DIVERGENCE-FREE FINITE ELEMENT METHODS FOR THE STOKES PROBLEM ON DOMAINS WITH CURVED BOUNDARY

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In this thesis, we propose finite element methods that yield divergence-free velocity approximations for the two dimensional Stokes problem on domains with curved boundary. In the first part, we propose and analyze an isoparametric method that is globally $\boldsymbol{H}(\mathrm{div})$-conforming. The corresponding pair is defined by mapping the Scott-Vogelius finite element space via a Piola transform. We use Stenberg's macro element technique to show that the method is stable and we also prove that the resulting method converges with optimal order, is divergence–free, and is pressure robust. In the second part, we build on our work from the first part and extend it to a globally $\boldsymbol{H}^1$-conforming isoparametric method by considering an enriched local reference space. We show that the enrichment procedure respects stability, optimal order convergence as well as the divergence-free property of the discrete velocity solution. Here, we also discuss the implementation of the proposed enriched velocity space. In the third part, we construct and analyze a boundary correction finite element method for the Stokes problem based on the Scott-Vogelius pair on Clough-Tocher splits. Here, we also introduce a Lagrange multiplier space to enforce boundary conditions and to mitigate the lack of pressure-robustness. We prove several inf-sup conditions leading to the well-posedness of the method. We also show that the resulting method converges with optimal order, and the velocity approximation is divergence–free.

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Preface

I would like to express my deepest gratitude to my advisor, Michael Neilan. He has been extremely supportive and encouraging since the very first day I met him. He devoted so much of his time to teach me the fundamentals of the field, patiently answered every single question I had in mind and has always been available for discussions. He is not only a great advisor but also an amazing researcher with whom I truly enjoyed working on a weekly basis. It was a lot of fun to discuss about the challenges we had together and eventually to come up with novel ideas to address them. He has taught me so much and this thesis would have not existed without him.

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1.0 Introduction

1.0.1 The Stokes problem and its finite element discretization

The Navier-Stokes equations (NSE) are a constrained system of partial differential equations (PDEs) that describe motion of a viscous fluid. There are different variants of NSE and they are used to describe different phenomenons of scientific and engineering interest such as modeling the weather, water flow, air flow and so on. One version of NSE is given by

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \end{aligned}$$

where $\boldsymbol{u}$, p, $\boldsymbol{f}$, ν denote the velocity, the pressure, the external force, the fluid viscosity, respectively, Ω is a domain in $\mathbb{R}^d$ with $d \in \{2, 3\}$, and $t \in (0, T)$, $T < \infty$.

While the NSE carry abundance of applications in real-world problems and are considered as the main step to fully understand the notion of turbulence, theoretical understanding of the solution(s) to NSE is still incomplete due to their mathematical complexity. Assuming that the velocity $\boldsymbol{u}$ is not time dependent and ignoring the non-linear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, we consider a more basic model to study the impact of the divergence constraint, which are the steady-state Stokes equations. Therefore, the Stokes equations can be regarded as a simpler version of NSE, and is given by

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.0.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.0.1b}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega. \tag{1.0.1c}$$

The case $\boldsymbol{g} = 0$ is often referred as the no-slip boundary condition.

To obtain the so-called variational formulation for the Stokes problem with no-slip boundary conditions, we consider an arbitrary test function $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, multiply (1.0.1a) by $\boldsymbol{v}$ and integrate over Ω, and we take an arbitrary test function $q \in L_0^2(\Omega)$ and perform the same operations on (1.0.1b). Doing so, the resulting equations read

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{1.0.2}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall q \in L_0^2(\Omega), \tag{1.0.3}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) = \nu(\boldsymbol{u}, \boldsymbol{v})$, $b(\boldsymbol{v}, q) = -(\nabla \cdot \boldsymbol{v}, q)$ and $(\cdot, \cdot)$ denotes the L^2 inner product on Ω.

Finite element methods (FEMs) are computational methods to numerically solve a given PDE. The main objective is to find an exact solution or possibly a "good" approximation to an exact solution by subdividing the domain into smaller elements (called finite elements) and using polynomial approximations on each element. More formally, a finite element is a triple (K, P, N) where $K \subset \mathbb{R}^d$ is a bounded closed set with non-empty interior and piecewise smooth boundary (the element domain), P is a finite dimensional space of functions on K (the space of shape functions) and N is a set of basis functions of the dual space of P (the set of nodal variables) [38, Definition 3.1.1]. Here, it is implied that the set N forms a unisolvent set over P, i.e., if $N = \{l_i\}_{i=1}^{\dim P}$, then for any given set of scalars $\{c_i\}_{i=1}^{\dim P}$ there exists a unique element $f \in P$ such that $l_i(f) = c_i$ for all $i \in \{1, ..., \dim P\}$.

Let $\boldsymbol{V}^h \subset \boldsymbol{H}_0^1(\Omega)$ denote a velocity finite element space (discrete velocity space) and $Q^h \subset L_0^2(\Omega)$ denote a pressure finite element space (discrete pressure space). Then, a finite element discretization for (1.0.1) seeks $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}^h \times Q^h$ satisfying

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}^h, \tag{1.0.4}$$

$$b(\boldsymbol{u}_h, q_h) = 0 \qquad \forall q_h \in Q^h. \tag{1.0.5}$$

It is a well-known fact in finite element theory that a problem of the type (1.0.4)-(1.0.5) has a unique solution if the bilinear form $a_h(\cdot,\cdot)$ is coercive, i.e., there exists a constant $C > 0$ such that $C\|\boldsymbol{v}_h\|^2_{\boldsymbol{V}^h} \leq a_h(\boldsymbol{v}_h, \boldsymbol{v}_h)$ for all $\boldsymbol{v}_h \in \boldsymbol{V}^h$ where $\|\cdot\|_{\boldsymbol{V}^h}$ denotes a norm of interest defined on $\boldsymbol{V}^h$, and if the pair $\boldsymbol{V}^h \times Q^h$ satisfies the so-called inf-sup condition, which is also known as the Ladyzenskaja-Babuska-Brezzi (LBB) condition. The inf-sup condition can be regarded as a compatibility criteria between the discrete spaces $\boldsymbol{V}^h$ and Q^h, and is given by

$$C\|q\|_{Q^h} \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}^h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}^h}} \quad \forall q_h \in Q^h, \tag{1.0.6}$$

where $C > 0$ is a constant independent of any mesh parameter and $\|\cdot\|_{Q^h}$ is a well-defined norm on Q^h.

Standard methods for the Stokes problem typically set $\|\boldsymbol{v}_h\|_{\boldsymbol{V}^h} := \|\boldsymbol{v}_h\|_{\boldsymbol{H}^1(\Omega)}$ and with this choice, the coercivity condition is immediately satisfied by the definition of the bilinear form $a(\cdot,\cdot)$. Therefore, for the sake of well-posedness of the method (1.0.4)-(1.0.5), one aims to construct a pair $\boldsymbol{V}^h \times Q^h$ that satisfies (1.0.6). In the literature, there have been several proposed pairs that satisfy the inf-sup condition such as Taylor-Hood elements, the MINI element [13], the Crouzeix-Raviart elements [10] and many more.

1.0.2 Divergence-free methods on and beyond polytopal domains

Other than the inf-sup criteria, which establishes the well-posedness of the discrete problem, there are two other desirable properties that one seeks for the Stokes problem; divergence-free property and pressure robustness. We say that a finite element method to (1.0.1) leads to a *divergence-free* solution if there holds $\nabla \cdot \boldsymbol{u}_h = 0$ in Ω. For commonly used methods, this is often equivalent to asking the divergence of every element in $\boldsymbol{V}^h$ to be an element of the space Q^h. Furthermore, we say that a finite element pair $(\boldsymbol{V}^h, Q^h)$ is *pressure-robust* if a gradient field in the forcing function only affects the discrete pressure solution.

There are various benefits that come with these two properties. Divergence-free methods are plausible not only because they maintain the consistency with the divergence-free property of the true solution $\boldsymbol{u}$, but they also enjoy many advantages such as robustness with respect to problem parameters, conservation of mass as well as improved accuracy. Moreover, for the standard methods, divergence-free property is also used (and often sufficient) to prove the pressure-robustness, which in turn is another desirable property as it leads to decoupling of errors. In fact, under the lack or pressure-robustness, it is possible to show that [44]

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega)} \le \inf_{\boldsymbol{v}_h \in \boldsymbol{V}^h} \|\nabla(\boldsymbol{u} - \boldsymbol{v}_h)\|_{L^2(\Omega)} + \nu^{-1} \inf_{q_h \in Q^h} \|p - q_h\|_{L^2(\Omega)},$$

i.e., the velocity error depends on the pressure error which is then scaled by the reciprocal of the viscosity. Therefore, pressure-robust methods can be quite advantageous in the existence of a large pressure and/or small viscosity. Indeed, there are several numerical examples and a detailed discussion in [44] that explicitly highlight the existence of poor estimates of the methods that only weakly impose the divergence-free constraint.

On simplicial meshes, the first stable, $\boldsymbol{H}^1$-conforming (i.e., $\boldsymbol{V}^h \subset \boldsymbol{H}_0^1(\Omega)$) and divergence-free finite element pairs were proposed by Scott-Vogelius [40]. Since then, there have been developed and analyzed several other divergence-free methods in the literature [18, 17, 46, 47, 29]. However, the task of building a stable, divergence-free and pressure robust method for the Stokes problem becomes even more challenging when the domain Ω has a curved boundary. In such cases, a direct use of traditional FEM techniques using affine elements is, by itself, not satisfactory enough as the order of convergence can be, at best, $\mathcal{O}(h^2)$ due to the geometric error. In order to address this issue, many different techniques have been developed and proposed in the literature such as isoparametric finite element methods, boundary correction methods, CutFEM, etc. The essential idea of these methods is to compensate the geometric error either by constructing curved elements in the computational domain or by incorporating additional terms in the variational formulation in order to reduce the inconsistency of the method. Even though these methods have been constructed in such a way that the resulting scheme maintains the optimal order of convergence, the divergence-free and pressure-robustness properties are lost in most of the existing schemes due to the additional complexity in the structure of the variational formulation to mitigate the geometric error.

1.0.3 Objective and outline of the thesis

In this thesis, we propose and analyze three different stable, optimally convergent, divergence-free finite element methods for the two-dimensional Stokes problem on domains with curved boundary. In each case, the corresponding method is based on the Scott-Vogelius pair on Clough-Tocher splits, i.e., simplicial triangulations obtained by connecting the vertices of each triangle in a given mesh to its barycenter (cf. Figure 1).

Two of the proposed methods belong to the class of isoparametric finite element methods. Isoparametric finite element methods were first introduced around fifty years ago, and they have been developed and used in the literature since then in order to find numerical solutions to PDEs on smooth domains. The essential idea of such methods is to use a polynomial mapping between the reference and physical domains so that the geometric discrepancy is comparable to approximation properties of the underlying finite element space. The implementation and analysis of isoparametric elements for second–order, scalar elliptic problems are well–established, and classical theories exist [48, 8, 24, 38, 39]. On the other hand, isoparametric elements for mixed problems, in particular the Stokes problem, is less developed [2, 45, 12], and existing applications suffer from the lack of divergence-free and pressure-robustness properties.

The third method we propose belongs to the family of boundary correction methods. Boundary correction methods is an example of unfitted methods, i.e., methods in which the computational domain does not conform to the physical domain Ω, and they were first introduced in [22]. Unlike isoparametric finite element methods, where the discrepancy between the physical and computational domains is mitigated geometrically with the use of curved elements, such methods instead use a polytopal approximation and benefit from the Taylor expansion by transferring the boundary conditions into the variational formulation in such a way that the resulting method still has optimal order convergence. One advantage of such methods is that, in the case of a dynamic problem with moving boundary, one does not necessarily need to remesh at each time step. These methods seem to be gaining in popularity and have recently been studied and improved in [9, 34, 26, 31, 30, 32].

In Chapter 2, we introduce an isoparametric method that is globally $\boldsymbol{H}(\mathrm{div})$-

conforming, stable and divergence-free. We study the error estimates of the method and show how the method can be extended to a pressure-robust scheme via a use of commuting projections. In Chapter 3, we build on our work in Chapter 2 and introduce an isoparametric method that is $\boldsymbol{H}^1$-conforming, stable, divergence-free and pressure-robust. As far as we are aware, this is the first isoparametric method for the Stokes problem with all of these attributes together. In Chapter 4, we construct and analyze a boundary correction method that is $\boldsymbol{H}^1$-conforming, stable and divergence-free. Here, we also show how the addition of a Lagrange multiplier space into the problem setting can improve the lack of pressure-robustness that results from the nature of boundary correction methods. To the best of our knowledge, this is the first $\boldsymbol{H}^1$-conforming, stable, divergence-free boundary correction method for the Stokes problem.

1.0.4 Some fixed notation

For the rest of the thesis, we use some fixed notations across the three chapters introduced below; $\hat{T}$ denotes the reference triangle that has vertices $(0,0)$, $(1,0)$ and $(0,1)$, and $\hat{T}^{ct}$ denotes its Clough-Tocher triangulation. We also let $\Omega \subset \mathbb{R}^2$ denote the physical domain and $\Omega_h \subset \mathbb{R}^2$ denote the computational domain.

For a non-negative integer k and an affine, regular, and simplicial triangulation $\mathcal{S}_h$, we define

$$\mathcal{P}_k(\mathcal{S}_h) = \{q \in L^2(S) : \ q|_K \in \mathcal{P}_k(K) \ \forall K \in \mathcal{S}_h\},$$

where $S = \mathrm{int}\big(\cup_{K\in\mathcal{S}_h} \bar{K}\big)$, and $\mathcal{P}_k(K)$ denotes the space of polynomials of degree $\leq k$ with domain K. We further define

$$\mathcal{P}_k^c(\mathcal{S}_h) = \mathcal{P}_k(\mathcal{S}_h) \cap H^1(S), \qquad \mathcal{P}_k^{c1}(\mathcal{S}_h) = \mathcal{P}_k(\mathcal{S}_h) \cap H^2(S).$$

As an example, $\mathcal{P}_k^c(\hat{T}^{ct})$ is the local kth-degree Lagrange finite element space, and $\mathcal{P}_k^{c1}(\hat{T}^{ct})$ is the local kth-degree C^1 finite element space, both of which are defined on the reference Clough-Tocher split. Analogous vector-valued spaces are denoted in boldface. For instance, $\boldsymbol{\mathcal{P}}_k^c(\hat{T}^{ct}) = [\mathcal{P}_k^c(\hat{T}^{ct})]^2$.

Remark 1.0.1. For the continuation of this thesis, we use C (with or without subscript) to denote a generic constant that is independent of the viscosity and any mesh size parameter.

2.0 Divergence-free Scott-Vogelius elements on curved domains

2.1 Introduction

In this chapter, we extend the isoparametric setting for the Stokes problem in two dimensions in order to develop a divergence-free, pressure-robust method. The basis of our construction is the lowest-order, two-dimensional Scott-Vogelius pair defined on Clough-Tocher splits. In this case, the velocity space is the space of continuous, piecewise quadratic polynomials, and the pressure space is the space of (discontinuous) piecewise linear polynomials. It is known, on affine Clough-Tocher meshes, that this pair is stable, and the corresponding scheme is divergence-free and pressure-robust. However, a direct application of the isoparametric paradigm to this pair leads to a method with neither of these desirable properties. Indeed, the Scott-Vogelius pair, defined by standard isoparametric mappings, is given by

$$\breve{\boldsymbol{V}}^h = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega_h) : \ \boldsymbol{v}|_K = \hat{\boldsymbol{v}} \circ F_K^{-1}, \ \exists \hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_2(\hat{T}) \ \forall K \in \mathcal{T}_h^{ct}\}, \tag{2.1.1a}$$

$$\breve{Q}^h = \{q \in L_0^2(\Omega_h) : \ q|_K = \hat{q} \circ F_K^{-1}, \ \exists \hat{q} \in \mathcal{P}_1(\hat{T}) \ \forall K \in \mathcal{T}_h^{ct}\}, \tag{2.1.1b}$$

where $F_K : \hat{T} \to K$ is a quadratic diffeomorphism, and $\mathcal{T}_h^{ct}$ is the Clough-Tocher refinement of a simplicial triangulation $\mathcal{T}_h$. The chain rule shows $\operatorname{div} \boldsymbol{v}_h \notin \breve{Q}^h$ for general $\boldsymbol{v}_h \in \breve{\boldsymbol{V}}^h$ (unless F_K is affine $\forall K \in \mathcal{T}_h$), and simple calculations show the exact enforcement of the divergence–free constraint and the pressure–robustness of the scheme using $\breve{\boldsymbol{V}}^h \times \breve{Q}^h$ is lost on curved elements.

Our construction to obtain a divergence-free, pressure-robust method in the above setting relies on two main ideas. First, unlike the traditional use of the isoparametric structure where the velocity and pressure spaces are defined through composition, we instead benefit from the Piola transform in the definition of the velocity space, and this modification is the key ingredient that leads to the desirable divergence-free property of the method. Moreover, we show that the resulting global velocity space is $\boldsymbol{H}^1$-conforming in the interior of the computational domain and $\boldsymbol{H}(\mathrm{div})$-conforming globally. The second main idea in our construction is to treat the Scott-Vogelius pair as a macro-element, rather than a finite element space defined on a refined Clough-Tocher triangulation. This is motivated by the stability analysis of the Scott-Vogelius pair, which is based on Stenberg's macro-element technique [11]. We adopt this technique to the isoparametric setting and show that the resulting pair is inf-sup stable, which leads to the well-posedness of the method.

The rest of this chapter is organized as follows. In the next section, we introduce the notation, state the properties of the quadratic diffeomorphisms, and provide some preliminary results. In Section 2.3, we define the local spaces of the velocity-pressure pair and study their characteristics. Here, we also prove a local inf-sup stability result. In Section 2.4, we introduce the global spaces and show that the resulting pair satisfies the inf-sup condition. We also show in this section that functions in the discrete velocity space are weakly continuous. We state the finite element method in Section 2.5 and show that the method is optimally convergent. In Section 2.6, we introduce a pressure-robust scheme through the use of commuting projections, and Section 2.7 provides numerical experiments that confirm the theoretical results.

2.2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a sufficiently smooth, bounded and open domain such that its boundary, $\partial\Omega$, is given by a finite number of local charts. We consider the Stokes problem with no-slip boundary condition:

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2.2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2.2.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{2.2.1c}$$

where $\nu > 0$ denotes a constant viscosity. Our construction of the computational mesh for the numerical solution of the above problem simply follows standard isoparametric framework [24, 38, 3, 8]. In detail, we begin with a shape regular, affine triangulation, $\tilde{\mathfrak{T}}_h$, with the boundary vertices of $\tilde{\mathfrak{T}}_h$ belonging to $\partial\Omega$ such that each $\tilde{T} \in \tilde{\mathfrak{T}}_h$ has at most two vertices on $\partial\Omega$, and $\tilde{\Omega}_h := \text{int}\Big(\cup_{\tilde{T}\in\tilde{\mathfrak{T}}_h} \overline{\tilde{T}}\Big)$ is an $\mathcal{O}(h^2)$ polygonal approximation to Ω, where $h = \max_{\tilde{T}\in\tilde{\mathfrak{T}}_h} \text{diam}(\tilde{T})$. Next, in order to compensate the geometric error between $\tilde{\Omega}_h$ and Ω, we assume a bijective map $G : \tilde{\Omega}_h \to \Omega$ satisfying $\|G\|_{W^{1,\infty}(\tilde{\Omega}_h)} \leq C$, with the additional property that $G|_{\tilde{T}}(x) = x$ at all vertices of $\tilde{T}$. In fact, G reduces to the identity mapping on any triangle $\tilde{T} \in \tilde{\mathfrak{T}}_h$ with no vertices on the boundary. Let G_h be the piecewise quadratic nodal interpolant of G such that $\|DG_h\|_{W^{1,\infty}(\tilde{T})} \leq C$ and $\|DG_h^{-1}\|_{W^{1,\infty}(\tilde{T})} \leq C$ for all $\tilde{T} \in \tilde{\mathfrak{T}}_h$, where the notation DH denotes the Jacobian of a regular mapping $H : \mathbb{R}^2 \to \mathbb{R}^2$. We then define the isoparametric triangulation and the computational domain, respectively, as follows:

$$\mathfrak{T}_h = \{G_h(\tilde{T}) : \ \tilde{T} \in \tilde{\mathfrak{T}}_h\}, \qquad \Omega_h := \text{int}\Big(\cup_{T\in\mathfrak{T}_h} \overline{T}\Big).$$

Let $F_{\tilde{T}} : \hat{T} \to \tilde{T}$ be an affine mapping. With the aid of the mappings G_h and $F_{\tilde{T}}$, we introduce the quadratic mapping $F_T : \hat{T} \to T$, defined by $F_T := G_h \circ F_{\tilde{T}}$, which satisfies [24, 3, 38]

$$
\begin{aligned}
&|F_T|_{W^{m,\infty}(\hat{T})} \le C h_T^m \quad 0 \le m \le 2, \qquad |F_T^{-1}|_{W^{m,\infty}(T)} \le C h_T^{-m} \quad 0 \le m \le 3, \\
&c_1 h_T^2 \le \det(DF_T) \le c_2 h_T^2,
\end{aligned} \tag{2.2.2}
$$

with $h_T = \operatorname{diam}(G_h^{-1}(T))$. It is important to note that $F_T = F_{\tilde{T}}$ at the vertices of $\hat{T}$. In fact, due to the above properties of the mappings G and G_h, if $e \subset \partial T$ is a straight edge, where $e = F_T(\hat{e})$ with $\hat{e} \subset \partial \hat{T}$, then $F_T|_{\hat{e}} = F_{\tilde{T}}|_{\hat{e}}$. Similarly, we also have $T = G_h(\tilde{T}) = \tilde{T}$ if $T \in \mathfrak{T}_h$ has all straight edges since G and G_h reduces to the identity mapping in this case. We denote the Clough-Tocher triangulation of $\hat{T}$ by $\hat{T}^{ct} = \{\hat{K}_i\}_{i=1}^3$, and we define the corresponding triangulations on $\tilde{T} \in \tilde{\mathfrak{T}}_h$ and $T \in \mathfrak{T}_h$, respectively, as follows (see also Figure 1):

$$
\tilde{T}^{ct} = \{F_{\tilde{T}}(\hat{K}) : \hat{K} \in \hat{T}^{ct}\}, \quad T^{ct} = \{F_T(\hat{K}) : \ \hat{K} \in \hat{T}^{ct}\}.
$$

Notice that the properties of the mapping F_T along with the shape regularity $\tilde{\mathfrak{T}}_h$ ensure that $|T|/|G_h^{-1}(T)| \le C$ and $|G_h^{-1}(T)|/|T| \le C$ for all $T \in \mathfrak{T}_h$. We let $\mathcal{E}_h^I$ denote the interior (straight) edges of $\mathfrak{T}_h$ and $\mathcal{E}_h^{I,\partial} \subset \mathcal{E}_h^I$ denote the set of interior edges that have only one endpoint on $\partial\Omega_h$. We also let $\boldsymbol{n}$ denote the outward unit normal vector of a domain which should be understood from the context. The notation $\boldsymbol{t}$ denotes the tangent vector, which is obtained by rotating $\boldsymbol{n}$ 90 degrees counterclockwise. The globally refined triangulations are defined as

$$
\tilde{\mathfrak{T}}_h^{ct} = \{\tilde{K} : \ \tilde{K} \in \tilde{T}^{ct}, \ \exists \tilde{T} \in \tilde{\mathfrak{T}}_h\}, \qquad \mathfrak{T}_h^{ct} = \{K : \ K \in T^{ct}, \ \exists T \in \mathfrak{T}_h\}.
$$

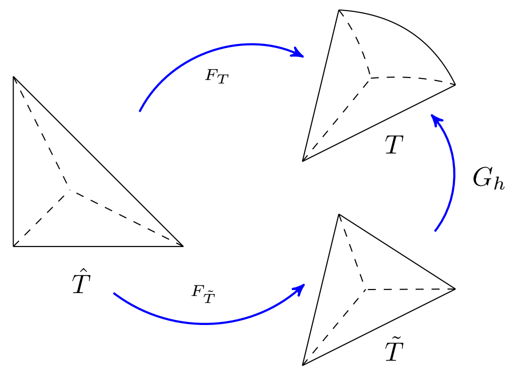

Figure 1: Illustration of each Clough-Tocher element and of the connection between the mappings $F_{\tilde{T}}$ and F_T.

Remark 2.2.1. It is important to note that the construction of the Clough-Tocher isoparametric mesh $\mathcal{T}_h^{ct}$ is based on mapping the reference *macro element* $\hat{T}^{ct}$, i.e., the isoparametric Clough-Tocher mesh $\mathcal{T}_h$ (or $\tilde{\mathcal{T}}_h$) is constructed through the reference macroelement $\hat{T}^{ct}$ via the mapping F_T (or $F_{\tilde{T}}$). Accordingly, the finite element spaces that are given in subsequent sections are defined on $\mathcal{T}_h$, not on $\mathcal{T}_h^{ct}$. Also, notice that, due to the isoparametric setting described above, this construction leads to curved interior edges in $\mathcal{T}_h^{ct}$ since interior edges of T^{ct} may be curved for $|T \cap \partial\Omega_h| > 0$ (where the notation $|S|$ denotes the one dimensional Lebesque measure of a measurable set S) as illustrated in Figure 1.

The next lemma introduces the Piola mapping and estimates its associated matrix and inverse.

Lemma 2.2.2. *For an arbitrarily given* $T \in \mathcal{T}_h$*, we define the matrix valued function* $A_T : \hat{T} \to \mathbb{R}^{2\times 2}$ *as*

$$A_T(\hat{x}) = \frac{DF_T(\hat{x})}{\det(DF_T(\hat{x}))}. \tag{2.2.3}$$

The Piola transform of a function $\hat{\boldsymbol{v}} : \hat{T} \to \mathbb{R}^2$ *is the function* $\boldsymbol{v} : T \to \mathbb{R}^2$ *given by*

$$\boldsymbol{v}(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x}), \qquad x = F_T(\hat{x}). \tag{2.2.4}$$

Then, there holds

$$(\nabla \cdot \boldsymbol{v})(x) = \frac{1}{\det(DF_T(\hat{x}))}(\hat{\nabla} \cdot \hat{\boldsymbol{v}})(\hat{x}). \tag{2.2.5}$$

Moreover, the matrix A_T *and its inverse satisfy the following estimates:*

$$|A_T|_{W^{m,\infty}(\hat{T})} \le C h_T^{m-1}, \qquad \textit{and} \qquad |A_T^{-1}|_{W^{m,\infty}(\hat{T})} \le \begin{cases} C h_T^{1+m} & m = 0, 1 \\ 0 & m \ge 2. \end{cases} \tag{2.2.6}$$

Proof. The equality in (2.2.5) is a “well-known” property of the Piola transform and its proof can be found, for example, in [27, Lemma 3.59]. For completeness, we provide a proof of this result. Let $p \in C_0^\infty(T)$ be arbitrary, where $C_0^\infty(T)$ denotes the set of infinitely many differentiable functions with compact support in T. Using integration by parts with change of variables, we then find

$$\begin{aligned} \int_T (\nabla \cdot \boldsymbol{v})\, p &= -\int_T \boldsymbol{v} \cdot \nabla p \\ &= -\int_{\hat{T}} \frac{DF_T(\hat{x})\hat{\boldsymbol{v}}}{\det(DF_T(\hat{x}))} \cdot \left(DF_T^{-T}\hat{\nabla}\hat{p}\right) \det(DF_T(\hat{x})) \\ &= -\int_{\hat{T}} \hat{\boldsymbol{v}} \cdot \hat{\nabla}\hat{p} \\ &= \int_{\hat{T}} \left(\hat{\nabla} \cdot \hat{\boldsymbol{v}}\right) \hat{p}. \end{aligned}$$

Transforming the right hand side of the last equality back to T and treating $\hat{\nabla} \cdot \hat{\boldsymbol{v}}$ as a scalar function, we get

$$\int_T (\nabla \cdot \boldsymbol{v})\, p = \int_T \left(\hat{\nabla} \cdot \hat{\boldsymbol{v}}\right) \frac{p}{\det(DF_T)}.$$

As the above equality holds for all $p \in C_0^\infty(T)$, we obtain the desired result.

Next, we estimate the bounds of A_T and its inverse. In order to ease the notation, let $\hat{g}(\hat{x}) := \det(DF_T(\hat{x}))$. Using (2.2.2) and that the mapping $DF_T \to \det(DF_T)$ is quadratic, we find that $|\hat{g}|_{W^{m,\infty}(\hat{T})} \leq Ch_T^{2+m}$. Let α denote any multi-index with $|\alpha| = m$. A direct use of the chain rule with $|\hat{g}|_{W^{m,\infty}(\hat{T})} \leq Ch_T^{2+m}$ and (2.2.2) yields

$$\begin{aligned}\Big|\frac{\partial^m}{\partial \hat{x}^\alpha}\frac{1}{\hat{g}}\Big| &\leq C \sum_{|\beta^{(1)}|+|\beta^{(2)}|+\cdots+|\beta^{(m)}|=m} \frac{|\partial^{|\beta^{(1)}|}\hat{g}/\partial\hat{x}^{\beta^{(1)}}|\cdots|\partial^{|\beta^{(m)}|}\hat{g}/\partial\hat{x}^{\beta^{(m)}}|}{|\hat{g}^{m+1}|} \\ &\leq C \sum_{|\beta^{(1)}|+|\beta^{(2)}|+\cdots+|\beta^{(m)}|=m} \frac{(h_T^{2+|\beta^{(1)}|})\cdots(h_T^{2+|\beta^{(m)}|})}{|\hat{g}^{m+1}|} \leq C\frac{h_T^{3m}}{|\hat{g}^{m+1}|} \leq Ch_T^{m-2}.\end{aligned}$$

Let $i, j \in \{1, 2\}$. Using the above inequality with the product rule and (2.2.2) shows

$$\begin{aligned}\Big|\frac{\partial^m (A_T)_{i,j}}{\partial \hat{x}^\alpha}\Big| &= \Big|\frac{\partial^m}{\partial \hat{x}^\alpha}\big((DF_T)_{i,j}/\hat{g}\big)\Big| \\ &\leq C \sum_{|\beta|+|\gamma|=m} \big|\partial^\beta (DF_T)_{i,j}/\partial^{|\beta|}\hat{x}\big|\big|\partial^\gamma \hat{g}^{-1}/\partial^{|\gamma|}\hat{x}\big| \\ &\leq C \sum_{|\beta|+|\gamma|=m} \big(h_T^{1+|\beta|}\big)\big(h_T^{|\gamma|-2}\big) \leq Ch_T^{m-1},\end{aligned}$$

and this proves the first inequality. In order to prove the second inequality, we first notice that $A_T^{-1} = \det(DF_T)(DF_T)^{-1} = \operatorname{adj}(DF_T)$. Moreover, recall that the entries of DF_T and $\operatorname{adj}(DF_T)$ are the same up to permutation and sign in two dimensions. Using this with (2.2.2) and that F_T is quadratic, we obtain

$$|A_T^{-1}|_{W^{m,\infty}(\hat{T})} = |DF_T|_{W^{m+1,\infty}(\hat{T})} \leq \begin{cases} Ch_T^{1+m} & m = 0, 1 \\ 0 & m \geq 2. \end{cases}$$

□

The following lemma will eventually be used to prove the $\boldsymbol{H}(\text{div})$-conformity of the global velocity space in the subsequent sections.

Lemma 2.2.3. *Let $\tilde{T} \in \tilde{\mathfrak{T}}_h$ and $T \in \mathfrak{T}_h$ such that $T = G_h(\tilde{T})$. Let $\hat{e}$ be an edge of $\hat{T}$ with outward unit normal $\hat{\boldsymbol{n}}$, and assume that the corresponding edge $e = F_T(\hat{e})$ on T is straight. Then,*

$$\det(DF_T(\hat{x}))(DF_T(\hat{x}))^{-\intercal}\hat{\boldsymbol{n}} = \det(DF_{\tilde{T}}(\hat{x}))(DF_{\tilde{T}}(\hat{x}))^{-\intercal}\hat{\boldsymbol{n}}$$

is constant on $\hat{e}$.

Proof. Let $\hat{\boldsymbol{t}}$ be the unit tangent vector of $\hat{e}$. A straight-forward calculation yields

$$\det(DF_T(\hat{x}))(DF_T(\hat{x}))^{-\intercal}\hat{\boldsymbol{n}} = \begin{pmatrix} -(DF_T(\hat{x})\hat{\boldsymbol{t}})_2 \\ (DF_T(\hat{x})\hat{\boldsymbol{t}})_1 \end{pmatrix}.$$

Next, recall that, due to the properties of the mapping F_T, F_T restricted to $\hat{e}$ is affine, and as a result, $(DF_T(\hat{x})\hat{\boldsymbol{t}})$ is constant on $\hat{e}$. This with $F_T|_{\hat{e}} = F_{\tilde{T}}|_{\hat{e}}$ completes the proof of the lemma. $\square$

The next lemma is a “classic” scaling result which can be found, for example, in [3].

Lemma 2.2.4. *Let $\hat{\boldsymbol{w}}(\hat{x}) = \boldsymbol{w}(x)$ for a sufficiently smooth $\boldsymbol{w} \in W^{m,p}(T)$ where $x = F_T(\hat{x})$. Then, $\hat{\boldsymbol{w}} \in W^{m,p}(\hat{T})$ for any $K \in T^{ct}$ and*

$$|\boldsymbol{w}|_{W^{m,p}(K)} \le C h_T^{2/p-m} \sum_{r=0}^{m} h_T^{2(m-r)} |\hat{\boldsymbol{w}}|_{W^{r,p}(\hat{K})},$$
$$|\hat{\boldsymbol{w}}|_{W^{m,p}(\hat{K})} \le C h_T^{m-2/p} \sum_{r=0}^{m} |\boldsymbol{w}|_{W^{r,p}(K)},$$

with $\hat{K} = F_T^{-1}(K)$.

Before we define the local spaces next, we introduce some more notation. We denote the set of s-dimensional simplices of the Clough-Tocher partition of a triangle T by $\Delta_s(T^{ct})$. For instance, $\Delta_0(\hat{T}^{ct})$ is the set of four vertices of $\hat{T}^{ct}$, and $\Delta_1(\hat{T}^{ct})$ is the set of six edges of $\hat{T}^{ct}$. Similarly, and with a slight abuse of notation, we let $\Delta_s(\hat{T})$ denote the set of s-dimensional simplices of $\hat{T}$.

2.3 Local spaces

Following the above notation, we first define the polynomial spaces on the reference triangle $\hat{T}$ without boundary conditions:

$$\hat{\boldsymbol{V}} := \boldsymbol{\mathcal{P}}_2^c(\hat{T}^{ct}), \quad \hat{Q} := \mathcal{P}_1(\hat{T}^{ct}).$$

For a given affine triangle $\tilde{T} \in \tilde{\mathcal{T}}_h$, we define the corresponding polynomial spaces via a direct composition through $F_{\tilde{T}}$, i.e.,

$$\tilde{\boldsymbol{V}}(\tilde{T}) = \{\tilde{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_2^c(\tilde{T}^{ct}) : \ \tilde{\boldsymbol{v}}|_{\partial\tilde{T}\cap\partial\tilde{\Omega}_h} = 0\}, \qquad \tilde{Q}(\tilde{T}) = \mathcal{P}_1(\tilde{T}^{ct}),$$

with $\tilde{x} = F_{\tilde{T}}(\hat{x})$. In other words, $\tilde{\boldsymbol{V}}(\tilde{T})$ is the local, quadratic Lagrange finite element space with respect to $\tilde{T}^{ct}$, and $\tilde{Q}(\tilde{T})$ is the space of discontinuous piecewise linear polynomials with respect to $\tilde{T}^{ct}$. We also define the analogous spaces with boundary conditions as follows:

$$\begin{aligned} &\hat{\boldsymbol{V}}_0 = \hat{\boldsymbol{V}} \cap \boldsymbol{H}_0^1(\hat{T}), && \hat{Q}_0 = \hat{Q} \cap L_0^2(\hat{T}), \\ &\tilde{\boldsymbol{V}}_0(\tilde{T}) = \tilde{\boldsymbol{V}}(\tilde{T}) \cap \boldsymbol{H}_0^1(\tilde{T}), && \tilde{Q}_0(\tilde{T}) = \tilde{Q}(\tilde{T}) \cap L_0^2(\tilde{T}). \end{aligned}$$

Next, for $T \in \mathcal{T}_h$ with possibly curved boundary, we benefit from the Piola transform in the definition of the local velocity space:

$$\boldsymbol{V}(T) = \{\boldsymbol{v} \in \boldsymbol{H}^1(T) : \ \boldsymbol{v}(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x}), \ \exists \hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}, \text{ and } \boldsymbol{v}|_{\partial T \cap \partial\Omega_h} = 0\},$$
$$\boldsymbol{V}_0(T) = \boldsymbol{V}(T) \cap \boldsymbol{H}_0^1(T),$$

where $x = F_T(\hat{x})$. For $T \in \mathcal{T}_h$, we define the local pressure space using a direct composition:

$$Q(T) = \{q \in L^2(T) : \ q(x) = \hat{q}(\hat{x}), \ \exists \hat{q} \in \hat{Q}\},$$
$$Q_0(T) = \{q \in L^2(T) : \ q(x) = \hat{q}(\hat{x}), \ \exists \hat{q} \in \hat{Q}_0\},$$

again with $x = F_T(\hat{x})$. Here, it is important to notice that when F_T and $F_{\tilde{T}}$ coincide, in which case F_T reduces to an affine mapping, there holds $\boldsymbol{V}(T) = \tilde{\boldsymbol{V}}(\tilde{T})$ and $Q(T) = \tilde{Q}(\tilde{T})$. Otherwise, due to the use of the quadratic mapping F_T, neither $\boldsymbol{V}(T)$ nor $Q(T)$ are necessarily piecewise polynomial spaces. Furthermore, for a given $\boldsymbol{v} \in \boldsymbol{V}(T)$ and a straight edge $e \subset \partial T$, $(\boldsymbol{v}|_e)(x)$ is not necessarily a polynomial even though F_T^{-1} is affine on e, and this is due to the use of the Piola transformation in the definition of the local velocity space. However, the next lemma ensures that the normal component of $\boldsymbol{v}$ when restricted on such an edge e is still a polynomial.

Lemma 2.3.1. *Given $\boldsymbol{v} \in \boldsymbol{V}(T)$ and a straight edge e of ∂T with unit normal $\boldsymbol{n}$. Then, $(\boldsymbol{v} \cdot \boldsymbol{n})|_e$ is a quadratic polynomial.*

Proof. By the definition of the local space $\boldsymbol{V}(T)$, we first write $\boldsymbol{v}(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x})$ for some $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$. Let $\hat{e} = F_T^{-1}(e)$ be the corresponding edge in $\partial\hat{T}$ with outward unit normal $\hat{\boldsymbol{n}}$. Then, by rewriting $\boldsymbol{v}$ and using the identity $\boldsymbol{n} = \frac{DF^{-\intercal}\hat{\boldsymbol{n}}}{|DF^{-\intercal}\hat{\boldsymbol{n}}|}$ (cf. [27]), we obtain

$$\boldsymbol{v}\cdot\boldsymbol{n} = \frac{DF_T(\hat{x})\hat{\boldsymbol{v}}}{\det(DF_T(\hat{x}))}\cdot\frac{DF^{-\intercal}\hat{\boldsymbol{n}}}{|DF^{-\intercal}\hat{\boldsymbol{n}}|} = \frac{\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}}{|\det(DF_T(\hat{x}))DF^{-\intercal}\hat{\boldsymbol{n}}|}.$$

Recall, by Lemma 2.2.3, that $(\det(DF_T)DF_T^{-\intercal}\hat{\boldsymbol{n}})$ is a constant vector. This, with the above equality, implies that $\boldsymbol{v}\cdot\boldsymbol{n}$ is a non-zero multiple of $\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}$. Finally, since $F_T|_{\hat{e}}$ is affine and $\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}$ is a quadratic polynomial on $\hat{e}$, we conclude $\boldsymbol{v}\cdot\boldsymbol{n}|_e$ is a quadratic polynomial on e. $\square$

Lemma 2.3.2. *Given $\boldsymbol{v} = A_T\hat{\boldsymbol{v}} \in \boldsymbol{V}(T)$ for some $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$. Then, there holds $\|\boldsymbol{v}\|_{H^1(T)} \le Ch_T^{-1}\|\hat{\boldsymbol{v}}\|_{H^1(\hat{T})}$.*

Proof. Using Lemma 2.2.2, and Lemma 2.2.4 along with the chain rule, we obtain

$$\begin{aligned}\|\boldsymbol{v}\|_{H^1(T)} &\le C(|A_T\hat{\boldsymbol{v}}|_{H^1(\hat{T})} + h_T\|A_T\hat{\boldsymbol{v}}\|_{L^2(\hat{T})})\\ &\le C(\|A_T\|_{L^\infty(\hat{T})}\|\hat{\boldsymbol{v}}\|_{H^1(\hat{T})} + \|A_T\|_{W^{1,\infty}(\hat{T})}\|\hat{\boldsymbol{v}}\|_{L^2(\hat{T})}) \le Ch_T^{-1}\|\hat{\boldsymbol{v}}\|_{H^1(\hat{T})}.\end{aligned}$$

$\square$

2.3.1 Degrees of freedom for $\boldsymbol{V}(T)$

Recall that the nodal degrees of freedom (DOFs) of the quadratic Lagrange finite element space on T^{ct} are given by function values at the four vertices, i.e., points in $\Delta_0(T^{ct})$, and function values at the six edge midpoints in T^{ct}. In this section, we show that these Lagrange DOFs also form a unisolvent set over $\boldsymbol{V}(T)$, and we study the interpolation error of the space $\boldsymbol{V}(T)$ based on these DOFs. First, let us introduce some notation. We denote the set of four vertices and six edge midpoints in $\hat{T}^{ct}$ by $\mathcal{N}_{\hat{T}}$. We then let $\mathcal{N}_T$ and $\mathcal{N}_{\tilde{T}}$ denote the corresponding sets on T^{ct} and $\tilde{T}^{ct}$, respectively, that is, $\mathcal{N}_T$ is the set of points a with $a = F_T(\hat{a})$ and $\mathcal{N}_{\tilde{T}}$ is the set of points $\tilde{a}$ with $\tilde{a} = F_{\tilde{T}}(\hat{a})$ for all $\hat{a} \in \mathcal{N}_{\hat{T}}$. Also, notice that in this case we have, by the definition of the mapping F_T, $a = G_h(\tilde{a})$ for $a \in \mathcal{N}_T$.

Lemma 2.3.3. *A function $\boldsymbol{v} \in \boldsymbol{V}(T)$ is uniquely determined by its values $\boldsymbol{v}(a)$ for all $a \in \mathcal{N}_T$.*

Proof. First, notice that the number of claimed DOFs is 20, which matches with the dimension of $\boldsymbol{V}(T)$. It is then sufficient to show that if $\boldsymbol{v} \in \boldsymbol{V}(T)$ vanishes on the DOFs, then $\boldsymbol{v} \equiv 0$. We write $\boldsymbol{v}(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x})$ for some $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$. Then, we obtain

$$0 = \boldsymbol{v}(a) = A_T(\hat{a})\hat{\boldsymbol{v}}(\hat{a}) \qquad \forall a \in \mathcal{N}_T.$$

Notice that since $A_T(\hat{a})$ is invertible, we have $\hat{\boldsymbol{v}}(\hat{a}) = 0$ for all $\hat{a} \in \mathcal{N}_{\hat{T}}$. Since the set $\mathcal{N}_{\hat{T}}$ forms a unisolvent over $\hat{\boldsymbol{V}}$, we conclude $\hat{\boldsymbol{v}} \equiv 0$, and hence $\boldsymbol{v} \equiv 0$. □

Lemma 2.3.4. *For all $\boldsymbol{v} \in \boldsymbol{V}(T)$, there holds*

$$\|\boldsymbol{v}\|^2_{H^1(T)} \leq C \sum_{a \in \mathcal{N}_T} |\boldsymbol{v}(a)|^2.$$

Proof. We again write $\boldsymbol{v}(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x})$ for some $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$. Then, we use the estimate $\|A_T^{-1}\|_{L^\infty(\hat{T})} \leq Ch_T$ from (2.2.6) together with the equivalence of norms in the finite dimensional setting to obtain

$$\begin{aligned}
\|\hat{\boldsymbol{v}}\|^2_{H^1(\hat{T})} &\leq C \sum_{\hat{a} \in \mathcal{N}_{\hat{T}}} |\hat{\boldsymbol{v}}(\hat{a})|^2 = C \sum_{\hat{a} \in \mathcal{N}_{\hat{T}}} |A_T^{-1}(\hat{a})A_T(\hat{a})\hat{\boldsymbol{v}}(\hat{a})|^2 \\
&\leq Ch_T^2 \sum_{\hat{a} \in \mathcal{N}_{\hat{T}}} |A_T(\hat{a})\hat{\boldsymbol{v}}(\hat{a})|^2 = Ch_T^2 \sum_{a \in \mathcal{N}_T} |\boldsymbol{v}(a)|^2.
\end{aligned}$$

Using the above inequality with Lemma 2.2.4, Lemma 2.2.2 and Lemma 2.3.2, we obtain

$$\|\boldsymbol{v}\|^2_{H^1(T)} \leq C\|A_T\hat{\boldsymbol{v}}\|^2_{H^1(\hat{T})} \leq C\|A_T\|^2_{W^{1,\infty}(\hat{T})}\|\hat{\boldsymbol{v}}\|^2_{H^1(\hat{T})} \leq Ch_T^{-2}\|\hat{\boldsymbol{v}}\|^2_{H^1(\hat{T})} \leq C\sum_{a\in\mathcal{N}_T}|\boldsymbol{v}(a)|^2.$$

$\square$

Lemma 2.3.5. *For a given* $T \in \mathcal{T}_h$*, consider the mapping* $\boldsymbol{I}_T : \boldsymbol{H}^3(T) \to \boldsymbol{V}(T)$ *that is uniquely defined by*

$$(\boldsymbol{I}_T\boldsymbol{u})(a) := \boldsymbol{u}(a) \qquad \forall a \in \mathcal{N}_T.$$

Then, there holds

$$\|\boldsymbol{u} - \boldsymbol{I}_T\boldsymbol{u}\|_{H^m(T)} \leq Ch_T^{3-m}\|\boldsymbol{u}\|_{H^3(T)} \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^3(T), \quad m = 0, 1.$$

Proof. For a given $\boldsymbol{u} \in \boldsymbol{H}^3(T)$, let us set $\boldsymbol{v} := \boldsymbol{I}_T\boldsymbol{u}$ in order to ease the notation. Next, we write $\boldsymbol{v}(x) = (A_T\hat{\boldsymbol{v}})(\hat{x})$, $\boldsymbol{u}(x) = (A_T\hat{\boldsymbol{u}})(\hat{x})$ for some $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{u}} \in \boldsymbol{H}^3(\hat{T})$. By the construction of $\boldsymbol{I}_T\boldsymbol{u}$, we get

$$(A_T\hat{\boldsymbol{v}})(\hat{a}) = (A_T\hat{\boldsymbol{u}})(\hat{a}) \qquad \forall \hat{a} \in \mathcal{N}_{\hat{T}}.$$

Using the invertibility of A_T, we then find $\hat{\boldsymbol{v}}(\hat{a}) = \hat{\boldsymbol{u}}(\hat{a})$ for all $\hat{a} \in \mathcal{N}_{\hat{T}}$. In other words, $\hat{\boldsymbol{v}}$ is the quadratic Lagrange nodal interpolant of $\hat{\boldsymbol{u}}$ with respect to the local triangulation $\hat{T}^{ct}$. It then follows from standard interpolation theory that

$$\|\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}\|_{H^m(\hat{T})} \leq C|\hat{\boldsymbol{u}}|_{H^3(\hat{T})}, \quad m = 0, 1.$$

The above inequality together with Lemmas 2.2.4 and 2.2.2 yields

$$\begin{aligned} |\boldsymbol{u} - \boldsymbol{v}|_{H^m(T)} \leq Ch_T^{1-m}\|A_T(\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}})\|_{H^m(\hat{T})} &\leq Ch_T^{1-m}\|A_T\|_{W^{m,\infty}(\hat{T})}\|\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}\|_{H^m(\hat{T})} \\ &\leq Ch_T^{-m}|\hat{\boldsymbol{u}}|_{H^3(\hat{T})}. \end{aligned}$$

Another use of Lemmas 2.2.2 and 2.2.4 shows

$$|\hat{\boldsymbol{u}}|_{H^3(\hat{T})} = |A_T^{-1} A_T \hat{\boldsymbol{u}}|_{H^3(\hat{T})} \le C\big(\|A_T^{-1}\|_{L^\infty(\hat{T})} |A_T \hat{\boldsymbol{u}}|_{H^3(\hat{T})} + |A_T^{-1}|_{W^{1,\infty}(\hat{T})} |A_T \hat{\boldsymbol{u}}|_{H^2(\hat{T})}\big)$$
$$\le C\big(h_T |A_T \hat{\boldsymbol{u}}|_{H^3(\hat{T})} + h_T^2 |A_T \hat{\boldsymbol{u}}|_{H^2(\hat{T})}\big) \le C h_T^3 \|\boldsymbol{u}\|_{H^3(T)}.$$

Combining the last two inequalities yields the desired result. □

2.3.2 A connection between local finite element spaces

In this section, we introduce an explicit correspondence between the two local spaces $\tilde{\boldsymbol{V}}(\tilde{T})$, $\boldsymbol{V}(T)$. This correspondence is simply based on the DOFs of $\boldsymbol{V}(T)$, which is described in the previous section, and it will be used to prove the global inf-sup stability in the subsequent section.

Definition 2.3.6. Let $\tilde{T} \in \tilde{\mathfrak{T}}_h$ and $T \in \mathfrak{T}_h$ with $T = G_h(\tilde{T})$.

(i) We introduce the operator $\boldsymbol{\Psi}_T : \tilde{\boldsymbol{V}}(\tilde{T}) \to \boldsymbol{V}(T)$ that is given through the DOFs of $\boldsymbol{V}(T)$ as

$$(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}})(a) := \tilde{\boldsymbol{v}}(\tilde{a}) \qquad \forall \tilde{a} \in \mathcal{N}_{\tilde{T}}, \quad \text{where } a = G_h(\tilde{a}).$$

(ii) We also introduce another operator $\Upsilon_T : \tilde{Q}(\tilde{T}) \to Q(T)$ through composition by

$$(\Upsilon_T \tilde{q})(x) := \tilde{q}(F_{\tilde{T}}(\hat{x})).$$

The next theorem further explores the connection between $\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}$ for an arbitrary $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}(\tilde{T})$.

Theorem 2.3.7.

(i) If F_T is affine, i.e., $F_T = F_{\tilde{T}}$, then $(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}})(x) = \tilde{\boldsymbol{v}}(\tilde{x})$.

(ii) *If $e \subset \partial T$ is a straight edge, in which case $e \subset \partial \tilde{T}$, then*

$$(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}) \cdot \boldsymbol{n}|_e = \tilde{\boldsymbol{v}} \cdot \boldsymbol{n}|_e.$$

(iii) *There holds $\|\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|_{H^1(T)} \le C \|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})}$.*

Proof. In order to ease the notation, let us set $\boldsymbol{v} := \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \in \boldsymbol{V}(T)$.

(i) If F_T is affine, then DF_T is constant, in which case A_T is also a constant. Then, by the construction of $\boldsymbol{V}(T)$, we find $\boldsymbol{V}(T) = \tilde{\boldsymbol{V}}(\tilde{T})$, and hence $\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$ by Lemma 2.3.3.

(ii) Suppose that $e \subset \partial T$ is a straight edge with outward unit normal $\boldsymbol{n}$, endpoints a_1 and a_2, and midpoint a_3. Then, by the construction of $\boldsymbol{v}$, we find

$$(\boldsymbol{v} \cdot \boldsymbol{n})(a_1) = (\tilde{\boldsymbol{v}} \cdot \boldsymbol{n})(a_1), \quad (\boldsymbol{v} \cdot \boldsymbol{n})(a_2) = (\tilde{\boldsymbol{v}} \cdot \boldsymbol{n})(a_2), \quad (\boldsymbol{v} \cdot \boldsymbol{n})(a_3) = (\tilde{\boldsymbol{v}} \cdot \boldsymbol{n})(a_3).$$

Recall from Lemma 2.3.3 that $\boldsymbol{v} \cdot \boldsymbol{n}|_e$ and $\tilde{\boldsymbol{v}} \cdot \boldsymbol{n}|_e$ are both quadratic polynomials, and the above equalities show that they coincide at three distinct points on e, which implies $\boldsymbol{v} \cdot \boldsymbol{n}|_e = \tilde{\boldsymbol{v}} \cdot \boldsymbol{n}|_e$.

(iii) Notice that if T is affine, i.e., $T = G_h(\tilde{T}) = \tilde{T}$, then the result trivially follows by (i). Suppose then that $|T \cap \partial \Omega_h| > 0$. Define $\hat{\tilde{\boldsymbol{v}}}(\hat{x}) := \tilde{\boldsymbol{v}}(\tilde{x})$ with $\tilde{x} = F_{\tilde{T}}(\hat{x})$. Using Lemma 2.3.4, construction of $\boldsymbol{v}$, equivalence of norms and a standard scaling argument with Poincaré inequality, we obtain

$$\begin{aligned}
\|\boldsymbol{v}\|_{H^1(T)}^2 \le C \sum_{a \in \mathcal{N}_T} |\boldsymbol{v}(a)|^2 = C \sum_{\tilde{a} \in \mathcal{N}_{\tilde{T}}} |\tilde{\boldsymbol{v}}(\tilde{a})|^2 \le C \sum_{\hat{a} \in \mathcal{N}_{\hat{T}}} |\hat{\tilde{\boldsymbol{v}}}(\hat{a})|^2 \\
\le C \|\hat{\tilde{\boldsymbol{v}}}\|_{H^1(\hat{T})}^2 \le C \left(h_{\tilde{T}}^{-2} \|\tilde{\boldsymbol{v}}\|_{L^2(\tilde{T})}^2 + |\tilde{\boldsymbol{v}}|_{H^1(\tilde{T})}^2 \right) \\
\le C |\tilde{\boldsymbol{v}}|_{H^1(\tilde{T})}^2 \le C \|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})}^2.
\end{aligned}$$

□

2.3.3 Local inf-sup stability

In this section, we prove a local inf-sup stability result of the pair $\boldsymbol{V}_0(T) \times Q_0(T)$. This result will then be used to prove the global inf-sup stability in the next section. As a starting point, we make use of the stability of the corresponding pair $\hat{\boldsymbol{V}}_0 \times \hat{Q}_0$ defined on the reference triangle. The proof of the following lemma can be found in, for instance, [1, 19]. For the sake of completeness, we also provide a proof of this result.

Lemma 2.3.8. *Given an arbitrary $\hat{q} \in \hat{Q}_0$, then there exists $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}_0$ such that $\hat{\nabla} \cdot \hat{\boldsymbol{v}} = \hat{q}$ and $\|\hat{\boldsymbol{v}}\|_{H^1(\hat{T})} \leq C\|\hat{q}\|_{L^2(\hat{T})}$.*

Proof. Notice that $\dim \hat{\boldsymbol{V}} = 20$, and so, due to the boundary conditions, we find $\dim \hat{\boldsymbol{V}}_0 = 20 - 6 \times 2 = 8$. Moreover, $\dim \hat{Q} = 9$ and then $\dim \hat{Q}_0 = 9 - 1 = 8$. Next, we show that if $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}_0$ and $\hat{\nabla} \cdot \hat{\boldsymbol{v}} = 0$, then $\hat{\boldsymbol{v}} = 0$; since $\hat{\boldsymbol{v}} \in H_0^1(\hat{T})$, we write $\hat{\boldsymbol{v}} = \hat{\mu}\hat{\boldsymbol{w}}$ with $\hat{\mu} \in \mathcal{P}_1^c(\hat{T}^{ct}) \cap H_0^1(\hat{T})$ such that $\hat{\mu}(\hat{b}) = 1$, where $\hat{b}$ denotes the barycenter of $\hat{T}$, and $\hat{\boldsymbol{w}} \in \boldsymbol{\mathcal{P}}_1^c(\tilde{T}^{ct})$. Then,

$$0 = \hat{\nabla} \cdot \hat{\boldsymbol{v}} = \hat{\nabla}\hat{\mu} \cdot \hat{\boldsymbol{w}} + \hat{\mu}\left(\hat{\nabla} \cdot \hat{\boldsymbol{w}}\right),$$

which implies that $\hat{\nabla}\hat{\mu} \cdot \hat{\boldsymbol{w}}|_{\partial\hat{T}} = 0$. Since $\hat{\nabla}\hat{\mu}$ is parallel to the outward unit normal vector, this further implies that $\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{n}}|_{\partial\hat{T}} = 0$, and so $\hat{\boldsymbol{w}}$ vanishes at all vertices of $\hat{T}$, and that it's piecewise linear polynomial on $\hat{T}^{ct}$ implies that $\hat{\boldsymbol{w}}|_{\partial\hat{T}} = 0$. Then, $\hat{\boldsymbol{w}} = \hat{\mu}\hat{\boldsymbol{c}}$ where $\hat{\boldsymbol{c}} \in \mathbb{R}^2$. Thus, $\hat{\boldsymbol{v}} = \hat{\mu}^2\hat{\boldsymbol{c}}$ and

$$0 = \hat{\nabla} \cdot \hat{\boldsymbol{v}} = 2\hat{\mu}\hat{\boldsymbol{c}} \cdot \hat{\nabla}\hat{\mu},$$

which implies that $\hat{\boldsymbol{c}} = 0$, and hence $\hat{\boldsymbol{v}} = 0$.

Next, by the rank nullity theorem, we find $\dim \hat{\nabla} \cdot \hat{\boldsymbol{V}}_0 = \dim \hat{\boldsymbol{V}}_0 = \dim \hat{Q}_0$, and since the inclusion $\hat{\nabla} \cdot \hat{\boldsymbol{V}}_0 \subset \hat{Q}_0$ holds, we conclude that $\hat{\nabla} \cdot \hat{\boldsymbol{V}}_0 = \hat{Q}_0$. This, with the equivalence of norms in the finite dimensional setting, yields the existence of $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}_0$ such that

$$\hat{\nabla} \cdot \hat{\boldsymbol{v}} = \hat{q} \text{ and } \|\hat{\boldsymbol{v}}\|_{\boldsymbol{H}^1(\hat{T})} \leq C\|\hat{\nabla} \cdot \hat{\boldsymbol{v}}\|_{L^2(\hat{T})} = \|\hat{q}\|_{L^2(\hat{T})},$$

and this completes the proof. □

The following theorem states the local inf-sup stability of the pair $\boldsymbol{V}_0(T) \times Q_0(T)$.

Theorem 2.3.9. *Given an arbitrary $q \in Q_0(T)$, then there exists $\boldsymbol{v} \in \boldsymbol{V}_0(T)$ such that*

$$(\nabla \cdot \boldsymbol{v})(x) = \frac{h_T^2 q(x)}{\det(DF_T(F_T^{-1}(x)))}, \quad \textit{and} \quad \|\boldsymbol{v}\|_{H^1(T)} \leq C\|q\|_{L^2(T)}.$$

Proof. Let $q \in Q_0(T)$ be arbitrarily given. Then, by the definition of the local spaces, there exists $\hat{q} \in \hat{Q}_0$ such that $q(x) = \hat{q}(\hat{x})$. Notice that $h_T^2 \hat{q}$ also belongs to $\hat{Q}_0$, then by Lemma 2.3.8, there exists $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}_0$ such that $\hat{\nabla} \cdot \hat{\boldsymbol{v}} = h_T^2 \hat{q}$ with $\|\hat{\boldsymbol{v}}\|_{H^1(\hat{T})} \leq C h_T^2 \|\hat{q}\|_{L^2(\hat{T})}$. Define $\boldsymbol{v}(x) := A_T \hat{\boldsymbol{v}}$ so that $\boldsymbol{v} \in \boldsymbol{V}_0(T)$. Then, using (2.2.5), we find

$$(\nabla \cdot \boldsymbol{v})(x) = \frac{(\hat{\nabla} \cdot \hat{\boldsymbol{v}})(\hat{x})}{\det(DF_T(\hat{x}))} = \frac{h_T^2 \hat{q}(\hat{x})}{\det(DF_T(\hat{x}))} = \frac{h_T^2 q(x)}{\det(DF_T(F_T^{-1}(x)))},$$

which proves the first equality in the statement of the theorem. In order to prove the stated inequality, we use Lemma 2.3.2 with a change of variables to find

$$\|\boldsymbol{v}\|_{H^1(T)} \leq C h_T^{-1} \|\hat{\boldsymbol{v}}\|_{H^1(\hat{T})} \leq C h_T \|\hat{q}\|_{L^2(\hat{T})} \leq C\|q\|_{L^2(T)}.$$

□

2.4 Global spaces

Having defined the local spaces, we are now ready to define the global spaces with the help of the mappings $\boldsymbol{\Psi}_T$ and Υ_T given in the Definition 2.3.6. We start with introducing the Scott–Vogelius pair on the affine triangulation $\tilde{\mathfrak{T}}_h$:

$$\tilde{\boldsymbol{V}}^h = \{\tilde{\boldsymbol{v}} \in \boldsymbol{H}_0^1(\tilde{\Omega}_h) : \ \tilde{\boldsymbol{v}}|_{\tilde{T}} \in \tilde{\boldsymbol{V}}(\tilde{T}), \ \forall \tilde{T} \in \tilde{\mathfrak{T}}_h\},$$
$$\tilde{Q}^h = \{\tilde{q} \in L_0^2(\tilde{\Omega}_h) : \ \tilde{q}|_{\tilde{T}} \in \tilde{Q}(\tilde{T}), \ \forall \tilde{T} \in \tilde{\mathfrak{T}}_h\}.$$

In order to define the global spaces $\boldsymbol{V}^h \times Q^h$, with respect to the triangulation $\mathfrak{T}_h$, through the Scott-Vogelius pair $\tilde{\boldsymbol{V}}^h \times \tilde{Q}^h$, we first define the following global mappings $\boldsymbol{\Psi}$ and Υ by patching the local mappings $\boldsymbol{\Psi}_T$ and Υ_T all together, i.e.,

$$\boldsymbol{\Psi}|_T = \boldsymbol{\Psi}_T, \qquad \Upsilon|_T = \Upsilon_T \qquad \forall T \in \mathfrak{T}_h.$$

We then introduce the global spaces by

$$\boldsymbol{V}^h : = \{\boldsymbol{v} : \ \boldsymbol{v} = \boldsymbol{\Psi}\tilde{\boldsymbol{v}}, \ \exists \tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h\}, \qquad Q^h := \{q : \ q = \Upsilon\tilde{q}, \ \exists \tilde{q} \in \tilde{Q}^h\}.$$

Remark 2.4.1. Notice that the functions in $\boldsymbol{V}^h$ can equivalently be described as functions that are locally in $\boldsymbol{V}(T)$ for every $T \in \mathfrak{T}_h$, continuous with respect to the DOFs given in Lemma 2.3.3, and vanish on the boundary of Ω_h.

We also remark that the space $\boldsymbol{V}^h$ is not $\boldsymbol{H}^1$-conforming. In more detail, consider an edge $e \subset T_1 \cap T_2$ such that $T_1, \ T_2 \in \mathfrak{T}_h$ with $|T_1 \cap \partial\Omega_h| > 0$ and $|T_2 \cap \partial\Omega_h| = 0$. Note that e is a straight edge in $\mathfrak{T}_h$. Let $\boldsymbol{v} \in \boldsymbol{V}^h$ be arbitrary and denote its restriction to T_i by $\boldsymbol{v}_i$ where $i = 1, 2$. Then, $\boldsymbol{v}_2|_e$ is a quadratic polynomial whereas $\boldsymbol{v}_1|_e$ is a rational function due to the use of Piola transformation. As a result, continuity on the shared edge e is, in general, only limited to the DOFs as noted earlier. Nevertheless, we show in the next theorem that the normal component of any element $\boldsymbol{v} \in \boldsymbol{V}^h$ is continuous across the interior edges.

Theorem 2.4.2.

(i) *There holds* $\boldsymbol{V}^h \subset \boldsymbol{H}_0(\mathrm{div};\Omega_h) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_h) :\ \nabla\cdot\boldsymbol{v} \in L^2(\Omega_h),\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega_h} = 0\}$.

(ii) *There holds* $q \in Q^h$ *if and only if* $q|_T \circ F_T \in \hat{Q}$ *for all* $T \in \mathfrak{T}_h$, *and*

$$\sum_{T\in\mathfrak{T}_h} 2|\tilde{T}| \int_T \frac{q}{\det(DF_T \circ F_T^{-1})} = 0.$$

Proof.

(i) Let $T_1, T_2 \in \mathfrak{T}_h$ such that $\emptyset \neq \partial T_1 \cap \partial T_2 =: e$, and let $\boldsymbol{n}$ be a unit normal of e. For an arbitrarily given $\boldsymbol{v} = \boldsymbol{\Psi}(\tilde{\boldsymbol{v}})$ for some $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$, we denote its restriction to T_i by $\boldsymbol{v}_i$ with $i = 1, 2$. Similarly, we denote the restriction of an arbitrary $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$ to $\tilde{T}_i$ by $\tilde{\boldsymbol{v}}_i$. Then, by Theorem 2.3.7 and the continuity of $\tilde{\boldsymbol{v}}$, we find

$$\boldsymbol{v}_1 \cdot \boldsymbol{n}|_e = \tilde{\boldsymbol{v}}_1 \cdot \boldsymbol{n}|_e = \tilde{\boldsymbol{v}}_2 \cdot \boldsymbol{n}|_e = \boldsymbol{v}_2 \cdot \boldsymbol{n}|_e,$$

which establishes the continuity of the normal component of $\boldsymbol{v}$ on the shared edges. This with that $\boldsymbol{v}|_{\partial T\cap\partial\Omega_h} = 0$ for all $T \in \mathfrak{T}_h$ by construction ensures that $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega_h)$.

(ii) Suppose that $q \in Q^h$, then there exists a unique $\tilde{q} \in \tilde{Q}^h$ such that $q = \Upsilon\tilde{q}$, i.e., $q|_T(F_T(\hat{x})) = \tilde{q}|_{\tilde{T}}(F_{\tilde{T}}(\hat{x}))$. Using change of variables, we find

$$\begin{aligned} 0 = \int_{\tilde{\Omega}_h} \tilde{q} = \sum_{\tilde{T}\in\tilde{\mathfrak{T}}_h} \int_{\tilde{T}} \tilde{q} = \sum_{\tilde{T}\in\tilde{\mathfrak{T}}_h} 2|\tilde{T}| \int_{\hat{T}} \tilde{q} \circ F_{\tilde{T}} &= \sum_{T\in\mathfrak{T}_h} 2|\tilde{T}| \int_{\hat{T}} q \circ F_T \\ &= \sum_{T\in\mathfrak{T}_h} 2|\tilde{T}| \int_T \frac{q}{\det(DF_T \circ F_T^{-1})}. \end{aligned}$$

With the above chain of equalities, the other direction of the statement is trivial.

□

2.4.1 Global inf-sup stability

In this section, we prove the inf-sup stability of the pair $\boldsymbol{V}^h \times Q^h$ by using the local inf-sup stability result given in Theorem 2.3.9 along with Stenberg's macro element technique. In order to be able to apply this technique, we first need to have an analogous result of quadratic-constant stability on Ω_h. For this purpose, we introduce the following spaces defined on $\tilde{\mathcal{T}}_h$ and $\mathcal{T}_h$, respectively:

$$\begin{aligned}
\tilde{Y}^h &:= \{q \in L_0^2(\tilde{\Omega}_h) : \ \tilde{q}|_T \in \mathcal{P}_0(\tilde{T}) \ \forall \tilde{T} \in \tilde{\mathcal{T}}_h\} \subset \tilde{Q}^h, \\
Y^h &:= \{q : \ q = \Upsilon(\tilde{q}), \ \exists \tilde{q} \in \tilde{Y}^h\} \subset Q^h.
\end{aligned}$$

The following lemma establishes the stability of the pair $\boldsymbol{V}^h \times Y^h$, which can be regarded as an intermediate step for the stability of the pair $\boldsymbol{V}^h \times Q^h$.

Lemma 2.4.3. *There holds*

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \ge \gamma_1 \|q\|_{L^2(\Omega_h)} \qquad \forall q \in Y^h,$$

where the gradient of $\boldsymbol{v}$ is understood piecewise with respect to $\mathcal{T}_h$. Here, $\gamma_1 > 0$ denotes a constant that is independent of any mesh parameter.

Proof. For a given $q \in Y^h$, we let $\tilde{q} \in \tilde{Y}^h$ be the piecewise constant function with respect to $\mathcal{T}_h$ such that $q = \Upsilon \tilde{q}$. Since q and $\tilde{q}$ are both piecewise constant, we then see that $q|_{\Omega_h \cap \tilde{\Omega}_h} = \tilde{q}|_{\Omega_h \cap \tilde{\Omega}_h}$. That q and $\tilde{q}$ are piecewise constant also implies

$$\int_T q = \frac{|T|}{|\tilde{T}|} \int_{\tilde{T}} \tilde{q}, \quad \text{and} \quad \|q\|_{L^2(T)}^2 = \frac{|T|}{|\tilde{T}|} \|\tilde{q}\|_{L^2(\tilde{T})}^2 \qquad \forall \tilde{T} \in \tilde{\mathcal{T}}_h,$$

with $T = G_h(\tilde{T})$. Thus, by the properties of G, we find $\|q\|_{L^2(\Omega_h)} \le C \|\tilde{q}\|_{L^2(\tilde{\Omega}_h)}$.

Next, we let $\tilde{\boldsymbol{w}} \in \boldsymbol{H}_0^1(\tilde{\Omega}_h)$ satisfy $\tilde{\nabla} \cdot \tilde{\boldsymbol{w}} = \tilde{q}$ with $\|\tilde{\nabla} \tilde{\boldsymbol{w}}\|_{L^2(\tilde{\Omega}_h)} \le C \|\tilde{q}\|_{L^2(\tilde{\Omega}_h)}$. The results in [6, Theorem 4.4] with the properties of the mapping G ensure that

$C > 0$ is independent of any mesh parameter. Furthermore, the stability proof of the piecewise quadratic-constant pair [4, 11] shows the existence of a $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$ such that

$$\int_{\tilde{e}} \tilde{\boldsymbol{v}} = \int_{\tilde{e}} \tilde{\boldsymbol{w}}, \quad \text{and} \quad \|\tilde{\nabla}\tilde{\boldsymbol{v}}\|_{L^2(\tilde{\Omega}_h)} \leq C\|\tilde{\nabla}\tilde{\boldsymbol{w}}\|_{L^2(\tilde{\Omega}_h)}.$$

Set $\boldsymbol{v} := \boldsymbol{\Psi}\tilde{\boldsymbol{v}}$ and recall, by Theorem 2.3.7, that we have $\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)} \leq C\|\tilde{\nabla}\tilde{\boldsymbol{v}}\|_{L^2(\tilde{\Omega}_h)}$. Another use of Theorem 2.3.7 with the divergence theorem shows that, on each $T \in \mathcal{T}_h$,

$$\int_T \nabla \cdot \boldsymbol{v} = \int_{\partial T} (\boldsymbol{v} \cdot \boldsymbol{n}) = \int_{\partial \tilde{T}} (\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{n}}) = \int_{\partial \tilde{T}} (\tilde{\boldsymbol{w}} \cdot \tilde{\boldsymbol{n}}) = \int_{\tilde{T}} \tilde{\nabla} \cdot \tilde{\boldsymbol{w}} = \int_{\tilde{T}} \tilde{q} = \frac{|\tilde{T}|}{|T|} \int_T q.$$

Since q is constant on T, we further find

$$\int_T (\nabla \cdot \boldsymbol{v}) q = \frac{|\tilde{T}|}{|T|} \int_T q^2 = \|\tilde{q}\|^2_{L^2(\tilde{T})}.$$

By summing both sides of the above equality over $T \in \mathcal{T}_h$, we obtain

$$\begin{aligned}\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q = \|\tilde{q}\|^2_{L^2(\tilde{\Omega}_h)} &\geq C\|\tilde{q}\|_{L^2(\tilde{\Omega}_h)}\|\tilde{\nabla}\tilde{\boldsymbol{w}}\|_{L^2(\tilde{\Omega}_h)} \geq C\|\tilde{q}\|_{L^2(\tilde{\Omega}_h)}\|\tilde{\nabla}\tilde{\boldsymbol{v}}\|_{L^2(\tilde{\Omega}_h)} \\ &\geq C\|\tilde{q}\|_{L^2(\tilde{\Omega}_h)}\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)} \geq C\|q\|_{L^2(\Omega_h)}\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}.\end{aligned}$$

Finally, dividing the above expression by $\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}$ yields the desired result. □

Theorem 2.4.4. *There holds*

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \geq C\|q\|_{L^2(\Omega_h)} \qquad \forall q \in Q^h,$$

where the gradient of $\boldsymbol{v}$ is understood piecewise with respect to $\mathcal{T}_h$.

Proof. Let $q \in Q^h$ be arbitrary. For each $T \in \mathcal{T}_h$, we define $\bar{q}_T \in \mathcal{P}_0(T)$ such that

$$\int_T \frac{(q - \bar{q}_T)}{\det(DF_T)} = 0,$$

and set $\bar{q}$ such that $\bar{q}|_T := \bar{q}_T$ for all $T \in \mathfrak{T}_h$. Notice then that this construction implies $(q-\bar{q})|_T \in Q_0(T)$ for all $T \in \mathfrak{T}_h$, and $\bar{q} \in Y^h$. As a result, by Theorem 2.3.9, for each $T \in \mathfrak{T}_h$, there exists $\boldsymbol{v}_{1,T} \in \boldsymbol{V}_0(T)$ such that

$$\nabla \cdot \boldsymbol{v}_{1,T} = \frac{h_T^2(q-\bar{q})}{\det(DF_T)}, \qquad \|\nabla \boldsymbol{v}_{1,T}\| \le C\|q-\bar{q}\|_{L^2(T)}.$$

Next, we set $\boldsymbol{v}_1$ such that $\boldsymbol{v}_1|_T := \boldsymbol{v}_{1,T}$ for all $T \in \mathfrak{T}_h$. Since $\boldsymbol{v}_{1,T}|_{\partial T} = 0$, this construction ensures that $\boldsymbol{v}_1 \in \boldsymbol{V}^h$. Using $\|\nabla \boldsymbol{v}_1\|_{L^2(\Omega_h)} \le C\|q-\bar{q}\|_{L^2(\Omega_h)}$ with (2.2.2), we find

$$\begin{aligned}
\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}_1)(q-\bar{q}) &= \sum_{T\in\mathfrak{T}_h} \int_T (\nabla \cdot \boldsymbol{v}_1)(q-\bar{q}) \\
&= \sum_{T\in\mathfrak{T}_h} \int_T \frac{h_T^2|q-\bar{q}|^2}{\det(DF_T)} \ge C \sum_{T\in\mathfrak{T}_h} \int_T |q-\bar{q}|^2 \\
&= C\|q-\bar{q}\|^2_{L^2(\Omega_h)} \\
&\ge C\|q-\bar{q}\|_{L^2(\Omega_h)}\|\nabla \boldsymbol{v}_1\|_{L^2(\Omega_h)}.
\end{aligned}$$

Recall that $\boldsymbol{v}_1|_{\partial T} = \boldsymbol{v}_{1,T}|_{\partial T} = 0$, and that $\bar{q}$ is constant on each T. The divergence theorem then shows,

$$\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}_1)\bar{q} = \sum_{T\in\mathfrak{T}_h} \int_T (\nabla \cdot \boldsymbol{v}_1)\bar{q} = \sum_{T\in\mathfrak{T}_h} \int_{\partial T} (\boldsymbol{v}_1 \cdot \boldsymbol{n})\bar{q} = 0.$$

Combining this last equality with the above inequality, we find the existence of a constant γ_0 independent of h such that

$$\gamma_0\|q-\bar{q}\|_{L^2(\Omega_h)} \le \sup_{\boldsymbol{v}\in\boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\Omega_h}(\nabla \cdot \boldsymbol{v})q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}}.$$

Next, using the stability of the $\boldsymbol{V}^h \times Y^h$ pair given in Lemma 2.4.3 with the triangle inequality, we find:

$$\begin{aligned}
\gamma_1 \|\bar{q}\|_{L^2(\Omega_h)} &\leq \sup_{\boldsymbol{v}\in \boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\Omega_h} (\nabla\cdot \boldsymbol{v})\bar{q}}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \\
&\leq \sup_{\boldsymbol{v}\in \boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\Omega_h} (\nabla\cdot \boldsymbol{v})q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} + \|q-\bar{q}\|_{L^2(\Omega_h)} \leq (1+\gamma_0^{-1}) \sup_{\boldsymbol{v}\in \boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\Omega_h} (\nabla\cdot \boldsymbol{v})q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}}.
\end{aligned}$$

Hence,

$$\|q\|_{L^2(\Omega_h)} \leq \|q-\bar{q}\|_{L^2(\Omega_h)} + \|\bar{q}\|_{L^2(\Omega_h)} \leq (\gamma_0^{-1} + \gamma_1^{-1}(1+\gamma_0^{-1})) \sup_{\boldsymbol{v}\in \boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\Omega_h} (\nabla\cdot \boldsymbol{v})q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}}.$$

□

2.4.2 Weak continuity of functions in $\boldsymbol{V}^h$

We already showed in Theorem 2.4.2 that the discrete velocity space, $\boldsymbol{V}^h$, is $\boldsymbol{H}_0(\mathrm{div};\Omega_h)$-conforming, and we noted that it is, however, lacking $\boldsymbol{H}^1(\Omega_h)$-conformity. Notice that the mentioned discontinuity only occurs on a shared edge of a curved triangle, i.e., a triangle $T \in \mathfrak{T}_h$ such that $|T \cap \partial\Omega_h| > 0$. However, by the construction of the space $\boldsymbol{V}^h$, we can still guarantee that the functions in $\boldsymbol{V}^h$ when restricted on any straight edge are single valued at three points (DOFs). In the next lemma, we exploit this property to show that the space $\boldsymbol{V}^h$ can be regarded as an approximate $\boldsymbol{H}_0^1(\Omega_h)$ function space.

Lemma 2.4.5. *There exists an operator $\boldsymbol{E}_h : \boldsymbol{V}^h \to \boldsymbol{H}_0^1(\Omega_h)$ such that for all $\boldsymbol{v} \in \boldsymbol{V}^h$,*

$$\|\boldsymbol{v} - \boldsymbol{E}_h\boldsymbol{v}\|_{L^2(T)} + h_T\|\nabla(\boldsymbol{v} - \boldsymbol{E}_h\boldsymbol{v})\|_{L^2(T)} \leq C h_T^2 \|\nabla \boldsymbol{v}\|_{L^2(T)} \qquad \forall T \in \mathfrak{T}_h. \tag{2.4.1}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^h$, then, by construction, there exists $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$ such that $\boldsymbol{v} = \boldsymbol{\Psi}\tilde{\boldsymbol{v}}$ and

$$\boldsymbol{v}|_T(a) = \tilde{\boldsymbol{v}}|_{\tilde{T}}(\tilde{a}) \qquad \forall a \in \mathcal{N}_T, \quad \forall T \in \mathcal{T}_h,$$

with $T = G_h(\tilde{T})$. We now define another function, $\boldsymbol{E}_h\boldsymbol{v}$, as follows:

$$\boldsymbol{E}_h\boldsymbol{v}|_T := (\tilde{\boldsymbol{v}} \circ F_{\tilde{T}} \circ F_T^{-1})|_T \qquad \forall T \in \mathcal{T}_h.$$

Notice that $\boldsymbol{E}_h\boldsymbol{v}$ is the function in the standard isoparametric quadratic Lagrange finite element space associated with $\tilde{\boldsymbol{v}}$. Using a chain of composition of continuous functions in the definition of $\boldsymbol{E}_h\boldsymbol{v}$ ensures that we have $\boldsymbol{E}_h\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega_h)$. Moreover, as F_T^{-1} is affine when restricted on straight edges, there holds $\tilde{\boldsymbol{v}} = \boldsymbol{E}_h\boldsymbol{v}$ on straight edges. As a result, we then have

$$\boldsymbol{E}_h\boldsymbol{v}|_T(a) = \boldsymbol{v}|_T(a) \qquad \forall a \in \mathcal{N}_T, \quad \forall T \in \mathcal{T}_h.$$

Next, we estimate the difference $\boldsymbol{v} - \boldsymbol{E}_h\boldsymbol{v}$ on an arbitrarily given $T \in \mathcal{T}_h$. Note that if $T \in \mathcal{T}_h$ is an affine triangle, then due to the properties of F_T, we have $\boldsymbol{v} = \boldsymbol{E}_h\boldsymbol{v}$ since, in this case, both functions are piecewise quadratic polynomials with matching DOFs. Hence, the desired estimate trivially holds in this situation. Let us then assume that $T \in \mathcal{T}_h$ has a curved boundary. Then, by construction, we have $\boldsymbol{v}|_{\partial T \cap \partial\Omega_h} = 0$. We again write $\boldsymbol{v}|_T(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x})$ with $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$, and we also define $\hat{\boldsymbol{w}} \in \hat{\boldsymbol{V}}$ such that $\hat{\boldsymbol{w}}(\hat{x}) := \boldsymbol{E}_h\boldsymbol{v}|_T(x)$ with $x = F_T(\hat{x})$. Then, there holds

$$A_T(\hat{a})\hat{\boldsymbol{v}}(\hat{a}) = \hat{\boldsymbol{w}}(\hat{a}) \qquad \forall \hat{a} \in \mathcal{N}_{\hat{T}}.$$

In other words, $\hat{\boldsymbol{w}}$ is the piecewise quadratic Lagrange interpolant of $A_T\hat{\boldsymbol{v}}$ on $\hat{T}^{ct}$. It then follows from the Bramble–Hilbert lemma that

$$\|A_T\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}\|_{H^m(\hat{K})} \le C|A_T\hat{\boldsymbol{v}}|_{H^3(\hat{K})} \qquad \forall \hat{K} \in \hat{T}^{ct}, \quad m = 0, 1.$$

Using the fact that $\hat{\boldsymbol{v}}|_{\hat{K}}$ is a quadratic polynomial along with Lemma 2.2.2, the product rule, and equivalence of norms, we obtain

$$\begin{aligned}
|A_T\hat{\boldsymbol{v}}|_{H^3(\hat{K})} &\le C\big(|A|_{W^{3,\infty}(\hat{K})}\|\hat{\boldsymbol{v}}\|_{L^2(\hat{K})} + |A|_{W^{2,\infty}(\hat{K})}|\hat{\boldsymbol{v}}|_{H^1(\hat{K})} + |A|_{W^{1,\infty}(\hat{K})}|\hat{\boldsymbol{v}}|_{H^2(\hat{K})}\big) \\
&\le C\|\hat{\boldsymbol{v}}\|_{H^2(\hat{K})} \le C\|\hat{\boldsymbol{v}}\|_{L^2(\hat{K})}.
\end{aligned}$$

Combining this last inequality with the previous inequality and using the estimate $\|A_T^{-1}\|_{L^\infty(\hat{T})} \le Ch_T$ from Lemma 2.2.2, we find

$$\|A_T\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}\|_{H^m(\hat{T})} \le Ch_T\|A_T\hat{\boldsymbol{v}}\|_{L^2(\hat{T})}, \qquad m = 0, 1.$$

Finally, the previous inequality together with Lemma 2.2.4 and the Poincaré inequality yields

$$\begin{aligned}
\|\boldsymbol{v} - \boldsymbol{E}_h\boldsymbol{v}\|_{H^m(T)} &\le Ch_T^{1-m}\|A_T\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}\|_{H^m(\hat{T})} \\
&\le Ch_T^{2-m}\|A_T\hat{\boldsymbol{v}}\|_{L^2(\hat{T})} \\
&\le Ch_T^{1-m}\|\boldsymbol{v}\|_{L^2(T)} \le Ch_T^{2-m}\|\nabla\boldsymbol{v}\|_{L^2(T)},
\end{aligned}$$

and this completes the proof of the Lemma. $\square$

Let us recall that $\mathcal{E}_h^I$ denotes the set of internal edges of $\mathcal{T}_h$. For a given $e = \mathcal{E}_h^I$, we write $e = \partial T_+ \cap \partial T_-$ for some $T_\pm \in \mathcal{T}_h$. Accordingly, we denote the outward unit normal of $\partial T_\pm$ restricted to e by $\boldsymbol{n}_\pm$, and we denote the restriction of a piecewise smooth function $\boldsymbol{v}$ to $T_\pm$ by $\boldsymbol{v}_\pm$. Then, we introduce the jump operator as

$$[\boldsymbol{v}]|_e := \boldsymbol{v}_+ \otimes \boldsymbol{n}_+ + \boldsymbol{v}_- \otimes \boldsymbol{n}_-,$$

with $(\boldsymbol{a} \otimes \boldsymbol{b})_{i,j} := a_i b_j$.

The next lemma estimates the bound of $[\boldsymbol{v}]|_e$ for a given $\boldsymbol{v} \in \boldsymbol{V}^h$ with a straight edge e, and it, together with Lemma 2.4.5, is heavily used for the convergence analysis in the subsequent section.

Lemma 2.4.6. *Given $e \in \mathcal{E}_h^I$ with $e = \partial T_+ \cap \partial T_-$ for some $T_\pm \in \mathcal{T}_h$. Then, for all $\boldsymbol{v} \in \boldsymbol{V}^h$, there holds*

$$\Big| \int_e [\boldsymbol{v}] \Big| \le C h_T^3 (\|\nabla \boldsymbol{v}\|_{L^2(T_+)} + \|\nabla \boldsymbol{v}\|_{L^2(T_-)}), \tag{2.4.2}$$

where $h_T = \max\{h_{T_+}, h_{T_-}\}$.

Proof. Notice that if both T_+ and T_- are affine, then, by full continuity on the shared edge e, we find $[\boldsymbol{v}]|_e = 0$, in which case the desired estimate trivially holds. Let us then assume, without loss of generality, that T_+ has a curved edge. Let a_1, a_2 denote the endpoints of e, and let a_3 denote the midpoint of e. Note that in this case, one of the endpoints a_1, a_2 lie on $\partial\Omega_h$. As noted earlier, the construction of the space $\boldsymbol{V}^h$, in particular, the definition of $\boldsymbol{\Psi}$, ensures that $[\boldsymbol{v}]|_e(a_i) = 0$, $i = 1, 2, 3$. Then, by the error of Simpson's rule, we find

$$\Big| \int_e [\boldsymbol{v}] \Big| \le C|e|^5 \big|[\boldsymbol{v}]\big|_{W^{4,\infty}(e)} \le C h_T^5 (|\boldsymbol{v}|_{W^{4,\infty}(K_+)} + |\boldsymbol{v}|_{W^{4,\infty}(K_-)}), \tag{2.4.3}$$

where $K_\pm \in T_\pm^{ct}$ satisfy $\partial K_+ \cap \partial K_- = e$. By writing $\boldsymbol{v}|_{K_\pm}(x) = (A_{T_\pm}\hat{\boldsymbol{v}}_\pm)|_{\hat{K}_\pm}(\hat{x})$ with $\hat{\boldsymbol{v}}_\pm \in \hat{\boldsymbol{V}}$, $\hat{K}_\pm = F_{T_\pm}^{-1}(K_\pm)$, and applying Lemmas 2.2.4 and 2.2.2 with the fact that $\boldsymbol{v}_\pm$ is a quadratic polynomial, we obtain

$$\begin{aligned}
|\boldsymbol{v}|_{W^{4,\infty}(K_\pm)} &\le C h_{T_\pm}^{-4} \sum_{r=0}^{4} h_{T_\pm}^{2(4-r)} |A_{T_\pm}\hat{\boldsymbol{v}}_\pm|_{W^{r,\infty}(\hat{K}_\pm)} \\
&\le C h_{T_\pm}^{4} \sum_{r=0}^{4} h_{T_\pm}^{-2r} \sum_{j=0}^{r} |A_{T_\pm}|_{W^{r-j,\infty}(\hat{T}_\pm)} |\hat{\boldsymbol{v}}_\pm|_{W^{j,\infty}(\hat{K}_\pm)} \\
&\le C h_{T_\pm}^{4} \sum_{r=0}^{4} h_{T_\pm}^{-2r} \sum_{j=0}^{2} h_{T_\pm}^{r-j-1} |\hat{\boldsymbol{v}}_\pm|_{W^{j,\infty}(\hat{K}_\pm)} \\
&\le C \sum_{j=0}^{2} h_{T_\pm}^{-j-1} |\hat{\boldsymbol{v}}_\pm|_{W^{j,\infty}(\hat{K}_\pm)} \le C h_{T_\pm}^{-3} \|\hat{\boldsymbol{v}}_\pm\|_{L^2(\hat{K}_\pm)},
\end{aligned}$$

where we again use equivalence of norms in the last inequality. Another use of the estimate $\|A_{T_\pm}^{-1}\|_{L^\infty(\hat{T})} \le C h_{T_\pm}$ with Lemma 2.2.4 yields

$$|\boldsymbol{v}|_{W^{4,\infty}(K_\pm)} \le C h_{T_\pm}^{-2} \|A_{T_\pm}\hat{\boldsymbol{v}}_\pm\|_{L^2(\hat{T})} \le C h_{T_\pm}^{-3} \|\boldsymbol{v}\|_{L^2(T_\pm)}.$$

We now combine this estimate with (2.4.3) and apply the Poincaré inequality (on T_+) to get

$$\Big| \int_e [\boldsymbol{v}] \Big| \le C h_T^2 \big(\|\boldsymbol{v}\|_{L^2(T_-)} + h_T \|\nabla \boldsymbol{v}\|_{L^2(T_+)} \big). \tag{2.4.4}$$

As the next step, we show $\|\boldsymbol{v}\|_{L^2(T_-)} \le C h_T \|\nabla \boldsymbol{v}\|_{L^2(T_-)}$. For this purpose, we set $\boldsymbol{w} := \boldsymbol{E}_h \boldsymbol{v}$, where $\boldsymbol{E}_h \boldsymbol{v}$ is given in Lemma 2.4.5. Using the triangle inequality with Lemma 2.4.5 , we first find

$$\|\boldsymbol{v}\|_{L^2(T_-)} \le \|\boldsymbol{v} - \boldsymbol{w}\|_{L^2(T_-)} + \|\boldsymbol{w}\|_{L^2(T_-)} \le \|\boldsymbol{w}\|_{L^2(T_-)} + C h_{T_-}^2 \|\nabla \boldsymbol{v}\|_{L^2(T_-)}.$$

Now, let $\hat{\boldsymbol{w}} \in \hat{\boldsymbol{V}}$ be such that $\hat{\boldsymbol{w}}(\hat{x}) = \boldsymbol{w}(x)$ with $x = F_{T_-}(\hat{x})$. Notice that $\boldsymbol{w}$ vanishes on $\partial\Omega_h$, so in particular $\boldsymbol{w}$ vanishes on at least one vertex of T_-. This then implies that

$$\hat{\boldsymbol{w}} \to \|\hat{\nabla}\boldsymbol{w}\|_{L^2(\hat{T})}$$

is a norm. Therefore, using Lemma 2.2.4 and equivalence of norms, we find

$$\|\boldsymbol{w}\|_{L^2(T_-)} \le Ch_T\|\hat{\boldsymbol{w}}\|_{L^2(\hat{T})} \le Ch_T\|\hat{\nabla}\hat{\boldsymbol{w}}\|_{L^2(\hat{T})} \le Ch_T\|\nabla\boldsymbol{w}\|_{L^2(T_-)}.$$

Thus, we have

$$\|\boldsymbol{v}\|_{L^2(T_-)} \le C\Big(h_T\|\nabla\boldsymbol{w}\|_{L^2(T_-)} + h_T^2\|\nabla\boldsymbol{v}\|_{L^2(T_-)}\Big) \le Ch_T\|\nabla\boldsymbol{v}\|_{L^2(T_-)}.$$

Finally, combining this estimate with (2.4.4) yields the desired estimate (2.4.2). □

2.5 Finite element method and convergence analysis

In this section, we introduce our finite element method and show that the introduced method yields exactly divergence-free velocity approximation. We also study the error estimates of the method and prove the optimal order of convergence for both velocity and pressure approximations.

2.5.1 A divergence-free method

Recall that, for a given function $\boldsymbol{f}$, the Stokes problem seeks for the solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega.$$

Assume that $\partial\Omega$ and $\boldsymbol{f}$ are sufficiently smooth so that $(\boldsymbol{u}, p) \in \boldsymbol{H}^3(\Omega) \times H^2(\Omega)$, and can be extended to $\mathbb{R}^2$ in such a way that $(\boldsymbol{u}, p) \in \boldsymbol{H}^3(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ with $\nabla \cdot \boldsymbol{u} = 0$ and $\|\boldsymbol{u}\|_{H^3(\mathbb{R}^2)} \le C\|\boldsymbol{u}\|_{H^3(\Omega)}$, $\|p\|_{H^2(\mathbb{R}^2)} \le C\|p\|_{H^2(\Omega)}$ (cf. [42]). Then, we accordingly extend $\boldsymbol{f}$ by

$$\boldsymbol{f} = -\nu\Delta\boldsymbol{u} + \nabla p,$$

which ensures that $\boldsymbol{f} \in \boldsymbol{H}^1(\mathbb{R}^2)$.

Let $\boldsymbol{f}_h \in \boldsymbol{L}^2(\Omega_h)$ denote a computable approximation of $\boldsymbol{f}|_\Omega$. For instance, one can consider $\boldsymbol{f}_h$ to be the (global) quadratic Lagrange nodal interpolant of $\boldsymbol{f}$. Our finite element method seeks for $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}^h \times Q^h$ such that

$$\int_{\Omega_h} \nu\nabla\boldsymbol{u}_h : \nabla\boldsymbol{v} - \int_{\Omega_h} (\nabla\cdot\boldsymbol{v})p_h = \int_{\Omega_h} \boldsymbol{f}_h \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}^h, \tag{2.5.1a}$$

$$\int_{\Omega_h} (\nabla\cdot\boldsymbol{u}_h)q = 0 \qquad \forall q \in Q^h, \tag{2.5.1b}$$

where the gradient is understood piecewise with respect to the triangulation. Note that the inf-sup stability proven in Lemma 2.4.4 together with the standard theory of mixed finite element methods ensures that the problem (2.5.1) is well-posed.

Theorem 2.5.1. *There exists a unique solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}^h \times Q^h$ *satisfying* (2.5.1).

Even though the inclusion $\nabla \cdot \boldsymbol{V}^h \subset Q^h$ does not hold, the next lemma shows that the finite element method (2.5.1) still yields exactly divergence–free velocity approximations.

Lemma 2.5.2. *If* $\boldsymbol{u}_h \in \boldsymbol{V}^h$ *satisfies* (2.5.1b), *then* $\nabla \cdot \boldsymbol{u}_h \equiv 0$ *in* Ω_h.

Proof. For each $T \in \mathcal{T}_h$, we write $\boldsymbol{u}_h|_T = A_T\hat{\boldsymbol{u}}_T$, for some $\hat{\boldsymbol{u}}_T \in \hat{\boldsymbol{V}}$. We then define q to be the following piecewise function:

$$q|_T(x) := \frac{1}{2|\tilde{T}|}(\hat{\nabla}\cdot\hat{\boldsymbol{u}}_T)(\hat{x}), \qquad x = F_T(\hat{x}), \qquad T = G_h(\tilde{T}),$$

for all $T \in \mathcal{T}_h$. Notice, by the property of the Piola transform given in equation (2.2.5), this construction yields

$$q|_T = \frac{\det(DF_T \circ F_T^{-1})}{2|\tilde{T}|}(\nabla \cdot \boldsymbol{u}_h|_T) \qquad \forall T \in \mathcal{T}_h,$$

and so, by the divergence theorem, we find

$$\sum_{T \in \mathcal{T}_h} 2|\tilde{T}| \int_T \frac{q}{\det(DF_T \circ F_T^{-1})} = \sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot \boldsymbol{u}_h = \int_{\partial \Omega_h} \boldsymbol{u}_h \cdot \boldsymbol{n} = 0,$$

which implies, by Theorem 2.4.2, that $q \in Q^h$. Next, we use this constructed $q \in Q^h$ as a test function in (2.5.1b) with (2.2.5) to find

$$\begin{aligned} 0 = \int_{\Omega_h} (\nabla \cdot \boldsymbol{u}_h) q = \sum_{T \in \mathcal{T}_h} \int_T (\nabla \cdot \boldsymbol{u}_h) q &= \sum_{T \in \mathcal{T}_h} \frac{1}{2|\tilde{T}|} \int_{\hat{T}} \frac{\hat{\nabla} \cdot \hat{\boldsymbol{u}}_T}{\det(DF_T)} (\hat{\nabla} \cdot \hat{\boldsymbol{u}}_T) \det(DF_T) \\ &= \sum_{T \in \mathcal{T}_h} \frac{1}{2|\tilde{T}|} \int_{\hat{T}} |\hat{\nabla} \cdot \hat{\boldsymbol{u}}_T|^2. \end{aligned}$$

Hence, $\hat{\nabla} \cdot \hat{\boldsymbol{u}}_T = 0$ for all $T \in \mathcal{T}_h$, and as a result $\nabla \cdot \boldsymbol{u}_h = 0$ in Ω_h. $\square$

2.5.2 Convergence analysis

We start with defining the following subspace of divergence-free functions:

$$\boldsymbol{X}^h := \{\boldsymbol{v} \in \boldsymbol{V}^h : \ \nabla \cdot \boldsymbol{v} = 0\} \not\subset \boldsymbol{X} := \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \ \nabla \cdot \boldsymbol{v} = 0\}.$$

Then, by Lemma 2.5.2, we notice that the discrete velocity solution satisfying (2.5.1) is uniquely determined by the following problem: Find $\boldsymbol{u}_h \in \boldsymbol{X}^h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) := \int_{\Omega_h} \nu \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v} = \int_{\Omega_h} \boldsymbol{f}_h \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{X}^h.$$

Using second Strang lemma (e.g., [38]) together with Theorem 2.4.4 and Lemma 2.3.5, we obtain

$$
\begin{aligned}
\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} &\le \inf_{\boldsymbol{w}\in\boldsymbol{X}^h} \nu\|\nabla(\boldsymbol{u}-\boldsymbol{w})\|_{L^2(\Omega_h)} + \sup_{\boldsymbol{v}\in\boldsymbol{X}^h\backslash\{0\}} \frac{a_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}} \qquad (2.5.2)\\
&\le C\inf_{\boldsymbol{w}\in\boldsymbol{V}^h} \nu\|\nabla(\boldsymbol{u}-\boldsymbol{w})\|_{L^2(\Omega_h)} + \sup_{\boldsymbol{v}\in\boldsymbol{X}^h\backslash\{0\}} \frac{a_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}}\\
&\le Ch^2\nu\|\boldsymbol{u}\|_{H^3(\Omega)} + \sup_{\boldsymbol{v}\in\boldsymbol{X}^h\backslash\{0\}} \frac{a_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}}.
\end{aligned}
$$

Next, we rewrite $\boldsymbol{f}$ as $-\nu\Delta\boldsymbol{u}+\nabla p$ and use the fact that $\boldsymbol{X}^h \subset \boldsymbol{H}_0(\mathrm{div};\Omega)$ to find

$$
\begin{aligned}
a_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}) &= \int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v} - a_h(\boldsymbol{u},\boldsymbol{v}) + \int_{\Omega_h}(\boldsymbol{f}_h-\boldsymbol{f})\cdot\boldsymbol{v}\\
&= -\int_{\Omega_h}\nu\Delta\boldsymbol{u}\cdot\boldsymbol{v} + \int_{\Omega_h}\nabla p\cdot\boldsymbol{v} - a_h(\boldsymbol{u},\boldsymbol{v}) + \int_{\Omega_h}(\boldsymbol{f}_h-\boldsymbol{f})\cdot\boldsymbol{v}\\
&= -\int_{\Omega_h}\nu\Delta\boldsymbol{u}\cdot\boldsymbol{v} - a_h(\boldsymbol{u},\boldsymbol{v}) + \int_{\Omega_h}(\boldsymbol{f}_h-\boldsymbol{f})\cdot\boldsymbol{v} \qquad \forall\boldsymbol{v}\in\boldsymbol{X}^h.
\end{aligned}
$$

We then integrate by parts and use that $\boldsymbol{v}$ is zero on $\partial\Omega_h$ to conclude

$$
-\int_{\Omega_h}\nu\Delta\boldsymbol{u}\cdot\boldsymbol{v} - a_h(\boldsymbol{u},\boldsymbol{v}) = \nu\sum_{e\in\mathcal{E}_h^I}\int_e \nabla\boldsymbol{u} : [\boldsymbol{v}].
$$

Remember that $\mathcal{E}_h^{I,\partial}$ denotes the set of edges in $\mathcal{E}_h^I$ that have one endpoint on $\partial\Omega_h$. Then, by the construction of $\boldsymbol{V}^h$ and the properties of the mapping F_T, we see that $[\boldsymbol{v}]|_e = 0$ for all $e\in\mathcal{E}_h^I\backslash\mathcal{E}_h^{I,\partial}$, i.e., for all e that can be written as a non-empty intersection of two affine triangles in $\mathcal{T}_h$. Thus, we rewrite the above equality as

$$
a_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}) = \nu\sum_{e\in\mathcal{E}_h^{I,\partial}}\int_e\nabla\boldsymbol{u}:[\boldsymbol{v}] + \int_{\Omega_h}(\boldsymbol{f}_h-\boldsymbol{f})\cdot\boldsymbol{v}. \qquad (2.5.3)
$$

The next lemma bounds the first term on the right hand side of the above equality.

Lemma 2.5.3. *There holds*

$$\nu \sum_{e\in\mathcal{E}_h^{I,\partial}} \int_e \nabla \boldsymbol{u} : [\boldsymbol{v}] \le C\nu h^2 \|\boldsymbol{u}\|_{H^3(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}^h.$$

Proof. Let $e \in \mathcal{E}_h^{I,\partial}$ and $G_e \in \mathbb{R}^{2\times 2}$ be the average of $\nabla \boldsymbol{u}$ on e. Then, using standard interpolation estimates with trace inequality, we find

$$h_e^{-1} \|\nabla \boldsymbol{u} - G_e\|_{L^2(e)}^2 \le C|\boldsymbol{u}|_{H^2(T)} \qquad h_e = \mathrm{diam}(e), \tag{2.5.4}$$

where T is such that $e \subset \partial T$. Moreover, notice also that we have $|G_e| \le C|\boldsymbol{u}|_{W^{1,\infty}(\Omega)}$. Let $\boldsymbol{E}_h \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega_h)$ satisfy (2.4.1). Then, by continuity of $\boldsymbol{E}_h \boldsymbol{v}$, we have $[\boldsymbol{E}_h \boldsymbol{v}]|_e = 0$ for all $e \in \mathcal{E}_h^I$. This with the triangle inequality yields

$$\begin{aligned} \nu \sum_{e\in\mathcal{E}_h^{I,\partial}} \int_e \nabla \boldsymbol{u} : [\boldsymbol{v}] &= \nu \sum_{e\in\mathcal{E}_h^{I,\partial}} \Big(\int_e (\nabla \boldsymbol{u} - G_e) : [\boldsymbol{v} - \boldsymbol{E}_h \boldsymbol{v}] + \int_e G_e : [\boldsymbol{v}] \Big) \\ &=: I_1 + I_2. \end{aligned} \tag{2.5.5}$$

Next, we bound I_1 and I_2. For the bound of I_1, we use the Cauchy–Schwarz inequality, the estimate (2.5.4) along with the trace inequality and Lemma 2.4.5:

$$\begin{aligned} I_1 &\le \nu \Big(\sum_{e\in\mathcal{E}_h^{I,\partial}} h_e^{-1} \|\nabla \boldsymbol{u} - G_e\|_{L^2(e)}^2 \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h^{I,\partial}} h_e \|[\boldsymbol{v} - \boldsymbol{E}_h \boldsymbol{v}]\|_{L^2(e)}^2 \Big)^{1/2} \\ &\le C\nu h^2 |\boldsymbol{u}|_{H^2(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}. \end{aligned} \tag{2.5.6}$$

In order to bound I_2, we use Lemma 2.4.6 with the estimate $|G_e| \le C|\boldsymbol{u}|_{W^{1,\infty}(\Omega)}$ and the Cauchy-Schwarz inequality:

$$I_2 = \nu \sum_{e\in\mathcal{E}_h^{I,\partial}} G_e : \int_e [\boldsymbol{v}] \tag{2.5.7}$$

$$\le C\nu|\boldsymbol{u}|_{W^{1,\infty}(\Omega)}\Big(\sum_{e\in\mathcal{E}_h^{I,\partial}} h_e\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^{I,\partial}} h_e^{-1}\Big|\int_e [\boldsymbol{v}]\Big|^2\Big)^{1/2}$$
$$\le C\nu h^{5/2}\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}\Big(\sum_{e\in\mathcal{E}_h^{I,\partial}} h_e\Big)^{1/2}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)} \le C\nu h^{5/2}\|\boldsymbol{u}\|_{H^3(\Omega)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}.$$

Combining (2.5.5)–(2.5.7) completes the proof. □

We now combine the estimates (2.5.2), (2.5.3) together with Lemma 2.5.3 in order to obtain the optimal order of convergence for the discrete velocity approximation. The analogous result for the discrete pressure approximation then follows from the inf-sup stability established in Theorem 2.4.4.

Theorem 2.5.4. *There holds*

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} \le C\big(h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}}\big), \tag{2.5.8}$$

where

$$|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}} = \sup_{\boldsymbol{v}\in\boldsymbol{X}^h\setminus\{0\}} \frac{\int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)\cdot\boldsymbol{v}}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}}.$$

Therefore if, for example, $\boldsymbol{f}_h$ is the nodal quadratic interpolant of $\boldsymbol{f}$, and $\boldsymbol{f}$ is sufficiently smooth, then there holds

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} \le C\big(h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \nu^{-1}h^3\|\boldsymbol{f}\|_{H^3(\Omega)}\big).$$

Moreover, the pressure approximation satisfies

$$\begin{aligned}\|p-p_h\|_{L^2(\Omega_h)} \le& C\big(\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} + \nu h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \inf_{q\in Q^h}\|p-q\|_{L^2(\Omega_h)} \\ &+ \|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega_h)}\big).\end{aligned} \tag{2.5.9}$$

Proof. The error estimate (2.5.8) immediately follows from combining (2.5.2), (2.5.3) with Lemma 2.5.3, so it remains to prove (2.5.9). For any $q \in Q^h$ and $\boldsymbol{v} \in \boldsymbol{V}^h$, we use (2.5.1), Lemma 2.5.3 to obtain

$$\begin{aligned}\int_{\Omega_h} (\nabla \cdot \boldsymbol{v})(p_h - q) &= a_h(\boldsymbol{u}_h, \boldsymbol{v}) - \int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q - \int_{\Omega_h} \boldsymbol{f}_h \cdot \boldsymbol{v} \\ &= a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}) - \int_{\Omega_h} (\nabla \cdot \boldsymbol{v})(q - p) - \int_{\Omega_h} (\boldsymbol{f}_h - \boldsymbol{f}) \cdot \boldsymbol{v} - \nu \sum_{e \in \mathcal{E}_h^{I,\partial}} \int_e \nabla \boldsymbol{u} : [\boldsymbol{v}] \\ &\leq C\big(\nu \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega_h)} + \nu h^2 \|\boldsymbol{u}\|_{H^3(\Omega)} + \|p - q\|_{L^2(\Omega_h)}\big) \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)} \\ &+ \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\Omega_h)} \|\boldsymbol{v}\|_{L^2(\Omega_h)}\big).\end{aligned}$$

Using the estimate (2.4.1) with the Poincaré inequality yields

$$\|\boldsymbol{v}\|_{L^2(\Omega_h)} \leq \|\boldsymbol{E}_h \boldsymbol{v}\|_{L^2(\Omega_h)} + \|\boldsymbol{v} - \boldsymbol{E}_h \boldsymbol{v}\|_{L^2(\Omega_h)} \leq C \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}.$$

Combining this last inequality with the previous inequality, we get

$$\begin{aligned}\int_{\Omega_h} (\nabla \cdot \boldsymbol{v})(p_h - q) \leq C\big(&\nu \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega_h)} + \nu h^2 \|\boldsymbol{u}\|_{H^3(\Omega)} \\ &+ \|p - q\|_{L^2(\Omega_h)} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\Omega_h)}\big) \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}.\end{aligned}$$

We then use the inf-sup condition given in Theorem 2.4.4 to obtain

$$\begin{aligned}C\|p_h - q\|_{L^2(\Omega_h)} &\leq \sup_{\boldsymbol{v} \in \boldsymbol{V}^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v})(p_h - q)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \\ &\leq C\big(\nu \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega_h)} + \nu h^2 \|\boldsymbol{u}\|_{H^3(\Omega)} + \|p - q\|_{L^2(\Omega_h)} \\ &+ \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\Omega_h)}\big).\end{aligned}$$

Finally, by applying the triangle inequality and taking the infimum over $q \in Q^h$, we obtain (3.5.3). $\square$

2.6 A pressure robust scheme

In this section, by assuming enough regularity for the solution $\boldsymbol{u}$, we show that it is possible to eliminate the term $\nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}}$ via an appropriate construction of $\boldsymbol{f}_h$, and as a result, that the method is pressure robust. In particular, we adopt and expand the recent results in [43] for Scott–Vogelius elements to construct commuting operators on meshes with curved boundary. First, we define the rot operator as $\operatorname{rot}\boldsymbol{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$, and we introduce the corresponding Hilbert space

$$\boldsymbol{H}(\operatorname{rot};\Omega_h) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_h) : \operatorname{rot}\boldsymbol{v} \in L^2(\Omega_h)\}.$$

The main objective of this section is to prove the following result.

Theorem 2.6.1. *There exists finite element spaces $\boldsymbol{W}^h \subset \boldsymbol{H}(\operatorname{rot};\Omega_h)$, $\Sigma^h \subset H_0^1(\Omega_h)$ with respect to $\mathcal{T}_h$, and operators $\boldsymbol{\Pi}_W : \boldsymbol{H}^2(\Omega) \to \boldsymbol{W}^h$ and $\Pi_\Sigma : H^3(\Omega) \to \Sigma^h$ such that*

$$\boldsymbol{\Pi}_W \nabla p = \nabla \Pi_\Sigma p \qquad \forall p \in H^3(\Omega). \tag{2.6.1}$$

Moreover, for any $\boldsymbol{f} \in \boldsymbol{H}^3(\Omega)$, there holds

$$\|\boldsymbol{f} - \boldsymbol{\Pi}_W \boldsymbol{f}\|_{L^2(\Omega_h)} \le Ch^2 \|\boldsymbol{f}\|_{H^3(\Omega)}, \tag{2.6.2}$$

where $\boldsymbol{f}$ in the left-hand side of the above inequality is an $\boldsymbol{H}^3$ extension of $\boldsymbol{f}|_\Omega$.

Before we present a proof of Theorem 2.6.1, we first state its immediate corollary.

Corollary 2.6.2. *Let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}^h \times Q^h$ be the solution of the finite element method* (2.5.1) *with $\boldsymbol{f}_h = \boldsymbol{\Pi}_W \boldsymbol{f}$. Then, there holds*

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega_h)} \le Ch^2 \|\boldsymbol{u}\|_{H^5(\Omega)}.$$

Proof. Recalling the estimate (2.5.8), it is sufficient to show $|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}} \leq C\nu h^2\|\boldsymbol{u}\|_{H^5(\Omega)}$. Remember that the extension of $\boldsymbol{f}|_\Omega$ is given by $\boldsymbol{f} = -\nu\Delta\boldsymbol{u}+\nabla p$. Thus, by rewriting $\boldsymbol{f}$, Theorem 2.6.1 and the integration by parts formula, for all $\boldsymbol{v} \in \boldsymbol{X}^h$ there holds,

$$
\begin{aligned}
\int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)\cdot\boldsymbol{v} &= \int_{\Omega_h}\big(-\nu(\Delta\boldsymbol{u}-\boldsymbol{\Pi}_W\Delta\boldsymbol{u})+(\nabla p-\boldsymbol{\Pi}_W\nabla p)\big)\cdot\boldsymbol{v} \\
&= \int_{\Omega_h}\big(-\nu(\Delta\boldsymbol{u}-\boldsymbol{\Pi}_W\Delta\boldsymbol{u})+\nabla(p-\Pi_\Sigma p)\big)\cdot\boldsymbol{v} \\
&= -\nu\int_{\Omega_h}(\Delta\boldsymbol{u}-\boldsymbol{\Pi}_W\Delta\boldsymbol{u})\cdot\boldsymbol{v},
\end{aligned}
$$

where we used that $\nabla\cdot\boldsymbol{v}=0$ and $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega_h}=0$. Hence, by (2.6.2), we obtain

$$
|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}} \leq C\nu\|\Delta\boldsymbol{u}-\boldsymbol{\Pi}_W\Delta\boldsymbol{u}\|_{L^2(\Omega_h)} \leq Ch^2\nu\|\Delta\boldsymbol{u}\|_{H^3(\Omega)} \leq C\nu h^2\|\boldsymbol{u}\|_{H^5(\Omega)}.
$$

□

2.6.1 Proof of Theorem 2.6.1: Preliminaries

As a first step of the proof of Theorem 2.6.1, we “rotate” the space $\boldsymbol{V}(T)$.

Definition 2.6.3. We introduce the following local spaces:

$$
\begin{aligned}
\boldsymbol{W}(T) &:= \{\boldsymbol{v}\in\boldsymbol{H}^1(T):\ \boldsymbol{v}(x) = (DF_T(\hat{x}))^{-\intercal}\hat{\boldsymbol{v}}(\hat{x}),\ \exists\hat{\boldsymbol{v}}\in\hat{\boldsymbol{V}}\}, \\
\boldsymbol{W}_0(T) &:= \boldsymbol{W}(T)\cap\boldsymbol{H}_0^1(T).
\end{aligned}
$$

Next, we define the matrix

$$
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

Notice that $\operatorname{rot}(S\boldsymbol{v}) = \nabla \cdot \boldsymbol{v}$, and $SDF_T S^{-1} = \det(DF_T)(DF_T)^{-\intercal}$. Therefore, if $\boldsymbol{v}(x) = (DF_T(\hat{x}))^{-\intercal}\hat{\boldsymbol{v}}(\hat{x})$, then using the above properties with (2.2.5), we have

$$\operatorname{rot}\boldsymbol{v}(x) = \operatorname{rot}\Big(S\frac{DF_T(\hat{x})S^{-1}\hat{\boldsymbol{v}}(\hat{x})}{\det(DF_T(\hat{x}))}\Big) = \nabla\cdot\Big(\frac{DF_T(\hat{x})S^{-1}\hat{\boldsymbol{v}}(\hat{x})}{\det(DF_T(\hat{x}))}\Big) = \frac{\hat{\operatorname{rot}}\,\hat{\boldsymbol{v}}(\hat{x})}{\det(DF_T(\hat{x}))}, \tag{2.6.3}$$

where we also used (2.2.5) in the last equality.

Remark 2.6.4. Notice that $\hat{\operatorname{rot}} : \hat{\boldsymbol{V}} \to \hat{Q}$ is a surjection. Indeed, let $\hat{q} \in \hat{Q}$. Then, there exists $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$ such that $\hat{\nabla}\cdot\hat{\boldsymbol{v}} = \hat{q}$. Then, let $\hat{\boldsymbol{w}} = S\hat{\boldsymbol{v}}$ so that $\hat{q} = \hat{\nabla}\cdot\hat{\boldsymbol{v}} = \hat{\operatorname{rot}}\,\hat{\boldsymbol{w}}$. Similar arguments show $\hat{\operatorname{rot}} : \hat{\boldsymbol{V}}_0 \to \hat{Q}_0$ is a bijection.

The next lemma identifies the DOFs of the space $\boldsymbol{W}(T)$.

Lemma 2.6.5. *Let $\{\hat{\alpha}_i\}_{i=1}^3, \{\hat{m}_i\}_{i=1}^3 \subset \mathcal{N}_{\hat{T}}$ be the vertices and edge midpoints of $\hat{T}$, respectively. We set $\alpha_i = F_T(\hat{\alpha}_i)$ and $m_i = F_T(\hat{m}_i)$ to be the corresponding points on T. Then, $\boldsymbol{v} \in \boldsymbol{W}(T)$ is uniquely determined by the values*

$$\boldsymbol{v}(\alpha_i),\ (\boldsymbol{v}\cdot\boldsymbol{n})(m_i) \qquad i = 1,2,3, \tag{2.6.4a}$$

$$\int_e \boldsymbol{v}\cdot\boldsymbol{t} \qquad \forall \textit{ edges of } T, \tag{2.6.4b}$$

$$\int_T (\operatorname{rot}\boldsymbol{v})q \qquad \forall q \in Q_0(T). \tag{2.6.4c}$$

Proof. Let $\boldsymbol{v}(x) = DF_T^{-\intercal}\hat{\boldsymbol{v}}$ for some $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$, and suppose that $\boldsymbol{v}$ vanishes on the DOFs. It then suffices to show that $\hat{\boldsymbol{v}} \equiv 0$. We clearly have $\hat{\boldsymbol{v}}(\hat{\alpha}_i) = 0$ for $i = 1,2,3$, then, by using the relation $\boldsymbol{t} = DF_T\hat{\boldsymbol{t}}/|DF_T\hat{\boldsymbol{t}}|$ [27] and a change of variables, we find

$$0 = \int_e \boldsymbol{v}\cdot\boldsymbol{t} = \int_{\hat{e}} \frac{(DF_T^{-\intercal}\hat{\boldsymbol{v}})\cdot(DF_T\hat{\boldsymbol{t}})}{|DF_T\hat{\boldsymbol{t}}|}|\det(DF_T)||DF^{-\intercal}\hat{\boldsymbol{n}}| = \int_{\hat{e}} \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{t}},$$

where we used the identity $|\det(DF_T)||DF^{-\intercal}\hat{\boldsymbol{n}}| = |DF_T\hat{\boldsymbol{t}}|$. The above equality with that fact that $\boldsymbol{v}\cdot\boldsymbol{t}$ vanishes at the two boundary points of the edge e yields $\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{t}}|_{\partial\hat{T}} = 0$. Similarly, using the relation $\boldsymbol{n} = DF_T^{-\intercal}\hat{\boldsymbol{n}}/|DF_T^{-\intercal}\hat{\boldsymbol{n}}|$, we find

$$0 = (\boldsymbol{v}\cdot\boldsymbol{n})(m_i) = \frac{\hat{\boldsymbol{v}}\cdot(DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})}{|DF_T^{-\intercal}\hat{\boldsymbol{n}}|}(\hat{m}_i).$$

Since $(DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{n}} = |DF_T^{-\intercal}\hat{\boldsymbol{n}}|^2 \neq 0$, we conclude $(DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})$ is not tangent to $\hat{\boldsymbol{t}}$. Thus, because $\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{t}}|_{\partial\hat{T}} = 0$, we obtain that $\hat{\boldsymbol{v}}|_{\partial\hat{T}} = 0$, i.e., $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}_0$.

Next, let $\hat{q} \in \hat{Q}_0$, and set $q(x) := \hat{q}(\hat{x})$ with $x = F_T(\hat{x})$ so that $q \in Q_0(T)$. We then use the equality $\operatorname{rot}\boldsymbol{v} = \frac{\hat{\operatorname{rot}}\,\hat{\boldsymbol{v}}}{\det(DF_T)}$ with a change of variables to find

$$0 = \int_T (\operatorname{rot}\boldsymbol{v})q = \int_{\hat{T}} (\hat{\operatorname{rot}}\,\hat{\boldsymbol{v}})\hat{q}.$$

Letting $\hat{q} = \hat{\operatorname{rot}}\,\hat{\boldsymbol{v}}$, we conclude $\hat{\operatorname{rot}}\,\hat{\boldsymbol{v}} = 0$. This implies $\hat{\boldsymbol{v}} \equiv 0$, and so $\boldsymbol{v} \equiv 0$. $\square$

We now introduce the local Clough-Tocher space on the reference element

$$\hat{\Sigma} = \{\hat{\sigma} \in H^2(\hat{T}) : \ \hat{\sigma}|_{\hat{K}} \in \mathcal{P}_3(\hat{K}) \ \forall \hat{K} \in \hat{T}^{ct}\}.$$

The dimension of $\hat{\Sigma}$ is 12, and any $\hat{\sigma} \in \hat{\Sigma}$ is uniquely determined by the values [37]

$$\hat{\nabla}\hat{\sigma}(\hat{\alpha}_i),\ \hat{\sigma}(\hat{\alpha}_i),\ (\hat{\nabla}\hat{\sigma}\cdot\hat{\boldsymbol{n}})(\hat{m}_i) \qquad i = 1,2,3, \tag{2.6.5}$$

where we again denote the vertices and edge midpoints of $\hat{T}$ by $\{\hat{\alpha}_i\}_{i=1}^3, \{\hat{m}_i\}_{i=1}^3$, respectively.

Next, we define the corresponding Clough–Tocher space on T through composition

$$\Sigma(T) = \{\sigma : \ \sigma(x) = \hat{\sigma}(\hat{x}),\ \exists\hat{\sigma} \in \hat{\Sigma}\}.$$

Notice that $\Sigma(T) \subset H^2(T)$. The following lemma shows that the above DOFs can be extended to $\Sigma(T)$.

Lemma 2.6.6. *A function $\sigma \in \Sigma(T)$ is uniquely determined by the values*

$$\nabla\sigma(\alpha_i),\ \sigma(\alpha_i),\ (\nabla\sigma\cdot\boldsymbol{n})(m_i) \qquad i = 1,2,3. \tag{2.6.6a}$$

Proof. We write $\sigma(x) = \hat\sigma(\hat x)$ with $\hat\sigma \in \hat\Sigma$, $x = F_T(\hat x)$. Notice that it is sufficient to show that if σ vanishes at the above DOFs, then $\hat\sigma$ vanishes on (2.6.5). Suppose then that σ vanishes at the above DOFs, then we clearly have

$$\hat\nabla\hat\sigma(\hat\alpha_i) = 0,\ \hat\sigma(\hat\alpha_i) = 0 \qquad i = 1,2,3.$$

Notice that this implies $\hat\sigma|_{\partial\hat T} = 0$, and therefore $\hat\nabla\hat\sigma\cdot\hat{\boldsymbol{t}}|_{\partial\hat T} = 0$. Next, we again use the chain rule and the relation $\boldsymbol{n} = DF^{-\intercal}\hat{\boldsymbol{n}}/|DF^{-\intercal}\hat{\boldsymbol{n}}|$ to find

$$0 = (\nabla\sigma\cdot\boldsymbol{n})(m_i) = \Big(\frac{1}{|DF_T^{-\intercal}\hat{\boldsymbol{n}}|}\hat\nabla\hat\sigma\cdot(DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})\Big)(\hat m_i).$$

In particular, we have $\big(\hat\nabla\hat\sigma\cdot(DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})\big)(\hat m_i) = 0$. Since

$$\big((DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{n}}\big)(\hat m_i) = |(DF_T\hat{\boldsymbol{n}})(\hat m_i)|^2 \neq 0,$$

we conclude that the vector $(DF_T^{-1}DF_T^{-\intercal}\hat{\boldsymbol{n}})(\hat m_i)$ is not tangent to $\hat e$. This with $(\hat\nabla\hat\sigma\cdot\hat{\boldsymbol{t}})(\hat m_i) = 0$ implies that $\hat\nabla\hat\sigma(\hat m_i) = 0$. Hence, $\hat\sigma \equiv 0$ and $\sigma \equiv 0$. $\square$

Remark 2.6.7. Notice that if $\sigma \in \Sigma(T)$ with $\sigma(x) = \hat\sigma(\hat x)$, then the chain rule shows that $\nabla\sigma(x) = (DF_T(\hat x))^{-\intercal}\hat\nabla\hat\sigma(\hat x)$, and so we conclude that $\nabla\sigma \in \boldsymbol{W}(T)$.

As a next step, we exploit the DOFs stated in Lemmas 2.6.5–2.6.6 in order to construct commuting operators with the desired properties stated in Theorem 2.6.1. Notice that an added difficulty of the desired construction arises from the fact that the operators are defined for functions with domain Ω, but map to functions with domain Ω_h. In order to overcome this mismatch, we again benefit from the mapping $G : \tilde\Omega_h \to \Omega$ and define, for each $T \in \mathcal{T}_h$ and edge e in $\mathcal{T}_h$,

$$T_R := G(G_h^{-1}(T)) \subset \Omega, \qquad e_R := G(G_h^{-1}(e)) \subset \bar\Omega,$$

where we recall that G_h is the quadratic interpolant of G. In other words, T_R is constructed by first mapping T to its associated affine element $\tilde{T} = G_h^{-1}(T) \in \tilde{\mathfrak{T}}_h$, and then mapping $\tilde{T}$ to $G(\tilde{T}) \subset \Omega$. Notice also that, by the properties of the quadratic interpolant G_h, we have $G(G_h^{-1}(\alpha_i)) = \alpha_i$ and $G(G_h^{-1}(m_i)) = m_i$ for all vertices and edge midpoints of T.

In the light of Lemma 2.6.5, we introduce the operator $\mathbf{\Pi}_W^T : \boldsymbol{H}^2(T_R) \to \boldsymbol{W}(T)$, which is uniquely determined by the following conditions:

$$(\mathbf{\Pi}_W^T \boldsymbol{v})(\alpha_i) = \boldsymbol{v}(\alpha_i) \qquad i = 1,2,3, \tag{2.6.7a}$$

$$(\mathbf{\Pi}_W^T \boldsymbol{v} \cdot \boldsymbol{n})(m_i) = (\boldsymbol{v} \cdot \boldsymbol{n})(m_i) \qquad i = 1,2,3, \tag{2.6.7b}$$

$$\int_e (\mathbf{\Pi}_W^T \boldsymbol{v}) \cdot \boldsymbol{t} = \int_{e_R} \boldsymbol{v} \cdot \boldsymbol{t}_{e_R} \qquad \forall \text{ edges of } T, \tag{2.6.7c}$$

$$\int_T (\operatorname{rot} \mathbf{\Pi}_W^T \boldsymbol{v}) q = \int_{T \cap T_R} (\operatorname{rot} \boldsymbol{v}) q \qquad \forall q \in Q_0(T), \tag{2.6.7d}$$

where $\boldsymbol{n}$ denotes the outward unit normal with respect to $e \subset \partial T$, $\boldsymbol{t}$ denotes the unit tangent of $e \subset \partial T$, and $\boldsymbol{t}_{e_R}$ denotes the unit tangent of $e_R \subset \partial T_R$.

Next, we use Lemma 2.6.6 to introduce the operator $\Pi_\Sigma^T : H^3(T_S) \to \Sigma(T)$, that is uniquely determined by the conditions

$$\Pi_\Sigma^T \sigma(\alpha_i) = \sigma(\alpha_i), \quad \nabla(\Pi_\Sigma \sigma)(\alpha_i) = \nabla \sigma(a), \qquad i = 1,2,3, \tag{2.6.8a}$$

$$\nabla(\Pi_\Sigma^T \sigma)(m_i) \cdot \boldsymbol{n}(m_i) = \nabla \sigma(m_i) \cdot \boldsymbol{n}(m_i) \qquad i = 1,2,3. \tag{2.6.8b}$$

Using the local spaces and the corresponding operators, we now define the global spaces as follows:

$$\boldsymbol{W}^h := \{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{rot}; \Omega_h) : \ \boldsymbol{v}|_T \in \boldsymbol{W}(T) \ \forall T \in \mathfrak{T}_h, \ \boldsymbol{v} \text{ is continuous on (2.6.4)}\},$$

$$\Sigma^h := \{\sigma \in H^1(\Omega_h) : \ \sigma|_T \in \Sigma(T) \ \forall T \in \mathfrak{T}_h, \ \sigma \text{ is continuous on (2.6.6)}\},$$

and we define the operators $\boldsymbol{\Pi}_W : H^2(\Omega) \to \boldsymbol{W}^h$, $\Pi_\Sigma : H^3(\Omega) \to \Sigma^h$ by

$$\boldsymbol{\Pi}_W \boldsymbol{v}|_T := \boldsymbol{\Pi}_W^T \boldsymbol{v}, \qquad \Pi_\Sigma \sigma|_T := \Pi_\Sigma^T \sigma, \qquad \forall T \in \mathcal{T}_h.$$

The last two steps to prove Theorem 2.6.1 are to show that these operators satisfy (2.6.1)–(2.6.2).

2.6.2 Proof of (2.6.1)

For a given $p \in H^3(\Omega)$, we set $\boldsymbol{\rho} = \boldsymbol{\Pi}_W \nabla p - \nabla \Pi_\Sigma p \in \boldsymbol{W}(T)$. We aim to show that $\boldsymbol{\rho} \equiv 0$. It then suffices to show that $\boldsymbol{\rho}$ vanishes at the given DOFs in Lemma 2.6.5 for each $T \in \mathcal{T}_h$.

First, we consider the interior DOFs of $\boldsymbol{W}(T)$. Using (2.6.7d) and the identity $\operatorname{rot} \nabla p = 0$, we have

$$\int_T (\operatorname{rot} \boldsymbol{\rho}) q = \int_T (\operatorname{rot} (\boldsymbol{\Pi}_W \nabla p)) q = \int_{T \cap T_R} (\operatorname{rot} (\nabla p)) q = 0 \qquad \forall q \in Q_0(T).$$

Let α_i be a vertex of T. Then, by (2.6.7a) and (2.6.8a), we find

$$\boldsymbol{\rho}(\alpha_i) = \boldsymbol{\Pi}_W \nabla p(\alpha_i) - \nabla \Pi_\Sigma p(\alpha_i) = 0.$$

Next, let m_i be an edge midpoint of T and let $\boldsymbol{n}$ be the outward unit normal at m_i. Then, (2.6.7b) and (2.6.8b) together yields

$$\boldsymbol{\rho}(m_i) \cdot \boldsymbol{n} = \boldsymbol{\Pi}_W \nabla p(m_i) \cdot \boldsymbol{n} - \nabla \Pi_\Sigma p(m_i) \cdot \boldsymbol{n} = 0.$$

Finally, let $e \subset \partial T$ be an edge of T with endpoints α_2 and α_1. Recalling that e_R also has endpoints α_2 and α_1, we use (2.6.8a) and (2.6.7c) to obtain

$$\begin{aligned}\int_e \boldsymbol{\rho} \cdot \boldsymbol{t} = \int_e \big(\boldsymbol{\Pi}_W \nabla p - \nabla \Pi_\Sigma p\big) \cdot \boldsymbol{t} &= \int_{e_R} \nabla p \cdot \boldsymbol{t}_{e_R} - \int_e (\nabla \Pi_\Sigma p) \cdot \boldsymbol{t} \\ &= p(\alpha_2) - p(\alpha_1) - \big((\Pi_\Sigma p)(\alpha_2) - (\Pi_\Sigma p)(\alpha_1)\big) = 0.\end{aligned}$$

Thus, $\boldsymbol{\rho}$ vanishes at all the DOFs in Lemma 2.6.5, and so $\boldsymbol{\rho} \equiv 0$.

2.6.3 Proof of (2.6.2)

We break up the proof of estimate (2.6.2) into three parts.

(i) We begin with extending $\boldsymbol{f}$ to $\mathbb{R}^2$ in such a way that $\|\boldsymbol{f}\|_{H^3(\mathbb{R}^2)} \leq C\|\boldsymbol{f}\|_{H^3(\Omega)}$. With this extension, we define $\boldsymbol{I}_W^T\boldsymbol{f} \in \boldsymbol{W}(T)$ uniquely by the conditions

$$(\boldsymbol{I}_W^T\boldsymbol{f})(\alpha_i) = \boldsymbol{f}(\alpha_i), \quad (\boldsymbol{I}_W^T\boldsymbol{f}\cdot\boldsymbol{n})(m_i) = (\boldsymbol{f}\cdot\boldsymbol{n})(m_i) \qquad i = 1,2,3, \tag{2.6.9}$$

$$\int_e (\boldsymbol{I}_W^T\boldsymbol{f})\cdot\boldsymbol{t} = \int_e \boldsymbol{f}\cdot\boldsymbol{t} \qquad \forall \text{ edges of } T, \tag{2.6.10}$$

$$\int_T (\operatorname{rot}\boldsymbol{I}_W^T\boldsymbol{f})q = \int_T (\operatorname{rot}\boldsymbol{f})q \qquad \forall q \in Q_0(T). \tag{2.6.11}$$

Notice the similarities between the two operators $\boldsymbol{\Pi}_W^T$, $\boldsymbol{I}_W^T$; they coincide at the vertex and edge midpoint DOFs, and they slightly differ at the remaining DOFs, which is due to the difference in the corresponding domains of the function spaces that the operators act on.

Next, we estimate $\|\boldsymbol{f}-\boldsymbol{I}_W^T\boldsymbol{f}\|_{L^2(T)}$. In order to ease the notation, we denote $\boldsymbol{I}_W^T\boldsymbol{f}$ by $\boldsymbol{v}$, and write

$$\boldsymbol{v}(x) = R_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x}), \qquad \boldsymbol{f}(x) = R_T(\hat{x})\hat{\boldsymbol{f}}(\hat{x}),$$

where $R_T(\hat{x}) = (DF_T(\hat{x}))^{-\intercal}$. Then, by the above construction of $\boldsymbol{v}$, we have

$$\hat{\boldsymbol{v}}(\hat{\alpha}_i) = \hat{\boldsymbol{f}}(\hat{\alpha}_i), \quad (\hat{\boldsymbol{v}}\cdot(R_T^\intercal\boldsymbol{n}))(\hat{m}_i) = (\hat{\boldsymbol{f}}\cdot(R_T^\intercal\boldsymbol{n}))(\hat{m}_i) \qquad i = 1,2,3. \tag{2.6.12}$$

Another use of change of variables yields (cf. proof of Lemma 2.6.5)

$$\int_{\hat{e}} \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{t}} = \int_e \boldsymbol{v}\cdot\boldsymbol{t} = \int_e \boldsymbol{f}\cdot\boldsymbol{t} = \int_{\hat{e}} \hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{t}}. \tag{2.6.13}$$

Let $q \in Q_0(T)$. We write $q(x) = \hat{q}(\hat{x})$ with $\hat{q} \in \hat{Q}_0$, $x = F_T(\hat{x})$. Then, by (2.6.3) and change of variables, we obtain

$$\int_{\hat{T}} (\hat{\mathrm{rot}}\hat{\boldsymbol{v}})\hat{q} = \int_{\hat{T}} (\det(DF_T)\mathrm{rot}\boldsymbol{v}) \circ F_T \hat{q} = \int_T \mathrm{rot}\boldsymbol{v} q = \int_T \mathrm{rot}\boldsymbol{f} q = \int_{\hat{T}} \hat{\mathrm{rot}}\hat{\boldsymbol{f}}\hat{q}. \tag{2.6.14}$$

Then, using (2.6.12)–(2.6.14) and a slight generalization of the Bramble–Hilbert lemma we have

$$\|\hat{\boldsymbol{f}} - \hat{\boldsymbol{v}}\|_{L^2(\hat{T})} \le C|\hat{\boldsymbol{f}}|_{H^3(\hat{T})}. \tag{2.6.15}$$

Therefore, using (2.2.2), (2.6.15), Lemma 2.2.4 along with the product rule, and recalling that $R_T^{-1} = DF_T^{\intercal}$, we obtain

$$\begin{aligned} \|\boldsymbol{f} - \boldsymbol{I}_W^T \boldsymbol{f}\|_{L^2(T)} &\le Ch_T \|R_T(\hat{\boldsymbol{f}} - \hat{\boldsymbol{v}})\|_{L^2(\hat{T})} \\ &\le C|\hat{\boldsymbol{f}}|_{H^3(\hat{T})} = C|R_T^{-1}R_T\hat{\boldsymbol{f}}|_{H^3(\hat{T})} \\ &\le C\big(\|R_T^{-1}\|_{L^\infty(\hat{T})}|R_T\hat{\boldsymbol{f}}|_{H^3(\hat{T})} + |R_T^{-1}|_{W^{1,\infty}(\hat{T})}|R_T\hat{\boldsymbol{f}}|_{H^2(\hat{T})}\big) \\ &\le Ch_T^3\|\boldsymbol{f}\|_{H^3(T)}. \end{aligned} \tag{2.6.16}$$

(ii) We now estimate $(\boldsymbol{\Pi}_W^T \boldsymbol{f} - \boldsymbol{I}_W^T \boldsymbol{f})$. First, we set $\boldsymbol{w} := \boldsymbol{\Pi}_W^T \boldsymbol{f} - \boldsymbol{I}_W^T \boldsymbol{f}$. Notice, by construction, that $\boldsymbol{w} \in \boldsymbol{W}(T)$. Moreover, we also have

$$\begin{aligned} &\boldsymbol{w}(\alpha_i) = 0, \quad (\boldsymbol{w}\cdot\boldsymbol{n})(m_i) = 0 && i = 1,2,3, \\ &\int_e \boldsymbol{w}\cdot\boldsymbol{t} = \int_{e_R} \boldsymbol{f}\cdot\boldsymbol{t}_{e_R} - \int_e \boldsymbol{f}\cdot\boldsymbol{t} && \forall \text{ edges of } T, \\ &\int_T (\mathrm{rot}\,\boldsymbol{w})q = \int_{T\cap T_R} (\mathrm{rot}\,\boldsymbol{f})q - \int_T (\mathrm{rot}\,\boldsymbol{f})q && \forall q \in Q_0(T). \end{aligned}$$

Next, we write $\boldsymbol{w}(x) = R_T(\hat{x})\hat{\boldsymbol{w}}(\hat{x})$ and use equivalence of norms to find

$$\|\hat{\boldsymbol{w}}\|_{H^m(\hat{T})}^2 \le C\Big(\sum_{i=1}^3 (|\hat{\boldsymbol{w}}(\hat{\alpha}_i)|^2 + |\hat{\boldsymbol{w}}(\hat{m}_i)|^2) + \sup_{\substack{\hat{q}\in\hat{Q}_0 \\ \|\hat{q}\|_{L^2(\hat{T})}=1}} \Big|\int_{\hat{T}} (\hat{\mathrm{rot}}\,\hat{\boldsymbol{w}})\hat{q}\Big|^2\Big) \tag{2.6.17}$$

$$= C\Big(\sum_{i=1}^{3} |\hat{\boldsymbol{w}}(\hat{m}_i)|^2 + \sup_{\substack{\hat{q}\in\hat{Q}_0 \\ \|\hat{q}\|_{L^2(\hat{T})}=1}} \Big|\int_{\hat{T}} (\hat{\mathrm{rot}}\,\hat{\boldsymbol{w}})\hat{q}\Big|^2\Big).$$

Next, we use the algebraic identity

$$\hat{\boldsymbol{w}}(\hat{m}_i) = \frac{1}{\boldsymbol{\alpha}^\perp \cdot \boldsymbol{\beta}}\Big(((\hat{\boldsymbol{w}}(\hat{m}_i)) \cdot \boldsymbol{\beta})\boldsymbol{\alpha}^\perp - ((\hat{\boldsymbol{w}}(\hat{m}_i)) \cdot \boldsymbol{\alpha})\boldsymbol{\beta}^\perp\Big), \tag{2.6.18}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^2$ are any linearly independent vectors and $\boldsymbol{\alpha}^\perp = S\boldsymbol{\alpha}$. We take $\boldsymbol{\alpha} = -\hat{\boldsymbol{t}}(\hat{m}_i)$ and $\boldsymbol{\beta} = R_T^\intercal(\hat{m}_i)\boldsymbol{n}(m_i)$, so that

$$\begin{aligned}|\boldsymbol{\alpha}^\perp \cdot \boldsymbol{\beta}| = |S\hat{\boldsymbol{t}}(\hat{m}_i) \cdot (R_T^\intercal(\hat{m}_i)\boldsymbol{n}(m_i))| = |(R_T(\hat{m}_i)\hat{\boldsymbol{n}}(\hat{m}_i)) \cdot \boldsymbol{n}(m_i)| &= |(R_T\hat{\boldsymbol{n}})(\hat{m}_i)| \\ &\neq 0,\end{aligned}$$

where we used the relation $\boldsymbol{n} = R_T\hat{\boldsymbol{n}}/|R_T\hat{\boldsymbol{n}}|$ in the last equality. We then use (2.6.18) and the identity $\hat{\boldsymbol{w}}(\hat{m}_i) \cdot \boldsymbol{\beta} = (\boldsymbol{w} \cdot \boldsymbol{n})(m_i) = 0$ to find

$$\begin{aligned}|\hat{\boldsymbol{w}}(\hat{m}_i)| &= \frac{1}{|R_T\hat{\boldsymbol{n}}(\hat{m}_i)|}\big|(\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}})(\hat{m}_i)SR_T^\intercal(\hat{m}_i)\boldsymbol{n}(m_i)\big| \le \frac{|R_T(\hat{m}_i)|}{|R_T\hat{\boldsymbol{n}}(\hat{m}_i)|}|(\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}})(\hat{m}_i)| \\ &\le |R_T(\hat{m}_i)||R_T^{-1}(\hat{m}_i)||(\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}})(\hat{m}_i)| \le C|(\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}})(\hat{m}_i)|,\end{aligned}$$

where we used $\frac{1}{|R_T\hat{\boldsymbol{n}}(\hat{m}_i)|} \le |R_T^{-1}(\hat{m}_i)|$ in the second last inequality, and (2.2.2) in the last inequality. We now use this estimate in (2.6.17) to conclude

$$\|\hat{\boldsymbol{w}}\|^2_{H^m(\hat{T})} \le C\Big(\sum_{i=1}^{3} |(\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}})(\hat{m}_i)|^2 + \sup_{\substack{\hat{q}\in\hat{Q}_0 \\ \|\hat{q}\|_{L^2(\hat{T})}=1}} \Big|\int_{\hat{T}} (\hat{\mathrm{rot}}\,\hat{\boldsymbol{w}})\hat{q}\Big|^2\Big).$$

Recall that $\hat{\boldsymbol{w}}$ vanishes on the vertices of $\hat{T}$, then we use Simpson's rule to find

$$\|\hat{\boldsymbol{w}}\|^2_{H^m(\hat{T})} \le C\Big(\sum_{\hat{e}\subset\partial\hat{T}} \Big|\int_{\hat{e}} \hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}}\Big|^2 + \sup_{\substack{\hat{q}\in\hat{Q}_0 \\ \|\hat{q}\|_{L^2(\hat{T})}=1}} \Big|\int_{\hat{T}} (\hat{\mathrm{rot}}\,\hat{\boldsymbol{w}})\hat{q}\Big|^2\Big). \tag{2.6.19}$$

We estimate the two terms on the right-hand side of (2.6.19) separately. First, we apply change of variables formula to get

$$\int_{\hat{e}} \hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}} = \int_{e} \boldsymbol{w} \cdot \boldsymbol{t} = \int_{e_R} \boldsymbol{f} \cdot \boldsymbol{t}_{e_R} - \int_{e} \boldsymbol{f} \cdot \boldsymbol{t}.$$

Then, we set $\Theta := G_h \circ G^{-1}$ so that $e = \Theta(e_R)$ and $T = \Theta(T_R)$. There holds [24, Proposition 3]

$$|\Theta(x) - x| = \mathcal{O}(h_T^3), \quad |D\Theta - I| = \mathcal{O}(h_T^2), \quad \boldsymbol{t}(\Theta(x)) = \frac{D\Theta \boldsymbol{t}_{e_R}}{|D\Theta \boldsymbol{t}_{e_R}|}(x), \tag{2.6.20}$$

where I denotes the identity matrix and $x \in \bar{T}_R$. With another use of change of variables and (2.6.20), we obtain

$$\int_{e} \boldsymbol{f} \cdot \boldsymbol{t} = \int_{e_R} |(D\Theta)\boldsymbol{t}_{e_R}| (\boldsymbol{f} \cdot \boldsymbol{t}) \circ \Theta = \int_{e_R} (\boldsymbol{f} \circ \Theta) \cdot (D\Theta \boldsymbol{t}_{e_R}).$$

Thus,

$$\begin{aligned}\int_{\hat{e}} \hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}} &= \int_{e_R} \big(\boldsymbol{f} \cdot \boldsymbol{t}_{e_R} - (\boldsymbol{f} \circ \Theta) \cdot (D\Theta \boldsymbol{t}_{e_R})\big) \\ &= \int_{e_R} \big(\boldsymbol{f} - (\boldsymbol{f} \circ \Theta)\big) \cdot \boldsymbol{t}_{e_R} - (\boldsymbol{f} \circ \Theta) \cdot (D\Theta \boldsymbol{t}_{e_R} - \boldsymbol{t}_{e_R})\big).\end{aligned}$$

We now use Taylor's Theorem together with (2.6.20) and a Sobolev embedding to conclude that

$$\Big|\int_{\hat{e}} \hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{t}}\Big| \le C\big(h_T^4 |\boldsymbol{f}|_{W^{1,\infty}(\mathbb{R}^2)} + h_T^3 \|\boldsymbol{f}\|_{L^\infty(\mathbb{R}^2)}\big) \le C h_T^3 \|\boldsymbol{f}\|_{H^3(\Omega)}. \tag{2.6.21}$$

Next, we estimate the second term in (2.6.19). For this purpose, we let $\hat{q} \in \hat{Q}_0$ with $\|\hat{q}\|_{L^2(\hat{T})} = 1$ and compute

$$\int_{\hat{T}} (\widehat{\operatorname{rot}}\, \hat{\boldsymbol{w}}) \hat{q} = \int_{T} \operatorname{rot} \boldsymbol{w} q = \int_{T \cap T_R} (\operatorname{rot} \boldsymbol{f}) q - \int_{T} (\operatorname{rot} \boldsymbol{f}) q = \int_{T \setminus T_R} (\operatorname{rot} \boldsymbol{f}) q,$$

where $q \in Q_0(T)$ with $q(x) = \hat{q}(\hat{x})$, $x = F_T(\hat{x})$. We then use the above equality together with (2.6.20), the estimate $\|q\|_{L^2(T)} \le Ch_T \|\hat{q}\|_{L^2(\hat{T})} \le Ch_T$ and the Cauchy-Schwarz inequality to find

$$\int_{\hat{T}} (\hat{\mathrm{rot}}\, \hat{\boldsymbol{w}})\hat{q} \le |T \backslash T_R| \|\mathrm{rot}\, \boldsymbol{f}\|_{L^\infty(\mathbb{R}^2)} \|q\|_{L^2(T)} \le Ch_T^3 \|\boldsymbol{f}\|_{H^3(\Omega)} \|q\|_{L^2(T)}$$
$$\le Ch_T^4 \|\boldsymbol{f}\|_{H^3(\Omega)}. \qquad (2.6.22)$$

Plugging the estimates (2.6.21)–(2.6.22) into (2.6.19) yields

$$\|\hat{\boldsymbol{w}}\|_{H^m(\hat{T})} \le Ch_T^3 \|\boldsymbol{f}\|_{H^3(\Omega)}.$$

Then, by (2.2.2) and Lemma 2.2.4, we find

$$\|\boldsymbol{\Pi}_W^T \boldsymbol{f} - \boldsymbol{I}_W^T \boldsymbol{f}\|_{L^2(T)} = \|\boldsymbol{w}\|_{L^2(T)} \le Ch_T \|R_T \hat{\boldsymbol{w}}\|_{L^2(\hat{T})} \le C \|\hat{\boldsymbol{w}}\|_{L^2(\hat{T})}$$
$$\le Ch_T^3 \|\boldsymbol{f}\|_{H^3(\Omega)}.$$

(iii) Therefore, by (2.6.16) and the triangle inequality, we have

$$\|\boldsymbol{f} - \boldsymbol{\Pi}_W^T \boldsymbol{f}\|_{L^2(T)} \le Ch_T^3 \|\boldsymbol{f}\|_{H^3(\Omega)}.$$

Finally, summing over $T \in \mathfrak{T}_h$ yields the estimate (2.6.2):

$$\|\boldsymbol{f} - \boldsymbol{\Pi}_W \boldsymbol{f}\|_{L^2(\Omega_h)} \le C\Big(\sum_{T \in \mathfrak{T}_h} h_T^6 \|\boldsymbol{f}\|_{H^3(\Omega)}^2\Big)^{1/2} \le Ch^2 \|\boldsymbol{f}\|_{H^3(\Omega)} \Big(\sum_{T \in \mathfrak{T}_h} h_T^2\Big)^{1/2}$$
$$\le Ch^2 \|\boldsymbol{f}\|_{H^3(\Omega)}.$$

2.7 Numerical experiments

In this section, we compute the finite element method (2.5.1) on the unit circle centered at the origin. We construct the source function such that the exact solution is

$$\boldsymbol{u} = \begin{pmatrix} (x_1^2+x_2^2-1)(8x_1^2x_2+x_1^2+5x_2^2-1) \\ -4x_1(x_1^2+x_2^2-1)(3x_1^2+x_2^2+x_2-1) \end{pmatrix}, \qquad p = 10(x_1^2+x_2^2-\frac{1}{2}). \quad (2.7.1)$$

We take the source approximation $\boldsymbol{f}_h$ to be the quadratic (nodal) Lagrange interpolant of $\boldsymbol{f}$, and the viscosity $\nu = 10^{-1}$. The errors are depicted in Figure 2–3 for mesh parameters $h = 2^{-j} \times 10^{-1}$ $(j = -1, 0, 1, 2, 3)$. For comparison purposes, we also plot the errors of the analogous Scott-Vogelius finite element method using affine approximations, i.e., method (2.5.1) with $\boldsymbol{V}^h \times Q^h$ replaced by $\tilde{\boldsymbol{V}}^h \times \tilde{Q}^h$. The numerical results show the asymptotic convergence rates

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega_h)} = \mathcal{O}(h^3), \quad \|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} = \mathcal{O}(h^2), \quad \|p-p_h\|_{L^2(\Omega_h)} = \mathcal{O}(h^2),$$

for the isoparametric approximations. These results are in agreement with the theoretical results stated in Theorem 2.5.4. In contrast, the numerics indicate the solution of the affine approximation, denoted by $(\boldsymbol{u}^{aff}, p_h^{aff}) \in \tilde{\boldsymbol{V}}^h \times \tilde{Q}^h$ satisfies the sub-optimal convergence rates

$$\|\boldsymbol{u}-\boldsymbol{u}_h^{aff}\|_{L^2(\tilde{\Omega}_h)} = \mathcal{O}(h^2), \quad \|\nabla(\boldsymbol{u}-\boldsymbol{u}_h^{aff})\|_{L^2(\tilde{\Omega}_h)} = \mathcal{O}(h^{3/2}), \quad \|p-p_h^{aff}\|_{L^2(\tilde{\Omega}_h)} = \mathcal{O}(h^{3/2}).$$

We also solve the finite element method (2.5.1) but with isoparametric spaces defined via the usual composition, i.e., with velocity-pressure pair (2.1.1). Numerical experiments indicate the method is stable and converges with optimal order. However, as Figure 3 shows, the method is not divergence–free (nor pressure robust).

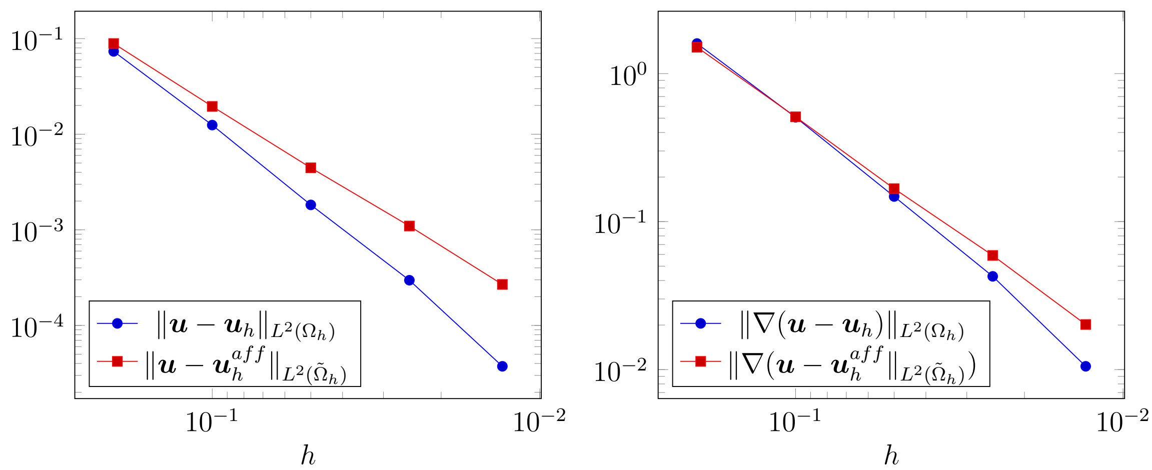

Figure 2: Velocity errors of the isoparametric Scott-Vogelius finite element method (2.5.1) (blue) and the affine Scott-Vogelius method (red).

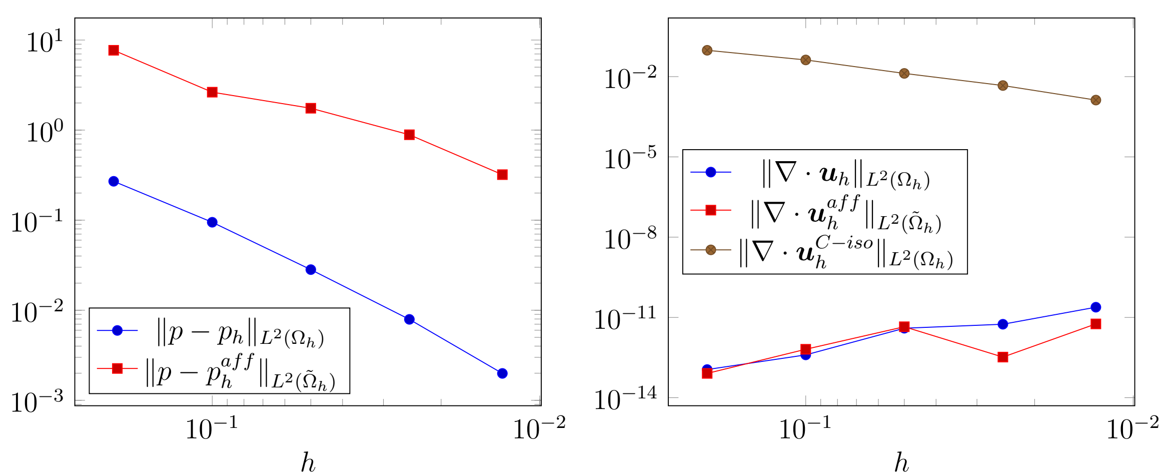

Figure 3: Left: Pressure errors of the isoparametric Scott-Vogelius finite element method (2.5.1) (blue) and the affine Scott-Vogelius method (red). Right: Divergence errors of the isoparametric Scott-Vogelius finite element method (blue), the affine Scott-Vogelius method (red), and the isoparametric Scott-Vogelius using the standard composition of isoparametric mappings (brown).

3.0 A stable and H^1-conforming divergence-free finite element pair for the Stokes problem using isoparametric mappings

3.1 Introduction

In this chapter, we construct and analyze a stable, $\boldsymbol{H}^1$-conforming and divergence-free method using isoparametric elements. As far as we are aware, this is the first isoparametric finite element scheme for the Stokes problem with all three of these properties. The construction is based on the work introduced in Chapter 2, and in particular, uses the lowest-order Scott–Vogelius finite element pair defined on Clough-Tocher partitions as its basis.

In more detail, recall that unlike the traditional use of isoparametric elements, our work presented in Chapter 2 uses the Piola transform in such a way that the function values at the nodal Lagrange degrees of freedom are preserved. While the use of Piola transform ensures the continuity of the normal component across the shared edges of the computational domain, we saw that this construction by itself does not result in an $\boldsymbol{H}^1$-conforming method. The key contribution of this chapter is to construct an $\boldsymbol{H}^1$-conforming, isoparametric method using the Piola transform, which potentially leads to improved error estimates in finite element schemes due to improved consistency.

The essential idea in our construction is similar to that given in the paper [25]. This paper constructs isoparametric C^1 elements on curved elements by first considering an enriched local reference space. Then, a subspace of this enriched space is extracted and mapped via composition to the computational domain in such a way that the function and the gradient values are preserved across the shared edges. Likewise, here we propose a local enriched space by adding divergence-free polynomials of higher degree to the reference macro element. We then extract and map a subspace of it using the Piola transformation in such a way that the resulting functions are single-valued when restricted to a shared edge, and the resulting spaces have the same dimensions as their affine counterparts. This modification using divergence-free elements not only leads to $\boldsymbol{H}^1$-conformity, but also preserves the desirable divergence-free property as well as inf-sup stability.

The rest of this chapter is organized as follows. In the next section, we recall the properties of the isoparametric framework as well as the Piola transform, and set the notation that is used throughout the following sections. In Section 3.3, we present several local spaces and state some of their properties. In this section, we also introduce the local mappings between the local spaces, which eventually leads to the definition of the local mapping that is used for the construction of the discrete velocity space (see Theorem 3.3.9). Here, we also study the behavior of this local mapping including its stability and approximation properties. In Section 3.4, we define the global mappings, the global spaces, and prove the inf-sup stability. In Section 3.5, we state the finite element method, prove the divergence-free property and that the method is of optimal order of convergence. Section 3.6 discusses the implementation of the method and presents numerical experiments which support the theoretical results.

3.2 Preliminaries

In this section, we introduce some notation and state some preliminary results. Most parts of the below set up coincide with that given in Section 2.2. However, for the sake of completeness and independence of each chapter, we recall the tools that are used in our isoparametric framework below.

We consider the Stokes problem with no-slip boundary condition introduced in (2.2.1). We again assume that $\Omega \subset \mathbb{R}^2$ is bounded, sufficiently smooth, open domain such that the boundary of Ω, $\partial\Omega$, is given by a finite number of local charts. $\tilde{\mathcal{T}}_h$ denotes a shape regular, affine triangulation of Ω such that the boundary vertices of $\tilde{\mathcal{T}}_h$ lies on $\partial\Omega$, and $\tilde{\Omega}_h := \mathrm{int}\Big(\cup_{\tilde{T} \in \tilde{\mathcal{T}}_h} \overline{\tilde{T}} \Big)$ is an $\mathcal{O}(h^2)$ polygonal approximation to Ω, where $h = \max_{\tilde{T} \in \tilde{\mathcal{T}}_h} \mathrm{diam}(\tilde{T})$. Moreover, we assume that $\tilde{\mathcal{T}}_h$ has at most two vertices on $\partial\Omega$.

Following the same isoparametric presented in Section 2.2, we consider the bijective map $G : \tilde{\Omega}_h \to \Omega$ introduced earlier and let G_h denote the piecewise quadratic nodal interpolant of G so that they satisfy the same properties as stated in Section 2.2. In this case, recall that the isoparametric triangulation and the computational domain are given, respectively, by

$$\mathcal{T}_h = \{G_h(\tilde{T}) : \ \tilde{T} \in \tilde{\mathcal{T}}_h\}, \qquad \Omega_h := \mathrm{int}\Big(\bigcup_{T \in \mathcal{T}_h} \overline{T} \Big).$$

Again, we let $F_{\tilde{T}} : \hat{T} \to \tilde{T}$ denote an affine mapping. Then, we introduce the quadratic mapping $F_T : \hat{T} \to T$, defined by $F_T := G_h \circ F_{\tilde{T}}$, which, we recall, satisfies (2.2.2).

Lastly, remember that $\hat{T}^{ct} = \{\hat{K}_i\}_{i=1}^3$ denotes the Clough-Tocher triangulation of the reference triangle $\hat{T}$, and $\tilde{T}^{ct}$ and T^{ct} denote the corresponding triangulations on $\tilde{T} \in \tilde{\mathcal{T}}_h$ and $T \in \mathcal{T}_h$, respectively, i.e.,

$$\tilde{T}^{ct} = \{F_{\tilde{T}}(\hat{K}) : \hat{K} \in \hat{T}^{ct}\}, \quad T^{ct} = \{F_T(\hat{K}) : \ \hat{K} \in \hat{T}^{ct}\}.$$

The globally refined triangulations are again given by

$$\tilde{\mathcal{T}}_h^{ct} = \{\tilde{K} : \ \tilde{K} \in \tilde{T}^{ct}, \ \exists \tilde{T} \in \tilde{\mathcal{T}}_h\}, \qquad \mathcal{T}_h^{ct} = \{K : \ K \in T^{ct}, \ \exists T \in \mathcal{T}_h\}.$$

3.3 Local spaces on macro elements

We begin this section by defining some polynomial spaces on the reference macro element $\hat{T}^{ct}$:

$$\hat{P} := \mathcal{P}_3^{c1}(\hat{T}^{ct}), \quad \hat{\boldsymbol{V}} := \boldsymbol{\mathcal{P}}_2^{c}(\hat{T}^{ct}), \quad \hat{Q} := \mathcal{P}_1(\hat{T}^{ct}),$$

i.e., $\hat{P}$ is the local C^1 Clough-Tocher element, $\hat{\boldsymbol{V}}$ is the vector-valued, quadratic Lagrange finite element space, and $\hat{Q}$ is the space of piecewise linear polynomials without continuity constraints. The next lemma reveals a connection between these spaces through the curl and the divergence operators.

Lemma 3.3.1. *The chain*

$$\mathbb{R} \xrightarrow{\subset} \hat{P} \xrightarrow{\widehat{\mathbf{curl}}} \hat{\boldsymbol{V}} \xrightarrow{\widehat{\mathrm{div}}} \hat{Q} \longrightarrow 0$$

forms an exact sequence, where $\widehat{\mathbf{curl}}\,\hat{z} = (\frac{\partial \hat{z}}{\partial \hat{x}_2}, -\frac{\partial \hat{z}}{\partial \hat{x}_1})^{\intercal}$.

Proof. First, notice that if $\hat{z} \in \hat{P}$ satisfies $\widehat{\mathbf{curl}}\,\hat{z} = 0$ then $\hat{z} \in \mathbb{R}$, and if $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$ is divergence–free, then clearly $\hat{\boldsymbol{v}} = \widehat{\mathbf{curl}}\,\hat{z}$ for some $\hat{z} \in \hat{P}$. Therefore, to prove the exactness property of the above chain, it remains to show that $\widehat{\mathrm{div}} : \hat{\boldsymbol{V}} \to \hat{Q}$ is surjective. To this end, first notice that $\widehat{\mathrm{div}}(\hat{\boldsymbol{V}}) \subset \hat{Q}$, and so it suffices to prove that $\widehat{\mathrm{div}}(\hat{\boldsymbol{V}})$ and $\hat{Q}$ have the same dimension. The well-known dimension formulas of these spaces are given by (cf. [7] and Lemma 3.3.2 below)

$$\dim \hat{P} = 12, \qquad \dim \hat{\boldsymbol{V}} = 20, \qquad \dim \hat{Q} = 9.$$

We use these formulas with the rank nullity theorem to find

$$\dim(\widehat{\mathrm{div}}(\hat{\boldsymbol{V}})) = \dim(\hat{\boldsymbol{V}}) - \dim(\widehat{\mathbf{curl}}\,(\hat{P})) = \dim(\hat{\boldsymbol{V}}) - \dim(\hat{P}) + 1 = \dim(\hat{Q}),$$

and this completes the proof of the exactness property. $\square$

Next, we define the analogous spaces on the affine macro element $\tilde{T}^{ct}$ by

$$\tilde{\boldsymbol{V}}(\tilde{T}) = \{\tilde{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_2^c(\tilde{T}^{ct}) : \ \tilde{\boldsymbol{v}}|_{\partial\tilde{T}\cap\partial\tilde{\Omega}_h} = 0\}, \qquad \tilde{Q}(\tilde{T}) = \mathcal{P}_1(\tilde{T}^{ct}).$$

For $T \in \mathcal{T}_h$, possibly with a curved edge, we again use the Piola transform in the definition of $\boldsymbol{V}(T)$, and use direct composition to construct $Q(T)$ as follows:

$$\begin{aligned} &\boldsymbol{V}(T) = \{\boldsymbol{v} \in \boldsymbol{H}^1(T) : \ \boldsymbol{v}(x) = A_T(\hat{x})\hat{\boldsymbol{v}}(\hat{x}), \ \exists \hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}, \text{ and } \boldsymbol{v}|_{\partial T\cap\partial\Omega_h} = 0\}, \\ &Q(T) = \{q \in L^2(T) : \ q(x) = \hat{q}(\hat{x}), \ \exists \hat{q} \in \hat{Q}\}, \end{aligned} \tag{3.3.1}$$

where $A_T(\hat{x}) = \frac{DF_T(\hat{x})}{\det(DF_T(\hat{x}))}$ and $x = F_T(\hat{x})$. Remember that if T is affine, i.e., if it only has straight edges, then A_T is constant, and therefore $\boldsymbol{V}(T) = \tilde{\boldsymbol{V}}(\tilde{T})$. Next, we define the variants of the above spaces with boundary conditions by

$$\begin{aligned} &\hat{\boldsymbol{V}}_0 = \hat{\boldsymbol{V}} \cap \boldsymbol{H}_0^1(\hat{T}), && \hat{Q}_0 = \hat{Q} \cap L_0^2(\hat{T}), \\ &\tilde{\boldsymbol{V}}_0(\tilde{T}) = \tilde{\boldsymbol{V}}(\tilde{T}) \cap \boldsymbol{H}_0^1(\tilde{T}), && \tilde{Q}_0(\tilde{T}) = \tilde{Q}(\tilde{T}) \cap L_0^2(\tilde{T}), \\ &\boldsymbol{V}_0(T) = \boldsymbol{V}(T) \cap \boldsymbol{H}_0^1(T), && Q_0(T) = \{q \in L^2(T) : \ q(x) = \hat{q}(\hat{x}), \ \exists \hat{q} \in \hat{Q}_0\}, \end{aligned}$$

where we recall that $L_0^2(S)$ is the space of functions in $L^2(S)$ with vanishing mean.

The main objective of this section is to construct a divergence-preserving, injective operator that maps functions in $\tilde{\boldsymbol{V}}(\tilde{T})$ onto a space of functions with domain T, where $T = G_h(\tilde{T})$, such that the operator respects function values on the shared edges and the range of the operator inherits the approximation properties of $\tilde{\boldsymbol{V}}(\tilde{T})$. The main idea of this construction is based on an enriching procedure used in the construction of isoparametric C^1 elements in [25], which we briefly describe next.

3.3.1 A local and enriched Clough-Tocher C^1 element

Here, we introduce a Clough-Tocher type, enriched element that is proposed by Mansfield in [25] in order to achieve C^1-continuity. The main idea is to consider an enriched local space of $\hat{P}$, and then extract and injectively map a 12-dimensional subspace of it in such a way that the function and gradient values are matched on the shared edges of the isoparametric mesh.

The mentioned enriched space consists of C^1 tricubic polynomials defined on the reference Clough-Tocher split, i.e., C^1 piecewise quartic polynomials that are cubic along all six edges in the split [5]. Following the notation introduced earlier, this space is given as

$$\hat{\mathbb{P}} := \{\hat{z} \in \mathcal{P}_4^{\mathrm{c1}}(\hat{T}^{\mathrm{ct}}) : \ \hat{z}|_{\hat{e}} \in \mathcal{P}_3(\hat{e}) \ \forall \hat{e} \in \Delta_1(\hat{T}^{\mathrm{ct}})\}.$$

We state the dimension counts and degrees of freedom for the spaces $\hat{P}$ and the enriched space $\hat{\mathbb{P}}$ in the next two lemmas. The first result is well-known and can be found in [7, Theorem 6.1.2] and [14, Theorem 1]. The proof of Lemma 3.3.3 is implicitly shown in [25, Theorem 1]. Here, we provide another proof of this result for the sake of completeness.

Lemma 3.3.2. *The space $\hat{P}$ is 12-dimensional, and a function $\hat{z} \in \hat{P}$ is uniquely determined by the degrees of freedom (DOFs)*

$$\hat{z}(\hat{a}), \ \hat{\nabla}\hat{z}(\hat{a}) \qquad \forall \hat{a} \in \Delta_0(\hat{T}), \tag{3.3.2a}$$

$$\frac{\partial \hat{z}}{\partial \hat{\boldsymbol{n}}_{\hat{e}}}(\hat{m}_{\hat{e}}) \qquad \forall \hat{e} \in \Delta_1(\hat{T}), \tag{3.3.2b}$$

where $\hat{m}_{\hat{e}}$ denotes the edge midpoint of $\hat{e}$, and $\hat{\boldsymbol{n}}_{\hat{e}}$ denotes the outward normal of $\partial\hat{T}$ restricted to $\hat{e}$.

Lemma 3.3.3. *The space* $\hat{\mathbb{P}}$ *is* 15*-dimensional, and a function* $\hat{z} \in \hat{\mathbb{P}}$ *is uniquely determined by the DOFs* (3.3.2a)–(3.3.2b) *and*

$$\frac{\partial^2 \hat{z}}{\partial \hat{\boldsymbol{t}}_{\hat{e}} \partial \hat{\boldsymbol{n}}_{\hat{e}}}(\hat{m}_{\hat{e}}) \qquad \forall \hat{e} \in \Delta_1(\hat{T}), \tag{3.1c}$$

where $\hat{\boldsymbol{t}}_{\hat{e}}$ *is the unit tangent of* $\hat{e}$*, obtained by rotating* $\hat{\boldsymbol{n}}_{\hat{e}}$ 90 *degrees clockwise.*

In particular, the space

$$\hat{\mathbb{W}} := \{\hat{z} \in \hat{\mathbb{P}} : \ \hat{z} \textit{ vanishes on } (3.3.2a)\text{–}(3.3.2b)\} \subset H^2(T) \cap H_0^1(T) \tag{3.3.3}$$

is three-dimensional, a function $\hat{z} \in \hat{\mathbb{W}}$ *is uniquely determined by its values* (3.1c)*, and there holds* $\hat{\mathbb{P}} = \hat{P} \oplus \hat{\mathbb{W}}$*.*

Proof. We begin with showing that $\dim \mathbb{P}(\hat{T}) \geq 15$. For this purpose, we first consider the intermediate space

$$\mathbb{P}^c(\hat{T}) := \{\hat{z} \in \mathcal{P}_4^c(\hat{T}^{\rm ct}) : \ \hat{z}|_{\hat{e}} \in \mathcal{P}_3(\hat{e}) \ \forall \hat{e} \in \Delta_1(\hat{T}^{\rm ct})\}.$$

Simple arguments show $\dim \mathbb{P}^c(\hat{T}) = 25$. Indeed, a function $\hat{z} \in \mathbb{P}^c(\hat{T})$ is uniquely determined by its values at the vertices in $\hat{T}^{ct}$ (4), its values at two interior points on each edge (12), and its values at three interior points of each sub-triangle (9).

Let $\hat{a}_0$ denote the barycenter of $\hat{T}^{ct}$, and label the vertices $\Delta_0(\hat{T}) = \{\hat{a}_{i0}\}_{i=1}^3$. We also label $\hat{T}^{ct} = \{\hat{K}_i\}_{i=1}^3$ such that $\hat{a}_{i0}$ is not a vertex of $\hat{K}_i$. Let $\hat{\ell}_i$ be the interior edge in $\hat{T}^{ct}$ that connects $\hat{a}_{i0}$ to $\hat{a}_0$, and let $\hat{a}_{i1}$, $\hat{a}_{i2}$ be two interior points of the edge $\hat{\ell}_i$, for $i = 1, 2, 3$ (see Figure 4). We set $\hat{\boldsymbol{n}}_{\ell_i}$ be a unit normal of $\hat{\ell}_i$, and $\hat{\boldsymbol{t}}_{\ell_i}$ be the unit tangent of $\hat{\ell}_i$, that is obtained by rotating $\hat{\boldsymbol{n}}_{\ell_i}$ 90 degrees clockwise.

For a function $\hat{z} \in \mathbb{P}^c(\hat{T})$, let us denote its restriction on $\hat{K}_i$ by $\hat{z}_i$. Suppose that $\hat{z} \in \mathbb{P}^c(\hat{T})$ satisfies the following 10 constraints:

$$\begin{aligned} \frac{\partial \hat{z}_{i+1}}{\partial \hat{\boldsymbol{n}}_{\hat{\ell}_i}}(\hat{a}_{ij}) &= \frac{\partial \hat{z}_{i+2}}{\partial \hat{\boldsymbol{n}}_{\hat{\ell}_i}}(\hat{a}_{ij}) \qquad i = 1,2,3,\ j = 0,1,2, \\ \frac{\partial \hat{z}_3}{\partial \hat{\boldsymbol{n}}_{\hat{\ell}_2}}(\hat{a}_0) &= \frac{\partial \hat{z}_1}{\partial \hat{\boldsymbol{n}}_{\hat{\ell}_2}}(\hat{a}_0), \end{aligned} \tag{3.3.4}$$

with the convention $\hat{z}_4 = \hat{z}_1$. These constraints, with the continuity of the tangential derivative on each $\hat{\ell}_i$, imply $\hat{\nabla}\hat{z}_3|_{\hat{\ell}_2} = \hat{\nabla}\hat{z}_1|_{\hat{\ell}_2}$. Then, using the continuity of $\hat{z}$, we have $\frac{\partial \hat{z}_1}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_1}}(\hat{a}_0) = \frac{\partial \hat{z}_3}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_1}}(\hat{a}_0) = \frac{\partial \hat{z}_2}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_1}}(\hat{a}_0)$. Since $\frac{\partial \hat{z}_1}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_1}}(\hat{a}_0) = \frac{\partial \hat{z}_2}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_1}}(\hat{a}_0)$ and $\frac{\partial \hat{z}_1}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_3}}(\hat{a}_0) = \frac{\partial \hat{z}_2}{\partial \hat{\boldsymbol{t}}_{\hat{\ell}_3}}(\hat{a}_0)$, again by continuity of $\hat{z}$, and $\{\hat{\boldsymbol{t}}_{\hat{\ell}_1}, \hat{\boldsymbol{t}}_{\hat{\ell}_3}\}$ spans $\mathbb{R}^2$, we conclude that $\hat{\nabla}\hat{z}_1(\hat{a}_0) = \hat{\nabla}\hat{z}_2(\hat{a}_0)$. This, when combined with (3.3.4), implies $\hat{\nabla}\hat{z}_1|_{\hat{\ell}_3} = \hat{\nabla}\hat{z}_2|_{\hat{\ell}_3}$. Similar arguments show $\hat{\nabla}\hat{z}_2|_{\hat{\ell}_1} = \hat{\nabla}\hat{z}_3|_{\hat{\ell}_1}$, and so $\hat{z} \in C^1(\hat{T})$. Thus, imposing the 10 constraints given in (3.3.4) on $\mathbb{P}^c(\hat{T})$ induces $\mathbb{P}(\hat{T})$, and so $\dim \mathbb{P}(\hat{T}) \geq \dim \mathbb{P}^c(\hat{T}) - 10 = 15$.

Next, suppose that $\hat{z}$ vanishes at the claimed 15 DOFs (3.3.2). This implies that $\hat{z} = \hat{\mu}^2\hat{p}$, where $\hat{\mu} \in \mathbb{P}_1^c(\hat{T}^{ct}) \cap H_0^1(\hat{T})$ with $\hat{\mu}(\hat{a}_0) = 1$, $\hat{p}$ is continuous, quadratic Lagrange with respect to $\hat{T}^{ct}$, and linear on each edge $\hat{e} \in \Delta_0(\hat{T})$. Notice that this actually implies that $\hat{p}$ is continuous and linear with respect to $\hat{T}^{ct}$, and so $\hat{z} = \hat{\mu}^2\hat{p}$ is cubic with respect to $\hat{T}^{ct}$, i.e., $\hat{z} \in \mathcal{P}_3^{c1}(\hat{T}^{ct})$. Then, by Lemma 3.3.2, we deduce that $\hat{z} \equiv 0$, and therefore $\dim \mathbb{P}(\hat{T}) = 15$, and (3.3.2) is a unisolvent set of DOFs for $\mathbb{P}(\hat{T})$.

Since (3.3.2)-(3.1c) represents a unisolvent set of DOFs for $\mathbb{P}(\hat{T})$, we conclude from the definition of $\hat{\mathbb{W}}$ that a function $\hat{z} \in \hat{\mathbb{W}}$ is uniquely determined by the values (3.1c). Therefore, $\dim \hat{\mathbb{W}} = 3$.

Moreover, by Lemma 3.3.2 and the definition of $\mathbb{W}(\hat{T})$, we also see that there holds $\mathcal{P}_3^{c1}(\hat{T}^{\mathrm{ct}}) \cap \mathbb{W}(\hat{T}) = \{0\}$, and so

$$\dim(\mathcal{P}_3^{c1}(\hat{T}^{\mathrm{ct}}) \oplus \mathbb{W}(\hat{T})) = \dim \mathcal{P}_3^{c1}(\hat{T}^{\mathrm{ct}}) + \dim \mathbb{W}(\hat{T}) = 12 + 3 = 15.$$

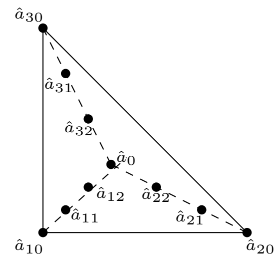

Figure 4: Node labeling convention 1.

The desired result now follows from $\mathcal{P}_3^{c1}(\hat{T}^{\mathrm{ct}}) \oplus \mathbb{W}(\hat{T}) \subset \mathbb{P}(\hat{T})$ and $\dim \mathbb{P}(\hat{T}) = 15$. $\square$

The next lemma exploits the DOFs of $\mathbb{W}(\hat{T})$ to construct a function in $\mathbb{W}(\hat{T})$ such that its normal derivative when restricted on $\partial\hat{T}$ coincides with a given function in $\mathcal{P}_3(\hat{T}^{ct})$ satisfying certain properties.

Lemma 3.3.4. *Let $\hat{w} \in \mathcal{P}_3(\hat{T}^{\mathrm{ct}})|_{\partial\hat{T}}$ such that $\hat{w}$ vanishes at the three vertices and the three edge midpoints of $\hat{T}$, i.e., $\hat{w}(\hat{a}) = 0$ and $\hat{w}(\hat{m}_{\hat{e}}) = 0$ for all $\hat{a} \in \Delta_0(\hat{T})$ and $\hat{e} \in \Delta_1(\hat{T})$. Then, there exists a unique $\hat{z} \in \hat{\mathbb{W}}$ such that*

$$\frac{\partial \hat{z}}{\partial \hat{\boldsymbol{n}}}\Big|_{\partial\hat{T}} = \hat{w}.$$

Proof. Let $\hat{w} \in \mathcal{P}_3(\hat{T}^{ct})|_{\partial\hat{T}}$ be a function that vanishes on $\Delta_0(\hat{T})$ and edge midpoints of $\hat{T}$. Using Lemma 3.3.3, we uniquely define $\hat{z} \in \hat{\mathbb{W}}$ by the conditions

$$\frac{\partial^2 \hat{z}}{\partial \hat{\boldsymbol{t}}_{\hat{e}} \partial \hat{\boldsymbol{n}}_{\hat{e}}}(\hat{m}_{\hat{e}}) = \frac{\partial \hat{w}}{\partial \hat{\boldsymbol{t}}_{\hat{e}}}(\hat{m}_{\hat{e}}) \quad \forall \hat{e} \in \Delta_1(\hat{T}).$$

For an edge $\hat{e} \in \Delta_1(\hat{T})$, set $\hat{r}_{\hat{e}} := \big(\frac{\partial \hat{z}}{\partial \hat{\boldsymbol{n}}_{\hat{e}}} - \hat{w}\big)|_{\hat{e}} \in \mathcal{P}_3(\hat{e})$. Then by the properties of $\hat{w}$ and the definition of $\hat{\mathbb{W}}$, $\hat{r}_{\hat{e}}$ vanishes at three distinct points on $\hat{e}$. Furthermore, by construction, and with an abuse of notation, we have $\hat{r}'_{\hat{e}}(\hat{m}_{\hat{e}}) = 0$. Notice that these conditions imply $\hat{r}_{\hat{e}} = 0$, and we conclude $\frac{\partial \hat{z}}{\partial \hat{\boldsymbol{n}}}|_{\partial\hat{T}} = \hat{w}|_{\partial T}$. $\square$

Next, we combine the local space $\hat{\boldsymbol{V}}$ with the image of the space $\hat{\mathbb{W}}$ under the curl operator to introduce a new enriched space.

3.3.2 A local and enriched Lagrange C^0 element

Using the space $\hat{\mathbb{W}}$ given by (3.3.3), we define the enriched (local) Lagrange space as

$$\hat{\mathbb{V}} = \hat{\boldsymbol{V}} + \widehat{\mathbf{curl}}\,\hat{\mathbb{W}}. \tag{3.3.5}$$

Proposition 3.3.5. *The sum in* (3.3.5) *is direct, in particular,* $\hat{\boldsymbol{V}} \cap \widehat{\mathbf{curl}}\,\hat{\mathbb{W}} = \{0\}$.

Proof. Let $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}} \cap \widehat{\mathbf{curl}}\,\hat{\mathbb{W}}$, then $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{V}}$ and is divergence-free. By Lemma 3.3.1, this implies $\hat{\boldsymbol{v}} = \widehat{\mathbf{curl}}\,\hat{z}_1$ for some $\hat{z}_1 \in \hat{P}$. On the other hand, because $\hat{\boldsymbol{v}} \in \widehat{\mathbf{curl}}\,\hat{\mathbb{W}}$, we can write $\hat{\boldsymbol{v}} = \widehat{\mathbf{curl}}\,\hat{z}_2$ for some $\hat{z}_2 \in \hat{\mathbb{W}}$ and so, $\widehat{\mathbf{curl}}\,\hat{z}_1 = \widehat{\mathbf{curl}}\,\hat{z}_2$, which further implies that $\hat{z}_1 = \hat{z}_2 + \hat{c}$ for some $\hat{c} \in \mathbb{R}$. Since $\hat{c} \in \hat{P}$ and $\hat{z}_1 = \hat{c}$ on the DOFs (3.3.2a)–(3.3.2b) by the construction of $\hat{\mathbb{W}}$, we conclude that $\hat{z}_1 \equiv \hat{c}$ by Lemma 3.3.2. Hence, $\hat{\boldsymbol{v}} \equiv 0$. □

3.3.3 Local mappings

Before we define the local mappings, we introduce some notation. Let $\{\hat{a}_i\}_{i=1}^{10}$ and $\{\tilde{a}_i\}_{i=1}^{10}$ denote the sets of vertices and edge midpoints with respect to $\hat{T}^{ct}$ and $\tilde{T}^{ct}$, respectively, labeled such that $\tilde{a}_i = F_{\tilde{T}}(\hat{a}_i)$. That is, $\{\hat{a}_i\}_{i=1}^{10}$ and $\{\tilde{a}_i\}_{i=1}^{10}$ are the locations of the Lagrange DOFs for $\hat{\boldsymbol{V}}$ and $\tilde{\boldsymbol{V}}(\tilde{T})$, respectively. With the help of Lemma 3.3.4 and [28, Lemma 3.3] (or see Lemma 2.3.3 in Chapter 2), we define three local mappings.

Definition 3.3.6. Let $\tilde{T} \in \tilde{\mathfrak{T}}_h$ and $T \in \mathfrak{T}_h$ with $T = G_h(\tilde{T})$.

(i) We define the bijection $\boldsymbol{\Psi}_T : \tilde{\boldsymbol{V}}(\tilde{T}) \to \boldsymbol{V}(T)$ uniquely determined by the conditions

$$(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}})(a_i) = \tilde{\boldsymbol{v}}(\tilde{a}_i) \quad i = 1, 2 \ldots, 10, \tag{3.3.6}$$

where $a_i = G_h(\tilde{a}_i) = F_T(\hat{a}_i)$.

(ii) We define the operator $\hat{\Theta}_{\tilde{T}} : \tilde{\boldsymbol{V}}(\tilde{T}) \to \hat{\mathbb{W}}$ uniquely determined by the conditions

$$\frac{\partial \hat{\Theta}_{\tilde{T}} \tilde{\boldsymbol{v}}}{\partial \hat{\boldsymbol{n}}}\Big|_{\partial \hat{T}} = A_T^{-1}\big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}}\big) \cdot \hat{\boldsymbol{t}}\big|_{\partial \hat{T}}. \tag{3.3.7}$$

(iii) We define the operator $\Upsilon_T : \tilde{Q}(\tilde{T}) \to Q(T)$ as

$$(\Upsilon_T \tilde{q})(x) = \tilde{q}(F_{\tilde{T}}(\hat{x})), \quad x = F_T(\hat{x}).$$

Remark 3.3.7. Notice that $\boldsymbol{\Psi}_T$ is the same operator as the one given in Definition 2.3.6. In particular, $\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}$ is of the form $(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}})(x) = (A_T \hat{\boldsymbol{v}}_0)(\hat{x})$, where $x = F_T(\hat{x})$, and $\hat{\boldsymbol{v}}_0 \in \hat{\boldsymbol{V}}$ uniquely satisfies $\hat{\boldsymbol{v}}_0(\hat{a}_i) = A_T^{-1}(\hat{a}_i)\tilde{\boldsymbol{v}}(\tilde{a}_i)$ $(i = 1, 2, \ldots, 10)$. We then immediately see that $A_T^{-1}\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T = \hat{\boldsymbol{v}}_0 \in \hat{\boldsymbol{V}}$. Moreover, as the entries of A_T^{-1} are linear polynomials on $\hat{T}$, there holds $A_T^{-1}\tilde{\boldsymbol{v}} \circ F_{\tilde{T}} \in \mathcal{P}_3^c(\hat{T}^{ct})$. This, together with the properties of $\boldsymbol{\Psi}_T$, ensures that the mapping $\hat{\Theta}_{\tilde{T}}$ is well-defined by Lemma 3.3.4 with $\hat{w} = A_T^{-1}\big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}}\big) \cdot \hat{\boldsymbol{t}}\big|_{\partial \hat{T}}$.

Remark 3.3.8. Notice, by Lemmas 3.3.3–3.3.4, that (3.3.7) is satisfied if and only if

$$\frac{\partial^2 \hat{\Theta}_{\tilde{T}} \tilde{\boldsymbol{v}}}{\partial \hat{\boldsymbol{n}}_{\hat{e}} \partial \hat{\boldsymbol{t}}_{\hat{e}}}(\hat{m}_{\hat{e}}) = \frac{\partial}{\partial \hat{\boldsymbol{t}}_{\hat{e}}}\Big(A_T^{-1}(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}}) \cdot \hat{\boldsymbol{t}}_{\hat{e}}\Big)(\hat{m}_{\hat{e}}) \quad \forall \hat{e} \in \Delta_1(\hat{T}).$$

The following theorem is the main result of this section and establishes the building block of the construction of a global continuous space using isoparametric Piola mappings.

Theorem 3.3.9. *We define the operator* $\boldsymbol{\Psi}_T^C : \tilde{\boldsymbol{V}}(\tilde{T}) \to \boldsymbol{H}^1(T)$ *as*

$$\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}} := \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} - (A_T \widehat{\mathbf{curl}}\, \hat{\Theta}_{\tilde{T}} \tilde{\boldsymbol{v}}) \circ F_T^{-1} \qquad \forall \tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}(\tilde{T}).$$

Then, there holds

$$\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}|_e = \tilde{\boldsymbol{v}}|_e \qquad \forall e \in \mathcal{E}_T^I, \quad \textit{and} \quad \boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}|_{\partial T \cap \partial \Omega_h} = 0,$$

where $\mathcal{E}_T^I$ *denotes the set of (straight) interior edges of* T.

Proof. In order to ease the notation, we set $\hat{z} := \hat{\Theta}_{\tilde{T}} \tilde{\boldsymbol{v}} \in \hat{\mathbb{W}}$ and use (3.3.7) to write

$$(\widehat{\mathbf{curl}}\, \hat{z}) \cdot \hat{\boldsymbol{t}}\big|_{\partial \hat{T}} = \frac{\partial \hat{z}}{\partial \hat{\boldsymbol{n}}}\big|_{\partial \hat{T}} = A_T^{-1} \big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}} \big) \cdot \hat{\boldsymbol{t}}\big|_{\partial \hat{T}}.$$

By the definition of $\boldsymbol{\Psi}_T$ and the properties of the Piola transform (see also Lemma 2.2.3), there holds

$$A_T^{-1} \big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}} \big) \cdot \hat{\boldsymbol{n}}\big|_{\hat{e}} = 0, \qquad \forall \hat{e} \in \mathcal{E}_{\hat{T}}^I := \{ F_T^{-1}(e) : \ e \in \mathcal{E}_T^I \}.$$

Moreover, since $\hat{z}|_{\partial \hat{T}} = 0$, there holds $(\widehat{\mathbf{curl}}\, \hat{z}) \cdot \hat{\boldsymbol{n}}|_{\partial \hat{T}} = 0$, and as a result, we have

$$\widehat{\mathbf{curl}}\, \hat{z}|_{\hat{e}} = A_T^{-1} \big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}} \big)\big|_{\hat{e}} \qquad \forall \hat{e} \in \mathcal{E}_{\hat{T}}^I. \tag{3.3.8}$$

Then, for $\hat{x} \in \mathcal{E}_{\hat{T}}^I$, we again use (3.3.7) to find with $x = F_T(\hat{x}) = F_{\tilde{T}}(\hat{x})$,

$$\begin{aligned} \boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}(x) &= \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}(x) - (A_T \widehat{\mathbf{curl}}\, \hat{z})(\hat{x}) \\ &= \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}(x) - \big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}} \big)(\hat{x}) \\ &= \tilde{\boldsymbol{v}}(x). \end{aligned}$$

Therefore, $\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}|_e = \tilde{\boldsymbol{v}}|_e$ for $e \in \mathcal{E}_T^I$.

Lastly, if $e = \partial T \cap \partial \Omega_h$ is a curved boundary edge, let $\tilde{e} = G_h^{-1}(e) = \partial \tilde{T} \cap \partial \tilde{\Omega}_h$ be the corresponding affine boundary edge. Then, by definition of $\tilde{\boldsymbol{V}}(\tilde{T})$, we have $\tilde{\boldsymbol{v}}|_{\tilde{e}} = 0$, which implies that $\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}|_e = 0$, and this further implies by (3.3.7) that $\widehat{\mathbf{curl}}\, \hat{z}|_{\hat{e}} = 0$. Hence, $\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}|_e = \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}|_e = 0$. $\square$

In what follows, we provide some estimates for the operator $\boldsymbol{\Psi}_T$, several properties of the operator $\boldsymbol{\Psi}_T^C$, and also study the approximation property of $\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}$ for a given $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}(\tilde{T})$.

Lemma 3.3.10. *Let $\tilde{T} \in \tilde{\mathfrak{T}}_h$ and $T \in \mathfrak{T}_h$ with $T = G_h(\tilde{T})$. There holds*

$$\|\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|_{H^1(T)} \leq C\|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})}, \quad \textit{and} \quad \|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})} \leq C\|\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|_{H^1(T)}. \tag{3.3.9}$$

Moreover, if T is affine, i.e., $\tilde{T} = T$, then $\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$.

Proof. The identity $\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$ on an affine triangle and the first estimate in (3.3.9) are shown in Theorem 2.3.7.

In order to prove the second estimate, notice again that the result trivially follows if T has only straight edges. We then assume that T has a curved edge, and write $(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}})(x) = (A_T \hat{\boldsymbol{v}}_0)(\hat{x})$, where $\hat{\boldsymbol{v}}_0 \in \hat{\boldsymbol{V}}$ is uniquely determined by the conditions $\hat{\boldsymbol{v}}_0(\hat{a}_i) = A_T^{-1}(\hat{a}_i)\tilde{\boldsymbol{v}}(\tilde{a}_i)$ for $i = 1, 2, \ldots, 10$. Then, a standard scaling argument together with Lemma 2.2.2 and equivalence of norms yield

$$\begin{aligned}
\|\tilde{\boldsymbol{v}}\|^2_{H^1(\tilde{T})} &\leq C \sum_{i=1}^{10} |\tilde{\boldsymbol{v}}(\tilde{a}_i)|^2 = C \sum_{i=1}^{10} |(A_T \hat{\boldsymbol{v}}_0)(\hat{a}_i)|^2 \\
&\leq C h_T^{-2} \sum_{i=1}^{10} |\hat{\boldsymbol{v}}_0(\hat{a}_i)|^2 \leq C h_T^{-2} \|\hat{\boldsymbol{v}}_0\|^2_{H^1(\hat{T})}.
\end{aligned}$$

Finally, we apply Lemmas 2.2.2–2.2.4 and the Poincaré inequality to conclude

$$\|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})} \leq C h_T^{-2} \|A_T^{-1}\|^2_{W^{1,\infty}(\hat{T})} \|A_T \hat{\boldsymbol{v}}_0\|^2_{H^1(\hat{T})} \leq C\|A_T \hat{\boldsymbol{v}}_0\|^2_{H^1(\hat{T})} \leq C\|\boldsymbol{\Psi} \tilde{\boldsymbol{v}}\|^2_{H^1(T)}.$$

□

Lemma 3.3.11. *Let $\tilde{T} \in \tilde{\mathfrak{T}}_h$ and $T \in \mathfrak{T}_h$ with $T = G_h(\tilde{T})$. Then, for all $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}(\tilde{T})$, there holds* *(i)* $\boldsymbol{\Psi}_T^C\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$ *if* T *is affine.*

(ii) $\operatorname{div}\boldsymbol{\Psi}_T^C\tilde{\boldsymbol{v}} = \operatorname{div}\boldsymbol{\Psi}_T\tilde{\boldsymbol{v}}$*.*

(iii) $\|\boldsymbol{\Psi}_T^C\tilde{\boldsymbol{v}}\|_{H^1(T)} \le C\|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})}$*.*

Proof. (i) If T is affine, then $\boldsymbol{\Psi}_T\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$ by Lemma 3.3.10. Also, notice from (3.3.7) and Lemma 3.3.4 that in this case we have $\frac{\partial\hat{\Theta}_{\tilde{T}}\tilde{\boldsymbol{v}}}{\partial\hat{\boldsymbol{n}}}\big|_{\partial\hat{T}} = 0$, which implies $\hat{\Theta}_{\tilde{T}}\tilde{\boldsymbol{v}} = 0$. Therefore, if T is affine, then $\boldsymbol{\Psi}_T^C\tilde{\boldsymbol{v}} = \boldsymbol{\Psi}_T\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}$.

(ii) The divergence-preserving property of the Piola transform (2.2.5) shows

$$\operatorname{div}\Big((A_T\widehat{\mathbf{curl}}\,\hat{\Theta}_{\tilde{T}}\tilde{\boldsymbol{v}})\circ F_T^{-1}\Big) = \Big(\frac{1}{\det(DF_T)}\widehat{\operatorname{div}}\,\widehat{\mathbf{curl}}\,(\hat{\Theta}_{\tilde{T}}\tilde{\boldsymbol{v}})\Big)\circ F_T^{-1} = 0.$$

The above identity immediately yields $\operatorname{div}\boldsymbol{\Psi}_T^C\tilde{\boldsymbol{v}} = \operatorname{div}\boldsymbol{\Psi}_T\tilde{\boldsymbol{v}}$.

(iii) Assume again that T is a boundary triangle with a curved edge as otherwise the result is trivial by (i). In order to ease the notation, we set $\hat{z} = \hat{\Theta}_{\tilde{T}}\tilde{\boldsymbol{v}} \in \hat{\mathbb{W}}$ so that

$$\boldsymbol{\Psi}_T^C\tilde{\boldsymbol{v}} = \boldsymbol{\Psi}_T\tilde{\boldsymbol{v}} - (A_T\widehat{\mathbf{curl}}\,\hat{z})\circ F_T^{-1}. \tag{3.3.10}$$

Notice from (3.3.9) that it suffices to estimate the term $\|\widehat{\mathbf{curl}}\,\hat{z}\|^2_{H^1(\hat{T})}$. Using Lemma 3.3.4, equivalence of norms, and (3.3.7) we get

$$\|\widehat{\mathbf{curl}}\,\hat{z}\|^2_{H^1(\hat{T})} \le \Big\|\frac{\partial\hat{z}}{\partial\hat{\boldsymbol{n}}}\Big\|^2_{L^2(\partial\hat{T})} \le C\|A_T^{-1}\|^2_{L^\infty(\hat{T})}\Big(\|\boldsymbol{\Psi}_T\tilde{\boldsymbol{v}}\circ F_T\|^2_{L^2(\hat{T})} + \|\tilde{\boldsymbol{v}}\circ F_{\tilde{T}}\|^2_{L^2(\hat{T})}\Big).$$

Then, by Lemmas 2.2.2–2.2.4, the estimate in (3.3.9) and the Poincaré inequality, we obtain

$$\begin{aligned}\|\widehat{\mathbf{curl}}\,\hat{z}\|^2_{H^1(\hat{T})} &\le Ch_T^2\Big(\|\boldsymbol{\Psi}_T\tilde{\boldsymbol{v}}\circ F_T\|^2_{L^2(\hat{T})} + \|\tilde{\boldsymbol{v}}\circ F_{\tilde{T}}\|^2_{L^2(\hat{T})}\Big)\\ &\le C\Big(\|\boldsymbol{\Psi}_T\tilde{\boldsymbol{v}}\|^2_{L^2(T)} + \|\tilde{\boldsymbol{v}}\|^2_{L^2(\tilde{T})}\Big) \le Ch_T^2\|\tilde{\boldsymbol{v}}\|^2_{H^1(\tilde{T})}.\end{aligned}$$

Finally, we plug in this estimate to (3.3.10) and again apply Lemmas 2.2.2–2.2.4 with (3.3.9) to conclude

$$
\begin{aligned}
\|\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}\|_{H^1(T)} &\le \|\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|_{H^1(T)} + C\|A_T \widehat{\mathbf{curl}}\, \hat{z}\|_{H^1(\hat{T})} \\
&\le C\big(\|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})} + \|A_T\|_{W^{1,\infty}(\hat{T})}\|\widehat{\mathbf{curl}}\, \hat{z}\|_{H^1(\hat{T})}\big) \\
&\le C\big(\|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})} + h_T^{-1}\|\widehat{\mathbf{curl}}\, \hat{z}\|_{H^1(\hat{T})}\big) \le C\|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{T})}.
\end{aligned}
$$

□

Lemma 3.3.12. *For $T \in \mathfrak{T}_h$ and $\boldsymbol{u} \in \boldsymbol{H}^3(T)$, let $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}(\tilde{T})$ be the unique function satisfying $\tilde{\boldsymbol{v}}(\tilde{a}_i) = \boldsymbol{u}(a_i)$ where $G_h(\tilde{a}_i) = a_i$ for $i = 1, 2, ..., 10$. Then, there holds*

$$
\|\boldsymbol{u} - \boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}\|_{H^m(T)} \le h_T^{3-m}\|\boldsymbol{u}\|_{H^3(T)} \qquad m = 0, 1. \tag{3.3.11}
$$

Proof. We write $\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}} = \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} - (A_T \widehat{\mathbf{curl}}\, \hat{z}) \circ F_T^{-1}$ with $\hat{z} = \hat{\Theta}_{\tilde{T}} \tilde{\boldsymbol{v}} \in \hat{\mathbb{W}}$. From Lemma 2.3.5, we have

$$
\|\boldsymbol{u} - \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|_{H^m(T)} \le h_T^{3-m}\|\boldsymbol{u}\|_{H^3(T)} \qquad m = 0, 1. \tag{3.3.12}
$$

By equivalence of norms and the triangle inequality, we also have

$$
\begin{aligned}
|\widehat{\mathbf{curl}}\, \hat{z}|^2_{H^m(\hat{T})} &\le C\|\widehat{\mathbf{curl}}\, \hat{z}\|^2_{L^2(\partial\hat{T})} \\
&\le C\big\|A_T^{-1}\big(\boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}}\big)\big\|^2_{L^2(\hat{T})} \\
&\le Ch_T^2\big(\|\hat{\boldsymbol{u}} - \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T\|^2_{L^2(\hat{T})} + \|\hat{\boldsymbol{u}} - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}}\|^2_{L^2(\hat{T})}\big),
\end{aligned} \tag{3.3.13}
$$

where $\hat{\boldsymbol{u}} \in \boldsymbol{H}^3(\hat{T})$ is defined as $\hat{\boldsymbol{u}}(\hat{x}) := \boldsymbol{u}(x)$ with $x = F_T(\hat{x})$. Using Lemma 2.2.4 and (3.3.12), we get

$$
\|\hat{\boldsymbol{u}} - \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}} \circ F_T\|^2_{L^2(\hat{T})} \le Ch_T^{-2}\|\boldsymbol{u} - \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|^2_{L^2(T)} \le Ch_T^4\|\boldsymbol{u}\|^2_{H^3(T)}. \tag{3.3.14}
$$

Next, using the properties of $\tilde{\boldsymbol{v}}$ together with standard approximation theory and Lemma 2.2.4, we obtain

$$\|\hat{\boldsymbol{u}} - \tilde{\boldsymbol{v}} \circ F_{\tilde{T}}\|^2_{L^2(\hat{T})} \le C|\hat{\boldsymbol{u}}|^2_{H^3(\hat{T})} \le Ch_T^4 \|\boldsymbol{u}\|^2_{H^3(T)}. \tag{3.3.15}$$

Combining (3.3.13)–(3.3.15) yields

$$|\widehat{\mathbf{curl}}\,\hat{z}|^2_{H^m(\hat{T})} \le Ch_T^6 \|\boldsymbol{u}\|^2_{H^3(T)},$$

and finally, we use (3.3.12) and Lemmas 2.2.4–2.2.2 with the triangle inequality to conclude

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{\Psi}_T^C \tilde{\boldsymbol{v}}\|_{H^m(T)} &\le \|\boldsymbol{u} - \boldsymbol{\Psi}_T \tilde{\boldsymbol{v}}\|_{H^m(T)} + \|(A_T \widehat{\mathbf{curl}}\,\hat{z}) \circ F_T^{-1}\|_{H^m(T)} \\
&\le C\big(h_T^{3-m}\|\boldsymbol{u}\|_{H^3(T)} + h_T^{1-m}\|A_T\widehat{\mathbf{curl}}\,\hat{z}\|_{H^m(\hat{T})}\big) \\
&\le C\big(h_T^{3-m}\|\boldsymbol{u}\|_{H^3(T)} + h_T^{-m}\|\widehat{\mathbf{curl}}\,\hat{z}\|_{H^m(\hat{T})}\big) \\
&\le Ch_T^{3-m}\|\boldsymbol{u}\|_{H^3(T)}.
\end{aligned}$$

□

3.4 Global spaces and inf-sup stability

We use the local mappings $\boldsymbol{\Psi}_T$, $\boldsymbol{\Psi}_T^C$, and Υ_T, for $T \in \mathfrak{T}_h$, introduced in the previous section in order to introduce the corresponding global mappings $\boldsymbol{\Psi}$, $\boldsymbol{\Psi}^C$, and Υ, which are defined by

$$\boldsymbol{\Psi}|_T := \boldsymbol{\Psi}_T, \qquad \boldsymbol{\Psi}^C|_T := \boldsymbol{\Psi}_T^C, \qquad \Upsilon|_T := \Upsilon_T, \quad \forall T \in \mathfrak{T}_h.$$

The Scott-Vogelius pair on the affine triangulation $\tilde{\mathfrak{T}}_h$ is defined as

$$\begin{aligned}
\tilde{\boldsymbol{V}}^h &= \{\tilde{\boldsymbol{v}} \in \boldsymbol{H}_0^1(\tilde{\Omega}_h):\ \tilde{\boldsymbol{v}}|_{\tilde{T}} \in \tilde{\boldsymbol{V}}(\tilde{T}),\ \forall \tilde{T} \in \tilde{\mathfrak{T}}_h\}, \\
\tilde{Q}^h &= \{\tilde{q} \in L_0^2(\tilde{\Omega}_h):\ \tilde{q}|_{\tilde{T}} \in \tilde{Q}(\tilde{T}),\ \forall \tilde{T} \in \tilde{\mathfrak{T}}_h\}.
\end{aligned}$$

Using this pair $\tilde{\boldsymbol{V}}^h \times \tilde{Q}^h$ and the global mappings $\boldsymbol{\Psi}^C$, Υ, we define the global velocity space and global pressure space, respectively, as follows:

$$\boldsymbol{V}_C^h := \{\boldsymbol{v} :\ \boldsymbol{v} = \boldsymbol{\Psi}^C \tilde{\boldsymbol{v}},\ \exists \tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h\}, \quad Q^h := \{q :\ q = \Upsilon \tilde{q},\ \exists \tilde{q} \in \tilde{Q}^h\}.$$

Lemma 3.4.1. *There holds* $\boldsymbol{V}_C^h \subset \boldsymbol{H}_0^1(\Omega_h)$.

Proof. By Theorem 3.3.9, we immediately see that if $\boldsymbol{v} \in \boldsymbol{V}_C^h$, then $\boldsymbol{v}$ vanishes on $\partial\Omega_h$. Then, it remains to show that any function $\boldsymbol{v} \in \boldsymbol{V}_C^h$ is single valued when restricted on any $e \in \mathcal{E}_T^I$ with $T \in \mathcal{T}_h$. Let $e = T_+ \cap T_- \neq \emptyset$ with $T_+, T_- \in \mathcal{T}_h$ and $\boldsymbol{v} \in \boldsymbol{V}_C^h$. We write $\boldsymbol{v} = \boldsymbol{\Psi}^C \tilde{\boldsymbol{v}}$ for some $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$. Let $\boldsymbol{v}_+, \boldsymbol{v}_-$ denote the restriction of $\boldsymbol{v}$ to T_+ and T_-, and $\tilde{\boldsymbol{v}}_+, \tilde{\boldsymbol{v}}_-$ denote the restriction of $\tilde{\boldsymbol{v}}$ to T_+ and T_-, respectively. By Theorem 3.3.9 and continuity of the functions in $\tilde{\boldsymbol{V}}^h$, we find

$$\boldsymbol{v}_+|_e = \tilde{\boldsymbol{v}}_+|_e = \tilde{\boldsymbol{v}}_-|_e = \boldsymbol{v}_-|_e,$$

which completes the proof. $\square$

We also recall the $\boldsymbol{H}(\mathrm{div};\Omega_h)$-conforming space (Theorem 2.4.2)

$$\boldsymbol{V}^h = \{\boldsymbol{v} :\ \boldsymbol{v} = \boldsymbol{\Psi} \tilde{\boldsymbol{v}},\ \exists \tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h\}.$$

The following theorem exploits the stability of the pair $\boldsymbol{V}^h \times Q^h$ (Theorem 2.4.4) to show that the conforming Stokes pair $\boldsymbol{V}_C^h \times Q^h$ is also inf-sup stable.

Theorem 3.4.2. *There exists* $\beta > 0$ *independent of* h *such that*

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_C^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \geq \beta \|q\|_{L^2(\Omega_h)} \qquad \forall q \in Q^h. \tag{3.4.1}$$

Proof. It is proven in Theorem 2.4.4 that the (nonconforming) pair $\boldsymbol{V}^h \times Q^h$ is inf-sup stable. In particular, there exists $C > 0$ such that for a given $q \in Q^h$, there exists $\boldsymbol{v} \in \boldsymbol{V}^h$ such that

$$\frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \geq C \|q\|_{L^2(\Omega_h)}. \tag{3.4.2}$$

Let $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$ be the unique function such that $\boldsymbol{v} = \boldsymbol{\Psi} \tilde{\boldsymbol{v}} \in \boldsymbol{V}^h$, and set $\boldsymbol{v}^C := \boldsymbol{\Psi}^C \tilde{\boldsymbol{v}}$. By Lemma 3.3.10–3.3.11, we have

$$\nabla \cdot \boldsymbol{v}^C = \nabla \cdot \boldsymbol{\Psi}^C \tilde{\boldsymbol{v}} = \nabla \cdot \boldsymbol{\Psi} \tilde{\boldsymbol{v}} = \nabla \cdot \boldsymbol{v},$$

and

$$\|\boldsymbol{v}^C\|_{H^1(\Omega_h)} \leq C \|\tilde{\boldsymbol{v}}\|_{H^1(\tilde{\Omega}_h)} \leq C \|\boldsymbol{v}\|_{H^1(\Omega)}.$$

Using these two identities in (3.4.2), we obtain

$$\|q\|_{L^2(\Omega_h)} \leq C \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}^C) q}{\|\boldsymbol{v}\|_{H^1(\Omega_h)}} \leq C \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}^C) q}{\|\boldsymbol{v}^C\|_{H^1(\Omega_h)}},$$

which yields the desired inf-sup stability result. □

3.5 Finite element method and convergence analysis

We assume the data is sufficiently regular so that the exact solution satisfies $(\boldsymbol{u}, p) \in \boldsymbol{H}^3(\Omega) \times H^2(\Omega)$. Since $\partial\Omega$ is assumed smooth, there exists extensions of the solution, still denoted by $(\boldsymbol{u}, p)$, such that $\nabla \cdot \boldsymbol{u} = 0$ in $\mathbb{R}^2$ and [42]

$$\|\boldsymbol{u}\|_{H^3(\mathbb{R}^2)} \leq C \|\boldsymbol{u}\|_{H^3(\Omega)}, \qquad \|p\|_{H^2(\mathbb{R}^2)} \leq C \|p\|_{H^2(\Omega)}.$$

We then again extend the source function via $\boldsymbol{f} = -\nu\Delta\boldsymbol{u} + \nabla p$ in $\mathbb{R}^2$. The proposed method seeks $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_C^h \times Q^h$ such that

$$\int_{\Omega_h} \nu\nabla\boldsymbol{u}_h : \nabla\boldsymbol{v} - \int_{\Omega_h} (\nabla\cdot\boldsymbol{v})p_h = \int_{\Omega_h} \boldsymbol{f}_h\cdot\boldsymbol{v} \qquad \forall\boldsymbol{v}\in\boldsymbol{V}_C^h, \tag{3.5.1a}$$

$$\int_{\Omega_h} (\nabla\cdot\boldsymbol{u}_h)q = 0 \qquad \forall q\in Q^h, \tag{3.5.1b}$$

where $\boldsymbol{f}_h \in \boldsymbol{L}^2(\Omega_h)$ is some computable approximation of $\boldsymbol{f}|_\Omega$.

The next lemma shows that the method (3.5.1) yields an exactly divergence-free velocity approximation.

Lemma 3.5.1. *Suppose $\boldsymbol{u}_h \in \boldsymbol{V}_C^h$ satisfies* (3.5.1b)*. Then, $\nabla\cdot\boldsymbol{u}_h \equiv 0$ in Ω_h.*

Proof. We write $\boldsymbol{u}_h = \boldsymbol{\Psi}^C\tilde{\boldsymbol{v}}$ for some $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}^h$. Then, by (3.5.1b) and Lemma 3.3.11 we have

$$\int_{\Omega_h} (\nabla\cdot\boldsymbol{u}_h)q = \int_{\Omega_h} (\nabla\cdot\boldsymbol{\Psi}^C\tilde{\boldsymbol{v}})q = \int_{\Omega_h} (\nabla\cdot\boldsymbol{\Psi}\tilde{\boldsymbol{v}})q = 0, \qquad \forall q\in Q^h.$$

Recall that we showed in Lemma 2.5.2 that the right hand side of the above equation implies that $\nabla\cdot\boldsymbol{\Psi}\tilde{\boldsymbol{v}} = 0$ in Ω_h. Hence, $\nabla\cdot\boldsymbol{\Psi}\tilde{\boldsymbol{v}} = \nabla\cdot\boldsymbol{\Psi}^C\tilde{\boldsymbol{v}} = \nabla\cdot\boldsymbol{u}_h = 0$ in Ω_h. □

Theorem 3.4.2 together with standard arguments from the mixed finite element theory ensure that the problem (3.5.1) is well-posed.

Theorem 3.5.2. *There exists a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_C^h \times Q^h$ satisfying* (3.5.1)*.*

The next theorem shows that the method (3.5.1) leads to optimally convergent, discrete velocity and pressure approximations.

Theorem 3.5.3. *There holds*

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} \leq C\big(h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}}\big), \tag{3.5.2}$$

where

$$|\boldsymbol{f}-\boldsymbol{f}_h|_{\boldsymbol{X}_C^{h*}} = \sup_{\boldsymbol{v}\in\boldsymbol{X}_C^h\setminus\{0\}} \frac{\int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)\cdot\boldsymbol{v}}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}}, \quad \boldsymbol{X}_C^h := \{\boldsymbol{v}\in\boldsymbol{V}_C^h :\ \nabla\cdot\boldsymbol{v}=0\}.$$

Moreover, the pressure approximation satisfies

$$\|p-p_h\|_{L^2(\Omega_h)} \leq C\big(\nu h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \inf_{q\in Q^h}\|p-q\|_{L^2(\Omega_h)} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega_h)}\big). \tag{3.5.3}$$

Proof. Notice that the error equation for the method (3.5.1) reads

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p-p_h) = \int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)\boldsymbol{v}_h \quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_C^h, \tag{3.5.4}$$

where $a_h(\boldsymbol{w},\boldsymbol{v}) := \int_{\Omega_h}\nu\nabla\boldsymbol{w} : \nabla\boldsymbol{v}$ and $b_h(\boldsymbol{v},q) := -\int_{\Omega_h}(\nabla\cdot\boldsymbol{v})q$, $\forall\boldsymbol{w},\boldsymbol{v}\in\boldsymbol{H}^1(\Omega_h)$ and $\forall q\in L^2(\Omega_h)$. Then, letting $\boldsymbol{v}_h\in\boldsymbol{X}_C^h$ be arbitrary and applying the integration by parts formula together with Lemma 3.5.1, we find

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) = \int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)\boldsymbol{v}_h \quad \forall\boldsymbol{v}_h\in\boldsymbol{X}_C^h, \tag{3.5.5}$$

where we also used that $\boldsymbol{v}_h$ vanishes on the boundary of Ω_h. We then let $\boldsymbol{w}_h\in\boldsymbol{X}_C^h$ be arbitrary, take $\boldsymbol{v}_h=\boldsymbol{w}_h-\boldsymbol{u}_h\in\boldsymbol{X}_C^h$ as a test function in (3.5.5), and use triangle inequality, the Cauchy-Schwarz inequality with the coercivity of the bilinear form $a_h(.,.)$ to obtain

$$\begin{aligned}\nu\|\nabla(\boldsymbol{u}_h-\boldsymbol{w}_h)\|^2_{L^2(\Omega_h)} &= a_h(\boldsymbol{w}_h-\boldsymbol{u},\boldsymbol{w}_h-\boldsymbol{u}_h) + \int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)(\boldsymbol{w}_h-\boldsymbol{u}_h)\\ &\leq \nu\|\nabla(\boldsymbol{w}_h-\boldsymbol{u})\|_{L^2(\Omega_h)}\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|_{L^2(\Omega_h)}\\ &\quad + \int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)(\boldsymbol{w}_h-\boldsymbol{u}_h),\end{aligned}$$

which, by dividing both sides of the above inequality by $\nu\|\nabla(\boldsymbol{u}_h-\boldsymbol{w}_h)\|_{L^2(\Omega_h)}$, leads

$$\|\nabla(\boldsymbol{u}_h-\boldsymbol{w}_h)\|_{L^2(\Omega_h)} \le \|\nabla(\boldsymbol{w}_h-\boldsymbol{u})\|_{L^2(\Omega_h)}\| + \nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{\boldsymbol{X}^{h*}}.$$

We then use the triangle inequality together with Theorem 3.4.2 and Lemma 3.3.12 to obtain

$$\begin{aligned}\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega_h)} &\le C(\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_C^h}\|\nabla(\boldsymbol{u}-\boldsymbol{v}_h)\|_{L^2(\Omega_h)} + \nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{\boldsymbol{X}^{h*}})\\ &\le C\big(h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{\boldsymbol{X}^{h*}}\big).\end{aligned}$$

It then remains to prove the pressure estimate. To this end, we let $q_h \in Q^h$, $\boldsymbol{v}_h \in \boldsymbol{V}_C^h$ be arbitrary, and use the triangle equality with (3.5.4) to write

$$\begin{aligned}b_h(\boldsymbol{v}_h, p_h-q_h) &\le b_h(\boldsymbol{v}_h, p-p_h) + b_h(\boldsymbol{v}_h, p-q_h)\\ &\le \int_{\Omega_h}(\boldsymbol{f}-\boldsymbol{f}_h)\boldsymbol{v}_h + a_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p-q_h).\end{aligned}$$

Applying the Cauchy-Schwarz inequality with the deduced velocity error to the above inequality, we obtain

$$b_h(\boldsymbol{v}_h, p_h-q_h) \le C\left(\nu h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \|p-q_h\|_{L^2(\Omega_h)} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega_h)}\right)\|\nabla\boldsymbol{v}_h\|_{L^2(\Omega_h)}.$$

Then, Theorem 3.4.2 with $q = p_h - q_h$ with the above inequality yields

$$\|p_h-q_h\|_{L^2(\Omega_h)} \le C\left(\nu h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \|p-q_h\|_{L^2(\Omega_h)} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega_h)}\right).$$

Finally, we apply the triangle inequality with the above inequality to conclude

$$\|p-p_h\|_{L^2(\Omega_h)} \le C\big(\nu h^2\|\boldsymbol{u}\|_{H^3(\Omega)} + \inf_{q\in Q^h}\|p-q\|_{L^2(\Omega_h)} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega_h)}\big).$$

□

Remark 3.5.4. Notice that assuming sufficient regularity on $\boldsymbol{u}$ and using the commuting projections introduced in Section 2.6 together with the properties of the space $\boldsymbol{X}_C^h$, it is again possible to construct an $\boldsymbol{f}_h$ such that $\nu^{-1}|\boldsymbol{f}-\boldsymbol{f}_h|_{X^{h*}} \le Ch^2\|\boldsymbol{u}\|_{H^5(\Omega)}$, and in particular, the method (3.5.1) is pressure-robust.

3.6 Numerics

3.6.1 Implementation aspects

In this section, we discuss the implementation of the computation of a basis for the velocity space $\boldsymbol{V}_C^h$. Recall that this space is defined as the image of the operator $\boldsymbol{\Psi}^C$ acting on the affine quadratic Lagrange finite element space $\tilde{\boldsymbol{V}}^h$. Therefore, for any basis of $\tilde{\boldsymbol{V}}^h$ restricted to some $\tilde{T} \in \tilde{\mathfrak{T}}_h$, say $\{\tilde{\boldsymbol{\varphi}}_i^{(k)}\}$, the set $\{\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{\varphi}}_i^{(k)}\}$ represents a basis of $\boldsymbol{V}_C^h$ restricted to $T = G_h(\tilde{T}) \in \mathfrak{T}_h$. In the following discussion, we take $\tilde{\boldsymbol{\varphi}}_i^{(k)}$ to be the canonical nodal basis. Notice that if T is affine, then $\boldsymbol{\Psi}_T^C \tilde{\boldsymbol{\varphi}}_i^{(k)} = \tilde{\boldsymbol{\varphi}}_i^{(k)}|_T$, and so the computation is standard. Then, in what follows, we study the case where T has a curved edge.

We start with some notation and assumptions. In order to ease the presentation, we set $B = A_T^{-1} = \mathrm{adj}(DF_T) : \hat{T} \to \mathbb{R}^{2\times 2}$. Recall that since $F_T : \hat{T} \to T$ is a quadratic mapping, the entries of B are linear polynomials. We denote the kth column of B by $\boldsymbol{\beta}^{(k)}$, i.e.,

$$\boldsymbol{\beta}_i^{(k)} = B_{i,k}.$$

Let $\hat{e}_1$ denote the edge of $\hat{T}$ connecting $(0,0)$ and $(1,0)$, $\hat{e}_2$ denote the edge of $\hat{T}$ connecting $(0,0)$ and $(0,1)$, and $\hat{e}_3$ denote the remaining edge. Without loss of generality and with an abuse of notation, we assume $F_T(\hat{e}_3) \subset \partial\Omega_h$, i.e., $\hat{e}_3$ is the pre-image of the curved edge of T. Also, let $\{\hat{a}_i\}_{i=1}^{10}$ denote the ten Lagrange DOFs of $\hat{\boldsymbol{V}}$ where, in order to ease the presentation, we assume $\hat{a}_1 = \hat{m}_{\hat{e}_1}$, $\hat{a}_2 = \hat{m}_{\hat{e}_2}$, and $\hat{a}_3 = (0,0)$ (see Figure 5). We also assume $\hat{a}_8, \hat{a}_9, \hat{a}_{10} \in \bar{\hat{e}}_3$, so that, due to the boundary conditions, these DOFs are not active. This labeling convention also implies that the unit tangent on $\hat{e}_k$ is $\hat{\boldsymbol{t}}_{\hat{e}_k} = \pm \boldsymbol{e}^{(k)}$ for $k = 1, 2$, where $\boldsymbol{e}^{(k)} = (\delta_{k1}, \delta_{k2})^\intercal$. Furthermore, let $\hat{K}_i \in \hat{T}^{ct}$, $i = 1, 2, 3$, be such that $\hat{K}_i$ does not contain $\hat{a}_i$.

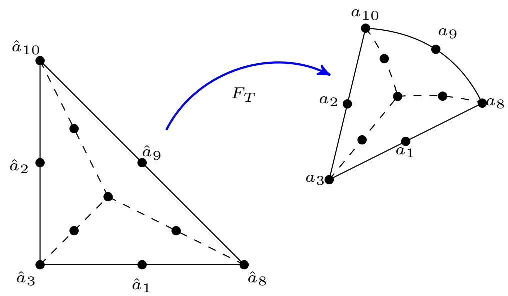

Figure 5: Node labeling convention 2.

Let $\hat{\varphi}_j$ denote the nodal Lagrange basis function of $\mathcal{P}_2^c(\hat{T}^{ct})$ corresponding to the node $\hat{a}_i$, i.e., $\hat{\varphi}_j(\hat{a}_i) = \delta_{i,j}$. Let $\hat{\boldsymbol{\varphi}}_j^{(k)} = \hat{\varphi}_j \boldsymbol{e}^{(k)}$, with $k = 1, 2$, so that $\{\hat{\boldsymbol{\varphi}}_j^{(k)}\}$ is a nodal basis of $\hat{\boldsymbol{V}}$. Likewise, we set $\tilde{\varphi}_j \in \mathcal{P}_2^c(\tilde{T}^{ct})$ as $\tilde{\varphi}_j(\tilde{x}) = \hat{\varphi}(\hat{x})$ with $\tilde{x} = F_{\tilde{T}}(\hat{x})$, and $\tilde{\boldsymbol{\varphi}}_j^{(k)} = \tilde{\varphi}_j \boldsymbol{e}^{(k)}$. Then, $\{\tilde{\boldsymbol{\varphi}}_j^{(k)}\}$ is a nodal basis of $\tilde{\boldsymbol{V}}(\tilde{T})$.

The construction of $\boldsymbol{\varphi}_j^{(k)} := \boldsymbol{\Psi}_T^C \tilde{\boldsymbol{\varphi}}_j^{(k)}$ is summarized in the following proposition.

Proposition 3.6.1. *Let $\{\hat{\tau}_1, \hat{\tau}_2\} \subset \hat{\mathbb{W}}$ be the nodal basis functions of $\hat{\mathbb{W}}$ satisfying (cf. Table* **??***)*

$$\frac{\partial^2 \hat{\tau}_j}{\partial \hat{\boldsymbol{n}}_{\hat{e}_i} \partial \boldsymbol{t}_{\hat{e}_i}}(\hat{m}_{\hat{e}_i}) = \delta_{i,j}, \quad i, j = 1, 2, \qquad \frac{\partial^2 \hat{\tau}_j}{\partial \hat{\boldsymbol{n}}_{\hat{e}_3} \partial \boldsymbol{t}_{\hat{e}_3}}(\hat{m}_{e_3}) = 0.$$

Define the functions $\hat{z}_j^{(k)} \in \hat{\mathbb{W}}$ as

$$\hat{z}_j^{(k)} = -\frac{\partial B_{j,k}}{\partial \hat{x}_j} \hat{\tau}_j \ \ \textit{for } j = 1, 2, \quad \hat{z}_3^{(k)} = -\frac{1}{2}(\hat{z}_1^{(k)} + \hat{z}_2^{(k)}), \quad \hat{z}_j^{(k)} = 0 \ \ \textit{for } 4 \le j \le 7.$$

Then, a nodal basis of $\boldsymbol{V}_C^h$ is given by the formula

$$\boldsymbol{\varphi}_j^{(k)} \circ F_T = A_T\big(\boldsymbol{\beta}^{(k)}(\hat{a}_j)\hat{\varphi}_j - \widehat{\mathbf{curl}}\, \hat{z}_j^{(k)}\big). \tag{3.6.1}$$

Proof. Set $\hat{z}_j^{(k)} = \hat{\Theta}_{\tilde{T}} \tilde{\boldsymbol{\varphi}}_j^{(k)}$, so that, by definition of $\boldsymbol{\Psi}_T^C$,

$$\boldsymbol{\varphi}_j^{(k)} = \boldsymbol{\Psi}_T \tilde{\boldsymbol{\varphi}}_j^{(k)} - (A_T \widehat{\mathbf{curl}}\, \hat{z}_j^{(k)}) \circ F_T^{-1}, \tag{3.6.2}$$

where we recall the function $\boldsymbol{\Psi}_T \tilde{\boldsymbol{\varphi}}_j^{(k)} \in \boldsymbol{V}(T)$ is uniquely defined by the conditions

$$\boldsymbol{\Psi}_T \tilde{\boldsymbol{\varphi}}_j^{(k)}(a_i) = \tilde{\boldsymbol{\varphi}}_j^{(k)}(\tilde{a}_i) = \delta_{i,j} \boldsymbol{e}^{(k)} \qquad i = 1, 2, \ldots, 10. \tag{3.6.3}$$

By the definition of $\boldsymbol{V}(T)$, we write $\boldsymbol{\Psi}_T \tilde{\boldsymbol{\varphi}}_j^{(k)}(x) = (A_T \hat{\boldsymbol{\psi}}_j^{(k)})(\hat{x})$ for some $\hat{\boldsymbol{\psi}}_j^{(k)} \in \hat{\boldsymbol{V}}$. In particular, from (3.6.3), we see that $\hat{\boldsymbol{\psi}}_j^{(k)} = \boldsymbol{\beta}^{(k)}(\hat{a}_j)\hat{\varphi}_j$. Combining this identity with (3.6.3) and (3.6.2) yields

$$\boldsymbol{\Psi}_T \tilde{\boldsymbol{\varphi}}_j^{(k)} \circ F_T = A_T \boldsymbol{\beta}^{(k)}(\hat{a}_j)\hat{\varphi}_j, \tag{3.6.4}$$

$$\boldsymbol{\varphi}_j^{(k)} \circ F_T = A_T \big(\boldsymbol{\beta}^{(k)}(\hat{a}_j)\hat{\varphi}_j - \widehat{\mathbf{curl}}\, \hat{z}_j^{(k)}\big). \tag{3.6.5}$$

Notice that, due to the boundary conditions and labeling convention, there holds $\tilde{\boldsymbol{\varphi}}_j^{(k)}|_{\partial \tilde{T}} = 0$ for $j = 4, 5, 6, 7$ and $k = 1, 2$. This implies $\hat{z}_j^{(k)} = 0$ (cf. Remark 3.3.8), and so we conclude by (3.6.5) that

$$\boldsymbol{\varphi}_j^{(k)} \circ F_T = A_T \boldsymbol{\beta}^{(k)}(\hat{a}_j)\hat{\varphi}_j \qquad \text{for } j = 4, 5, 6, 7,\ k = 1, 2.$$

Therefore, it remains to discuss the construction of $\hat{z}_j^{(k)}$ for $j = 1, 2, 3,\ k = 1, 2$.

Recall by Remark 3.3.8 that the function $\hat{z}_j^{(k)}$ is uniquely determined by the conditions

$$c_{j,i}^{(k)} := \frac{\partial^2 \hat{z}_j^{(k)}}{\partial \hat{\boldsymbol{n}}_{\hat{e}_i} \partial \boldsymbol{t}_{\hat{e}_i}}(\hat{m}_{\hat{e}_i}) = \frac{\partial}{\partial \hat{\boldsymbol{t}}_{\hat{e}_i}} \Big(B(\boldsymbol{\Psi}_T \tilde{\boldsymbol{\varphi}}_j^{(k)} \circ F_T - \tilde{\boldsymbol{\varphi}}_j^{(k)} \circ F_{\tilde{T}}) \cdot \hat{\boldsymbol{t}}_{\hat{e}_i} \Big)(\hat{m}_{\hat{e}_i}) \quad i = 1, 2, 3.$$

Next, notice due to the boundary conditions and labeling convention that the very right-hand side of the above expression is zero in the case $i = 3$. Thus, we have

$$\hat{z}_j^{(k)} = c_{j,1}^{(k)} \hat{\tau}_1 + c_{j,2}^{(k)} \hat{\tau}_2. \tag{3.6.6}$$

$$\hat{\tau}_1 = \begin{cases} x_1(2x_2-1+x_1)(4x_1^2-2x_1x_2-2x_2^2-x_1+2x_2), & \text{on } \hat{K}_1 \\ x_2^2(2x_2-1+x_1)(6x_1-6x_2+1), & \text{on } \hat{K}_2 \\ (2x_2-1+x_1)(12x_1+6x_2-5)(-1+x_1+x_2)^2, & \text{on } \hat{K}_3 \end{cases}$$

$$\hat{\tau}_2 = \begin{cases} -x_1^2(2x_1+x_2-1)(6x_1-6x_2-1), & \text{on } \hat{K}_1 \\ -x_2(2x_1+x_2-1)(2x_1^2+2x_1x_2-4x_2^2-2x_1+x_2), & \text{on } \hat{K}_2 \\ (6x_1+12x_2-5)(2x_1+x_2-1)(-1+x_1+x_2)^2, & \text{on } \hat{K}_3. \end{cases}$$

Table 1: Formulas for two nodal basis functions of the space introduced in (3.3.3).

Using the identity $\hat{\boldsymbol{t}}_{\hat{e}_i} = \pm \boldsymbol{e}^{(i)}$ for $i = 1, 2$ and (3.6.4), we get

$$\begin{aligned} c_{j,i}^{(k)} &= \frac{\partial}{\partial \hat{x}_i}\Big(B(A_T\boldsymbol{\beta}^{(k)}(\hat{a}_j)\hat{\varphi}_j - \hat{\boldsymbol{\varphi}}_j^{(k)}) \cdot \boldsymbol{e}^{(i)}\Big)(\hat{a}_i) \\ &= \frac{\partial}{\partial \hat{x}_i}\Big(\hat{\varphi}_j\big(\boldsymbol{\beta}^{(k)}(\hat{a}_j) - B\boldsymbol{e}^{(k)}\big) \cdot \boldsymbol{e}^{(i)}\Big)(\hat{a}_i) \\ &= \frac{\partial}{\partial \hat{x}_i}\Big(\hat{\varphi}_j\big(B_{i,k}(\hat{a}_j) - B_{i,k}\big)\Big)(\hat{a}_i) \qquad i = 1, 2. \end{aligned}$$

We then use the product rule and the property $\hat{\varphi}_j(\hat{a}_i) = \delta_{i,j}$ to find

$$c_{j,i}^{(k)} = \frac{\partial \hat{\varphi}_j}{\partial \hat{x}_i}(\hat{a}_i)\big(B_{i,k}(\hat{a}_j) - B_{i,k}(\hat{a}_i)\big) - \delta_{i,j}\frac{\partial B_{i,k}}{\partial \hat{x}_i}.$$

In particular, because $\frac{\partial \hat{\varphi}_j}{\partial \hat{x}_i}(\hat{a}_i) = 0$ for $i \neq j$, $i, j \in \{1, 2\}$, there holds

$$c_{j,i}^{(k)} = \frac{\partial^2 \hat{z}_j^{(k)}}{\partial \hat{\boldsymbol{n}}_{\hat{e}_i}\partial \boldsymbol{t}_{\hat{e}_i}}(\hat{a}_i) = -\delta_{i,j}\frac{\partial B_{i,k}}{\partial \hat{x}_i} \quad \text{for } j = 1, 2. \tag{3.6.7}$$

The case $j = 3$ reads

$$c_{3,i}^{(k)} = \frac{\partial^2 \hat{z}_3^{(k)}}{\partial \hat{\boldsymbol{n}}_{\hat{e}_i}\partial \boldsymbol{t}_{\hat{e}_i}}(\hat{a}_i) = \frac{\partial \hat{\varphi}_3}{\partial \hat{x}_i}(\hat{a}_i)\big(B_{i,k}(\hat{a}_3) - B_{i,k}(\hat{a}_i)\big).$$

A direct calculation shows $\frac{\partial \hat{\varphi}_3}{\partial \hat{x}_i}(\hat{a}_i) = -1$ for $i = 1, 2$, and therefore,

$$c_{3,i}^{(k)} = \big(B_{i,k}(\hat{a}_i) - B_{i,k}(\hat{a}_3)\big) = \frac{1}{2}\frac{\partial B_{i,k}}{\partial \hat{x}_i}. \tag{3.6.8}$$

The statements (3.6.6)–(3.6.8) combined with (3.6.5) yields the desired result (3.6.1).

□

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	rate	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2}$	rate	$\|p-p_h\|_{L^2}$	rate	$\|\operatorname{div}\boldsymbol{u}_h\|_{L^2}$
0.2000	$2.938e-01$		$6.144e+00$		$2.001e+00$		$6.422e-13$
0.1000	$4.656e-02$	2.66	$1.656e+00$	1.89	$7.717e-01$	1.38	$1.222e-12$
0.0500	$5.795e-03$	3.01	$4.729e-01$	1.81	$2.919e-01$	1.40	$6.504e-13$
0.0250	$9.042e-04$	2.68	$1.371e-01$	1.79	$1.073e-01$	1.44	$2.174e-11$
0.0125	$1.171e-04$	2.95	$3.527e-02$	1.96	$2.613e-02$	2.04	$6.509e-11$
0.00625	$1.440e-05$	3.02	$8.759e-03$	2.01	$6.128e-03$	2.09	$2.642e-10$

Table 2: Errors of the finite element method (3.5.1) with $\Omega = B_1(0)$, $\nu = 10^{-1}$, and exact solution (3.6.9). Norms are taken with respect to the domain Ω_h.

3.6.2 Numerical experiments

We compute the finite element method (3.5.1) on the unit circle centered at the origin. We construct the source function such that the exact solution is

$$\boldsymbol{u} = \mathbf{curl}\left((1-x_1^2-x_2^2)^2\sin(5x_1+2x_2)\right), \qquad p = x_1^2+x_2^2+\sin(10\pi(x_1^2+x_2^2)) - \frac{1}{2}. \tag{3.6.9}$$

We take the source approximation $\boldsymbol{f}_h$ to be the quadratic (nodal) Lagrange interpolant of $\boldsymbol{f}$. Table 2 provides the errors of the discrete solution on a sequence of refined quasi-uniform meshes with viscosity $\nu = 10^{-1}$. The numerical experiments show second-order convergence of the velocity and pressure in the H^1 and L^2-norms, respectively, which is in agreement with the theoretical results given in Theorem 3.5.3. In addition, we observe third-order convergence of discrete velocity function in the L^2-norm.

4.0 A divergence-free finite element method for the Stokes problem with boundary correction

4.1 Introduction

In this chapter, we develop and analyze a boundary correction finite element method for the Stokes problem in two dimensions based on the Scott-Vogelius pair on Clough-Tocher splits. The discrete velocity space is the space of continuous, piecewise polynomials of degree k with $k \geq 2$, and the discrete pressure space is given by the space of (discontinuous) piecewise polynomials of degree $k-1$. We also introduce a Lagrange multiplier space consisting of continuous piecewise polynomials of degree k with respect to the boundary partition in order to enforce the boundary conditions as well as to improve the lack of pressure-robustness. We show that the resulting method is well posed, divergence-free and has optimal order convergence.

In more detail, we start with a background mesh that completely covers Ω and then form the computational mesh as those elements in the background mesh that are fully contained in $\bar{\Omega}$. We then follow the general framework of boundary correction methods for the Stokes problem, and use a standard Nitsche-based formulation, where the Dirichlet boundary conditions are enforced via penalization. Due to the discrepancy between the computational and physical domain, boundary conditions are corrected with a use of Taylor's theorem through the boundary transfer operator introduced in (4.2.2) in order to reduce the inconsistency of the scheme and to preserve the optimal order of convergence.

Although the above description is standard for the Poisson problem (cf. [22, 26, 31, 30, 32]), its application to the Stokes problem raises some difficulties. First, as explained in [20], the standard proof of inf-sup stability in the continuous setting, which is needed for the discrete result, is based on a decomposition of the computational domain into a finite number of strictly star shaped domains, and the number of star shaped domains is generally unbounded as $h \to 0$. Therefore, the desired inf-sup stability for the Stokes pair is not clear as the computational domain explicitly depends on the mesh parameter h. One way to address this issue is to use pressure stabilization [26, 30], which comes at the cost of additional consistency errors and poor conservation properties. In this project, we instead establish inf-sup stability by designing the computational mesh in such a way that it preserves the macro element structure and then applying the framework proposed in [20].

Another difficulty of boundary correction methods for the Stokes problem is that they inherently lack pressure-robustness due to the weak enforcement of the boundary conditions via penalization. In particular, a divergence-free method for the Stokes problem with weak enforcement of the boundary conditions is not pressure-robust as divergence-free functions with non-zero normal boundary conditions are not L^2-orthogonal to gradients. Therefore, use of integration by parts formula with such functions is not sufficient by itself anymore to eliminate the pressure term. We address this issue by introducing an additional Lagrange multiplier that enforces the boundary conditions of the normal component of the velocity. This results in a weakly coupled velocity error estimate where the velocity error's dependence on the viscosity is compensated by a higher-order power of the mesh parameter h, and therefore, the lack of pressure-robustness is mitigated by an additional power of h in the error analysis (see also Theorem 4.5.5).

The rest of the chapter is organized as follows. In the next section, we state the Stokes problem, the computational mesh, and the boundary transfer operator. In Section 4.3, we state the finite element method and show that the method yields exactly divergence–free velocity approximations. Section 4.4 proves several inf-sup conditions and the well-posedness of the method. In Section 4.5, we show that the method is optimally convergent provided the exact solution is sufficiently smooth. Section 4.6 provides some numerical experiments that confirm the theoretical results.

4.2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We consider the Stokes problem

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{4.2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{4.2.1b}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega, \tag{4.2.1c}$$

where the viscosity $\nu > 0$ is assumed to be constant. For the sake of simplicity, and without loss of generality, we assume that $\boldsymbol{g} = 0$ through the rest of this chapter as the extension to non-homogeneous case is relatively straight-forward [41].

Moreover, we assume the domain has smooth boundary $\partial\Omega$. We denote the outward unit normal by $\boldsymbol{n}$, and we let ϕ denote the signed distance function of $\partial\Omega$ such that $\phi(x) < 0$ for $x \in \Omega$ and $\phi(x) \geq 0$ otherwise. Notice, in this case, that there holds $\boldsymbol{n} = \nabla\phi/|\nabla\phi|$ on $\partial\Omega$. For a positive number τ, we define the tubular region around $\partial\Omega$ by $\Gamma_\tau := \{x \in \mathbb{R}^2 : |\phi(x)| \leq \tau|\}$. By [16, Lemma 14.16], there exists $\tau_0 > 0$ such that the closest point projection $\boldsymbol{p} : \Gamma_{\tau_0} \to \partial\Omega$ is well defined and satisfies $\boldsymbol{p}(x) = x - \phi(x)\boldsymbol{n}(\boldsymbol{p}(x))$ for all $x \in \Gamma_{\tau_0}$ [15].

Let $S \subset \mathbb{R}^2$ be a polygon such that $\Omega \subset S$, and let $\mathcal{S}_h$ be a shape-regular simplicial triangulation of S. Then, we define the computational mesh and the computational domain, respectively, by

$$\mathcal{T}_h = \{T \in \mathcal{S}_h : \ \bar{T} \subset \bar{\Omega}\}, \quad \Omega_h = \mathrm{int}\Big(\bigcup_{T \in \mathcal{T}_h} \bar{T} \Big) \subset \Omega.$$

We again denote the Clough-Tocher refinement of $\mathcal{T}_h$ by $\mathcal{T}_h^{ct}$. We also let $\mathcal{E}_h^B$ denote the set of boundary edges of $\mathcal{T}_h$, which is also the set of boundary edges of $\mathcal{T}_h^{ct}$. For a piecewise smooth function q with respect to $\mathcal{E}_h^B$, we write, with an abuse of notation,

$$\int_{\partial \Omega_h} q = \sum_{e \in \mathcal{E}_h^B} \int_e q\,.$$

Let $\boldsymbol{n}_h$ denote the outward unit normal with respect to the computational boundary $\partial\Omega_h$. For $K \in \mathcal{T}_h^{ct}$, we set $h_K = \mathrm{diam}(K)$ and $h = \max_{K \in \mathcal{T}_h^{ct}} h_K$. Similarly, for $e \in \mathcal{E}_h^B$, we set $h_e = \mathrm{diam}(e)$.

Remark 4.2.1. Let $\mathcal{S}_h^{ct}$ denote the Clough-Tocher refinement of the background mesh $\mathcal{S}_h$. We emphasize that $\mathcal{T}_h^{ct} \subset \mathcal{S}_h^{ct}$, however,

$$\mathcal{T}_h^{ct} \neq \{K \in \mathcal{S}_h^{ct} : \ \bar{K} \subset \bar{\Omega}\}.$$

In particular, we respect the macro-element structure that is needed to prove the stability of the Scott-Vogelius pair.

4.2.1 Boundary transfer operator

The essential idea of boundary correction methods is to incorporate the boundary information on Ω to Ω_h via a well-defined mapping $M : \partial\Omega_h \to \partial\Omega$ that assigns each point on the computational boundary to physical one. With such a mapping in hand, we define the transfer direction and the transfer length, respectively, by

$$\mathfrak{d}(x) = (M - I)x, \quad \delta(x) = |\mathfrak{d}(x)| \qquad x \in \partial\Omega_h.$$

Here, we emphasize that different choices for the mapping M have been proposed in the literature. For instance, one common choice is to define M to be the closest point projection, i.e., $M = \boldsymbol{p}$, in which case, assuming Ω_h is close enough to Ω, the direction of the vector $\mathfrak{d}(x)$ coincides with that of $\boldsymbol{n}(x)$. In particular, there holds $\mathfrak{d}(x) = -\phi(x)\boldsymbol{n}(\boldsymbol{p}(x))$ and $\delta(x) = |\phi(x)|$. Another common choice is to take the transfer direction to be parallel to the outward unit normal of the computational boundary, i.e., $\mathfrak{d}/\delta = \boldsymbol{n}_h$. This latter choice of the transfer direction vector yields a simpler implementation in the numerical method. On the other hand, there holds $\delta(x) \geq |\phi(x)|$ with possible large discrepancies between $\delta(x)$ and $|\phi(x)|$.

In our analysis, instead of explicitly defining the mapping M, we rather require M to only satisfy the following assumption

$$\max_{e \in \mathcal{E}_h^B} h_e^{-1}\delta_e \leq c_\delta < 1, \qquad \text{for } c_\delta \text{ sufficiently small,} \tag{A}$$

where $\delta_e := \max_{x \in \bar{e}} \delta(x)$. In other words, we ask that the distance between Ω and Ω_h be of order h with a sufficiently small constant. We also note that a similar assumption have been made in [22, 34, 15, 31, 30, 32]. As stated in [31, Remark 3], this assumption can be satisfied in practice by shifting the location of the nodes on the computational boundary along the direction $\boldsymbol{n}$. However, the numerical experiments presented in Section 4.6 indicate that the smallness assumption of c_δ can be relaxed, and a shifting of nodes on the computational boundary is not needed to ensure stability.

Next, we set $\boldsymbol{d} := \mathfrak{d}/\delta$. Then, for a given function $\boldsymbol{v}$, we define the boundary transfer operator, $S_h\boldsymbol{v}$, as the k^{th}-order Taylor expansion of $\boldsymbol{v}$ (assuming enough regularity on $\boldsymbol{v}$) in the direction of $\boldsymbol{d}$, i.e.,

$$(S_h\boldsymbol{v})(x) = \sum_{j=0}^{k} \frac{1}{j!}(\delta(x))^j \frac{\partial^i \boldsymbol{v}}{\partial \boldsymbol{d}^j}(x). \tag{4.2.2}$$

4.3 A divergence–free finite element method

We define the global velocity and pressure spaces with respect to the Clough-Tocher triangulation $\mathcal{T}_h^{ct}$, respectively, by

$$\boldsymbol{V}^h = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_h) :\ \boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}_k(K)\ \forall K \in \mathcal{T}_h^{ct},\ \int_{\partial\Omega_h} \boldsymbol{v}\cdot\boldsymbol{n}_h = 0\},$$
$$Q^h = \{q \in L^2(\Omega_h) :\ q|_K \in \mathcal{P}_{k-1}(K)\ \forall K \in \mathcal{T}_h^{ct}\}.$$

The analogous spaces with boundary conditions are then given by

$$\mathring{\boldsymbol{V}}^h = \boldsymbol{V}^h \cap \boldsymbol{H}_0^1(\Omega_h), \qquad \mathring{Q}^h = Q^h \cap L_0^2(\Omega_h).$$

Moreover, we also introduce a Lagrange multiplier space and its variant with zero mean constraint as

$$X^h = \{\mu \in C(\partial\Omega_h) :\ \mu|_e \in \mathcal{P}_k(e)\ \forall e \in \mathcal{E}_h^B\}, \quad \mathring{X}^h = \{\mu \in X^h :\ \int_{\partial\Omega_h} \mu = 0\}.$$

Next, we define the following bilinear forms:

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = \nu\Big(\int_{\Omega_h} \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - \int_{\partial\Omega_h} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_h} \cdot \boldsymbol{v} + \int_{\partial\Omega_h} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_h} \cdot (S_h \boldsymbol{u})$$
$$+ \sum_{e\in\mathcal{E}_h^B} \int_e \frac{\sigma}{h_e}(S_h\boldsymbol{u}) \cdot (S_h\boldsymbol{v})\Big),$$

and

$$b_h(\boldsymbol{v}, (q, \mu)) = -\int_{\Omega_h} (\nabla \cdot \boldsymbol{v})q + \int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)\mu,$$
$$b_h^e(\boldsymbol{v}, (q, \mu)) = -\int_{\Omega_h} (\nabla \cdot \boldsymbol{v})q + \int_{\partial\Omega_h} ((S_h\boldsymbol{v}) \cdot \boldsymbol{n}_h)\mu,$$

where $\sigma > 0$ is the penalty parameter.

Note that the bilinear form $a_h(\cdot, \cdot)$ is a modification of the standard Nitche bilinear form associated with the Laplace operator where the original functions values are replaced by the action of the boundary transfer operator for the "symmetry" and the "penalty" terms [36]. In this framework, such a modification is needed to improve the consistency of the method due to the geometric error between the physical and computational domains. In particular, we see that, due to the positive sign in front of the third term in the bilinear form $a_h(\cdot, \cdot)$, the bilinear form is actually based on a non-symmetric version of Nitsche's method. However, notice that the analogous bilinear form that is based on a symmetric version of Nitsche's method still does not result in a symmetric bilinear form [22, 26], and the non-symmetric version leads more flexibility on the penalty parameter σ to ensure stability as we show in the next section (cf. Lemma 4.4.3).

Our finite element method seeks $(\boldsymbol{u}_h, p_h, \lambda_h) \in \boldsymbol{V}^h \times \mathring{Q}^h \times \mathring{X}^h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) + b_h(\boldsymbol{v}, (p_h, \lambda_h)) = \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}^h, \tag{4.3.1a}$$

$$b_h^e(\boldsymbol{u}_h, (q, \mu)) = 0 \qquad\qquad \forall (q, \mu) \in \mathring{Q}^h \times \mathring{X}^h. \tag{4.3.1b}$$

Here, we emphasize that the condition $\int_{\partial\Omega_h} \boldsymbol{v} \cdot \boldsymbol{n}_h = 0$ in the definition of the discrete velocity space is necessary to impose the divergence-free property of the discrete velocity solution as the next lemma shows. Moreover, it is this condition that forces λ_h to be in $\mathring{X}^h$ so that λ_h can not be a non-zero constant as otherwise we would have $b_h(\boldsymbol{v}, (0, 1)) = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}^h$, and as a result, the problem (4.3.1) would not be well-posed.

Lemma 4.3.1 (Divergence–free property)**.** *If* $(\boldsymbol{u}_h, p_h, \lambda_h) \in \boldsymbol{V}^h \times \mathring{Q}^h \times \mathring{X}^h$ *satisfies* (4.3.1)*, then* $\nabla \cdot \boldsymbol{u}_h \equiv 0$ *in* Ω_h*.*

Proof. Notice that the definition of the Stokes pair $\boldsymbol{V}^h \times \mathring{Q}^h$, in particular the constraint $\int_{\partial\Omega_h} \boldsymbol{v} \cdot \boldsymbol{n}_h = 0$ with the divergence theorem, shows $\nabla \cdot \boldsymbol{u}_h \in \mathring{Q}^h$. Then, letting $q = \nabla \cdot \boldsymbol{u}_h$ and $\mu = 0$ in (4.3.1b) yields

$$0 = b_h^e(\boldsymbol{u}_h, (\nabla \cdot \boldsymbol{u}_h, 0)) = -\|\nabla \cdot \boldsymbol{u}_h\|_{L^2(\Omega_h)}^2.$$

Hence, $\nabla \cdot \boldsymbol{u}_h \equiv 0$. □

4.4 Stability and continuity estimates

We begin this section by defining three H^1-type norms on $\boldsymbol{V}^h + \boldsymbol{H}^{k+1}(\Omega_h)$:

$$\|\boldsymbol{v}\|_h^2 = \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}^2 + \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|S_h \boldsymbol{v}\|_{L^2(e)}^2,$$

$$\|\boldsymbol{v}\|_{1,h}^2 = \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}^2 + \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|\boldsymbol{v}\|_{L^2(e)}^2,$$

$$|||\boldsymbol{v}|||_h^2 = \|\boldsymbol{v}\|_h^2 + \sum_{e\in\mathcal{E}_h^B} h_e \|\nabla \boldsymbol{v}\|_{L^2(e)}^2.$$

Moreover, we also define a $H^{-1/2}$-type norm on the Lagrange multiplier space $\mathring{X}^h$:

$$\|\mu\|_{-1/2,h}^2 = \sum_{e\in\mathcal{E}_h^B} h_e \|\mu\|_{L^2(e)}^2.$$

Finally, we define the norm on $\mathring{Q}^h \times \mathring{X}^h$ as

$$\|(q,\mu)\| := \|q\|_{L^2(\Omega_h)} + \|\mu\|_{-1/2,h}.$$

The next lemma provides some estimates related to the boundary transfer operator $S_h\boldsymbol{v}$ that are heavily used throughout the rest of this chapter for analysis purposes.

Lemma 4.4.1. *Assuming* (A)*, there holds for all* $\boldsymbol{v} \in \boldsymbol{V}^h$*,*

$$\begin{aligned}
\sum_{e\in\mathcal{E}_h^B} h_e^{-1} \|S_h\boldsymbol{v} - \boldsymbol{v}\|_{L^2(e)}^2 &\le C c_\delta^2 \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}^2, \\
\sum_{e\in\mathcal{E}_h^B} h_e^{-1} \|S_h\boldsymbol{v}\|_{L^2(e)}^2 &\le C \|\boldsymbol{v}\|_{1,h}^2.
\end{aligned} \tag{4.4.1}$$

In particular, $\|\cdot\|_h$*,* $\|\cdot\|_{1,h}$*, and* $|||\cdot|||_h$ *are equivalent on* $\boldsymbol{V}^h$*.*

Proof. Using the trace and inverse inequalities, the shape-regularity of $\mathcal{T}_h$ and (A), for all $e \in \mathcal{E}_h^B$, there holds

$$h_e^{-1} \int_e |\delta|^{2j} \Big|\frac{\partial^j \boldsymbol{v}}{\partial \boldsymbol{d}^j}\Big|^2 \le C\delta_e^{2j} h_e^{-2j} \|\nabla \boldsymbol{v}\|_{L^2(T_e)}^2 \le C c_\delta^{2j} \|\nabla \boldsymbol{v}\|_{L^2(T_e)}^2, \tag{4.4.2}$$

where $T_e \in \mathcal{T}_h$ satisfies $e \subset \partial T_e$ and $j = 1, 2, ..., k$. Notice that the estimate (4.4.2) directly implies the first inequality in (4.4.1). The estimate (4.4.2) also leads

$$\sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|S_h \boldsymbol{v}\|_{L^2(e)}^2 \leq C \sum_{e \in \mathcal{E}_h^B} \sum_{j=0}^{k} h_e^{-1} \int_e |\delta|^{2j} \Big|\frac{\partial^j \boldsymbol{v}}{\partial \boldsymbol{d}^j}\Big|^2 \leq C \|\boldsymbol{v}\|_{1,h}^2,$$

which proves the second inequality in (4.4.1). Moreover, notice that the second inequality immediately yields $\|\boldsymbol{v}\|_h \leq C\|\boldsymbol{v}\|_{1,h}$ due to the definition of these norms. Furthermore, standard arguments involving the trace and inverse inequalities show $\|\boldsymbol{v}\|_h \leq |||\boldsymbol{v}|||_h \leq C\|\boldsymbol{v}\|_h$ on $\boldsymbol{V}^h$. Therefore, in order to complete the proof, it suffices to show $\|\boldsymbol{v}\|_{1,h} \leq C\|\boldsymbol{v}\|_h$. For this purpose, we once again use (4.4.2) with the triangle inequality to obtain

$$\begin{aligned}
\sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|\boldsymbol{v}\|_{L^2(e)}^2 &\leq 2 \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|S_h \boldsymbol{v}\|_{L^2(e)}^2 + 2 \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|S_h \boldsymbol{v} - \boldsymbol{v}\|_{L^2(e)}^2 \\
&\leq 2 \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|S_h \boldsymbol{v}\|_{L^2(e)}^2 + C \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \sum_{j=1}^{k} \int_e |\delta|^{2j} \Big|\frac{\partial^j \boldsymbol{v}}{\partial \boldsymbol{d}^j}\Big|^2 \\
&\leq 2 \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|S_h \boldsymbol{v}\|_{L^2(e)}^2 + C \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}^2.
\end{aligned}$$

This inequality implies $\|\boldsymbol{v}\|_{1,h} \leq C\|\boldsymbol{v}\|_h$ and completes the proof. □

4.4.1 Continuity and coercivity estimates of bilinear forms

Lemma 4.4.2. *Assuming* (A), *there holds*

$$|a_h(\boldsymbol{v},\boldsymbol{w})| \le c_2(1+\sigma)\nu|||\boldsymbol{v}|||_h|||\boldsymbol{w}|||_h \qquad \forall \boldsymbol{v}\in \boldsymbol{V}^h+\boldsymbol{H}^{k+1}(\Omega_h),\ \forall \boldsymbol{w}\in\boldsymbol{V}^h, \tag{4.4.3}$$

$$\big|b_h(\boldsymbol{v},(q,\mu))\big| \le C\|\boldsymbol{v}\|_{1,h}\|(q,\mu)\| \qquad \forall (q,\mu)\in \mathring{Q}^h\times\mathring{X}^h, \tag{4.4.4}$$

$$\big|b_h(\boldsymbol{v},(q,\mu)) - b_h^e(\boldsymbol{v},(q,\mu))\big| \le Cc_\delta\|\boldsymbol{v}\|_{1,h}\|(q,\mu)\| \qquad \forall \boldsymbol{v}\in\boldsymbol{V}^h,\ \forall (q,\mu)\in \mathring{Q}^h\times\mathring{X}^h, \tag{4.4.5}$$

$$\big|b_h^e(\boldsymbol{v},(q,\mu))\big| \le C(1+c_\delta)\|\boldsymbol{v}\|_{1,h}\|(q,\mu)\| \quad \forall (q,\mu)\in \mathring{Q}^h\times\mathring{X}^h. \tag{4.4.6}$$

Proof. A similar proof of the estimate of (4.4.3) is given in [30, Proposition 2]. Here, we provide a proof of this result for the sake of completeness. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
|a_h(\boldsymbol{v},\boldsymbol{w})| &= \nu\Big(\int_{\Omega_h}\nabla\boldsymbol{v}:\nabla\boldsymbol{w} - \int_{\partial\Omega_h}\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}_h}\cdot\boldsymbol{w} + \int_{\partial\Omega_h}\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}_h}\cdot(S_h\boldsymbol{v}) \\
&\quad + \sum_{e\in\mathcal{E}_h^B}\int_e\frac{\sigma}{h_e}(S_h\boldsymbol{v})\cdot(S_h\boldsymbol{w})\Big) \\
&\le \nu\Big(\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_h)} + \|h^{1/2}\nabla\boldsymbol{v}\cdot\boldsymbol{n}_h\|_{L^2(\partial\Omega_h)}\|h^{-1/2}\boldsymbol{w}\|_{L^2(\partial\Omega_h)} \\
&\quad + \|h^{1/2}\nabla\boldsymbol{w}\cdot\boldsymbol{n}_h\|_{L^2(\partial\Omega_h)}\|h^{-1/2}S_h\boldsymbol{v}\|_{L^2(\partial\Omega_h)} + \sigma\|h^{-1/2}S_h\boldsymbol{v}\|_{L^2(\partial\Omega_h)}\|h^{-1/2}S_h\boldsymbol{w}\|_{L^2(\partial\Omega_h)}\Big).
\end{aligned}$$

By the definition of the norm $|||\cdot|||_h$, the trace inequality and equivalence between the norm $|||\cdot|||_h$ and the norm $\|\cdot\|_{1,h}$, we have

$$\|h^{1/2}\nabla\boldsymbol{v}\cdot\boldsymbol{n}_h\|_{L^2(\partial\Omega_h)}\|h^{-1/2}\boldsymbol{w}\|_{L^2(\partial\Omega_h)} \le C|||\boldsymbol{v}|||_h|||\boldsymbol{w}|||_h,$$

and

$$\|h^{1/2}\nabla\boldsymbol{w}\cdot\boldsymbol{n}_h\|_{L^2(\partial\Omega_h)} \le C|||\boldsymbol{w}|||_h.$$

Combining all the above estimates yields the desired continuity property (4.4.3) of the bilinear form $a_h(\cdot,\cdot)$.

The continuity estimate of $b_h(\cdot,\cdot)$ (4.4.4) immediately follows from the Cauchy-Schwarz inequality.

The third estimate (4.4.5) follows from the definition of the forms, the Cauchy-Schwarz inequality, and (4.4.1):

$$\begin{aligned}\big|b_h(\boldsymbol{v},(q,\mu)) - b_h^e(\boldsymbol{v},(q,\mu))\big| &= \Big|\sum_{e\in\mathcal{E}_h^B}\int_e \big((\boldsymbol{v}-S_h\boldsymbol{v})\cdot\boldsymbol{n}_h\big)\mu\Big| \\ &\le \Big(\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|\boldsymbol{v}-S_h\boldsymbol{v}\|_{L^2(e)}^2\Big)^{1/2}\|\mu\|_{-1/2,h} \\ &\le Cc_\delta\|\boldsymbol{v}\|_{1,h}\|\mu\|_{-1/2,h}.\end{aligned}$$

Lastly, the estimate (4.4.6) now follows directly from the estimates (4.4.4) and (4.4.5) using the triangle inequality. □

Lemma 4.4.3. *Suppose that Assumption* (A) *is satisfied for c_δ sufficiently small. Then there holds,*

$$c_1\nu\|\boldsymbol{v}\|_{1,h}^2 \le a_h(\boldsymbol{v},\boldsymbol{v}) \qquad \forall\boldsymbol{v}\in\boldsymbol{V}^h,$$

for $c_1>0$ independent of h and ν, and for any positive penalty parameter $\sigma>0$.

Proof. By definition of the bilinear form $a_h(\cdot,\cdot)$,

$$a_h(\boldsymbol{v},\boldsymbol{v}) = \nu\Big(\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}^2 + \sum_{e\in\mathcal{E}_h^B}\Big(\int_e\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}_h}\cdot(S_h\boldsymbol{v}-\boldsymbol{v}) + \frac{\sigma}{h_e}\|S_h\boldsymbol{v}\|_{L^2(e)}^2\Big)\Big).$$

A discrete trace inequality with (4.4.1) yields

$$\Big|\sum_{e\in\mathcal{E}_h^B}\int_e\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}_h}\cdot(S_h\boldsymbol{v}-\boldsymbol{v})\Big| \le Cc_\delta\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}^2. \tag{4.4.7}$$

Therefore, we find

$$a_h(\boldsymbol{v},\boldsymbol{v}) \ge \nu\Big((1-Cc_\delta)\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}^2 + \sum_{e\in\mathcal{E}_h^B}\frac{\sigma}{h_e}\|S_h\boldsymbol{v}\|_{L^2(e)}^2\Big)\Big) \ge C\nu\|\boldsymbol{v}\|_h^2 \ge C\nu\|\boldsymbol{v}\|_{1,h}^2$$

for c_δ sufficiently small and for $\sigma>0$. □

In what follows, we prove several inf-sup stability results, and then combine these results to obtain the main stability result.

4.4.2 Inf-sup stability I

In this section we prove the discrete inf-sup (LBB) condition for the Stokes pair $\mathring{\boldsymbol{V}}^h \times \mathring{Q}^h$ with stability constants independent of h. In the case of a fixed polygonal domain, the LBB stability for this pair is well-known (cf. [1, 33, 19]). However, the extension of these results to an unfitted domain Ω_h is not straightforward. Indeed, the proofs in [1, 33, 19] (directly or indirectly) rely on the Nečas inequality:

$$\mathfrak{c}_h \|q\|_{L^2(\Omega_h)} \le \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega_h)\backslash\{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \qquad \forall q \in L_0^2(\Omega_h)$$

for some $\mathfrak{c}_h > 0$ depending on the domain Ω_h, and as explained in [20], it is unclear if the constant $\mathfrak{c}_h$ in this inequality is independent of h.

Our idea to obtain this result is based on combining the local stability result of the Scott-Vogelius pair with that of $\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_0$ pair, similar to the idea presented in the proof of Theorem 2.4.4. For this purpose, we first define the analogous local spaces with boundary conditions for a macro element $T \in \mathcal{T}_h$ by

$$\begin{aligned} \boldsymbol{V}_0(T) &= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(T) : \ \boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}_k(K) \ \forall K \subset T, \ K \in \mathcal{T}_h^{ct}\}, \\ Q_0(T) &= \{q \in L_0^2(T) : \ q|_K \in \mathcal{P}_{k-1}(K) \ \forall K \subset T, \ K \in \mathcal{T}_h^{ct}\}. \end{aligned}$$

Next, we state a local surjectivity of the divergence operator acting on these spaces.

Lemma 4.4.4. *For every $q \in Q_0(T)$, there exists $\boldsymbol{v} \in \boldsymbol{V}_0(T)$ such that $\nabla \cdot \boldsymbol{v} = q$ and $\|\nabla \boldsymbol{v}\|_{L^2(T)} \le \beta_T^{-1} \|q\|_{L^2(T)}$. Here, the constant $\beta_T > 0$ depends only on the shape-regularity of T.*

Proof. The proof for the general case can be found in [19, Theorem 3.1]. Here, we provide a proof for the sake of completeness.

We define $\boldsymbol{Z}(T) := \{\boldsymbol{v} \in \boldsymbol{V}_0(T) : \quad \nabla \cdot \boldsymbol{v} = 0\}$. Because $\boldsymbol{v} \in \boldsymbol{H}_0^1(T)$ is divergence-free and $\boldsymbol{v}$ is a piecewise polynomial of order k with respect to $K \subset T$, $K \in \mathcal{T}_h^{ct}$, there exists $\Psi \in H_0^2(T)$ such that $\boldsymbol{v} = \mathbf{curl}\,\Psi$ and Ψ is a piecewise polynomial of degree $k+1$ with respect to $K \subset T$, $K \in \mathcal{T}_h^{ct}$.

Define $\Sigma(T) := \{\Psi \in H_0^2(T) : \quad \Psi \in \mathcal{P}_{k+1}(K), \quad \forall K \subset T, \; K \in \mathcal{T}_h^{ct}\}$. Notice that there holds $\boldsymbol{Z}(T) = \mathbf{curl}\,\Sigma(T)$. Notice also that if $\Psi \in \Sigma(T)$ and $\mathbf{curl}\,\Psi = 0$, then, due to the boundary conditions, we find that $\Psi = 0$ on T. In other words, the curl operator has trivial kernel on $\Sigma(T)$, and so, by the rank nullity theorem, we have $\dim \Sigma(T) = \dim \mathbf{curl}\,\Sigma(T)$. Then, this result with another use of the rank nullity theorem and the fact that $\dim \Sigma(T) = \frac{3}{2}k^2 - \frac{9}{2}k + 3$ (cf. [14]) yield

$$
\begin{aligned}
\dim \operatorname{div} \boldsymbol{V}_0(T) &= \dim \boldsymbol{V}_0(T) - \dim \boldsymbol{Z}(T) \\
&= \dim \boldsymbol{V}_0(T) - \dim \Sigma(T) \\
&= 2(1 + 3(k-1) + \frac{3}{2}(k-1)(k-2)) - (\frac{3}{2}k^2 - \frac{9}{2}k + 3) \\
&= \frac{3}{2}k^2 + \frac{3}{2}k - 1 \\
&= 3 \cdot \frac{1}{2}k(k+1) - 1 \\
&= \dim Q_0(T),
\end{aligned}
$$

which shows that the divergence operator is surjective from $\boldsymbol{V}_0(T)$ to $Q_0(T)$.

Next, we use a scaling argument to show that given $q \in Q_0(T)$, there exists $\boldsymbol{v} \in \boldsymbol{V}_0(T)$ such that $\nabla \cdot \boldsymbol{v} = q$ and $\|\nabla \boldsymbol{v}\|_{L^2(T)} \le \beta_T^{-1} \|q\|_{L^2(T)}$. To this end, we first prove an analogous result based on the reference triangle $\hat{T}$.

Claim: For any $\hat{q} \in \mathcal{P}_{k-1}(\hat{T}^{ct}) \cap L_0^2(\hat{T})$, there exists a $\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_k(\hat{T}^{ct}) \cap H_0^1(\hat{T})$ such that $\hat{\nabla} \cdot \hat{\boldsymbol{v}} = \hat{q}$ and $\|\hat{\nabla} \hat{\boldsymbol{v}}\|_{L^2(\hat{T})} \le C \|\hat{q}\|_{L^2(\hat{T})}$.

Proof of the claim: We define $\hat{\boldsymbol{Z}} := \{\hat{\boldsymbol{w}} \in \boldsymbol{\mathcal{P}}_k(\hat{T}^{ct}) \cap H_0^1(\hat{T}) : \hat{\nabla} \cdot \hat{\boldsymbol{w}} = 0\}$, and $\hat{\boldsymbol{Z}}^{\perp} := \{\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_k(\hat{T}^{ct}) \cap H_0^1(\hat{T}) : \int_{\hat{T}} \hat{\nabla}\hat{\boldsymbol{v}} : \hat{\nabla}\hat{\boldsymbol{w}} = 0, \forall \hat{\boldsymbol{w}} \in \hat{\boldsymbol{Z}}\}$. If $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{Z}}^{\perp}$ with $\|\hat{\nabla} \cdot \hat{\boldsymbol{v}}\|_{L^2(\hat{T})} = 0$, then $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{Z}} \cap \hat{\boldsymbol{Z}}^{\perp} = \{0\}$. In other words, $\|\hat{\nabla} \cdot \hat{\boldsymbol{v}}\|_{L^2(\hat{T})}$ is a norm on the space $\hat{\boldsymbol{Z}}^{\perp}$. Therefore, by the equivalence of norms, we have $\|\hat{\nabla}\hat{\boldsymbol{v}}\|_{L^2(\hat{T})} \leq C\|\hat{\nabla} \cdot \hat{\boldsymbol{v}}\|_{L^2(\hat{T})}$, for all $\hat{\boldsymbol{v}} \in \hat{\boldsymbol{Z}}^{\perp}$.

Next, using the surjectivity of the divergence operator, we let $\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_k(\hat{T}^{ct}) \cap H_0^1(\hat{T})$ such that $\hat{\nabla} \cdot \hat{\boldsymbol{v}} = \hat{q}$. Then, we uniquely write $\hat{\boldsymbol{v}} = \hat{\boldsymbol{z}} + \hat{\boldsymbol{v}}^{\perp}$, where $\hat{\boldsymbol{z}} \in \hat{\boldsymbol{Z}}$ and $\hat{\boldsymbol{v}}^{\perp} \in \hat{\boldsymbol{Z}}^{\perp}$. Notice that because $\hat{\nabla} \cdot \hat{\boldsymbol{z}} = 0$, we have $\hat{\nabla} \cdot \hat{\boldsymbol{v}}^{\perp} = \hat{\nabla} \cdot \hat{\boldsymbol{v}} = \hat{q}$, and therefore,

$$\|\hat{\nabla}\hat{\boldsymbol{v}}^{\perp}\|_{L^2(\hat{T})} \leq C\|\hat{\nabla} \cdot \hat{\boldsymbol{v}}^{\perp}\|_{L^2(\hat{T})} = C\|\hat{q}\|_{L^2(\hat{T})},$$

which completes the proof of the claim.

Let $q \in \mathcal{P}_{k-1}(T^{ct}) \cap L_0^2(T)$ and set $\hat{q} : \hat{T} \to \mathbb{R}$ such that $\hat{q}(\hat{x}) := q(x)$ with $x = F_T(\hat{x})$. Here, F_T is a linear map of the form $F_T(\hat{x}) = A\hat{x} + b$ where $A \in \mathbb{R}^{2\times 2}$ is a constant matrix with $|A|_{L^\infty(\hat{T})} \leq Ch_T$ and $|A^{-1}|_{L^\infty(\hat{T})} \leq Ch_T$ and $b \in \mathbb{R}^2$ is a constant vector. Notice that $\hat{q} \in \mathcal{P}_{k-1}(\hat{T}^{ct}) \cap L_0^2(\hat{T})$, and so by the above claim, there exists $\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{P}}_k(\hat{T}^{ct}) \cap H_0^1(\hat{T})$ such that $\hat{\nabla} \cdot \widehat{\boldsymbol{v}} = \hat{q}$ and $\|\hat{\nabla}\hat{\boldsymbol{v}}\|_{L^2(\hat{T})} \leq C\|\hat{q}\|_{L^2(\hat{T})}$.

Next, we set $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(T^{ct}) \cap H_0^1(T)$ such that $\boldsymbol{v}(x) := A\hat{\boldsymbol{v}}(\hat{x})$, $x = F_T(\hat{x})$. Then, $(\nabla \cdot \boldsymbol{v})(x) = (\hat{\nabla} \cdot \hat{\boldsymbol{v}})(\hat{x}) = \hat{q}(\hat{x}) = q(x)$. This result with a change of variables show

$$\begin{aligned}
\|\nabla \boldsymbol{v}\|_{L^2(T)}^2 = \int_T |\nabla \boldsymbol{v}|^2 &= 2|T| \int_{\hat{T}} |A\hat{\nabla}\hat{\boldsymbol{v}}A^{-1}|^2 \\
&\leq C|T| \|\hat{\nabla}\hat{\boldsymbol{v}}\|_{L^2(\hat{T})}^2 \\
&\leq C|T| \|\hat{q}\|_{L^2(\hat{T})} \\
&\leq C\|q\|_{L^2(T)}.
\end{aligned}$$

Setting $\beta_T = C^{-1}$ gives the desired result. $\square$

The next lemma states the recent stability result of the $\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_{k-2}$ pair on unfitted domains (cf. [20, Theorem 1, Section 6.3, and Remark 1]).

Lemma 4.4.5. *Define the space of piecewise polynomials of degree $(k-2)$ with respect to the mesh $\mathcal{T}_h$:*

$$\mathring{Y}^h = \{q \in L_0^2(\Omega_h) : \ q|_T \in \mathcal{P}_{k-2}(T) \ \forall T \in \mathcal{T}_h\} \subset \mathring{Q}^h.$$

There exist $\beta_0 > 0$ and $h_0 > 0$ such that for $h \le h_0$, there holds

$$\sup_{\boldsymbol{v} \in \mathring{\boldsymbol{V}}^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \ge \beta_0 \|q\|_{L^2(\Omega_h)} \qquad \forall q \in \mathring{Y}^h.$$

Following similar arguments as in the proof of Theorem 2.4.4, we combine Lemmas 4.4.4 - 4.4.5 in order to obtain the stability of the pair $\mathring{\boldsymbol{V}}^h \times \mathring{Q}^h$.

Lemma 4.4.6. *There exists $\beta_1 > 0$ independent of h such that*

$$\sup_{\boldsymbol{v} \in \mathring{\boldsymbol{V}}^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) q}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}} \ge \beta_1 \|q\|_{L^2(\Omega_h)} \qquad \forall q \in \mathring{Q}^h,$$

for $h \le h_0$.

Proof. Let $q \in \mathring{Q}^h$ and $\bar{q} \in \mathring{Y}^h$ be its piecewise average, i.e., $\bar{q}|_T = |T|^{-1} \int_T q$ for all $T \in \mathcal{T}_h$. Then, we have $(q - \bar{q})|_T \in Q_0(T)$ for all $T \in \mathcal{T}_h$, and so, by Lemma 4.4.4, there exists $\boldsymbol{v}_{1,T} \in \boldsymbol{V}_0(T)$ such that $\nabla \cdot \boldsymbol{v}_{1,T} = (q - \bar{q})|_T$ and $\|\nabla \boldsymbol{v}\|_{L^2(T)} \le \beta_T^{-1} \|q\|_{L^2(T)}$. Defining $\boldsymbol{v}_1 \in \mathring{\boldsymbol{V}}^h$ as $\boldsymbol{v}_1|_T := \boldsymbol{v}_{1,T}$ $\forall T \in \mathcal{T}_h$, we have $\nabla \cdot \boldsymbol{v}_1 = (q - \bar{q})$ in Ω_h and $\|\nabla \boldsymbol{v}_1\|_{L^2(\Omega_h)} \le \beta_*^{-1} \|q - \bar{q}\|_{L^2(\Omega_h)}$, where $\beta_* = \min_{T \in \mathcal{T}_h} \beta_T$. These results together with Lemma 4.4.5 and the triangle inequality yield

$$\beta_0 \|\bar{q}\|_{L^2(\Omega_h)} \le \sup_{\boldsymbol{v} \in \mathring{\boldsymbol{V}}^h \setminus \{0\}} \frac{\int_{\Omega_h} (\nabla \cdot \boldsymbol{v}) \bar{q}}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}}$$

$$
\begin{aligned}
&\leq \sup_{\boldsymbol{v}\in \mathring{\boldsymbol{V}}^h\setminus\{0\}} \frac{\int_{\Omega_h}(\nabla\cdot\boldsymbol{v})q}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}} + \|q-\bar{q}\|_{L^2(\Omega_h)} \\
&\leq (1+\beta_*^{-1}) \sup_{\boldsymbol{v}\in \mathring{\boldsymbol{V}}^h\setminus\{0\}} \frac{\int_{\Omega_h}(\nabla\cdot\boldsymbol{v})q}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}}.
\end{aligned}
$$

Therefore,

$$
\|q\|_{L^2(\Omega_h)} \leq \|q-\bar{q}\|_{L^2(\Omega_h)} + \|\bar{q}\|_{L^2(\Omega_h)} \leq \left(\beta_*^{-1}+\beta_0^{-1}(1+\beta_*^{-1})\right) \sup_{\boldsymbol{v}\in \mathring{\boldsymbol{V}}^h\setminus\{0\}} \frac{\int_{\Omega_h}(\nabla\cdot\boldsymbol{v})q}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega_h)}}.
$$

Setting $\beta_1 = \left(\beta_*^{-1}+\beta_0^{-1}(1+\beta_*^{-1})\right)^{-1}$ gives the desired result. $\square$

4.4.3 Inf-sup stability II

In this section, we prove the inf-sup stability for the Lagrange multiplier part of the bilinear form $b_h(\cdot,\cdot)$.

Lemma 4.4.7. *Assume the triangulation $\mathcal{T}_h$ is quasi-uniform. Then, there holds*

$$
\sup_{\boldsymbol{v}\in \boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\partial\Omega_h}(\boldsymbol{v}\cdot\boldsymbol{n}_h)\mu}{\|\boldsymbol{v}\|_{1,h}} \geq \beta_2\|\mu\|_{-1/2,h} \qquad \forall \mu\in \mathring{X}^h, \tag{4.4.8}
$$

for some $\beta_2>0$ independent of h.

Proof. We label the boundary edges as $\{e_j\}_{j=1}^N = \mathcal{E}_h^B$, and denote the boundary vertices by $\{a_j\}_{j=1}^N = \mathcal{V}_h^B$, labeled such that e_j has vertices a_j and a_{j+1}, with the convention that $a_{N+1}=a_1$. For a boundary edge $e\in\mathcal{E}_h^B$, let $\mathcal{M}_h^e = \{m_j\}_{j=1}^{k-1}$ denote the canonical interior degrees of freedom on the edge e, and set $\mathcal{M}_h^B = \cup_{e\in\mathcal{E}_h^B}\mathcal{M}_h^e$. Let $\boldsymbol{n}_j$ denote the normal vector of $\partial\Omega_h$ restricted to the edge e_j, and let $\boldsymbol{t}_j$ denote the tangent vector obtained by rotating $\boldsymbol{n}_j$ 90 degrees clockwise. Without loss of generality, we assume that $\boldsymbol{t}_j$ is parallel to $a_{j+1}-a_j$. We further denote the set of boundary corner vertices by $\mathcal{V}_h^C$, i.e., if $a_j\in\mathcal{V}_h^C$, then the outward unit normals

$\boldsymbol{n}_j, \boldsymbol{n}_{j-1}$ of the edges touching a_j are linearly independent. The set of flat boundary vertices are defined as $\mathcal{V}_h^F = \mathcal{V}_h^B \backslash \mathcal{V}_h^C$. Note that $\boldsymbol{n}_j = \boldsymbol{n}_{j-1}$ and $\boldsymbol{t}_j = \boldsymbol{t}_{j-1}$ for $a_j \in \mathcal{V}_h^F$.

We let $h_I \in X^h$ denote the continuous, piecewise linear polynomial with respect to the partition $\mathcal{E}_h^B$ satisfying $h_I(a_j) = \frac{1}{2}(h_{e_{j-1}} + h_{e_j})$. For a given $\mu \in \mathring{X}^h$, we let $P_h(h_I\mu) \in \mathring{X}^h$ be the L^2-projection of $h_I\mu$, i.e.,

$$\int_{\partial\Omega_h} P_h(h_I\mu)\kappa = \int_{\partial\Omega_h} h_I\mu\kappa \qquad \forall\kappa \in \mathring{X}^h.$$

We then define $\boldsymbol{v} \in \boldsymbol{V}^h$ by the conditions

$$\begin{aligned}
&(\boldsymbol{v}\cdot\boldsymbol{n}_j)(a_j) = P_h(h_I\mu)(a_j), && (\boldsymbol{v}\cdot\boldsymbol{n}_{j-1})(a_j) = P_h(h_I\mu)(a_j) && \forall a_j \in \mathcal{V}_h^C,\\
&(\boldsymbol{v}\cdot\boldsymbol{n}_j)(a_j) = P_h(h_I\mu)(a_j), && (\boldsymbol{v}\cdot\boldsymbol{t}_j)(a_j) = 0 && \forall a_j \in \mathcal{V}_h^F,\\
&(\boldsymbol{v}\cdot\boldsymbol{n}_j)(m_j) = P_h(h_I\mu)(m_j), && (\boldsymbol{v}\cdot\boldsymbol{t}_j)(m_j) = 0 && \forall m_j \in \mathcal{M}_h^e,\ \forall e \in \mathcal{E}_h^B.
\end{aligned} \tag{4.4.9}$$

All other (Lagrange) degrees of freedom of $\boldsymbol{v}$ are set to zero.

Since $(\boldsymbol{v}\cdot\boldsymbol{n}_j - P_h(h_I\mu))|_{e_j}$ is a polynomial of degree k on each $e_j \in \mathcal{E}_h^B$, and $\boldsymbol{v}\cdot\boldsymbol{n}_j = P_h(h_I\mu)$ at $(k+1)$ distinct points on e_j, we have $\boldsymbol{v}\cdot\boldsymbol{n}_j - P_h(h_I\mu)|_{e_j} = 0$. Thus, by shape regularity,

$$\int_{\partial\Omega_h} (\boldsymbol{v}\cdot\boldsymbol{n}_h)\mu = \int_{\partial\Omega_h} P_h(h_I\mu)\mu = \int_{\partial\Omega_h} h_I\mu^2 \geq C\|\mu\|_{-1/2,h}^2. \tag{4.4.10}$$

In order to complete the proof, it remains to show that $\|\boldsymbol{v}\|_{1,h} \leq C\|\mu\|_{-1/2,h}$.

For $K \in \mathcal{T}_h^{ct}$, let $\mathcal{V}_K^B, \mathcal{V}_K^C, \mathcal{V}_K^F, \mathcal{M}_K^B$ be the sets of elements in $\mathcal{V}_h^B, \mathcal{V}_h^C, \mathcal{V}_h^F, \mathcal{M}_h^B$ contained in $\bar{K}$, respectively. By a standard scaling argument and (4.4.9), we get $(m = 0, 1)$

$$\|\boldsymbol{v}\|_{H^m(K)}^2 \leq C \sum_{c_j \in \mathcal{V}_K^B \cup \mathcal{M}_K^B} h_{e_j}^{2-2m} |\boldsymbol{v}(c_j)|^2 \tag{4.4.11}$$

$$= C\Big(\sum_{a_j \in \mathcal{V}_K^C} h_{e_j}^{2-2m} |\boldsymbol{v}(a_j)|^2 + \sum_{c_j \in \mathcal{V}_K^F \cup \mathcal{M}_K^B} h_{e_j}^{2-2m} |P_h(h_I \mu)(c_j)|^2 \Big).$$

Claim: $|\boldsymbol{v}(a_j)| \leq C|P_h(h_I\mu)(a_j)|$ for all $a_j \in \mathcal{V}_K^C$, where $C > 0$ is uniformly bounded and independent of h, $\boldsymbol{n}_j$ and $\boldsymbol{n}_{j-1}$.

Proof of the claim: Assume that $\mathcal{V}_K^C$ is non-empty for otherwise the proof is trivial. For $a_j \in \mathcal{V}_K^C$, we write $\boldsymbol{v}(a_j)$ in terms of the basis $\{\boldsymbol{t}_j, \boldsymbol{t}_{j-1}\}$, use (4.4.9), and apply some elementary vector identities:

$$\begin{aligned}
\boldsymbol{v}(a_j) &= \frac{1}{\boldsymbol{t}_{j-1} \cdot \boldsymbol{n}_j} (\boldsymbol{v} \cdot \boldsymbol{n}_j)(a_j) \boldsymbol{t}_{j-1} + \frac{1}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}} (\boldsymbol{v} \cdot \boldsymbol{n}_{j-1})(a_j) \boldsymbol{t}_j \\
&= P_h(h_I\mu)(a_j) \Big(\frac{1}{\boldsymbol{t}_{j-1} \cdot \boldsymbol{n}_j} \boldsymbol{t}_{j-1} + \frac{1}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}} \boldsymbol{t}_j \Big) \\
&= P_h(h_I\mu)(a_j) \Big(\frac{\boldsymbol{t}_j - \boldsymbol{t}_{j-1}}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}} \Big).
\end{aligned} \tag{4.4.12}$$

We next show that $\big|\frac{\boldsymbol{t}_j - \boldsymbol{t}_{j-1}}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}}\big|$ is bounded. To this end, we write $\boldsymbol{t}_j = (\cos(\theta_j), \sin(\theta_j))^\intercal$ with $\theta_{j-1}, \theta_j \in [-\pi, \pi]$, so that

$$\frac{\boldsymbol{t}_j - \boldsymbol{t}_{j-1}}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}} = \frac{(\cos(\theta_j) - \cos(\theta_{j-1}), \sin(\theta_j) - \sin(\theta_{j-1}))^\intercal}{\sin(\theta_j - \theta_{j-1})}.$$

Since

$$\begin{aligned}
\lim_{\theta_j \to \theta_{j-1}} \frac{(\cos\theta_j - \cos\theta_{j-1}, \sin\theta_j - \sin\theta_{j-1})^\intercal}{\sin(\theta_j - \theta_{j-1})} &= \lim_{\theta_j \to \theta_{j-1}} \frac{(-\sin\theta_j, \cos\theta_j)^\intercal}{\cos(\theta_j - \theta_{j-1})} \\
&= (-\sin\theta_{j-1}, \cos\theta_{j-1})^\intercal,
\end{aligned}$$

and due to the shape regularity of the mesh, we conclude $\big|\frac{\boldsymbol{t}_j - \boldsymbol{t}_{j-1}}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}}\big|$ is bounded in the case $|\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}| \ll 1$, i.e., for "nearly flat boundary vertices". Therefore, $\big|\frac{\boldsymbol{t}_j - \boldsymbol{t}_{j-1}}{\boldsymbol{t}_j \cdot \boldsymbol{n}_{j-1}}\big| \leq C$ on shape-regular triangulations for some $C > 0$ independent of h and $\{\boldsymbol{n}_{j-1}, \boldsymbol{n}_j\}$. With (4.4.12), this yields $|\boldsymbol{v}(a_j)| \leq C|P_h(h_I\mu)(a_j)|$ for all $a_j \in \mathcal{V}_K^C$, which concludes the proof of the claim.

Applying the claim to (4.4.11) and a scaling argument yields

$$\|\boldsymbol{v}\|_{H^m(K)}^2 \le C \sum_{c_j \in \mathcal{V}_K^B \cup \mathcal{M}_K^B} h_{e_j}^{2-2m} |P_h(h_I\mu)(c_j)|^2 \le C \sum_{\substack{e \in \mathcal{E}_h^B \\ a_j \in \bar{e}:\ a_j \in \mathcal{V}_K^B}} h_e^{1-2m} \|P_h(h_I\mu)\|_{L^2(e)}^2.$$

Therefore, by the trace and inverse inequalities together with the shape-regularity of $\mathcal{T}_h^{ct}$, we obtain

$$\begin{aligned}
\|\boldsymbol{v}\|_{1,h}^2 &= \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}^2 + \sum_{e \in \mathcal{E}_h^B} \frac{1}{h_e} \|\boldsymbol{v}\|_{L^2(e)}^2 \\
&\le \|\nabla \boldsymbol{v}\|_{L^2(\Omega_h)}^2 + C \sum_{K \in \mathcal{T}_h^{ct}} h_K^{-2} \|\boldsymbol{v}\|_{L^2(K)}^2 \le C \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|P_h(h_I\mu)\|_{L^2(e)}^2.
\end{aligned}$$

Finally, using the L^2-stability of $P_h(h_I\mu)$ and the quasi-uniform assumption, we have

$$\begin{aligned}
\|\boldsymbol{v}\|_{1,h}^2 &\le C \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \|P_h(h_I\mu)\|_{L^2(e)}^2 \\
&\le C h^{-1} \|P_h(h_I\mu)\|_{L^2(\partial\Omega_h)}^2 \le C h^{-1} \|h_I\mu\|_{L^2(\partial\Omega_h)}^2 \le C \|\mu\|_{-1/2,h}^2.
\end{aligned} \tag{4.4.13}$$

Combining this estimate with (4.4.10) yields the desired inf-sup condition (4.4.8). □

Remark 4.4.8. Notice that the proof of Lemma 4.4.7, and in particular the proof of the claim, relies on the continuity properties of the Lagrange multiplier space at nearly flat corner vertices.

4.4.4 Main stability estimates

Here, we combine Lemmas 4.4.6 and 4.4.7 in order to obtain the inf-sup stability for the bilinear form $b_h(\cdot,\cdot)$. We then show that this result implies the inf-sup stability for the bilinear form with boundary correction $b_h^e(\cdot,\cdot)$.

Theorem 4.4.9. *Assume $\mathcal{T}_h$ is quasi-uniform. Then there exists $\beta > 0$ depending only on β_1 and β_2 such that*

$$\beta\|(q,\mu)\| \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}^h\setminus\{0\}} \frac{b_h(\boldsymbol{v},(q,\mu))}{\|\boldsymbol{v}\|_{1,h}} \qquad \forall (q,\mu)\in \mathring{Q}^h\times\mathring{X}^h. \tag{4.4.14}$$

Proof. We use Lemmas 4.4.6 and 4.4.7 and follow the arguments in [21, Theorem 3.1].

Let $(q,\mu)\in\mathring{Q}^h\times\mathring{X}^h$. The statement (4.4.8) implies the existence of $\boldsymbol{v}_2\in\boldsymbol{V}^h$ such that $\|\boldsymbol{v}_2\|_{1,h}\leq 1$ and

$$\int_{\partial\Omega_h}(\boldsymbol{v}_2\cdot\boldsymbol{n}_h)\mu \geq \beta_2\|\mu\|_{-1/2,h}.$$

By Lemma 4.4.6, there exists $\boldsymbol{v}_1\in\mathring{\boldsymbol{V}}^h$ satisfying $\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_h)} = \|\boldsymbol{v}_1\|_{1,h}\leq 1$ and

$$-\int_{\Omega_h}(\nabla\cdot\boldsymbol{v}_1)q \geq \beta_1\|q\|_{L^2(\Omega_h)}.$$

Set $\boldsymbol{v} = c\boldsymbol{v}_1+\boldsymbol{v}_2$ for some $c>0$, so that $\|\boldsymbol{v}\|_{1,h}\leq(1+c)$, and

$$\begin{aligned}
-\int_{\Omega_h}(\nabla\cdot\boldsymbol{v})q &\geq c\beta_1\|q\|_{L^2(\Omega_h)} - \|\nabla\cdot\boldsymbol{v}_2\|_{L^2(\Omega_h)}\|q\|_{L^2(\Omega_h)}\\
&\geq c\beta_1\|q\|_{L^2(\Omega_h)} - \sqrt{2}\|\nabla\boldsymbol{v}_2\|_{L^2(\Omega_h)}\|q\|_{L^2(\Omega_h)}\\
&\geq c\beta_1\|q\|_{L^2(\Omega_h)} - \sqrt{2}\|\boldsymbol{v}_2\|_{1,h}\|q\|_{L^2(\Omega_h)}\\
&= \big(c\beta_1-\sqrt{2}\big)\|q\|_{L^2(\Omega_h)}.
\end{aligned}$$

Moreover, since $\boldsymbol{v}_1|_{\partial\Omega_h} = 0$, we also have

$$\int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)\mu = \int_{\partial\Omega_h} (\boldsymbol{v}_2 \cdot \boldsymbol{n}_h)\mu \geq \beta_2 \|\mu\|_{-1/2,h}.$$

Combining the above inequalities, we find

$$\begin{aligned} b_h(\boldsymbol{v}, (q,\mu)) &\geq \big(c\beta_1 - \sqrt{2}\big)\|q\|_{L^2(\Omega_h)} + \beta_2\|\mu\|_{-1/2,h} \\ &\geq (1+c)^{-1}\Big(\big(c\beta_1 - \sqrt{2}\big)\|q\|_{L^2(\Omega_h)} + \beta_2\|\mu\|_{-1/2,h}\Big)\|\boldsymbol{v}\|_{1,h}, \end{aligned}$$

and choosing $c > 0$ sufficiently large yields the desired result. □

Corollary 4.4.10. *Provided Assumption* (A) *is satisfied and the mesh* $\mathcal{T}_h$ *is quasi-uniform, there exists* $\beta_e > 0$ *independent of* h *such that there holds*

$$\beta_e\|(q,\mu)\| \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}^h\setminus\{0\}} \frac{b_h^e(\boldsymbol{v},(q,\mu))}{\|\boldsymbol{v}\|_{1,h}} \qquad \forall (q,\mu) \in \mathring{Q}^h \times \mathring{X}^h. \tag{4.4.15}$$

Proof. Combining Theorem 4.4.9 and Lemma 4.4.2 with the triangle inequality, we immediately find

$$\beta\|(q,\mu)\| \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}^h\setminus\{0\}} \frac{b_h^e(\boldsymbol{v},(q,\mu))}{\|\boldsymbol{v}\|_{1,h}} + Cc_\delta\|(q,\mu)\| \qquad \forall (q,\mu) \in \mathring{Q}^h \times \mathring{X}^h.$$

This result implies (4.4.15) for c_δ sufficiently small with $\beta_e = \beta - Cc_\delta$. □

The next theorem provides a stability result and ensures that the problem (4.3.1) is well posed.

Theorem 4.4.11. *Let* $(\boldsymbol{u}_h, p_h, \lambda_h) \in \boldsymbol{V}^h \times \mathring{Q}^h \times \mathring{X}^h$ *satisfy* (4.3.1)*. Then, provided* c_δ *in Assumption* (A) *is sufficiently small and the mesh* $\mathcal{T}_h$ *is quasi-uniform, there holds*

$$\nu\|\boldsymbol{u}_h\|_{1,h} + \|(p_h,\lambda_h)\| \leq C\|\boldsymbol{f}\|_{-1,h}, \tag{4.4.16}$$

where $\|\boldsymbol{f}\|_{-1,h} = \sup_{\boldsymbol{v}\in\boldsymbol{V}^h\setminus\{0\}} \frac{\int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|_{1,h}}$*. Consequently, there exists a unique solution to* (4.3.1)*.*

Proof. Setting $\boldsymbol{v} = \boldsymbol{u}_h$ in (4.3.1a), $(q, \mu) = (p_h, \lambda_h)$ in (4.3.1b), and subtracting the resulting expressions yields

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{u}_h + \int_{\partial\Omega_h} \big((S_h \boldsymbol{u}_h - \boldsymbol{u}_h) \cdot \boldsymbol{n}_h\big)\lambda_h.$$

We then apply the coercivity result in Lemma 4.4.3, the Cauchy-Schwarz inequality, and (4.4.1) to get

$$\nu c_1 \|\boldsymbol{u}_h\|_{1,h}^2 \leq \|\boldsymbol{f}\|_{-1,h} \|\boldsymbol{u}_h\|_{1,h} + C c_\delta \|\boldsymbol{u}_h\|_{1,h} \|\lambda_h\|_{-1/2,h}. \tag{4.4.17}$$

On the other hand, we use inf-sup stability (4.4.14) with (4.3.1a) to conclude

$$\begin{aligned} \beta \|(p_h, \lambda_h)\|_{-1/2,h} &\leq \sup_{\boldsymbol{v} \in \boldsymbol{V}^h \setminus \{0\}} \frac{b_h(\boldsymbol{v}, (p_h, \lambda_h))}{\|\boldsymbol{v}\|_{1,h}} \\ &\leq \sup_{\boldsymbol{v} \in \boldsymbol{V}^h \setminus \{0\}} \frac{\int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v} - a_h(\boldsymbol{u}_h, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1,h}}. \end{aligned}$$

Using the continuity estimate (4.4.3) with the above inequality yields

$$\beta \|\lambda_h\|_{-1/2,h} \leq \beta \|(p_h, \lambda_h)\| \leq \|\boldsymbol{f}\|_{-1,h} + C(1+\sigma)\nu \|\boldsymbol{u}_h\|_{1,h}. \tag{4.4.18}$$

Inserting this estimate into (4.4.17), we obtain

$$\nu\big(c_1 - C c_\delta \beta^{-1}(1+\sigma)\big)\|\boldsymbol{u}_h\|_{1,h} \leq (1 + C c_\delta \beta^{-1}) \|\boldsymbol{f}\|_{-1,h}.$$

Therefore, $\|\boldsymbol{u}_h\|_{1,h} \leq C\nu^{-1}\|\boldsymbol{f}\|_{-1,h}$ for a sufficiently small c_δ. This, combined with (4.4.18), yields the desired stability result (4.4.16).

□

4.5 Convergence analysis

In this section, we show that the finite element method (4.3.1) leads to optimally convergent solutions provided that the exact solution is sufficiently smooth. Throughout this section, we assume that the hypotheses of Theorem 4.4.11 are satisfied, i.e., Assumption (A) is satisfied and the mesh $\mathcal{T}_h$ is quasi-uniform.

4.5.1 Consistency estimates

Notice that due to the homogeneous Dirichlet boundary conditions imposed on $\partial\Omega$, we have $S_h\boldsymbol{u}+R_h\boldsymbol{u} = 0$, where $R_h\boldsymbol{u}$ denotes the Taylor remainder. The following lemma bounds the boundary correction operator acting on the exact velocity function $\boldsymbol{u}$. The result essentially follows from an estimate on $R_h\boldsymbol{u}$ and can be proven using similar arguments in [32, Proposition 3] (also see [22]). For this reason, we just give a sketch of the proof.

Lemma 4.5.1. *For any $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, there holds*

$$\sum_{e\in\mathcal{E}_h^B} h_e^{-1} \int_e \left|S_h\boldsymbol{u}\right|^2 \le Ch^{2k}\|\boldsymbol{u}\|^2_{H^{k+1}(\Omega)}.$$

Proof. Let $e \in \mathcal{E}_h^B$ be a boundary edge with endpoints a_1, a_2, and let $x(t)$ be a parametrization of e given by $x(t) := a_1 + th_e^{-1}(a_2 - a_1)$ with $0 \le t \le h_e$. Then, we define a 2D parameterization $\varphi(t,s) := x(t) + s\boldsymbol{d}(x(t))$ with $0 \le t \le h_e$ and $0 \le s \le \delta(x(t))$. The Taylor remainder estimation with $S_h\boldsymbol{u} + R_h\boldsymbol{u} = 0$ yields

$$|S_h\boldsymbol{u}(x(t))| = |R_h\boldsymbol{u}(x(t))| = \frac{1}{k!}\Big|\int_0^{\delta(x(t))} \frac{\partial^{k+1}\boldsymbol{u}}{\partial\boldsymbol{d}^{k+1}}(\varphi(t,s))(\delta(x(t)) - s)^k\Big|.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$|S_h \boldsymbol{u}(x(t))| \le C\delta(x(t))^{k+1/2}\Big(\int_0^{\delta(x(t))} \big|\frac{\partial^{k+1}\boldsymbol{u}}{\partial \boldsymbol{d}^{k+1}}(\varphi(t,s))\big|^2\Big)^{1/2},$$

and therefore

$$\begin{aligned} h_e^{-1}\|S_h\boldsymbol{u}\|^2_{L^2(e)} &\le Ch_e^{-1}\delta_e^{2k+1}\int_0^{h_e}\int_0^{\delta(x(t))}\big|\frac{\partial^{k+1}\boldsymbol{u}}{\partial \boldsymbol{d}^{k+1}}(\varphi(t,s))\big|^2 \\ &\le Ch_e^{2k}\int_0^{h_e}\int_0^{\delta(x(t))}\big|\frac{\partial^{k+1}\boldsymbol{u}}{\partial \boldsymbol{d}^{k+1}}(\varphi(t,s))\big|^2, \end{aligned}$$

where we used Assumption (A) in the last inequality. The estimate in Lemma 4.5.1 now follows from a change of variables (cf. [35, 32]) and summing over $e \in \mathcal{E}_h^B$. □

Using the above lemma and previous results, we next compute the consistency error of the scheme.

Lemma 4.5.2. *There holds for all* $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega)\cap \boldsymbol{H}_0^1(\Omega)$,

$$\Big| -\nu\int_{\Omega_h}\Delta\boldsymbol{u}\cdot\boldsymbol{v} - a_h(\boldsymbol{u},\boldsymbol{v})\Big| \le C\nu h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|\boldsymbol{v}\|_{1,h} \qquad \forall \boldsymbol{v}\in\boldsymbol{V}^h. \tag{4.5.1}$$

If $\nabla\cdot\boldsymbol{u} = 0$ *in* Ω, *then*

$$\big|b_h^e(\boldsymbol{u},(q,\mu))\big| \le Ch^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|(q,\mu)\| \qquad \forall (q,\mu)\in \mathring{Q}^h\times\mathring{X}^h.$$

Proof. We integrate-by-parts to write

$$\Big| -\nu\int_{\Omega_h}\Delta\boldsymbol{u}\cdot\boldsymbol{v} - a_h(\boldsymbol{u},\boldsymbol{v})\Big| = \nu\Big|\sum_{e\in\mathcal{E}_h^B}\int_e \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}_h}\cdot(S_h\boldsymbol{u}) + \sum_{e\in\mathcal{E}_h^B}\frac{\sigma}{h_e}\int_e (S_h\boldsymbol{u})\cdot(S_h\boldsymbol{v})\Big|.$$

Next, we estimate the two terms on the right hand side of the above equality by using the Cauchy-Schwarz inequality, trace and inverse inequalities, along with Lemmas 4.4.1 and 4.5.1 as follows:

$$\begin{aligned}\Big|\sum_{e\in\mathcal{E}_h^B}\int_e \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_h}\cdot(S_h\boldsymbol{u})\Big| &\le \Big(\sum_{e\in\mathcal{E}_h^B} h_e\int_e \Big|\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_h}\Big|^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\int_e |S_h\boldsymbol{u}|^2\Big)^{1/2}\\ &\le Ch^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|\boldsymbol{v}\|_{1,h},\end{aligned}$$

and

$$\begin{aligned}\Big|\sum_{e\in\mathcal{E}_h^B}\frac{\sigma}{h_e}\int_e (S_h\boldsymbol{u})\cdot(S_h\boldsymbol{v})\Big| &\le \sigma\Big(\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\int_e |S_h\boldsymbol{u}|^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\int_e |S_h\boldsymbol{v}|^2\Big)^{1/2}\\ &\le Ch^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|\boldsymbol{v}\|_{1,h},\end{aligned}$$

which proves the first estimate (4.5.1).

Assuming that $\boldsymbol{u}$ is divergence-free in Ω, another use of the Cauchy-Schwarz inequality with Lemma 4.5.1 yields

$$\big|b_h^e(\boldsymbol{u},(q,\mu))\big| = \Big|\sum_{e\in\mathcal{E}_h^B}\int_e (S_h\boldsymbol{u}\cdot\boldsymbol{n}_h)\mu\Big| \le Ch^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|\mu\|_{-1/2,h},$$

and this completes the proof. □

4.5.2 Approximation properties of the kernel

We define the discrete kernel as

$$\boldsymbol{Z}^h = \{\boldsymbol{v} \in \boldsymbol{V}^h : \; b_h^e(\boldsymbol{v}, (q,\mu)) = 0, \; \forall (q,\mu) \in \mathring{Q}^h \times \mathring{X}^h\}.$$

Note that if $\boldsymbol{v} \in \boldsymbol{Z}^h$, then $\nabla \cdot \boldsymbol{v} = 0$ in Ω_h (cf. Lemma 4.3.1), and as a result, there holds

$$\int_{\partial\Omega_h} ((S_h \boldsymbol{v}) \cdot \boldsymbol{n}_h)\mu = 0 \qquad \forall \mu \in \mathring{X}^h. \tag{4.5.2}$$

In this section, we show that the kernel $\boldsymbol{Z}^h$ has optimal order approximation properties with respect to divergence-free smooth functions. To this end, we define the orthogonal complement of $\boldsymbol{Z}^h$ as

$$\boldsymbol{Z}_\perp^h := \{\boldsymbol{v} \in \boldsymbol{V}^h : \; (\boldsymbol{v}, \boldsymbol{w})_{1,h} = 0 \;\; \forall \boldsymbol{w} \in \boldsymbol{Z}^h\},$$

where $(\cdot,\cdot)_{1,h}$ is the inner product on $\boldsymbol{V}^h$ that induces the norm $\|\cdot\|_{1,h}$.

Define $B^e : \boldsymbol{Z}_\perp^h \to (\mathring{Q}^h \times \mathring{X}^h)'$ as $B^e(\boldsymbol{v})(q,\mu) := b_h^e(\boldsymbol{v}, (q,\mu))$, $\forall \boldsymbol{v} \in \boldsymbol{Z}_\perp^h$ and $\forall (q,\mu) \in \mathring{Q}^h \times \mathring{X}^h$, where the notation $(\mathring{Q}^h \times \mathring{X}^h)'$ denotes the dual space of $\mathring{Q}^h \times \mathring{X}^h$.

The next lemma is a well-known implication of the inf-sup stability established in Corollary 4.4.10 (cf. [38, Lemma 12.5.10]).

Lemma 4.5.3. *$B^e : \boldsymbol{Z}_\perp^h \to (\mathring{Q}^h \times \mathring{X}^h)'$ is an isomorphism. Moreover, there holds*

$$\beta_e \|\boldsymbol{w}\|_{1,h} \le \sup_{(q,\mu)\in \mathring{Q}^h \times \mathring{X}^h \setminus \{0\}} \frac{b_h^e(\boldsymbol{w}, (q,\mu))}{\|(q,\mu)\|} \qquad \forall \boldsymbol{w} \in \boldsymbol{Z}_\perp^h.$$

The following theorem states the approximation properties of the discrete kernel.

Theorem 4.5.4. *For any $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ with $\nabla \cdot \boldsymbol{u} = 0$, there holds*

$$\inf_{\boldsymbol{w}\in \boldsymbol{Z}^h} |||\boldsymbol{u} - \boldsymbol{w}|||_h \le C h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}. \tag{4.5.3}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^h$ be arbitrary. By Lemma 4.5.3, there exists $\boldsymbol{y} \in \boldsymbol{Z}_\perp^h$ such that

$$b_h^e(\boldsymbol{y}, (q, \mu)) = b_h^e(\boldsymbol{u} - \boldsymbol{v}, (q, \mu)) \qquad \forall (q, \mu) \in \mathring{Q}^h \times \mathring{X}^h,$$

with $\|\boldsymbol{y}\|_{1,h} \leq C\beta_e^{-1}\|\boldsymbol{u} - \boldsymbol{v}\|_{1,h}$, where $C > 0$ is the continuity constant of the bilinear form b_h^e (cf. (4.4.6)). Another use of Lemma 4.5.3 implies the existence of $\boldsymbol{z} \in \boldsymbol{Z}_\perp^h$ satisfying

$$b_h^e(\boldsymbol{z}, (q, \mu)) = -b_h^e(\boldsymbol{u}, (q, \mu)) \qquad \forall (q, \mu) \in \mathring{Q}^h \times \mathring{X}^h.$$

Then $\boldsymbol{w} := \boldsymbol{v} + \boldsymbol{y} + \boldsymbol{z} \in \boldsymbol{Z}^h$, and

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{w}\|_{1,h} &\leq \|\boldsymbol{u} - \boldsymbol{v}\|_{1,h} + \|\boldsymbol{y}\|_{1,h} + \|\boldsymbol{z}\|_{1,h} \\ &\leq (1 + C\beta_e^{-1})\|\boldsymbol{u} - \boldsymbol{v}\|_{1,h} + \|\boldsymbol{z}\|_{1,h}.\end{aligned}$$

Moreover, using Lemmas 4.5.3 and 4.5.2, we find

$$\beta_e \|\boldsymbol{z}\|_{1,h} \leq \sup_{(q,\mu) \in \mathring{Q}^h \times \mathring{X}^h \setminus \{0\}} \frac{b_h^e(\boldsymbol{u}, (q, \mu))}{\|(q, \mu)\|} \leq Ch^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)},$$

and so, by Lemma 4.4.1,

$$\begin{aligned}|||\boldsymbol{u} - \boldsymbol{w}|||_h &\leq |||\boldsymbol{u} - \boldsymbol{v}|||_h + C\|\boldsymbol{v} - \boldsymbol{w}\|_{1,h} \leq C\big(|||\boldsymbol{u} - \boldsymbol{v}|||_h + \|\boldsymbol{u} - \boldsymbol{v}\|_{1,h} + \|\boldsymbol{u} - \boldsymbol{w}\|_{1,h}\big) \\ &\leq C(1 + \beta_e^{-1})\big(|||\boldsymbol{u} - \boldsymbol{v}|||_h + \|\boldsymbol{u} - \boldsymbol{v}\|_{1,h} + h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\big) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}^h.\end{aligned}$$

Letting $\boldsymbol{v}$ be the nodal interpolant of $\boldsymbol{u}$ yields the desired result. $\square$

Next theorem studies the error estimates of the proposed scheme (4.3.1), and is the main result of this section.

Theorem 4.5.5. *Suppose that the solution to* (4.2.1) *has sufficient regularity so that* $(\boldsymbol{u}, p) \in \boldsymbol{H}^{k+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega) \times H^1(\Omega)$*. Furthermore, without loss of generality, assume that* $p|_{\Omega_h} \in L_0^2(\Omega_h)$*. Then,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \le C\big(h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \nu^{-1} \inf_{\mu \in X^h} \|p - \mu\|_{-1/2,h}\big), \tag{4.5.4a}$$

$$\|p - p_h\|_{L^2(\Omega_h)} \le C(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \inf_{\mu \in X^h} \|p - \mu\|_{-1/2,h} + \inf_{q_h \in \mathring{Q}^h} \|p - q_h\|_{L^2(\Omega)}), \tag{4.5.4b}$$

$$\|\mathring{p} - \lambda_h\|_{-1/2,h} \le C\Big(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \inf_{\mu \in X^h} \|p - \mu\|_{-1/2,h}\big), \tag{4.5.4c}$$

where $\mathring{p} := p - \frac{1}{|\partial\Omega_h|} \int_{\partial\Omega_h} p$*. In particular, if* $p \in H^{k+1}(\Omega)$ *there holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \le C\big(h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \nu^{-1} h^{k+1} \|p\|_{H^{k+1}(\Omega)}\big), \tag{4.5.5a}$$

$$\|p - p_h\|_{L^2(\Omega_h)} \le C\big(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^k \|p\|_{H^k(\Omega)}\big), \tag{4.5.5b}$$

$$\|\mathring{p} - \lambda_h\|_{-1/2,h} \le C\big(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^{k+1} \|p\|_{H^{k+1}(\Omega)}\big). \tag{4.5.5c}$$

Remark 4.5.6. Theorem 4.5.5 shows that the inclusion of the Lagrange multiplier in the method yields an additional power of h in the velocity error, which compensates its dependence on the inverse of the viscosity and mitigates the lack of pressure robustness..

Proof. Let $\boldsymbol{w} \in \boldsymbol{Z}^h$ be arbitrary. For all $\boldsymbol{v} \in \boldsymbol{Z}^h$ and $\mu \in X^h$, we use (4.3.1a), the integration by parts formula and the equality $\int_{\partial\Omega_h} \boldsymbol{v} \cdot \boldsymbol{n}_h = 0$ to write

$$\begin{aligned} a_h(\boldsymbol{u}_h - \boldsymbol{w}, \boldsymbol{v}) &= \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v} - a_h(\boldsymbol{w}, \boldsymbol{v}) - b_h(\boldsymbol{v}, (p_h, \lambda_h)) \\ &= -\nu \int_{\Omega_h} \Delta \boldsymbol{u} \cdot \boldsymbol{v} - a_h(\boldsymbol{w}, \boldsymbol{v}) - \int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)(\lambda_h - p) \\ &= -\nu \int_{\Omega_h} \Delta \boldsymbol{u} \cdot \boldsymbol{v} - a_h(\boldsymbol{w}, \boldsymbol{v}) - \int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)(\mu - p) \end{aligned}$$

$$- \int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)(\lambda_h - \mathring{\mu}),$$

where $\mathring{\mu} = \mu - \frac{1}{|\partial\Omega_h|} \int_{\partial\Omega_h} \mu \in \mathring{X}^h$.

Therefore, using Lemma 4.5.2, the continuity of $a_h(\cdot,\cdot)$ (cf. (4.4.3)), and the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
a_h(\boldsymbol{u}_h - \boldsymbol{w}, \boldsymbol{v}) &\leq C\big(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p-\mu\|_{-1/2,h}\big)\|\boldsymbol{v}\|_{1,h} + a_h(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{v}) \\
&\quad - \int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)(\lambda_h - \mathring{\mu})\big) \\
&\leq C\big(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \nu(1+\sigma)|||\boldsymbol{u} - \boldsymbol{w}|||_h + \|p-\mu\|_{-1/2,h}\big)\|\boldsymbol{v}\|_{1,h} \\
&\quad - \int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)(\lambda_h - \mathring{\mu})\big).
\end{aligned}$$

We then use (4.5.2), the Cauchy-Schwarz inequality and (4.4.1) to obtain

$$\begin{aligned}
\int_{\partial\Omega_h} (\boldsymbol{v} \cdot \boldsymbol{n}_h)(\lambda_h - \mathring{\mu}) &= \int_{\partial\Omega_h} \big((\boldsymbol{v} - S_h\boldsymbol{v}) \cdot \boldsymbol{n}_h\big)(\lambda_h - \mathring{\mu}) \\
&\leq C c_\delta \|\boldsymbol{v}\|_{1,h} \|\lambda_h - \mathring{\mu}\|_{-1/2,h}.
\end{aligned}$$

Setting $\boldsymbol{v} = \boldsymbol{u}_h - \boldsymbol{w}$, applying the coercivity of $a_h(\cdot,\cdot)$ and Theorem 4.5.4, we get

$$c_1\nu\|\boldsymbol{u}_h - \boldsymbol{w}\|_{1,h} \leq C\big(\nu(1+\sigma)h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p-\mu\|_{-1/2,h} + c_\delta\|\lambda_h - \mathring{\mu}\|_{-1/2,h}\big) \tag{4.5.6}$$

for $\boldsymbol{w} \in \boldsymbol{Z}^h$ satisfying (4.5.3).

Next, let $P_h \in \mathring{Q}^h$ be the L^2-projection of p and note that, due to the definitions of the finite element spaces, $\int_{\Omega_h} (\nabla \cdot \boldsymbol{v})(p - P_h) = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}^h$. This identity, along with the inf-sup stability estimate given in Theorem 4.4.9 yields

$$\beta\|(p_h - P_h, \lambda_h - \mathring{\mu})\| \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}^h\setminus\{0\}} \frac{b_h(\boldsymbol{v}, (p_h - P_h, \lambda_h - \mathring{\mu}))}{\|\boldsymbol{v}\|_{1,h}}$$

$$= \sup_{\boldsymbol{v}\in \boldsymbol{V}^h\setminus\{0\}} \frac{b_h(\boldsymbol{v},(p_h-p,\lambda_h-\mathring{\mu}))}{\|\boldsymbol{v}\|_{1,h}}.$$

We then use (4.3.1a) and Lemma 4.5.2 in order to bound the numerator:

$$\begin{aligned}
b_h(\boldsymbol{v},(p_h-p,\lambda_h-\mathring{\mu})) &= b_h(\boldsymbol{v},(p_h,\lambda_h)) - b_h(\boldsymbol{v},(p,\mathring{\mu}))\\
&= \int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v} - a_h(\boldsymbol{u}_h,\boldsymbol{v}) + \int_{\Omega_h} (\nabla\cdot\boldsymbol{v})p\\
&\quad - \int_{\partial\Omega_h} (\boldsymbol{v}\cdot\boldsymbol{n}_h)\mu\\
&\le C\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|\boldsymbol{v}\|_{1,h} + a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})\\
&\quad - \int_{\partial\Omega_h} (\boldsymbol{v}\cdot\boldsymbol{n}_h)(\mu-p).
\end{aligned}$$

Therefore, by continuity and the Cauchy-Schwarz inequality, we have

$$\beta\|(p_h-P_h,\lambda_h-\mathring{\mu})\| \le C\big(\nu h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + c_2\nu(1+\sigma)|||\boldsymbol{u}-\boldsymbol{u}_h|||_h + \|p-\mu\|_{-1/2,h}\big) \tag{4.5.7}$$

$$\begin{aligned}
&\le C\big(\nu h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + c_2\nu(1+\sigma)\big(|||\boldsymbol{u}-\boldsymbol{w}|||_h + \|\boldsymbol{u}_h-\boldsymbol{w}\|_{1,h}\big) + \|p-\mu\|_{-1/2,h}\big)\\
&\le C\big(\nu(1+\sigma)h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + c_2\nu(1+\sigma)\|\boldsymbol{u}_h-\boldsymbol{w}\|_{1,h} + \|p-\mu\|_{-1/2,h}\big).
\end{aligned}$$

Inserting this estimate into (4.5.6), we find

$$\nu\big(c_1 - C\beta^{-1}c_2(1+\sigma)c_\delta\big)\|\boldsymbol{u}_h-\boldsymbol{w}\|_{1,h} \le C\nu(1+\sigma)h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + C\|p-\mu\|_{-1/2,h}. \tag{4.5.8}$$

Notice that there also holds $\inf_{\boldsymbol{w}\in \boldsymbol{Z}^h}\|\boldsymbol{u}-\boldsymbol{w}\|_{1,h} \le Ch^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}$. Using this inequality with the triangle inequality, and assuming sufficiently small c_δ, we obtain

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h} \le C\big(h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \nu^{-1}\inf_{\mu\in X^h}\|p-\mu\|_{-1/2,h}\big),$$

which establishes the velocity estimate (4.5.4a).

In order to obtain the estimate for the pressure approximation (4.5.4b), we use the triangle inequality and the approximation properties of the L^2-projection:

$$\|p - p_h\|_{L^2(\Omega_h)} \le \|p_h - P_h\|_{L^2(\Omega_h)} + \inf_{q_h \in \mathring{Q}^h} \|p - q_h\|_{L^2(\Omega_h)}.$$

Inserting (4.5.7) and (4.5.8) into the right-hand side yields the desired bound for the pressure. Likewise, combining (4.5.7) and (4.5.8) yields

$$\|\mathring{p} - \lambda_h\|_{-1/2,h} \le C\Big(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \inf_{\mu \in X^h} \big(\|p - \mu\|_{-1/2,h} + \|\mathring{p} - \mathring{\mu}\|_{-1/2,h}\big)\Big).$$

Applications of the Cauchy-Schwarz inequality show $\|\mathring{p} - \mathring{\mu}\|_{-1/2,h} \le C\|p - \mu\|_{-1/2,h}$ on quasi-uniform meshes, and therefore (4.5.4c) holds.

Next, we estimate the term $\inf_{\mu \in X^h} \|p - \mu\|_{-1/2,h}$ for $p \in H^{k+1}(\Omega)$. With an abuse of notation, let μ_I denote the kth degree nodal Lagrange interpolant of p on Ω_h with respect to $\mathcal{T}_h^{ct}$. Notice that $\mu_I|_{\partial\Omega_h} \in X^h$. Applying a trace inequality, followed by standard interpolation estimates and shape regularity of $\mathcal{T}_h^{ct}$, we obtain for each $e \in \mathcal{E}_h^B$,

$$\|p - \mu_I\|_{L^2(e)}^2 \le C\big(h_e^{-1}\|p - \mu_I\|_{L^2(T_e)}^2 + h_e\|\nabla(p - \mu_I)\|_{L^2(T_e)}^2\big) \le C h_e^{2k+1}\|p\|_{H^{k+1}(T_e)}^2,$$

where $T_e \in \mathcal{T}_h^{ct}$ satisfies $e \subset \partial T_e$. Therefore, we conclude from the definition of $\|\cdot\|_{-1/2,h}$ that

$$\inf_{\mu \in X^h} \|p - \mu\|_{-1/2,h} \le C h^{k+1}\|p\|_{H^{k+1}(\Omega)}. \tag{4.5.9}$$

Finally, the estimates (4.5.5a)-(4.5.5c) follow from (4.5.4a)-(4.5.4c), interpolation estimates, and (4.5.9). $\square$

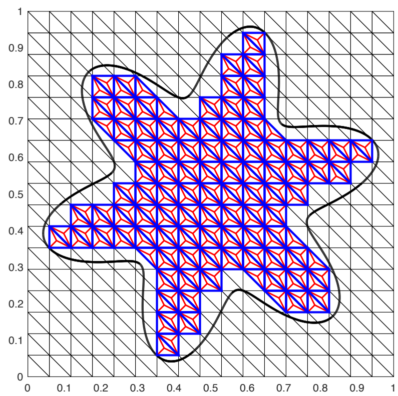

Figure 6: Physical domain and computational mesh with $h = \frac{1}{24}$.

4.6 Numerical experiments

In this section, we perform some numerical experiments of the method (4.3.1). In the series of tests, the domain is defined via a level set function [23]

$$\Omega = \{x \in \mathbb{R}^2 : \ \phi(x) < 0\}, \text{ where } \quad \phi = r - 0.3723423423343 - 0.1\sin(6\theta), \quad (4.6.1)$$

where $r = \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}$, and $\theta = \tan^{-1}((x_2 - 0.5)/(x_1 - 0.5))$. We take $k = 2$, $S = (0,1)^2$, and the background mesh $\mathcal{S}_h$ to be a sequence of type I triangulations of S, i.e., a mesh obtained by drawing diagonals of a cartesian mesh; cf. Figure 6. For all tests, the Nitsche penalty parameter in the bilinear form $a_h(\cdot,\cdot)$ is set $\sigma = 40$.

We solve an auxiliary 2×2 nonlinear system at each quadrature point of each boundary edge of $\mathcal{T}_h^{ct}$ to obtain the extension direction $\boldsymbol{d}$. In particular, for each quadrature point $x \in \partial\Omega_h$, we seek $x_* \in \partial\Omega$ such that

$$\phi(x_*) = 0, \quad (\nabla\phi(x_*))^{\perp} \cdot (x - x_*) = 0,$$

and set $\boldsymbol{d} = (x - x_*)/|x - x_*|$ and $\delta(x) = |x - x_*|$. Notice that the first equation ensures that x_* is on the boundary $\partial\Omega$, and the second equation gurantees that $\boldsymbol{d}$ is parallel to the outward unit normal of $\partial\Omega$ at x_*.

The data is chosen such that the exact solution to the Stokes problem is given by

$$\boldsymbol{u} = \begin{pmatrix} 2(x_1^2 - x_1 + \frac{1}{4} + x_2^2 - x_2)(2x_2 - 1) \\ -2(x_1^2 - x_1 + \frac{1}{4} + x_2^2 - x_2)(2x_1 - 1) \end{pmatrix}, \quad p = 10(x_1^2 - x_2^2)^2. \tag{4.6.2}$$

In this case, Theorem 4.5.5 predicts the convergence rates

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(\Omega_h)} = \mathcal{O}(h^2 + \nu^{-1}h^3), \qquad \|p - p_h\|_{L^2(\Omega_h)} = \mathcal{O}(h^2). \tag{4.6.3}$$

The velocity and pressure errors are plotted in Figure 7 for mesh parameters $h = 2^{-j}$ ($j = 1, 2, 3, 4, 5, 6$) and viscosities $\nu = 10^{-k}$ ($k = 1, 3, 5$). The results show that, for the moderately sized viscosities $\nu = 10^{-1}$ and $\nu = 10^{-3}$, the L^2 and H^1 velocities converge with the optimal order three and two, respectively. While we observe larger velocity errors for viscosity value $\nu = 10^{-5}$, we obtain fourth and third order convergence in the L^2 and H^1 norms in this case (cf. Figure 7). Notice that this behavior is consistent with the theoretical estimate (4.6.3). Finally, the numerical experiments show second order convergence for the pressure approximation and divergence errors comparable to machine epsilon.

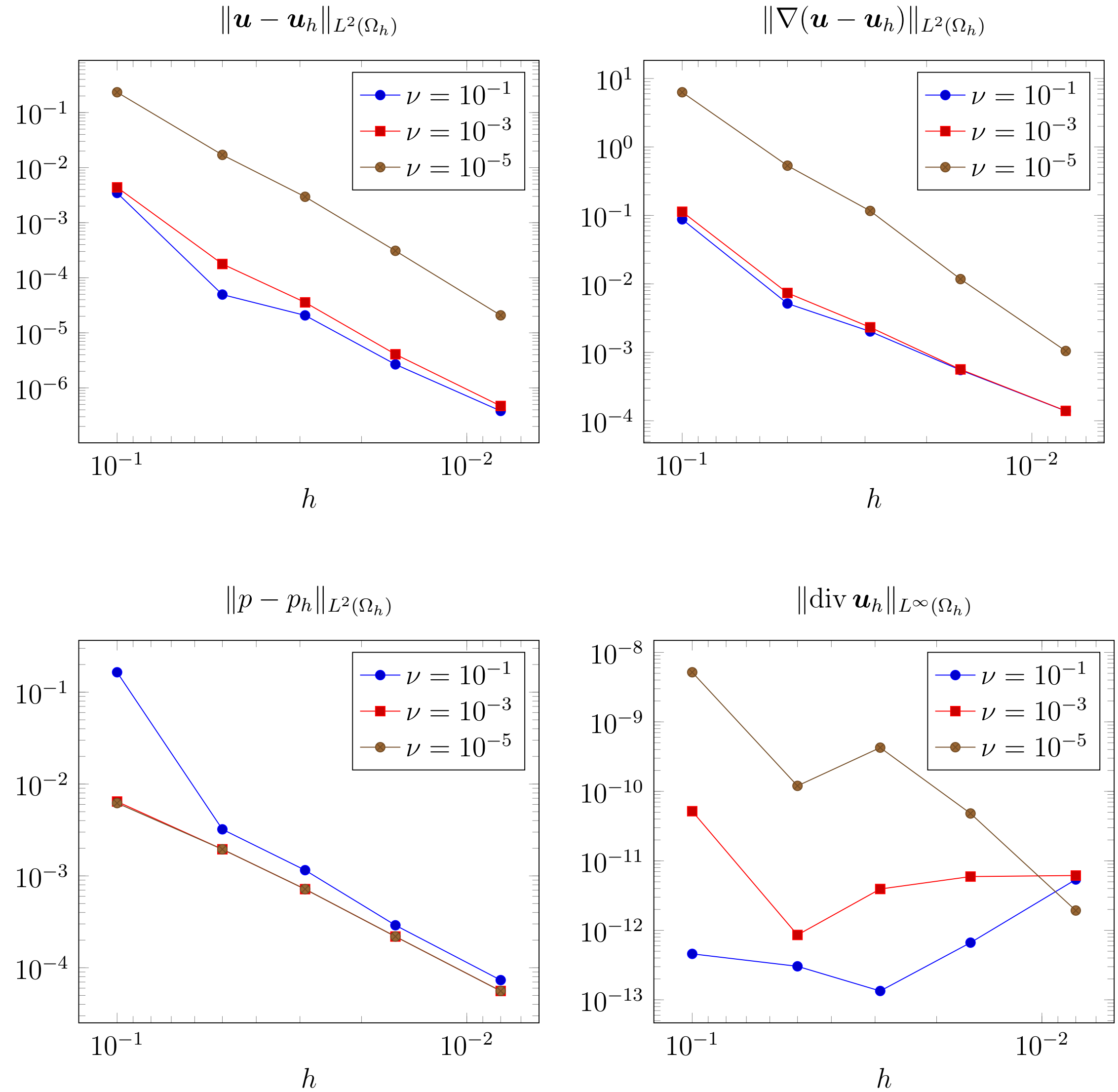

Figure 7: Errors for the velocity and pressure for a sequence of meshes on domain (4.6.1) and exact solution (4.6.2).

5.0 Conclusions

In this thesis, we constructed divergence-free finite element methods for the two dimensional Stokes problem on non-polytopal domains. In Chapter 2, we constructed an $\boldsymbol{H}(\text{div})$-conforming isoparametric method based on the lowest order Scott-Vogelius pair on Clough-Tocher refinements. We showed how an appropriate use of the Piola transform in the construction of the discrete velocity space yields a divergence-free discrete velocity solution, and we benefited from Stenberg's macro-element technique to establish the desired inf-sup stability. In Chapter 3, we extended our work to an $\boldsymbol{H}^1$-conforming isoparametric method through an enrichment process. The enrichment process was constructed in such a way that it "corrects" the tangential component of the functions in the discrete velocity space, maintains stability, optimal order of convergence and respects the divergence-free property. In either case, we showed that it is possible to obtain a pressure-robust scheme via a use of commuting projections. In Chapter 4, we built a uniformly stable and divergence-free method for the Stokes problem on unfitted meshes using boundary correction. We showed that while the method is not pressure-robust, a use of Lagrange multiplier enforcing the normal boundary conditions can improve the affect of the pressure contribution in the velocity error.

We emphasize that while we used the lowest order Scott-Vogelius pair in Chapter 2, the generalization of the proposed method to an arbitrary polynomial degree $k \geq 3$ is almost straight-forward. For instance, Lemma 2.4.6 uses the Simpson's rule to estimate the jump of functions in the discrete velocity space, and one can instead use the corresponding Gauss-Lobatto rule for $k \geq 3$ to obtain an analogous result. We also mention that the generalization of this method to the three dimensional setting is, however, non-trivial. In particular, there is no analogous quadrature rule of the Gauss-Lobatto rules in higher dimensions, and as a result, one can show that the direct extension of the proposed method is only sub-optimally convergent by an order of $\sqrt{h}$. It is also worth mentioning that the extension of the method proposed in Chapter 3 to an arbitrary polynomial degree is not straight-forward. In particular, Remark 3.3.7 together with Lemma 3.3.4 suggests that the corresponding enriched space should contain elements of degree $2k-1$ with $k \geq 2$, and it is not clear how to generalize the existing construction to such cases.

The presentation of the proposed method in Chapter 4 is also confined to the two dimensional setting, however many of the results extend to the three dimensional setting as well. For example, the proof of inf-sup stability given in Lemma 4.4.6 applies mutatis mutandis to the the three-dimensional Scott-Vogelius pair. On the other hand, inf-sup stability of the velocity-Lagrange multiplier pairing (cf. Lemma 4.4.7), and its dependence on the geometry of the computational mesh is less obvious in this case.

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