2.6) We only give a sketch of the proof of the existence part. For the proof of uniqueness of the cocharacter upto the action of Weyl group, see [35, Corollary 6.17]. We show that $\mathcal{E}$ has a reduction to T from which the claim will follow. To do so, note that by Lemma 2.5 it is enough to find a B-reduction of $\mathcal{E}$ of T-type μ with $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Phi^+$.

Let K denote the function field of $\mathbb{P}^1$. Since $\mathcal{E}$ is assumed to be Zariski locally trivial, $\mathcal{E}_K$ is a trivial principal G-bundle over $\mathrm{Spec}(K)$, thus by Lemma 2.1, $\mathcal{E}(G/B)$ has a section over $\mathrm{Spec}(K)$. Since G/B is proper, this section extends to whole of $\mathbb{P}^1$ by Propostion 2.1. Therefore by Lemma 2.1, $\mathcal{E}$ has a reduction to a principal B-bundle.

Let $\sigma : \mathbb{P}^1 \to \mathcal{E}(G/B)$ be a section of $\mathcal{E}(G/B) \to \mathbb{P}^1$. Then under the bijection in Lemma 2.1, σ corresponds to a reduction $\sigma^*\mathcal{E}$ of $\mathcal{E}$ to B. More explicitly, $\sigma^*\mathcal{E}$ is the pullback of the principal B-bundle $\mathcal{E} \to \mathcal{E}/B$ along σ, where we note that $\mathcal{E}(G/B) \cong \mathcal{E}/B$. For a character χ of B, let $\chi_*\sigma^*\mathcal{E}$ denote the line bundle associated to the principal B-bundle $\sigma^*\mathcal{E}$ through the character χ. Set

$$n(\chi, \sigma) := \deg \chi_*\sigma^*\mathcal{E}.$$

We note that $n(\chi, \sigma) = \langle \chi, \lambda_\sigma \rangle$, where λ_σ is the T-type of $\sigma^*\mathcal{E}$.

Let $\omega_1, \ldots, \omega_l$ be the fundamental weights of G corresponding to the pair (B, T). Let s be a positive integer such that $s\omega_1, \ldots, s\omega_l$ are characters of B. We claim that the set of integers of the form $n(s\omega_i, \sigma)$ is bounded from above as σ varies over all B-reductions of $\mathcal{E}$. Indeed, fix i and let V^i be the irreducible representation of G with highest weight $s\omega_i$. Let $V^i_{s\omega_i}$ denote the highest weight space of V^i. Since $V^i_{s\omega_i}$ is B-invariant, we can consider the line bundle $(s\omega_i)_*\sigma^*\mathcal{E} = \sigma^*\mathcal{E} \times^B V^i_{s\omega_i}$ of degree $n(s\omega_i, \sigma)$. Note that $(s\omega_i)_*\sigma^*\mathcal{E}$ is a line subbundle of the vector bundle $(\sigma^*\mathcal{E}) \times^B V^i = \big((\sigma^*\mathcal{E}) \times^B G\big) \times^G V^i = \mathcal{E}(V^i)$. Now we need the following lemma ([27, Lemma 13]):

Lemma 2.7. *Let E be a vector bundle over a smooth projective irreducible curve X over k. Then there exists an integer $n(E)$ such that for every coherent subsheaf $F \subset E$, we have $\deg(F) \leq n(E)$.*

Proof. Our lemma will proceed using induction on the rank of the vector bundle E. If E is a line bundle, then one can take $n(E) = \max(0, \deg(E))$ since any non-zero coherent subsheaf of E is locally free of rank one of smaller degree. Now suppose E has rank > 1. Take any rational section of E and let E_1 be the corresponding line subbundle of E. Then we have a short exact sequence of vector bundles

$$0 \to E_1 \to E \to E/E_1 \to 0.$$

Note that E/E_1 is a vector bundle since X is a curve. Now let F be a coherent subsheaf of E. Then we have an exact sequence

$$0 \to E_1 \cap F \to F \to F/(E_1 \cap F) \to 0.$$

Since degree is addtive, we have

$$\deg(F) = \deg(E_1 \cap F) + \deg(F/(E_1 \cap F)).$$

By inductive hypothesis, we have

$$\deg(E_1 \cap \mathcal{F}) + \deg(F/(E_1 \cap F)) \leq n(E_1) + n(E/E_1) =: n(E).$$

□

We return to the proof of Theorem 2.6. By Lemma 2.7, the set of integers of $n(s\omega_i, \sigma)$ is bounded from above as σ varies over all B-reductions of $\mathcal{E}$. Let σ be such that $n(s\omega_i, \sigma)$ are maximal in the following sense: there exists no σ' with

$$n(s\omega_i, \sigma') \geq n(s\omega_i, \sigma) \text{ for all } i$$

and

$$n(s\omega_{i_0}, \sigma') > n(s\omega_{i_0}, \sigma) \text{ for some } i_0.$$

We claim that $n(\alpha, \sigma) \geq 0$ for all $\alpha \in \Pi$. Let $\alpha \in \Pi$ and let P_α be the minimal parabolic k-subgroup corresponding to α. Let T_α denote the identity component of $\ker(\alpha)$, i.e, $T_\alpha = (\ker(\alpha)_{\mathrm{red}})^\circ$ and let U_{P_α} denote the unipotent radical of P_α. Consider $P_\alpha/(T_\alpha \cdot U_{P_\alpha})$, it is a connected semisimple group of rank 1, therefore by [40, Theorem 7.2.4], $P_\alpha/(T_\alpha \cdot U_{P_\alpha}) \simeq SL_2$ or PSL_2. Moreover, under the surjective morphism $P_\alpha \to P_\alpha/(T_\alpha \cdot U_{P_\alpha})$, the Borel k-subgroups of G that are contained in P_α are in one-to-one correspondence with Borel k-subgroups of $P_\alpha/(T_\alpha \cdot U_{P_\alpha})$. Thus if we consider a reduction of the principal SL_2 or PSL_2-bundle $\sigma^*\mathcal{E}(P_\alpha/(T_\alpha \cdot U_{P_\alpha}))$ to a Borel k-subgroup, then it gives a reduction of the G-bundle $\mathcal{E}$ to a Borel k-subgroup of G contained in P_α. Using explicit calculations with SL_2 and PSL_2, one can show ([35, Theorem 4.2]) that there exists a reduction σ' of the G-bundle $\mathcal{E}$ to a Borel k-subgroup of G contained in P_α such that if $n(\alpha, \sigma) < 0$, then $n(s\omega_i, \sigma') = n(s\omega_i, \sigma)$, $i \neq i_0$ and $n(s\omega_{i_0}, \sigma') > n(s\omega_{i_0}, \sigma)$, where ω_{i_0} is the fundamental weight corresponding to α. This contradicts the maximality of σ and thus $n(\alpha, \sigma) \geq 0$ for all $\alpha \in \Pi$. Since $n(\chi, \sigma)$ is additive in χ, we get that $n(\alpha, \sigma) \geq 0$ for all $\alpha \in \Phi^+$. Now by Lemma 2.5, we get that $\mathcal{E}$ has a T-reduction and this finishes the sketch of the proof of the existence part of Theorem 2.6. $\square$

In the case $k = \mathbb{F}_q$, every principal G-bundle is isomorphic to exactly one $\mathcal{E}_\mu$, $\mu \in X_+(T)$. This follows from the following pair of results (see [25] and [20, Theorem 3.8a)] for proofs):

Theorem 2.8. *(Lang) Let $\mathbb{F}_q$ be the finite field with q elements and let H be a connected affine algebraic group over $\mathbb{F}_q$. Then every principal H-bundle over* $\mathrm{Spec}(\mathbb{F}_q)$ *is trivial.*

Theorem 2.9. *Let k be any field. Then the principal G-bundles over $\mathbb{P}^1$ that can be trivialized locally in the Zariski topology can be identified with the principal G-bundles over $\mathbb{P}^1$ that are trivial when restricted to the point $\{\infty\}$, i.e, the following sequence of pointed sets is exact:*

$$H^1_{Zar}(\mathbb{P}^1, G) \to H^1(\mathbb{P}^1, G) \xrightarrow{ev_\infty} H^1(k, G),$$

where for any principal G-bundle $\mathcal{E}$, $ev_\infty(\mathcal{E})$ is the fiber $\mathcal{E}_{\{\infty\}}$ of $\mathcal{E}$ at ∞.

Remark 2.2. *The statement that every principal G-bundle over $\mathbb{P}^1$ is Zariski-locally trivial holds for more general fields. Recall that a field k is of dimesnion ≤ 1 if $Br K = 0$ for every algebraic extension K of k ([34, Proposition 1.5.25]). Let k be a perfect field of $\dim k \leq 1$. Then a theorem of Steinberg (see [43, Theorem 1.9]) says that $H^1(k, G) = \{*\}$. Therefore by the above theorem, every principal G-bundle over $\mathbb{P}^1$ is Zariski-locally trivial in this case.*

For algebraic curves of positive genus, we have the following result when $k = \overline{k}$:

Proposition 2.2. *Let X be a smooth connected projective curve over $k = \overline{k}$. Then any principal G-bundle $\mathcal{E}$ over X is locally trivial in the Zariski topology.*

The main ingredient in proving this result is Tsen's theorem [26, Theorem 17]:

Theorem 2.10. *(Tsen's theorem) Every principal G-bundle over $\mathrm{Spec}(K)$ is trivial, where K is the function field of a smooth connected projective curve over an algebraically closed field.*

Proof. (of Proposition 2.2) Since any principal T-bundle is locally trivial in the Zariski topology (see [31, Proposition 4.9, Chapter III]), it is enough to show that $\mathcal{E}$ has a reduction to B and that every principal B-bundle has a reduction to T.

Let K be the function field of X. By Tsen's theorem, $\mathcal{E}_K$ is a trivial principal G-bundle over $\mathrm{Spec}(K)$. Thus by Lemma 2.1, $\mathcal{E}(G/B)$ has a section over $\mathrm{Spec}(K)$. Since G/B is proper, this section extends to whole of X by Propostion 2.1. Therefore by Lemma 2.1 $\mathcal{E}$ has a reduction to a principal B-bundle.

Since principal T-bundles are Zariski locally trivial, it is enough to show that $\mathcal{F}$ admits a reduction to T over every affine open subset $\mathrm{Spec}(A) \subset X$. This is very similiar to the proof of Lemma 2.5 using exact sequence of cohomology groups along with the fact that $H^1(\mathrm{Spec}(A), \mathcal{F}(\mathbb{G}_a)) = 1$. $\square$

Remark 2.3. *Theorem 2.10 is a particular case of Grothendieck-Serre conjecture in dimension one [33].*

2.3.4 Examples of principal G-bundles over $\mathbb{P}^1$.

Let us give examples of the Grothendieck-Harder theorem in the classical cases.

1. $G = GL_n$: Over any scheme, principal GL_n-bundles can be identified with vector bundles of rank n. Any vector bundle over $\mathbb{P}^1$ of rank n is isomorphic to exactly one vector bundle of the form:

$$\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n), \quad a_i \in \mathbb{Z}, a_1 \geq \ldots \geq a_n.$$

2. $G = Sp_{2n}$ $(n \geq 2)$: Any principal Sp_{2n}-bundle over $\mathbb{P}^1$ is Zariski locally trivial [38, 4.4 (c)]. Moreover, principal Sp_{2n}-bundle over $\mathbb{P}^1$ can be regarded as vector bundles with extra structures. In this case, the corresponding vector bundles are of the form

$$\big(\mathcal{O}(a_1) \oplus \mathcal{O}(-a_1)\big) \oplus \ldots \oplus \big(\mathcal{O}(a_n) \oplus \mathcal{O}(-a_n)\big), \quad a_i \in \mathbb{Z}, a_1 \geq \ldots \geq a_n$$

equipped with a non-degenerate skew-symmetric form induced by the perfect pairing between $\mathcal{O}(a_i)$ and $\mathcal{O}(-a_i)$, $1 \leq i \leq n$.

3. $G = SO_{2n}$ $(n \geq 3)$: (char $k \neq 2$) Consider the even dimensional special orthogonal group SO_{2n}, which is the subgroup of SL_{2n} preserving the non-degenrate quadratic form $q(x_1, \ldots, x_{2n}) = x_1 x_{n+1} + \ldots + x_n x_{2n}$. Principal SO_{2n}-bundles over $\mathbb{P}^1$ which are Zariski locally trivial can be identified with vector bundles of the form

$$\big(\mathcal{O}(a_1) \oplus \mathcal{O}(-a_1)\big) \oplus \ldots \oplus \big(\mathcal{O}(a_n) \oplus \mathcal{O}(-a_n)\big), \quad a_i \in \mathbb{Z}, a_1 \geq \ldots \geq a_n \geq 0$$

equipped with a non-degenerate quadratic form given by the orthogonal sum of the hyperbolic form on $\mathcal{O}(a_i) \oplus \mathcal{O}(-a_i)$ induced by the perfect pairing between $\mathcal{O}(a_i)$ and $\mathcal{O}(-a_i)$, $1 \leq i \leq n$.

4. $G = SO_{2n+1}$ ($n \geq 2$): (char $k \neq 2$) Consider the odd dimensional special orthogonal group SO_{2n+1}, which is the subgroup of SL_{2n+1} preserving the non-degenrate quadratic form $q(x_{0,}x_1, \ldots, x_{2n}) = x_0^2 + x_1x_{n+1} + \ldots + x_nx_{2n}$. Principal SO_{2n+1}-bundles over $\mathbb{P}^1$ which are Zariski locally trivial can be identified with vector bundles of the form

$$\mathcal{O} \oplus \big(\mathcal{O}(a_1) \oplus \mathcal{O}(-a_1)\big) \oplus \ldots \oplus \big(\mathcal{O}(a_n) \oplus \mathcal{O}(-a_n)\big), \quad a_i \in \mathbb{Z}, a_1 \geq \ldots \geq a_n \geq 0$$

equipped with a non-degenerate quadratic form given by the orthogonal sum of the quadratic form x_0^2 on $\mathcal{O}$ and the hyperbolic form on $\mathcal{O}(a_i) \oplus \mathcal{O}(-a_i)$ induced by the perfect pairing between $\mathcal{O}(a_i)$ and $\mathcal{O}(-a_i)$, $1 \leq i \leq n$.

3.0 Main Results.

In this chapter we formulate the main results of this thesis. In the special case $G = GL_n$ and $k = \mathbb{F}_q$, they give a counting result of Mellit [29, Section 5.4].

3.1 Coproduct.

Let H be a split connected reductive group over $\mathbb{F}_q$ with a split maximal torus T_H and let B_H be a Borel $\mathbb{F}_q$-subgroup containing T_H. Let $\Pi_H \subset X^*(T_H)$ denote the corresponding set of simple roots of H. For $J \subset \Pi_H$, let P_J denote the standard parabolic $\mathbb{F}_q$-subgroup of H and let L_J denote the standard Levi factor of P_J (see Section 2.1.3). Let W_H denote the Weyl group of H relative to T_H and $J_1, J_2 \subset \Pi_H$. We let W_i denote the subgroup of W_H generated by s_α, $\alpha \in J_i$, $i = 1, 2$. We need the following notation:

Notation. It is known that every double coset in $W_1 \backslash W_H / W_2$ has a unique minimal length representative (see [7, Proposition 2.7.3]) and we denote this set of representatives by $D^H_{J_1,J_2}$.

Let $\mathcal{P}(\Pi_H)$ denote the set of subsets of Π_H. We let $\mathbb{Z}[\mathcal{P}(\Pi_H)]$ denote the lattice of functions on $\mathcal{P}(\Pi_H)$ taking values in $\mathbb{Z}$. For any $f \in \mathbb{Z}[\mathcal{P}(\Pi_H)]$, define

$$\Delta_H(f) : \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H) \to \mathbb{Z},$$

which is given by

$$\Delta_H(f)(J_1, J_2) := \sum_{w \in D^H_{J_1,J_2}} f(J_1 \cap w \cdot J_2).$$

We will call $\Delta_H(f)$ the coproduct of f. We have:

$$\Delta_H : \mathbb{Z}[\mathcal{P}(\Pi_H)] \to \mathbb{Z}[\mathcal{P}(\Pi_H)] \otimes \mathbb{Z}[\mathcal{P}(\Pi_H)] \cong \mathbb{Z}[\mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H)].$$

3.1.1 Generalized Springer and generalized Steinberg varieties.

For any $J \subset \Pi_H$, let $Sp_H(J)$ denote the generalized Springer variety of H with respect to J, which is defined as the following scheme of pairs:

$$Sp_H(J) := \{(n, P) : P \text{ is } \mathbb{F}_q\text{-conjugate to } P_J, n \text{ is nilpotent}, n \in \mathrm{Lie}(P)\}.$$

In particular, P is a parabolic subgroup defined over $\mathbb{F}_q$. For any two subsets $J_1, J_2 \subset \Pi_H$, let $St_H(J_1, J_2)$ denote the generalized Steinberg variety of H with respect to J_1 and J_2, which is defined as the scheme of triples (n, P, Q), where P is $\mathbb{F}_q$-conjugate to P_{J_1}, Q is $\mathbb{F}_q$-conjugate to P_{J_2}, n is nilpotent such that $n \in \mathrm{Lie}(P) \cap \mathrm{Lie}(Q)$. In particular, P and Q are parabolic subgroups defined over $\mathbb{F}_q$.

Observe that $Sp_H(J) \cong St_H(\Pi_H, J)$. Define

$$[Sp_H] : \mathcal{P}(\Pi_H) \to \mathbb{Z}, \quad J \mapsto |Sp_H(J)|$$

and define

$$[St_H] : \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H) \to \mathbb{Z}, \quad (J_1, J_2) \mapsto |St_H(J_1, J_2)|.$$

Let Φ_H denote the root system of H with respect to T_H and let Φ_H^+ denote the set of positive roots with respect to B_H and T_H. For $J \subset \Pi_H$, let Φ_J denote the root system of L_J with respect to T_H and let Φ_J^+ denote the set of positive roots with respect to $B_H \cap L_J$ and T_H.

Notation. Let M be an affine algebraic group over $\mathbb{F}_q$ and let $\mathfrak{m}$ be the associated Lie algebra. Recall that the rank of M is the dimension of a maximal torus of M or equivalently the dimension of a Cartan subalgebra of $\mathfrak{m}$. We will denote the rank of M by $\mathrm{rk}(M)$ or $\mathrm{rk}(\mathfrak{m})$.

The following theorem gives an explicit formula for the number of points of generalized Steinberg varieties:

Theorem 3.1. *With notations as above, we have*

(i) $|Sp_H(J)| = q^{|\Phi_J^+|+|\Phi_H^+|} \sum_{w \in W_H/W_J} q^{l(w)}$, *where* $l(w)$ *represents the minimal length of the elements in* wW_J *and also,* $|\Phi_J^+| + |\Phi_H^+| = \dim(P_J) - \mathrm{rk}(P_J)$.

(ii) $\Delta_H([Sp_H]) = [St_H]$.

We give the proof of Theorem 3.1 in Chapter 4.

3.2 Stratification of triples.

Definition. Fix a k-rational point x of $\mathbb{P}^1$. For $J \subset \Pi$, a *parabolic structure* on a principal G-bundle $\mathcal{E}$ over $\mathbb{P}^1$ at x of type J is a choice of a k-rational point P_x of $\mathcal{E}_x/P_J$ where $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x.

Let $\mu \in X_+(T)$ and let $\mathcal{E}_\mu = \mu_*\mathcal{O}(1)^\times$ be as in Section 2.3. Let $\mathrm{ad}(\mathcal{E}_\mu)$ denote the adjoint vector bundle over $\mathbb{P}^1$ associated to $\mathcal{E}_\mu$. Recall that $\mathrm{ad}(\mathcal{E}_\mu) = \mathcal{E} \times^G \mathfrak{g} = (\mathcal{E} \times \mathfrak{g})/G$ for the right action of G on $\mathcal{E} \times \mathfrak{g}$ given by $g \cdot (e, x) = (e \cdot g, \mathrm{Ad}_{g^{-1}} \cdot x)$. Note that $\mathrm{ad}(\mathcal{E}_\mu) = \mathcal{O}(1)^\times \times^{\mathbb{G}_m} \mathfrak{g}$, i.e, it is the quotient of $\mathcal{O}(1)^\times \times \mathfrak{g}$ under the action of $\mathbb{G}_m$ given by $g \cdot (e, f) = (e \cdot g, \mathrm{Ad}_{\mu(g)^{-1}}(f))$, $e \in \mathcal{O}(1)^\times, f \in \mathfrak{g}, g \in \mathbb{G}_m$. The sheaf of sections of the adjoint vector bundle $\mathrm{ad}(\mathcal{E}_\mu)$ form a sheaf of Lie algebras and thus $H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ has the structure of a Lie algebra. Nilpotent elements of the Lie algebra $H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ are called nilpotent sections of $\mathrm{ad}(\mathcal{E}_\mu)$.

Let $\mu \in X_+(T)$ and $J_0, J_\infty \subset \Pi$, define $\mathcal{T}rip_\mu(J_0, J_\infty)$ to be the scheme parameterizing triples (P_0, P_∞, Ψ) such that Ψ is a nilpotent section of $\mathrm{ad}(\mathcal{E}_\mu)$, P_0 (resp. P_∞) is a parabolic structure at 0 (resp. ∞) of type J_0 (resp. J_∞) and $\Psi_0 \in \mathrm{Lie}(P_0)$, $\Psi_\infty \in \mathrm{Lie}(P_\infty)$ (we will explain the meaning of this condition later). We note that $\mathcal{T}rip_\mu(J_0, J_\infty)$ is a scheme because it is the closed subscheme of $\mathcal{E}_0/P_{J_0} \times \mathcal{E}_\infty/P_{J_\infty} \times H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ given by three closed conditions, which are: Ψ is nilpotent, $\Psi_0 \in \mathrm{Lie}(P_0)$, $\Psi_\infty \in \mathrm{Lie}(P_\infty)$.

Now let us explain the meaning of $\mathrm{Lie}(P_x)$, $x = 0, \infty$ in the definition of $\mathcal{T}rip_\mu(J_0, J_\infty)$. For $x = 0, \infty$, we view $(\mathcal{E}_\mu)_x$ as a principal G-bundle over the point x and we let $\mathrm{Aut}((\mathcal{E}_\mu)_x)$ denote the k-group scheme whose R-valued points are the principal $G \times \mathrm{Spec}(R)$-bundle automorphisms of $(\mathcal{E}_\mu)_x \times \mathrm{Spec}(R)$. Since $\mathcal{E}_\mu$ is a pushforward of the $\mathbb{G}_m$-bundle $\mathcal{O}(1)^\times$, $(\mathcal{E}_\mu)_x$ is a trivial principal G-bundle over the point x and therefore $\mathrm{Aut}((\mathcal{E}_\mu)_x)$ can be non-canonically identified with G. Now, $\mathrm{Aut}((\mathcal{E}_\mu)_x)$ acts on $(\mathcal{E}_\mu)_x/P_{J_x}$ and the stabilizer of P_x is a parabolic subgroup of $\mathrm{Aut}((\mathcal{E}_\mu)_x)$. We denote by $\mathrm{Lie}(P_x)$ the Lie algebra of this stabilizer. This is a parabolic subalgebra of $\mathrm{Lie}(\mathrm{Aut}((\mathcal{E}_\mu)_x)) = \mathrm{ad}(\mathcal{E}_\mu)_x$.

Since $\mathcal{O}(1)^\times$ is a principal $\mathbb{G}_m$-bundle over $\mathbb{P}^1$, $\mathbb{G}_m$ acts on $\mathcal{E}_\mu = (\mathcal{O}(1)^\times \times G)/\mathbb{G}_m$ by acting on the first component. This gives a $\mathbb{G}_m$-action on the parabolic structures and on $\mathrm{ad}(\mathcal{E}_\mu)$, which gives a $\mathbb{G}_m$-action on $H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$. On combining these actions, we get a

$\mathbb{G}_m$-action:

$$\mathbb{G}_m \curvearrowright \mathcal{T}rip_\mu(J_0, J_\infty). \tag{1}$$

In this thesis, in the case when $k = \mathbb{F}_q$ we want to count the number of $\mathbb{F}_q$-points of $\mathcal{T}rip_\mu(J_0, J_\infty)$ for each $\mu \in X_+(T), J_0, J_\infty \subset \Pi$. For this, we would like to apply the Bialynicki–Birula decomposition to $\mathcal{T}rip_\mu(J_0, J_\infty)$ with respect to the $\mathbb{G}_m$-action (1). Note that it is not immediate in this case because $\mathcal{T}rip_\mu(J_0, J_\infty)$ is neither smooth nor projective in general but nevertheless we will prove below Theorem 3.2, which allows to reduce counting $|\mathcal{T}rip_\mu(J_0, J_\infty)|$ to counting points of the generalized Steinberg varieties.

Notation. For $J \subset \Pi$, denote by $W_J \subset W$ the subgroup generated by $s_\alpha, \alpha \in J$, here s_α denotes the reflection corresponding to α. For any $\mu \in X_*(T)$, let $\Pi_\mu \subset \Pi$ denote the set of simple roots that are annihilated by μ and denote by L_μ the identity component of the centralizer of $\mu(\mathbb{G}_m)$ in G. Since $\mathrm{Lie}(L_\mu) = \mathrm{Lie}(L_{\Pi_\mu})$ ([30, Theorem 13.33] and Section 2.1.3), $L_\mu = L_{\Pi_\mu}$. We note that Π_μ is the set of simple roots of L_μ corresponding to T and $B \cap L_\mu$.

Example. In the special case $G = GL_n$, if μ is of the form

$$t \mapsto \mathrm{diag}(\underbrace{t^{m_1}, \dots, t^{m_1}}_{i_1 \text{ times}}, \dots, \underbrace{t^{m_s}, \dots, t^{m_s}}_{i_s \text{ times}}), \quad m_i \neq m_j \text{ for } i \neq j, m_j \in \mathbb{Z} \text{ for } 1 \leq j \leq s,$$

then $L_\mu \simeq GL_{i_1} \times \ldots \times GL_{i_s}$.

Notation. Let X be a scheme over k and let H be an affine algebraic group over k acting on X. We will denote the fixed point locus of this action by X^H.

Theorem 3.2. *Keep notations as above. Let $\mathbb{G}_m$ act on $\mathcal{T}rip_\mu(J_0, J_\infty)$ as in* (1)*. Then there exists a stratification of $\mathcal{T}rip_\mu(J_0, J_\infty)$ by locally closed subsets as:*

$$\mathcal{T}rip_\mu(J_0, J_\infty) = \bigsqcup_{\substack{w \in W_{\Pi_\mu} \backslash W / W_{J_0} \\ w' \in W_{\Pi_\mu} \backslash W / W_{J_\infty}}} \mathcal{T}rip_\mu(J_0, J_\infty)^+_{w,w'}$$

and a decomposition of $\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}$ as:

$$\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m} = \bigsqcup_{\substack{w \in W_{\Pi_\mu} \backslash W / W_{J_0} \\ w' \in W_{\Pi_\mu} \backslash W / W_{J_\infty}}} \mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{w,w'},$$

where $\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{w,w'}$ are the connected components of $\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}$ with morphisms

$$\mathcal{T}rip_\mu(J_0, J_\infty)^+_{w,w'} \to \mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{w,w'},$$

which are given by the limit map as $t \to 0$ and are affine fibrations for $w \in W_{\Pi_\mu}\backslash W/W_{J_0}$, $w' \in W_{\Pi_\mu}\backslash W/W_{J_\infty}$ of relative dimensions $\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)$*.*

Moreover, the schemes $\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{\overline{w},\overline{w'}}$ are isomorphic to the generalized Steinberg varieties $St_{L_\mu}(\Pi_\mu \cap \overline{w} \cdot J_0, \Pi_\mu \cap \overline{w'} \cdot J_\infty)$, $\overline{w} \in D^G_{\Pi_\mu, J_0}$, $\overline{w'} \in D^G_{\Pi_\mu, J_\infty}$.

The proof of Theorem 3.2 will be given in Chapter 6.

Upto this point, the base field k in Theorem 3.2 was arbitrary. Now let $k = \mathbb{F}_q$. For $\mu \in X_+(T)$, define $\pi_\mu : \mathbb{Z}[\mathcal{P}(\Pi_\mu)] \to \mathbb{Z}[\mathcal{P}(\Pi)]$ as:

$$\pi_\mu(f)(J) := \sum_{w \in D^G_{\Pi_\mu, J}} f(\Pi_\mu \cap w \cdot J), \quad f \in \mathbb{Z}[\mathcal{P}(\Pi_\mu)].$$

and define $[\mathcal{T}rip_\mu] : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathbb{Z}$ as:

$$[\mathcal{T}rip_\mu](J_0, J_\infty) := |\mathcal{T}rip_\mu(J_0, J_\infty)|$$

As an easy corollary of Theorem 3.2, we get:

Corollary 3.2.1. *Keeping the above notations, we have:*

$$[\mathcal{T}rip_\mu] = q^{\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)} (\pi_\mu \otimes \pi_\mu)([St_{L_\mu}]).$$

Proof. Let $J_0, J_\infty \subset \Pi$. From Theorem 3.2, we have

$$\begin{aligned} [\mathcal{T}rip_\mu](J_0, J_\infty) &= \sum_{\substack{w \in W_{\Pi_\mu}\backslash W/W_{J_0} \\ w' \in W_{\Pi_\mu}\backslash W/W_{J_\infty}}} |\mathcal{T}rip_\mu(J_0, J_\infty)^+_{w,w'}| \\ &= \sum_{\substack{w \in W_{\Pi_\mu}\backslash W/W_{J_0} \\ w' \in W_{\Pi_\mu}\backslash W/W_{J_\infty}}} q^{\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)} |\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{w,w'}|. \end{aligned}$$

Since the schemes $\mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{\overline{w},\overline{w'}}$ are isomorphic to the generalized Steinberg varieties $St_{L_\mu}(\Pi_\mu \cap \overline{w} \cdot J_0, \Pi_\mu \cap \overline{w'} \cdot J_\infty)$, $\overline{w} \in D^G_{\Pi_\mu, J_0}$, $\overline{w'} \in D^G_{\Pi_\mu, J_\infty}$ (see Theorem 3.2), we have

$$[\mathcal{T}rip_\mu](J_0, J_\infty) = q^{\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)} \sum_{\substack{w \in D^G_{\Pi_\mu, J_0} \\ w' \in D^G_{\Pi_\mu, J_\infty}}} |St_{L_\mu}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty)|.$$

Now the corollary follows from the defintion of π_μ. $\square$

Remark 3.1. *(i) For deducing Corollary 3.2.1 from Theorem 3.2, it is crucial that all fibers of the morphism* $\mathcal{T}rip_\mu(J_0, J_\infty)^+_{w,w'} \to \mathcal{T}rip_\mu(J_0, J_\infty)^{\mathbb{G}_m}_{w,w'}$ *have the same dimension.*

(ii) Notice that π_μ *is an instance of* Δ_G. *More precisely, let* $f \in \mathbb{Z}[\mathcal{P}(\Pi_\mu)]$ *and let* $\tilde{f}$ *be any extension of* f *to* $\mathcal{P}(\Pi)$, *i.e,* $\tilde{f} \in \mathbb{Z}[\mathcal{P}(\Pi)]$ *and* $\tilde{f}_{|\mathcal{P}(\Pi_\mu)} = f$. *Then we have* $\pi_\mu(f) = \Delta_G(\tilde{f})(\Pi_\mu, \cdot)$.

More explicitly, we have the following corollary.

Corollary 3.2.2. *Keep notations as above. Then* $|\mathcal{T}rip_\mu(J_0, J_\infty)|$ *is equal to*

$$q^{|\Phi^+_{\Pi_\mu}|+\sum_{\langle\alpha,\mu\rangle>0}\left(\langle\alpha,\mu\rangle+1\right)} \sum_{\substack{w\in D^G_{\Pi_\mu,J_0}\\ w'\in D^G_{\Pi_\mu,J_\infty}}} \sum_{w''\in D^{L_\mu}_{\Pi_\mu\cap w\cdot J_0,\Pi_\mu\cap w'\cdot J_\infty}} q^{\left|\Phi^+_{\Pi_\mu\cap w\cdot J_0\cap w''\cdot(\Pi_\mu\cap w'\cdot J_\infty)}\right|} A(\mu, w, w', w''; q),$$

where $\Phi_{\Pi_\mu\cap w\cdot J_0\cap w''\cdot(\Pi_\mu\cap w'\cdot J_\infty)}$ *is the root system of* $L_{\Pi_\mu\cap w\cdot J_0\cap w''\cdot(\Pi_\mu\cap w'\cdot J_\infty)}$ *with respect to* T *and*

$$A(\mu, w, w', w''; q) = \sum_{w'''\in D^{L_\mu}_{\emptyset,\Pi_\mu\cap w\cdot J_0\cap w''\cdot(\Pi_\mu\cap w'\cdot J_\infty)}} q^{l(w''')}.$$

In particular, we see that $|\mathcal{T}rip_\mu(J_0, J_\infty)|$ *is a polynomial in* q *with non-negative integer coefficients.*

To prove Corollary 3.2.2, we need the following result (see [35, Proposition 5.2]), which describes $\mathrm{Aut}(\mathcal{E}_\mu)$ as a scheme:

Fact 3.1. *Let* $\mathcal{E}_\mu$ *be as above. Then* $\mathrm{Aut}(\mathcal{E}_\mu)$ *is isomorphic as a scheme to*

$$L_\mu \times \prod_{\alpha\in\Phi:\langle\alpha,\mu\rangle>0} H^0(\mathbb{P}^1, \mathcal{O}(\langle\alpha,\mu\rangle)).$$

Proof. (of Corollary 3.2.2) By Theorem 3.1(ii) and Corollary 3.2.1, we get

$$[\mathcal{T}rip_\mu](J_0, J_\infty) = q^{\dim(\mathrm{Aut}(\mathcal{E}_\mu))-\dim(L_\mu)} \sum_{\substack{w\in D^G_{\Pi_\mu,J_0}\\ w'\in D^G_{\Pi_\mu,J_\infty}}} \Delta_{L_\mu}([Sp_{L_\mu}])(\Pi_\mu\cap w\cdot J_0, \Pi_\mu\cap w'\cdot J_\infty). \quad (2)$$

Using the defintion of Δ_{L_μ}, $[\mathcal{T}rip_\mu](J_0, J_\infty)$ is equal to

$$q^{\dim(\mathrm{Aut}(\mathcal{E}_\mu))-\dim(L_\mu)} \sum_{\substack{w\in D^G_{\Pi_\mu,J_0}\\ w'\in D^G_{\Pi_\mu,J_\infty}}} \sum_{w''\in D^{L_\mu}_{\Pi_\mu\cap w\cdot J_0,\Pi_\mu\cap w'\cdot J_\infty}} |Sp_{L_\mu}(\Pi_\mu\cap w\cdot J_0\cap w''\cdot(\Pi_\mu\cap w'\cdot J_\infty))|.$$

Now the corollary follows from Theorem 3.1(i) and Fact 3.1. ∎

It follows from definitions that $\mathcal{T}rip_0(J_0, J_\infty) = St_G(J_0, J_\infty)$ and $\mathcal{T}rip_0(\Pi, \Pi) = \mathcal{N}(\mathfrak{g})$, the nilpotent cone of $\mathfrak{g}$. In particular, we see that even in the trivial case $\mu = 0$, $J_0 = J_\infty = \Pi$, $\mathcal{T}rip_\mu(J_0, J_\infty)$ is neither smooth nor projective.

We note the following corollary.

Corollary 3.2.3. *Keep notations as above and assume that $\mu \in X_+(T)$ is a central cocharacter. Then $[\mathcal{T}rip_\mu] = [St_G]$.*

Proof. It follows from Corollary 3.2.2 that $[\mathcal{T}rip_\mu] = [\mathcal{T}rip_0]$. □

3.3 Comparison between different groups.

In this section, we let $k = \mathbb{F}_q$. We will compare $|\mathcal{T}rip_\mu(J_0, J_\infty)|$ for different groups below. For this, we introduce the following notation.

Notation. Let H, T_H, B_H, Π_H be as in Section 3.1. Let $\nu \in X_+(T_H)$ and let $\mathcal{E}_\nu$ denote the principal H-bundle over $\mathbb{P}^1$ induced by ν. For $J_0, J_\infty \subset \Pi_H$, as before we let $\mathcal{T}rip_{\mu,H}(J_0, J_\infty)$ denote the scheme parameterizing triples (P_0, P_∞, Ψ) such that Ψ is a nilpotent section of $\mathrm{ad}(\mathcal{E}_\nu)$, P_0 (resp. P_∞) is a parabolic structure at 0 (resp. ∞) of type J_0 (resp. J_∞) and $\Psi_0 \in \mathrm{Lie}(P_0)$, $\Psi_\infty \in \mathrm{Lie}(P_\infty)$. Again as before, define

$$[\mathcal{T}rip_{\nu,H}] : \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H) \to \mathbb{Z}$$

by

$$[\mathcal{T}rip_{\nu,H}](J_0, J_\infty) := |\mathcal{T}rip_{\nu,H}(J_0, J_\infty)|$$

Consider the following two situations:

(i) Recall G, T, B, Π from Section 2.1.2. Let $G' := [G, G]$ be the derived group of G. Let $j : G' \to G$ be the natural inclusion. Denote the split maximal torus $T \cap G'$ of G' by T' and the Borel $\mathbb{F}_q$-subgroup $B \cap G'$ of G' by B'. Let $\mu' \in X_+(T')$, we have $\mu := j \circ \mu' \in X_+(T)$. Since the root systems of G and G' are isomorphic, we will consider $[\mathcal{T}rip_{\mu',G'}]$ and $[\mathcal{T}rip_{\mu,G}]$ as functions with domain $\Pi \times \Pi$.

(ii) Recall that a morphism $u : G_1 \to G_2$ of connected affine algebraic groups over $\mathbb{F}_q$ is called a central isogeny if it is a finite flat surjection such that $\ker(u)$ is central in G_1 (see [10, Definition 3.3.9]). Now let $u : G_1 \to G_2$ be a central isogeny of split connected reductive groups over $\mathbb{F}_q$. Let T_1 be a split maximal torus of G_1 and let B_1 be a Borel $\mathbb{F}_q$-subgroup of G_1 containing T_1. Then $T_2 := u(T_1)$ is a split maximal torus of G_2 and $B_2 := u(B_1)$ is a Borel $\mathbb{F}_q$-subgroup of G_2 containing T_2 (see [9, Section 3.3]). Let $\mu_1 \in X_+(T_1)$, we have $\mu_2 := u \circ \mu_1 \in X_+(T_2)$. Since the root systems of G_1 and G_2 are isomorphic, we will consider $[\mathcal{T}rip_{\mu_1,G_1}]$ and $[\mathcal{T}rip_{\mu_2,G_2}]$ as functions with domain $\Pi_1 \times \Pi_1$, where Π_1 is the set of simple roots of G_1 with respect to (B_1, T_1).

We have the following:

Corollary 3.2.4. *(a) With notations as in* (i) *above, we have*

$$[\mathcal{T}rip_{\mu',G'}] = [\mathcal{T}rip_{\mu,G}].$$

(b) With notations as in (ii) *above, we have*

$$[\mathcal{T}rip_{\mu_1,G_1}] = [\mathcal{T}rip_{\mu_2,G_2}].$$

We give the proof of Corollary 3.2.4 in Chapter 6.

As special cases, we may take $G = GL_n$ and $G' = SL_n$ in Corollary 3.2.4 (a) and $SL_n \to PGL_n \cong SL_n/\mu_n$ or $Spin_n \to SO_n$ in Corollary 3.2.4 (b).

4.0 Generalized Steinberg Varieties.

In this chapter, we give a proof of Theorem 3.1. Recall that for any scheme X over $\mathbb{F}_q$, we denote the number of $\mathbb{F}_q$-rational points of X by $|X|$.

We now prove a simple lemma that will be used several times in the thesis:

Lemma 4.1. *Let M be an affine algebraic group over $\mathbb{F}_q$ and let M' be a connected $\mathbb{F}_q$-subgroup of M. Then $|M/M'| = |M|/|M'|$.*

Proof. Let $x : \mathrm{Spec}(\mathbb{F}_q) \to M/M'$ be an $\mathbb{F}_q$-rational point of M/M' and let $M \xrightarrow{\pi} M/M'$ be the natural morphim giving M a structure of a principal M'-bundle over M/M'. Pulling back the principal M'-bundle $M \xrightarrow{\pi} M/M'$ along x, we get a principal M'-bundle $x^*M \to \mathrm{Spec}(\mathbb{F}_q)$. Recall that a theorem of Lang ([25]) asserts that for any connected affine algebraic group H over a finite field K, every principal H-bundle over $\mathrm{Spec}(K)$ is trivial, thus we get that $x^*M \to \mathrm{Spec}(\mathbb{F}_q)$ is a trivial principal M'-bundle and so, $x^*M \cong M'$. Since $\mathbb{F}_q$-rational points of M map to $\mathbb{F}_q$-rational points of M/M' under π, the number of $\mathbb{F}_q$-rational points of M mapping to x is equal to $|M'|$ and the lemma follows. $\square$

Notation. Let M be an affine algebraic group over $\mathbb{F}_q$ and let $\mathfrak{m}$ be the associated Lie algebra. We will denote the nilpotent cone of $\mathfrak{m}$ by $\mathcal{N}(\mathfrak{m})$.

We will need the following proposition later.

Proposition 4.1. *Let k be a perfect field. Let M be a connected affine algebraic group over k and let $\mathcal{R}_u(M)$ denote the k-unipotent radical of M. Then $M/\mathcal{R}_u(M)$ is a connected reductive k-group.*

Proof. Consider the $\overline{k}$-group $M_{\overline{k}}/\mathcal{R}_u(M_{\overline{k}})$. We claim that $M_{\overline{k}/\mathcal{R}_u(M_{\overline{k}})}$ is a connected reductive group over $\overline{k}$. To see this, consider the natural projection $\pi : M_{\overline{k}} \to M_{\overline{k}}/\mathcal{R}_u(M_{\overline{k}})$, assume that there exists a non-trivial connected, unipotent, normal subgroup U of $M_{\overline{k}}/\mathcal{R}_u(M_{\overline{k}})$, then $\pi^{-1}(U)^\circ$ satisfies the same properties and strictly contains $\mathcal{R}_u(M_{\overline{k}})$, which contradicts the fact that $\mathcal{R}_u(M_{\overline{k}})$ is the unipotent radical of $M_{\overline{k}}$. Next, we need the following result [8, Proposition 1.1.9(1)]:

Fact 4.1. *Let G be a connected affine algebraic group over k and let K/k be a separable extension of fields. Then we have $\mathcal{R}_{u,k}(G)_K = \mathcal{R}_{u,K}(G_K)$ inside G_K.*

We return to the proof of Proposition 4.1. Since $(M/\mathcal{R}_u(M))_{\overline{k}} \cong M_{\overline{k}}/\mathcal{R}_u(M)_{\overline{k}}$ and $\mathcal{R}_u(M_{\overline{k}}) = \mathcal{R}_u(M)_{\overline{k}}$ inside $M_{\overline{k}}$, we get that $\mathcal{R}_u(M/\mathcal{R}_u(M))_{\overline{k}} = \{1\}$. As a consequence, we have $\mathcal{R}_u(M/\mathcal{R}_u(M)) = \{1\}$, therefore $M/\mathcal{R}_u(M)$ is a connected reductive group over k. $\square$

Remark 4.1. *When k is not necessarily perfect, then $M/\mathcal{R}_u(M)$ is only a pseudo-reductive group (see [8] for the theory of pseudo-reductive groups).*

The following proposition is proved in [41, (7)] in the case of connected reductive groups over $\mathbb{F}_q$. We deduce the statement in the general case using the case of connected reductive groups over $\mathbb{F}_q$.

Proposition 4.2. *Let M be an arbitrary connected affine algebraic group over $\mathbb{F}_q$ and $\mathfrak{m}$ be its Lie algebra. Then $|\mathcal{N}(\mathfrak{m})| = q^{\dim(\mathfrak{m})-\mathrm{rk}(\mathfrak{m})}$.*

Proof. The case of connected reductive groups over $\mathbb{F}_q$ is proved in [41, (7)]. We claim that the general case follows from the case of connected reductive groups over $\mathbb{F}_q$. Indeed, let $\mathcal{R}_u(M)$ denote the $\mathbb{F}_q$-unipotent radical of M. Then by Proposition 4.1, $M/\mathcal{R}_u(M)$ is a connected reductive group over $\mathbb{F}_q$. Now let $\mathfrak{u}$ denote the Lie algebra of $\mathcal{R}_u(M)$, we have $\mathrm{Lie}(M/\mathcal{R}_u(M)) = \mathfrak{m}/\mathfrak{u}$.

We need a simple lemma:

Lemma 4.2. *With notations as above, we have*

$$|\mathcal{N}(\mathfrak{m})| = q^{\dim(\mathfrak{u})}|\mathcal{N}(\mathfrak{m}/\mathfrak{u})|.$$

Proof. Consider the natural projection $\mathfrak{m} \xrightarrow{\pi} \mathfrak{m}/\mathfrak{u}$. We will prove that $\mathcal{N}(\mathfrak{m}) = \pi^{-1}(\mathcal{N}(\mathfrak{m}/\mathfrak{u}))$ from which the lemma would follow easily. Since π maps nilpotent elements of $\mathfrak{m}$ to nilpotent elements of $\mathfrak{m}/\mathfrak{u}$, we get $\pi(\mathcal{N}(\mathfrak{m})) \subset \mathcal{N}(\mathfrak{m}/\mathfrak{u})$. Now suppose $x \in \pi^{-1}(\mathcal{N}(\mathfrak{m}/\mathfrak{u}))$, using Jordan decomposition write $x = x_s + x_n$, where x_s is a semisimple element, x_n is a nilpotent element and $[x_s, x_n] = 0$. Assume on the contrary that $x_s \neq 0$. Since π is a Lie algebra morphism, $\pi(x) = \pi(x_s) + \pi(x_n)$ is the Jordan decomposition of $\pi(x)$. Since $x_s \notin \mathfrak{u}$, we have $\pi(x_s) \neq 0$. Therefore, $\pi(x) \notin \mathcal{N}(\mathfrak{m}/\mathfrak{u})$, which is a contradiction. Thus, we have $\mathcal{N}(\mathfrak{m}) = \pi^{-1}(\mathcal{N}(\mathfrak{m}/\mathfrak{u}))$.

Since π is clearly surjective, we get $|\mathcal{N}(\mathfrak{m})| = |\mathfrak{u}||\mathcal{N}(\mathfrak{m}/\mathfrak{u})|$. Now the lemma follows from $|\mathfrak{u}| = q^{\dim(\mathfrak{u})}$. $\square$

We return to the proof of Proposition 4.2. Since the statement of Proposition 4.2 is known for reductive groups (see [41, (7)]), we obtain

$$|\mathcal{N}(\mathfrak{m}/\mathfrak{u})| = q^{\dim(\mathfrak{m}/\mathfrak{u})-\mathrm{rk}(\mathfrak{m}/\mathfrak{u})}.$$

Since $\mathrm{rk}(\mathfrak{m}) = \mathrm{rk}(\mathfrak{m}/\mathfrak{u})$, we get

$$|\mathcal{N}(\mathfrak{m}/\mathfrak{u})| = q^{\dim(\mathfrak{m}/\mathfrak{u})-\mathrm{rk}(\mathfrak{m})}. \tag{3}$$

By applying Lemma 4.2 to (3), we get

$$|\mathcal{N}(\mathfrak{m})| = q^{\dim(\mathfrak{u})}q^{\dim(\mathfrak{m}/\mathfrak{u})-\mathrm{rk}(\mathfrak{m})} = q^{\dim(\mathfrak{m})-\mathrm{rk}(\mathfrak{m})}.$$

This finishes the proof of Proposition 4.2. $\square$

4.1 Proof of Theorem 3.1(i).

Let H be a split reductive group over $\mathbb{F}_q$ with a split maximal torus T_H and let B_H be a Borel $\mathbb{F}_q$-subgroup containing T_H. Let $\Pi_H \subset X^*(T_H)$ denote the corresponding set of simple roots of H. Let W_H denote the Weyl group of H relative to T_H. For any $J \subset \Pi_H$, let P_J be the corresponding standard parabolic $\mathbb{F}_q$-subgroup of H. Let L_J and U_J be the Levi factor and the unipotent radical of P_J, respectively and let W_J be the corresponding subgroup of W_H.

The number of points of the generalized Springer variety of H corresponding to $J \subset \Pi_H$ is given by

$$|Sp_H(J)| = \frac{|H|}{|P_J|}|\mathcal{N}(\mathrm{Lie}(P_J)| = \frac{|H|}{|P_J|}q^{\dim(P_J)-\mathrm{rk}(P_J)}, \tag{4}$$

where the first equality holds because the normalizer of P_J is itself and the fact that if P is a parabolic subgroup of G conjugate over $\mathbb{F}_q$ to P_J, then $\mathcal{N}(\mathrm{Lie}(P)) \cong \mathcal{N}(\mathrm{Lie}(P_J))$. The second equality follows from Proposition 4.2.

Since H/P_J has a stratification by locally closed subsets as $\bigsqcup_{w \in W_H/W_J} \mathbb{A}^{l(w)}$ (see [4, Proposition 3.16]), where $l(w)$ represents the minimal length of the elements in wW_J, using Lemma 4.1 we get that $|H|/|P_J| = \sum_{w \in W_H/W_J} q^{l(w)}$, which gives

$$|Sp_H(J)| = q^{|\Phi_J^+|+|\Phi_H^+|} \sum_{w \in W_H/W_J} q^{l(w)}.$$

This finishes the proof of part (i) of Theorem 3.1.

4.2 Proof of Theorem 3.1(ii).

In the proof of part (ii) of Theorem 3.1, we will need another formula for $|Sp_H(J)|$, which we now give. First we need a lemma.

Lemma 4.3. *Let U be a connected unipotent group over k. Assume that k is perfect. Then $U \simeq \mathbb{A}^{\dim(U)}$ as schemes over k.*

Recall that a connected solvable group M over k is k-split ([40, Section 14.1]) if there exists a sequence

$$\{e\} = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$$

of closed, connected, normal k-subgroups such that the quotients M_i/M_{i-1} are k-isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$ over k. Lemma 4.3 is an easy consequence of the following two facts (see [40, Corollary 14.2.7 and Corollary 14.3.10]):

Fact 4.2. *Let M be a connected solvable group over k that is k-split. Then M is isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$ as k-schemes with $r = \dim(M/\mathcal{R}_u(M))$ and $s = \dim(\mathcal{R}_u(M)))$. In particular, if in addition M is unipotent, then $M \simeq \mathbb{A}^{\dim(M)}$ as schemes over k.*

Fact 4.3. *Let M be a connected solvable group over k. Assume that k is perfect. Then $\mathcal{R}_u(M)$ is k-split.*

Let us return to the proof of Theorem 3.1(ii). Let $J \subset \Pi_H$ be as in the statement of Theorem 3.1. Let U_J denote the unipotent radical of P_J. Then, we have $|U_J| = q^{\dim(U_J)}$ by

Lemma 4.3. Since $P_J \cong L_J \times U_J$ as schemes over $\mathbb{F}_q$, we have $|P_J| = |L_J||U_J|$. Substituting this in (4) gives

$$|Sp_H(J)| = \frac{|H|}{|L_J|} q^{\dim(L_J) - \mathrm{rk}(L_J)}. \tag{5}$$

Now Proposition 4.2 gives

$$|Sp_H(J)| = |H| \frac{|\mathcal{N}(\mathrm{Lie}(L_J))|}{|L_J|}. \tag{6}$$

For any $J_i \subset \Pi_H$, let $P_i := P_{J_i}$ be the corresponding standard parabolic $\mathbb{F}_q$-subgroup of H, $i = 1, 2$. Let $L_i := L_{J_i}$ and $U_i := U_{J_i}$ be the Levi factor and the unipotent radical of P_i respectively, and let $W_i := W_{J_i}$ be corresponding subgroup of W_H, $i = 1, 2$. Consider the natural action of $H(\mathbb{F}_q)$ on $St_H(J_1, J_2)$. Since the normalizer of P_1 in $H(\mathbb{F}_q)$ is $P_1(\mathbb{F}_q)$, the number of points of $St_H(J_1, J_2)$ is given by

$$\begin{aligned} |St_H(J_1, J_2)| &= \frac{|H|}{|P_1|} \sum_{h \in H(\mathbb{F}_q)/P_2(\mathbb{F}_q)} |\mathcal{N}(\mathrm{Lie}(P_1 \cap h \cdot P_2))| \\ &= \frac{|H|}{|P_1|} |P_1| \sum_{h \in P_1(\mathbb{F}_q)\backslash H(\mathbb{F}_q)/P_2(\mathbb{F}_q)} \frac{|\mathcal{N}(\mathrm{Lie}(P_1 \cap h \cdot P_2))|}{|P_1 \cap h \cdot P_2|} \end{aligned}$$

where the second equality follows from the following easy lemma.

Lemma 4.4. *Let A be a finite abstract group and let B and C be subgroups of A. Then for any $x \in A$, we have*

$$|BxC| = \frac{|B||C|}{|B \cap xCx^{-1}|}.$$

We will need the following fact:

Proposition 4.3. *Keep notations as above. Then we have a natural bijection*

$$P_1(\mathbb{F}_q)\backslash H(\mathbb{F}_q)/P_2(\mathbb{F}_q) \cong W_1 \backslash W_H / W_2.$$

Proof. (Sketch) The proposition follows from [14, Theorem 65.21] , [30, Theorem 21.91] and the well-known fact that $H(\mathbb{F}_q)$ is a finite group with a BN-pair [13] for $B = B_H(\mathbb{F}_q), N = N_{T_H}(\mathbb{F}_q)$, where N_{T_H} is the normalizer of T_H in H. □

We return to the proof of Theorem 3.1. By Proposition 4.3, Lemma 4.1 and Proposition 4.2, we get

$$|St_H(J_1, J_2)| = |H| \sum_{w \in W_1 \backslash W_H / W_2} \frac{q^{\dim(P_1 \cap w \cdot P_2) - \mathrm{rk}(P_1 \cap w \cdot P_2)}}{|P_1 \cap w \cdot P_2|}.$$

Next, we have the following decomposition (the statement is easily reduced to $\overline{\mathbb{F}}_q$ in which case it is given by [15, Proposition 2.15]):

$$P_1 \cap w \cdot P_2 = (L_1 \cap w \cdot L_2)(L_1 \cap w \cdot U_2)(U_1 \cap w \cdot L_2)(U_1 \cap w \cdot U_2), \tag{7}$$

which is a direct product of varieties over $\mathbb{F}_q$. By Lemma 4.3, we obtain

$$\begin{aligned} |St_H(J_1, J_2)| &= |H| \sum_{w \in W_1 \backslash W_H / W_2} \frac{q^{\dim(L_1 \cap w \cdot L_2) - \mathrm{rk}(L_1 \cap w \cdot L_2)}}{|L_1 \cap w \cdot L_2|} \\ &= |H| \sum_{w \in W_1 \backslash W_H / W_2} \frac{|\mathcal{N}(\mathrm{Lie}(L_1 \cap w \cdot L_2))|}{|L_1 \cap w \cdot L_2|}. \end{aligned}$$

where we use Proposition 4.2 for the second equality. Recall $D^H_{J_1,J_2}$ from Section 3.1 and let $w \in D^H_{J_1,J_2}$. In this case, we also have the following decomposition (the statement is easily reduced to $\overline{\mathbb{F}}_q$ in which it is given by [7, Theorem 2.8.7]):

$$P_1 \cap w \cdot P_2 = (L_{J_1 \cap w \cdot J_2})(L_1 \cap w \cdot U_2)(U_1 \cap w \cdot L_2)(U_1 \cap w \cdot U_2) \tag{8}$$

By (7), (8) and the fact that $L_{J_1 \cap w \cdot J_2} \subset L_1 \cap w \cdot L_2$, we get $L_{J_1 \cap w \cdot J_2} = L_1 \cap w \cdot L_2$, which gives

$$|St_H(J_1, J_2)| = |H| \sum_{w \in D^H_{J_1,J_2}} \frac{|\mathcal{N}(\mathrm{Lie}(L_{J_1 \cap w \cdot J_2}))|}{|L_{J_1 \cap w \cdot J_2}|}. \tag{9}$$

Recalling that Δ_H is given by

$$\Delta_H(f)(J_1, J_2) = \sum_{w \in D^H_{J_1,J_2}} f(J_1 \cap w \cdot J_2),$$

we get from (6) and (9) that

$$\Delta_H([Sp_H]) = [St_H].$$

This finishes the proof of part *(ii)* of Theorem 3.1. □

4.3 More on coproduct.

In this section, we would like to prove a few properties of Δ_H that are of independent interest and will be used later in Chapter 7 in the case of GL_n. First we need some definitions.

Definition. Let $J_1, J_2 \subset \Pi_H$, we say J_1 and J_2 are associates whenever $\Phi_{J_2} = w \cdot \Phi_{J_1}$ for some $w \in W_H$. This gives an equivalence relation on $\mathcal{P}(\Pi_H)$, which we denote by $\sim_H$. Let $f \in \mathbb{Z}[\mathcal{P}(\Pi_H)]$, we say f is associate invariant if $f(J_1) = f(J_2)$ whenever J_1 and J_2 are associates.

Let $\mathcal{O}$ be an equivalence class of $\sim_H$. Let $\delta_{\mathcal{O}} \in \mathbb{Z}[\mathcal{P}(\Pi_H)]$ be the function on $\mathcal{P}(\Pi_H)$ that takes the value 1 on J if $J \in \mathcal{O}$ and 0 otherwise. Let us fix a representative $J_{\mathcal{O}}$ in each equivalence class $\mathcal{O}$. We say that a function of two variables is associate invariant if it is associate invariant in each variable. The following lemma states that Δ_H preserves associate invariant functions.

Lemma 4.5. *Keep notations as above. Then*

$$\Delta_H(\delta_{\mathcal{O}}) = \sum_{(\mathcal{O}_1,\mathcal{O}_2)\in\left(\mathcal{P}(\Pi_H)/\sim\right)\times\left(\mathcal{P}(\Pi_H)/\sim\right)} n_{\mathcal{O}}^{\mathcal{O}_1,\mathcal{O}_2}\delta_{\mathcal{O}_1}\otimes\delta_{\mathcal{O}_2}, \tag{10}$$

where

$$n_{\mathcal{O}}^{\mathcal{O}_1,\mathcal{O}_2} = \left|\{w \in W_{J_{\mathcal{O}_1}}\backslash W_H/W_{J_{\mathcal{O}_2}} : \Phi_{J_{\mathcal{O}_1}} \cap w\cdot\Phi_{J_{\mathcal{O}_2}} = w'\cdot\Phi_{J_{\mathcal{O}}} \text{ for some } w' \in W_H\}\right|.$$

In particular, Δ_H preserves the subspace of associate invariant functions.

Proof. First we rewrite the coproduct Δ_H for associate invariant functions. Set $\mathcal{R}(J) := \Phi_J$, so that $\mathcal{R}$ is a bijection from $P(\Pi_H)$ onto the set of root systems of all Levi subgroups of H containing T_H. Let $f \in \mathbb{Z}[\mathcal{P}(\Pi_H)]$ be associate invariant. Then for $(J_1, J_2) \in \mathcal{P}(\Pi_H) \times \mathcal{P}(\Pi_H)$,

$$\Delta_H(f)(J_1,J_2) = \sum_{w\in W_{J_1}\backslash W_H/W_{J_2}} f\big(\mathcal{R}^{-1}(\Phi_{J_1}\cap w\cdot\Phi_{J_2})\big). \tag{11}$$

We note that in this reformulation of Δ_H for associate invariant functions the summands does not depend on a particular choice of the element of a double coset. For any $J_1, J_2 \in \mathcal{P}(\Pi_H)$, $\Delta_H(\delta_{\mathcal{O}})$ evaluated at (J_1, J_2) is equal to

$$\sum_{w \in W_{J_1} \backslash W_H / W_{J_2}} \delta_{\mathcal{O}}\big(\mathcal{R}^{-1}(\Phi_{J_1} \cap w \cdot \Phi_{J_2})\big),$$

which in turn is equal to

$$\big|\{w \in W_{J_1} \backslash W_H / W_{J_2} : \Phi_{J_1} \cap w \cdot \Phi_{J_2} = w' \cdot \Phi_{J_{\mathcal{O}}} \text{ for some } w' \in W_H\}\big|.$$

On the other hand, RHS of (10) evaluated at (J_1, J_2) is equal to $n_{\mathcal{O}}^{\mathcal{O}_1, \mathcal{O}_2}$, where $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) is the equivalence class of J_1 (resp. J_2). There exists $w_1, w_2 \in W_H$ such that $\Phi_{J_1} = w_1 \cdot \Phi_{J_{\mathcal{O}_1}}$, $\Phi_{J_2} = w_2 \cdot \Phi_{J_{\mathcal{O}_2}}$ and so, $W_{J_1} = w_1 W_{J_{\mathcal{O}_1}} w_1^{-1}$ and $W_{J_2} = w_2 W_{J_{\mathcal{O}_2}} w_2^{-1}$. Now the lemma follows from the bijection

$$W_{J_{\mathcal{O}_1}} \backslash W_H / W_{J_{\mathcal{O}_2}} \to W_{J_1} \backslash W_H / W_{J_2}, \quad W_{J_{\mathcal{O}_1}} w W_{J_{\mathcal{O}_2}} \mapsto W_{J_1}(w_1 w w_2^{-1}) W_{J_2}.$$

This finishes the proof of Lemma 4.5. □

Remark 4.2. *The proof of Lemma 4.5 suggests that* (11) *may be a better definition for* Δ_H *as it does not use [7, Proposition 2.7.3]. In fact, it may be even better to view* f *as a function on the set of root systems of the Levi subgroups. Moreover, using this formulation it is easy to see that* Δ_H *is co-commutative for associate invariant functions.*

We have the following corollary.

Corollary 4.5.1. *Let* $[Sp_H]$ *and* $[St_H]$ *be as in Section 3.1. Then* $[Sp_H]$ *and* $[St_H]$ *are associate invariant functions.*

Proof. Let $J, J' \in \Pi_H$ be such that $J \sim_H J'$. Then we have $L_J \simeq L_{J'}$ and as a consequence of (6), it follows that $[Sp_H]$ is associate invariant. Now Lemma 4.5 together with Theorem 3.1(ii) imply that $[St_H]$ is associate invariant in each variable. □

Assume that $H = H_1 \times \ldots \times H_n$. For $k = 1, \ldots, n$, let Π_k be the set of simple roots of H_k with respect to some maximal torus and a Borel subgroup containing it. We can identify Π_H with the disjoint union $\bigsqcup_k \Pi_k$. Thus, $\mathcal{P}(\Pi_H) = \prod_k \mathcal{P}(\Pi_k)$ and $\mathbb{Z}[\mathcal{P}(\Pi_H)] = \bigotimes_k \mathbb{Z}[\mathcal{P}(\Pi_k)]$. Under this isomorphism, the following lemma follows from the definitions.

Lemma 4.6. *Keep notations as above. Then*

$$[St_H] = [St_{H_1}] \otimes \ldots \otimes [St_{H_n}].$$

5.0 Bialynicki–Birula decomposition.

In this chapter we recall the Bialynicki–Birula decomposition. We will use these facts in the next chapter to give a proof of Theorem 3.2.

Definition. Let X and Z be two schemes. A morphism $\phi : X \to Z$ is called an affine fibration of relative dimension d if for every $z \in Z$, there is a Zariski open neighborhood U of z such that $X_U \cong U \times \mathbb{A}^d$ and this isomorphism identifies $\phi_{|U} : X_U \to Z$ with the projection on the first factor.

A morphism $\phi : X \to Z$ is called a trivial affine fibration of relative dimension d if $X \cong Z \times \mathbb{A}^d$ and this isomorphism identifies $\phi : X \to Z$ with the projection on the first factor.

We use the following result (see [6, Theorem 3.2]), known as the Bialynicki–Birula decomposition which is key to our calculation:

Fact 5.1. *(Bialynicki–Birula, Hesselink, Iversen). Let X be a smooth, projective scheme over k equipped with a $\mathbb{G}_m$-action. Then the following holds:*

(i) *The fixed point locus $X^{\mathbb{G}_m}$ is a closed subscheme of X and is smooth over k.*

(ii) *There exists a numbering $X^{\mathbb{G}_m} = \bigsqcup_{i=1}^n Z_i$ of the connected components of $X^{\mathbb{G}_m}$, and a filtration of X by closed subschemes:*

$$X = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

and affine fibrations $\phi_i : X_i - X_{i-1} \to Z_i$.

(iii) *The relative dimension of ϕ_i is the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space of X at an arbitrary closed point $z \in Z_i$ and* $\dim(Z_i) = \dim(T^{\mathbb{G}_m}_{z,X})$.

In particular, we obtain a stratification of X by locally closed subsets $X_i^+ := X_i - X_{i-1}$.

Definition. Let Y be a separated scheme over k. Let $\phi : \mathbb{A}^1\backslash\{0\} \to Y$ be a morphism. If ϕ extends to a morphism $\widetilde{\phi} : \mathbb{A}^1 \to Y$, we say that $\lim_{t\to 0} \phi(t)$ exists and we set it equal to $\widetilde{\phi}(0)$. Since Y is separated over k and $\mathbb{A}^1$ is reduced, the extension $\tilde{\phi}$ is unique. Note that if, moreover, Y is proper over k then an extension of ϕ always exists.

Remark 5.1. *(see [6, Section 3]) The Bialynicki–Birula decomposition is explicit in the sense that the locally closed subscheme X_i^+ is the set of all points $x \in X$ such that $\lim_{t\to 0} t \cdot x \in Z_i$ where $(t, x) \mapsto t \cdot x$ is the $\mathbb{G}_m$-action. Moreover, the map $\phi_i : X_i^+ \to Z_i$ is then given by $x \mapsto \lim_{t\to 0} t \cdot x$.*

Example. Consider the $\mathbb{G}_m$-action on $\mathbb{P}^n$ given by:

$$t \cdot [x_0 : \ldots : x_i : \ldots : x_n] = [t^0 x_0 : \ldots : t^i x_i : \ldots : t^n x_n], \quad t \in \mathbb{G}_m, [x_0 : \ldots : x_i : \ldots : x_n] \in \mathbb{P}^n.$$

This action has $n + 1$ fixed points, namely $p_i = [0 : \ldots : 0 : \underbrace{1}_{i} : 0 : \ldots : 0]$, $0 \leq i \leq n$. For $0 \leq i \leq n$, over the i-th coordinate chart $U_i = \{[x_0 : \ldots : x_i : \ldots : x_n] : x_i \neq 0\}$, this action is

$$t \cdot [x_0 : \ldots : 1 : \ldots : x_n] = [t^{-i} x_0 : \ldots : 1 : \ldots : t^{n-i} x_n].$$

Therefore, $X_i = \{[0 : \ldots : 0 : 1 : x_{i+1} \ldots : x_n]\} \simeq \mathbb{A}^{n-i}$, $0 \leq i \leq n$ and we have the following decomposition, which is analogous to the CW-decomposition of the classical projective space:

$$\mathbb{P}^n = \mathbb{A}^0 \sqcup \ldots \sqcup \mathbb{A}^i \sqcup \ldots \sqcup \mathbb{A}^n.$$

Let k be a field. Let S be a smooth separated scheme over k equipped with a $\mathbb{G}_m$-action. By [8, Proposition A.8.10], $S^{\mathbb{G}_m}$ is smooth over k. By a smooth equivariant compactification of S, we will mean a scheme $\overline{S}$ that is smooth and projective over k, S is an open and dense subscheme of $\overline{S}$ and $\overline{S}$ is equipped with a $\mathbb{G}_m$-action that extends the $\mathbb{G}_m$-action on S. The following proposition is a consequence of Fact 5.1.

Proposition 5.1. *Assume that there is a smooth equivariant compactification $\overline{S}$ of S. Let S^{fin} be the subset of S consisting of points x in S for which $\lim_{t\to 0} t\cdot x$ exists in S. Then S^{fin} is a constructible subset of S and there exists a stratification of S^{fin} by locally closed subsets as:*

$$S^{\mathrm{fin}} = \bigsqcup_{\alpha\in I} S_\alpha^+$$

and a decomposition of $S^{\mathbb{G}_m}$ as:

$$S^{\mathbb{G}_m} = \bigsqcup_{\alpha\in I} S_\alpha^{\mathbb{G}_m},$$

where S_α are the connected components of $S^{\mathbb{G}_m}$, $\alpha \in I$. Moreover, there are affine fibrations $lim_\alpha : S_\alpha^+ \to S_\alpha^{\mathbb{G}_m}$ given by the limit map as $t \to 0$.

Proof. By Fact 5.1 applied to $\overline{S}$, we get a stratification of $\overline{S}$ by locally closed subsets as:

$$\overline{S} = \bigsqcup_{\alpha\in I} \overline{S}_\alpha^+$$

and a decomposition of $\overline{S}^{\mathbb{G}_m}$ as:

$$\overline{S}^{\mathbb{G}_m} = \bigsqcup_{\alpha\in I} \overline{S}_\alpha^{\mathbb{G}_m}$$

where $\overline{S}_\alpha^{\mathbb{G}_m}$ are the connected components of $\overline{S}^{\mathbb{G}_m}$, $\alpha \in I$. Moreover, we get retractions $lim_\alpha : \overline{S}_\alpha^+ \longrightarrow \overline{S}_\alpha^{\mathbb{G}_m}$, $\alpha \in I$. Note that these retractions are given by the limit map as $t \to 0$ (see Remark 5.1).

Now by base change of $\overline{S}_\alpha^+ \xrightarrow{lim_\alpha} \overline{S}_\alpha^{\mathbb{G}_m}$ along $\overline{S}_\alpha^{\mathbb{G}_m} \cap S \to \overline{S}_\alpha^{\mathbb{G}_m}$, we get a scheme say S_α^+ with a retraction to $S_\alpha^{\mathbb{G}_m} := \overline{S}_\alpha^{\mathbb{G}_m} \cap S$, which is an affine fibration and we denote it again by $lim_\alpha : S_\alpha^+ \to S_\alpha^{\mathbb{G}_m}$. Next, we claim that $S_\alpha^+ \subset S$. Indeed, since $\overline{S}\backslash S$ is projective and $\mathbb{G}_m$-stable, lim_α preserves $\overline{S}\backslash S$ and hence $S_\alpha^+ \subset S$. Thus $S^{\mathrm{fin}} = \bigsqcup_{\alpha\in I} S_\alpha^+$ and S^{fin} is constructible since $\overline{S}_\alpha^+$, $\alpha \in I$ are locally closed subsets of $\overline{S}$.

Now we show that $S_\alpha^{\mathbb{G}_m}$ are the connected components of $S^{\mathbb{G}_m}$. Since $\overline{S}$ is projective, $\overline{S}^{\mathbb{G}_m}$ is noetherian. Thus there are finitely many irreducible components of $\overline{S}^{\mathbb{G}_m}$. Since $\overline{S}^{\mathbb{G}_m}$ is smooth ([8, Proposition A.8.10]), $\overline{S}_\alpha^{\mathbb{G}_m}$ is irreducible and we get that $\overline{S}_\alpha^{\mathbb{G}_m} \cap S$ is irreducible. This gives us that $\overline{S}_\alpha^{\mathbb{G}_m} \cap S$, $\alpha \in I$ are the connected components of $S^{\mathbb{G}_m}$ since the number of connected components of $\overline{S}^{\mathbb{G}_m}$ is finite.

This finishes the proof of Proposition 5.1. $\square$

In the case of an equivariant vector bundle over a smooth projective scheme equipped with a $\mathbb{G}_m$-action, we can say a bit more about the strata in Proposition 5.1. Let k be a field and let X be a smooth projective scheme over k equipped with a $\mathbb{G}_m$-action. Let $\pi : E \to X$ be an equivariant vector bundle over X. Compactify E by considering the projectivization $\mathbb{P}\big(E \oplus (X \times \mathbb{A}^1)\big) =: \overline{E}$. We extend the given $\mathbb{G}_m$-action on E to a $\mathbb{G}_m$-action on $\overline{E}$ by letting $\mathbb{G}_m$ act trivially on $\mathbb{A}^1$ and via the given $\mathbb{G}_m$-action on X. Since a projectivization of a vector bundle over a smooth scheme is smooth, $\overline{E}$ is smooth. Thus $\overline{E}$ is a smooth equivariant compactification of E.

Now let us consider the Bialynicki–Birula decomposition of X. By Fact 5.1, X has a stratification by locally closed subsets as:

$$X = \bigsqcup_{\alpha \in I} X_\alpha^+$$

and a decomposition of $X^{\mathbb{G}_m}$ as:

$$X^{\mathbb{G}_m} = \bigsqcup_{\alpha \in I} X_\alpha^{\mathbb{G}_m}$$

where $X_\alpha^{\mathbb{G}_m}$ are the connected components of $X^{\mathbb{G}_m}$, $\alpha \in I$.

Since $\mathbb{G}_m$ acts trivially on $X_\alpha^{\mathbb{G}_m}$ and π is $\mathbb{G}_m$-equivariant, $\mathbb{G}_m$ acts on the vector bundle $\pi^{-1}(X_\alpha^{\mathbb{G}_m}) \to X_\alpha^{\mathbb{G}_m}$ fibrewise. Therefore, $\pi^{-1}(X_\alpha^{\mathbb{G}_m})$ decomposes according to the characters of $\mathbb{G}_m$,

$$\pi^{-1}(X_\alpha^{\mathbb{G}_m}) = \oplus_{n \in \mathbb{Z}} V_{\alpha,n},$$

where $V_{\alpha,n}$ is the subbundle of $\pi^{-1}(X_\alpha^{\mathbb{G}_m})$ on which $t \in \mathbb{G}_m$ acts via multiplication by t^n. We have the following proposition.

Proposition 5.2. *Keep notations as above and as in Proposition 5.1. Then E^{fin} is a constructible subset of E and there exists a stratification of E^{fin} by locally closed subsets as:*

$$E^{\text{fin}} = \bigsqcup_{\alpha \in I} E_\alpha^+$$

and a decomposition of $E^{\mathbb{G}_m}$ as:

$$E^{\mathbb{G}_m} = \bigsqcup_{\alpha \in I} V_{\alpha,0},$$

where $V_{\alpha,0}$ are the connected components of $E^{\mathbb{G}_m}$, $\alpha \in I$ and there are affine fibrations $lim_\alpha : E_\alpha^+ \to V_{\alpha,0}$ given by the limit map as $t \to 0$.

Proof. Notice that we have $E^{\mathbb{G}_m} = \bigsqcup_{\alpha \in I} V_{\alpha,0}$. Since $V_{\alpha,0} = \left(\pi^{-1}(X_\alpha^{\mathbb{G}_m})\right)^{\mathbb{G}_m}$, $V_{\alpha,0}$ is closed, $\alpha \in I$. Moreover, since $V_{\alpha,0}$, $\alpha \in I$ are connected, we get that $V_{\alpha,0}$, $\alpha \in I$ are the connected components of $E^{\mathbb{G}_m}$. It remains to use Proposition 5.1. $\square$

6.0 Counting triples.

This chapter will be devoted to the proof of Theorem 3.2. Let G, T, B, Π, W, Φ be as in Section 2.1.2 and let μ, J_0, J_∞ be as in the statement of Theorem 3.2. Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G. Since μ, J_0 and J_∞ are fixed in the statement of Theorem 3.2, we will denote $\mathcal{T}rip_\mu(J_0, J_\infty)$ by $\mathcal{T}rip$ in the proof of Theorem 3.2.

6.1 Strategy of the proof.

In this section, we outline the strategy of the proof of Theorem 3.2. Let $\mathbb{G}_m$ act on $\mathfrak{g}$ via μ, so $t \in \mathbb{G}_m$ acts trivially on $\mathfrak{h}$ and via multiplication by $t^{\langle\alpha,\mu\rangle}$ on the root spaces $\mathfrak{g}_\alpha$. Let $\mathfrak{g}^0 := \mathfrak{h} \oplus_{\langle\alpha,\mu\rangle=0} \mathfrak{g}_\alpha, \mathfrak{g}^+ := \oplus_{\langle\alpha,\mu\rangle>0}\mathfrak{g}_\alpha$ and $\mathfrak{g}^- := \oplus_{\langle\alpha,\mu\rangle<0}\mathfrak{g}_\alpha$. Then we get the following $\mathbb{G}_m$-stable decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^-. \tag{12}$$

Note that we have $\mathfrak{g}^0 = \mathrm{Lie}(L_\mu)$.

For $J \subset \Pi$, define $\mathcal{B}_J$ to be the scheme of pairs (P, v) such that $P \in G/P_J, v \in \mathrm{Lie}(P)$, where we identify G/P_J with the scheme of parabolic subgroups of G that are conjugate to P_J. Note that $\mathcal{B}_J$ is vector bundle over G/P_J (see Lemma 6.2 for the proof), in fact, it is a vector subbundle of the trivial vector bundle $G/P_J \times \mathfrak{g}$ over G/P_J. As vector bundles over smooth schemes are smooth, we get that $\mathcal{B}_J$ is smooth. Note that G acts in a natural way on $G/P_J \times \mathfrak{g}$ preserving $\mathcal{B}_J$. Pulling back this action along $\mu : \mathbb{G}_m \to T \to G$, we get an action

$$\mathbb{G}_m \curvearrowright \mathcal{B}_J. \tag{13}$$

We introduce the following object for our proof of Theorem 3.2.

Definition. Let $\mathcal{Q}uad$ be the closed subscheme of $\mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}$ consisting of quadruples $(P_0, v_0, P_\infty, v_\infty)$ such that v_0 and v_∞ are nilpotent and with respect to the decomposition (12), the $\mathfrak{g}^-$-components of v_0 and v_∞ are zero and their $\mathfrak{g}^0$-components are equal.

Note that $\mathcal{Q}uad$ depends on μ, J_0 and J_∞.

Remark 6.1. *The requirement of v_0 and v_∞ being nilpotent in the definition of $\mathcal{Q}uad$ is equivalent to the requirement of the $\mathfrak{g}^0$-components of v_0 and v_∞ being nilpotent.*

Recall $\mathcal{B}_J^{\text{fin}}$ from Proposition 5.1. Since $\mathcal{B}_J$ is an equivariant vector bundle over G/P_J, we stratify $\mathcal{B}_J^{\text{fin}}$ by applying Proposition 5.2 on $\mathcal{B}_J$. We obtain the required stratification of $\mathcal{T}rip$ in the following manner: trivialize the fibers of the line bundle $\mathcal{O}(1)$ at 0 and ∞ to identify $\text{ad}(\mathcal{E}_\mu)_0$ and $\text{ad}(\mathcal{E}_\mu)_\infty$ with $\mathfrak{g}$, now evaluating the nilpotent sections at 0 and ∞ gives us a $\mathbb{G}_m$-equivariant morphism $\mathcal{T}rip \to \mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}$ with $\mathbb{G}_m$ acting diagonally on $\mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}$. We will see in Lemma 6.1 that this evaluation morphism is a trivial affine fibration onto its image, which is equal to $\mathcal{Q}uad$. Thus it is enough to stratify $\mathcal{Q}uad$. We show that for points in $\mathcal{Q}uad$, the limit exists in $\mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}$ as $t \to 0$ (see Lemma 6.3), so $\mathcal{Q}uad \subset \mathcal{B}_{J_0}^{\text{fin}} \times \mathcal{B}_{J_\infty}^{\text{fin}}$. We will see in Lemma 6.4 that intersecting the strata of $\mathcal{B}_{J_0}^{\text{fin}} \times \mathcal{B}_{J_\infty}^{\text{fin}}$ with $\mathcal{Q}uad$, we obtain a stratification of $\mathcal{Q}uad$.

6.2 Reduction to $\mathcal{Q}uad$.

Now we consider evaluations of the nilpotent sections of $\text{ad}(\mathcal{E}_\mu)$ at 0 and ∞ and then use them to reduce Theorem 3.2 to finding a stratification of $\mathcal{Q}uad$. Recall that as $\mathcal{O}(1)^\times$ is a $\mathbb{G}_m$-bundle over $\mathbb{P}^1$, $\mathbb{G}_m$ acts on $\text{ad}(\mathcal{E}_\mu) = \mathcal{O}(1)^\times \times^{\mathbb{G}_m} \mathfrak{g}$ (Section 2.2) and this gives an action:

$$\mathbb{G}_m \curvearrowright H^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)). \tag{14}$$

First, we describe sections of the adjoint bundle $\text{ad}(\mathcal{E}_\mu)$ over $\mathbb{P}^1$. Since $\text{ad}(\mathcal{E}_\mu) = \mathcal{O}(1)^\times \times^{\mathbb{G}_m} \mathfrak{g}$, the $\mathbb{G}_m$-stable decomposition (12) of $\mathfrak{g}$ gives a $\mathbb{G}_m$-stable decomposition of $\text{ad}(\mathcal{E}_\mu)$ as

$$\text{ad}(\mathcal{E}_\mu) = \text{ad}(\mathcal{E}_\mu)^0 \oplus \text{ad}(\mathcal{E}_\mu)^+ \oplus \text{ad}(\mathcal{E}_\mu)^-,$$

where $\text{ad}(\mathcal{E}_\mu)^0 := \mathcal{O}(1)^\times \times^{\mathbb{G}_m} \mathfrak{g}^0, \text{ad}(\mathcal{E}_\mu)^+ := \mathcal{O}(1)^\times \times^{\mathbb{G}_m} \mathfrak{g}^+$ and $\text{ad}(\mathcal{E}_\mu)^- := \mathcal{O}(1)^\times \times^{\mathbb{G}_m} \mathfrak{g}^-$. Since $\text{ad}(\mathcal{E}_\mu)^-$ is a direct sum of the line bundles $\mathcal{O}(m)$, $m < 0$ and $H^0(\mathbb{P}^1, \mathcal{O}(m)) = 0$ for $m < 0$, we get

$$H^0(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)) = \mathfrak{g}^0 \oplus H^0\big(\mathbb{P}^1, \text{ad}(\mathcal{E}_\mu)^+\big).$$

Thus

$$H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu)) = \mathfrak{g}^0 \oplus \Big(\oplus_{\alpha:\langle\alpha,\mu\rangle>0} H^0\big(\mathbb{P}^1, \mathcal{O}(\langle\alpha,\mu\rangle)\big)\Big). \tag{15}$$

For $x = 0, \infty$, $\mathrm{ad}(\mathcal{E}_\mu)_x$ has a structure of a Lie algebra and for $\Psi \in H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$, denote the value of Ψ at x by Ψ_x, which is an element of $\mathrm{ad}(\mathcal{E}_\mu)_x$.

We get the following $\mathbb{G}_m$-stable decomposition of $\mathrm{ad}(\mathcal{E}_\mu)_x$:

$$\mathrm{ad}(\mathcal{E}_\mu)_x = \mathrm{ad}(\mathcal{E}_\mu)_x^0 \oplus \mathrm{ad}(\mathcal{E}_\mu)_x^+ \oplus \mathrm{ad}(\mathcal{E}_\mu)_x^-, \quad x = 0, \infty.$$

Remark 6.2. *By trivializing the fibers of the $\mathbb{G}_m$-bundle $\mathcal{O}(1)^\times$ at 0 and ∞, we identify $(\mathcal{E}_\mu)_x/P_{J_x}$ with G/P_{J_x} and we get a $\mathbb{G}_m$-equivariant isomorphism (which is fixed from now on) $ad(\mathcal{E}_\mu)_x \cong \mathfrak{g}$, which maps $ad(\mathcal{E}_\mu)_x^0$ isomorphically onto $\mathfrak{g}^0$, $x = 0, \infty$. We note that the isomorphism $ad(\mathcal{E}_\mu)_x^0 \cong \mathfrak{g}^0$ is independent of the trivialization. From now on, we will use the isomorphism $ad(\mathcal{E}_\mu)_x \cong \mathfrak{g}$ to identify elements of $ad(\mathcal{E}_\mu)_x$ with those of $\mathfrak{g}$, $x = 0, \infty$.*

The $\mathbb{G}_m$-action (13) on $\mathcal{B}_{J_x}$, $x = 0, \infty$ gives a $\mathbb{G}_m$-action on $\mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}$ by $\mathbb{G}_m$ acting diagonally. Since the decomposition (12) is $\mathbb{G}_m$-stable, we get an action

$$\mathbb{G}_m \curvearrowright \mathcal{Q}uad. \tag{16}$$

Using Remark 6.2, we consider the evaluation morphism at 0 and ∞ as taking values in $\mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}$:

$$ev^{0,\infty} : \mathcal{T}rip \to \mathcal{B}_{J_0} \times \mathcal{B}_{J_\infty}, \quad (P_0, P_\infty, \Psi) \mapsto (P_0, \Psi_0, P_\infty, \Psi_\infty).$$

Consider the evaluation map at 0 and ∞,

$$eval : H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu)) \to \mathfrak{g} \oplus \mathfrak{g}, \quad \Psi \mapsto (\Psi_0, \Psi_\infty).$$

Notice that for $\Psi \in \mathfrak{g}^0$, $eval(\Psi) = (\Psi, \Psi)$. Since $\Psi \in H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ is nilpotent if and only if the $\mathfrak{g}^0$-component of Ψ is nilpotent and the evaluation map $H^0(\mathbb{P}^1, \mathcal{O}(m)) \to \mathbb{A}^1 \times \mathbb{A}^1$, $\phi \mapsto (\phi_0, \phi_\infty)$ is surjective for $m > 0$, the image of $ev^{0,\infty}$ is equal to $\mathcal{Q}uad$.

The next lemma relates $\mathcal{T}rip$ and $\mathcal{Q}uad$ via evaluation at 0 and ∞.

Lemma 6.1. *The evaluation morphism*

$$ev^{0,\infty} : \mathcal{T}rip \to \mathcal{Q}uad$$

is $\mathbb{G}_m$-equivariant and a trivial affine fibration of relative dimension $\sum_{\langle\alpha,\mu\rangle>0}\Big(\langle\alpha,\mu\rangle - 1\Big)$. Moreover, $ev^{0,\infty}$ gives the following commutative triangle:

$$\begin{tikzcd}
\mathcal{T}rip \arrow[rr, "\sim"] \arrow[dr, "ev^{0,\infty}"'] & & \mathcal{Q}uad \times W \arrow[dl, "pr_1"] \\
& \mathcal{Q}uad &
\end{tikzcd}$$

where W is a $\mathbb{G}_m$-representation with $\mathbb{G}_m$ acting by positive weights and all morphisms in the above triangle are $\mathbb{G}_m$-equivariant. In particular, $ev^{0,\infty} : \mathcal{T}rip \to \mathcal{Q}uad$ induces an isomorphism

$$ev^{0,\infty} : \mathcal{T}rip^{\mathbb{G}_m} \xrightarrow{\sim} \mathcal{Q}uad^{\mathbb{G}_m}. \tag{17}$$

Proof. Put $\mathfrak{g}_{0,\infty}$ to be the affine space consisting of pairs $(v_0, v_\infty) \in \mathfrak{g} \oplus \mathfrak{g}$ such that $\mathfrak{g}^0$-components of v_x are equal, $\mathfrak{g}^-$-components of v_x are 0, $x = 0, \infty$. Since the image of *eval* lies inside $\mathfrak{g}_{0,\infty}$, we will consider *eval* with codomain $\mathfrak{g}_{0,\infty}$,

$$eval : H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu)) \to \mathfrak{g}_{0,\infty}.$$

Since the evaluation map $H^0(\mathbb{P}^1, \mathcal{O}(m)) \to \mathbb{A}^1 \times \mathbb{A}^1$, $\phi \mapsto (\phi_0, \phi_\infty)$ is surjective for $m > 0$ and $eval(v) = (v, v)$ for $v \in \mathfrak{g}^0$, the morphism *eval* is surjective.

Let $W := \ker(eval)$. Notice that W is a $\mathbb{G}_m$-representation acting by positive weights. Since $\mathbb{G}_m$ is reductive, we get a $\mathbb{G}_m$-equivariant isomorphism:

$$H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu)) \cong W \times \mathfrak{g}_{0,\infty}.$$

Denote the nilpotent elements of $H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ by $H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))^{nil}$. Let $\mathfrak{g}_{0,\infty}^{nil}$ denote the set of elements $(v_0, v_\infty) \in \mathfrak{g}_{0,\infty}$ such that the $\mathfrak{g}^0$-components of v_0 and v_∞ are nilpotent.

Since $\Psi \in H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ is nilpotent if and only if the $\mathfrak{g}^0$-component of Ψ is nilpotent, we get a $\mathbb{G}_m$-equivariant isomorphism

$$H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))^{nil} \cong W \times \mathfrak{g}^{nil}_{0,\infty}.$$

Since $ev^{0,\infty} : \mathcal{T}rip \to \mathcal{Q}uad$ is the pullback of $H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))^{nil} \xrightarrow{eval} \mathfrak{g}^{nil}_{0,\infty}$ along the natural projection $\mathcal{Q}uad \to \mathfrak{g}^{nil}_{0,\infty}$, we get a $\mathbb{G}_m$ -equivariant isomorphism

$$\mathcal{T}rip \cong W \times \mathcal{Q}uad.$$

The statement about relative dimension follows from the fact that $\Psi \in H^0(\mathbb{P}^1, \mathrm{ad}(\mathcal{E}_\mu))$ is nilpotent if and only if the $\mathfrak{g}^0$-component (12) of Ψ is nilpotent, (15), $eval(\Psi) = (\Psi, \Psi)$ for $\Psi \in \mathfrak{g}^0$ and by the fact that the evaluation map $H^0(\mathbb{P}^1, \mathcal{O}(m)) \to \mathbb{A}^1 \times \mathbb{A}^1$, $\phi \mapsto (\phi_0, \phi_\infty)$ has nullity $m-1$ for $m > 0$. This finishes the proof of Lemma 6.1. □

Thus we have reduced the problem of finding a stratification of $\mathcal{T}rip$ to finding a stratification of $\mathcal{Q}uad$.

6.3 Stratification of $\mathcal{B}_J^{\mathrm{fin}}$.

First let us give a quick proof that $\mathcal{B}_J$ is in fact a vector bundle over G/P_J.

Lemma 6.2. *Consider the natural morphism $\mathcal{B}_J \to G/P_J$. Then $\mathcal{B}_J$ becomes a vector bundle over G/P_J.*

Proof. Consider $G \times^{P_J} \mathrm{Lie}(P_J) := (G \times \mathrm{Lie}(P_J))/P_J$, which is the quotient of $G \times P_J$ for the twisted action of P_J on $G \times \mathrm{Lie}(P_J)$ given by $p \cdot (g, v) = (g \cdot p^{-1}, \mathrm{Ad}_p(v))$. Then $G \times^{P_J} \mathrm{Lie}(P_J)$ becomes a vector bundle over G/P_J via $(g, v) \mapsto gP_Jg^{-1}$. The assignment

$$F : (g, v) \mapsto (gP_Jg^{-1}, \mathrm{Ad}_g(v))$$

gives a G-equivariant isomorphism $G \times^{P_J} \mathrm{Lie}(P_J) \to \mathcal{B}_J$ of schemes over G/P_J, where G acts on the first factor via left multiplication on $G \times^{P_J} \mathrm{Lie}(P_J)$. This isomorphism gives $\mathcal{B}_J$ a structure of a vector bundle over G/P_J. □

The following example of the Bialynicki–Birula decomposition will be important to us.

Let $\mathbb{G}_m$ act on G/P_J via μ. We have an explicit description of the connected components of the fixed point locus given by the following result (the statement follows by reducing to $\overline{k}$ and by noting that the proof of [19, Lemma 1] works for any algebraically closed field):

Fact 6.1. *Recall from Section 3.2 that L_μ is the identity component of the centralizer of $\mu(\mathbb{G}_m)$ in G. Then*

$$(G/P_J)^{\mathbb{G}_m} = \bigsqcup_{w \in W_{\Pi_\mu} \backslash W/W_J} Z_w$$

with Z_w the orbit of $w \cdot P_J$ under L_μ. In particular, the connected components Z_i of the fixed point locus $(G/P_J)^{\mathbb{G}_m}$ appearing in the Bialynicki–Birula decomposition of G/P_J (Fact 5.1(ii)) are in one to one correspondence with the elements of $W_{\Pi_\mu} \backslash W/W_J$.

Note that $Z_w \cong L_\mu/(L_\mu \cap w \cdot P_J)$ (see [30, Proposition 7.12]), which is a partial flag variety of the Levi subgroup L_μ of G defined over k. From Fact 6.1 we get:

$$(G/P_J)^{\mathbb{G}_m} \cong \bigsqcup_{w \in W_{\Pi_\mu} \backslash W/W_J} L_\mu/(L_\mu \cap w \cdot P_J).$$

Let $\pi : \mathcal{B}_J \to G/P_J$ be the projection. Note that π is $\mathbb{G}_m$-equivariant where $\mathbb{G}_m$ acts on $\mathcal{B}_J$ as in (13). Thus $\mathcal{B}_J$ is an equivariant vector bundle over the smooth projective scheme G/P_J. By Proposition 5.2, we have a stratification of $\mathcal{B}_J^{\text{fin}}$ by locally closed subsets as:

$$\mathcal{B}_J^{\text{fin}} = \bigsqcup_{w \in W_{\Pi_\mu} \backslash W/W_J} \mathcal{B}_{J,w}^{+} \tag{18}$$

and a decomposition of $\mathcal{B}_J^{\mathbb{G}_m}$ as

$$\mathcal{B}_J^{\mathbb{G}_m} = \bigsqcup_{w \in W_{\Pi_\mu} \backslash W/W_J} V_{w,0}, \tag{19}$$

where $V_{w,0}$ are the connected components of $\mathcal{B}_J^{\mathbb{G}_m}$, $w \in W_{\Pi_\mu} \backslash W/W_J$. Moreover, there are affine fibrations $lim_w : \mathcal{B}_{J,w}^{+} \to V_{w,0}$ given by the limit map as $t \to 0$.

Remark 6.3. *We can describe $V_{w,0}$ more explicitly, it is isomorphic to $\mathcal{B}_{\Pi_\mu \cap w\cdot J}$, where the underlying group is L_μ. Indeed, identify $L_\mu/(L_\mu \cap w \cdot P_J)$ with the scheme of parabolic subgroups of L_μ that are conjugate to $L_\mu \cap w \cdot P_J$. By Fact 6.1, we obtain*

$$V_{w,0} \cong \{(P', v) : P' \in L_\mu/(L_\mu \cap w \cdot P_J), v \in \mathrm{Lie}(P')\}, \tag{20}$$

where the above isomorphism is given by $(P, v) \mapsto (P \cap L_\mu, v)$. Note that if, for some $v' \in \mathfrak{g}$ we have $Ad_{\mu(t)} \cdot v' = v'$ for all $t \in \mathbb{G}_m$, then $v' \in \mathrm{Lie}(L_\mu)$. Therefore, $v \in \mathrm{Lie}(P) \cap \mathrm{Lie}(L_\mu) = \mathrm{Lie}(P \cap L_\mu)$. Thus $(P, v) \mapsto (P \cap L_\mu, v)$ is a well-defined morphism.

The next proposition gives the relative dimension of lim_w.

Proposition 6.1. *The relative dimension of the affine fibration $lim_w : \mathcal{B}^+_{J,w} \to V_{w,0}$ is $(\dim G - \dim L_\mu)/2$.*

Proof. To calculate the relative dimension of $lim_w : \mathcal{B}^+_{J,w} \to V_{w,0}$, we will use Fact 5.1(iii) on $\overline{\mathcal{B}}_J$ (this gives us the desired relative dimension because lim_w is obtained by base change of the affine fibration that we get by applying the Bialynicki–Birula decomposition on $\overline{\mathcal{B}}_J$).

Let $\underline{a} = (w \cdot P_J, 0) \in V_{w,0}(k)$. Since $\underline{a}$ is a $\mathbb{G}_m$-fixed point (see Fact 6.1), we get an action

$$\mathbb{G}_m \curvearrowright T_{\underline{a}}(\overline{\mathcal{B}}_J) = T_{\underline{a}}(\mathcal{B}_J),$$

where $T_{\underline{a}}(\overline{\mathcal{B}}_J) = T_{\underline{a}}(\mathcal{B}_J)$ because $\mathcal{B}_J$ is an open subscheme of $\overline{\mathcal{B}}_J$. Let $T^+_{\underline{a}}(\mathcal{B}_J)$ (resp. $T^-_{\underline{a}}(\mathcal{B}_J)$) denote the positive (resp. negative) eigenspace of the $\mathbb{G}_m$-action on the tangent space of $\mathcal{B}_J$ at $\underline{a}$ and let $T^0_{\underline{a}}(\mathcal{B}_J)$ denote the fixed eigenspace of the $\mathbb{G}_m$-action of the tangent space of $\mathcal{B}_J$ at $\underline{a}$. Since $\underline{a} \in V_{w,0}(k)$, the relative dimension of the affine fibration $lim_w : \mathcal{B}^+_{J,w} \to V_{w,0}$ is equal to $\dim T^+_{\underline{a}}(\mathcal{B}_J)$ by Fact 5.1(iii), so it suffices to calculate $\dim T^+_{\underline{a}}(\mathcal{B}_J)$. Note that $T_{\underline{a}}(\mathcal{B}_J)$ is $\mathbb{G}_m$-equivariantly isomorphic to $\mathfrak{g}/\mathrm{Lie}(w \cdot P_J) \oplus \mathrm{Lie}(w \cdot P_J)$. Since L_μ is in the centralizer of $\mu(\mathbb{G}_m)$, we see that $\mathrm{Ad}_{\mu(t)}$ acts on $\mathrm{Ad}_w(\mathfrak{g}_\alpha)$ via multiplication by $t^{\langle w\cdot\alpha,\mu\rangle}$, $t \in \mathbb{G}_m$, $\alpha \in \Phi$ and acts trivially on $\mathrm{Ad}_w(\mathfrak{h})$. Thus, $T_{\underline{a}}(\mathcal{B}_J)$ is $\mathbb{G}_m$-equivariantly isomorphic to $\mathfrak{g}$, which gives

$$\dim T^+_{\underline{a}}(\mathcal{B}_J) = (\dim G - \dim L_\mu)/2.$$

□

6.4 Stratification of $\mathcal{Q}uad$.

We will now work towards obtaining a stratification of $\mathcal{Q}uad$ by using the stratification (18) of $\mathcal{B}_J^{\mathrm{fin}}$. Once we have such a stratification, Theorem 3.2 will be an easy consequence of it as explained in Section 6.1. First, let us show that the $\mathcal{Q}uad$ is contained in $\mathcal{B}_{J_0}^{\mathrm{fin}} \times \mathcal{B}_{J_\infty}^{\mathrm{fin}}$.

Lemma 6.3. *Keep notations as above. We have $\mathcal{Q}uad \subset \mathcal{B}_{J_0}^{\mathrm{fin}} \times \mathcal{B}_{J_\infty}^{\mathrm{fin}}$.*

Proof. Note that it is enough to show that $\mathcal{Q}uad$ is contained in the constructible subset $\mathcal{B}_{J_0}^{\mathrm{fin}} \times \mathcal{B}_{J_\infty}^{\mathrm{fin}}$ at the level of closed points. Let K be a finite extension of k. Let $(P_0, v_0, P_\infty, v_\infty) \in \mathcal{Q}uad(K)$, then $(P_0, v_0) \in \mathcal{B}_{J_0}(K)$ and $(P_\infty, v_\infty) \in \mathcal{B}_{J_\infty}(K)$. The lemma will follow if we show $\lim_{t\to 0} t \cdot (P_x, v_x)$ exists in $\mathcal{B}_{J_x}$, $x = 0, \infty$.

Since G/P_{J_x} is a projective scheme, we get that $\lim_{t\to 0} t \cdot P_x$ exists, $x = 0, \infty$. By defintion of $\mathcal{Q}uad$, $\mathfrak{g}^-$-component (12) of v_x is 0 and therefore $\lim_{t\to 0} t \cdot v_x$ exists and is equal to the $\mathfrak{g}^0$-component (12) of v_x, $x = 0, \infty$. Thus, $\lim_{t\to 0} t \cdot (P_0, v_0)$ exists in $\mathcal{B}_{J_0}$ and $\lim_{t\to 0} t \cdot (P_\infty, v_\infty)$ exists in $\mathcal{B}_{J_\infty}$. ∎

Recall $W_{\Pi_\mu}, W_{J_0}, W_{J_\infty}, L_\mu$ from Section 3.2. For $w \in W_{\Pi_\mu}\backslash W/W_{J_0}$, $w' \in W_{\Pi_\mu}\backslash W/W_{J_\infty}$, recall $V_{w,0}$, $V_{w',0}$, lim_w, $lim_{w'}$ from Section 6.3 and put

$$\mathcal{Q}uad_{w,w'}^{\mathbb{G}_m} := (V_{w,0} \times V_{w',0}) \cap \mathcal{Q}uad.$$

Let $\mathcal{Q}uad_{w,w'}^+$ be the pullback of $lim_w \times lim_{w'} : \mathcal{B}_{J_0,w}^+ \times \mathcal{B}_{J_\infty,w'}^+ \longrightarrow V_{w,0} \times V_{w',0}$ along $\mathcal{Q}uad_{w,w'}^{\mathbb{G}_m} \to V_{w,0} \times V_{w',0}$, that is, we have the following cartesian square:

$$\begin{tikzcd}
\mathcal{Q}uad_{w,w'}^+ \arrow[r] \arrow[d] & \mathcal{B}_{J_0,w}^+ \times \mathcal{B}_{J_\infty,w'}^+ \arrow[d, "lim_w \times lim_{w'}"] \\
\mathcal{Q}uad_{w,w'}^{\mathbb{G}_m} \arrow[r] & V_{w,0} \times V_{w',0}
\end{tikzcd}$$

Let us denote the left vertical arrow in the above diagram again by $lim_w \times lim_{w'}$. Next, we would like to show that the schemes $\mathcal{Q}uad_{w,w'}^+$ give a stratification of $\mathcal{Q}uad$, which is the content of the next lemma.

Lemma 6.4. *Keep notations as above. We have $\mathcal{Q}uad_{w,w'}^+ \subset \mathcal{Q}uad$.*

Proof. Note that it is enough to show that $\mathcal{Q}uad^{+}_{w,w'}$ is contained in $\mathcal{Q}uad$ at the level of closed points. Let K be a finite extension of k. Let $(P_0, v_0, P_\infty, v_\infty) \in \mathcal{Q}uad^{+}_{w,w'}(K)$, then we have $\lim_{t\to 0} t \cdot (P_0, v_0, P_\infty, v_\infty) \in \mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}(K)$. In particular, $\lim_{t\to 0} t \cdot v_x$ exists in $\mathfrak{g}$ and is equal to the $\mathfrak{g}^0$-component (12) of v_x, $x = 0, \infty$. Since for any $v \in \mathfrak{g}$, $\lim_{t\to 0} t \cdot v$ exists in $\mathfrak{g}$ if and only if the $\mathfrak{g}^-$-component (12) of v is zero, the $\mathfrak{g}^-$-components of v_0 and v_∞ are zero. Moreover, since $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ is the closed subscheme of $\mathcal{Q}uad$ consisting of quadruples (P_0, n, P_∞, n) such that $P_0 \in Z_w$ (resp. $P_\infty \in Z_{w'}$), n is a nilpotent element of $\mathfrak{g}$ such that $n \in \mathrm{Lie}(P_0)$ and $n \in \mathrm{Lie}(P_\infty)$, we get that the $\mathfrak{g}^0$-components (12) of v_0 and v_∞ are equal and nilpotent. The lemma now follows from Remark 6.1. □

The following lemma identifies the schemes $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ with the generalized Steinberg varieties.

Lemma 6.5. *Keep notations as above. Then the schemes $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ are isomorphic to the generalized Steinberg varieties $St_{L_\mu}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty)$, $w \in D^G_{\Pi_\mu, J_0}, w' \in D^G_{\Pi_\mu, J_\infty}$.*

Proof. Notice that $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ is the closed subscheme of $\mathcal{Q}uad$ consisting of quadruples (P_0, n, P_∞, n) such that $P_0 \in Z_w$ (resp. $P_\infty \in Z_{w'}$), n is a nilpotent element of $\mathfrak{g}$ such that $n \in \mathrm{Lie}(P_0)$ and $n \in \mathrm{Lie}(P_\infty)$ (note that the $\mathfrak{g}^+$ and $\mathfrak{g}^-$-components of n are 0 since $\mathbb{G}_m$ acts trivially on $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$). Thus we have

$$\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'} \cong St_{L_\mu}(\Pi_\mu \cap w \cdot J_0, \Pi_\mu \cap w' \cdot J_\infty), \tag{21}$$

where the above isomorphism is given by $(P_0, n, P_\infty, n) \mapsto (n, P_0 \cap L_\mu, P_\infty \cap L_\mu)$. □

Next, we show that the generalized Steinberg varieties (see Section 3.1) are connected.

Lemma 6.6. *Recall Π_H, J_1, J_2 and $St_H(J_1, J_2)$ from Section 3.1. Then $St_H(J_1, J_2)$ is connected.*

Proof. We show that $St_H(J_1, J_2)$ is geometrically connected, that is, $St_H(J_1, J_2)_K$ is connected where K is the algebraic closure of k. Note that natural projection $St_H(J_1, J_2)_K \to St_H(J_1, J_2)$ is surjective as surjective morphisms are preserved under base change [42, Lemma 29.9.4]. Thus we will have that $St_H(J_1, J_2)$ is connected.

Since closed points of $St_H(J_1, J_2)_K$ are dense in $St_H(J_1, J_2)_K$ and the connected components are closed, it suffices to show that all the closed points of $St_H(J_1, J_2)_K$ are contained in the same connected component. Let $(n, P, Q) \in St_H(J_1, J_2)(K)$. Consider the morphism

$$\phi : \mathbb{A}^1_K \to St_H(J_1, J_2)_K, \quad t \mapsto (t \cdot n, P, Q).$$

Since $\mathbb{A}^1_K$ is connected, the image of ϕ is connected. Therefore, (n, P, Q) and $(0, P, Q)$ are contained in the same connected component of $St_H(J_1, J_2)_K$. Since $H/P_{J_1} \times H/P_{J_2}$ is geometrically connected (see [34, Proposition 5.2.4] and use the fact that quotient commutes with field extensions), each closed point of $St_H(J_1, J_2)_K$ is contained in the connected component containing $\{0\} \times_K (H/P_{J_1})_K \times_K (H/P_{J_2})_K$. This finishes the proof of the lemma. $\square$

Thus by Lemma 6.3 and Lemma 6.4 we get a stratification of $\mathcal{Q}uad$ by locally closed subsets as:

$$\mathcal{Q}uad = \bigsqcup_{\substack{w \in W_{\Pi_\mu} \backslash W / W_{J_0} \\ w' \in W_{\Pi_\mu} \backslash W / W_{J_\infty}}} \mathcal{Q}uad^+_{w,w'} \tag{22}$$

and a decomposition of the fixed point locus $\mathcal{Q}uad^{\mathbb{G}_m}$ as:

$$\mathcal{Q}uad^{\mathbb{G}_m} = \bigsqcup_{\substack{w \in W_{\Pi_\mu} \backslash W / W_{J_0} \\ w' \in W_{\Pi_\mu} \backslash W / W_{J_\infty}}} \mathcal{Q}uad^{\mathbb{G}_m}_{w,w'},$$

where $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ are the connected components (see Lemma 6.5 and Lemma 6.6) of $\mathcal{Q}uad^{\mathbb{G}_m}$. Moreover, we have retractions

$$lim_w \times lim_{w'} : \mathcal{Q}uad^+_{w,w'} \to \mathcal{Q}uad^{\mathbb{G}_m}_{w,w'},$$

which are affine fibrations.

Finally we calculate the relative dimension of the affine fibration $lim_w \times lim_{w'}$.

Corollary 6.6.1. *The relative dimension of the affine fibration*

$$lim_w \times lim_{w'} : \mathcal{Q}uad^+_{w,w'} \to \mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$$

is equal to $\dim G - \dim L_\mu$.

Proof. Since the affine fibration $lim_w \times lim_{w'} : \mathcal{Q}uad^+_{w,w'} \to \mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ is obtained by base change of the affine fibration $lim_w \times lim_{w'} : \mathcal{B}^+_{J_0,w} \times \mathcal{B}^+_{J_\infty,w'} \to V_{w,0} \times V_{w',0}$, the corollary follows from Proposition 6.1. $\square$

6.5 Completing the proof of Theorem 3.2.

Recall $ev^{0,\infty}$ defined in Lemma 6.1. For each $w \in W_{\Pi_\mu}\backslash W/W_{J_0}$, $w' \in W_{\Pi_\mu}\backslash W/W_{J_\infty}$, put

$$\mathcal{T}rip^{+}_{w,w'} := \left(ev^{0,\infty}\right)^{-1}\left(\mathcal{Q}uad^{+}_{w,w'}\right).$$

Since $\mathcal{Q}uad^{+}_{w,w'}$, $w \in W_{\Pi_\mu}\backslash W/W_{J_0}$, $w' \in W_{\Pi_\mu}\backslash W/W_{J_\infty}$ form a stratification of $\mathcal{Q}uad$, we get a stratification of $\mathcal{T}rip$ by locally closed subsets as:

$$\mathcal{T}rip = \bigsqcup_{\substack{w\in W_{\Pi_\mu}\backslash W/W_{J_0}\\ w'\in W_{\Pi_\mu}\backslash W/W_{J_\infty}}} \mathcal{T}rip^{+}_{w,w'}.$$

Now let $\left(ev^{0,\infty}\right)_{w,w'} := ev^{0,\infty}|_{\mathcal{T}rip_\mu(J_0,J_\infty)^{+}_{w,w'}}$. Consider the morphism

$$\left(lim_w \times lim_{w'}\right) \circ \left(ev^{0,\infty}\right)_{w,w'} : \mathcal{T}rip^{+}_{w,w'} \to \mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}.$$

Lemma 6.1 and Corollary 6.6.1 have the following consequence.

Lemma 6.7. *The morphism* $\left(lim_w \times lim_{w'}\right) \circ \left(ev^{0,\infty}\right)_{w,w'}$ *is an affine fibration of relative dimension* $\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)$*.*

Proof. Since $lim_w \times lim_{w'}$ is a trivial affine fibration (see Lemma 6.1) and $\left(ev^{0,\infty}\right)_{w,w'}$ is an affine fibration (see Corollary 6.6.1), their composition $\left(lim_w \times lim_{w'}\right) \circ \left(ev^{0,\infty}\right)_{w,w'}$ is an affine fibration.

Now let us calculate the required relative dimension. By Fact 3.1, we have

$$\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu) = \dim\Big(\prod_{\alpha\in\Phi:\langle\alpha,\mu\rangle>0} H^0(\mathbb{P}^1, \mathcal{O}(\langle\alpha,\mu\rangle))\Big) = \sum_{\langle\alpha,\mu\rangle>0}\Big(\langle\alpha,\mu\rangle+1\Big).$$

As $\left(ev^{0,\infty}\right)_{w,w'}$ is of relative dimension $\sum_{\langle\alpha,\mu\rangle>0}\left(\langle\alpha,\mu\rangle-1\right)$ (see Lemma 6.1) and $lim_w \times lim_{w'}$ is of relative dimesnsion $\dim G - \dim L_\mu$ (see Corollary 6.6.1), we see that $\left(lim_w \times lim_{w'}\right) \circ \left(ev^{0,\infty}\right)_{w,w'}$ is of relative dimension

$$\dim G - \dim L_\mu + \sum_{\langle\alpha,\mu\rangle>0}\Big(\langle\alpha,\mu\rangle-1\Big).$$

Now the lemma follows by noting that $\dim G - \dim L_\mu = 2|\{\alpha\in\Phi : \langle\alpha,\mu\rangle > 0\}|$. $\square$

We define $\mathcal{T}rip^{\mathbb{G}_m}_{w,w'}$ to be the subscheme of $\mathcal{T}rip^{\mathbb{G}_m}$ corresponding to $\mathcal{Q}uad^{\mathbb{G}_m}_{w,w'}$ in (17). Thus, Lemma 6.7 gives the required affine fibration in Theorem 3.2:

$$\mathcal{T}rip^{+}_{w,w'} \to \mathcal{T}rip^{\mathbb{G}_m}_{w,w'}.$$

This finishes the proof of Theorem 3.2. □

6.5.1 Connection with calculation of volumes.

Definition. Let $\mathfrak{X}$ be a groupoid having finitely many isomorphism classes of objects and finite automorphism groups. We define the *volume* of the groupoid $\mathfrak{X}$ as

$$[\mathfrak{X}] = \sum_{\xi \in \mathfrak{X}/\sim} \frac{1}{\#\operatorname{Aut}(\xi)}$$

where the sum is taken over the set of isomorphism classes of objects of $\mathfrak{X}$, and for an isomorphism class of objects ξ, $\operatorname{Aut}(\xi)$ is the automorphism group of any representative of ξ. In case $\mathfrak{X} = X$ is a set, $[\mathfrak{X}]$ is just the number of elements of X.

We need a simple lemma which compares volumes of groupoids:

Lemma 6.8. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two groupoids having finitely many isomorphism classes of objects and finite automorphism groups and let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism such that ϕ is surjective at the level of isomorphism classes of objects. Then*

$$[\mathfrak{X}] = \sum_{\eta \in \mathfrak{Y}/\sim} \frac{[\operatorname{Fib}(\eta)]}{\#\operatorname{Aut}(\eta)}$$

where $\operatorname{Fib}(\eta)$ *is the groupoid defined as:*

$$Ob(\operatorname{Fib}(\eta)) = \{(x, f) : x \in Ob(X), f : \phi(x) \to \eta\}$$

and for (x, f), $(x', f') \in Ob(\operatorname{Fib}(\eta))$,

$$Mor((x, f), (x', f')) = \{g : x \to x' : f' \circ \phi(g) = f\}.$$

Proof. It is clear that the lemma can be reduced to the case when $\mathcal{Y}$ has a single isomorphism class of objects, say η. Thus we have to prove

$$\sum_{\xi \in \mathfrak{X}/\sim} \frac{1}{\#\operatorname{Aut}(\xi)} = \frac{1}{\#\operatorname{Aut}(\eta)} \sum_{\zeta \in \operatorname{Fib}(\eta)/\sim} \frac{1}{\#\operatorname{Aut}(\zeta)}.$$

For each isomorphism class of object $\xi \in \mathfrak{X}/\sim$, choose a representative x_ξ. Now let $(x, f) \in \operatorname{Fib}(\eta)$, then there is a unique $\xi \in \mathfrak{X}/\sim$ such that $(x, f) \cong (x_\xi, f')$ for some $f' : \phi(x) \to \eta$. Thus the required sum can be rewritten as

$$\sum_{\xi \in \mathfrak{X}/\sim} \frac{1}{\#\operatorname{Aut}(\xi)} = \frac{1}{\#\operatorname{Aut}(\eta)} \sum_{\xi \in \mathfrak{X}/\sim} \sum_{[(x_\xi, f)] \in \operatorname{Fib}(\eta)/\sim} \frac{1}{\#\operatorname{Aut}([x, f])}.$$

Thus its enough to prove

$$\frac{1}{\#\operatorname{Aut}(\xi)} = \frac{1}{\#\operatorname{Aut}(\eta)} \sum_{[(x_\xi, f)] \in \operatorname{Fib}(\eta)/\sim} \frac{1}{\#\operatorname{Aut}([x, f])}$$

for all $\xi \in \mathfrak{X}/\sim$. Now for any x_ξ, we get the natural group morphism $\phi_{x_\xi} : \operatorname{Aut}(x_\xi) \to \operatorname{Aut}(\phi(x_\xi))$. We have by definition that $\#\operatorname{Aut}([(x_\xi, f)]) = \ker(\phi_{x_\xi})$.

Now fix x_ξ as above. Consider the action of $\phi_{x_\xi}(\operatorname{Aut}(x_\xi))$ on $\operatorname{Mor}(\phi(x_\xi), \eta)$ by precomposing. Then

$$\#\{[(x_\xi, f)] : [(x_\xi, f)] \in \operatorname{Fib}(\eta)/\sim\} = \phi_{x_\xi}(\operatorname{Aut}(x_\xi)) \backslash \operatorname{Mor}(\phi(x_\xi), \eta).$$

Thus we want to show

$$\frac{1}{\#\operatorname{Aut}(x_\xi)} = \frac{1}{\#\operatorname{Aut}(\eta)} \cdot \frac{1}{\#\phi_{x_\xi}(\operatorname{Aut}(x_\xi)) \backslash \operatorname{Mor}(\phi(x_\xi), \eta)} \cdot \frac{1}{\#\ker(\phi_{x_\xi})}.$$

The above identity holds by noting that size of the stabilizer of any $f \in \operatorname{Mor}(\phi(x_\xi), \eta)$ equals $\#\ker(\phi_{x_\xi})$. $\square$

Consider the case when $k = \mathbb{F}_q$. Define a nilpotent parabolic pair of type $(G, \mathbb{P}^1, \{0, \infty\})$ to be a collection $(\mathcal{E}, P_0, P_\infty, \Psi)$, where $\mathcal{E}$ is a principal G-bundle over $\mathbb{P}^1$, P_x is a parabolic structure on $\mathcal{E}$ at x, Ψ is a nilpotent section of $\mathrm{ad}(\mathcal{E})$ such that $\Psi_0 \in \mathrm{Lie}(P_0)$ and $\Psi_\infty \in \mathrm{Lie}(P_\infty)$. We will denote the groupoid of nilpotent parabolic pairs by $\mathcal{P}air^{nilp}(G, \mathbb{P}^1, \{0, \infty\})$. Then $\mathcal{P}air^{nilp}(G, \mathbb{P}^1, \{0, \infty\})$ decomposes into subgroupoids according to the type of parabolic structures at 0 and ∞. We denote these subgroupoids by $\mathcal{P}air^{nilp}_{J_0, J_\infty}(G, \mathbb{P}^1, \{0, \infty\})$, $J_0, J_\infty \subset \Pi$.

For $\mu \in X_+(T), J_0, J_\infty \subset \Pi$, let $\mathcal{P}air^{nilp,\mu}_{J_0, J_\infty}(G, \mathbb{P}^1, \{0, \infty\})$ denote the subgroupoid of $\mathcal{P}air^{nilp}_{J_0, J_\infty}(G, \mathbb{P}^1, \{0, \infty\})$ such that the underlying principal G-bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{E}_\mu$. Explicitly knowing $|\mathcal{T}rip_\mu(J_0, J_\infty)|$ allows us to calculate the volume of the groupoid $\mathcal{P}air^{nilp,\mu}_{J_0, J_\infty}(G, \mathbb{P}^1, \{0, \infty\})$. More concretely, by Lemma 6.8 we have

$$[\mathcal{P}air^{nilp,\mu}_{J_0, J_\infty}(G, \mathbb{P}^1, \{0, \infty\})] = \frac{|\mathcal{T}rip_\mu(J_0, J_\infty)|}{|\operatorname{Aut}(\mathcal{E}_\mu)|},$$

where $[\mathfrak{X}]$ denotes the volume of any groupoid $\mathfrak{X}$.

6.6 Proof of Corollary 3.2.4.

In this section, we will give the proof of Corollary 3.2.4. First we need the following notation:

Notation. For any affine algebraic group H over $\mathbb{F}_q$ and a cocharacter μ of a maximal torus, we will denote the centralizer of $\mu(\mathbb{G}_m)$ in H by $Z_H(\mu)$.

Now we will prove Corollary 3.2.4. We need a lemma:

Lemma 6.9. *Keep notations as in Section 3.3. Then we have*

$$[L_\mu, L_\mu] = [L_{\mu'}, L_{\mu'}].$$

In particular, the root systems of L_μ and $L_{\mu'}$ are isomorphic.

Proof. We have $L_\mu = Z_G(\mu)^\circ$ and

$$L_{\mu'} = Z_{G'}(\mu')^\circ = (Z_G(\mu) \cap G')^\circ = (L_\mu \cap G')^\circ \tag{23}$$

Clearly by (23), we have $[L_{\mu'}, L_{\mu'}] \subset [L_\mu, L_\mu]$. Now we show the other inclusion. Since $G' = [G, G]$, we have $[L_\mu, L_\mu] \subset G'$. Thus we have $[[L_\mu, L_\mu], [L_\mu, L_\mu]] \subset [G', G']$. Since derived group of any connected reductive group over $\mathbb{F}_q$ is perfect (see [8, Proposition 1.2.6]), $[[L_\mu, L_\mu], [L_\mu, L_\mu]] = [L_\mu, L_\mu]$ and hence $[L_\mu, L_\mu] \subset [G', G']$. Combining it with the fact that $[L_\mu, L_\mu]$ is connected, we get that

$$[L_\mu, L_\mu] \subset (L_\mu \cap G')^\circ$$

Now (23) gives us that $[L_\mu, L_\mu] \subset L_{\mu'}$ and hence we have the other inclusion $[L_\mu, L_\mu] \subset [L_{\mu'}, L_{\mu'}]$. This finishes the proof of Lemma 6.9. □

We return to the proof of Corollary 3.2.4. By Lemma 6.9, root systems of L_μ and $L_{\mu'}$ are isomorphic, which gives us that $[Sp_{L_\mu}] = [Sp_{L_{\mu'}}]$ as the number of points of the generalized Springer variety depend only on the root system of the underlying affine algebraic group (see Theorem 3.1(i)), hence $\Delta_{L_\mu}([Sp_{L_\mu}]) = \Delta_{L_{\mu'}}([Sp_{L_{\mu'}}])$ (see the definition of coproduct in 3.1). Now Corollary 3.2.4 (a) follows from the equality $\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu) = \dim(\mathrm{Aut}(\mathcal{E}_{\mu'})) - \dim(L_{\mu'})$ (see Theorem 3.2 and Fact 3.1).

Now we give a proof of Corollary 3.2.4 (b). Keep notations as in Section 3.3. We have $u_{|L_{\mu_1}} : L_{\mu_1} \to L_{\mu_2}$ is a flat surjective morphism (see [8, Corollary 2.1.9]). Morover, $u_{|L_{\mu_1}} : L_{\mu_1} \to L_{\mu_2}$ is finite as the restriction of a finite morphism to closed subschemes is again a finite morphism. Clearly, $\ker(u_{|L_{\mu_1}})$ is central in L_{μ_1} as $\ker(u)$ is central in G_1. Hence, $u_{|L_{\mu_1}} : L_{\mu_1} \to L_{\mu_2}$ is a central isogeny. So we get that the root systems of L_{μ_1} and L_{μ_2} are isomorphic (see [9, Proposition 3.4.1]), which gives us that $[Sp_{L_{\mu_1}}] = [Sp_{L_{\mu_2}}]$ as the number of points of the generalized Springer variety depend only on the root system of the underlying affine algebraic group (see Theorem 3.1(i)), hence $\Delta_{L_{\mu_1}}([Sp_{L_\mu}]) = \Delta_{L_{\mu_2}}([Sp_{L_{\mu'}}])$ (see the definition of coproduct in 3.1). Now Corollary 3.2.4 (b) follows from the equality $\dim(\mathrm{Aut}(\mathcal{E}_{\mu_1})) - \dim(L_{\mu_1}) = \dim(\mathrm{Aut}(\mathcal{E}_{\mu_2})) - \dim(L_{\mu_2})$ (see Theorem 3.2 and Fact 3.1).

This finishes the proof of Corollary 3.2.4 (b). □

7.0 Special case of vector bundles over $\mathbb{P}^1$.

In this chapter, we work over $k = \mathbb{F}_q$ and derive the Mellit's result [29, Section 5.4] using our method. Let us recall the notions of lambda rings, plethystic substitutions and plethystic exponentials from [29, Section 2.1, Section 2.2].

7.1 Symmetric functions.

Fix a base ring R. Let $f \in R[x_1, \ldots, x_n]$. We say that f is a symmetric polynomial if f remains unchanged when the variables $x_1, \ldots, x_n$ are permuted, i.e,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n), \quad \sigma \in S_n.$$

We denote the ring of symmetric polynomials in the variables $x_1, \ldots, x_n$ with coefficients in R by $\mathrm{Sym}_R[x_1, \ldots, x_n]$. Consider the morphism of R-algebras

$$\pi_n : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_{n-1}], \quad f(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_{n-1}, 0) \quad n \geq 2.$$

Note that π_n preserves symmetric polynomials, thus we get a direct system of the rings of symmetric polynomials. We define the ring of symmetric functions in the sequence of variables $(x_1, x_2, \ldots)$ with coefficients in R as the direct limit $\varinjlim \mathrm{Sym}_R[x_1, \ldots, x_n]$ and we will denote it by $\mathrm{Sym}_R[X]$, where $X = (x_1, x_2, \ldots)$. In other words, a symmetric function in X is a sequence $(f_n)_{n\geq 1}$ of symmetric polynomials, $f_n \in \mathrm{Sym}_R[x_1, \ldots, x_n]$ such that $\pi_n(f_n) = f_{n-1}$ for all n. Note that there is a well-defined notion of the degree of a symmetric function since π_n preserves the degrees of symmetric polynomials. We will denote the degree d component of $\mathrm{Sym}_R[X]$ by $\mathrm{Sym}^d_R[X]$.

We will denote the ring of symmetric functions with coefficients in R that are symmetric in the two sequences of variables $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ by $\mathrm{Sym}_R[X, Y]$. We have

$$\mathrm{Sym}_R[X, Y] \simeq \mathrm{Sym}_R[X] \otimes_R \mathrm{Sym}_R[Y].$$

We will denote the bidegree (d, d) component of $\mathrm{Sym}_R[X, Y]$ by $\mathrm{Sym}^d_R[X, Y]$.

Now let us give examples of symmetric functions, which will be used later in this chapter.

Example. For $n \in \mathbb{N}$, consider the following symmetric functions:

$$h_n(X) = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n} \text{ and } p_n(X) = \sum_i x_i^n.$$

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_l)$ be a finite sequence of positive integers. We define the following three types of symmetric functions:

- Complete homogeneous functions: $h_\nu(X) = \prod_{i=1}^l h_{\nu_i}(X)$.
- Power sum functions: $p_\nu(X) = \prod_{i=1}^l p_{\nu_i}(X)$.
- Monomial symmetric functions: $m_\nu(X) = \sum x_{i_1}^{\nu_1} \ldots x_{i_l}^{\nu_l}$, where the sum is taken over all distinct monomials of the form $x_{i_1}^{\nu_1} \ldots x_{i_l}^{\nu_l}$ such that $i_s \neq i_t$ for $s \neq t$.

We will need the following fact about $\mathrm{Sym}_R[X]$.

Fact 7.1. *If $\mathbb{Q} \subset R$, then $\mathrm{Sym}_R[X]$ is isomorphic to the polynomial ring $R[p_1, p_2, \ldots]$, given by $p_n(X) \mapsto p_n$, $n \in \mathbb{N}$. In particular, $\{p_\lambda : \lambda$ is a partition$\}$ forms an R-module basis of $\mathrm{Sym}_R[X]$.*

Definition. Let Λ be a ring such that $\mathbb{Q} \subset \Lambda$. A lambda ring structure on Λ is a collection of ring homomorphisms $p_n : \Lambda \to \Lambda$, $n \in \mathbb{Z}_{>0}$ satisfying:

1. $p_1(x) = x$, $x \in \Lambda$ and
2. $p_m(p_n(x)) = p_{mn}(x)$, $m, n \in \mathbb{Z}_{>0}$, $x \in \Lambda$.

In other words, giving a lambda ring structure on Λ is equivalent to giving a monoid homomorphism $\mathbb{Z}_{>0} \to \mathrm{End}_{Rings}(\Lambda)$. By a lambda ring, we will mean a ring together with a lambda ring structure.

Remark 7.1. *In the above defintion, we require Λ to contain $\mathbb{Q}$ because of the fact that $\{p_\lambda : \lambda$ is a partition$\}$ forms an R-basis of $Sym_R[X]$ when R contains $\mathbb{Q}$ (Fact 7.1).*

When our base ring R is itself a lambda ring, then we define a lambda ring structure on $\mathrm{Sym}_R[X]$ as follows: note that our ring is freely generated as an R-algebra by $p_m(X)$ since $R \supset \mathbb{Q}$ (Fact 7.1). Thus for each $n \in \mathbb{Z}_{>0}$, there is a unique homomorphism p_n :

$\mathrm{Sym}_R[X] \to \mathrm{Sym}_R[X]$ whose restriction to R is given by the lambda ring structure on R and $p_n(p_m(X)) = p_{nm}(X)$ for all $m \in \mathbb{Z}_{>0}$. We define the lambda ring structure on $\mathrm{Sym}_R[X, Y]$ similarly.

The lambda ring structure that we consider on $\mathbb{Q}(q)$ is defined as:

$$p_n : \mathbb{Q}(q) \to \mathbb{Q}(q), \quad n \in \mathbb{N}$$

$$r \mapsto r, \quad q \mapsto q^n, \quad r \in \mathbb{Q}.$$

The lambda ring structure that we consider on $\mathbb{Q}[[q^{-1}]]$ is defined as:

$$p_n : \mathbb{Q}[[q^{-1}]] \to \mathbb{Q}[[q^{-1}]], \quad n \in \mathbb{N}$$

$$r \mapsto r, \quad q^{-1} \mapsto q^{-n}, \quad r \in \mathbb{Q}.$$

The lambda ring structure that we consider on $\mathbb{Q}[[q^{-1}]][[t]]$ is defined as:

$$p_n : \mathbb{Q}[[q^{-1}]][[t]] \to \mathbb{Q}[[q^{-1}]][[t]], \quad n \in \mathbb{N}$$

$$r \mapsto r, \quad q^{-1} \mapsto q^{-n}, \quad t \mapsto t^n, \quad r \in \mathbb{Q}.$$

Note that $\mathbb{Q}(q)[[t]]$ is a sub lambda ring of $\mathbb{Q}[[q^{-1}]][[t]]$, that is, the inclusion

$$\mathbb{Q}(q)[[t]] \hookrightarrow \mathbb{Q}[[q^{-1}]][[t]]$$

is equivariant with respect to the action of the monoid $\mathbb{Z}_{>0}$.

Definition. Let Λ be a lambda ring containing $\mathbb{Q}$. Let $F \in \mathrm{Sym}_{\mathbb{Q}}[X]$ and $x \in \Lambda$. We define the plethystic action of F on x as follows: write F as a polynomial in power sum symmetric functions, say $F = f(p_1, p_2, \ldots)$ for some $f \in \mathbb{Q}[p_1, p_2, \ldots]$, we set

$$F[x] = f(p_1(x), p_2(x), \ldots).$$

The plethystic action satisfies the following properties:

$$(FG)[x] = F[x]G[x], \quad (F+G)[x] = F[x]+G[x], \quad r[x] = r, \quad F, G \in \mathrm{Sym}_{\mathbb{Q}}[X], r \in \mathbb{Q}, x \in \Lambda.$$

For each $x \in \Lambda$, the plethystic action $F \mapsto F[x]$ gives a homomorphism of $\mathbb{Q}$-algebras from $\mathrm{Sym}_{\mathbb{Q}}[X]$ to Λ.

We will need the following lemma later:

Lemma 7.1. *Let $h_n(X) \in \mathrm{Sym}_{\mathbb{Q}(q)}[X]$ be the complete homogeneous symmetric function. Then we have $h_n[qA] = q^n h_n[A]$ for any $A \in \mathrm{Sym}_{\mathbb{Q}(q)}[X]$.*

Proof. Let $A \in \mathrm{Sym}_{\mathbb{Q}(q)}[X]$. Let us recall one of the Newton's identity that expresses complete homogeneous symmetric polynomials in terms of power sum functions:

$$h_n(X) = \sum_{\substack{m_1,\dots,m_n \geq 0:\\ \sum im_i = n}} \prod_{i=1}^{n} \frac{p_i(X)^{m_i}}{m_i! i^{m_i}}.$$

By the properties of the plethystic action mentioned above, we have

$$h_n[qA] = \sum_{\substack{m_1,\dots,m_n \geq 0:\\ \sum im_i = n}} \prod_{i=1}^{n} \frac{p_i^{m_i}}{m_i! i^{m_i}}[qA] = \sum_{\substack{m_1,\dots,m_n \geq 0:\\ \sum im_i = n}} \prod_{i=1}^{n} \frac{(p_i[qA])^{m_i}}{m_i! i^{m_i}} = \sum_{\substack{m_1,\dots,m_n \geq 0:\\ \sum im_i = n}} \prod_{i=1}^{n} \frac{q^{im_i}(p_i[A])^{m_i}}{m_i! i^{m_i}}$$

$$= q^n \sum_{\substack{m_1,\dots,m_n \geq 0:\\ \sum im_i = n}} \prod_{i=1}^{n} \frac{(p_i[A])^{m_i}}{m_i! i^{m_i}} = q^n h_n[A].$$

$\square$

Definition. Let R be a base ring such that $\mathbb{Q} \subset R$. Let Λ be a topological lambda ring containing R, that is, a lambda ring equipped with a topology such that $p_n : \Lambda \to \Lambda$ is continuous for all $n \geq 1$. For $x \in \Lambda$, define $\mathrm{Exp}[x]$ as:

$$\mathrm{Exp}[x] = \exp\left(\sum_{n=1}^{\infty} \frac{p_n[x]}{n}\right)$$

provided that the right hand side converges.

7.2 Counting vector bundles over $\mathbb{P}^1$ with nilpotent endomorphisms preserving flags at 0 and ∞.

Let $\Xi_n := \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ denote the set of simple roots of GL_n relative to the diagonal torus T_n and the Borel subgroup B_n consisting of upper-triangular matrices. Consider the standard full flag $E_\bullet = \{E_j\}$ in $\mathbb{F}_q^n$. Let $J \subset \Xi_n$. Recall from Section 2.1.3 that P_J denotes the standard parabolic subgroup of GL_n corresponding to the subset J. Then P_J is the stabilizer in GL_n of the flag obtained by removing from $E_\bullet$ the terms E_j for $e_j - e_{j+1} \in J$. From now on, we identify $\mathcal{P}(\Xi_n)$ with the set of standard parabolic subgroups of GL_n via $J \mapsto P_J$.

Let Π_n denote the set of partitions of $\{1, \cdots, n\}$. For any partition $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_l) \in \Pi_n$, set

$$J(\nu) := \{e_i - e_{i+1} : i \neq \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \ldots + \nu_l = n, 1 \leq i \leq n-1\}.$$

This gives a inclusion from Π_n to $\mathcal{P}(\Xi_n)$, $\nu \mapsto P_{J(\nu)}$, where the image consists of stabilizers in GL_n of standard partial flags with jumps given by partitions. If we compose this map with the map that associates to each standard parabolic subgroup its Levi factor, then we get a bijection between the set of partitions of n and $GL_n(\mathbb{F}_q)$-conjugacy classes of Levi $\mathbb{F}_q$-subgroups of GL_n. Define $\mu(\nu) : \mathbb{G}_m \to T_n$ as:

$$t \mapsto \operatorname{diag}(\overbrace{t^l, \ldots, t^l}^{\nu_1 \text{ times}}, \ldots, \overbrace{t, \ldots, t}^{\nu_l \text{ times}}).$$

Recall L_μ from Section 3.2. We set $L_\nu := L_{\mu(\nu)}$. Notice that we have $L_\nu \cong GL_{\nu_1} \times \ldots \times GL_{\nu_l}$ (see Section 3.2).

Before proceeding, we make the following convention.

Convention 7.1. *We identify symmetric functions of degree n with the associate invariant functions on $\mathcal{P}(\Xi_n)$ by identifying m_λ with $\delta_{[J(\lambda)]}$, $\lambda \in \Pi_n$.*

Let $\mu \in X_+(T_n)$. Recall from Section 3.2 that $[\mathcal{T}rip_\mu]$ is the function on $\mathcal{P}(\Xi_n) \times \mathcal{P}(\Xi_n)$ that counts the number of $\mathbb{F}_q$-points of $\mathcal{T}rip_\mu(J_0, J_\infty)$, $(J_0, J_\infty) \in \mathcal{P}(\Xi_n) \times \mathcal{P}(\Xi_n)$. Since $[St_{L_\mu}]$ is an associate invariant function (Lemma 4.5) and $\pi_\mu = \Delta_{GL_n}(\Pi_\mu, \cdot)$ (Remark 3.1(iii)), $(\pi_\mu \otimes \pi_\mu)([St_{L_\mu}])$ is associate invariant by Corollary 4.5.1. Now using Corollary 3.2.1, we consider $[\mathcal{T}rip_\mu]$ as a symmetric function (see Convention 7.1). Thus we can write $[\mathcal{T}rip_\mu]$ as:

$$[\mathcal{T}rip_\mu] = \sum_{(\nu^0,\nu^\infty)\in\Pi_n\times\Pi_n} |\mathcal{T}rip_\mu(J(\nu^0), J(\nu^\infty))| m_{\nu^0}(X) m_{\nu^\infty}(Y).$$

Notice that $[\mathcal{T}rip_\mu] \in \mathrm{Sym}_{\mathbb{Q}(q)}[X,Y]$ by Corollary 3.2.2. Let $\mu \in X_+(T_n)$, define the symmetric function $C_\mu[X,Y;q]$ as:

$$C_\mu[X,Y;q] = \frac{[\mathcal{T}rip_\mu]}{|\operatorname{Aut}(\mathcal{E}_\mu)|}$$

and consider

$$\Omega^{\leq 0}_{n,(0,\infty)}(\mathbb{P}^1)[X,Y;q,t] = \sum_{\mu\in X_+(T_n)^*} t^{-\deg(\mathcal{E}_\mu)} C_\mu[X,Y;q],$$

where $X_+(T_n)^*$ consists of cocharaters $\mu : \mathbb{G}_m \to T_n$ of the form

$$t \mapsto \operatorname{diag}(\overbrace{t^{-d_1},\ldots,t^{-d_1}}^{\mu_1 \text{ times}},\ldots,\overbrace{t^{-d_m},\ldots,t^{-d_m}}^{\mu_m \text{ times}}), \quad 0 \leq d_1 < \ldots < d_m, \sum_{i=1}^m \mu_i = n, \mu_i > 0\,\forall i.$$

Explicitly, $\Omega^{\leq 0}_{n,(0,\infty)}(\mathbb{P}^1)[X,Y;q,t]$ is equal to

$$\sum_{\mu\in X_+(T_n)^*} t^{-\deg(\mathcal{E}_\mu)} \sum_{(\nu^0,\nu^\infty)\in\Pi_n\times\Pi_n} \frac{|\mathcal{T}rip_\mu(J(\nu^0), J(\nu^\infty))|}{|\operatorname{Aut}(\mathcal{E}_\mu)|} m_{\nu^0}(X) m_{\nu^\infty}(Y).$$

Notice that $\Omega^{\leq 0}_{n,(0,\infty)}(\mathbb{P}^1)[X,Y;q,t]$ defined above is the same as the one considered in [29, Section 5.4]. Now using our techniques, we would like to re-derive the following result of Mellit [29, Section 5.4].

Proposition 7.1. *The following holds as formal series in t with coefficients in the completion of* $\mathrm{Sym}_{\mathbb{Q}(q)}[X,Y]$*:*

$$\sum_{n=0}^\infty \Omega^{\leq 0}_{n,(0,\infty)}(\mathbb{P}^1)[X,Y;t] = \operatorname{Exp}\left[\frac{XY}{(q-1)(1-t)}\right], \quad \textit{where } XY = \sum_{i,j} x_i y_j.$$

The proof of Proposition 7.1 will be given in Section 7.2.2.

7.2.1 Reproducing kernel.

Let us briefly mention how the plethystic exponential occuring in the right hand side of Proposition 7.1 is used in the proof of [29, Theorem 5.5].

Notation. We will denote the set of all partions by $\mathcal{P}$. For any $\lambda \in \mathcal{P}$, we will denote the size of the partition λ by $|\lambda|$.

Let us consider $\mathrm{Sym}_R[X]$ where $R \supset \mathbb{Q}$. Then the Hall scalar product on $\mathrm{Sym}_R[X]$ is defined as:

$$\big(h_\mu(X), m_\lambda(X)\big) = \delta_{\lambda,\mu},$$

where $\lambda, \mu \in \mathcal{P}$.

Let $(\alpha_\lambda(X))_{\lambda\in\mathcal{P}}$ be a basis of $\mathrm{Sym}_R[X]$ such that $\deg(\alpha_\lambda(X)) = |\lambda|$ and let $(\beta_\lambda(X))_{\lambda\in\mathcal{P}}$ be the dual basis. We define the reproducing kernel to be the infinite sum $\sum_{\lambda\in\mathcal{P}} \alpha_\lambda(X)\beta_\lambda(X)$ (this makes sense in the completion of $\mathrm{Sym}_R[X,Y]$). Then we have

$$\mathrm{Exp}[XY] = \sum_{\lambda\in\mathcal{P}} \alpha_\lambda(X)\beta_\lambda(X).$$

In particular, the infinite sum is independent of the basis $(\alpha_\lambda(X))_{\lambda\in\mathcal{P}}$. One of the main properties of the reproducing kernel is the following: if $(\alpha'_\lambda(X))_{\lambda\in\mathcal{P}}$ and $(\beta'_\lambda(X))$ are such that $\deg(\alpha'_\lambda(X)) = |\lambda| = \deg(\beta'_\lambda(X))$ and

$$\mathrm{Exp}[XY] = \sum_{\lambda\in\mathcal{P}} \alpha'_\lambda(X)\beta'_\lambda(X),$$

then $(\alpha'_\lambda(X))_{\lambda\in\mathcal{P}}$ and $(\beta'_\lambda(X))$ are dual basis of $\mathrm{Sym}_R[X,Y]$.

Now let us consider the ring $\mathrm{Sym}_{\mathbb{Q}(q,t)}[X]$. Define a q,t-scalar product on $\mathrm{Sym}_{\mathbb{Q}(q,t)}[X]$ as:

$$\big\langle f(X), g(X)\big\rangle_{q,t} = \big(f\big[X\big], g\big[(q-1)(1-t)X\big]\big).$$

Definition. The modified Macdonald polynomials $\widetilde{H}_\lambda\big[X;q,t\big] \in \mathrm{Sym}_{\mathbb{Q}(q,t)}[X]$, $\lambda \in \mathcal{P}$ are the unique symmetric functions defined by the following three properties:

- orthogonality: $\big\langle \widetilde{H}_\lambda\big[X;q,t\big], \widetilde{H}_\mu\big[X;q,t\big]\big\rangle_{q,t} = 0$ if $\lambda \neq \mu$.
- normalization: $\widetilde{H}_\lambda\big[1\big] = 1$.

- upper-triangularity: $\widetilde{H}_\lambda\big[(t-1)X\big] \in M_{\leq\lambda}$, where $M_{\leq\lambda}$ is the span of the monomial symmetric functions $m_\mu(X)$, $\mu \leq \lambda$.

Mellit in the proof of [29, Theorem 5.5] used the above property of the reproducing kernel to identify certain unknown functions (namely, $F_{\lambda,q}[X;t]$ in [29, Theorem 5.6]) with the modified Macdonald polynomials. We refer the reader to [29] for details.

7.2.2 Proof of Proposition 7.1.

Recall from Section 3.1 that $[Sp_{GL_n}]$ is the function on $\mathcal{P}(\Xi_n)$ that counts the number of $\mathbb{F}_q$-points of $Sp_{GL_n}(J)$, $J \in \mathcal{P}(\Xi_n)$. Using Corollary 4.5.1 and Convention 7.1, we consider $[Sp_{GL_n}]$ as a symmetric function.

As a first step in proving Proposition 7.1, we prove the following:

Proposition 7.2. *The following holds in* $\mathrm{Sym}_{\mathbb{Q}(q)}[X]$*:*

$$h_n\left[\frac{X}{q-1}\right] = \frac{1}{|GL_n|}[Sp_{GL_n}].$$

Proof. By (5), the desired equality can be rewritten as:

$$h_n\left[\frac{X}{q-1}\right] = \sum_{\nu\in\Pi_n} \frac{q^{\dim(L_\nu)}}{q^n|L_\nu|} m_\nu(X). \tag{24}$$

We have the following idenitity(see [28, Chapter 4, Section 2]) in $\mathrm{Sym}_{\mathbb{Q}(q)}[X,Y]$:

$$h_n(XY) = \sum_{\nu\in\Pi_n} m_\nu(X)h_\nu(Y). \tag{25}$$

Then the specialization $x_i \mapsto x_i, y_j \mapsto q^{-(j-1)}, i,j \in \mathbb{N}$ gives a homomorphism of lambda rings $\mathrm{Sym}_{\mathbb{Q}(q)}[X,Y] \to \mathrm{Sym}_{\mathbb{Q}[[q^{-1}]]}[X]$. Thus this specialization commutes with the plethystic action and we have

$$h_n\left[X\left(1+\frac{1}{q}+\cdots\frac{1}{q^j}+\cdots\right)\right] = \sum_{\nu\in\Pi_n} m_\nu(X)h_\nu\left[1+\frac{1}{q}+\cdots\frac{1}{q^j}+\cdots\right] \quad \text{in } \mathrm{Sym}_{\mathbb{Q}[[q^{-1}]]}[X]$$

$$h_n\left[\frac{qX}{q-1}\right] = \sum_{\nu\in\Pi_n} m_\nu(X)h_\nu\left[\frac{q}{q-1}\right] \quad \text{in } \mathrm{Sym}_{\mathbb{Q}[[q^{-1}]]}[X].$$

Since the terms of the above identity lie in $\mathrm{Sym}_{\mathbb{Q}(q)}[X]$, the equality holds in $\mathrm{Sym}_{\mathbb{Q}(q)}[X]$.

Since $h_\nu[qA] = q^{|\nu|}h_\nu[A]$ for any $A \in \mathrm{Sym}_{\mathbb{Q}(q)}[X]$ (see Lemma 7.1), we get

$$h_n\left[\frac{X}{q-1}\right] = \sum_{\nu\in\Pi_n} m_\nu(X)h_\nu\left[\frac{1}{q-1}\right]. \tag{26}$$

Now we need to calculate $h_\nu\left[\frac{1}{q-1}\right]$, this follows from the following lemma:

Lemma 7.2. *The following holds in $\mathbb{Q}(q)$:*

$$h_n\left[\frac{1}{1-q}\right] = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Proof. Let $H(w) := \sum_{r\geq 0} h_r(X)w^r \in (\mathrm{Sym}_{\mathbb{Q}(q)}[X])[[w]]$ be the generating function for the homogeneous symmetric functions and let $P(w) := \sum_{r\geq 1} p_r(X)w^{r-1} \in (\mathrm{Sym}_{\mathbb{Q}(q)}[X])[[w]]$ be the generating function for the power sum symmetric functions. Then we have the following well-known identity in $(\mathrm{Sym}_{\mathbb{Q}(q)}[X])[[w]]$:

$$H(w) = exp\left(\int P(w)dw\right).$$

Now,

$$P\left[\frac{1}{1-q}\right] = \sum_{r\geq 1} p_r\left[\frac{1}{1-q}\right]w^{r-1} = \sum_{r\geq 1}\frac{1}{1-q^r}w^{r-1}$$

$$= \sum_{r\geq 1} w^{r-1}\left(\sum_{m\geq 0}(q^r)^m\right) = \sum_{m\geq 0}\frac{q^m}{1-wq^m}. \tag{27}$$

By (27) we have

$$H\left[\frac{1}{1-q}\right] = exp\left(\int P\left[\frac{1}{1-q}\right]dw\right) = exp\left(\int \sum_{m\geq 0}\frac{q^m}{1-wq^m}dw\right) = \prod_{m\geq 0}\frac{1}{1-wq^m}.$$

Now the lemma follows from [28, Chapter I, Section 2, Example 4]. $\square$

We return to the proof of Proposition 7.2. The specialization $q \mapsto 1/q, x_i \mapsto x_i, i \in \mathbb{N}$ gives an automorphism of lambda rings $\mathrm{Sym}_{\mathbb{Q}(q)}[X] \to \mathrm{Sym}_{\mathbb{Q}(q)}[X]$. Thus, this specialization commutes with the plethystic action on $\mathrm{Sym}_{\mathbb{Q}(q)}[X]$ and we have

$$h_n\left[\frac{1}{1-\frac{1}{q}}\right] = \frac{1}{(1-\frac{1}{q})(1-\frac{1}{q^2})\cdots(1-\frac{1}{q^n})} = \frac{qq^2\cdots q^n}{(q-1)(q^2-1)\cdots(q^n-1)}$$

Since $h_n[qA] = q^n h_n[A]$ for any $A \in \mathrm{Sym}_{\mathbb{Q}(q)}[X]$ (see Lemma 7.1), we get

$$h_n\left[\frac{1}{q-1}\right] = \frac{1}{q^n}\frac{qq^2\cdots q^n}{(q-1)(q^2-1)\cdots(q^n-1)}.$$

Now (26) gives

$$h_n\left[\frac{X}{q-1}\right] = \sum_{\nu\in\Pi_n} m_\nu(X)\prod_{i=1}^k h_{\nu_i}\left[\frac{1}{q-1}\right] = \sum_{\nu\in\Pi_n} m_\nu(X)\frac{1}{q^n}\frac{\prod_{i=1}^k qq^2\cdots q^{\nu_i}}{\prod_{i=1}^k (q-1)(q^2-1)\cdots(q^{\nu_i}-1)}. \tag{28}$$

The coefficient of $m_\nu(X)$ in equation (28) is equal to

$$\frac{1}{q^n}\frac{\prod_{i=1}^k (qq^2\cdots q^{\nu_i})^2}{\prod_{i=1}^k q^{\nu_i}(q^{\nu_i}-q^{\nu_i-1})\cdots(q^{\nu_i}-1)} = \frac{q^{\sum \nu_i^2}q^{\sum \nu_i}}{q^{2n}\prod_{i=1}^k |GL_{\nu_i}(\mathbb{F}_q)|} = \frac{1}{q^n}\frac{q^{\sum\nu_i^2}}{\prod_{i=1}^k |GL_{\nu_i}(\mathbb{F}_q)|}.$$

Since $L_\nu \cong GL_{\nu_1}\times\ldots\times GL_{\nu_l}$, Proposition 7.2 follows. □

Next, consider one of the two standard coproducts on symmetric functions:

$$\Delta^n : \mathrm{Sym}^n_{\mathbb{Z}}[X] \to \mathrm{Sym}^n_{\mathbb{Z}}[X]\otimes\mathrm{Sym}^n_{\mathbb{Z}}[Y] = \mathrm{Sym}^n_{\mathbb{Z}}[X,Y], \quad f(X)\mapsto f(XY).$$

Let Δ'_{GL_n} denote the restriction of Δ_{GL_n} to the associate invariant functions. We would like to show that Δ^n agrees with Δ'_{GL_n} by identifying symmetric functions of degree n with the associate invariant functions on $\mathcal{P}(\Xi_n)$ (see Convention 7.1). First we need a notation.

Notation. For any sequence of positive integers $\alpha = (\alpha_1,\ldots,\alpha_m)$ such that $\sum_{j=1}^m \alpha_j = n$, we will denote the subgroup $S_{\alpha_1}\times\ldots\times S_{\alpha_m}$ of S_n by S_α.

We have the following proposition.

Proposition 7.3. *Keep notations as above. Then we have $\Delta'_{GL_n} = \Delta^n$.*

Proof. Since m_ν, $\nu \in \Pi_n$ form a basis of $\mathrm{Sym}^n_{\mathbb{Z}}[X]$, it is enough to check that Δ'_{GL_n} agrees with Δ^n on this basis. We re-write the conclusion of Lemma 4.5 for GL_n. Let ν be a partition of n. It gives an equivalence relation $\sim_\nu$ on $\{1, \ldots, n\}$, where $i \sim_\nu j$ if and only if there exists a t such that $\nu_1 + \ldots + \nu_t \leq i, j < \nu_1 + \ldots + \nu_{t+1}$. Note that S_n acts on $\{1, \ldots, n\}$ and thus on the equivalence relations. For an equivalence relation $\sim$ on $\{1, \ldots, n\}$, we will write $Part(\sim) \in \Pi_n$ for the corresponding partition of n, that is, the ordered sequence of sizes of equivalence classes. By Lemma 4.5, we have

$$\Delta'_{GL_n}(m_\nu) = \sum_{\lambda,\mu \in \Pi_n} n_\nu^{\lambda,\mu} m_\lambda \otimes m_\mu,$$

where

$$n_\nu^{\lambda,\mu} = \big|\{w \in S_\lambda \backslash S_n / S_\mu : Part(\sim_\lambda \cap w(\sim_\mu)) = \nu\}\big|.$$

Thus we have

$$n_\nu^{\lambda,\mu} = \sum_{\substack{w \in S_n: \\ Part(\sim_\lambda \cap w(\sim_\mu)) = \nu}} \frac{|w^{-1} S_\lambda w \cap S_\mu|}{|S_\mu||S_\lambda|}. \tag{29}$$

Now consider the coproduct Δ^n. We have

$$\Delta^n(m_\nu) = \sum_{[(i_1,j_1),\ldots,(i_n,j_n)]} (X_{i_1} Y_{j_1}) \ldots (X_{i_n} Y_{j_n})$$

where the sum is over all multisets $[(i_t, j_t)]$, where the multiplicities of elements are given by ν.

The group S_n is acting naturally on length n sequences. Let (μ) be the standard sequence

$$\underbrace{1, \ldots, 1}_{\mu_1 \text{ times}}, \underbrace{2, \ldots, 2}_{\mu_2 \text{ times}}, \ldots.$$

In $\Delta^n(m_\nu)$, $X^\lambda Y^\mu$ occurs as:

$$\sum_{j_1,\ldots,j_n} \frac{1}{|\text{orbit of } S_\lambda \text{ on } j_1, \ldots, j_n|} (X_1 Y_{j_1} \ldots X_1 Y_{j_{\lambda_1}})(X_2 Y_{j_{\lambda_1+1}} \ldots X_2 Y_{j_{\lambda_2}}) \ldots,$$

where the summation is over all sequences $j_1, \ldots, j_n$ such that $[j_1, \ldots, j_n] = [(\mu)]$ and $Part(\sim_\lambda \cap \sim_j) = \nu$, where $\sim_j$ denotes the equivalence relation $t \sim_j s$ iff $j_t = j_s$. Let

$\widetilde{n}_{\nu}^{\lambda,\mu}$ denote the coefficient of $m_\lambda \otimes m_\mu$ in $\Delta^n(m_\nu)$. The condition $[j_1,\ldots,j_n] = [(\mu)]$ is equivalent to the existence of $w \in S_n$ such that $w\cdot(\mu) = (j_1,\ldots,j_n)$, in which case $w\cdot(\sim_\mu) = \sim_j$. Since there are exactly $|S_\mu|$ such w, we get

$$\widetilde{n}_{\nu}^{\lambda,\mu} = \sum_{\substack{w\in S_n:\\ Part(\sim_\lambda \cap w(\sim_\mu))=\nu}} \frac{|w^{-1}S_\lambda w \cap S_\mu|}{|S_\mu||S_\lambda|},$$

which agrees with (29). This finshes the proof of Proposition 7.3. $\square$

Recall the vector bundle $\mathcal{E}$ of rank n over $\mathbb{P}^1$ in [29, Section 5.4], which is defined as:

$$\mathcal{E} = \mathcal{O}(-d_1)^{\mu_1} \oplus \ldots \oplus \mathcal{O}(-d_m)^{\mu_m}, \quad 0 \le d_1 < \ldots \le d_m, \mu_i > 0, 1 \le i \le m, \sum_{i=1}^m \mu_i = n.$$

Let $\mu : \mathbb{G}_m \to T_n$ be the cocharacter of the form

$$t \mapsto \mathrm{diag}(\overbrace{t^{-d_1},\ldots,t^{-d_1}}^{\mu_1 \text{ times}},\ldots,\overbrace{t^{-d_m},\ldots,t^{-d_m}}^{\mu_m \text{ times}}), \quad 0 \le d_1 < \ldots < d_m, \sum_{i=1}^m \mu_i = n, \mu_i > 0\, \forall i.$$

Then we have $\mu \in X_+(T_n)$ and we get that $\mathcal{E} = \mathcal{E}_\mu$ (see Section 2.2).

Let us write $\mu = (\widetilde{\mu}_1,\ldots\widetilde{\mu}_m)$, where $\widetilde{\mu}_k : \mathbb{G}_m \to T_{\mu_k}$, $1 \le k \le m$ is the cocharacter

$$t \mapsto \mathrm{diag}(\overbrace{t^{-d_k},\ldots,t^{-d_k}}^{\mu_k \text{ times}}).$$

We have $\widetilde{\mu}_k \in X_+(T_{\mu_k})$. The following is a key factorization result, which is a corollary of Theorem 3.2:

Corollary 7.2.1. *For the vector bundle $\mathcal{E}$ over $\mathbb{P}^1$, we have*

$$C_\mu[X,Y;q] = \prod_{k=1}^m C_{\widetilde{\mu}_k}[X,Y;q].$$

Proof. Let f_k be an associate invariant function on $\mathcal{P}(\Xi_{\mu_k})$, where Ξ_{μ_k} is the set of simple roots of GL_{μ_k}, $1 \le k \le m$. According to our Convention 7.1, f_k is viewed as an element of $\mathrm{Sym}_{\mathbb{Z}}^{\mu_k}[X]$. However, we can also view f_k as a symmetric polynomial f'_k in the variables $x_{\mu_1+\ldots+\mu_{k-1}+1}$, $\ldots$, $x_{\mu_1+\ldots+\mu_k}$. Recall the map π_μ defined in Section 3.2. In the case of GL_n, this map relates products for the two different interpretations of the associate invariant functions f_k, $1 \le k \le m$ in the following way:

Lemma 7.3. *Keep notations as above. Then*

$$f_1 \dots f_k = \pi_\mu(f_1' \dots f_k').$$

Proof. Note that in the case of GL_n the map π_μ is the symmetrization map. $\square$

We return to the proof of Corollary 7.2.1. Recall from Section 3.1 that $[St_{GL_{\mu_k}}]$ is the function on $\mathcal{P}(\Xi_{\mu_k}) \times \mathcal{P}(\Xi_{\mu_k})$ that counts the number of $\mathbb{F}_q$-points of $St_{GL_{\mu_k}}(J_1, J_2)$, $(J_1, J_2) \in \mathcal{P}(\Xi_{\mu_k}) \times \mathcal{P}(\Xi_{\mu_k})$. Using Corollary 4.5.1, we consider $[St_{GL_{\mu_k}}]$ as a symmetric function (see Convention 7.1). We can also view $[St_{GL_{\mu_k}}]$ as a symmetric function $[St_{GL_{\mu_k}}]'$ in the variables $x_{\mu_1+\ldots+\mu_{k-1}+1}, \dots, x_{\mu_1+\ldots+\mu_k}$. We have

$$C_\mu[X, Y; q] = q^{\dim(\mathrm{Aut}(\mathcal{E}_\mu)) - \dim(L_\mu)} \frac{(\pi_\mu \otimes \pi_\mu)\big([St_{L_\mu}]\big)}{|\,\mathrm{Aut}(\mathcal{E}_\mu)|} = (\pi_\mu \otimes \pi_\mu)\big([St_{L_\mu}]\big) \Big/ \prod_k |GL_{\mu_k}|,$$

where the first equality follows from Corollary 3.2.1 and the second equality follows from Fact 3.1. Now by Lemma 4.6, we get

$$C_\mu[X, Y; q] = (\pi_\mu \otimes \pi_\mu)\Big(\prod_k [St_{GL_{\mu_k}}]'\Big) \Big/ \prod_k |GL_{\mu_k}|.$$

Using Lemma 7.3 in each variable, we get

$$(\pi_\mu \otimes \pi_\mu)\Big(\prod_k [St_{GL_{\mu_k}}]'\Big) \Big/ \prod_k |GL_{\mu_k}| = \prod_k [St_{GL_{\mu_k}}]/|GL_{\mu_k}| = \prod_k C_{\tilde{\mu}_k}[X, Y; q],$$

where we used Corollary 3.2.3 for the second equality. This finishes the proof of Corollary 7.2.1. $\square$

Let us illustrate the Corollary 7.2.1 with the following example.

Example. Consider the rank 2 vector bundle $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. Let $\mu \in X_+(T_2)^*$ be the following cocharacter

$$t \mapsto \operatorname{diag}(1, t^{-1}),$$

here $\mu_1 = \mu_2 = 1$ and $d_1 = 0, d_2 = 1$. The cocharacters $\widetilde{\mu}_1, \widetilde{\mu}_2 : \mathbb{G}_m \to \mathbb{G}_m$ are $t \mapsto 1$ and $t \mapsto t^{-1}$ respectively. Then $\mathcal{E} \simeq \mathcal{E}_\mu$, $L_\mu \simeq \mathbb{G}_m \times \mathbb{G}_m$ and we have that $C_{\widetilde{\mu}_1}[X, Y; q] \times C_{\widetilde{\mu}_2}[X, Y; q]$ is equal to

$$\left(\frac{|\mathcal{T}rip_{\widetilde{\mu}_1}(\emptyset, \emptyset)|}{|\operatorname{Aut}(\mathcal{E}_{\widetilde{\mu}_1})|} m_{(1)}(X) m_{(1)}(Y)\right)\left(\frac{|\mathcal{T}rip_{\widetilde{\mu}_2}(\emptyset, \emptyset)|}{|\operatorname{Aut}(\mathcal{E}_{\widetilde{\mu}_2})|} m_{(1)}(X) m_{(1)}(Y)\right)$$

$$= \left(\frac{1}{q-1}\left(\sum_i x_i\right)\left(\sum_j y_j\right)\right)\left(\frac{1}{q-1}\left(\sum_i x_i\right)\left(\sum_j y_j\right)\right) = \frac{1}{(q-1)^2}\left(\sum_i x_i\right)^2\left(\sum_j y_j\right)^2.$$

On the other hand,

$$C_\mu[X, Y; q] = \sum_{(\nu^0, \nu^\infty) \in \Pi_2 \times \Pi_2} \frac{|\mathcal{T}rip_\mu(J(\nu^0), J(\nu^\infty))|}{|\operatorname{Aut}(\mathcal{E}_\mu)|} m_{\nu^0}(X) m_{\nu^\infty}(Y)$$

$$= \frac{|\mathcal{T}rip_\mu(\emptyset, \emptyset)|}{|\operatorname{Aut}(\mathcal{E}_\mu)|} m_{(1,1)}[X] m_{(1,1)}[Y] + \frac{|\mathcal{T}rip_\mu(\emptyset, \{e_1 - e_2\})|}{|\operatorname{Aut}(\mathcal{E}_\mu)|} m_{(1,1)}[X] m_{(2)}[Y]$$

$$+ \frac{|\mathcal{T}rip_\mu(\{e_1 - e_2\}, \emptyset)|}{|\operatorname{Aut}(\mathcal{E}_\mu)|} m_{(2)}[X] m_{(1,1)}[Y] + \frac{|\mathcal{T}rip_\mu(\{e_1 - e_2\}, \{e_1 - e_2\})|}{|\operatorname{Aut}(\mathcal{E}_\mu)|} m_{(2)}[X] m_{(2)}[Y]$$

$$= \frac{4q^2}{q^2(q-1)^2}\left(\sum\nolimits_{i<j} x_i x_j\right)\left(\sum\nolimits_{i'<j'} y_{i'} y_{j'}\right) + \frac{2q^2}{q^2(q-1)^2}\left(\sum\nolimits_{i<j} x_i x_j\right)\left(\sum\nolimits_{i'} y_{i'}^2\right)$$

$$+ \frac{2q^2}{q^2(q-1)^2}\left(\sum\nolimits_{i'} x_{i'}^2\right)\left(\sum\nolimits_{i'<j'} y_{i'} y_{j'}\right) + \frac{q^2}{q^2(q-1)^2}\left(\sum\nolimits_{i} x_i^2\right)\left(\sum\nolimits_{i'} y_{i'}^2\right).$$

Thus, we get

$$C_\mu[X, Y; q] = C_{\widetilde{\mu}_1}[X, Y; q] \times C_{\widetilde{\mu}_2}[X, Y; q].$$

We have the following corollary.

Corollary 7.3.1. *Keep notations as in Corollary 7.2.1. Then*

$$C_\mu[X, Y; q] = \prod_{k=1}^{m} h_{\mu_k}\left[\frac{XY}{q-1}\right].$$

Proof. By Corollary 7.2.1, we have

$$C_\mu[X,Y;q] = \prod_{k=1}^{m} C_{\widetilde{\mu}_k}[X,Y;q] = \prod_{k=1}^{m} \frac{[St_{GL_{\mu_k}}]}{|GL_{\mu_k}|} = \prod_{k=1}^{m} \frac{\Delta_{GL_{\mu_k}}\big([Sp_{GL_{\mu_k}}]\big)}{|GL_{\mu_k}|},$$

where the second equality follows from Corollary 3.2.3 and the last equality follows from Theorem 3.1 (ii). By Proposition 7.3, we have

$$C_\mu[X,Y;q] = \prod_{k=1}^{m} \frac{\Delta^{\mu_k}\big([Sp_{GL_{\mu_k}}]\big)}{|GL_{\mu_k}|} = \prod_{k=1}^{m} h_{\mu_k}\left[\frac{XY}{q-1}\right],$$

where the second equality follows from Proposition 7.2. □

Now we are ready to prove Proposition 7.1. We have

$$\sum_{n=0}^{\infty} \Omega^{\leq 0}_{n,(0,\infty)}(\mathbb{P}^1)[X,Y;q,t] = \sum_{n=0}^{\infty} \sum_{\mu\in X_+(T_n)^*} t^{-\deg(\mathcal{E}_\mu)} C_\mu[X,Y;q]$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{\mu=(\mu_1,\ldots,\mu_m),\\ d=(0\leq d_1<d_2<\ldots<d_m):\\ \sum \mu_k=n}} t^{\sum_{k=1}^{m} d_k\mu_k} h_\mu\left[\frac{XY}{q-1}\right],$$

where the second equality follows from Corollary 7.3.1. The above is equal to

$$\prod_{d=0}^{\infty}\sum_{k=0}^{\infty} t^{dk} h_k\left[\frac{XY}{q-1}\right] = \prod_{d=0}^{\infty} \operatorname{Exp}\left[\frac{t^d XY}{q-1}\right] = \operatorname{Exp}\left[\frac{XY}{q-1}\sum_{d=0}^{\infty} t^d\right] = \operatorname{Exp}\left[\frac{XY}{(q-1)(1-t)}\right].$$

This finishes the proof of Proposition 7.1. □

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