Mathematical and numerical modeling of flow and transport in
fluid–poroelastic structure interaction

by

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The thesis focuses on the analysis and simulation of fluid–poroelastic structure interaction (FPSI), which describes the interaction between free fluid and a neighboring poroelastic medium through an interface. The Stokes or time-dependent Navier-Stokes equations govern the flow in the fluid region. The Biot system describes the flow within the poroelastic media and the structure deformation. The two regions are coupled via mass conservation, balance of normal stress, balance of momentum, and the Beaver-Joseph-Saffman slip with friction condition. A mixed Darcy formulation is employed, and a pressure Lagrange multiplier on interface is introduced to impose weakly continuity of flux. The Stokes–Biot model is further coupled with an advection-diffusion transport equation for the solute concentration.

First, we discuss a Lagrange multiplier method for the fully dynamic Navier-Stokes–Biot model. The existence, uniqueness, and stability of the solution are obtained. We further study the well-posedness of a fully discrete scheme approximating the model based on suitable finite element spaces and derive error estimates. Numerical simulations illustrate the order of convergence and the feasibility of the numerical method to model a benchmark problem involving Newtonian blood flow.

We extend the Newtonian model to a non-Newtonian fluid via the Carreau-Yasuda model for shear-thinning rheology. We study the effect of poroelasticity of blood vessels and non-Newtonian blood rheology on important clinical markers such as wall shear stress and relative residence time. We further conduct numerical experiments illustrating the blood flow in stenotic vessels.

Finally, we discuss a two-way coupled Stokes–Biot–transport model. The convective term in the transport equation, which depends on the fluid velocity and the concentration-dependent viscosity, makes the coupled system non-linear. We show well-posedness for a linearized formulation. Next, the convergence of a fixed-point iterative algorithm is analyzed.
to obtain a solution of the original non-linear model. We study the well-posedness of the non-linear formulation and the finite element approximation. Error estimates are derived for the semi-discrete problem. A series of computational experiments are conducted to illustrate the theoretical convergence rates and to explore the method’s applicability to model fluid flow and solute transport within networks of channels.

**Keywords:** FPSI, Navier-Stokes–Biot model, coupled flow and transport, non-linear convective transport, Lagrange multiplier, mixed Darcy formulation, convergence analysis.
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1.0 Introduction

1.1 Motivations and overview

Fluid-poroelastic structure interaction (FPSI) refers to the interaction of a free fluid with a neighboring poroelastic medium through an interface [1, 3, 23, 57, 66]. This multiphysics phenomenon has a wide range of applications in petroleum engineering, hydrology, environmental sciences, and biomechanical engineering [41, 46, 61, 71]. Examples include groundwater and oil/gas flow through fractured or deformable aquifers and reservoirs, as well as the interaction of arterial flow with the poroelastic arterial wall. Furthermore, the flow process may be coupled with transport phenomena, with the substance of interest propagating both in the free fluid region and through the poroelastic structures [2, 44, 73, 77]. Typical examples include tracking and cleaning up groundwater contaminants, leakage of subsurface radioactive waste, modeling drug delivery in blood circulation, and transport of low-density lipoprotein (LDL), which is the leading cause of atherosclerosis [32, 39, 61].

We adopt the time-dependent incompressible Navier-Stokes (or Stokes) equations to model the fluid flow and the Biot system for the flow in the poroelastic material. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation, which describes the average velocity of the fluid in the pores [13, 76]. The two regions are coupled via dynamic and kinematic interface conditions, including balance of forces, continuity of normal velocity, and a slip with friction tangential velocity condition, also known as the Beavers-Joseph-Saffman (BJS) condition [3, 51, 66]. We observe that these multiphysics models exhibit both features of coupled Stokes–Darcy flows [41, 43, 47, 48, 89] and fluid-structure interaction (FSI) problem [10, 75, 90]. The Stokes–Biot system can also be coupled with an advection-diffusion equation to model the transport of chemical species within the fluid [2, 27]. The coupling between flow and transport is completed via the velocity field and the concentration.

Works studying the fluid–poroelastic structure interaction (FPSI) problem, considering the Stokes–Biot or Navier-Stokes–Biot coupled problem, include [1, 3, 7, 66]. A fully dynamic
Stokes–Biot model is first considered in [80]. By rewriting the model into a parabolic system, the author presents theoretical analysis, including the well-posedness of the coupled Stokes–Biot model. In [7], a variational multiscale finite element method is developed for the Navier-Stokes–Biot system. In [7], to deal with the coupling of the two underlying models numerically, both monolithic and iterative partition methods are proposed. Readers are also referred to [57], where a coupled displacement–pressure interface problem is solved in the monolithic approach. Meanwhile, the drained split and fixed stress Biot splitting methods are further analyzed. Inspired by Nitsche’s method, a non-standard partitioning strategy is adopted in [23] by adding appropriate interface operators to the variational formulation. In [3, 4, 5, 65], a mixed formulation for the Darcy equation is considered so that no additional regularity is required on the data, which is widely accepted and used in practical engineering applications. Proved and confirmed in [42, 68], the mixed formulation is able to provide a locally and globally mass conservative flow approximation and accurately render the Darcy velocity and structure pressure. However, a trade-off is that an essential type of normal velocity condition on the boundary of the structure subdomain needs to be enforced weakly. To address this problem, Nitsche’s interior penalty method is utilized in [23, 24] to impose the continuity of the normal flux weakly. In [3, 66], a Lagrange multiplier method is introduced along with the interface to avoid the presence of the extra penalty parameter. The so-called Lagrange multiplier formulation can provide up to the machine precision accuracy on a matching grid, while on a nonmatching grid, it is also a convergent algorithm. Note that more than one Lagrange multiplier can be used along with the interface; see [66] for details. In [66], a fully mixed Biot formulation which is locking-free and reduces the number of degrees of freedom is used, resulting in the introduction of two Lagrange multipliers: the traces of the structure velocity and the Darcy pressure.

In this thesis, we focus on two main new directions in FPSI modeling, the Navier-Stokes-Biot model and the two-way coupled FPSI-transport model, which is discussed later. We apply the models to simulating cardiovascular flows, drug delivery, and LDL transport. In Chapter 2, we develop and analyze a new Navier-Stokes-Biot model, which extends the work in [3, 28]. Unlike [3], which considers the Stokes flow, typical in flows where the velocities are slow and the Reynolds number is small, we utilize the time-dependent Navier-Stokes
equation and a fully dynamic Biot system which involves the second time derivative of the displacement. This model is more suitable for describing the blood flow in an aorta. On the other hand, unlike [28], our model employs a mixed finite formulation for the Darcy flow, which results in a locally mass conservative flow approximation in the poroelastic region. The mixed flow formulation results in an essential type of continuity of normal velocity via the interface. To impose it weakly, we introduce a Lagrange multiplier, which enforces the condition in a stable and accurate way and avoids the need for a penalty parameter [3, 20]. Out of consideration for the multiscale computation, the introduction of Lagrange multiplier benefits in reducing the coupled model to an interface problem and attaining efficient parallel domain decomposition algorithms.

We discuss and analyze the fully dynamic incompressible Navier-Stokes–Biot model problem in Chapter 2, which is organized as follows. In Section 2.1, we present the mathematical model with the appropriate interface, boundary, and initial conditions, set up the assumptions on data, and come up with the main result. The following section is devoted to the well-posedness proof of the main theorem of this chapter. In Section 2.3, we present the fully discrete numerical scheme and show the existence, uniqueness, and stability of the fully discrete solution. A detailed error analysis is presented in Section 2.4, which enlightens the expected rates with different choices of finite element spaces in the following section. Numerical experiments which confirm our theoretical results and applications to blood flow with real-world parameters are discussed in Section 2.5.

In the blood flow applications in Section 2.5, we note that the shear-thinning property of the blood cannot be captured by the Newtonian fluid assumptions. Due to changes in lifestyle such as inadequate dietary habits, smoking, staying up late, and being sedentary, cardiovascular diseases have become a primary concern of modern society. There has been considerable evidence that hydrodynamic factors would play an essential role in identifying, diagnosing, and understanding the development and progression of arterial lesions. Motivated by this, we focus on investigating the prototype problem arising in blood flow in Chapter 3. We study fully dynamic blood flow models for the interaction of an incompressible Newtonian or non-Newtonian fluid and fluid within a poroelastic or an elastic vessel. This is a complex problem with predicting, modeling, or controlling processes in blood rheology. Arterial flow
is not only affected by the poroelastic nature of the arterial wall [7, 15, 24, 25] but also the geometrical complexity of the vessel [52, 72]. Therefore, it is crucial building a mathematical model that correctly simulates the interaction of a free viscous fluid with a porous material and accounts for the medium’s elasticity in various computational domains.

Related literature on blood flow problems is rich [6, 23, 24, 41, 52, 72]. In [52], based on three-dimensional patient-specific stenotic vessels, the influences of the degrees of stenosis on Newtonian and non-Newtonian behavior of blood have been studied. The limitations of the blood flow simulations in [52] are the absence of turbulence in the stenotic models and a fluid-structure interaction model. In [40], non-Newtonian blood behavior on LDL (low-density lipoprotein) accumulation is analyzed, and fluid-multilayer arteries are adopted for healthy and stenotic vessel models. Numerical investigation of non-Newtonian modeling effects on unsteadying periodic flows in a two-dimensional vessel with multiple idealized stenoses of different degrees has been learned in [72]. However, to our knowledge, few works have considered that the arterial wall is porous and deformable and the non-Newtonian property of blood flow in different vessel geometries simultaneously.

Earlier works [10, 40] on fluid-structure interaction (FSI) models focused on establishing and exploring rheological phenomena through the porous arterial walls. It has been categorized that there are three distinct wall layer groups: wall-free model, fluid-wall single layer model, and fluid-wall multilayer model [39, 70]. However, they don’t account for the natural deformation of arterial walls. Instead of a porous vessel, we will employ a coupled Biot system which includes the second derivative of displacement to govern the fluids in deformable and porous structures. The fluids-structure regions are coupled through dynamic and kinematic interface conditions and the Beavers-Joseph-Saffman slip with friction conditions. In addition, we study how the structure parameter permeability and Lamé coefficients affect blood flow. Since the position of the interface between fluid and poroelastic domains is time-dependent, numerical methods need to be able to track down the corresponding movements of the structure and allow various quantities to be calculated on moving mesh. One commonly used method to tackle this interface difficulty is the arbitrary Lagrangian-Eulerian (ALE) [55, 56], also known as the boundary-fitted approach. ALE method is adaptive and allows the fluid mesh to match the structure displacement along with the interface, which is
suitable in our blood flow models.

On the other hand, blood is comprised of different types of elements, such as red blood cells, platelets, proteins, water, etc., and will exhibit complex rheological properties [40, 52, 61]. Much biophysical research has confirmed that blood shows a shear-thinning behavior: viscosity decreases with increasing shear rates, and it will reach approximately a constant value. However, the assumption of Newtonian fluids is generally used and accepted for blood flow studies in large-sized vessels [50, 53]. But so far, no universal agreement has been reached about the proper model to describe the viscous properties of blood, especially for medium-sized or small-sized arteries. Viscosity models to describe the blood’s non-Newtonian properties include the Power law, the Cross model, and the Carreau-Yasuda model [14, 52, 59, 69]. The Power law model only contains two parameters and is able to derive analytical solutions in various flow conditions. However, if the deformation goes to zero, the viscosity in the flow region would go to infinity, making this model less representative in certain applications [14]. The Cross model is an empirical equation that describes pseudoplastic flow with asymptotic viscosities at zero and infinity shear rates [38]. Since the Cross model is deduced empirically, this model may not fit the data exactly. In Chapter 3, we utilize the Carreau-Yasuda model. Even though the Carreau-Yasuda model is an empirical equation and it has more parameters than the Cross model and the Power model, it is well suited for fluids that are beginning to shear thin, and in most situations is used to describe emulsions, biopolymer solutions, and protein solutions [14], which is more suitable in the blood flow simulations. In addition to comparing the difference between poroelastic and elastic models, we also study how non-Newtonian properties affect blood flow characteristics.

Last but not least, we apply our simulation to a relatively complicated geometry. Due to the deposition of lipid, cholesterol, and some other substances, there is a high risk that stenosis would be initiated preferentially in arteries and regions with high curvature or bifurcations [52, 72], which will cause significant changes in flow structure and consequently lead to large changes in fluid loading on vessels. Such plaques or arterial constrictions usually disturb normal blood flow through the artery. A few studies have even been carried out on multiple stenoses [72, 86], using the momentum integral method and Galerkin finite element method. In [85], a theoretical analysis of the pulsatile flow of blood in stenosed arteries was studied.
On the contrary, in [84], the consequences of multiple stenoses on pressure were investigated experimentally. It has been confirmed that once mild stenosis is developed inside the arterial lumen, the resulting flow disorder further affects the development of arterial deformability and gradually changes the regional blood rheology [72, 85]. Few of these studies worked on the coupled fluids–structure model, namely the FPSI model, using non-Newtonian viscosity. We enhance our numerical simulations by considering the poroelastic nature of the arterial wall and the non-Newtonian behavior of blood flow in a stenotic geometry.

Chapter 3 uses time-dependent Navier-Stokes equations to model the fluid region and a fully dynamic Biot system to govern the poroelastic structure. We focus on differences between Newtonian and non-Newtonian fluids using the Carreau-Yasuda model. In addition to the prototype benchmark problem, we also conduct simulations for stenosis cases. This chapter considers four different parts essential to the blood flow problem: 1. the difference between elasticity and poroelasticity models; 2. how the parameter permeability affects the dynamic characteristics; 3. the differences between Newtonian and non-Newtonian blood rheology in numerical simulations; 4. practicability of coupled elastic and poroelastic models in stenotic regions.

The outline of Chapter 3 is presented as follows: In Section 3.1, we describe two mathematical models: Navier-Stokes/Elasticity (NSE/E) model and Navier-Stokes/Poroelasticity (NSE/P) model, together with the appropriate interface, boundary, and initial conditions. Section 3.2 is devoted to the numerical simulations of blood flow models with real-world parameters. We will mainly focus on the differences between the four types of models: Newtonian NSE/E, Newtonian NSE/P, non-Newtonian NSE/E, non-Newtonian NSE/P models, and several effects on blood rheology. We summarize and make conclusions in Section 3.2.6.

Inspired and motivated by the fact that there are different elements dissolved in the blood, the process of drug delivery, and the accumulation of LDL along with the vessels, we realize that the solute concentration is another critical hydrodynamic factor. Therefore, in Chapter 4, the Stokes–Biot system is further coupled with an advection-diffusion equation for modeling the transport of chemical species within the fluid. Realistic simulations for flow, transport and chemical reactions present computational challenges. This type of multiphysics problem is challenging due to three main reasons. First, this process typically
involves modeling the interaction between the fluid in the free fluid region and within the
poroelastic area, which is a typical FPSI problem [3, 10, 46, 60]. Second, the flow process
is coupled with transport phenomena which in turn affect the flow, because of the species
of interest propagating and diffusing in both the fluid region and poroelastic medium. Ex-
amples include evaluating leakage of subsurface radioactive waste and simulating proppant
injection in hydraulic fracturing [2, 83]. In this area of applications, in general, people use
a solid concrete matrix to preserve the radioactive and acid solute. However, due to erosion
and potential deformations, fractures are formed, resulting in the necessity of modeling in
hydraulic fractures, where the convection is much faster than in the structure matrix. Third,
the flow is fully coupled since the viscosity of the fluid depends on the concentration of the
concerned species non-linearly. For example, the non-Newtonian behavior of blood needs to
be considered in small vessels, and the viscosity of blood is affected by the concentration of
low-density lipoprotein (LDL), which is the leading cause of atherosclerosis [39, 40].

There is extensive literature on the coupled transport problem [2, 8, 27, 35, 36, 44, 77, 83].
In [83], the non-symmetric interior penalty Galerkin (NIPG) method is used for the transport
equation. The continuity equation and Darcy equation are utilized to govern the miscible
displacement in porous media. To address the non-linear dispersion operator in the transport
equation, a cut-off operator is introduced. A general coupling of miscible displacement in
porous media with the surface flow and transport, in other words, a fully coupled Navier-
Stokes/Darcy–transport model, is analyzed in [27]. Note that in [27], there is no assumption
on the boundedness of the diffusion–dispersion matrix in the porous areas. In [2], to avoid
the presence of a cut-off operator, the $L^\infty$-bounds for the computed Stokes–Biot velocity
are established so that the velocity can be directly incorporated into the transport scheme.
However, in this loose one-way coupling, the velocity field obtained from the Stokes–Biot
problem becomes input data for the transport equation. To improve this one-way coupling,
in Chapter 4, we focus on the analysis of a two-way coupled system, namely the fully coupled
Stokes–Biot–transport model. In a two-way coupled model, the viscosity of the fluid depends
on the concentration of the dissolved substance. At the same time, the concentration is also
affected by the fluid velocity. To the best of our knowledge, the general two-way coupled
Stokes–Biot–transport problem has not been studied in the literature.
The aim of Chapter 4 is to establish the well-posedness of the fully coupled Stokes–Biot–Transport system. The difficulty of this model is on a whole new level due to the non-linear equations and the non-linear coupling between them. Therefore, the first step of this paper is to linearize the coupled model. We retain the remaining coupling terms. Direct analysis of the linearized model is still tricky. Therefore, under the framework of the Galerkin approximation, we obtain the existence and uniqueness of the approximated solution in a discretized-space and continuous-in-time sense via the theory of Differential-Algebraic Equations (DAEs) [19] and an ODE theory [37]. Afterward, the suitable stability results, i.e., \( a \text{ priori} \) estimates, are obtained to ensure the passing to the limit procedure via weak compactness. Based on the established well-posedness and stability of the linearized model, we introduce an iterative algorithm to approximate the non-linear terms we linearized before. Under the assumption that the concentration and Stokes–Biot velocity admit enough smoothness and using the stability results we derived previously, we can construct a well-defined iterative sequence which is also a contraction in a small time interval. Via the Banach Fixed Point Iteration Theorem [30], we show that the sequence converges to the solution of the original non-linear problem in appropriate norms. However, due to the contractive procedure, one problem is that we can only obtain the desired convergence result locally in time, in a small time interval. Therefore, we proceed by using the piecewise continuation in time to extend the local result globally in any finite time interval. The other problem is that since we are using the mixed formulation for the Darcy velocity, we need the convergence of the sequence under a higher regularity. We manage to address this problem by estimating terms in \( L^1 \)-norm. We mention some literature using iterative schemes. In [9, 21, 65], one can find several iterative techniques for solving different coupled systems.

As a summary of this chapter, the well-posedness of a mixed weak formulation of the fully coupled Stokes–Biot–transport system using a Lagrange multiplier on the interface is proved under a natural hypothesis on the regularity of the non-linear convective term and viscosity. We obtain the error estimates of the semi-discretized continuous-in-time formulation. A series of numerical simulations are provided.

The outline of Chapter 4 is shown as follows: In Section 4.1, we present the mathematical model together with the proper interface, boundary, initial conditions, and the Lagrange
multiplier weak formulation. In Section 4.2, the well-posedness and stability of the linearized formulation are established. In Section 4.3, we use the fixed point iterative method to obtain the well-posedness of the original coupled system. More importantly, we manage to provide the convergence of the Darcy velocity in the $H(\text{div})$ norm, which is essential in the mixed formulation construction. Based on Section 4.3, we obtain the well-posedness of the Galerkin problem of the original problem for future use in Section 4.4. A detailed error analysis is presented in Section 4.5. Section 4.6 is devoted to the numerical experiments that confirm our theoretical convergence results and applications to the flow and transport model in an irregularly shaped channel network.

1.2 Preliminaries

1.2.1 Notations

In this section, we introduce some Sobolev spaces and fix some notations. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a domain with Lipschitz boundary. We denote by $\Gamma$ the boundary of $\Omega$. Throughout this thesis, we use the standard notation for the Lebesgue spaces $L^p(\Omega)$, for $p \in [1, +\infty)$, equipped with the following norm

$$\|\phi\|_{L^p(\Omega)}^p = \int_\Omega |\phi|^p dA.$$ 

We consider the Sobolev spaces $W^{k,p}(\Omega)$, for $k \geq 0$, with the norm and corresponding seminorm

$$\|\phi\|_{W^{k,p}(\Omega)}^p = \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha \phi|^p dA, \quad |\phi|_{W^{k,p}(\Omega)} = \sum_{|\alpha| = k} \int_\Omega |\partial^\alpha \phi|^p dA.$$ 

If $k = 0$, we write $W^{0,p}(\Omega) = L^p(\Omega)$ instead. Also for $p = 0$, we use $W^{k,2}(\Omega) = H^k(\Omega)$ with the norm $\|\cdot\|_{H^k(\Omega)}$. We denote by $W^{-k,p'}(\Omega)$ the dual space of $W^{k,p}(\Omega)$, where $p'$ is the conjugation of $p$, satisfying $1/p + 1/p' = 1$. Note that we use $L^p(\Omega)$, $H(\Omega)$, and $W^{k,p}(\Omega)$ for the corresponding vector functions spaces.
We further define the space $H(\text{div}; \Omega) := \left\{ \mathbf{v} \in L^2(\Omega) : \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}$ endowed with the norm

$$\|\mathbf{v}\|_{H(\text{div}; \Omega)}^2 := \|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2.$$ 

The $L^2(\Omega)$ inner product is denoted by $(\cdot, \cdot)_\Omega$ for scalar, vector, and tensor valued functions. For the boundary $\Gamma$, we use $(\cdot, \cdot)_\Gamma$ for the $L^2(\Gamma)$ inner product or duality pairing. We also denote by $C$ a generic positive constant independent of discretization parameters.

For the time-dependent function $\phi$, we in addition introduce the Bochner spaces equipped with norms:

$$\|\phi\|_{L^2(0,T;X)}^2 := \int_0^T \|\phi(t)\|_X^2 \, dt, \quad \|\phi\|_{H^1(0,T;X)}^2 := \int_0^T \left( \|\phi(t)\|_X^2 + \|\partial_t \phi(t)\|_X^2 \right) \, dt,$$

$$\|\phi\|_{L^\infty(0,T;X)} := \text{ess sup}_{t \in [0,T]} \|\phi(t)\|_X, \quad \|\phi\|_{W^{1,\infty}(0,T;X)} := \text{ess sup}_{t \in [0,T]} \left\{ \|\phi(t)\|_X, \|\partial_t \phi(t)\|_X \right\}.$$ 

For the time discretization in $[0,T]$, let $N$ be the number of time steps and $\Delta t = T/N$, $t_n = n\Delta t$, $0 \leq n \leq N$. Let $d_t u^n = (u^n - u^{n-1})/\Delta t$ be the first order (backward) discrete time derivative, where $u^n = u(t_n)$. We introduce the discrete-in-time norms:

$$\|\phi\|_{l^2(0,T;X)}^2 := \Delta t \sum_{n=1}^N \|\phi^n\|_X^2, \quad \|\phi\|_{l^\infty(0,T;X)} := \max_{0 \leq n \leq N} \|\phi^n\|_X.$$ 

1.2.2 Important inequalities

- (Hölder/Cauchy-Schwarz) For any $\phi \in L^p(\Omega)$ and $\psi \in L^{p'}(\Omega)$,

$$\|\phi \psi\|_{L^1(\Omega)} \leq \|\phi\|_{L^p(\Omega)} \|\psi\|_{L^{p'}(\Omega)}. \quad (1.2.1)$$

- (Trace) For any $\phi \in W^{1,p}(\Omega)$,

$$\|\phi\|_{L^p(\partial\Omega)} \leq C\|\phi\|_{W^{1,p}(\Omega)}. \quad (1.2.2)$$

- (Young’s) For any real numbers $a$, $b$, and $\epsilon > 0$,

$$ab \leq \frac{\epsilon a^p}{p} + \frac{b^{p'}}{\epsilon^{p'}p'}. \quad (1.2.3)$$
• (Gronwall) Let \( u(t), h(t), \) and \( f(t) \) be continuous functions and let \( g(t) \) be integrable in \([a, T]\). If \( h(t) \geq 0, g(t) \geq 0, f(t) \) is non-decreasing, and \( u(t) + h(t) \leq f(t) + \int_a^t g(\tau)u(\tau)\,d\tau \forall t \in [a, T] \), then

\[
u(t) + h(t) \leq f(t) \exp \left( \int_a^t g(\tau)\,d\tau \right) \forall t \in [a, T]. \tag{1.2.4}\]

• (Discrete Gronwall) Let \( \tau > 0, B \geq 0, \) and let \( a_n, b_n, c_n, d_n, \ n \geq 0, \) be non-negative sequences such that \( a_0 \leq B \) and

\[
a_n + \tau \sum_{l=1}^n b_l \leq \tau \sum_{l=1}^{n-1} d_l a_l + \tau \sum_{l=1}^n c_l + B, \quad n \geq 1. \tag{1.2.5}\]

Then,

\[
a_n + \tau \sum_{l=1}^n b_l \leq \exp(\tau \sum_{l=1}^{n-1} d_l) \left( \tau \sum_{l=1}^n c_l + B \right), \quad n \geq 1. \tag{1.2.6}\]
2.0 A Lagrange multiplier method for the fully dynamic Navier-Stokes–Biot system

In this chapter, we focus on analyzing the time-dependent Naiver-Stokes–Biot coupled system, which involves the second time derivative of displacement in the Biot subsystem. We start by introducing and describing the model problem. In Section 2.1 and Section 2.3, we provide the well-posedness of the solution in three different senses. First, the well-posedness of a continuous-in-time but discretized-in-space model is obtained in Section 2.2.2. Next, utilizing the results that the solution of the semi-discretized model is bounded, we push forward to obtain the well-posedness of the continuous case. Finally, we discuss the fully discretized model in Section 2.3, both in time and space. To confirm the feasibility of using Lagrange multiplier methods in FPSI simulations, we test the convergence rate separately in lower and higher order finite element spaces and simulate a benchmark blood flow problem in a 2-D vessel.

2.1 Navier-Stokes–Biot model problem

Let \( \Omega \subset \mathbb{R}^d, d = 2, 3, \) be a union of non-overlapping polygonal regions \( \Omega_f \) and \( \Omega_p. \) Here \( \Omega_f \) is a free fluid region governed by the time-dependent Navier-Stokes equations and \( \Omega_p \) is a poroelastic region governed by the Biot system. Let \( \Gamma_{fp} = \partial \Omega_f \cap \partial \Omega_p \) be the interface between the two regions. Let \( (u_*, p_*) \) be the velocity-pressure pair in \( \Omega_*, * = f, p, \eta_p \) be the displacement in \( \Omega_p, \rho_f \) be the fluid density, \( \mu_f \) be the fluid viscosity, and \( f_* \) be a body force. Let \( D(u_f) \) and \( \sigma_f(u_f, p_f) \) denote the deformation rate tensor and the Cauchy stress tensor, respectively:

\[
D(u_f) = \frac{1}{2} (\nabla u_f + \nabla u_f^T), \quad \sigma_f(u_f, p_f) = -p_f I + 2\mu_f D(u_f). \tag{2.1.1}
\]

In the free fluid region \( \Omega_f, \) \( (u_f, p_f) \) satisfy the incompressible Navier-Stokes equations

\[
\rho_f \partial_t u_f - \nabla \cdot \sigma_f(u_f, p_f) + \rho_f u_f \cdot \nabla u_f = f_f \quad \text{in} \ \Omega_f \times (0,T], \tag{2.1.2}
\]
\[ \nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f \times (0, T], \quad (2.1.3) \]

where \( T > 0 \) is the final time. Next, let \( \sigma_e(\eta_p) \) and \( \sigma_p(\eta_p, p_p) \) be the elastic and poroelastic stress tensors, respectively:

\[
\sigma_e(\eta_p) = \lambda_p (\nabla \cdot \eta_p) I + 2 \mu_p \mathbf{D}(\eta_p), \quad \sigma_p(\eta_p, p_p) = \sigma_e(\eta_p) - \alpha p_p I. \quad (2.1.4)
\]

The fully dynamic Biot system [13, 28] in \( \Omega_p \) is as follows:

\[
\rho_p \partial_{tt} \eta_p - \nabla \cdot \sigma_p(\eta_p, p_p) = f_p \quad \text{in } \Omega_p \times (0, T], \quad (2.1.5)
\]

\[
\mu_f K^{-1} \mathbf{u}_p + \nabla p_p = 0 \quad \text{in } \Omega_p \times (0, T], \quad (2.1.6)
\]

\[
\partial_t (s_0 p_p + \alpha \nabla \cdot \eta_p) + \nabla \cdot \mathbf{u}_p = q_p \quad \text{in } \Omega_p \times (0, T], \quad (2.1.7)
\]

where \( 0 < \lambda_{\text{min}} \leq \lambda_p \leq \lambda_{\text{max}} \) and \( 0 < \mu_{\text{min}} \leq \mu_p \leq \mu_{\text{max}} \) are the Lamé coefficients, \( \alpha \) is the Biot-Willis constant, \( s_0 \geq 0 \) is a storage coefficient, \( q_p \) is an external source or sink term, and \( K \) is a symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants \( 0 < k_{\text{min}} \leq k_{\text{max}}, \forall \xi \in \mathbb{R}^d \),

\[
k_{\text{min}} \xi^T \xi \leq \xi^T K(x) \xi \leq k_{\text{max}} \xi^T \xi, \quad \forall x \in \Omega_p. \quad (2.1.8)
\]

The interface conditions coupling the two regions are:

\[
\mathbf{u}_f \cdot \mathbf{n}_f + (\partial_t \eta_p + \mathbf{u}_p) \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0, T], \quad (2.1.9)
\]

\[
-(\sigma_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p \quad \text{on } \Gamma_{fp} \times (0, T], \quad (2.1.10)
\]

\[
\sigma_f \mathbf{n}_f + \sigma_p \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0, T], \quad (2.1.11)
\]

\[
-(\sigma_f \mathbf{n}_f) \cdot \tau_{f,j} = \mu_f \alpha_{BJS} \sqrt{K_j^{-1}} (\mathbf{u}_f - \partial_t \eta_p) \cdot \tau_{f,j} \quad \text{on } \Gamma_{fp} \times (0, T]. \quad (2.1.12)
\]

where \( \mathbf{n}_f \) and \( \mathbf{n}_p \) are the outward unit normal vectors to \( \partial \Omega_f \) and \( \partial \Omega_p \), respectively, \( \tau_{f,j}, 1 \leq j \leq d - 1 \), is an orthogonal system of unit tangential vectors on the interface \( \Gamma_{fp} \), \( K_j = (K \tau_{f,j}) \cdot \tau_{f,j} \), and \( \alpha_{BJS} \geq 0 \) is friction coefficient. The conditions (2.1.9)–(2.1.12) represent, respectively, mass conservation, balance of normal stress, balance of momentum, and the
Beavers-Joseph-Saffman (BJS) slip with friction condition [11]. We consider homogeneous boundary conditions for simplicity:

\[ u_f = 0 \text{ on } \Gamma_f \times (0, T], \quad \eta_p = 0 \text{ on } \Gamma_p \times (0, T], \]

\[ p_p = 0, \quad \text{on } \Gamma_p^D \times (0, T], \quad u_p \cdot n_p = 0 \text{ on } \Gamma_p^N \times (0, T], \]

where \( \Gamma_f = \partial \Omega_f \cap \partial \Omega, \Gamma_p = \partial \Omega_p \cap \partial \Omega, \) and \( \Gamma_p = \Gamma_p^D \cup \Gamma_p^N. \) To avoid the issue with restricting the mean value of the pressure, we assume that \( |\Gamma_p^D| > 0. \) We further assume that \( \Gamma_p^D \) is not adjacent to the interface \( \Gamma_{fp}, \) i.e., \( \text{dist} (\Gamma_p^D, \Gamma_{fp}) \geq s \) for some \( s > 0, \) which is used to simplify the space for \( u_p \cdot n_p. \) Finally, the system is supplemented by a set of homogeneous initial conditions:

\[ p_p(x, 0) = 0, \quad \eta_p(x, 0) = 0, \quad \partial_t \eta_p(x, 0) = 0 \text{ in } \Omega_p, \quad u_f(x, 0) = 0 \text{ in } \Omega_f. \]

Let the functional spaces for the solution variables be \( V_f \times W_f \) for \( (u_f, p_f), \) \( V_p \times W_p \) for \( (u_p, p_p), \) and \( X_p \) for \( \eta_p, \) with

\[ V_f = \{ v_f \in H^1(\Omega_f) : v_f = 0 \text{ on } \Gamma_f \}, \quad W_f = L^2(\Omega_f), \]

\[ V_p = \{ v_p \in H(\text{div}; \Omega_p) : v_p \cdot n_p = 0 \text{ on } \Gamma_p^N \}, \quad W_p = L^2(\Omega_p), \]

\[ X_p = \{ \xi_p \in H^1(\Omega_p) : \xi_p = 0 \text{ on } \Gamma_p \}. \]

The weak formulation is obtained by multiplying equations (2.1.2)–(2.1.3) and (2.1.5)–(2.1.7) with suitable test functions, integrating by parts, and utilizing the interface and boundary conditions. We introduce the following bilinear forms related to the Navier-Stokes, Darcy, elasticity, and divergence operators:

\[ a_f(u_f, v_f) := (2\mu_f \mathbf{D}(u_f), \mathbf{D}(v_f))_{\Omega_f}, \quad a_p^d(u_p, v_p) := (\mu_f K^{-1} u_p, v_p)_{\Omega_p}, \]

\[ a_p^e(\eta_p, \xi_p) := (2\mu_p \mathbf{D}(\eta_p), \mathbf{D}(\xi_p))_{\Omega_p} + (\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p)_{\Omega_p}, \quad b_*(v, \omega) = - (\nabla \cdot v, \omega)_{\Omega_*}. \]

We use test function \( v_f \in V_f \) and \( w_f \in W_f \) defined on \( \Omega_f. \) We first write the weak form for the Navier-Stokes equation (2.1.2) and (2.1.3), respectively:

\[ \int_{\Omega_f} f_f \cdot v_f dA = \int_{\Omega_f} \rho_f \partial_t u_f \cdot v_f dA - \int_{\Omega_f} (\nabla \cdot \sigma_f) \cdot v_f dA + \int_{\Omega_f} (\rho_f u_f \cdot \nabla u_f) \cdot v_f dA \]
Similarly, we use test function \( \xi \in X_p \) for equation (2.1.5) to obtain

\[
\int_{\Omega_p} f_p \cdot \xi_p dA = \int_{\Omega_p} \rho_p \partial_t \eta_p \cdot \xi_p dA - \int_{\Omega_p} (\nabla \cdot \sigma_p) \cdot \xi_p dA
\]

\[
= \int_{\Omega_p} \rho_p \partial_t \eta_p \cdot \xi_p dA + \int_{\Omega_p} 2\mu_p D(\eta_p) : D(\xi_p) dA + \int_{\Omega_p} \lambda_p (\nabla \cdot \eta_p)(\nabla \cdot \xi_p) dA
\]

\[
+ \int_{\Omega_p} \alpha_p (\nabla \cdot \xi_p) dA - \int_{\Gamma_{fp}} (\sigma_p n_p) \cdot \xi_p ds
\]

\[
= (\rho_p \partial_t \eta_p, \xi_p)_{\Omega_p} + a^c_p(\eta_p, \xi_p) + \alpha b_p(\xi_p, p_p) - \langle \sigma_p n_p, \xi_p \rangle_{\Gamma_{fp}}. \tag{2.1.15}
\]

For the Darcy equation (2.1.6) in the Biot system, we use test function \( v_p \in V_p \)

\[
0 = \int_{\Omega_p} \mu_f K^{-1} u_p \cdot v_p dA - \int_{\Omega_p} p_p (\nabla \cdot v_p) dA + \int_{\Gamma_{fp}} p_p (n_p \cdot v_p) ds
\]

\[
= a^d_p(u_p, v_p) + b_p(v_p, p_p) + \langle p_p, v_p \cdot n_p \rangle_{\Gamma_{fp}}. \tag{2.1.16}
\]

For the third equation (2.1.7), we use test function \( w_p \in W_p \) to have

\[
\int_{\Omega_p} q_p \cdot w_p dA = \int_{\Omega_p} (s_0 \partial_t p_p) w_p dA + \int_{\Omega_p} \alpha (\nabla \cdot \eta_p) w_p dA + \int_{\Omega_p} (\nabla \cdot u_p) w_p dA
\]

\[
= (s_0 \partial_t p_p, w_p)_{\Omega_p} - \alpha b_p(\partial_t \eta_p, w_p) - b_p(u_p, w_p). \tag{2.1.17}
\]

Integration by parts for the terms involving \( -\nabla \cdot \sigma_f(u_f, p_f) \) in (2.1.2), \( -\nabla \cdot \sigma_p(\eta_p, p_p) \) in (2.1.5), and \( \nabla p_p \) in (2.1.6) results in the interface term

\[
I_{\Gamma_{fp}} = - \langle \sigma_f n_f, v_f \rangle_{\Gamma_{fp}} - \langle \sigma_p n_p, \xi_p \rangle_{\Gamma_{fp}} + \langle p_p, v_p \cdot n_p \rangle_{\Gamma_{fp}}.
\]
In analysis of this thesis, we use the mixed formulation for the Darcy equation, which means the structure velocity-pressure pair \((u_p, p_p)\) are assumed to be in \(H(\text{div}, \Omega_p) \times L^2(\Omega_p)\). This cause a further question: insufficient regularity of \(p_p\) so that \(\langle p_p, v_p \cdot n_p \rangle_{\Gamma_{fp}}\) is well-defined. Following [3], we use the balance of normal stress condition (2.1.10) to introduce the following Lagrange multiplier, which will be used to impose the mass conservation condition (2.1.9):

\[
\lambda = - (\sigma_f n_f) \cdot n_f = p_p \text{ on } \Gamma_{fp}. \quad (2.1.18)
\]

Next, we use the fact that \(\{n_f, \tau_{f,j}\}\), for \(j = 1, \cdots, n-1\) are orthogonal basis on \(\Gamma_{fp}\) and split as follows:

\[
\langle \sigma_f n_f, v_f \rangle_{\Gamma_{fp}} = \langle (\sigma_f n_f) \cdot n_f, v_f \rangle_{\Gamma_{fp}} + \sum_{j=1}^{n-1} \langle (\sigma_f n_f) \tau_{f,j}, v_f \rangle_{\Gamma_{fp}} = \langle (\sigma_f n_f) \cdot n_f, v_f \cdot n_f \rangle_{\Gamma_{fp}} + \sum_{j=1}^{n-1} \langle (\sigma_f n_f) \cdot \tau_{f,j}, v_f \cdot \tau_{f,j} \rangle_{\Gamma_{fp}} \quad (2.1.19)
\]

Combining with (2.1.18) and (2.1.12), we have

\[
\langle \sigma_f n_f, v_f \rangle_{\Gamma_{fp}} + \langle p_p, v_f \cdot n_f \rangle_{\Gamma_{fp}} = \left\langle \mu_f \alpha_{BJS} \sqrt{K_j^{-1}} (u_f - \partial_t \eta_p) \cdot \tau_{f,j}, v_f \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}}. \quad (2.1.20)
\]

Similarly, we use the fact that \(\tau_{f,j} = - \tau_{p,j}\) to obtain

\[
\langle \sigma_p n_p, \xi_p \rangle_{\Gamma_{fp}} + \langle p_p, \xi_p \cdot n_p \rangle_{\Gamma_{fp}} = \left\langle \mu_f \alpha_{BJS} \sqrt{K_j^{-1}} (u_f - \partial_t \eta_p) \cdot \tau_{f,j}, - \xi_p \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}}. \quad (2.1.21)
\]

Therefore, using (2.1.20) and (2.1.21), we summarize

\[
I_{\Gamma_{fp}} = a_{BJS}(u_f, \partial_t \eta_p; v_f, \xi_p) + b_T(v_f, v_p, \xi_p; \lambda),
\]

where we use the notations

\[
a_{BJS}(u_f, \eta_p; v_f, \xi_p) = \sum_{j=1}^{d-1} \left\langle \mu_f \alpha_{BJS} \sqrt{K_j^{-1}} (u_f - \eta_p) \cdot \tau_{f,j}, (v_f - \xi_p) \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}},
\]

\[
b_T(v_f, v_p, \xi_p; \mu) = \langle v_f \cdot n_f + (\xi_p + v_p) \cdot n_p, \mu \rangle_{\Gamma_{fp}}. \quad (2.1.21)
\]

We associate the following seminorm with \(a_{BJS}\):

\[
|v_f - \xi_p|_{BJS}^2 = a_{BJS}(v_f, \xi_p; v_f, \xi_p).
\]
For the continuity of $b_T$, note that $\mathbf{v}_f \in \mathbf{V}_f \subset H^1(\Omega_f)$, $\xi_p \in \mathbf{X}_p \subset H^1(\Omega_p)$, and $\mathbf{v}_p \in \mathbf{V}_p \subset H(\text{div};\Omega_p)$, thus $\mathbf{v}_p \cdot \mathbf{n}|_{\Gamma_{fp}}$ is less regular than $\mathbf{v}_f \cdot \mathbf{n}|_{\Gamma_{fp}}$ and $\xi_p \cdot \mathbf{n}|_{\Gamma_{fp}}$. Therefore we need to take $\lambda \in \Lambda := (\mathbf{V}_p \cdot \mathbf{n}|_{\Gamma_{fp}})'$. Since $\mathbf{v}_p \cdot \mathbf{n} \in H^{-1/2}(\partial \Omega_p)$ and $\text{dist}(\Gamma_p^D, \Gamma_{fp}) \geq s > 0$, it is easy to see that [1] $\mathbf{v}_p \cdot \mathbf{n} \in H^{-1/2}(\Gamma_{fp})$. Therefore we set $\Lambda = H^{1/2}(\Gamma_{fp})$. We will use the notation

$$\| \cdot \|_{\mathbf{v}_p} = \| \cdot \|_{H(\text{div};\Omega_p)} \quad \text{and} \quad \| \cdot \|_\Lambda = \| \cdot \|_{H^{1/2}(\Gamma_{fp})}.$$  

The Lagrange multiplier weak formulation is as follows.

**LMWF1** For $t \in [0, T]$, find $\mathbf{u}_f(t) \in \mathbf{V}_f$, $p_f(t) \in W_f$, $\mathbf{u}_p(t) \in \mathbf{V}_p$, $p_p(t) \in W_p$, $\eta_p(t) \in \mathbf{X}_p$, and $\lambda(t) \in \Lambda$, such that $\mathbf{u}_f(0) = 0$, $p_p(0) = 0$, $\eta_p(0) = 0$, $\partial_t \eta_p(0) = 0$, and, for a.e. $t \in (0, T]$ and for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in W_f$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in W_p$, $\xi_p \in \mathbf{X}_p$, and $\mu \in \Lambda$,

\[
\begin{align*}
(\rho_f \partial_t \mathbf{u}_f, \mathbf{v}_f)_{\Omega_f} &+ a_f(\mathbf{u}_f, \mathbf{v}_f) + (\rho_f \mathbf{u}_f \cdot \nabla \mathbf{u}_f, \mathbf{v}_f)_{\Omega_f} + b_f(\mathbf{v}_f, p_f) \\
+ (\rho_p \partial_t \eta_p, \xi_p)_{\Omega_p} &+ a_p^e(\eta_p, \xi_p) + a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) \\
+ a_{BJS}(\mathbf{u}_f, \partial_t \eta_p; \mathbf{v}_f, \xi_p) + b_T(\mathbf{v}_f, \mathbf{v}_p, \xi_p; \lambda) &=(\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p, \xi_p)_{\Omega_p}, \\
(s_0 \partial_t p_p, w_p)_{\Omega_p} &- a_p(\partial_t \eta_p, w_p) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f) = (q_p, w_p)_{\Omega_p}, \\
b_T(\mathbf{u}_f, \mathbf{u}_p, \partial_t \eta_p; \mu) &= 0.
\end{align*}
\tag{2.1.22}
\tag{2.1.23}
\tag{2.1.24}
\]

We end this section with the following well-known inequalities which will be used repeatedly in this thesis:

- **(Poincaré)** There exist $P_f > 0$ and $P_p > 0$ such that

\[
\forall \mathbf{v}_f \in \mathbf{V}_f, \quad \| \mathbf{v}_f \|_{L^2(\Omega_f)} \leq P_f \| \mathbf{v}_f \|_{H^1(\Omega_f)}; \tag{2.1.25}
\]

\[
\forall \xi_p \in \mathbf{X}_p, \quad \| \xi_p \|_{L^2(\Omega_p)} \leq P_p \| \xi_p \|_{H^1(\Omega_f)}; \tag{2.1.26}
\]

- **(Sobolev)** There exists $S_f > 0$ such that

\[
\forall \mathbf{v}_f \in \mathbf{V}_f, \quad \| \mathbf{v}_f \|_{L^4(\Omega_f)} \leq S_f \| \mathbf{v}_f \|_{H^1(\Omega_f)}; \tag{2.1.27}
\]

- **(Korn)** There exist $K_f > 0$ and $K_p > 0$ such that

\[
\forall \mathbf{v}_f \in \mathbf{V}_f, \quad \| \mathbf{v}_f \|_{H^1(\Omega_f)} \leq K_f \| \mathbf{D}(\mathbf{v}_f) \|_{L^2(\Omega_f)}, \tag{2.1.28}
\]

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∀\(\xi_p \in X_p, \quad |\xi_p|_{H^1(\Omega_p)} \leq K_p \|D(\xi_p)\|_{L^2(\Omega_p)};\) \hfill (2.1.29)

Using the above inequalities and properties of the coefficients \(\mu_f, K, \lambda_p \) and \(\mu_p\), it is easy to see that the bilinear forms \(a_f, a_d^p, a_e^p\) are continuous and coercive in the appropriate norms. In particular, there exist positive constants \(c^f, C^f, c^e, C^e, C^p, C^e\) such that ∀\(v_f, q_f \in V_f, \forall q_p \in V_p, \) and ∀\(\xi_p, \zeta_p \in X_p,\)

\[c^f \|v_f\|_{H^1(\Omega_f)}^2 \leq a_f(v_f, v_f), \quad a_f(v_f, q_f) \leq C^f \|v_f\|_{H^1(\Omega_f)} \|q_f\|_{H^1(\Omega_f)}, \quad (2.1.30)\]

\[c^p \|v_p\|_{L^2(\Omega_p)}^2 \leq a_d^p(v_p, v_p), \quad a_d^p(v_p, q_p) \leq C_p \|v_p\|_{L^2(\Omega_p)} \|q_p\|_{L^2(\Omega_p)}, \quad (2.1.31)\]

\[c^e \|\xi_p\|_{H^1(\Omega_p)}^2 \leq a_e^p(\xi_p, \xi_p), \quad a_e^p(\xi_p, \zeta_p) \leq C^e \|\xi_p\|_{H^1(\Omega_p)} \|\zeta_p\|_{H^1(\Omega_p)}, \quad (2.1.32)\]

In addition, \(a_{BJS}\) is non-negative and, due to the trace inequality, continuous: there exits \(C_{BJS} > 0\) such that ∀\(v_f, q_f \in V_f, \xi_p, \zeta_p \in X_p,\)

\[a_{BJS}(v_f, \xi_p; q_f, \zeta_p) \leq C_{BJS}(\|v_f\|_{H^1(\Omega_f)} + \|\xi_p\|_{H^1(\Omega_p)})(\|q_f\|_{H^1(\Omega_f)} + \|\zeta_p\|_{H^1(\Omega_p)}). \quad (2.1.33)\]

Furthermore, there exist \(C^\Gamma > 0\) and \(C^\alpha > 0\) such that ∀\(u_f, v_f \in V_f, v_p \in V_p, \xi_p \in X_p, \mu \in \Lambda,\)

\[b_\Gamma(v_f, v_p, \xi_p; \mu) \leq C^\Gamma(\|v_f\|_{H^1(\Omega_f)} + \|v_p\|_{V_p} + \|\xi_p\|_{H^1(\Omega_p)})\|\mu\|_{\Lambda}, \quad (2.1.34)\]

\[(\rho_f u_f \cdot \nabla u_f, v_f)_{\Omega_f} \leq C^\alpha \|u_f\|_{H^1(\Omega_f)}^2 \|v_f\|_{H^1(\Omega_f)}, \quad (2.1.35)\]

where (2.1.34) follows from the trace and normal trace inequalities and (2.1.35) follows from the Cauchy-Schwarz inequality and (2.1.27).
2.2 Well posedness of (LMWF1)

The analysis of (LMWF1) is done in several steps. First, we consider a divergence-free Navier-Stokes formulation with the fluid pressure $p_f$ eliminated. We next introduce its Galerkin finite element approximation and show that it can be reduced to a system of ordinary differential equations (ODEs). Using the ODE theory, we establish that the Galerkin problem has a unique solution on a subinterval $[0, T_1]$ of $[0, T]$. After establishing a priori bounds for the Galerkin solution (under a small data condition), we show existence on the entire time interval $[0, T]$, which allows us to pass to the limit from the Galerkin solution to the weak solution of the divergence-free weak formulation. Finally, using an inf-sup condition, we recover and bound the fluid pressure $p_f$.

2.2.1 A divergence-free Lagrange multiplier variational formulation

Let

$$V_f^0 = \{ v_f \in V_f : \nabla \cdot v_f = 0 \}.$$  

Any solution of (LMWF1) is also a solution of the following divergence-free weak formulation.

(LMWF2) For $t \in [0, T]$, find $u_f(t) \in V_f^0$, $u_p(t) \in V_p$, $p_p(t) \in W_p$, $\eta_p(t) \in X_p$, and $\lambda(t) \in \Lambda$, such that $u_f(0) = 0$, $p_p(0) = 0$, $\eta_p(0) = 0$, $\partial t \eta_p(0) = 0$, and, for a.e. $t \in (0, T]$ and for all $v_f \in V_f^0$, $v_p \in V_p$, $w_p \in W_p$, $\xi_p \in X_p$, and $\mu \in \Lambda$,

\[
\begin{align*}
(\rho_f \partial_t u_f, v_f)_{\Omega_f} &+ a_f(u_f, v_f) + (\rho_f u_f \cdot \nabla u_f, v_f)_{\Omega_f} \\
+ (\rho_p \partial_t \eta_p, \xi_p)_{\Omega_p} &+ a_p^e(\eta_p, \xi_p) + \alpha b_p(\xi_p, p_p) + a_p^d(u_p, v_p)_{\Omega_p} + b_p(v_p, p_p) \\
+ a_{BJS}(u_f, \partial_t \eta_p; v_f, \xi_p) &+ b_T(v_f, v_p, \xi_p; \lambda) = (f_f, v_f)_{\Omega_f} + (f_p, \xi_p)_{\Omega_p}, \\
(s_0 \partial_t p_p, w_p)_{\Omega_p} &- \alpha b_p(\partial_t \eta_p, w_p) - b_p(u_p, w_p) = (q_p, w_p)_{\Omega_p}, \\
b_T(u_f, u_p, \partial_t \eta_p; \mu) &= 0.
\end{align*}
\]

2.2.1 A divergence-free Lagrange multiplier variational formulation

We next consider the well-posedness of (LMWF2).
2.2.2 A Galerkin approximation of (LMWF2)

To construct a finite-dimensional approximation of (LMWF2), one can consider either subsets of the infinite dimensional Hilbert bases of the functional spaces or a finite element approximation. Since our analysis requires discrete inf-sup condition for the Darcy pressure and the Lagrange multiplier, cf. (2.2.4), we employ the latter approach. We note that this construction is solely for the purpose of the analysis. Different choice of finite element spaces in the Navier-Stokes region will be made for the numerical method. Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular and quasi-uniform partitions of $\Omega_f$ and $\Omega_p$, respectively, both consisting of affine elements with maximal element diameter $h$. The two partitions may be non-matching at the interface $\Gamma_{fp}$. In $\Omega_f$, let $V_{f,h} \subset V_f$ be a conforming finite element space and let $V_{f,h}^0 = \{v_{f,h} \in V_{f,h} : \nabla \cdot v_{f,h} = 0\} \subset V_f^0$ be the divergence-free subspace. In $\Omega_p$, let $V_{p,h} \times W_{p,h} \subset V_p \times W_p$ be any inf-sup stable mixed finite element elements, such as the Raviart-Thomas or Brezzi-Douglas-Marini spaces [16], with $V_{p,h}$ containing polynomials of degree $k \geq 1$. In turn, let $X_{p,h} \subset X_p$ be a conforming finite element space. Finally, we take a conforming finite element space $\Lambda_h \subset \Lambda$ with continuous piecewise polynomials of degree $k$ defined on the trace of $\mathcal{T}_h^p$ on $\Gamma_{fp}$. Let the spaces $V_{f,h}^0$, $V_{p,h}$, $W_{p,h}$, $X_{p,h}$, and $\Lambda_h$ have bases

\[
\{\phi_{u,i}^{N_{uf}}\}_{i=1}^{N_{uf}}, \{\phi_{p,i}^{N_{up}}\}_{i=1}^{N_{up}}, \{\phi_{p,i}^{N_{np}}\}_{i=1}^{N_{np}}, \{\phi_{\lambda,i}^{N_\lambda}\}_{i=1}^{N_\lambda}, \text{ and } \{\phi_{\alpha,b}\}.
\]

The following inf-sup condition has been established in [43, Lemma 4.7]. There exists a constant $\beta_p > 0$ independent of $h$ such that

\[
\inf_{(0,0) \neq (w_{p,h},\mu_h) \in W_{p,h} \times \Lambda_h} \sup_{0 \neq v_{p,h} \in V_{p,h}} \frac{b_p(v_{p,h},w_{p,h}) + \langle v_{p,h} \cdot n, \mu_h \rangle_{\Gamma_{fp}}}{\|v_{p,h}\|_V \|w_{p,h}\|_{L^2(\Omega_p)} + \|\mu_h\|_{\Lambda_h}} \geq \beta_p. \tag{2.2.4}
\]

We consider the following finite-dimensional Galerkin problem.

(GP) For $t \in [0,T]$, find $u_{f,h}(t) \in V_{f,h}^0$, $u_{p,h}(t) \in V_{p,h}$, $p_{p,h}(t) \in W_{p,h}$, $\eta_{p,h}(t) \in X_{p,h}$, and $\lambda_h(t) \in \Lambda_h$, such that $u_{f,h}(0) = 0$, $p_{p,h}(0) = 0$, $\eta_{p,h}(0) = 0$, $\partial_t \eta_{p,h}(0) = 0$, and, for a.e. $t \in (0,T]$ and for all $v_{f,h} \in V_{f,h}^0$, $v_{p,h} \in V_{p,h}$, $w_{p,h} \in W_{p,h}$, $\xi_{p,h} \in X_{p,h}$, and $\mu_h \in \Lambda_h$,

\[
\begin{align*}
&\left(\rho_f \partial_t u_{f,h}, v_{f,h}\right)_{\Omega_f} + a_f(u_{f,h}, v_{f,h}) + \left(\mu_h, v_{f,h}\right)_{\Omega_f} \\
&+ \left(\rho_p \partial_t \eta_{p,h}, \xi_{p,h}\right)_{\Omega_p} + a_p(\eta_{p,h}, \xi_{p,h}) + \alpha b_p(\xi_{p,h}, p_{p,h}) + a_p^d(u_{p,h}, v_{p,h}) + b_p(v_{p,h}, p_{p,h}) \\
&+ a_{BJS}(u_{f,h}, \partial_t \eta_{p,h}, v_{f,h}, \xi_{p,h}) + b_T(v_{f,h}, v_{p,h}, \xi_{p,h}; \lambda_h) = (f_f, v_{f,h})_{\Omega_f} + (f_p, \xi_{p,h})_{\Omega_p}, \tag{2.2.5}
\end{align*}
\]
We write (2.2.5)–(2.2.7) in a matrix form. Denote, for $1 \leq i, j, l \leq k$,

\[
(M_f)_{ij} = (\phi_{u_{f,j}}, \phi_{u_{f,i}})_{\Omega_f}, \quad (M_s)_{ij} = (\phi_{\eta_{p,j}}, \phi_{\eta_{p,i}})_{\Omega_p}, \quad (M_p)_{ij} = (\phi_{p_{p,j}}, \phi_{p_{p,i}})_{\Omega_p},
\]

\[
(A_f)_{ij} = a_f(\phi_{u_{f,j}}, \phi_{u_{f,i}}), \quad (A_e)_{ij} = a_e(\phi_{p_{e,j}}, \phi_{p_{e,i}}), \quad (A_p)_{ij} = a_p(\phi_{u_{p,j}}, \phi_{u_{p,i}}),
\]

\[
N_{ijl} = (\rho_f \phi_{u_{f,l}} \cdot \nabla \phi_{u_{f,j}}, \phi_{u_{f,i}})_{\Omega_f}, \quad (B_p)_{ij} = b_p(\phi_{u_{p,j}}, \phi_{p_{p,i}}), \quad (B_{e\Gamma})_{ij} = b_{e\Gamma}(0, \phi_{u_{p,j}}, 0; \phi_{e\Gamma}),
\]

\[
(A_{BJS}^{Bf})_{ij} = a_{BJS}(\phi_{u_{f,j}}, 0; \phi_{u_{f,i}}, 0), \quad (A_{BJS}^{Be})_{ij} = a_{BJS}(\phi_{u_{f,j}}, 0; 0, \phi_{u_{f,i}}),
\]

In order to obtain an ODE system, we introduce as an explicit variable the structure velocity $\mathbf{u}_{s,h} = \partial_t \Omega_{p,h}$. Taking $\mathbf{u}_{f,h}(\mathbf{x}, t) = \sum_{j=1}^{k} u_{f,j}(t) \phi_{u_{f,j}}, \quad \mathbf{u}_{p,h}(\mathbf{x}, t) = \sum_{j=1}^{k} \eta_{p,j}(t) \phi_{p_{p,j}}, \quad \mathbf{u}_{s,h}(\mathbf{x}, t) = \sum_{j=1}^{k} u_{s,j}(t) \phi_{u_{p,j}}$, and $\lambda_h(\mathbf{x}, t) = \sum_{j=1}^{k} \lambda_j(t) \phi_{\lambda,j}$ in (2.2.5)–(2.2.7) and denoting the time-dependent coefficient vectors as $\mathbf{u}_f, \mathbf{u}_e, \mathbf{u}_p, \mathbf{p}_p, \mathbf{p}_e$, and $\mathbf{\lambda}$, results in the system of differential-algebraic equations

\[
\rho_f M_f \partial_t \mathbf{u}_f + A_f \mathbf{u}_f + (N_f) \mathbf{u}_f + A_{BJS}^{Bf} \mathbf{u}_s + (A_{BJS}^{Be})^T \mathbf{u}_s + B_{f,\Gamma}^T \mathbf{\lambda} = \mathcal{F}_u,
\]

\[
\partial_t \mathbf{\eta}_p - \mathbf{u}_s = 0,
\]

\[
\rho_p M_s \partial_t \mathbf{u}_s + A_e \mathbf{\eta}_p + \alpha B_{e\Gamma}^T \mathbf{p}_p + A_{BJS}^{Be} \mathbf{u}_f + A_{BJS}^{Be} \mathbf{u}_s + B_{e,\Gamma}^T \mathbf{\lambda} = \mathcal{F}_{\eta},
\]

\[
A_p \mathbf{u}_p + B_{pp}^T \mathbf{p}_p + B_{p,\Gamma}^T \mathbf{\lambda} = 0,
\]

\[
s_0 M_p \partial_t \mathbf{p}_p - \alpha B_{e\Gamma} \mathbf{u}_s - B_{pp} \mathbf{u}_p = \mathcal{F}_{p},
\]

\[
B_{f,\Gamma} \mathbf{u}_f + B_{p,\Gamma} \mathbf{u}_p + B_{e,\Gamma} \mathbf{u}_s = 0,
\]
where \((\mathcal{F}_f)_i = (f, \phi_{u,f,i})\alpha_f\), \((\mathcal{F}_p)_i = (f, \phi_{u_p,i})\alpha_p\), and \((\mathcal{F}_p)_i = (q, \phi_{q_p,i})\alpha_p\), \(1 \leq i \leq k\).

Due to (2.1.8), the matrix \(A_p\) is nonsingular, so \(\bar{u}_p\) can be eliminated from (2.2.11):

\[
\bar{u}_p = -A_p^{-1}B_{pp}^T\bar{p}_p - A_p^{-1}B_{p,\Gamma}^T\bar{\lambda}. \tag{2.2.14}
\]

Substituting (2.2.14) into (2.2.12) results in

\[
s_0M_p\partial_t\bar{p}_p - \alpha B_{ep}\bar{u}_s + B_{pp}A_p^{-1}B_{pp}^T\bar{p}_p + B_{pp}A_p^{-1}B_{p,\Gamma}^T\bar{\lambda} = \mathcal{F}_p, \tag{2.2.15}
\]

Also, substituting (2.2.14) into (2.2.13) and differentiating in time gives

\[
-B_{f,\Gamma}\partial_t\bar{u}_f - B_{e,\Gamma}\partial_t\bar{u}_s + B_{p,\Gamma}A_p^{-1}B_{pp}^T\partial_t\bar{p}_p + B_{p,\Gamma}A_p^{-1}B_{p,\Gamma}^T\partial_t\bar{\lambda} = 0. \tag{2.2.16}
\]

The system (2.2.8)–(2.2.10), (2.2.15), (2.2.16) can be written in a matrix-vector form as

\[
E\partial_t X(t) + HX(t) + \mathcal{N}(X(t)) = R(t), \tag{2.2.17}
\]

where \(\mathcal{N}(X) = (N\bar{u}_f)\bar{u}_f\),

\[
X(t) = \begin{pmatrix}
\bar{u}_f(t) \\
\bar{u}_p(t) \\
\bar{u}_s(t) \\
\bar{p}_p(t) \\
\bar{\lambda}(t)
\end{pmatrix}, \quad R(t) = \begin{pmatrix}
\mathcal{F}_u_f \\
0 \\
\mathcal{F}_p \\
0
\end{pmatrix},
\]

\[
E = \begin{pmatrix}
\rho_f M_f & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & \rho_p M_s & 0 & 0 \\
0 & 0 & 0 & s_0 M_p & 0 \\
-B_{f,\Gamma} & -B_{e,\Gamma} & B_{p,\Gamma}A_p^{-1}B_{pp}^T & B_{p,\Gamma}A_p^{-1}B_{p,\Gamma}^T & 0
\end{pmatrix}_{5 \times 5},
\]
Due to (2.2.4), \( B_{p,\Gamma}^T \overline{\mu}_h \neq 0 \) for \( \overline{\mu}_h \neq 0 \), implying that \( B_{p,\Gamma} A_p^{-1} B_{p,\Gamma}^T \) is positive definite. Therefore the matrix \( E \) is lower block-triangular with invertible diagonal blocks, so \( E \) is invertible. Then the matrix equation (2.2.17) can be rewritten as

\[
\partial_t X(t) = E^{-1}(R(t) - HX(t) - N(X(t))) := g(X(t)). \tag{2.2.18}
\]

**Lemma 2.2.1.** Let \( f \in C^0(0, T; L^2(\Omega_f)), f_p \in C^0(0, T; L^2(\Omega_p)), \) and \( q_p \in C^0(0, T; L^2(\Omega_p)) \). Then there exists \( T_1 \in (0, T] \) such that there exists a unique solution of (2.2.5)–(2.2.7) in \( [0, T_1] \) satisfying \( u_{f,h}(0) = 0, u_{p,h}(0) = 0, p_{p,h}(0) = 0, \eta_{p,h}(0) = 0, \partial_t \eta_{p,h}(0) = 0, \) and \( \lambda_h(0) = 0. \) Moreover, \( u_{f,h}, u_{p,h}, p_{p,h}, \partial_t \eta_{p,h}, \) and \( \lambda_h \) belong to \( C^1(0, T_1) \).

**Proof.** Due to the data assumption and the continuity bounds (2.1.30)–(2.1.35), the function \( g(X) \) on the right hand side of (2.2.18) is continuous in time and locally Lipschitz in \( X \). Therefore it follows from the ODE theory [37] that there exists a unique maximal solution \( X \) of (2.2.18) in the interval \( [0, T_1] \) for some \( 0 < T_1 \leq T \) with \( X \in C^1(0, T_1) \) and satisfying \( X(0) = 0. \) Next, we integrate (2.2.16) in time from 0 to any \( t \in (0, T_1] \) and use the zero initial conditions, obtaining

\[
-B_{f,\Gamma} \overline{u}_f - B_{e,\Gamma} \overline{u}_s + B_{p,\Gamma} A_p^{-1} B_{pp}^T \overline{p}_p + B_{p,\Gamma} A_p^{-1} B_{p,\Gamma}^T \lambda = 0.
\]

We recover \( \overline{u}_p(t) \) from (2.2.14), concluding that (2.2.11)–(2.2.13) hold. This results in a solution of (2.2.8)–(2.2.13). Finally, setting \( \eta_{p,h}(t) = \int_0^t \partial_t \eta_{s,h} \) gives a unique solution of (2.2.5)–(2.2.7) in \( [0, T_1] \) satisfying \( u_{f,h}(0) = 0, u_{p,h}(0) = 0, p_{p,h}(0) = 0, \eta_{p,h}(0) = 0, \partial_t \eta_{p,h}(0) = 0, \) and \( \lambda_h(0) = 0, \) where \( u_{f,h}, u_{p,h}, p_{p,h}, \partial_t \eta_{p,h}, \) and \( \lambda_h \) belong to \( C^1(0, T_1) \).
Remark 2.2.1. Due to the presence of the non-linear term $N(X)$ in $g(X)$, only local solvability for (GP) is established. In the following section we obtain a priori bounds for the solution under small data assumption, which allows us to establish global existence and uniqueness in $[0,T]$.

2.2.4 Existence, uniqueness, and stability of the solution of (GP)

For notational convenience, we define the following quantities that depend on the data:

$$C_1(t) = \frac{P_f^2 K_f^2}{\mu_f} \|f_f\|_{L^2(\Omega_f)}^2 + \frac{1}{\rho_p} \|f_p\|_{L^2(\Omega_p)}^2 + \frac{2}{k_{min} \beta_p^2} \|q_p\|_{L^2(\Omega_p)},$$

(2.2.19)

$$C_2(t) = \frac{P_f^2 K_f^2}{\mu_f} \|\partial_t f_f\|_{L^2(\Omega_f)}^2 + \|\partial_t f_p\|_{L^2(\Omega_p)}^2 + \frac{2}{k_{min} \beta_p^2} \|\partial_t q_p\|_{L^2(\Omega_p)},$$

(2.2.20)

$$C_3 = \frac{1}{\rho_f} \|f_f(0)\|_{L^2(\Omega_f)}^2 + \frac{1}{\rho_p} \|f_p(0)\|_{L^2(\Omega_p)}^2 + \frac{1}{\rho_0} \|g_p(0)\|_{L^2(\Omega_p)}^2.$$  

(2.2.21)

Theorem 2.2.1. Suppose that $f_f \in H^1(0,T;L^2(\Omega_f))$, $f_p \in H^1(0,T;L^2(\Omega_p))$, $q_p \in H^1(0,T;L^2(\Omega_p))$. In addition, assume that the following small data condition holds:

$$\exp(T) \left( \|C_1\|_{L^1(0,T)} + \frac{1}{2} \|C_2\|_{L^1(0,T)} + \frac{1}{2} \|C_3\|_{L^1(0,T)} \right) + \frac{1}{2} \|C_1\|_{L^\infty(0,T)} < \frac{\mu_f^2}{4\rho_f^2 S_4^2 K_f^6}.$$  

(2.2.22)

Then, the Galerkin problem (GP) has a unique solution in the interval $[0,T]$ satisfying $u_{f,h}(0) = 0$, $u_{p,h}(0) = 0$, $p_{p,h}(0) = 0$, $\eta_{p,h}(0) = 0$, $\partial_t \eta_{p,h}(0) = 0$, and $\lambda_h(0) = 0$. The solution satisfies

$$\rho_f \|u_{f,h}\|_{L^\infty(0,T;L^2(\Omega_f))}^2 + 2\mu_f \|D(u_{f,h})\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|u_{f,h} - \partial_t \eta_{p,h}\|_{L^2(0,T;\mathbb{A}BJS)}^2 + \rho_p \|\partial_t \eta_{p,h}\|_{L^\infty(0,T;L^2(\Omega_p))}^2 + 2\mu_p^{1/2} \|D(\eta_{p,h})\|_{L^2(0,T;L^2(\Omega_p))}^2 + \|\lambda_p^{1/2} \nabla \cdot \eta_{p,h}\|_{L^\infty(0,T;L^2(\Omega_p))}^2 + s_0 \|p_{p,h}\|_{L^\infty(0,T;W_p)}^2 + \|K^{-1/2} u_{p,h}\|_{L^2(0,T;L^2(\Omega_p))}^2 + \frac{1}{2} \|P_{p,h}\|_{L^2(0,T;W_p)} + \frac{k_{min} \beta_p^2}{2} \|\lambda_h\|_{L^2(0,T;\Lambda)}^2 \leq \exp(T) \|C_1\|_{L^1(0,T)}.$$  

(2.2.23)

Furthermore, it holds that

$$\rho_f \|\partial_t u_{f,h}\|_{L^\infty(0,T;L^2(\Omega_f))}^2 + \mu_f \|D(\partial_t u_{f,h})\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\partial_t u_{f,h} - \partial_t \eta_{p,h}\|_{L^2(0,T;\mathbb{A}BJS)}^2 + \rho_p \|\partial_t \eta_{p,h}\|_{L^\infty(0,T;L^2(\Omega_p))}^2 + 2\mu_p^{1/2} \|D(\partial_t \eta_{p,h})\|_{L^2(0,T;L^2(\Omega_p))}^2.$$  

(2.2.24)
\[ + \| \lambda_p^{1/2} \nabla \cdot \partial_t \eta_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))}^2 \] 
\[ + s_0 \| \partial_t p_{p,h} \|_{L^\infty(0,T;W^1_p)}^2 + \| K^{-1/2} \partial_t u_{p,h} \|_{L^2(0,T;L^2(\Omega_p))} \]
\[ + \frac{k_{\min} \beta^2}{2} \| \partial_t p_{p,h} \|_{L^2(0,T;W^1_p)}^2 + k_{\min} \beta^2 \| \partial_t \lambda \|_{L^2(0,T;\Lambda)}^2 \]
\[ \leq \exp(T)(\| C_2 \|_{L^1(0,T)} + C_3), \] (2.24)

\[ \| \nabla \cdot u_{p,h} \|_{L^2(\Omega_p)} \leq s_0 \| \partial_t p_{p,h} \|_{L^2(\Omega_p)} + \alpha \| \nabla \cdot \partial_t \eta_{p,h} \|_{L^2(\Omega_p)} + \| q_p \|_{L^2(\Omega_p)} \] for a.e. \( t \in (0,T) \), (2.25)

and
\[ \| D(u_{f,h}) \|_{L^\infty(0,T;L^2(\Omega_f))} < \frac{\mu_f}{2 \rho_f S_f^2 K_f^3}. \] (2.26)

**Proof.** Existence and uniqueness of a solution of (GP) in the subinterval \([0, T_1]\) was established in Lemma 2.2.1. We next verify the bounds (2.23)–(2.26) in the subinterval \([0, T],\) which implies global existence and uniqueness in \([0, T]\).

**Proof of (2.23).** Taking \((v_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \lambda_h) = (u_{f,h}, u_{p,h}, p_{p,h}, \partial_t \eta_{p,h}, \lambda_h)\) in (2.25)–(2.27) and summing gives the following energy equation:

\[ (\rho_f \partial_t u_{f,h}, u_{f,h})_{\Omega_f} + a_f(u_{f,h}, u_{f,h}) + (\rho_f u_{f,h} \cdot \nabla u_{f,h}, u_{f,h})_{\Omega_f} + (\rho_f \partial_t \eta_{p,h}, \partial_t \eta_{p,h})_{\Omega_p} \]
\[ + a_p^e(\eta_{p,h}, \partial_t \eta_{p,h}) + |u_{f,h} - \partial_t \eta_{p,h}|_{BJS}^2 + a_p^d(u_{p,h}, u_{p,h}) + (s_0 \partial_t p_{p,h}, p_{p,h})_{\Omega_p} \]
\[ = (f_f, u_{f,h})_{\Omega_f} + (f_p, \partial_t \eta_{p,h})_{\Omega_p} + (q_p, p_{p,h})_{\Omega_p}. \] (2.27)

For the non-linear term we use the Cauchy–Schwarz, Sobolev (2.1.27), and Korn’s (2.1.28) inequalities:

\[ (\rho_f u_{f,h} \cdot \nabla u_{f,h}, u_{f,h})_{\Omega_f} \leq \rho_f \| \nabla u_{f,h} \|_{L^2(\Omega_f)}^2 \| u_{f,h} \|_{L^2(\Omega_f)}^2 \leq \rho_f S_f^2 K_f^3 \| D(u_{f,h}) \|_{L^2(\Omega_f)}, \] (2.28)

Since \( u_{f,h}(0) = 0 \) and \( D(u_{f,h}) \) is continuous in time, there exists a time \( \tilde{T}, 0 < \tilde{T} \leq T_1 \), such that for any \( 0 \leq t \leq \tilde{T}, \) it holds that

\[ \| D(u_{f,h}) \|_{L^2(\Omega_f)} < \frac{\mu_f}{2 \rho_f S_f^2 K_f^3}. \] (2.29)
We will show at the end of the proof that (2.2.29) holds in \([0, T_1]\), under the small data condition (2.2.22). The terms on the right hand side in (2.2.27) are bounded using the Poincaré (2.1.25), Korn’s (2.1.28), the Cauchy-Schwarz, and Young’s inequalities:

\[
(f_f, u_{f,h})_{\Omega_f} + (f_p, \partial \eta_{p,h})_{\Omega_p} + (q_p, p_{p,h})_{\Omega_p} \leq \frac{P_f^2 K_f^2}{2 \mu_f} \|f_f\|_{L^2(\Omega_f)}^2 + \frac{\mu_f}{2} \|D(u_{f,h})\|_{L^2(\Omega_f)}^2 \tag{2.2.30}
+ \frac{1}{2\rho_p} \|f_p\|_{L^2(\Omega_p)}^2 + \frac{\rho_p}{2} \|\partial \eta_{p,h}\|_{L^2(\Omega_p)}^2 + \frac{1}{2\epsilon} \|q_p\|_{L^2(\Omega_p)}^2 + \frac{\epsilon}{2} \|p_{p,h}\|_{L^2(\Omega_p)}^2,
\]

where \(\epsilon > 0\) will be chosen later. Combining (2.2.27)–(2.2.30) gives

\[
\frac{1}{2} \frac{d}{dt} \left( \rho_f \|u_{f,h}\|_{L^2(\Omega_f)}^2 + \rho_p \|\partial \eta_{p,h}\|_{L^2(\Omega_p)}^2 + a_p^c(\eta_{p,h}, \eta_{p,h}) + s_0 \|p_{p,h}\|_{L^2(\Omega_p)}^2 \right)
+ 2\mu_f \|D(u_{f,h})\|_{L^2(\Omega_f)}^2 + |u_{f,h} - \partial \eta_{p,h}|_{BJS}^2 + \|K^{-1/2}u_{p,h}\|_{L^2(\Omega_p)}^2
\leq \mu_f \|D(u_{f,h})\|_{L^2(\Omega_f)}^2 + \frac{P_f^2 K_f^2}{2 \mu_f} \|f_f\|_{L^2(\Omega_f)}^2 + \frac{1}{2\rho_p} \|f_p\|_{L^2(\Omega_p)}^2 + \frac{\rho_p}{2} \|\partial \eta_{p,h}\|_{L^2(\Omega_p)}^2
+ \frac{1}{2\epsilon} \|q_p\|_{L^2(\Omega_p)}^2 + \frac{\epsilon}{2} \|p_{p,h}\|_{L^2(\Omega_p)}^2
\]

Integrating in time from 0 to \(t \in (0, T_1]\), we obtain

\[
\rho_f \|u_{f,h}\|_{L^2(\Omega_f)}^2 + \rho_p \|\partial \eta_{p,h}\|_{L^2(\Omega_p)}^2 + a_p^c(\eta_{p,h}, \eta_{p,h}) + s_0 \|p_{p,h}\|_{L^2(\Omega_p)}^2
+ \int_0^t \left(2\mu_f \|D(u_{f,h})\|_{L^2(\Omega_f)}^2 + 2|u_{f,h} - \partial \eta_{p,h}|_{BJS}^2 + 2\|K^{-1/2}u_{p,h}\|_{L^2(\Omega_p)}^2 \right) \tag{2.2.31}
\leq \int_0^t \left( \frac{P_f^2 K_f^2}{\mu_f} \|f_f\|_{L^2(\Omega_f)}^2 + \frac{1}{\rho_p} \|f_p\|_{L^2(\Omega_p)}^2 + \frac{1}{\epsilon} \|q_p\|_{L^2(\Omega_p)}^2 + \epsilon \|p_{p,h}\|_{L^2(\Omega_p)}^2 \right) + \int_0^t \rho_p \|\partial \eta_{p,h}\|_{L^2(\Omega_p)}^2.
\]

Next, we use the inf-sup condition (2.2.4) and the equation obtained from (2.2.5) with test function \(v_{p,h}\), to control \(\|p_{p,h}\|_{L^2(\Omega_p)}\) and \(\|\lambda_h\|_\Lambda\):

\[
\beta_p(\|p_{p,h}\|_{L^2(\Omega_p)} + \|\lambda_h\|_\Lambda) \leq \sup_{0 \neq v_{p,h} \in V_{p,h}} \frac{b_p(v_{p,h}, p_{p,h}) + \langle v_{p,h}, n_p, \lambda_h \rangle_{\Gamma_{fp}}}{\|v_{p,h}\|_{V_p}} \tag{2.2.32}
= \sup_{0 \neq v_{p,h} \in V_{p,h}} \frac{-a_p^c(u_{p,h}, v_{p,h})}{\|v_{p,h}\|_{V_p}} \leq k_{min}^{-1/2} \|K^{-1/2}u_{p,h}\|_{L^2(\Omega_p)}.
\]

Combining (2.2.31) and (2.2.32), taking \(\epsilon = \frac{k_{min}^2 \beta_p^2}{2}\), and using Gronwall’s inequality (1.2.4) results in

\[
\rho_f \|u_{f,h}\|_{L^2(\Omega_f)}^2 + \rho_p \|\partial \eta_{p,h}\|_{L^2(\Omega_p)}^2 + a_p^c(\eta_{p,h}, \eta_{p,h}) + s_0 \|p_{p,h}\|_{L^2(\Omega_p)}^2
+ \int_0^t \left(2\mu_f \|D(u_{f,h})\|_{L^2(\Omega_f)}^2 + 2|u_{f,h} - \partial \eta_{p,h}|_{BJS}^2 + \|K^{-1/2}u_{p,h}\|_{L^2(\Omega_p)}^2 \right) \tag{2.2.33}
\]
\[
\begin{align*}
&\quad + \int_0^t \left( \frac{k_{\text{min}}^2}{2} \|p_{p,h}\|_{L^2(\Omega)}^2 + k_{\text{min}}^2 \|\lambda_h\|_\lambda^2 \right) \\
&\quad \leq \exp(t) \int_0^t \left( \frac{P^2 K_f^2}{\mu_f} \|f\|_{L^2(\Omega)}^2 + \frac{1}{\rho_p} \|f_p\|_{L^2(\Omega)}^2 + \frac{2}{k_{\text{min}}^2} \|q_p\|_{L^2(\Omega)}^2 \right),
\end{align*}
\]
which implies (2.2.23) in \((0,T_1]\).

**Proof of (2.2.24) and (2.2.25).** Differentiating in time (2.2.5), (2.2.6), and (2.2.7), and taking \(v_{f,h} = \partial_t u_{f,h}, v_{p,h} = \partial_t u_{p,h}, w_{p,h} = \partial_t p_{p,h}, \xi_{p,h} = \partial_t \eta_{p,h}, \) and \(\mu_h = \partial_t \lambda_h,\) we obtain

\[
\begin{align*}
(\rho_f & \partial_t u_{f,h}, \partial_t u_{f,h})_{\Omega_f} + a_f(\partial_t u_{f,h}, \partial_t u_{f,h}) + (\rho_f \partial_t u_{f,h} \cdot \nabla u_{f,h}, \partial_t u_{f,h})_{\Omega_f} \\
&\quad + (\rho_f u_{f,h} \cdot \nabla \partial_t u_{f,h}, \partial_t u_{f,h})_{\Omega_f} + (\rho_p \partial_t p_{p,h}, \partial_t \eta_{p,h})_{\Omega_p} + a_p^f(\partial_t \eta_{p,h}, \partial_t \eta_{p,h}) \\
&\quad + |\partial_t u_{f,h} \cdot \eta_{p,h}|_{BJS}^2 + a_p^d(\partial_t u_{p,h}, \partial_t u_{p,h}) + (s_0 \partial_t p_{p,h}, \partial_t p_{p,h})_{\Omega_p} \\
&\quad = (\partial_t f_f, \partial_t u_{f,h})_{\Omega_f} + (\partial_t f_p, \partial_t \eta_{p,h})_{\Omega_p} + (\partial_t q_p, \partial_t p_{p,h})_{\Omega_p}. \quad (2.2.34)
\end{align*}
\]
For the two non-linear terms, similarly to (2.2.28), using the Cauchy-Schwarz, Sobolev (2.1.27), and Korn’s (2.1.28) inequalities, we have

\[
\begin{align*}
(\rho_f & \partial_t u_{f,h} \cdot \nabla u_{f,h}, \partial_t u_{f,h})_{\Omega_f} + (\rho_f u_{f,h} \cdot \nabla \partial_t u_{f,h}, \partial_t u_{f,h})_{\Omega_f} \\
&\quad \leq 2 \rho_f S_f^2 K_f^3 \|D(u_{f,h})\|_{L^2(\Omega_f)} \|D(\partial_t u_{f,h})\|_{L^2(\Omega_f)}^2 \leq \mu_f \|D(\partial_t u_{f,h})\|_{L^2(\Omega_f)}^2, \quad (2.2.35)
\end{align*}
\]
using (2.2.29) in the last inequality. Similarly to (2.2.30), we bound the terms on the right hand side in (2.2.34) using the Poincaré (2.1.25), Korn’s (2.1.28), the Cauchy-Schwarz, and Young’s inequalities:

\[
\begin{align*}
(\partial_t f_f, \partial_t u_{f,h})_{\Omega_f} + (\partial_t f_p, \partial_t \eta_{p,h})_{\Omega_p} + (\partial_t q_p, \partial_t p_{p,h})_{\Omega_p} \\
&\quad \leq \frac{P^2 K_f^2}{2 \mu_f} \|\partial_t f_f\|_{L^2(\Omega_f)}^2 + \frac{\mu_f}{2} \|D(\partial_t u_{f,h})\|_{L^2(\Omega_f)}^2 + \frac{1}{2 \rho_p} \|\partial_t f_p\|_{L^2(\Omega_p)}^2 + \frac{\rho_p}{2} \|\partial_t \eta_{p,h}\|_{L^2(\Omega_p)}^2 \\
&\quad + \frac{1}{2 \epsilon} \|\partial_t q_p\|_{L^2(\Omega_p)}^2 + \frac{\epsilon}{2} \|\partial_t p_{p,h}\|_{L^2(\Omega_p)}^2, \quad (2.2.36)
\end{align*}
\]
with $\epsilon > 0$ to be chosen later. Next, the inf-sup condition (2.2.4) and the time-differentiated equation (2.2.5) with test function $v_{p,h}$ imply

$$\beta_{p}(\|\partial_t p_{p,h}\|_{L^2(\Omega_p)} + \|\partial_t \lambda_{h}\|_{A}) \leq \sup_{0 \neq v_{p,h} \in V_{p,h}} \frac{b_p(v_{p,h}, \partial_t p_{p,h}) + \langle v_{p,h} \cdot n_p, \partial_t \lambda_{h} \rangle_{f_p}}{\|v_{p,h}\|_{V_p}}$$

$$= \sup_{0 \neq v_{p,h} \in V_{p,h}} \frac{-a_p^d(\partial_t u_{p,h}, v_{p,h})}{\|v_{p,h}\|_{V_p}} \leq k_{min}^{-1/2}\|K^{-1/2} \partial_t u_{p,h}\|_{L^2(\Omega_p)}.$$  (2.2.37)

We combine (2.2.34)–(2.2.37), take $\epsilon = \frac{k_{min} \beta_{p}^2}{2}$, integrate in time from 0 to $t \in (0, T_1]$, and use Gronwall’s inequality (1.2.4) for the term $\int_{0}^{t} \rho_p \|\partial_t \eta_{p,h}\|_{L^2(\Omega_p)}^2$, see the arguments leading to (2.2.33) for details. We obtain

$$\rho_f \|\partial_t u_{f,h}\|_{L^2(\Omega_f)}^2 + \rho_p \|\partial_t \eta_{p,h}\|_{L^2(\Omega_p)}^2 + a_p^d(\partial_t \eta_{p,h}, \partial_t \eta_{p,h}) + s_0 \|\partial_t p_{p,h}\|_{L^2(\Omega_p)}^2$$

$$+ \int_{0}^{t} \left( \mu_f \|D(\partial_t u_{f,h})\|_{L^2(\Omega_f)}^2 + 2\|\partial_t u_{f,h} - \partial_t \eta_{p,h}\|_{BJS}^2 + \|K^{-1/2} \partial_t u_{p,h}\|_{L^2(\Omega_p)}^2 \right)$$

$$+ \int_{0}^{t} \left( \frac{k_{min} \beta_{p}^2}{2}\|\partial_t p_{p,h}\|_{L^2(\Omega_p)}^2 + k_{min} \beta_{p}^2 \|\partial_t \lambda_{h}\|_{A}^2 \right)$$

$$\leq \exp(t) \int_{0}^{t} \left( \frac{P^2 K^2}{\mu_f} \|\partial_t f_{p}\|_{L^2(\Omega_f)}^2 + \frac{1}{\rho_p} \|\partial_t f_{p}\|_{L^2(\Omega_p)}^2 + \frac{2}{k_{min} \beta_{p}^2} \|\partial_t q_{p}\|_{L^2(\Omega_p)}^2 \right)$$

$$+ \exp(t) \left( \rho_f \|\partial_t u_{f,h}(0)\|_{L^2(\Omega_f)}^2 + \rho_p \|\partial_t \eta_{p,h}(0)\|_{L^2(\Omega_p)}^2 + s_0 \|\partial_t p_{p,h}(0)\|_{L^2(\Omega_p)}^2 \right). \quad (2.2.38)$$

To complete the estimate we need to control the initial terms. Recall from Lemma 2.2.1 that $u_{f,h}$, $u_{p,h}$, $p_{p,h}$, $\partial_t \eta_{p,h}$, and $\lambda_{h}$ belong to $C^1(0, T_1)$. Therefore (2.2.5)–(2.2.7) hold at $t = 0$. Taking $v_{f,h} = 0$, $v_{p,h} = 0$, and $\xi_{p,h} = \partial_t \eta_{p,h}(0)$ in (2.2.5) at $t = 0$, together with the initial conditions $u_{f,h}(0) = 0$, $\eta_{p,h}(0) = 0$, $\partial_t \eta_{p,h}(0) = 0$, $p_{p,h}(0) = 0$, and $\lambda_{h}(0) = 0$, gives

$$\rho_p(\partial_t \eta_{p,h}(0), \partial_t \eta_{p,h}(0))_{\alpha_p} = (f_{p}(0), \partial_t \eta_{p,h}(0))_{\alpha_p},$$

implying

$$\|\partial_t \eta_{p,h}(0)\|_{L^2(\Omega_p)} \leq \frac{1}{\rho_p} \|f_{p}(0)\|_{L^2(\Omega_p)}. \quad (2.2.39)$$

Similarly, taking $v_{f,h} = \partial_t u_{f,h}(0)$, $v_{p,h} = 0$, and $\xi_{p,h} = 0$ in (2.2.5) and $w_{p,h} = \partial_t p_{p,h}(0)$ in (2.2.6) and at $t = 0$ gives

$$\|\partial_t u_{f,h}(0)\|_{L^2(\Omega_f)} \leq \frac{1}{\rho_f} \|f_{p}(0)\|_{L^2(\Omega_f)}, \quad (2.2.40)$$

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Combining (2.2.38)-(2.2.41) results in (2.2.44). Bound (2.2.25) follows from taking \( w_{p,h} = \nabla \cdot u_{p,h} \) in (2.2.6).

**Proof of** (2.2.26). We prove the bound by contradiction. Assume that there exists \( \bar{T} \in (0, T] \) such that

\[
\forall t \in [0, \bar{T}), \quad \| D(u_{f,h})(t) \|_{L^2(\Omega_f)} < \frac{\mu_f}{2 J_f S_f^2 K_f^3} \quad \text{and} \quad \| D(u_{f,h})(\bar{T}) \|_{L^2(\Omega_f)} = \frac{\mu_f}{2 J_f S_f^2 K_f^3}.
\]

(2.2.42)

We will prove that this is impossible under the small data condition (2.2.22).

We note that, due to assumption (2.2.42), bounds (2.2.23) and (2.2.24) hold in \([0, \bar{T}]\).

Using the energy equation (2.2.27), we have

\[
2 \mu_f \| D(u_{f,h}) \|_{L^2(\Omega_f)}^2 + \| K^{-1/2} u_{p,h} \|_{\Omega_f}^2 \leq (f, f, u_{f,h})_{\Omega_f} + (f, \partial_t \eta_{p,h}, u_{f,h})_{\Omega_f} - s_0 (p, p_{f,h}, u_{f,h})_{\Omega_f} + \rho_f (\partial_t u_{f,h}, u_{f,h})_{\Omega_f} - (\rho_f u_{f,h} \cdot \nabla u_{f,h}, u_{f,h})_{\Omega_f} - (\rho_f \partial_t \eta_{p,h}, \partial_t \eta_{p,h})_{\Omega_f} - s_0 (p_{f,h}, p_{f,h})_{\Omega_f}
\]

\[
\leq \frac{1}{2} \left( \rho_f \| u_{f,h} \|_{L^2(\Omega_f)}^2 + \rho_f \| \partial_t \eta_{p,h} \|_{L^2(\Omega_f)}^2 + a_{p}^e(\eta_{p,h}, \eta_{p,h}) + s_0 \| p_{f,h} \|_{L^2(\Omega_f)}^2 \right) + \frac{\mu_f}{2} \| D(u_{f,h}) \|_{L^2(\Omega_f)}^2 + \frac{1}{2} \| f \|_{L^2(\Omega_f)}^2 + \frac{\rho_f}{2} \| \partial_t \eta_{p,h} \|_{L^2(\Omega_f)}^2 + \frac{\rho_p}{2} \| \partial_t \eta_{p,h} \|_{L^2(\Omega_f)}^2 + \frac{\rho_p}{2} \| \partial_t \eta_{p,h} \|_{L^2(\Omega_f)}^2 + \frac{1}{2} \| \eta \|_{H^1(\Omega_f)}^2 + \frac{\rho_p s_0}{2} \| p_{f,h} \|_{L^2(\Omega_f)}^2 + \frac{\rho_p s_0}{2} \| p_{f,h} \|_{L^2(\Omega_f)}^2 + \frac{\rho_f S_f^2 K_f^3}{2} \| D(u_{f,h}) \|_{L^2(\Omega_f)}^3,
\]

where we used Young’s inequality, (2.2.28), and (2.2.30) to obtain the last inequality. Next, using assumption (2.2.42) and the inf-sup condition (2.2.32), and taking \( \epsilon = k_{min} \beta_p^2 \), we obtain \( \forall t \in [0, \bar{T}] \),

\[
\mu_f \| D(u_{f,h}) \|_{L^2(\Omega_f)}^2 + \frac{1}{2} \| K^{-1/2} u_{p,h} \|_{\Omega_f}^2 \leq \rho_f \| u_{f,h} \|_{L^2(\Omega_f)}^2 + \rho_f \| \partial_t \eta_{p,h} \|_{L^2(\Omega_f)}^2 + \frac{1}{2} \| f \|_{L^2(\Omega_f)}^2 + \frac{\mu_f}{2} \| D(u_{f,h}) \|_{L^2(\Omega_f)}^2 + \frac{1}{2} \| \eta \|_{H^1(\Omega_f)}^2 + \frac{\rho_p s_0}{2} \| p_{f,h} \|_{L^2(\Omega_f)}^2 + \frac{\rho_p s_0}{2} \| p_{f,h} \|_{L^2(\Omega_f)}^2 + \frac{\rho_f S_f^2 K_f^3}{2} \| D(u_{f,h}) \|_{L^2(\Omega_f)}^3.
\]
Using (2.2.23) and (2.2.24) in \([0, \bar{T}]\) results in, for all \(t \in [0, \bar{T}]\),

\[
\mu_f \| D(u_{f,h}(t)) \|_{L^2(\Omega_f)} \leq \exp(\bar{T}) \left( \| C_1 \|_{L^1(0,T)} + \frac{1}{2} \| C_2 \|_{L^1(0,T)} + \frac{1}{2} C_3 \right) + \frac{1}{2} \| C_1 \|_{L^\infty(0,T)}.
\]

Combined with the small data condition (2.2.22), this implies that for all \(t \in [0, \bar{T}]\),

\[
\| D(u_{f,h}(t)) \|_{L^2(\Omega_f)} < \frac{\mu_f}{2 \rho_f S_f^2 K_f^3}, \tag{2.2.43}
\]

which contradicts (2.2.42). Therefore (2.2.26) holds for all \(t \in [0, T_1]\).

Finally, the a priori bounds (2.2.23)–(2.2.26) in \([0, T_1]\) imply global existence and uniqueness of a solution of (GP) in \([0, T]\), with the bounds (2.2.23)–(2.2.26) holding in \([0, T]\).

\[
\square
\]

### 2.2.5 Existence, uniqueness, and stability of the solution of (LMWF1)

In the analysis we will utilize the following compactness result.

**Lemma 2.2.2.** [81] Let \(X, Y\) and \(B\) be Banach spaces such that \(X \subset B \subset Y\) where the embedding of \(X\) into \(B\) is compact. Let \(F\) be a bounded set in \(L^p(0,T; X)\) where \(1 \leq p < \infty\) and the set \(\{\partial_t f\}_{f \in F}\) is bounded in \(L^1(0,T; Y)\). Then \(F\) is relatively compact in \(L^p(0,T; B)\).

We will also use to following continuous inf-sup conditions.

**Lemma 2.2.3.** [16] There exist constants \(\beta_p > 0\) and \(\beta_f > 0\) such that

\[
\inf_{(0,0) \neq (w_p, \mu) \in W_p \times \Lambda} \sup_{0 \neq v_p \in V_p} \frac{b_p(v_p, w_p) + \langle v_p \cdot n_p, \mu \rangle_{\Gamma_{f_p}}}{\| v_p \|_{V_p}(\| w_p \|_{L^2(\Omega_p)} + \| \mu \|_{\Lambda})} \geq \beta_p, \tag{2.2.44}
\]

\[
\inf_{0 \neq w_f \in W_f} \sup_{0 \neq v_f \in V_f} \frac{b_f(v_f, w_f)}{\| v_f \|_{H^1(\Omega_f)}\| w_f \|_{L^2(\Omega_f)}} \geq \beta_f. \tag{2.2.45}
\]

We now present the main result of this section.

\[30\]
Theorem 2.2.2. Suppose that $f_f \in H^1(0,T; L^2(\Omega_f))$, $f_p \in H^1(0,T; L^2(\Omega_p))$, $q_p \in H^1(0,T; L^2(\Omega_p))$ and that the small data condition (2.2.22) holds. Then, the weak formulation (LMWF1) has a unique solution in the interval $[0,T]$ satisfying $u_f(0) = 0$, $u_p(0) = 0$, $p_p(0) = 0$, $\eta_p(0) = 0$, $\partial_t \eta_p(0) = 0$, and $\lambda(0) = 0$. The solution satisfies

$$\rho_f \|u_f\|_{L^2(0,T;L^2(\Omega_f))}^2 + 2\mu_f \|D(u_f)\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|u_f - \partial_t \eta_p\|_{L^2(0,T;H^{1,JS})}^2 + \rho_p \|\partial_t \eta_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + 2\|\mu_p^{1/2}D(\eta_p)\|_{L^\infty(0,T;L^2(\Omega_p))}^2 + \|\lambda_p^{1/2}\nabla \cdot \eta_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + s_0\|\partial_t p_p\|_{L^2(0,T;W_p)}^2 + \|K^{-1/2}u_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + \frac{k_{\text{min}}^2}{2}\|p_p\|_{L^2(0,T;W_p)}^2 + k_{\text{min}}^2\|\lambda\|_{L^2(0,T;L^2(\Omega_p))}^2 \leq \exp(T)\|C_1\|_{L^1(0,T)}.$$ (2.2.46)

Furthermore, it holds that

$$\rho_f \|\partial_t u_f\|_{L^2(0,T;L^2(\Omega_f))}^2 + \mu_f \|D(\partial_t u_f)\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\partial_t u_f - \partial_t \eta_p\|_{L^2(0,T;H^{1,JS})}^2 + \rho_p \|\partial_t \eta_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + 2\|\mu_p^{1/2}D(\partial_t \eta_p)\|_{L^\infty(0,T;L^2(\Omega_p))}^2 + \|\lambda_p^{1/2}\nabla \cdot \partial_t \eta_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + s_0\|\partial_t p_p\|_{L^2(0,T;W_p)}^2 + \|K^{-1/2}\partial_t u_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + \frac{k_{\text{min}}^2}{2}\|\partial_t p_p\|_{L^2(0,T;W_p)}^2 + k_{\text{min}}^2\|\partial_t \lambda\|_{L^2(0,T;\Omega_p)}^2 \leq \exp(T)(\|C_2\|_{L^1(0,T)} + C_3).$$ (2.2.47)

$$\|\nabla \cdot u_p\|_{L^2(\Omega_p)} \leq s_0\|\partial_t p_p\|_{L^2(\Omega_p)} + \alpha\|\nabla \cdot \partial_t \eta_p\|_{L^2(\Omega_p)} + \|q_p\|_{L^2(\Omega_p)} \quad \text{for a.e. } t \in (0,T),$$ (2.2.48)

$$\|D(u_f)\|_{L^\infty(0,T;L^2(\Omega_f))} < \frac{\mu_f}{2\rho_f S_f^2 K_f^3},$$ (2.2.49)

and, for a.e. $t \in (0,T)$,

$$\|p_f\|_{L^2(\Omega_f)} \leq C \left( \|\partial_t u_f\|_{L^2(\Omega_f)} + \|u_f\|_{H^1(\Omega_f)} + \|\partial_t \eta_p\|_{H^1(\Omega_p)} + \|\lambda\|_{L^2(\Omega_p)} + \|f_f\|_{L^2(\Omega_f)} \right).$$ (2.2.50)
Proof. We consider sequences of the solution components of (GP) with \( h \to 0 \). From Theorem 2.2.1, bounds (2.2.23) and (2.2.24), we have that

\[
\|u_{f,h}\|_{H^1(0,T;H^1(\Omega_f))} < \infty, \quad \|\eta_{p,h}\|_{H^1(0,T;H^1(\Omega_p))} < \infty, \quad \|\partial_t \eta_{p,h}\|_{L^2(0,T;L^2(\Omega_p))} < \infty;
\]

\[
\|u_{p,h}\|_{H^1(0,T;H(\text{div};\Omega_p))} < \infty, \quad \|p_{p,h}\|_{H^1(0,T;L^2(\Omega_p))} < \infty, \quad \|\lambda_h\|_{H^1(0,T;\Lambda)} < \infty.
\]

Let \( H_0^1(0,T) = \{ \varphi \in H^1(0,T) : \varphi(0) = 0 \} \). Consider the reflexive Hilbert spaces

\[
\nabla f = L^2(0,T;V_0^0) \cap H^1_0(0,T;H^1(\Omega_f)),
\]

\[
\nabla_p = L^2(0,T;X_p) \cap H^1_0(0,T;H^1(\Omega_p)) \cap H^2(0,T;L^2(\Omega_p)),
\]

\[
\left(\nabla f, \nabla_p \right) = L^2(0,T;V_0^0) \cap H^1_0(0,T;H^1(\Omega_f)) \cap H^2(0,T;L^2(\Omega_p)), \quad \bar{\nabla} = H^1_0(0,T;\Lambda)
\]

Due to the boundness of the sequences of the Galerkin solution, there exist subsequences, still denoted by \( \{u_{f,h}, \eta_{p,h}, u_{p,h}, p_{p,h}, \lambda_h\} \) such that

\[
u_{f,h} \rightharpoonup u_f \text{ in } \nabla f, \quad \eta_{p,h} \rightharpoonup \eta_p \text{ in } \nabla_p, \quad u_{p,h} \rightharpoonup u_p \text{ in } \nabla_p, \quad p_{p,h} \rightharpoonup p_p \text{ in } \bar{\nabla}, \quad \lambda_h \rightharpoonup \lambda \text{ in } \bar{\nabla},
\]

where \( \rightharpoonup \) means weak convergence. Note that the limit functions satisfy the appropriate initial and boundary conditions.

We need strong convergence for the non-linear term in (2.2.5). From Sobolev embedding, we know that \( H^1(\Omega_f) \hookrightarrow L^4(\Omega_f) \subset L^2(\Omega_f) \). Therefore the subsequence \( \{u_{f,h}\} \) is bounded in \( L^4(0,T;H^1(\Omega_f)) \) and \( \{\partial_t u_{f,h}\} \) is bounded in \( L^2(0,T;L^2(\Omega_f)) \). Lemma 2.2.2 implies that \( \{u_{f,h}\} \) has a subsequence, still denoted the same, that converges strongly to \( u_f \) in \( L^4(0,T;L^4(\Omega_f)) \). Next, multiplying (2.2.5)–(2.2.7) by an arbitrary \( \phi(t) \in L^2(0,T) \) and integrating in time, we have, for all \( v_{f,h} \in V_{f,h}^0, v_{p,h} \in V_{p,h}, w_{p,h} \in W_{p,h}, \xi_{p,h} \in X_{p,h} \), and \( \mu_h \in \Lambda_h \),

\[
\int_0^T (\rho_f \partial_t u_{f,h}, \phi(t)v_{f,h})_{\Omega_f} + \int_0^T a_{f}(u_{f,h}, \phi(t)v_{f,h}) + \int_0^T (\rho_f u_{f,h} \cdot \nabla u_{f,h}, \phi(t)v_{f,h})_{\Omega_f} \\
+ \int_0^T b_f(\phi(t)v_{f,h}, p_{f,h}) + \int_0^T (\rho_p \partial_t \eta_{p,h}, \phi(t)\xi_{p,h})_{\Omega_f} + \int_0^T a_p(\eta_{p,h}, \phi(t)\xi_{p,h})_p \\
+ \int_0^T \alpha_b p(\phi(t)\xi_{p,h}, p_{p,h}) + \int_0^T a_{p}(u_{p,h}, \phi(t)v_{p,h}) + \int_0^T b_p(\phi(t)v_{p,h}, p_{p,h}) \\
+ \int_0^T a_{HJS}(u_{f,h}, \partial_t \eta_{p,h}, \phi(t)v_{f,h}, \phi(t)\xi_{p,h}) + \int_0^T \partial_t(\phi(t)v_{f,h}, \phi(t)v_{p,h}, \phi(t)\xi_{p,h}; \lambda_h)
\]

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\[
\int_0^T (f_f, \phi(t)v_{f,h})_{\Omega_f} + \int_0^T (f_p, \phi(t)\xi_{p,h})_{\Omega_p},
\]
(2.2.51)
\[
\int_0^T (s_0 \partial_t p_{p,h}, \phi(t)w_{p,h})_{\Omega_p} - \int_0^T \alpha b_p(\partial_t \eta_{p,h}, \phi(t)w_{p,h}) - \int_0^T b_p(u_{p,h}, \phi(t)w_{p,h}) = \int_0^T (q_p, \phi(t)w_{p,h})_{\Omega_p},
\]
(2.2.52)
\[
\int_0^T b_T(u_{f,h}, u_{p,h}, \partial_t \eta_{p,h}, \phi(t)\mu_h) = 0.
\]
(2.2.53)

Fixing \( h \) for the test functions, letting \( h \to 0 \) for the solution functions, and using the above weak and strong convergence results, we obtain, for all \( v_{f,h} \in V_{f,h}^0, v_{p,h} \in V_{p,h}, w_{p,h} \in W_{p,h}, \xi_{p,h} \in X_{p,h}, \) and \( \mu_h \in \Lambda_h, \)
\[
\int_0^T (\rho_f \partial_t u_f, \phi(t)v_{f,h})_{\Omega_f} + \int_0^T a_f(u_f, \phi(t)v_{f,h}) + \int_0^T (\rho_f u_f \cdot \nabla u_f, \phi(t)v_{f,h})_{\Omega_f}
\]
+ \[
\int_0^T b_f(\phi(t)v_{f,h}, p_f) + \int_0^T (\rho_p \partial_t \eta_{p,h}, \phi(t)\xi_{p,h})_{\Omega_p} + \int_0^T a^c_{p,h}(\eta_{p,h}, \phi(t)\xi_{p,h})
\]
+ \[
\int_0^T a_b(u_f, \partial_t \eta_{p,h}, \phi(t)v_{f,h}, \phi(t)\xi_{p,h}) + \int_0^T b_T(\phi(t)v_{f,h}, \phi(t)v_{p,h}, \phi(t)\xi_{p,h}; \lambda) = \int_0^T (f_f, \phi(t)v_{f,h})_{\Omega_f} + \int_0^T (f_p, \phi(t)\xi_{p,h})_{\Omega_p},
\]
(2.2.54)
\[
\int_0^T (s_0 \partial_t p_{p,h}, \phi(t)w_{p,h})_{\Omega_p} - \int_0^T \alpha b_p(\partial_t \eta_{p,h}, \phi(t)w_{p,h}) - \int_0^T b_p(u_{p,h}, \phi(t)w_{p,h}) = \int_0^T (q_p, \phi(t)w_{p,h})_{\Omega_p},
\]
(2.2.55)
\[
\int_0^T b_T(u_f, u_p, \partial_t \eta_p, \phi(t)\mu_h) = 0.
\]
(2.2.56)

Since \( V_{f,h}^0, V_{p,h}, W_{p,h}, X_{p,h}, \) and \( \Lambda_h \) are dense in \( V_{f,h}^0, V_p, W_p, X_p, \) and \( \Lambda, \) respectively, and since \( \phi(t) \in L^2(0, T) \) is arbitrary, we conclude that (2.2.1)–(2.2.3) hold. The proof of bounds (2.2.46)–(2.2.49) is the same as the proof of bounds (2.2.23)–(2.2.26) from Theorem 2.2.1, using the continuous inf-sup condition (2.2.44).

**Uniqueness of the solution.** Let \((u_{f,1}, u_{p,1}, p_{p,1}, \eta_{p,1}, \lambda_1)\) and \((u_{f,2}, u_{p,2}, p_{p,2}, \eta_{p,2}, \lambda_2)\) be two solutions of (LMWF2) Then, for \( \tilde{u}_f = u_{f,1} - u_{f,2}, \tilde{u}_p = u_{p,1} - u_{p,2}, \tilde{p}_p = p_{p,1} - p_{p,2}, \)
\( \tilde{\eta}_p = \eta_{p,1} - \eta_{p,2} \) and \( \tilde{\lambda} = \lambda_1 - \lambda_2 \), we have, for all \( \mathbf{v}_f \in \mathbf{V}_f^0, \mathbf{v}_p \in \mathbf{V}_p, w_p \in W_p, \xi_p \in \mathbf{X}_p \), and \( \mu \in \Lambda \),

\[
\begin{align*}
(\rho_f \partial_t \tilde{\mathbf{u}}_f, \mathbf{v}_f)_{\Omega_f} + a_f(\tilde{\mathbf{u}}_f, \mathbf{v}_f) + (\rho_f \mathbf{u}_{f,1} \cdot \nabla \mathbf{u}_{f,1}, \mathbf{v}_f)_{\Omega_f} - (\rho_f \mathbf{u}_{f,2} \cdot \nabla \mathbf{u}_{f,2}, \mathbf{v}_f)_{\Omega_f} \\
+ (\rho_p \partial_t \tilde{\mathbf{p}}, \xi_p)_{\Omega_p} + a_p^e(\tilde{\mathbf{p}}, \xi_p) + \alpha b_p(\xi_p, \tilde{\mathbf{p}}_p) + a_p^d(\tilde{\mathbf{u}}_p, \mathbf{v}_p)_{\Omega_p} + b_p(\mathbf{v}_p, \tilde{\mathbf{p}}_p) \\
+ a_{BJS}(\tilde{\mathbf{u}}_f, \partial_t \tilde{\mathbf{p}}, \mathbf{v}_f, \xi_p) + b_f(\mathbf{v}_f, \mathbf{v}_p, \xi_p; \tilde{\lambda}) = 0, \quad (2.2.57) \\
(s_0 \partial_t \tilde{\mathbf{p}}, w_p)_{\Omega_p} - \alpha b_p(\partial_t \tilde{\mathbf{p}}, w_p) - b_p(\tilde{\mathbf{u}}_p, w_p) = 0, \quad (2.2.58) \\
b_f(\tilde{\mathbf{u}}_f, \tilde{\mathbf{u}}_p, \partial_t \tilde{\eta}_p; \mu) = 0. \quad (2.2.59)
\end{align*}
\]

Take \( \mathbf{v}_f = \tilde{\mathbf{u}}_f, \mathbf{v}_p = \tilde{\mathbf{u}}_p, w_p = \tilde{\mathbf{p}}_p, \xi_p = \partial_t \tilde{\mathbf{p}}_p \), and \( \mu = \tilde{\lambda} \) in (2.2.57)–(2.2.59) and combine the equations:

\[
(\rho_f \partial_t \tilde{\mathbf{u}}_f, \tilde{\mathbf{u}}_f)_{\Omega_f} + a_f(\tilde{\mathbf{u}}_f, \tilde{\mathbf{u}}_f) + (\rho_p \partial_t \tilde{\mathbf{p}}, \partial_t \tilde{\mathbf{p}})_{\Omega_p} + a_p^e(\tilde{\mathbf{p}}, \partial_t \tilde{\mathbf{p}}) + |\tilde{\mathbf{u}}_f - \partial_t \tilde{\mathbf{p}}|^2_{BJS} \\
+ a_p^d(\tilde{\mathbf{u}}_p, \tilde{\mathbf{u}}_p) + (s_0 \partial_t \tilde{\mathbf{p}}_p, \tilde{\mathbf{p}}_p)_{\Omega_p} = -(\rho_f \mathbf{u}_{f,1} \cdot \nabla \mathbf{u}_{f,1}, \tilde{\mathbf{u}}_f)_{\Omega_f} + (\rho_f \mathbf{u}_{f,2} \cdot \nabla \mathbf{u}_{f,2}, \tilde{\mathbf{u}}_f)_{\Omega_f}. \quad (2.2.60)
\]

We control the right hand side of (2.2.60) as follows:

\[
- (\rho_f \mathbf{u}_{f,1} \cdot \nabla \mathbf{u}_{f,1}, \tilde{\mathbf{u}}_f)_{\Omega_f} + (\rho_f \mathbf{u}_{f,2} \cdot \nabla \mathbf{u}_{f,2}, \tilde{\mathbf{u}}_f)_{\Omega_f} \\
= -(\rho_f \tilde{\mathbf{u}}_f \cdot \nabla \mathbf{u}_{f,1}, \tilde{\mathbf{u}}_f)_{\Omega_f} - (\rho_f \mathbf{u}_{f,2} \cdot \nabla \tilde{\mathbf{u}}_f, \tilde{\mathbf{u}}_f)_{\Omega_f} \\
\leq \rho_f S_f^2 K_f^2 \| \mathbf{D}(\mathbf{u}_f) \|_{L^2(\Omega_f)}^2 \| \mathbf{D}(\mathbf{u}_{f,1}) \|_{L^2(\Omega_f)} + \| \mathbf{D}(\mathbf{u}_{f,2}) \|_{L^2(\Omega_f)}^2 \leq \frac{1}{2} a_f(\tilde{\mathbf{u}}_f, \tilde{\mathbf{u}}_f),
\]

using that \( \mathbf{u}_{f,1} \) and \( \mathbf{u}_{f,2} \) satisfy (2.2.49). Combining this with (2.2.60) and integrating in time from 0 to \( t \in (0, T] \), we obtain

\[
\rho_f \| \tilde{\mathbf{u}}_f \|_{L^2(\Omega_f)}^2 + \rho_p \| \partial_t \tilde{\mathbf{p}} \|_{L^2(\Omega_p)}^2 + a_p^e(\tilde{\mathbf{p}}, \tilde{\mathbf{p}}) + s_0 \| \tilde{\mathbf{p}}_p \|_{L^2(\Omega_p)}^2 + \int_0^t (a_f(\tilde{\mathbf{u}}_f, \tilde{\mathbf{u}}_f) + 2 a_p^d(\tilde{\mathbf{u}}_p, \tilde{\mathbf{u}}_p)) \leq 0,
\]

which implies \( \tilde{\mathbf{u}}_f = 0, \tilde{\mathbf{p}} = 0, \tilde{\mathbf{u}}_p = 0, \) and \( \tilde{\mathbf{p}}_p = 0 \). The inf-sup condition (2.2.44), along with (2.2.57) for test function \( \mathbf{v}_p \), implies that \( \tilde{\lambda} = 0 \). This completes the proof of uniqueness.
**Recovery of** $p_f$. To recover the Navier-Stokes pressure $p_f$, which has been eliminated in (LMWF2), we utilize the inf-sup condition (2.2.45). For each $t \in (0, T]$, define the linear functional $\mathcal{F} : V_f \to \mathbb{R}$ as follows:

$$
\mathcal{F}(v_f) = (\rho_f \partial_t u_f, v_f)_{\Omega_f} + a_f(u_f, v_f) + (\rho_f u_f \cdot \nabla u_f, v_f)_{\Omega_f} \\
+ b_f(v_f, 0, \lambda) + a_{BJS}(u_f, \partial_t \eta_p; v_f, 0) - (f_f, v_f)_{\Omega_f}, \quad \forall v_f \in V_f.
$$

The functional $\mathcal{F}$ is continuous and $\mathcal{F}(v) = 0$ on $V_f^0$ for a.e. $t \in (0, T)$, from (2.2.1). Then, by (2.2.45), we conclude that, for a.e. $t \in (0, T)$, there exists a unique $p_f \in W_f$ such that for all $v_f \in V_f$,

$$
b_f(v_f, p_f) = \mathcal{F}(v_f). \tag{2.2.61}
$$

Therefore $(u_f, p_f, u_p, p_p, \eta_p, \lambda)$ is a unique solution of (LMWF1). Furthermore, (2.2.45) and (2.2.61) imply

$$
\|p_f\|_{L^2(\Omega_f)} \leq \frac{1}{\beta_f} \left( \rho_f \|\partial_t u_f\|_{L^2(\Omega_f)} + 2\mu_f \|D(u_f)\|_{L^2(\Omega_f)} + \rho_f S^2 f^2 K^2 \|D(u_f)\|_{L^2(\Omega_f)}^2 \\
+ C^r \|\lambda\|_\Lambda + C^{BJS} (\|u_f\|_{H^1(\Omega_f)} + \|\partial_t \eta_p\|_{H^1(\Omega_p)}) + \|f_f\|_{L^2(\Omega_f)} \right). \tag{2.2.62}
$$

where we used the continuity bounds (2.1.33) and (2.1.34). Using (2.2.49), we have

$$
\rho_f S^2 f^2 K^2 \|D(u_f)\|_{L^2(\Omega_f)}^2 \leq \frac{H_f^2}{2K_f} \|D(u_f)\|_{L^2(\Omega_f)}^2,
$$

which, together with (2.2.62), implies (2.2.50). This completes the proof of the theorem. 

\[\square\]
2.3 Fully discretized numerical scheme

For the spatial discretization, we consider the subdomain finite element partitions $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ introduced in Section 2.2.2, as well as the finite element spaces for the Biot system, $V_{p,h} \times W_{p,h} \subset V_p \times W_p$ for Darcy and $X_{p,h} \subset X_p$ for elasticity. We will also the Lagrange multiplier finite element space $\Lambda_h \subset \Lambda$ defined on the trace of $\mathcal{T}_h^p$ on $\Gamma_{fp}$. Note that in this section, $\Lambda_h$ can be conforming spaces $\Lambda_h \subset \Lambda$, equipped with $H^{1/2}$-norm, or non-conforming spaces $\tilde{\Lambda}_h := V_{p,h} \cdot n\Gamma_{fp}$. We discuss the details of the non-conforming space $\tilde{\Lambda}_h$ in Section 2.4. For now, we focus on using conforming $\Lambda_h$. We recall that these spaces satisfy the inf-sup condition (2.2.4). For the Navier-Stokes discretization, we consider any conforming inf-sup stable Stokes pair $V_{f,h} \times W_{f,h} \subset V_f \times W_f$, such as the MINI elements or the Taylor-Hood spaces [16]. For the time discretization we use the backward Euler’s method. Let $N$ be the number of time steps and $\Delta t = T/N$, $t_n = n\Delta t$, $0 \leq n \leq N$. Let $d_t u^n = (u^n - u^{n-1})/\Delta t$ be the first order (backward) discrete time derivative, where $u^n = u(t_n)$. The numerical scheme is as follows:

(FDNS): Given $u_{f,h}^0 = u_{f,h}(0) = 0$, $\eta_{p,h}^0 = \eta_{p,h}(0) = 0$, $\eta_{p,h}^{-1} = 0$, $u_{p,h}^0 = u_{p,h}(0) = 0$ and $p_{p,h}^0 = p_{p,h}(0) = 0$, find $\{(u^n_{f,h}, p^n_{f,h}, u^n_{p,h}, p^n_{p,h}, \eta^n_{p,h}, \lambda^n_{h})\}_{n=1}^N \subset V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h$ such that for $0 \leq n \leq N - 1$ and for all $v_{f,h} \in V_{f,h}$, $w_{f,h} \in W_{f,h}$, $v_{p,h} \in V_{p,h}$, $w_{p,h} \in W_{p,h}$, $\xi_{p,h} \in X_{p,h}$, and $\mu_h \in \Lambda_h$,

\[
\begin{align*}
\left(\rho_f \frac{u_{f,h}^{n+1} - u_{f,h}^n}{\Delta t}, v_{f,h}\right)_{\Omega_f} + a_f(u_{f,h}^{n+1}, v_{f,h}) + (\rho_f u_{f,h}^n \cdot \nabla u_{f,h}^{n+1}, v_{f,h})_{\Omega_f} + b_f(v_{f,h}, p_{f,h}^{n+1}) \\
+ a_p^d(u_{f,h}^{n+1}, v_{p,h}) + b_p(v_{p,h}, p_{p,h}^{n+1}) + b_T(v_{f,h}, v_{p,h}, \xi_{p,h}; \lambda_{h}^{n+1}) \\
+ a_{BJS}(u_{f,h}^{n+1}, \xi_{p,h}) = (f_{f,h}^{n+1}, v_{f,h})_{\Omega_f} + (f_{p,h}^{n+1}, \xi_{p,h})_{\Omega_p},
\end{align*}
\]

(2.3.1)

\[
\begin{align*}
\left(s_0 \frac{p_{p,h}^{n+1} - p_{p,h}^n}{\Delta t}, w_{p,h}\right)_{\Omega_p} - \alpha b_p\left(\frac{\eta_{p,h}^{n+1} - \eta_{p,h}^n}{\Delta t}, w_{p,h}\right) - b_p(u_{p,h}^{n+1}, w_{p,h}) \\
- b_f(u_{f,h}^{n+1}, w_{f,h}) = (q_{p,h}^{n+1}, w_{p,h})_{\Omega_p},
\end{align*}
\]

(2.3.2)

\[
\begin{align*}
b_T\left(u_{f,h}^{n+1}, p_{f,h}^{n+1}, \eta_{p,h}^{n+1} - \eta_{p,h}^n; \mu_h\right) = 0.
\end{align*}
\]

(2.3.3)
Note that setting $\eta_{p,h}^{-1} = 0$ provides an approximation to $\partial_t \eta(0) = 0$. Also, in (2.3.1), we utilize a linearization $(\rho_f u_{f,h}^n \cdot \nabla u_{f,h}^{n+1}, v_{f,h})_\Omega_f$, instead of directly using the non-linear term. We state the following inf-sup condition in discrete sense:

**Lemma 2.3.1.** [64] There exists a constant $\beta_f > 0$ such that

$$\inf_{q_{f,h} \in W_{f,h}} \sup_{v_{f,h} \in V_{f,h}} \frac{b_f(v_{f,h}, q_{f,h})}{\|v_{f,h}\|_{V_f} \|q_{f,h}\|_{W_f}} \geq \beta_f. \quad (2.3.4)$$

### 2.3.1 Well posedness of the fully discrete model (FDNS)

Remember that in Section 2.2.4, we define quantities $C_1(t)$, $C_2(t)$, and $C_3$, which depend on the data $f_f$, $f_p$, and $q_p$. Since we are using backward Euler’s method, we further define the following quantity:

$$C_4 := \frac{1}{\rho_f} \|f_f^1\|_{L^2(\Omega_f)}^2 + \frac{1}{\rho_p} \|f_p^1\|_{L^2(\Omega_p)}^2 + \frac{1}{s_0} \|q_p^1\|_{L^2(\Omega_p)}^2. \quad (2.3.5)$$

We state the main results for the formulation (2.3.1)–(2.3.3).

**Theorem 2.3.1.** Suppose that $f_f$, $f_p$, and $q_p$ satisfy the discrete small data condition

$$\exp(T) \left( \|C_1\|_{L^2(0,T)} + \frac{1}{2} \|C_2\|_{L^2(0,T)} + \frac{1}{2} C_4 \right) + \frac{1}{2} \|C_1\|_{L^\infty(0,T)} < \frac{\mu_3^2}{4 \rho_f^2 S_f^2 K_f^6}. \quad (2.3.6)$$

The fully discrete problem (FDNS) exists an unique solution $\{(u_{f,h}^n, p_{f,h}^n, u_{p,h}^n, p_{p,h}^n, \eta_{p,h}^n, \lambda_{h}^n)\}_{n \geq 1}$.

**Proof.** First, we have the following equation which is used repeatedly in the proof:

$$\int_{\Omega} \phi^n d_t \phi^n = d_t \|\phi^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \Delta t \|d_t \phi^n\|_{L^2(\Omega)}^2. \quad (2.3.7)$$

Let us now prove the existence of the numerical solution. The proof follows the idea in the semi-discrete case. However, instead of directly showing the existence, we utilize the induction. We first assume the existence of $\{(u_{f,h}^j, p_{f,h}^j, u_{p,h}^j, p_{p,h}^j, \eta_{p,h}^j, \lambda_{h}^j)\}$, for $j \leq n$, and obtain some stability results, especially the bound for $D(u_{f,h}^n)$. Next, by using a fixed point theory, we show the existence for $n + 1$ step.
We assume that \( \{(u^n_{f,h}, p^n_{f,h}, u^n_{p,h}, p^n_{p,h}, \eta^n_{p,h}, \lambda^n_{p,h})\}_{n \geq 1} \) exist. Under such assumption, we first show \( \{(u^n_{f,h}, p^n_{f,h}, u^n_{p,h}, p^n_{p,h}, \eta^n_{p,h}, \lambda^n_{p,h})\}_{n \geq 1} \) satisfying the following bounds, which facilitate the future existence proof.

\[
\rho_f \| u^n_{f,h} \|^2_{L^2(\Omega_f)} + \mu_f \Delta t \sum_{j=0}^{n-1} \| D(u^{j+1}_{f,h}) \|^2_{L^2(\Omega_f)} + \Delta t \sum_{j=0}^{n-1} \| u^{j+1}_{f,h} - d_t \eta^{j+1}_{p,h} \|^2_{a_{BJS}} \\
+ \rho_p \| d_t \eta^n_{p,h} \|^2_{L^2(\Omega_p)} + 2 \| \mu_p^{1/2} D(\eta^n_{p,h}) \|^2_{L^2(\Omega_p)} + \| \lambda_p^{1/2} \nabla \cdot \eta^n_{p,h} \|^2_{L^2(\Omega_p)} + s_0 \| p^n_{p,h} \|^2_{L^2(\Omega_p)} \\
+ \Delta t \sum_{j=0}^{n-1} \| K^{-1/2} u^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} + \frac{k_{\text{min}}^2 \beta_p}{2} \Delta t \sum_{j=0}^{n-1} \| p^{j+1}_{p,h} \|^2_{W_p} + k_{\text{min}}^2 \beta_p \Delta t \sum_{j=0}^{n-1} \| \lambda^{j+1}_{p,h} \|^2 \lambda \\
+ \Delta t^2 \sum_{j=0}^{n-1} \left( \rho_f \| d_t u^{j+1}_{f,h} \|^2_{L^2(\Omega_f)} + 2 \| \mu_p^{1/2} D(d_t \eta^{j+1}_{p,h}) \|^2_{L^2(\Omega_p)} + \| \lambda_p^{1/2} \nabla \cdot d_t \eta^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} \\
+ s_0 \| d_t p^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} \right) \leq \exp(T) \left( \frac{P_f^2 K_f^2}{\mu_f} \| f_f \|^2_{L^2(0,T;L^2(\Omega_f))} + \frac{1}{\rho_p} \| f_p \|^2_{L^2(0,T;L^2(\Omega_p))} + \frac{2}{k_{\text{min}}^2 \beta_p^2} \| q_p \|^2_{L^2(0,T;L^2(\Omega_p))} \right). \tag{2.3.8}
\]

\[
\rho_f \| d_t u^n_{f,h} \|^2_{L^2(\Omega_f)} + \mu_f \Delta t \sum_{j=1}^{n-1} \| D(d_t u^{j+1}_{f,h}) \|^2_{L^2(\Omega_f)} + \Delta t \sum_{j=1}^{n-1} \| d_t u^{j+1}_{f,h} - d_t \eta^{j+1}_{p,h} \|^2_{a_{BJS}} \\
+ \rho_p \| d_t \eta^n_{p,h} \|^2_{L^2(\Omega_p)} + 2 \| \mu_p^{1/2} D(d_t \eta^n_{p,h}) \|^2_{L^2(\Omega_p)} + \| \lambda_p^{1/2} \nabla \cdot d_t \eta^n_{p,h} \|^2_{L^2(\Omega_p)} + s_0 \| d_t p^n_{p,h} \|^2_{L^2(\Omega_p)} \\
+ \Delta t \sum_{j=1}^{n-1} \| K^{-1/2} d_t u^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} + \frac{k_{\text{min}}^2 \beta_p}{2} \Delta t \sum_{j=1}^{n-1} \| d_t p^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} \\
+ k_{\text{min}}^2 \beta_p \Delta t \sum_{j=1}^{n-1} \| d_t \lambda^{j+1}_{p,h} \|^2_{\lambda} + \Delta t^2 \sum_{j=1}^{n-1} \left( \rho_f \| d_t u^{j+1}_{f,h} \|^2_{L^2(\Omega_f)} + 2 \| \mu_p^{1/2} D(d_t \eta^{j+1}_{p,h}) \|^2_{L^2(\Omega_p)} \\
+ \| \lambda_p^{1/2} \nabla \cdot d_t \eta^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} + s_0 \| d_t p^{j+1}_{p,h} \|^2_{L^2(\Omega_p)} \right) \leq \exp(T) \left( \frac{P_f^2 K_f^2}{\mu_f} \| f_f \|^2_{L^2(0,T;L^2(\Omega_f))} + \frac{1}{\rho_p} \| f_p \|^2_{L^2(0,T;L^2(\Omega_p))} + \frac{2}{k_{\text{min}}^2 \beta_p^2} \| q_p \|^2_{L^2(0,T;L^2(\Omega_p))} \right) \\
+ \frac{1}{\rho_f} \| f_f \|^2_{L^2(\Omega_f)} + \frac{1}{\rho_p} \| f_p \|^2_{L^2(\Omega_p)} + \frac{1}{s_0} \| q_p \|^2_{L^2(\Omega_p)} \right). \tag{2.3.9}
\]

\[
\| D(u^n_{f,h}) \|_{L^2(\Omega_f)} \leq \frac{\mu_f}{2 \rho_f S_f^2 K_f^2}. \tag{2.3.10}
\]

Note that in (2.3.9), we manually define \( \eta_{p,h}^{-1} = 0 \) for the simplicity of notation.
Before proving (2.3.8) and (2.3.9), we assume for $0 \leq j \leq n - 1$,

$$
\|D(u_{f,h}^j)\|_{L^2(\Omega_t)} \leq \frac{\mu_f}{2\rho_f S_f^2 K_f^2}.
$$

(2.3.11)

We finish the proof of (2.3.10) with induction strategy.

**Proof of (2.3.8).** In (2.3.1)–(2.3.3), considering at time step $j + 1$, $0 \leq j \leq n - 1$, taking the test function $v_{f,h} = u_{f,h}^{j+1}$, $w_{f,h} = p_{f,h}^{j+1}$, $v_{p,h} = u_{p,h}^{j+1}$, $w_{p,h} = p_{p,h}^{j+1}$, $\xi_{p,h} = d_t \eta_{p,h}^{j+1}$, $\mu_h = \lambda_h^j$ and adding them together, we obtain the energy equality after combining with equation (2.3.7):

$$
\begin{align*}
\frac{1}{2} d_t \left( \rho_f \|u_{f,h}^{j+1}\|_{L^2(\Omega_t)}^2 + s_0 \|p_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2 + a^p(v_{p,h}, \eta_{p,h}^{j+1}) \right) + \frac{\rho_p}{2\Delta t} \left( \|d_t \eta_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2 \right) \\
- \|d_t \eta_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2 + \frac{\Delta t}{2} \left( \rho_f \|d_t u_{f,h}^{j+1}\|_{L^2(\Omega_t)}^2 + s_0 \|d_t p_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2 + a^p(d_t \eta_{p,h}^{j+1}, d_t \eta_{p,h}^{j+1}) \right) \\
+ a_f(u_{f,h}^{j+1}, u_{f,h}^{j+1}) + a_f(u_{f,h}^{j+1}, u_{p,h}^{j+1}) + |u_{f,h}^{j+1} - d_t \eta_{f,h}^{j+1}|^2 \\
= (f_j^{j+1}, u_{f,h}^{j+1})_{\Omega_f} + (q_p^{j+1}, d_t \eta_{p,h}^{j+1})_{\Omega_p} + (p_{p,h}^{j+1}, p_{p,h}^{j+1})_{\Omega_p} - (\rho_f u_{f,h}^{j+1} \cdot \nabla u_{f,h}^{j+1}, u_{f,h}^{j+1})_{\Omega_f}.
\end{align*}
$$

(2.3.12)

For the last term on the right hand side of above equation, using (2.3.10), Cauchy-Schwarz, Sobolev (2.1.27), and Korn (2.1.28), we have

$$
|\langle \rho_f u_{f,h}^{j+1} \cdot \nabla u_{f,h}^{j+1}, u_{f,h}^{j+1} \rangle_{\Omega_f} | \leq \frac{\mu_f}{2} \|D(u_{f,h}^{j+1})\|_{L^2(\Omega_f)}^2.
$$

(2.3.13)

The rest two terms on the right hand side can be bounded as follows by utilizing Cauchy-Schwarz-Young inequality:

$$
\begin{align*}
(f_j^{j+1}, u_{f,h}^{j+1})_{\Omega_f} + (q_p^{j+1}, p_{p,h}^{j+1})_{\Omega_p} + (f_j^{j+1}, d_t \eta_{p,h}^{j+1})_{\Omega_p} \\
\leq \frac{\mu_f}{2\rho_p} \|f_j^{j+1}\|_{L^2(\Omega_f)}^2 + \frac{\mu_f}{2\rho_p} \|u_{f,h}^{j+1}\|_{L^2(\Omega_t)}^2 + \frac{1}{\beta_p^2 \kappa_{\text{min}}^2} \|q_p^{j+1}\|_{L^2(\Omega_p)}^2 + \frac{1}{\beta_p^2 \kappa_{\text{min}}^2} \|p_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2 \\
+ \frac{1}{2\rho_p} \|f_j^{j+1}\|_{L^2(\Omega_f)}^2 + \frac{\rho_p}{2} \|d_t \eta_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2.
\end{align*}
$$

(2.3.14)

Then summing up over the time index $j = 0, 1, \cdots, n - 1$, using initial conditions, and multiplying $2\Delta t$ on both sides of the inequality, we have

$$
\begin{align*}
\rho_f \|u_{f,h}^n\|_{L^2(\Omega)}^2 + \rho_f \Delta t^2 \sum_{j=0}^{n-1} \|d_t u_{f,h}^{j+1}\|_{L^2(\Omega_t)}^2 + \mu_f \Delta t \sum_{j=0}^{n-1} \|D(u_{f,h}^{j+1})\|_{L^2(\Omega_t)}^2 + \rho_P \|d_t \eta_{p,h}^n\|_{L^2(\Omega_p)}^2 \\
+ a^p(\eta_{p,h}, \eta_{p,h}) + \Delta t^2 \sum_{j=0}^{n-1} a^p(\eta_{p,h}, d_t \eta_{p,h}^{j+1}) + s_0 \|p_{p,h}^n\|_{L^2(\Omega_p)}^2 + s_0 \Delta t^2 \sum_{j=0}^{n-1} \|d_t p_{p,h}^{j+1}\|_{L^2(\Omega_p)}^2
\end{align*}
$$

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+ 2\Delta t \sum_{j=0}^{n-1} \left( a^d_p(u^j_{p,h}, u^j_{p,h}) + |u^j_{f,h} - d_t \eta^j_{p,h}|^2_{\alpha_{BJS}} \right)
\leq \Delta t \sum_{j=0}^{n-1} \left( \frac{P^q_{2} K^2}{\mu_f} ||f^j_{f,h}||^2_{L^2(\Omega_f)} + \frac{2}{\beta^2_{p} k_{min}} ||q^j_{p,h}||^2_{L^2(\Omega_p)} \right) + \frac{\beta^2_{p} k_{min}}{2} \Delta t \sum_{j=0}^{n-1} ||p^j_{p,h}||^2_{L^2(\Omega_p)}
+ \Delta t \sum_{j=0}^{n-1} \left( \frac{1}{\rho_p} ||f^j_{p,h}||^2_{L^2(\Omega_p)} + \rho_p ||d_t \eta^j_{p,h}||^2_{L^2(\Omega_p)} \right).
\tag{2.3.15}

From the discrete inf-sup inequality (2.2.32), we have
\[ \beta^2_{p} k_{min} \sum_{j=0}^{n-1} (||f^j_{p,h}||^2_{L^2(\Omega_p)} + ||\lambda^j_{h}||^2_{\Lambda}) \leq \sum_{j=0}^{n-1} K^{-1/2} ||u^j_{p,h}||^2_{L^2(\Omega_p)}. \tag{2.3.16} \]

Combining (2.3.15)–(2.3.16) and using discrete Gronwall’s inequality, gives
\[
\rho_f ||u^n_{f,h}||^2_{L^2(\Omega_f)} + \rho_f \Delta t \sum_{j=0}^{n-1} ||d_t u^j_{f,h}||^2_{L^2(\Omega_f)} + \mu_f \Delta t \sum_{j=0}^{n-1} ||D(u^j_{f,h})||^2_{L^2(\Omega_f)} + \rho_p ||d_t \eta^n_{p,h}||^2_{L^2(\Omega_p)}
+ 2 ||\mu_p^{1/2} D(\eta^n_{p,h})||^2_{L^2(\Omega_p)} + ||\lambda^1_{p,h} \cdot \nabla \eta^n_{p,h}||^2_{L^2(\Omega_p)} + \Delta t \sum_{j=0}^{n-1} a^e_p(d_t \eta^j_{p,h}, d_t \eta^j_{p,h}) + s_0 ||p^n_{p,h}||^2_{L^2(\Omega_p)}
+ s_0 \Delta t \sum_{j=0}^{n-1} ||d_t p^j_{p,h}||^2_{L^2(\Omega_p)} + \frac{\beta^2_{p} k_{min}}{2} \Delta t \sum_{j=0}^{n-1} ||p^j_{p,h}||^2_{L^2(\Omega_p)} + \Delta t \sum_{j=0}^{n-1} ||K^{-1/2} u^j_{p,h}||^2_{L^2(\Omega_p)}
+ \beta^2_{p} k_{min} \Delta t \sum_{j=0}^{n-1} ||\lambda^j_{h}||^2_{\Lambda} + 2 \Delta t \sum_{j=0}^{n-1} ||u^j_{f,h} - d_t \eta^j_{p,h}||^2_{\alpha_{BJS}}
\leq \exp(T) \Delta t \sum_{j=0}^{n-1} \left( \frac{P^q_{2} K^2}{\mu_f} ||f^j_{f,h}||^2_{L^2(\Omega_f)} + \frac{1}{\rho_p} ||f^j_{p,h}||^2_{L^2(\Omega_p)} + \frac{2}{\beta^2_{p} k_{min}} ||q^j_{p,h}||^2_{L^2(\Omega_p)} \right). \tag{2.3.17} \]

which implies (2.2.7).

**Proof of (2.3.9).** First of all, let us introduce the following equations which are the discrete time derivative version of (2.3.1)–(2.3.3): for 1 \leq j \leq n - 1
\[
\left( \rho_f \frac{d_t u^j_{f,h} - d_t u^j}{\Delta t}, v_{f,h} \right)_{\Omega_f} + a_f(d_t u^j_{f,h}, v_{f,h}) + \left( \rho_f \frac{u^j_{f,h} \cdot \nabla u^j_{f,h} - u^j_{f,h} \cdot \nabla u^j_{f,h}}{\Delta t}, v_{f,h} \right)_{\Omega_f}
+ b_f(v_{f,h}, d_t p^j_{f,h}) + \left( \rho_p \frac{d_t \eta^j_{p,h} - 2d_t \eta^j_{p,h} + d_t \eta^j_{p,h}}{\Delta t}, \xi_{p,h} \right)_{\Omega_p} + a^e_p(d_t \eta^j_{p,h}, \xi_{p,h})
+ \alpha b_p(\xi_{p,h}, d_t p^j_{p,h}) + a^d_p(d_t u^j_{p,h}, v_{p,h}) + b_p(v_{p,h}, d_t p^j_{p,h}) + b_f(v_{f,h}, v_{p,h}, \eta_{p,h}, d_t \lambda^j_{h})
+ a_{BJS}(d_t u^j_{f,h}, \frac{d_t \eta^j_{p,h} - d_t \eta^j_{p,h}}{\Delta t}; v_{f,h}, \xi_{p,h}) = (d_t f^j_{f,h}, v_{f,h})_{\Omega_f} + (d_t f^j_{p,h}, \xi_{p,h})_{\Omega_p}. \tag{2.3.18} \]
First, we need to deal with the last term on the right hand side of the above equation.

Taking the test function \( v_{f,h} = d_t u_{j+1}^{f,h}, w_{f,h} = d_t p_{j+1}^{f,h}, \psi_{f,h} = \frac{\Omega_{p}}{\mu_{h}} \) in (2.3.18)–(2.3.20) and adding them together, we obtain the energy equality after combining with equation (2.3.7):

\[
\begin{align*}
\frac{1}{2} d_t \left( \rho_f d_t u_{j+1}^{f,h} \right)_{\Omega_f} + s_0 \left( d_t p_{j+1}^{f,h} \right)_{\Omega_p} + a_p^e (d_t \eta_{p,h}^{j+1}, d_t \eta_{p,h}^{j}) + \rho_p \frac{1}{2 \Delta t} \left( \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^{j}}{\Delta t} \right)_{\Omega_p} \\
- \frac{\Delta t}{2} \left( \rho_f \frac{d_t u_{j+1}^{f,h} - d_t u_{j}^{f,h}}{\Delta t} \right)_{\Omega_p} + a_p^e \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^{j}}{\Delta t} \left( \eta_{p,h}^{j+1} - \eta_{p,h}^{j} \right)_{\Omega_p} + a_f (d_t u_{j+1}^{f,h}, d_t u_{j}^{f,h}) \\
+ \frac{\Delta t}{2} \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^{j}}{\Delta t} \left( \eta_{p,h}^{j+1} - \eta_{p,h}^{j} \right)_{\Omega_p} + \frac{\Delta t}{2} \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^{j}}{\Delta t} \left( \eta_{p,h}^{j+1} - \eta_{p,h}^{j} \right)_{\Omega_p} \right)_{\Omega_f}.
\end{align*}
\]

First, we need to deal with the last term on the right hand side of the above equation.

Combining with (2.3.10), we have

\[
\begin{align*}
\left( \rho_f \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^{j}}{\Delta t} \right)_{\Omega_f} \\
= \left( \frac{\rho_f u_{j+1}^{f,h} \cdot \nabla u_{j+1}^{f,h} - u_{j+1}^{f,h} \cdot \nabla u_{j}^{f,h}}{\Delta t} \right)_{\Omega_f} + \left( \frac{\rho_f u_{j+1}^{f,h} \cdot \nabla u_{j+1}^{f,h} - u_{j+1}^{f,h} \cdot \nabla u_{j}^{f,h}}{\Delta t} \right)_{\Omega_f} \\
\leq \rho_f \| u_{j+1}^{f,h} \|_{L^4(\Omega_f)} \| \nabla u_{j+1}^{f,h} \|_{L^2(\Omega_f)} + \rho_f \| u_{j+1}^{f,h} \|_{L^4(\Omega_f)} \| \nabla u_{j+1}^{f,h} \|_{L^2(\Omega_f)} + \frac{\mu_f}{2} \| D(d_t u_{j+1}^{f,h}) \|_{L^2(\Omega_f)} + \frac{\mu_f}{2} \| D(d_t u_{j+1}^{f,h}) \|_{L^2(\Omega_f)} \\
\leq \frac{3 \mu_f}{4} \| D(d_t u_{j+1}^{f,h}) \|_{L^2(\Omega_f)} + \frac{\mu_f}{4} \| D(d_t u_{j+1}^{f,h}) \|_{L^2(\Omega_f)}. \end{align*}
\]
Note that we have the second order discrete time derivative. Summing (2.3.21) over \( j \) from 1 to \( n - 1 \), times \( 2\Delta t \) on the both side the resulting equation, and utilizing (2.3.22), we obtain the following inequality after using inf-sup condition:

\[
\begin{align*}
\rho_f &\left| \frac{d_t u_{f,h}^n}{\Delta t} \right|_L^2(\Omega) + s_0 \left| \frac{d_t p_{p,h}^n}{\Delta t} \right|_L^2(\Omega) + a_p \left| d_t \eta_{p,h}^n, d_t \eta_{p,h}^n \right|_p + \rho_p \left| \frac{d_t \eta_{p,h}^n - d_t \eta_{p,h}^{n-1}}{\Delta t} \right|_L^2(\Omega) \nonumber \\
&+ \left| \frac{d_t u_{f,h}^{j+1} - d_t u_{f,h}^j}{\Delta t} \right|_L^2(\Omega) + s_0 \left| \frac{d_t p_{p,h}^{j+1} - d_t p_{p,h}^j}{\Delta t} \right|_L^2(\Omega) \\
&+ a_p \left( \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^j}{\Delta t} + \frac{d_t \eta_{p,h}^{j+1} - d_t \eta_{p,h}^j}{\Delta t} \right) + \Delta t \sum_{j=1}^{n-1} \left| K \frac{1}{2} d_t u_{f,h}^{j+1} \right|_L^2(\Omega) \\
&+ \frac{\mu_f \Delta t}{2} \left| D \left( d_t u_{f,h}^n \right) \right|_L^2(\Omega) + \frac{\mu_f \Delta t}{2} \left| D \left( d_t u_{f,h}^{j+1} \right) \right|_L^2(\Omega) + \frac{k_{\min} \beta_p^2 \Delta t}{2} \sum_{j=1}^{n-1} \left| d_t p_{p,h}^j \right|_L^2(\Omega) \nonumber \\
&+ \frac{2 \Delta t}{k_{\min} \beta_p^2} \left| d_t q_{p,h}^{n+1} \right|_L^2(\Omega) + \rho_p \Delta t \sum_{n=1}^N \left| \frac{d_t \eta_{p,h}^{n+1} - d_t \eta_{p,h}^n}{\Delta t} \right|_L^2(\Omega). \quad (2.3.23)
\end{align*}
\]

The next step is to bound the extra terms at time step \( j = 1 \) on the right hand side of the above inequality. We utilize the initial conditions. Let us go back the original formulation (2.3.1)–(2.3.3) at \( j = 0 \). Instead, we take the test functions \( v_{f,h} = d_t u_{f,h}^n, w_{f,h} = \frac{p_{f,h}}{\Delta t}, v_{p,h} = d_t u_{p,h}^n, w_{p,h} = d_t p_{p,h}^n, \xi_{p,h} = \frac{d_t \eta_{p,h}^n - d_t \eta_{p,h}^0}{\Delta t}, \) and \( \mu_h = d_t \lambda_{h}^1 \). Since the homogeneous initial conditions \( u_{f,h}^0 = 0, u_{p,h}^0 = 0, \eta_{p,h}^0 = 0, \eta_{p,h}^{-1} = 0, p_{p,h} = 0, \) and \( \lambda_{h}^0 = 0 \), we have

\[
(\rho_f u_{f,h}^n \cdot \nabla u_{f,h}^n, u_{f,h}^n)_{\Omega_f} = 0, \quad b_f(d_t u_{f,h}^n, p_{f,h}^n) = b_f(u_{f,h}^n, \frac{p_{f,h}^n}{\Delta t}).
\]

\[
b_p(d_t u_{p,h}^n, p_{p,h}^n) = b_p(u_{p,h}^n, d_t p_{p,h}^n),
\]

\[
\alpha b_p \left( \frac{d_t \eta_{p,h}^n - d_t \eta_{p,h}^0}{\Delta t}, p_{p,h}^n \right) = \alpha b_p(d_t \eta_{p,h}^n, d_t p_{p,h}^n),
\]

\[
b_\Gamma(d_t u_{f,h}^n, d_t u_{p,h}^n, \frac{d_t \eta_{p,h}^n - d_t \eta_{p,h}^0}{\Delta t}, \lambda_h) = b_\Gamma \left( u_{f,h}^n, u_{p,h}^n, d_t \eta_{p,h}^n, d_t \lambda_h \right).
\]
Therefore, we have the following:

\[
\rho_f \|d_t u_{f,h}^1\|_{L^2(\Omega_t)}^2 + s_0 \|d_t p_{p,h}\|_{L^2(\Omega_t)}^2 + \alpha''(d_t \eta_{p,h}^1, d_t \eta_{p,h}^1) + \rho_p \|d_t \eta_{p,h}^1 - d_t \eta_{p,h}^0\|_{L^2(\Omega_t)}^2 \\
+ 2\mu_f \Delta t \|D(d_t u_{f,h}^1)\|_{L^2(\Omega_t)}^2 \\
\leq \frac{1}{\rho_f} \|f_f^1\|_{L^2(\Omega_t)}^2 + \frac{1}{\rho_p} \|f_p^1\|_{L^2(\Omega_t)}^2 + \frac{1}{s_0} \|q_p^1\|_{L^2(\Omega_t)}^2.
\]  
(2.3.24)

After using discrete Gronwall’s lemma 1.2.5 and combining with (2.3.23) and (2.3.24), we obtain (2.3.9).

**Proof of (2.3.10).** Utilizing (2.3.8) and (2.3.9), we show briefly the process to prove (2.3.10). Note that we assume that for \(0 \leq j \leq n - 1\), (2.3.10) holds. To complete the induction statement, we need to show for \(j = n\), and in particular \(j = 1\), (2.3.10) holds. Starting with (2.3.1) at time step \(n\) again, we take the test functions \(v_{f,h} = u_{f,h}^j\), \(w_{f,h} = p_{f,h}^n\), \(v_{p,h} = u_{p,h}^n\), \(w_{p,h} = p_{p,h}^n\), \(\xi_{p,h} = d_t \eta_{p,h}^n\), and \(\mu_h = \lambda_h^n\) to obtain

\[
2\mu_f \|D(u_{f,h}^n)\|_{L^2(\Omega_t)}^2 + \|K^{-1/2}u_{p,h}^n\|_{L^2(\Omega_t)}^2 \\
\leq (f_f^n, u_{f,h}^n)_{\Omega_t} + (f_p^n, d_t \eta_{p,h}^n)_{\Omega_t} + (q_p^n, p_{p,h}^n)_{\Omega_t} - (\rho_f u_{f,h}^{n-1} \cdot \nabla u_{f,h}^n, u_{f,h}^n)_{\Omega_t} + (\rho_f d_t u_{f,h}^n, u_{f,h}^n)_{\Omega_t} \\
- (\rho_p d_t \eta_{p,h}^n - d_t \eta_{p,h}^{n-1})_{\Omega_t} + \alpha''(\eta_{p,h}^n, d_t \eta_{p,h})_{\Omega_t} + (s_0 d_t p_{p,h}^n, p_{p,h}^n)_{\Omega_t} \\
\leq \frac{1}{2} \left( \rho_f \|u_{f,h}^n\|_{L^2(\Omega_t)}^2 + \rho_p \|d_t \eta_{p,h}^n\|_{L^2(\Omega_t)}^2 + (\alpha''(\eta_{p,h}^n, \eta_{p,h}^n) + s_0 \|p_{p,h}^n\|_{L^2(\Omega_t)}^2) \\
+ \frac{1}{2} \left( \rho_f \|d_t u_{f,h}^n\|_{L^2(\Omega_t)}^2 + \rho_p \|d_t \eta_{p,h}^n - d_t \eta_{p,h}^{n-1}\|_{L^2(\Omega_t)}^2 + \alpha''(d_t \eta_{p,h}^n, d_t \eta_{p,h}^n) \\
+ s_0 \|d_t p_{p,h}^n\|_{L^2(\Omega_t)}^2 \right) + \frac{p_f^2 K^2}{2 \mu_f} \|f_f^n\|_{L^2(\Omega_t)}^2 + \frac{\mu_f^2}{2} \|D(u_{f,h}^n)\|_{L^2(\Omega_t)}^2 + \frac{1}{2 \rho_p} \|f_p^n\|_{L^2(\Omega_t)}^2 \\
+ \frac{\rho_p}{2} \|d_t \eta_{p,h}^n\|_{L^2(\Omega_t)}^2 + \frac{2}{\beta_p^2 k_{min}} \|q_p^n\|_{L^2(\Omega_t)}^2 + \|p_{p,h}^n\|_{L^2(\Omega_t)}^2 \\
+ \rho_f \|D(u_{f,h}^{n-1})\|_{L^2(\Omega_t)}^2 \|D(u_{f,h}^n)\|_{L^2(\Omega_t)}^2.
\]  
(2.3.25)

Therefore, combining with (2.3.8) and (2.3.9), we obtain for \(n \geq 2\)

\[
\mu_f \|D(u_{f,h}^n)\|_{L^2(\Omega_t)}^2 \leq \exp(T) \left( \|C_1\|_{C^2(0,T)} + \frac{1}{2} \|C_2\|_{C^2(0,T)} + \frac{1}{2} C_4 \right) + \frac{1}{2} \|C_1\|_{L^\infty(0,T)}.  
\]  
(2.3.26)

On the other hand, multiplying (2.3.24) \(\Delta t\) on the both sides of the inequality, we have

\[
\mu_f \|D(u_{f,h}^1)\|_{L^2(\Omega_t)}^2 \leq \frac{\Delta t}{2 \rho_f} \|f_f^1\|_{L^2(\Omega_t)}^2 + \frac{\Delta t}{2 \rho_p} \|f_p^1\|_{L^2(\Omega_t)}^2 + \frac{\Delta t}{2 s_0} \|q_p^1\|_{L^2(\Omega_t)}^2.
\]
Combining (2.3.26) and (2.3.27) with the discrete small data condition (2.3.6), we obtain (2.3.10).

**Proof of existence.** Next, we prove the existence of the numerical solution, by using induction strategy. We provide the proof for existence of \(\{(u^{n+1}_{f,h}, p^{n+1}_{f,h}, u^{n+1}_{p,h}, P^{n+1}_{p,h}, \eta^{n+1}_{p,h}, \lambda^{n+1}_{h})\}_{n \geq 2}\) under the assumption that \(\{(u^{n}_{f,h}, p^{n}_{f,h}, u^{n}_{p,h}, P^{n}_{p,h}, \eta^{n}_{p,h}, \lambda^{n}_{h})\}_{n \geq 2}\) are given. By using the Brouwer’s fixed point Theorem, we can show the existence. The proof to show the existence of \(\{(u^{1}_{f,h}, p^{1}_{f,h}, u^{1}_{p,h}, P^{1}_{p,h}, \eta^{1}_{p,h}, \lambda^{1}_{h})\}\) is simpler and will be outlined later. We introduce a mapping \(\mathcal{F}_n\): \(V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h \rightarrow V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h\) defined for \((\tilde{u}_{f,h}, \tilde{p}_{f,h}, \tilde{u}_{p,h}, \tilde{P}_{p,h}, \tilde{\eta}_{p,h}, \tilde{\lambda}_h)\) as follows: for all \((v_{f,h}, p_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \mu_h) \in V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h,\)

\[
\exp\left(\frac{T}{2}\right) C_4.
\]  

(2.3.27)

\[
\mathcal{F}_n(\tilde{u}_{f,h}, \tilde{p}_{f,h}, \tilde{u}_{p,h}, \tilde{P}_{p,h}, \tilde{\eta}_{p,h}, \tilde{\lambda}_h, (v_{f,h}, p_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \mu_h))
\]

\[
= \left(\rho_f \frac{\tilde{u}_{f,h} - u^n_{f,h}}{\Delta t}, v_{f,h}\right)_{\Omega_f} + a_f(\tilde{u}_{f,h}, v_{f,h}) + (\rho_f u^n_{f,h} \cdot \nabla \tilde{u}_{f,h}, v_{f,h})_{\Omega_f} + b_f(v_{f,h}, \tilde{p}_{f,h})
\]

\[
+ \left(\rho_p \frac{\tilde{\eta}_{p,h} - 2\eta^n_{p,h} + \eta^{n-1}_{p,h}}{\Delta t}, \frac{\xi_{p,h} - \eta^n_{p,h}}{\Delta t}\right)_{\Omega_p} + a_p^d(\eta_{p,h}, \frac{\xi_{p,h} - \eta^n_{p,h}}{\Delta t})
\]

\[
+ \alpha b_p\left(\frac{\xi_{p,h} - \eta^n_{p,h}}{\Delta t}, \frac{p^{n+1}_{p,h}}{\Delta t}\right) + a_p^d(\tilde{u}_{p,h}, v_{p,h}) + b_p(v_{p,h}, \tilde{p}_{p,h}) + b_{\Gamma}(v_{f,h}, \tilde{v}_{p,h}, \frac{\xi_{p,h} - \eta^n_{p,h}}{\Delta t})_{\Omega_p}
\]

\[
+ a_{BJS}\left(\tilde{u}_{f,h}, \frac{\tilde{\eta}_{p,h} - \eta^n_{p,h}}{\Delta t}; v_{f,h}, \frac{\xi_{p,h} - \eta^n_{p,h}}{\Delta t}\right) + \left(\frac{\tilde{p}_{p,h} - p^n_{p,h}}{\Delta t}, w_{p,h}\right)_{\Omega_p}
\]

\[
- \alpha b_p\left(\frac{\tilde{\eta}_{p,h} - \eta^n_{p,h}}{\Delta t}, w_{p,h}\right) - b_p(\tilde{u}_{p,h}, w_{p,h}) - b_{\Gamma}\left(\tilde{u}_{f,h}, \tilde{u}_{p,h}, \frac{\tilde{\eta}_{p,h} - \eta^n_{p,h}}{\Delta t}, \mu_h\right)
\]

\[
- b_f(\tilde{u}_{f,h}, w_{f,h}) - (\tilde{f}^{n+1}_f, v_{f,h})_{\Omega_f} - \left(\tilde{f}^{n+1}_p, \frac{\xi_{p,h} - \eta^n_{p,h}}{\Delta t}\right)_{\Omega_p} - (q^{n+1}_p, w_{p,h})_{\Omega_p}.
\]  

(2.3.28)

\(\mathcal{F}_n\) is a well-defined and continuous map from \(V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h\) into themselves by Riesz representation Theorem. Based on the stability result (2.3.8), we have the following:

\[
\mathcal{F}_n(\tilde{u}_{f,h}, \tilde{p}_{f,h}, \tilde{u}_{p,h}, \tilde{P}_{p,h}, \tilde{\eta}_{p,h}, \tilde{\lambda}_h, (u^{1}_{f,h}, p^{1}_{f,h}, u^{1}_{p,h}, P^{1}_{p,h}, \eta^{1}_{p,h}, \lambda^{1}_{h}))
\]

\[
= \left(\rho_f \frac{\tilde{u}_{f,h} - u^0_{f,h}}{\Delta t}, \tilde{u}_{f,h}\right)_{\Omega_f} + a_f(\tilde{u}_{f,h}, \tilde{u}_{f,h}) + (\rho_f u^0_{f,h} \cdot \nabla \tilde{u}_{f,h}, \tilde{u}_{f,h})_{\Omega_f} + a_p^d(\tilde{u}_{p,h}, \tilde{u}_{p,h})
\]
The Brouwer's fixed point Theorem [17] implies that \( F \) with the radius defined to be
\[
\left| \frac{\tilde{u}_{f,h} - \frac{\eta_{p,h}^n}{\Delta t}}{\rho_p} - (f_{p+1}^n, \frac{\eta_{p,h}^n - \eta_{p,h}^{n-1}}{\Delta t})_{\Omega_p} - (q_{p+1}, \tilde{p}_{p,h})_{\Omega_p} \right| \geq \frac{\rho_f}{2\Delta t} \left\| \tilde{u}_{f,h} \right\|_{L^2(\Omega_t)}^2 + \mu_f \left\| D(\tilde{u}_{f,h}) \right\|_{L^2(\Omega_t)}^2 + \frac{\rho_p}{2\Delta t} \left\| \frac{\eta_{p,h}^n - \eta_{p,h}^{n-1}}{\Delta t} \right\|_{L^2(\Omega_p)}^2 + \frac{1}{\rho_p k_{\min}} \left\| \eta_p^0 \right\|_{L^2(\Omega_p)}^2.
\]
we conclude that \( F_n(\tilde{u}_{f,h}, \tilde{p}_{f,h}, \tilde{u}_{f,h}, \tilde{p}_{h}, \eta_{p,h}, \tilde{\lambda}_h) \geq 0 \) for all \((\tilde{u}_{f,h}, \tilde{p}_{f,h}, \tilde{u}_{f,h}, \tilde{p}_{p,h}, \eta_{p,h}, \tilde{\lambda}_h)\) such that
\[
\Delta t \left| \frac{\tilde{u}_{f,h} - \frac{\eta_{p,h}^n}{\Delta t}}{\rho_p} - (f_{p+1}^n, \frac{\eta_{p,h}^n - \eta_{p,h}^{n-1}}{\Delta t})_{\Omega_p} - (q_{p+1}, \tilde{p}_{p,h})_{\Omega_p} \right| \geq \frac{\rho_f}{2\Delta t} \left\| \tilde{u}_{f,h} \right\|_{L^2(\Omega_t)}^2 + \mu_f \left\| D(\tilde{u}_{f,h}) \right\|_{L^2(\Omega_t)}^2 + \frac{\rho_p}{2\Delta t} \left\| \frac{\eta_{p,h}^n - \eta_{p,h}^{n-1}}{\Delta t} \right\|_{L^2(\Omega_p)}^2 + \frac{1}{\rho_p k_{\min}} \left\| \eta_p^0 \right\|_{L^2(\Omega_p)}^2.
\]
with the radius defined to be
\[
R_n^2 = \rho_f \left\| \tilde{u}_{f,h} \right\|_{L^2(\Omega_t)}^2 + 2\mu_f \Delta t \left\| D(\tilde{u}_{f,h}) \right\|_{L^2(\Omega_t)}^2 + \rho_p \left\| \frac{\eta_{p,h}^n - \eta_{p,h}^{n-1}}{\Delta t} \right\|_{L^2(\Omega_p)}^2 + \frac{\rho_p}{\Delta t} \left\| \eta_p^0 \right\|_{L^2(\Omega_p)}^2.
\]
The Brouwer's fixed point Theorem implies that \( F_n \) has a zero, and we denote it by \((u_{f,h}^{n+1}, p_{f,h}^{n+1}, u_{p,h}^{n+1}, p_{p,h}^{n+1}, \eta_{p,h}^{n+1}, \lambda_{h}^{n+1})\), the existence of which is obtained for \( n \geq 1 \) by using induction. The remaining question is to show the existence of \((u_{f,h}^{1}, p_{f,h}^{1}, u_{p,h}^{1}, p_{p,h}^{1}, \eta_{p,h}^{1}, \lambda_{h}^{1})\).

We follow a similar procedure. Note that the mapping \( F_1 \) is defined differently since we need to cooperate with the discrete initial conditions: \( u_{f,h}^0 = 0, \eta_{p,h}^0 = 0, \eta_{p,h}^{-1} = 0, p_{p,h}^0 = 0, \) and \( \lambda_{h}^0 = 0 \). The mapping \( F_1 \) is defined as follows: for any \((v_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \mu_{h}) \in V_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_{h}, \)
\[
(F_1(\tilde{u}_{f,h}, \tilde{u}_{p,h}, \tilde{p}_{p,h}, \eta_{p,h}, \tilde{\lambda}_h), (v_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \mu_{h})) = (\rho_f \frac{\tilde{u}_{f,h}}{\Delta t}, v_{f,h} \Omega_t + a_f(\tilde{u}_{f,h}, v_{f,h}) + \langle \tilde{p}_{p,h}, v_{p,h} \rangle_{\Omega_p} + a_p^e(\tilde{\eta}_{p,h}, \xi_{p,h}) + b_f(v_{f,h}, \tilde{p}_{f,h}) \)
\]
between two solutions. Next, we utilize the induction strategy and assume that

Assume there are two solutions

Proof of uniqueness.

This inequality implies that the map \( F_1 \) has a zero in the ball of radius \( R_1 \) defined as

\[
R_1^2 := \frac{P_f^2 K_2^2 \Delta t}{\mu_f} \| f_1 \|^2_{L^2(\Omega_f)} + \frac{\Delta t^2}{2 \rho_p} \| f_p \|^2_{L^2(\Omega_p)} + \frac{2 \Delta t}{\rho_p k_{\min}} \| q_p \|^2_{L^2(\Omega_p)}.
\]  

(2.3.34)

**Proof of uniqueness.** Assume there are two solution \( \{(u_{f,1}, p_{f,1}, u_{p,1}, p_{p,1}, \eta_{p,1}, \lambda_{1,1})\} \) and \( \{(u_{f,2}, p_{f,2}, u_{p,2}, p_{p,2}, \eta_{p,2}, \lambda_{2,2})\} \). Let \( \{(\tilde{u}_{f,1}, \tilde{p}_{f,1}, \tilde{u}_{p,1}, \tilde{p}_{p,1}, \tilde{\eta}_{p,1}, \tilde{\lambda}_{1,1})\}_{i \geq 1} \) be the difference between two solutions. Next, we utilize the induction strategy and assume that \( \{(u_{f,j}, p_{f,j}, u_{p,j}, p_{p,j}, \eta_{p,j}, \lambda_{j,j})\}_{1 \leq j \leq n} \) is unique. Under such assumption, we prove the uniqueness of \( \{(u_{f,j+1}, p_{f,j+1}, u_{p,j+1}, p_{p,j+1}, \eta_{p,j+1}, \lambda_{j+1})\} \). Therefore, we have for any \( (v_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \mu_h) \in V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h \), we have

\[
(\rho_f \frac{\tilde{u}_{f,h}^{n+1}}{\Delta t}, v_{f,h})_{\Omega_f} + a_f(\tilde{u}_{f,h}^{n+1}, v_{f,h}) + (\rho_f \mathbf{u}_{f,h}^{n+1} \cdot \nabla \hat{u}_{f,h}^{n+1}, v_{f,h})_{\Omega_f} + (\rho_p \frac{\eta_{p,h}^{n+1}}{\Delta t}, \xi_{p,h})_{\Omega_p} + a_p(\tilde{\eta}_{p,h}^{n+1}, \xi_{p,h}) + b_p(v_{p,h}, \tilde{p}_{p,h}^{n+1}) + a_p^{ij}(\tilde{u}_{p,h}^{n+1}, \hat{u}_{p,h}^{n+1}, \hat{v}_{h}^{n+1}) + a_p^{ij}(\tilde{u}_{p,h}^{n+1}, \hat{u}_{p,h}^{n+1}, \hat{v}_{h}^{n+1}) = 0,
\]  

(2.3.35)

\[
(s_0 \frac{\tilde{p}_{p,h}^{n+1}}{\Delta t}, w_{p,h})_{\Omega_p} - \alpha b_p(\frac{\eta_{p,h}^{n+1}}{\Delta t}, \hat{v}_{h}^{n+1}) - b_p(\tilde{u}_{p,h}^{n+1}, w_{p,h}) - b_f(\tilde{u}_{f,h}^{n+1}, w_{f,h}) = 0,
\]  

(2.3.36)

\[
b_f(\tilde{u}_{f,h}^{n+1}, \tilde{u}_{p,h}^{n+1}, \frac{\tilde{\eta}_{p,h}^{n+1}}{\Delta t}; \mu_h) = 0.
\]  

(2.3.37)
We take test functions to be $v_{f,h} = \tilde{u}_{n+1}^{f,h}$, $w_{f,h} = \tilde{p}_{n+1}^{f,h}$, $v_{p,h} = \tilde{u}_{n+1}^{p,h}$, $w_{p,h} = \tilde{p}_{n+1}^{p,h}$, $\xi_{p,h} = \tilde{\eta}_{n+1}^{p,h}/\Delta t$, and $\mu_h = \tilde{\lambda}_{n+1}$ to have
\[
\rho_f \frac{\Delta t}{2} \| \tilde{u}_{f,h}^{n+1} \|^2_{L^2(\Omega_f)} + a_f(\tilde{u}_{f,h}^{n+1}, \tilde{u}_{f,h}^{n+1}) + \rho_p \frac{\Delta t}{2} \| \tilde{\eta}_{p,h}^{n+1} \|^2_{L^2(\Omega_p)} + \frac{1}{\Delta t} a_p^*(\tilde{\eta}_{p,h}^{n+1}) + \tilde{\eta}_{p,h}^{n+1} \eta_{p,h}^{n+1} \\
+ a_p^d(\tilde{u}_{p,h}^{n+1}, \tilde{u}_{p,h}^{n+1}) + |\tilde{u}_{f,h}^{n+1} - \tilde{\eta}_{p,h}^{n+1} \Delta t|^2_{a_{BJS}} + \frac{s_0}{\Delta t} \| \tilde{p}_{p,h}^{n+1} \|^2_{L^2(\Omega_p)} \\
= - (\rho_f u_{f,h}^n \cdot \nabla \tilde{u}_{f,h}^{n+1}, \tilde{u}_{f,h}^{n+1})_{\Omega_f}. \tag{2.3.38}
\]

For the right hand side of the above equation, we use the inequality (2.3.10) to obtain
\[
(\rho_f u_{f,h}^n \cdot \nabla \tilde{u}_{f,h}^{n+1}, \tilde{u}_{f,h}^{n+1})_{\Omega_f} \leq \frac{\mu_f}{2} \| D(\tilde{u}_{f,h}^{n+1}) \|^2_{L^2(\Omega_f)}, \tag{2.3.39}
\]
which can be hidden to the left hand side. We conclude that $u_{f,h}^{n+1} = 0$, $u_{p,h}^{n+1} = 0$, $p_{p,h}^{n+1} = 0$, and $\eta_{p,h}^{n+1} = 0$. Then, after combining with inf-sup condition (2.2.4) and (2.3.1), we obtain the uniqueness of $\lambda_{n+1}$ and $p_{f,h}^{n+1}$. The remaining step is to ensure the uniqueness of $(u_{f,h}^1, p_{f,h}^1, u_{p,h}^1, p_{p,h}^1, \eta_{p,h}^1, \lambda_{h}^1)$. Note that since $u_{f,h}^0 = 0$, we have
\[
(\rho_f u_{f,h}^0 \cdot \nabla \tilde{u}_{f,h}^1, \tilde{u}_{f,h}^1)_{\Omega_f} = 0. \tag{2.3.40}
\]
Therefore, we have
\[
\rho_f \frac{\Delta t}{2} \| \tilde{u}_{f,h}^1 \|^2_{L^2(\Omega_f)} + a_f(\tilde{u}_{f,h}^1, \tilde{u}_{f,h}^1) + \rho_p \frac{\Delta t}{2} \| \tilde{\eta}_{p,h}^1 \|^2_{L^2(\Omega_p)} + \frac{1}{\Delta t} a_p^*(\tilde{\eta}_{p,h}^1) + \tilde{\eta}_{p,h}^1 \eta_{p,h}^1 \\
+ a_p^d(\tilde{u}_{p,h}^1, \tilde{u}_{p,h}^1) + |\tilde{u}_{f,h}^1 - \tilde{\eta}_{p,h}^1 \Delta t|^2_{a_{BJS}} + \frac{s_0}{\Delta t} \| \tilde{p}_{p,h}^1 \|^2_{L^2(\Omega_p)} = 0. \tag{2.3.41}
\]
We complete the proof of uniqueness.

\[\square\]

The following Theorem is a direct result from the proof of Theorem 2.3.1, (2.3.8), (2.3.9), and (2.3.10). To avoid duplication, we omit the proof.
Theorem 2.3.2. Under the assumption in Theorem 2.3.1, the solution \( \{(u_{j,h}^n, p_{j,h}^n, u_{p,h}^n, p_{p,h}^n, \eta_{p,h}^n, \lambda_{p,h}^n)\}_{n \geq 1} \) of the fully discrete problem (FDNS) satisfies:

\[
\begin{align*}
\rho_f \| u_{f,h} \|_{L^\infty(0,T;L^2(\Omega_f))}^2 &+ \mu_f \| D(u_{f,h}) \|_{L^2(0,T;L^2(\Omega_f))}^2 + |u_{f,h} - d_t \eta_{p,h}|_{L^2(0,T;a_{BJS})}^2 \\
&+ \rho_p \| d_t \eta_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))}^2 + 2\| \mu_p^{1/2} D(\eta_{p,h}) \|_{L^\infty(0,T;L^2(\Omega_p))}^2 + \| \lambda_p^{1/2} \nabla \cdot \eta_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))}^2 \\
&+ s_0 \| p_{p,h} \|_{L^2(0,T;W_p)}^2 + \| K^{-1/2} u_{p,h} \|_{L^2(0,T;L^2(\Omega_p))}^2 + \frac{k_{\min}^2}{2} \| d_t p_{p,h} \|_{L^2(0,T;W_p)}^2 + \frac{k_{\min}^2}{2} \| d_t \lambda_{p,h} \|_{L^2(0,T;\Lambda)}^2 \\
&\leq \exp(T) \| C_1 \|_{L^2(0,T)} + C_4.
\end{align*}
\]

Furthermore, it holds that

\[
\begin{align*}
\rho_f \| d_t u_{f,h} \|_{L^\infty(0,T;L^2(\Omega_f))}^2 &+ \mu_f \| D(d_t u_{f,h}) \|_{L^2(0,T;L^2(\Omega_f))}^2 + |d_t u_{f,h} - d_t \eta_{p,h}|_{L^2(0,T;a_{BJS})}^2 \\
&+ \rho_p \| d_t \eta_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))}^2 + 2\| \mu_p^{1/2} D(d_t \eta_{p,h}) \|_{L^\infty(0,T;L^2(\Omega_p))}^2 + \| \lambda_p^{1/2} \nabla \cdot d_t \eta_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))}^2 \\
&+ s_0 \| d_t p_{p,h} \|_{L^2(0,T;W_p)}^2 + \| K^{-1/2} d_t u_{p,h} \|_{L^2(0,T;L^2(\Omega_p))}^2 + \frac{k_{\min}^2}{2} \| d_t p_{p,h} \|_{L^2(0,T;W_p)}^2 \\
&+ k_{\min}^2 \| d_t \lambda_{p,h} \|_{L^2(0,T;\Lambda)}^2 \\
&\leq \exp(T) \left( \| C_2 \|_{L^2(0,T)} + C_4 \right).
\end{align*}
\]

\[
\| D(u_{f,h}) \|_{L^\infty(0,T;L^2(\Omega_f))} \leq \frac{\mu_f}{2 \rho_f S_f^2 K_f^3}.
\]

\[
\| p_{f,h} \|_{L^2(0,T;L^2(\Omega_f))} \leq \frac{1}{\beta_f} \left( \rho_f \| d_t u_{f,h} \|_{L^2(0,T;L^2(\Omega_f))} + 4 \| D(u_{f,h}) \|_{L^2(0,T;L^2(\Omega_f))} + \sigma \| \lambda_{p,h} \|_{L^2(0,T;\Lambda)} \\
+ 2 K_p \| D(d_t \eta_{p,h}) \|_{L^2(0,T;L^2(\Omega_p))} + \| f_f \|_{L^2(0,T;L^2(\Omega_f))} \right).
\]
2.4 Error analysis

In this section, we analyze the error due to discretization both in space and time. We denote by $k_f$ and $s_f$ the degrees of polynomials in the spaces $V_{f,h}$ and $W_{f,h}$, respectively. Let $k_p$ and $s_p$ be the degrees of polynomials in the spaces $V_{p,h}$ and $W_{p,h}$, respectively. Let $k_s$ be the polynomial degree in $X_{p,h}$. Note that in this section, we use non-conforming finite element space for the Lagrange multiplier space $\tilde{\Lambda}_h$, as shown in (2.4.1). In this section, instead of using conforming finite element space $\Lambda_h$, we utilize the following non-conforming discrete Lagrange multiplier space [3]:

$$\tilde{\Lambda}_h = V_{p,h} \cdot n_p|_{\Gamma_{fp}}, \quad (2.4.1)$$

which consists of discontinuous piecewise polynomials. We equip $\tilde{\Lambda}_h$ with the norm $\|\nu_h\|_{\tilde{\Lambda}_h}^2 = \|\nu_h\|^2_{L^2(\Gamma_{fp})} + |\nu_h|_{\tilde{\Lambda}_h}^2$. $|\nu_h|_{\tilde{\Lambda}_h}$ is a semi-norm [1, 3], defined in $\tilde{\Lambda}_h$ as

$$|\nu_h|_{\tilde{\Lambda}_h}^2 = a_p^d(u_{ph}^*(\nu_h), u_{ph}^*(\nu_h)),$$

where $(u_{ph}^*(\nu_h), p_{ph}^*(\nu_h)) \in V_{p,h} \times W_{p,h}$ is the mixed finite element solution to a Darcy problem with Dirichlet data $\nu_h$ on the interface $\Gamma_{fp}$. The norm $\|\nu_h\|_{\tilde{\Lambda}_h}$ can be viewed as a discrete version of the $H^{1/2}$-norm. Note that $H^{1/2}(\Gamma_{fp})$ is dense in $L^2(\Gamma_{fp})$. Therefore, in Theorem 2.2.1, (2.2.23)–(2.2.24) also hold for the non-conforming space $\tilde{\Lambda}_h = V_{p,h} \cdot n_p|_{\Gamma_{fp}}$, equipped with $L^2(\Gamma_{fp})$ norm. The necessity of changing into non-conforming space $\tilde{\Lambda}_h$ is to simplify the following error estimates.

2.4.1 Preliminaries

Let $Q_{f,h}$, $Q_{p,h}$ and $Q_{\lambda,h}$ be the $L^2$-projection operators onto $W_{f,h}$, $W_{p,h}$, and $\tilde{\Lambda}_h$, respectively, satisfying:

\begin{align*}
(p_f - Q_{f,h}p_f, w_{f,h})_{\Omega_f} &= 0, \quad \forall w_{f,h} \in W_{f,h}, \quad (2.4.2) \\
(p_p - Q_{p,h}p_p, w_{p,h})_{\Omega_p} &= 0, \quad \forall w_{p,h} \in W_{p,h}, \quad (2.4.3) \\
(\lambda - Q_{\lambda,h}\lambda, \mu_h)_{\Gamma_{fp}} &= 0, \quad \forall \mu_h \in \tilde{\Lambda}_h. \quad (2.4.4)
\end{align*}
These operators have the approximation properties [33]:

\[ \|p_f - Q_{ph}p_f\|_{L^2(\Omega_f)} \leq Ch^{r_f} \|p_f\|_{H^{r_f}(\Omega_f)}, \quad 0 \leq r_f \leq s_f + 1, \]  
\[ \|p_p - Q_{ph}p_p\|_{L^2(\Omega_p)} \leq Ch^{r_p} \|p_p\|_{H^{r_p}(\Omega_p)}, \quad 0 \leq r_p \leq s_p + 1, \]  
\[ \|\lambda - Q_{ph}\lambda\|_{L^2(\Gamma_{fp})} \leq Ch^{r_{\lambda}} \|\lambda\|_{H^{r_{\lambda}}(\Gamma_{fp})}, \quad 0 \leq r_{\lambda} \leq k_p + 1. \]  

Since we choose the discrete Lagrange multiplier space to be \( \tilde{\Lambda}_h = V_{p,h} \cdot n_p |_{\Gamma_{fp}} \), we have for any \( v_{p,h} \in V_{p,h} \),

\[ \langle \lambda - \lambda_h, v_{p,h} \cdot n_p \rangle_{\Gamma_{fp}} = 0. \]  
(2.4.8)

Therefore, we have

\[ \|\lambda_h - Q_{ph}\lambda\|_{\tilde{\Lambda}_h} = \|\lambda_h - Q_{ph}\lambda\|_{L^2(\Gamma_{fp})}. \]  
(2.4.9)

Next, we consider a Stokes-like projection operator [3], \((S_{fh}, R_{fh}) : V_f \to V_{f,h} \times W_{f,h}\), defined for all \( v_f \in V_f \) by

\[ a_f(S_{fh}v_f, v_{f,h}) - b_f(v_{f,h}, R_{fh}v_f) = a_f(v_f, v_{f,h}), \quad \forall v_{f,h} \in V_{f,h}, \]
\[ b_f(S_{fh}v_f, w_{f,h}) = b_f(v_f, w_{f,h}), \quad \forall w_{f,h} \in W_{f,h}. \]

Let \( \Pi_{ph} \) be the MFE interpolant onto \( V_{p,h} \) satisfying for all \( v_p \in H^1(\Omega_p) \)[16],

\[ (\nabla \cdot \Pi_{ph}v_p, w_{p,h}) = (\nabla \cdot v_p, w_{p,h}), \quad \forall w_{p,h} \in W_{p,h}, \]  
(2.4.10)
\[ \langle \Pi_{ph}v_p \cdot n_p, v_{p,h} \cdot n_p \rangle_{\Gamma_{fp}} = \langle v_p \cdot n_p, v_{p,h} \cdot n_p \rangle_{\Gamma_{fp}}, \quad \forall v_{p,h} \in V_{p,h}, \]  
(2.4.11)
\[ \|v_p - \Pi_{ph}v_p\|_{L^2(\Omega_p)} \leq Ch^{r_{kp}} \|v_p\|_{H^{r_{kp}}(\Omega_p)}, \quad 1 \leq r_{kp} \leq k_p + 1. \]  
(2.4.12)

Let \( S_{sh} \) be the Scott-Zhang interpolant from \( X_p \) onto \( X_{p,h} \), satisfying [79]:

\[ \|\xi_p - S_{sh}\xi_p\|_{L^2(\Omega_p)} + h |\xi_p - S_{sh}\xi_p|_{H^1(\Omega_p)} \leq C h^{r_{ks}} \|\xi_p\|_{H^{r_{ks}}(\Omega_p)}, \quad 1 \leq r_{ks} \leq k_s + 1. \]  
(2.4.13)

Next, consider an operator onto the space that satisfies the weak continuity of normal velocity condition. Let

\[ U = \{(v_f, v_p, \xi_p) \in V_f \times H^1(\Omega_p) \times X_p : \quad b_T(v_f, v_p, \xi_p; \mu) = 0, \forall \mu \in \Lambda\}. \]
Consider its discrete analog
\[ \mathbf{U}_h = \left\{ (v_{f,h}, v_{p,h}, \xi_{p,h}) \in \mathbf{V}_{f,h} \times \mathbf{V}_{p,h} \times \mathbf{X}_{p,h} : b_T(v_{f,h}, v_{p,h}, \xi_{p,h}; \mu_h) = 0, \forall \mu_h \in \tilde{\Lambda}_h \right\}. \]

An interpolation operator \( I_h : \mathbf{U} \to \mathbf{U}_h \) is constructed in [3, Section 5] as a triple
\[ I_h(v_f, v_p, \xi_p) = (I_{fh}v_f, I_{ph}v_p, I_{sh}\xi_p), \]

where \( I_{fh} = S_{fh}, I_{ph} \) is based on a correction of \( \Pi_{ph} \) designed to satisfy the continuity of normal velocity. The interpolant \( I_h \) has the following properties:

\begin{align*}
\quad b_T(I_{fh}v_f, I_{ph}v_p, I_{sh}\xi_p; \mu_h) = 0, & \forall \mu_h \in \tilde{\Lambda}_h, \quad (2.4.14) \\
\quad b_f(I_{fh}v_f - v_f, w_{f,h}) = 0, & \forall w_{f,h} \in W_{f,h}, \quad (2.4.15) \\
\quad b_p(I_{ph}v_p - v_p, w_{p,h}) = 0, & \forall w_{p,h} \in W_{p,h}. \quad (2.4.16)
\end{align*}

The approximation properties of the component of \( I_h \) are established in [3, Lemma 5.1]: for all sufficiently smooth \( v_f, v_p, \) and \( \xi_p, \)

\begin{align*}
\|v_f - I_{fh}v_f\|_{H^1(\Omega_f)} & \leq Ch^{r_{k_f}} \|v_f\|_{H^{r_{k_f}+1}(\Omega_f)}, \quad 0 \leq r_{k_f} \leq k_f. \quad (2.4.17) \\
\|v_p - I_{ph}v_p\|_{L^2(\Omega_p)} & \leq C \left(h^{r_{k_p}} \|v_p\|_{H^{r_{k_p}+1}(\Omega_p)} + h^{r_{k_f}} \|v_f\|_{H^{r_{k_f}+1}(\Omega_f)} + h^{r_{k_s}} \|\xi_p\|_{H^{r_{k_s}+1}(\Omega_p)} \right), \\
& \quad 1 \leq r_{k_p} \leq k_p + 1, 0 \leq r_{k_f} \leq k_f, 0 \leq r_{k_s} \leq k_s. \quad (2.4.18) \\
\|\nabla \cdot (v_p - I_{ph}v_p)\|_{L^2(\Omega_p)} & \leq Ch^{r_{k_p}} \|v_p\|_{H^{r_{k_p}+1}(\Omega_p)}, \quad 0 \leq r_{k_p} \leq k_p. \quad (2.4.19) \\
\|\xi_p - I_{sh}\xi_p\|_{L^2(\Omega_p)} + h\|\xi_p - I_{sh}\xi_p\|_{H^1(\Omega_p)} & \leq Ch^{r_{k_s}} \|\xi_p\|_{H^{r_{k_s}+1}(\Omega_p)}, \quad 1 \leq r_{k_s} \leq k_s + 1. \quad (2.4.20)
\end{align*}
2.4.2 Error estimates

We proceed with estimating the error between (FDNS) and (LMVF1). We use the same notations as in Section 2.3, $\Delta t = T/N$, $t_n = n\Delta t$, $0 \leq n \leq N$ and $d_t u^n = (u^n - u^{n-1})/\Delta t$. We introduce the errors for all variables at $t_n$ and split them into approximation and discretization errors:

\[ E_f^n := u_f^n - u_{f,h}^n = (u_f^n - I_{fh} u_f^n) + (I_{fh} u_f^n - u_{f,h}^n) = \chi_f^n + \phi_{f,h}^n, \]
\[ E_p^n := u_p^n - u_{p,h}^n = (u_p^n - I_{ph} u_p^n) + (I_{ph} u_p^n - u_{p,h}^n) = \chi_p^n + \phi_{p,h}^n, \]
\[ E_s^n := \eta_p^n - \eta_{s,h}^n = (\eta_p^n - I_{sh} \eta_p^n) + (I_{sh} \eta_p^n - \eta_{s,h}^n) = \chi_s^n + \phi_{s,h}^n, \]
\[ E_{f_p}^n := p_f^n - p_{f,h}^n = (p_f^n - Q_{fh} p_f^n) + (Q_{fh} p_f^n - p_{f,h}^n) = \chi_{f,p}^n + \phi_{f,p,h}^n, \]
\[ E_{p_p}^n := p_p^n - p_{p,h}^n = (p_p^n - Q_{ph} p_p^n) + (Q_{ph} p_p^n - p_{p,h}^n) = \chi_{p,p}^n + \phi_{p,p,h}^n, \]
\[ E_{\lambda}^n := \lambda^n - \lambda_{h}^n = (\lambda^n - Q_{\lambda h} \lambda^n) + (Q_{\lambda h} \lambda^n - \lambda_{h}^n) = \chi_{\lambda}^n + \phi_{\lambda,h}^n. \]

Note that the initial conditions $u_f(0) = 0$, $u_p(0) = 0$, $\eta_p(0) = 0$, $\partial_t \eta_p(0) = 0$, and $\lambda(0) = 0$. Since $Q_{ph}$ and $Q_{\lambda h}$ are $L^2$-projection operators, we have $\chi_{f,p}^0 = 0$ and $\chi_{\lambda}^0 = 0$. According to the definition of the interpolation operators $I_{h}$, we have $\chi_{f}^0 = 0$, $\chi_{p}^0 = 0$, and $\chi_{s}^0 = 0$. Therefore, we have $\phi_{f,h}^0 = 0$, $\phi_{p,h}^0 = 0$, $\phi_{s,h}^0 = 0$, $\phi_{p,p,h}^0 = 0$, and $\phi_{\lambda,h}^0 = 0$. We especially have $\phi_{s,h}^{-1} = 0$ to estimate the second order discrete time derivative.

**Theorem 2.4.1.** Assume enough smoothness of the solution of (LMWF1). Assume the small data condition (2.2.22) and its discrete version (2.3.6) hold. There exists a constant $C$ independent of the mesh size $h$ and time step $\Delta t$ such that

\[
\rho_f \|\phi_{f,h}^N\|_{L^2(\Omega_f)}^2 + \frac{\mu_f \Delta t}{4} \|D(\phi_{f,h}^N)\|_{L^2(\Omega_f)}^2 + \frac{\mu_f \Delta t}{4} \sum_{n=0}^{N-1} \|D(\phi_{f,h}^{n+1})\|_{L^2(\Omega_f)}^2 + \frac{c^e \Delta t}{2} \sum_{n=0}^{N-1} \|\phi_{p,h}^{n+1}\|_{L^2(\Omega_p)}^2
\]
\[+ \frac{\beta_p^2 \Delta t}{3C} \sum_{n=0}^{N-1} \left( \|\phi_{p,h}^{n+1}\|_{L^2(\Omega_p)}^2 + \|\phi_{\lambda,h}^{n+1}\|_{L_\lambda}^2 \right) + \frac{1}{2} \|d_t \phi_{s,h}^N\|_{L^2(\Omega_p)}^2 + \rho_p \|d_t \phi_{s,h}^N\|_{L^2(\Omega_p)}^2
\]
\[+ s_0 \|\phi_{s,h}^N\|_{L^2(\Omega_p)}^2 + \Delta t \sum_{n=0}^{N-1} |\phi_{f,h}^{n+1} - d_t \phi_{s,h}^{n+1}|_{a_{BJS}}^2 + \Delta t^2 \sum_{n=0}^{N-1} \left( \rho_f \|d_t \phi_{f,h}^{n+1}\|_{L^2(\Omega_f)}^2
\]
\[+ \rho_p \|d_t \phi_{s,h}^{n+1} - d_t \phi_{p,h}^n\|_{L^2(\Omega_p)}^2 + s_0 \|d_t \phi_{p,h}^n\|_{L^2(\Omega_p)}^2 + c^e (d_t \phi_{s,h}^{n+1}, d_t \phi_{s,h}^{n+1}) \right)
\]
\[\leq C \exp(T) \left( h_{\min}^{2k_f,2k_p + 2,2s_p,2,2k_s} + \Delta t^2 \right). \]
Proof. Subtracting (2.3.1)–(2.3.2) from (2.1.22)–(2.1.23) at \( t_{n+1} \) and summing the two equations, we obtain the error equation

\[
(\rho_f \partial_t u_f^{n+1} - \rho_f u_f^{n+1}, v_{f,h}) \alpha_j + a_f(E_{f,h}^{n+1}, v_{f,h}) + a_p(E_s^{n+1}, \xi_{p,h}) + a_p(E_p^{n+1}, v_{p,h}) \\
+ (\rho_f u_f^{n+1} \cdot \nabla u_f^{n+1} - \rho_f u_{f,h}^{n+1} \cdot \nabla u_{f,h}^{n+1}, v_{f,h}) \alpha_j + (\rho_p \partial_t \eta_p - \rho_p \eta_p^{n+1} - 2 \eta_p^{n+1} + \eta_p^{n-1}, \xi_{p,h}) \alpha_p \\
+ \alpha_b(p_t \xi_{p,h}, E_{pp}^{n+1}) + b_f(v_{f,h}, E_{fp}^{n+1}) + b_p(v_{p,h}, E_{pp}^{n+1}) + b_T(v_{f,h}, v_{p,h}, \xi_{p,h}; E_{\chi}^{n+1}) \\
+ (s_0 \partial_t p_p^{n+1} - s_0 P_{p,h}^{n+1} - P_{p,h}^{n}, w_{p,h}) \alpha_p + a_{BJS}(E_f^{n+1}, \partial_t \eta_p^{n+1} - \eta_p^{n+1}, \xi_{f,h}, \xi_{p,h}) \\
- \alpha_b(p \partial_t \eta_p^{n+1} - \eta_p^{n+1}, \eta_p^{n+1} - \eta_p^{n-1}, \xi_{p,h}) \alpha_p \\
= 0. \quad (2.4.23)
\]

In above equation, taking \( v_{f,h} = \phi_{f,h}^{n+1}, w_{f,h} = \phi_{fp,h}^{n+1}, v_{p,h} = \phi_{p,h}^{n+1}, w_{p,h} = \phi_{pp,h}^{n+1}, \) and \( \xi_{p,h} = (\phi_{s,h}^{n+1} - \phi_{s,h}^{n}) / \Delta t, \) we obtain

\[
(\rho_f \partial_t u_f^{n+1} - \rho_f u_f^{n+1}, \phi_{f,h}^{n+1}) \alpha_j + a_f(\phi_{f,h}^{n+1}, \phi_{f,h}^{n+1}) + \rho_p(\phi_{s,h}^{n+1} - 2 \phi_{s,h}^{n} + \phi_{s,h}^{n-1}, \phi_{s,h}^{n+1} - \phi_{s,h}^{n}) \alpha_p \\
+ a_p^e(\phi_{s,h}^{n+1}, \phi_{s,h}^{n+1}) + a_p^d(\phi_{p,h}^{n+1}, \phi_{p,h}^{n+1}) + |\phi_{f,h}^{n+1} - \phi_{f,h}^{n}|^2 a_{BJS} \\
+ (s_0 \partial_t p_p^{n+1} - s_0 P_{p,h}^{n+1} - P_{p,h}^{n}, w_{p,h}) \alpha_p + b_f(\chi_f^{n+1}, \phi_{fp,h}^{n+1}) \\
- \rho_p(\partial_t \eta_p^{n+1} - \eta_p^{n+1}, \eta_p^{n+1} - \eta_p^{n-1}, \phi_{pp,h}^{n+1}) \alpha_p + b_T(\phi_{f,h}^{n+1}, \phi_{p,h}^{n+1}, \phi_{s,h}^{n+1} - \phi_{s,h}^{n}) \alpha_p \\
+ \alpha_b(p \partial_t \eta_p^{n+1} - \eta_p^{n+1}, \phi_{pp,h}^{n+1}) \alpha_p - b_T(\phi_{f,h}^{n+1}, \phi_{p,h}^{n+1}, \phi_{s,h}^{n+1} - \phi_{s,h}^{n}) \alpha_p \\
- a_{BJS}(\chi_f^{n+1}, \partial_t \eta_p^{n+1} - \eta_p^{n+1}, \phi_{fp,h}^{n+1}, \phi_{s,h}^{n+1} - \phi_{s,h}^{n}) \alpha_p \\
- s_0 (\partial_t p_p^{n+1} - \eta_p^{n+1}, \phi_{pp,h}^{n+1}) \alpha_p - b_f(\chi_f^{n+1}, \phi_{fp,h}^{n+1}) = 0. \quad (2.4.24)
\]

The following terms can be simplified, due to the projection operator, shown in (2.4.2)–(2.4.4) and (2.4.14)–(2.4.16)

\[
b_f(\chi_f^{n+1}, \phi_{fp,h}^{n+1}) = b_p(\chi_p^{n+1}, \phi_{pp,h}^{n+1}) = b_p(\phi_{p,h}^{n+1}, \chi_{pp}) = 0.
\]
Combining with (2.2.22) and (2.3.6), we have

\[
 b_t(\phi_{f,h}^{n+1}, \phi_{p,h}^{n+1} : \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^n}{\Delta t} ; \phi_{\lambda,h}^n) = 0,
\]

\[
 b_t(\phi_{f,h}^{n+1}, \phi_{p,h}^{n+1} : \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^n}{\Delta t} ; \chi_{\lambda}^n) = \left( \frac{\phi_{f,h}^{n+1} \cdot n_f + \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^n}{\Delta t} \cdot n_p, \chi_{\lambda}^n}{\Gamma_{fp}} \right).
\]

We deal with the terms only containing \(\phi_{f,h}^{n+1}\) on the right hand side of the above equation.

By using Taylor expansion, for some \(t_1^{n+1} \in [t^n, T]\), we have the following:

\[
 (\rho_f \partial_t u_f^{n+1} - \rho_f \frac{I_{fh} u_{f}^{n+1} - I_{fh} u_f^n}{\Delta t}, \phi_{f,h}^{n+1})_{\Omega_f} \\
 = (\rho_f \partial_t u_f^{n+1} - \frac{u_f^{n+1} - u_f^n}{\Delta t}, \phi_{f,h}^{n+1})_{\Omega_f} + (\rho_f \frac{\chi_{f}^{n+1} - \chi_f^n}{\Delta t}, \phi_{f,h}^{n+1})_{\Omega_f} \\
 = (\rho_f \frac{\Delta t}{2} \partial_t u_f(t_1^{n+1}), \phi_{f,h}^{n+1})_{\Omega_f} + (\rho_f \frac{\chi_{f}^{n+1} - \chi_f^n}{\Delta t}, \phi_{f,h}^{n+1})_{\Omega_f}. \tag{2.4.25}
\]

Therefore, after using Cauchy-Schwarz and Young’s inequalities, we have

\[
 - (\rho_f \partial_t u_f^{n+1} - \rho_f \frac{I_{fh} u_{f}^{n+1} - I_{fh} u_f^n}{\Delta t}, \phi_{f,h}^{n+1})_{\Omega_f} - a_f(\chi_f^{n+1}, \phi_{f,h}^{n+1}) - b_f(\phi_{f,h}^{n+1}, \chi_{fp}^{n+1}) \\
 - (\phi_{f,h}^{n+1} \cdot n_f, \chi_{\lambda}^{n+1})_{\Gamma_{fp}} \\
 \leq \frac{\mu_f}{4} \|D(\phi_{f,h}^{n+1})\|_{L^2(\Omega_f)}^2 + C \left( \left\| \Delta t \partial_t u_f(t_1^{n+1}) \right\|_{L^2(\Omega_f)}^2 + \left\| \partial_t \chi_f^{n+1} \right\|_{L^2(\Omega_f)}^2 + \left\| \chi_{fp}^{n+1} \right\|_{L^2(\Omega_f)}^2 \\
 + \left\| \chi_{\lambda}^{n+1} \right\|_{L^2(\Omega_f)}^2 \right). \tag{2.4.26}
\]

For the skew term containing \(\phi_{f,h}^{n+1}\), and again using Taylor expansion, we have for some \(t_2^{n+1} \in [t^n, T]\),

\[
 (\rho_f u_f^{n+1} \cdot \nabla u_f^{n+1} - \rho_f u_f^{n} \cdot \nabla u_f^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f} \\
 = (\rho_f (u_f^{n+1} - u_f^n) \cdot \nabla u_f^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f} + (\rho_f u_f^n \cdot \nabla u_f^{n+1} - \rho_f u_f^n \cdot \nabla u_f^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f} \\
 = (\rho_f \Delta t \partial_t u_f(t_2^{n+1}) \cdot \nabla u_f^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f} + (\rho_f u_f^n \cdot \nabla \chi_f^{n+1} + \rho_f u_f^n \cdot \nabla \phi_{f,h}^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f} \\
 + (\rho_f \chi_f^n \cdot \nabla u_f^{n+1} + \rho_f \phi_{f,h}^{n+1} \cdot \nabla u_f^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f}. \tag{2.4.27}
\]

Combining with (2.2.22) and (2.3.6), we have

\[
 (\rho_f u_f^{n+1} \cdot \nabla u_f^{n+1} - \rho_f u_f^n \cdot \nabla u_f^{n+1}, \phi_{f,h}^{n+1})_{\Omega_f} \\
 \leq \rho_f \left\| \Delta t \partial_t u_f(t_2^{n+1}) \right\|_{L^2(\Omega_f)} \left\| \nabla u_f^{n+1} \right\|_{L^2(\Omega_f)} \left\| \phi_{f,h}^{n+1} \right\|_{L^2(\Omega_f)}
\]
We use the Taylor expansion to have for some $t_{n+1}$ and $t_{n+1}^4$ in $[t_{n+1}, T]$,

$$
\eta_p^n = \eta_p^{n+1} - \Delta t \partial_t \eta_p^{n+1} + \frac{\Delta t^2}{2} \partial_{tt} \eta_p^{n+1} - \frac{\Delta t^3}{6} \partial_{ttt} \eta_p(t_{n+1}^4);
$$

$$
\eta_p^{n-1} = \eta_p^{n+1} - 2 \Delta t \partial_t \eta_p^{n+1} + 2 \Delta t^2 \partial_{tt} \eta_p^{n+1} - \frac{4 \Delta t^3}{3} \partial_{ttt} \eta_p(t_{n+1}^4).
$$

Combining the two equations above, we have

$$
\rho_p(\partial_t \eta_p^{n+1} - \frac{I_{sh} \eta_{p,h}^{n+1} - 2 I_{sh} \eta_{p,h}^{n} + I_{sh} \eta_{p,h}^{n-1}}{\Delta t^2} \partial_t \eta_p^{n+1} - \frac{\Delta t^3}{6} \partial_{ttt} \eta_p(t_{n+1}^4));
$$

$$
= \rho_p(\frac{\Delta t}{3} \partial_{tt} \eta_p(t_{n+1}^4) - \frac{4 \Delta t}{3} \partial_{ttt} \eta_p(t_{n+1}^4), \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^{n}}{\Delta t} \Omega_p)
$$

$$
+ \rho_p(\frac{\chi_s^{n+1} - 2 \chi_s^n + \chi_s^{n-1}}{\Delta t^2}, \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^{n}}{\Delta t} \Omega_p) (2.4.29)
$$

For the BJS term on the right hand side of (2.4.24), we can have some $t_{5,n+1}^1$ in $[t_{n+1}, T]$ such that

$$
\alpha_{BJS}(x_{f}^{n+1}, \partial_{tt} x_{p,h}^{n+1} - \frac{I_{sh} \eta_{p,h}^{n+1} - 2 I_{sh} \eta_{p,h}^{n} + I_{sh} \eta_{p,h}^{n-1}}{\Delta t} \partial_{tt} \eta_p^{n+1} - \frac{\Delta t^3}{6} \partial_{ttt} \eta_p(t_{n+1}^4)),
$$

$$
= \alpha_{BJS}(x_{f}^{n+1}, \frac{\Delta t}{2} \partial_{tt} \eta_p(t_{n+1}^4); \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^{n}}{\Delta t} \Omega_p)
$$

$$
+ \alpha_{BJS}(x_{f}^{n+1}, \frac{\chi_s^{n+1} - 2 \chi_s^n + \chi_s^{n-1}}{\Delta t}, \frac{\phi_{s,h}^{n+1} - \phi_{s,h}^{n}}{\Delta t} \Omega_p) (2.4.30)
$$

Let us keep the remaining terms on the right hand side of (2.4.24) containing $(\phi_{s,h}^{n+1} - \phi_{s,h}^{n})/\Delta t$. Next, we have for some $t_{6,n+1}^1$ in $[t_{n+1}, T]$ such that

$$
\rho_p(\partial_{tt} x_{p,h}^{n+1} - \frac{Q_{ph} \eta_{p,h}^{n+1} - Q_{ph} \eta_{p,h}^{n}}{\Delta t} \phi_{s,h}^{n+1} \Omega_p)
$$

$$
= \rho_p(\frac{\Delta t}{2} \partial_{tt} x_{p,h}^{n+1} \phi_{s,h}^{n+1}, \Omega_p) + \rho_p(\frac{\chi_{pp}^{n+1} - \chi_{pp}^n}{\Delta t} \phi_{s,h}^{n+1} \Omega_p) (2.4.31)
$$
For the terms on the left hand side of (2.4.24) containing discrete time derivative, we write
\[
\frac{\rho_f}{\Delta t} \frac{\phi_{f,h}^{n+1} - \phi_{f,h}^n}{\Delta t} \phi_{f,h}^{n+1}(\Omega_f) = \frac{\rho_f}{2} d_t \| \phi_{f,h}^{n+1} \|_{L^2(\Omega_f)}^2 + \frac{\rho_f \Delta t}{2} \| d_t \phi_{f,h}^{n+1} \|_{L^2(\Omega_f)}^2.
\] (2.432)

This pattern applies to the other discrete time derivative terms on the left hand side of (2.4.24). Finally, we utilize the inf-sup condition (2.2.4) to obtain
\[
\beta_p(\phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)} + \phi_{\chi h}^{n+1} \|_{\Lambda_h}) \leq \sup_{0 \neq v_{p,h} \in \mathbf{V}_{p,h}} \frac{b_p(v_{p,h}, \phi_{pp,h}^{n+1}) + \langle v_{p,h} \cdot n_p, \phi_{\chi h}^{n+1} \rangle_{\Gamma_f}}{\| v_{p,h} \|_{\mathbf{V}_p}}.
\] (2.433)

Note that \( b_p(v_{p,h}, \chi_{pp}^{n+1}) = 0 \) and \( \langle v_{p,h} \cdot n_p, \chi_{\lambda}^{n+1} \rangle_{\Gamma_f} = 0 \). Namely, we have
\[
\beta_p(\phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)} + \phi_{\chi h}^{n+1} \|_{\Lambda_h}) \leq C \left( \| \phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)}^2 + \| \phi_{\chi h}^{n+1} \|_{\Lambda_h}^2 \right).
\] (2.434)

Summing (2.4.24) from 0 to \( N-1 \), timing \( 2\Delta t \) on the both sides of the equation, using all the previous estimates, and utilizing Cauchy-Schwarz and Young’s inequalities, we obtain
\[
\rho_f \| \phi_{f,h}^N \|_{L^2(\Omega_f)}^2 + \frac{\mu_f \Delta t}{4} \| D(\phi_{f,h}^n) \|_{L^2(\Omega_f)}^2 + \frac{\mu_f \Delta t}{4} \sum_{n=0}^{N-1} \| D(\phi_{f,h}^{n+1}) \|_{L^2(\Omega_f)}^2 + \frac{\epsilon_f \Delta t}{2} \sum_{n=0}^{N-1} \| \phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)}^2
\] \[+ \frac{c_e \beta_p \Delta t}{3c_e} \sum_{n=0}^{N-1} \left( \| \phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)}^2 + \| \phi_{\chi h}^{n+1} \|_{\Lambda_h}^2 \right) + a_p(\phi_{s,h}^N, \phi_{s,h}^N) + \rho_p \| d_t \phi_{s,h}^N \|_{L^2(\Omega_p)}^2 + s_0 \| \phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)}^2
\] \[+ \Delta t \sum_{n=0}^{N-1} \| \phi_{f,h}^{n+1} - d_t \phi_{s,h}^{n+1} \|_{A_{BJS}}^2 + \Delta t^2 \sum_{n=0}^{N-1} \left( \rho_f \| d_t \phi_{f,h}^{n+1} \|_{L^2(\Omega_f)}^2 + \rho_p \| d_t \phi_{f,h}^{n+1} \|_{L^2(\Omega_p)}^2 \right)
\] \[+ s_0 \| d_t \phi_{pp,h}^{n+1} \|_{L^2(\Omega_p)}^2 + a_p(\phi_{s,h}^n, \phi_{s,h}^n)
\] \[\leq C \Delta t \sum_{n=0}^{N-1} \left( \| \Delta t \partial_t \mathbf{u}_{f}(t_{n+1}^+)^2 \|_{L^2(\Omega_f)}^2 + \| \partial_t \mathbf{x}_{f}^{n+1} \|_{L^2(\Omega_f)}^2 + \| D(\mathbf{x}_{f}^{n+1}) \|_{L^2(\Omega_f)}^2 \right)
\] \[+ \| \chi_{fp}^{n+1} \|_{L^2(\Omega_f)}^2 + \| \chi_{x}^{n+1} \|_{H^1(\Omega_f)}^2 + \| d_t \chi_{s}^{n+1} \|_{L^2(\Omega_f)}^2 + \| d_t \chi_{s}^{n+1} \|_{H^1(\Omega_p)}^2 + \| d_t \chi_{p}^{n+1} \|_{L^2(\Omega_p)}^2
\] \[+ \| \Delta t \partial_t \mathbf{n}_{p}(t_{n+1}^+) \|_{L^2(\Omega_p)}^2 + \| \Delta t \partial_t \mathbf{n}_{p}(t_{n+1}^+) \|_{H^1(\Omega_p)}^2 + s_0 \| d_t \mathbf{n}_{p}(t_{n+1}^+) \|_{L^2(\Omega_p)}^2
\] \[+ \| \chi_{p}^{n+1} \|_{L^2(\Omega_p)}^2 + \| \chi_{\lambda}^{n+1} \|_{\Lambda_h}^2 + \rho_f \Delta t \sum_{n=0}^{N-1} \| d_t \phi_{s,h}^{n+1} \|_{L^2(\Omega_p)}^2
\] \[+ 2\Delta t \sum_{n=0}^{N-1} \left( a_p(\chi_{s}^{n+1}, d_t \phi_{s,h}^{n+1}) + a_b(\phi_{s,h}^{n+1}, \chi_{pp}^{n+1}) + \langle d_t \phi_{s,h}^{n+1} \cdot n_p, \chi_{\lambda}^{n+1} \rangle_{\Gamma_f} \right).
\] (2.435)
For the last three terms on the right hand side of the above inequality, we use the following as an example

\[
\Delta t \sum_{n=0}^{N-1} a^e_p(\chi^{n+1}_s, d_t \Phi^{n+1}_{s,h}) = \sum_{n=0}^{N-1} a^e_p(\chi^{n+1}_s, \Phi^{n+1}_{s,h} - \Phi^n_{s,h})
\]

\[
= a^e_p(\chi^N_s, \Phi^N_{s,h}) - \Delta t \sum_{n=0}^{N-1} a^e_p(d_t \chi^{n+1}_s, \Phi^n_{s,h}).
\] (2.4.36)

Therefore, after using the initial condition \( \Phi^0_{s,h} = 0 \), we have

\[
\Delta t \sum_{n=0}^{N-1} \left( a^e_p(\chi^{n+1}_s, d_t \Phi^{n+1}_{s,h}) + \alpha b_p(d_t \Phi^{n+1}_{s,h}, \chi^{n+1}_{pp}) + \langle d_t \Phi^{n+1}_{s,h} \cdot n_p, \chi^{n+1}_\lambda \rangle_{\Gamma fp} \right)
\]

\[
\leq \left( a^e_p(\chi^N_s, \Phi^N_{s,h}) + \alpha b_p(\Phi^N_{s,h}, \chi^N_{pp}) + \langle \Phi^N_{s,h} \cdot n_p, \chi^N_\lambda \rangle_{\Gamma fp} \right)
\]

\[
- \Delta t \sum_{n=0}^{N-1} \left( a^e_p(d_t \chi^{n+1}_s, \Phi^n_{s,h}) + \alpha b_p(\Phi^n_{s,h}, d_t \chi^{n+1}_{pp}) + \langle \Phi^n_{s,h} \cdot n_p, d_t \chi^{n+1}_\lambda \rangle_{\Gamma fp} \right)
\]

\[
\leq \frac{1}{2} a^e_p(\Phi^N_{s,h}, \Phi^N_{s,h}) + C \left( \| \chi^N_s \|^2_{H^1(\Omega_p)} + \| \chi^N_{pp} \|^2_{L^2(\Omega_p)} + \| \chi^N_\lambda \|^2_{\Lambda_h} \right) + \frac{1}{2} \Delta t \sum_{n=0}^{N-1} a^e_p(\Phi^{n+1}_{s,h}, \Phi^{n+1}_{s,h})
\]

\[
+ \frac{1}{2} \Delta t \sum_{n=0}^{N-1} \left( \| d_t \chi^{n+1}_s \|^2_{H^1(\Omega_p)} + \| d_t \chi^{n+1}_{pp} \|^2_{L^2(\Omega_p)} + \| d_t \chi^{n+1}_\lambda \|^2_{\Lambda_h} \right).
\] (2.4.37)

Applying discrete Gronwall’s lemma 1.2.5 and using the approximations (2.4.35) and (2.4.37), we conclude the Theorem 2.4.22.

We conclude the following Theorem as a direct result of Theorem 2.4.1.

**Theorem 2.4.2.** Assume enough smoothness of the solution of (LMWF1). Assume that the small data condition (2.2.22) and its discrete version (2.3.6) hold. There exists a constant \( C \) independent of the mesh size \( h \) and time step \( \Delta t \) such that

\[
\| u_f - u_{f,h} \|_{L^\infty(0,T;L^2(\Omega_f))} + \| u_f - u_{f,h} \|_{L^2(0,T;H^1(\Omega_f))} + \| u_p - u_{p,h} \|_{L^2(0,T;L^2(\Omega_p))}
\]

\[
+ \| p_p - p_{p,h} \|_{L^2(0,T;L^2(\Omega_p))} + \sqrt{S_0} \| p_p - p_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))} + \| \lambda - \lambda_h \|_{L^2(0,T;\Lambda_h)}
\]

\[
+ \| d_t \eta_p - d_t \eta_{p,h} \|_{L^\infty(0,T;L^2(\Omega_p))} + \| \eta_p - \eta_{p,h} \|_{L^2(0,T;H^1(\Omega_p))}
\]

\[
+ \| (u_f - d_t \eta_p) - (u_{f,h} - d_t \eta_{p,h}) \|_{L^2(0,T;H_{ABJS})} + \sqrt{\Delta t} \left( \| u_f - u_{f,h} \|_{L^\infty(0,T;H^1(\Omega_f))} \right)
\]

\[
+ \| d_t u_f - d_t u_{f,h} \|_{L^2(0,T;L^2(\Omega_f))} + \| d_t \eta_p - d_t \eta_{p,h} \|_{L^2(0,T;L^2(\Omega_p))}
\]

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Furthermore, we have
\[ \| p_f - p_{f,h} \|_{I^2(0,T;L^2(\Omega_f))} \leq \beta_f \left( \| d_t u_f - d_t u_{f,h} \|_{I^2(0,T;L^2(\Omega_f))} + \| u_f - u_{f,h} \|_{I^2(0,T;H^1(\Omega_f))} \right. \\
+ \left. \| \lambda - \lambda_h \|_{I^2(0,T;H^1(\Omega_f))} + \| d_t \eta_p - d_t \eta_{p,h} \|_{I^2(0,T;H^1(\Omega_p))} \right), \]
(2.4.39)
and,
\[ \| \nabla \cdot (u_p - u_{p,h}) \|_{I^2(0,T;L^2(\Omega_p))} \leq C \left( \| d_t \eta_p - d_t \eta_{p,h} \|_{I^2(0,T;H^1(\Omega_p))} \right. \\
+ \left. \| d_t p - d_t p_{p,h} \|_{I^2(0,T;L^2(\Omega_p))} \right). \]
(2.4.40)

**Proof.** Inequality (2.4.38) is a direct result of Theorem 2.4.1. Therefore, we skip the details. We focus on the remaining two results (2.4.39) and (2.4.40) and start with the error equation (2.4.23) and take \( \xi_{p,h} = 0, v_{p,h} = 0, \) and \( w_{p,h} = 0 \) to obtain
\[ (\rho_f \partial_t u_f^{n+1} - \rho_f \frac{u_f^{n+1} - u_f^{n,h}}{\Delta t}, v_{f,h})_{\Omega_f} + a_f(E_f^{n+1}, v_{f,h}) + b_f(v_{f,h}, E_f^{n+1}) + b_r(v_{f,h}, 0; E_f^{n+1}) \\
+ (\rho_f \nabla u_f^{n+1} \cdot \nu u_f^{n+1} - \rho_f u_f^{n,h} \cdot \nabla u_f^{n+1}, v_{f,h})_{\Omega_f} + a_{BJS}(E_f^{n+1}, \partial \eta_p^{n+1} - \frac{\eta_p^{n+1} - \eta_p^n}{\Delta t}; v_{f,h}, 0) \\
= 0, \]
(2.4.41)
Next, we the discrete inf-sup condition (2.3.1) and combine with above equation to have
\[ \beta_f \| E_{fp}^{n+1} \|_{L^2(\Omega_f)} \leq \sup_{0 \neq v_{f,h} \in V_{f,h}} \frac{b_f(v_{f,h}, E_{fp}^{n+1})}{\| v_{f,h} \|_{V_{f}}} \\
= \sup_{0 \neq v_{f,h} \in V_{f,h}} \frac{-(\rho_f \partial_t u_f^{n+1} - \rho_f \frac{u_f^{n+1} - u_f^{n,h}}{\Delta t}, v_{f,h})_{\Omega_f} - a_f(E_f^{n+1}, v_{f,h})}{\| v_{f,h} \|_{V_{f}}} \\
+ \sup_{0 \neq v_{f,h} \in V_{f,h}} \frac{-b_r(v_{f,h}, 0; E_f^{n+1}) - (\rho_f u_f^{n+1} \cdot \nabla u_f^{n+1} - \rho_f u_f^{n,h} \cdot \nabla u_f^{n+1}, v_{f,h})_{\Omega_f}}{\| v_{f,h} \|_{V_{f}}} \\
+ \sup_{0 \neq v_{f,h} \in V_{f,h}} \frac{-a_{BJS}(E_f^{n+1}, \partial \eta_p^{n+1} - \frac{\eta_p^{n+1} - \eta_p^n}{\Delta t}; v_{f,h}, 0)}{\| v_{f,h} \|_{V_{f}}}.
\]
(2.4.42)
Therefore, we have the following inequality after utilizing trace inequality,

\[
\beta f \| E_{f p}^{n+1} \|_{L^2(\Omega_f)} \leq C \left( \| \partial_t u_{f h}^{n+1} - \rho f \frac{u_{f h}^{n+1} - u_f^n}{\Delta t} \|_{L^2(\Omega_f)} + \| \Delta t \partial_t u_f(t_i^{n+1}) \|_{H^1(\Omega_f)} \right. \\
+ \| D(E_f^{n+1}) \|_{L^2(\Omega_f)} + \| E_f^{n+1} \|_{H^1(\Omega_f)} + \| E_f^n \|_{H^1(\Omega_f)} + \| E_{\lambda}^{n+1} \|_{L^2(\Gamma_f)} \\
+ \left. \| \partial_t \eta_{p h}^{n+1} - \frac{\eta_{p h}^{n+1} - \eta_{p h}^n}{\Delta t} \|_{H^1(\Omega_p)} \right),
\]  

(2.4.43)

which results in (2.4.39). Again, we start with (2.4.23). But this time, we choose \( v_{f h} = 0 \), \( w_{f h} = 0 \), \( v_{p h} = 0 \), \( \xi_{p h} = 0 \), and \( w_{p h} = \nabla \cdot E_{p h}^{n+1} \) to have

\[
\| \nabla \cdot E_{p h}^{n+1} \|_{L^2(\Omega_p)} \\
= (s_0 \partial_t p_{p h}^{n+1} - s_0 \frac{p_{p h}^{n+1} - p_{p h}^n}{\Delta t}, \nabla \cdot E_{p h}^{n+1})_{\Omega_p} - \alpha b_p (\partial_t \eta_{p h}^{n+1} - \frac{\eta_{p h}^{n+1} - \eta_{p h}^n}{\Delta t}, \nabla \cdot E_{p h}^{n+1}).
\]  

(2.4.44)

Therefore, after using Cauchy-Schwarz and Young’s inequality, we have

\[
\| \nabla \cdot E_{p h}^{n+1} \|_{L^2(\Omega_p)} \leq C \left( \| s_0 \partial_t p_{p h}^{n+1} - s_0 \frac{p_{p h}^{n+1} - p_{p h}^n}{\Delta t} \|_{L^2(\Omega_p)} + \| \partial_t \eta_{p h}^{n+1} - \frac{\eta_{p h}^{n+1} - \eta_{p h}^n}{\Delta t} \|_{H^1(\Omega_p)} \right),
\]  

(2.4.45)

which proves (2.4.40). We finish the proof of this theorem.

\[ \square \]

**Remark 2.4.1.** Note that to have the estimates in (2.4.39) and (2.4.40), the higher regularities \( \| d_t \eta_p - d_t \eta_{p h} \|_{L^2(0,T;H^1(\Omega_p))} \) and \( \| d_t p_p - d_t p_{p h} \|_{L^2(0,T;L^2(\Omega_p))} \) are needed. The higher regularity can be obtained by introducing the discrete second time derivative of (2.3.1)–(2.3.3), which is similar to the process of proving Theorem 2.3.1. To avoid duplications, we skip the details.
2.5 Numerical results

In this section, we present the results of several numerical experiments in 2 dimensions. The fully discrete method has been implemented by using the finite element package FreeFem++ [54]. In Section 2.5.1, we present the convergence of a fully discrete scheme. In Section 2.5.2, we analyze a prototype problem arising from cardiovascular mechanics and geomechanics to confirm the practicability of our numerical model. Note that, in the simulation region, shown in Figure 2.5.1, we use finer mesh than the plots.

Figure 2.5.1: Computational domains for Example 1 (left) and Example 2 (right).

2.5.1 Example 1: convergence test

The convergence test confirms the theoretical convergence rates for the problem using an analytical solution. The domain is \( \Omega = [0, 1] \times [-1, 1] \). We associate the upper half with the Navier-Stokes flow, while the lower half represents the flow in the poroelastic structure governed by the Biot system. The appropriate interface conditions are enforced on the interface \( y = 0 \). The solution in the Navier-Stokes region is

\[
\begin{align*}
\mathbf{u}_f &= \pi \cos(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}, \\
p_f &= e^t \sin(\pi x) \cos(\frac{\pi y}{2}) + 2\pi \cos(\pi t).
\end{align*}
\]

The Biot solutions are chosen accordingly to satisfy the interface conditions (2.1.9)-(2.1.12):

\[
\begin{align*}
\mathbf{u}_p &= \pi e^t \begin{pmatrix} -\cos(\pi x) \cos(\frac{\pi y}{2}) \\ \frac{1}{2} \sin(\pi x) \sin(\frac{\pi y}{2}) \end{pmatrix}, \\
p_p &= e^t \sin(\pi x) \cos(\frac{\pi y}{2}), \\
\eta_p &= \sin(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}.
\end{align*}
\]
The right hand side functions $f_f$, $q_f$, $f_p$ and $q_p$ are computed from (2.1.2)–(2.1.7) using the above solution. The model problem is then complemented with the appropriate Dirichlet boundary conditions and initial data. We study the convergence for two choices of finite element spaces. The lower order choice is the $P_1-P_0$ for Navier-Stokes, the Raviart-Thomas $RT_0-P_0$ and continuous Lagrangian $P_0$ elements for the Biot system, and piecewise constant Lagrange multiplier $P_0$. In this case, the analysis in the previous section implies the first order of convergence for all variables. However, in Table 2.5.1, we observe higher convergence rate for the error of displacement, $e_s$ in norm $l^\infty(H^1(\Omega_p))$. The total simulation time for this test case is $T = 1 \times 10^{-1}$ s and the time step is $\Delta t = 2.5 \times 10^{-4}$ s. The higher order choice is the Taylor-Hood $P_2-P_1$ for Navier-Stokes, the Raviart-Thomas $RT_1-P_1^{dc}$ and $P_2$ for Biot, and $P_1^{dc}$ for the Lagrange multiplier, with $k_f = 2$, $s_f = 1$, $k_p = 1$, $s_p = 1$, and $k_s = 2$, in which case second order convergence rate for all variables is expected. For this case, the time step needs to be sufficiently small so that the time discretization error does not affect the convergence rates. These theoretical results are verified by the rates shown in Table 2.5.1.

2.5.2 Example 2: blood flow

In this section, we focus on the Newtonian fluid in the free fluid region, namely constant viscosity, since the blood flow is very commonly considered as Newtonian fluid in large diameter vessels, such as the aorta. We consider a classical benchmark problem. For simplicity, we adopt a two-dimensional geometrical model, which consists of two poroelastic structures and a fluid channel. In this case, a time-dependent pressure as follows drives the blood flow:

$$p_{in}(t) = \begin{cases} \frac{P_{max}}{2} (1 - \cos(\frac{2\pi t}{T_{max}})) & \text{if } t \leq T_{max}; \\ 0 & \text{if } t > T_{max}, \end{cases}$$

(2.5.1)

where $P_{max} = 13,334$ dyne/cm$^2$ and $T_{max} = 0.003$ s. For poroelastic structure, additionally, we modify the governing equation to better represent the behavior of an artery:

$$\rho_p \partial_t \eta_p + \xi \eta_p - \nabla \cdot \sigma_p(\eta_p, p_p) = 0, \quad \text{in } \Omega_p \times (0,T];$$

(2.5.2)
\( P_1 - P_0, \mathcal{T}_0 - P_0, P_1, P_0 \)

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<th>( |e_s|_{L^\infty(\Omega_p)} ) error rate</th>
<th>( |e_{p}|<em>{L^2(\Gamma</em>{fp})} ) error rate</th>
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<td>2.018E-01</td>
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Table 2.5.1: Example 1: relative numerical errors and convergence rates for \( \{u_f, p_f, u_p, p_p, \eta_p, \lambda\} \) with the lower order spaces.

The additional term \( \xi \eta_p \) comes from the axially symmetric formulation, accounting for the recoil due to the circumferential strain. In other words, it acts like a spring term to keep the top and bottom structure displacements connected. Shown in Figure 3.1.1, denote the inlet and outlet boundaries by \( \Gamma_{in}^f = \{(0, y)| - R < y < R\} \) and \( \Gamma_{out}^f = \{(L, y)| - R < y < R\} \).
\[ P_2 - P_1, \mathcal{RT}_1 - P_1^{dc}, P_2, P_1 \]

\[ \| e_f \|_{H^1(\Omega_f)} \quad \| e_{fp} \|_{L^2(\Omega_f)} \quad \| e_p \|_{L^2(\Omega_p)} \quad \| \nabla \cdot e_p \|_{L^2(\Omega_p)} \]

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<th>( | e_{\lambda} |<em>{L^2(\Gamma</em>{fp})} )</th>
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<td>9.885E-01</td>
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<td>( 1/32 )</td>
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<td>( 1/64 )</td>
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Table 2.5.2: Example 1: relative numerical errors and convergence rates for \( \{ u_f, p_f, u_p, p_p, \eta_p, \lambda \} \) with the higher order spaces.

We prescribe the normal stress at the inlet and outlet boundary as follows:

\[ \sigma_f n_f^{in} = -p_{in}(t)n_f^{in}, \text{ on } \Gamma_f^{in} \times (0, T]; \quad \sigma_f n_f^{out} = 0, \text{ on } \Gamma_f^{out} \times (0, T]; \quad (2.5.3) \]

where \( n_f^{in} \) and \( n_f^{out} \) are the outward unit normals on the inlet/outlet fluid boundaries. Even though these inflow and outflow boundary conditions are not physiologically optimal since
the flow distribution and pressure field are often unknown, they are common in blood flow models. Shown in Figure.3.1.1, denote the inlet and outlet poroelastic structure boundaries, respectively, by $\Gamma_{in}^p = \{(0, y)| -R - r_p < y < R \text{ or } R < y < R + r_p\}$ and $\Gamma_{out}^p = \{(L, y)| -R - r_p < y < R \text{ or } R < y < R + r_p\}$, where $r_p$ is the poroelastic wall thickness. We assume that the poroelastic structure is fixed at the inlet and outlet boundaries, namely:

$$\eta_p = 0, \quad \text{on } \Gamma_{in}^p \cup \Gamma_{out}^p \times (0,T),$$

(2.5.4)

and for the external structure boundary $\Gamma_{ext}^p = \{(x, y)| 0 < x < L, y = R + r_p \text{ or } y = -R - r_p\}$, we are using the external ambient pressure:

$$n_p \cdot \sigma_e n_p = 0, \quad \text{on } \Gamma_{ext}^p \times (0,T),$$

(2.5.5)

also the displacement in tangential direction of the exterior boundary $\Gamma_{ext}^p$ is zero:

$$\eta_p \cdot \tau_p = 0, \quad \text{on } \Gamma_{ext}^p \times (0,T),$$

(2.5.6)

Additionally, for the fluids in the poroelastic medium, we impose the following drained boundary conditions:

$$u_p = 0, \quad \text{on } \Gamma_{in}^p \cup \Gamma_{out}^p \times (0,T).$$

(2.5.7)

On the fluid pressure in the structure region, we impose the following boundary condition:

$$p_p = 0, \quad \text{on } \Gamma_{ext}^p \cup \Gamma_{in}^p \cup \Gamma_{out}^p \times (0,T).$$

(2.5.8)

The reference values of the parameters used in this model fall within the range of physical values for blood flow and are reported in Table.3.2.1. The propagation of the pressure wave is analyzed over the time zone $[0, 0.006]$ s. The final time is selected so that the pressure wave can reach the outflow section.

Some visualizations of the solutions, calculated using the numerical settings addressed before, are reported on the top panel of Figure.2.5.2. In Figure.2.5.2, pressure waves within the channel are presented, together with corresponding the velocity and the deformation of the fluid region at the times $t = 1.8, 3.6, 5.4$ ms. For visualization purposes, the vertical deformation is magnified 40 times. As a supplement to the previous theoretical work and convergence test, we use two structure regions, with an additional thin layer in the arterial
Wall, to have a more practical model. The three plots in Figure.2.5.2 show that the variable
inflow pressure together with the fluid-structure interaction generates a wave in the structure
from left to right. This simulation is similar to the ones obtained in [23], using a different
numerical approach to model the fluid-structure interface of blood flow. The velocity arrows
in Figure.2.5.2 are scaled proportionable to the magnitude of the fluid velocity at the fluids
region. In Figure.2.5.4 top panel, we show the Darcy velocity $u_p \cdot n$ in the normal direction
along with the interface between the fluid region and the top structure region at the times
t = 1.8, 3.6, 5.4 ms. The peaks of the Darcy velocity $u_p \cdot n$ coincide with the ones of structure

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<th>Symbol</th>
<th>Values</th>
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</tr>
<tr>
<td>Length</td>
<td>cm</td>
<td>$L$</td>
<td>6</td>
</tr>
<tr>
<td>wall thickness</td>
<td>cm</td>
<td>$r_p$</td>
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</tr>
<tr>
<td>Total time</td>
<td>s</td>
<td>$T$</td>
<td>0.006</td>
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<tr>
<td>wall density</td>
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<td>$\rho_p$</td>
<td>1.1</td>
</tr>
<tr>
<td>Spring coeff.</td>
<td>dyn/cm⁴</td>
<td>$\xi$</td>
<td>$5 \times 10^7$</td>
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<td>Fluid density</td>
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<td>$\rho_f$</td>
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<tr>
<td>Dyn. viscosity</td>
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<td>cm²</td>
<td>$K$</td>
<td>$diag(5,5) \times 10^{-9}$</td>
</tr>
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<td>Lamé coeff.</td>
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<td>$\mu_p$</td>
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<td>Biot-Willis constant</td>
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</table>

Table 2.5.3: Geometry, poroelasticity and fluid parameters.
displacement in the normal direction $\eta_p \cdot n$ in Figure.2.5.3 middle panel and the corresponding peak of the arterial pressure. We also notice that the Darcy velocity $u_p \cdot n$ decreases when the fluid penetrates into the arterial wall gradually, which is a consequence of equation (2.1.7). This equation describes that $\nabla \cdot u$ is not locally preserved, and it depends on the rate of change in pressure and the displacement. The poroelastic coupling results in this phenomenon. In the bottom panel of Figure.2.5.4, we can detect consistent lags of the peaks of the Darcy velocity in the tangential direction. We compare Darcy velocity $u_p \cdot \tau$ with the velocity in the free fluid region $u_f \cdot \tau$ in the tangential direction at time $t = 1.8, 3.6, 5.4$ ms, presented in Figure.2.5.5. The tangential Darcy velocity $u_p \cdot \tau$ is much smaller than tangential free fluid velocity $u_f \cdot \tau$ at the same time and position. Also, in the BJS condition (2.1.12), $u_p \cdot \tau$ is neglected since it is relatively small than $u_f \cdot \tau$, which is confirmed in Figure.2.5.4 bottom panel and Figure.2.5.5. Since we neglect the effect of $u_p \cdot \tau$, this would result in the consistent lags of peaks presenting in Figure.2.5.4 bottom panel.

Additionally, what attracts our attention are the negative values of $u_f \cdot \tau$, which means there are backward flows, the opposite direction with the blood flow, on the arterial wall. For example, in the second plot of Figure.2.5.5, we can observe clear negative directional fluid velocity in the tangential direction at lengths from 0 to about 1.5 cm on the arterial wall. And similar negative $u_f \cdot \tau$ happens in the third plot of Figure.2.5.5 from length 1.5
Figure 2.5.3: Structure displacement in the normal direction $\eta_p \cdot n$ along with the arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms.

Figure 2.5.4: Top panel: Darcy velocity $u_p \cdot n$ in the normal direction; bottom panel: Darcy velocity $u_p \cdot \tau$ in the tangential direction along with the arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms.

cm to 3.5 cm. It is reasonable and practical to have such negative values, which confirm the applicability of our simulation. First of all, at time 3.6 ms, according to the second velocity-pressure plot in Figure 2.5.2, the highest pressure (the light yellow part) concentrates at the

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middle position of the arterial wall, and the pressure (with a negative sign) at the left side is much smaller than the middle part. There is a positive pressure gradient. And the blood would flow from the higher pressure position to the lower pressure position. Also, notice that $p_m(t) = 0$ for $t \geq 0.003$ s, namely, there isn’t driven force afterward. On the other hand, we don’t observe backward flow in the fluid region according to Figure 2.5.2 since inertia is the dominant force that can cancel out the backflow trend. However, along with the arterial, the fluid velocity is relatively small compared to the velocity in the fluid region, and the pressure gradient effect can not be balanced. It is the viscosity that plays the leading role and results in the backward flow, namely negative values of $u_f \cdot \tau$ at the beginning part of the arterial wall. In conclusion, all the referred values confirmed the practicability of our numerical model.
3.0 Non-Newtonian and poroelastic effects in simulations of arterial flows

In the previous chapter, we conduct a blood flow simulation using the fully dynamic Navier-Stokes–Biot model. This simulation triggers some natural questions. Is there a better model to use in describing the daily yet complex fluid flow process? Can we improve the existing model and come up with a relatively complete simulation to present more rheological characteristics of blood? Can the Lagrange multiplier method, studied in Chapter 2, deal with complex simulation domains and still be stable? What can we tell from the blood flow simulation? All the related questions will be answered in Chapter 3. In Section 3.2.1, we compare the difference between a poroelastic and a pure elastic model. In Section 3.2.2, instead of using constant viscosity, we expand our model to suit with shear-thinning properties of blood. We simulate the blood flow in a stenotic vessel in Section 3.2.5. We also analyze the effects of structure parameters, including the permeability of poroelastic models and the lamé coefficients of both poroelastic and elastic models.

3.1 Models and methods

3.1.1 Simulations domains

We first focus on the prototype benchmark problem arising from FSI modeling of blood flows [23, 24, 45] in Section 3.2.1 and Section 3.2.2 and then consider an ideal stenotic model in Section 3.2.5. We show the two simulation domains in Figure 3.1.1.

Note that Figure 3.1.1 (a) is the computational domain \( \Omega \) used in Section 3.2.1 and Section 3.2.2, where \( \star \in \{p, e\} \). (b) is the computational domain \( \Omega \) used in Section 3.2.5. The red areas are structure regions \( \Omega_\star \), and the grey areas are fluid regions \( \Omega_f \). The mesh we are using in the simulation is non-matching and much finer than what is shown in (a) and (b). We exaggerate the thickness of the structure region \( \Omega_\star \) in Figure 3.1.2 to have a better view and understanding.
As shown in Figure 3.1.1, we consider a Lipschitz rectangular domain $\Omega \subset \mathbb{R}^2$, which is subdivided into three non-overlapping regions: fluid region $\Omega_f$ in the middle, and we use $\Omega_e$ to represent the structure regions for the elastic model, and correspondingly, $\Omega_p$ stands for the poroelastic model. For notation purposes, we use $\Omega_\star, \star \in \{p, e\}$ to represent the structure regions. Let $\Gamma_{f, \star} = \partial \Omega_{f} \cap \partial \Omega_\star$ denote the nonempty interfaces between these regions. For the fluid region, we denote the inlet and outlet boundaries by $\Gamma_{in}^f = \{(0, y)| - R < y < R\}$ and $\Gamma_{out}^f = \{(L, y)| - R < y < R\}$. And for the structure region with $\star \in \{p, e\}$, we denote the inlet and outlet poroelastic/elastic structure boundaries, respectively, by $\Gamma_{in}^\star = \{(0, y)| - R - r_\star < y < R \text{ or } R < y < R + r_\star\}$ and $\Gamma_{in}^\star = \{(L, y)| - R - r_\star < y < R \text{ or } R < y < R + r_\star\}$, where $r_\star$ is the poroelastic/elastic wall thickness. In addition, we let $\Gamma_{ext}^\star = \{(x, y)| 0 < x < L, y = R + r_\star \text{ or } y = -R - r_\star\}$ be the external structure boundaries.
Finally, we denote by \( n \) the unit outward normal vector, which points outward from the fluid domain on \( \partial \Omega_f \). And we denote by \( t \) the unit tangential vector, which points toward the direction of the blood flow on the interface \( \Gamma_{f,\star} \) and other horizontal exterior boundaries \( \Gamma_{\star}^{ext} \) and upward on the vertical boundaries \( \Gamma_{\star}^{in}, \Gamma_{\star}^{out}, \Gamma_{\star}^{\infty} \), and \( \Gamma_{\star}^{\infty} \).

We focus on studying the propagation of a single pressure wave whose amplitude is comparable to the pressure difference between the systolic and diastolic phases of the heartbeat. A time-dependent pressure as follows drive the blood flow:

\[
p_{in}(t) = \begin{cases} 
P_{\max} \frac{1}{2} \left(1 - \cos \left( \frac{2 \pi t}{T_{\max}} \right) \right), & \text{if } t \leq T_{\max}; \\ 0, & \text{if } t > T_{\max}, \end{cases}
\]

where \( P_{\max} = 13,334 \text{ dyn/cm}^2 \) and \( T_{\max} = 0.003 \text{ s.} \)

### 3.1.2 Navier-Stokes/Elasticity model problem

We assume that the flow in \( \Omega_f \) is governed by the time-dependent Navier-Stokes equations with density \( \rho_f \) and viscosity \( \mu_f \), which are written in the following stress-velocity-pressure formulation:

\[
\rho_f \left( \partial_t \mathbf{u}_f + (\nabla \mathbf{u}_f) \mathbf{u}_f \right) - \nabla \cdot \mathbf{\sigma}_f = 0, \quad \nabla \cdot \mathbf{u}_f = 0, \quad \text{in } \Omega_f \times (0,T],
\]

where \( \mathbf{u}_f \) is the fluid velocity, \( \mathbf{\sigma}_f = -p_f \mathbf{I} + 2 \mu_f \mathbf{D}(\mathbf{u}_f) \) is the stress tensor, and \( \mathbf{D}(\mathbf{u}_f) := \frac{1}{2} (\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T) \). We adopt commonly used boundary conditions in blood flow models and prescribe the normal stress at the inlet and outlet boundaries \([24, 23]\) as follows:

\[
\mathbf{\sigma}_f n = -p_{in}(t)n, \quad \text{on } \Gamma_{f}^{in} \times (0,T],
\]

\[
\mathbf{\sigma}_f n = 0, \quad \text{on } \Gamma_{f}^{out} \times (0,T].
\]

We then state the elastic model which is used to govern the structure region. Let \( \mathbf{\eta}_e \) be the displacement in \( \Omega_e \), and let \( \mathbf{\sigma}_e \) be the elastic stress tensor defined as follows:

\[
\mathbf{\sigma}_e := \lambda_e \nabla \cdot (\mathbf{\eta}_e) \mathbf{I} + 2 \mu_e \mathbf{D}(\mathbf{\eta}_e), \quad \text{in } \Omega_e \times (0,T],
\]

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where \(0 < \lambda_{\text{min}} \leq \lambda_e(x) \leq \lambda_{\text{max}}\) and \(0 < \mu_{\text{min}} \leq \mu_e(x) \leq \mu_{\text{max}}\) are the Lamé parameters and are determined by Young’s modulus which we discuss later [1]. We use the following governing equation to better represent the behavior of an artery [24]:

\[
\rho_e \partial_{tt} \eta_e + \xi \eta_e - \nabla \cdot (\sigma_e) = 0, \quad \text{in } \Omega_e \times (0, T].
\] (3.1.5)

where \(\rho_e > 0\) is the wall density, \(\xi > 0\) is the spring coefficient. The term \(\xi \eta_e\) comes from the axially symmetric formulation, accounting for the recoil due to the circumferential strain [23]. In other words, it acts like a spring term to keep the top and bottom structure displacements connected. For boundary conditions, we assume that elastic structure is fixed at the inlet and outlet boundaries, namely:

\[
\eta_e = 0, \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \times (0, T].
\] (3.1.6)

For the external structure boundary \(\Gamma_{e}^{\text{ext}}\), we assume that the external ambient pressure and the displacement in the tangential direction of the exterior boundary \(\Gamma_{e}^{\text{ext}}\) is zero:

\[
n \cdot \sigma_e n = 0, \quad \text{and } \eta_e \cdot t = 0, \quad \text{on } \Gamma_{e}^{\text{ext}} \times (0, T].
\] (3.1.7)

Next, we introduce the transmission conditions on the top interface as well as the bottom interface \(\Gamma_{fe}\) [11]:

\[
u_f = \partial_t \eta_e, \quad \sigma_f n = \sigma_e n, \quad \text{on } \Gamma_{fe} \times (0, T].
\] (3.1.8)

The first equation in (3.1.8) corresponds to the continuity of the velocity vector, and the second one represents the continuity of the normal stress vector on \(\Gamma_{fe}\).

Finally, the above coupling system is complemented by a set of initial conditions:

\[
u_f(x, 0) = 0 \quad \text{and } \quad p_f(x, 0) = p_{\text{in}}(0), \quad \text{in } \Omega_f,
\]

\[
\eta_e(x, 0) = 0 \quad \text{and } \quad \partial_t \eta_e(x, 0) = 0, \quad \text{in } \Omega_e.
\]
3.1.3 Navier-Stokes/Poroelasticity model problem

The Navier-Stokes equations are exactly the same as (3.1.2). As for the poroelastic region, let \( \sigma^p_e \) and \( \sigma_p \) be the elastic and poroelastic stress tensors, respectively,

\[
\sigma^p_e := \lambda_p \nabla \cdot (\eta_p) I + 2 \mu_p D(\eta_p) \quad \text{and} \quad \sigma_p := \sigma^p_e - \alpha_p p_p I \quad \text{in} \quad \Omega_p \times (0, T],
\]

where \( 0 < \lambda_{\min} \leq \lambda_p(x) \leq \lambda_{\max} \) and \( 0 < \mu_{\min} \leq \mu_p(x) \leq \mu_{\max} \) are the Lamé parameters and \( 0 \leq \alpha_p \leq 1 \) is the Biot-Willis constant. The poroelasticity region \( \Omega_p \) is governed by the fully dynamic Biot system [3, 13]:

\[
\begin{align*}
\rho_p \partial_t \eta_p + \xi \eta_p - \nabla \cdot (\sigma_p) &= 0, \quad \mu_f K^{-1} u_p + \nabla p_p &= 0, \quad \text{in} \quad \Omega_p \times (0, T], \\
\partial_t (s_0 p_p + \alpha_p \nabla \cdot \eta_p) + \nabla \cdot (u_p) &= 0, \quad \text{in} \quad \Omega_p \times (0, T],
\end{align*}
\]

where \((u_p, p_p)\) is the velocity-pressure pair in \( \Omega_p \), \( s_0 \geq 0 \) is a storage coefficient and \( K \) the symmetric and uniformly positive definite permeability tensor, satisfying, for some constants \( 0 < k_{\min} \leq k_{\max} \),

\[
\forall \xi \in \mathbb{R}^n \quad k_{\min} |\xi|^2 \leq \xi^t K^{-1}(x) \xi \leq k_{\max} |\xi|^2 \quad \forall x \in \Omega_p.
\]

And we complement the boundary conditions for \( \eta_p \) and \( \sigma_p \):

\[
\eta_p = 0, \quad \text{on} \quad \Gamma^\text{in}_p \cup \Gamma^\text{out}_p \times (0, T].
\]

\[
n \cdot \sigma_p n = 0, \quad \text{and} \quad \eta_p \cdot t = 0, \quad \text{on} \quad \Gamma^\text{ext}_p \times (0, T].
\]

Additionally, for the fluids in the poroelastic medium, we impose the following boundary conditions:

\[
u_p \cdot n = 0, \quad \text{on} \quad \Gamma^\text{in}_p \cup \Gamma^\text{out}_p \times (0, T],
\]

\[
p_p = 0, \quad \text{on} \quad \Gamma^\text{ext}_p \times (0, T].
\]

Next, we introduce the transmission conditions on the interface \( \Gamma_{fp} \) [3]:

\[
u_f \cdot n_f + (\partial_t \eta_p + u_p) \cdot n_p = 0 \quad \text{on} \quad \Gamma_{fp} \times (0, T],
\]

\[
- (\sigma_f n_f) \cdot n_f = p_p \quad \text{on} \quad \Gamma_{fp} \times (0, T],
\]

\[
\sigma_f n_f + \sigma_p n_p = 0 \quad \text{on} \quad \Gamma_{fp} \times (0, T].
\]
\[-(\sigma_f n_f) \cdot \tau_{f,j} = \mu_f \alpha_{BJS} \sqrt{K_f^{-1} (u_f - \partial_t \eta_p)} \cdot \tau_{f,j} \quad \text{on } \Gamma_{fp} \times (0, T). \quad (3.1.18)\]

where \(\alpha_{BJS} \geq 0\) is an experimentally determined friction coefficient. The first (3.1.15) and second (3.1.16) equations correspond to the conservation of mass and balance of momentum on \(\Gamma_{fp}\), respectively, whereas the third one represents the balance of normal stress and the last one represents Beaver-Joseph-Saffman (BJS) slip with friction condition, respectively.

Finally, the above coupling system is complemented by a set of initial conditions,

\[ u_f(x, 0) = 0, \quad p_f(x, 0) = p_{in}(0), \quad \text{in } \Omega_f, \]

\[ p_p(x, 0) = p_{in}(0), \quad \eta_p(x, 0) = 0 \quad \text{and} \quad \partial_t \eta_p(x, 0) = 0, \quad \text{in } \Omega_p. \]

### 3.1.4 Discretized models

For the time discretization, we consider the backward Euler method with a semi-implicit way. We indicate with \(z^n\) the approximation of a generic function \(z(t)\) evaluated at \(t^n = n \Delta t\), \(n = 1, 2, \cdots N\). At each time step \(t^n\), we have the following discretized in time Navier-Stokes equation in the fluid region \(\Omega_f\):

\[ \rho_f \frac{u_f^{n+1} - u_f^n}{\Delta t} + \rho_f (\nabla u_f^{n+1}) u_f^{n+1} - \nabla \cdot \sigma_f^{n+1} = 0, \]

\[ \nabla \cdot u_f^{n+1} = 0. \quad (3.1.19)\]

For the elastic model, we have the following for the governing equation in the structure region \(\Omega_e\):

\[ \rho_e \frac{\eta_e^{n+1} - 2\eta_e^n + \eta_e^{n-1}}{\Delta t^2} + \xi \eta_e^{n+1} - \nabla \cdot \sigma_e^{n+1} = 0. \quad (3.1.20)\]

While for the poroelastic model, we have the following for the Biot system in the structure region \(\Omega_p\):

\[ \rho_p \frac{\eta_p^{n+1} - 2\eta_p^n + \eta_p^{n-1}}{\Delta t^2} + \xi \eta_p^{n+1} - \nabla \cdot \sigma_p^{n+1} = 0, \]

\[ \rho_f K^{-1} u_f^{n+1} + \nabla p_p^{n+1} = 0, \quad (3.1.21)\]

\[ s_0 \frac{p_p^{n+1} - p_p^n}{\Delta t} + \alpha_p \frac{\nabla \cdot \eta_p^{n+1} - \nabla \cdot (\eta_p^n)}{\Delta t} + \nabla \cdot u_p^{n+1} = 0. \]
All the numerical tests in the following sections are implemented by the finite element library Freefem++[54]. For the discretization of space, we choose the Taylor-Hood $\mathcal{P}_1 - \mathcal{P}_0$ finite elements for variables $(u_f, p_f)$ in $\Omega_f$. For the elastic model, we use continuous Lagrangian $\mathcal{P}_1$ for variable $\eta_e$ in $\Omega_e$, while for the poroelastic model, we utilize Raviart-Thomas $\mathcal{RT}_0 - \mathcal{P}_0$ for $(u_p, p_p)$ and $\mathcal{P}_2$ for $p_p$ in $\Omega_p$. In NSE/E and NSE/P models, a Lagrange multiplier method is employed to impose the normal stress on the interface and continuity of flux condition, respectively [3]. We have finished the convergent test for our numerical methods in [3]. We set the time discretization parameter $\Delta t = 6 \times 10^{-5}s$ in Section 3.2.1 and 3.2.2; and $\Delta t = 3 \times 10^{-5}s$ in Section 3.2.5.

3.2 Numerical applications and discussions

In this section, in order to study non-Newtonian and poroelastic effects in blood flow simulations, we present numerical results for ten different cases. We fix the line styles for these cases in the following sections, as shown in Figure.3.2.1. Also, note that for the velocity and pressure waves, and viscosity plots in all the following sections, we use solid dark blue to simply present the structure region $\Omega_e$ for elastic models. In the structure region $\Omega_e$, elastic models don’t have velocity, pressure, and viscosity as variables.

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Figure 3.2.1: Line styles for different cases.
3.2.1 Newtonian models

In this section, we focus on the Newtonian fluids in the computational domain shown in Figure.3.1.1 part (a). With Newtonian fluids assumption, parameter viscosity, a proportional constant between shear stress and shear rate, is enough to describe the blood rheological behavior. In this case, it is important to fully understand the effects of poroelasticity and elasticity, and the effects of different permeability $K$ on NSE/P models. The reference values of the parameters used in this study fall within the range of physical values for blood flow and are reported in Table.3.2.1. The Lamé coefficients are determined from Young’s modulus $E$ and the Poisson’s ratio $\tilde{\nu}$ via the following relationship [3]:

$$\lambda_* = \frac{E \tilde{\nu}}{(1 + \tilde{\nu})(1 - 2\tilde{\nu})}, \quad \mu_* = \frac{E}{2(1 + \tilde{\nu})},$$  \hspace{1cm} (3.2.1)

The propagation of the pressure wave is analyzed over the time zone $[0, 0.006]$ s. The final time is selected so that the pressure wave can reach the outflow section. We introduce the following cases:

- case 1: Newtonian NSE/E model;
- case 2: Newtonian NSE/P model, with $K = diag(1, 1) \times 10^{-9}$;
- case 3: Newtonian NSE/P model, with $K = diag(1, 1) \times 10^{-7}$.

Note that the permeability values are within the physical range for arterial walls established in the literature [32, 40, 41].
Table 3.2.1: Geometry, elasticity, poroelasticity and fluid parameters. Note that $\star \in \{p, e\}$.

Some visualization of the solutions is reported in this section. Velocity and pressure waves through the channel are presented in Figure.3.2.2 and Figure.3.2.3, together with the corresponding deformation at time $t = 1.8, 3.6, 5.4$ ms. Note that for case 1, there is no Darcy velocity. We are using solid blue to present structure $\Omega_e$. This pattern follows in all the following plots of elastic models. For visualization purposes, deformations of 2D plots in this and the next section are magnified 40 times. Figure.3.2.2 shows that the variable inflow velocity together with the fluid-structure iteration generates a wave from left to right.
More differences are noticeable at time $t = 3.6$ and $t = 5.4$ ms. In particular, for case 3, the flow has a smaller velocity magnitude and takes more time to reach the outflow region. Correspondingly in Figure 3.2.3, we can observe a clear increase of pressure on the interface for case 3. We look into more quantities closely on the top interfaces.

Figure 3.2.2: Fluid velocity magnitude together with velocity arrows scaled with the magnitude at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 1, case 2 and case 3.

Figure 3.2.3: Pressure waves at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 1, case 2 and case 3.

Wall shear stress (WSS) is an important index for the risk of plaque rupture in the blood flow dynamic model [29, 52] and it is interesting to evaluate the possible effects of
different models and different permeability values on this index. We continue to pick up time $t = 1.8, 3.6, 5.4$ ms for comparison, as shown in Figure 3.2.4. The wall shear stress is defined as follows [58]:

$$WSS = \sigma_f \mathbf{n} \cdot \mathbf{t}.$$  \hspace{1cm} (3.2.2)

Figure 3.2.4: Wall shear stress $(\sigma_f \mathbf{n}) \cdot \mathbf{t}$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 1, case 2, and case 3.

From the definition in (3.2.2), WSS is only a local physical quantity at a specific time. Therefore, to consider WSS over a period of time, we are going to introduce three more quantities: time averaged wall shear stress (TAWSS), oscillatory shear index (OSI), and relative residence time (RRT). For TAWSS, OSI, and RRT, functions of space on the lumen boundary [52, 58], the definitions are as follows:

$$TAWSS(x) = \frac{1}{T} \int_0^T |\sigma_f \mathbf{n} \cdot \mathbf{t}(t, x)| dt;$$  \hspace{1cm} (3.2.3)

$$OSI(x) = \frac{1}{2} \left( 1 - \frac{\int_0^T |\sigma_f \mathbf{n} \cdot \mathbf{t}(t, x)| dt}{\int_0^T |\sigma_f \mathbf{n} \cdot \mathbf{t}(t, x)| dt} \right);$$  \hspace{1cm} (3.2.4)

$$RRT(x) = \frac{1}{(1 - 2 OSI(x))TAWSS(x)} = \frac{T}{\int_0^T |\sigma_f \mathbf{n} \cdot \mathbf{t}(t, x)| dt}. $$  \hspace{1cm} (3.2.5)

We focus on RRT since high RRT distribution is emerging as an appropriate tool for identifying the possible regions of atheromatous concentrations and potential arterial lesions [52, 82].

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Figure 3.2.5: The magnitude of wall shear $|\sigma_f n \cdot t|$ stress along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 1, case 2, and case 3.

Figure 3.2.6: RRT along with the top arterial wall for case 1, case 2, and case 3.

We can see from Figure 3.2.6 that RRT is greatly influenced by permeability. Even though the behaviors of case 1 and case 2 are very similar at different time spots, we observe differences in values between case 1 and case 2.
Figure 3.2.7: Displacement in the normal direction $\eta_\star \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 1, case 2, and case 3.

In Figure 3.2.7, we present the displacement in the normal direction $\eta_\star \cdot n$ along with the top arterial wall. We observe that the peaks of it coincide with the peaks of velocity magnitude and pressure. Notice that the peaks of $\eta_p \cdot n$ in case 3 are overall smaller than case 1 and case 2. For larger permeability $K$ in case 3, the peaks of normal displacement would be smaller.

Figure 3.2.8: Velocity in the normal direction along with the top arterial wall $u_f \cdot n$ at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 1, case 2, and case 3.
Figure 3.2.9: Darcy velocity in the normal direction $u_p \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 2 and case 3.

We present fluid velocity in normal $u_f \cdot n$ on the interface in Figure.3.2.8 and the Darcy velocity in the normal direction $u_p \cdot n$ along with the interface, which is known as filtration velocity, for case 2 and case 3 in Figure.3.2.9. Note that the absence of filtration velocity in case 1 is due to the NSE/E model. For both cases, the peaks of filtration velocity coincide with the ones of structure displacement in the normal direction $\eta_p \cdot n$ in Figure.3.2.7. We also notice that the Darcy velocity $u_p \cdot n$ in both cases decreases when the fluid penetrates into the arterial wall gradually, which is a consequence of the third equation in (3.1.10). This equation describes that $\nabla \cdot u_p$ is not locally preserved, and it depends on the rate of change in pressure and displacement. Note that in Figure.3.2.9 the normal velocity of case 2 is not zero. It is small compared with case 3 since a relatively small permeability. At time $t = 1.8, 3.6, 5.4$ ms, the value of $u_p \cdot n$ in case 3 is much bigger than the one of case 2. In conclusion, larger permeability would result in larger filtration velocity.

3.2.2 Non-Newtonian models

In this section, we compare the effects of non-Newtonian property and the effects of permeability on NSE/P models. We use the same computational domain as in the Newtonian models shown in Figure.3.1.1 part (a). In non-Newtonian fluids, viscosity can change with force, velocity, or temperature to either more liquid or more solid. Most commonly, the viscosity of non-Newtonian fluids depends on shear rate or shear rate history, showing a
shear-thinning property. Even though in computational fluid dynamics, it is widely accepted to assume the blood flow is Newtonian, the non-Newtonian behavior of blood needs to be taken into consideration, especially in small vessels, like the capillaries [52, 67]. Moreover, for the middle-sized vessels, for example, the carotid or coronary vessels, we are not completely clear whether it is validated to assume the Newtonian property of blood. For these reasons, we investigate how non-Newtonian behavior affects the blood flow characteristics. There are different non-Newtonian models, such as Power Law [59], Casson [69], and Carreau-Yasuda model [52]. In this section, we perform simulations using the Carreau-Yasuda model to describe the non-Newtonian blood rheology. For the non-Newtonian fluid, instead of constant viscosity $\mu_f$, we use the following non-linear viscosity in the fluid regions for both NSE/E and NSE/P models [18, 31, 52]. In this paper, we perform simulation using the Carreau-Yasuda model to describe the non-Newtonian blood rheology. For the non-Newtonian fluid, instead of constant viscosity $\mu_f$, we use the following non-linear viscosity in the fluid regions for both NSE/E and NSE/P models [52, 18, 31],

$$\nu(x, y, t) = \nu_{\text{inf}} + (\nu_0 - \nu_{\text{inf}})(1 + (\delta \dot{\gamma}(x, y, t)^a)^{\frac{n-1}{2}}),$$

(3.2.6)

where $\dot{\gamma}(x, y, t) = \sqrt{\frac{1}{2} \text{D}(u_f) : \text{D}(u_f)}$ is the shear rate. The values of parameters defined in (3.2.6) are chosen as $\delta = 1.902 s$, $n = 0.22$, $a = 1.25$, $\nu_0 = 0.56$ Poi, $\nu_{\text{inf}} = 0.035$ Poi.

For NSE/P model, we use the above shear rate in the fluid region while we use $\dot{\gamma}(x, y, t) = \sqrt{\text{u}_p \cdot \text{u}_p}$ as shear rate for structure region.

Let us consider the following cases:

- case 4: non-Newtonian NSE/E model;
- case 5: non-Newtonian NSE/P model, with $K = \text{diag}(5, 5) \times 10^{-11}$;
- case 6: non-Newtonian NSE/P model, with $K = \text{diag}(1, 1) \times 10^{-9}$.

We do not pick $K = \text{diag}(1, 1) \times 10^{-7}$ as in case 3 since the flow rate of non-Newtonian fluids with this permeability is too small. When $t = 0.006 s$, this non-Newtonian flow would barely reach the middle position of the vessel therefore it loses the necessity in our comparison models.

In Figure 3.2.10, we present the fluids velocity and the velocity arrow for case 4, case 5 and case 6 at time $t = 1.8, 3.6, 5.4$ ms. Note that there are more fluid residuals along with
the lumen than in the Newtonian cases. The velocity in case 6 is overall smaller than the other two cases at the same time, as well as the deformation. Note that for case 4, there is no pressure variable in the structure region $\Omega_e$. We also observe similar phenomena for the pressure waves of case 4, case 5, and case 6 in Figure.3.2.11.

Figure 3.2.10: Fluid velocity magnitude together with scaled velocity arrows at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5 and case 6.

Figure 3.2.11: Pressure waves at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5 and case 6.

In Figure.3.2.12, we present the viscosity of case 4, case 5 and case 6 at time $t = 1.8, 3.6, 5.4$ ms. For case 4 with NSE/E model, since there is no Darcy flow in the structure
region, we use solid blue instead, and the color, in this case has nothing to do with the viscosity value. The viscosity of case 5 is larger than case 6 at the structure regions near the interface and, we detect the shear-thinning phenomena for case 6. At the time $t = 5.4$ ms, the dynamic viscosity of case 6 in the fluid region is overall larger than the other two cases.

![Viscosity at different times for case 4, case 5, case 6.](image)

**Figure 3.2.12**: Viscosity at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5, case 6.

In Figure.3.2.13 and Figure.3.2.14, we present the wall shear stress $\sigma_f \cdot n \cdot t$ and its magnitude $|\sigma_f \cdot n \cdot t|$ respectively for case 4, case 5 and case 6 at time $t = 1.8, 3.6, 5.4$ ms along with the interface.

![Wall shear stress for different cases.](image)

**Figure 3.2.13**: The shear stress $\sigma_f \cdot n \cdot t$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5, and case 6.
Figure 3.2.14: The shear stress $|\sigma f \cdot n| t$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5, and case 6.

In Figure 3.2.15, we report the RRT distribution along with the lumen for case 4, case 5, and case 6. The peak of case 5 is lower than case 4 even if they show similar dynamic behaviors, such as fluids velocity and pressure. As expected, the peak of case 6 is the highest and is localized ahead of case 4 and case 5.

Figure 3.2.15: RRT along with the top arterial wall for case 4, case 5 and case 6.

In Figure 3.2.16, we show the displacement in the normal direction $\eta_\ast \cdot n$ along with the lumen. In Figure 3.2.18, we present the filtration velocity for case 5 and case 6 at time $t = 1.8, 3.6, 5.4$ ms. Overall, larger permeability corresponds to larger filtration velocities.
Figure 3.2.16: Displacement in the normal direction $\eta \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5, and case 6.

Figure 3.2.17: Velocity in the normal direction $u_f \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 4, case 5, and case 6.

Figure 3.2.18: Darcy velocity in the normal direction $u_p \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 5, case 6.
3.2.3 Comparisons between Newtonian and non-Newtonian models

In this section, we focus on the comprehensive comparison between NSE/E and NSE/P models and between Newtonian and non-Newtonian models. In Figure 3.2.19, we present the WSS at a different time for all six cases above. To understand the effects of poroelasticity, we compare case 1 and case 2, and case 4 and case 5. We do not observe significant differences between these cases; therefore, we conclude that for relatively small permeability cases, NSE/E model is sufficient to describe WSS. We then compare case 1 with case 4 and case 2 with case 6 to study non-Newtonian effects, and we draw the conclusion that Newtonian models would typically generate smaller WSS. For Newtonian fluids, larger permeability would result in smaller WSS. However, for non-Newtonian models, the peak of case 6 with larger permeability is the highest. So for the non-Newtonian case with larger permeability, the WSS would also be larger. Finally, comparing case 3 with case 5, the permeability of case 3 is $2 \times 10^3$ times larger than case 5. However, the difference between case 3 and case 5 is not dramatic. Thus non-Newtonian properties have a larger influence on WSS than permeability.

Figure 3.2.19: The shear stress $\sigma_f n \cdot t$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for all six cases.
Figure 3.2.20: The shear stress $|\sigma f \cdot t|$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for all six cases.

In Figure 3.2.21, we present the RRT for all six cases. To study non-Newtonian effects, we compare case 2 and case 6, both of which are NSE/P models with the same permeability. We can see that Newtonian models would typically generate smaller RRT. As for the effects of permeability, we can see that in both Newtonian and non-Newtonian fluids, larger permeability $K$ corresponds to larger RRT. Overall, we can see from the plots that RRT is greatly affected by different types of models and their permeabilities.

Figure 3.2.21: RRT along with the top arterial wall for all six cases.

In Figure 3.2.22, we present the displacement in normal direction $\eta \cdot n$ for all six cases above. We draw the conclusion that larger permeability would result in smaller displacement in the normal direction for both Newtonian and non-Newtonian models. In particular, non-Newtonian cases would generate smaller normal displacements.
Figure 3.2.22: Displacement in the normal direction $\eta_s \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for all six cases.

From Figure.3.2.23, the non-Newtonian property of fluids and poroelasticity don’t affect much the fluids velocity in normal directions along with the interface.

Figure 3.2.23: Velocity in the normal direction $u_f \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for all six cases.

In Figure.3.2.24, we conclude that permeability $K$ affects the filtration velocity greatly. At the time $t = 1.8$ ms, the peak of case 3 is the highest since it has the largest permeability $K$. At the time $t = 3.6, 5.4$ ms, even though the permeability of case 3 is $10^2$ times larger than case 6, the filtration velocity is still comparable, which means the non-Newtonian property has a large impact on the filtration velocity. And it is important to include the non-Newtonian characteristic of blood.
Figure 3.2.24: Darcy velocity in the normal direction $\mathbf{u}_p \cdot \mathbf{n}$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 2, case 3, case 5 and case 6.

### 3.2.4 Conclusions of healthy structures

Our comparisons show that along with the interface:

- For the velocity and pressure fields, NSE/P model with larger permeability presents smaller quantities. Similarly, non-Newtonian models tend to generate smaller velocity, pressure, and deformations. And the fluids would be more viscous.
- Small differences between NSE/E and NSE/P models are found in the quantification of WSS. Great differences are found between non-Newtonian and Newtonian models. Non-Newtonian models tend to generate larger WSS.
- RRT is affected dramatically by both non-Newtonian behaviors of blood flow and the poroelasticity of the vessel structures. To accurately compute RRT, we believe it is important to consider not only the non-Newtonian property but also the structure characteristics.
- No significant differences are found in the fluids velocities in the normal direction.
- Displacement in the normal direction is affected greatly by the non-Newtonian property, especially for NSE/P models. And non-Newtonian models with large permeability would result in a smaller normal displacement.
- Both non-Newtonian and poroelasticity make a difference in the filtration velocity. Non-Newtonian models with larger permeability would generate larger filtration velocity.
3.2.5 Non-Newtonian stenosis model

In this section, we change our computational geometry to an ideal stenosis region shown in Figure.3.1.1 part (b). Stenosis is characterized by local arterial narrowing, which is initially due to the deposition of lipid, cholesterol, and some other substances on the endothelium [72, 74]. It is important to understand the non-Newtonian fluid dynamical properties of the blood flow using a coupled model in stenotic cases. Abrupt geometrical changes along with the arterial wall may cause flow separation and appearance of the recirculation zone at the post-stenotic region [87]. For this stenotic model, we use the following cosine function to describe the interfaces: for the top interface,

\[ y = \begin{cases} 
0.5, & \text{for } 0 \leq x \leq 2 \text{ and } 4 \leq x \leq 6; \\
0.4 + 0.1 \cdot \cos(\pi(x-2)), & \text{for } 2 \leq x \leq 4,
\end{cases} \tag{3.2.7} \]

for the bottom interface, symmetrically, we use

\[ y = \begin{cases} 
-0.5, & \text{for } 0 \leq x \leq 2 \text{ and } 4 \leq x \leq 6; \\
-0.4 + 0.1 \cdot \cos(\pi(x-3)), & \text{for } 2 \leq x \leq 4.
\end{cases} \tag{3.2.8} \]

Note that for the plots along with the interface of this section, the x-axis would be [0, 6.048] cm.

We still use the same non-linear viscosity in the fluid and structure regions as in equation (3.2.6) as well as the same inflow/outflow boundary conditions shown in Section 3.1. In this section, we investigate how the build-up area affects non-Newtonian blood rheology. And in the meantime, we still focus on the difference between NSE/P and NSE/E models. In addition, we add the variation of Lamé coefficients to fit the stenosis case better. To do so, we introduce the following non-Newtonian cases:

- **case 7**: NSE/E model with constant Lamé coefficients pair shown in Table.3.2.1 for structures;
- **case 8**: NSE/E model with piecewise constant Lamé coefficients pair;
- **case 9**: NSE/P model with constant Lamé coefficients pair shown in Table.3.2.1 and constant permeability \( K = \text{diag}(1,1) \times 10^{-9} \) for structures;
• case 10: NSE/P mode with piecewise constant Lamé coefficients pair and permeability.

We start by describing the piecewise constant Lamé coefficients in detail. For the Lamé coefficients in case 8 and case 10, we still adopt \( \lambda^* = 4.28 \times 10^6 \) and \( \mu^* = 1.07 \times 10^6 \) in the non-stenosis area, namely when \( x \in [0, 6], y \in [0.5, 0.6] \) or \( y \in [-0.6, -0.5] \). While for the stenosis area, we keep the same Poisson’s ratio \( \tilde{\nu} \) but decrease Young’s modulus \( E \) from \( 2.996 \times 10^6 \) into \( 2.996 \times 10^4 \). For case 8 and case 10, we use the following Lamé coefficients for the stenosis area only:

\[
\lambda^*(x, y) = \begin{cases} 
4.28 \times 10^4, & \text{for } 2 \leq x \leq 4 \text{ and } 0.4 + 0.1 \cdot \cos(\pi(x - 2)) \leq y \leq 0.5; \\
4.28 \times 10^4, & \text{for } 2 \leq x \leq 4 \text{ and } -0.5 \leq y \leq -0.4 + 0.1 \cdot \cos(\pi(x - 3)); 
\end{cases}
\]

\[
\mu^*(x, y) = \begin{cases} 
1.07 \times 10^4, & \text{for } 2 \leq x \leq 4 \text{ and } 0.4 + 0.1 \cdot \cos(\pi(x - 2)) \leq y \leq 0.5; \\
1.07 \times 10^4, & \text{for } 2 \leq x \leq 4 \text{ and } -0.5 \leq y \leq -0.4 + 0.1 \cdot \cos(\pi(x - 3)); 
\end{cases}
\]

Normally the build-up stenosis area would be not only softer but also more permeable. We use the following piecewise constant function for case 10 for the stenotic area:

\[
K(x, y) = \begin{cases} 
\text{diag}(1, 1) \times 10^{-7}, & \text{for } 2 \leq x \leq 4 \text{ and } 0.4 + 0.1 \cdot \cos(\pi(x - 2)) \leq y \leq 0.5; \\
\text{diag}(1, 1) \times 10^{-7}, & \text{for } 2 \leq x \leq 4 \text{ and } -0.5 \leq y \leq -0.4 + 0.1 \cdot \cos(\pi(x - 3)); 
\end{cases}
\]

And for the remaining structure regions in case 10, we still use \( K = \text{diag}(1, 1) \times 10^{-9} \).
Figure 3.2.25: Fluid velocity magnitude together with scaled velocity arrows at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 7, case 8, case 9 and case 10.

In Figure 3.2.25 and Figure 3.2.27, we present the fluids velocity and pressure waves along with the vessel at time $t = 1.8, 3.6, 5.4$ ms. Note that we still use moving mesh in the simulation for this section. However, we don’t magnify the deformation as in the former sections to avoid confusion with geometries. From the plots, we note that the velocity waves are strongly influenced by stenosis. In Figure 3.2.27, we can observe further differences in pressure fields among four stenotic cases. In particular, we can detect clear increases of $p_p$ in the stenotic areas in case 10. Since we use a piecewise permeability $K$ for the stenosis in case 10, more fluids are allowed in the stenosis areas, which would result in higher pressure bands in the structure areas.
Figure 3.2.26: Darcy velocity magnitude together with the scaled velocity arrow in the stenotic area for the top structure at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 9 and case 10.

Figure 3.2.27: Pressure waves in the vessel at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 7, case 8, case 9 and case 10.
Figure 3.2.28: Pressure waves in the stenotic area for the top structure at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 9 and case 10.

In Figure.3.2.26 and Figure.3.2.28, we also display Darcy velocity and pressure in the top stenotic area for case 9 and case 10 separately. Note that in Figure.3.2.26 for better observation, the length of the velocity arrow in case 9 is magnified 40 times while we only magnify case 10 for 5 times. From Figure.3.2.26, we can see the fluids would concentrate on the interface for case 9. Less filtration would be found in case 9, even for the stenotic area. But for case 10, fluids can actually penetrate through the stenotic area. In Figure.3.2.28, we detect a clear discontinuity of pressure fields of case 10 compared with case 9.

In Figure.3.2.29, we present the fluid viscosity in the fluid and structure regions. More differences are observed for the top build-up structure areas shown in Figure.3.2.30. Being consistent with Figure.3.2.26, the decrease of viscosity would only show up along with the interface of case 9. We can see a clear decrease in the viscosity of case 10 through the structure.
In Figure.3.2.31, the WSS for case 7, case 8, case 9, and case 10 are shown, respectively. As expected, at three different times, the WSS of case 7 and case 8 are close to each other, while case 9 is similar to case 10. Comparing case 9 and case 10, we can make a conclusion that the discontinuities of permeability and Lamé do not affect the WSS much. In conclusion, it is the poroelasticity that makes a difference in WSS. In Figure.3.2.33, we present the RRT for four stenosis cases. We use different plots since the scales of RRT for four cases are
at difference. We lose some information about case 7 and case 8 if we put them together. The same pattern is also applied to some of the following plots. There are huge differences between NSE/E and NSE/P models. The peaks of NSE/P models would be much higher than that in NSE/E models. The peaks of case 7 and case 8 are only attaining $0.45 \, \text{s}^{-1}$. While for case 9, the peak could reach approximately $30 \, \text{s}^{-1}$. In addition, for NSE/P models, the peaks of RRT would typically show up before the stenosis part.

Figure 3.2.31: The wall shear stress $\sigma_f \mathbf{n} \cdot \mathbf{t}$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 7, case 8, case 9 and case 10.

Figure 3.2.32: The magnitude of the wall shear stress $|\sigma_f \mathbf{n} \cdot \mathbf{t}|$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 7, case 8, case 9 and case 10.
Figure 3.2.33: RRT along with the top arterial wall. The left plot is for case 7, case 8. The right plot is for case 9 and case 10.

In Figure 3.2.34, we present the displacement in the normal direction $\eta_s \cdot n$ along with the arterial wall at time $t = 1.8, 3.6, 5.4$ ms. Comparing case 7 with case 8, we note that smaller Lamé coefficients would contribute to larger displacement in the normal direction. While when we compare case 8 with case 10, NSE/P model results in smaller normal displacement. We conclude that both Lamé coefficients and permeability $K$ affect the normal displacement. Another interesting phenomenon is that at time $t = 3.6$ ms, unlike the previous cases with healthy structures, the peaks of $\eta_s \cdot n$ of all four stenotic cases, showing up at the beginning part of stenosis, do not coincide with the peaks of pressure $p_f$.

Figure 3.2.34: Displacement in the normal direction $\eta_s \cdot n$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 7, case 8, case 9 and case 10.

In fig.3.2.35, we present the fluid velocity in the normal direction $u_f \cdot n$ along with the arterial wall at different times for case 7, case 8, case 9, and case 10, respectively.
Figure 3.2.35: Fluid velocity in the normal direction $\mathbf{u}_f \cdot \mathbf{n}$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 7, case 8, case 9 and case 10.

In Figure 3.2.36 and Figure 3.2.37, we present the Darcy velocities in the normal $\mathbf{u}_p \cdot \mathbf{n}$ and tangential $\mathbf{u}_p \cdot \mathbf{t}$ directions along with the interface for case 9 and case 10, respectively. As expected, there are remarkable differences in the filtration velocities $\mathbf{u}_p \cdot \mathbf{n}$ for these NSE/P cases. For case 9 with constant permeability, the magnitude of filtration velocity $\mathbf{u}_p \cdot \mathbf{n}$ is much smaller than that of case 10, and so is $\mathbf{u}_p \cdot \mathbf{t}$. In addition, when we compare case 9 with case 6, which has differences only in the geometry, Darcy velocities of case 9 in both directions are much smaller than case 6. In conclusion, on the one hand, similar to the previous healthy structure cases, larger permeability would result in larger Darcy velocities in both directions; on the other hand, since the change of geometry, the appearance of stenosis would result in smaller Darcy velocity in both directions.

3.2.6 Conclusions of stenotic structures

From the stenosis models, we draw the following conclusions:

- Compared with the healthy vessel structure, the stenotic geometry of arteries has a vital effect on blood flow patterns. Flow variables (velocities and pressure fields) are affected dramatically. The general flow rates are large, and the pressure loss is slight even for larger permeability.
- Poroelasticity affects the WSS dramatically. WSS of stenotic models is in general, larger than the healthy structure models.
Figure 3.2.36: Darcy velocity in the normal direction $\mathbf{u}_p \cdot \mathbf{n}$ along with the top arterial wall at time $t=1.8$ ms, $t=3.6$ ms, $t=5.4$ ms for case 9 and case 10. The top panel of this figure is for case 9, and the bottom panel is case 10.

- Great differences between NSE/E and NSE/P models are found in the quantification of RRT for the stenotic geometry. For NSE/P models, the peak of RRT typically appears at the prior-stenotic area. It is the combination of permeability and Lamé coefficients that make a difference. There would be smaller risks of vessel lesions for softer and more permeable build-ups. NSE/E model would typically generate a smaller RRT. We believe that the poriferous property of the vessel should not be neglected to calculate RRT accurately. Models without considering the structure would typically underestimate the arterial lesion.

- For stenosis models, the displacement in the normal direction is not affected much by the permeability $K$, but by the Lamé coefficients. The peak of normal displacement would generally occur at the prior-stenosis regions, and it would not coincide with the peak of pressure fields.

- For fluids velocity in the normal direction along with the arterial wall, no significant
differences are detected.

- For Darcy velocities in normal and tangential directions along with the lumen, they are greatly influenced by permeability $K$. NSE/P models with larger permeability would result in larger Darcy velocities in both normal and tangential directions. Compared with healthy structure, the stenotic geometry tends to impede the penetration on the interface.
Multicomponent transport phenomena in poroelastic media are often studied in civil, environmental, and technical engineering [2, 61, 73]. For example, a simulated drug delivery process without hemodiagnosis significantly improves patients’ experience. The simulation of underground water contaminant transport can benefit and guide the cleaning-up both economically and human costly. However, a reliable and efficient numerical simulation of such processes is not a fully tackled task. Typically, the model equations are strongly coupled such that inaccuracies in one unknown directly affect all other unknowns. Chapter 4 explores the two-way coupled Stokes-Biot–transport model with non-linear fluid viscosities. In sections 4.1 and 4.1.2, we specify the governing PDEs and the associated weak formulation. We study the well-posedness of a linearized version in Section 4.2. Using non-linear analysis, we obtain the existence and uniqueness of the original solution, followed by the formal error analysis. In Section 4.6, the convergence test for algorithm robustness and the transport phenomena of mercury dissolved in water are presented.

4.1 The coupled model and its weak formulation

4.1.1 Stokes–Biot–transport model

In this chapter, again let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a union of non-overlapping polygonal regions $\Omega_f$ and $\Omega_p$. Here $\Omega_f$ is a free fluid region with flow governed by the Stokes equations, and $\Omega_p$ is a poroelastic region governed by the Biot system. Let $\Gamma_{fp} = \partial \Omega_f \cap \partial \Omega_p$ denote the (non-empty) interface between these regions and let $\Gamma_f = \partial \Omega_f \setminus \Gamma_{fp}$ and $\Gamma_p = \partial \Omega_p \setminus \Gamma_{fp}$ denote the external parts on the boundary $\partial \Omega$. We denote by $\mathbf{n}_f$ and $\mathbf{n}_p$ the unit normal vectors that point outward from $\partial \Omega_f$ and $\partial \Omega_p$, respectively, noting that $\mathbf{n}_f = -\mathbf{n}_p$ on $\Gamma_{fp}$. Let $(\mathbf{u}_*, p_*)$ be the velocity-pressure pair in $\Omega_*$, $* \in \{f, p\}$. Let $\eta_p$ be the displacement in
\( \Omega_p, \mu(c) \) be the fluid viscosity that depends on the concentration \( c \) of a substance of a solute transported by the fluid, \( \mathbf{f}_f \) be the body force, and \( q_* \) be an external source or sink terms. In the free fluid region \( \Omega_f \), the velocity \( \mathbf{u}_f \) and pressure \( p_f \) satisfy the Stokes equations

\[
- \nabla \cdot \sigma_f = \mathbf{f}_f, \quad \nabla \cdot \mathbf{u}_f = q_f \quad \text{in} \quad \Omega_f \times (0, T],
\]

\[
\mathbf{u}_f = 0 \quad \text{on} \quad \Gamma^D_f \times (0, T], \quad \sigma_f \mathbf{n}_f = 0 \quad \text{on} \quad \Gamma^N_f \times (0, T],
\]

where \( \sigma_f = -p_f \mathbf{I} + 2\mu(c) \mathbf{e}(\mathbf{u}_f) \) is the Cauchy stress tensor, \( \mathbf{e}(\mathbf{u}_f) = \frac{1}{2} (\nabla \mathbf{u}_f + \nabla \mathbf{u}_f^T) \) stands for the deformation rate tensor, \( T > 0 \) is the final time, and \( \Gamma_f = \Gamma^D_f \cup \Gamma^N_f \). We assume that \( |\Gamma^D_f| > 0 \). In the poroelastic region \( \Omega_p \) we let \( \Gamma_p = \Gamma^D_p \cup \Gamma^N_p = \tilde{\Gamma}^D_p \cup \tilde{\Gamma}^N_p \) and consider the quasi-static Biot system

\[
- \nabla \cdot \sigma_p = \mathbf{f}_p, \quad \mu(c) K^{-1} \mathbf{u}_p + \nabla p_p = 0 \quad \text{in} \quad \Omega_p \times (0, T],
\]

\[
\frac{\partial}{\partial t} (s_0 p_p + \alpha \nabla \cdot \mathbf{\eta}_p) + \nabla \cdot \mathbf{u}_p = q_p \quad \text{in} \quad \Omega_p \times (0, T],
\]

\[
p_p = 0 \quad \text{on} \quad \Gamma^D_p \times (0, T], \quad \mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma^N_p \times (0, T],
\]

\[
\mathbf{\eta}_p = 0 \quad \text{on} \quad \tilde{\Gamma}^D_p \times (0, T], \quad \sigma_p \mathbf{n}_p = 0 \quad \text{on} \quad \tilde{\Gamma}^N_p \times (0, T],
\]

where the poroelastic stress tensor \( \sigma_p \) and the Cauchy stress tensor \( \sigma_e \) are defined, respectively, as

\[
\sigma_p = \sigma_e - \alpha p_p \mathbf{I}, \quad \sigma_e = \lambda_p (\nabla \cdot \mathbf{\eta}_p) \mathbf{I} + 2\mu_p \mathbf{e}(\mathbf{\eta}_p),
\]

where

\[
0 < \lambda_{min} \leq \lambda_p(x) \leq \lambda_{max}, \quad 0 < \mu_{min} \leq \mu_p(x) \leq \mu_{max}
\]

are the Lamé parameters. In addition, \( \alpha > 0 \) is the Biot-Willis constant, \( s_0 > 0 \) is a storage coefficient and \( K \) is a symmetric and uniformly positive definite rock permeability tensor, satisfying for some constants \( 0 < k_{min} \leq k_{max}, \forall \xi \in \mathbb{R}^d \),

\[
k_{min} \xi^T \xi \leq \xi^T K(x) \xi \leq k_{max} \xi^T \xi \quad \forall x \in \Omega_p.
\]

We assume that \( |\Gamma^D_p| > 0 \) and \( |\tilde{\Gamma}^D_p| > 0 \). In addition, to simplify the characterization of the normal trace of the Darcy velocity on \( \Gamma_{fp} \), we assume that \( \text{dist}(\Gamma^D_p, \Gamma_{fp}) \geq s > 0 \). We assume
that \( \mu(\cdot) \) belongs to \( C^1(\mathbb{R}^+ \cup \{0\}) \) and there exists positive constants \( \mu_L \) and \( \mu_U \) such that for any \( x \in \mathbb{R}^+ \cup \{0\} \),
\[
0 < \mu_L \leq \mu(x) \leq \mu_U \quad \text{and} \quad |\mu'(x)| \leq \mu_U.
\]
(4.1.5)

We further assume that both \( \mu(\cdot) \) and \( \mu'(\cdot) \) are Lipschitz continuous, i.e., for all \( x, y \in \mathbb{R}^+ \), there exists positive constants \( L_1 \) and \( L_2 \) such that
\[
|\mu(x) - \mu(y)| \leq L_1 |x - y| \quad \text{and} \quad |\mu'(x) - \mu'(y)| \leq L_2 |x - y|.
\]
(4.1.6)

The coupling conditions on the fluid-poroelasticity interface \( \Gamma_{fp} \times (0, T] \) are mass conservation, balance of force, balance of normal stress, and the Beavers-Joseph-Saffman (BJS) slip with friction condition [11]:
\[
\begin{align*}
\mathbf{u}_f \cdot \mathbf{n}_f + \left( \frac{\partial \mathbf{n}_p}{\partial t} + \mathbf{u}_p \right) \cdot \mathbf{n}_p &= 0, & -\mathbf{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f &= p_p, \\
\mathbf{\sigma}_f \mathbf{n}_f + \mathbf{\sigma}_p \mathbf{n}_p &= 0, & -(\mathbf{\sigma}_f \mathbf{n}_f) \cdot \mathbf{\tau}_{f,j} &= \mu(c_0) \alpha_{BJS} \sqrt{K_j^{-1}} \left( \mathbf{u}_f - \frac{\partial \mathbf{n}_p}{\partial t} \right) \cdot \mathbf{\tau}_{f,j},
\end{align*}
\]
(4.1.7)

where \( \tau_{f,j}, 1 \leq j \leq d-1 \), is an orthogonal system of the unit tangential vectors on \( \Gamma_{fp} \), \( K_j := (K \mathbf{\tau}_{f,j}) \cdot \mathbf{\tau}_{f,j}, \alpha_{BJS} \geq 0 \) is an experimentally determined friction coefficient, and \( \mu(c_0(x)) \) is the viscosity on the interface, where \( c_0(x) \) is the initial concentration, cf. (4.1.11).

We set the initial condition as follows:
\[
p_p(x, 0) = p_{p,0}(x) \quad \text{in} \quad \Omega_p.
\]

The initial data of the remaining variables are constructed to satisfy a compatibility condition. The details are discussed in Theorem 2.2.4.

The Stokes–Biot model is further coupled with the following transport equation for the solute concentration \( c(\mathbf{x}, t) \) in \( \Omega \):
\[
\begin{align*}
\phi \frac{\partial c}{\partial t} + \nabla \cdot (c \mathbf{u} - \mathbf{D}(\mathbf{u}) \nabla c) &= q \tilde{c} \quad \text{in} \quad \Omega \times (0, T], \\
c &= 0 \quad \text{on} \quad \Gamma_{in} \times (0, T], \quad (\mathbf{D}(\mathbf{u}) \nabla c) \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma_{out} \times (0, T],
\end{align*}
\]
(4.1.8)

where \( \mathbf{n} \) is the unit outward normal vector to \( \partial \Omega \) and \( \partial \Omega = \Gamma_{in} \cup \Gamma_{out} \). We assume that \( |\Gamma_{in}| > 0, \Gamma_{out} \cap \Gamma_f^N = \emptyset \) and \( \Gamma_{out} \cap \Gamma_p^D = \emptyset \). In addition, \( 0 < \phi_* \leq \phi(\mathbf{x}) \leq \phi^* \leq 1 \) is the
porosity of the medium in $\Omega_p$, while in $\Omega_f$ it is set to be 1, $u$ is the velocity field over the entire domain $\Omega$, defined as $u|_{\Omega_f} = u_f$, $u|_{\Omega_p} = u_p$, $q$ is the source term given by $q|_{\Omega_f} = q_f$, $q|_{\Omega_p} = q_p$, respectively, whereas $\tilde{c}$ is defined as follows:

$$\tilde{c} = \begin{cases} 
\text{injected concentration } c_w, & \text{for } q > 0, \\
\text{resident concentration } c, & \text{for } q < 0.
\end{cases}$$

(4.1.9)

Letting $q^+ = \max\{q, 0\}$ and $q^- = \min\{q, 0\}$ results in $q\tilde{c} = q^+c_w + q^-c$. $D(u)$ denotes the diffusion-dispersion tensor in (4.1.8), which is a non-linear function of the velocity that combines the effects of molecular diffusion and mechanical dispersion, defined as:

$$D(u) = \begin{cases} 
\phi_d m I & \text{in } \Omega_f, \\
\phi_d m I + |u| [\alpha_l E(u) + \alpha_t (I - E(u))], & \text{in } \Omega_p,
\end{cases}$$

(4.1.10)

where $d_m$ is the molecular diffusivity, the constants $\alpha_l > 0$ and $\alpha_t > 0$ are the longitudinal and transverse dispersion, respectively, and $E(u)$ is the projection tensor onto the direction of $u$ with entries $(E(u))_{i,j} = \frac{u_i u_j}{|u|^2}$. Here, for a vector $\zeta \in \mathbb{R}^d$, $|\zeta|^2 = \zeta^T \zeta$. The transport model is complemented by the initial condition for the concentration,

$$c(x, 0) = c_0(x) \text{ in } \Omega.$$ 

(4.1.11)

**Remark 4.1.1.** The choice of homogeneous inflow condition $c = 0$ on $\Gamma_{in}$ is made merely for simplicity of the presentation. The analysis easily extends to the case $c = c_{in}$ on $\Gamma_{in}$.

**Remark 4.1.2.** Note that in Chapters 2 and 3, the operator $D(u_f)$ is defined to be the deformation rate which is associated with stress tensors. However, starting from Chapter 4, in order to follow the notation used massively, we change $D(u)$ to denote the diffusion-dispersion tensor and use $\epsilon(u_f)$ to represent the deformation rate.
4.1.2 Weak formulation

We start by introducing the functional spaces for the Stokes region $\Omega_f$, the Biot region $\Omega_p$ and the transport region $\Omega = \Omega_f \cup \Omega_p$:

\[
V_f := \left\{ v_f \in H^1(\Omega_f) : v_f = 0 \text{ on } \Gamma^D_f \right\}, \quad W_f := L^2(\Omega_f),
\]

\[
V_p := \left\{ v_p \in H(\text{div}; \Omega_p) : v_p \cdot n_p = 0 \text{ on } \Gamma^N_p \right\}, \quad W_p := L^2(\Omega_p),
\]

\[
X_p := \left\{ \xi_p \in H^1(\Omega_p) : \xi_p = 0 \text{ on } \Gamma_p \right\},
\]

\[
X_c := \left\{ \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_{in} \right\},
\]

where $H(\text{div}; \Omega_p) := \left\{ v \in L^2(\Omega_p) : \nabla \cdot v \in L^2(\Omega_p) \right\}$ endowed with the norm

\[
\| v \|^2_{H(\text{div}; \Omega_p)} := \| v \|^2_{L^2(\Omega_p)} + \| \nabla \cdot v \|^2_{L^2(\Omega_p)}.
\]

The above spaces are equipped with the norms

\[
\| v_f \|_{V_f} := \| v_f \|_{H^1(\Omega_f)}, \quad \| w_f \|_{W_f} := \| w_f \|_{L^2(\Omega_f)},
\]

\[
\| v_p \|_{V_p} := \| v_p \|_{H(\text{div}; \Omega_p)}, \quad \| w_p \|_{W_p} := \| w_p \|_{L^2(\Omega_p)},
\]

\[
\| \xi_p \|_{X_p} := \| \xi_p \|_{H^1(\Omega_p)}, \quad \| \psi \|_{X_c} := \| \psi \|_{H^1(\Omega)}.
\]

We derive the Stokes–Biot weak formulation by testing (4.1.1)–(4.1.2) with arbitrary $v_f \in V_f$, $w_f \in W_f$, $\xi_p \in X_p$, $v_p \in V_p$, and $w_p \in W_p$, and integrating by parts the terms involving $\nabla \cdot \sigma_f$, $\nabla \cdot \sigma_p$, and $\nabla p$. This results in the following functionals related to the Stokes, Darcy, and elasticity operators:

\[
a_f(u_f, v_f; c) := (2\mu(c)\epsilon(u_f), \epsilon(v_f))_{\Omega_f}, \quad a^d_p(u_p, v_p; c) := (\mu(c)K^{-1}u_p, v_p)_{\Omega_p},
\]

\[
a_p(\eta_p, \xi_p) := (2\mu_p\epsilon(\eta_p), \epsilon(\eta_p))_{\Omega_p} + (\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p)_{\Omega_p},
\]

\[
b_* (v_*, w_*) := -(\nabla \cdot v_*, w_*)_{\Omega_*}, \quad * \in \{ f, p \}, \quad b_0(\xi_p, w_p) := -(\nabla \cdot \xi_p, w_p)_{\Omega_p},
\]

and the interface term:

\[
I_{\Gamma fp} := -\langle \sigma_f n_f, v_f \rangle_{\Gamma fp} - \langle \sigma_p n_p, \xi_p \rangle_{\Gamma fp} + \langle p_p, v_p \cdot n_p \rangle_{\Gamma fp}.
\]
Introducing the Lagrange multiplier

\[ \lambda = - (\sigma_f n_f) \cdot n_f = p_p \quad \text{on} \quad \Gamma_{fp}, \]

and using the second, third and fourth equations in (4.1.7), we obtain

\[ I_{\Gamma_{fp}} = a_{BJS}(u_f, \partial_t \eta_p; v_f, \xi_p) + b_T(v_f, v_p, \xi_p; \lambda), \]

where

\[ a_{BJS}(u_f, \eta_p; v_f, \xi_p) := \alpha_{BJS} \sum_{j=1}^{d-1} \left\langle \mu(c_0) \sqrt{K_j^{-1}} (u_f - \eta_p) \cdot \tau_{f,j}, (v_f - \xi_p) \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}}, \]

\[ b_T(v_f, v_p, \xi_p; \lambda) := \langle v_f \cdot n_f + (\xi_p + v_p) \cdot n_p, \lambda \rangle_{\Gamma_{fp}}. \]

The continuity of normal flux condition, cf. the first equation in (4.1.7), is imposed weakly using a test function \( \nu \in \Lambda \). Finally, the transport equation (4.1.8) is tested by an arbitrary \( \psi \in X_c \). We obtain the following coupled Stokes–Biot–transport weak formulation.

\[ (P) \quad \text{Given} \quad f_f, f_p, q_p, q_f, q, \text{ and } c_w, \text{ find} \quad (u_f, p_f, \eta_p, u_p, p_p, \lambda, c) : [0, T] \rightarrow V_f \times W_f \times X_p \times V_p \times W_p \times \Lambda \times X_c, \text{ such that for a.e. } t \in (0, T) \text{ and for all } v_f \in V_f, w_f \in W_f, \xi_p \in X_p, \]

\[ v_p \in V_p, w_p \in W_p, \nu \in \Lambda, \text{ and } \psi \in X_c, \]

\[ a_f(u_f, v_f; c) + b_f(v_f, p_f) + a_{BJS}(u_f, \partial_t \eta_p; v_f, 0) + b_T(v_f, 0, 0; \lambda) = (f_f, v_f)_{\Omega_f}, \quad (4.1.12a) \]

\[ - b_f(u_f, w_f) = (q_f, w_f)_{\Omega_f}, \quad (4.1.12b) \]

\[ a^c_p(\eta_p, \xi_p) + ab_v(\xi_p, p_p) + a_{BJS}(u_f, \partial_t \eta_p; 0, \xi_p) + b_T(0, 0, \xi_p; \lambda) = (f_p, \xi_p)_{\Omega_p}, \quad (4.1.12c) \]

\[ a^d_p(u_p, v_p; c) + b_p(v_p, p_p) + b_T(0, v_p, 0; \lambda) = 0, \quad (4.1.12d) \]

\[ s_0(\partial_t p_p, w_p)_{\Omega_p} - ab_v(\partial_t \eta_p, w_p) - b_p(u_p, w_p) = (q_p, w_p)_{\Omega_p}, \quad (4.1.12e) \]

\[ b_T(u_f, u_p, \partial_t \eta_p; \nu) = 0, \quad (4.1.12f) \]

\[ (\phi \partial_t c, \psi) + (D(u) \nabla c, \nabla \psi) - (cu, \nabla \psi) - (q-c, \psi) = (q^+ c_w, \psi). \quad (4.1.12g) \]

We next collect several properties of the functionals, which will be helpful in the analysis.

We define the following seminorm for \( v_f \in V_f \) and \( \xi_p \in X_p \) on the interface \( \Gamma_{fp} \):

\[ |v_f - \xi_p|^2_{BJS} = \sum_{j=1}^{d-1} \left\langle (v_f - \xi_p) \cdot \tau_{f,j}, (v_f - \xi_p) \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}}. \]
We also employ the following notation:

\[ \| (v_f, v_p) \|_{W_f \times V_p}^2 = \| v_f \|_{W_f}^2 + \| v_p \|_{V_p}^2, \quad \| (w_f, w_p, \nu) \|_{W_f \times W_p \times \Lambda}^2 = \| w_f \|_{W_f}^2 + \| w_p \|_{W_p}^2 + \| \nu \|_{\Lambda}^2. \]

**Lemma 4.1.1.** There exist positive constants \( c_f, c_f', c_p, c_p', c_e, c_e', c_I, \) and \( c_I' \) such that for all \( u_f, v_f \in V_f, u_p, v_p \in V_p, \eta_p, \xi_p \in X_p \), there hold

\[
\begin{align*}
&c_f \| v_f \|_{H^1(\Omega_f)}^2 \leq a_f(v_f, v_f; c), \quad a_f(u_f, v_f; c) \leq c_f' \| u_f \|_{H^1(\Omega_f)} \| v_f \|_{H^1(\Omega_f)}, \\
&c_p \| v_p \|_{L^2(\Omega_p)}^2 \leq a_p(v_p, v_p; c), \quad a_p(u_p, v_p; c) \leq c_p' \| u_p \|_{L^2(\Omega_p)} \| v_p \|_{L^2(\Omega_p)}, \\
&c_e \| \xi_p \|_{H^1(\Omega_p)}^2 \leq a_e(\xi_p, \xi_p), \quad a_e(\eta_p, \xi_p) \leq c_e' \| \eta_p \|_{H^1(\Omega_p)} \| \xi_p \|_{H^1(\Omega_p)}, \\
&c_I \| v_f - \xi_p \|_{BJS}^2 \leq a_{BJS}(v_f, \xi_p; v_f, \xi_p), \quad a_{BJS}(u_f, \eta_p; v_f, \xi_p) \leq c_I \| u_f - \eta_p \|_{BJS} \| v_f - \xi_p \|_{BJS}.
\end{align*}
\]

**Proof.** We recall Korn inequality: \( \exists K_f > 0 \text{ such that } \forall v_f \in V_f, \| \varepsilon(v_f) \|_{L^2(\Omega_f)} \geq K_f \| v_f \|_{H^1(\Omega_f)}. \)

Similarly, \( \exists K_p > 0 \text{ such that } \forall \xi_p \in X_p, \| \varepsilon(\xi_p) \|_{L^2(\Omega_p)} \geq K_p \| \xi_p \|_{H^1(\Omega_p)}. \) Using (4.1.3), (4.1.4), and (4.1.5), we obtain the results with constants \( c_f = 2 \mu_L K_f^2, \quad c_f' = 2 \mu_U, \quad c_p = \mu_L / k_{\max}, \quad c_p' = 2 \mu_{\min} K_p^2, \quad c_e = 2 \mu_{\min} \mu_L + \lambda_{\max}, \quad c_e' = 2 \mu_{\max} + \lambda_{\max}, \quad c_I = \alpha_{BJS} \mu_L / \sqrt{k_{\max}}, \) and \( c_I' = \alpha_{BJS} \mu_U / \sqrt{k_{\min}}. \)

**Lemma 4.1.2.** There exists a constant \( \beta_1 > 0 \) such that for all \( (w, \nu) \in \mathcal{W} \times \Lambda, \) there holds

\[
\sup_{0 \neq (v_f, v_p) \in V_f \times V_p} \frac{b_f(v_f, w_f) + b_p(v_p, w_p) + b_T(v_f, v_p, 0; \nu)}{\| (v_f, v_p) \|_{V_f \times V_p}} \geq \beta_1 \| (w_f, w_p, \nu) \|_{W_f \times W_p \times \Lambda}.
\]

**Proof.** It follows from a slight adaption of the arguments in Lemmas 3.1 and 3.2 in [43].

**Lemma 4.1.3.** For \( u \in L^\infty(0, T; L^\infty(\Omega_p)), \) it holds that for any \( \zeta \in \mathbb{R}^d, \)

\[
\forall (x, t) \in \Omega_p \times (0, T], \quad \phi^* d_m |\zeta|^2 \leq D(u) \zeta \cdot \zeta \leq (\phi^* d_m + (\alpha_l + \alpha_t) |u|) |\zeta|^2. \quad (4.1.14)
\]

For \( u \in W^{1,\infty}(0, T; L^\infty(\Omega_p)), \) it holds that for any \( \zeta \in \mathbb{R}^d, \)

\[
\partial_t D(u) \zeta \cdot \zeta \leq 5(\alpha_t + \alpha_l) |\partial_t u| |\zeta|^2. \quad (4.1.15)
\]
Proof. Bound (4.1.14) follows from the definition of $D$ (4.1.10) and the fact that the eigenvalues of $E(u)$ are 1 and 0. For the inequality (4.1.15), we have in $\Omega_p$,

$$\partial_t D(u) = \partial_t |u| [\alpha_t E(u) + \alpha_t (I - E(u))] + (\alpha_t - \alpha_t) |u| \partial_t E(u). \quad (4.1.16)$$

Since

$$\partial_t |u| = \frac{u \cdot \partial_t u}{|u|} \leq |\partial_t u|,$$

we have

$$\partial_t |u| [\alpha_t E(u) + \alpha_t (I - E(u))] \zeta \cdot \zeta \leq (\alpha_t + \alpha_t) |\partial_t u| \zeta^2. \quad (4.1.17)$$

Next, using that

$$\partial_t (E(u)) = \partial_t \left( \frac{uu^T}{|u|^2} \right) = \frac{(\partial_t u) u^T + u (\partial_t u)^T}{|u|^2} - \frac{2(u \cdot \partial_t u)}{|u|^2} E(u),$$

we obtain

$$(\alpha_t - \alpha_t) |u| \partial_t E(u) \zeta \cdot \zeta \leq 4(\alpha_t + \alpha_t) |\partial_t u| \zeta^2. \quad (4.1.18)$$

Combining (4.1.16)–(4.1.18) gives (4.1.15).

Direct analysis of $(P)$ is difficult due to its fully-coupled nature and non-linearities. In the next section, we study a linearized and decoupled version of the problem and return to the well-posedness of $(P)$ in Section 4.3.

4.2 A linearized problem

In this section, we introduce and analyze a linearized weak formulation related to $(P)$. To that end, we first replace the terms $a_f(u_f, v_f; c)$ in (4.1.12a) and $a_p^d(u_p, v_p; c)$ in (4.1.12d) with $a_f(u_f, v_f; \gamma)$ and $a_p^d(u_p, v_p; \gamma)$, respectively, for some given $\gamma \in L^\infty(0, T; L^\infty(\Omega))$. Next, in the transport equation (4.1.12g), we replace the terms $(c u, \nabla \psi)$ and $(D(u) \nabla c, \nabla \psi)$ with $(c \theta, \nabla \psi)$ and $(D(\theta) \nabla c, \nabla \psi)$, respectively, for some given vector $\theta \in L^\infty(0, T; L^\infty(\Omega))$. We consider the following decoupled linearized problems.
(LP1) Given \( f_f, f_p, q_p, q_f, \) and \( \gamma; \) find \((u_f, p_f, \eta_p, u_p, p_p, \lambda) : [0, T] \rightarrow V_f \times W_f \times X_p \times V_p \times W_p \times \Lambda,\) such that for a.e. \( t \in (0, T) \) and for all \( v_f \in V_f, \ w_f \in W_f, \ \xi_p \in X_p, \ v_p \in V_p, \ w_p \in W_p, \) and \( \nu \in \Lambda,\)

\[
a_f(u_f, v_f; \gamma) + b_f(v_f, p_f) + a_{BJS}(u_f, \partial_t \eta_p; v_f, 0) + b_T(v_f, 0, 0; \lambda) = (f_f, v_f)_{\Omega_f}, \tag{4.2.1a}
\]

\[
-b_f(u_f, w_f) = (q_f, w_f)_{\Omega_f}, \tag{4.2.1b}
\]

\[
a_p^e(\eta_p, \xi_p) + \alpha_b(\xi_p, p_p) + a_{BJS}(u_f, \partial_t \eta_p; 0, \xi_p) + b_T(0, 0, \xi_p; \lambda) = (f_p, \xi_p)_{\Omega_p}, \tag{4.2.1c}
\]

\[
a_p^d(u_p, v_p; \gamma) + b_p(v_p, p_p) + b_T(0, v_p, 0; \lambda) = 0, \tag{4.2.1d}
\]

\[
s_0(\partial_t p_p, w_p)_{\Omega_p} - \alpha_b(\partial_t \eta_p, w_p) - b_p(u_p, w_p) = (q_p, w_p)_{\Omega_p}, \tag{4.2.1e}
\]

\[
b_T(u_f, u_p, \partial_t \eta_p; \nu) = 0. \tag{4.2.1f}
\]

(LP2) Given \( q, c_w, \) and \( \theta, \) find \( c : [0, T] \rightarrow X_c, \) such that for a.e. \( t \in (0, T) \) and for all \( \psi \in X_c,\)

\[
(\phi \partial_t c, \psi) + (D(\theta) \nabla c, \nabla \psi) - (c \theta, \nabla \psi) - (q^- c, \psi) = (q^+ c_w, \psi). \tag{4.2.2}
\]

In order to prove the well-posedness of problems (LP1) and (LP2), similarly to [21], we introduce their semi-discrete Galerkin approximations using discretization in space. We prove the solvability of (LP1) using the theory of differential algebraic equations (DAE), while for (LP2), we use the theory of ordinary differential equations (ODE). We then obtain stability estimates for the Galerkin solutions and employ weak compactness and pass to the limit to obtain the existence and stability of solutions of (LP1) and (LP2).

### 4.2.1 Semi-discrete continuous-in-time Galerkin approximation

Let \( T_h^f \) and \( T_h^p \) be shape-regular partitions of \( \Omega_f \) and \( \Omega_p, \) respectively, both consisting of affine elements with maximal element diameter \( h. \) The two partitions match at the interface \( \Gamma_{fp}. \) Let \( T_h = T_h^f \cup T_h^p. \) For the discretization of the fluid velocity and pressure we choose inf-sup stable finite element spaces \( V_{fh} \subset V_f \) and \( W_{fh} \subset W_f, \) such as the MINI elements or the Taylor-Hood spaces. For the discretization of the Darcy equation, we choose \( V_{ph} \subset V_p \) and \( W_{ph} \subset W_p \) to be any inf-sup stable mixed finite element spaces, such as the Raviart-Thomas or Brezzi-Douglas-Marini spaces [16]. In turn, let \( X_{ph} \subset X_p \) and \( X_{ch} \subset X_c \) be
conforming Lagrangian finite element spaces. Finally, we take a conforming finite element space $\Lambda_h \subset \Lambda$, defined on $T_h|_{\Gamma_{fp}}$, which is chosen as follows. If the normal trace of the space $V_{ph}$ contains piecewise polynomials in $P_k$ on simplices or $Q_k$ on cubes with $k \geq 1$, we take $\Lambda_h$ to be the space of continuous piecewise polynomials in $P_k$ or $Q_k$ on the trace of the neighboring subdomain grids. Here $P_k$ denotes the polynomials of a total degree up to $k$ and $Q_k$ stands for polynomials of a degree up to $k$ in each variable. In the case $k = 0$, we take $\Lambda_h$ to be the space of continuous piecewise polynomials in $P_1$ on a grid obtained by coarsening by two the trace of the subdomain grids. This choice is made to ensure inf-sup stability for the interface bilinear form $b_\Gamma$, stated in the following lemma.

**Lemma 4.2.1.** There exists a constant $\beta_2 > 0$ such that for all $(w_{fh}, w_{ph}, \nu_h) \in W_{fh} \times W_{ph} \times \Lambda_h$, there holds

$$\sup_{0 \neq (v_{fh}, v_{ph}) \in V_f \times V_p} \frac{b_f(v_{fh}, w_{fh}) + b_p(v_{ph}, w_{ph}) + b_\Gamma(v_{fh}, v_{ph}, 0; \nu_h)}{\| (v_{fh}, v_{ph}) \|_{V_f \times V_p}} \geq \beta_2 \| (w_{fh}, w_{ph}, \nu_h) \|_{W_{fh} \times W_{ph} \times \Lambda}.$$  \hspace{1cm} (4.2.3)

**Proof.** It follows from the proofs of [43, Lemma 4.7] for $k \geq 1$ and [49, Lemma 5.1] for $k = 0$. \hfill \Box

We consider the following linear and fully decoupled semi-discrete problems.

**LGP1** Given $f_f$, $f_p$, $q_p$, $q_f$, and $\gamma$, find $(u_{fh}, p_{fh}, \eta_{ph}, u_{ph}, p_{ph}, \lambda_h) : [0, T] \to V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \Lambda_h$, such that for a.e. $t \in (0, T]$ and for all $v_{fh} \in V_{fh}$, $w_{fh} \in W_{fh}$, $\xi_{ph} \in X_{ph}$, $v_{ph} \in V_{ph}$, $w_{ph} \in W_{ph}$, and $\nu_h \in \Lambda_h$,

$$a_f(u_{fh}, v_{fh}; \gamma) + b_f(v_{fh}, p_{fh}) + a_{BJS}(u_{fh}, \partial_t \eta_{ph}, v_{fh}, 0) + b_\Gamma(v_{fh}, 0, 0; \lambda) = (f_f, v_{fh})_{\Omega_f},$$ \hspace{1cm} (4.2.4a)

$$- b_f(u_{fh}, w_{fh}) = (q_f, w_{fh})_{\Omega_f},$$ \hspace{1cm} (4.2.4b)

$$a_p^c(\eta_{ph}, \xi_{ph}) + \alpha b_e(\xi_{ph}, p_{ph}) + a_{BJS}(u_{fh}, \partial_t \eta_{ph}; 0, \xi_{ph}) + b_\Gamma(0, 0, \xi_{ph}; \lambda_h) = (f_p, \xi_{ph})_{\Omega_p},$$ \hspace{1cm} (4.2.4c)

$$a_p^d(u_{ph}, v_{ph}; \gamma) + b_p(v_{ph}, p_{ph}) + b_\Gamma(0, v_{ph}, 0; \lambda_h) = 0,$$ \hspace{1cm} (4.2.4d)

$$s_0(\partial_t p_{ph}, w_{ph})_{\Omega_p} - \alpha b_e(\partial_t \eta_{ph}, w_{ph}) - b_p(u_{ph}, w_{ph}) = (q_p, w_{ph})_{\Omega_p},$$ \hspace{1cm} (4.2.4e)

$$s_0(\partial_t p_{ph}, w_{ph})_{\Omega_p} - \alpha b_e(\partial_t \eta_{ph}, w_{ph}) - b_p(u_{ph}, w_{ph}) = (q_p, w_{ph})_{\Omega_p},$$ \hspace{1cm} (4.2.4f)
\[ b_T(u_{fh}, u_{ph}, \partial_t \eta_{ph}, \nu_h) = 0. \] (4.2.4g)

(LGP2) Given \( q, c_w, \) and \( \theta, \) find \( c_h : [0, T] \rightarrow X_{ch}, \) such that for a.e. \( t \in (0, T] \) and for all \( \psi_h \in X_{ch}, \)

\[ (\phi \partial_t c_h, \psi_h) + (D(\theta) \nabla c_h, \nabla \psi_h) - (c_h \theta, \nabla \psi_h) - (q^- c_h, \psi_h) = (q^+ c_w, \psi_h). \] (4.2.5)

Let \( \{ \varphi_{u,i} \}, \{ \varphi_{u,p,i} \}, \{ \varphi_{\eta,i} \}, \{ \varphi_{p,i} \}, \{ \varphi_{c,i} \}, \) and \( \{ \varphi_{c,i} \} \) be the basis of \( V_{fh}, V_{ph}, X_{ph}, W_{fh}, W_{ph}, \Lambda_h, \) and \( X_{ch}, \) respectively. We introduce the following matrices:

\[
(M_p)_{ij} = (\varphi_{p,p,j}, \varphi_{p,p,i}) \eta_p; \quad (M_c)_{ij} = (\varphi_{c,j}, \varphi_{c,i});
\]

\[
(A_f)_{ij} = a_f(\varphi_{u,i,j}, \varphi_{u,i,j}; \gamma); \quad (A^d_p)_{ij} = a^d_p(\varphi_{u,p,j}, \varphi_{u,p,i}; \gamma); \quad (A^e_p)_{ij} = a^e_p(\varphi_{\eta,j}, \varphi_{\eta,i});
\]

\[
(A^f_{BJS})_{ij} = a_{BJS}(\varphi_{u,i,j}, 0; \varphi_{u,i,j}, 0); \quad (A^e_{BJS})_{ij} = a_{BJS}(\varphi_{u,i,j}, 0; 0, \varphi_{\eta,i});
\]

\[
(A^c_{BJS})_{ij} = a_{BJS}(0, \varphi_{\eta,j}, 0, \varphi_{\eta,i});
\]

\[
(B_f)_{ij} = b_f(\varphi_{u,i,j}, \varphi_{p,i,j}); \quad (B_p)_{ij} = b_p(\varphi_{u,p,j}, \varphi_{p,p,i}); \quad (B_c)_{ij} = b_c(\varphi_{\eta,j}, \varphi_{p,p,i});
\]

\[
(B_{f,\Gamma})_{ij} = b_r(\varphi_{u,i,j}, 0, \varphi_{\lambda,i}); \quad (B_{p,\Gamma})_{ij} = b_r(0, \varphi_{u,p,j}, 0, \varphi_{\lambda,i}); \quad (B_{c,\Gamma})_{ij} = b_r(0, 0, \varphi_{\eta,j}, \varphi_{\lambda,i});
\]

\[
(D^1_c)_{ij} = (D(\theta) \nabla \varphi_{c,j}, \nabla \varphi_{c,i}); \quad (D^2_c)_{ij} = -(\theta \varphi_{c,j}, \nabla \varphi_{c,i}); \quad (D^- c)_{ij} = -(q^- \varphi_{c,j}, \varphi_{c,i}).
\]

For the data on the right hand side of (LGP1) and (LGP2), let

\[
(F_f)_{i} := (f, \varphi_{u,i}) \Omega_f; \quad (F_p)_{i} := (f, \varphi_{p,i}) \Omega_p; \quad (F_q)_{i} := (q, \varphi_{p,i}) \Omega_f;
\]

\[
(F_{\eta})_{i} := (q, \varphi_{\eta,i}) \Omega_p; \quad (F_{c}) := (q^+ c_w, \varphi_{c,i}) \Omega.
\]

Taking in (4.2.4b)–(4.2.4g), \( u_{fh}(x, t) = \sum_i u_{f,i}(t) \varphi_{u,i}, \) \( u_{ph}(x, t) = \sum_i u_{p,i}(t) \varphi_{p,i}, \) \( \eta_{ph}(x, t) = \sum_i \eta_{p,i}(t) \varphi_{p,i}, \) \( p_{fh}(x, t) = \sum_i p_{f,i}(t) \varphi_{f,i}, \) \( p_{ph}(x, t) = \sum_i p_{p,i}(t) \varphi_{p,i}, \) \( \text{and} \lambda_h(x, t) = \sum_i \lambda_i(t) \varphi_{\lambda,i} \) with (time-dependent) coefficient vectors \( \bar{u}_f, \bar{u}_p, \bar{\eta}_p, \bar{p}_f, \bar{p}_p, \) and \( \bar{\lambda}, \) leads to the system

\[
A_f \bar{u}_f + B_f \bar{p}_f + A^f_{BJS} \bar{u}_f + (A^e_{BJS})^T \partial_t \bar{\eta}_p + B^T_{f,\Gamma} \bar{\lambda} = F_f, \quad (4.2.6a)
\]

\[
- B_f \bar{u}_f = F_q \quad (4.2.6b)
\]

\[
A^c_p \bar{\eta}_p + \alpha B^T_{c} \bar{p}_p + A^e_{BJS} \bar{u}_f + A^{ee}_{BJS} \partial_t \bar{\eta}_p + B^T_{c,\Gamma} \bar{\lambda} = F_p, \quad (4.2.6c)
\]
A_p^d \mathbf{u}_p + B_p^T \mathbf{p}_p + B_p^T \lambda = 0, \hspace{1cm} (4.2.6d)

s_0 M_p \partial_t \mathbf{p}_p - \alpha B_e \partial_t \mathbf{u}_p - B_p \mathbf{u}_p = \mathcal{F}_p, \hspace{1cm} (4.2.6e)

B_{f,\Gamma} \mathbf{u}_f + B_{p,\Gamma} \mathbf{u}_p + B_{e,\Gamma} \partial_t \mathbf{p}_p = 0. \hspace{1cm} (4.2.6f)

This is a DAE system

\[ \mathbf{G} \partial_t X(t) + \mathbf{H} X(t) = R(t), \hspace{1cm} (4.2.7) \]

where

\[
\begin{pmatrix}
\mathbf{u}_f(t) \\
\mathbf{p}_f(t) \\
\mathbf{p}_p(t) \\
\mathbf{u}_p(t) \\
\bar{\mathbf{p}}_p(t) \\
\bar{\lambda}(t)
\end{pmatrix}, \quad
\begin{pmatrix}
\mathcal{F}_f \\
\mathcal{F}_{qf} \\
\mathcal{F}_{fp} \\
0 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & (A_{BJS}^{fe})^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{BJS}^e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha B_e & 0 & s_0 M_p & 0 & 0 \\
0 & -B_{e,\Gamma} & 0 & 0 & 0 & 0
\end{pmatrix}_{6 \times 6},
\]

\[
\mathbf{H} = \begin{pmatrix}
A_f + A_{BJS}^{fe} (B_f)^T & 0 & 0 & 0 & (B_{f,\Gamma})^T \\
-B_f & 0 & 0 & 0 & 0 \\
A_{BJS}^e & 0 & A_p^e & \alpha (B_e)^T & (B_{e,\Gamma})^T \\
0 & 0 & 0 & A_p^d & (B_p)^T & (B_{p,\Gamma})^T \\
0 & 0 & -B_p & 0 & 0 \\
-B_{f,\Gamma} & 0 & 0 & -B_{p,\Gamma} & 0 & 0
\end{pmatrix}_{6 \times 6}.
\]

Next, we take \( c_h(x, t) = \sum_i c_i(t) \varphi_{c,i} \) in (4.2.5) with a coefficient vector \( \tilde{c} \). Denoting \( \tilde{D} := D^1_c + D^2_c + D^-_c \), we obtain the ODE system

\[ M_c \partial_t \varphi(t) + \tilde{D} \varphi(t) = \mathcal{F}_c(t). \hspace{1cm} (4.2.8) \]
4.2.2 Existence and uniqueness of a solution of the semi-discrete Galerkin problems (LGP1) and (LGP2)

We next discuss the solvability of the initial value problems associated with (4.2.7) and (4.2.8). Assuming that \( c_0 \in H^1(\Omega) \), we take initial data \( c_{h,0} = I_{ch}c_0 \), where \( I_{ch} : H^1(\Omega) \to X_{ch} \) is the Scott-Zhang interpolant \([79]\). Initial data compatible with the DAE system (4.2.7) is constructed in the proof of the following Theorem.

**Theorem 4.2.1.** Let \( f_f \in L^\infty(0,T;\mathbb{V}'_f) \), \( f_p \in L^\infty(0,T;\mathbb{V}'_p) \), \( q_f \in L^\infty(0,T;W'_f) \), \( q_p \in L^\infty(0,T;W'_p) \), and \( \gamma \in L^\infty(0,T;L^\infty(\Omega)) \). Assume that \( p_{p,0} \in H_p \), where \( H_p := \left\{ w_p \in H^1(\Omega_p) : \frac{1}{\mu(c_0)}K\nabla w_p \in H^1(\Omega_p), \frac{1}{\mu(c_0)}K\nabla w_p \cdot n_p = 0 \text{ on } \Gamma^N_p, \ w_p = 0 \text{ on } \Gamma^D_p \right\} \).

Then there exists initial data \( (u_{fh,0},p_{fh,0},\eta_{ph,0},u_{ph,0},p_{ph,0},\lambda_{h,0}) \in (\mathbb{V}_{fh} \times W_{fh} \times X_{ph} \times \mathbb{V}_{ph} \times W_{ph} \times \Lambda_h) \) such that the Galerkin problem (LGP1) with initial conditions \( p_{ph}(0) = p_{ph,0} \) and \( \eta_{ph}(0) = \eta_{ph,0} \) has a unique solution. Moreover, the solution satisfies \( u_{fh}(0) = u_{fh,0}, p_{fh}(0) = p_{fh,0}, u_{ph}(0) = u_{ph,0}, \) and \( \lambda_{h}(0) = \lambda_{h,0} \).

Let \( q^- \in L^\infty(0,T;L^\infty(\Omega)) \), \( q^+c_w \in L^\infty(0,T;X'_c) \), \( \theta \in L^\infty(0,T;L^\infty(\Omega)) \), and \( c_0 \in H^1(\Omega) \). Then the Galerkin problem (LGP2) with initial condition \( c_h(0) = c_{h,0} \) has a unique solution.

**Proof.** First, we note that the continuity of the functionals established in Lemma 4.1.1 and the assumed regularity of the data guarantee that all terms in (LGP1) and (LGP2) are well defined, resulting in well defined DAE system (4.2.7) and ODE system (4.2.8). Moreover, (4.2.7) is linear with constant coefficient matrices \( G \) and \( H \) and continuous \( R(t) \). Similarly, (4.2.8) is linear with constant coefficient matrices \( M_c \) and \( \tilde{D} \) and continuous \( F_c(t) \).

We begin with proving the solvability (LGP1). The proof is organized in three main steps: (1) prove the non-singularity of the matrix \( G + H \) and utilize the DAE theory to establish the existence of a solution; (2) construct compatible initial data for all variables; (3) derive an energy estimate to obtain uniqueness of the solution under a suitable initial condition.

**Existence for (LGP1).** The existence of a solution of the system (4.2.7) is established using the DAE theory, cf. Theorem 2.3.1 in [19]. For the associated initial value problem,
we need to construct initial data for all variables that are consistent with the DAE system. To address this issue, we consider an auxiliary DAE system by introducing a new variable $u_{sh} \in X_{ph}$ and the equation

\[(u_{sh}, v_{sh})_{\Omega_p} = (\partial_t \eta_{ph}, v_{sh})_{\Omega_p}, \quad \forall v_{sh} \in X_{ph}, \quad \text{(4.2.9)}\]

and replacing $\partial_t \eta_{ph}$ with $u_{sh}$ in (4.2.4b), (4.2.4d), (4.2.4f), and (4.2.4g). Let $u_{sh}(x, t) = \sum_i u_{s,i}(t) \varphi_{\eta_{p,i}}$, and $u_s$ be the coefficient vector. This results in the extended DAE system

\[\tilde{G} \partial_t \tilde{X}(t) + \tilde{H} \tilde{X}(t) = \tilde{R}(t), \quad \text{(4.2.10)}\]

where $\tilde{X} = [u_f \ u_p \ \eta_p \ \bar{p}_f \ \bar{p}_p \ \bar{\lambda} \ \bar{u}_s]^T$. Note that any solution of (4.2.10) also solves (2.2.18). According to Theorem 2.3.1 in [19] and [91], (4.2.10) has a solution if the matrix $\omega \tilde{G} + \tilde{H}$ is non-singular for some $\omega \in \mathbb{R}$. We prove that $\tilde{G} + \tilde{H}$ is non-singular by showing that the system $(\tilde{G} + \tilde{H})\tilde{X} = 0$ has only the zero solution. After eliminating $\bar{u}_s$, this system results in $(G + H)X = 0$, which implies

\[X^T(G + H)X = a_f(u_{fh}, u_{fh}; \gamma) + a_p^d(u_{ph}, u_{ph}; \gamma) + a_p^e(\eta_{ph}, \eta_{ph}) + a_{BSJ}(u_{fh}, \eta_{ph}; u_{fh}, \eta_{ph}) + s_0(p_{ph}, p_{ph})\Omega_p = 0. \quad \text{(4.2.11)}\]

Lemma 4.1.1 implies that $u_{fh} = 0$, $u_{ph} = 0$, $\eta_{ph} = 0$, and $p_{ph} = 0$. The inf-sup condition (4.2.3) with $(w_{fh}, w_{ph}, \nu_h) = (p_{fh}, p_{ph}, \lambda_h)$, combined with (4.2.4b) and (4.2.4e), gives $p_{fh} = 0$ and $\lambda_h = 0$, while (4.2.9) implies that $u_{sh} = 0$. We conclude that (4.2.10) has a solution.

**Construction of initial data for (LGP1).** To construct discrete initial data that are consistent with the DAE system (4.2.10), we first construct data for the continuous weak formulation (4.2.1). We then define the discrete data using elliptic projections.

Recall that $p_{p,0} \in H_p$ is given. Let $u_{p,0} := -\frac{1}{\mu(c_0)} K \nabla p_{p,0}$. We have that $u_{p,0} \in H^1(\Omega_p) \cap V_p$. Letting $\lambda_0 := p_{p,0}|_{\Gamma_{fp}} \in \Lambda$, we conclude that

\[a_p^d(u_{p,0}, v_p; c_0) + b_p(v_p, p_{p,0}) + \langle v_p \cdot n_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0, \quad \forall v_p \in V_p. \quad \text{(4.2.12)}\]
Next, let \( (u_{f,0}, p_{f,0}) \in V_f \times W_f \) be the solution to the Stokes problem

\[
a_f(u_{f,0}, v_f; c_0) + b_f(v_f, p_{f,0}) + \alpha_{BJS} \sum_{j=1}^{d-1} \left\langle \mu(c_0) \sqrt{K_j^{-1}} u_{p,0} \cdot \tau_{f,j}, v_f \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}} \\
+ \left\langle v_f \cdot n_f, \lambda_0 \right\rangle_{\Gamma_{fp}} = (f_f(0), v_f)_{\Omega_f}, \quad \forall v_f \in V_f,
\]

\[
- b_f(u_{f,0}, w_f) = (q_f(0), w_f)_{\Omega_f}, \quad \forall w_f \in W_f,
\]

where \( u_{p,0} \) and \( \lambda_0 \) are given as data. Next, solve an elasticity problem for \( \eta_{p,0} \in X_p \):

\[
a_p^e(\eta_{p,0}, \xi_p) + \alpha b_e(\xi_p, p_{p,0}) - \alpha_{BJS} \sum_{j=1}^{d-1} \left\langle \mu(c_0) \sqrt{K_j^{-1}} u_{p,0} \cdot \tau_{f,j}, \xi_p \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}} \\
+ \left\langle \xi_p \cdot n_p, \lambda_0 \right\rangle_{\Gamma_{fp}} = (f_p(0), \xi_p)_{\Omega_p}, \quad \forall \xi_p \in X_p,
\]

where \( p_{p,0}, u_{p,0}, \) and \( \lambda_0 \) are given as data. Finally, let

\[
u_{s,0} := E_p(u_{f,0} - u_{p,0})|_{\Gamma_{fp}} \in H^1(\Omega_p),
\]

where \( E_p : H^{1/2}(\Gamma_{fp}) \rightarrow H^1(\Omega_p) \) is a continuous extension. The definition of \( \nu_{s,0} \) implies that

\[
b_T(u_{f,0}, u_{p,0}, u_{s,0}; \nu) = 0, \quad \forall \nu \in \Lambda,
\]

and enables us to replace \( u_{p,0} \cdot \tau_{f,j} \) in the BJS terms in (4.2.13a) and (4.2.14) with \( (u_{f,0} - u_{s,0}) \cdot \tau_{f,j} \).

We proceed with the construction of the discrete initial data. Let \( I_{sh} : H^1(\Omega_p) \rightarrow X_{ph} \) is the Scott-Zhang interpolant. Next, let \( (u_{f,h,0}, p_{f,h,0}, u_{p,h,0}, p_{p,h,0}; \lambda_{h,0}) \in V_{fh} \times W_{fh} \times V_{ph} \times W_{ph} \times \Lambda_h \) be the solution to the Stokes–Darcy problem: for all \( v_{fh} \in V_{fh}, w_{fh} \in W_{fh}, v_{ph} \in V_{ph}, w_{ph} \in W_{ph}, \) and \( \nu_h \in \Lambda_h, \)

\[
a_f(u_{f,h,0}, v_{fh}; c_0) + b_f(v_{fh}, p_{f,h,0}) + a_{BJS}(u_{f,h,0}, u_{sh,0}; v_{fh}, 0) + b_T(v_{fh}, 0; \lambda_{h,0}) \\
= a_f(u_{f,0}, v_{fh}; c_0) + b_f(v_{fh}, p_{f,0}) + a_{BJS}(u_{f,0}, u_{s,0}; v_{fh}, 0) + b_T(v_{fh}, 0; \lambda_0) \\
= (f_f(0), v_{fh})_{\Omega_f},
\]

\[
- b_f(u_{f,h,0}, w_{fh}) = - b_f(u_{f,0}, w_{fh}) = (q_f(0), w_{fh})_{\Omega_f},
\]

\[
a^d_p(u_{p,h,0}, v_{ph}; c_0) + b_p(v_{ph}, p_{p,h,0}) + b_T(0, v_{ph}, 0; \lambda_{h,0}) \\
= a^d_p(u_{p,0}, v_{ph}; c_0) + b_p(v_{ph}, p_{p,0}) + b_T(0, v_{ph}, 0; \lambda_0) = 0,
\]

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\[-b_p(u_{ph,0}, w_{ph}) = -b_p(u_p, w_{ph}) = -(\nabla \cdot \mu(c_0)^{-1}K\nabla p_{p,0}, w_{ph})\Omega_p,\]  
(4.2.17d)

\[b_T(u_{fh,0}, u_{ph,0}, u_{sh,0}; \nu_h) = b_T(u_{f,0}, u_{p,0}, u_{s,0}; \nu_h) = 0.\]  
(4.2.17e)

This is a well-posed Stokes–Darcy problem, due the inf-sup condition (4.2.3), using the theory of saddle point problems [16], see [64, 43] for details. Finally, let \(\eta_{ph,0} \in X_{ph}\) be the solution to the elasticity problem: for all \(\xi_{ph} \in X_{ph}\),

\[a_p^e(\eta_{ph,0}, \xi_{ph}) + ab_e(\xi_{ph}, P_{ph,0}) - a_{BJS}(u_{fh,0}, u_{sh,0}; 0, \xi_{ph}) + b_T(0, 0, \xi_{ph}; \lambda_h, 0) = a_p^e(\eta_{p,0}, \xi_{ph}) + ab_e(\xi_{ph}, P_{ph,0}) - a_{BJS}(u_{f,0}, u_{s,0}; 0, \xi_{ph}) + b_T(0, 0, \xi_{ph}; \lambda_0) = (f_p(0), \xi_{ph})\Omega_p.\]  
(4.2.18)

Based on (4.2.17)–(4.2.18), we have constructed initial data \((u_{fh,0}, p_{fh,0}, \eta_{ph,0}, u_{ph,0}, P_{ph,0}, \lambda_h, 0, u_{sh,0})\) for the extended DAE system (4.2.10), which satisfy all equations in (4.2.10) at \(t = 0\), except for the equations with time derivatives, cf. (4.2.6e) and (4.2.9). With the constructed initial data, these equations at \(t = 0\) determine \(\partial_t p_{ph}(0)\) and \(\partial_t \eta_{ph}(0)\). Since \(\partial_t p_{ph}\) appears only in (4.2.6e) and \(\partial_t \eta_{ph}(0)\) appears only in (4.2.9), requiring the two equations to hold at \(t = 0\) does not lead to inconsistency with the rest of the equations. Therefore there exists a solution to (4.2.10) satisfying \((u_{fh}(0), p_{fh}(0), \eta_{ph}(0), u_{ph}(0), p_{ph}(0), \lambda_h(0), u_{sh}(0)) = (u_{fh,0}, p_{fh,0}, \eta_{ph,0}, u_{ph,0}, P_{ph,0}, \lambda_h, 0, u_{sh,0})\). Using (4.2.9) to eliminate \(u_{sh}\) from the system results in a solution to the original DAE system (4.2.7) satisfying \((u_{fh}(0), p_{fh}(0), \eta_{ph}(0), u_{ph}(0), p_{ph}(0), \lambda_h(0)) = (u_{fh,0}, p_{fh,0}, \eta_{ph,0}, u_{ph,0}, P_{ph,0}, \lambda_h, 0)\).

**Uniqueness of the solution to (LGP1).** Assume that there exist two solutions to LGP1 satisfying \(p_{ph}(0) = p_{ph,0}\) and \(\eta_{ph}(0) = \eta_{ph,0}\). Let \((\tilde{u}_{fh}, \tilde{p}_{fh}, \tilde{\eta}_{ph}, \tilde{u}_{ph}, \tilde{p}_{ph}, \tilde{\lambda}_h)\) be the difference between the two solutions. These functions satisfy (4.2.4b)–(4.2.4g) with zero data on the right-hand side, as well as \(\tilde{p}_{ph}(0) = 0\) and \(\tilde{\eta}_{ph}(0) = 0\). Taking \((v_{fh}, w_{fh}, \xi_{ph}, v_{ph}, w_{ph}, \nu_h) = (\tilde{u}_{fh}, \tilde{p}_{fh}, \partial_t \tilde{\eta}_{ph}, \tilde{u}_{ph}, \tilde{p}_{ph}, \tilde{\lambda}_h)\) and combining the equations, we obtain

\[s_0(\partial_t \tilde{p}_{ph}, \tilde{p}_{ph})\Omega_p + a_p^d(\tilde{u}_{ph}, \tilde{u}_{ph}; \gamma) + a_{BJS}(\tilde{u}_{fh}, \partial_t \tilde{\eta}_{ph}; \tilde{u}_{fh}, \partial_t \tilde{\eta}_{ph}) = 0.\]  
(4.2.19)
Next, we integrate (4.2.19) from 0 to $t \in (0, T]$ and use Lemma 4.1.1 to deduce
\[
s_0 \| \bar{p}_{ph}(t) \|_{L^2(\Omega_p)}^2 + \| \bar{\eta}_{ph}(t) \|_{H^1(\Omega_p)}^2 + \int_0^t \left( \| \bar{u}_{fh} \|_{H^1(\Omega)}^2 + \| \bar{u}_{ph} \|_{L^2(\Omega_p)}^2 + |\bar{u}_{fh} - \partial_t \bar{\eta}_{ph}|_{BJS}^2 \right) \, ds = 0,
\]
which implies that $\bar{p}_{ph}(t) = 0$, $\bar{\eta}_{ph}(t) = 0$, $\bar{u}_{fh}(t) = 0$, and $\bar{u}_{ph}(t) = 0$. The inf-sup condition (4.2.3), together with (4.2.4b) and (4.2.4e), gives $\bar{p}_{fh}(t) = 0$ and $\bar{\lambda}_h(t) = 0$, concluding the uniqueness proof.

**Existence and uniqueness of a solution to (LGP2).** Next, we establish the solvability of the transport equation (4.2.5), which leads to the system (4.2.8). Since the matrix $M_c$ is invertible, we can rewrite (4.2.8) as
\[
\partial_t \bar{c}(t) = M_c^{-1} F_c(t) - M_c^{-1} \bar{D} \bar{c}(t) := g(t, \bar{c}(t)).
\]
(4.2.20)
Note that $g(t, \bar{c}(t))$ is continuous in $t$, due to the smoothness assumption on $q^-, q^+ c_w$, and $\theta$, and it is linear in $\bar{c}$. Therefore, it follows from the ODE theory, see e.g. Picard-Lindel"of Theorem, Theorem 1.3.1 in [37], that there exists a unique maximal solution $\bar{c}(t)$ in $(0, T]$ satisfying $c_h(0) = c_{h,0}$.

\[
\square
\]

### 4.2.3 Stability of (LGP1) and (LGP2)

**Theorem 4.2.2.** Assume that $f_f \in L^\infty(0, T; V_f')$, $f_p \in H^1(0, T; V_p')$, $q_f \in L^\infty(0, T; W_f')$, $q_p \in L^\infty(0, T; W_p')$, $\gamma \in L^\infty(0, T; L^\infty(\Omega))$, and $p_{p,0} \in H_p$. Then, for the solution of (LGP1), there exists a positive constant $C$ independent of $h$ such that
\[
\sqrt{s_0} \| p_{ph} \|_{L^\infty(0, T; W_p)} + \| \eta_{ph} \|_{L^\infty(0, T; X_p)} + \| u_{fh} \|_{L^2(0, T; V_f)} + \| p_{fh} \|_{L^2(0, T; W_f)}
\]
\[
+ \| u_{ph} \|_{L^2(0, T; L^2(\Omega_p))} + \| p_{ph} \|_{L^2(0, T; W_p)} + \| \lambda_h \|_{L^2(0, T; A)} + \| u_{fh} - \partial_t \eta_{ph} \|_{L^2(0, T; BJS)}
\]
\[
\leq C \sqrt{\exp(T)} \left( \| f_f \|_{L^2(0, T; V_f')} + \| f_p \|_{H^1(0, T; X_p')} + \| q_p \|_{L^2(0, T; W_p')} + \| q_f \|_{L^2(0, T; W_f')}
\]
\[
+ \| p_{p,0} \|_{H^1(\Omega_p)} + \| \mu(c_0)^{-1} K \nabla p_{p,0} \|_{H^1(\Omega_p)} \right).
\]
(4.2.21)
Moreover, if \( f_f \in H^1(0, T; \mathbf{V}_f') \), \( f_p \in H^2(0, T; \mathbf{X}_p') \), \( q_f \in H^1(0, T; W_f') \), \( q_p \in H^1(0, T; W_p') \), and \( \gamma \in W^{1, \infty}(0, T; L^\infty(\Omega)) \), it holds that

\[
\sqrt{s_0} \| \partial_t p_{ph} \|_{L^\infty(0,T;W_p)} + \| \partial_t \eta_{ph} \|_{L^\infty(0,T;\mathbf{x}_p)} + \| \partial_t u_{fh} \|_{L^2(0,T;\mathbf{V}_f)} + \| \partial_t p_{fh} \|_{L^2(0,T;W_f)} \\
+ \| \partial_t u_{ph} \|_{L^2(0,T;L^2(\Omega_p))} + \| \nabla \cdot u_{ph} \|_{L^\infty(0,T;L^2(\Omega_p))} + \| \partial_t p_{ph} \|_{L^2(0,T;W_p)} \\
+ \| \partial_t \lambda_h \|_{L^2(0,T;\mathbf{A})} + \| \partial_t u_{fh} - \partial_t \eta_{ph} \|_{L^2(0,T;BJS)} \\
\leq C \sqrt{\exp(T)} \left( \| f_f \|_{H^1(0,T;\mathbf{V}_f')} + \| f_p \|_{H^2(0,T;\mathbf{X}_p')} + \| q_f \|_{H^1(0,T;W_f')} + \| q_p \|_{H^1(0,T;W_p')} \\
+ \frac{1}{\sqrt{s_0}} (\| p_w \|_{H^1(\Omega)} + \| \mu(c_0)^{-1} K \nabla p_w \|_{H^1(\Omega)} + \| f_f(0) \|_{\mathbf{V}_f} + \| q_f(0) \|_{W_f'}) \right). \tag{4.2.22}
\]

Assume that \( q^w \in L^\infty(0, T; L^\infty(\Omega)) \), \( q^c_w \in L^\infty(0, T; X'_p) \), \( \theta \in L^\infty(0, T; L^\infty(\Omega)) \), and \( c_0 \in H^1(\Omega) \). Then, for the solution of \( \text{LGP2} \), it holds that

\[
\| c_h \|_{L^\infty(0,T;L^2(\Omega))} + \| c_h \|_{L^2(0,T;\mathbf{x}_c)} \leq C \sqrt{\exp(T)} \left( \| c_0 \|_{H^1(\Omega)} + \| q^c_w \|_{L^2(0,T;X'_p)} \right). \tag{4.2.23}
\]

Furthermore, if \( q^c_w \in L^\infty(0, T; L^2(\Omega)) \) and \( \theta \in W^{1,\infty}(0, T; L^\infty(\Omega)) \), it holds that

\[
\| \partial_t c_h \|_{L^2(0,T;L^2(\Omega))} + \| c_h \|_{L^\infty(0,T;\mathbf{x}_c)} \leq C \sqrt{\exp(T)} \left( \| c_0 \|_{H^1(\Omega)} + \| q^c_w \|_{L^2(0,T;L^2(\Omega))} \right). \tag{4.2.24}
\]

**Proof.** Before proceeding with the proof, we note that (4.2.21) does not provide control on \( \nabla \cdot u_{ph} \). In (4.2.22) we obtain control on \( \partial_t p_{ph} \) and \( \partial_t \eta_{ph} \), which allows us to also bound \( \nabla \cdot u_{ph} \). The latter bound is necessary for compactness in \( \| u_{ph} \|_{L^2(0,T;\mathbf{V}_p)} \) and passing to the limit.

Taking \((v_{fh}, w_{fh}, \xi_{ph}, v_{ph}, w_{ph}, \nu_h) = (u_{fh}, p_{fh}, \partial_t \eta_{ph}, u_{ph}, p_{ph}, \lambda_h) \) in (4.2.4b)-(4.2.4g), summing the equations, and integrating from 0 to \( t \in (0, T] \) gives

\[
\begin{align*}
\frac{s_0}{2} \| p_{ph}(t) \|_{W_p}^2 + & \frac{1}{2} a_p^e(\eta_{ph}(t), \eta_{ph}(t)) \\
+ & \int_0^t \left( a_f(u_{fh}, u_{fh}; \gamma) + a_p^f(u_{ph}, u_{ph}; \gamma) + a_{BJS}(u_{fh}, \partial_t \eta_{ph}, u_{fh}, \partial_t \eta_{ph}) \right) ds \\
= & \frac{s_0}{2} \| p_{ph}(0) \|_{W_p}^2 + \frac{1}{2} a_p^e(\eta_{ph}(0), \eta_{ph}(0)) \\
+ & \int_0^t \left( (f_f, u_{fh}) \Omega_f + (f_p, \partial_t \eta_{ph}) \Omega_p + (q_p, p_{ph}) \Omega_p + (q_f, p_{fh}) \Omega_f \right) ds. \tag{4.2.25}
\end{align*}
\]
For the term $\int_0^t (f_p, \partial_t \eta_{ph})_{\Omega_p} ds$ we employ the identity

$$\int_0^t (f_p, \partial_t \eta_{ph})_{\Omega_p} ds = (f_p, \eta_{ph})_{\Omega_p} \big|_0^t - \int_0^t (\partial_t f_p, \eta_{ph})_{\Omega_p} ds. \quad (4.2.26)$$

We combine (4.2.25) and (4.2.26) and use the coercivity and continuity of $a_f, a_f^d, a_f^e$ and $a_{BJS}$ from Lemma 4.1.1 and the Cauchy-Schwarz and Young’s inequalities to obtain, for $\epsilon > 0$,

$$s_0 \| p_{ph}(t) \|_{W_p}^2 + \| \eta_{ph}(t) \|_{X_p}^2 + \int_0^t \left( \| u_{fh} \|_{V_f}^2 + \| u_{ph} \|_{L^2(\Omega_p)}^2 + \| u_{fh} - \partial_t \eta_{ph} \|_{L^2(\Omega_p)}^2 \right) ds$$

$$\leq C \left( \| p_{ph}(0) \|_{W_p}^2 + \| \eta_{ph}(0) \|_{X_p}^2 + \| f_p(0) \|_{L^2(\Omega_p)}^2 \right) + \int_0^t \left( \| \eta_{ph} \|_{X_p}^2 + \| \partial_t f_p \|_{X_p}^2 \right) ds$$

$$+ \epsilon \int_0^t \left( \| u_{fh} \|_{V_f}^2 + \| p_{ph} \|_{W_p}^2 + \| p_{fh} \|_{W_f}^2 \right) ds + \frac{C}{\epsilon} \int_0^t \left( \| f_p \|_{V_f}^2 + \| q_{ph} \|_{W_p}^2 + \| q_{fh} \|_{W_f}^2 \right) ds$$

$$+ \epsilon \| \eta_{ph}(t) \|_{X_p}^2 + \frac{C}{\epsilon} \| f_p(t) \|_{X_p}^2. \quad (4.2.27)$$

Using the inf-sup condition (4.2.3), combined with (4.2.4b) and (4.2.4e), we obtain

$$\beta_2 \| (p_{fh}, p_{ph}, \lambda_h) \|_{W_f \times W_p \times \Lambda} \leq \sup_{0 \neq (v_{fh}, v_{ph}) \in V_f \times V_p} \frac{b_f(v_{fh}, p_{fh}) + b_p(v_{ph}, p_{ph}) + b_1(v_{fh}, v_{ph}, 0; \lambda_h)}{\| (v_{fh}, v_{ph}) \|_{V_f \times V_p}}$$

$$= \sup_{0 \neq (v_{fh}, v_{ph}) \in V_f \times V_p} \frac{(f_p, v_{fh})_{\Omega_f} - a_f(u_{fh}, v_{fh}; \gamma) - a_{p}^d(u_{ph}, v_{ph}; \gamma) - a_{BJS}(u_{fh}, \partial_t \eta_{ph}; v_{fh}, 0)}{\| (v_{fh}, v_{ph}) \|_{V_f \times V_p}}$$

$$\leq C \left( \| f_p \|_{V_f} + \| u_{fh} \|_{V_f} + \| u_{ph} \|_{L^2(\Omega_p)} + \| u_{fh} - \partial_t \eta_{ph} \|_{L^2(\Omega_p)} \right), \quad (4.2.28)$$

where we use the continuity bounds from Lemma 4.1.1 and the trace inequality

$$\| \varphi \|_{H^{1/2}(\Gamma_{fh})} \leq C \| \varphi \|_{H^1(\Omega_f)} \quad \forall \varphi \in \{ f, p \}. \quad (4.2.29)$$

This yields

$$\int_0^t \left( \| p_{fh} \|_{W_f}^2 + \| p_{ph} \|_{W_p}^2 + \| \lambda_h \|_{\Lambda}^2 \right) ds$$

$$\leq C \int_0^t \left( \| f_p \|_{V_f}^2 + \| u_{fh} \|_{V_f}^2 + \| u_{ph} \|_{L^2(\Omega_p)}^2 + \| u_{fh} - \partial_t \eta_{ph} \|_{L^2(\Omega_p)}^2 \right) ds. \quad (4.2.30)$$

Combining (4.2.30) with (4.2.27), taking $\epsilon$ small enough, and using Gronwall’s inequality for the term $\int_0^t \| \eta_{ph} \|_{X_p}^2 ds$, we obtain

$$s_0 \| p_{ph}(t) \|_{W_p}^2 + \| \eta_{ph}(t) \|_{X_p}^2$$
\[ \begin{multline*} + \int_0^t \left( \| u_{fh} \|_{V_f}^2 + \| u_{ph} \|_{L^2(\Omega_p)}^2 + \| u_{fh} - \partial_t \eta_{ph} \|_{B_{JS}}^2 + \| p_{fh} \|_{W_j}^2 + \| p_{ph} \|_{W_p}^2 + \| \lambda_h \|_A^2 \right) \, ds \\
leq C \exp(T) \left( \int_0^t \left( \| f_f \|_{V_f}^2 + \| \partial_t f_p \|_{X_p}^2 + \| q_p \|_{W_p}^2 + \| q_f \|_{W_j}^2 \right) \, ds + \| f_p(0) \|_{X_p}^2 \right. \\
+ \| p_{ph}(0) \|_{W_p}^2 + \| \eta_{ph}(0) \|_{X_p} + \| f_p(0) \|_{X_p}^2 \right). \tag{4.2.31} \end{multline*} \]

It remains to bound the initial values. Recall that \((u_{fh}(0), p_{fh}(0), \eta_{ph}(0), u_{ph}(0), p_{ph}(0), \lambda_h(0)) = (u_{fh0}, p_{fh0}, \eta_{ph0}, u_{ph0}, p_{ph0}, \lambda_h0)\), the initial data constructed in Theorem 2.2.4. The definition of \(u_{sh}(4.2.15)\), the continuity of the extension \(E_p\), and the trace inequality (4.2.29) imply that
\[ \| u_{sh,0} \|_{H^1(\Omega_p)} \leq C(\| u_{f,0} - u_{p,0} \|_{H^{1/2}(\Gamma_p)}, \leq C(\| u_{f,0} \|_{H^1(\Omega_f)} + \| u_{p,0} \|_{H^1(\Omega_p)}). \]

Recalling that \(u_{p,0} = -\mu(c_0)^{-1} K \nabla p_{p,0} \) and \(\lambda_0 = p_{p,0}|_{\Gamma_p}^\prime\), the stability of the Stokes problem \((4.2.13)\), and the continuity of the Scott-Zhang interpolant \(I_{sh}\) imply that
\[ \| u_{sh,0} \|_{H^1(\Omega_p)} \leq C(\| p_{p,0} \|_{H^1(\Omega_f)} + \| \mu(c_0)^{-1} K \nabla p_{p,0} \|_{H^1(\Omega_p)} + \| f_f(0) \|_{V_f} + \| q_f(0) \|_{W_j}^2). \tag{4.2.32} \]

Then, the stability of the discrete Stokes–Darcy problem \((4.2.17)\) [64, 43] and the discrete elasticity problem \((4.2.18)\) imply
\[ \begin{align*} &\| u_{fh}(0) \|_{V_f}^2 + \| u_{ph}(0) \|_{V_p}^2 + \| \eta_{ph}(0) \|_{X_p}^2 + \| p_{fh}(0) \|_{W_j}^2 + \| p_{ph}(0) \|_{W_p}^2 + \| \lambda_h(0) \|_A^2 \\
&\leq C \left( \| p_{p,0} \|_{H^1(\Omega_p)}^2 + \| \mu(c_0)^{-1} K \nabla p_{p,0} \|_{H^1(\Omega_p)}^2 + \| f_f(0) \|_{V_f}^2 + \| q_f(0) \|_{W_j}^2 \right). \tag{4.2.33} \end{align*} \]

Combining (4.2.31) and (4.2.33) gives (4.2.21).

Next, we derive (4.2.22), assuming higher regularity of the data. Differentiating in time \((4.2.4b)–(4.2.4g)\) and taking \((v_{fh}, w_{fh}, \xi_{ph}, v_{ph}, w_{ph}, \nu_h) = (\partial_t u_{fh}, \partial_t p_{fh}, \partial_t \eta_{ph}, \partial_t u_{ph}, \partial_t p_{ph}, \partial_t \lambda_h)\), we obtain
\[ \begin{align*} &a_f(\partial_t u_{fh}, \partial_t u_{fh}; \gamma) + a_p^d(\partial_t u_{ph}, \partial_t u_{ph}; \gamma) + a_p^s(\partial_t \eta_{ph}, \partial_t \eta_{ph}) \\
&+ a_{BJS}(\partial_t u_{fh}, \partial_t \eta_{ph} - \partial_t u_{fh} \eta_{ph}) + s_0(\partial_t p_{ph}, p_{ph}) \Omega_p \\
&= (\partial_t q_f, \partial_t p_{fh}) \Omega_f + (\partial_t q_p, \partial_t p_{ph}) \Omega_p + (\partial_t f_f, \partial_t u_{fh}) \Omega_f + \partial_t (\partial_t f_p, \partial_t \eta_{ph}) \Omega_p - (\partial_t f_p, \partial_t \eta_{ph}) \Omega_p \end{align*} \]
\[-(2\mu'(\gamma) \partial_t \gamma \epsilon(u_{fh}), \partial_t \epsilon(u_{fh}))_{\Omega_f} - (\mu'(\gamma) \partial_t \gamma K^{-1} u_{ph}, \partial_t u_{ph})_{\Omega_p}.\] (4.2.34)

Integrating (4.2.34) from 0 to \(t \in (0, T]\) and using Lemma 4.1.1, (4.1.5), \(\gamma \in W^{1,\infty}(0, T; L^\infty(\Omega))\), and the Cauchy-Schwarz and Young’s inequalities, we obtain, for \(\epsilon > 0\),

\[
s_0 \|\partial_t p_{ph}(t)\|^2_{W_p} + \|\partial_t \eta_{ph}(t)\|^2_{X_p} + \int_0^t \left( \|\partial_t u_{fh}\|^2_{V_f} + \|\partial_t u_{ph}\|^2_{L^2(\Omega_p)} + \|\partial_t u_{fh} - \partial_t \eta_{ph}\|^2_{BJS} \right) \, ds 
\leq C \left( s_0 \|\partial_t p_{ph}(0)\|^2_{W_p} + \|\partial_t \eta_{ph}(0)\|^2_{X_p} + \int_0^t \left( \|\partial_t \eta_{ph}\|^2_{X_p} + \|\partial_t f_p(t)\|^2_{X_p} \right) \, ds \right) 
+ \frac{\epsilon}{\epsilon} \left( \int_0^t \left( \|\partial_t q_f\|^2_{W_f'} + \|\partial_t q_p\|^2_{W_p'} + \|\partial_t f_p\|^2_{V_f'} + \|\partial_t u_{fh}\|^2_{V_f} + \|u_{ph}\|^2_{L^2(\Omega_p)} \right) \, ds \right) 
+ \|\partial_t f_p(t)\|^2_{X_p}. \] (4.2.35)

Next, we use the inf-sup condition (4.2.3) for \((\partial_t p_{fh}, \partial_t p_{ph}, \partial_t \lambda_h)\), combined with the time differentiated (4.2.4b) and (4.2.4c), to obtain, similarly to (4.2.28)–(4.2.30),

\[
\int_0^t \left( \|\partial_t p_{fh}\|^2_{W_f} + \|\partial_t p_{ph}\|^2_{W_p} + \|\partial_t \lambda_h\|^2_{X_p} \right) \, ds \leq C \int_0^t (\|\partial_t f_f\|^2_{V_f} + \|\partial_t u_{fh}\|^2_{V_f}) \, ds 
+ \|\partial_t u_{ph}\|^2_{L^2(\Omega_p)} + \|\partial_t u_{fh} - \partial_t \eta_{ph}\|^2_{BJS} + \|u_{fh}\|^2_{V_f} + \|u_{ph}\|^2_{L^2(\Omega_p)}} \, ds, \] (4.2.36)

where we also used (4.1.5) and \(\gamma \in W^{1,\infty}(0, T; L^\infty(\Omega))\). Taking \(\epsilon\) small enough in (4.2.35) and combining it with (4.2.31) and (4.2.36), and using Gronwall’s inequality for the term \(\int_0^t \|\partial_t \eta_{ph}\|^2_{X_p}\), we get

\[
s_0 \|\partial_t p_{ph}(t)\|^2_{W_p} + \|\partial_t \eta_{ph}(t)\|^2_{X_p} + \int_0^t \left( \|\partial_t u_{fh}\|^2_{V_f} + \|\partial_t u_{ph}\|^2_{L^2(\Omega_p)} + \|\partial_t u_{fh} - \partial_t \eta_{ph}\|^2_{BJS} \right) \, ds 
+ \|\partial_t P_{fh}\|^2_{W_f} + \|\partial_t P_{ph}\|^2_{W_p} + \|\partial_t \lambda_h\|^2_{X_p} \right) ds \leq C \exp(T) \left( s_0 \|\partial_t p_{ph}(0)\|^2_{W_p} + \|\partial_t \eta_{ph}(0)\|^2_{X_p} + \|\partial_t f_p(0)\|^2_{X_p} + \|\partial_t f_p(t)\|^2_{X_p} + \|f_p(t)\|^2_{X_p} \right) \right) 
+ \|\partial_t q_p\|^2_{W_p} + \|\partial_t f_p\|^2_{V_f} + \|f_p(t)\|^2_{W_f} \right) ds \right) \, ds. \] (4.2.37)
Recalling the initial data bound (4.2.33), we need to control additionally $\|\partial_t p_{ph}(0)\|_{W^p}$ and $\|\partial_t \eta_{ph}(0)\|_{X_p}$. Using (4.2.9) at $t = 0$, $\partial_t \eta_{ph}(0) = u_{sh}(0) = u_{sh,0}$. Then (4.2.32) gives

$$
\|\partial_t \eta_{ph}(0)\|^2_{X_p} \leq C \left( \|p_{ph,0}\|^2_{H^1(\Omega_p)} + \|\mu(c_0)^{-1} K \nabla p_{ph,0}\|^2_{H^1(\Omega_p)} + \|\mathbf{f}(0)\|^2_{V'} + \|\mathbf{q}(0)\|^2_{W''} \right).
$$

(4.2.38)

To bound $\|\partial_t p_{ph}(0)\|_{W^p}$, we test (4.2.4f) at $t = 0$ with $w_{ph} = \partial_t p_{ph}(0)$ and use (4.2.33), obtaining

$$
s_0 \|\partial_t p_{ph}(0)\|^2_{W^p} \leq C \left( \|p_{ph,0}\|^2_{H^1(\Omega_p)} + \|\mu(c_0)^{-1} K \nabla p_{ph,0}\|^2_{H^1(\Omega_p)} + \|\mathbf{f}(0)\|^2_{V'} + \|\mathbf{q}(0)\|^2_{W''} \right).
$$

(4.2.39)

Finally, to complete the estimate, we bound $\nabla \cdot \mathbf{u}_{ph}$. To that end, we test (4.2.4f) with $w_{ph} = \nabla \cdot \mathbf{u}_{ph}$, which is permissible choice, since stable Darcy pairs satisfy $\nabla \cdot \mathbf{V}_{ph} = W_{ph}$. We obtain

$$
\|\nabla \cdot \mathbf{u}_{ph}\|_{L^2(\Omega_p)} \leq C \left( \|q_{ph}\|_{W^p} + s_0 \|\partial_t p_{ph}\|_{W^p} + \|\partial_t \eta_{ph}\|_{H^1(\Omega_p)} \right).
$$

(4.2.40)

Estimate (4.2.22) follows from combining (4.2.37)–(4.2.40) with (4.2.33).

For the transport equation, taking $\psi_h = c_h$ in equation (4.2.5), we have

$$
\frac{1}{2} \frac{d}{dt} \|\phi^{1/2} c_h\|_{L^2(\Omega)}^2 + (\mathbf{D}(\mathbf{\theta}) \nabla c_h, \nabla c_h) - (c_h \mathbf{\theta}, \nabla c_h) - (q^- c_h, c_h) = (q^- c_w, c_h).
$$

(4.2.41)

Since $-(q^- c_h, c_h) \geq 0$, this term can be dropped. Using that $\mathbf{\theta} \in L^\infty(0,T; L^\infty(\Omega))$ and Young’s inequality, we write, with some $\epsilon > 0$,

$$(c_h \mathbf{\theta}, \nabla c_h) + (q^+ c_w, c_h) \leq \epsilon \left( \|\nabla c_h\|_{L^2(\Omega)}^2 + \|c_h\|_{X_c}^2 \right) + \frac{C}{\epsilon} \left( \|c_h\|_{L^2(\Omega)}^2 + \|q^+ c_w\|_{X_c}^2 \right).
$$

(4.2.42)

Using (4.1.14) and Poincaré inequality in (4.2.41) and taking $\epsilon$ small enough in (4.2.42) results in

$$
\frac{1}{2} \frac{d}{dt} \|\phi^{1/2} c_h\|_{L^2(\Omega)}^2 + \|c_h\|_{X_c}^2 \leq C \left( \|c_h\|_{L^2(\Omega)}^2 + \|q^+ c_w\|_{X_c}^2 \right).
$$

Integrating from 0 to $t \in (0,T]$ and using that $\phi \geq \phi_\star > 0$ and Gronwall’s inequality, we arrive at

$$
\|c_h(t)\|_{L^2(\Omega)}^2 + \int_0^t \|c_h\|_{X_c}^2 \, ds \leq C \exp(T) \left( \|c_h(0)\|_{L^2(\Omega)}^2 + \int_0^t \|q^+ c_w\|_{X_c}^2 \, ds \right),
$$

(4.2.43)
which combined with \( c_h(0) = c_{h,0} = I_{ch}c_0 \) and the continuity of \( I_{ch} \) in \( H^1(\Omega) \), yields (4.2.23).

It remains to prove the higher regularity bound (4.2.24). We take \( \psi_h = \partial_t c_h \) in (4.2.5) to obtain
\[
(\phi \partial_t c_h, \partial_t c_h) + \frac{1}{2} \frac{d}{dt} \left( (\mathbf{D}(\theta) \nabla c_h, \nabla c_h) - \frac{1}{2} (\partial_t \mathbf{D}(\theta) \nabla c_h, \nabla c_h) - \frac{d}{dt} (c_h \theta, \nabla c_h) \right) + (\partial_t c_h \theta, \nabla c_h) + (c_h \partial_t \theta, \nabla c_h) - (q^- c_h, \partial_t c_h) = (q^+ c_w, \partial_t c_h).
\]

Next, we integrate the above equation from 0 to \( t \in (0, T] \), and use (4.1.14) and (4.1.15), the Cauchy-Schwarz and Young’s inequalities, and \( \theta \in W^{1,\infty}(0, T; L^\infty(\Omega)) \), to obtain, for some \( \epsilon > 0 \),
\[
\int_0^t \|\partial_t c_h\|_{L^2(\Omega)}^2 ds + \|\nabla c_h(t)\|_{L^2(\Omega)}^2 \leq C \left( \|c_h(0)\|_{H^1(\Omega)}^2 + \int_0^t \left( \|\nabla c_h\|_{L^2(\Omega)}^2 + \|c_h\|_{L^2(\Omega)}^2 \right) ds \right) + \epsilon \left( \|\nabla c_h(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\partial_t c_h\|_{L^2(\Omega)}^2 ds \right) + C \frac{1}{\epsilon} \left( \|c_h(t)\|_{L^2(\Omega)}^2 + \int_0^t \left( \|\nabla c_h\|_{L^2(\Omega)}^2 + \|c_h\|_{L^2(\Omega)}^2 + \|q^+ c_w\|_{L^2(\Omega)}^2 \right) ds \right).
\]

Taking \( \epsilon \) small enough and using (4.2.23) gives (4.2.24).

\( \square \)

### 4.2.4 Existence and uniqueness of a solution to (LP1) and (LP2)

Before stating the result, we recall that the we are given initial conditions \( p_p(0) = p_{p,0} \) and \( c(0) = c_0 \). In the proof of Theorem 2.2.4, we constructed initial data for the rest of the variables that satisfy the weak formulation at \( t = 0 \), cf. (4.2.12)–(4.2.16).

**Theorem 4.2.3.** Let \( f_f \in H^1(0, T; V_f') \), \( f_p \in H^2(0, T; X_p') \), \( q_f \in H^1(0, T; W_f') \), \( q_p \in H^1(0, T; W'_p) \), \( \gamma \in W^{1,\infty}(0, T; L^\infty(\Omega)) \), and \( p_{p,0} \in H_p \). Then there exists a unique solution to (LP1), \( u_f \in H^1(0, T; V_f) \), \( p_f \in H^1(0, T; W_f) \), \( \eta_p \in W^{1,\infty}(0, T; X_p) \), \( u_p \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V_p) \), \( p_p \in W^{1,\infty}(0, T; W_p) \), \( \lambda \in H^1(0, T; \Lambda) \), satisfying \( p_p(0) = p_{p,0} \) and \( \eta_p(0) = \eta_{p,0} \). Moreover, \( u_f(0) = u_{f,0} \), \( p_f(0) = p_{f,0} \), \( u_p(0) = u_{p,0} \), and \( \lambda(0) = \lambda_0 \).

Let \( q^- \in L^\infty(0, T; L^\infty(\Omega)) \), \( q^+ c_w \in L^\infty(0, T; L^2(\Omega)) \), \( \theta \in W^{1,\infty}(0, T; L^\infty(\Omega)) \), and \( c_0 \in H^1(\Omega) \). Then, there exists a unique solution to (LP2), \( c \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; X_c) \), with \( c(0) = c_0 \).
Proof. We pass to the limit in the sequences defined by (LGP1) and (LGP2). Theorem 4.2.2 implies that \( \{u_{fh}\} \) is bounded in \( H^1(0,T; H^1(\Omega_f)) \), \( \{p_{fh}\} \) is bounded in \( H^1(0,T; L^2(\Omega_f)) \), \( \{\eta_{ph}\} \) is bounded in \( W^{1,\infty}(0,T; H^1(\Omega_p)) \), \( \{u_{ph}\} \) is bounded in \( H^1(0,T; L^2(\Omega_p)) \cap L^\infty(0,T; H(\text{div}; \Omega_p)) \), \( \{\lambda_h\} \) is bounded in \( H^1(0,T; H^{1/2}(\Gamma_{fp})) \), and \( \{c_h\} \) is bounded in \( H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \). By the Eberlein–Šmulian Theorem [34], there exist \( u_f \in H^1(0,T; H^1(\Omega_f)) \), \( p_f \in H^1(0,T; L^2(\Omega_f)) \), \( \eta_p \in W^{1,\infty}(0,T; H^1(\Omega_p)) \), \( u_p \in H^1(0,T; L^2(\Omega_p)) \cap L^\infty(0,T; H(\text{div}; \Omega_p)) \), \( p_p \in W^{1,\infty}(0,T; L^2(\Omega_p)) \), \( \lambda \in H^1(0,T; H^{1/2}(\Gamma_{fp})) \), and \( c \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \), such that, up to a subsequence,

\[
\begin{align*}
  u_{fh} & \rightarrow u_f \text{ in } H^1(0,T; H^1(\Omega_f)), & p_{fh} & \rightarrow p_f \text{ in } H^1(0,T; L^2(\Omega_f)), \\
  \eta_{ph} & \rightarrow \eta_p \text{ in } H^1(0,T; H^1(\Omega_p)), & p_{ph} & \rightarrow p_p \text{ in } H^1(0,T; L^2(\Omega_p)), \\
  u_{ph} & \rightarrow u_p \text{ in } H^1(0,T; L^2(\Omega_p)) \cap L^2(0,T; H(\text{div}; \Omega_p)), & \lambda_h & \rightarrow \lambda \text{ in } H^1(0,T; H^{1/2}(\Gamma_{fp})), \\
  c_h & \rightarrow c \text{ in } L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)),
\end{align*}
\]

where \( \rightarrow \) denotes weak convergence. Fixing a set of test functions \( (v_{fh}, w_{fh}, \xi_{ph}, v_{ph}, w_{ph}, \nu_h) \in C^0(0,T; V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \Lambda_h) \) in (4.2.4) and \( \psi_h \in C^0(0,T; X_{ch}) \) in (4.2.5), integrating in time from 0 to \( T \), and taking \( h \rightarrow 0 \), we conclude that \( (u_f, p_f, \eta_p, u_p, p_p, \lambda) \) and \( c \) satisfy the time-integrated versions of (4.2.1) and (4.2.2), respectively, with this choice of test functions. Since the spaces \( C^0(0,T; V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \Lambda_h) \) and \( C^0(0,T; X_{ch}) \) are dense in \( L^2(0,T; V_f \times W_f \times X_p \times V_p \times W_p \times \Lambda) \) and \( L^2(0,T; X_c) \), respectively, we deduce that (4.2.1) and (4.2.2) hold for a.e. \( t \in (0,T] \). It remains to show that the initial conditions are satisfied. For any \( w_p \in W_p \) we write

\[
(p_{p,0} - p_p(0), w_p)_{\Omega_p} = (p_{p,0} - p_{ph,0}, w_p)_{\Omega_p} + (p_{ph,0} - p_p(0), w_p)_{\Omega_p},
\]

where we used that \( p_{ph,0} = p_{ph,0} \). As \( h \rightarrow 0 \), \( (p_{p,0} - p_{ph,0}, w_p)_{\Omega_p} \rightarrow 0 \), due to the approximation property of the Stokes–Darcy elliptic projection (4.2.17), and \( (p_{ph,0} - p_p(0), w_p)_{\Omega_p} \rightarrow 0 \) from the weak convergence \( p_{ph} \rightarrow p_p \) in \( H^1(0,T; L^2(\Omega_p)) \). Therefore \( p_p(0) = p_{p,0} \). A similar argument shows that \( u_f(0) = u_{f,0}, p_f(0) = p_{f,0}, u_p(0) = u_{p,0}, \) and \( \lambda(0) = \lambda_0 \). In a similar way we conclude that \( \eta_p(0) = \eta_{p,0} \) and \( c(0) = c_0 \), using the approximation properties of the elasticity elliptic projection (4.2.18) and the Scott-Zhang interpolant \( I_{ch} \), respectively.

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The uniqueness of the solution to (4.2.1) satisfying $p_{ph}(0) = p_{p,0}$ and $\eta_p(0) = \eta_{p,0}$, and of the solution to (4.2.2) satisfying $c(0) = c_0$ follow from the uniqueness arguments in the proof of Theorem 2.2.4, using the continuous inf-sup condition (4.1.13).

\[ \square \]

### 4.3 Analysis of the original non-linear problem (P)

The analysis of the original fully coupled non-linear problem (P) is done by considering an iteration of decoupled linearized problems (LP1) and (LP2) and utilizing the Banach fixed point Theorem [30]. We first obtain a local in time solution, then extend it on small time intervals until the solution is obtained globally on $(0, T]$. Similar techniques have been used in [21, 88]. The iterative algorithm is as follows: given $c^0$, $u_j^0$ and $u_p^0$, for $m \geq 1$, solve

**(LP1)**: Find $(u_j^m, p_j^m, \eta_p^m, u_p^m, p_p^m, \lambda^m) : [0, T] \to V_f \times V_p \times X_p \times W_f \times W_p \times \Lambda$ such that $p_p^m(0) = p_{p,0}$, $\eta_p^m(0) = \eta_{p,0}$, and for a.e. $t \in (0, T]$ and for all $v_f \in V_f$, $w_f \in W_f$, $\xi_p \in X_p$, $v_p \in V_p$, $u_p \in W_p$, and $\nu \in \Lambda$,

\[
\begin{align*}
  a_f(u_j^m, v_f; c^{m-1}) &+ b_f(v_f, p_j^m) + a_{BJS}(u_j^m, \partial_t \eta_p^m, v_f, 0) + b_T(v_f, 0, 0; \lambda^m) &\quad \text{(4.3.1a)} \\
  &= (f_f, v_f)_{\Omega_f}, &\quad \text{(4.3.1b)} \\
  - b_f(u_j^m, w_f) &= (q_f, w_f)_{\Omega_f}, &\quad \text{(4.3.1c)} \\
  a_p^d(u_p^m, v_p; c^{m-1}) &+ b_p(v_p, p_p^m) + b_T(0, 0, v_p, 0; \lambda^m) = 0, &\quad \text{(4.3.1e)} \\
  s_0(\partial_t p_p^m, w_p)_{\Omega_p} &- \alpha b_s(\partial_t \eta_p^m, w_p) - b_p(u_p^m, w_p) = (q_p, w_p)_{\Omega_p}, &\quad \text{(4.3.1f)} \\
  b_T(u_j^m, u_p^m, \partial_t \eta_p^m, \nu) &= 0. &\quad \text{(4.3.1g)}
\end{align*}
\]

**(LP2)**: Find $c^m : [0, T] \to X_c$ such that $c^m(0) = c_0$, and for a.e. $t \in (0, T]$ and for all $\psi \in X_c$,

\[
(\phi \partial_t c^m, \psi) + (D(u^{m-1}) \nabla c^m, \nabla \psi) - (c^m u^{m-1}, \nabla \psi) - (q^c c^m, \psi) = (q^c c_w, \psi). \quad \text{(4.3.2)}
\]
where \( u^m|_{\Omega_f} := u_f^m \) and \( u^m|_{\Omega_p} := u_p^m \). Recall that the initial conditions have been constructed in Theorem 2.2.4. We make the following assumption on the velocity \( u^m \) and concentration \( c^m \).

**Assumption 4.3.1.** For all \( m \geq 0 \), \( u_f^m \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega_f)) \), \( u_p^m \in W^{1,\infty}(0, T; L^\infty(\Omega_p)) \), and \( c^m \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega)) \). Denote

\[
\beta_u := \max \left\{ \sup_m \| u_f^m \|_{W^{1,\infty}(0, T; W^{1,\infty}(\Omega_f))}, \sup_m \| u_p^m \|_{W^{1,\infty}(0, T; L^\infty(\Omega_p))} \right\}
\]

and

\[
\beta_c := \sup_m \{ \| c^m \|_{W^{1,\infty}(0, T; W^{1,\infty}(\Omega))} \}.
\]

The assumption is satisfied with sufficient smooth data and domain boundary. We ensure the practicability of this assumption at the end of this section. Readers are referred to [21], Appendix A, for details.

**Remark 4.3.1.** The well-posedness of \((LP1m)\) and \((LP2m)\) are ensured by Theorem 4.2.3 and Assumption 4.3.1. For each fixed \( m \geq 1 \), \((LP1m)\) and \((LP2m)\) are actually two separate linear system. Second, under Assumption 4.3.1, we can repeat the same procedures as the proof of Theorem 4.2.3 to obtain the solvability and stability of this linear system \((LP1m)\) and \((LP2m)\) for every \( m \geq 1 \), respectively.

**Remark 4.3.2.** Note that in Chapter 4, the superscript \( m \) of \( u^m \) denotes the terms of a sequence. In Chapter 2, the superscript \( n \) in \( u^n \) denotes the number of time steps, namely \( u^n = u(t_n) \).

We come up with the main result of the original non-linear problem.

**Theorem 4.3.1.** For each

\[
f_f \in H^1(0, T; V'_f), \quad f_p \in H^1(0, T; X'_p), \quad q_f \in H^1(0, T; W'_f), \quad q_p \in H^1(0, T; W'_p),
\]

\[
q^+ c_w, L^2(0, T; X'_f), \quad \text{and} \quad q^- \in L^\infty(0, T; L^\infty(\Omega)),
\]

under Assumption 4.3.1, the algorithm (4.3.1)-(4.3.2) defines a unique iterative sequence \((u_f^m, u_p^m, \eta_p^m, p_f^m, p_p^m, \lambda^m, c^m)\) that converges to the weak solution \((u_f, u_p, \eta_p, p_f, p_p, \lambda, c)\) of \((P)\), satisfying

\[
(u_f, u_p, \eta_p) \in L^\infty(0, T; V_f) \times (L^\infty(0, T; L^2(\Omega_p)) \cap L^2(0, T; V_p)) \times H^1(0, T; X_p);
\]
\[(p_f, p_p, \lambda) \in L^\infty(0, T; W_f) \times H^1(0, T; W_p) \times L^\infty(0, T; \Lambda);\]
\[c \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; X_c).\]

**Proof.** The iteration \((u^m_f, p^m_f, \eta^m_p, u^m_p, p^m_p, \lambda^m, c^m)\) is well-defined for all \(m \geq 1\) due to Theorem 4.2.3 and the resulting sequence is unique. It remains to show the convergence of the iteration to the solution of \((P)\) in suitable norms. For \(m \geq 2\), denote by \((e^m_{u_f}, e^m_{p_f}, e^m_{\eta_p}, e^m_{u_p}, e^m_{p_p}, e^m_\lambda, e^m_c)\) the difference between the solutions of \((LP1_m)\) and \((LP2_m)\) at the \((m)th\) step and the \((m-1)th\) step, namely \(e^m_{u_f} := u^m_f - u^{m-1}_f\), with analogous notation for the remaining variables.

Subtracting (4.3.1)–(4.3.2) with \((m)th\) step and \((m-1)th\) step, we obtain: \((e^m_{u_f}, e^m_{p_f}, e^m_{\eta_p}, e^m_{u_p}, e^m_{p_p}, e^m_\lambda, e^m_c) : [0, T] \to V_f \times W_f \times X_p \times V_p \times W_p \times \Lambda \times X_c\) such that for all \((v_f, w_f, \xi_p, v_p, w_p, \nu, \psi)\) \(\in V_f \times W_f \times X_p \times V_p \times W_p \times \Lambda \times X_c\) and for a.e. \(t \in (0, T)\):

\[
a_f(e^m_{u_f}, v_f; c^{m-1}) + \left(2(\mu(c^{m-1}) - \mu(c^{m-2}))(u^m_f, - \epsilon(v_f)\right)_{\Omega_f} = a_{BJS}(e^m_{u_f}, \partial_t e^m_{\eta_p}; \nu_f, v_f, 0)
\]
\[+ b_f(v_f, e^m_f) + b_T(v_f, 0, 0; e^m) = 0, \tag{4.3.3a}\]
\[b_f(e^m_{u_f}, w_f) = 0, \tag{4.3.3b}\]
\[a^c_p(e^m_{\eta_p}, \xi_p) + a_{bc}(\xi_p, e^m_p) + a_{BJS}(e^m_{u_f}, \partial_t e^m_{\eta_p}; 0, \xi_p) + b_T(0, 0, \xi_p; e^m_\lambda) = 0, \tag{4.3.3c}\]
\[a^d_p(e^m_{u_p}, v_p; c^{m-1}) + \left((\mu(c^{m-1}) - \mu(c^{m-2}))K^{-1}u^{m-1}_p, v_p\right)_{\Omega_p} + b_p(v_p, e^m_{p_p})
\]
\[+ b_T(0, v_p, 0; e^m_\lambda) = 0, \tag{4.3.3d}\]
\[s_0(\partial_t e^m_{p_p}, w_p)_{\Omega_p} - b_p(e^m_{u_p}, w_p) - ab_c(\partial_t e^m_{\eta_p}, w_p) = 0, \tag{4.3.3e}\]
\[b_T(e^m_{u_f}, e^m_{u_p}, \partial_t e^m_{\eta_p}; \nu) = 0, \tag{4.3.3f}\]
\[(\phi \partial_t e^m_c, \psi) + (D(u^{m-1}) \nabla e^m_c, \nabla \psi) + ((D(u^{m-1}) - D(u^{m-2})) \nabla e^{m-1}, \nabla \psi)
\]
\[\left(- e^m_c u^m_f, \nabla \psi\right) - (c^{m-1} u^{m-1}_u, \nabla \psi) - (q^c e^m_c, \psi) = 0, \tag{4.3.3g}\]

together with homogeneous initial conditions for all the variables. The proof consists of five steps: (1) derive bounds on \((e^m_{u_f}, e^m_{p_f}, e^m_{\eta_p}, e^m_{u_p}, e^m_{p_p}, e^m_\lambda, e^m_c)\) on a small time interval; (2) find bounds on \(e^m_c\) on a small time interval; (3) using these bounds, obtain a local in time convergence using a fixed point argument; (4) find the bounds for \((e^m_{u_f}, e^m_{p_f}, e^m_{\eta_p}, e^m_{u_p}, e^m_{p_p}, e^m_\lambda, e^m_c)\)
with a higher regularity based on the contraction iteration we have at step (3); (5) extend the convergence results globally in $(0, T)$.

- **Step 1**: Bound $(\mathbf{e}_{m}^{f}, \mathbf{e}_{m}^{p}, \mathbf{e}_{m}^{w}, \mathbf{e}_{m}^{\epsilon}, \mathbf{e}_{m}^{\lambda})$ by $e_{c}^{m-1}$.

Taking $(\mathbf{v}_{f}, \mathbf{w}_{f}, \mathbf{v}_{p}, \mathbf{w}_{p}, \nu) = (\mathbf{e}_{m}^{f}, \mathbf{e}_{m}^{p}, \partial_{t} \mathbf{e}_{m}^{w}, \mathbf{e}_{m}^{w}, \mathbf{e}_{m}^{\lambda})$ in (4.3.3a)–(4.3.3f), we deduce that

\[
\frac{s_{0}}{2} \frac{d}{dt}\|e_{m}^{f}\|_{L^{2}(\Omega_{p})}^{2} + \frac{1}{2} \frac{d}{dt} a_{p}(e_{m}^{f}, e_{m}^{f}) + a_{f}(e_{m}^{f}, e_{m}^{f}, c^{m-1}) + a_{p}(e_{m}^{w}, e_{m}^{w}; c^{m-1}) + a_{BJS}(e_{m}^{f}, \partial_{t} e_{m}^{w}, e_{m}^{w}, \partial_{t} e_{m}^{w})
= \left(2(\mu(c^{m-2}) - \mu(c^{m-1}))\|e_{m}^{f}\|_{L^{2}(\Omega_{f})}, e(e_{m}^{f})\right)_{\Omega_{f}}
\]

\[
+ \left((\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}u_{m}^{p-1}, e_{m}^{m}\right)_{\Omega_{p}}. \tag{4.3.4}
\]

Using Assumption 4.3.1 to control $\|e(u_{m}^{f-1})\|_{L^{2}(0,T;L^{2}(\Omega_{f}))}$ by $C$ and the Lipschitz continuity of the viscosity function $\mu(\cdot)$ (cf. (4.1.6)), we have

\[
\left(2(\mu(c^{m-2}) - \mu(c^{m-1}))\|e_{m}^{f-1}\|_{L^{2}(\Omega_{f})}, e(e_{m}^{f})\right)_{\Omega_{f}} \leq C \|c^{m-2} - c^{m-1}\|_{L^{2}(\Omega_{f})} \|e_{m}^{m}\|_{L^{2}(\Omega_{f})}. \tag{4.3.5}
\]

Similarly, as the other term on the right hand side of (4.3.4), we have

\[
\left((\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}u_{m}^{p-1}, e_{m}^{m}\right)_{\Omega_{p}} \leq C \|c^{m-2} - c^{m-1}\|_{L^{2}(\Omega_{p})} \|e_{m}^{m}\|_{L^{2}(\Omega_{p})}. \tag{4.3.6}
\]

Note that both in (4.3.5) and (4.3.6), the constant $C$ depends on the Lipschitz constant $L_{1}$ in (4.1.6). Next, combining with (4.3.4)–(4.3.6), using the coercivity of $a_{f}, a_{p}, a_{c}, a_{BJS}$ from Lemma 4.1.1, and integrating over time from $0$ to $t$, as well as utilizing the Cauchy-Schwarz and Young’s inequalities for some positive $\epsilon > 0$, we get

\[
\|e_{m}^{f}(t)\|_{L^{2}(\Omega_{p})}^{2} + \|e_{m}^{w}(t)\|_{H^{1}(\Omega_{p})}^{2} + \int_{0}^{t} \left(\|e_{m}^{f}\|_{H^{1}(\Omega_{f})}^{2} + \|e_{m}^{w}\|_{L^{2}(\Omega_{p})}^{2} + \|e_{m}^{w} - \partial_{t} e_{m}^{w}\|_{BJS}^{2}\right) ds
\leq C \int_{0}^{t} \left(\|e_{c}^{m-1}\|_{L^{2}(\Omega_{f})}^{2} + \|e_{c}^{m-1}\|_{L^{2}(\Omega_{p})}^{2}\right) ds + \epsilon \int_{0}^{t} \left(\|e_{m}^{f}\|_{H^{1}(\Omega_{f})}^{2} + \|e_{m}^{w}\|_{L^{2}(\Omega_{p})}^{2}\right) ds. \tag{4.3.7}
\]

Next, using the inf-sup condition (4.1.13) (cf. Lemma 4.1.2), (4.3.3a), and Assumption 4.3.1, we deduce that

\[
\|(e_{m}^{f-1}, e_{m}^{p-1}, e_{m}^{m})\|_{W_{f} \times W_{p} \times \Lambda}
\]
Thus, for the first term on the right hand side of (4.3.11), we have

\[ \epsilon \]

which yields

\[ \int_0^t \left( \|e_p^m\|^2_{L^2(\Omega_f)} + \|e_p^m\|^2_{L^2(\Omega_p)} + \|e_c^m\|^2_{L^2(\Omega)} \right) ds \]

\[ \leq \int_0^t \left( \|e_{u_f}^m\|^2_{H^1(\Omega_f)} + \|e_{\eta_p}^m\|^2_{H^1(\Omega_p)} + \|e_c^m\|^2_{H^1(\Omega)} \right) ds. \]

(4.3.9)

Thus, taking \( \epsilon \) small enough in (4.3.7), and using (4.3.9), we find

\[ \|e_p^m(t)\|^2_{L^2(\Omega_p)} + \|e_{u_f}^m(t)\|^2_{H^1(\Omega_f)} + \int_0^t \left( \|e_{u_f}^m\|^2_{H^1(\Omega_f)} + \|e_{u_p}^m\|^2_{H^1(\Omega_p)} \right) ds \]

\[ \leq C \int_0^t \|e_c^{m-1}\|^2_{L^2(\Omega)} ds. \]

(4.3.10)

- **Step 2:** Bound \( e_c^m \) by \( e_{u_f}^{m-1} \) and \( e_{u_p}^{m-1} \).

Taking \( \psi = e_c^m \) in (4.3.3g), gives

\[ \phi \partial_t e_c^m + (D(u^{m-1}) \nabla e_c^m, \nabla e_c^m) = (q^c e_c^m, e_c^m) \]

\[ = ([D(u^{m-2}) - D(u^{m-1})] \nabla c^{m-1}, \nabla e_c^m) + (u^{m-1} e_c^m, \nabla e_c^m) + (e_{u_f}^{m-1} e_c^m, \nabla e_c^m). \]

(4.3.11)

Note that the term \(-(q^c e_c^m, e_c^m)\) on the left hand side of (4.3.11) is non-negative which can be dropped directly. Next, according to Theorem 3.1 in [44], we have the following property for the \((i,j)\)-entry of \( |u_1| E(u_1) - |u_2| E(u_2) \),

\[ \left| \frac{(u_1)_i(u_1)_j}{|u_1|} - \frac{(u_2)_i(u_2)_j}{|u_2|} \right| \leq 3|u_1 - u_2|. \]

Thus, for the first term on the right hand side of (4.3.11), we have

\[ \left| ([D(u^{m-2}) - D(u^{m-1})] \nabla c^{m-1}, \nabla e_c^m) \right| \leq C \|\nabla c^{m-1}\|_{L^\infty(\Omega)} \|e_{u_f}^{m-1}\|_{L^2(\Omega)} \|\nabla e_c^m\|_{L^2(\Omega)}. \]

(4.3.12)
Note that $\nabla c^{m-1} \in L^\infty(0, T; L^\infty(\Omega))$ due to Assumption 4.3.1. For the remaining terms of the right hand side of (4.3.11), we use that $u^{m-1} \in L^\infty(0, T; L^\infty(\Omega))$, $c^{m-1} \in L^\infty(0, T; L^\infty(\Omega))$, as well as the combination of Cauchy-Schwarz and Young’s inequalities to get

$$
\frac{d}{dt} \|e^m_c\|^2_{L^2(\Omega)} + \|\nabla e^m_c\|^2_{L^2(\Omega)} \leq C \left( \beta_c \|e^{m-1}_u\|^2_{L^2(\Omega)} + \|e^m_c\|^2_{L^2(\Omega)} \right). 
$$

(4.3.13)

Next, adding $\|e^m_c\|^2_{L^2(\Omega)}$ on the both sides of the above inequality, integrating over time from 0 to $t$ and using Gronwall’s inequality, we obtain

$$
\|e^m_c(t)\|^2_{L^2(\Omega)} + \int_0^t \left( \|e^m_c\|^2_{L^2(\Omega)} + \|\nabla e^m_c\|^2_{L^2(\Omega)} \right) ds 
\leq C \exp(T) \int_0^t \left( \|e^{m-1}_u\|^2_{L^2(\Omega)} + \|e^{m-1}_c\|^2_{L^2(\Omega)} \right) ds .
$$

(4.3.14)

- **Step 3:** Prove $(e^m_u, e^m_p, e^m_{\eta_p}, e^m_{u_p}, e^m_{p_p}, e^m_{\lambda}, e^m_c) \to 0$ as $m \to \infty$ under suitable norms by fixed point argument on $(0, t_1]$, for some $0 < t_1 \leq T$ to be fixed later.

We integrate (4.3.14) from 0 to $t_1$ again, and use (4.3.10), to deduce

$$
\int_0^{t_1} \|e^m_c\|^2_{L^2(\Omega)} ds \leq t_1 C \exp(T) \int_0^{t_1} \left( \|e^{m-1}_u\|^2_{L^2(\Omega)} + \|e^{m-1}_c\|^2_{L^2(\Omega)} \right) ds 
\leq t_1 C \exp(T) \int_0^{t_1} \|e^{m-2}_c\|^2_{L^2(\Omega)} ds ,
$$

(4.3.15)

where the constant $0 < C < \infty$ is independent of the fixed point iteration step $m$ and the local time $t_1$. Thus, if we choose $t_1$ such that $q := t_1 C \exp(T) < 1$ in (4.3.15), we conclude that the mapping $e^{m-2}_c \mapsto e^m_c$ is a contraction on $(0, t_1]$. The Banach fixed point Theorem [30], together with the estimates (4.3.10), (4.3.14), and (4.3.15) imply the following convergence as $m \to \infty$,

$$(e^m_u, e^m_{\eta_p}, e^m_{u_p}) \to 0 \quad \text{in} \quad L^2(0, t_1; V_f) \times L^2(0, t_1; L^2(\Omega_p)) \times L^\infty(0, t_1; X_p) ;$$

$$(e^m_p, e^m_{p_p}, e^m_{\lambda}) \to 0 \quad \text{in} \quad L^2(0, t_1; W_f) \times L^\infty(0, t_1; W_p) \times L^2(0, t_1; \Lambda) ;$$

$$e^m_c \to 0 \quad \text{in} \quad L^\infty(0, t_1; L^2(\Omega)) \cap L^2(0, t_1; X_c).$$

(4.3.16)

- **Step 4:** Obtain convergence of $(e^m_u, e^m_{\eta_p}, e^m_{u_p}, e^m_{p_p}, e^m_{\lambda}, e^m_c)$ with higher regularity bounds.
From Step 3, we know that \((e_m^n, e_p^m, e_{\eta_p}^m, e_{\nu_p}^m, e_{p_p}^m, e_{c_p}^m, e_c^m) \to 0\) in the norms given in (4.3.16), when \(m \to \infty\). However, this is not enough since at least we also need \(\nabla \cdot e_{u_p}^m \to 0\) to obtain that the limit of \(\{u_p^m\}\), \(u_p\) is indeed in \(H(\text{div}; \Omega_p)\). First, we note that Cauchy-Schwarz inequality implies
\[
\left(\int_0^{t_1} \int_{\Omega_f} |e(u_{f_j})| ds\right)^2 \leq t_1 \int_0^{t_1} \left(\int_{\Omega_f} |e(u_{f_j})| ds\right)^2 ds
\]
\[
\leq T |\Omega_f| \int_0^{t_1} \|e(u_{f_j})\|^2_{L^2(\Omega_f)} ds \to 0, \text{ as } m \to \infty. \tag{4.3.17}
\]
Similarly, we have for \(e_{u_p}^m\) and \(e_{c_p}^m\), as \(m \to \infty\)
\[
\left(\int_0^{t_1} \int_{\Omega_p} |e_{u_p}^m| ds\right)^2 \leq T |\Omega_p| \int_0^{t_1} \|e_{u_p}^m\|^2_{L^2(\Omega_p)} ds \to 0, \tag{4.3.18}
\]
\[
\left(\int_0^{t_1} \int_{\Omega_p} |\nabla e_{c_p}^m| ds\right)^2 \leq T |\Omega| \int_0^{t_1} \|\nabla e_{c_p}^m\|^2_{L^2(\Omega)} ds \to 0. \tag{4.3.19}
\]
Next, differentiating (4.3.3a)–(4.3.3f), adding them together, and choosing the test functions to be \(v_j = e_{u_j}^m, w_f = \partial_t e_{u_j}^m, \xi_p = \partial_t e_{\eta_p}^m, v_p = e_{u_p}^m, w_p = \partial_t e_{c_p}^m, \nu = \partial_t e_c^m\); we obtain the differentiated error equation:
\[
a_f(\partial_t e_{u_j}^m, e_{u_j}^m, e_{c_p}^{m-1}) + \left(2\partial_t \mu(c^{m-1}) e_{u_j}^m, e_{u_j}^m, e_{u_j}^m\right)_{\Omega_f} + a_p^t(\partial_t e_{u_p}^m, e_{u_p}^m, e_{c_p}^{m-1})
\]
\[
+ \left(\partial_t \mu(c^{m-1}) K^{-1} e_{u_p}^m, e_{u_p}^m\right)_{\Omega_p} + a_p^c(\partial_t e_{\eta_p}^m, \partial_t e_{\eta_p}^m) + a_{BJS}(\partial_t e_{u_p}^m, \partial_t e_{\eta_p}^m, e_{u_p}^m, \partial_t e_{\eta_p}^m)
\]
\[
+ \left(2\partial_t (\mu(c^{m-1}) - \mu(c^{m-2})) e(u_{j-1}^m), e(u_{j-1}^m)\right)_{\Omega_f} + \left(2(\mu(c^{m-1}) - \mu(c^{m-2})) \partial_t e(u_{j-1}^m), e_{u_j}^m\right)_{\Omega_f}
\]
\[
+ \left(\partial_t (\mu(c^{m-1}) - \mu(c^{m-2})) K^{-1} u_p^{m-1}, e_{u_p}^m\right)_{\Omega_p} + \left((\mu(c^{m-1}) - \mu(c^{m-2})) K^{-1} \partial_t u_p^{m-1}, e_{u_p}^m\right)_{\Omega_p}
\]
\[
+ s_0(\partial_t e_{c_p}^m, \partial_t e_{c_p}^m)_{\Omega_p} = 0. \tag{4.3.20}
\]
We integrate over time from 0 to \(t_1\) the above equation to get
\[
\|e_{u_j}^m(t_1)\|^2_{H^1(\Omega_f)} + \|e_{u_p}^m(t_1)\|^2_{L^2(\Omega_p)} + \|e_{u_j}^m - \partial_t e_{\eta_p}^m(t_1)\|^2_{BJS} + s_0 \int_0^{t_1} \|\partial_t e_{c_p}^m\|^2_{L^2(\Omega_p)} ds
\]
\[
+ \int_0^{t_1} \|\partial_t e_{\eta_p}^m\|^2_{H^1(\Omega_p)} ds
\]
\[
\leq C \left(\int_0^{t_1} \left(2\partial_t \mu(c^{m-1}) e(u_{j-1}^m), e(u_{j-1}^m)\right)_{\Omega_f} + \left(\partial_t \mu(c^{m-1}) K^{-1} e_{u_p}^m, e_{u_p}^m\right)_{\Omega_p}\right) ds
\]
\[
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\]
\[
+ \int_0^{t_1} \left( 2\partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))\epsilon(u_f^{m-1}, \epsilon(e_{u_f}^m))_{\Omega_f} + \left( \partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}u_p^{m-1}, e_{u_p}^m \right)_{\Omega_p} \right) ds \\
+ \int_0^{t_1} \left( 2\mu(c^{m-2}) - \mu(c^{m-1}))\partial_t(\epsilon(u_f^{m-1}, \epsilon(e_{u_f}^m))_{\Omega_f} + \left( \partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}u_p^{m-1}, e_{u_p}^m \right)_{\Omega_p} \right) ds \\
+ \left( \mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\partial_t u_p^{m-1}, e_{u_p}^m \right)_{\Omega_p} \right) ds.
\]  

(4.3.21)

For the first two terms on the right hand side of (4.3.21), we have
\[
\int_0^{t_1} \left( 2\partial_t(\mu(c^{m-1})\epsilon(e_{u_f}^m), \epsilon(e_{u_f}^m))_{\Omega_f} + \left( \partial_t(\mu(c^{m-1})K^{-1}e_{u_p}^m, e_{u_p}^m \right)_{\Omega_p} \right) ds \\
\leq C_\beta \int_0^{t_1} \left( \|e_{u_f}^m\|_{H^1(\Omega_f)}^2 + \|e_{u_p}^m\|_{L^2(\Omega_p)}^2 \right) ds \to 0, \text{ as } m \to \infty.
\]  

(4.3.22)

For the second and third terms on the right hand side of (4.3.21), combing with (4.3.17) and (4.3.18), we have
\[
\int_0^{t_1} \left( 2\partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))\epsilon(u_f^{m-1}, \epsilon(e_{u_f}^m))_{\Omega_f} + \left( \partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}u_p^{m-1}, e_{u_p}^m \right)_{\Omega_p} \right) ds \\
\leq C T \beta \left( \int_0^{t_1} \int_{\Omega_f} |\epsilon(e_{u_f}^m)| ds + \int_0^{t_1} \int_{\Omega_p} |e_{u_p}^m| ds \right) \to 0, \text{ as } m \to \infty.
\]  

(4.3.23)

For the last two terms on the right hand side of (4.3.21), by using the Cauchy-Schwarz and Young’s inequalities, we have
\[
\int_0^{t_1} \left( 2\mu(c^{m-2}) - \mu(c^{m-1}))\partial_t(\epsilon(u_f^{m-1}, \epsilon(e_{u_f}^m))_{\Omega_f} + \left( \mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\partial_t u_p^{m-1}, e_{u_p}^m \right)_{\Omega_p} \right) ds \\
\leq C \beta \int_0^{t_1} \left( \|e_{c}^{m-1}\|_{L^2(\Omega_f)}^2 + \|e_{u_f}^m\|_{L^2(\Omega_f)}^2 + \|e_{u_p}^m\|_{L^2(\Omega_p)}^2 \right) ds \to 0, \text{ as } m \to \infty.
\]  

(4.3.24)

Combining (4.3.22)–(4.3.24) with (4.3.21), we obtain
\[
\|e_{u_f}^m(t_1)\|_{H^1(\Omega_f)} + \|e_{u_p}^m(t_1)\|_{L^2(\Omega_p)} + \|e_{c}^{m-1} - \partial_t e_{\eta_p}^m\|_{B.I.S} + s_0 \int_0^{t_1} \|\partial_t e_{\eta_p}^m\|_{L^2(\Omega_p)} ds \\
+ \int_0^{t_1} \|\partial_t e_{\eta_p}^m\|_{H^1(\Omega_p)} ds \to 0, \text{ as } m \to \infty.
\]  

(4.3.25)
Next, take \( w_f = 0 \) and \( w_p = \nabla \cdot e^{m}_{u_p} \) in (4.3.3e) and integrate over time from 0 to \( t_1 \), which results in

\[
\int_0^{t_1} \| \nabla \cdot e^{m}_{u_p} \|_{L^2(\Omega)}^2 \, ds \leq C \int_0^{t_1} \left( \| \partial_t e^{m}_{p} \|_{L^2(\Omega)}^2 + \| \partial_t e^{m}_{\eta_p} \|_{H^1(\Omega)}^2 \right) \, ds \to 0, \quad \text{as } m \to \infty. \quad (4.3.26)
\]

At the same time, from (4.3.8), we also have

\[
\| e^{m}_{p} \|_{L^2(\Omega)}^2 + \| e^{m}_{p} \|_{L^2(\Omega_p)}^2 + \| e^{m}_{\eta_p} \|_{L^2(\Omega_p)}^2 \leq C \left( \| e^{m}_{u_f} \|_{H^1(\Omega_f)}^2 + \| e^{m}_{u_p} \|_{L^2(\Omega_p)}^2 + \| e^{m}_{u_f} - \partial_t e^{m}_{\eta_p} \|_{BJS}^2 + \| e^{m-1}_{\eta} \|_{L^2(\Omega)}^2 \right) \to 0, \quad \text{as } m \to \infty. \quad (4.3.27)
\]

We proceed with the transport error equation (4.3.3g). Taking the test function \( \psi = \partial_t e^{m}_c \) gives

\[
(\phi \partial_t e^m_c, \partial_t e^m_c) + (D(\mathbf{u}^{m-1}) \nabla e^m_c, \nabla \partial_t e^m_c) - (q^{-} e^m_c, \partial_t e^m_c) = \left( (D(\mathbf{u}^{m-2}) - D(\mathbf{u}^{m-1})) \nabla c^{m-1}, \nabla \partial_t e^m_c \right) + (\mathbf{u}^{m-1} \nabla e^m_c, \nabla \partial_t e^m_c)
\]

\[
+ (e^{m-1} \nabla e^m_c, \nabla \partial_t e^m_c). \quad (4.3.28)
\]

For the second term on the left hand side of (4.3.28), we write

\[
(D(\mathbf{u}^{m-1}) \nabla e^m_c, \nabla \partial_t e^m_c) = \frac{1}{2} \frac{d}{dt} (D(\mathbf{u}^{m-1}) \nabla e^m_c, \nabla e^m_c) - \frac{1}{2} (\partial_t D(\mathbf{u}^{m-1}) \nabla e^m_c, \nabla e^m_c). \quad (4.3.29)
\]

Then, with similar expression for the rest of the terms in (4.3.28), results in

\[
(\phi \partial_t e^m_c, \partial_t e^m_c) + \frac{1}{2} \frac{d}{dt} (D(\mathbf{u}^{m-1}) \nabla e^m_c, \nabla e^m_c) - \frac{1}{2} (\partial_t D(\mathbf{u}^{m-1}) \nabla e^m_c, \nabla e^m_c) - (q^{-} e^m_c, \partial_t e^m_c) = \frac{d}{dt} \left( (D(\mathbf{u}^{m-2}) - D(\mathbf{u}^{m-1})) \nabla c^{m-1}, \nabla e^m_c \right) - (\partial_t (D(\mathbf{u}^{m-2}) - D(\mathbf{u}^{m-1})) \nabla c^{m-1}, \nabla e^m_c)
\]

\[
- (D(\mathbf{u}^{m-2}) - D(\mathbf{u}^{m-1})) \nabla \partial_t c^{m-1}, \nabla e^m_c) + \frac{d}{dt} (\mathbf{u}^{m-1} \nabla e^m_c, \nabla e^m_c) - (\partial_t \mathbf{u}^{m-1} \nabla e^m_c, \nabla e^m_c)
\]

\[
- (\mathbf{u}^{m-1} \partial_t e^m_c, \nabla e^m_c) + \frac{d}{dt} (e^{m-1} \nabla e^m_c, \nabla e^m_c) - (\partial_t e^{m-1} \nabla e^m_c, \nabla e^m_c) - (e^{m-1} \partial_t e^{m-1}, \nabla e^m_c). \]

Next, we integrate over time from 0 to \( t_1 \), drop some positive terms on the left hand side, move some terms to the right hand side, use Assumption 4.3.1, the regularity of \( q^{-} \), the Cauchy-Schwarz, trace, and Young's inequalities to obtain

\[
\int_0^{t_1} \| \partial_t e^m_c \|_{L^2(\Omega)}^2 \, ds + \| \nabla e^m_c(t_1) \|_{L^2(\Omega)}^2 \leq C \left( \int_0^{t_1} \| \nabla e^m_c \|_{L^2(\Omega)}^2 \, ds + \| e^m_c \|_{L^2(\Omega)}^2 + \int_0^{t_1} \| e^m_c \|_{L^2(\Omega)}^2 \, ds \right)
\]

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Then, combining (4.3.30) with (4.3.19) and (4.3.25), as well as the results from Step 3, we conclude that

$$
\int_0^{t_1} \left( \|e^{m-1}_u\|_{L^2(\Omega_p)}^2 + \|e^{m-1}_f\|_{H^1(\Omega_f)}^2 + \int_0^{t_1} \left( \|e^{m-1}_{u_p}\|_{L^2(\Omega_p)}^2 + \|e^{m-1}_f\|_{H^1(\Omega_f)}^2 \right) \right) ds.
$$

(4.3.30)

$$\int_0^{t_1} \|\nabla e^m\|_{L^2(\Omega)}^2 ds + \|\nabla e^m(t_1)\|_{L^2(\Omega)}^2 \to 0, \text{ as } m \to \infty.
$$

(4.3.31)

Summarizing (4.3.25)–(4.3.27) and (4.3.31) implies that as \( m \to \infty \),

$$(e^m_{u_f}, e^m_{u_p}, e^m_{\eta_p}) \to 0 \quad \text{in} \quad L^\infty(0, t_1; V_f) \times (L^\infty(0, t_1; L^2(\Omega_p)) \cap L^2(0, t_1; V_p)) \times H^1(0, t_1; X_p);$$

$$(e^m_{p_f}, e^m_{p_p}, e^m_\lambda) \to 0 \quad \text{in} \quad L^\infty(0, t_1; W_f) \times H^1(0, t_1; W_p) \times L^\infty(0, t_1; \Lambda);$$

$$e^m_c \to 0 \quad \text{in} \quad H^1(0, t_1; L^2(\Omega)) \cap L^\infty(0, t_1; X_c).$$

(4.3.32)

Due to (4.3.32), we can take \( m \to \infty \) in (LP1) and (LP2) to conclude that \((u_f^m, p_f^m, \eta_p^m, u_p^m, p_p^m, \lambda^m, c^m)\) converges to a solution \((u_f, p_f, \eta_p, u_p, p_p, \lambda, c)\) of the weak formulation \((P)\) on \([0, t_1]\).

**Step 5:** Extend the local result on \((0, t_1]\) globally in time to \((0, T]\).

Step 4 establishes convergence of the iteration on the time interval \((0, t_1]\). We now extend this result globally in time. Let \([0, T] = \sum_{k=1}^N [t_{k-1}, t_k]\) satisfying \( t_0 = 0, t_N = T \) and \( \sup_{1 \leq k \leq N} |t_k - t_{k-1}| \leq q/(C \exp(T)) \) and we denote \( \Delta t_k = t_k - t_{k-1} \).

Then, for \( k = 2, \cdots, N \), we consider the iteration based on (LP1) and (LP2) on the interval \((t_{k-1}, t_k]\) with initial data \((u_f(t_{k-1}), p_f(t_{k-1}), \eta_p(t_{k-1}), u_p(t_{k-1}), p_p(t_{k-1}), \lambda(t_{k-1}), c(t_{k-1}))\).

The solution of \((P)\) at \( t = t_{k-1} \) obtained in Step 4. The argument from Step 1–3 implies that \((u_f^m, p_f^m, \eta_p^m, u_p^m, p_p^m, \lambda^m, c^m)\) converges to a solution of \((P)\) on \((t_{k-1}, t_k] \) for \( k = 2, \cdots, N \), which establishes an extension of a solution of \((P)\) on \((0, T]\).

Furthermore, we consider the terms containing viscosities in the Stokes–Biot subsystem \((LP1m)\). The limits of \( (\mu(c^m-1)K^{-1}u^m_p, v_p)_{\Omega_p} \) and \( (2\mu(c^m-1)e(u^m_f), e(v_f))_{\Omega_f} \) have shown that as \( m \to \infty \),

$$c^m \to c, \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \quad \text{strongly.}$$

(4.3.33)
Therefore, we have, as $m \to \infty$,

\begin{equation}
    c^m \to c, \text{ a.e. in } \Omega \times (0,T].
\end{equation}

(4.3.34)

For any $v_p \in L^2(\Omega_p)$ and $v_f \in H^1(\Omega_f)$, combining the fact that $\mu(c)$ is Lipschitz, we have, as $m \to \infty$,

\begin{align*}
    \mu(c^m)v_p &\to \mu(c)v_p, \text{ a.e. in } \Omega_p \times (0,T], \\
    \mu(c^m)e(v_f) &\to \mu(c)e(v_f), \text{ a.e. in } \Omega_f \times (0,T].
\end{align*}

In addition, we have for any $v_p \in L^2(\Omega_p)$ and $v_f \in H^1(\Omega_f)$,

\begin{align*}
    |\mu(c^m)v_p| &\leq \mu_U|v_p|, \text{ a.e. in } \Omega_p \times (0,T], \\
    |\mu(c^m)v_f| &\leq \mu_U|v_f|, \text{ a.e. in } \Omega_f \times (0,T].
\end{align*}

Therefore, by Lebesgue dominated convergence Theorem [78, Page 88], we have as $m \to \infty$,

\begin{align*}
    \mu(c^m)v_p &\to \mu(c)v_p, \text{ in } L^2(0,T;L^2(\Omega_p)), \\
    \mu(c^m)v_f &\to \mu(c)v_f, \text{ in } L^2(0,T;H^1(\Omega_f)).
\end{align*}

Also, from (4.3.16) and the extended in time results in this step, we know that when $m \to \infty$, $u^m_p \to u_p$ in $L^2(0,T;L^2(\Omega_p))$ and $u^m_f \to u_f$ in $L^2(0,T;H^1(\Omega_f))$. Therefore, as $m \to \infty$

\begin{align*}
    \int_0^T (\mu(c^m)K^{-1}u^m_p, v_p)_{\Omega_p} &\to \int_0^T (\mu(c)K^{-1}u_p, v_p)_{\Omega_p}, \\
    \int_0^T (2\mu(c^m)e(u^m_f), e(v_f))_{\Omega_f} &\to \int_0^T (2\mu(c)e(u_f), e(v_f))_{\Omega_f}.
\end{align*}

(4.3.35) (4.3.36)

Finally, for the transport subproblem (LP2m), we have for any $\psi \in X_c$:

\begin{equation}
    |D(u^{m-1}^{\infty})\nabla \psi| \leq \phi^*(d_m + \alpha_l \beta_u)|\nabla \psi| \quad \text{and} \quad |\psi u^m| \leq \beta_u|\psi|.
\end{equation}

Again, after using Lebesgue dominated convergence Theorem [78, Page 88], similarly, we obtain when $m \to \infty$

\begin{align*}
    \int_0^T (D(u^{m-1}^{\infty})\nabla c^m, \nabla \psi) &\to \int_0^T (D(u)\nabla c, \nabla \psi), \\
    \int_0^T (c^m u^{m-1}, \nabla \psi) &\to \int_0^T (cu, \nabla \psi).
\end{align*}
The next step is to prove uniqueness of the solution of problem \((P)\).

**Theorem 4.3.2.** The solution \((u_f, p_f, \eta_p, u_p, p_p, \lambda, c)\) of the problem \((P)\), equations (4.1.12)–(4.1.12g), is unique.

**Proof.** Note that with the data \(c\), the limit of \((u^m_f, p^m_f, \eta^m_p, u^m_p, p^m_p, \lambda^m)\) in suitable norms, \((u_f, p_f, \eta_p, u_p, p_p, \lambda)\) also solves the Stokes–Biot subproblem \((LP1m)\). Similarly, with the data \(u\), the limit of \(c^m, c\) solves the transport subproblem \((LP2m)\). Therefore, Assumption 4.3.1 holds for \(u_f, u_p, p\), and \(c\), respectively.

Assume that there are two solutions of (4.1.12a)–(4.1.12e), \((u^1_f, p^1_f, \eta^1_p, u^1_p, p^1_p, \lambda^1)\) and \((u^2_f, p^2_f, \eta^2_p, u^2_p, p^2_p, \lambda^2)\), both of which satisfy the same initial conditions. We denote by \((e_u, e_p, e_{\eta_p}, e_{u_p}, e_{p_p}, e_{\lambda})\) as the difference of the two solutions. Similarly, we assume there are two solutions \(c^1\) and \(c^2\) of \((P)\), (4.1.12g). The solutions have the same initial conditions. Denote by \(e_c\) the difference of \(c^1\) and \(c^2\). We repeat the process similar to the proof of Theorem 4.3.1, Step 1–Step 3, to have on a small interval \((0, t_1)\],

\[
\int_0^{t_1} \|e_c\|^2_{L^2(\Omega)} \, ds \leq t_1 C \exp(T) \int_0^{t_1} \|e_c\|^2_{L^2(\Omega)} \, ds. \tag{4.3.37}
\]

Take \(t_1\) small enough such that \(t_1 C \exp(T) < 1\). This implies that on \((0, t_1]\), \(e_c = 0\), namely the solution \(c\) of the transport subproblem is unique. Next, we repeat the proof of Theorem 4.3.1, Step 1, to have the following:

\[
\|e_{p_p}(t_1)\|^2_{L^2(\Omega_p)} + \|e_{\eta_p}(t_1)\|^2_{H^1(\Omega_p)} + \int_0^{t_1} \left( \|e_u\|^2_{H^1(\Omega_f)} + \|e_{u_p}\|^2_{L^2(\Omega_p)} + \|e_{p_p}\|^2_{L^2(\Omega_p)} + \|e_{\lambda}\|^2_{\Lambda} + \|e_{u_f} - \partial_\tau e_{\eta_p}\|^2_{BJS} \right) \, ds \leq C \int_0^{t_1} \|e_c\|^2_{L^2(\Omega)} \, ds. \tag{4.3.39}
\]

Combining (4.3.37) and (4.3.38), we conclude that on the interval \((0, t_1]\), the solution \((u_f, p_f, \eta_p, u_p, p_p, \lambda)\) of the Stokes–Biot subproblem is unique. The final step is to extend the uniqueness result globally in time from \((0, T]\). We repeat the process in the proof of Theorem 4.3.1, Step 5, to obtain the global uniqueness.
4.4 Analysis of the semi-discrete problem (DP)

In the previous sections, we discussed the Galerkin problems (LGP1) and (LGP2) of the linearized problem (LP1) and (LP2), respectively. In this section, we first introduce the Galerkin problem (DP) of the original non-linear problem (P) and then confirm the well-posedness of the Galerkin problem via the justification of a discretized version of Assumption 4.3.1. In Section 4.2.4, in order to pass to the limit of the Galerkin solution, we employ conforming finite element space $\Lambda_h \in \Lambda$, which is defined on $T_h|_{\Gamma_{fp}}$ and equipped with $H^{1/2}$-norm. In this section and the following sections, instead of using conforming finite element space, we utilize the following non-conforming discrete Lagrange multiplier space:\[3\]:

$$\tilde{\Lambda}_h = V_{ph} \cdot n_p|_{\Gamma_{fp}},$$

(4.4.1)

which consists of discontinuous piecewise polynomials. We equip $\tilde{\Lambda}_h$ with the norm $\|\nu_h\|_{\tilde{\Lambda}_h}^2 = \|\nu_h\|_{L^2(\Gamma_{fp})}^2 + |\nu_h|_{\tilde{\Lambda}_h}^2$, $|\nu_h|_{\tilde{\Lambda}_h}$ is a semi-norm [2, 3, 47], defined in $\tilde{\Lambda}_h$ as

$$|\nu_h|_{\tilde{\Lambda}_h}^2 = a_d^d(u_{ph}^*(\nu_h), u_{ph}^*(\nu_h)),$$

where $(u_{ph}^*(\nu_h), p_{ph}^*(\nu_h)) \in V_{ph} \times W_{ph}$ is the mixed finite element solution to a Darcy problem with Dirichlet data $\nu_h$ on the interface $\Gamma_{fp}$. The norm $\|\nu_h\|_{\tilde{\Lambda}_h}$ can be viewed as a discrete version of the $H^{1/2}$-norm. Note that $H^{1/2}(\Gamma_{fp})$ is dense in $L^2(\Gamma_{fp})$. Therefore, in Theorem 4.2.2, (4.2.21) and (4.2.22) also hold for the non-conforming space $\tilde{\Lambda}_h = V_{ph} \cdot n_p|_{\Gamma_{fp}}$, equipped with $L^2(\Gamma_{fp})$ norm. The necessity of changing into non-conforming space $\tilde{\Lambda}_h$ is to ensure the following error estimates. We will discuss the details in Section 4.5, seeing Remark 4.5.1.

Since we change the discrete space for the Lagrange multiplier into $\tilde{\Lambda}_h$, we need a new version of the discrete inf-sup condition, which is shown [3, Lemma 3.1].

**Lemma 4.4.1.** There exists a constant $\beta_3 > 0$ such that for all $(w_{fh}, w_{ph}, \nu_h) \in W_{fh} \times W_{ph} \times \tilde{\Lambda}_h$, there holds

$$\sup_{0 \neq (v_{fh}, v_{ph}) \in V_f \times V_p} \frac{b_f(v_{fh}, w_{fh}) + b_p(v_{ph}, w_{ph}) + b_T(v_{fh}, v_{ph}, 0; \nu_h)}{\|(v_{fh}, v_{ph})\|_{V_f \times V_p}} \geq \beta_3 \|(w_{fh}, w_{ph}, \nu_h)\|_{W_{fh} \times W_{ph} \times \tilde{\Lambda}_h}. \tag{4.4.2}$$
Let us introduce the following semi-discrete weak formulation of (P):

\((\text{DP})\): Find \((u_{fh}, p_{fh}, \eta_{ph}, u_{ph}, p_{ph}, \lambda_h, c_h) : [0, T] \to V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \tilde{A}_h \times X_{ch}\) such that for a.e. \(t \in (0, T)\) and for all \((v_{fh}, w_{fh}, \xi_{ph}, v_{ph}, w_{ph}, \psi_h) \in V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \tilde{A}_h \times X_{ch},\)

\[
af(u_{fh}, v_{fh}; c_h) + bf(v_{fh}, p_{fh}) + ab_{BJS}(u_{fh}, \partial_t \eta_{ph}; v_{fh}, 0) + b_{\Gamma}(v_{fh}, 0, 0; \lambda_h) = (f_f, v_{fh})_{\Omega_f},
\]

\[
bf(u_{fh}, w_{fh}) = (q_f, w_{fh})_{\Omega_f},
\]

\[
a_{\eta}^{\eta}(\eta_{ph}, \xi_{ph}) + \alpha b_{\epsilon}(\xi_{ph}, p_{ph}) + ab_{BJS}(u_{fh}, \partial_t \eta_{ph}; 0, \xi_{ph}) + b_{\Gamma}(0, 0, \xi_{ph}; \lambda_h) = (f_p, \xi_{ph})_{\Omega_p},
\]

\[
a_{\lambda}^{\lambda}(u_{ph}, v_{ph}; c_h) + b_{\lambda}(v_{ph}, p_{ph}) + b_{\Gamma}(0, v_{ph}, 0; \lambda_h) = 0,
\]

\[
s_{\epsilon}(\partial_t p_{ph}, w_{ph}) - b_{\epsilon}(u_{ph}, w_{ph}) - \alpha b_{\epsilon}(\partial_t \eta_{ph}, w_{ph}) = (q_{\phi}, w_{ph})_{\Omega_p},
\]

\[
b_{\Gamma}(u_{fh}, u_{ph}, \partial_t \eta_{ph}; \psi_h) = 0,
\]

\[
\phi \partial_t e_{h} + (D(u_{h}) \nabla c_{h}, \nabla \psi_h) - (c_{h} u_{h}, \nabla \psi_h) - (q^{+} c_{h}, \psi_h) = (q^{+} c_{w}, \psi_h),
\]

In Section 4.3, we obtain the well-posedness of the original non-linear problem (P). Still, we cannot drop the direct well-posedness result of (DP) since the continuous case depends highly on Assumption 4.3.1. Therefore, we introduce the discretized version of Assumption 4.3.1 and then confirm the feasibility of this discrete assumption based on employing an auxiliary problem. The benefit of this process is that we can avoid using the cut-off operator in the error estimates section. Next, we confirm the feasibility of the above assumption. Let us consider an auxiliary problem of the Stokes–Biot subproblem of (DP) as follows:

\((\text{LDP1})\): Find \((u_{fh}, p_{fh}, \eta_{ph}, u_{ph}, p_{ph}, \lambda_h) : [0, T] \to V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \tilde{A}_h \times X_{ch}\) such that for a.e. \(t \in (0, T)\) and for all \((v_{fh}, w_{fh}, \xi_{ph}, v_{ph}, w_{ph}, \psi_h) \in V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \tilde{A}_h \times X_{ch},\)

\[
a_f(u_{fh}, v_{fh}; \gamma_h) + b_f(v_{fh}, p_{fh}) + a_{BJS}(u_{fh}, \partial_t \eta_{ph}; v_{fh}, 0) + b_{\Gamma}(v_{fh}, 0, 0; \lambda_h) = (f_f, v_{fh})_{\Omega_f},
\]

\[
bf(u_{fh}, w_{fh}) = (q_f, w_{fh})_{\Omega_f},
\]
Note that in the auxiliary problem, we assume that the data \( \gamma \) has a unique solution (as follows):

\[
a^p_p(\eta_{ph}, \xi_{ph}) + \alpha b_e(\xi_{ph}, p_{ph}) + a_{BJS}(u_{fh}, \partial_t \eta_{ph}; 0, \xi_{ph}) + b_t(0, 0, \xi_{ph}; \lambda_h) \\
= (f_p, \xi_{ph})_{\Omega_p},
\]

(4.4.4d)

\[
a^p_p(u_{ph}, v_{ph}; c_h) + b_p(v_{ph}, p_{ph}) + b_t(0, v_{ph}, 0; \lambda_h) = 0,
\]

(4.4.4e)

\[
s_0(\partial_t p_{ph}, w_{ph})_{\Omega_p} - b_p(u_{ph}, w_{ph}) - \alpha b_e(\partial_t \eta_{ph}, w_{ph}) = (q_p, w_{ph})_{\Omega_p},
\]

(4.4.4f)

\[
b_t(u_{fh}, u_{ph}, \partial_t \eta_{ph}; \nu_h) = 0,
\]

(4.4.4g)

Note that we use the same data terms \( f_p, f_p, q_p \) and \( q_p \) as in (DP) and assume that \( \gamma_h \) has high enough smoothness. Actually, we have that the auxiliary problem (LDP1) is the same as (LGP1), except instead of directly using data \( \gamma \), we use \( \gamma_h \). In (4.4.4e), the two non-linear terms \( a_f \) and \( a_p^d \) have \( \mu(\gamma_h) \) as the viscosity and we assume sufficient smoothness for \( \gamma_h \).

Next, let us consider the auxiliary problem (LDP2) of the transport subproblem of (DP) as follows:

(LDP2): Find \( c_h : [0, T] \to X_{ch} \) such that for all \( \psi_h \in X_{ch} \) and for a.e. \( t \in (0, T) \):

\[
(\phi \partial_t c_h, \psi_h) + (D(\theta_h) \nabla c_h, \nabla \psi_h) - (c_h \theta_h, \nabla \psi_h) - (q^- c_h, \psi_h) = (q^+ c_h, \psi_h).
\]

(4.4.5)

Note that in the auxiliary problem, we assume the data \( \theta_h \in W^{1,\infty}(0, T; L^\infty(\Omega)) \).

**Assumption 4.4.1.** Suppose that for the Galerkin problem (LDP1) and (LDP2), we have \( (u_{fh}, u_{ph}) \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega_f)) \times W^{1,\infty}(0, T; L^\infty(\Omega_p)) \), and \( c_h \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega)) \).

The proof for the solvability and stability of (LDP1) and (LDP2) is similar to Theorem 2.2.4 and Theorem 4.2.2 since the data \( \gamma_h \) and \( \theta_h \) have the same smoothness as in Assumption 4.3.1. We come up with the following well-posedness results.

**Theorem 4.4.1.** Under the same assumption of Theorem 2.2.4 and Theorem 4.2.2, there exist initial data \( (u_{fh,0}, p_{fh,0}, \eta_{ph,0}, u_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in (V_{fh} \times W_{fh} \times X_{ph} \times V_{ph} \times W_{ph} \times \tilde{\Lambda}_h) \) such that the Galerkin problem (LDP1) with initial conditions \( p_{ph,0} = p_{ph,0} \) and \( \eta_{ph,0} = \eta_{ph,0} \), has a unique solution \( (u_{fh}, u_{ph}, \eta_{ph}, p_{fh}, p_{ph}, \lambda_h) \). Moreover, the solution satisfies \( u_{fh}(0) = u_{fh,0}, p_{fh}(0) = p_{fh,0}, u_{ph}(0) = u_{ph,0}, \) and \( \lambda_h(0) = \lambda_{h,0} \). It holds, for some positive constant \( C \),

\[
\sqrt{s_0} \| p_{ph} \|_L^\infty(0,T;W_p) + \| \eta_{ph} \|_L^\infty(0,T;X_p) + \| u_{fh} \|_L^2(0,T;V_f) + \| p_{fh} \|_L^2(0,T;W_f)
\]
Furthermore, it holds that

\[
\sqrt{s_0} \partial_t p_{ph} \leq C \left( \| f_f \|_{H^1(0, T; \mathbf{L}_2)} + \| f_p \|_{L^2(0, T; \mathbf{L}_2) \Omega_p} + \| q_p \|_{L^2(0, T; \mathbf{L}_2) \Omega_p} \right)
\]

and,

\[
\sqrt{s_0} \partial_t p_{ph} \leq C \left( \| \partial_t \lambda_h \|_{L^2(0, T; \lambda_h)} + \| \partial_t u_{fh} \|_{L^2(0, T; \mathbf{X}_p)} + \| \nabla \cdot u_{ph} \|_{L^2(0, T; \mathbf{L}_2(\Omega_p))} \right)
\]

For the solution \( c_h \) of (LDP2), it holds that

\[
\| c_h \|_{L^\infty(0, T; \mathbf{L}_2(\Omega))} + \| c_h \|_{L^2(0, T; \mathbf{X}_c)} \leq C \sqrt{\exp(T)} \left( \| c_0 \|_{H^1(\Omega)} + \| q^+ c_w \|_{L^2(0, T; \mathbf{X}_c)} \right).
\]

Furthermore, it holds that

\[
\| \partial_t c_h \|_{L^2(0, T; \mathbf{L}_2(\Omega))} + \| c_h \|_{L^\infty(0, T; \mathbf{X}_c)} \leq C \sqrt{\exp(T)} \left( \| c_0 \|_{H^1(\Omega)} + \| q^+ c_w \|_{L^2(0, T; \mathbf{L}_2(\Omega))} \right).
\]

**Proof.** The well-posedness proof of (LDP1) is similar to the proof of Theorem 2.2.4 and Theorem 4.2.2. Instead of using inf-sup Lemma 4.2.1, we need to use the new inf-sup condition Lemma 4.4.1 in the uniqueness and stability analysis. To avoid duplications, we omit the details to obtain the stabilities and only present the different parts. For the uniqueness part, we can have the differences between two solutions \((\tilde{u}_{fh}, \tilde{u}_{ph}, \tilde{\lambda}_h)\) are zeros.

For the remaining variables \((\tilde{p}_{fh}, \tilde{p}_{ph}, \tilde{\lambda}_h)\), using (4.4.1), we have

\[
\beta_0 \| (\tilde{p}_{fh}, \tilde{p}_{ph}, \tilde{\lambda}_h) \|_{W_f \times W_p \times \lambda_h} \leq \sup_{0 \neq (v_{fh}, v_{ph}) \in \mathbf{V}_f \times \mathbf{V}_p} \frac{b_f(v_{fh}, \tilde{p}_{fh}) + b_p(v_{ph}, \tilde{p}_{ph}) + b_T(v_{fh}, v_{ph}, 0; \tilde{\lambda}_h)}{\| (v_{fh}, v_{ph}) \|_{\mathbf{V}_f \times \mathbf{V}_p}}
\]

\[
= \sup_{0 \neq (v_{fh}, v_{ph}) \in \mathbf{V}_f \times \mathbf{V}_p} \frac{-a_f(\tilde{u}_{fh}, v_{fh}; \gamma h) - a_{d}(\tilde{u}_{ph}, v_{ph}; \gamma h) - a_{BJS}(\tilde{u}_{fh}, \partial_t \tilde{\lambda}_{ph}; v_{fh}, 0)}{\| (v_{fh}, v_{ph}) \|_{\mathbf{V}_f \times \mathbf{V}_p}}
\]

\[
= 0.
\]
In the proof of stability, using (4.4.1), we have
\[
\beta_3 \| (p_{fh}, p_{ph}, \lambda_h) \|_{W_f \times W_p \times \Lambda_h} \leq \sup_{0 \neq (v_{fh}, v_{ph}) \in V_f \times V_p} \frac{b_f(v_{fh}, p_{fh}) + b_p(v_{ph}, p_{ph}) + b_t(v_{fh}, v_{ph}, 0; \lambda_h)}{\| (v_{fh}, v_{ph}) \|_{V_f \times V_p}}
\]
\[
\leq C \left( \| f \|_{V_f'} + \| u_{fh} \|_{H^1(\Omega_f)} + \| u_{ph} \|_{L^2(\Omega_p)} + | u_{fh} - \partial_t \eta_{ph} |_{BJS} \right).
\]
(4.4.11)
A detailed proof can be found in [3, Theorem 3.1, Theorem 4.1].

The next step is to confirm the practicability of Assumption 4.4.1. In a similar way to prove the stability results in Theorem 4.2.2, (4.2.21) and (4.2.22), and Theorem 4.4.1, under the assumption that we have sufficient smooth data \( f_f, f_p, q_f, q_p \) and \( \gamma_h \), we are able to show the following:

\[
\| u_{fh} \|_{W^{1, \infty}(0,T; H^1(\Omega_f))} \leq M_f, \quad \| u_{ph} \|_{W^{1, \infty}(0,T; L^2(\Omega_p))} \leq M_p,
\]
(4.4.12)
where \( M_f \) and \( M_p \) depend on the data \( f_f, f_p, q_f, q_p \) and \( \gamma_h \). Next, we use inverse inequality [33, Theorem 3.1.2, Theorem 3.1.3] to obtain

\[
\| u_{fh} \|_{W^{1, \infty}(0,T; W^{1, \infty}(\Omega_f))} \leq M_f h^{-d/2}, \quad \| u_{ph} \|_{W^{1, \infty}(0,T; L^\infty(\Omega_p))} \leq M_p h^{-d/2},
\]
(4.4.13)
where, defined in Section 4.2.1, \( h \) is the maximal element diameter. In a similar way, we also have

\[
\| c_h \|_{W^{1, \infty}(0,T; W^{1, \infty}(\Omega))} \leq M_c h^{-d/2},
\]
(4.4.14)
where \( M_c \) depends on the data \( q^c_w \) and \( \theta_h \). Note that even though in (4.4.13) and (4.4.14), the bounds depend on \( h \) with negative power, however we have that \( M_f h^{-d/2}, M_p h^{-d/2}, \) and \( M_c h^{-d/2} \) are finite for any fixed \( h \). In conclusion, Assumption 4.4.1 is justified and well-established.

Finally, after confirming Assumption 4.4.1, we come up with the well-posedness results of the Galerkin problem (DP).
Theorem 4.4.2. For each

\[ f_f \in H^1(0, T; V_f'), \quad f_p \in H^1(0, T; V_p'), \quad q_f \in H^1(0, T; W_f'), \quad q_p \in H^1(0, T; W_p'), \]

\[ q^\perp c_w \in L^2(0, T; X'_c), \quad \text{and} \quad q^- \in H^1(0, T; L^\infty(\Omega)), \]

the Galerkin problem (DP) has a unique solution \((u_{fh}, u_{ph}, \eta_{ph}, p_{fh}, p_{ph}, \lambda_h, c_h)\) satisfying

\[
(u_{fh}, u_{ph}, \eta_{ph}) \in L^\infty(0, T; V_{fh}) \times (L^\infty(0, T; L^2(\Omega_p)) \cap L^2(0, T; V_{ph})) \times H^1(0, T; X_{ph});
\]

\[
(p_{fh}, p_{ph}, \lambda_h) \in L^\infty(0, T; W_{fh}) \times (L^\infty(0, T; W_{ph}) \cap H^1(0, T; W_{ph})) \times L^\infty(0, T; \tilde{\Lambda}_h);
\]

\[
c_h \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; X_{ch}).
\]

Proof. The proof is in the same routine of the existence and uniqueness proof for Theorem 4.2.3. Instead of using the smooth linearizing terms \(\theta\) and \(\gamma\), we use the Assumption 4.4.1.

\[ \square \]

4.5 Error analysis

In this section, we analyze the error due to discretization in space. We denote by \(k_f\) and \(s_f\) the degrees of polynomials in the spaces \(V_{fh}\) and \(W_{fh}\), respectively. Let \(k_p\) and \(s_p\) be the degrees of polynomials in the spaces \(V_{ph}\) and \(W_{ph}\), respectively. Let \(k_s\) and \(k_c\) be the polynomial degree in \(X_{ph}\) and \(X_{ch}\). Note that in this section, we still use non-conforming finite element space for the Lagrange multiplier space \(\tilde{\Lambda}_h\), as shown in (4.4.1).
4.5.1 Approximation error

Let $Q_{fh}$, $Q_{ph}$ and $Q_{\lambda h}$ be the $L^2$-projection operators onto $W_{fh}$, $W_{ph}$, and $\tilde{\Lambda}_h$, respectively, satisfying:

$$
(p_f - Q_{fh} p_f, w_{fh})_{\Omega_f} = 0, \quad \forall w_{fh} \in W_{fh}, \quad (4.5.1)
$$

$$
(p_p - Q_{ph} p_p, w_{ph})_{\Omega_p} = 0, \quad \forall w_{ph} \in W_{ph}, \quad (4.5.2)
$$

$$
\langle \lambda - Q_{\lambda h} \lambda, \mu_h \rangle_{\Gamma_{fp}} = 0, \quad \forall \mu_h \in \tilde{\Lambda}_h. \quad (4.5.3)
$$

These operators have the approximation properties [33]:

$$
\|p_f - Q_{fh} p_f\|_{L^2(\Omega_f)} \leq C h^{r_{sf}} \|p_f\|_{H^{r_{sf}}(\Omega_f)}, \quad 0 \leq r_{sf} \leq s_f + 1, \quad (4.5.4)
$$

$$
\|p_p - Q_{ph} p_p\|_{L^2(\Omega_p)} \leq C h^{r_{sp}} \|p_p\|_{H^{r_{sp}}(\Omega_p)}, \quad 0 \leq r_{sp} \leq s_p + 1, \quad (4.5.5)
$$

$$
\|\lambda - Q_{\lambda h} \lambda\|_{L^2(\Gamma_{fp})} \leq C h^{r_{k\lambda}} \|\lambda\|_{H^{r_{k\lambda}}(\Gamma_{fp})}, \quad 0 \leq r_{k\lambda} \leq k_p + 1. \quad (4.5.6)
$$

Since we choose the discrete Lagrange multiplier space to be $\tilde{\Lambda}_h = V_{ph} \cdot n_{fp}$, we have for any $v_{ph} \in V_{ph}$,

$$
\langle \lambda - \lambda_h, v_{ph} \cdot n_{fp} \rangle_{\Gamma_{fp}} = 0. \quad (4.5.7)
$$

Therefore, we have

$$
\|\lambda_h - Q_{\lambda h} \lambda\|_{\tilde{\Lambda}_h} = \|\lambda_h - Q_{\lambda h} \lambda\|_{L^2(\Gamma_{fp})}. \quad (4.5.8)
$$

Next, we consider a Stokes–like projection operator [3], $(S_{fh}, R_{fh}) : V_f \rightarrow V_{fh} \times W_{fh}$, defined for all $v_f \in V_f$ by

$$
a_f(S_{fh} v_f, v_{fh}; c) - b_f(v_{fh}, R_{fh} v_f) = a_f(v_f, v_{fh}; c), \quad \forall v_{fh} \in V_{fh},
$$

$$
b_f(S_{fh} v_f, w_{fh}) = b_f(v_f, w_{fh}), \quad \forall w_{fh} \in W_{fh}.
$$

Let $\Pi_{ph}$ be the MFE interpolant onto $V_{ph}$ satisfying for all $v_p \in H^1(\Omega_p)$[16],

$$
(\nabla \cdot \Pi_{ph} v_p, w_{ph}) = (\nabla \cdot v_p, w_{ph}), \quad \forall w_{ph} \in W_{ph}, \quad (4.5.9)
$$

$$
\langle \Pi_{ph} v_p \cdot n_p, v_{ph} \cdot n_p \rangle_{\Gamma_{fp}} = \langle v_p \cdot n_p, v_{ph} \cdot n_p \rangle_{\Gamma_{fp}}, \quad \forall v_{ph} \in V_{ph}, \quad (4.5.10)
$$

$$
\|v_p - \Pi_{ph} v_p\|_{L^2(\Omega_p)} \leq C h^{r_{kp}} \|v_p\|_{H^{r_{kp}}(\Omega_p)}, \quad 1 \leq r_{kp} \leq k_p + 1. \quad (4.5.11)
$$
Let $S_{sh}$ be the Scott-Zhang interpolant from $X_p$ onto $X_{ph}$, satisfying [79]:

$$\|\xi_p - S_{sh}\xi_p\|_{L^2(\Omega_p)} + h\|\xi_p - S_{sh}\xi_p\|_{H^1(\Omega_p)} \leq C h^{r_{ks}}\|\xi_p\|_{H^{r_{ks}}(\Omega_p)}, \quad 1 \leq r_{ks} \leq k_s + 1.$$  \hspace{1cm} (4.5.12)

Let $I_{ch}$ be the Scott-Zhang interpolant from $X_c$ onto $X_{ch}$, satisfying [79]:

$$\|\psi - I_{ch}\psi\|_{L^2(\Omega)} + h\|\psi - I_{ch}\psi\|_{H^1(\Omega)} \leq C h^{r_{kc}}\|\psi\|_{H^{r_{kc}}(\Omega)}, \quad 1 \leq r_{kc} \leq k_c + 1.$$  \hspace{1cm} (4.5.13)

Next, consider an operator onto the space that satisfies the weak continuity of normal velocity condition. Let

$$U = \{(v_f, v_p, \xi_p) \in V_f \times H^1(\Omega_p) \times X_p : b_{\Gamma}(v_f, v_p, \xi_p; \mu) = 0, \forall \mu \in \Lambda\}.$$  

Consider its discrete analog

$$U_h = \{(v_{fh}, v_{ph}, \xi_{ph}) \in V_{fh} \times V_{ph} \times X_{ph} : b_{\Gamma}(v_{fh}, v_{ph}, \xi_{ph}; \mu_h) = 0, \forall \mu_h \in \tilde{\Lambda}_h\}.$$  

An interpolation operator $I_h : U \rightarrow U_h$ is constructed in [3, Section 5] as a triple

$$I_h(v_f, v_p, \xi_p) = (I_{fh}v_f, I_{ph}v_p, I_{sh}\xi_p),$$  

where $I_{fh} = S_{fh}, I_{sh} = S_{sh},$ and $I_{ph}$ is based on a correction of $\Pi_{ph}$ designed to satisfy the continuity of normal velocity. The interpolant $I_h$ has the following properties:

$$b_{\Gamma}(I_{fh}v_f, I_{ph}v_p, I_{sh}\xi_p; \mu_h) = 0, \quad \forall \mu_h \in \tilde{\Lambda}_h,$$  \hspace{1cm} (4.5.14)

$$b_f(I_{fh}v_f - v_f, w_{fh}) = 0, \quad \forall w_{fh} \in W_{fh},$$  \hspace{1cm} (4.5.15)

$$b_p(I_{ph}v_p - v_p, w_{ph}) = 0, \quad \forall w_{ph} \in W_{ph}.$$  \hspace{1cm} (4.5.16)

The approximation properties of the component of $I_h$ are established in [3, Lemma 5.1]: for all sufficiently smooth $v_f$, $v_p$, and $\xi_p$,

$$\|v_f - I_{fh}v_f\|_{H^1(\Omega_f)} \leq C h^{r_{kf}}\|v_f\|_{H^{r_{kf}+1}(\Omega_f)}, \quad 0 \leq r_{kf} \leq k_f.$$  \hspace{1cm} (4.5.17)

$$\|v_p - I_{ph}v_p\|_{L^2(\Omega_p)} \leq C \left(h^{r_{kp}}\|v_p\|_{H^{r_{kp}}(\Omega_p)} + h^{r_{kf}}\|v_f\|_{H^{r_{kf}+1}(\Omega_f)} + h^{r_{ks}}\|\xi_p\|_{H^{r_{ks}+1}(\Omega_p)}\right),$$  \hspace{1cm} (4.5.18)

$$1 \leq r_{kp} \leq k_p + 1, \quad 0 \leq r_{kf} \leq k_f, \quad 0 \leq r_{ks} \leq k_s.$$  

$$\|\nabla \cdot (v_p - I_{ph}v_p)\|_{L^2(\Omega_p)} \leq C h^{r_{kp}}\|v_p\|_{H^{r_{kp}+1}(\Omega_p)}, \quad 0 \leq r_{kp} \leq k_p.$$  \hspace{1cm} (4.5.19)

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\[ \| \xi_p - I_{sh} \xi_p \|_{L^2(\Omega_p)} + h \| \xi_p - I_{sh} \xi_p \|_{H^1(\Omega_p)} \leq C h^{r_s} \| \xi_p \|_{H^{rk_s}(\Omega_p)}, \quad 1 \leq r_s \leq k_s + 1. \] (4.5.20)

We proceed with estimating the error between (P) and (DP). We introduce the errors for all variables and split them into approximation and discretization errors:

\[ E_f := u_f - u_{fh} = (u_f - I_{fh} u_f) + (I_{fh} u_f - u_{fh}) := \chi_f + \phi_{fh}, \] (4.5.21a)

\[ E_p := u_p - u_{ph} = (u_p - I_{ph} u_p) + (I_{ph} u_p - u_{ph}) := \chi_p + \phi_{ph}, \] (4.5.21b)

\[ E_s := \eta_p - \eta_{ph} = (\eta_p - I_{sh} \eta_p) + (I_{sh} \eta_p - \eta_{ph}) := \chi_s + \phi_{sh}, \] (4.5.21c)

\[ E_{fp} := p_f - p_{fh} = (p_f - Q_{fh} p_f) + (Q_{fh} p_f - p_{fh}) := \chi_{fp} + \phi_{fph}, \] (4.5.21d)

\[ E_{pp} := p_p - p_{ph} = (p_p - Q_{ph} p_p) + (Q_{ph} p_p - p_{ph}) := \chi_{pp} + \phi_{pph}, \] (4.5.21e)

\[ E_{\lambda} := \lambda - \lambda_h = (\lambda - Q_{ch} \lambda) + (Q_{ch} \lambda - \lambda_h) := \chi_{\lambda} + \phi_{\lambda h}, \] (4.5.21f)

\[ E_c := c - c_h = (c - I_{ch} c) + (I_{ch} c - c_h) := \chi_c + \phi_{ch}. \] (4.5.21g)

In addition, we denote

\[ E_u := u - u_h, \quad \text{with} \quad E_u|_{\Omega_s} = E_s, \quad * \in \{ f, p \}. \] (4.5.22)

### 4.5.2 Error estimates for the Stokes–Biot problem

In this section, we derive the error estimates for the approximation of the Stokes–Biot sub-problem, which depends on \( E_c \).

**Theorem 4.5.1. (a priori error estimate for the Stokes–Biot problem)** Assuming sufficient smoothness for the solution of the continuous problem (P), then the solution of the semi-discrete problem (DP) (4.4.3e)–(4.4.3h) with initial data constructed in Theorem 2.2.4, satisfies

\[
\| \eta_p - \eta_{ph} \|_{L^\infty(0,T;H^1(\Omega_p))} + \sqrt{\alpha} \| p_f - p_{ph} \|_{L^\infty(0,T;L^2(\Omega_p))} + \| u_f - u_{fh} \|_{L^2(0,T;H^1(\Omega_f))} \\
+ \| \lambda - \lambda_h \|_{L^2(0,T;H^1(\Omega_f))} + \| u_p - u_{ph} \|_{L^2(0,T;L^2(\Omega_p))} + \| (u_f - \partial_t \eta_p) - (u_{fh} - \partial_t \eta_{ph}) \|_{L^2(0,T;BJS)} \\
+ \| p_f - p_{fh} \|_{L^2(0,T;L^2(\Omega_f))} + \| p_p - p_{ph} \|_{L^2(0,T;L^2(\Omega_p))} \\
\leq C \sqrt{\exp(T)} \left( M_1 h^{\min\{k_f, s_f + 1, k_p + 1, s_p + 1, k_s\}} + \| E_c \|_{L^2(0,T;L^2(\Omega))} \right),
\] (4.5.23)
where

\[ M_1 := \|u_f\|_{L^2(0,T;H^{k_f+1}(\Omega_f))} + \|u_p\|_{L^2(0,T;H^{k_p+1}(\Omega_p))} + \|p_f\|_{L^2(0,T;H^{k_f+1}(\Gamma_f))} \]
\[ + \|p_p\|_{L^\infty(0,T;H^{k_p+1}(\Omega_p))} + \|\partial_t p_p\|_{L^2(0,T;H^{k_p+1}(\Omega_p))} \]
\[ + \|\eta_p\|_{L^\infty(0,T;H^{k_p+1}(\Omega_p))} + \|\partial_t \eta_p\|_{L^2(0,T;H^{k_p+1}(\Gamma_f))} \]
\[ + \|\lambda\|_{L^\infty(0,T;H^{k_p+1}(\Gamma_f))} + \|\partial_t \lambda\|_{L^2(0,T;H^{k_p+1}(\Gamma_f))} . \]

Proof. Subtracting (4.4.3e)–(4.4.3h) from (4.1.12a)–(4.1.12f) and summing the two equations, we obtain the following error equation for the flow subproblem:

\[
\begin{align*}
& a_f(E_f, v_{fh}; c_h) + a_p^d(E_p, v_{ph}; c_h) + a_p^e(E_s, \xi_{ph}) + a_{BJS}(E_f, \partial_t E_s; v_{fh}, \xi_{ph}) + s_0(\partial_t E_{pp}, w_{ph})\Omega_p \\
& + (2(\mu(c) - \mu(c_h))c)\epsilon(u_f, \epsilon(v_{fh}))\Omega_f + (\mu(c) - \mu(c_h))K^{-1}u_p, v_{ph})\Omega_p + b_f(v_{fh}, E_{fp}) \\
& + b_p(v_{ph}, E_{pp}) + \alpha b_e(\xi_{ph}, E_{pp}) + b_T(v_{fh}, v_{ph}, \xi_{ph}, E_\lambda) - b_f(E_f, w_{fh}) - b_p(E_p, w_{ph}) \\
& - \alpha b_e(\partial_t E_s, w_{ph}) = 0 .
\end{align*}
\]

(4.5.24)

We next set \( v_{fh} = \phi_{fh}, w_{fh} = \phi_{fpph}, \xi_{ph} = \partial_t \phi_{sh}, v_{ph} = \phi_{ph}, w_{ph} = \phi_{pph} \) in (4.5.24) and split the error as the approximation and discretization error shown as in (4.5.21). Some terms can be simplified due to the interpolant property of projection (4.5.14)–(4.5.16):

\[
\begin{align*}
& b_f(x_f, \phi_{fph}) = b_p(x_p, \phi_{ppph}) = b_p(\phi_{ph}, x_{pp}) = s_0(x_{pp}, \phi_{ppph})\Omega_p = 0, \\
& b_{\Gamma_{fp}}(\phi_{fh}, \phi_{ph}, \partial_t \phi_{sh}; \phi_{fph}) = 0.
\end{align*}
\]

(4.5.25)  (4.5.26)

We also use that \( \tilde{\Lambda}_h = V_{ph} \cdot n_p|_{\Gamma_{fp}} \), to have

\[
\langle \phi_{ph} \cdot n_p, \chi_\lambda \rangle_{\Gamma_{fp}} = 0 .
\]

(4.5.27)

Rearranging terms and use the result above, the error equation becomes

\[
\begin{align*}
& \|\phi_{fh}\|^2_{H^1(\Omega_f)} + \|\phi_{ph}\|^2_{L^2(\Omega_p)} + \|\phi_{fh} - \partial_t \phi_{sh}\|^2_{L^2(\Omega_f)} + \frac{1}{2} d dt \ a^e_p(\phi_{sh}, \phi_{sh}) + \frac{s_0}{2} d dt \|\phi_{pph}\|^2_{L^2(\Omega)} \\
& \leq -a_f(x_f, \phi_{fh}; c_h) - a^e_p(x_s, \partial_t \phi_{sh}) - a^d_p(x_p, \phi_{ph}; c_h) - a_{BJS}(x_f, \partial_t x_s; \phi_{fh}, \partial_t \phi_{sh}) \\
& - (2(\mu(c) - \mu(c_h))c)\epsilon(u_f, \epsilon(\phi_{fh}))\Omega_f - (\mu(c) - \mu(c_h))K^{-1}u_p, \phi_{ph})\Omega_p - b_f(\phi_{fh}, x_{fh}) \\
& - \alpha b_e(\partial_t \phi_{sh}, x_{pp}) + \alpha b_e(\partial_t x_s, \phi_{ppph}) - \langle \phi_{fh} \cdot n_f + \partial_t \phi_{sh} \cdot n_p, \chi_\lambda \rangle_{\Gamma_{fp}} .
\end{align*}
\]

(4.5.28)
We proceed with bounding the terms on the right hand side of (4.5.28). Using the Cauchy-Schwarz and Young’s inequalities, we first have for positive \( \epsilon \)

We proceed with bounding the terms on the right hand side of (4.5.28). Using the Cauchy-Schwarz and Young’s inequalities and trace inequality for the BJS term, we have for positive \( \epsilon \)

Next, for the viscous terms on the right hand side, combining with Assumption 4.3.1 and 4.4.1, we have for some \( \epsilon > 0 \)

We bound the rest terms that do not involve \( \partial_t \phi_{\text{sh}} \) as follows:

Combining (4.5.29)–(4.5.31), integrating over time from 0 to \( t \), utilizing the coercivity of the functionals on the left hand side of (4.5.28), and taking \( \epsilon \) small enough, we obtain

\[
\begin{align*}
\| \phi_{\text{sh}}(t) \|^2_{H^1(\Omega_p)} + \| \phi_{pph}(t) \|^2_{L^2(\Omega_p)} + & \int_0^t \left( \| \phi_{fh} \|^2_{H^1(\Omega)} + \| \phi_{ph} \|^2_{L^2(\Omega_p)} + |\phi_{fh} - \partial_t \phi_{sh}|^2_{BJS} \right) ds \\
\leq \quad & C \left( \int_0^t \left( \| \chi_{f} \|^2_{L^2(\Omega)} + \| \partial_t \chi_s \|^2_{H^1(\Omega_p)} + \| \chi_{\lambda} \|^2_{L^2(\Gamma_{fp})} + \| \chi_{e} \|^2_{L^2(\Gamma_{fp})} + \| \chi_{pp} \|^2_{L^2(\Omega_p)} + \right) ds \\
+ \quad & \int_0^t \left( \| E_{c} \|^2_{L^2(\Omega)} + \| \phi_{pph} \|^2_{L^2(\Omega_p)} \right) ds + \int_0^t \left( a_p(\chi_s ; \partial_t \phi_{sh}) + \right. \\
& \left. \alpha b_c(\partial_t \chi_s , \phi_{pph}) + \langle \partial_t \phi_{sh} \cdot n_p , \chi_{\lambda} \rangle_{\Gamma_{fp}} \right) ds + \| \phi_{sh}(0) \|^2_{H^1(\Omega_p)} + \| \phi_{pph}(0) \|^2_{L^2(\Omega_p)} \right) .
\end{align*}
\]
Next, we use integration by parts in above inequality containing $\partial_t\phi_{sh}$ as follow

$$
\int_0^t \left( a^e_p(\mathbf{x}_s, \partial_t \phi_{sh}) + \alpha_b(\partial_t \phi_{sh}, \chi_{pp}) + \langle \partial_t \phi_{sh} \cdot \mathbf{n}_p, \chi \rangle_{\Gamma_{fp}} \right) \, ds
= a^e_p(\mathbf{x}_s, \phi_{sh})|_0^t + \alpha_b(\phi_{sh}, \chi_{pp})|_0^t + \langle \phi_{sh} \cdot \mathbf{n}_p, \chi \rangle_{\Gamma_{fp}}|_0^t
- \int_0^t \left( a^e_p(\partial_t \mathbf{x}_s, \phi_{sh}) + \alpha_b(\phi_{sh}, \partial_t \chi_{pp}) + \langle \phi_{sh} \cdot \mathbf{n}_p, \partial_t \chi \rangle_{\Gamma_{fp}} + \langle \phi_{sh} \cdot \partial_t \mathbf{n}_p, \chi \rangle_{\Gamma_{fp}} \right) \, ds
\leq \epsilon \|\phi_{sh}\|_{\mathbf{H}^1(\Omega_p)}^2 + C \left( \|\mathbf{x}_s\|_{\mathbf{H}^1(\Omega_p)}^2 + \|\chi \|_{L^2(\Gamma_{fp})}^2 + \|\chi_{pp}\|_{L^2(\Omega_p)}^2 + \int_0^t \|\partial_t \chi_{sh}\|_{\mathbf{H}^1(\Omega_p)}^2 \, ds + \int_0^t \|\phi_{sh}\|_{\mathbf{H}^1(\Omega_p)}^2 \, ds \right).
$$

Combining (4.5.33) with (4.5.32), taking $\epsilon$ small enough and using Gronwall’s Lemma, we have

$$
\|\phi_{sh}(t)\|_{\mathbf{H}^1(\Omega_p)}^2 + \|\phi_{pph}(t)\|_{L^2(\Omega_p)}^2 + \int_0^t \left( \|\phi_{fh}\|_{\mathbf{H}^1(\Omega_f)}^2 + \|\phi_{ph}\|_{\mathbf{H}^2(\Omega_p)}^2 + \|\phi_{fh} - \partial_t \phi_{sh}\|_{\mathbf{H}^2(\Gamma_{fp})}^2 \right) \, ds
\leq C \exp(T) \left( \|\mathbf{x}_s\|_{\mathbf{H}^1(\Omega_p)}^2 + \|\chi \|_{L^2(\Gamma_{fp})}^2 + \|\chi_{pp}\|_{L^2(\Omega_p)}^2 \right) + \int_0^t \|\nabla \chi\|_{L^2(\Omega)}^2 \, ds
+ \int_0^t \left( \|\partial_t \mathbf{x}_s\|_{\mathbf{H}^1(\Omega_p)}^2 + \|\partial_t \chi_{pp}\|_{L^2(\Omega_p)}^2 + \|\partial_t \chi \|_{L^2(\Gamma_{fp})}^2 + \|\chi_{fh}\|_{L^2(\Omega_f)}^2 + \|\chi \|_{L^2(\Gamma_{fp})}^2 \right)
+ \|\mathbf{x}_s\|_{\mathbf{H}^1(\Omega_f)}^2 + \|\mathbf{x}_p\|_{L^2(\Omega_p)}^2 \, ds + \|\phi_{sh}(0)\|_{\mathbf{H}^1(\Omega_p)}^2 + \|\phi_{pph}(0)\|_{L^2(\Omega_p)}^2 \right).
$$

Note that for the initial conditions, $\phi_{pph}(0) = Q_{ph}p, 0 - p_{ph,0} = 0$ and $\phi_{sh}(0) = I_{sh}\eta_{p,0} - \eta_{ph,0}$. According to the construction of $p_{ph,0}$ and $\eta_{ph,0}$, we have

$$
\|\phi_{pph}(0)\|_{L^2(\Omega_p)}^2 = 0, \quad \|\phi_{sh}(0)\|_{\mathbf{H}^1(\Omega_p)}^2 \leq \|\mathbf{x}_s(0)\|_{\mathbf{H}^1(\Omega_p)}^2 + h^{k_2} \|\eta_{p,0}\|_{\mathbf{H}^1(\Omega_p)}^2.
$$

Next, we use the inf-sup condition Lemma 4.4.1 to obtain the bounds for $\phi_{fh}$, $\phi_{pph}$ and $\phi_{sh}$. From (4.5.24), we have

$$
\|\phi_{fh}, \phi_{pph}, \phi_{sh}\|_{W_{1,1} \times W_{1,1} \times \Lambda_h}
\leq \sup_{0 \neq (v_{fh}, v_{ph}) \in \mathbf{V}_h \times \mathbf{V}_p} \frac{|b_f(v_{fh}, \phi_{fh}) + b_p(v_{ph}, \phi_{pph}) + b_T(v_{fh}, v_{ph}, 0; \phi_{sh})|}{\|\mathbf{v}_{fh}, \mathbf{v}_{ph}\|_{\mathbf{V}_h \times \mathbf{V}_p}}
= \sup_{0 \neq (v_{fh}, v_{ph}) \in \mathbf{V}_h \times \mathbf{V}_p} \frac{-a_f(E_f, v_{fh}, c_h) - (2(\mu(c) - \mu(c)) \mathbf{e}(u_f), \mathbf{e}(v_{fh}))/\Omega_f - b_p(v_{ph}, \chi_{pp})}{\|\mathbf{v}_{fh}, \mathbf{v}_{ph}\|_{\mathbf{V}_h \times \mathbf{V}_p}}
$$
+ \frac{-a_p^h(E_p, v_{ph}; c) - ((\mu(c) - \mu(c_h))K^{-1}u_p, v_{ph})_{\Omega_p} - a_{BJS}(E_f, \partial_t E_s; v_{fh}, 0) - b_f(v_{fh}, \chi_{fp})}{\| (v_{fh}, v_{ph}) \times \mathbf{v} \|} \\
\leq \left( \| \phi_{fh} \|_{H^1(\Omega_f)}^2 + \| \phi_{ph} \|_{L^2(\Omega_p)}^2 + \| \phi_{fh} - \partial_t \phi_{sh} \|_{BJS}^2 + \| \partial_t \chi_s \|_{\dot{H}^1(\Omega_p)}^2 \\
+ \| \chi_f \|_{H^1(\Omega_f)}^2 + \| \chi_p \|_{L^2(\Omega_p)}^2 + \| \chi_{fp} \|_{L^2(\Gamma_{fp})}^2 + \| \chi \|_{L^2(\Gamma_{fp})}^2 \right). 
}\quad (4.5.36)

We integrate (4.5.36) over time from 0 to t and take \( \epsilon \) small enough, use Gronwall’s Lemma, and combine with (4.5.34) to obtain
\begin{align}
\| \phi_{sh}(t) \|_{H^1(\Omega_p)}^2 &+ \| \phi_{ph} \|_{L^2(\Omega_p)}^2 \\
&+ \int_0^t \left( \| \phi_{fh} \|_{H^1(\Omega_f)}^2 + \| \phi_{ph} \|_{L^2(\Omega_p)}^2 + \| \phi_{fh} - \partial_t \phi_{sh} \|_{BJS}^2 \right) ds \\
&+ \int_0^t \left( \| \phi_{ph} \|_{L^2(\Omega_f)}^2 + \| \phi_{ph} \|_{L^2(\Omega_p)}^2 + \| \phi_{ph} \|_{L^2(\Gamma_{fp})}^2 \right) ds \\
&\leq C \exp(T) \left( \int_0^t \left( \| \partial_t \chi_s \|_{H^1(\Omega_p)}^2 + \| \partial_t \chi_{pp} \|_{L^2(\Omega_p)}^2 + \| \chi_{pp} \|_{L^2(\Omega_p)}^2 + \| \partial_t \chi \|_{L^2(\Gamma_{fp})}^2 + \| \chi_{fp} \|_{L^2(\Omega_f)}^2 \\
+ \| \chi \|_{L^2(\Gamma_{fp})}^2 + \| \chi_s \|_{H^1(\Omega_p)}^2 + \| \chi_p \|_{L^2(\Omega_p)}^2 \right) ds + \| \phi_{sh}(0) \|_{H^1(\Omega_p)}^2 \\
+ \int_0^t \| \partial_t \chi_s \|_{L^2(\Omega_p)}^2 ds \right). 
\end{align} 
\quad (4.5.37)

After utilizing the bound for the initial condition (4.5.35), the above inequality implies (4.5.23) after combining with the approximation properties (4.5.4)–(4.5.6), (4.5.17)–(4.5.18), and using the triangular inequality.

\[ \square \]

**Remark 4.5.1.** In Section 4.4, we begin to use non-conforming space \( \bar{\Lambda}_h \). If we continue using conforming space \( \Lambda_h \), (4.5.27) doesn’t hold any longer. As a consequence, on the right hand side of (4.5.28), there will be an extra term \( \langle \phi_{ph}, \mathbf{n}_p, \chi_\lambda \rangle_{\Gamma_{fp}} \) which needs to be controlled. According to trace inequality, we end up having an extra \( \int_0^t \| \nabla \cdot \phi_{ph} \|_{L^2(\Omega_p)}^2 \) on the right hand side of (4.5.37), but only \( \int_0^t \| \phi_{ph} \|_{L^2(\Omega_p)}^2 \) on the left hand side. However, we will not be able to handle this term until we finish the error analysis for the coupled system in Theorem 4.5.3, (4.5.45).
4.5.3 Error estimates for the transport problem

In this section, we derive an estimates for $E_c$, which depends on $E_u$.

**Theorem 4.5.2.** (A priori error estimate for the transport subproblem) Assuming sufficient smoothness of $c$, the solution of the semi-discrete transport subproblem (4.4.3i) with $c_h(0) = I_h c_0$ satisfies for $C > 0$, independent on size of the mesh $h$,

$$\|c - c_h\|_{L_\infty(0,T;L^2(\Omega))} + \|c - c_h\|_{L^2(0,T;L^2(\Omega))} + \|\nabla c - \nabla c_h\|_{L^2(0,T;L^2(\Omega))} \leq C \sqrt{\exp(T)} \left( h^k \|c\|_{L^2(0,T;H_{kc_0}^1(\Omega))} + \|E_f\|_{L^2(0,T;H^1(\Omega_f))} + \|E_p\|_{L^2(0,T;L^2(\Omega_p))}\right).$$  \(4.5.38\)

**Proof.** Take the difference between (4.1.12g) and (4.4.3i) to obtain the following error equation:

$$(\phi \partial_t E_c, \psi_h) - (c E_u, \nabla \psi_h) - (E_c u_h, \nabla \psi_h) + (D(u_h) \nabla E_c, \nabla \psi_h)$$

$$+ ((D(u) - D(u_h)) \nabla c, \nabla \psi_h) - (q^- E_c, \psi_h) = 0. \quad (4.5.39)$$

Next, we take the test function $\psi_h = \phi_{ch}$, and rearrange the equation as follows:

$$(\phi \partial_t \phi_{ch}, \phi_{ch}) + (D(u_h) \nabla \phi_{ch}, \nabla \phi_{ch}) - (q^- \phi_{ch}, \phi_{ch})$$

$$= -((\phi \partial_t \chi_c, \phi_{ch}) + (\phi_{ch} u_h, \nabla \phi_{ch}) + (\chi_c u_h, \nabla \phi_{ch}) + (c E_u, \nabla \phi_{ch}) - (D(u_h) \nabla \chi_c, \nabla \phi_{ch})$$

$$+ ((D(u) - D(u_h)) \nabla c, \nabla \phi_{ch}) + (q^- \chi_c, \phi_{ch}). \quad (4.5.40)$$

First of all, we use the Cauchy-Schwarz and Young’s inequalities for the right hand side of (4.5.40) and drop a non-negative term $-(q^- \phi_{ch}, \phi_{ch})$ on the left hand side. Namely, for $\epsilon > 0$, we have

$$\frac{d}{dt} \|\phi_{ch}\|_{L^2(\Omega)}^2 + \|\nabla \phi_{ch}\|_{L^2(\Omega)}^2 \leq \epsilon \|\nabla \phi_{ch}\|_{L^2(\Omega)}^2 + C\left(\|\phi_{ch}\|_{L^2(\Omega)}^2 + \|\chi_c\|_{L^2(\Omega)}^2 + \|\partial_t \chi_c\|_{L^2(\Omega)}^2 + \|\nabla \chi_c\|_{L^2(\Omega)}^2 + \|E_u\|_{L^2(\Omega)}^2\right). \quad (4.5.41)$$

Integrating (4.5.41) over time, taking $\epsilon$ small enough, and using Gronwall’s inequality, we obtain

$$\|\phi_{ch}(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \phi_{ch}\|_{L^2(\Omega)}^2 ds.$$
\[ C \exp(T) \left( \int_0^t \left( \| \chi_c \|_{L^2(\Omega)}^2 + \| \partial_t \chi_c \|_{L^2(\Omega)}^2 + \| \nabla \chi_c \|_{L^2(\Omega)}^2 \right) ds \right)
+ \int_0^t \| E_f \|_{L^2(\Omega_f)}^2 ds + \int_0^t \| E_p \|_{L^2(\Omega_p)}^2 ds + \| \phi_{ch}(0) \|_{L^2(\Omega)}^2 ) \].

(4.5.42)

Note that we have the initial condition of \( \phi_{ch} \) to be zero, \( \phi_{ch}(0) = 0 \). We can simply set \( c_h(0) = I_{ch}c_0 \). Combining the interpolant property (4.5.13) and (4.5.42), we obtain the estimate result (4.5.38).

4.5.4 Error estimates for the coupled Stokes–Biot–transport problem

We now state and prove our final result on the error estimate of the coupled Stokes–Biot–Transport problem. The following Theorem is a corollary from Theorem 4.5.1 and Theorem 4.5.2.

**Theorem 4.5.3.** (Error estimates for the coupled system) Let the assumptions in Theorem 4.5.1 and Theorem 4.5.2 hold. Then, there exists a constant \( c \), independent of \( h \), such that

\[
\| \eta_p - \eta_{ph} \|_{L^\infty(0,T;H^1(\Omega_p))} + \sqrt{S_0} \| p_p - p_{ph} \|_{L^\infty(0,T;L^2(\Omega_p))} + \| u_f - u_{fh} \|_{L^2(0,T;H^1(\Omega_f))}
+ \| \lambda - \lambda_h \|_{L^2(0,T;\tilde{A}_h)} + \| u_p - u_{ph} \|_{L^2(0,T;L^2(\Omega_p))} + \| (u_f - \partial_t \eta_p) - (u_{fh} - \partial_t \eta_{ph}) \|_{L^2(0,T;BJS)}
+ \| P_f - P_{fh} \|_{L^2(0,T;L^2(\Omega_f))} + \| P_p - P_{ph} \|_{L^2(0,T;L^2(\Omega_p))}
\leq C \sqrt{\exp(T)} h^{\min\{k_f,s_f+1,k_p+1,s_p+1,k_s,k_c\}},
\] (4.5.43)

and

\[
\| c - c_h \|_{L^\infty(0,T;L^2(\Omega))} + \| \nabla c - \nabla c_h \|_{L^2(0,T;L^2(\Omega))}
\leq C \sqrt{\exp(T)} h^{\min\{k_f,s_f+1,k_p+1,s_p+1,k_s,k_c\}}.
\] (4.5.44)

Furthermore,

\[
\| \nabla \cdot (u_p - u_{ph}) \|_{L^2(0,T;L^2(\Omega_p))} \leq C \sqrt{\exp(T)} h^{\min\{k_f,2,s_f+1,k_p+1/2,s_p+1,k_s,k_c\}}.
\] (4.5.45)
Proof. Combining (4.5.23) and (4.5.38), we obtain

$$
\|E_c\|_{L^2(\Omega)}^2 \leq C \int_0^t \|E_c\|_{L^2(\Omega)}^2 \, ds + C \ell \min\{2k_f, 2k_p + 2k_p + 2.2k_p + 2k_c, 2k_c\}.
$$

Using Gronwall’s inequality, we arrive at (4.5.44). Substituting (4.5.44) into (4.5.23) results in (4.5.43). Next, based on (4.5.43) and (4.5.44), we are able to have the error estimate for \( \nabla \cdot (u_p - u_{ph}) \). Subtracting the time differentiated (4.4.3e)–(4.4.3h) from time differentiated (4.1.12a)–(4.1.12f), summing the two equations, choosing the test functions to be \( \mathbf{v}_{fh} = \mathbf{\phi}_{fh} \), \( w_{fh} = \partial_t \phi_{fh}, \eta_{p} = \partial_t \phi_{sh}, \mathbf{v}_{ph} = \mathbf{\phi}_{ph}, w_{ph} = \partial_t \phi_{pph}, \) and \( \eta_{h} = \partial_t \phi_{\lambda h} \), we obtain

$$
\frac{1}{2} \frac{d}{dt} a_f(\mathbf{\phi}_{fh}, \mathbf{\phi}_{fh}; c_h) + \frac{1}{2} \frac{d}{dt} a_p^e(\partial_t \phi_{sh}, \partial_t \phi_{sh}) + \frac{1}{2} \frac{d}{dt} a_{BJS}(\mathbf{\phi}_{fh}, \partial_t \phi_{sh}, \mathbf{\phi}_{fh}, \partial_t \phi_{sh}) + s_0 \|\partial_t \phi_{pph}\|_{L^2(\Omega_p)}^2
= -2(\mu(c) - \mu(c_h)) \mathbf{e}(\partial_t \mathbf{u}_f), \mathbf{e}(\mathbf{\phi}_{fh})_{\Omega_f} - a_f(\partial_t \chi_f, \mathbf{\phi}_{fh}; c_h) - (\partial_t \mu(c) \mathbf{e}(\mathbf{\phi}_{fh}), \mathbf{e}(\mathbf{\phi}_{fh}))_{\Omega_f}
- (\mu(c) - \mu(c_h)) K^{-1} \partial_t \mathbf{u}_p, \mathbf{\phi}_{ph})_{\Omega_p} - ((\partial_t \mu(c) - \partial_t \mu(c_h)) K^{-1} \mathbf{u}_p, \mathbf{\phi}_{ph} + a_p^d(\partial_t \chi_p, \mathbf{\phi}_{ph}; c_h)
- \frac{1}{2} (\partial_t \mu(c) \mathbf{K}^{-1} \mathbf{\phi}_{ph}, \mathbf{\phi}_{ph})_{\Omega_p} - (\partial_t \mu(c) \mathbf{K}^{-1} \mathbf{\chi}_p, \mathbf{\phi}_{ph})_{\Omega_p} - a_p^e(\partial_t \chi_s, \partial_t \mathbf{\phi}_{sh})
- b_f(\mathbf{\phi}_{fh}, \partial_t \chi_{fp}) - a_b(\partial_t \phi_{sh}, \partial_t \chi_{pp}) + a_b(\partial_t \chi_{s}, \partial_t \phi_{pph})
- a_{BJS}(\partial_t \chi_f, \partial_t \chi_s, \mathbf{\phi}_{fh}, \partial_t \mathbf{\phi}_{sh}) - \langle \mathbf{\phi}_{fh} \cdot \mathbf{n}_f + \partial_t \phi_{sh} \cdot \mathbf{n}_p, \partial_t \chi_{\lambda} \rangle_{\Gamma_{fp}}. \quad (4.5.46)
$$

First, we integrate over time from 0 to \( t \) the two terms \( (2(\partial_t \mu(c) - \partial_t \mu(c_h))) \mathbf{e}(\mathbf{u}_f), \mathbf{e}(\mathbf{\phi}_{fh})_{\Omega_f} \) and \( ((\partial_t \mu(c) - \partial_t \mu(c_h)) \mathbf{K}^{-1} \mathbf{u}_p, \mathbf{\phi}_{ph})_{\Omega_p} \) to obtain

$$
\left| \int_0^t (2(\partial_t \mu(c) - \partial_t \mu(c_h)) \mathbf{e}(\mathbf{u}_f), \mathbf{e}(\mathbf{\phi}_{fh})_{\Omega_f} \right| \leq C \int_0^t \int_{\Omega_f} |\mathbf{e}(\mathbf{\phi}_{fh})| \, ds,
$$

$$
\left| \int_0^t ((\partial_t \mu(c) - \partial_t \mu(c_h)) \mathbf{K}^{-1} \mathbf{u}_p, \mathbf{\phi}_{ph})_{\Omega_p} \right| \leq C \int_0^t \int_{\Omega_p} |\mathbf{\phi}_{ph}| \, ds.
$$

Next, we integrate (4.5.46) over time from 0 to \( t \) and use Cauchy-Schwarz and Young’s inequalities to have for some \( \epsilon > 0 \)

$$
\|\mathbf{\phi}_{fh}(t)\|_{H^1(\Omega_f)}^2 + \|\phi_{sh}(t)\|_{L^2(\Omega_p)}^2 + |(\mathbf{\phi}_{fh} - \partial_t \phi_{sh}(t))|_{B_{\mathbf{JS}}}^2 + s_0 \int_0^t \|\partial_t \phi_{pph}\|_{L^2(\Omega_p)}^2 \, ds
+ \int_0^t \|\partial_t \phi_{sh}\|_{H^1(\Omega_p)}^2 \, ds
$$
\[ \leq \epsilon \int_{0}^{t} \| \partial_{t} \phi_{sh} \|_{H^{1}(\Omega_{p})}^2 + C \left( \int_{0}^{t} \left( \| \phi_{fh} \|_{H^{1}(\Omega_{f})}^2 + \| \phi_{ph} \|_{L^{2}(\Omega_{p})}^2 \right) ds + \int_{0}^{t} \int_{\Omega_{f}} |\epsilon(\phi_{fh})| ds \right) \]
\[ + \int_{0}^{t} \int_{\Omega_{p}} |\phi_{ph}| ds + \int_{0}^{t} \left( \| \partial_{t} \chi_{f} \|_{H^{1}(\Omega_{f})}^2 + \| \chi_{f} \|_{H^{1}(\Omega_{f})}^2 + \| \partial_{t} \chi_{p} \|_{L^{2}(\Omega_{p})}^2 + \| \chi_{p} \|_{L^{2}(\Omega_{p})}^2 \right) \]
\[ + \| \partial_{t} \chi_{s} \|_{H^{1}(\Omega_{p})}^2 + \| \partial_{t} \chi_{s} \|_{H^{2}(\Omega_{p})}^2 + \| \partial_{t} \chi_{fp} \|_{L^{2}(\Omega_{f})}^2 + \| \partial_{t} \chi_{pp} \|_{L^{2}(\Omega_{p})}^2 \]
\[ + \| \partial_{t} \chi_{pp} \|_{L^{2}(\Gamma_{fp})}^2 ) ds \right) + \int_{0}^{t} \| E_{c} \|_{L^{2}(\Omega_{f})}^2 ds \right). \quad (4.5.47) \]

Note that we have
\[ \left( \int_{0}^{t} \int_{\Omega_{f}} |\epsilon(\phi_{fh})| ds \right)^2 \leq C T |\Omega_{f}| \int_{0}^{t} \int_{\Omega_{f}} |\epsilon(\phi_{fh})|^2 ds; \]
\[ \left( \int_{0}^{t} \int_{\Omega_{p}} |\phi_{ph}| ds \right)^2 \leq C T |\Omega_{p}| \int_{0}^{t} \int_{\Omega_{p}} |\phi_{ph}|^2 ds. \]

Next, taking \( \epsilon \) in (4.5.47) small enough, using Gronwall’s inequality, and combining with (4.5.43) and (4.5.44), we have
\[ s_{0} \int_{0}^{t} \| \partial_{t} \phi_{pph} \|_{L^{2}(\Omega_{p})}^2 ds + \int_{0}^{t} \| \partial_{t} \phi_{sh} \|_{H^{1}(\Omega_{p})}^2 ds \leq C \exp(T) h^{\min\{k_{f}, 2s_{f} + 2, k_{p} + 1, 2s_{p} + 2, k_{s}, 2k_{c}\}}. \quad (4.5.48) \]

We next subtract (4.4.3g) from (4.1.12e), take \( w_{fh} = 0 \) and \( \phi_{ph} = \nabla \cdot \phi_{ph} \), and integrate over time from 0 to \( t \), to obtain
\[ \int_{0}^{t} \| \nabla \cdot \phi_{ph} \|_{L^{2}(\Omega_{p})}^2 ds \leq C \int_{0}^{t} \left( \| \nabla \cdot \chi_{f} \|_{L^{2}(\Omega_{p})}^2 + \| \partial_{t} \phi_{sh} \|_{H^{1}(\Omega_{p})}^2 + \int_{0}^{t} \| \partial_{t} \chi_{s} \|_{H^{1}(\Omega_{p})}^2 \right) ds + \int_{0}^{t} \| \partial_{t} \phi_{pph} \|_{L^{2}(\Omega_{p})}^2 + \| \partial_{t} \chi_{pp} \|_{L^{2}(\Omega_{p})}^2 ) ds. \quad (4.5.49) \]

Finally, combine (4.5.48), (4.5.49) with (4.5.43) and (4.5.44) to have
\[ \int_{0}^{t} \| \nabla \cdot \phi_{ph} \|_{L^{2}(\Omega_{p})}^2 ds \leq C \exp(T) h^{\min\{k_{f}, 2s_{f} + 2, k_{p} + 1, 2s_{p} + 2, k_{s}, 2k_{c}\}}. \quad (4.5.50) \]

The bound (4.5.45) follows from (4.5.50) and (4.5.19), and the triangular inequality.
4.6 Numerical results

In this section, we present numerical results from two computational experiments in two dimensions. The method has been implemented using the finite element package FreeFem++[54]. In the first test, we verify the theoretical convergence rates for the problem with an analytical solution. The second example shows the applicability of the method to modeling flows in an irregularly shaped channel with physically realistic parameters.

Figure 4.6.1: Computational domains for Example 1 (left) and Example 2 (right).

4.6.1 Example 1: convergence test

The convergence test confirms the theoretical convergence rates for the problem using an analytical solution. We first introduce the discrete-in-time norms. Let $\Delta t$ be the time step, $T = N\Delta t$ the total time, $t_n = n\Delta t$, and $\varphi^n = \varphi(t_n)$. The domain is $\Omega = [0, 1] \times [-1, 1]$, namely $\Omega_p = [0, 1] \times [-1, 0]$ and $\Omega_f = [0, 1] \times [0, 1]$. We associate the upper half in the grey area with the Stokes flow, while the lower half in the red area represents the flow in the poroelastic structure governed by the Biot system, seeing Figure.4.6.1 (a). Note that in the actual simulations, the mesh we use is finer than the plot in Figure.4.6.1 (a). The
appropriate interface conditions are enforced on interface $y = 0$. The solution in the Stokes region $\Omega_f$ is

$$\mathbf{u}_f = \pi \cos(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}, \quad p_f = e^t \sin(\pi x) \cos\left(\frac{\pi y}{2}\right) + 2\pi \cos(\pi t).$$

The Biot solutions $\{\mathbf{u}_p, p_p, \eta_p\}$ are chosen accordingly to satisfy the interface conditions (4.1.7):

$$\mathbf{u}_p = \frac{\pi e^t}{\mu(c)} \begin{pmatrix} -\cos(\pi x) \cos\left(\frac{\pi y}{2}\right) \\ \frac{1}{2} \sin(\pi x) \sin\left(\frac{\pi y}{2}\right) \end{pmatrix}, \quad p_p = e^t \sin(\pi x) \cos\left(\frac{\pi y}{2}\right), \quad \eta_p = \sin(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}.$$

The transport solution $c$ is set to be

$$c = t(\cos(\pi x) + \cos(\pi y))/\pi,$$

with the diffusion-dispersion tensor $\mathbf{D} = 10^{-3}\mathbf{I}$ in the fluid domain $\Omega_f$, while $\mathbf{D}(|\mathbf{u}|) = 10^{-3}\mathbf{I} + |\mathbf{u}|\mathbf{I}$ in the structure region $\Omega_p$. Note that we take the longitudinal and transverse dispersion $\alpha_l = \alpha_t = 1$. We choose porosity $\phi = 1$ in $\Omega_f$ and $\phi = 0.9$ in $\Omega_p$. We set the permeability $K$ to be 1. The viscosity $\mu(c)$ is chosen to be

$$\mu(c) = 2 - \frac{1}{c + 1}.$$ 

On the interface $\Gamma_{fp}$, we choose the constant viscosity $\mu_f = \mu(c(0)) = 1$.

The right hand side functions $f_f$, $q_f$, $f_p$ and $q_p$ are computed from (4.1.1)–(4.1.2) using the above solution. The model problem is then complemented with the appropriate Dirichlet boundary conditions and initial data. We study the convergence for two choices of finite element spaces. The higher order choice is the Taylor-Hood $\mathcal{P}_2 - \mathcal{P}_1$ for Stokes, the Raviart-Thomas $\mathcal{RT}_1 - \mathcal{P}_1^{dc}$ and $\mathcal{P}_2$ for Biot, and $\mathcal{P}_1^{dc}$ for the Lagrange multiplier, in which case second order convergence rate for all variables is expected. For the lower order choice, we use $\mathcal{P}_1 - \mathcal{P}_0$ for Stokes, the Raviart-Thomas $\mathcal{RT}_0 - \mathcal{P}_0$ and $\mathcal{P}_1$ for Biot, and $\mathcal{P}_0$ for the Lagrange multiplier. The transport equation is discretized using $\mathcal{P}_2$ for the higher order choice and $\mathcal{P}_1$ for the lower order choice. The total simulation time for this test case is $T = 1 \times 10^{-4}$s and the time step is $\Delta t = 1 \times 10^{-5}$s. For this case, the time step needs to be
sufficiently small so that the time discretization error does not affect the convergence rates. The theoretical results are verified by the rates shown in Table 4.6.1, 4.6.2 and 4.6.3. In particular, we calculate the convergence rate for $\nabla \cdot u$ in the norm $L^2(0,T; L^2(\Omega_p))$.

**Remark 4.6.1.** Note that in this section, the true solution of the concentration $c$ may be negative at some point. It does not have any physical meaning. It is out for the purpose of testing the robustness of the algorithm.

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</tr>
<tr>
<td>1/256</td>
<td>2.587E-02 1.0</td>
<td>8.093E-03 2.0</td>
<td>1.417E-02 1.0</td>
</tr>
</tbody>
</table>

Table 4.6.1: Example 1: relative numerical errors and convergence rates for \{$u_f, p_f, u_p, p_p, \eta_p, \lambda$\} with the lower order spaces.
\(P_2 - P_1, \mathcal{RT}_1 - P_1^{dc}, P_2, P_1^{dc}\)

<table>
<thead>
<tr>
<th>(h)</th>
<th>(|E_f|_{H^1(\Omega_f)}) error</th>
<th>(|E_{fp}|_{L^2(\Omega_p)}) error</th>
<th>(|E_p|_{L^2(\Omega_p)}) error</th>
<th>(|\nabla \cdot E_p|_{L^2(\Omega_p)}) error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>5.830E-04 1.9</td>
<td>2.504E-03 1.9</td>
<td>3.133E-02 1.2</td>
<td>1.994E-01 1.3</td>
</tr>
<tr>
<td>1/32</td>
<td>1.147E-04 2.3</td>
<td>5.954E-04 2.1</td>
<td>1.130E-02 1.5</td>
<td>6.508E-02 1.6</td>
</tr>
<tr>
<td>1/64</td>
<td>2.555E-05 2.2</td>
<td>1.465E-04 2.0</td>
<td>2.887E-03 2.0</td>
<td>1.669E-02 1.9</td>
</tr>
<tr>
<td>1/128</td>
<td>6.557E-06 2.0</td>
<td>4.000E-05 1.9</td>
<td>6.084E-04 2.2</td>
<td>4.172E-03 2.0</td>
</tr>
<tr>
<td>1/256</td>
<td>1.583E-06 1.9</td>
<td>1.108E-05 1.9</td>
<td>1.527E-04 2.1</td>
<td>1.123E-03 1.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(h)</th>
<th>(|E_{pp}|_{L^\infty(\Omega_p)}) error</th>
<th>(|E_s|_{H^1(\Omega_p)}) error</th>
<th>(|E_\lambda|<em>{L^2(\Gamma</em>{fp})}) error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>1.097E-03 –</td>
<td>9.885E-01 –</td>
<td>7.529E-03 –</td>
</tr>
<tr>
<td>1/16</td>
<td>2.753E-03 2.0</td>
<td>2.436E-01 2.0</td>
<td>1.877E-03 2.0</td>
</tr>
<tr>
<td>1/32</td>
<td>6.856E-04 2.0</td>
<td>6.063E-02 2.0</td>
<td>4.692E-04 2.0</td>
</tr>
<tr>
<td>1/64</td>
<td>1.693E-04 2.0</td>
<td>1.514E-02 2.0</td>
<td>1.174E-04 2.0</td>
</tr>
<tr>
<td>1/128</td>
<td>4.170E-05 2.0</td>
<td>3.785E-03 2.0</td>
<td>2.934E-05 2.0</td>
</tr>
<tr>
<td>1/256</td>
<td>1.037E-05 2.0</td>
<td>9.461E-03 2.0</td>
<td>7.335E-06 2.0</td>
</tr>
</tbody>
</table>

Table 4.6.2: Example 1: relative numerical errors and convergence rates for \(\{u_f, p_f, u_p, p_p, \eta_p, \lambda\}\) with the higher order spaces.
Table 4.6.3: Example 1: relative numerical errors and convergence rates for concentration $c$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|E_c|_{\mathcal{L}^2(\Omega)}$</th>
<th>$|E_c|_{L^\infty(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>4.509E-01</td>
<td>1.153E-01</td>
</tr>
<tr>
<td>$1/16$</td>
<td>2.242E-01 1.0</td>
<td>2.427E-02 2.2</td>
</tr>
<tr>
<td>$1/32$</td>
<td>1.130E-01 1.0</td>
<td>5.826E-03 2.1</td>
</tr>
<tr>
<td>$1/64$</td>
<td>5.663E-02 1.0</td>
<td>1.442E-03 2.0</td>
</tr>
<tr>
<td>$1/128$</td>
<td>2.833E-02 1.0</td>
<td>3.618E-04 2.0</td>
</tr>
<tr>
<td>$1/256$</td>
<td>1.417E-02 1.0</td>
<td>9.927E-05 1.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|E_c|_{\mathcal{L}^2(\Omega)}$</th>
<th>$|E_c|_{L^\infty(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>1.330E-02</td>
<td>6.163E-02</td>
</tr>
<tr>
<td>$1/16$</td>
<td>2.120E-02 2.6</td>
<td>1.761E-02 1.8</td>
</tr>
<tr>
<td>$1/32$</td>
<td>6.412E-03 1.7</td>
<td>4.413E-03 2.0</td>
</tr>
<tr>
<td>$1/64$</td>
<td>1.610E-03 2.0</td>
<td>1.165E-03 1.9</td>
</tr>
<tr>
<td>$1/128$</td>
<td>4.008E-04 2.0</td>
<td>2.913E-04 1.9</td>
</tr>
<tr>
<td>$1/256$</td>
<td>1.083E-04 1.9</td>
<td>7.281E-05 2.0</td>
</tr>
</tbody>
</table>

4.6.2 Example 2: flow through poroelastic media with channel network

In Figure 4.6.1 part (b), we show the square computational domain $\Omega = [-1,1] \times [-1,1]$, which includes an irregularly shaped network of channels. This example is designed to illustrate the robustness of the method with respect to geometry. We use the transport equation to model the concentration of mercury, which enters the domain filled with water.
along with the inflow boundary. The flow is driven from left to right via a pressure drop of 1 kPa.

First, the Lamé coefficients are determined by Young’s modulus $E$ and the Poisson’s ratio $\nu$, via the following relationship [3, 62]:

$$\lambda_p = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu_p = \frac{E}{2(1 + \nu)}.$$  

For the non-linear viscosity function $\mu(c)$ of the solvent concentration, we use the famous quarter-power mixing law [63]:

$$\mu(c) = (c \mu_s^{-0.25} + (1 - c) \mu_0^{-0.25})^{-4},$$  

(4.6.1)

where $\mu_s$ is the viscosity of the solute, mercury, and $\mu_0$ is the viscosity of the solvent, water.

The boundary conditions are as follows:

$$p_p = 1, \text{ on } \Gamma_{p, left}; \quad p_p = 0, \text{ on } \Gamma_{p, right}; \quad u_p \cdot n_p = 0, \text{ on } \Gamma_{p, top} \cup \Gamma_{p, bottom};$$

$$\sigma_p n_p = 0, \text{ on } \Gamma_{p, left}; \quad \eta_p = 0, \text{ on } \Gamma_{p, right}; \quad \eta_p \cdot \tau_p = 0, \text{ on } \Gamma_{p, top} \cup \Gamma_{p, bottom};$$

$$(\sigma_p n_p) \cdot n_p = 0, \text{ on } \Gamma_{p, top} \cup \Gamma_{p, bottom}; \quad (\sigma_f n_f) \cdot n_f = 0, \text{ on } \Gamma_{f, right} \cup \Gamma_{f, top};$$

$$u_f \cdot n_f = 0.2 \text{ and } u_f \cdot \tau_f = 0, \text{ on } \Gamma_{f, right} \cup \Gamma_{f, left} \cup \Gamma_{f, top};$$

$$(cu - D \nabla c) \cdot n = (c_{in} u) \cdot n, \quad c_{in} = 1, \text{ on } \Gamma_{p, left};$$

$$(D \nabla c) \cdot n = 0, \quad \text{ on } \partial \Omega \setminus \Gamma_{p, left}.$$

We set the initial conditions accordingly:

$$\eta_{p,0} = 0, \quad p_{p,0} = 1.$$

Recall in (4.1.10), the diffusion-dispersion operator is defined as:

$$D(u) = d_m I + |u| \left\{ \alpha_l E(u) + \alpha_t (I - E(u)) \right\}.$$  

In this channel example, we set $d_m = 10^{-6}$ m/s, $\alpha_l = \alpha_t = 0$ for the fluid region $\Omega_f$, namely $D = 10^{-6} I$. In the poroelastic region $\Omega_p$, we choose $d_m = 5 \times 10^{-4}$ m/s, and $\alpha_l = \alpha_t = 5 \times 10^{-4}$. In Table.4.6.4, we present the physical realistic parameters.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Symbol</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width</td>
<td>m</td>
<td>$L$</td>
<td>2</td>
</tr>
<tr>
<td>Height</td>
<td>m</td>
<td>$H$</td>
<td>2</td>
</tr>
<tr>
<td>Mass storativity</td>
<td>kPa$^{-1}$</td>
<td>$s_0$</td>
<td>$5 \times 10^{-2}$</td>
</tr>
<tr>
<td>Lamé coeff.</td>
<td>kPa</td>
<td>$\lambda_p$</td>
<td>$5/18 \times 10^7$</td>
</tr>
<tr>
<td>Lamé coeff.</td>
<td>kPa</td>
<td>$\mu_p$</td>
<td>$5/12 \times 10^7$</td>
</tr>
<tr>
<td>Solvent viscosity</td>
<td>kPa s</td>
<td>$\mu_s$</td>
<td>$1.074 \times 10^{-6}$</td>
</tr>
<tr>
<td>Biot-Willis constant</td>
<td></td>
<td>$\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>Porosity(in $\Omega_f$)</td>
<td></td>
<td>$\phi$</td>
<td>1.0</td>
</tr>
<tr>
<td>Porosity(in $\Omega_p$)</td>
<td></td>
<td>$\phi$</td>
<td>0.4</td>
</tr>
<tr>
<td>BJS coeff.</td>
<td></td>
<td>$\alpha_{BJS}$</td>
<td>1.0</td>
</tr>
<tr>
<td>Dynamic viscosity</td>
<td>kPa s</td>
<td>$\mu_0$</td>
<td>$8.90 \times 10^{-7}$</td>
</tr>
<tr>
<td>Permeability</td>
<td>m$^2$</td>
<td>$K$</td>
<td>$\text{diag}(1,1) \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 4.6.4: Geometry, elasticity, poroelasticity, fluid and transport parameters.

The total simulation time is $T = 25$s and with a time step of size $\Delta t = 0.01$s. For the space discretization, we use the Taylor-Hood finite element spaces $P_2 - P_1$ for the fluid velocity and pressure $(u_f, p_f)$; while for the Darcy velocity and structure pressure $(u_p, p_p)$, we are utilizing the Raviart-Thomas $RT_1 - P_1^{dc}$; for the displacement $\eta_p$, we are using the continuous Lagrangian $P_2$, and discontinuous $P_1^{dc}$ for the Lagrange multiplier $\lambda$. We use the continuous piecewise linears $P_2$ for the concentration $c$.

In Figure.4.6.2 part (a) and (c), we present the computed velocity fields in the poroelastic and fracture regions and the structure displacement in the Figure.4.6.2 part (b), at the final time $t = 25$ s. Six snapshots of the concentration solution at different times are shown in Figure.4.6.3. The Darcy velocity $u_p$ and structure pressure $p_p$ are relatively larger between the left inflow boundary and the fractures, which leads to the larger displacement $\eta_p$ in this
area. The Stokes velocity $\mathbf{u}_f$ in the channel network is an order of magnitude larger than in the poroelastic area. It is interesting to see that the fluid in the network at each bifurcated area has a preferential path to the relatively wider channel to reach the outflow boundary. As expected, the mercury follows the flow and tends to propagate much faster in the free fluid region than in the poroelastic region with much slighter diffusion. From Figure 4.6.3 part (c), (d), and (e), we can also detect that the mercury propagates in the simulation domain not only from the fluid region but also from the poroelastic media. With very subtle diffusion in the right channel, we can observe two coexisting mercury traces transported via channels. The two streams are close with each other, but they are not mixing, not until the final stage of the simulation, around time $t = 25$ s. In Figure 4.6.4, we show the viscosity of the fluid, which presents similar behaviors of the concentration. The lower bound of the color range in Figure 4.6.4 is the viscosity of water and the upper bound is of mercury.
Figure 4.6.3: Computed concentration solution.
Figure 4.6.4: Computed viscosity of the fluid.
5.0 Conclusions

In this thesis, we have studied the Lagrange multiplier method and its applicability to model the interaction problem between a fluid region and a poroelastic medium. We develop and analyze the coupled Navier-Stokes–Biot or Stokes–Biot model, which is utilized to describe the FPSI problem. The theoretical analysis includes the well-posedness results in semi-discretized and fully-discretized senses. We employ the compactness theory to pass to the limit to obtain the well-posedness of the continuous weak formulation under the baseline stability results we obtained in the semi-discretized formulation. Numerical experiments are conducted to validate the error analysis and test the robustness of our methods in simulating a benchmark blood flow model. We further fully coupled the proposed model with the transport equation, which describes the solute concentration.

First, via a Lagrange multiplier, coming from the essential type of velocity interface condition in the mixed Darcy formulation, we obtain the well-posedness of the method. Instead of direct analysis, we first derive a divergence-free semi-discretized formulation, the fluid pressure of which has been eliminated. Then, we recover the existence, uniqueness, and stability of the fluid pressure under the help of an inf-sup condition. We then come up with a fully discretized formulation. Before jumping into the error analysis and simulations, we ensure the solvability and stability of the fully discretized model. Different in the way we derive the well-posedness of the semi-discretized model, we use an induction strategy to fulfill the analysis in the fully-discretized formulation. Computational experiments are designed to illustrate the practicability and effectiveness of the method.

Second, we extend the benchmark blood flow to fluid possessing the so-called shear-thinning property. Furthermore, we compare our fully dynamic Naiver-Stokes–Biot model with a pure elastic model. Important indexes, including wall shear stress (WSS) and relative residence time (RRT), are introduced to evaluate the vessel lesion. After conducting serval simulations in different simulation domains, including the most exciting scenario: stenotic vessels having different permeability and Lamé coefficients, we conclude and summarize the impact of non-Newtonian elastic and poroelastic models and parameters on the index WSS.
and RRT.

Finally, we introduce the fully coupled FPSI-transport model problem. For the FPSI subproblem, again, we use the mixed Darcy formulation and avoid an extra penalty parameter. For the transport subproblem, a popular way to discretize the equation is to use the discontinuous Galerkin framework, for example, non-symmetric interior penalty Galerkin. However, we use continuous Galerkin, which can be extended to a discontinuous Galerkin setting. Without using a cut-off operator, we obtain the theoretical results, including the existence, uniqueness, and stability of the formulation, both in continuous and semi-discretized senses. The non-linear analysis is used since the fluid viscosity depends non-linearly on the concentration. Namely, the coupling between FPSI and the transport equation is non-linear. Besides that, the diffusion-dispersion operator in the transport equation itself is non-linear. All these non-linear factors perplex the analysis. After obtaining certain results of a linearized formulation, we employ a fixed point iteration to ensure that the solution of the linearized weak formulation converges to the original non-linear formulation. A convergence test and a simulation of the transport of mercury in an irregular channel network confirm the effectiveness and the practicability.

One direction for future work is on the fully dynamic Navier-Stokes–Biot equations further coupled with the transport equation. This FPSI-transport coupling has a wide range of applications, including drug delivery, low-density lipoprotein accumulations, and contaminants cleaning up. To conduct simulations in these settings for a relatively long time, numerical accuracy, stability, and efficiency become extremely crucial. There are three aspects we can work on and try in the future.

First, we can start with a fully mixed formulation, namely a mixed elasticity formulation, instead of a pure mixed Darcy equation used in this thesis for the efficiency of the solution and to relax the hypotheses on the corresponding discrete subspaces. Possible methods are the cell-centered finite volume method and the multipoint stress-flux mixed finite element method [66]. Those methods can provide accurate stress approximations with continuous normal components across element edges or faces and decrease the number of freedom degrees.

Aiming at fast numerical simulations and utilizing large or even adaptive time steps
require higher-order time discretization methods. In Chapter 2 and Chapter 3, we use the classic semi and fully implicit backward Euler’s method of first-order convergence. For the time discretization strategy, the so-called time filter method can bring in a second ordered convergence rate. The curvature-reducing time filter with adding one line in the code gives an immediately increased accuracy and induces a technique akin to BDF2 and is worth trying. Another noteworthy strategy that suits our needs for adaptive time steps can be found in [22]. This method is based on the kinematically coupled $\beta$–scheme and the refactorized Cauchy’s one-legged $\theta$–like method.

For spatial discretization, discontinuous Galerkin methods are good fits for the presence of the transport equation since it exhibits local mass conservation and reduces the numerical diffusions [2, 83]. Another different strategy is to design and use corresponding preconditioners [12, 26], which are numerically and computationally efficient and robust with respect to a various range of parameters. Potential choices are the Riesz map (diagonal) preconditioner and block preconditioner. The related question is how to choose and investigate the optimal preconditioner.
We present FreeFem++ code for the convergence test used in Section 2.5.

```cpp
// MACRO:
macro div(ax,ay) (dx(ax)+dy(ay))/
macro sigmaEnx(ax,ay) (muS*(2.0*dx(ax)*N.x + (dx(ay)+dy(ax))*N.y) +
lambdaS*(div(ax,ay))*N.x)/
macro sigmaEny(ax,ay) (muS*(2.0*dy(ay)*N.y + (dx(ay)+dy(ax))*N.x) +
lambdaS*(div(ax,ay))*N.y)/
macro sigmaFnx(ax,ay) (muF*(dx(ax)*N.x + dy(ax)*N.y))/
macro sigmaFny(ax,ay) (muF*(dy(ay)*N.y + dx(ax)*N.x))/
macro sigmasymFnx(ax,ay) (ci*muF*(dx(ax)*N.x + dy(ax)*N.y))/
macro sigmasymFny(ax,ay) (ci*muF*(dy(ay)*N.y + dx(ax)*N.x))/

macro cdot(ax,ay,bx,by) (ax*bx+ay*by)/
macro tgx(ax,ay) (ax-cdot(ax,ay,N.x,N.y)*N.x)/
macro tgy(ax,ay) (ay-cdot(ax,ay,N.x,N.y)*N.y)/

int Ttx = -1;
int Tty = 0;

// time:
real T=0.0001;
real delt=0.000001;
int pr=1;
real t=0;
func NN=T/delt;

// FLAGS:
bool debug=false; // true for debugging, mesh plots etc.
bool plotflag=1; // true for making .vtk files
bool converg=1; // true for convergence test (output made in reverse order,
                 // from finer mesh to coarser)
bool intresid=0; // true for interface residual (output made in reverse order,
                 // from finer mesh to coarser)
int timedep=1;
```
int m,n,l;
if(converg){
    m = 64;
    l = 2;
} else {
    m=32;
    l=m;
}

int number = log(real(m/l))/log(2.0) + 1;
cout << "Number of steps: " << number << endl;

// SAVE SOLUTIONS:
#include "SaveVTK2d.edp";
int mmm = 1;
mesh[int] thS(number);
mesh[int] thF(number);
mesh[int] thL(number);

thS[0] = square(mmm,mmm);
thS[0] = movemesh(thS[0],[x,y-1.0]);

thF[0] = square(mmm,mmm);

//plot(thS[0],thF[0],wait=1);

for(int i=1;i<number;++i){
    thS[i] = splitmesh(thS[i-1],2);
thF[i] = splitmesh(thF[i-1],2);
}

for(int i=0;i<number;++i){
    //thL[i] = emptymesh(thF[i]);
    thL[i] = thF[i];
}

int nMeshes = number;
int count=0;

// initialize arrays for errors
real[int] error1(nMeshes);
real[int] error2(nMeshes);
real[int] error3(nMeshes);
real[int] error4(nMeshes);
real[int] error44(nMeshes);
real[int] error5(nMeshes);
real[int] error6(nMeshes);
real[int] error7(nMeshes);

error1 = 0;
error2 = 0;
error3 = 0;
error4 = 0;
error44 = 0;
error5 = 0;
error6 = 0;
error7 = 0;

real[int] abs1(nMeshes);
real[int] abs4(nMeshes);
real[int] abs44(nMeshes);
real[int] abs6(nMeshes);
real[int] abs7(nMeshes);

abs1 = 0;
abs4 = 0;
abs44 = 0;
abs6 = 0;
abs7 = 0;

real[int] error2tmp(NN);
real[int] error3tmp(NN);
real[int] error5tmp(NN);
real[int] error7tmp(NN);

error2tmp = 0;
error3tmp = 0;
error5tmp = 0;

real[int] cond13left(nMeshes);
real[int] cond13right(nMeshes);
real[int] displright(nMeshes);

int meshcount = 0;
// h-TEST LOOP:
for(int n=1;n<=m;n*=2){

t=0;

string namefluid="/paraview"+string(n)+"/fluid";
string namesolid1="/paraview"+string(n)+"/structure1_";
string nameq1="/paraview"+string(n)+"/Darcy1_";
string namesolid2="/paraview"+string(n)+"/structure2_";
string nameq2="/paraview"+string(n)+"/Darcy2_";
mesh ThF = square(n,n,flags=3);
mesh ThS1 = square(4*n,4*n,flags=3);
ThS1 = movemesh(ThS1, [x,y-1]);

mesh ThL = emptymesh(ThS1);

cout << "lala"<< endl;

string filename = "mesh"+string(n)+".eps";

meshcount++;

if(debug){
    plot(ThF,ThS1, wait=true);
    plot(ThL,ThS1, wait=true);

    int nbtriangles=ThF.nt;
    for (int i=0;i<nbtriangles;i++)
        for (int j=0; j <3; j++)
            cout << i << " " << j << " Th[i][j] = " << ThF[i][j] << " x = " << ThF[i][j].x << " , y= " << ThF[i][j].y << " , label=" << ThF[i][j].label << endl;
};

// FINITE ELEMENT SPACES:
// fluid:
fe space VFh(ThF,[P2,P2,P1]);
// structure:
fe space VM1h(ThS1,[RT1,P1dc]);
// displacement
fe space VS1h(ThS1,[P2,P2]);
// lagrange:
fe space LLh(ThL, P1);

VFh [uFx,uFy,pF], [vFx,vFy,wF], [uFoldx,uFoldy,pFold],
    [uFprevx,uFprevy,pFprev], [uFxV,uFyV,pFV], [uFxTrue,uFyTrue,pFTrue];
VM1h [uP1x,uP1y,pP1], [vP1x,vP1y,wP1], [uP1oldx,uP1oldy,pP1old],
    [uP1xv,uP1yV,pP1V], [uP1xTrue,uP1yTrue,pP1True];
VS1h [eta1x,eta1y], [ksi1x,ksi1y], [eta1oldx,eta1oldy],
    [eta1Voldx,eta1Voldy],[eta1Initialx,eta1Initialy],[etaoldx, etaoldy],
    [eta1xTrue,eta1yTrue] ,[eta1Vx,eta1yV],[eta1primex,eta1primey];
LLh LAMBDA, MU, LAMBDAold, LAMBDAV;

////////////////////////////
//define constants, coefficients
////////////////////////////
real tol =1e-5;
int maxiter = 50;
real tol1=1e-5;
int maxitera=50;
// DATA
// structure:
func rohS=1.0;
func ES = 1.0;
func sigmaS=1.0;
func lambdaS = 1.0;
func muS = 1.0;

// Brinkman:
func rohF=1.0;
real muF = 1.0;
// Darcy:
real rohP=1.0;
real alfa=1.0;
real alfabjs=1.0;
real s0=1.0;
real Kxx=1.0;
real Kyy=1.0;
real kappaxx=muF/Kxx;
real kappayy=muF/Kyy;
real coef=9.869233e-13;
// BJS:
real bjs=1.0;
// stabilization:
real stab=0.0;
real stabf=0.0;
// injection:
real VolFlowRate=25;
real ld=0;
real p0=0.0;
real time=0.0;

func ufx0 = pi*cos(pi*t)*(-3*x*cos(y));
func ufy0 = pi*cos(pi*t)*(y+1);

func duf11 = pi*cos(pi*t)*(-3);
func duf12 = pi*cos(pi*t)*(-sin(y));
func duf21 = 0;
func duf22 = pi*cos(pi*t);

func upx0 = -exp(t)*pi*cos(pi*x)*cos(pi*y/2);
func upy0 = exp(t)*pi/2*sin(pi*x)*sin(pi*y/2);

func divup0=(5/4)*(pi^2)*(exp(t))*sin(pi*x)*cos(pi*y/2);
func etax0 = sin(pi*t)*(-3*x*cos(y));
func etay0 = sin(pi*t)*(y+1);

func etaxprime0=pi*cos(pi*t)*(-3*x*cos(y));
func etayprime0 = pi*cos(pi*t)*(y+1);

func deta11 = sin(pi*t)*(-3);
func deta12 = sin(pi*t)*(-sin(y));
func deta21 = 0;
func deta22 = sin(pi*t);

func pp0sol = exp(t)*sin(pi*x)*cos(pi*y/2);
func pf0sol = pp0sol + 2*pi*cos(pi*t);

func ffx=-pi^2*sin(pi*t)*(-3*x+cos(y))+pi*cos(pi*t)*cos(y)
+pi*exp(t)*cos(pi*x)*cos(pi*y/2)+(pi*cos(pi*t))^2*(9*x-3*cos(y)
-sin(y))*(y+1));

func ffy =-pi^2*sin(pi*t)*(y+1)-(pi/2)*exp(t)*sin(pi*x)*sin((pi*y)/2)
+((pi*cos(pi*t))^2)*2*(y+1));

func qf = -2*pi*cos(pi*t);
func fpx = -pi^2*sin(pi*t)*(-3*x+cos(y))+sin(pi*t)*cos(y) +
pi*exp(t)*cos(pi*x)*cos((pi*y)/2);

func fpy = -pi^2*sin(pi*t)*(y+1)-(pi*exp(t)*sin(pi*x)*sin(pi*y/2))/2;

func qp = exp(t)*cos(pi*y/2)*sin(pi*x) - 2*pi*cos(pi*t) +
(5*pi^2*exp(t)*cos(pi*y/2)*sin(pi*x))/4;

func real force(real x, real y){
    if(x^2+y^2<=50) return (1d);
    else return (0.0);
}

func real initPp(real x, real y){
    return(p0);
}

func real initu(real x, real y){
    return(0);
}

func real initv(real x, real y){
    return(0);
func real initPf(real x, real y){
    return(p0);
}

///////////////////////////////////////////////////////////////////////////////////////
//Matrix formulation
///////////////////////////////////////////////////////////////////////////////////////
// 1-2.INJECTION TERM: (f_f,vF)+(q_f,wF)
varf BCin([uFx,uFy,pF],[vFx,vFy,wF],solver=UMFPACK,init=1)=
    int2d(ThF)(ffx*vFx + ffy*vFy) + int2d(ThF)(qf*wF) + on(2,3,4,uFx=ufx0, uFy=ufy0);

///////////////////////////////////////////////////////////////////////////////////////
// 3.INJECTION TERM: (q_p,wp)+boundary terms from integration by parts
varf BCinM1([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],solver=UMFPACK,init=1)=
    int2d(ThS1)(qp*wP1) - int1d(ThS1,1,2,4)(cdot(pp0sol,pp0sol,vP1x*N.x,vP1y*N.y));

///////////////////////////////////////////////////////////////////////////////////////
// 4.INJECTION TERM: (f_p,ksi)+boundary terms from integration by parts
varf BCinS1([eta1x,eta1y],[ksi1x,ksi1y],solver=UMFPACK,init=1)=
    int2d(ThS1)(fpx*ksi1x + fpy*ksi1y) + on(1,2,4,eta1x=etax0, eta1y=etay0);

///////////////////////////////////////////////////////////////////////////////////////
// 5. for the initial step. extra source term from the first derivative
varf sourceterm1([eta1primex,eta1primey],[ksi1x,ksi1y], solver=UMFPACK,init=1)=
    int2d(ThS1) (rohP/(delt)*(etaxprime0*ksi1x + etayprime0*ksi1y));

varf BCinL(LAMBDA,MU)=intalledges(ThL)(1.e-15*pp0sol*MU);

///////////////////////////////////////////////////////////////////////////////////////
// 1.a_p^e(eta_p,ksi_p)
varf AS1sum([eta1x,eta1y],[ksi1x,ksi1y],init=1)=
    int2d(ThS1)(2.0*muS*( dx(eta1x)*dx(ksi1x) + dy(eta1y)*dy(ksi1y) ))
    +int2d(ThS1)(muS*( (dy(eta1x) + dx(eta1y))*dy(ksi1x) + (dy(eta1x) + dx(eta1y))*dx(ksi1y) ))
    +int2d(ThS1)((lambdaS)*(dx(ksi1x)*dx(eta1x)+dy(ksi1y)*dx(eta1x)
    +dx(ksi1x)*dy(eta1y)+dy(ksi1y)*dy(eta1y)) + on(1,2,4,eta1x=etax0, eta1y=etay0);
matrix AS1=AS1sum(VS1h,VS1h);

///////////////////////////////////////////////////////////////////////////////////////
//2.rohP(double prime eta_p, ksi_p)
varf FDSSsum([eta1x, eta1y], [ksi1x, ksi1y], init=1)=
int2d(ThS1)(timedep*rohP/(delt^2))*cdot(eta1x, eta1y, ksi1x, ksi1y));
matrix FDSS=FDSSsum(VS1h, VS1h);

//5.a_BJS(0, prime eta_p;0, ksi_p)
varf ABJS31sum([eta1x, eta1y], [ksi1x, ksi1y], init=1)=
int1d(ThS1, 3)(bjs*(1.0/delt)*cdot(eta1x, eta1y, Ttx, Tty)*cdot(ksi1x, ksi1y, Ttx, Tty));
matrix ABJS31=ABJS31sum(VS1h, VS1h);

matrix Block11=ABJS31+AS1+FDSS;

//3.b_gemma(0, 0, ksi_p;LAMBDA)
varf BG3T1sum([LAMBDA], [ksi1x, ksi1y], init=1)=
int1d(ThL, 3)(LAMBDA*cdot(ksi1x, ksi1y, N.x, N.y));
matrix BG3T1=BG3T1sum(LLh, VS1h);

//4. alpha b_p(ksi_p, p_p)
varf BSP1sum([uP1x, uP1y, pP1], [ksi1x, ksi1y], init=1)=
-int2d(ThS1)(alfa*pP1*div(ksi1x, ksi1y));
matrix BSP1=BSP1sum(VM1h, VS1h);

//6. a_BJS(u_f, 0;0, ksi_p)
varf ABJS21sum([uFx, uFy, pF], [ksi1x, ksi1y], init=1)=
-int1d(ThS1, 3)(bjs*cdot(uFx, uFy, Ttx, Tty)*cdot(ksi1x, ksi1y, Ttx, Tty));
matrix ABJS21=ABJS21sum(VFh, VS1h);

//7. b_gemma(0, u_p, 0;mu)
varf BG21sum([uP1x, uP1y, pP1], [mu], init=1)=
int1d(ThL, 3)(mu*(1/delt)*cdot(uP1x, uP1y, N.x, N.y));
matrix BG21=BG21sum(VM1h, LLh);

//Stable term
varf TECH1sum([LAMBDA], [mu], init=1)=
talledges(ThL)(1.e-30*LAMBDA*MU); // + int1d(ThL, 2, 1, 4)(1.e-15*LAMBDA*MU);
matrix TECH1=TECH1sum(LLh, LLh);

//8.b_gemma(0, u_p, 0;mu)
varf BG11sum([uFx, uFy, pF], [mu], init=1)=
-int1d(ThL, 3)(mu*cdot(uFx, uFy, N.x, N.y));
matrix BG11=BG11sum(VFh, LLh);

//9.b_gemma(u_f, 0;0, mu)
varf BG11sum([uFx, uFy, pF], [mu], init=1)=
-int1d(ThL, 3)(mu*cdot(uFx, uFy, N.x, N.y));
matrix BG11=BG11sum(VFh, LLh);
// 10. \(-\alpha b_p(prime \eta_p,w_p)\)
varf BSPT1sum([eta1x,eta1y],[vP1x,vP1y,wP1],init=1) =
  int2d(ThS1)((alfa/delt)*wP1*div(eta1x,eta1y));
matrix BSPT1=BSPT1sum(VS1h,VM1h);

// 11. \(b_{\text{gemma}}(0,v_p,0;\lambda)\)
varf BG2T1sum([\Lambda],[vP1x,vP1y,wP1],init=1) =
  int1d(ThL,3)(\Lambda*dot(vP1x,vP1y,N.x,N.y));
matrix BG2T1=BG2T1sum(LLh,VM1h);

// 12. \(a_p^{\text{d}}(u_p,v_p)\)
varf AQ1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1) =
  int2d(ThS1)(dot(uP1x,uP1y,vP1x,vP1y)) + int2d(ThS1)(1.e-10*pP1*wP1);
matrix AQ1=AQ1sum(VM1h,VM1h);

// 13. \(-b_p(u_p,w_p)\)
varf BPQ1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1) =
  int2d(ThS1)(1*wP1*div(uP1x,uP1y));
matrix BPQ1=BPQ1sum(VM1h,VM1h);

// 14. \(b_p(v_p,p_p)\)
varf BPQT1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1) =
  -int2d(ThS1)(1*pP1*div(vP1x,vP1y));
matrix BPQT1=BPQT1sum(VM1h,VM1h);

// 15. \((prime p_p,w_p)\)
varf MASSP1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1) =
  int2d(ThS1)((s0/delt)*(wP1*pP1));
matrix MASSP1=MASSP1sum(VM1h,VM1h);

matrix Block33=AQ1+BPQ1+BPQT1+MASSP1;

// 16. \(a_{\text{BJS}}(0,prime \eta_p; v_f,0)\)
varf ABJS2T1sum([eta1x,eta1y],[vFx,vFy,wF],init=1) =
  -int1d(ThF,1)(bjs*(1.0/delt)*dot(eta1x,eta1y,Ttx,Tty) * dot(vFx,vFy,Ttx,Tty));
matrix ABJS2T1=ABJS2T1sum(VS1h,VFh);

// 18. \(b_{\text{gemma}}(v_f,0,0;\lambda)\)
varf BG1T1sum([\Lambda],[vFx,vFy,wF],init=1) =
  -int1d(ThL,3)(\Lambda*dot(vFx,vFy,N.x,N.y));
matrix BG1T1=BG1T1sum(LLh,VFh);

// 19. \(a_f(u_f,v_f)\)
varf AFsum([uFx,uFy,pF],[vFx,vFy,wF],init=1) =
  int2d(ThF)(2.0*muF*(dx(uFx)*dx(vFx) + dy(uFy)*dy(vFy)));
\[ +\int_{\Omega} (\mu_F \cdot \nabla (\nabla \phi) \cdot \nabla \psi) \, d\Omega \]

\[ \text{matrix } AF = AFsum(VFh, VFh) \]

//20. \(-b_f(u_f, w_f)\)
\[ \text{varf } BPFsum([uFx, uFy, pF], [vFx, vFy, wF], \text{init}=1) = \]
\[ \int_{\Omega} (wF \cdot \text{div}(uFx, uFy)) \, d\Omega \]
\[ \text{matrix } BPF = BPFsum(VFh, VFh) \]

//21. \(b_f(v_f, p_f)\)
\[ \text{varf } BPFTsum([uFx, uFy, pF], [vFx, vFy, wF], \text{init}=1) = \]
\[ -\int_{\Omega} (pF \cdot \text{div}(vFx, vFy)) \, d\Omega \]
\[ \text{matrix } BPFT = BPFTsum(VFh, VFh) \]

//22. \((\text{prime } u_f, v_f)\)
\[ \text{varf } MASSFsum([uFx, uFy, pF], [vFx, vFy, wF], \text{init}=1) = \]
\[ \int_{\Omega} ((t_{\text{m}} \cdot \rho_F / \Delta t) \cdot (\nabla \phi \cdot \nabla \chi) + \text{on}(2, 3, 4, uFx = uFx0, uFy = uFy0)) \, d\Omega \]
\[ \text{matrix } MASSF = MASSFsum(VFh, VFh) \]

//17. \(a_{BJS}(u_f, 0; v_f, 0)\)
\[ \text{varf } ABJS1sum([uFx, uFy, pF], [vFx, vFy, wF], \text{init}=1) = \]
\[ \int_{\Omega} (b_{\text{JS}} \cdot (\nabla \phi \cdot \nabla \chi)) \, d\Omega \]
\[ \text{matrix } ABJS1 = ABJS1sum(VFh, VFh) \]

matrix Block44 = AF + BPF + BPFT + MASSF + ABJS1;

matrix mono = [
[Block11, BG3T1, BSP1, ABJS21],
[BG31, TECH1, BG21, BG11],
[BSPT1, BG2T1, Block33, 0],
[ABJS2T1, BG1T1, 0, Block44];

matrix monoInitial = mono;

throw 178;
matrix Block11old=ABJS31old+FDSSold;

//7. b_gemma(0,0,prime eta; mu)
varf BG31sumold([eta1oldx,eta1oldy],[MU],init=1)=
    int1d(ThL,3)(MU*(1/delt)*cdot(eta1oldx,eta1oldy,N.x,N.y));
matrix BG31old=BG31sumold(VS1h,LLh);
matrix A21=0*BG31old;

//10. -alpha b_p(prime eta_p,w_p)
varf BSPT1sumold([eta1oldx,eta1oldy],[vP1x,vP1y,wP1],init=1)=
    int2d(ThS1)((alfa/delt)*wP1*div(eta1oldx,eta1oldy));
matrix BSPT1old=BSPT1sumold(VS1h,VM1h);
matrix A31=0*BSPT1old;

//15. (prime p_p,w_p)
varf MASSP1sumold([uP1oldx,uP1oldy,pP1old],[vP1x,vP1y,wP1],init=1)=
    int2d(ThS1)((s0/delt)*(wP1*pP1old));
matrix MASSP1old=MASSP1sumold(VM1h,VM1h);
matrix A33=0*MASSP1old;

//16. a_BJS(0,prime eta_p; v_f,0)
varf ABJS2T1sumold([eta1oldx,eta1oldy],[vFx,vFy,wF],init=1)=
    -int1d(ThF,1)(bjs*(1.0/delt)*cdot(eta1oldx,eta1oldy,Ttx,Tty)*cdot(vFx,vFy,Ttx,Tty));
matrix ABJS2T1old=ABJS2T1sumold(VS1h,VFh);
matrix A41=0*ABJS2T1old;

//22. (prime u_f,v_f)
varf MASSFsumold([uFoldx,uFoldy,pFold],[vFx,vFy,wF],init=1)=
    int2d(ThF)((timedep*rohF/delt)*cdot(uFoldx,uFoldy,vFx,vFy));
matrix MASSFold=MASSFsumold(VFh,VFh);
matrix A44=0*MASSFold;

//EXTRA Stable
matrix TECH1old=0*TECH1;

matrix monoold=[
    [Block11old, 0, 0, 0],
    [BG31old, TECH1old, 0, 0],
    [BSPT1old, 0, MASSP1old, 0],
    [ABJS2T1old, 0, 0, MASSFold]
];

//initial term
varf FDSSsumoldIni([eta1Initialx,eta1Initialy],[ksi1x,ksi1y],init=1)=
    int2d(ThS1)(timedep*(rohP/delt^2)*cdot(eta1Initialx, eta1Initialy, ksi1x,
                                                             ksi1y));
matrix FDSSoldIni=FDSSsumoldIni(VS1h,VS1h);

matrix Block11Ini=ABJS31old+FDSSoldIni;

matrix monoInitialV=[
[Block11Ini, 0, 0, 0 ],
[BG31old, TECH1old, 0, 0 ],
[BSPT1old, 0, MASSP1old, 0 ],
[ABJS2T1old, 0, 0, MASSFold]
];

/////////////////////////////////////////////////////////////////////////
//second derivative, Vold matrix formulation
/////////////////////////////////////////////////////////////////////////
varf FDSSsumVold([eta1Voldx,eta1Voldy],[ksi1x,ksi1y],init=1)=
-int2d(ThS1)(timedep*rohP/((delt)^2)*cdot(eta1Voldx,eta1Voldy,ksi1x,ksi1y));
matrix FDSSVold=FDSSsumVold(VS1h,VS1h);

matrix monoVold=[
[FDSSVold, 0, 0, 0 ],
[A21, TECH1old, 0, 0 ],
[A31, 0, A33, 0 ],
[A41, 0, 0, A44]
];

//vector of RHS
real[int] xxf(Block44.n),xxfold(Block44.n),xxfVold(Block44.n),
xxfstep1(Block44.n),xxfstep0(Block44.n),xxfInitial(Block44.n),
xxfmono(Block44.n),bf(Block44.n),bfb(Block44.n),bowf(Block44.n);

real[int] xxm1(A33.n), xxm1Initial(A33.n), xxm1step1(A33.n),
xxm1step0(A33.n),xxm1oldd(A33.n),xxm1Vold(A33.n),
xxm1mono(A33.n),bm1(A33.n),bwm1(A33.n),bowm1(A33.n);

real[int] xx11(AS1.n), xx11Initial(AS1.n), xx11step1(AS1.n),
xx11step0(AS1.n),xx11oldd(AS1.n),xx11Vold(AS1.n),
xx11mono(AS1.n),bs1(AS1.n),bbs1(AS1.n),bowls1(AS1.n);

real[int] xx11(TECH1.n),xx11Initial(TECH1.n), xx11step1(TECH1.n),
xx11step0(TECH1.n),xx11oldd(TECH1.n), xx11Vold(TECH1.n),
xx11mono(TECH1.n);

real[int] pfakem1(A33.n),pfakes1(AS1.n),pfakel(TECH1.n);

pfakem1=0;
pfakes1=0;
pfakel=0;
cout << pfakem1.n << " " << pfakes1.n << " " << pfakel.n << " " << endl;

t=0;
int br=1;

[uFx,uFy,pF]=[ufx0,ufy0,pf0sol];
[uFprevx,uFprevy,pFprev]=[ufx0,ufy0,pf0sol];
[uP1x,uP1y,pP1]=[upx0,upy0,pp0sol];
[eta1x,eta1y] = [etax0,etay0];
[eta1primex,eta1primey]=[etaxprime0,etayprime0];
LAMBDA = pp0sol;

int[int] fforder1 = [1,0];
int[int] fforder2 = [1,0,1];

// INITIALIZATION

xxfold=uFx[];
xxm1old=uP1x[];
xxs1old=eta1x[];
xxl1old=LAMBDA[];

xxfInitial=0;
xxm1Initial=0;
xxs1Initial=0;
xxl1Initial=0;

real[int] xxVold=[xxs1old,xxl1old,xxm1old,xxfold];
real[int] xxInitial=[xxs1Initial,xxl1Initial,xxm1Initial,xxfInitial];
real[int] xxStep1=xxInitial;
real[int] xxStep0=[xxs1old,xxl1old,xxm1old,xxfold];

xxStep1=0.0;
xxInitial=0.0;

varf lfd(unused,VS1h)=sourceterm1;
varf l(unused,VFh)=BCin;
varf lM1(unused,VM1h)=BCinM1;
varf lS1(unused,VS1h)=BCinS1;
varf LL(unused,LLh)=BCinL;
real [int] Pinveclfd = lfd(0, VS1h);

t = delt;

real [int] Pinvec = l(0, VFh);
real [int] PinvecM1 = lM1(0, VM1h);
real [int] PinvecS1 = lS1(0, VS1h);
real [int] Pintitial = PinvecS1 + Pinveclfd;
real [int] Stableterm = LL(0, LLh);

real [int] b1 = [Pintitial, Stableterm, PinvecM1, Pinvec];
b1 += (monoInitialV * xxVold);

real epsln = 10;
int iter = 0;
while (iter < 1) {
    // Nonlinear flow term
    varf BPFTsum([uFx, uFy, pF], [vFx, vFy, wF], init=1) = -int2d(ThF)(pF*div(vFx, vFy)) + int2d(ThF)((uFprevx)*dx(uFx)*vFx + (uFprevy)*dy(uFx)*vFx + ((uFprevx)*dx(uFy) + (uFprevy)*dy(uFy))*vFy);

    matrix BPFT = BPFTsum(VFh, VFh);

    matrix Block44 = ABJS1 + AF + BPF + BPFT + MASSF;

    matrix mono = [
        [Block11, BG3T1, BSP1, ABJS21],
        [BG31, TECH1, BG21, BG11],
        [BSPT1, BG2T1, Block33, 0],
        [ABJS2T1, BG1T1, 0, Block44]
    ];

    set(mono, solver=sparesolver);
    xxInitial = mono^-1*b1;
    [xxs1mono, xxl1mono, xxm1mono, xxfmono] = xxInitial;

    uFx[] = xxfmono;

    epsln = sqrt(int2d(ThF)((uFx - uFprevx)^2 + (uFy - uFprevy)^2));

    iter += 1;
    xxVold = xxInitial;
    uFprevx[] = uFx[];
    cout << "Epsilon: " << epsln << endl;
xxStep1=xxInitial;

real[int] XX=xxStep1;
real[int] XXold=xxStep1;
real[int] temp=xxStep1;
real[int] XXVold=xxStep0;
XX=0.0;
temp=0.0;
pfakel=0;

error4[count] = 0;
error1[count] = 0;
error5[count] = 0;

t=delt;
for (int k=1;k<=NN;++k){

cout << k << " iterations out of " << NN << endl;

t=t+delt;

real[int] Pinvec= 1(0,VFh);
real[int] PinvecM1= 1M1(0,VM1h);
real[int] PinvecS1= 1S1(0,VS1h);
real[int] Stableterm=LL(0,LLh);

real[int] b=[PinvecS1,Stableterm,PinvecM1,Pinvec];
b+=(monoVold*XXVold);
b+=(monoold*XXold);

temp=XXold;

varf BPFTsum([uFx,uFy,pF],[vFx,vFy,wF],init=1)=
-int2d(ThF)(pF*div(vFx,vFy))+int2d(ThF)((uFprevx)*dx(uFx)*vFx
+(uFprevy)*dy(uFx)*vFx+((uFprevx)*dx(uFy)+(uFprevy)*dy(uFy))*vFy);

matrix BPFT=BPFTsum(VFh,VFh);

matrix Block44= ABJS1+AF+BPF+BPFT+MASSF;

matrix mono=[
[Block11, BG3T1, BSP1, ABJS21],
[BG31, TECH1, BG21, BG11 ],
[BSPT1, BG2T1, Block33, 0 ],
[ABJS2T1, BG1T1, 0, Block44]
]
set(mono,solver=sparsesolver);
XX = mono^-1 * b;

[xxs1mono, xxl1mono, xxm1mono, xxfmono]=XX;
uFx[]=xxfmono;
XXold=XX;
uFprevx[]= uFx[];

XXVold=temp;
XXold=XX;
temp=0.0;
[xxs1mono, xxl1mono, xxm1mono, xxfmono]=XX;
uFx[]=xxfmono;
uP1x[]=xxm1mono;
etax[]=xxs1mono;
LAMBDA[]=xxl1mono;

[uFxTrue,uFyTrue,pFTrue]=[ufx0,ufy0,pf0sol];
[uP1xTrue,uP1yTrue,pP1True]=[upx0,upy0,pp0sol];
[etaxTrue,etayTrue] = [etax0,etay0];

//plot([etaxTrue,etayTrue]);
//plot(pFTrue);
//plot(pP1True);
//plot([uFxTrue,uFyTrue]);
//plot([uP1xTrue,uP1yTrue]);
//plot([etax,etay]);
//plot([uP1x,uP1y],wait=1);
//plot([uFx,uFy],wait=1);
//plot(pF,wait=1);
//plot(pP1True);

if(k % 10 == 0)
cout << k << " iterations out of " << NN << endl;

if(k%pr==0&plotflag){
br=br+1;
savevtk("fractureTrue"+string(n)+"_"+string(br)+".vtk", ThF,
[ufx0,ufy0,0],pf0sol,order=fforder1,dataname="Velocity Pressure");
savevtk("structureTrue"+string(n)+"_"+string(br)+".vtk", ThS1, [upx0,upy0,0],pp0sol,[etax0,etay0,0], order=fforder2, dataname="Velocity Pressure Displacement");

savevtk("fracture"+string(n)+"_"+string(br)+".vtk", ThF, [uFx,uFy,0],pF,order=fforder1,dataname="Velocity Pressure");
savevtk("structure"+string(n)+"_"+string(br)+".vtk", ThS1, [uP1x,uP1y,0],pP1,[eta1x,eta1y,0], order=fforder2, dataname="Velocity Pressure Displacement");

}

VFh [ttx,tty,ttp] = [ufx0,ufy0,pf0sol];
VFh [eufx,eufy,epf] = [ufx0 - uFx, ufy0 - uFy, pf0sol - pF];
VFh [rufx,rufy,rfp] = [ufx0, uf0y, pf0sol];

VM1h [eup1x,eup1y,ep1] = [upx0 - uP1x, upy0 - uP1y, pp0sol - pP1];
VM1h [rup1x,rup1y,rp1] = [upx0, upy0, pp0sol];

VS1h [deta11x,deta12y] = [dx(eta1x),dy(eta1x)];
VS1h [deta21x,deta22y] = [dx(eta1y),dy(eta1y)];
VS1h [eeta1x,eeta1y] = [etax0 - eta1x, etay0 - eta1y];
VS1h [reta1x,reta1y] = [etax0, etay0];

if (plotflag)
{
    savevtk("GradDispl"+string(n)+"_"+string(br)+".vtk", ThS1, [deta11x,deta12y,0],pP1,[deta21x,deta22y,0],order=fforder2,dataname="Grad1 P Grad2");
    savevtk("GradDisplTrue"+string(n)+"_"+string(br)+".vtk", ThS1, [deta11,deta12,0],pP1,[deta21,deta22,0],order=fforder2,dataname="Grad1 P Grad2");
    savevtk("ErrorDispl"+string(n)+"_"+string(br)+".vtk", ThS1, [eeta1x,eeta1y,0],pP1,order=fforder1,dataname="Error P ");
    savevtk("ErrorStokes"+string(n)+"_"+string(br)+".vtk", ThF, [eufx,eufy,0],epf,order=fforder1,dataname="ErrorVel ErrorPres ");
    savevtk("ErrorDarcy"+string(n)+"_"+string(br)+".vtk", ThS1, [eup1x,eup1y,0],ep1,order=fforder1,dataname="ErrorVel ErrorPres ");
}

LLh elambda = 1.0;
LLh rlambda = 1.0;
if (1)
{
error1[count] += int2d(ThF)( (dx(uFx) - duf11)^2 + (dy(uFy) - duf22)^2 + 
(dx(uFx) - duf21)^2 + (dy(uFx) - duf12)^2);
abs1[count] += int2d(ThF)( duf11^2 + duf12^2 + duf21^2 + duf22^2);

error4[count] += int2d(ThS1)( (uP1x - upx0)^2 + (uP1y - upy0)^2);
abs4[count] += int2d(ThS1)( upx0^2 + upy0^2);

error44[count] += int2d(ThS1)( (dx(uP1x)+dy(uP1y)-divup0)^2+(uP1x - upx0)^2 + 
(uP1y - upy0)^2);
abs44[count] += int2d(ThS1)( (divup0)^2+upx0^2 + upy0^2);

error6[count] += int1d(ThL,3)( (LAMBDA-pp0sol)^2);
abs6[count] += int1d(ThL,3)( pp0sol^2);

error7[count] += (int2d(ThF)((pF - pf0sol)^2 ));
abs7[count] += int2d(ThF)( pf0sol^2);

error2tmpl[k-1] = (int2d(ThS1)( (pp0sol - pP1)^2 )) / (int2d(ThS1)( pp0sol^2 ));

error5tmpl[k-1] = (int2d(ThF)( (ufx0 - uFx)^2 + (ufy0 - uFy)^2 )) / 
(int2d(ThF)( ufx0^2 + ufy0^2 ));

error3tmpl[k-1] = ((int2d(ThS1)( (dx(eta1x) - deta11)^2 + (dy(eta1y) - 
deta22)^2 + (dx(eta1y) - deta21)^2 + (dy(eta1x) - deta12)^2 )) 
)/(int2d(ThS1)( deta11^2 + deta22^2 + deta12^2 + deta21^2 ));

error2[count] = error2tmp.max;
error3[count] = error3tmp.max;
error5[count] = error5tmp.max;

count+=1;
}

// OUTPUT ERRORS:
real[int] err1(nMeshes);
real[int] err2(nMeshes);
real[int] err3(nMeshes);
real[int] err4(nMeshes);
real[int] err44(nMeshes);
real[int] err5(nMeshes);
real[int] err6(nMeshes);
real[int] err7(nMeshes);
// initialize rate arrays
real[int] rate1(nMeshes);
real[int] rate2(nMeshes);
real[int] rate3(nMeshes);
real[int] rate4(nMeshes);
real[int] rate44(nMeshes);
real[int] rate5(nMeshes);
real[int] rate6(nMeshes);
real[int] rate7(nMeshes);
real[int] rate83(nMeshes);

for (int k=0; k<error1.n; ++k) {
    cout.precision(10);
    cout.scientific << error1(k) << " " << error4(k) << " " << error7(k) << endl;
}

for (int k=0; k<error1.n; ++k) {
    // Fluid velocity H1 in space L2 in time
    err1(k) = sqrt(error1(k)/abs1(k));
    // Fluid pressure L2 in space L2 in time
    err7(k) = sqrt(error7(k)/abs7(k));
    // Darcy velocity L2 in space L2 in time
    err4(k) = sqrt(error4(k)/abs4(k));
    // divergence of Darcy velocity L2 in space L2 in time
    err44(k) = sqrt(error44(k)/abs44(k));
    // Darcy pressure L2 in space l-infinity in time
    err2(k) = sqrt(error2(k));
    // Displacement H1 in space l-infinity
    err3(k) = sqrt(error3(k));
    // Fluid velocity L2 in space l-infinity in time
    err5(k) = sqrt(error5(k));
    err6(k) = sqrt(error6(k)/abs6(k));

    if (k == 0) {
        rate1(k) = 0.0;
        rate2(k) = 0.0;
        rate3(k) = 0.0;
        rate4(k) = 0.0;
        rate44(k) = 0.0;
        rate5(k) = 0.0;
        rate7(k) = 0.0;
        rate83(k) = 0.0;
    } else {
        rate1(k) = log(err1(k-1)/err1(k))/log(2.0);
        rate2(k) = log(err2(k-1)/err2(k))/log(2.0);
rate3(k) = log(err3(k-1)/err3(k))/log(2.0);
rate4(k) = log(err4(k-1)/err4(k))/log(2.0);
rate44(k) = log(err44(k-1)/err44(k))/log(2.0);
rate5(k) = log(err5(k-1)/err5(k))/log(2.0);
rate7(k) = log(err7(k-1)/err7(k))/log(2.0);
rate83(k)=log(err6(k-1)/err6(k))/log(2.0);
}
}

// OUTPUT ERRORS:
if(converg){
    matrix errors=[[err1), (rate1), (err2), (rate2), (err3), (rate3), (err4), (rate4), (err5), (rate5), (err7), (rate7)];
    {
        ofstream errOut("errorsrates.txt");
        errOut<<errors;
    }
    matrix errors1=[[error1), (error2), (error3), (error4), (error5), (error7)];
    {
        ofstream errout("errors.txt");
        errout << errors1;
    }
}

// OUTPUT INTERFACE RESIDUALS:
if(intresid){
    matrix flux=[[cond13left]];  
    {
        ofstream fluxOut("flux.txt");
        fluxOut<<flux;
    }
}

// Print results
cout << "================================================================" << endl;
cout << "Errors and rates" << endl;
cout << "|u_f(H1)|" << " rate ">
<< "|p_f(L2)|" << " rate "
<< "|u_p(L2)|" << " rate "
<< "|div(u_p)(L2)|" << " rate "
<< "|p_p(L2)|" << " rate "
<< "|eta(H1)|" << " rate "
<< "|LAMBDA(L2)|" << " rate "
<< endl;
for (int i=0; i<err1.n; i++){
    // Stokes velocity
}
cout.precision(3);
cout.scientific << err1[i] << " ";
cout.precision(1);
cout.fixed << rate1[i] << " ";
// Stokes pressure
cout.precision(3);
cout.scientific << err7[i] << " ";
cout.precision(1);
cout.fixed << rate7[i] << " ";
// Darcy velocity
cout.precision(3);
cout.scientific << err4[i] << " ";
cout.precision(1);
cout.fixed << rate4[i] << " ";
// divergence of Darcy velocity
cout.precision(3);
cout.scientific << err44[i] << " ";
cout.precision(1);
cout.fixed << rate44[i] << " ";
// Darcy pressure
cout.precision(3);
cout.scientific << err2[i] << " ";
cout.precision(1);
cout.fixed << rate2[i] << " ";
// Displacement
cout.precision(3);
cout.scientific << err3[i] << " ";
cout.precision(1);
cout.fixed << rate3[i] << " ";
// multiplier
cout.precision(3);
cout.scientific << err6[i] << " ";
cout.precision(1);
cout.fixed << rate83[i] << " ";

cout << endl;
}

cout << "================================================================" << endl;
Bibliography


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