Interface Problems in Two-Phase Magnetohydrodynamic Flows

by

Tian Jing

B.S., Nanjing Normal University, 2017

Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2023

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Tian Jing

It was defended on

March 16, 2023

and approved by

Prof. Dehua Wang, Department of Mathematics

Prof. Armin Schikorra, Department of Mathematics

Prof. Ming Chen, Department of Mathematics

Prof. Ian Tice, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh

Interface Problems in Two-Phase Magnetohydrodynamic Flows

Tian Jing, PhD

University of Pittsburgh, 2023

We study the motion of two incompressible, conductive fluids in a magnetic field. The viscosity and surface tension are considered. The study includes the existence of varifold solutions, strong solutions, and their weak-strong uniqueness. To obtain varifold solutions, we approximate the equations using the Galerkin method. Using solution operators and the Schauder fixed-point theorem, we can obtain the approximate solutions. The weak convergence method is then used for studying the limit of approximate solutions. Varifolds are used for describing the interface. To find a strong solution, we apply the Hanzawa transformation to the equations, which are transformed into a fixed-interface problem for a short time. The new equations are divided into principal parts and nonlinear parts, which are studied separately. The solution is obtained using the fixed-point theory of contraction mappings. When the strong solution exists, all varifold solutions coincide with it. This is proved by estimating the error between strong and varifold solutions using the relative entropy. An inequality of the relative entropy is derived and controlled by utilizing the Gronwall's inequality.

Keywords: 3-D MHD, two-phase, varifold solutions, strong solutions, weak-strong uniqueness.

Table of Contents

Preface

This dissertation is devoted to the well-posedness theory of the two-phase magnetohydrodynamic equations, which includes the global existence of varifold solutions, the local existence of strong solutions, and their weak strong uniqueness. These problems have been challenges for me. I spent a lot of time and energy overcoming these difficulties. During my study, I have received a lot of help from many people. I would like to express my gratitude and appreciation to them.

I would like to thank my advisors, Professor Dehua Wang and Professor Armin Schikorra. I have got a lot of valuable advice from them and they were always willing to discuss with me when I have confusions. They also supported me with research assistant positions, which enabled me to focus on research.

I would like to thank the rest of the members of the thesis committee: Professor Ming Chen (University of Pittsburgh) and Professor Ian Tice (Carnegie Mellon University), for their valuable suggestions on my research projects and my thesis.

I would like to thank Professor Helmut Abels for being willing to discuss a lot with me on the two-phase problem and related areas, as well as for many valuable suggestions on the development of my academic career. I would also like to thank Professor Mathias Wilke, Professor Matthias Köhne, Professor Julian Fischer and Dr. Sebastian Hensel for spending time answering my questions about their papers.

I would like to thank the Department of Mathematics and the University of Pittsburgh for offering me the chance to work toward my dream of being an academic researcher.

Finally, I would like to thank my family for their support and love all the time, which encourages me to keep exploring my research area and my life.

1.0 Introduction

Magnetohydrodynamics (MHD) concerns the motion of electrically conducting fluids in an electromagnetic field. It has a very wide range of applications in many physical areas. For example, the motion of liquid metals, the magnetic field of the Earth, and the activities of cosmic stars.

The two-phase MHD equation is a system of equations that describes the motion of two conductive fluids in a magnetic field. In our study, we focus on fluids that are incompressible, viscous and resistive. When a conductive fluid moves in a magnetic field, an electric current is generated in the fluid. The motion of this charged fluid can be affected by the Lorentz force due to the magnetic field. Meanwhile, the charged fluid itself generates its own magnetic field around it, which will conversely affect the magnetic field in the whole region. The interface between the two fluids is moving along the fluids. The surface tension on the interface is also considered in our study. The equations consist of the Navier-Stokes equations and the magnetic equations.

The study of the two-phase MHD equation is a development of the research on Navier-Stokes equations and MHD equations. Thus, it is still a young and developing area. The Navier-Stokes equations have been widely studied for many decades. The study on two-phase Navier-Stokes equations is a relatively young branch. When the surface tension is considered, it brings more difficulty to the solving of the equations. In 1993, Plotnikov proved the existence of varifold solutions to the two-phase Navier-Stokes equation in $\mathbb{R}^2$. In 2007, Abels studied the existence of varifold solutions in $\mathbb{R}^2$ and $\mathbb{R}^3$. In 2013, Prüss, et al. studied analytic solutions to the two-phase Navier-Stokes equation in a bounded domain $\Omega \subseteq \mathbb{R}^3$. In 2020, Fischer and Hensel proved the weak-strong uniqueness of the two-phase Navier-Stokes equation.

In 2010, Padula and Solonnikov studied the local existence of solutions to fluid-vacuum MHD equations in bounded domains with surface tension considered [24]. In 2014, Secchi and Trakhinin studied the well-posedness of ideal compressible MHD equations [32]. In 2019, Gu and Wang proved the local existence of solutions to ideal MHD equations without surface tension [11]. In 2021, Wang and Xin proved the global existence of solutions to the fluid-vacuum model in a slab-shaped region [39]. In 2022, Trakhinin and Wang studied the local existence theory of the compressible model with perfect conductivity [38].

In Chapter 2, we study the existence theory of varifold solutions to the two-phase MHD equations. We obtain varifold solutions by first approximating the equations using the Galerkin method and then studying their weak limit. Due to the existence of the magnetic field, it is hard to obtain approximate solutions using the theory of monotone operators. In order to overcome this issue, we utilize solution operators to reduce the unknown variables. These operators can be obtained by independently solving the magnetic equation and the transport equation. An operator is then constructed using the Galerkin approximate equations and solution operators. The approximate solution is obtained using the Schauder fixed-point theorem. Next, we obtain the weak convergence of approximate solutions using the Banach-Alaoglu theorem. Stronger convergence can then be obtained using the Arzela-Ascoli theorem and the Aubin-Lions lemma. Finally, we represent the weak limit of the mean curvature terms with the help of varifolds, which completes the varifold solution.

In Chapter 3, we establish the local existence theory of strong solutions to the two-phase MHD equations. The Hanzawa transformation is applied to transform the free interface into a fixed interface for a short time. The transformed equations are divided into the principal part and the nonlinear part. In the principal part, we use the theory of two-phase Stokes equations and the theory of parabolic equations to solve the linearized problem with arbitrary source terms. The study of the nonlinear part is mainly focused on the estimate of its Fréchet derivative. The equations can then be rewritten using an operator, which is a contraction mapping in some specific set. The solution is then obtained by finding the fixed point of the operator.

In Chapter 4, we prove that when the unique strong solution to the two-phase MHD equations exists, all varifold solutions coincide with it. We prove the weak-strong uniqueness by controlling the error between the strong solution and a varifold solution using the relative entropy. The construction of the relative entropy for MHD equations is inspired by [10]. A relative entropy inequality is derived by combining the energy inequalities and equations with specific test functions. The Gronwall's inequality is utilized to obtain the estimate of the relative entropy, which implies the weak-strong uniqueness if the initial error is 0.

The rest of the thesis will be organized in the following structure. In Chapter 2, we establish the global existence of varifold solutions to the two-phase MHD equations using the Galerkin approximation and the weak convergence method. In Chapter 3, we prove the local existence of the strong solution using the Hanzawa transformation and the contraction mapping theory. In Chapter 4, we establish the weak-strong uniqueness of strong and varifold solutions in $\mathbb{R}^3$ using the relative entropy method.

2.0 Existence of Varifold Solutions to the Two-Phase MHD Equations

2.1 Introduction and Main Results

In this chapter, we study the two-phase magnetohydrodynamic (MHD) problem of two immiscible Newtonian fluids which are incompressible, viscous and conducting, in a three-dimensional bounded, simply connected smooth domain $\Omega \subseteq \mathbb{R}^3$. The domains of the two fluids are denoted by open sets $\Omega^+(t)$ and $\Omega^-(t)$. The interface between them is defined as $\Gamma(t) := \partial\Omega^+(t) \setminus \partial\Omega$. The sets $\Omega^+(t)$, $\Omega^-(t)$ and $\Gamma(t)$ give a partition of Ω. We assume that the density equals to 1 everywhere and consider the following equations:

$$\partial_t u + u \cdot \nabla u - (\nabla \times B) \times B - \nu^{\pm} \triangle u + \nabla p = 0 \quad \text{in } \Omega^{\pm}(t), \tag{2.1.1}$$

$$\partial_t B - \nabla \times (u \times B) + \nabla \times (\sigma \nabla \times B) = 0 \quad \text{in } \Omega, \tag{2.1.2}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega^{\pm}(t), \tag{2.1.3}$$

$$\operatorname{div} B = 0 \quad \text{in } \Omega, \tag{2.1.4}$$

$$- \llbracket 2\nu(\chi) Du - pI \rrbracket \, n = \kappa H n \quad \text{on } \Gamma(t), \tag{2.1.5}$$

$$V_\Gamma = n \cdot u \quad \text{on } \Gamma(t), \tag{2.1.6}$$

$$u|_{\partial\Omega} = 0, \; B|_{\partial\Omega} = 0, \tag{2.1.7}$$

$$u|_{t=0} = u_0, \; B|_{t=0} = B_0, \tag{2.1.8}$$

where $u \in \mathbb{R}^3$ is the velocity, $B \in \mathbb{R}^3$ the magnetic field, $\sigma > 0$ the magnetic diffusion coefficient of both fluids, ν^+, $\nu^- \geq 0$ the viscosity coefficients of the two fluids, $\kappa \geq 0$ the surface tension coefficient; The quantities V_Γ, n, H are all defined pointwisely on the interface $\Gamma(t)$, where V_Γ denotes the velocity of the interface, n the normal vector, H the mean curvature; The term $Du := (\nabla u + \nabla u^T)/2$ is the strain rate tensor and $|Du|$ is the shear rate. In order to study the positions of $\Omega^+(t)$ and $\Omega^-(t)$, we consider the indicator function of $\Omega^+(t)$, i.e. $\chi(t) := \chi_{\Omega^+(t)}$. Let ν be such that $\nu(1) = \nu^+$ and $\nu(0) = \nu^-$. Then we can use $\nu(\chi(t, x))$ for the viscosity. The notation $\llbracket f \rrbracket$ denotes the jump of f across $\Gamma(t)$.

We briefly review some related results. When there is no magnetic field B, the problem becomes the two-phase Navier-Stokes equations. The problem of varifold solutions was first studied by Plotnikov [25]. In his paper, the case of two incompressible non-Newtonian fluids with surface tension has been considered in $\mathbb{R}^2$. In the seminal work [1], Abels proved the existence of varifold solutions in more general cases, where the viscosity coefficients depend on the shear rate $|Du|$. From [1], there exists a weak solution when $\kappa = 0$ and a measure-valued varifold solution when $\kappa > 0$. For the case of $\kappa > 0$, the equations have been studied in $\mathbb{R}^2$ and $\mathbb{R}^3$. When the viscosity coefficients are constants, Yeressian [40] has proved the existence of varifold solutions in $\mathbb{R}^3$. When the strong solution exists, Fischer and Hensel proved the weak-strong uniqueness in [10] with the technique of relative entropy. For interested readers we also refer to [22, 29, 31].

Since the problem with $\kappa > 0$ has been studied in $\mathbb{R}^2$ and $\mathbb{R}^3$ in [1] and [40] for the Navier-Stokes equations, in this work we are interested in the case of bounded domains Ω for the magnetohydrodynamics, for which the both viscosity coefficients $\nu^{\pm}$ are also taken to be (different) constants. In [1] and [40] the approximate equations are derived by mollifying the original equations. In the case of a bounded domain, it will be complicated to mollify the equations near the boundary of the domain. Thus, we will use the Galerkin method to construct the approximate solutions in this work.

We first give the definitions of varifold solutions and weak solutions based on the definitions in [1]. The space $\mathbb{R}^d$ is replaced by Ω in an appropriate way. Some boundary conditions are also included.

Definition 2.1.1 (Varifold solution)**.** Let $u_0, B_0 \in L^2(\Omega)$ such that $\mathrm{div} u_0 = \mathrm{div} B_0 = 0$ weakly. Let $Q_T := \Omega \times (0,T)$. Let $\Omega_0^+ \subseteq \Omega$ be a bounded domain such that $\chi_0 = \chi_{\Omega_0^+}$ is of finite perimeter. A quadruple (u, B, χ, V) with

$$
\begin{aligned}
&u \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)),\\
&B \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)),\\
&\mathrm{div} u = \mathrm{div} B = 0,\\
&\chi \in L^\infty([0,T]; BV(\Omega; \{0,1\})),\\
&V \in L^\infty([0,T]; \mathcal{M}(\Omega \times \mathbb{S}^2)),
\end{aligned}
$$

is called a varifold solution to the two-phase flow problem (2.1.1)-(2.1.8) with the initial data (u_0, B_0, χ_0) if

(1).

$$
\begin{aligned}
&-(u_0, \varphi(0))_\Omega - (u, \partial_t \varphi)_{Q_T} - (u \otimes u, \nabla \varphi)_{Q_T} + (B \otimes B, \nabla \varphi)_{Q_T} \\
&\quad + (2\nu(\chi) Du, D\varphi)_{Q_T} + \kappa \int_0^T \langle \delta V(t), \varphi(t) \rangle \, dt = 0
\end{aligned}
\tag{2.1.9}
$$

is satisfied for all $\varphi \in C_c^\infty([0,T) \times \Omega)$ with $\mathrm{div}\varphi = 0$;

(2).

$$
-(B_0, \varphi(0))_\Omega - (B, \partial_t \varphi)_{Q_T} - (u \otimes B, \nabla \varphi)_{Q_T} + (B \otimes u, \nabla \varphi)_{Q_T} + \sigma (\nabla B, \nabla \varphi)_{Q_T} = 0 \tag{2.1.10}
$$

is satisfied for all $\varphi \in C_c^\infty([0,T) \times \Omega)$ with $\mathrm{div}\varphi = 0$;

(3). For almost every $t \in [0,T]$,

$$
\int_{\Omega \times \mathbb{S}^2} s \cdot \psi(x) dV(t)(x,s) = -\int_\Omega \psi d\nabla \chi(t) \tag{2.1.11}
$$

is satisfied for all $\psi \in C_0(\Omega)$;

(4). The indicator function χ is the unique renormalized solution of

$$
\begin{aligned}
&\partial_t \chi + u \cdot \nabla \chi = 0 \quad \text{in} \ \ (0,T) \times \Omega, \\
&\chi|_{t=0} = \chi_0 \quad \text{in} \ \ \Omega;
\end{aligned}
\tag{2.1.12}
$$

(5). The generalized energy inequality

$$
\begin{aligned}
&\frac{1}{2} \|u(t)\|_{L^2}^2 + \frac{1}{2} \|B(t)\|_{L^2}^2 + \kappa \|V(t)\|_{\mathcal{M}(\Omega \times \mathbb{S}^2)} + 2 \int_0^t \int_\Omega \nu(\chi) |Du|^2 dx ds \\
&\quad + \sigma \int_0^t \|\nabla B(s)\|_{L^2}^2 ds \le \frac{1}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} \|B_0\|_{L^2}^2 + \kappa \|\nabla \chi_0\|_{\mathcal{M}(\Omega)}
\end{aligned}
\tag{2.1.13}
$$

holds for almost every $t \in [0,T]$;

Remark 2.1.1. The notation $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{Q_T}$ stands for the inner product in $L^2(\Omega)$ and $L^2(Q_T)$. For details about the renormalized solutions, see Proposition 2.2 in [1]. The term δV in (4.1.4) is the first variation of the measure V. The definitions of δV and $\langle \delta V(t), \cdot \rangle$ are in Section 2.2.4. The initial energy is:

$$
E_0 := \frac{1}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} \|B_0\|_{L^2}^2 + \kappa \|\nabla \chi_0\|_{\mathcal{M}}. \tag{2.1.14}
$$

Definition 2.1.2 (Weak solution)**.** Let (u, B, χ, V) be a varifold solution of the two-phase flow problem (2.1.1)-(2.1.8) with the initial data (u_0, B_0, χ_0) as in definition 2.1.1. Then the triple (u, B, χ) is called a weak solution if for almost every $t \in [0, T]$, the equality

$$\langle \delta V(t), \varphi \rangle = -\left\langle H_{\chi(t)}, \varphi \right\rangle := \int_\Omega P_\tau : \nabla\varphi d|\nabla\chi(t)|$$

holds for all $\varphi \in C_c^\infty(\Omega)$ with $\mathrm{div}\varphi = 0$. Here $P_\tau := I - n \otimes n$ and $n := \nabla\chi(t)/\left|\nabla\chi(t)\right|$.

Remark 2.1.2. The term χ contains all the information to define the mean curvature functional H_χ; see Section 2.2.3 for details. The term $\nabla\chi(t)$ is a vector-valued Radon measure on Ω and $|\nabla\chi(t)|$ is the total variation measure of $\nabla\chi(t)$. Thus, the normal vector n can be defined using the Radon-Nikodym derivative. See Section 2.2 for details. The varifold solution is weaker than the weak solution, since the weak limits of some terms are represented by measures.

The main result of this work is given as the following:

Theorem 2.1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded, smooth and simply connected domain; $u_0, B_0 \in L^2(\Omega)$ satisfy $\mathrm{div}u_0 = \mathrm{div}B_0 = 0$; and $\chi_0 := \chi_{\Omega_0^+}$, where $\Omega_0^+ \subseteq \Omega$ is a simply connected C^2-domain such that $\overline{\Omega_0^+} \subseteq \Omega$. Then for any $T > 0$, there exists a varifold solution to the two-phase flow problem* (2.1.1)-(2.1.8) *on* $[0, T]$ *with the initial data* (u_0, B_0, χ_0).

The proof of theorem 2.1.1 will be in the spirit of [1] with some new ideas to deal with the bounded domain Ω and the magnetic field B. We shall use the Galerkin method to construct the approximate solutions in a bounded domain. Due to the extra term B in the equations, we cannot use the method of monotone operators in [1, 42, 43] to solve the approximate equations. Instead, we will rewrite the approximate equations using operators and solve the equations by finding the fixed points of the operators. In fact, if our velocity u is from certain function spaces, then the quantity B and χ are uniquely decided by u. Thus, there exist solution operators that map each u to $B(u)$ and $\chi(u)$. These operators have some good properties of continuity and boundedness, which will contribute to showing the compactness of the fixed-point operator; see [1] and [12] for more details. Due to the free interface $\Gamma(t)$, it is hard to prove the Lipschitz continuity of the operators. Thus, we cannot use the classical contraction mapping theorem to prove the existence of the fixed-points. In order to overcome this difficulty, we firstly prove the compactness of the operators and then use the Schauder fixed-point theorem.

The rest of this chapter is organized as follows. We firstly list some useful background knowledge in Section 2.2. In Section 2.3, we will study the Galerkin approximate equations on $[0,T]$ and prove that the approximate solutions exists globally on $[0,T]$. Then we give a uniform energy estimate for all the approximate solutions. Finally, we will study the limits of the approximate solutions in Section 2.4.

2.2 Preliminary

2.2.1 Function spaces

We recall some definitions of function spaces. Given a bounded domain $\Omega \subseteq \mathbb{R}^d$. The space $C^k(\Omega)$ denotes the functions with continuous partial derivatives until order k. The subspace $C_b^k(\Omega) \subseteq C^k(\Omega)$ consists of bounded functions with bounded derivatives up to order k. The space $C^k(\overline{\Omega})$ is the subspace of $C^k(\Omega)$, such that for each $f \in C^k(\overline{\Omega})$, we can find $F \in C^k(\mathbb{R}^d)$ with $f = F$ on $\overline{\Omega}$. The space $C_{c,\sigma}^\infty(\Omega)$ consists of functions in $C_c^\infty(\Omega)$ which are divergence-free. The following result will be useful.

Proposition 2.2.1 ([41], Appendix (24d)). *For a compact set $K \subseteq \mathbb{R}^d$, for any $k \in \mathbb{Z}$ and $k \geq 0$, we have the compact embedding $C^{k+1}(K) \hookrightarrow\hookrightarrow C^k(K)$.*

For a Banach space X and $1 \leq p \leq \infty$, the Bochner space $L^p([0,T];X)$ is the space of functions $u(t)$ from $[0,T]$ to X such that

$$\|u\|_{L^p([0,T];X)} := \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{\frac{1}{p}} < \infty \quad \text{for} \;\; 1 \leq p < \infty, \text{ or}$$

$$\|u\|_{L^\infty([0,T];X)} := \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X < \infty \quad \text{for} \;\; p = \infty.$$

The space $C([0,T];X)$ consists of functions $u(t)$ from $[0,T]$ to X such that for any $t_0 \in [0,T]$

$$\lim_{t\to t_0} \|u(t) - u(t_0)\|_X = 0.$$

For more details on function spaces, we refer to [23]. For Sobolev spaces and the embedding theorems, we refer to [8, 18].

For a locally compact separable metric space X, we denote the space of finite Radon measures on X by $\mathcal{M}(X)$. We recall that $\mathcal{M}(X) = (C_0(X))^*$.

We denote by $L^\infty_w([0,T], \mathcal{M}(X))$ the space of functions $f : [0,T] \to \mathcal{M}(X)$ such that: for all $\varphi \in L^1([0,T], C_0(X))$ the duality $(f,\varphi)(t)$ is measurable on $[0,T]$; $\|f\|_{\mathcal{M}(X)}(t)$ is measurable on $[0,T]$; and $\operatorname{ess\,sup}_{[0,T]} \|f\|_{\mathcal{M}(X)}(t) < \infty$. More details can be found in [1, 40].

2.2.2 Sets of finite perimeter

In order to study the free interface, we recall some topics about the sets with less regular boundaries. Most of the topics here can be found in the geometric measure theory. We refer to [5, 19] for interested readers.

Definition 2.2.1 ([5], definition 3.1)**.** For an open set $\Omega \subseteq \mathbb{R}^d$. A function $u \in L^1(\Omega)$ is of bounded variation in Ω if there exist finite Radon measures $\lambda_1, \cdots \lambda_d \in \mathcal{M}(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi d\lambda_i$$

holds for all $\varphi \in C_c^\infty(\Omega)$ and $i = 1, \cdots, d$. The space $BV(\Omega)$ consists of functions of bounded variation in Ω.

The variations of functions are useful when we study the space $BV(\Omega)$.

Definition 2.2.2 ([5], definition 3.4)**.** Given a function $u \in L^1_{loc}(\Omega)^m$ with $\Omega \subseteq \mathbb{R}^d$. Its variation in Ω is defined as

$$\mathcal{V}(u,\Omega) := \sup\left\{ \sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div}\varphi^\alpha dx : \varphi \in C_c^1(\Omega)^{md},\ \|\varphi\|_{L^\infty(\Omega)} \le 1 \right\}.$$

When studying the boundary of a set E, we usually consider its indicator function χ_E. The perimeter of E is defined using the variation of χ_E. See [5, 9] for details.

Definition 2.2.3 ([5], definition 3.35)**.** Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $E \subseteq \mathbb{R}^d$ a Lebesgue measurable set. The perimeter of E in Ω is defined as:

$$\mathcal{P}(E,\Omega) := \sup\left\{ \int_E \mathrm{div}\varphi dx : \varphi \in C_c^1(\Omega)^d,\ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

The set E is of finite perimeter in Ω if $\mathcal{P}(E,\Omega) < \infty$.

Remark 2.2.1. By this definition, only $\partial E \cap \Omega$ will be counted into the perimeter of E.

2.2.3 Mean curvature functional

When calculating the weak form of (2.1.1), we will get a so-called mean curvature functional. This functional is dependent on the interface $\Gamma(t)$, and thus will be denoted by $H_{\Gamma(t)}$ or $H_{\chi(t)}$. When everything is smooth enough, we have the following formula:

$$\left\langle H_{\Gamma(t)}, \varphi(t) \right\rangle := \int_{\Gamma(t)} Hn \cdot \varphi d\mathcal{H}^{d-1}(x),$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. On the interface Γ the tangential divergence of a function $\varphi \in C^1(\Gamma)^d$ is defined as

$$\mathrm{div}^\Gamma \varphi := \mathrm{div}\varphi - n \otimes n : \nabla\varphi = (I - n \otimes n) : \nabla\varphi,$$

where $n(x)$ denotes the normal vector. Note that $P_\tau := I - n \otimes n$ is the orthogonal projection onto the tangent space, which is defined pointwisely on Γ. The mean curvature is defined as (see [5, 19, 33])

$$H := -\mathrm{div}^\Gamma \varphi.$$

According to the generalized divergence theorem (see [19]),

$$\int_\Gamma \mathrm{div}^\Gamma \varphi d\mathcal{H}^{d-1} = \int_\Gamma Hn \cdot \varphi d\mathcal{H}^{d-1} + \int_{\partial\Gamma} \varphi \cdot n_{\partial\Gamma} d\mathcal{H}^{d-2} \tag{2.2.1}$$

holds for all $\varphi \in C_c^1(\mathbb{R}^d)^d$. In our problem, the set $\overline{\Omega^+(t)}$ will always stay in the interior of Ω. Thus, $\Gamma(t)$ will be a closed surface and will not have any edge, i.e. the $(d-2)$-dimensional boundary. As a result, the second term on the right-hand side in (2.2.1) vanishes. Now when the surface Γ becomes less regular, as long as a measure theoretical normal vector exists, we can use the generalized divergence theorem to replace $Hn \cdot \varphi$ with $\mathrm{div}^{\Gamma}\varphi$, and study the generalized mean curvature functional.

In order to study the convergence of the mean curvature terms, we modify lemma 2.4 from [1] and obtain the following lemma.

Lemma 2.2.1 ([1], lemma 2.4, revised)**.** *Let Ω be a bounded, simply connected, smooth domain. Let Ω_0^+ be a bounded, simply connected C^2-domain such that $\overline{\Omega_0^+} \subseteq \Omega$. Suppose that u, $v \in C([0,T]; C_b^2(\Omega))$ with $\mathrm{div}u = \mathrm{div}v = 0$ and $u \to v$ in $C([0,T]; C^1(\Omega))$. Then*

$$\int_{\Gamma_u(t)} f(x, n_x) d\mathcal{H}^{d-1}(x) \to \int_{\Gamma_v(t)} f(x, n_x) d\mathcal{H}^{d-1}(x) \tag{2.2.2}$$

uniformly on $[0,T]$. Here $\Gamma_u(t)$ and $\Gamma_v(t)$ are interfaces obtained from u and v.

The details on how the velocity determines the interface are stated in Section 2.3.3. The convergence of the flow mappings is still valid when the domain $\mathbb{R}^d$ in [1] is replaced by a bounded domain Ω. Thus, using the same argument as in [1], we can apply a local parameterization to $\Gamma_0 = \partial\Omega_0^+$ to prove the lemma.

2.2.4 Varifolds

For the problem we study, even the measure theoretical normal vectors might not be guaranteed to exist all the time. In this case, we have to use varifolds to describe the surfaces. We refer to the definition in [1] and give an analogue one for the case of the bounded domain Ω. A measure V is called a general $(d-1)$-varifold if it is a finite Radon measure on $\Omega \times \mathbb{S}^{d-1}$, i.e. $V \in \mathcal{M}(\Omega \times \mathbb{S}^{d-1})$. The varifold V can be understood as assigning different weight to vectors in Ω and $\mathbb{S}^{d-1}$. In another word, it tells the possibility of a point to be on the interface and the possibility of a vector to be the normal vector. For interested readers we refer to [20, 21, 33].

The first variation of V is defined as

$$\langle \delta V, \varphi \rangle := \int_{\Omega \times \mathbb{S}^{d-1}} (I - s \times s) : \nabla \varphi dV(x, s)$$

for any $\varphi \in C_0^1(\Omega)$; see [33] chapter 8 or [3]. This allows us to replace the functional $H_{\chi_E(t)}$ with $-\delta V(t)$, and study the mean curvature functional when the interface is less regular.

2.2.5 Compact operators

We recall some theory about compact operators; see [41] for details. The compact operators will be used to solve the Galerkin approximate equations.

Definition 2.2.4 ([41], definition 2.9)**.** Given two Banach spaces X and Y. An operator $T : D(T) \subset X \to Y$ is called a compact operator if it is continuous and maps bounded sets into precompact sets.

The following proposition is important when proving the compactness of an operator.

Proposition 2.2.2 ([41], Appendix (24g))**.** *The set $M \subseteq C(\overline{\Omega})$ is precompact if and only if*
(1) $\sup_{f\in M} \sup_{x\in\overline{\Omega}} |f(x)| < \infty$.
(2) For every $\varepsilon > 0$, there exists $\delta > 0$, such that $\sup_{f\in M} |f(x) - f(y)| < \varepsilon$ *for every $x, y \in \overline{\Omega}$ and $|x - y| < \delta$.*

We give a specific version of the Arzela-Ascoli theorem.

Theorem 2.2.1 ([41], Appendix (24i))**.** *Let X be a Banach space. The set $A \subseteq C([0, T]; X)$ is precompact if and only if*
(1) For all $t \in [0, T]$, the set $\{f(t) : f \in A\}$ is precompact in X.
(2) For all $t \in [0, T]$ and $\varepsilon > 0$, there exists $\delta > 0$, such that $\sup_{f\in A} \|f(t) - f(s)\|_X < \varepsilon$ *for all $s \in [0, T]$ and $|t - s| < \delta$.*

At last, we recall the Schauder fixed-point theorem of compact operators.

Theorem 2.2.2 ([41], theorem 2.A)**.** *Let X be a Banach space. Suppose $A \subseteq X$ is nonempty, bounded, closed and convex. Given a compact operator $T : A \to A$. There exists a fixed point of T in A.*

2.3 The Galerkin Method

2.3.1 Weak formula and energy estimate

We test (2.1.1) with $\varphi \in C_c^\infty([0,T)\times\Omega)$ such that $\mathrm{div}\varphi = 0$. For the first and second terms we simply integrate them by parts; for the third term we recall the equality $(\nabla \times B)\times B = B\cdot\nabla B - \nabla(|B|^2)/2$ and then integrate by parts; the fifth term will vanish; the calculation of the fourth term will generate the mean curvature functional:

$$
\begin{aligned}
&\int_\Omega \nu(\chi)\triangle u\varphi = \int_\Omega \nu(\chi)\mathrm{div}(\nabla u + \nabla u^T)\varphi \\
=&\int_\Omega \nu(\chi)\sum_j\sum_i \partial_i(\partial_i u_j + \partial_j u_i)\varphi_j \\
=&\int_\Omega \nu(\chi)\sum_j\sum_i \partial_i((\partial_i u_j + \partial_j u_i)\varphi_j) - \int_\Omega \nu(\chi)\sum_j\sum_i(\partial_i u_j + \partial_j u_i)\partial_i\varphi_j \\
=&2(\nu^+ - \nu^-)\int_\Gamma n\cdot(Du\varphi) - 2\int_\Omega \nu(\chi)Du : D\varphi \\
=&\kappa\int_\Gamma Hn\cdot\varphi - 2(\nu(\chi)Du, D\varphi)_\Omega.
\end{aligned}
$$

Thus, we obtain the weak formula of (2.1.1):

$$
\begin{aligned}
&-(u_0,\varphi(0))_\Omega - (u,\partial_t\varphi)_{Q_T} - (u\otimes u,\nabla\varphi)_{Q_T} + (B\otimes B,\nabla\varphi)_{Q_T} \\
&\quad + 2(\nu(\chi)Du, D\varphi)_{Q_T} - \kappa\int_0^T\int_{\Gamma(t)} Hn\cdot\varphi d\mathcal{H}^2 = 0.
\end{aligned}
\tag{2.3.1}
$$

Testing (2.1.2) with $\varphi \in C_c^\infty([0,T)\times\Omega)$ such that $\mathrm{div}\varphi = 0$. Using the fact that

$$
\nabla\times(\nabla\times B) = \nabla(\mathrm{div}B) - \triangle B, \quad \text{and} \quad \nabla\times(u\times B) = -(u\cdot\nabla)B + (B\cdot\nabla)u,
$$

we obtain

$$
-(B_0,\varphi(0))_\Omega - (B,\partial_t\varphi)_{Q_T} - (u\otimes B,\nabla\varphi)_{Q_T} + (B\otimes u,\nabla\varphi)_{Q_T} + \sigma(\nabla B,\nabla\varphi)_{Q_T} = 0. \tag{2.3.2}
$$

Now we derive the energy estimate. Suppose all the functions are smooth enough. Testing (2.1.1) and (2.1.2) on Ω with $\varphi = u$ and $\varphi = B$ respectively, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + (B\otimes B,\nabla u)_\Omega + 2(\nu(\chi)Du, Du)_\Omega - \kappa\int_{\Gamma(t)} Hn\cdot u d\mathcal{H}^2 = 0,
$$

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 - (B \otimes B, \nabla u)_\Omega + \sigma \|\nabla B\|_{L^2}^2 = 0.$$

Adding these two equations, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + 2(\nu(\chi)Du, Du)_\Omega \\ +\sigma \|\nabla B\|_{L^2}^2 - \kappa \int_{\Gamma(t)} Hn \cdot u d\mathcal{H}^2 = 0. \end{aligned}$$

From the derivation of (1.9) in [1], we have

$$\frac{d}{dt}\mathcal{H}^2(\Gamma(t)) = -\int_{\Gamma(t)} HV_\Gamma d\mathcal{H}^2 = -\int_{\Gamma(t)} Hn \cdot u d\mathcal{H}^2.$$

Note that by Korn's inequality, there exists $c > 0$ such that

$$2(\nu(\chi)Du, Du)_\Omega \geq c \|\nabla u\|_{L^2}^2 .$$

Finally, we obtain the energy inequality:

$$\begin{aligned} \frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{1}{2}\|B(t)\|_{L^2}^2 + \kappa\mathcal{H}^2(\Gamma(t)) + c\|\nabla u\|_{L^2([0,T]\times\Omega)}^2 + \sigma\|\nabla B\|_{L^2([0,T]\times\Omega)}^2 \\ \leq \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{2}\|B_0\|_{L^2}^2 + \kappa\mathcal{H}^2(\Gamma_0). \end{aligned} \tag{2.3.3}$$

This estimate drives us to look for a solution (u, B, Γ) such that

$$u \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)), \quad B \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)),$$

and $\mathcal{H}^2(\Gamma(t))$ is bounded on $[0, T]$.

2.3.2 Approximate equations

In order to use the Galerkin method, we pick the eigenfunctions of the Stokes operator to be a basis. The existence of this basis is from the following theorem:

Theorem 2.3.1 ([30], theorem 2.24). *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain. Let A be the Stokes operator, i.e. $Au := -\mathbb{P}\triangle u$ where $\mathbb{P}$ is the Helmholtz projection. There exists a set of functions $\mathcal{N} = \{\eta_1, \eta_2, \cdots, \}$ such that*
(1) the functions form an orthonormal basis of $\mathbb{H}(\Omega)$;
(2) the functions form an orthogonal basis of $\mathbb{V}(\Omega)$;
(3) the functions belong to $D(A) \cap C^\infty(\overline{\Omega})$ and they are eigenfunctions of A with positive, nondecreasing eigenvalues which goes to infinity.

Here $\mathbb{H}(\Omega)$ denotes the closure of $\{\varphi \in C_c^\infty(\Omega) : \mathrm{div}\varphi = 0\}$ under the L^2 norm and $\|\cdot\|_{\mathbb{H}(\Omega)} := \|\cdot\|_{L^2(\Omega)}$. The space $\mathbb{V}(\Omega) := \mathbb{H}(\Omega) \cap H_0^1(\Omega)$ and $\|\cdot\|_{\mathbb{V}(\Omega)} := \|\cdot\|_{H^1(\Omega)}$. Note that all the eigenfunctions η_j have trace 0. For details of the Stokes operator, see [35, 36].

Let $G_n := \mathrm{span}\{\eta_1, \cdots, \eta_n\}$. For each $t \in [0, T]$, we consider the approximate equation:

$$\begin{aligned}(u_n(t), \eta)_\Omega - (u_0, \eta)_\Omega - \int_0^t (u_n \otimes u_n, \nabla\eta)_\Omega ds + \int_0^t (B_n \otimes B_n, \nabla\eta)_\Omega ds \\ +2\int_0^t (\nu(\chi_n) Du_n, D\eta)_\Omega ds + \kappa \int_0^t \int_\Omega P_\tau : \nabla\eta d\left|\nabla\chi_n(s)\right| ds = 0,\end{aligned} \tag{2.3.4}$$

for all $\eta \in G_n$. We define the functionals M and N on G_n and rewrite the equation. Let

$$\langle M(u), \eta\rangle := \int_\Omega u \cdot \eta,$$

$$\begin{aligned}\langle N(u, \chi, B), \eta\rangle := (u \otimes u, \nabla\eta)_\Omega - (B \otimes B, \nabla\eta)_\Omega \\ - 2(\nu(\chi) Du, D\eta)_\Omega + \kappa \int_\Omega P_\tau : \nabla\eta d\left|\nabla\chi\right|.\end{aligned}$$

By integrating $N(u, \chi, B)$ from 0 to t, we define

$$\left\langle \int_0^t N ds, \eta \right\rangle := \int_0^t \langle N(s), \eta\rangle\, ds.$$

Now we can rewrite the equation as:

$$\langle M(u_n(t)), \eta\rangle = \langle M(u_0), \eta\rangle + \int_0^t \langle N(u_n, \chi_n, B_n), \eta\rangle\, ds \tag{2.3.5}$$

for all $\eta \in G_n$. It remains to represent χ and B with u using the solution operators, i.e. $\chi(u)$ and $B(u)$.

2.3.3 Solution operators $\chi(u)$ and $B(u)$

Suppose $u \in C([0,T];C^2(\overline{\Omega}))$ and Ω_0^+ is a simply connected C^2-domain with $\overline{\Omega_0^+} \subseteq \Omega$. For each $x \in \overline{\Omega}$, we consider the ODE

$$\begin{aligned} &\frac{d}{dt}X(t,x) = u\left(t, X(t,x)\right), \\ &X(0,x) = x. \end{aligned} \tag{2.3.6}$$

By the Picard-Lindelöf theorem there exists a unique solution locally in time. Since the solution will not blow up as stated in remark 2.3.1, we can always extend it to $[0,T]$. When we start from different initial values on $\overline{\Omega}$, the solutions will not intersect. Thus, we obtain a function $X(t,x) : [0,T] \times \overline{\Omega} \to \overline{\Omega}$, which is a bijection on $\overline{\Omega}$ for each fixed t. We call $X(t,x)$ the flow mapping, and denote it by $X_t(x)$ in some cases. We will also use $X_u(t,x)$ or $X_{u,t}(x)$ if needed to emphasize the velocity field that generates this flow mapping.

Remark 2.3.1. When $x \in \partial\Omega$, the Picard iterating always generate constant functions equal to x. Thus, we can obtain a unique solution $X(t,x) \equiv x$ on $[0,T]$. When $x \in \Omega$, the local solution will not exceed Ω, so it can still be extended to $[0,T]$. In both cases, the proof of uniqueness can be done by the Gronwall's inequality.

Note that $u \in C([0,T];C_0^2(\overline{\Omega}))$. Similarly to the proof of theorem 2.10 in [37], we can prove that $X \in C([0,T];C^2(\Omega))$. Letting $\chi(x,t) := \chi_0(X_t^{-1}(x))$, then we have obtained the indicator function χ using the velocity u.

We now estimate the variation of $\chi(x,t)$. From [4] Exercise 3.2, the Jacobian $J(X_t) \equiv 1$. By changing of variable, we have

$$\int_\Omega \chi(x,t)\mathrm{div}\varphi(x)dx = \int_\Omega \chi(X_t(y),t)\mathrm{div}\varphi(X_t(y))dy. \tag{2.3.7}$$

Similarly to the argument in [1], we integrate by parts. Let $A = (a_{ij})_{3\times 3}$ be the matrix inverse of ∇X_t, i.e. $A(y) := (\nabla_y X_t(y))^{-1}$. Let $\widetilde{\varphi}(y) = A^T(y)\varphi(X_t(y))$ with A^T being the transpose of A. For the gradient of $\widetilde{\varphi}(y)$, i.e. $\nabla_y\left(A^T(y)\varphi(X_t(y))\right)$, we consider its trace:

$$\begin{aligned} &\mathrm{Tr}\nabla_y\left(A^T\varphi(X_t(y))\right) \\ &= \sum_i\sum_j \partial_{y_i}a_{ji}\cdot\varphi_j(X_t(y)) + \sum_i\sum_j a_{ji}\cdot\partial_{y_i}\varphi_j(X_t(y)) \\ &= I_1 + I_2. \end{aligned}$$

Simplifying I_2, we obtain

$$
\begin{aligned}
I_2 =& \sum_i \sum_j \sum_k a_{ji} \partial_k \varphi_j (X_t(y)) \cdot \partial_{y_i} X_{t,k}(y) \\
=& \sum_j \sum_k \partial_k \varphi_j (X_t(y)) \cdot \left(\sum_i a_{ji} \partial_{y_i} X_{t,k}(y) \right) \\
=& \sum_j \sum_k \partial_k \varphi_j (X_t(y)) \cdot \left(\sum_i \left((\nabla X_t)^{-1} \right)_{ji} (\nabla X_t)_{ik} \right) \\
=& \sum_j \partial_j \varphi_j (X_t(y)) = \mathrm{div} \varphi (X_t(y)).
\end{aligned}
$$

Continuing with (2.3.7), we have

$$
\begin{aligned}
&\int_\Omega \chi(X_t(y), t) \mathrm{div} \varphi(X_t(y)) dy = \int_\Omega \chi(X_t(y), t) I_2 dy \\
&= \int_\Omega \chi(X_t(y), t) \mathrm{Tr} \nabla_y (A^T \varphi(X_t(y))) - \int_\Omega \chi(X_t(y), t) I_1 \\
&= \int_\Omega \chi(X_t(y), t) \mathrm{Tr} \nabla_y (A^T \varphi(X_t(y))) - \int_\Omega \chi(X_t(y), t) \sum_i \sum_j \partial_{y_i} a_{ji} \cdot \varphi_j(X_t(y)) \\
&= \int_\Omega \chi_0(y) \mathrm{Tr} \nabla_y (\widetilde{\varphi}(y)) - \int_\Omega \chi_0(y) \sum_i \sum_j \partial_{y_i} a_{ji} \cdot \varphi_j(X_t(y)) \\
&= I_3 - I_4.
\end{aligned}
$$

Since $\chi_0 \in BV(\Omega)$, we have

$$
\begin{aligned}
|I_3| =& \left| \int_\Omega \chi_0(y) \mathrm{div}_y \widetilde{\varphi}(t, y) \right| \leq \|\nabla \chi_0\|_{\mathcal{M}(\Omega)} \|\widetilde{\varphi}\|_{L^\infty([0,T]\times\Omega)} \\
\leq& C \|\nabla \chi_0\|_{\mathcal{M}(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \|\nabla A\|_{L^\infty([0,T]\times\Omega)} \\
\leq& \|\chi_0\|_{BV(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \beta(\|u\|_{C([0,T];C^2(\overline{\Omega}))}).
\end{aligned}
$$

The notation $\beta(\cdot)$ denotes a continuous function. In $\|\nabla A\|_{L^\infty(\Omega)}$, we firstly find the Euclidean norm $|\nabla A|$ and then find the L^∞ norm of $|\nabla A|$. The situations later will be treated in the same way. We then estimate I_4:

$$
\begin{aligned}
|I_4| \leq& C \|\nabla A\|_{L^\infty([0,T]\times\Omega)} \|\chi_0\|_{L^1(\Omega)} \|\varphi(X_t(y))\|_{L^\infty(\Omega)} \\
\leq& \|\chi_0\|_{BV(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \beta(\|u\|_{C([0,T];C^2(\overline{\Omega}))}).
\end{aligned}
$$

We still use the notation $\beta(\cdot)$, so it represents different continuous functions in different contexts. The estimates of I_3 and I_4 implies

$$\left|\int_\Omega \chi(X_t(y),t)\mathrm{div}\varphi(X_t(y))dy\right| \leq \|\chi_0\|_{BV(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\beta(\|u\|_{C([0,T];C^2(\overline{\Omega}))}).$$

Thus, we have

$$\mathcal{V}(\chi(t),\Omega) \leq \|\chi_0\|_{BV(\Omega)}\beta(\|u\|_{C([0,T];C^2(\overline{\Omega}))}).$$

Noticing that $\|\chi(t)\|_{L^1(\Omega)} = \|\chi_0\|_{L^1(\Omega)}$ and $\|\nabla\chi(t)\|_{\mathcal{M}(\Omega)} \leq \mathcal{V}(\chi(t),\Omega)$, one has

$$\begin{aligned}\|\chi(t)\|_{BV(\Omega)} &= \|\chi(t)\|_{L^1(\Omega)} + \|\nabla\chi(t)\|_{\mathcal{M}(\Omega)}\\ &\leq \|\chi_0\|_{L^1(\Omega)} + \mathcal{V}(\chi(t),\Omega) \leq \beta(\|u\|_{C([0,T];C^2(\overline{\Omega}))})\|\chi_0\|_{BV(\Omega)}.\end{aligned}$$

Remark 2.3.2. In order to control $(\nabla X_t)^{-1}$ with $\beta(\|u\|_{C([0,T];C^2(\overline{\Omega}))})$, we only need to consider ∇X_t. This is because $\det(\nabla X_t) = 1$, which implies that $(\nabla X_t)^{-1} = \mathrm{adj}(\nabla X_t)$. We take the derivatives of the following equation:

$$X(t,x) = x + \int_0^t u(s,X(s,x))ds, \tag{2.3.8}$$

and then we use the Gronwall's inequality. In order to estimate $\partial_{y_i}a_{ji}$, we take the derivatives of the equation $(\nabla X_t)^{-1} = \mathrm{adj}(\nabla X_t)$. It remains to estimate the second derivatives of $X(t,x)$, which can be solved similarly by the Gronwall's inequality.

Remark 2.3.3. When $\|u\|_{C([0,T];C^2(\overline{\Omega}))} \leq R$ we have $\|\chi(u)\|_{L^\infty([0,T];BV(\Omega))} \leq C(R)$.

Now we study the operator $B(\cdot)$. We recall the lemma 3.2 from [12].

Lemma 2.3.1 ([12], lemma 3.2). *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded C^3-domain and $u \in C([0,T];C_0^2(\overline{\Omega}))$. There exists a unique solution operator $B(\cdot)$, such that $B(u)$ solves* (2.1.2), (2.1.4) *and* (2.1.8) *in the weak sense. Given any bounded set $A \subseteq C([0,T];C_0^2(\overline{\Omega}))$, the image $B(A)$ is bounded in $L^2([0,T];H_0^1(\Omega)) \cap L^\infty([0,T];L^2(\Omega))$ and $B(\cdot)$ is continuous on A.*

We will show later that the condition $u \in C([0,T];C_0^2(\overline{\Omega}))$ will be guaranteed. Thus, we can always use the operator $B(\cdot)$ when solving the approximate equations. Note that when $\|u\|_{C([0,T];C^2(\overline{\Omega}))} \leq R$, we have $\|B(u)\|_{L^2([0,T];H_0^1(\Omega))} + \|B(u)\|_{L^\infty([0,T];L^2(\Omega))} \leq C(R)$.

2.3.4 Estimating the operator $N(u, \chi, B)$

Substituting $\chi(u)$ and $B(u)$, we obtain $N(u) = N(u, \chi(u), B(u))$. We estimate its operator norm. For convenience, we denote the formula by

$$\langle N(u), \eta \rangle = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \int_\Omega u \otimes u : \nabla \eta dx, \\
I_2 &= -\int_\Omega B \otimes B : \nabla \eta dx, \\
I_3 &= -2 \int_\Omega \nu(\chi) Du : D\eta dx, \\
I_4 &= \kappa \int_\Omega P_\tau : \nabla \eta d|\nabla \chi|.
\end{aligned}$$

We estimate the integrals as follows. Notice that the C^1 norm is equivalent to the G_n norm in the finite-dimensional space G_n. Thus, we obtain

$$|I_1| \leq \int_\Omega |u|^2 |\nabla \eta| \leq \|\eta\|_{C^1(\Omega)} \|u\|^2_{L^2(\Omega)} \leq C \|u\|^2_{G_n} \|\eta\|_{G_n}.$$

Similarly, we estimate I_2 and I_3 as

$$|I_2| \leq \|\eta\|_{C^1(\Omega)} \|B\|^2_{L^2(\Omega)} \leq C \|B\|^2_{L^2(\Omega)} \|\eta\|_{G_n},$$

$$|I_3| \leq C \|Du\|_{L^2(\Omega)} \|D\eta\|_{L^2(\Omega)} \leq C \|u\|_{G_n} \|\eta\|_{G_n}.$$

Using the fact that $|(P_\tau)_{ij}| = |\delta_{ij} - n_i n_j| \leq 1$, we have

$$|P_\tau : \nabla \eta| \leq |P_\tau| \, |\nabla \eta| \leq C \, |\nabla \eta| \leq C \|\eta\|_{G_n},$$

and then we obtain

$$|I_4| \leq \|\chi\|_{BV(\Omega)} \|P_\tau : \nabla \eta\|_{L^\infty(\Omega)} \leq C \|\chi\|_{BV(\Omega)} \|\eta\|_{G_n}.$$

Thus, the operator norm of $N(u)$ is estimated as the following:

$$\|N(u)\|_{G_n^*}(t) \leq C \left(\|u\|^2_{G_n} + \|u\|_{G_n} + \|B(u)\|^2_{L^2(\Omega)} + \|\chi(u)\|_{BV(\Omega)} \right)(t). \tag{2.3.9}$$

2.3.5 Operator for the fixed-point method

In order to construct the operator for the fixed-point method, we consider the equation

$$M(u(t)) = M(u_0) + \int_0^t N(u(s))ds, \tag{2.3.10}$$

which can be rewritten as:

$$u(t) = M^{-1}(M(u_0)) + M^{-1}\left(\int_0^t N(u(s))ds\right). \tag{2.3.11}$$

In order to prove M is invertible, we suppose $M(\eta) = 0$ in G_n^*, then we have $\|\eta\|_{L^2(\Omega)}^2 = \langle M(\eta), \eta\rangle = 0$. Since η is a continuous function, we have $\eta = 0$. Thus, $M : G_n \to G_n^*$ is invertible.

Remark 2.3.4. When $u_0 \in L^2(\Omega)$, the functional $M(u_0) \in G_n^*$ is still well defined.

Let $\widetilde{u}_0 := M^{-1}(M(u))$, we define

$$K(u(t)) := M^{-1}\left(M(u_0) + \int_0^t N(u(s))ds\right) = \widetilde{u}_0 + M^{-1}\left(\int_0^t N(u(s))ds\right). \tag{2.3.12}$$

For convenience, we define the set:

$$A_{a,b} := \{u \in C([0,a], G_n) : \|u\|_{L^\infty([0,a];G_n)} \le b\}.$$

Remark 2.3.5. Since all norms are equivalent in G_n, we pick an arbitrary norm and fix it to be our $\|\cdot\|_{G_n}$. We will consider the properties of $K(\cdot)$ on $A_{T^*,R}$. In fact, $K(\cdot)$ becomes a compact operator on $A_{T^*,R}$ for suitable T^* and R.

Given $u \in A_{T,R}$. From (2.3.9) and Section 2.3.3, we have the following estimate:

$$\|N(u)\|_{L^\infty([0,T];G_n^*)} \leq C(R). \tag{2.3.13}$$

We now study the properties of K. Firstly, we study the continuity of K with respect to t. In fact, from (2.3.13), we have

$$\begin{aligned} &\|K(u)(t) - K(u)(s)\|_{G_n} \\ &\leq \left\|M^{-1}\right\|_{\mathcal{L}(G_n^*,G_n)} \int_s^t \|N(u)\|_{G_n^*}(r)dr \\ &\leq C(R)|t-s|. \end{aligned} \tag{2.3.14}$$

Thus, $K(u) \in C([0,T], G_n)$.

Secondly, we study the boundedness of $K(u)$. Still using the estimate in (2.3.13), we obtain

$$\|K(u)\|_{G_n}(t) \leq \|\widetilde{u}_0\|_{G_n} + C(R)t. \tag{2.3.15}$$

We choose $R > \|\widetilde{u}_0\|_{G_n}$ and T^* small enough, such that

$$\|\widetilde{u}_0\|_{G_n} + C(R)T^* < R.$$

Then the operator K maps $A_{T^*,R}$ into $A_{T^*,R}$.

Thirdly, we show that $K(\cdot)$ is a continuous operator on $A_{T^*,R}$. We fix $v \in A_{T^*,R}$ and let $u \in A_{T^*,R}$ be such that $u \to v$ in $C([0,T^*]; G_n)$. Since

$$\|K(u) - K(v)\|_{G_n}(t) \leq \left\|M^{-1}\right\|_{\mathcal{L}(G_n^*,G_n)} \int_0^t \|N(u) - N(v)\|_{G_n^*}(s)ds,$$

we need to estimate $\|N(u) - N(v)\|_{G_n^*}(t)$. For any $t \in [0,T^*]$, we consider

$$\langle N(u) - N(v), \eta\rangle = I_1 + I_2 + I_3 + I_4, \tag{2.3.16}$$

where

$$\begin{aligned} I_1 &= \int_\Omega (u \otimes u - v \otimes v) : \nabla\eta, \\ I_2 &= -\int_\Omega (B_u \otimes B_u - B_v \otimes B_v) : \nabla\eta, \\ I_3 &= -2\int_\Omega (\nu(\chi_u)Du - \nu(\chi_v)Dv) : \nabla\eta, \\ I_4 &= \kappa\left(\int_\Omega P_\tau : \nabla\eta d\,|\nabla\chi_u| - \int_\Omega P_\tau : \nabla\eta d\,|\nabla\chi_v|\right). \end{aligned} \tag{2.3.17}$$

Here we denote $B(u)$ and $B(v)$ by B_u and B_v for short. The variable t is ignored for convenience when there is no ambiguity. The terms I_1 to I_4 are estimated as follows. For I_1, we have

$$\begin{aligned}
|I_1| \leq & \int_\Omega |u \otimes (u-v) + (u-v) \otimes v| \, |\nabla \eta| dx \\
\leq & \int_\Omega |u||u-v||\nabla \eta| dx + \int_\Omega |u-v||v||\nabla \eta| dx \\
\leq & \|u\|_{L^2(\Omega)} \|u-v\|_{L^2(\Omega)} \|\nabla \eta\|_{L^\infty(\Omega)} + \|u-v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \|\nabla \eta\|_{L^\infty(\Omega)} \\
\leq & C \left(\|u\|_{G_n} + \|v\|_{G_n} \right) \|u-v\|_{G_n} \|\eta\|_{G_n} \\
\leq & CR \|u-v\|_{G_n} \|\eta\|_{G_n} .
\end{aligned}$$

Thus, $\sup_{[0,T^*]} |I_1(t)| \leq CR \|u-v\|_{C([0,T^*];G_n)} \|\eta\|_{G_n}$. Similarly to the estimate of I_1, we obtain

$$\begin{aligned}
|I_2| \leq & \int_\Omega |B_u||B_u - B_v||\nabla \eta| + \int_\Omega |B_u - B_v||B_v||\nabla \eta| \\
\leq & C \left(\|B_u\|_{L^2} + \|B_v\|_{L^2} \right) \|B_u - B_v\|_{L^2} \|\eta\|_{G_n} \\
\leq & C(R) \|B_u - B_v\|_{L^2} \|\eta\|_{G_n} .
\end{aligned}$$

From lemma 2.3.1, we have $\sup_{[0,T^*]} |I_2(t)| \leq C(\|u-v\|_{C([0,T^*];G_n)}) \|\eta\|_{G_n}$. Moreover, the constant $C(\|u-v\|_{C([0,T^*];G_n)}) \to 0$ when $\|u-v\|_{C([0,T^*];G_n)} \to 0$. For I_3 we obtain

$$\begin{aligned}
|I_3| \leq & 2 \int_\Omega |\nu(\chi_u) Du - \nu(\chi_v) Dv| \, |\nabla \eta| \\
\leq & 2 \int_\Omega |\nu(\chi_u)| \, |Du - Dv||\nabla \eta| + 2 \int_\Omega |\nu(\chi_u) - \nu(\chi_v)| \, |Dv||\nabla \eta| \\
\leq & C \int_\Omega |\nabla (u-v)||\nabla \eta| + C \int_\Omega \left| \nu^+ \chi_u + \nu^-(1-\chi_u) - \nu^+ \chi_v - \nu^-(1-\chi_v) \right| |\nabla v||\nabla \eta| \\
\leq & C \|u-v\|_{G_n} \|\eta\|_{G_n} + C \|v\|_{G_n} \|\eta\|_{G_n} \int_\Omega |\chi_u - \chi_v| .
\end{aligned}$$

The key point is to prove that $\int_\Omega |\chi_u - \chi_v| \to 0$ as $u \to v$ in $C([0,T^*];G_n)$. In fact, $u \to v$ in $C([0,T^*];G_n)$ implies $u \to v$ in $C([0,T^*];C^1(\overline{\Omega}))$. Thus, similarly to the argument in [1], we obtain $X_u \to X_v$ in $C([0,T^*];C^1(\Omega))$. Let $\Omega_u^+(t) := X_u(t,\Omega_0^+)$, $\Omega_v^+(t) := X_v(t,\Omega_0^+)$ and $\Gamma_u(t) := X_u(t,\Gamma_0)$, $\Gamma_v(t) := X_v(t,\Gamma_0)$. Notice that

$$\int_\Omega |\chi_u - \chi_v|\, dx = \left|\Omega_u^+ \triangle \Omega_v^+\right|$$

where $\triangle$ denotes the symmetric difference of sets. For any $\varepsilon > 0$, if $\|X_u - X_v\|_{C([0,T^*];C^1(\overline{\Omega}))} < \varepsilon$, then $\Gamma_u(t) \subseteq B(\Gamma_v(t),\varepsilon)$ for every $t \in [0,T^*]$. Here $B(\Gamma_v(t),\varepsilon)$ is the ε-neighborhood of $\Gamma_v(t)$. Since $v \in C([0,T^*];G_n) \subseteq C([0,T^*];C^2(\Omega))$, we obtain that the flow mapping $X_v(t,x) \in C([0,T^*];C^2(\Omega))$. Since our Γ_0 is a C^2 surface, we can apply a local parameterization to Γ_0. By composing with $X_v(t,x)$, it will naturally give us a local parameterization of $\Gamma_v(t)$. Suppose that $\varphi(a_1,a_2)$ is a C^2-diffeomorphism from an open set $D \subseteq \mathbb{R}^2$ to a local piece of $\Gamma_v(t)$. Using the normal vector $n(\varphi(a_1,a_2))$, the function

$$\psi(a_1,a_2,a_3) := \varphi(a_1,a_2) + a_3 n(\varphi(a_1,a_2))$$

gives us a diffeomorphism from $D \times (-\varepsilon,\varepsilon)$ to an open set in $B(\Gamma_v(t),\varepsilon)$. This allows us to obtain a local parameterization of $B(\Gamma_v(t),\varepsilon)$. Notice that both $D \times (-\varepsilon,\varepsilon)$ and $\psi(D \times (-\varepsilon,\varepsilon))$ are monotone increasing sets as ε increases. Thus, we can obtain the boundedness of the integrands and then use the Lebesgue dominated convergence theorem. When $\varepsilon \to 0$, by calculating the integrals, we have $|B(\Gamma_v(t),\varepsilon)| \to 0$ uniformly in t. Thus, $\|u - v\|_{C([0,T^*];G_n)} \to 0$ implies

$$\sup_{t\in[0,T^*]} \int_\Omega |\chi_u - \chi_v|\,(t) dx \to 0.$$

Hence, $\sup_{[0,T^*]} |I_3(t)| \le C(\|u-v\|_{C([0,T^*];G_n)}) \|\eta\|_{G_n}$. Similarly to the constant term in I_2, the constant $C(\|u-v\|_{C([0,T^*];G_n)}) \to 0$ as $\|u-v\|_{C([0,T^*];G_n)} \to 0$.

In order to estimate I_4, we consider the functional $F_u(t)$ such that

$$\langle F_u(t), \eta\rangle := \left\langle H_{\chi_u(t)}, \eta\right\rangle - \left\langle H_{\chi_v(t)}, \eta\right\rangle.$$

Thus, $I_4(t) = \kappa\langle F_u(t), \eta\rangle$. Suppose $\|u-v\|_{C([0,T^*];G_n)} \to 0$, we need to prove that

$$\|F_u\|_{L^\infty([0,T^*];G_n^*)} \to 0.$$

Note that $\{\eta \in G_n : \|\eta\|_{G_n} = 1\}$ is a subset of $A := \{\eta \in G_n : \|\eta\|_{C^2(\Omega)} \leq C\}$ for a suitable C. Thus, it is sufficient to show

$$\sup_{t\in[0,T^*]} \sup_{\eta\in A} |\langle F_u(t), \eta\rangle| \to 0. \tag{2.3.18}$$

Note that since $u, v \in C([0,T^*], G_n)$, the interfaces $\Gamma_u(t)$ and $\Gamma_v(t)$ are both C^2-surfaces for all $t \in [0,T^*]$. Since Γ_0 is compact, by applying a local parameterization and using the partition of unity, we can consider the integrals locally. Let $\varphi(a_1, a_2)$ be the C^2-diffeomorphism from an open set $D \subseteq \mathbb{R}^2$ to a local piece on Γ_0. The function $X_u(t, \varphi(a_1, a_2))$ allows us to calculate the normal vector

$$n_u(X_u(t, \varphi(a_1, a_2))) \in C([0,T^*]; C^1(D)).$$

When $u \to v$ in $C([0,T^*]; C^1(\Omega))$, we have $X_u \to X_v$ in $C([0,T^*]; C^1(\Omega))$. Thus, $n_u \to n_v$ in $C([0,T^*] \times D)$. Similarly, the Jacobians $J(X_u(t, \varphi(a_1, a_2)))$ goes to $J(X_v(t, \varphi(a_1, a_2)))$ in $C([0,T^*] \times D)$, and the test functions $\nabla\eta(X_u(t, \varphi(a_1, a_2)))$ goes to $\nabla\eta(X_v(t, \varphi(a_1, a_2)))$ in $C([0,T^*] \times D)$ as well. Then (2.3.18) is obtained by calculating the integrals.

Thus, for all $\eta \in G_n$, we have

$$|I_4| = \kappa\, |\langle F_u(t), \eta\rangle| \leq C(\|u - v\|_{C([0,T^*];G_n)})\, \|\eta\|_{G_n}.$$

The constant $C(\|u - v\|_{C([0,T^*];G_n)})$ goes to 0 as $u \to v$ in $C([0,T^*]; G_n)$. From the estimates of I_1 to I_4, we have

$$\|N(u) - N(v)\|_{C([0,T^*];G_n^*)} \leq C(\|u - v\|_{C([0,T^*];G_n)}) \to 0. \tag{2.3.19}$$

Finally, we obtain

$$\begin{aligned}
&\|K(u) - K(v)\|_{C([0,T^*];G_n)} \\
&\leq \left\|M^{-1}\right\|_{\mathcal{L}(G_n^*, G_n)} T^* \, \|N(u) - N(v)\|_{C([0,T^*];G_n^*)} \\
&\leq C(\|u - v\|_{C([0,T^*];G_n)}) \to 0,
\end{aligned}$$

which implies that $K(\cdot)$ is continuous on $A_{T^*,R}$.

Now we prove that $K(A_{T^*,R})$ is precompact. Given any $v \in K(A_{T^*,R})$ and $t \in [0, T^*]$, there exists $u \in A_{T^*,R}$ such that $v = Ku$. From (2.3.15), we have

$$\|v\|_{G_n}(t) = \|Ku\|_{G_n}(t) \leq \|\tilde{u}_0\|_{G_n} + C(R)T^*.$$

Thus, the set $\{v(t) : v \in K(A_{T^*,R})\} \subseteq G_n$ is precompact since it is bounded and G_n is finite dimensional. From (2.3.14), the functions in $K(A_{T^*,R})$ are equicontinuous. Thus, from proposition 2.2.2, $K(A_{T^*,R})$ is precompact. Since $A_{T^*,R}$ is already bounded, the operator K maps all the bounded subsets of $A_{T^*,R}$ into precompact sets. Thus, from definition 2.2.4, we obtain that K is a compact operator.

It remains to verify the properties of the set $A_{T^*,R}$ in $C([0, T^*]; G_n)$. Since $u(t) \equiv 0$ is in $A_{T^*,R}$, the set is non-empty. From the definition of $A_{T^*,R}$, we know it is closed and bounded. For the convexity of $A_{T^*,R}$, picking any u and v in $A_{T^*,R}$ and any $0 \leq \theta \leq 1$, we have

$$\|\theta u + (1-\theta)v\|_{G_n}(t) \leq \theta \|u\|_{G_n}(t) + (1-\theta)\|v\|_{G_n}(t) \leq R.$$

Thus, $A_{T^*,R}$ is convex.

From theorem 2.2.2, there exists a solution $u_n(t) \in C([0, T^*]; G_n)$. Replacing the initial value u_0 by $u(T^*)$ and repeating the steps above, we can increase the value of T^*. Currently, we can only guarantee that there will be a limit when we increase T^*. Thus, the maximum interval would be either $[0, T^*)$ or $[0, T]$, where $T^* \leq T$. When T^* is excluded from the interval, it actually means the solution will go to infinity when t is approaching T^*. This will not happen in our problem, as shown in the following section.

2.3.6 Extending the solution to $[0, T]$

Assuming that $T^* < T$, we derive a contradiction using the energy estimate. For each fixed n, we need to prove that $\sup_{[0,T]} \|u_n\|_{G_n} \leq C$, which is equivalent to $\sup_{[0,T]} \|u_n\|_{L^2} \leq C$. Since u_n is the solution of the approximate equation, we take the derivative of (2.3.4) with respect to the variable t. Substituting η with $u_n(t)$, and using (2.8) in [1], we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \kappa\frac{d}{dt}\|\nabla\chi_n\|_{\mathcal{M}(\Omega)} + (B_n \otimes B_n, \nabla u_n)_\Omega + 2(\nu(\chi_n)Du_n, Du_n)_\Omega = 0. \qquad (2.3.20)$$

Integrating from 0 to t, we have

$$\begin{aligned}&\frac{1}{2}\|u_n(t)\|_{L^2}^2+\kappa\|\nabla\chi_n(t)\|_{\mathcal{M}(\Omega)}+\int_0^t(B_n\otimes B_n,\nabla u_n)_\Omega ds\\&+2\int_0^t(\nu(\chi_n)Du_n,Du_n)_\Omega ds=\frac{1}{2}\|u_0\|_{L^2}^2+\kappa\|\nabla\chi_0\|_{\mathcal{M}(\Omega)}.\end{aligned} \tag{2.3.21}$$

For each u_n, the solution operator $B(\cdot)$ gives us a weak solution of (2.1.2). Thus, by testing (2.1.2) with $\varphi = B_n$ on $\Omega\times[0,t]$, we obtain

$$\frac{1}{2}\|B_n(t)\|_{L^2(\Omega)}^2-\frac{1}{2}\|B_0\|_{L^2(\Omega)}^2-\int_0^t(B_n\otimes B_n,\nabla u_n)_\Omega ds+\sigma\int_0^t\|\nabla B_n\|_{L^2(\Omega)}^2=0. \tag{2.3.22}$$

Using the same argument as in the energy estimate, for some $c>0$, we have

$$\begin{aligned}&\frac{1}{2}\|u_n(t)\|_{L^2(\Omega)}^2+\frac{1}{2}\|B_n(t)\|_{L^2(\Omega)}^2+\kappa\|\nabla\chi_n(t)\|_{\mathcal{M}(\Omega)}+c\int_0^t\|\nabla u_n(s)\|_{L^2(\Omega)}^2ds\\&+\sigma\int_0^t\|\nabla B_n(s)\|_{L^2(\Omega)}^2ds\le\frac{1}{2}\|u_0\|_{L^2(\Omega)}^2+\frac{1}{2}\|B_0\|_{L^2(\Omega)}^2+\kappa\|\nabla\chi_0\|_{\mathcal{M}(\Omega)}=E_0\end{aligned} \tag{2.3.23}$$

for any $t\in[0,T^*]$. Thus, $\sup_{[0,T^*]}\|u_n(t)\|_{G_n}\le\sup_{[0,T^*]}C\|u_n(t)\|_{L^2(\Omega)}\le C$.

From the ODE theory, we know that if the maximum interval of a solution is $[0,T^*)$, then the solution must blow up at T^*. We will use the same argument. Picking an increasing sequence $t_m\in[0,T^*)$ such that $t_m\to T^*$, we consider the sequence $\{u_n(t_m)\}_{m=1}^\infty\subset G_n$. Since $\sup_{[0,T]}\|u_n\|_{G_n}(t)\le C$ and $\dim G_n<\infty$, we can find a subsequence, still denoted by t_m, such that $u_n(t_m)\to a\in G_n$, as $m\to\infty$. We only need to prove that $\lim_{t\to T^*}\|u_n(t)-a\|_{G_n}=0$. Then the solution $u_n(t)$ can be continuously extended to $[0,T^*]$. Using the Schauder fixed point theorem again, with T^* being the new initial time, we will get a contradiction. It then follows that $T^*=T$. Now we assume that $\lim_{t\to T^*}u_n(t)\neq a$, then there exists an $\varepsilon_0>0$, such that for all $\delta>0$, there exists $T^*-\delta<s<T^*$, such that $\|u_n(s)-a\|_{G_n}>\varepsilon_0$. Meanwhile, there exists an m, such that $T^*-\delta<t_m<T^*$ and $\|u_n(t_m)-a\|_{G_n}<\varepsilon_0/2$. Thus, we obtain

$$\|u_n(s)-u_n(t_m)\|_{G_n}\ge\|u_n(s)-a\|_{G_n}-\|u_n(t_m)-a\|_{G_n}>\varepsilon_0/2.$$

Recall that

$$\|u_n(s) - u_n(t_m)\|_{G_n} = \left\| M^{-1} \int_{t_m}^{s} N(u_n) \right\|_{G_n} \le \|M^{-1}\| \int_{t_m}^{s} \|N(u_n)\|_{G_n^*}$$
$$\le C|s - t_m| < C\delta.$$

Let δ be small enough such that $C\delta < \varepsilon_0/2$, then we get a contradiction. Thus, $\|u_n(t) - a\|_{G_n} \to 0$ as $t \to T^*$.

Consequently, we have found a solution $u_n \in C([0,T]; G_n)$. Using the solution operators, we obtain the corresponding $B_n := B(u_n)$ and $\chi_n := \chi(u_n)$. The energy inequality

$$\begin{aligned} &\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|B_n(t)\|^2_{L^2(\Omega)} + \kappa \|\nabla \chi_n(t)\|_{\mathcal{M}(\Omega)} \\ &+ c\|\nabla u_n\|^2_{L^2([0,T];L^2(\Omega))} + \sigma \|\nabla B_n\|^2_{L^2([0,T];L^2(\Omega))} \le E_0 \end{aligned} \tag{2.3.24}$$

holds for all $t \in [0,T]$.

2.4 Passing the Limit

In this section, we study the limits of u_n, B_n and χ_n. Recall that

$$u_n \in C([0,T]; G_n), \quad B_n \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega)), \quad \chi_n \in L^\infty([0,T]; BV(\Omega)),$$

and $\text{div} u_n = \text{div} B_n = 0$. From the energy inequality (2.3.24), we have the following estimates:

$$\begin{aligned} \|u_n\|_{L^\infty([0,T];L^2(\Omega))} &\le \sqrt{2E_0}, \\ \|u_n\|_{L^2([0,T];H_0^1(\Omega))} &\le \sqrt{2TE_0 + E_0/c}, \\ \|B_n\|_{L^\infty([0,T];L^2(\Omega))} &\le \sqrt{2E_0}, \\ \|B_n\|_{L^2([0,T];H_0^1(\Omega))} &\le \sqrt{2TE_0 + E_0/\sigma}, \\ \|\nabla \chi_n\|_{L^\infty([0,T];\mathcal{M}(\Omega))} &\le E_0/\kappa, \\ \|\chi_n\|_{L^\infty([0,T];BV(\Omega))} &\le |\Omega| + E_0/\kappa. \end{aligned} \tag{2.4.1}$$

2.4.1 Limits of u_n, B_n and χ_n

From the embedding theorems, we have

$$\mathcal{M}(\Omega) \hookrightarrow H^{-3}(\Omega).$$

From the estimates in (2.4.1) and the Banach-Alaoglu theorem (see [6]), we have

$$\begin{aligned}
u_n &\rightharpoonup^* u && \text{in } L^\infty([0,T];L^2(\Omega)),\\
u_n &\rightharpoonup v && \text{in } L^2([0,T];H_0^1(\Omega)),\\
B_n &\rightharpoonup^* B && \text{in } L^\infty([0,T];L^2(\Omega)),\\
B_n &\rightharpoonup G && \text{in } L^2([0,T];H_0^1(\Omega)),\\
\chi_n &\rightharpoonup^* \chi && \text{in } L^\infty([0,T];L^\infty(\Omega)),\\
\nabla\chi_n &\rightharpoonup^* \zeta && \text{in } L^\infty([0,T];H^{-3}(\Omega)),
\end{aligned}$$

for suitable subsequences.

In order to pass the limit in nonlinear terms, we need to obtain stronger convergence properties of u_n. We begin by showing an improved version of lemma A.3 in [13].

Lemma 2.4.1. *Let $\Omega \subseteq \mathbb{R}^3$ be bounded. Suppose that $u_n \rightharpoonup^* u$ in $L^\infty([0,T];L^2(\Omega))$, and for any $\varphi \in \mathbb{H}(\Omega)$,*

$$\sup_{t\in[0,T]} \left| \int_\Omega u_n\varphi dx - \int_\Omega u\varphi dx \right| \to 0. \tag{2.4.2}$$

Then we have $u_n \to u$ in $C([0,T];\mathbb{V}^*(\Omega))$.

Proof. Assume that u_n do not converge to u in $C^0([0,T];\mathbb{V}^*)$, then there exists $\varepsilon_0 > 0$ and $t_n \in [0,T]$, such that

$$\|u_n - u\|_{\mathbb{V}^*}(t_n) > \varepsilon_0. \tag{2.4.3}$$

Thus, there exist $\varphi_n \in \mathbb{V}(\Omega)$ with $\|\varphi_n\|_{\mathbb{V}} \equiv 1$, such that

$$|\langle u_n, \varphi_n\rangle(t_n) - \langle u, \varphi_n\rangle(t_n)| > \frac{\varepsilon_0}{2}, \tag{2.4.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pair on a space and its dual. Since $\varphi_n \in \mathbb{V} = \mathbb{H} \cap H_0^1$ and $H_0^1 \hookrightarrow\hookrightarrow L^2$, there exists $\varphi \in L^2(\Omega)$ such that $\varphi_n \to \varphi$ in $L^2(\Omega)$. Using the fact that $\|\varphi_n\|_{H_0^1(\Omega)} \leq C$, we can obtain $\varphi \in \mathbb{V}(\Omega)$. Now we have

$$\sup_{t\in[0,T]} |\langle u_n - u, \varphi_n \rangle| \leq \sup_{t\in[0,T]} |\langle u_n - u, \varphi_n - \varphi \rangle| + \sup_{t\in[0,T]} |\langle u_n - u, \varphi \rangle| . \tag{2.4.5}$$

The first term on the right-hand side goes to 0 since u_n and u are bounded in $L^\infty([0,T]; L^2(\Omega))$, and $\varphi_n \to \varphi$ in $L^2(\Omega)$; the second term goes to 0 by the condition (2.4.2) of this lemma. This contradicts with (2.4.4), which completes the proof. $\square$

In order to use the lemma above, we still need to verify (2.4.2). We recall that elements η_i form an orthonormal basis of $\mathbb{H}(\Omega)$ and an orthogonal basis of $\mathbb{V}(\Omega)$. Let η be a finite linear combination of η_i. For all sufficiently large n, the Galerkin approximate equations

$$\begin{aligned} \int_\Omega u_n(t)\eta - \int_\Omega u_0\eta = \int_0^t \int_\Omega u_n \otimes u_n : \nabla\eta - \int_0^t \int_\Omega B_n \otimes B_n : \nabla\eta \\ - \int_0^t \int_\Omega \nu(\chi_n) Du_n : D\eta - \kappa \int_0^t \int_\Omega P_\tau : \nabla\eta d\,|\nabla\chi_n| . \end{aligned} \tag{2.4.6}$$

all hold for this η. Considering the terms $f_n(t) := \int_\Omega u_n(t)\eta dx$, we claim that f_n have a uniformly convergent subsequence. First, we prove that $f_n(t)$ are equicontinuous. Given $0 \leq s < t \leq T$. Since η is fixed, from (2.4.6) we have

$$\begin{aligned} |f_n(t) - f_n(s)| &\leq C \int_s^t \|u_n\|_{L^2}^2 + \|B_n\|_{L^2}^2 + \|\nabla u_n\|_{L^1} + \|\nabla\chi_n\|_{\mathcal{M}} \\ &\leq \left(\|u_n\|_{L^\infty L^2}^2 + \|B_n\|_{L^\infty L^2}^2 + \|\nabla\chi_n\|_{L^\infty\mathcal{M}}\right)|t-s| + \int_0^T \chi_{[s,t]} \|\nabla u_n\|_{L^1} \\ &\leq C\,|t-s| + C\sqrt{t-s}, \end{aligned} \tag{2.4.7}$$

which implies that f_n is equicontinuous on $[0,T]$. By letting $s = 0$ we can show that f_n is uniformly bounded. Thus, by the Arzela-Ascoli theorem, there exists a subsequence, still denoted by f_n, such that

$$\sup_{t\in[0,T]} |f_n(t) - g(t)| \to 0 \tag{2.4.8}$$

for some $g(t) \in C[0,T]$, as $n \to \infty$.

We recall that $u_n \rightharpoonup u$ in $L^2([0,T]; H_0^1(\Omega))$. Now we want to show that

$$g = \int_\Omega u\eta \tag{2.4.9}$$

almost everywhere on $[0,T]$. Given any $\varphi \in L^2[0,T]$. Since $\eta\varphi \in L^2([0,T]; L^2(\Omega))$ and $u_n \rightharpoonup^* u$ in $L^\infty([0,T]; L^2(\Omega))$, we have

$$\int_0^T f_n \varphi dt = \int_0^T \int_\Omega u_n \eta \varphi dx dt \to \int_0^T \int_\Omega u\eta\varphi dx dt, \tag{2.4.10}$$

which implies that $f_n \rightharpoonup \int_\Omega u\eta$ weakly in $L^2[0,T]$. Since $f_n \to g$ in $C[0,T]$, we also have $f_n \rightharpoonup g$ weakly in $L^2[0,T]$, which proves (2.4.9).

Now we prove that (2.4.2) holds for all $\varphi \in \mathbb{H}(\Omega)$. Picking η_1, using the argument above we can find a subsequence of u_n, denoted by u_{1n}, such that $\int_\Omega u_{1n}\eta_1 \to \int_\Omega u\eta_1$ in $C[0,T]$. Now picking η_2 and using the same argument, we can obtain a subsequence of u_{1n}, denoted by u_{2n}, such that $\int_\Omega u_{2n}\eta_2 \to \int_\Omega u\eta_2$ in $C[0,T]$. Repeating these steps we can obtain u_{mn} for any $m, n \in \mathbb{N}$. Notice that for each m, the sequence u_{nn} is a subsequence of u_{mn} after finitely many terms. Thus, the convergence

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_\Omega u_{nn}\eta_k - u\eta_k \right| = 0 \tag{2.4.11}$$

holds for any $k \in \mathbb{N}$. The argument in (2.4.2) then follows the fact that $\{\eta_1, \eta_2, \cdots\}$ is a basis of $\mathbb{H}(\Omega)$.

Now we estimate $\partial_t B_n$. Picking $\varphi \in C_c^\infty(\Omega)$ with $\text{div}\varphi = 0$, we have

$$\int_\Omega \partial_t B_n \varphi = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_\Omega u_n \nabla B_n \varphi, \\ I_2 &= \int_\Omega B_n \nabla u_n \varphi, \\ I_3 &= \sigma \int_\Omega \nabla B_n : \nabla\varphi. \end{aligned}$$

The estimates are given as the following:

$$\begin{aligned}
|I_1| &\leq \|u_n\|_{L^3(\Omega)} \|\nabla B_n\|_{L^2(\Omega)} \|\varphi\|_{L^6(\Omega)} \leq C \|u_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|u_n\|_{H_0^1(\Omega)}^{\frac{1}{2}} \|B_n\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}, \\
|I_2| &\leq \|B_n\|_{L^3(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} \|\varphi\|_{L^6(\Omega)} \leq C \|B_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|B_n\|_{H_0^1(\Omega)}^{\frac{1}{2}} \|u_n\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}, \\
|I_3| &\leq C \|\nabla B_n\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \leq C \|B_n\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\partial_t B_n\|_{H^{-1}(\Omega)} \leq\ & C \|u_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|u_n\|_{H_0^1(\Omega)}^{\frac{1}{2}} \|B_n\|_{H_0^1(\Omega)} \\
& + C \|B_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|B_n\|_{H_0^1(\Omega)}^{\frac{1}{2}} \|u_n\|_{H_0^1(\Omega)} + C \|B_n\|_{H_0^1(\Omega)}.
\end{aligned}$$

Integrating with respect to t, we obtain

$$\begin{aligned}
&\int_0^T \|\partial_t B_n\|_{H^{-1}(\Omega)} \leq C \|u_n\|_{L^\infty([0,T];L^2(\Omega))}^{\frac{1}{2}} \int_0^T \|u_n\|_{H_0^1(\Omega)}^{\frac{1}{2}} \|B_n\|_{H_0^1(\Omega)} \, dt \\
&+ C \|B_n\|_{L^\infty([0,T];L^2(\Omega))}^{\frac{1}{2}} \int_0^T \|B_n\|_{H_0^1(\Omega)}^{\frac{1}{2}} \|u_n\|_{H_0^1(\Omega)} \, dt + C \|B_n\|_{L^1([0,T];H_0^1(\Omega))} \\
&\leq C \|u_n\|_{L^\infty([0,T];L^2(\Omega))}^{\frac{1}{2}} \|u_n\|_{L^2([0,T];H_0^1(\Omega))}^{\frac{1}{2}} \|B_n\|_{L^2([0,T];H_0^1(\Omega))} \\
&+ C \|B_n\|_{L^\infty([0,T];L^2(\Omega))}^{\frac{1}{2}} \|B_n\|_{L^2([0,T];H_0^1(\Omega))}^{\frac{1}{2}} \|u_n\|_{L^2([0,T];H_0^1(\Omega))} + C \|B_n\|_{L^2([0,T];H_0^1(\Omega))}.
\end{aligned}$$

Thus, $\|\partial_t B_n\|_{L^1([0,T];H^{-1}(\Omega))} \leq C$. By the Aubin-Lions lemma, we can find a suitable subsequence, still denoted by B_n, such that $B_n \to K$ strongly in $L^2([0,T];L^2(\Omega))$ for some $K \in L^2([0,T];L^2(\Omega))$. We can prove the uniqueness of limits using the following argument.

Proposition 2.4.1. *For the strong and weak limits B, G and K of B_n, we have $B = G = K$.*

Proof. Since $L^2([0,T];H_0^1(\Omega)) \subseteq L^2([0,T];L^2(\Omega))$, $B_n \rightharpoonup G$ in $L^2([0,T];H_0^1(\Omega))$ implies that $B_n \rightharpoonup G$ in $L^2([0,T];L^2(\Omega))$. By the uniqueness of the limit, we have $G = K$. Since $L^\infty([0,T];L^2(\Omega)) \subseteq L^2([0,T];L^2(\Omega))$, $B_n \rightharpoonup^* B$ in $L^\infty([0,T];L^2(\Omega))$ implies that $B_n \rightharpoonup^* B$ in $L^2([0,T];L^2(\Omega))$. Similarly to the first step, we have $B = K$. Thus, $B = G = K$. $\square$

We now consider the transport equation $\partial_t \chi_n + u_n \cdot \nabla \chi_n = 0$. For $\varphi \in C_c^\infty(\Omega)$, we have

$$\left| \int_\Omega \partial_t \chi_n \varphi \right| = \left| \int_\Omega \chi_n u_n \cdot \nabla \varphi \right| \leq \|u_n\|_{L^2(\Omega)} \|\varphi\|_{H_0^1(\Omega)} .$$

Thus, $\|\partial_t \chi_n\|_{H^{-1}(\Omega)} \leq \|u_n\|_{L^2(\Omega)}$, which implies that

$$\|\partial_t \chi_n\|_{L^2([0,T];H^{-1}(\Omega))} \leq \|u_n\|_{L^2([0,T];H_0^1(\Omega))} \leq C.$$

Therefore, using the same argument as in [1], Section 5.2, we can find a suitable subsequence, still denoted by χ_n, such that $\chi_n \to \chi$ strongly in $L^2([0,T]; L^2(\Omega))$. Now we prove that $\zeta = \nabla \chi$ in the weak sense. For almost every $t \in [0,T]$ and $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \chi_n \mathrm{div} \varphi dx = - \int_\Omega \varphi \cdot d\nabla \chi_n$$

for any n. Since $\mathrm{div}\varphi \in L^1(\Omega)$, we have

$$\int_\Omega \chi_n(t) \mathrm{div}\varphi dx \longrightarrow \int_\Omega \chi(t) \mathrm{div}\varphi dx,$$
$$\int_\Omega \varphi \cdot d\nabla \chi_n(t) \longrightarrow \int_\Omega \varphi \cdot d\zeta(t),$$

which implies that $\nabla \chi = \zeta$ for almost every $t \in [0,T]$.

Therefore, we finally obtain

$$\begin{aligned}
u_n &\rightharpoonup^* u && \text{in } L^\infty([0,T]; L^2(\Omega)),\\
u_n &\rightharpoonup u && \text{in } L^2([0,T]; H_0^1(\Omega)),\\
u_n &\to u && \text{in } C([0,T]; \mathbb{V}^*(\Omega)),\\
B_n &\rightharpoonup^* B && \text{in } L^\infty([0,T]; L^2(\Omega)),\\
B_n &\rightharpoonup B && \text{in } L^2([0,T]; H_0^1(\Omega)),\\
B_n &\to B && \text{in } L^2([0,T]; L^2(\Omega)),\\
\chi_n &\rightharpoonup^* \chi && \text{in } L^\infty([0,T]; L^\infty(\Omega)),\\
\nabla\chi_n &\rightharpoonup^* \nabla\chi && \text{in } L^\infty([0,T]; H^{-3}(\Omega)),\\
\chi_n &\to \chi && \text{in } L^2([0,T]; L^2(\Omega)).
\end{aligned}$$

2.4.2 Varifold limit of H_{χ_n}

Since we cannot pass the limit directly when dealing with H_{χ_n}, we will represent them with varifolds, and then consider the weak limit of the varifolds. Recall that $\Gamma_k(t) := X_{u_k}(t, \Gamma_0)$, where $\Gamma_0 = \partial\Omega_0^+$. Using the same argument as in [1], we consider the varifold $V_k(t)$ corresponding to $\Gamma_k(t)$, i.e.

$$\langle V_k(t), \varphi\rangle := \int_\Omega \varphi\left(x, n_k(t,x)\right) d\left|\nabla\chi_k(t)\right|$$

for any $\varphi \in C_0(\Omega \times \mathbb{S}^2)$, where $n_k(t,x) := \nabla\chi_k(t,x)/\left|\nabla\chi_k(t,x)\right|$. Since

$$\left|\langle V_k(t), \varphi\rangle\right| \leq \left\|\nabla\chi_k(t)\right\|_{\mathcal{M}(\Omega)} \left\|\varphi\right\|_{C_0(\Omega\times\mathbb{S}^2)},$$

we have

$$\left\|V_k(t)\right\|_{\mathcal{M}(\Omega\times\mathbb{S}^2)} \leq \left\|\nabla\chi_k(t)\right\|_{\mathcal{M}(\Omega)}.$$

Now for all $\varphi \in L^1([0,T]; C_0(\Omega\times\mathbb{S}^2))$, we define

$$\langle V_k, \varphi\rangle := \int_0^T \int_\Omega \varphi\left(t, x, n_k(t,x)\right) d\left|\nabla\chi_k(t)\right| dt.$$

Then we have

$$\left\|V_k\right\|_{L^\infty_w([0,T];\mathcal{M}(\Omega\times\mathbb{S}^2))} \leq \left\|\nabla\chi_k\right\|_{L^\infty_w([0,T];\mathcal{M}(\Omega))}.$$

Since V_k is bounded in $L^\infty_w([0,T]; \mathcal{M}(\Omega\times\mathbb{S}^2)$ and $\mathcal{M}(\Omega\times\mathbb{S}^2) \hookrightarrow H^{-3}(\Omega\times\mathbb{S}^2)$, by the same argument as in [1], there exists $V \in L^\infty_w([0,T]; \mathcal{M}(\Omega\times\mathbb{S}^2))$, such that

$$V_k \rightharpoonup^* V \quad \text{in } L^\infty([0,T]; H^{-3}(\Omega\times\mathbb{S}^2)).$$

For $\psi \in C_c^\infty([0,T) \times \Omega)$, letting $\varphi(x,s,t) = \mathrm{div}\psi - s \otimes s : \nabla\psi$, we have $\varphi \in L^1([0,T]; C_0(\Omega \times \mathbb{S}^2))$. The regularity of $n_k(x) := -\nabla\chi_k / |\nabla\chi_k|$ is guaranteed since $\nabla\chi_k$ are approximating solutions. Thus, we have

$$
\begin{aligned}
&-\int_0^T \left\langle H_{\chi_k(t)}, \psi(t) \right\rangle = \int_0^T \int_\Omega P_\tau : \nabla\psi d\, |\nabla\chi_k|\, dt \\
&= \int_0^T \int_\Omega \big(\mathrm{div}\psi - n_k \otimes n_k : \nabla\psi\big) d\, |\nabla\chi_k|\, dt \\
&= \int_0^T \int_{\Omega\times\mathbb{S}^2} \big(\mathrm{div}\psi - s \otimes s : \nabla\psi\big) dV_k dt \\
&\to \int_0^T \int_{\Omega\times\mathbb{S}^2} \big(\mathrm{div}\psi - s \otimes s : \nabla\psi\big) dV dt \\
&= \int_0^T \left\langle \delta V(t), \psi(t) \right\rangle dt
\end{aligned}
\tag{2.4.12}
$$

as $k \to \infty$. Letting $\varphi(x,s,t) = s\psi(x,t)$, we have

$$
\begin{aligned}
\int_0^T \left\langle \nabla\chi_k, \psi \right\rangle &= -\int_0^T \int_{\partial^*\{\chi_k=1\}} \psi \cdot n_k d\mathcal{H}^2 dt \\
= -\int_0^T \int_\Omega \psi \cdot n_k d\, |\nabla\chi_k|\, dt &= -\int_0^T \int_{\Omega\times\mathbb{S}^2} \psi \cdot s dV_k dt.
\end{aligned}
$$

Since $\nabla\chi_k \rightharpoonup^* \nabla\chi$ in $L^\infty([0,T]; H^{-3}(\Omega))$ and $V_k \rightharpoonup^* V$ in $L^\infty([0,T]; H^{-3}(\Omega \times \mathbb{S}^2))$, we have

$$
\begin{aligned}
\int_0^T \left\langle \nabla\chi_k, \psi \right\rangle &\to \int_0^T \left\langle \nabla\chi, \psi \right\rangle, \\
\int_0^T \int_{\Omega\times\mathbb{S}^2} \psi \cdot s dV_k dt &\to \int_0^T \int_{\Omega\times\mathbb{S}^2} \psi \cdot s dV dt.
\end{aligned}
$$

Therefore, using the fact that $C_c^\infty(\Omega)$ is dense in $C_0(\Omega)$, the equation

$$
\int_{\Omega\times\mathbb{S}^2} \psi \cdot s dV = -\int_\Omega \psi d\nabla\chi. \tag{2.4.13}
$$

holds for all $\psi \in C_0(\Omega)$.

2.4.3 Passing the limit

The weak formula represented by the varifolds is

$$(\partial_t u_n, \varphi)_{Q_T} - (u_n \otimes u_n, \nabla\varphi)_{Q_T} + (B_n \otimes B_n, \nabla\varphi)_{Q_T}$$
$$+2(\nu(\chi_n)Du_n, D\varphi)_{Q_T} + \kappa \int_0^T \langle \delta V_n(t), \varphi(t)\rangle \, dt = 0,$$

where $\varphi \in C_c^\infty([0,T) \times \Omega)$ and $\mathrm{div}\varphi = 0$. We have

$$\begin{aligned} &\left|\int_0^T \int_\Omega (u_n \otimes u_n - u \otimes u) : \nabla\varphi\right| \\ &\leq \left|\int_0^T \int_\Omega u_n \otimes (u_n - u) : \nabla\varphi\right| + \left|\int_0^T \int_\Omega (u_n - u) \otimes u : \nabla\varphi\right| =: I_1 + I_2. \end{aligned} \tag{2.4.14}$$

In I_1, the integrand equals to $(u_n - u) \cdot (u_n \nabla\varphi)$. Since u_n and φ are smooth, by direct calculation we have $\mathrm{div}(u_n \nabla\varphi) = 0$, which implies $u_n \nabla\varphi \in \mathbb{V}(\Omega)$. Recall that $\|\cdot\|_{\mathbb{V}} := \|\cdot\|_{H_0^1}$ and u_n is bounded in $L^2([0,T]; H_0^1(\Omega))$, so we have

$$\begin{aligned} I_1 &= \int_0^T \langle u_n - u, \psi\rangle_{\mathbb{V}^*,\mathbb{V}} \leq \|u_n - u\|_{L^\infty\mathbb{V}^*} \|u_n \nabla\varphi\|_{L^1\mathbb{V}} \\ &\leq C \|u_n - u\|_{L^\infty\mathbb{V}^*} \|u_n \nabla\varphi\|_{L^2 H_0^1} \to 0. \end{aligned} \tag{2.4.15}$$

The second term goes to 0 since $u_n \rightharpoonup u$ weakly in $L^2([0,T]; H_0^1)$ and $u\nabla\varphi \in L^2([0,T]; H_0^1)$.

For the term $(B_n \otimes B_n, \nabla\varphi)_{Q_T}$, we have

$$\begin{aligned} &\left|\int_0^T \int_\Omega B_n \otimes B_n : \nabla\varphi - B \otimes B : \nabla\varphi dxdt\right| \\ &\leq \|B_n\|_{L^2(Q_T)} \|B_n - B\|_{L^2(Q_T)} \|\nabla\varphi\|_{L^\infty(Q_T)} + \|B\|_{L^2(Q_T)} \|B_n - B\|_{L^2(Q_T)} \|\nabla\varphi\|_{L^\infty(Q_T)}, \end{aligned}$$

which converges to 0. For the viscosity term, we have

$$\begin{aligned} &|(\nu(\chi_n)Du_n, D\varphi)_{Q_T} - (\nu(\chi)Du, D\varphi)_{Q_T}| \\ =&\left|\int_0^T \int_\Omega \big(\chi_n \nu^+ Du_n + (1-\chi_n)\nu^- Du_n\big) : D\varphi - \int_0^T \int_\Omega \big(\chi \nu^+ Du + (1-\chi)\nu^- Du\big) : D\varphi\right| \\ \leq&\nu^+ \left|\int_0^T \int_\Omega \chi_n Du_n : D\varphi - \chi Du : D\varphi\right| + \nu^- \left|\int_0^T \int_\Omega (1-\chi_n)Du_n : D\varphi - (1-\chi)Du : D\varphi\right|. \end{aligned}$$

We only need to show the convergence of the first term, since the second term can be shown by the same argument. For the first term, we have

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \chi_n Du_n : D\varphi - \chi Du : D\varphi\right| \\
&\le \int_0^T\int_\Omega |\chi_n-\chi|\,|Du_n|\,|D\varphi| + \left|\int_0^T\int_\Omega Du_n : (\chi D\varphi) - \int_0^T\int_\Omega Du : (\chi D\varphi)\right|.
\end{aligned} \tag{2.4.16}$$

Since $u_n \rightharpoonup u$ in $L^2([0,T];H_0^1(\Omega))$ and $\varphi \in C_c^\infty([0,T)\times\Omega)$, we have $Du_n \rightharpoonup Du$ in $L^2([0,T];L^2(\Omega))$ and $\chi D\varphi \in L^2([0,T];L^2(\Omega))$. Thus, the second term goes to 0. Since $\left|Du_n\right|\left|D\varphi\right| \in L^2([0,T];L^2(\Omega))$ and $\chi_n \to \chi$ strongly in $L^2([0,T];L^2(\Omega))$, we have

$$\int_0^T\int_\Omega \left|\chi_n-\chi\right|\left|Du_n\right|\left|D\varphi\right| \le C\left\|\chi_n-\chi\right\|_{L^2L^2} \to 0,$$

which finishes the proof.

For the transport equation,

$$(\chi_0,\varphi(x,0))_\Omega + (\chi_n,\partial_t\varphi)_{Q_T} + (\chi_n, u_n\cdot\nabla\varphi)_{Q_T} = 0.$$

Since $\partial_t\varphi \in L^1([0,T];L^1(\Omega))$, we have $(\chi_n,\partial_t\varphi)_{Q_T} \to (\chi,\partial_t\varphi)_{Q_T}$. For the third term, we have

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \chi_n u_n\cdot\nabla\varphi - \int_0^T\int_\Omega \chi u\cdot\nabla\varphi\right| \\
&\le \left|\int_0^T\int_\Omega \chi_n u_n\cdot\nabla\varphi - \int_0^T\int_\Omega \chi_n u\cdot\nabla\varphi\right| + \left|\int_0^T\int_\Omega \chi_n u\cdot\nabla\varphi - \int_0^T\int_\Omega \chi u\cdot\nabla\varphi\right| \\
&\le \left\|\chi_n\right\|_{L^2(Q_T)}\left\|u_n-u\right\|_{L^2(Q_T)}\left\|\nabla\varphi\right\|_{L^\infty(Q_T)} + \left\|\chi_n-\chi\right\|_{L^2(Q_T)}\left\|u\right\|_{L^2(Q_T)}\left\|\nabla\varphi\right\|_{L^\infty(Q_T)},
\end{aligned}$$

which goes to 0 as $n\to\infty$.

From Section 2.4.2, we have proved that

$$\int_0^T \langle\delta V_n(t),\varphi\rangle\,dt \to \int_0^T \langle\delta V(t),\varphi\rangle\,dt.$$

Therefore, by letting $n\to\infty$, we obtain

$$\begin{aligned}
&-(u_0,\varphi(0))_\Omega - (u,\partial_t\varphi)_{Q_T} - (u\otimes u,\nabla\varphi)_{Q_T} + (B\otimes B,\nabla\varphi)_{Q_T} \\
&+ 2(\nu(\chi)Du, D\varphi)_{Q_T} + \kappa\int_0^T \langle\delta V(t),\varphi(t)\rangle\,dt = 0
\end{aligned}$$

for all $\varphi \in C_c^\infty([0,T) \times \Omega)$ with $\mathrm{div}\varphi = 0$. From (2.4.13), we have

$$\int_{\Omega\times\mathbb{S}^2} \psi \cdot s dV = -\int_\Omega \psi d\nabla\chi$$

for all $\psi \in C_0(\Omega)$. From proposition 2.2 in [1], χ is the unique renormalized solution of

$$\begin{aligned}
&\partial_t\chi + u \cdot \nabla\chi = 0 \quad \text{in } Q_T,\\
&\chi|_{t=0} = \chi_0 \quad \text{in } \Omega.
\end{aligned}$$

It remains to prove that (u, B, χ, V) satisfies the generalized energy inequality. Since $u_n \to u$, $B_n \to B$ in $L^2([0,T]; L^2(\Omega))$. For suitable subsequences, we have $u_n(t) \to u(t)$, $B_n(t) \to B(t)$ in $L^2(\Omega)$ for almost every $t \in [0,T]$. From theorem 1.1.1 in [7], we have

$$\begin{aligned}
\|u(t)\|_{L^2} &\leq \liminf_{n\to\infty} \|u_n(t)\|_{L^2},\\
\|B(t)\|_{L^2} &\leq \liminf_{n\to\infty} \|B_n(t)\|_{L^2}.
\end{aligned}$$

In Section 2.4.2 we have proved $\|V_n(t)\|_{\mathcal{M}(\Omega\times\mathbb{S}^2)} \leq \|\nabla\chi_n(t)\|_{\mathcal{M}(\Omega)}$. Given any $\varphi \in C_0(\Omega)$, we have

$$\begin{aligned}
\left|\int_{\Omega\times\mathbb{S}^2} \varphi dV(t)\right| = \left|\lim_{n\to\infty}\int_{\Omega\times\mathbb{S}^2} \varphi dV_n(t)\right| &\leq \liminf_{n\to\infty} \|\varphi\|_{L^\infty} \|V_n(t)\|_{\mathcal{M}(\Omega\times\mathbb{S}^2)}\\
&\leq \liminf_{n\to\infty} \|\varphi\|_{L^\infty} \|\nabla\chi_n(t)\|_{\mathcal{M}(\Omega)}.
\end{aligned}$$

Thus,

$$\|V(t)\|_{\mathcal{M}(\Omega\times\mathbb{S}^2)} \leq \liminf_{n\to\infty} \|\nabla\chi_n(t)\|_{\mathcal{M}(\Omega)}.$$

Since $\nabla B_n \rightharpoonup \nabla B$ in $L^2([0,T]; L^2(\Omega))$. For all $t \in [0,T]$, we still have $\nabla B_n \rightharpoonup \nabla B$ in $L^2([0,t]; L^2(\Omega))$. Thus,

$$\int_0^t \|\nabla B\|_{L^2}^2 \leq \liminf_{n\to\infty} \int_0^t \|\nabla B_n\|_{L^2}^2.$$

For the viscosity term, notice that

$$\begin{aligned}
f_n &:= (\nu(\chi_n)Du_n - \nu(\chi_n)Du) : (Du_n - Du)\\
&= \left(\chi_n\nu^+ + (1-\chi_n)\nu^-\right) |Du_n - Du|^2 \geq 0.
\end{aligned}$$

From $Du_n \rightharpoonup Du$ in $L^2([0,T];L^2(\Omega))$, we have

$$\int_0^t \int_\Omega f_n \leq C.$$

Thus, using Fatou's lemma, we obtain

$$\begin{aligned}
0 &\leq \int_0^t \int_\Omega \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_0^t \int_\Omega f_n \\
&= \liminf_{n\to\infty} \int_0^t \int_\Omega \nu(\chi_n) Du_n : Du_n + \lim_{n\to\infty} \int_0^t \int_\Omega -\nu(\chi_n) Du_n : Du \\
&+ \lim_{n\to\infty} \int_0^t \int_\Omega -\nu(\chi_n) Du : Du_n + \lim_{n\to\infty} \int_0^t \int_\Omega \nu(\chi_n) Du : Du \\
&= \liminf_{n\to\infty} \int_0^t \int_\Omega \nu(\chi_n) Du_n : Du_n - \int_0^t \int_\Omega \nu(\chi) Du : Du.
\end{aligned} \tag{2.4.17}$$

This is because (2.4.16) yields $\chi_n Du_n \rightharpoonup \chi Du$ in $L^2([0,T];L^2(\Omega))$, which implies that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \int_\Omega -\nu(\chi_n) Du_n : Du = \lim_{n\to\infty} \int_0^t \int_\Omega -\nu(\chi_n) Du : Du_n \\
&= -\int_0^t \int_\Omega \nu(\chi) Du : Du.
\end{aligned}$$

Since $|Du|^2 \in L^1([0,T];L^1(\Omega))$ and $\chi_n \rightharpoonup^* \chi$ in $L^\infty([0,T];L^\infty(\Omega))$, we have

$$\lim_{n\to\infty} \int_0^t \int_\Omega \nu(\chi_n) Du : Du = \int_0^t \int_\Omega \nu(\chi) Du : Du.$$

Thus, the lower-semicontinuity in (2.4.17) has been proved.

Recall that (2.3.21) and (2.3.22) give us

$$\begin{aligned}
&\frac{1}{2} \|u_n(t)\|_{L^2}^2 + \frac{1}{2} \|B_n(t)\|_{L^2}^2 + \kappa \|\nabla \chi_n(t)\|_{\mathcal{M}} + 2 \int_0^t \int_\Omega \nu(\chi_n) Du_n : Du_n dx ds \\
&\qquad + \sigma \int_0^t \|\nabla B_n\|_{L^2}^2 ds \leq \frac{1}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} \|B_0\|_{L^2}^2 + \kappa \|\nabla \chi_0\|_{\mathcal{M}}.
\end{aligned} \tag{2.4.18}$$

Taking the $\liminf$ on (2.4.18), and using the fact that $\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n)$, we finally obtain

$$\begin{aligned}
&\frac{1}{2} \|u(t)\|_{L^2}^2 + \frac{1}{2} \|B(t)\|_{L^2}^2 + \kappa \|V(t)\|_{\mathcal{M}(\Omega\times\mathbb{S}^2)} + 2 \int_0^t \int_\Omega \nu(\chi) Du : Du dx ds \\
&\qquad + \sigma \int_0^t \|\nabla B\|_{L^2}^2 ds \leq \frac{1}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} \|B_0\|_{L^2}^2 + \kappa \|\nabla \chi_0\|_{\mathcal{M}},
\end{aligned} \tag{2.4.19}$$

which finishes the proof of the energy inequality.

Therefore, we have finished the proof of theorem 2.1.1.

3.0 Existence of Strong Solutions to the Two-Phase MHD Equations

3.1 Introduction and Main Results

In this chapter, we study the existence of strong solutions to the two-phase magnetohydrodynamic (MHD) equations. The two immiscible fluids are incompressible, viscous and resistive. They occupy an open, bounded, simply connected C^3 domain Ω. The positions of the inner and outer fluids are represented by open sets $\Omega^+(t)$ and $\Omega^-(t)$ respectively, and the fluid-fluid interface is denoted by the set $\Gamma(t)$. These three sets are disjoint and we have $\Omega^+(t) \cup \Gamma(t) \cup \Omega^-(t) = \Omega$. In this work we assume that $\Gamma(t)$ and $\Omega^+(t)$ do not touch the boundary $\partial\Omega$. We consider the following equations:

$$\partial_t u + (u \cdot \nabla)u - (\nabla \times B) \times B + \nabla p - \nu^{\pm} \triangle u = 0 \quad \text{in } \Omega \setminus \Gamma(t), \tag{3.1.1}$$

$$\partial_t B - \nabla \times (u \times B) + \nabla \times (\sigma \nabla \times B) = 0 \quad \text{in } \Omega, \tag{3.1.2}$$

$$\mathrm{div} u = 0 \quad \text{in } \Omega \setminus \Gamma(t), \tag{3.1.3}$$

$$\mathrm{div} B = 0 \quad \text{in } \Omega, \tag{3.1.4}$$

$$-\left[\!\left[2\nu^{\pm} \tilde{D} u - pI \right]\!\right] n = \kappa H n \quad \text{on } \Gamma(t), \tag{3.1.5}$$

$$V_\Gamma = u \cdot n \quad \text{on } \Gamma(t), \tag{3.1.6}$$

$$u|_{\partial\Omega} = 0, \; B|_{\partial\Omega} = 0, \tag{3.1.7}$$

$$u|_{t=0} = u_0, \; B|_{t=0} = B_0. \tag{3.1.8}$$

The terms u, B and p stand for the velocity, magnetic field and pressure. The density of both fluids is assumed to be equal to 1 everywhere and the magnetic diffusion coefficient σ remains a constant. The viscosity coefficient ν takes different constant values ν^+ and ν^- in two fluids, which is sometimes written as $\nu(\chi)$ to emphasize its dependency on the fluid position. The indicator function $\chi := \chi_{\Omega^+(t)}$ expresses the position of the internal fluid. The surface tension coefficient is $\kappa > 0$. The mean curvature of the interface is H. The outward (pointing to $\partial\Omega$) normal vector and the speed of the interface are denoted by n and V_Γ. The term $\tilde{D}u := (\nabla u + \nabla u^\top)/2$ stands for the strain rate tensor. The notation $[\![f]\!]$ stands for the jump of f across the interface Γ, i.e. for all $t \geq 0$ and all $x \in \Gamma(t)$,

$$[\![f]\!](x) := \lim_{\varepsilon \to 0^+} f(x + \varepsilon n(x)) - \lim_{\varepsilon \to 0^+} f(x - \varepsilon n(x));$$

when f does not have enough continuity, its one-side limits at $\Gamma(t)$ are considered in the sense of trace. For more details about this model, we refer to [1, 10, 14].

3.1.1 Related research

When the magnetic field vanishes, the problem turns into the two-phase Navier-Stokes equations. In [27], Prüss and Simonett studied the problem in $\mathbb{R}^{n+1}$, where the interface can be expressed as the graph of a function defined on $\mathbb{R}^n$. For any time interval, if the initial values satisfy some smallness conditions (dependent on the time interval), then the unique strong solution exists. In [28] also by Prüss and Simonett, a different type of existence theory is obtained. The smallness condition in [28] is only required for the initial interface, which implies the local existence of the solution. The same equations have been studied in a bounded domain by Köhne, Prüss and Wilke in [15]. Moreover, Abels and Wilke have studied the two-phase Navier–Stokes-Mullins–Sekerka system in [2].

There is also much research on global solutions to two-phase flows with surface tension considered. In [25], Plotnikov has studied the two-phase Navier-Stokes equations for incompressible non-Newtonian fluids in $\mathbb{R}^2$. The case of incompressible non-Newtonian fluids in $\mathbb{R}^3$ has been studied by Abels in [1]. In [40], Yeressian has studied the case of Newtonian fluids in $\mathbb{R}^3$. The weak-strong uniqueness of strong solutions and varifold solutions to the two-phase incompressible Navier-Stokes equations has been studied by Fischer and Hensel in [10].

3.1.2 Main results

Based on the settings in [15], we give the definition of strong solutions to the two-phase MHD equations.

Definition 3.1.1 (Strong solution)**.** Let $q \geq 1$ be a fixed number. Let $u_0 \in W^{2-\frac{2}{q},q}(\Omega \setminus \Gamma_0)$ and $B_0 \in W^{2-\frac{2}{q},q}(\Omega)$ with Γ_0 a $W^{3-\frac{2}{q},q}$ surface in Ω. We call (u, B, p, Γ) a strong solution to the two-phase MHD equations (3.1.1)-(3.1.8) on $[0, T]$ if:

1. $u \in W^{1,q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{2,q}(\Omega \setminus \Gamma(t))) \cap C([0,T] \times \Omega)$;
2. $B \in W^{1,q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{2,q}(\Omega))$;
3. $p \in L^q([0,T]; \dot{W}^{1,q}(\Omega \setminus \Gamma(t)))$;
4. $\Gamma(t)$ is the graph of a height function h on some C^3 reference surface Σ and $h \in W^{2-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap W^{1,q}([0,T]; W^{2-\frac{1}{q},q}(\Sigma)) \cap L^q([0,T]; W^{3-\frac{1}{q},q}(\Sigma))$ i.e. $\Gamma(t) = \{x + h(t,x)n(x) : x \in \Sigma\}$;
5. There exists a function $\tilde{p} \in W^{\frac{1}{2}-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{1-\frac{1}{q},q}(\Sigma))$ such that $[\![p]\!]\,(t, x + h(t,x)n_\Sigma(x)) = \tilde{p}(t,x)$ almost everywhere on $[0,T] \times \Sigma$;
6. For almost every $t \in [0,T]$, the equations (3.1.1)-(3.1.8) are satisfied almost everywhere on Ω or $\Gamma(t)$.

When the initial interface Γ_0 is smooth enough, we have the following result on existence.

Theorem 3.1.1. *Let $q > 5$ be a fixed number and Ω be a bounded C^3 domain. Given initial value*

$$u_0 \in W^{2-\frac{2}{q},q}(\Omega \setminus \Gamma_0) \cap C(\Omega), \quad B_0 \in W^{2-\frac{2}{q},q}(\Omega), \quad \text{and} \quad \Gamma_0 \in C^3,$$

which satisfies the following compatibility conditions:

1. $\mathrm{div} u_0 = 0$ *in* $\Omega \setminus \Gamma_0$, $\mathrm{div} B_0 = 0$ *in* Ω*;*
2. $u_0 = B_0 = 0$ *on* $\partial\Omega$*;*
3. Γ_0 *is a closed interface and* $\Gamma_0 \cap \partial\Omega = \emptyset$*.*
4. $(I - n_{\Gamma_0} \otimes n_{\Gamma_0}) \left[\!\left[\nu^{\pm} \left(\nabla u_0 + \nabla u_0^{\top}\right)\right]\!\right] n_{\Gamma_0} = 0$*;*

Then there exists $T > 0$ such that the original problem (3.1.1) - (3.1.8) *has a unique strong solution on $[0,T]$. The reference surface $\Sigma = \Gamma_0$.*

When the initial interface has less regularity, we have the following result.

Theorem 3.1.2. *Let $q > 5$ be a fixed number and Ω be a bounded C^3 domain. Let Σ be an arbitrary closed C^3 surface in Ω such that $\Sigma \cap \partial\Omega = \emptyset$. For all $M_0 > 0$, there exists $\varepsilon_0(\Sigma, M_0) > 0$, such that for all admissible initial value (u_0, B_0, Γ_0) in Definition*

3.1.1 with $\|u_0\|_{W^{2-\frac{2}{q},q}(\Omega\setminus\Gamma_0)} \le M_0$*,* $\|B_0\|_{W^{2-\frac{2}{q},q}(\Omega)} \le M_0$*,* $\|h_0\|_{W^{3-\frac{2}{q},q}(\Sigma)} \le M_0$*,* $\|u_0\|_\infty \le M_0$*,* $\|B_0\|_\infty \le M_0$*, and* $\|h_0\|_{C^2(\Sigma)} < \varepsilon_0$*, there exists* $T(\Sigma, M_0, \varepsilon_0) > 0$*, such that the problem* (3.1.1) - (3.1.8) *has a unique strong solution on* $[0,T]$*. Here* h_0 *denotes the height function of* Γ_0 *on the reference surface* Σ*.*

The main difficulty of this work comes from the coupling of fluid equations and magnetic equations, which changes the structure of both the principal part and the nonlinear part of the transformed two-phase Navier-Stokes equations in e.g. [15]. The new principal part is divided into two parts: the two-phase Stokes equations and the parabolic equations. Using the maximal regularity theory of these two problems, we obtain a solution operator for the principal part of the two-phase MHD equations. The remaining nonlinear terms and lower-order terms are carefully estimated. A contraction mapping is then constructed and the equation is solved by finding the fixed point of the contraction mapping.

We will organize this chapter as follows. In Section 3.2, we review some basic background knowledge. In Section 3.3, we use the Hanzawa transformation to transform the free-interface problem into a fixed-interface problem. The new equations will be separated into the linear (principal) part and the nonlinear part. In Section 3.4, we study the solvability of the linear part. Then we express the nonlinear part using an operator and estimate its Fréchet derivative in Section 3.5. Finally, we prove the main theorem in Section 3.6.

3.2 Preliminary

3.2.1 Notations

In complicated formulas, we use $[\,\cdot\,]$ to denote the values of variables, e.g. we use $f\,[g(x)]$ to express $(f \circ g)(x)$.

The gradient ∇f of a scalar function f is considered as a column vector by default. When f is a vector-valued function, the gradient of each entry is viewed as a column vector in the matrix by default, i.e. $(\nabla f)_{ij} := \partial_i f_j$. Notice that it then implies the formula

$$\nabla(f \circ g) = (\nabla g)\left((\nabla f) \circ g\right).$$

We denote the r-neighborhood of a point x by $B(x;r)$. For a set A, we define

$$B(A;r) := \bigcup_{x\in A} B(x;r).$$

For a function f, we define

$$f(A) := \bigcup_{x\in A} f(x).$$

Since the viscosity coefficient ν remains a constant in each fluid, we may use notations $\nu(\chi)$, $\nu^{\pm}$ or simply ν in formulas. We remind readers that ν is discontinuous at the interface.

To simplify statements, we write $a \lesssim b$ if $a \le Cb$ for some constant $C > 0$ which is independent of any parameter.

We will frequently use the symmetric gradient $\tilde{D}F := \left(\nabla F + \nabla F^{\top}\right)/2$ for vector-valued functions.

The projection matrix on a surface S is denoted by $\mathcal{P}_S := I - n_S \otimes n_S$, where n_S is the normal vector of S.

3.2.2 Function spaces

3.2.2.1 Continuous and differentiable functions

In this problem, we mainly consider two types of domains in $\mathbb{R}^3$: a bounded, open, 3-dimensional domain Ω; and a closed, 2-dimensional surface Σ. We say Σ is C^k if it can be locally parameterized using C^k functions. We say Ω is a C^k domain if its boundary $\partial\Omega$ is a C^k surface.

Let f be a function from $[0,T]$ to a Banach space X. For any $t_0 \in [0,T]$, we say f is continuous at t_0 if

$$\lim_{t\to t_0} \|f(t) - f(t_0)\|_X = 0.$$

If f is continuous on $[0,T]$ then we say $f \in C([0,T];X)$. We say $g(t_0) \in X$ is the derivative of f at t_0 if

$$\lim_{t\to t_0} \left\| \frac{f(t) - f(t_0)}{t - t_0} - g(t_0) \right\|_X = 0.$$

If f has a continuous derivative $g \in C([0,T];X)$, then we say $f \in C^1([0,T];X)$. Similarly, we can define the space $C^k([0,T];X)$ for $k \in \mathbb{Z}$, $k \geq 0$. We will frequently use the special case that $X = C^m(\Omega)$ and $X = C^m(\Sigma)$ for $m \in \mathbb{Z}$, $m \geq 0$.

Remark 3.2.1. For vector-valued or matrix-valued functions in spaces $C^k([0,T];C^m(\Omega))$ or $C^k([0,T];C^m(\Sigma))$, we define their norms by taking the $C^k([0,T];C^m)$ norm of each entry and then taking the vector norm or the matrix norm.

3.2.2.2 Lebesgue and Sobolev spaces

We will use $\|f\|_\infty$ to abbreviate $\|f\|_{L^\infty([0,T];L^\infty(\Omega))}$, $\|f\|_{L^\infty([0,T];L^\infty(\Sigma))}$, $\|f\|_{L^\infty(\Omega)}$ or $\|f\|_{L^\infty(\Sigma)}$, depending on context.

Let $\Omega \subseteq \mathbb{R}^n$. Given $s \in (0,1)$ and $q \in [1,\infty)$, we say a function f is in the fractional Sobolev space $W^{s,q}(\Omega)$ if

$$\|f\|_{W^{s,q}(\Omega)} := \|f\|_{L^q(\Omega)} + [f]_{W^{s,q}(\Omega)} < \infty,$$

where $[f]_{W^{s,q}(\Omega)}$ is the Gagliardo seminorm defined as

$$[f]_{W^{s,q}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}} dydx\right)^{\frac{1}{q}}.$$

For a Banach-space-valued function, i.e. $f : [0,T] \to X$ where X is a Banach space, the Gagliardo seminorm is defined as

$$[f]_{W^{s,q}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{\|f(x)-f(y)\|_X^q}{|x-y|^{n+sq}} dydx\right)^{\frac{1}{q}},$$

which enables us to define the space $W^{s,q}([0,T];X)$. When $s \in (0,+\infty) \setminus \mathbb{Z}$, the Sobolev norm is defined as

$$\|f\|_{W^{s,q}(\Omega)} := \|f\|_{W^{\lfloor s \rfloor,q}(\Omega)} + [f]_{W^{s-\lfloor s \rfloor,q}(\Omega)}.$$

For a compact hypersurface Σ in $\mathbb{R}^n$, the Sobolev space $W^{s,q}(\Sigma)$ can be defined similarly since Σ can be locally mapped to Euclidean spaces.

3.2.3 Calculus on surfaces

Let f be a function defined in an open domain $\Omega \subseteq \mathbb{R}^m$. Let Σ be a hypersurface in Ω whose normal vector field is denoted by n. For all $x \in \Sigma$, the surface gradient $\nabla_\Sigma f$ is defined as

$$\nabla_\Sigma f := \nabla f - (n \cdot \nabla f)n = (I - n \otimes n)\nabla f,$$

which is the projection of ∇f onto $T_x\Sigma$. If F is a vector-valued function, then we can define the surface divergence by

$$\mathrm{div}_\Sigma F := \mathrm{tr}(\nabla_\Sigma F).$$

Thus, we can also define the Laplace–Beltrami operator $\triangle_\Sigma$ by

$$\triangle_\Sigma f := \mathrm{div}_\Sigma \, \nabla_\Sigma f.$$

In fact, the surface derivatives only depend on the value of the function on Σ, which is discussed in e.g. [5, Remark 7.26].

Using the surface gradient $\nabla_\Sigma n_\Sigma$ of the normal vector field n_Σ, the Weingarten tensor of Σ is defined as $L_\Sigma := -\nabla_\Sigma n_\Sigma$, which is a matrix-valued function defined on Σ. For each $x \in \Sigma$, we have $L_\Sigma n_\Sigma = 0$; and L_Σ is an isomorphism on $T_x\Sigma$. The principal curvatures of Σ at x are eigenvalues of $L_\Sigma[x]$. The mean curvature $H_\Sigma[x]$ can be represented as $H_\Sigma[x] = \mathrm{tr} L_\Sigma[x]$.

For more details on the calculus on surfaces, we refer to [5, 19].

3.2.4 Nearest point projection

For the reader's convenience, we restate the theorem of nearest point projection in [34, Section 2.12.3] with some modification.

Theorem 3.2.1. *Let Σ be a compact, $(m-1)$-dimensional, C^k manifold in $\mathbb{R}^m$. There exists $\varrho_0(\Sigma) > 0$ and a C^{k-1} projection mapping $\Pi : B(\Sigma; \varrho_0) \to \Sigma$, such that for all $x \in B(\Sigma; \varrho_0)$:*

1. $x - \Pi(x) \perp T_{\Pi(x)}\Sigma$;
2. $\mathrm{dist}(x, \Sigma) = |x - \Pi(x)|$;
3. *for all $y \in \Sigma$ and $y \neq x$ we have* $\mathrm{dist}(x, \Sigma) < |x - y|$;
4. *for all $y \in \Sigma$ and $\lambda \in (0, \varrho_0)$, $\Pi(y + \lambda n(y)) = y$.*

3.2.5 Fréchet derivative

In this problem, we need to frequently study the derivatives of operators, which are called Fréchet derivatives. Given Banach spaces X and Y and an operator $F : X \to Y$. Suppose that for $x \in X$ there exists a linear operator $A : X \to Y$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(x+h) - F(x) - Ah\|_Y}{\|h\|_X} = 0, \tag{3.2.1}$$

then we say that F is Fréchet differentiable at x. The linear operator A is called the Fréchet derivative of F at x and is denoted by $DF(x)$.

Suppose that DF exists in an open neighborhood U of $x \in X$, then we have an operator $DF : U \to \mathcal{L}(X;Y)$ and the second derivative D^2F at x can be defined using the same way. Notice that $D^2F(x) \in \mathcal{L}(X;\mathcal{L}(X;Y))$. Similarly, we can obtain the n-th derivative D^nF. The space

$$\mathcal{L}(X;\mathcal{L}(X;\cdots\mathcal{L}(X;Y))) \tag{3.2.2}$$

is equal to the space of multilinear operators

$$\mathcal{L}^{(n)}(X \times \cdots \times X;Y), \tag{3.2.3}$$

which is usually abbreviated to $\mathcal{L}^{(n)}(X^n;Y)$.

The product rule and chain rule are still valid for Fréchet derivatives. Given $F : X \to Y_1$ and $G : X \to Y_2$. Suppose that the product is well-defined, i.e. there exists a bilinear mapping $Y_1 \times Y_2 \to Z$ which is called the "product". Then the value of $D(FG)$ at $x \in X$ is a linear mapping, i.e. $D(FG)\,[x] \in \mathcal{L}(X;Z)$, such that for each $h \in X$ we have

$$D(FG)\,[x]\,h = (DF\,[x]\,h)\,(G\,[x]) + (F\,[x])\,(DG\,[x]\,h)\,. \tag{3.2.4}$$

Given $F : X \to Y$ and $G : Y \to Z$, the Fréchet derivative of $G \circ F : X \to Z$ at an arbitrary $x \in X$ is a linear operator $D(G \circ F)[x] \in \mathcal{L}(X;Z)$. For all $h \in X$ we have

$$D(G \circ F)\,[x]\,h = DG\,[F(x)]\,(DF\,[x]\,h). \tag{3.2.5}$$

The proofs of these properties can be found in [41, Section 4.3].

3.3 Transformation of the Problem

In this section, we apply the Hanzawa transformation to the original equations (3.1.1) - (3.1.8). This allows us to turn the free-interface problem into a fixed-interface problem. We refer to [15, 29] for more details about this method.

3.3.1 Representation of the free interface

The key point of this part is to represent the free interface $\Gamma(t)$ using a fixed reference surface and a height function. This method is called normal parameterization.

By the assumption that Γ_0 is a $W^{3-\frac{2}{q}}$ surface and $q > 5$, we obtain that Γ_0 is a C^2 manifold. From [29, Section 2.3], we know that for all sufficiently small ε, there exists an analytic manifold Σ and a function $h_0 \in C^2(\Sigma)$, such that

$$\Gamma_0 = \{x + h_0(x)n_\Sigma(x) : x \in \Sigma\} \quad \text{and} \quad \|h_0\|_{C^2(\Sigma)} < \varepsilon. \tag{3.3.1}$$

By the theory of nearest point projection introduced in Section 3.2, there exists a tubular neighborhood $B(\Sigma; \varrho_0)$, such that the mapping

$$\Lambda(x, r) := x + rn_\Sigma(x) \tag{3.3.2}$$

is a diffeomorphism from $\Sigma \times (-\varrho_0, \varrho_0)$ to $B(\Sigma; \varrho_0)$. Its inverse mapping is

$$\Lambda^{-1}(x) = (\Pi(x), d(x)), \tag{3.3.3}$$

where $\Pi(x)$ is the projection of x onto Σ, and $d(x)$ is the signed distance between x and Σ, where the positive direction of Σ is defined to have the same direction as the exterior normal vector n_Σ.

Since there is a positive distance between Γ_0 and $\partial\Omega$, we may let ϱ_0 be sufficiently small such that $B(\Sigma; \varrho_0) \Subset \Omega$. By replacing r with a height function $h : \Sigma \to (-\varrho_0, \varrho_0)$, the mapping $\Lambda(x, h(x)) := x + h(x)n_\Sigma(x)$ defines a surface in $B(\Sigma; \varrho_0)$.

For every height function h, we can define a diffeomorphism in Ω using the same idea as in [15]. We define

$$\Theta_h(x) := \begin{cases} x + \eta(d(x)/\varrho_0)h(\Pi(x))n_\Sigma(\Pi(x)), & x \in B(\Sigma;\varrho_0), \\ x, & x \notin B(\Sigma;\varrho_0). \end{cases} \tag{3.3.4}$$

where η is a smooth cut-off function on $\mathbb{R}$ such that $0 \leq \eta \leq 1$; $\eta(s) = 1$ when $|s| < 1/3$, and $\eta(s) = 0$ when $|s| > 2/3$. Notice that the cutoff function can be ignored in the set

$$B(\Sigma;\varrho) \quad \text{with} \quad \varrho := \frac{\varrho_0}{3}. \tag{3.3.5}$$

For convenience, we define the displacement of the point x under the diffeomorphism by

$$\theta_h(x) := \Theta_h(x) - x. \tag{3.3.6}$$

Remark 3.3.1. In order to make Θ_h a bijection, we need to guarantee that $\|h\|_{C^0(\Sigma)}$ is sufficiently small. As an example, we consider a mapping in $\mathbb{R}^2$. Suppose that Σ is the x-axis and $h(x) = b > 0$, then $\Theta(x,y) = (x, y + b\eta(y/\varrho_0))$. To make Θ a bijection, it is necessary that the function $f(y) := y + b\eta(y/\varrho_0)$ should be a bijection, which requires b, i.e. $|h|$, to be sufficiently small.

Remark 3.3.2. The derivative of the distance function is

$$\nabla d(x) = n(\Pi(x)). \tag{3.3.7}$$

To calculate the derivative of the projection mapping Π, we take the derivative of $x - \Pi(x) = d(x)n(\Pi(x))$, which implies

$$\begin{aligned} I - \nabla\Pi(x) &= d(x)\nabla_x \left(n\left[\Pi(x)\right]\right) + \nabla_x d\left[x\right] \otimes n\left[\Pi(x)\right] \\ &= d(x)\nabla\Pi\left[x\right]\left(\nabla_\Sigma n\right)\left[\Pi(x)\right] + n\left[\Pi(x)\right] \otimes n\left[\Pi(x)\right]. \end{aligned} \tag{3.3.8}$$

Recall that $L_\Sigma := -\nabla_\Sigma n$, so we have

$$I - n\left[\Pi(x)\right] \otimes n\left[\Pi(x)\right] = \nabla\Pi(x)\left(I - d(x)L_\Sigma\left[\Pi(x)\right]\right), \tag{3.3.9}$$

which implies

$$\nabla\Pi(x) = \mathcal{P}_\Sigma\left[\Pi(x)\right]\left(I - d(x)L_\Sigma\left[\Pi(x)\right]\right)^{-1}. \tag{3.3.10}$$

We refer to [29, Section 2.3] or [34, Section 2.12.3] for more details on the nearest point projection.

3.3.2 Regular terms

Using the diffeomorphism Θ_h, the original problem with interface $\Gamma(t)$ can be transformed into equations with interface Σ. The interface $\Gamma(t)$ can be determined as long as the function $h(t,x) : [0,T] \times \Sigma$ can be obtained. In order to do this, we need to change the variables in (3.1.1)-(3.1.8) with the help of Θ_h. We shall use similar arguments as in [15] with more details included for completeness.

We first consider (3.1.1), which is equivalent to

$$\left(\partial_t u + (u\cdot\nabla)u - (B\cdot\nabla)B + \frac{1}{2}\nabla(|B|^2) + \nabla p - \nu(\chi)\triangle u\right)\circ\Theta_h = 0 \tag{3.3.11}$$

since Θ_h is a bijection. We use $\Theta_h^{-1}(t,x)$ to denote the inverse mapping of $\Theta_{h(t)}(\cdot)$ at time t, which implies $\Theta_h^{-1}(t,\Theta_h(t,x)) = x$. Given any function f in Ω, we define

$$\overline{f}(t,x) := f(t,\Theta_h(t,x)), \tag{3.3.12}$$

where f can be u, B, p, χ, etc. In this chapter, we will ignore the subscript h in Θ_h when there is no confusion. For any fixed t, by the definition of Θ we have $\Theta_h^{-1}(t,\cdot) = (\Theta_{h(t,\cdot)})^{-1}$, which implies that

$$f(t,x) = \overline{f}(t,\Theta_h^{-1}(t,x)). \tag{3.3.13}$$

As opposed to partial derivatives with respect to specific variables, we will use ∂_0 temporarily to denote the partial derivative with respect to the position of the time variable. This helps distinguish $\partial_0 u(t,\Theta(t,x))$ and $\partial_t u(t,\Theta(t,x))$.

Using similar arguments as in [15], we write the transformed equations in terms of the new variables. More details can be found in [29, Section 1.3]. For the time derivative, we have

$$\partial_t u(t,x) = \partial_t(\overline{u}(t,\Theta^{-1}(t,x))) = (\partial_0)\overline{u}\circ\Theta^{-1} + \sum_i((\partial_i\overline{u})\circ\Theta^{-1})\partial_t\Theta_i^{-1}, \tag{3.3.14}$$

which implies

$$\partial_t u\circ\Theta = \partial_0\overline{u} + (\partial_t\Theta^{-1}\circ\Theta)\nabla\overline{u} = \partial_t\overline{u} + (\partial_t\Theta^{-1}\circ\Theta)\nabla\overline{u}. \tag{3.3.15}$$

Next, we calculate the formula of each entry of $\nabla \overline{u}$, which is

$$\begin{aligned}\partial_\alpha u_\beta(t,x) = \partial_{x_\alpha} u_\beta(t,x) = \partial_{x_\alpha} \overline{u}_\beta(t, \Theta^{-1}(t,x)) = \sum_i ((\partial_i \overline{u}_\beta) \circ \Theta^{-1}) \partial_{x_\alpha} \Theta_i^{-1} \\ = \sum_i ((\partial_i \overline{u}_\beta) \circ \Theta^{-1})(((\nabla\Theta)^{-1})_{\alpha i} \circ \Theta^{-1}).\end{aligned} \tag{3.3.16}$$

Taking the time derivative of the equation $\Theta(t, \Theta^{-1}(t,x)) = x$, we obtain

$$(\partial_0 \Theta) \circ \Theta^{-1} + \partial_t \Theta^{-1} \left((\nabla\Theta) \circ \Theta^{-1} \right) = 0. \tag{3.3.17}$$

Composing (3.3.17) with Θ, we obtain

$$\left(\partial_t \Theta^{-1} \right) \circ \Theta = -\partial_t \Theta \ \left(\nabla\Theta \right)^{-1}. \tag{3.3.18}$$

From (3.3.18) and the equality $(\nabla\Theta)^{-1} = I - (I + \nabla\theta)^{-1} \nabla\theta$, we obtain

$$\left(\partial_t \Theta^{-1} \right) \circ \Theta = -\partial_t \Theta (\nabla\Theta)^{-1} = -\partial_t \theta \left(I - (I + \nabla\theta)^{-1} \nabla\theta \right). \tag{3.3.19}$$

Similarly, we obtain from (3.3.16) that

$$(\nabla u) \circ \Theta = (\nabla\Theta)^{-1} \nabla \overline{u} = \nabla \overline{u} - \left((I + \nabla\theta)^{-1} \nabla\theta \right) \nabla \overline{u}, \tag{3.3.20}$$

$$(\mathrm{div} u) \circ \Theta = \left((\nabla\Theta)^{-1} \right)^\top : \nabla \overline{u} = \mathrm{div} \overline{u} - \left((I + \nabla\theta)^{-1} \nabla\theta \right) : \nabla \overline{u}. \tag{3.3.21}$$

Taking one more derivative on (3.3.16), we obtain the Laplacian of each entry of u:

$$\begin{aligned}\triangle u_\beta &= \sum_\alpha \partial_{x_\alpha} \partial_{x_\alpha} \overline{u}_\beta(t, \Theta^{-1}(t,x)) = \sum_\alpha \partial_{x_\alpha} \left(\sum_i ((\partial_i \overline{u}_\beta) \circ \Theta^{-1}) \partial_{x_\alpha} \Theta_i^{-1} \right) \\ &= \sum_\alpha \sum_i \partial_{x_\alpha} \left((\partial_i \overline{u}_\beta) \circ \Theta^{-1} \right) \partial_{x_\alpha} \Theta_i^{-1} + \sum_\alpha \sum_i \left((\partial_i \overline{u}_\beta) \circ \Theta^{-1} \right) \partial_{x_\alpha x_\alpha} \Theta_i^{-1} \\ &= \sum_\alpha \sum_i \sum_j \left((\partial_{ji} \overline{u}_\beta) \circ \Theta^{-1} \right) \partial_{x_\alpha} \Theta_i^{-1} \partial_{x_\alpha} \Theta_j^{-1} + \sum_i \triangle \left(\Theta_i^{-1} \right) \left((\partial_i \overline{u}_\beta) \circ \Theta^{-1} \right) \\ &= \left(\left(\nabla\Theta^{-1} \right)^\top \left(\nabla\Theta^{-1} \right) \right) : \left(\left(\nabla^2 \overline{u}_\beta \right) \circ \Theta^{-1} \right) + \triangle\Theta^{-1} \cdot \left((\nabla \overline{u}_\beta) \circ \Theta^{-1} \right).\end{aligned} \tag{3.3.22}$$

The subscript in (3.3.22) can be removed to obtain the vector equality

$$(\triangle u) \circ \Theta = \left(\left(\left(\nabla\Theta^{-1} \right)^\top \left(\nabla\Theta^{-1} \right) \right) \circ \Theta \right) : \nabla^2 \overline{u} + \left(\left(\triangle\Theta^{-1} \right) \circ \Theta \right) \cdot (\nabla \overline{u}). \tag{3.3.23}$$

In the first term of the right-hand side of (3.3.23), using the formula of inverse functions, we have

$$\Big(\left(\nabla\Theta^{-1}\right)^{\top}\left(\nabla\Theta^{-1}\right)\Big)\circ\Theta=(\nabla\Theta)^{-\top}\,(\nabla\Theta)^{-1}\,, \tag{3.3.24}$$

which implies that

$$\begin{aligned}(\triangle u)\circ\Theta&=\Big((\nabla\Theta)^{-\top}\,(\nabla\Theta)^{-1}\Big):\nabla^2\overline{u}+\left(\left(\triangle\Theta^{-1}\right)\circ\Theta\right)\cdot(\nabla\overline{u})\\&=\triangle\overline{u}+\Big((\nabla\Theta)^{-\top}\,(\nabla\Theta)^{-1}-I\Big):\nabla^2\overline{u}+\left(\left(\triangle\Theta^{-1}\right)\circ\Theta\right)\cdot(\nabla\overline{u}).\end{aligned} \tag{3.3.25}$$

Using exactly the same argument for u, we can transform the terms which contain B and p. Using also (3.3.18), we can rewrite (3.3.11) as

$$\begin{aligned}&\partial_t\overline{u}-\partial_t\Theta\;(\nabla\Theta)^{-1}\,\nabla\overline{u}+\overline{u}\left((\nabla\Theta)^{-1}\right)\nabla\overline{u}-\overline{B}\left((\nabla\Theta)^{-1}\right)\nabla\overline{B}\\&+\frac{1}{2}\left((\nabla\Theta)^{-1}\right)\nabla\left(\left|\overline{B}\right|^2\right)+\left((\nabla\Theta)^{-1}\right)\nabla\overline{p}-\nu(\overline{\chi})\left((\nabla\Theta)^{-\top}(\nabla\Theta)^{-1}\right):\nabla^2\overline{u}\\&-\nu(\overline{\chi})\left(\left(\triangle\Theta^{-1}\right)\circ\Theta\right)\cdot(\nabla\overline{u})=0.\end{aligned} \tag{3.3.26}$$

Using the fact that $\nabla\times(u\times B)=-(u\cdot\nabla)B+(B\cdot\nabla)u$ and $\nabla\times(\nabla\times B)=\nabla(\mathrm{div}B)-\triangle B$, we rewrite (3.1.2) as

$$(\partial_t B+(u\cdot\nabla)B-(B\cdot\nabla)u-\sigma\triangle B)\circ\Theta=0. \tag{3.3.27}$$

Using the same arguments as in (3.3.15) , (3.3.16) and (3.3.25), we can obtain the representation of $\partial_t B$, ∇B and $\triangle B$, which imply

$$\begin{aligned}&\partial_t\overline{B}-\partial_t\Theta\,(\nabla\Theta)^{-1}\nabla\overline{B}+\overline{u}(\nabla\Theta)^{-1}\nabla\overline{B}-\overline{B}(\nabla\Theta)^{-1}\nabla\overline{u}\\&-\sigma\left((\nabla\Theta)^{-\top}(\nabla\Theta)^{-1}\right):\nabla^2\overline{B}-\sigma\left(\left(\triangle\Theta^{-1}\right)\circ\Theta\right)\cdot(\nabla\overline{B})=0.\end{aligned} \tag{3.3.28}$$

The divergence-free conditions (3.1.3) and (3.1.4) can be treated in the same way using (3.3.21), which implies

$$0=\mathrm{div}u=\mathrm{div}\overline{u}-\left((I+\nabla\theta)^{-1}\nabla\theta\right):\nabla\overline{u}, \tag{3.3.29}$$

$$0=\mathrm{div}B=\mathrm{div}\overline{B}-\left((I+\nabla\theta)^{-1}\nabla\theta\right):\nabla\overline{B}. \tag{3.3.30}$$

We will show in Section 3.3.4 that (3.3.30) can actually be ignored in the transformed problem.

3.3.3 Geometric terms

In the previous section, we have obtained the transformation of (3.1.1) to (3.1.4). It remains to transform (3.1.5) and (3.1.6), which require representations of tangent vectors, normal vectors and curvatures. To treat these terms, we follow the arguments in e.g. [15,29]. We include some details for convenience. In this section, we will temporarily do calculations in $\mathbb{R}^d$, in order to maintain a clear structure.

3.3.3.1 Tangent and normal vectors

Suppose that Σ is a C^k surface, then it can be locally parameterized by a C^k function Φ, i.e. for all $x \in \Sigma$ there exists a neighborhood $B(x;a) \cap \Sigma$ such that there exists a domain $D \subseteq \mathbb{R}^{d-1}$ and a diffeomorphism $\Phi(s)$ from D to $B(x;a) \cap \Sigma$, where $s = (s_1, \cdots, s_{d-1}) \in D$. Let $x = \Phi(s) \in \Sigma$, then the tangent vectors at x are

$$\tau_i^\Sigma(s) = \partial_i \Phi(s), \quad i = 1, \cdots d-1, \tag{3.3.31}$$

which form a basis of the tangent space $T_x\Sigma$. We will also use notations $\tau^\Sigma_{(i),k}$ and $\tau^{(i)}_{\Sigma,k}$ to denote the k-th entry of the i-th vector. These $d-1$ vectors depend on the choice of Φ. Thus, we directly view τ_i^Σ as a function defined in D. The normal vector n_Σ is independent of Φ. Thus, it can be viewed as a function defined on Σ.

To simplify calculations, we introduce another basis $\{\tau_\Sigma^1, \cdots, \tau_\Sigma^{d-1}\}$ of the tangent plane $T_x\Sigma$. The new basis satisfies $\tau_\Sigma^i \cdot \tau_j^\Sigma = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We ignore the name of surfaces in superscripts or subscripts when there is no ambiguity. Suppose $\xi = \sum c_i \tau^i = \sum c^i \tau_i$, then we have $c_i = \xi \cdot \tau_i$ and $c^i = \xi \cdot \tau^i$. We refer readers to [29] for more details.

For every height function h, its corresponding surface is $\Gamma_h(t) := \Theta_h(t, \Sigma)$, which can be parameterized using $\Theta_h \circ \Phi$. Its tangent vectors at $y := \Theta(x) = \Theta(\Phi(s))$ are

$$\tau_i^\Gamma(s) = \partial_{s_i} \left(\Theta(\Phi(s))\right) = \sum_j \left(\partial_i \Phi_j\right)\left(\partial_j \Theta \circ \Phi\right). \tag{3.3.32}$$

To find the normal vector n_Γ, we first seek for $\alpha \in T_x\Sigma$ such that $n_\Sigma - \alpha$ is perpendicular to $T_{\Theta(x)}\Gamma$. We refer readers to [29, Section 2.2.2] for more details. For convenience, we include some key steps in the derivation of α. When $x \in B(\Sigma; \varrho)$ with $\varrho = \varrho/3$, we have $\Theta_h(x) = x + h(x)n_\Sigma(x)$, which implies

$$\Theta_h(\Phi(s)) = \Phi(s) + h(\Phi(s))n_\Sigma(\Phi(s)). \tag{3.3.33}$$

Taking derivatives of the equation (3.3.33), we have

$$\begin{aligned} \tau_i^\Gamma(s) &= \partial_{s_i}\Theta(\Phi(s)) = \partial_{s_i}\left(\Phi(s) + \theta(\Phi(s))\right) \\ &= \partial_i\Phi(s) + \partial_{s_i}(h \circ \Phi)\,[s]\, n_\Sigma\,[\Phi\,[s]] + h\,[\Phi\,[s]]\,\partial_{s_i}(n_\Sigma \circ \Phi)\,[s]\,. \end{aligned} \tag{3.3.34}$$

In this work, we let the parameterization $\Phi : D \to \Sigma$ be fixed once it has been chosen. To make calculations concise, for any function f on Σ we will abbreviate $f \circ \Phi$ to f when there is no confusion. We will also use the notation $\partial_i f$ to represent the derivative $\partial_{s_i}(f(\Phi(s))$ when there is no confusion. This follows the convention in [29].

Now we simplify (3.3.34). Since $|n_\Sigma| \equiv 1$, we have

$$2\partial_{s_i}n_\Sigma(\Phi(s)) \cdot n_\Sigma(\Phi(s)) = \partial_{s_i}(|n_\Sigma(\Phi(s))|^2) = 0. \tag{3.3.35}$$

In fact, the vectors $\partial_{s_1}n, \cdots, \partial_{s_{d-1}}n$ form a new basis of $T_{\Phi(s)}\Sigma$. Using the Weingarten tensor L_Σ, which satisfies

$$L_\Sigma\,[\Phi(s)]\,\tau_i^\Sigma\,[s] := -\partial_{s_i}(n_\Sigma \circ \Phi)\,[s]\,, \tag{3.3.36}$$

we can simplify (3.3.34) to

$$\tau_i^\Gamma = (I - hL_\Sigma)\,\tau_i^\Sigma + \partial_i h\, n_\Sigma. \tag{3.3.37}$$

Next, we simplify the formula of α using (3.3.37). Since $n_\Sigma - \alpha$ is required to be perpendicular to $T_{\Theta(\Phi(s))}\Gamma$, we have $(n_\Sigma - \alpha) \perp \tau_i^\Gamma$ for all $1 \le i \le d-1$, which implies

$$\begin{aligned} 0 &= (n_\Sigma - \alpha) \cdot \tau_i^\Gamma = (n_\Sigma - \alpha) \cdot \left(\tau_i^\Sigma + \partial_i h\, n_\Sigma + h\partial_i\, n_\Sigma\right) \\ &= n_\Sigma \cdot \tau_i^\Sigma + n_\Sigma \cdot (\partial_i h\, n_\Sigma) + n_\Sigma \cdot (h\partial_i\, n_\Sigma) - \alpha \cdot \tau_i^\Sigma - \alpha \cdot (\partial_i h\, n_\Sigma) - \alpha \cdot (h\partial_i\, n_\Sigma) \\ &= 0 + \partial_i h + 0 - \alpha \cdot \tau_i^\Sigma - 0 - \alpha \cdot (h\,\partial_i n_\Sigma) = \partial_i h - \alpha \cdot \left((I - hL_\Sigma)\,\tau_i^\Sigma\right). \end{aligned} \tag{3.3.38}$$

Since $I - hL_\Sigma$ is a symmetric linear transformation (see e.g. Section 2.2 in [29]), we have

$$\partial_i h = ((I - hL_\Sigma)\,\alpha) \cdot \tau_i^\Sigma. \tag{3.3.39}$$

Notice that (3.3.39) is the abbreviation of:

$$\partial_{s_i}(h(\Phi(s))) = ((I - hL_\Sigma)\,\alpha) \cdot \partial_i \Phi(s), \tag{3.3.40}$$

which can be solved by letting

$$(I - hL_\Sigma)\,\alpha = \nabla_\Sigma h, \quad \text{i.e.} \quad \alpha = (I - hL_\Sigma)^{-1} \nabla_\Sigma h. \tag{3.3.41}$$

The surface gradient ∇_Σ, also called the tangential gradient, is explained in Section 3.2. We refer interested readers to [5, 19] for details about gradient, divergence and other differential operators on surfaces. Now we have obtained that

$$n_\Gamma(s) = \frac{n_\Sigma - (I - hL_\Sigma)^{-1} \nabla_\Sigma h}{\left|n_\Sigma - (I - hL_\Sigma)^{-1} \nabla_\Sigma h\right|} =: \left(n_\Sigma - (I - hL_\Sigma)^{-1} \nabla_\Sigma h\right) \beta, \tag{3.3.42}$$

where $\beta(s)$ is a scalar function that renormalizes the vector, as used in [29].

3.3.3.2 Mean curvature

In this section, we express the mean curvature H_Γ in terms of Σ and h. We recall that the calculations are in the space $\mathbb{R}^d$. We start with the representation of $\nabla_\Gamma f$ for an arbitrary function f defined on Γ. From equation (2.47) in [29, Section 2.2.3], we have the representation of tangent vectors on Γ:

$$\tau_\Gamma^i = \mathcal{P}_\Gamma (I - hL_\Sigma)^{-1} \tau_\Sigma^i\,, \quad 1 \le i \le d-1. \tag{3.3.43}$$

Notice that for any function f on Γ (parameterized by $\Theta_h \circ \Phi$) we have

$$\left(\sum_{j=1}^{d-1} \partial_j f \tau_\Gamma^j\right) \tau_i^\Gamma = \partial_i f, \tag{3.3.44}$$

which implies

$$\nabla_\Gamma f = \sum_{j=1}^{d-1} \partial_j f \tau_\Gamma^j = \sum_{j=1}^{d-1} \mathcal{P}_\Gamma (I - hL_\Sigma)^{-1} \tau_\Sigma^j \partial_j f = \mathcal{P}_\Gamma (I - hL_\Sigma)^{-1} \nabla_\Sigma (f \circ \Theta). \tag{3.3.45}$$

From the definition of mean curvature in Section 2.2.5 of [29], we have

$$H_\Gamma := -\mathrm{div}_\Gamma n_\Gamma = -\sum_{i=1}^{d-1} \tau_\Gamma^i \cdot \partial_i n_\Gamma. \tag{3.3.46}$$

We also refer to Section 7.3 in [5] for more details. Now it remains to calculate $\partial_i n_\Gamma$ in (3.3.46). From (3.3.42) we have

$$\partial_i n_\Gamma(s) = \partial_i \beta \, \left(n_\Sigma - (I - hL_\Sigma)^{-1} \nabla_\Sigma h\right) + \beta \, \partial_i \left(n_\Sigma - (I - hL_\Sigma)^{-1} \nabla_\Sigma h\right). \tag{3.3.47}$$

Using the same argument as in [29, Section 2.2.5], we obtain the formula of the mean curvature

$$\begin{aligned}
H_\Gamma &= -\sum_{i=1}^{d-1} \tau_\Gamma^i \cdot \partial_i n_\Gamma \\
&= -\sum_{i=1}^{d-1} \tau_\Gamma^i \cdot \left(\partial_i \beta \left(\frac{n_\Gamma}{\beta}\right)\right) - \sum_{i=1}^{d-1} \tau_\Gamma^i \cdot \left(\beta \partial_i \left(n_\Sigma - (I - hL_\Sigma)^{-1} \nabla_\Sigma h\right)\right) \\
&= 0 + \sum_{i=1}^{d-1} \left(\mathcal{P}_\Gamma \left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^i\right) \cdot \left(\beta \left(L_\Sigma \tau_i^\Sigma + \partial_i \alpha\right)\right) \\
&= \sum_{i=1}^{d-1} \left(\left(I - n_\Gamma \otimes n_\Gamma\right) \left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^i\right) \cdot \left(\beta \left(L_\Sigma \tau_i + \partial_i \alpha\right)\right) \\
&= \sum_{i=1}^{d-1} \left(\left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^i\right) \cdot \left(\beta \left(L_\Sigma \tau_i^\Sigma + \partial_i \alpha\right)\right) \\
&\quad - \sum_{i=1}^{d-1} \left(n_\Gamma \cdot \left(\left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^i\right)\right) \left(\beta n_\Gamma \cdot \left(L_\Sigma \tau_i^\Sigma + \partial_i \alpha\right)\right).
\end{aligned} \tag{3.3.48}$$

Still from [29, Section 2.2.5], we have the following equalities:

$$\sum_{i=1}^{d-1} \left(\left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^i\right) \cdot \left(L_\Sigma \tau_i^\Sigma\right) = \mathrm{tr}\left(\left(\left(I - hL_\Sigma\right)^{-1} L_\Sigma\right)\right), \tag{3.3.49}$$

$$\sum_{i=1}^{d-1} \left(\left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^i\right) \cdot \partial_i \alpha = \mathrm{tr}\left(\left(I - hL_\Sigma\right)^{-1} \nabla_\Sigma \alpha\right), \tag{3.3.50}$$

$$n_\Gamma \cdot \left(\left(I - hL_\Sigma\right)^{-1} \tau_\Sigma^j\right) = -\beta \left(\left(\left(I - hL_\Sigma\right)^{-1} \alpha\right)\right)^j, \tag{3.3.51}$$

where $1 \le j \le d-1$.

Remark 3.3.3. Since $\tau_i^\Sigma \cdot \tau_\Sigma^j = \delta_{ij}$, we combine them with the normal vector n_Σ and obtain (the notation Σ is ignored for convenience):

$$\begin{pmatrix} \tau_{(1),1} & \cdots & \tau_{(1),d} \\ \vdots & \ddots & \vdots \\ \tau_{(d-1),1} & \cdots & \tau_{(d-1),d} \\ n_1 & \cdots & n_d \end{pmatrix} \begin{pmatrix} \tau_1^{(1)} & \cdots & \tau_1^{(d-1)} & n_1 \\ \vdots & \ddots & \vdots & \vdots \\ \tau_d^{(1)} & \cdots & \tau_d^{(d-1)} & n_d \end{pmatrix} = I_{d\times d}. \tag{3.3.52}$$

The left-hand side is commutative by the property of inverse matrices, which implies

$$\sum_{k=1}^{d-1} \tau_i^{(k)} \tau_{(k),j} = \delta_{ij} - n_i n_j = (I - n \otimes n)_{ij} \tag{3.3.53}$$

for all $1 \le i, j \le d-1$. We recall that for any vector $\eta \in \mathbb{R}^d$ the matrix $\nabla_\Sigma \eta$ is defined as $(\nabla_\Sigma \eta_1, \cdots \nabla_\Sigma \eta_d)$, where each $\nabla_\Sigma \eta_i$ is viewed as a column vector. This implies that

$$\begin{aligned} \sum_{i=1}^{d-1} \tau^{(i)} \cdot \left(\nabla_\Sigma \eta \tau_{(i)}\right) &= \sum_{i=1}^{d-1} \sum_{k=1}^{d} \sum_{j=1}^{d} \tau_k^{(i)} \left(\nabla_\Sigma \eta\right)_{kj} \tau_{(i),j} \\ = \sum_{k=1}^{d} \sum_{j=1}^{d} \left(\sum_{i=1}^{d-1} \tau_k^{(i)} \tau_{(i),j} \right) \left(\nabla_\Sigma \eta\right)_{kj} &= (I - n \otimes n) : \nabla_\Sigma \eta \\ &= \mathrm{tr} \nabla_\Sigma \eta - n^\top \nabla_\Sigma \eta \, n = \mathrm{tr} \nabla_\Sigma \eta, \end{aligned} \tag{3.3.54}$$

which can be utilized in the derivation of (3.3.48), (3.3.49) and (3.3.50).

Notice that the symmetric operator $(I - hL_\Sigma)^{-1}$ in (3.3.51) maps tangent vectors to tangent vectors and maps n_Σ to n_Σ. Thus, for all $1 \le i \le d-1$ we have

$$\begin{aligned} & n_\Gamma \cdot \left((I - hL_\Sigma)^{-1} \tau_\Sigma^i \right) = \beta \left(n_\Sigma - \alpha \right) \cdot \left((I - hL_\Sigma)^{-1} \tau_\Sigma^i \right) = -\beta \alpha \cdot \left((I - hL_\Sigma)^{-1} \tau_\Sigma^i \right) \\ & = -\beta \left((I - hL_\Sigma)^{-1} \alpha \right) \cdot \tau_\Sigma^i =: -\beta \left((I - hL_\Sigma)^{-1} \alpha \right)^i, \end{aligned} \tag{3.3.55}$$

and

$$\begin{aligned} & n_\Gamma \cdot \left(L_\Sigma \tau_i + \partial_i \alpha \right) = \beta \left(n_\Sigma - \alpha \right) \cdot \left(L_\Sigma \tau_i + \partial_i \alpha \right) \\ & = \beta \left(n_\Sigma \cdot L_\Sigma \tau_i + n_\Sigma \cdot \partial_i \alpha - \alpha \cdot L_\Sigma \tau_i - \alpha \cdot \partial_i \alpha \right) \\ & = \beta \left(0 + n_\Sigma \cdot \partial_i \alpha + \alpha \cdot \partial_i n_\Sigma - \alpha \cdot \partial_i \alpha \right) = \beta \left(\partial_i \left(n_\Sigma \cdot \alpha \right) - \alpha \cdot \partial_i \alpha \right) = -\beta \alpha \cdot \partial_i \alpha. \end{aligned} \tag{3.3.56}$$

Notice that $n_\Sigma \cdot \alpha = 0$ since α is a tangent vector of Σ. From (3.3.48), (3.3.55) and (3.3.56), we have

$$\begin{aligned} H_\Gamma &= \beta \mathrm{tr}\left(\left(I - hL_\Sigma\right)^{-1} \left(L_\Sigma + \nabla_\Sigma \alpha\right)\right) - \sum_{i=1}^{d-1} \beta^3 \left(\left(I - hL_\Sigma\right)^{-1} \alpha\right)^i \left(\alpha \cdot \partial_i \alpha\right) \\ &= \beta \mathrm{tr}\left(\left(I - hL_\Sigma\right)^{-1} \left(L_\Sigma + \nabla_\Sigma \alpha\right)\right) - \beta^3 \left(\left(I - hL_\Sigma\right)^{-1} \alpha\right) \nabla_\Sigma \alpha \, \alpha^\top. \end{aligned} \tag{3.3.57}$$

Remark 3.3.4. For convenience, we let $\xi := \left(I - hL_\Sigma\right)^{-1} \alpha$ and ignore the notation Σ in tangent vectors. The term $\nabla_\Sigma \alpha \, \alpha^\top$ in (3.3.57) is obtained by the following calculation.

$$\begin{aligned} &\sum_{i=1}^{d-1} \left(\left(I - hL_\Sigma\right)^{-1} \alpha\right)^i \left(\alpha \cdot \partial_i \alpha\right) = \sum_{i=1}^{d-1} \xi^i \left(\alpha \cdot \partial_i \alpha\right) = \sum_{i=1}^{d-1} \left(\xi \cdot \tau^{(i)}\right) \left(\alpha \cdot \partial_i \alpha\right) \\ &= \sum_{i=1}^{d-1} \left(\sum_{k=1}^{d} \xi_k \tau_k^{(i)} \right) \left(\alpha \cdot \partial_i \alpha\right). \end{aligned} \tag{3.3.58}$$

We remind the readers that $\partial_i \alpha$ is the abbreviation of $\partial_i(\alpha \circ \Phi)$ by our convention. We view the surface gradient of vector α as the matrix such that

$$\left(\nabla_\Sigma \alpha\right)_{ij} := \left(\nabla_\Sigma \alpha_j\right)_i. \tag{3.3.59}$$

Notice that we assume all vectors to be written as row vectors, then for all $1 \leq k \leq d-1$ we have

$$\begin{aligned} \partial_k \alpha &= \left(\partial_k \alpha_1, \cdots, \partial_k \alpha_d\right) \\ &= \left(\nabla_\Sigma \alpha_1 \cdot \tau_k, \cdots, \nabla_\Sigma \alpha_d \cdot \tau_k\right) = \tau_k \nabla_\Sigma \alpha \end{aligned} \tag{3.3.60}$$

Thus, we rewrite the last term in (3.3.58) as

$$\begin{aligned} &\sum_{i=1}^{d-1} \left(\sum_{k=1}^{d} \xi_k \tau_k^{(i)} \right) \left(\alpha \cdot \partial_i \alpha\right) = \sum_{i=1}^{d-1} \left(\sum_{k=1}^{d} \xi_k \tau_k^{(i)} \right) \left(\sum_{j=1}^{d} \alpha_j \left(\tau_{(i)} \cdot \left(\nabla_\Sigma \alpha_j\right)\right)\right) \\ &= \sum_{i=1}^{d-1} \left(\sum_{k=1}^{d} \xi_k \tau_k^{(i)} \right) \left(\sum_{j=1}^{d} \alpha_j \left(\sum_{s=1}^{d} \tau_{(i),s} \left(\nabla_\Sigma \alpha_j\right)_s \right)\right) \\ &= \sum_{k=1}^{d} \sum_{j=1}^{d} \sum_{s=1}^{d} \sum_{i=1}^{d-1} \tau_k^{(i)} \tau_{(i),s} \xi_k \alpha_j \left(\nabla_\Sigma \alpha_j\right)_s = \sum_{k=1}^{d} \sum_{j=1}^{d} \xi_k \alpha_j \left(\nabla_\Sigma \alpha_j\right)_k \\ &= \xi \nabla_\Sigma \alpha \alpha^\top, \end{aligned} \tag{3.3.61}$$

where the fourth equality is guaranteed by (3.3.53).

3.3.3.3 Transformation of equations

Using the formulas of n_Γ and H_Γ in (3.3.42) and (3.3.57), we are able to transform equations (3.1.5) and (3.1.6). Composing Θ with (3.1.5), we have

$$-\left(\left[\!\left[\nu(\chi)\left(\nabla u+\nabla u^\top\right)-pI\right]\!\right] n_\Gamma\right)\circ\Theta=(\kappa H_\Gamma n_\Gamma)\circ\Theta. \tag{3.3.62}$$

Due to the effect of Θ, the equation (3.3.62) is defined on Σ rather than $\Gamma(t)$. We calculate the projections of (3.3.62) to $n_\Sigma(x)$ and $T_x\Sigma$ respectively using the same arguments as in [15, Section 2] with some details included for convenience. We recall that $n_\Gamma\cdot n_\Sigma=\beta$. Taking the inner product of (3.3.62) and n_Σ/β, we obtain the projection of the equation onto the normal vector

$$-\left(\left[\!\left[\left(\nu(\chi)\left(\nabla u+\nabla u^\top\right)\circ\Theta\right)\right]\!\right]\left(n_\Sigma-\left(I-hL_\Sigma\right)^{-1}\nabla_\Sigma h\right)\right)\cdot n_\Sigma+[\![p]\!]=\kappa H_\Gamma. \tag{3.3.63}$$

From $\nabla u\circ\Theta=\nabla\overline{u}-\mathcal{M}_1\nabla\overline{u}$ (see Section 3.3.4 for definitions of $\mathcal{M}_i$, $i=0,\cdots,4$) we have

$$\begin{aligned}&\left(\nu(\chi)\left(\nabla u+\nabla u^\top\right)\right)\circ\Theta\\&=\nu(\overline{\chi})\left(\nabla\overline{u}-\mathcal{M}_1\nabla\overline{u}+(\nabla\overline{u})^\top-(\mathcal{M}_1\nabla\overline{u})^\top\right).\end{aligned} \tag{3.3.64}$$

Thus, we can rewrite (3.3.63) as

$$\begin{aligned}&[\![p]\!]-\kappa H_\Gamma\\&=\left(\left[\!\left[\left(\nu(\overline{\chi})\left(\nabla\overline{u}+\nabla\overline{u}^\top\right)\right)\right]\!\right]n_\Sigma\right)\cdot n_\Sigma-\left(\left[\!\left[\left(\nu(\overline{\chi})\left(\nabla\overline{u}+\nabla\overline{u}^\top\right)\right)\right]\!\right]\mathcal{M}_0\nabla_\Sigma h\right)\cdot n_\Sigma\\&\quad-\left(\left[\!\left[\nu(\overline{\chi})\left(\mathcal{M}_1\nabla\overline{u}+(\mathcal{M}_1\nabla\overline{u})^\top\right)\right]\!\right]\left(n_\Sigma-\mathcal{M}_0\nabla_\Sigma h\right)\right)\cdot n_\Sigma\\&=:\left(\left[\!\left[\nu(\overline{\chi})\left(\nabla\overline{u}+\nabla\overline{u}^\top\right)\right]\!\right]n_\Sigma\right)\cdot n_\Sigma+\mathcal{G}_1.\end{aligned} \tag{3.3.65}$$

Letting

$$\mathcal{G}_2:=\kappa(H_\Gamma-DH_\Gamma\,[0]\,h), \tag{3.3.66}$$

where $DH_\Gamma\,[0]$ is the Fréchet derivative of H_Γ at $h=0$. Then we can linearize H_Γ using the equality

$$\kappa H_\Gamma=\kappa DH_\Gamma\,[0]\,h+\mathcal{G}_2.$$

The equation (3.3.65) can then be written as

$$[\![p]\!]-\kappa DH_\Gamma\,[0]\,h-\mathcal{G}_2=\left(\left[\!\left[\nu(\overline{\chi})\left(\nabla\overline{u}+\nabla\overline{u}^\top\right)\right]\!\right]n_\Sigma\right)\cdot n_\Sigma+\mathcal{G}_1, \tag{3.3.67}$$

which is the normal projection of (3.3.62). Next, we calculate the tangential projection of (3.3.62). We first notice that for a symmetric matrix $A = (a_{ij})_{d\times d}$ and the normal vector n, we have for any $1 \le i \le d$ that

$$(\mathcal{P}An)_i = \sum_{k=1}^{d}\sum_{j=1}^{d}(\delta_{ik} - n_i n_k)a_{kj}n_j = \sum_{j=1}^{d} a_{ij}n_j - n_i \sum_{k=1}^{d} n_k \left(\sum_{j=1}^{d} a_{kj}n_j\right), \tag{3.3.68}$$

i.e. $\mathcal{P}An = An - ((An)\cdot n)n$. Now we let $A := [\![\nu(\chi)(\nabla u + \nabla u^\top)]\!] \circ \Theta$ and recall that $\alpha = \mathcal{M}_0 \nabla_\Sigma h$. We cancel κH_Γ and $[\![p]\!]$ by calculating (3.3.62) $-$ (3.3.63)n_Γ, which gives us

$$A(n_\Sigma - \alpha) = \left((A(n_\Sigma - \alpha))\cdot n_\Sigma\right)(n_\Sigma - \alpha) \tag{3.3.69}$$

Using (3.3.69) (in the third equality below), we have

$$\begin{aligned} &\mathcal{P}_\Sigma A(n_\Sigma - \alpha) = (I - n_\Sigma \otimes n_\Sigma)A(n_\Sigma - \alpha) \\ &= A(n_\Sigma - \alpha) - \left((A(n_\Sigma - \alpha))\cdot n_\Sigma\right) n_\Sigma \\ &= \left((A(n_\Sigma - \alpha))\cdot n_\Sigma\right)(n_\Sigma - \alpha) - \left((A(n_\Sigma - \alpha))\cdot n_\Sigma\right) n_\Sigma \\ &= -\left((A(n_\Sigma - \alpha))\cdot n_\Sigma\right)\alpha. \end{aligned} \tag{3.3.70}$$

For convenience, we abbreviate $\nu(\chi)$ and $\nu(\overline{\chi})$ to ν and $\overline{\nu}$ respectively when there is no confusion. Substituting with

$$\begin{aligned} A &= [\![\nu(\chi)(\nabla u + \nabla u^\top)]\!] \circ \Theta \\ &= \left[\!\!\left[\overline{\nu}(\nabla\overline{u} + \nabla\overline{u}^\top - \mathcal{M}_1\nabla\overline{u} - (\mathcal{M}_1\nabla\overline{u})^\top)\right]\!\!\right], \end{aligned} \tag{3.3.71}$$

we obtain

$$\begin{aligned} &\mathcal{P}_\Sigma \left[\!\!\left[\overline{\nu}(\nabla\overline{u} + \nabla\overline{u}^\top - \mathcal{M}_1\nabla\overline{u} - (\mathcal{M}_1\nabla\overline{u})^\top)\right]\!\!\right](n_\Sigma - \alpha) \\ &= -\left(\left(\left[\!\!\left[\overline{\nu}(\nabla\overline{u} + \nabla\overline{u}^\top - \mathcal{M}_1\nabla\overline{u} - (\mathcal{M}_1\nabla\overline{u})^\top)\right]\!\!\right](n_\Sigma - \alpha)\right)\cdot n_\Sigma\right)\alpha \end{aligned} \tag{3.3.72}$$

Expanding the brackets in the left-hand side of (3.3.72) and rearranging the terms, we have

$$\begin{aligned} &\mathcal{P}_\Sigma [\![\overline{\nu}\left(\nabla\overline{u} + \nabla\overline{u}^\top\right)]\!] n_\Sigma \\ &= \mathcal{P}_\Sigma \left[\!\!\left[\overline{\nu}(I - \mathcal{M}_1)\nabla\overline{u} + \overline{\nu}\left((I - \mathcal{M}_1)\nabla\overline{u}\right)^\top\right]\!\!\right] \mathcal{M}_0\nabla_\Sigma h + \mathcal{P}_\Sigma [\![\overline{\nu}\left(\mathcal{M}_1\nabla\overline{u} + (\mathcal{M}_1\nabla\overline{u})^\top\right)]\!] n_\Sigma \\ &\quad - \left(\left(\left[\!\!\left[\overline{\nu}(I - \mathcal{M}_1)\nabla\overline{u} + \overline{\nu}\left((I - \mathcal{M}_1)\nabla\overline{u}\right)^\top\right]\!\!\right](n_\Sigma - \mathcal{M}_0\nabla_\Sigma h)\right)\cdot n_\Sigma\right)\mathcal{M}_0\nabla_\Sigma h \\ &=: -\mathcal{G}_3. \end{aligned} \tag{3.3.73}$$

Now we can combine the tangential and normal projections by letting (3.3.67)n_Σ + (3.3.73), which gives us the transformation of (3.1.5). From (3.3.20), (3.3.67), (3.3.73) and (3.3.57), we obtain

$$-\left[\!\left[\left(\nu(\overline{\chi})\left(\nabla\overline{u}+\nabla\overline{u}^{\top}\right)\right)\right]\!\right] n_\Sigma + [\![p]\!]\, n_\Sigma - \kappa\left(DH\left[0\right]h\right)n_\Sigma = \left(\mathcal{G}_1+\mathcal{G}_2\right)n_\Sigma + \mathcal{G}_3. \tag{3.3.74}$$

In (3.5.124), we will prove that $DH_\Gamma\left[0\right] = (\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma)$, which enables us to obtain the final version of the transformed equation

$$-\left[\!\left[\left(\nu(\overline{\chi})\left(\nabla\overline{u}+\nabla\overline{u}^{\top}\right)\right)\right]\!\right] + [\![p]\!]\, n_\Sigma - \kappa\triangle_\Sigma h = \left(\mathcal{G}_1+\mathcal{G}_2+\kappa\mathrm{tr}L_\Sigma^2 h\right)n_\Sigma + \mathcal{G}_3. \tag{3.3.75}$$

Now we transform (3.1.6). By [29, Section 2.5.2], the velocity of the interface satisfies

$$\beta\partial_t h = V_\Gamma\circ\Theta_t^{-1} = \left(u\cdot n_\Gamma\right)\circ\Theta_t^{-1} = \beta\overline{u}\cdot\left(n_\Sigma - \mathcal{M}_0\nabla_\Sigma h\right), \tag{3.3.76}$$

which implies that $\partial_t h = \overline{u}\cdot\left(n_\Sigma - \mathcal{M}_0\nabla_\Sigma h\right)$ and can then be rewritten as

$$\partial_t h - \overline{u}\cdot n_\Sigma + b\cdot\nabla_\Sigma h = \left(I-\mathcal{M}_0\right)\nabla_\Sigma h\cdot\overline{u} + (b-\overline{u})\nabla_\Sigma h. \tag{3.3.77}$$

The term $b \in W^{1-\frac{1}{2q},q}([0,T];L^q(\Sigma)) \cap L^q([0,T];W^{2-\frac{1}{q},q}(\Sigma))$ is an auxiliary function, which will be specially selected in later sections. For details on V_Γ and the derivation of (3.3.77), we refer to Section 2.2.5 in [29].

3.3.4 Transformed equations

Similarly as in [15], we abbreviate some terms which will be frequently used in later calculations. In (3.3.41), we define

$$\mathcal{M}_0 := (I - hL_\Sigma)^{-1}. \tag{3.3.78}$$

In (3.3.20), we define

$$\mathcal{M}_1 := \left((I + \nabla\theta)^{-1}\nabla\theta\right). \tag{3.3.79}$$

In (3.3.25), we define

$$\mathcal{M}_2 := \left(\triangle\Theta^{-1}\right) \circ \Theta, \quad \text{and} \quad \mathcal{M}_4 := (\nabla\Theta)^{-\top} (\nabla\Theta)^{-1} - I. \tag{3.3.80}$$

In (3.3.19), we define

$$\mathcal{M}_3 := \partial_t\theta \left(I - (I + \nabla\theta)^{-1}\nabla\theta\right). \tag{3.3.81}$$

We also remind the readers that the abbreviations $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in the transformation of the surface tension equation are defined in (3.3.65), (3.3.66) and (3.3.73).

Notice that (3.1.2) has the form $\partial_t B = \nabla \times F$. Thus, taking the divergence of the equation, we obtain

$$\partial_t \mathrm{div} B = 0. \tag{3.3.82}$$

Since the solution $\overline{B}$ in the transformed equations satisfies $\mathrm{div} B_0 = 0$, the equation (3.1.4) will always be satisfied and thus can be removed from the transformed problem.

Notice that the viscosity $\overline{\nu}$ in the transformed problem is independent of time since the interface in the transformed problem is a fixed surface. Finally, using (3.3.26), (3.3.28), (3.3.29), (3.3.75) and (3.3.77), the equations (3.1.1) - (3.1.8) can be transformed to

$$\begin{aligned}\partial_t\overline{u} + \nabla\overline{p} - \overline{\nu}\triangle\overline{u} \;&=\; -\frac{1}{2}\nabla\left(\left|\overline{B}\right|^2\right) - \overline{u}\nabla\overline{u} + \overline{B}\nabla\overline{B} + \mathcal{M}_3\nabla\overline{u} + \overline{u}\mathcal{M}_1\nabla\overline{u} \\ &- \overline{B}\mathcal{M}_1\nabla\overline{B} + \frac{1}{2}\mathcal{M}_1\nabla\left(\left|\overline{B}\right|^2\right) + \mathcal{M}_1\nabla\overline{p} + \overline{\nu}\mathcal{M}_4 : \nabla^2\overline{u} + \overline{\nu}\mathcal{M}_2\cdot(\nabla\overline{u}),\end{aligned} \tag{3.3.83}$$

$$\begin{aligned}\partial_t\overline{B} - \sigma\triangle\overline{B} \;&=\; -\overline{u}\nabla\overline{B} + \overline{B}\nabla\overline{u} + \overline{u}\mathcal{M}_1\nabla\overline{B} - \overline{B}\mathcal{M}_1\nabla\overline{u} \\ &+ \mathcal{M}_3\nabla\overline{B} + \sigma\mathcal{M}_4 : \nabla^2\overline{B} + \sigma\mathcal{M}_2\cdot(\nabla\overline{B}),\end{aligned} \tag{3.3.84}$$

$$\mathrm{div}\overline{u} = \mathcal{M}_1 : \nabla\overline{u}, \tag{3.3.85}$$

$$\left[\!\!\left[-2\nu(\chi)\tilde{D}\overline{u}\right]\!\!\right] n_\Sigma + [\![\overline{p}]\!]\, n_\Sigma - \kappa(\triangle_\Sigma h) n_\Sigma = \left(\mathcal{G}_1 + \mathcal{G}_2 + \kappa \mathrm{tr} L_\Sigma^2 h\right) n_\Sigma + \mathcal{G}_3, \tag{3.3.86}$$

$$\partial_t h - \overline{u} \cdot n_\Sigma + b \cdot \nabla_\Sigma h = \left(I - \mathcal{M}_0\right) \nabla_\Sigma h \cdot \overline{u} + (b - \overline{u}) \nabla_\Sigma h, \tag{3.3.87}$$

$$\overline{u}|_{\partial\Omega} = 0,\ \overline{B}|_{\partial\Omega} = 0, \quad \overline{u}|_{t=0} = \overline{u}_0 := u \circ \Theta,\ \overline{B}|_{t=0} = \overline{B}_0 := B \circ \Theta. \tag{3.3.88}$$

For convenience, we ignore the bars in $\overline{u}$, $\overline{B}$ and $\overline{p}$ when there is no confusion. The height function h only exists in the transformed equations, so it always appears without a bar.

3.4 Linear Part

In this section, we consider the linear part of the equations (3.3.83) - (3.3.88), which can be rewritten as the following linear problem. For convenience, we ignore the bars over variables.

$$\partial_t u + \nabla p - \nu(\chi) \triangle u = g_1 \quad \text{in } \Omega \setminus \Sigma, \tag{3.4.1}$$

$$\mathrm{div} u = g_3 \quad \text{in } \Omega \setminus \Sigma, \tag{3.4.2}$$

$$[\![-\nu(\chi)\left(\nabla u + \nabla u^\top\right) + pI]\!]\, n_\Sigma - \kappa(\triangle_\Sigma h) n_\Sigma = g_4 \quad \text{on } \Sigma \tag{3.4.3}$$

$$[\![u]\!] = 0 \quad \text{on } \Sigma \tag{3.4.4}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{3.4.5}$$

$$\partial_t h - u \cdot n_\Sigma + b \cdot \nabla_\Sigma h = g_5 \quad \text{on } \Sigma \tag{3.4.6}$$

$$u(0) = u_0 \quad \text{in } \Omega \setminus \Sigma, \tag{3.4.7}$$

$$h(0) = h_0 \quad \text{on } \Sigma. \tag{3.4.8}$$

$$\partial_t B - \sigma \triangle B = g_2 \quad \text{in } \Omega, \tag{3.4.9}$$

$$B = 0 \quad \text{on } \partial\Omega, \tag{3.4.10}$$

$$B(0) = B_0 \quad \text{in } \Omega. \tag{3.4.11}$$

The linear problem can be divided into two sub-problems (3.4.1) - (3.4.8) and (3.4.9) - (3.4.11), which can be solved using the theory of two-phase Stokes problems in [15, 29] and parabolic problems in [17, Theorem 9.1].

Remark 3.4.1. We remark that the condition $\text{div}B = 0$ in the original problem can be removed in the transformed problem by the argument in Section 3.3.4.

The equations (3.4.1)-(3.4.8), can be solved using [15, Theorem 1]. We restate the theorem with some simplification.

Theorem 3.4.1 ([15] Theorem 1, simplified)**.** *Let Ω be a C^3 domain in $\mathbb{R}^d$. Let $T > 0$ and $q > d + 2$. Let $\Sigma \subseteq \Omega$ be a closed C^3-hypersurface. Suppose*

$$b \in W^{1-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{2-\frac{1}{q},q}(\Sigma)); \tag{3.4.12}$$

Let the source terms and initial values be as follows:

1. $g_1 \in L^q([0,T] \times \Omega)$;
2. $u_0 \in W^{2-2/q,q}(\Omega \setminus \Sigma)$, $u_0 = 0$ *on* $\partial\Omega$;
3. $g_3 \in L^q([0,T]; W^{1,q}(\Omega \setminus \Sigma))$, $\text{div}u_0 = g_3(0) = 0$;
4. $g_4 \in W^{\frac{1}{2}-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{1-\frac{1}{q},q}(\Sigma))$;
5. $\mathcal{P}_\Sigma \left[\!\left[2\nu \tilde{D} u_0 \right]\!\right] = \mathcal{P}_\Sigma g_4(0)$;
6. $g_5 \in W^{1-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{2-\frac{1}{2q},q}(\Sigma))$;
7. $h_0 \in W^{3-\frac{2}{q},q}(\Sigma)$.

Then there exists a unique solution (u, p, h) to (3.4.1)-(3.4.8)*, such that*

1. $u \in W^{1,q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{2,q}(\Omega \setminus \Sigma))$;
2. $p \in L^q([0,T]; \dot{W}^{1,q}(\Omega \setminus \Sigma))$;
3. $[\![p]\!] \in W^{\frac{1}{2}-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{1-\frac{1}{2q},q}(\Sigma))$;
4. $h \in W^{2-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap W^{1,q}([0,T]; W^{2-\frac{1}{q},q}(\Sigma)) \cap L^q([0,T]; W^{3-\frac{1}{q},q}(\Sigma))$;
5. *The mapping* $(g_1, g_3, g_4, g_5, u_0, h_0, b) \mapsto (u, p, [\![p]\!], h)$ *is continuous.*

For the principal part of the magnetic equations, we use the theory of parabolic equations in bounded domains, which can be found in e.g. [17, Theorem 9.1]. We also refer to [17, Chapter IV, Section 4] and [16, Chapter 8, Section 3] for more details on the localization and flattening of bounded domains. We will consider the case that $\partial\Omega$ is at least C^3, which is stronger than the requirement of O^2 in [17]. We refer to [17, Chapter I, Section 1] for definitions of O^l, H^l and C^l spaces when the domain is a surface.

Theorem 3.4.2 ([17] Chapter IV Theorem 9.1, simplified)**.** *Let* Ω *be a* C^2 *domain. Let* $T > 0$ *and* $q > 3/2$*. Suppose* $g_2 \in L^q([0,T] \times \Omega)$*,* $B_0 \in W^{2-2/q,q}(\Omega)$ *and* $B_0|_{\partial\Omega} = 0$*. Then there exists a unique solution*

$$B \in W^{1,q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{2,q}(\Omega)) \tag{3.4.13}$$

to (3.4.9)-(3.4.11)*. The solution has the estimate*

$$\|B\|_{W^{1,q}([0,T];L^q(\Omega))} + \|B\|_{L^q([0,T];W^{2,q}(\Omega))} \le C(T) \left(\|g_2\|_{L^q([0,T]\times\Omega)} + \|B_0\|^{2-2/q}_{L^q(\Omega)} \right), \tag{3.4.14}$$

where the constant $C(T)$ *is bounded when* T *is finite.*

Consequently, we can obtain a continuous solution operator defined as follows.

Definition 3.4.1. Given $T > 0$ and $q > 5$. Given (u_0, B_0, h_0, b) such that

1. $u_0 \in W^{2-\frac{2}{q},q}(\Omega \setminus \Sigma) \cap C(\Omega)$, $u_0 = 0$ on $\partial\Omega$;
2. $B_0 \in W^{2-\frac{2}{q},q}(\Omega)$, $B_0 = 0$ on $\partial\Omega$;
3. $h_0 \in W^{3-\frac{2}{q},q}(\Sigma)$;
4. $b \in W^{1-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{2-\frac{1}{q},q}(\Sigma))$.

Given source terms $(g_1, g_2, g_3, g_4, g_5)$ such that

1. $g_1 \in L^q([0,T] \times \Omega)$;
2. $g_2 \in L^q([0,T] \times \Omega)$;
3. $g_3 \in L^q([0,T]; W^{1,q}(\Omega \setminus \Sigma))$, $\quad \mathrm{div} u_0 = g_3(0) = 0$;
4. $g_4 \in W^{1/2-1/(2q),q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{1-1/q,q}(\Sigma))$, $\quad \mathcal{P}_\Sigma g_4(0) = \mathcal{P}_\Sigma \left[\!\left[2\nu \tilde{D} u_0 \right]\!\right]$;
5. $g_5 \in W^{1-1/(2q),q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{2-\frac{1}{q},q}(\Sigma))$.

We define the solution operator $S_{(u_0,B_0,h_0,b)}$, or simply S if there is no confusion, by

$$S_{(u_0,B_0,h_0,b)}(g_1, g_2, g_3, g_4, g_5) := (u, B, p, \varpi, h), \tag{3.4.15}$$

where (u, p, ϖ, h) is the solution to (3.4.1) - (3.4.8) with $\varpi = [\![p]\!]$ an auxiliary variable; and B is the solution to (3.4.9) - (3.4.11).

From Theorem 3.4.1 and Theorem 3.4.2, we know that $S_{(u_0,B_0,h_0,b)}$ is a continuous operator. When the initial value vanishes, the solution operator becomes linear, which implies that $S_{(0,0,0,b)}$ is a bounded linear operator.

3.5 Nonlinear Part

In this section, we estimate the terms on the right-hand side of (3.3.83) - (3.3.88) by calculating and estimating their Fréchet derivatives.

3.5.1 Equations and spaces

For convenience, we define the linear parts and the nonlinear parts of the transformed equations by L_i and G_i similarly as in [15]. The bars over variables are ignored for convenience. We define

$$L_1 := \partial_t u + \nabla p - \nu(\chi)\triangle u, \tag{3.5.1}$$

$$L_2 := \partial_t B - \sigma\triangle B, \tag{3.5.2}$$

$$L_3 := \mathrm{div} u, \tag{3.5.3}$$

$$L_4 := [\![-2\nu(\chi)Du]\!]\, n_\Sigma + \varpi n_\Sigma - \kappa(\triangle_\Sigma h)n_\Sigma, \tag{3.5.4}$$

$$L_5 := \partial_t h - u\cdot n_\Sigma + b\cdot\nabla_\Sigma h, \tag{3.5.5}$$

$$\begin{aligned} G_1 :=& -\frac{1}{2}\nabla\left(|B|^2\right) - u\nabla u + B\nabla B + \mathcal{M}_3\nabla u + u\mathcal{M}_1\nabla u - B\mathcal{M}_1\nabla B \\ &+ \frac{1}{2}\mathcal{M}_1\nabla\left(|B|^2\right) + \mathcal{M}_1\nabla p + \nu(\chi)\mathcal{M}_4 : \nabla^2 u + \nu(\chi)\mathcal{M}_2\cdot(\nabla u), \end{aligned} \tag{3.5.6}$$

$$G_2 := -u\nabla B + B\nabla u + u\mathcal{M}_1\nabla B - B\mathcal{M}_1\nabla u + \mathcal{M}_3\nabla B + \sigma\mathcal{M}_4 : \nabla^2 B + \sigma\mathcal{M}_2\cdot(\nabla B), \tag{3.5.7}$$

$$G_3 = \mathcal{M}_1 : \nabla u, \tag{3.5.8}$$

$$G_4 = \left(\mathcal{G}_1 + \mathcal{G}_2 + \kappa(\mathrm{tr}L_\Sigma^2)h\right) n_\Sigma + \mathcal{G}_3, \tag{3.5.9}$$

$$G_5 = ((I - \mathcal{M}_0)\nabla_\Sigma h)\cdot u + (b-u)\cdot\nabla_\Sigma h. \tag{3.5.10}$$

For convenience, we rewrite the formulas of $\mathcal{M}_0$ to $\mathcal{M}_4$:

$$\mathcal{M}_0 = (I - hL_\Sigma)^{-1}, \tag{3.5.11}$$

$$\mathcal{M}_1 = (I + \nabla\theta)^{-1}\nabla\theta, \tag{3.5.12}$$

$$\mathcal{M}_2 = \left(\triangle\Theta^{-1}\right)\circ\Theta, \tag{3.5.13}$$

$$\mathcal{M}_3 = \partial_t\theta \left(I - (I + \nabla\theta)^{-1}\nabla\theta\right), \tag{3.5.14}$$

$$\mathcal{M}_4 = (\nabla\Theta)^{-\top} (\nabla\Theta)^{-1} - I. \tag{3.5.15}$$

To make the arguments concise, we abbreviate some common function spaces. Based on the settings in [15], we define

$$u \in \mathcal{W}_1^T := W^{1,q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{2,q}(\Omega \setminus \Sigma)), \tag{3.5.16}$$

$$B \in \mathcal{W}_2^T := W^{1,q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{2,q}(\Omega)), \tag{3.5.17}$$

$$p \in \mathcal{W}_3^T := L^q([0,T]; \dot{W}^{1,q}(\Omega \setminus \Sigma)), \tag{3.5.18}$$

$$\varpi \in \mathcal{W}_4^T := W^{\frac{1}{2}-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{1-\frac{1}{q},q}(\Sigma)), \tag{3.5.19}$$

$$h \in \mathcal{W}_5^T := W^{2-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap W^{1,q}([0,T]; W^{2-\frac{1}{q},q}(\Sigma)) \cap L^q([0,T]; W^{3-\frac{1}{q},q}(\Sigma)). \tag{3.5.20}$$

For convenience, we define

$$\mathcal{W}_6^T := W^{\frac{1}{2},q}([0,T]; L^q(\Omega)) \cap L^q([0,T]; W^{1,q}(\Omega \setminus \Sigma)), \tag{3.5.21}$$

which is the space that ∇u and ∇B belong to. Similarly to [15], we define the solution space

$$\mathcal{W}^T := \left\{(u, B, p, \varpi, h) \in \mathcal{W}_1^T \times \mathcal{W}_2^T \times \mathcal{W}_3^T \times \mathcal{W}_4^T \times \mathcal{W}_5^T : [\![p]\!] = \varpi\right\}. \tag{3.5.22}$$

We denote by $\mathcal{S}_i$ the space that source terms belong to, i.e.

$$G_1 \in \mathcal{S}_1^T := L^q([0,T]; L^q(\Omega)), \tag{3.5.23}$$

$$G_2 \in \mathcal{S}_2^T := L^q([0,T]; L^q(\Omega)), \tag{3.5.24}$$

$$G_3 \in \mathcal{S}_3^T := W^{1,q}([0,T]; \dot{W}^{-1,q}(\Omega)) \cap L^q([0,T]; W^{1,q}(\Omega \setminus \Sigma)), \tag{3.5.25}$$

$$G_4 \in \mathcal{S}_4^T := W^{\frac{1}{2}-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{1-\frac{1}{q},q}(\Sigma)), \tag{3.5.26}$$

$$G_5 \in \mathcal{S}_5^T := W^{1-\frac{1}{2q},q}([0,T]; L^q(\Sigma)) \cap L^q([0,T]; W^{2-\frac{1}{q},q}(\Sigma)). \tag{3.5.27}$$

Then we define

$$\mathcal{S}^T := \mathcal{S}_1^T \times \mathcal{S}_2^T \times \mathcal{S}_3^T \times \mathcal{S}_4^T \times \mathcal{S}_5^T. \tag{3.5.28}$$

We will also frequently use the following spaces:

$$\begin{aligned} \mathcal{C}_0^T &:= C^0([0,T];C^0(\Omega)), \\ \mathcal{C}_1^T &:= C^0([0,T];C^1(\Omega)) \cap C^1([0,T];C^0(\Omega)), \\ \mathcal{C}_2^T &:= C^0([0,T];C^2(\Omega)) \cap C^1([0,T];C^1(\Omega)). \end{aligned} \tag{3.5.29}$$

For convenience, we also define

$$\mathcal{C}_3^T := C^0([0,T];C^1(\Omega)), \quad \text{and} \quad \mathcal{C}_4^T := C^0([0,T];C^2(\Omega)). \tag{3.5.30}$$

The spaces $\mathcal{C}_i$, $i = 0,1,2,3,4$, can also be defined on $[0,T] \times \Sigma$, which can be expressed by replacing Ω with Σ. We will ignore the notation of domain (e.g. Ω or Σ) and the time variable T when there is no confusion. For all spaces Z with the form $X([0,T];Y)$, we use

$$\mathring{Z}^T \quad \text{and} \quad \mathring{X}([0,T];Y) \tag{3.5.31}$$

to denote the subspace of elements whose initial values on the time interval $[0,T]$ are 0 in the sense of limit or trace. If Z is the intersection of spaces, i.e.

$$Z^T = X_1([0,T];Y_1) \cap \cdots \cap X_k([0,T];Y_k) \tag{3.5.32}$$

for some $k \in \mathbb{N}$. Then we use $\mathring{Z}^T$ to denote

$$\mathring{Z}^T = \mathring{X}_1([0,T];Y_1) \cap \cdots \cap \mathring{X}_k([0,T];Y_k). \tag{3.5.33}$$

3.5.2 Fréchet derivatives of $\mathcal{M}_i$ and their estimates

In this section, we assume that $\Omega \subseteq \mathbb{R}^m$ for $m \geq 2$. We will estimate the terms $\mathcal{M}_i$, $i = 0,1,2,3,4$ in (3.5.11) - (3.5.15), which are introduced in [15]. We will also calculate and estimate their Fréchet derivatives $D\mathcal{M}_i$. The estimates are studied on a generic time interval $[0,T] \subseteq [0,T_0]$ with $T_0 > 0$ a fixed number. We temporarily ignore the parameter T in the notations of function spaces for convenience.

3.5.2.1 $\mathcal{M}_0$ and $D\mathcal{M}_0$

We recall that $\mathcal{M}_0 := (I - hL_\Sigma)^{-1}$. Since Σ is a fixed surface, its Weingarten tensor $L_\Sigma := -\nabla_\Sigma n_\Sigma$ is a fixed, matrix-valued function on Σ. The entries of $I - hL_\Sigma$ are

$$I - hL_\Sigma = \begin{pmatrix} 1 - hl_{11} & -hl_{12} & \cdots & -hl_{1m} \\ -hl_{21} & 1 - hl_{22} & \cdots & -hl_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -hl_{m1} & -hl_{m2} & \cdots & 1 - hl_{mm} \end{pmatrix}. \tag{3.5.34}$$

The determinant of $I - hL_\Sigma$ is in the form of $1 + hP(h)$, where $P(h)$ denotes a polynomial of h. The entries of the adjugate matrix $\mathrm{adj}(I - hL_\Sigma)$ can all be represented using polynomials $Q_{ij}(h)$. From the inverse matrix formula $A^{-1} = \mathrm{adj}(A)/\det(A)$, we have

$$(I - hL_\Sigma)^{-1}_{ij} = \frac{Q_{ij}(h)}{1 + hP(h)}. \tag{3.5.35}$$

Let ∂ denote either the time or spatial derivative, we have

$$\partial\,(I - hL_\Sigma)^{-1}_{ij} = \frac{\partial\,(Q_{ij}(h))\,(1 + hP(h)) - (Q_{ij}(h))\,\partial\,(1 + hP(h))}{(1 + hP(h))^2} = \frac{\tilde{Q}_{ij}(h, \partial h)}{1 + h\tilde{P}(h)}. \tag{3.5.36}$$

Let $\delta_0(\Sigma) > 0$ be sufficiently small and assume without loss of generality that $\delta_0 < 1$, then for all $\|h\|_{\mathcal{C}_0} < \delta_0$ we have

$$\left\|1 + h\tilde{P}(h)\right\|_{\mathcal{C}_0} \geq \frac{1}{2}, \tag{3.5.37}$$

which implies that

$$\left\|(I - hL_\Sigma)^{-1}\right\|_{\mathcal{C}_0} \leq C(m, \Sigma). \tag{3.5.38}$$

Taking higher-order derivatives, we obtain

$$\partial^k\,(I - hL_\Sigma)^{-1}_{ij} = \frac{\overline{Q}_{ij}(h, \partial h, \cdots, \partial^k h)}{1 + h\overline{P}(h)}, \tag{3.5.39}$$

where $\overline{P}$ and $\overline{Q}_{ij}$ are also polynomials. Similarly, in (3.5.39), we can stay away from the singularities of the denominator by letting $\|h\|_{\mathcal{C}_0} < \delta_0$ for some $\delta_0(m, \Sigma, k) \in (0, 1)$. Moreover, let $M > 0$ be an upper bound of the derivatives of h, then we can bound the numerator as well. Specially, for $k = 0, 1, 2$, there exists $\delta_0(m, \Sigma, k) \in (0, 1)$ such that for all $\|h\|_{\mathcal{C}_0} < \delta_0$ and $\|h\|_{\mathcal{C}_k} < M$ we have

$$\|\mathcal{M}_0\|_{\mathcal{C}_k} = \left\|(I - hL_\Sigma)^{-1}\right\|_{\mathcal{C}_k} \leq C(m, \Sigma, M). \tag{3.5.40}$$

Next, we estimate the Fréchet derivative $D\mathcal{M}_0$. We decompose $\mathcal{M}_0$ into $F : h \mapsto I - hL_\Sigma$ and $G : A \mapsto A^{-1}$, where A is an invertible $m \times m$ matrix. Then we can calculate their Fréchet derivatives using the definition, which implies

$$DF\,[h]\,\varphi = -\varphi L_\Sigma \quad \text{and} \quad DG\,[A]\,H = -A^{-1}HA^{-1} \tag{3.5.41}$$

for all φ in the same space as for h and all H in the same space as for A. The term DG is obtained by considering

$$(A + H)^{-1} - A^{-1} = A^{-1}\left((I + HA^{-1})^{-1} - I\right) \tag{3.5.42}$$

and expanding $(I + HA^{-1})^{-1}$ using power series. Thus, we have

$$\begin{aligned} &D\mathcal{M}_0\,[h]\,\varphi = D(G \circ F)\,[h]\,\varphi = DG\,[F\,[h]]\,DF\,[h]\,\varphi \\ &= (I - hL_\Sigma)^{-1}\,(\varphi L_\Sigma)\,(I - hL_\Sigma)^{-1} = \mathcal{M}_0 L_\Sigma \mathcal{M}_0 \varphi. \end{aligned} \tag{3.5.43}$$

Therefore, from (3.5.40) we know that for $k = 0, 1, 2$, there exists $\delta_0(m, \Sigma, k) \in (0, 1)$, such that for all $\|h\|_{\mathcal{C}_0} < \delta_0$, $\|h\|_{\mathcal{C}_k} < M$ and $\varphi \in \mathring{\mathcal{C}}_k$, we have

$$\begin{aligned} &\|D\mathcal{M}_0\,[h]\,\varphi\|_{\mathring{\mathcal{C}}_k} = \|\mathcal{M}_0 \varphi L_\Sigma \mathcal{M}_0\|_{\mathring{\mathcal{C}}_k} \\ &\leq \|\mathcal{M}_0\|^2_{\mathcal{C}_k} \|L_\Sigma\|_{C^k(\Sigma)} \|\varphi\|_{\mathring{\mathcal{C}}_k} \leq C(m, \Sigma, M) \|\varphi\|_{\mathring{\mathcal{C}}_k}. \end{aligned} \tag{3.5.44}$$

When $k = 1$, we also have for all $0 \leq T \leq T_0$ that

$$\begin{aligned} &\|D\mathcal{M}_0\,[h]\,\varphi\|_{\mathring{\mathcal{S}}_4^T} \leq C(m, T_0, \Sigma, q) \|\mathcal{M}_0\|^2_{\mathcal{C}_1^T} \|L_\Sigma\|_{C^1(\Sigma)} \|\varphi\|_{\mathring{\mathcal{S}}_4^T} \\ &\leq C(m, T_0, \Sigma, q, M) \|\varphi\|_{\mathring{\mathcal{S}}_4^T}. \end{aligned} \tag{3.5.45}$$

3.5.2.2 $\mathcal{M}_1$ and $D\mathcal{M}_1$

Now we estimate $\mathcal{M}_1 = (I + \nabla\theta)^{-1}\nabla\theta$ in (3.5.12). We recall that

$$\theta_h(x) = \eta\left(\frac{d(x)}{\varrho_0}\right) h(\Pi(x))\nu_\Sigma(\Pi(x))$$

when $x \in B(\Sigma; \varrho_0)$ and $\theta_h(x) = 0$ if $x \notin B(\Sigma; \varrho_0)$.

Since Σ is a fixed surface, we can write θ as

$$\theta(x) = \mathbb{h}(x)\mathbb{n}(x), \tag{3.5.46}$$

where $\mathbb{n} := \eta(d(x)/\varrho_0)n_\Sigma(\Pi(x))$ is an extension of the normal vector field n_Σ to Ω, which is supported in $B(\Sigma; \varrho_0)$; and $\mathbb{h}$ is an extension of h from Σ to Ω by letting $\mathbb{h}(x) := h(\Pi(x))$ for $x \in B(\Sigma; \varrho_0)$ and $\mathbb{h}(x) := 0$ otherwise. Thus, we have

$$\nabla\theta = \nabla(\mathbb{h}\mathbb{n}) = \nabla\mathbb{h} \otimes \mathbb{n} + \mathbb{h}\nabla\mathbb{n}, \tag{3.5.47}$$

which implies that

$$\begin{aligned}
&\|\nabla\theta\|_{\mathcal{C}_1} := \|\nabla\theta\|_{C^1([0,T];C(\Omega))} + \|\nabla\theta\|_{C([0,T];C^1(\Omega))} \\
&= \|\nabla(\mathbb{h}\mathbb{n})\|_{C^1([0,T];C(B(\Sigma;\varrho_0)))} + \|\nabla(\mathbb{h}\mathbb{n})\|_{C([0,T];C^1(B(\Sigma;\varrho_0)))} \\
&\leq C(\Sigma)\, \|\mathbb{h}\|_{C^1([0,T];C^1(B(\Sigma;\varrho_0)))} + C(\Sigma)\, \|\mathbb{h}\|_{C([0,T];C^2(B(\Sigma;\varrho_0)))} \\
&\leq C(\Sigma)\, \|\mathbb{h}\|_{\mathcal{C}_2} \leq C(\Sigma)\, \|h\|_{\mathcal{C}_2} .
\end{aligned} \tag{3.5.48}$$

Using similar arguments and the fact that Σ is a C^3 surface, we obtain

$$\begin{aligned}
&\|\nabla\theta\|_{\mathcal{S}_5} = \|\nabla\mathbb{h} \otimes \mathbb{n}\|_{\mathcal{S}_5} + \|\mathbb{h}\nabla\mathbb{n}\|_{\mathcal{S}_5} \\
&\leq C(\Sigma)\, \|\nabla\mathbb{h}\|_{\mathcal{S}_5} + C(\Sigma)\, \|\mathbb{h}\|_{\mathcal{S}_5} \leq C(\Sigma)\, \|\mathbb{h}\|_{\mathcal{W}_5} \leq C(\Sigma)\, \|h\|_{\mathcal{W}_5} .
\end{aligned} \tag{3.5.49}$$

Letting $F : \mathbb{h} \to (I + \nabla\theta)^{-1}$, for $\varphi \in \mathring{\mathcal{W}}_5$ we have

$$DF\,[\mathbb{h}]\,\varphi = -\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1} \nabla\left(\varphi\mathbb{n}\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1} \tag{3.5.50}$$

The entries of $I+\nabla(\mathbb{h}\mathbb{n})$ are

$$\begin{pmatrix} 1+\partial_1\mathbb{h}\mathbb{n}_1+\mathbb{h}\partial_1\mathbb{n}_1 & \partial_1\mathbb{h}\mathbb{n}_2+\mathbb{h}\partial_1\mathbb{n}_2 & \cdots & \partial_1\mathbb{h}\mathbb{n}_m+\mathbb{h}\partial_1\mathbb{n}_m \\ \partial_2\mathbb{h}\mathbb{n}_1+\mathbb{h}\partial_2\mathbb{n}_1 & 1+\partial_2\mathbb{h}\mathbb{n}_2+\mathbb{h}\partial_2\mathbb{n}_2 & \cdots & \partial_2\mathbb{h}\mathbb{n}_m+\mathbb{h}\partial_2\mathbb{n}_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_m\mathbb{h}\mathbb{n}_1+\mathbb{h}\partial_m\mathbb{n}_1 & \partial_m\mathbb{h}\mathbb{n}_2+\mathbb{h}\partial_m\mathbb{n}_2 & \cdots & 1+\partial_m\mathbb{h}\mathbb{n}_m+\mathbb{h}\partial_m\mathbb{n}_m \end{pmatrix}. \tag{3.5.51}$$

Since $\nabla\mathbb{n}$ only depends on Σ, it is a fixed matrix. Thus, the determinant of $I+\nabla(\mathbb{h}\mathbb{n})$ is

$$\det\left(I+\nabla(\mathbb{h}\mathbb{n})\right)=1+\eth\mathbb{h}P(\eth\mathbb{h}). \tag{3.5.52}$$

To simplify the statement, we temporarily use the notation $\eth\mathbb{h}$ when:

1. this term is either h or its derivatives;
2. in all cases, the term can be controlled using the norm in the context.

Without loss of generality, we also slightly abuse the notation $\eth\mathbb{h}P(\eth\mathbb{h})$ to denote the sum of multiple terms with this structure, e.g.

$$\mathbb{h}(1+\mathbb{h}\partial_{x_i}\mathbb{h})+\partial_{x_i}\mathbb{h}(1+\mathbb{h}).$$

This notation does not bring trouble as long as the term $\eth\mathbb{h}$ can be controlled by the needed norm in the context. The entries of the adjugate matrix of $I+\nabla(\mathbb{h}\mathbb{n})$ are all in the form of $P(\mathbb{h},\partial\mathbb{h})$. Thus, all entries of the matrix $\left(I+\nabla(\mathbb{h}\mathbb{n})\right)^{-1}$ can be expressed as

$$\frac{Q_{ij}(\eth\mathbb{h})}{1+\eth\mathbb{h}\,P(\eth\mathbb{h})}. \tag{3.5.53}$$

Moreover, their time or spatial derivatives, i.e.

$$\partial_t\left(\frac{Q_{ij}(\eth\mathbb{h})}{1+\eth\mathbb{h}\,P(\eth\mathbb{h})}\right) \quad \text{or} \quad \partial_{x_k}\left(\frac{Q_{ij}(\eth\mathbb{h})}{1+\eth\mathbb{h}\,P(\eth\mathbb{h})}\right),$$

can also be expressed using the same formula as in (3.5.53). We recall that the polynomial fractions in (3.5.53) only depend on Σ. Thus, when $\|h\|_{\mathcal{C}_3}\le\delta_0$ for some sufficiently small $\delta_0(\Sigma)$, we are able to exclude all singularities. Suppose we also have $\|h\|_{\mathcal{C}_2}\le M$ for some $M>0$, then we have

$$\left\|\left(I+\nabla(\mathbb{h}\mathbb{n})\right)^{-1}\right\|_{\mathcal{C}_1}\le C(m,\Sigma,M). \tag{3.5.54}$$

We remark that the $\mathcal{C}_3$ norm is enough in the condition $\|h\|_{\mathcal{C}_3} \leq \delta_0$, since taking derivatives of polynomial fractions does not create higher-order derivatives in their denominators.

To get the formula of $D\mathcal{M}_1$, we still need to calculate the Fréchet derivative of $G : \mathbb{h} \mapsto \nabla\theta$, i.e. $DG\,[\mathbb{h}]\,\varphi = \nabla(\varphi\mathbb{n})$. Using the product rule of Fréchet derivatives, we then obtain that

$$\begin{aligned} D(FG)\,[\mathbb{h}]\,\varphi &= ((DF)\,[\mathbb{h}]\,\varphi)\,(G\,[\mathbb{h}]) + (F\,[\mathbb{h}])\,((DG)\,[\mathbb{h}]\,\varphi) \\ &= -\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\varphi\mathbb{n}\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\mathbb{h}\mathbb{n}\right) + \left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\varphi\mathbb{n}\right) \\ &= \left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\varphi\mathbb{n}\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\mathbb{h}\mathbb{n}\right), \end{aligned} \tag{3.5.55}$$

where the last equality is obtained using $-(I + \nabla\theta)^{-1}\nabla\theta + I = (I + \nabla\theta)^{-1}\nabla\theta$. Since there is no singularity in $\nabla(\mathbb{h}\mathbb{n})$, we have

$$\|\nabla(\mathbb{h}\mathbb{n})\|_{\mathcal{C}_1} \leq C(m, \Sigma)\,\|\mathbb{h}\|_{\mathcal{C}_2}\,. \tag{3.5.56}$$

The term $\nabla(\varphi\mathbb{n})$ can be treated in the same way since $\mathbb{h}$ and φ are in the same space. Finally, letting $H : h \mapsto \mathbb{h}$, we have $D\mathcal{M}_1 = D_h\mathcal{M}_1 = D((FG) \circ H)$ and

$$\begin{aligned} &D((FG) \circ H)\,[h]\,\psi = D(FG)\,[H\,[h]]\,DH\,[h]\,\psi \\ &= (DF\,[H\,[h]]\,DH\,[h]\,\psi)\,(G\,[H\,[h]]) + (F\,[H\,[h]])\,(DG\,[H\,[h]]\,DH\,[h]\,\psi) \\ &= -\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left((\psi \circ \Pi)\mathbb{n}\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\mathbb{h}\mathbb{n}\right) + \left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left((\psi \circ \Pi)\mathbb{n}\right) \end{aligned} \tag{3.5.57}$$

for all $\psi \in \mathring{\mathcal{C}}_2$. Consequently, we obtain the following estimates for $\mathcal{M}_1$ and $D\mathcal{M}_1$.

Proposition 3.5.1. *There exists* $\delta_0(m, \Sigma) \in (0, 1)$*, such that for all* $\|h\|_{\mathcal{C}_3} < \delta_0$*, if* $\|h\|_{\mathcal{C}_1} < M$ *for some* $M > 0$ *then*

$$\|\mathcal{M}_1\,[h]\|_{\mathcal{C}_1} \leq C(m, \Sigma, M)\,\|h\|_{\mathcal{C}_2}\,; \tag{3.5.58}$$

and for all $\varphi \in \mathring{\mathcal{C}}_2$ *we have*

$$\|D\mathcal{M}_1\,[h]\,\varphi\|_{\mathring{\mathcal{C}}_1} \leq C(m, \Sigma, M)\,\|\varphi\|_{\mathring{\mathcal{C}}_2}\,. \tag{3.5.59}$$

Remark 3.5.1. For the operator $H : h \mapsto \mathbb{h}$, its Fréchet derivative is $DH\,[h] : \varphi \mapsto \varphi \circ \Pi$. Notice that H is a linear operator.

3.5.2.3 $\mathcal{M}_2$ and $D\mathcal{M}_2$

Now we study $\mathcal{M}_2$ in (3.5.13). Letting $\mathbb{i} : x \to x$ be the identity mapping in Ω, we have $\Theta = \mathbb{i} + \mathbb{h}\mathbb{n}$ and $\nabla\Theta = I + \nabla\theta = I + \nabla(\mathbb{h}\mathbb{n})$. Using the inverse function theorem, we have

$$\begin{aligned}
\triangle\Theta_k^{-1} &= \sum_i \partial_i(\nabla\Theta_k^{-1})_i = \sum_i \partial_i \left((\nabla\Theta)_{ik}^{-1} \circ \Theta^{-1}\right) \\
&= \sum_i \sum_j \partial_i\Theta_j^{-1} \left(\partial_j(\nabla\Theta)_{ik}^{-1}\right) \circ \Theta^{-1} \\
&= \sum_i \sum_j \left((\nabla\Theta)_{ij}^{-1} \circ \Theta^{-1}\right) \left(\left(\partial_j(\nabla\Theta)_{ik}^{-1}\right) \circ \Theta^{-1}\right),
\end{aligned} \tag{3.5.60}$$

which implies that

$$(\mathcal{M}_2)_k = \left(\triangle\Theta_k^{-1}\right) \circ \Theta = \sum_i \sum_j (\nabla\Theta)_{ij}^{-1} \left(\partial_j(\nabla\Theta)_{ik}^{-1}\right). \tag{3.5.61}$$

Here we abbreviated $\left((\nabla\Theta)^{-1}\right)_{ij}$ to $(\nabla\Theta)_{ij}^{-1}$ for convenience. In order to estimate $(\triangle\Theta^{-1}) \circ \Theta$, we only need to estimate its entries $\left(\triangle\Theta_k^{-1}\right) \circ \Theta$, which only requires the estimate of $(\nabla\Theta)_{ij}^{-1} \left(\partial_j(\nabla\Theta)_{ik}^{-1}\right)$. We only need to estimate one entry, i.e. for a fixed choice of (i, j, k), since all entries have the same structure.

Letting $F : \mathbb{h} \mapsto (\nabla\Theta)_{ij}^{-1}$ and $G : \mathbb{h} \mapsto \partial_j(\nabla\Theta)_{ik}^{-1}$. Similar to the arguments for $\mathcal{M}_1$, we obtain

$$DF\,[\mathbb{h}]\,\varphi = -\left((\nabla(\mathbb{i} + \mathbb{h}\mathbb{n}))^{-1}\,\nabla(\varphi\mathbb{n})\,(\nabla(\mathbb{i} + \mathbb{h}\mathbb{n}))^{-1}\right)_{ij}, \tag{3.5.62}$$

$$DG\,[\mathbb{h}]\,\varphi = -\partial_j \left((\nabla(\mathbb{i} + \mathbb{h}\mathbb{n}))^{-1}\,\nabla(\varphi\mathbb{n})\,(\nabla(\mathbb{i} + \mathbb{h}\mathbb{n}))^{-1}\right)_{ik}. \tag{3.5.63}$$

Notice that $\nabla(\mathbb{i} + \mathbb{h}\mathbb{n}) = I + \nabla(\mathbb{h}\mathbb{n})$. Then we have

$$\begin{aligned}
&D(FG)\,[\mathbb{h}]\,\varphi = (DF\,[\mathbb{h}]\,\varphi)\,(G\,[\mathbb{h}]) + (F\,[\mathbb{h}])\,(DG\,[\mathbb{h}]\,\varphi) \\
&= -\left((I + \nabla(\mathbb{h}\mathbb{n}))^{-1}\,\nabla(\varphi\mathbb{n})\,(I + \nabla(\mathbb{h}\mathbb{n}))^{-1}\right)_{ij} \left(\partial_j\,(I + \nabla(\mathbb{h}\mathbb{n}))_{ik}\right) \\
&\quad - \left((I + \nabla(\mathbb{h}\mathbb{n}))^{-1}\right)_{ij} \partial_j \left((I + \nabla(\mathbb{h}\mathbb{n}))^{-1}\,\nabla(\varphi\mathbb{n})\,(I + \nabla(\mathbb{h}\mathbb{n}))^{-1}\right)_{ik} \\
&=: I_1 + I_2.
\end{aligned} \tag{3.5.64}$$

In term I_1, we have

$$\partial_j\,(I + \nabla(\mathbb{h}\mathbb{n}))_{ik} = \partial_{ji}^2\mathbb{h}\mathbb{n}_k + \partial_i\mathbb{h}\partial_j\mathbb{n}_k + \partial_j\mathbb{h}\partial_i\mathbb{n}_k + \mathbb{h}\partial_{ji}^2\mathbb{n}_k, \tag{3.5.65}$$

which implies

$$\|\partial_j \left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)_{ik}\|_{\mathcal{C}_0} \le C(m, \Sigma)\,\|h\|_{\mathcal{C}_4}\,. \tag{3.5.66}$$

When $\|h\|_{\mathcal{C}_3} < \delta_0$ for some sufficiently small $\delta_0(\Sigma)$, using the same argument as in the derivation of (3.5.54), we have

$$\left\|\left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)^{-1}\right\|_{\mathcal{C}_0} \le C(m, \Sigma). \tag{3.5.67}$$

We remark that conditions like $\|h\|_{\mathcal{C}_1} < M$ are not required in the current estimation. Using (3.5.67), we obtain that

$$\begin{aligned}
&\left\|\left(\left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)^{-1} \nabla(\varphi\mathbb{n})\left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)^{-1}\right)_{ik}\right\|_{\mathring{\mathcal{C}}_0} \\
&\lesssim \left\|\left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)^{-1} \nabla(\varphi\mathbb{n})\left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)^{-1}\right\|_{\mathring{\mathcal{C}}_0} \\
&\lesssim \left\|\left(I + \nabla\left(\mathbb{h}\mathbb{n}\right)\right)^{-1}\right\|_{\mathcal{C}_0}^2 \|\nabla(\varphi\mathbb{n})\|_{\mathring{\mathcal{C}}_0} \\
&\le C(m, \Sigma)\,\|\varphi\|_{\mathring{\mathcal{C}}_3}
\end{aligned} \tag{3.5.68}$$

for all $\varphi \in \mathring{\mathcal{C}}_3$, which completes the estimate

$$\|I_1\|_{\mathcal{C}_0} \le C(m, \Sigma, M)\,\|\varphi\|_{\mathring{\mathcal{C}}_3}\,. \tag{3.5.69}$$

In order to estimate I_2, we first obtain that

$$\|I + \nabla(\mathbb{h}\mathbb{n})\|_{\mathcal{C}_0} \le C(m, \Sigma)\left(1 + \|h\|_{\mathcal{C}_3}\right). \tag{3.5.70}$$

To estimate the term $\partial_j \left(\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1} \nabla(\varphi\mathbb{n})\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\right)_{ik}$, we first notice that for three matrices A, B and C we have

$$\begin{aligned}
&\|\partial_j(ABC)_{ik}\|_{\mathcal{C}_0} = \|(\partial_j ABC)_{ik} + (A\partial_j BC)_{ik} + (AB\partial_j C)_{ik}\|_{\mathcal{C}_0} \\
&\lesssim \|\partial_j A\|_{\mathcal{C}_0} \|B\|_{\mathcal{C}_0} \|C\|_{\mathcal{C}_0} + \|A\|_{\mathcal{C}_0} \|\partial_j B\|_{\mathcal{C}_0} \|C\|_{\mathcal{C}_0} + \|A\|_{\mathcal{C}_0} \|B\|_{\mathcal{C}_0} \|\partial_j C\|_{\mathcal{C}_0}\,,
\end{aligned} \tag{3.5.71}$$

i.e. the entry of the product of matrices can be estimated using the matrix norm.

Remark 3.5.2. We remind the reader that for vector-valued or matrix-valued functions in $\mathcal{C}_k$, the norm is defined by first taking the $\mathcal{C}_k$ norm of components, and then taking the matrix or vector norm.

We recall that we already have

$$\|\nabla(\varphi \mathbb{m})\|_{\mathring{\mathcal{C}}_3} \leq C(\Sigma) \|\varphi\|_{\mathring{\mathcal{C}}_4}, \tag{3.5.72}$$

$$\|\partial_j (\nabla(\varphi \mathbb{m}))\|_{\mathring{\mathcal{C}}_0} \leq C(\Sigma) \|\varphi\|_{\mathring{\mathcal{C}}_4}. \tag{3.5.73}$$

It remains to estimate the matrix norm of $\partial_j (I + \nabla(\mathbb{h}\mathbb{m}))^{-1}$, which can be obtained using the same argument as in the derivation of (3.5.54). In fact, there exists $\delta_0(\Sigma) \in (0,1)$, such that for all $\|h\|_{\mathcal{C}_3} < \delta_0$, if $\|h\|_{\mathcal{C}_4} < M$ for some $M > 0$, then we have

$$\left\|\partial_j \left((I + \nabla(\mathbb{h}\mathbb{m}))^{-1}\right)\right\|_{\mathcal{C}_0} \leq C(m, \Sigma, M), \tag{3.5.74}$$

which completes the estimate

$$\|I_2\|_{\mathring{\mathcal{C}}_0} \leq C(m, \Sigma, M) \|\varphi\|_{\mathring{\mathcal{C}}_4}. \tag{3.5.75}$$

Consequently, we have the following estimates.

Proposition 3.5.2. *There exists $\delta_0(\Sigma) \in (0,1)$, such that for all $\|h\|_{\mathcal{C}_3} < \delta_0$, if $\|h\|_{\mathcal{C}_4} < M$ for some $M > 0$, then*

$$\|\mathcal{M}_2\|_{\mathcal{C}_0} \leq C(m, \Sigma, M); \tag{3.5.76}$$

and for all $\varphi \in \mathring{\mathcal{C}}_4$ we have

$$\|D\mathcal{M}_2 [h] \varphi\|_{\mathring{\mathcal{C}}_0} \leq C(m, \Sigma, M) \|\varphi\|_{\mathring{\mathcal{C}}_4}. \tag{3.5.77}$$

3.5.2.4 $\mathcal{M}_3$ and $D\mathcal{M}_3$

The term $\mathcal{M}_3 = \partial_t\theta\,(I - (I+\nabla\theta)^{-1}\nabla\theta)$ can be written as

$$\mathcal{M}_3 = \partial_t\,(\mathbb{h}\mathbb{n})\,(I - \mathcal{M}_1). \tag{3.5.78}$$

Letting $F : \mathbb{h} \mapsto I - \mathcal{M}_1$ and $G : \mathbb{h} \mapsto \partial_t\,(\mathbb{h}\mathbb{n})$. Using the same calculation as in (3.5.55), we have for all $\varphi \in \mathring{\mathcal{C}}_1$ that

$$DF\,[\mathbb{h}]\,\varphi = -\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\varphi\mathbb{n}\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}. \tag{3.5.79}$$

Since $\partial_t\theta = \partial_t(\mathbb{h}\mathbb{n}) = \partial_t\mathbb{h}\mathbb{n}$, we have

$$DG\,[\mathbb{h}]\,\varphi = \partial_t\varphi\mathbb{n}. \tag{3.5.80}$$

Thus, the Fréchet derivative of $\mathcal{M}_3$ is

$$\begin{aligned} D(GF)\,[\mathbb{h}]\,\varphi &= (G\,[\mathbb{h}])\,(DF\,[\mathbb{h}]\,\varphi) + (DG\,[\mathbb{h}]\,\varphi)\,(F\,[\mathbb{h}]) \\ &= (\partial_t\mathbb{h}\mathbb{n})\left(-\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla\left(\varphi\mathbb{n}\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\right) \\ &\quad + (\partial_t\varphi\mathbb{n})\left(I - \left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla(\mathbb{h}\mathbb{n})\right). \end{aligned} \tag{3.5.81}$$

Composing with the mapping $h \mapsto \mathbb{h} = h \circ \Pi$, we have the following estimates.

Proposition 3.5.3. *There exists $\delta_0(\Sigma) \in (0,1)$, such that for all $\|h\|_{\mathcal{C}_3} < \delta_0$, if $\|h\|_{\mathcal{C}_1} < M$ for some $M > 0$, then*

$$\|\mathcal{M}_3\,[h]\|_{\mathcal{C}_0} \leq C(m,\Sigma,M)\,\|h\|_{\mathcal{C}_1}\,; \tag{3.5.82}$$

and for all $\varphi \in \mathring{\mathcal{C}}_1$,

$$\|D\mathcal{M}_3\,[h]\,\varphi\|_{\mathring{\mathcal{C}}_0} \leq C(m,\Sigma,M)\,\|\varphi\|_{\mathring{\mathcal{C}}_1}. \tag{3.5.83}$$

3.5.2.5 $\mathcal{M}_4$ and $D\mathcal{M}_4$

In order to estimate $\mathcal{M}_4 = (\nabla\Theta)^{-\top}(\nabla\Theta)^{-1} - I$, we first rewrite it as

$$\begin{aligned} \mathcal{M}_4 &= (\nabla\Theta)^{-\top}\left(I - \nabla\Theta^\top\nabla\Theta\right)(\nabla\Theta)^{-1} \\ &= -(\nabla\Theta)^{-\top}\left(\nabla\theta^\top\nabla\theta + \nabla\theta^\top + \nabla\theta\right)(\nabla\Theta)^{-1}, \end{aligned} \tag{3.5.84}$$

which allows us to separate the variable h.

Next, we study the Fréchet derivative. We define

$$F : A \mapsto A^\top A - I, \tag{3.5.85}$$

where A is a matrix. Since

$$F(A+H) - F(A) = A^\top H + H^\top A + H^\top H, \tag{3.5.86}$$

we have

$$DF\,[A]\,H = A^\top H + H^\top A. \tag{3.5.87}$$

We recall that for the inverse matrix operator $G : A \mapsto A^{-1}$ we have

$$DG\,[A]\,H = -A^{-1}HA^{-1}. \tag{3.5.88}$$

Thus, we have

$$D(F\circ G)\,[A]\,H = DF\,[G\,[A]]\,DG\,[A]\,H = -A^{-\top}\left(A^{-1}H + \left(A^{-1}H\right)^\top\right)A^{-1}. \tag{3.5.89}$$

For the operator $K : \mathbb{h} \mapsto \nabla\Theta = I + \nabla\theta$, we have $DK\,[\mathbb{h}]\,\varphi = \nabla(\varphi\mathbb{h})$. This implies that

$$\begin{aligned} &D(F\circ G\circ K)\,[\mathbb{h}]\,\varphi = D(F\circ G)\,[K(\mathbb{h})]\,DK\,[\mathbb{h}]\,\varphi \\ &= -\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-\top}\left(\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla(\varphi\mathbb{n}) + \left(\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}\nabla(\varphi\mathbb{n})\right)^\top\right)\left(I + \nabla(\mathbb{h}\mathbb{n})\right)^{-1}. \end{aligned} \tag{3.5.90}$$

Composing $F\circ G\circ K$ with the mapping $h \mapsto \mathbb{h}$ and using the same argument as in the estimate of $\mathcal{M}_1$, we obtain the following estimate.

Proposition 3.5.4. *There exists $\delta_0(\Sigma) \in (0,1)$, such that for all $\|h\|_{\mathcal{C}_3} < \delta_0$,*

$$\|\mathcal{M}_4[h]\|_{\mathcal{C}_0} \leq C(m,\Sigma)\|h\|_{\mathcal{C}_3}; \tag{3.5.91}$$

and for all $\varphi \in \mathring{\mathcal{C}}_3$,

$$\|D\mathcal{M}_4[h]\varphi\|_{\mathring{\mathcal{C}}_0} \leq C(m,\Sigma)\|\varphi\|_{\mathring{\mathcal{C}}_3}. \tag{3.5.92}$$

Therefore, we have obtained the estimates of terms $\mathcal{M}_0$ to $\mathcal{M}_4$. In the remaining part of this chapter, we will fix a value for δ_0 such that all these estimates can be satisfied, which can be done by taking the minimum of finite many values.

Remark 3.5.3. Notice that the variable we consider is h when we calculate the Fréchet derivatives, rather than the point $x \in \Omega$. Thus, the function $\Theta(x) = x + \mathbb{h}(x)\mathbb{n}(x)$ is understood as a mapping $h \mapsto \mathbb{h} \mapsto \mathbb{i} + \mathbb{h}\mathbb{n}$, where $\mathbb{i} : x \to x$ and $\mathbb{n}$ are fixed functions defined in Ω. The Fréchet derivative is the derivative of an operator, so it is taken with respect to h rather than x.

3.5.3 Fréchet derivatives of $\mathcal{G}_i$ and their estimates

In this section, we estimate nonlinear terms $\mathcal{G}_i$, $i = 1,2,3$ in the surface tension equation, which are defined in (3.3.65) (3.3.66) and (3.3.73). These terms have been studied in e.g. [15,28]. In our calculation, we add more details for completeness. We remind the readers that we still calculate in $\mathbb{R}^m$ instead of $\mathbb{R}^3$ in order to have clearer structures. For convenience, we do not mark the dependency on m in constant terms.

$$\begin{aligned}\mathcal{G}_1 := &-\left(\left[\!\left[\nu(\chi)\left(\nabla u + \nabla u^\top\right)\right]\!\right]\mathcal{M}_0\nabla_\Sigma h\right)\cdot n_\Sigma \\ &-\left(\left[\!\left[\nu(\chi)\left(\mathcal{M}_1\nabla u + (\mathcal{M}_1\nabla u)^\top\right)\right]\!\right](n_\Sigma - \mathcal{M}_0\nabla_\Sigma h)\right)\cdot n_\Sigma\end{aligned} \tag{3.5.93}$$

$$\mathcal{G}_2 := \kappa H_\Gamma - \kappa DH_\Gamma[0]\,h, \tag{3.5.94}$$

$$\begin{aligned}\mathcal{G}_3 = &\,\mathcal{P}_\Sigma\left[\!\left[\nu(\chi)(I-\mathcal{M}_1)\nabla u + \nu(\chi)\left((I-\mathcal{M}_1)\nabla u\right)^\top\right]\!\right]\mathcal{M}_0\nabla_\Sigma h \\ &+ \mathcal{P}_\Sigma\left[\!\left[\nu(\chi)\mathcal{M}_1\nabla u + \nu(\chi)\left(\mathcal{M}_1\nabla u\right)^\top\right]\!\right]n_\Sigma \\ &-\left(\left(\left[\!\left[\nu(\chi)(I-\mathcal{M}_1)\nabla u + \nu(\chi)\left((I-\mathcal{M}_1)\nabla u\right)^\top\right]\!\right](n_\Sigma - \mathcal{M}_0\nabla_\Sigma h)\right)\cdot n_\Sigma\right)\mathcal{M}_0\nabla_\Sigma h.\end{aligned} \tag{3.5.95}$$

3.5.3.1 Estimate of α

We recall that in (3.3.41) we defined the auxiliary vector $\alpha := \mathcal{M}_0 \nabla_\Sigma h$ to help calculate n_Γ. Suppose that $\|h\|_{\mathcal{C}_2} < M$. Since Σ is fixed, from (3.5.40), there exists $\delta_0(\Sigma) \in (0,1)$ such that

$$\|\alpha\|_{\mathcal{C}_1} \le C \|\mathcal{M}_0\|_{\mathcal{C}_1} \|\nabla_\Sigma h\|_{\mathcal{C}_1} \le C(\Sigma, M) \|h\|_{\mathcal{C}_2} . \tag{3.5.96}$$

Moreover, we have

$$\|\alpha\|_{\mathcal{S}_5} \le C(T_0, \Sigma) \|\mathcal{M}_0\|_{\mathcal{C}_2} \|\nabla_\Sigma h\|_{\mathcal{S}_5} \le C(T_0, \Sigma, M) \|h\|_{\mathcal{W}_5} . \tag{3.5.97}$$

Now we estimate the Fréchet derivative $D\alpha$. Given $z = (u, B, p, \varpi, h) \in \mathcal{W}$ and $\varphi = (\varphi_u, \varphi_B, \varphi_p, \varphi_\varpi, \varphi_h) \in \mathring{\mathcal{W}}$, we have

$$\begin{aligned}
(D\alpha [z]) \varphi &= D_h [h] (\mathcal{M}_0 \nabla_\Sigma h) \varphi_h \\
&= (D_h \mathcal{M}_0 \varphi_h) (\nabla_\Sigma h) + (\mathcal{M}_0) (D_h (\nabla_\Sigma h) \varphi_h) \\
&= (\mathcal{M}_0 \varphi_h L_\Sigma \mathcal{M}_0) (\nabla_\Sigma h) + \mathcal{M}_0 (\nabla_\Sigma \varphi_h) .
\end{aligned} \tag{3.5.98}$$

Here we use D_h to express that the derivative is taken only with respect to h. Suppose that $\|h\|_{\mathcal{C}_2} < M$. From (3.5.40), there exists $\delta_0(\Sigma) \in (0,1)$ such that if $\|h\|_{\mathcal{C}_0} < \delta_0$, then

$$\begin{aligned}
\|D\alpha\varphi\|_{\mathring{\mathcal{C}}_1} &= \|(\mathcal{M}_0 \varphi_h L_\Sigma \mathcal{M}_0) (\nabla_\Sigma h) + \mathcal{M}_0 (\nabla_\Sigma \varphi_h)\|_{\mathring{\mathcal{C}}_1} \\
&\lesssim \|\mathcal{M}_0\|_{\mathcal{C}_1}^2 \|\nabla_\Sigma h\|_{\mathcal{C}_1} \|\varphi_h\|_{\mathring{\mathcal{C}}_1} + \|\mathcal{M}_0\|_{\mathcal{C}_1} \|\nabla_\Sigma \varphi_h\|_{\mathring{\mathcal{C}}_1} \\
&\le C(\Sigma, M) \left(1 + \|h\|_{\mathcal{C}_2}\right) \|\varphi_h\|_{\mathring{\mathcal{C}}_2} .
\end{aligned} \tag{3.5.99}$$

Moreover, we have

$$\begin{aligned}
\|D\alpha\varphi\|_{\mathring{\mathcal{S}}_5} &= \|(\mathcal{M}_0 \varphi_h L_\Sigma \mathcal{M}_0) (\nabla_\Sigma h) + \mathcal{M}_0 (\nabla_\Sigma \varphi_h)\|_{\mathring{\mathcal{S}}_5} \\
&\le C(T_0, \Sigma) \|\mathcal{M}_0\|_{\mathcal{C}_2}^2 \|\nabla_\Sigma h\|_{\mathcal{S}_5} \|\varphi_h\|_{\mathring{\mathcal{C}}_2} + C(T_0, \Sigma) \|\mathcal{M}_0\|_{\mathcal{C}_2} \|\nabla_\Sigma \varphi_h\|_{\mathring{\mathcal{S}}_5} \\
&\le C(T_0, \Sigma, M) \left(1 + \|h\|_{\mathcal{W}_5}\right) \|\varphi_h\|_{\mathring{\mathcal{W}}_5} ,
\end{aligned} \tag{3.5.100}$$

where we used the embedding theory in [28, Proposition 5.1] in the last inequality.

Remark 3.5.4. In order to make calculations concise, we will frequently use notations like $D(\nabla_\Sigma h)$ without additionally using DF by defining $F(h) := \nabla_\Sigma h$. The simplified notation denotes the derivative of the mapping $h \mapsto \nabla_\Sigma h$ at h. This is similar to notations like $(x^2)'$ or $d(x^2)$ in calculus. Moreover, we will not distinguish D and D_h when there is no confusion.

3.5.3.2 Estimate of $\mathcal{G}_1$

We still assume that $\|h\|_{\mathcal{C}_4} < \delta_0$ for a sufficiently small δ_0 and $\|h\|_{\mathcal{C}_2} \le M$. We rewrite $\mathcal{G}_1$ as

$$\begin{aligned}\mathcal{G}_1 &= -\left(\left[\!\left[\nu(\chi)\left(\nabla u + \nabla u^\top\right)\right]\!\right] \mathcal{M}_0 \nabla_\Sigma h\right) \cdot n_\Sigma \\ &\quad - \left(\left[\!\left[\nu(\chi)\left(\mathcal{M}_1 \nabla u + (\mathcal{M}_1 \nabla u)^\top\right)\right]\!\right] (n_\Sigma - \mathcal{M}_0 \nabla_\Sigma h)\right) \cdot n_\Sigma \\ &=: I_1 + I_2.\end{aligned} \tag{3.5.101}$$

In the term I_1, the operator $F : u \mapsto \left[\!\left[\nu(\chi)\left(\nabla u + \nabla u^\top\right)\right]\!\right]$ is linear, which implies

$$DF\,[u]\,\varphi = \left[\!\left[\nu(\chi)\left(\nabla \varphi_u + \nabla \varphi_u^\top\right)\right]\!\right] \tag{3.5.102}$$

for all $\varphi = (\varphi_u, \varphi_B, \varphi_p, \varphi_\varpi, \varphi_h) \in \mathring{\mathcal{W}}$. Thus, the Fréchet derivative DI_1 at $z = (u, B, p, \varpi, h)$ dependents only on u and h, i.e.

$$\begin{aligned}DI_1 \varphi &= -\left(\left(DF\varphi_u\right) \mathcal{M}_0 \nabla_\Sigma h + F\,[u]\, D\left(\mathcal{M}_0 \nabla_\Sigma h\right) \varphi_h\right) \cdot n_\Sigma \\ &= -\left(\left(DF\varphi_u\right) \alpha + F\,[u]\left(D\alpha\varphi_h\right)\right) \cdot n_\Sigma.\end{aligned} \tag{3.5.103}$$

This implies the following estimate:

$$\begin{aligned}&\|DI_1\varphi\|_{\mathring{\mathcal{S}}_4^T} \le C(T_0,\Sigma)\,\|DF\varphi_u\|_{\mathring{\mathcal{S}}_4^T}\left(\|\alpha\|_{\mathcal{S}_4^T} + \|\alpha\|_\infty\right) + C(T_0,\Sigma)\,\|F\,[u]\|_{\mathcal{S}_4^T}\,\|D\alpha\varphi_h\|_{\mathring{\mathcal{C}}_1^T} \\ &\le C(T_0,\Sigma)\,\|\varphi_u\|_{\mathring{\mathcal{W}}_1^T}\left(\|h\|_{\mathcal{W}_5^T} + \|h\|_\infty\right) + C(T_0,\Sigma)\,\|u\|_{\mathcal{W}_1^T}\,\|\varphi_h\|_{\mathring{\mathcal{C}}_2^T} \\ &\le C(\Sigma,T_0)\left(\|z\|_{\mathcal{W}^T} + \|h\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}^T},\end{aligned} \tag{3.5.104}$$

where the constants are fixed for all $T \in (0, T_0]$.

To estimate I_2, we define $G : z \mapsto \mathcal{M}_1 \nabla u$, whose derivative is

$$DG\,[z]\,\varphi = \left(D\mathcal{M}_1 \varphi_h\right) \nabla u + \mathcal{M}_1 \nabla \varphi_u. \tag{3.5.105}$$

Then we obtain

$$\begin{aligned}DI_2\,[z]\,\varphi &= -\left(\left[\!\left[\nu\left(DG\varphi_h + (DG\varphi_h)^\top\right)\right]\!\right](n_\Sigma - \mathcal{M}_0\nabla_\Sigma h)\right) \cdot n_\Sigma \\ &\quad + \left(\left[\!\left[\nu\left(G + G^\top\right)\right]\!\right]\left(D\left(\mathcal{M}_0 \nabla_\Sigma h\right)\varphi_h\right)\right) \cdot n_\Sigma \\ &= -\left(\left[\!\left[\nu\left(DG\varphi_h + (DG\varphi_h)^\top\right)\right]\!\right](n_\Sigma - \alpha)\right) \cdot n_\Sigma + \left(\left[\!\left[\nu\left(G + G^\top\right)\right]\!\right]\left(D\alpha\varphi_h\right)\right) \cdot n_\Sigma.\end{aligned} \tag{3.5.106}$$

Thus, using the estimates of α and $D\alpha$, we have for all $T \in (0, T_0]$ that

$$
\begin{aligned}
\|DI_2\,[z]\,\varphi\|_{\mathring{\mathcal{S}}_4^T} \ &\leq (\Sigma, M, T_0) \left\| \left[\!\left[\nu \left(DG\varphi_h + (DG\varphi_h)^\top \right) \right]\!\right] \right\|_{\mathring{\mathcal{S}}_4} \|(n_\Sigma - \alpha)\|_{\mathcal{C}_1} \\
&\quad + (\Sigma, M, T_0) \left\| \left[\!\left[\nu \left(G + G^\top \right) \right]\!\right] \right\|_{\mathcal{S}_4} \|(D\alpha\varphi_h)\|_{\mathring{\mathcal{C}}_1} \\
&\leq C(\Sigma, M, T_0) \left(1 + \|h\|_{\mathcal{C}_2}\right) \|DG\varphi_h\|_{\mathring{\mathcal{W}}_6} + C(\Sigma, M, T_0) \|G\|_{\mathcal{W}_6} \|\varphi_h\|_{\mathring{\mathcal{C}}_2} \\
&\leq C(\Sigma, M, T_0) \left(1 + \|h\|_{\mathcal{C}_2}\right) \left(\|(D\mathcal{M}_1\varphi_h)\, \nabla u\|_{\mathring{\mathcal{W}}_6} + \|\mathcal{M}_1 \nabla \varphi_u\|_{\mathring{\mathcal{W}}_6} \right) \\
&\quad + C(\Sigma, M, T_0) \|\mathcal{M}_1 \nabla u\|_{\mathcal{W}_6} \|\varphi_h\|_{\mathring{\mathcal{C}}_2} \\
&\leq C(\Sigma, M, T_0) \|D\mathcal{M}_1 \varphi_h\|_{\mathring{\mathcal{C}}_1} \|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, M, T_0) \left(\|\mathcal{M}_1\|_{\mathcal{W}_6} + \|\mathcal{M}_1\|_\infty \right) \|\nabla \varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\quad + C(\Sigma, M, T_0) \|\mathcal{M}_1\|_{\mathcal{C}_1} \|\nabla u\|_{\mathcal{W}_6} \|\varphi_h\|_{\mathring{\mathcal{C}}_2} \\
&\leq C(\Sigma, M, T_0) \|\varphi_h\|_{\mathring{\mathcal{C}}_2} \|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, M, T_0) \left(\|\nabla_\Sigma h\|_{\mathcal{W}_6} + \|\nabla_\Sigma h\|_\infty \right) \|\nabla \varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\quad + C(\Sigma, M, T_0) \|h\|_{\mathcal{C}_2} \|\nabla u\|_{\mathcal{W}_6} \|\varphi_h\|_{\mathring{\mathcal{C}}_2} \\
&\leq C(\Sigma, T_0, M) \|\varphi_h\|_{\mathring{\mathcal{W}}_5} \|u\|_{\mathcal{W}_1} + C(\Sigma, M, T_0) \left(\|h\|_{\mathcal{W}_5} + \|\nabla_\Sigma h\|_\infty \right) \|\varphi_u\|_{\mathring{\mathcal{W}}_1} \\
&\quad + C(\Sigma, T_0, M) \|h\|_{\mathcal{C}_2} \|u\|_{\mathcal{W}_1} \|\varphi_h\|_{\mathring{\mathcal{W}}_5} \\
&\leq C(\Sigma, T_0, M) \left(\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty \right) \|\varphi\|_{\mathring{\mathcal{W}}^T},
\end{aligned}
\tag{3.5.107}
$$

where and the last inequality is guaranteed by (3.5.58) and (3.5.59). For convenience, we ignored the parameter T in some notations of function spaces.

From (3.5.104) and (3.5.106), we finally obtain

$$
\|D\mathcal{G}_1\,[z]\,\varphi\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, M, T_0) \left(\|z\|_{\mathcal{W}^T} + \|h\|_{\mathcal{C}_4^T} \right) \|\varphi\|_{\mathring{\mathcal{W}}^T}. \tag{3.5.108}
$$

3.5.3.3 Estimate of $\mathcal{G}_2$

As a preparation of the estimate of $\mathcal{G}_2$, we first calculate the derivative of the mean curvature H_Γ in (3.3.57):

$$
H_\Gamma = \beta \mathrm{tr} \left(\mathcal{M}_0 \left(L_\Sigma + \nabla_\Sigma \alpha \right) \right) - \beta^3 \left(\mathcal{M}_0 \alpha \right) \nabla_\Sigma \alpha \alpha^\top \tag{3.5.109}
$$

For the function $F(s) := 1/s$, we have

$$
DF\,[s]\,r = -\frac{r}{s^2}. \tag{3.5.110}
$$

For the Euclidean norm $G(x) := |x| = \sqrt{x_1^2 + \cdots + x_n^2}$, its Fréchet derivative is

$$DG\,[x]\,v = \frac{x \cdot v}{|x|}. \tag{3.5.111}$$

Letting $H(h) := n_\Sigma - \mathcal{M}_0 \nabla_\Sigma h$, we have for all $\varphi \in \mathring{\mathcal{W}}_5^T$ that

$$DH\,[h]\,\varphi = -\left(D\mathcal{M}_0\,[h]\,\varphi\right)\left(\nabla_\Sigma h\right) - \left(\mathcal{M}_0\,[h]\right)\left(\nabla_\Sigma \varphi\right). \tag{3.5.112}$$

Since $\beta(h) = F \circ G \circ H(h)$, we have

$$\begin{aligned} D\beta\,[h]\,\varphi &= DF\,[G(H(h))]\,DG\,[H(h)]\,DH\,[h]\,\varphi \\ &= \frac{(n_\Sigma - \mathcal{M}_0 \nabla_\Sigma h) \cdot \left(\left(D\mathcal{M}_0\,[h]\,\varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0\,[h]\,\nabla_\Sigma \varphi\right)}{\left|n_\Sigma - \mathcal{M}_0 \nabla_\Sigma h\right|^3} \\ &= \beta^3\left(n_\Sigma - \alpha\right) \cdot \left(\left(D\mathcal{M}_0 \varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0 \nabla_\Sigma \varphi\right), \end{aligned} \tag{3.5.113}$$

which implies

$$D(\beta^3)\,[h]\,\varphi = 3\beta^2\left(D\beta\,[h]\,\varphi\right) = 3\beta^5\left(n_\Sigma - \alpha\right) \cdot \left(\left(D\mathcal{M}_0 \varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0 \nabla_\Sigma \varphi\right). \tag{3.5.114}$$

Now we calculate the derivative of $E : h \mapsto \operatorname{tr}\left(\mathcal{M}_0\left(L_\Sigma + \nabla_\Sigma \alpha\right)\right)$. Letting $F : h \mapsto L_\Sigma + \nabla_\Sigma \alpha$, we have

$$DF\,[h]\,\varphi = \nabla_\Sigma\big(\left(D\mathcal{M}_0\,[h]\,\varphi\right)\left(\nabla_\Sigma h\right) + \left(\mathcal{M}_0\,[h]\right)\left(\nabla_\Sigma \varphi\right)\big). \tag{3.5.115}$$

Letting $G : h \mapsto \mathcal{M}_0 F$, we have

$$\begin{aligned} DG\,[h]\,\varphi &= \left(D\mathcal{M}_0\,[h]\,\varphi\right)\left(F\,[h]\right) + \left(\mathcal{M}_0\,[h]\right)\left(DF\,[h]\,\varphi\right) \\ &= \left(D\mathcal{M}_0 \varphi\right)\left(L_\Sigma + \nabla_\Sigma \alpha\right) + \mathcal{M}_0\left(\nabla_\Sigma\left(\left(D\mathcal{M}_0 \varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0 \nabla_\Sigma \varphi\right)\right). \end{aligned} \tag{3.5.116}$$

Letting $H : A \mapsto tr(A)$, we have

$$\begin{aligned} DE\,[h]\,\varphi &= DH\,[G(h)]\,DG\,[h]\,\varphi \\ &= \operatorname{tr}\left(\left(D\mathcal{M}_0 \varphi\right)\left(L_\Sigma + \nabla_\Sigma \alpha\right) + \mathcal{M}_0\left(\nabla_\Sigma\left(\left(D\mathcal{M}_0 \varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0 \nabla_\Sigma \varphi\right)\right)\right). \end{aligned} \tag{3.5.117}$$

Using (3.5.112), we have

$$\begin{aligned} D\left(\mathcal{M}_0 \alpha\right)[h]\,\varphi &= D\left(\mathcal{M}_0 \mathcal{M}_0 \nabla_\Sigma h\right)[h]\,\varphi \\ &= \left(D\mathcal{M}_0 \varphi\right)\left(\mathcal{M}_0 \nabla_\Sigma h\right) + \left(\mathcal{M}_0\right)\left(D\left(\mathcal{M}_0 \nabla_\Sigma h\right)\varphi\right). \end{aligned} \tag{3.5.118}$$

Using (3.5.112) and (3.5.115), we obtain

$$\begin{aligned}
&D\left(\nabla_\Sigma\alpha\alpha^\top\right)[h]\,\varphi = \left(D\left(\nabla_\Sigma\alpha\right)[h]\,\varphi\right)\left(\alpha^\top[h]\right) + \left(\nabla_\Sigma\alpha\,[h]\right)\left(\left(D\left(\alpha^\top\right)\right)[h]\,\varphi\right)\\
&= \left(\nabla_\Sigma\left(\left(D\mathcal{M}_0\,[h]\,\varphi\right)\left(\nabla_\Sigma h\right) + \left(\mathcal{M}_0\,[h]\right)\left(\nabla_\Sigma\varphi\right)\right)\right)\left(\alpha^\top[h]\right)\\
&+ \left(\nabla_\Sigma\alpha\,[h]\right)\left(\left(D\mathcal{M}_0\,[h]\,\varphi\right)\left(\nabla_\Sigma h\right) + \left(\mathcal{M}_0\,[h]\right)\left(\nabla_\Sigma\varphi\right)\right)^\top.
\end{aligned} \tag{3.5.119}$$

Thus, we have

$$\begin{aligned}
&DH_\Gamma\,[h]\,\varphi = \left(\left(D\beta\right)\varphi\right)\mathrm{tr}\left(\mathcal{M}_0\left(L_\Sigma + \nabla_\Sigma\alpha\right)\right) + \left(\beta\right)\left(D\left(\mathrm{tr}\left(\mathcal{M}_0\left(L_\Sigma + \nabla_\Sigma\alpha\right)\right)\right)\varphi\right)\\
&- \left(D\left(\beta^3\right)\varphi\right)\left(\mathcal{M}_0\alpha\right)\left(\nabla_\Sigma\alpha\alpha^\top\right)\\
&- \left(\beta^3\right)\left(D\left(\mathcal{M}_0\alpha\right)\varphi\right)\left(\nabla_\Sigma\alpha\alpha^\top\right) - \left(\beta^3\right)\left(\mathcal{M}_0\alpha\right)\left(D\left(\nabla_\Sigma\alpha\alpha^\top\right)\varphi\right)\\
&= \beta^3\left(n_\Sigma - \alpha\right)\left(\left(D\mathcal{M}_0\varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0\nabla_\Sigma\varphi_h\right)\left(\mathrm{tr}\left(\left(\mathcal{M}_0\right)\left(L_\Sigma + \nabla_\Sigma\alpha\right)\right)\right)\\
&+ \beta\left(\mathrm{tr}\left(\left(D\mathcal{M}_0\varphi\right)\left(L_\Sigma + \nabla_\Sigma\alpha\right) + \mathcal{M}_0\left(\nabla_\Sigma\left(\left(D\mathcal{M}_0\varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0\nabla_\Sigma\varphi_h\right)\right)\right)\right)\\
&- 3\beta^5\left(n_\Sigma - \alpha\right)\cdot\left(\left(D\mathcal{M}_0\varphi\right)\left(\nabla_\Sigma h\right) + \mathcal{M}_0\nabla_\Sigma\varphi_h\right)\left(\mathcal{M}_0\alpha\right)\left(\nabla_\Sigma\alpha\alpha^\top\right)\\
&- \beta^3\left(\left(D\mathcal{M}_0\varphi\right)\left(\mathcal{M}_0\nabla_\Sigma h\right) + \left(\mathcal{M}_0\right)\left(D\left(\mathcal{M}_0\nabla_\Sigma h\right)\varphi\right)\right)\left(\nabla_\Sigma\alpha\alpha^\top\right)\\
&- \beta^3\mathcal{M}_0\alpha\left(\nabla_\Sigma\left(\left(D\mathcal{M}_0\varphi\right)\nabla_\Sigma h + \mathcal{M}_0\nabla_\Sigma\varphi_h\right)\alpha^\top + \nabla_\Sigma\alpha\left(\left(D\mathcal{M}_0\varphi\right)\nabla_\Sigma h + \mathcal{M}_0\nabla_\Sigma\varphi_h\right)^\top\right)\\
&=: I_1 + I_2 - I_3 - I_4 - I_5.
\end{aligned} \tag{3.5.120}$$

When $h = 0$, we have $\mathcal{M}_0 = I$ and $\nabla_\Sigma h = 0$, which implies $\alpha = 0$ and $\beta = 1/\left\|n_\Sigma\right\| = 1$. Thus, the term I_1 turns to

$$I_1\,[0] = n_\Sigma\left(0 + \nabla_\Sigma\varphi\right)\mathrm{tr}L_\Sigma = 0 \tag{3.5.121}$$

since n_Σ is perpendicular to $\nabla_\Sigma\varphi$. The term I_2 turns to

$$I_2\,[0] = \mathrm{tr}\left(\varphi L_\Sigma\left(L_\Sigma + 0\right) + \nabla_\Sigma\left(0 + \nabla_\Sigma\varphi\right)I\right) = \left(\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma\right)\varphi. \tag{3.5.122}$$

By definition, we have $\alpha\,[h] = 0$ when $h = 0$. Thus, we obtain

$$I_3\,[0] = I_4\,[0] = I_5\,[0] = 0 \tag{3.5.123}$$

with no need for calculation. Therefore, we have

$$DH_\Gamma\,[0] = \mathrm{tr}L_\Sigma^2 + \triangle_\Sigma. \tag{3.5.124}$$

Since the mapping $F : h \mapsto DH_\Gamma[0]\,h$ is linear, its derivative is $DF[h] = DH_\Gamma[0]$ for all h. Thus, the Fréchet derivative of $\mathcal{G}_2$ is

$$D\mathcal{G}_2[h]\,\varphi = D(H_\Gamma[h] - DH_\Gamma[0]\,h)\varphi = I_1 + I_2 - I_3 - I_4 - I_5 - \left(\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma\right)\varphi \tag{3.5.125}$$

with I_1 to I_5 defined in (3.5.120). It remains to estimate $\|I_1\|$, $\|I_3\|$, $\|I_4\|$, $\|I_5\|$ and $\|I_2 - (\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma)\,\varphi\|$ term by term.

For I_1, we first notice that the terms with structures like $n_\Sigma \cdot \nabla_\Sigma f$ or $n_\Sigma \cdot \mathcal{M}_0 \nabla_\Sigma f$ will vanish. This is because the tangential gradient ∇_Σ of a function is a tangent vector, which is perpendicular to the normal vector n_Σ; moreover, the matrix $\mathcal{M}_0$ maps tangent vectors to tangent vectors. Thus, we obtain

$$\begin{aligned}&(n_\Sigma - \alpha)\left((D\mathcal{M}_0\varphi)(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi\right)\\ &= n_\Sigma\,(D\mathcal{M}_0\varphi)(\nabla_\Sigma h) - \alpha\left((D\mathcal{M}_0\varphi)(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi\right),\end{aligned} \tag{3.5.126}$$

which implies that

$$\begin{aligned}I_1 &= \beta^3\,(n_\Sigma - \alpha)\left((D\mathcal{M}_0\varphi)(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi\right)\left(\mathrm{tr}\left(\mathcal{M}_0\left(L_\Sigma + \nabla_\Sigma\alpha\right)\right)\right)\\ &= \beta^3\left(n_\Sigma\,(D\mathcal{M}_0\varphi)(\nabla_\Sigma h) - \alpha\left((D\mathcal{M}_0\varphi)(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi\right)\right)\left(\mathrm{tr}\left(\mathcal{M}_0\left(L_\Sigma + \nabla_\Sigma\alpha\right)\right)\right)\\ &=: \beta^3 I_{11} I_{12}.\end{aligned} \tag{3.5.127}$$

Using [28, (5.3)], we have

$$\|I_1\|_{\mathring{\mathcal{S}}_4} \leq C(\Sigma, T_0)\left\|\beta^3\right\|_{\mathcal{C}_1}\|I_{11}\|_{\mathring{\mathcal{S}}_4}\left(\|I_{12}\|_\infty + \|I_{12}\|_{\mathcal{S}_4}\right). \tag{3.5.128}$$

From the formula of β in (3.3.42), we can find a sufficiently small $\delta_0(\Sigma) \in (0,1)$, such that for all $\|h\|_{\mathcal{C}_3} < \delta_0$, if $\|h\|_{\mathcal{C}_2} < M$ for some $M > 0$ then

$$\left\|\beta^3\right\|_{\mathcal{C}_1} \leq C(\Sigma, M). \tag{3.5.129}$$

Without loss of generality, we can always use the same notation δ_0 and M in the derivation of (3.5.129). This is because we can always choose the smallest δ_0 and the largest M from finite many candidates. From [28, (5.4)], we have for all $T \in (0, T_0]$ that

$$\begin{aligned}
&\|I_{11}\|_{\mathring{\mathcal{S}}_4^T} = \|n_\Sigma\,(D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h) - \alpha\,((D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi_h)\|_{\mathring{\mathcal{S}}_4} \\
&\leq \|n_\Sigma\,(D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h)\|_{\mathring{\mathcal{S}}_4} + \|\alpha\,((D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi_h)\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0)\,\|D\mathcal{M}_0\varphi\|_{\mathring{\mathcal{S}}_4}\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right) \\
&\quad + C(\Sigma, T_0)\left(\|\alpha\|_{\mathcal{S}_4} + \|\alpha\|_\infty\right)\|(D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h) + \mathcal{M}_0\nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0, M)\,\|\varphi_h\|_{\mathring{\mathcal{S}}_4}\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right) \\
&\quad + C(\Sigma, T_0, M)\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right)\left(\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right)\|\varphi_h\|_{\mathring{\mathcal{S}}_4} + \|\nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{S}}_4}\right) \\
&\leq C(\Sigma, T_0, M)\left(\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}.
\end{aligned} \tag{3.5.130}$$

The parameter T in the notations of function spaces is ignored for convenience. When $\|h\|_{\mathcal{C}_0^{T_0}} \leq \delta_0$ and $\|h\|_{\mathcal{C}_1^{T_0}} \leq M$, for all $T \in (0, T_0]$ we have

$$\begin{aligned}
\|I_{12}\|_{\mathcal{S}_4^T} &\leq C(T_0, \Sigma)\,\|\mathcal{M}_0\|_{\mathcal{C}_1}\|L_\Sigma + \nabla_\Sigma\alpha\|_{\mathcal{S}_4} \\
&\leq C(T_0, \Sigma)\,\|\mathcal{M}_0\|_{\mathcal{C}_1}\left(\|L_\Sigma\|_{C^1(\Sigma)} + \|\nabla_\Sigma\alpha\|_{\mathcal{S}_4}\right) \\
&\leq C(\Sigma, T_0, M)\left(1 + \|\alpha\|_{\mathcal{S}_5}\right) \leq C(\Sigma, T_0, M)\left(1 + \|h\|_{\mathcal{W}_5}\right) \\
&\leq C(\Sigma, T_0, M)\left(1 + \|z\|_{\mathcal{W}^T}\right).
\end{aligned} \tag{3.5.131}$$

Moreover, we have

$$\|I_{12}\|_{\mathcal{C}_0} \lesssim \|\mathcal{M}_0\|_{\mathcal{C}_0}\|L_\Sigma + \nabla_\Sigma\alpha\|_{\mathcal{C}_0} \leq C(\Sigma, M). \tag{3.5.132}$$

Thus, when $\|h\|_{\mathcal{C}_0} < \delta_0$ and $\|h\|_{\mathcal{C}_1} < M$, for all $\varphi \in \mathring{\mathcal{S}}_4$ and $T \in (0, T_0]$ we have

$$\|I_1\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, T_0, M)\left(1 + \|z\|_{\mathcal{W}^T}\right)\left(\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{S}}_4^T}. \tag{3.5.133}$$

Similarly to the derivation of I_1, we remove those vanishing products of perpendicular terms in I_3 and I_4, which implies

$$\|I_3\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, T_0, M)\left(1 + \|z\|_{\mathcal{W}^T}\right)\left(\|h\|_{\mathcal{W}_5^T} + \|h\|_{\mathcal{C}_4^T}\right)\|\varphi\|_{\mathring{\mathcal{W}}_5^T} \tag{3.5.134}$$

and

$$
\begin{aligned}
&\|I_4\|_{\mathring{\mathcal{S}}_4^T} \le C(\Sigma,T_0,M)\,\|(D\mathcal{M}_0\varphi)\,(\mathcal{M}_0\nabla_\Sigma h)+(\mathcal{M}_0)\,(D\,(\mathcal{M}_0\nabla_\Sigma h)\,\varphi)\|_{\mathring{\mathcal{C}}_1^T}\,\|\nabla_\Sigma\alpha\|_{\mathcal{S}_4^T}\,\left\|\alpha^\top\right\|_{\mathcal{C}_1^T}\\
&\le C(\Sigma,T_0,M)\left(1+\|h\|_{\mathcal{C}_2^T}\right)\|h\|_{\mathcal{C}_2^T}\,\|\varphi\|_{\mathring{\mathcal{W}}^T}\,\|z\|_{\mathcal{W}^T}
\end{aligned}
\tag{3.5.135}
$$

provided that $\|h\|_{\mathcal{C}_0^{T_0}} < \delta_0$ and $\|h\|_{\mathcal{C}_2^{T_0}} < M$. For the term I_5, we have to be careful with the selection of norms in order to estimate higher-order derivatives with our nonredundant regularity. We have for all $T \in (0,T_0]$ that

$$
\begin{aligned}
&\|I_5\|_{\mathring{\mathcal{S}}_4^T} \le C(\Sigma,T_0,M)\,\|\mathcal{M}_0\alpha\|_{\mathcal{C}_1}\left(\|\alpha\|_{\mathcal{S}_4}+\|\alpha\|_\infty\right)\|\nabla_\Sigma\left((D\mathcal{M}_0\varphi_h)(\nabla_\Sigma h)+\mathcal{M}_0\nabla_\Sigma\varphi_h\right)\|_{\mathring{\mathcal{S}}_4}\\
&\quad + C(\Sigma,T_0,M)\,\|\mathcal{M}_0\alpha\|_{\mathcal{C}_1}\,\|\nabla_\Sigma\alpha\|_{\mathcal{S}_4}\,\|(D\mathcal{M}_0\varphi_h)(\nabla_\Sigma h)+\mathcal{M}_0\nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{C}}_1}\\
&\le C(\Sigma,T_0,M)\,\|h\|_{\mathcal{C}_2}\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4}+\|\nabla_\Sigma h\|_\infty\right)\|(D\mathcal{M}_0\varphi_h)(\nabla_\Sigma h)+\mathcal{M}_0\nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{S}}_5}\\
&\quad + C(\Sigma,T_0,M)\,\|h\|_{\mathcal{C}_2}\,\|h\|_{\mathcal{W}_5}\,\|(D\mathcal{M}_0\varphi_h)(\nabla_\Sigma h)+\mathcal{M}_0\nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{C}}_1}\\
&\le C(\Sigma,T_0,M)\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4}+\|\nabla_\Sigma h\|_\infty\right)\left(\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\,\|h\|_{\mathcal{W}_5}+\|\varphi_h\|_{\mathring{\mathcal{W}}_5}\right)\\
&\quad + C(\Sigma,T_0,M)\,\|h\|_{\mathcal{W}_5}\left(\|\varphi_h\|_{\mathring{\mathcal{C}}_1}\,\|h\|_{\mathcal{C}_2}+\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\right)\\
&\le C(\Sigma,T_0,M)\left(\|h\|_{\mathcal{C}_3^T}+\|z\|_{\mathcal{W}^T}\right)\left(1+\|h\|_{\mathcal{C}_2^T}+\|z\|_{\mathcal{W}^T}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}\,.
\end{aligned}
\tag{3.5.136}
$$

It remains to estimate $I_2 - (\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma)\,\varphi_h$, which can be written as

$$
\begin{aligned}
&I_2-\left(\mathrm{tr}L_\Sigma^2+\triangle_\Sigma\right)\varphi\\
&=\beta\left(\mathrm{tr}\left((D\mathcal{M}_0\varphi)\,(L_\Sigma+\nabla_\Sigma\alpha)+(\nabla_\Sigma\left((D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h)+\mathcal{M}_0\nabla_\Sigma\varphi\right))\,\mathcal{M}_0\right)\right)-\left(\mathrm{tr}L_\Sigma^2+\triangle_\Sigma\right)\varphi\\
&=tr\left(\beta\,(D\mathcal{M}_0\varphi)\,(L_\Sigma+\nabla_\Sigma\alpha)-L_\Sigma^2\varphi\right)+tr\left(\beta\left(\nabla_\Sigma\left((D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h)+\mathcal{M}_0\nabla_\Sigma\varphi\right)\right)\mathcal{M}_0-\nabla_\Sigma^2\varphi\right)\\
&=tr\left(\beta\,(D\mathcal{M}_0\varphi)\,L_\Sigma-L_\Sigma^2\varphi\right)+tr\left(\beta\,(D\mathcal{M}_0\varphi)\,\nabla_\Sigma\alpha\right)\\
&\quad+tr\left(\beta\nabla_\Sigma\left((D\mathcal{M}_0\varphi)\,(\nabla_\Sigma h)\right)\mathcal{M}_0\right)+tr\left(\beta\nabla_\Sigma\left(\mathcal{M}_0\nabla_\Sigma\varphi\right)\mathcal{M}_0-\nabla_\Sigma^2\varphi\right)\\
&=:tr\left(I_{21}+I_{22}+I_{23}+I_{24}\right).
\end{aligned}
\tag{3.5.137}
$$

We still assume that $\|h\|_{\mathcal{C}_0} < \delta_0$, $\|h\|_{\mathcal{C}_2} < M$ and $T \in (0,T_0]$. We recall that the product of two bounded and Lipschitz continuous functions is still Lipschitz continuous. Thus, the estimate of I_{21} to I_{24} can be simplified to the estimate of their factors. For I_{21}, we have for all $T \in (0, T_0]$ that

$$\begin{aligned} &\|I_{21}\|_{\mathring{\mathcal{S}}_4^T} = \left\|\beta\left(D\mathcal{M}_0\varphi\right)L_\Sigma - L_\Sigma^2\varphi\right\|_{\mathring{\mathcal{S}}_4^T} \\ &\le \left\|\beta\left(D\mathcal{M}_0\varphi\right)L_\Sigma - \beta L_\Sigma^2\varphi\right\|_{\mathring{\mathcal{S}}_4^T} + \left\|\beta L_\Sigma^2\varphi - L_\Sigma^2\varphi\right\|_{\mathring{\mathcal{S}}_4^T} \\ &\le C(\Sigma, T_0)\|\beta\|_{\mathcal{C}_1}\|\mathcal{M}_0 L_\Sigma \mathcal{M}_0 - L_\Sigma\|_{\mathcal{S}_4}\|\varphi_h\|_{\mathring{\mathcal{C}}_1} + C(\Sigma, T_0)\|\beta - 1\|_{\mathcal{S}_4}\|\varphi_h\|_{\mathring{\mathcal{C}}_1}. \end{aligned} \tag{3.5.138}$$

From the structure of β in (3.3.42) and the fact that $|n_\Sigma| = 1$, there exists $\delta_0(\Sigma) \in (0, 1)$, such that when $\|h\|_{\mathcal{C}_3} < \delta_0$ and $\|h\|_{\mathcal{C}_2} < M$ we have

$$\|n_\Sigma - \alpha\|_{\mathcal{C}_0} \ge \frac{1}{2} \quad \text{and} \quad \|\beta\|_{\mathcal{C}_1} \le C(\Sigma, M).$$

Then we can obtain that

$$|\beta - 1| = \left|\frac{1}{|n_\Sigma - \alpha|} - 1\right| = \left|\frac{|n_\Sigma| - |n_\Sigma - \alpha|}{|n_\Sigma - \alpha|}\right| \le \frac{|\alpha|}{|n_\Sigma - \alpha|} \le C\,|\alpha|. \tag{3.5.139}$$

It then follows that

$$\|\beta - 1\|_{\mathcal{S}_4} \le C(\Sigma, M, T_0)\|\nabla_\Sigma h\|_{\mathcal{S}_4} \tag{3.5.140}$$

and

$$\|\beta - 1\|_\infty \le C(\Sigma)\|\nabla_\Sigma h\|_\infty. \tag{3.5.141}$$

Moreover, from the structure of $\mathcal{M}_0$ in (3.5.35), we have

$$\|\mathcal{M}_0 - I\|_{\mathcal{S}_4} = \|\mathcal{M}_0 L_\Sigma h\|_{\mathcal{S}_4} \le C(\Sigma, M, T_0)\|h\|_{\mathcal{S}_4}, \tag{3.5.142}$$

which implies that

$$\begin{aligned} &\|\mathcal{M}_0 L_\Sigma \mathcal{M}_0 - L_\Sigma\|_{\mathcal{S}_4^T} \\ &\le \|(\mathcal{M}_0 - I)L_\Sigma \mathcal{M}_0\|_{\mathcal{S}_4} + \|L_\Sigma(\mathcal{M}_0 - I)\|_{\mathcal{S}_4} \\ &\le C(T_0, \Sigma)\|L_\Sigma \mathcal{M}_0\|_{\mathcal{C}_1}\|\mathcal{M}_0 - I\|_{\mathcal{S}_4} + C(T_0, \Sigma)\|L_\Sigma\|_{C^1(\Sigma)}\|\mathcal{M}_0 - I\|_{\mathcal{S}_4} \\ &\le C(\Sigma, T_0, M)\|\mathcal{M}_0 - I\|_{\mathcal{S}_4} \le C(\Sigma, T_0, M)\|h\|_{\mathcal{S}_4^T}. \end{aligned} \tag{3.5.143}$$

From (3.5.138), (3.5.140) and (3.5.143), we have for all $T \in (0, T_0]$ that

$$\|I_{21}\|_{\mathring{\mathcal{S}}_4^T} \le C(\Sigma, M, T_0)\|h\|_{\mathcal{W}^T}\|\varphi_h\|_{\mathring{\mathcal{W}}^T}. \tag{3.5.144}$$

Using estimates in [28, Lemma 5.2], we obtain the estimate of I_{22}:

$$\begin{aligned}
&\|I_{22}\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, T_0)\, \|\beta\|_{\mathcal{C}_1^T}\, \|D\mathcal{M}_0\varphi_h\|_{\mathring{\mathcal{S}}_4^T} \left(\|\nabla_\Sigma \alpha\|_{\mathcal{S}_4^T} + \|\nabla_\Sigma \alpha\|_{\mathcal{C}_0^T}\right) \\
&\leq C(\Sigma, T_0, M) \left(\|h\|_{\mathcal{W}_5^T} + \|h\|_{\mathcal{C}_4^T}\right) \|\varphi_h\|_{\mathring{\mathcal{S}}_4^T}\,.
\end{aligned} \tag{3.5.145}$$

Since $\varphi_h \in \mathring{\mathcal{W}}_5$, the term I_{23} have the estimate

$$\begin{aligned}
&\|I_{23}\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, T_0)\, \|\beta\|_{\mathcal{C}_1}\, \|\nabla_\Sigma\left(\left(D\mathcal{M}_0\varphi\right)\left(\nabla_\Sigma h\right)\right)\|_{\mathring{\mathcal{S}}_4}\, \|\mathcal{M}_0\|_{\mathcal{C}_1} \\
&\leq C(\Sigma, T_0, M)\, \|\nabla_\Sigma\left(\left(D\mathcal{M}_0\varphi\right)\left(\nabla_\Sigma h\right)\right)\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0, M)\, \|\nabla_\Sigma\left(D\mathcal{M}_0\varphi\right)\left(\nabla_\Sigma h\right)\|_{\mathring{\mathcal{S}}_4} + C(\Sigma, T_0, M)\, \|\left(D\mathcal{M}_0\varphi\right)\nabla_\Sigma\left(\nabla_\Sigma h\right)\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0, M)\, \|\nabla_\Sigma\left(D\mathcal{M}_0\varphi\right)\|_{\mathring{\mathcal{S}}_4} \left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right) + C(\Sigma, T_0, M)\, \|D\mathcal{M}_0\varphi\|_{\mathring{\mathcal{C}}_1} \left\|\nabla_\Sigma^2 h\right\|_{\mathcal{S}_4} \\
&\leq C(\Sigma, T_0, M)\, \|D\mathcal{M}_0\varphi\|_{\mathring{\mathcal{S}}_5} \left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right) + C(\Sigma, T_0, M)\, \|\varphi\|_{\mathring{\mathcal{C}}_1}\, \|h\|_{\mathcal{W}_5} \\
&\leq C(\Sigma, T_0, M)\left(\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty\right) \|\varphi\|_{\mathring{\mathcal{W}}^T}\,.
\end{aligned} \tag{3.5.146}$$

Similarly, we obtain the estimate of I_{24}:

$$\begin{aligned}
&\|I_{24}\|_{\mathring{\mathcal{S}}_4^T} \leq \left\|\beta\left(\nabla_\Sigma\left(\mathcal{M}_0\nabla_\Sigma\varphi_h\right)\mathcal{M}_0 - \nabla_\Sigma^2\varphi_h\right)\right\|_{\mathring{\mathcal{S}}_4} + \left\|\left(\beta - 1\right)\nabla_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(T_0)\, \|\beta\|_{\mathcal{C}_1}\, \|\left(\mathcal{M}_0\nabla_\Sigma\varphi_h\right)\mathcal{M}_0 - \nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{S}}_4} + C(T_0)\left(\|\beta - 1\|_{\mathcal{S}_4} + \|\beta - 1\|_\infty\right) \left\|\nabla_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0, M)\, \|h\|_{\mathcal{S}_4}\, \|\nabla_\Sigma\varphi_h\|_{\mathring{\mathcal{S}}_4} + C(\Sigma, T_0, M)\left(\|\nabla_\Sigma h\|_{\mathcal{S}_4} + \|\nabla_\Sigma h\|_\infty\right) \left\|\nabla_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{W}_5^T} + \|\nabla_\Sigma h\|_\infty\right) \|\varphi\|_{\mathring{\mathcal{W}}^T}\,.
\end{aligned} \tag{3.5.147}$$

From the estimates of I_{21} to I_{24}, we obtain the estimate of $I_2 - \left(\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma\right)\varphi_h$. For all $T \in (0, T_0]$ we have

$$\left\|I_2 - \left(\mathrm{tr}L_\Sigma^2 + \triangle_\Sigma\right)\varphi_h\right\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, T_0, M)\left(\|z\|_{\mathcal{W}^T} + \|h\|_{\mathcal{C}_4^T}\right) \|\varphi\|_{\mathring{\mathcal{W}}^T}\,. \tag{3.5.148}$$

Consequently, from (3.5.125) and the estimates in (3.5.133), (3.5.134), (3.5.135), (3.5.136) and (3.5.148), we obtain for all $T \in (0, T_0]$ that

$$\|D\mathcal{G}_2\left[z\right]\varphi\|_{\mathring{\mathcal{S}}_4^T} \leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_4^T} + \|z\|_{\mathcal{W}^T}\right)\left(1 + \|h\|_{\mathcal{C}_2^T} + \|z\|_{\mathcal{W}^T}\right) \|\varphi\|_{\mathring{\mathcal{W}}^T}\,. \tag{3.5.149}$$

3.5.3.4 Estimate of $\mathcal{G}_3$

Now we study $\mathcal{G}_3$. We abbreviate it to

$$\begin{aligned}
\mathcal{G}_3 &= \mathcal{P}_\Sigma \left[\!\!\left[\nu(\chi)((I-\mathcal{M}_1)\nabla u + ((I-\mathcal{M}_1)\nabla u)^\top)\right]\!\!\right] \mathcal{M}_0 \nabla_\Sigma h \\
&\quad + \mathcal{P}_\Sigma \left[\!\!\left[\nu(\chi)\left(\mathcal{M}_1 \nabla u + (\mathcal{M}_1 \nabla u)^\top\right)\right]\!\!\right] n_\Sigma \\
&\quad - \left(\left(\left[\!\!\left[\nu(\chi)\left((I-\mathcal{M}_1)\nabla u + ((I-\mathcal{M}_1)\nabla u)^\top\right)\right]\!\!\right](n_\Sigma - \mathcal{M}_0 \nabla_\Sigma h)\right)\cdot n_\Sigma\right) \mathcal{M}_0 \nabla_\Sigma h \\
&=: I_1 + I_2 - I_3.
\end{aligned} \tag{3.5.150}$$

To estimate I_1, we first calculate the Fréchet derivative of $F : z \mapsto (I-\mathcal{M}_1)\nabla u$:

$$DF[z]\varphi = (-D\mathcal{M}_1 \varphi_h)(\nabla u) + (I-\mathcal{M}_1)(\nabla \varphi_u), \tag{3.5.151}$$

which implies that

$$\begin{aligned}
DI_1[z]\varphi &= \mathcal{P}_\Sigma \Big[\!\!\Big[\nu\big((-D\mathcal{M}_1\varphi_h)(\nabla u) + (I-\mathcal{M}_1)(\nabla\varphi_u)\big) \\
&\qquad + \nu\big((-D\mathcal{M}_1\varphi_h)(\nabla u) + (I-\mathcal{M}_1)(\nabla\varphi_u)\big)^\top\Big]\!\!\Big] \mathcal{M}_0 \nabla_\Sigma h \\
&\quad + \mathcal{P}_\Sigma \left[\!\!\left[\nu(\chi)\left((I-\mathcal{M}_1)\nabla u + ((I-\mathcal{M}_1)\nabla u)^\top\right)\right]\!\!\right] \left((\mathcal{M}_0 \varphi_h L_\Sigma \mathcal{M}_0)(\nabla_\Sigma h) + \mathcal{M}_0(\nabla_\Sigma \varphi_h)\right) \\
&=: I_{11} + I_{12}.
\end{aligned} \tag{3.5.152}$$

Notice that the estimation of $[\![A + A^\top]\!]$ is equivalent to the estimation of $[\![A]\!]$. Moreover, the projection matrix $\mathcal{P}_\Sigma$ is fixed and thus will only become a constant in the estimation. Suppose that $\|h\|_{\mathcal{C}_4} < \delta_0$ for some sufficiently small $\delta_0(\Sigma)$ and $\|h\|_{\mathcal{C}_2} < M$ for some M. In I_{11}, we first obtain

$$\begin{aligned}
&\|[\![(-D\mathcal{M}_1\varphi_h)(\nabla u) + (I-\mathcal{M}_1)(\nabla\varphi_u)]\!]\|_{\mathring{\mathcal{S}}_4} \\
&\lesssim \|(-D\mathcal{M}_1\varphi_h)(\nabla u) + (I-\mathcal{M}_1)(\nabla\varphi_u)\|_{\mathcal{W}_6} \\
&\le C(\Sigma, T_0)\,\|D\mathcal{M}_1\varphi_h\|_{\mathring{\mathcal{C}}_1}\,\|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, T_0)\,\|I-\mathcal{M}_1\|_{\mathcal{C}_1}\,\|\nabla\varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\le C(\Sigma, T_0, M)\,\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\,\|u\|_{\mathcal{W}_1} + C(\Sigma, T_0, M)\,\|\varphi_u\|_{\mathring{\mathcal{W}}_1} \\
&\le C(\Sigma, T_0, M)\,(1 + \|z\|_{\mathcal{W}})\,\|\varphi\|_{\mathring{\mathcal{W}}}.
\end{aligned} \tag{3.5.153}$$

The estimation of $\mathcal{M}_0 \nabla_\Sigma h =: \alpha$ has been studied in Section 3.5.3.1. To estimate I_{12}, we use the same argument as in (3.5.153) to obtain

$$\|[\![(I-\mathcal{M}_1)\,\nabla u]\!]\|_{\mathcal{S}_4} \leq C(\Sigma, T_0, M)\,\|u\|_{\mathcal{W}_1}\,. \tag{3.5.154}$$

Moreover, we have

$$\|(\mathcal{M}_0 \varphi_h L_\Sigma \mathcal{M}_0)\,(\nabla_\Sigma h) + \mathcal{M}_0\,(\nabla_\Sigma \varphi_h)\|_{\mathring{\mathcal{C}}_1} \leq C(\Sigma, T_0, M)\,\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\,. \tag{3.5.155}$$

Therefore, by (3.5.153), (3.5.154) and (3.5.155) we have

$$\|DI_1\,[z]\,\varphi\|_{\mathring{\mathcal{S}}_4} \leq C(\Sigma, T_0, M)\,(1+\|z\|_{\mathcal{W}})\,\|\varphi\|_{\mathring{\mathcal{W}}}\left(\|h\|_{\mathcal{W}_5} + \|\nabla_\Sigma h\|_\infty\right) + C(\Sigma, T_0, M)\,\|u\|_{\mathcal{W}_1}\,\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\,. \tag{3.5.156}$$

Using almost the same argument as for I_1, we obtain the Fréchet derivative of I_2 and I_3:

$$DI_2\varphi = \mathcal{P}_\Sigma\left[\!\!\left[\nu\left((D\mathcal{M}_1\varphi_h)\,(\nabla u) + \mathcal{M}_1 \nabla \varphi_u + \left((D\mathcal{M}_1\varphi_h)\,(\nabla u) + \mathcal{M}_1\nabla\varphi_u\right)^\top\right)\right]\!\!\right] n_\Sigma, \tag{3.5.157}$$

and

$$\begin{aligned}
DI_3\varphi \;=\; & \Bigg(\Bigg(\Bigg[\!\!\Bigg[\,\nu\big(\,(-D\mathcal{M}_1\varphi_h)\,(\nabla u) + (I-\mathcal{M}_1)\,(\nabla\varphi_u)\,\big)\\
& +\nu\big(\,(-D\mathcal{M}_1\varphi_h)\,(\nabla u) + (I-\mathcal{M}_1)\,(\nabla\varphi_u)\,\big)^\top\Bigg]\!\!\Bigg]\,(n_\Sigma - \mathcal{M}_0\nabla_\Sigma h)\\
& + \left[\!\!\left[\nu(I-\mathcal{M}_1)\nabla u + \nu\left((I-\mathcal{M}_1)\nabla u\right)^\top\right]\!\!\right]\left((\mathcal{M}_0\varphi_h L_\Sigma\mathcal{M}_0)\,(\nabla_\Sigma h) + \mathcal{M}_0\,(\nabla_\Sigma\varphi_h)\right)\cdot n_\Sigma\Bigg)\mathcal{M}_0\nabla_\Sigma h\\
& + \left(\left(\left[\!\!\left[\nu(I-\mathcal{M}_1)\nabla u + \nu\left((I-\mathcal{M}_1)\nabla u\right)^\top\right]\!\!\right](n_\Sigma - \mathcal{M}_0\nabla_\Sigma h)\right)\cdot n_\Sigma\right)\left(D\,(\mathcal{M}_0\nabla_\Sigma h)\,\varphi_h\right)\\
& =: A_1 + A_2.
\end{aligned} \tag{3.5.158}$$

We remind the readers to be careful with the brackets in the term A_1 due to its complexity. Using the estimates in [28, Proposition 5.1 (a)], we have

$$\begin{aligned}
&\|DI_2\varphi\|_{\mathring{\mathcal{S}}_4} \lesssim \|[\![(D\mathcal{M}_1\varphi_h)(\nabla u) + \mathcal{M}_1\nabla\varphi_u]\!]\|_{\mathring{\mathcal{S}}_4} \\
&\lesssim \|(D\mathcal{M}_1\varphi_h)(\nabla u) + \mathcal{M}_1\nabla\varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\leq C(\Sigma, T_0)\|D\mathcal{M}_1\varphi_h\|_{\mathring{\mathcal{C}}_1}\|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, T_0)\left(\|\mathcal{M}_1\|_{\mathcal{W}_6} + \|\mathcal{M}_1\|_\infty\right)\|\nabla\varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\leq C(\Sigma, T_0, M)\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, T_0, M)\left(\|\nabla_\Sigma h\|_{\mathcal{W}_6} + \|\nabla_\Sigma h\|_\infty\right)\|\nabla\varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\leq C(\Sigma, T_0, M)\|\varphi_h\|_{\mathring{\mathcal{W}}_5}\|u\|_{\mathcal{W}_1} + C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{W}_5} + \|\nabla_\Sigma h\|_\infty\right)\|\varphi_u\|_{\mathring{\mathcal{W}}_1}.
\end{aligned} \tag{3.5.159}$$

To estimate I_3, we need the following estimates. For the term A_1 in (3.5.158), we abbreviate it to

$$\left(\left([\![\nu\left(I_{31} + I_{31}^\top\right)]\!]\, I_{32} + [\![\nu\left(I_{33} + I_{33}^\top\right)]\!]\, I_{34}\right)\cdot n_\Sigma\right) I_{35}, \tag{3.5.160}$$

where

$$\begin{aligned}
I_{31} &:= (-D\mathcal{M}_1\varphi_h)(\nabla u) + (I - \mathcal{M}_1)(\nabla\varphi_u), \\
I_{32} &:= n_\Sigma - \mathcal{M}_0\nabla_\Sigma h, \\
I_{33} &:= (I - \mathcal{M}_1)\nabla u, \\
I_{34} &:= (\mathcal{M}_0\varphi_h L_\Sigma\mathcal{M}_0)(\nabla_\Sigma h) + \mathcal{M}_0(\nabla_\Sigma\varphi_h), \\
I_{35} &:= \mathcal{M}_0\nabla_\Sigma h.
\end{aligned} \tag{3.5.161}$$

For I_{31} we have

$$\begin{aligned}
&\|(D\mathcal{M}_1\varphi_h)(\nabla u) + (I - \mathcal{M}_1)(\nabla\varphi_u)\|_{\mathring{\mathcal{W}}_6} \\
&\leq C(\Sigma, T_0)\|D\mathcal{M}_1\varphi_h\|_{\mathring{\mathcal{C}}_1}\|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, T_0)\|I - \mathcal{M}_1\|_{\mathcal{C}_1}\|\nabla\varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\leq C(\Sigma, T_0, M)\|\varphi_h\|_{\mathring{\mathcal{C}}_2}\|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, T_0, M)\|\nabla\varphi_u\|_{\mathring{\mathcal{W}}_6} \\
&\leq C(\Sigma, T_0, M)\|\varphi_h\|_{\mathring{\mathcal{W}}_5}\|\nabla u\|_{\mathcal{W}_6} + C(\Sigma, T_0, M)\|\varphi_u\|_{\mathring{\mathcal{W}}_1}.
\end{aligned} \tag{3.5.162}$$

For I_{32} we have

$$\|n_\Sigma - \mathcal{M}_0\nabla_\Sigma h\|_{\mathcal{C}_1} \leq C(\Sigma, M). \tag{3.5.163}$$

For I_{33} we have

$$\|(I - \mathcal{M}_1)\nabla u\|_{\mathcal{W}_6} \leq C(\Sigma, T_0)\|I - \mathcal{M}_1\|_{\mathcal{C}_1}\|\nabla u\|_{\mathcal{W}_6} \leq C(\Sigma, T_0, M)\|u\|_{\mathcal{W}_1}. \tag{3.5.164}$$

For I_{34} we have

$$
\begin{aligned}
&\left\|\left(\mathcal{M}_0\varphi_h L_\Sigma \mathcal{M}_0\right)\left(\nabla_\Sigma h\right)+\mathcal{M}_0\left(\nabla_\Sigma \varphi_h\right)\right\|_{\mathring{\mathcal{C}}_1}\\
&\leq \left\|\mathcal{M}_0\varphi_h L_\Sigma \mathcal{M}_0\right\|_{\mathring{\mathcal{C}}_1}\left\|\nabla_\Sigma h\right\|_{\mathcal{C}_1}+\left\|\mathcal{M}_0\right\|_{\mathcal{C}_1}\left\|\nabla_\Sigma \varphi_h\right\|_{\mathring{\mathcal{C}}_1} \qquad (3.5.165)\\
&\leq C(\Sigma,M)\left\|\varphi_h\right\|_{\mathring{\mathcal{C}}_1}\|h\|_{\mathcal{C}_2}+C(\Sigma,M)\left\|\varphi_h\right\|_{\mathring{\mathcal{C}}_2}.
\end{aligned}
$$

For I_{35} we have

$$
\left\|\mathcal{M}_0\nabla_\Sigma h\right\|_{\mathcal{C}_1}\lesssim \left\|\mathcal{M}_0\right\|_{\mathcal{C}_1}\left\|\nabla_\Sigma h\right\|_{\mathcal{C}_1}\leq C(\Sigma,M)\|h\|_{\mathcal{C}_2}. \qquad (3.5.166)
$$

Thus, we have

$$
\begin{aligned}
&\left\|\left(\left(\left[\!\left[\nu\left(I_{31}+I_{31}^\top\right)\right]\!\right] I_{32}+\left[\!\left[\nu\left(I_{33}+I_{33}^\top\right)\right]\!\right] I_{34}\right)\cdot n_\Sigma\right) I_{35}\right\|_{\mathring{\mathcal{S}}_4}\\
&\leq C(\Sigma,T_0)\left(\left\|\left[\!\left[I_{31}\right]\!\right]\right\|_{\mathring{\mathcal{S}}_4}\left\|I_{32}\right\|_{\mathcal{C}_1}+\left\|\left[\!\left[I_{33}\right]\!\right]\right\|_{\mathcal{S}_4}\left\|I_{34}\right\|_{\mathring{\mathcal{C}}_1}\right)\left(\left\|I_{35}\right\|_{\mathcal{S}_4}+\left\|I_{35}\right\|_\infty\right)\\
&\leq C(\Sigma,T_0)\left(\left\|I_{31}\right\|_{\mathring{\mathcal{W}}_6}\left\|I_{32}\right\|_{\mathcal{C}_1}+\left\|I_{33}\right\|_{\mathcal{W}_6}\left\|I_{34}\right\|_{\mathring{\mathcal{C}}_1}\right)\left(\left\|I_{35}\right\|_{\mathcal{S}_4}+\left\|I_{35}\right\|_\infty\right)\\
&\leq C(\Sigma,T_0,M)\Big(\left\|\varphi_h\right\|_{\mathring{\mathcal{W}}_5}\|\nabla u\|_{\mathcal{W}_6}+\left\|\varphi_u\right\|_{\mathring{\mathcal{W}}_1}+\|u\|_{\mathcal{W}_1}\left\|\varphi_h\right\|_{\mathring{\mathcal{C}}_2}\|h\|_{\mathcal{C}_2}\\
&\quad+\|u\|_{\mathcal{W}_1}\left\|\varphi_h\right\|_{\mathring{\mathcal{C}}_2}\Big)\left(\left\|\nabla_\Sigma h\right\|_{\mathcal{S}_4}+\left\|\nabla_\Sigma h\right\|_\infty\right)\\
&\leq C(\Sigma,T_0,M)\left(\|\varphi\|_{\mathring{\mathcal{W}}}\|z\|_{\mathcal{W}}+\|\varphi\|_{\mathring{\mathcal{W}}}+\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}\|h\|_{\mathcal{C}_2}+\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}\right)\left(\|h\|_{\mathcal{W}_5}+\left\|\nabla_\Sigma h\right\|_\infty\right)\\
&\leq C(\Sigma,T_0,M)\left(1+\|z\|_{\mathcal{W}}\right)\left(\|h\|_{\mathcal{W}_5}+\left\|\nabla_\Sigma h\right\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}}.
\end{aligned}
\qquad (3.5.167)
$$

The term A_2 in (3.5.158) can be abbreviated to

$$
\left(\left(\left[\!\left[\nu\left(I_{33}+I_{33}^\top\right)\right]\!\right] I_{32}\right)\cdot n_\Sigma\right)\left(D\alpha\varphi_h\right), \qquad (3.5.168)
$$

where α is defined in (3.5.98) and estimated in (3.5.96) and (3.5.99). Thus, we have

$$
\begin{aligned}
&\left\|\left(\left(\left[\!\left[\nu\left(I_{33}+I_{33}^\top\right)\right]\!\right] I_{32}\right)\cdot n_\Sigma\right)\left(D\alpha\varphi_h\right)\right\|_{\mathring{\mathcal{S}}_4}\leq C(\Sigma,T_0)\left\|\left[\!\left[I_{33}\right]\!\right]\right\|_{\mathcal{S}_4}\left\|I_{32}\right\|_{\mathcal{C}_1}\left\|D\alpha\varphi_h\right\|_{\mathring{\mathcal{C}}_1}\\
&\leq C(\Sigma,T_0,M)\left\|I_{33}\right\|_{\mathcal{W}_6}\left\|\varphi_h\right\|_{\mathring{\mathcal{C}}_2}\leq C(\Sigma,T_0,M)\|u\|_{\mathcal{W}_1}\left\|\varphi_h\right\|_{\mathring{\mathcal{C}}_2}\leq C(\Sigma,T_0,M)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}.
\end{aligned}
\qquad (3.5.169)
$$

From (3.5.167) and (3.5.169), we obtain

$$
\left\|DI_3\varphi\right\|_{\mathring{\mathcal{S}}_4}\leq C(\Sigma,T_0,M)\left(1+\|z\|_{\mathcal{W}}\right)\left(\|h\|_{\mathcal{W}_5}+\left\|\nabla_\Sigma h\right\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}}+C(\Sigma,T_0,M)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}. \qquad (3.5.170)
$$

Consequently, the estimate of $D\mathcal{G}_3$ follows (3.5.156), (3.5.159) and (3.5.170). For all $T\in(0,T_0]$ we have

$$
\left\|D\mathcal{G}_3[z]\,\varphi\right\|_{\mathring{\mathcal{S}}_4^T}\leq C(\Sigma,T_0,M)\left(1+\|z\|_{\mathcal{W}^T}\right)\left(\|z\|_{\mathcal{W}^T}+\left\|\nabla_\Sigma h\right\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}. \qquad (3.5.171)
$$

3.5.4 Estimates of nonlinear terms

In this section, we estimate the nonlinear operators G_i using the $\mathcal{S}_i$ norm with $i = 1, \cdots, 5$, using similar ideas as in [15, 28]. Similar to the arguments before, we still assume $\|h\|_{\mathcal{C}_4} < \delta_0$ for some sufficiently small $\delta_0(\Sigma)$ and $\|h\|_{\mathcal{C}_2} < M$. We will also ignore the parameter T in function spaces when there is no confusion.

3.5.4.1 Term G_1

We start with the Fréchet derivative of G_1. We will use the $\mathcal{S}_1$ norm, i.e. the $L^q([0,T];L^q(\Omega))$ norm. For the 1st term we have

$$D\left(\frac{1}{2}\nabla\left(|B|^2\right)\right)\varphi = D(\nabla B \cdot B)\varphi = \nabla\varphi_B \cdot B + \nabla B \cdot \varphi_B. \tag{3.5.172}$$

Its estimate is

$$\begin{aligned}
&\left\|\left(D\frac{1}{2}\nabla\left(|B|^2\right)\right)\varphi\right\|_{\mathring{\mathcal{S}}_1} \lesssim \|\nabla\varphi_B\|_{\mathring{\mathcal{C}}_0}\|B\|_{\mathcal{S}_1} + \|\nabla B\|_{\mathcal{S}_1}\|\nabla\varphi_B\|_{\mathring{\mathcal{C}}_0}\\
&\lesssim \|\varphi_B\|_{\mathring{\mathcal{C}}_1}\|B\|_{\mathcal{S}_1} + \|B\|_{\mathcal{W}_6}\|\varphi_B\|_{\mathring{\mathcal{C}}_0}\\
&\lesssim \|\varphi_B\|_{\mathring{\mathcal{W}}_2}\|B\|_{\mathcal{W}_2} + \|B\|_{\mathcal{W}_2}\|\varphi_B\|_{\mathring{\mathcal{W}}_2} \le C(\Sigma, T_0)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}.
\end{aligned} \tag{3.5.173}$$

The 2nd and 3rd terms can be estimated using exactly the same argument and function spaces. For the 4th term we have

$$D\left(\mathcal{M}_3\nabla u\right)\varphi = \left(D\mathcal{M}_3\varphi_h\right)\nabla u + \mathcal{M}_3\nabla\varphi_u. \tag{3.5.174}$$

Using (3.5.83), we have

$$\begin{aligned}
&\|D\left(\mathcal{M}_3\nabla u\right)\varphi\|_{\mathring{\mathcal{S}}_1} \lesssim \|D\mathcal{M}_3\varphi_h\|_{\mathring{\mathcal{C}}_0}\|\nabla u\|_{\mathcal{S}_1} + \|\mathcal{M}_3\|_{\mathcal{S}_1}\|\nabla\varphi_u\|_{\mathring{\mathcal{C}}_0}\\
&\le C(\Sigma, T_0, M)\|\varphi_h\|_{\mathring{\mathcal{C}}_1}\|u\|_{\mathcal{W}_1} + C(\Sigma, T_0, M)\|\partial_t h\|_{\mathcal{S}_1}\|\varphi_u\|_{\mathring{\mathcal{W}}_1}\\
&\le C(\Sigma, T_0, M)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}.
\end{aligned} \tag{3.5.175}$$

For the 5th term we have

$$D\left(u\mathcal{M}_1\nabla u\right)\varphi = \varphi_u\mathcal{M}_1\nabla u + u\left(D\mathcal{M}_1\varphi_h\right)\nabla u + u\mathcal{M}_1\nabla\varphi_u. \tag{3.5.176}$$

Its estimate is

$$
\begin{aligned}
&\|D(u\mathcal{M}_1\nabla u)\varphi\|_{\mathring{\mathcal{S}}_1} \\
&\lesssim \|\varphi_u\|_{\mathring{\mathcal{C}}_0}\|\mathcal{M}_1\|_{\mathcal{C}_0}\|\nabla u\|_{\mathcal{S}_1} + \|u\|_\infty \|D\mathcal{M}_1\varphi_h\|_{\mathring{\mathcal{C}}_0}\|\nabla u\|_{\mathcal{S}_1} + \|u\|_{\mathcal{S}_1}\|\mathcal{M}_1\|_{\mathcal{C}_0}\|\nabla\varphi_u\|_{\mathring{\mathcal{C}}_0} \\
&\leq C(\Sigma, T_0, M)\left(\|\varphi_u\|_{\mathring{\mathcal{C}}_0}\|h\|_{\mathcal{C}_3}\|u\|_{\mathcal{W}_1} + \|u\|_\infty\|\varphi_h\|_{\mathring{\mathcal{C}}_3}\|u\|_{\mathcal{W}_1} + \|u\|_{\mathcal{W}_1}\|h\|_{\mathcal{C}_3}\|\varphi_u\|_{\mathring{\mathcal{C}}_3}\right) \\
&\leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_3} + \|u\|_\infty\right)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}.
\end{aligned}
\tag{3.5.177}
$$

Using the same argument, we obtain the estimate of the 6th term:

$$
\|D(B\mathcal{M}_1\nabla B)\varphi\|_{\mathring{\mathcal{S}}_1} \leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_1} + \|B\|_\infty\right)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}. \tag{3.5.178}
$$

For the 7th term we have

$$
D\left(\frac{1}{2}\mathcal{M}_1\nabla\left(|B|^2\right)\right)\varphi = (D\mathcal{M}_1\varphi)\nabla BB + \mathcal{M}_1\nabla\varphi_B B + \mathcal{M}_1\nabla B\varphi_B. \tag{3.5.179}
$$

Estimating by terms, we obtain

$$
\begin{aligned}
&\|(D\mathcal{M}_1\varphi)\nabla BB\|_{\mathring{\mathcal{S}}_1} \lesssim \|D\mathcal{M}_1\varphi_h\|_{\mathring{\mathcal{C}}_0}\|\nabla B\|_{\mathcal{S}_1}\|B\|_\infty \\
&\leq C(\Sigma, T_0, M)\|\varphi_h\|_{\mathring{\mathcal{W}}_5}\|B\|_{\mathcal{W}_2}\|B\|_\infty \leq C(\Sigma, T_0, M)\|\varphi\|_{\mathring{\mathcal{W}}}\|z\|_{\mathcal{W}}\|B\|_\infty,
\end{aligned}
\tag{3.5.180}
$$

$$
\begin{aligned}
&\|\mathcal{M}_1\nabla\varphi_B B\|_{\mathring{\mathcal{S}}_1} \lesssim \|\mathcal{M}_1\|_{\mathcal{C}_0}\|\nabla\varphi_B\|_{\mathring{\mathcal{C}}_0}\|B\|_{\mathcal{S}_1} \\
&\leq C(\Sigma, T_0, M)\|h\|_{\mathcal{C}_3}\|\varphi_B\|_{\mathring{\mathcal{W}}_2}\|B\|_{\mathcal{W}_2} \leq C(\Sigma, T_0, M)\|h\|_{\mathcal{C}_3}\|\varphi\|_{\mathring{\mathcal{W}}}\|z\|_{\mathcal{W}},
\end{aligned}
\tag{3.5.181}
$$

and

$$
\begin{aligned}
&\|\mathcal{M}_1\nabla B\varphi_B\|_{\mathring{\mathcal{S}}_1} \lesssim \|\mathcal{M}_1\|_{\mathcal{C}_0}\|\varphi_B\|_{\mathring{\mathcal{C}}_0}\|\nabla B\|_{\mathcal{S}_1} \\
&\leq C(\Sigma, T_0, M)\|h\|_{\mathcal{C}_3}\|\varphi_B\|_{\mathring{\mathcal{W}}_2}\|B\|_{\mathcal{W}_2} \leq C(\Sigma, T_0, M)\|h\|_{\mathcal{C}_3}\|\varphi\|_{\mathring{\mathcal{W}}}\|z\|_{\mathcal{W}}.
\end{aligned}
\tag{3.5.182}
$$

Thus, we have

$$
\left\|D\left(\frac{1}{2}\mathcal{M}_1\nabla\left(|B|^2\right)\right)\varphi\right\|_{\mathring{\mathcal{S}}_1} \leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_3} + \|B\|_\infty\right)\|z\|_{\mathcal{W}}\|\varphi\|_{\mathring{\mathcal{W}}}. \tag{3.5.183}
$$

For the 8th term we have

$$
D(\mathcal{M}_1\nabla p)\varphi = (D\mathcal{M}_1\varphi_h)\nabla p + \mathcal{M}_1\nabla\varphi_p. \tag{3.5.184}
$$

For any $\varphi \in \mathring{\mathcal{W}}$ we have the estimate

$$\begin{aligned}
&\left\| D\left(\mathcal{M}_1 \nabla p\right) \varphi \right\|_{\mathring{\mathcal{S}}_1} \lesssim \left\| D\mathcal{M}_1 \varphi_h \right\|_{\mathring{\mathcal{C}}_0} \left\| \nabla p \right\|_{\mathcal{S}_1} + \left\| \mathcal{M}_1 \right\|_\infty \left\| \nabla \varphi_p \right\|_{\mathcal{S}_1} \\
&\leq C(\Sigma, T_0, M) \left\| \varphi_h \right\|_{\mathring{\mathcal{C}}_1} \left\| p \right\|_{\mathcal{W}_3} + C(\Sigma, T_0, M) \left\| \nabla_\Sigma h \right\|_\infty \left\| \varphi_p \right\|_{\mathring{\mathcal{W}}_3} \\
&\leq C(\Sigma, T_0, M) \left\| \varphi_h \right\|_{\mathring{\mathcal{W}}_5} \left\| p \right\|_{\mathcal{W}_3} + C(\Sigma, T_0, M) \left\| \nabla_\Sigma h \right\|_\infty \left\| \varphi_p \right\|_{\mathring{\mathcal{W}}_3} \\
&\leq C(\Sigma, T_0, M) \left(\left\| z \right\|_{\mathcal{W}} + \left\| \nabla_\Sigma h \right\|_\infty \right) \left\| \varphi \right\|_{\mathring{\mathcal{W}}}.
\end{aligned} \tag{3.5.185}$$

For the 9th term, notice that the viscosity ν is a fixed function in the transformed equations since the interface has been pulled to a fixed interface. Thus, we have

$$D\left(\nu \mathcal{M}_4 : \nabla^2 u\right) \varphi = \nu \left(D\mathcal{M}_4 \varphi_h\right) : \nabla^2 u + \nu \mathcal{M}_4 : \nabla^2 \varphi_u. \tag{3.5.186}$$

For any $\varphi \in \mathring{\mathcal{W}}$ we have the estimate

$$\begin{aligned}
&\left\| D\left(\nu \mathcal{M}_4 : \nabla^2 u\right) \varphi \right\|_{\mathcal{S}_1} \lesssim \left\| D\mathcal{M}_4 \varphi_h \right\|_{\mathring{\mathcal{C}}_0} \left\| \nabla^2 u \right\|_{\mathcal{S}_1} + \left\| \mathcal{M}_4 \right\|_\infty \left\| \nabla^2 \varphi_u \right\|_{\mathcal{S}_1} \\
&\leq C(\Sigma, T_0, M) \left\| \varphi_h \right\|_{\mathring{\mathcal{C}}_3} \left\| u \right\|_{\mathcal{W}_1} + C(\Sigma, T_0, M) \left\| \nabla_\Sigma h \right\|_\infty \left\| \varphi_u \right\|_{\mathring{\mathcal{W}}_1} \\
&\leq C(\Sigma, T_0, M) \left\| \varphi_h \right\|_{\mathring{\mathcal{W}}_5} \left\| u \right\|_{\mathcal{W}_1} + C(\Sigma, T_0, M) \left\| \nabla_\Sigma h \right\|_\infty \left\| \varphi_u \right\|_{\mathring{\mathcal{W}}_1} \\
&\leq C(\Sigma, T_0, M) \left(\left\| z \right\|_{\mathcal{W}} + \left\| \nabla_\Sigma h \right\|_\infty \right) \left\| \varphi \right\|_{\mathring{\mathcal{W}}}.
\end{aligned} \tag{3.5.187}$$

For the 10th term we have

$$D\left(\nu \mathcal{M}_2 \cdot (\nabla u)\right) \varphi = \nu \left(D\mathcal{M}_2 \varphi_h\right) \cdot (\nabla u) + \nu \mathcal{M}_2 \cdot \left(\nabla \varphi_u\right). \tag{3.5.188}$$

For any $\varphi \in \mathring{\mathcal{W}}$ we have the estimate

$$\begin{aligned}
&\left\| D\left(\nu \mathcal{M}_2 \cdot (\nabla u)\right) \varphi \right\|_{\mathcal{S}_1} \lesssim \left\| D\mathcal{M}_2 \varphi_h \right\|_{\mathring{\mathcal{C}}_0} \left\| \nabla u \right\|_{\mathcal{S}_1} + \left\| \mathcal{M}_2 \right\|_\infty \left\| \nabla \varphi_u \right\|_{\mathcal{S}_1} \\
&\leq C(\Sigma, T_0, M) \left\| \varphi_h \right\|_{\mathring{\mathcal{C}}_4} \left\| u \right\|_{\mathcal{W}_1} + C(\Sigma, T_0, M) \left\| \nabla_\Sigma h \right\|_\infty \left\| \varphi_u \right\|_{\mathring{\mathcal{W}}_1} \\
&\leq C(\Sigma, T_0, M) \left(\left\| z \right\|_{\mathcal{W}} + \left\| \nabla_\Sigma h \right\|_\infty \right) \left\| \varphi \right\|_{\mathring{\mathcal{W}}}.
\end{aligned} \tag{3.5.189}$$

Combining the estimates of all these 10 terms, we obtain

$$\left\| DG_1 [z] \varphi \right\|_{\mathcal{S}_1} \leq C(\Sigma, T_0, M) \left(\left\| \nabla_\Sigma h \right\|_\infty + \left(1 + \left\| u \right\|_\infty + \left\| B \right\|_\infty\right) \left\| z \right\|_{\mathcal{W}} \right) \left\| \varphi \right\|_{\mathring{\mathcal{W}}}. \tag{3.5.190}$$

3.5.4.2 Term G_2

Now we estimate G_2 using the $\mathcal{S}_2$ norm, which is equal to the $\mathcal{S}_1$ norm. The 1st and 2nd terms $u\nabla B$ and $B\nabla u$ can be treated using the same argument as in (3.5.172) and (3.5.173) since their structures and spaces are exactly the same. Thus, we have

$$\|D\,(u\nabla B)\,\varphi\|_{\mathcal{S}_2} \leq C\,\|z\|_{\mathcal{W}}\,\|\varphi\|_{\mathring{\mathcal{W}}} \tag{3.5.191}$$

and

$$\|D\,(B\nabla u)\,\varphi\|_{\mathcal{S}_2} \leq C\,\|z\|_{\mathcal{W}}\,\|\varphi\|_{\mathring{\mathcal{W}}}\,. \tag{3.5.192}$$

The 3rd and 4th terms $u\mathcal{M}_1\nabla B$ and $B\mathcal{M}_1\nabla u$ can be treated using the same argument as in (3.5.176) and (3.5.177), which implies

$$\begin{aligned}
&\|D\,(u\mathcal{M}_1\nabla B)\,\varphi\|_{\mathcal{S}_2}\\
&\leq C(\Sigma, T_0, M)\,\|\varphi_u\|_{\mathring{\mathcal{C}}_0}\,\|h\|_{\mathcal{C}_3}\,\|B\|_{\mathcal{W}_2} + C(\Sigma, T_0, M)\,\|u\|_{\infty}\,\|\varphi_h\|_{\mathring{\mathcal{C}}_3}\,\|B\|_{\mathcal{W}_2}\\
&\quad + C(\Sigma, T_0, M)\,\|u\|_{\mathcal{W}_1}\,\|h\|_{\mathcal{C}_3}\,\|\varphi_B\|_{\mathring{\mathcal{C}}_3}\\
&\leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_3} + \|u\|_{\infty}\right)\|z\|_{\mathcal{W}}\,\|\varphi\|_{\mathring{\mathcal{W}}}\,,
\end{aligned} \tag{3.5.193}$$

and

$$\|D\,(B\mathcal{M}_1\nabla u)\,\varphi\|_{\mathcal{S}_2} \leq C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_3} + \|B\|_{\infty}\right)\|z\|_{\mathcal{W}}\,\|\varphi\|_{\mathring{\mathcal{W}}}\,. \tag{3.5.194}$$

The 5th, 6th and 7th terms $\mathcal{M}_3\nabla B$, $\sigma\mathcal{M}_4 : \nabla^2 B$, and $\sigma\mathcal{M}_2\cdot\nabla B$ can be treated using the same argument as in (3.5.175), (3.5.187), and (3.5.189), which implies

$$\|D\,(\mathcal{M}_3\nabla B)\,\varphi\|_{\mathcal{S}_2} \leq C(\Sigma, T_0, M)\,\|z\|_{\mathcal{W}}\,\|\varphi\|_{\mathring{\mathcal{W}}}\,, \tag{3.5.195}$$

$$\left\|D\left(\sigma\mathcal{M}_4 : \nabla^2 B\right)\varphi\right\|_{\mathcal{S}_2} \leq C(\Sigma, T_0, M)\,(\|z\|_{\mathcal{W}} + \|\nabla_\Sigma h\|_{\infty})\,\|\varphi\|_{\mathring{\mathcal{W}}}\,, \tag{3.5.196}$$

and

$$\|D\,(\sigma\mathcal{M}_2\cdot(\nabla B))\,\varphi\|_{\mathcal{S}_2} \leq C(\Sigma, T_0, M)\,(\|z\|_{\mathcal{W}} + \|\nabla_\Sigma h\|_{\infty})\,\|\varphi\|_{\mathring{\mathcal{W}}}\,. \tag{3.5.197}$$

Consequently, we have the same estimate as in (3.5.190):

$$\|DG_2\,[z]\,\varphi\|_{\mathcal{S}_2} \leq C(\Sigma, T_0, M)\,(\|\nabla_\Sigma h\|_{\infty} + (1 + \|u\|_{\infty} + \|B\|_{\infty})\,\|z\|_{\mathcal{W}})\,\|\varphi\|_{\mathring{\mathcal{W}}}\,. \tag{3.5.198}$$

3.5.4.3 Term G_3

For term G_3, we only need to estimate $\mathcal{M}_1 : \nabla u$. Its Fréchet derivative is

$$DG_3\varphi = D\left(\mathcal{M}_1 : \nabla u\right)\varphi = \left(D\mathcal{M}_1\varphi_h\right) : \nabla u + \mathcal{M}_1 : \nabla\varphi_u. \tag{3.5.199}$$

Suppose that $\|h\|_{\mathcal{C}_4} < \delta_0$ and $\|h\|_{\mathcal{C}_2} < M$. For all $\varphi \in \mathring{\mathcal{W}}$, using similar ideas as in [28, (5.20)], we have

$$\begin{aligned}
&\|DG_3\varphi\|_{\mathring{\mathcal{S}}_3} = \|D\left(\mathcal{M}_1 : \nabla u\right)\varphi\|_{\mathring{\mathcal{S}}_3} \\
&\leq \|D\left(\mathcal{M}_1 : \nabla u\right)\varphi\|_{\mathring{W}^{1,q}([0,T];\dot{W}^{-1,q}(\Omega))} + \|D\left(\mathcal{M}_1 : \nabla u\right)\varphi\|_{L^q([0,T];W^{1,q}(\Omega))} \\
&\leq C(\Sigma, T_0, M)\left(\|\nabla_\Sigma h\|_\infty + \|h\|_{\mathcal{W}_5^T} + \|u\|_{\mathcal{W}_1^T}\right)\left(\|\varphi_u\|_{\mathring{\mathcal{W}}_1^T} + \|\varphi_h\|_{\mathring{\mathcal{W}}_5^T}\right) \\
&\quad + C\|D\mathcal{M}_1\varphi_h\|_{\mathring{\mathcal{C}}_3^T}\|\nabla u\|_{\mathcal{W}_1^T} + C\|\mathcal{M}_1\|_{\mathcal{C}_3^T}\|\nabla\varphi_u\|_{\mathring{\mathcal{W}}_1^T} \\
&\leq C(\Sigma, T_0, M)\left(\|\nabla_\Sigma h\|_\infty + \|z\|_{\mathcal{W}}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}.
\end{aligned} \tag{3.5.200}$$

3.5.4.4 Term G_4

For term G_4, all its components have been studied in Section 3.5.3. We estimate its Fréchet derivative

$$DG_4\varphi = \left(D\mathcal{G}_1\varphi_h + D\mathcal{G}_2\varphi_h + \kappa(\mathrm{tr}L_\Sigma^2)\varphi_h\right)n_\Sigma + D\mathcal{G}_3\varphi_h. \tag{3.5.201}$$

When $\|h\|_{\mathcal{C}_4} < \delta_0$ and $\|h\|_{\mathcal{C}_2} < M$, using (3.5.108), (3.5.149) and (3.5.171), we obtain for all $\varphi \in \mathring{\mathcal{W}}$ and all $T \in (0, T_0]$ that

$$\begin{aligned}
&\|DG_4\varphi\|_{\mathring{\mathcal{S}}_4^T} \lesssim \|D\mathcal{G}_1\varphi_h\|_{\mathring{\mathcal{S}}_4} + \|D\mathcal{G}_2\varphi_h\|_{\mathring{\mathcal{S}}_4} + \left\|\mathrm{tr}L_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4} + \|D\mathcal{G}_3\varphi_h\|_{\mathring{\mathcal{S}}_4} \\
&\leq C(\Sigma, T_0, M)\left(\|z\|_{\mathcal{W}} + \|h\|_{\mathcal{C}_4}\right)\|\varphi\|_{\mathcal{W}} \\
&\quad + C(\Sigma, T_0, M)\left(\|h\|_{\mathcal{C}_4} + \|z\|_{\mathcal{W}}\right)\left(1 + \|h\|_{\mathcal{C}_2} + \|z\|_{\mathcal{W}}\right)\|\varphi\|_{\mathring{\mathcal{W}}} + C\left\|\mathrm{tr}L_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4} \\
&\quad + C(\Sigma, T_0, M)\left(1 + \|z\|_{\mathcal{W}}\right)\left(\|h\|_{\mathcal{C}_4} + \|z\|_{\mathcal{W}}\right)\|\varphi\|_{\mathring{\mathcal{W}}} \\
&\leq C(\Sigma, T_0, M)\left(1 + \|h\|_{\mathcal{C}_2^T} + \|z\|_{\mathcal{W}^T}\right)\left(\|h\|_{\mathcal{C}_4^T} + \|z\|_{\mathcal{W}^T}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T} + C\left\|\mathrm{tr}L_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4^T}.
\end{aligned} \tag{3.5.202}$$

3.5.4.5 Term G_5

In the term G_5, we recall that b is a fixed auxiliary function for linearity. The details of the term b can be found in [15]. For all $\varphi \in \mathring{\mathcal{W}}$ we have

$$\begin{aligned}
DG_5\varphi =& -((D\mathcal{M}_0\varphi_h)(\nabla_\Sigma h)) \cdot u + ((I - \mathcal{M}_0)\nabla_\Sigma \varphi_h) \cdot u \\
&+ ((I - \mathcal{M}_0)\nabla_\Sigma h) \cdot \varphi_u - \varphi_u \cdot \nabla_\Sigma h + (b - u) \cdot \nabla_\Sigma \varphi_h \\
=:& -I_1 + I_2 + I_3 - I_4 + I_5.
\end{aligned} \tag{3.5.203}$$

Suppose that $\|h\|_{\mathcal{C}_4} < \delta_0$ and $\|h\|_{\mathcal{C}_2} < M$. From [26, Proposition 5.1 (a) and Lemma 5.5], we have for all $T \in (0, T_0]$ that

$$\begin{aligned}
&\|I_1\|_{\mathring{\mathcal{S}}_5^T} \le C(\Sigma, T_0) \left\|u^\top (D\mathcal{M}_0\varphi_h)\right\|_{\mathring{\mathcal{S}}_5} \left(\|\nabla_\Sigma h\|_{\mathcal{S}_5} + \|\nabla_\Sigma h\|_\infty\right) \\
&\le C(T_0) \|u\|_{\mathcal{S}_5} \|D\mathcal{M}_0\varphi_h\|_{\mathring{\mathcal{C}}_2} \left(\|\nabla_\Sigma h\|_{\mathcal{S}_5} + \|\nabla_\Sigma h\|_\infty\right) \\
&\le C(\Sigma, T_0, M) \|u\|_{\mathcal{S}_5} \|\varphi_h\|_{\mathring{\mathcal{C}}_2} \left(\|\nabla_\Sigma h\|_{\mathcal{S}_5} + \|\nabla_\Sigma h\|_\infty\right) \\
&\le C(\Sigma, T_0, M) \|u\|_{\mathcal{W}_1} \|\varphi_h\|_{\mathring{\mathcal{W}}_5} \left(\|h\|_{\mathcal{W}_5} + \|\nabla_\Sigma h\|_\infty\right) \\
&\le C(\Sigma, T_0, M) \|z\|_{\mathcal{W}^T} (\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty) \|\varphi\|_{\mathring{\mathcal{W}}^T} .
\end{aligned} \tag{3.5.204}$$

Since $I - \mathcal{M}_0 = (I - hL_\Sigma)^{-1} ((I - hL_\Sigma) - I) = -\mathcal{M}_0 L_\Sigma h$, we have

$$\begin{aligned}
&\|I_2\|_{\mathring{\mathcal{S}}_5^T} = \|((\mathcal{M}_0 L_\Sigma h)\nabla_\Sigma \varphi_h) \cdot u\|_{\mathring{\mathcal{S}}_5} \\
&\le C(\Sigma, T_0, M) \left(\|h\|_{\mathcal{S}_5} + \|h\|_\infty\right) \left(\|u\|_{\mathcal{S}_5} + \|u\|_\infty\right) \|\nabla_\Sigma \varphi_h\|_{\mathring{\mathcal{S}}_5} \\
&\le C(\Sigma, T_0, M) (\|z\|_{\mathcal{W}^T} + \|h\|_\infty) (\|z\|_{\mathcal{W}^T} + \|u\|_\infty) \|\varphi\|_{\mathring{\mathcal{W}}^T} .
\end{aligned} \tag{3.5.205}$$

Similarly, we obtain the estimates for I_3, I_4 and I_5:

$$\begin{aligned}
&\|I_3\|_{\mathring{\mathcal{S}}_5^T} = \|((\mathcal{M}_0 L_\Sigma h)\nabla_\Sigma h) \cdot \varphi_u\|_{\mathring{\mathcal{S}}_5} \\
&\le C(\Sigma, T_0, M) \left(\|h\|_{\mathcal{S}_5} + \|h\|_\infty\right) \left(\|\nabla_\Sigma h\|_{\mathcal{S}_5} + \|\nabla_\Sigma h\|_\infty\right) \|\varphi_u\|_{\mathring{\mathcal{S}}_5} \\
&\le C(\Sigma, T_0, M) \left(\|h\|_{\mathcal{S}_5} + \|h\|_\infty\right) \left(\|h\|_{\mathcal{W}_5} + \|\nabla_\Sigma h\|_\infty\right) \|\varphi_u\|_{\mathring{\mathcal{S}}_5} \\
&\le C(\Sigma, T_0, M) (\|z\|_{\mathcal{W}^T} + \|h\|_\infty) (\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty) \|\varphi\|_{\mathring{\mathcal{W}}^T} ,
\end{aligned} \tag{3.5.206}$$

$$\begin{aligned}
&\|I_4\|_{\mathring{\mathcal{S}}_5^T} \le C(\Sigma, T_0) \left(\|\nabla_\Sigma h\|_{\mathcal{S}_5} + \|\nabla_\Sigma h\|_\infty\right) \|\varphi_u\|_{\mathring{\mathcal{S}}_5} \\
&\le C(\Sigma, T_0) (\|z\|_{\mathcal{W}^T} + \|\nabla_\Sigma h\|_\infty) \|\varphi\|_{\mathring{\mathcal{W}}^T} ,
\end{aligned} \tag{3.5.207}$$

and

$$\begin{aligned}\|I_5\|_{\mathring{\mathcal{S}}_5^T} &\leq C(\Sigma,T_0)\left(\|b-u\|_{\mathcal{S}_5}+\|b-u\|_\infty\right)\|\varphi_h\|_{\mathring{\mathcal{S}}_5}\\ &\leq C(\Sigma,T_0)\left(\|b-u\|_{\mathcal{S}_5^T}+\|b-u\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}.\end{aligned} \tag{3.5.208}$$

Consequently, we have

$$\begin{aligned}\|DG_5\varphi\|_{\mathring{\mathcal{S}}_5} \leq{}& C(\Sigma,T_0,M)\left(1+\|z\|_{\mathcal{W}}+\|h\|_\infty+\|u\|_\infty\right)\left(\|z\|_{\mathcal{W}}+\|h\|_{\mathcal{C}_4}\right)\|\varphi\|_{\mathring{\mathcal{W}}}\\ &+C(\Sigma,T_0)\left(\|b-u\|_{\mathcal{S}_5}+\|b-u\|_\infty\right)\|\varphi\|_{\mathring{\mathcal{W}}}.\end{aligned} \tag{3.5.209}$$

3.5.4.6 Estimate of operator G

Combining (3.5.190), (3.5.198), (3.5.200), (3.5.202) and (3.5.209), we have the following estimate.

Proposition 3.5.5. *Given any C^3 surface Σ, there exists $\delta_0(\Sigma)\in(0,1)$ sufficiently small, such that for all $T_0>0$ and all $z=(u,B,p,\varpi,h)\in\mathcal{W}$, if*

1. $\|h\|_{\mathcal{C}_4^{T_0}}<\delta_0$;
2. $\|h\|_{\mathcal{C}_2^{T_0}}\leq M$ *for some* $M>0$;

then for all $T\in(0,T_0]$ we have the estimate

$$\begin{aligned}&\|DG[z]\varphi\|_{\mathring{\mathcal{S}}^T}\\ &\leq C(\Sigma,T_0,M)\left(1+\|u\|_{L^\infty([0,T]\times\Omega)}+\|B\|_{L^\infty([0,T]\times\Omega)}+\|h\|_{\mathcal{C}_2^T}+\|z\|_{\mathcal{W}^T}\right)\\ &\qquad\cdot\left(\|h\|_{\mathcal{C}_4^T}+\|z\|_{\mathcal{W}^T}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}\\ &\quad+C\left\|\mathrm{tr}L_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4^T}+C(\Sigma,T_0)\left(\|b-u\|_{\mathcal{S}_5^T}+\|b-u\|_{L^\infty([0,T]\times\Sigma)}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}\end{aligned} \tag{3.5.210}$$

for all $\varphi=(\varphi_u,\varphi_B,\varphi_p,\varphi_\varpi,\varphi_h)\in\mathring{\mathcal{W}}^T$.

3.6 Local Existence

In this section, we study the existence of strong solutions. Without loss of generality, we fix $T_0 > 0$ and only consider time intervals $[0, T] \subseteq [0, T_0]$. We first consider the case that the initial interface is a C^3 surface. In this case, we choose the initial interface itself to be the reference surface, which implies that the initial height function $h_0 = 0$. Next, we study the case that the initial interface is a $W^{3-2/q}$ surface and is close to some C^3 surface in the sense of C^2 norm.

3.6.1 C^3 initial interface

Suppose that we have the initial condition

$$u_0 \in W^{2-\frac{2}{q},q}(\Omega \setminus \Gamma_0) \cap C(\Omega), \quad B_0 \in W^{2-\frac{2}{q},q}(\Omega), \quad \textit{and} \quad \Gamma_0 \in C^3,$$

which satisfy:

1. $\mathrm{div} u_0 = 0$ in $\Omega \setminus \Gamma_0$; $\mathrm{div} B_0 = 0$ in Ω;
2. $u_0 = B_0 = 0$ on $\partial\Omega$;
3. Γ_0 is a closed interface and $\Gamma_0 \cap \partial\Omega = \emptyset$;
4. $\mathcal{P}_{\Gamma_0} \left[\!\left[\nu \tilde{D} u_0 \right]\!\right] n_{\Gamma_0} = 0$.

Letting the reference surface be $\Sigma := \Gamma_0$, then we have $h_0 = 0$, $\overline{u}_0 = u_0$ and $\overline{B_0} = B_0$. The solution can be obtained by first finding an auxiliary solution with initial value u_0 and B_0, then finding the remaining part, which has an initial value of 0.

Picking an arbitrary $T_0 > 0$, using the same argument as in [15, Theorem 2], we can extend the initial value u_0 to a function $u_b \in \mathcal{W}_1^{T_0}$. Letting b be the restriction of u_b to Σ, i.e.

$$b := u_b|_{[0,T_0]\times\Sigma}. \tag{3.6.1}$$

We recall the solution operator S in Section 3.4, which allows us to define the auxiliary solution by

$$z_\alpha := (u_\alpha, B_\alpha, p_\alpha, \varpi_\alpha, h_\alpha) := S_{(u_0,B_0,0,b)}(0,0,0,0,0). \tag{3.6.2}$$

Next, similarly as in [15], we consider the equation

$$L(z + z_\alpha) = G(z + z_\alpha), \quad z(0) = 0. \tag{3.6.3}$$

with $z \in \mathring{\mathcal{W}}$. The sum $z + z_\alpha$ will then solve the transformed equations with initial value $(u_0, B_0, h_0) = (u_0, B_0, 0)$. The equation (3.6.3) implies

$$Lz = G(z + z_\alpha) - Lz_\alpha.$$

When $t = 0$, the right-hand side turns to $G(z_0) - L(z_0)$. The compatibility conditions for $G(z_0) - L(z_0)$ are $\mathrm{div} u_0 = 0$ and $(I - n_{\Gamma_0} \otimes n_{\Gamma_0}) \left[\!\left[\nu \left(\nabla u_0 + \nabla u_0^\top \right) \right]\!\right] n_{\Gamma_0} = 0$, which are exactly included in the requirements of the initial conditions. Thus, using the solution operator $S_{(0,0,0,b)}$ and the fact that $Lz_\alpha = 0$, we obtain the equation

$$z = S_{(0,0,0,b)} G(z + z_\alpha) =: K(z). \tag{3.6.4}$$

It remains to find the fixed point of K in the space $\mathring{\mathcal{W}}$, which can be done by a contraction mapping argument. Let $r_0 > 0$ be a sufficiently large fixed number. For $r \in (0, r_0]$ and $T \in (0, T_0]$, we define

$$\mathcal{B}_r^T := \{ w \in \mathring{\mathcal{W}}^T : \|w\|_{\mathcal{W}^T} \leq r \}. \tag{3.6.5}$$

Our goal is to show that K is a contraction mapping on $\mathcal{B}_r^T$ for suitable r and T.

In the auxiliary solution z_α, we have $h_\alpha(0) = h_0 = 0$, which implies that $\|h_\alpha(0)\|_{C^2} = 0$. Moreover, by [28, Proposition 5.1] we have $h_\alpha \in \mathcal{W}_5^{T_0} \hookrightarrow \mathcal{C}_2^{T_0}$, which implies that $h_\alpha \in \mathcal{C}_4^T = C([0, T]; C^2(\Sigma))$. Thus, given any $\varepsilon > 0$, there exists a sufficiently small $T_1 > 0$ such that $\|h_\alpha(t)\|_{C^2(\Sigma)} < \varepsilon$ on $[0, T_1]$, i.e. $\|h_\alpha\|_{\mathcal{C}_4^{T_1}} < \varepsilon$.

Given any $z_1, z_2 \in \mathcal{B}_r^{T_0}$, we have the estimate

$$\begin{aligned} \|K(z_1) - K(z_2)\|_{\mathcal{W}^T} &\leq \|S\| \, \|G(z_1 + z_\alpha) - G(z_2 + z_\alpha)\| \\ &\leq C \|S\| \sup_{0 \leq c \leq 1} \|DG[cz_1 + (1-c)z_2 + z_\alpha]\|_{\mathcal{S}^T} \|z_1 - z_2\|_{\mathring{\mathcal{W}}^T} . \end{aligned} \tag{3.6.6}$$

Letting $z_\xi := cz_1 + (1-c)z_2$ for abbreviation, we estimate the operator DG.

Proposition 3.6.1. *The term* $1 + \|u\|_\infty + \|B\|_\infty + \|h\|_{\mathcal{C}_2} + \|z\|_{\mathcal{W}}$ *in* (3.5.210) *is bounded on* $\mathcal{B}_{r_0}^{T_0}$.

Proof. We start with the estimation of $\|u_\xi\|_{L^\infty([0,T_0]\times\Omega)}$ and $\|u_\alpha\|_{L^\infty([0,T_0]\times\Omega)}$.

Since $u_\alpha \in \mathcal{W}_1^{T_0} \hookrightarrow C([0,T_0];C^1(\Omega\setminus\Sigma))$ and $u_0 \in W^{2-\frac{2}{q},q} \hookrightarrow C^{1,\alpha}$ with $\alpha = 1-(n+2)/q$ (see e.g. [15, Section 4]). There exists $M_1(T_0) > 0$ such that

$$\|u_\alpha\|_{L^\infty([0,T_0]\times\Omega)} < M_1. \tag{3.6.7}$$

Since $u_\xi \in \mathcal{B}_r^{T_0}$, we have by [28, Proposition 5.1 (a)] that

$$\|u_\xi\|_{\mathring{C}([0,T_0];C^1(\Omega\setminus\Sigma))} \le C\,\|u_\xi\|_{\mathring{\mathcal{W}}^{T_0}} \le Cr_0, \tag{3.6.8}$$

where the constant C is independent of T_0. Thus, we have

$$\|u_\xi + u_\alpha\|_{L^\infty([0,T_0]\times\Omega)} \le Cr_0 + M_1. \tag{3.6.9}$$

Using the same argument and [28, Proposition 5.1 (d)], we can also obtain

$$\|B_\xi + B_\alpha\|_{L^\infty([0,T_0]\times\Omega)} \le Cr_0 + M_1 \tag{3.6.10}$$

and

$$\|h_\xi + h_\alpha\|_{\mathcal{C}_2^{T_0}} \le Cr_0 + M_1, \tag{3.6.11}$$

where we still use the notation M_1 without loss of generality. For the term $\|z\|_{\mathcal{W}}$ we have

$$\|z\|_{\mathcal{W}^T} \le \|z_\alpha\|_{\mathcal{W}^T} + \|z_\xi\|_{\mathring{\mathcal{W}}^T} \le C(z_0, T_0, \Sigma) + r_0$$

for all $T \in (0, T_0]$, which completes the proof. $\square$

Next, using the same idea as in [15] we claim that the norm of $\|h\|_{\mathcal{C}_4^T} + \|z\|_{\mathcal{W}^T}$ in (3.5.210) can be as small as we need by picking a sufficiently small $T \in (0, T_0]$.

Proposition 3.6.2. *There exists a constant C_1, such that for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ and $r(\varepsilon) > 0$, such that for all $z_\xi \in \mathcal{B}_r^T$ we have*

$$\|h_\xi + h_\alpha\|_{\mathcal{C}_4^T} + \|z_\xi + z_\alpha\|_{\mathcal{W}^T} < C_1\varepsilon. \tag{3.6.12}$$

Proof. Since $z_\alpha \in \mathcal{W}$, which consists of Sobolev spaces, we obtain

$$\lim_{T \to 0} \|z_\alpha\|_{\mathcal{W}^T} = 0 \tag{3.6.13}$$

using the Lebesgue dominated convergence theorem. Thus, we can pick a sufficiently small $T(\varepsilon) \le T_0$ such that

$$\|z_\alpha\|_{\mathcal{W}^T} < \varepsilon. \tag{3.6.14}$$

Without loss of generality, we assume $r < \varepsilon$, which implies that

$$\|z_\xi + z_\alpha\|_{\mathcal{W}^T} < \varepsilon + r < 2\varepsilon. \tag{3.6.15}$$

Since $h_\xi \in \mathring{\mathcal{W}}_5^{T_0}$, for all $T \in (0, T_0]$, we have

$$\|h_\xi\|_{\mathring{\mathcal{C}}_2^T} \le \|h_\xi\|_{\mathring{\mathcal{C}}_2^{T_0}} \le C \|h_\xi\|_{\mathring{\mathcal{W}}_5^{T_0}} < Cr < C\varepsilon. \tag{3.6.16}$$

where the constant C is independent of T or T_0.

Since $h_\alpha(0) = 0$ and $h_\alpha \in C([0, T_0]; C^2(\Sigma)) =: \mathcal{C}_4^{T_0}$, there exists a sufficiently small $T(\varepsilon)$ such that for all $t \in [0, T]$ we have

$$\|h_\alpha(t)\|_{C^2(\Sigma)} < \varepsilon, \tag{3.6.17}$$

which implies

$$\|h_\alpha\|_{\mathring{\mathcal{C}}_4^T} < \varepsilon. \tag{3.6.18}$$

This completes the proof.

$\square$

Proposition 3.6.3. *Let $q > 5$ be fixed. Let Σ be a compact C^3 surface in $\mathbb{R}^3$. For all $\varepsilon > 0$, there exists $T(q, \Sigma, \varepsilon) > 0$, such that*

$$\left\|\mathrm{tr} L_\Sigma^2 \varphi_h\right\|_{\mathring{\mathcal{S}}_4^T} < \varepsilon \|\varphi_h\|_{\mathring{\mathcal{S}}_4^T} \tag{3.6.19}$$

for all $\varphi_h \in \mathring{\mathcal{W}}_5^T$.

Proof. For convenience, we abbreviate φ_h by φ and let

$$r := 1 - \frac{1}{q}, \quad s := \frac{1}{2} - \frac{1}{2q}, \quad \text{and} \quad f := \mathrm{tr} L_\Sigma^2.$$

From the definition of $\mathcal{S}_4$, we have

$$\left\| \mathrm{tr} L_\Sigma^2 \, \varphi \right\|_{\mathring{\mathcal{S}}_4^T} = \| f\varphi \|_{\mathring{\mathcal{S}}_4^T} \leq \| f\varphi \|_{W^{s,q}([0,T];L^q(\Sigma))} + \| f\varphi \|_{L^q([0,T];W^{r,q}(\Sigma))} . \tag{3.6.20}$$

Step 1:

Given any $t \in [0,T]$, we estimate $\| f\varphi \|_{W^{r,q}(\Sigma)} (t)$. First, we have

$$\| f\varphi \|_{L^q(\Sigma)} \leq \| f \|_{L^\infty(\Sigma)} \| \varphi \|_{L^q(\Sigma)} . \tag{3.6.21}$$

Next, we estimate the Gagliardo seminorm:

$$\begin{aligned} &[f\varphi]_{W^{r,q}(\Sigma)} := \left(\int_\Sigma \int_\Sigma \frac{|f(x)\varphi(x) - f(y)\varphi(y)|^q}{|x-y|^{n+rq}} dydx \right)^{\frac{1}{q}} \\ &\leq C(q) \left(\int_\Sigma \int_\Sigma \frac{|f(x)|^q |\varphi(x) - \varphi(y)|^q + |f(x) - f(y)|^q |\varphi(y)|^q}{|x-y|^{n+rq}} dydx \right)^{\frac{1}{q}} \\ &\leq C(q) \| f \|_{L^\infty(\Sigma)} [\varphi]_{W^{r,q}(\Sigma)} + C(q) [f]_{W^{r,q}(\Sigma)} \| \varphi \|_{L^\infty(\Sigma)} . \end{aligned} \tag{3.6.22}$$

Thus, we have

$$\begin{aligned} \| f\varphi \|_{W^{r,q}(\Sigma)} &:= \| f\varphi \|_{L^q(\Sigma)} + [f\varphi]_{W^{r,q}(\Sigma)} \\ &\leq C(q) \| f \|_{L^\infty(\Sigma)} \| \varphi \|_{W^{r,q}(\Sigma)} + C(q) [f]_{W^{r,q}(\Sigma)} \| \varphi \|_{L^\infty(\Sigma)} . \end{aligned} \tag{3.6.23}$$

Taking the L^q norm on $[0,T]$ and notice that f is independent of t, we have

$$\begin{aligned} &\| f\varphi \|_{L^q([0,T];W^{r,q}(\Sigma))} \leq C(q) \| f \|_{L^q([0,T];L^\infty(\Sigma))} \| \varphi \|_{L^\infty([0,T];W^{r,q}(\Sigma))} \\ &\quad + C(q) \| f \|_{L^q([0,T];W^{r,q}(\Sigma))} \| \varphi \|_{L^\infty([0,T]\times\Sigma)} \\ &\leq C(q,\Sigma) T^{\frac{1}{q}} \| f \|_{L^\infty(\Sigma)} \| \varphi \|_{\mathring{\mathcal{W}}_5^T} + C(q,\Sigma) T^{\frac{1}{q}} \| f \|_{W^{r,q}(\Sigma)} \| \varphi \|_{\mathring{\mathcal{W}}_5^T} \end{aligned} \tag{3.6.24}$$

where we used the embedding theory in [28, Proposition 5.1] in the 2nd inequality.

Step 2:

Now we estimate $\|f\varphi\|_{W^{s,q}([0,T];L^q(\Sigma))}$. From (3.6.23) and (3.6.24), we have

$$\begin{aligned}&\|f\varphi\|_{L^q([0,T];L^q(\Sigma))} \leq C(q)\|f\|_{L^q([0,T];L^\infty(\Sigma))}\|\varphi\|_{L^\infty([0,T];W^{r,q}(\Sigma))}\\&\leq C(q,\Sigma)T^{\frac{1}{q}}\|f\|_{L^\infty(\Sigma)}\|\varphi\|_{\mathring{\mathcal{W}}_5^T}.\end{aligned} \tag{3.6.25}$$

It remains to estimate the Gagliardo seminorm. Since f is independent of time, we have

$$\begin{aligned}&[f\varphi]_{W^{s,q}([0,T];L^q(\Sigma))} := \left(\int_0^T\int_0^T \frac{\|f\varphi(t)-f\varphi(\tau)\|^q_{L^q(\Sigma)}}{|t-\tau|^{1+sq}}dtd\tau\right)^{\frac{1}{q}}\\&\leq \left(\int_0^T\int_0^T \frac{\|f\|^q_{L^q(\Sigma)}\|\varphi(t)-\varphi(\tau)\|^q_{L^\infty(\Sigma)}}{|t-\tau|^{1+sq}}dtd\tau\right)^{\frac{1}{q}}\\&\leq \left(\int_0^T\int_0^T \frac{\|f\|^q_{L^q(\Sigma)}\|\varphi\|^q_{\mathring{C}^1([0,T];C^1(\Sigma))}|t-\tau|^q}{|t-\tau|^{1+sq}}dtd\tau\right)^{\frac{1}{q}}\\&\leq \left(\|f\|^q_{L^q(\Sigma)}\|\varphi\|^q_{\mathring{C}^1([0,T];C^1(\Sigma))}\int_0^T\int_0^T |t-\tau|^{\frac{q-1}{2}}dtd\tau\right)^{\frac{1}{q}}\\&\leq C(\Sigma,T_0)\|f\|_{L^\infty(\Sigma)}\|\varphi\|_{\mathring{\mathcal{W}}_5^T}\left(\int_0^T\int_0^T T^{\frac{q-1}{2}}dtd\tau\right)^{\frac{1}{q}}\\&\leq C(\Sigma,T_0)T^{\frac{q+3}{2q}}\|f\|_{L^\infty(\Sigma)}\|\varphi\|_{\mathring{\mathcal{W}}_5^T}.\end{aligned} \tag{3.6.26}$$

From (3.6.25) and (3.6.26), we obtain

$$\|f\varphi\|_{W^{s,q}([0,T];L^q(\Sigma))} \leq C(q,\Sigma,T_0)\left(T^{\frac{1}{q}}+T^{\frac{q+3}{2q}}\right)\|f\|_{L^\infty(\Sigma)}\|\varphi\|_{\mathring{\mathcal{W}}_5^T}. \tag{3.6.27}$$

Consequently, from (3.6.24) and (3.6.27), by assuming $T<1$ without any loss of generality, we have

$$\begin{aligned}&\left\|\mathrm{tr}L_\Sigma^2\varphi\right\|_{\mathcal{S}_4^T} \leq C(q,\Sigma,T_0)T^{\frac{1}{q}}\left(\left\|\mathrm{tr}L_\Sigma^2\right\|_{L^\infty(\Sigma)}+\left\|\mathrm{tr}L_\Sigma^2\right\|_{W^{r,q}(\Sigma)}\right)\|\varphi\|_{\mathring{\mathcal{W}}_5^T}\\&\leq C_1(q,\Sigma,T_0)T^{\frac{1}{q}}\|\varphi\|_{\mathring{\mathcal{W}}_5^T}.\end{aligned} \tag{3.6.28}$$

For any $\varepsilon>0$, a sufficiently small T such that $C_1T^{1/q}<\varepsilon$ completes the proof. □

Using the same idea as in [15, 28], we obtain the following result.

Proposition 3.6.4. *There exists a constant $C(\Sigma)$ such that for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ and $r(\varepsilon) > 0$, such that for all $z_\xi \in \mathcal{B}_r^T$ we have*

$$\|b - u\|_{\mathcal{S}_5^T} + \|b - u\|_{\mathring{C}([0,T];C(\Sigma))} < C(\Sigma)\varepsilon. \tag{3.6.29}$$

Proof. Notice that $\mathcal{S}_5$ consists of Sobolev spaces and Lebesgue spaces. Without loss of generality, we require $r < \varepsilon$. Using the same arguments as in (3.6.13) and (3.6.14), we can find a sufficiently small T, such that

$$\|b - u\|_{\mathcal{S}_5^T} \leq \|b - u_\alpha\|_{\mathcal{S}_5^T} + \|u_\xi\|_{\mathring{\mathcal{S}}_5^T} \leq \varepsilon + r < 2\varepsilon. \tag{3.6.30}$$

On the other hand, for a sufficiently small T, we have

$$\|b - u\|_\infty \leq \|b - u_\alpha\|_{\mathring{C}([0,T];C(\Sigma))} + C(T_0, \Sigma)\, \|u_\xi\|_{\mathring{\mathcal{W}}_1^T} < \varepsilon + C(\Sigma) r, \tag{3.6.31}$$

which finishes the proof. □

Consequently, we can obtain the smallness of DG, which implies the existence of a strong solution.

Proof of Theorem 3.1.1. From Proposition 3.6.1, Proposition 3.6.2, Proposition 3.6.3 and Proposition 3.6.4, we can let T and r be sufficiently small such that

$$\|K(z_1) - K(z_2)\|_{\mathring{\mathcal{W}}} \leq \frac{1}{2}\, \|z_1 - z_2\|_{\mathring{\mathcal{W}}} \tag{3.6.32}$$

for all z_1, $z_2 \in \mathcal{B}_r^T$. Thus, by the contraction mapping theorem, the operator K has a unique fixed point $z_\beta \in \mathcal{B}_r^T$, which implies that

$$\overline{z} := (\overline{u}_\gamma, \overline{B}_\gamma, \overline{p}_\gamma, \varpi_\gamma, h_\gamma) := z_\alpha + z_\beta$$

is the unique solution to the transformed equations on $[0, T]$. Here we return the bars to transformed terms.

Now we recover the original solution. Since $\Sigma \in C^3$ and

$$h \in \mathcal{W}_5 \hookrightarrow C^1([0,T]; C^1(\Sigma)) \cap C^0([0,T]; C^2(\Sigma)),$$

the diffeomorphism $\Theta_h \in C^1([0,T]; C^1(\Sigma)) \cap C^0([0,T]; C^2(\Sigma))$. Thus, in the original solution, the terms $u = \overline{u} \circ \Theta_h^{-1}$, $B = \overline{B} \circ \Theta_h^{-1}$ and $p = \overline{p} \circ \Theta_h^{-1}$ have the same regularity as $\overline{u}$, $\overline{B}$ and $\overline{p}$ and the equations are satisfied almost everywhere on corresponding domains. The jump of the pressure $[\![p]\!] = \varpi \circ \Theta_h^{-1}$ also satisfies the requirement in Definition 3.1.1. This finishes the proof of the theorem.

$\square$

3.6.2 $W^{3-\frac{2}{q}}$ initial interface

In this section, we prove Theorem 3.1.2. Suppose that we have a C^3 surface Σ and the nearest point projection property is valid in $B(\Sigma; \varrho_0)$ for some $\varrho_0(\Sigma) > 0$. Let $M_0 > 0$ be an arbitrary number and $\varepsilon_0 > 0$ a number to be determined later. Let (u_0, B_0, Γ_0) be an initial value and h_0 be the corresponding height function, such that:

1. $u_0 \in W^{2-\frac{2}{q},q}(\Omega \setminus \Gamma_0) \cap C(\Omega)$, $\quad B_0 \in W^{2-\frac{2}{q},q}(\Omega)$, $\quad$ and Γ_0 is a $W^{3-\frac{2}{q},q}$ surface;
2. $\|u_0\|_{W^{2-\frac{2}{q},q}(\Omega\setminus\Gamma_0)} \leq M_0$, $\quad \|u_0\|_{L^\infty(\Omega)} \leq M_0$, $\quad \|B_0\|_{W^{2-\frac{2}{q},q}(\Omega)} \leq M_0$, $\quad \|B_0\|_{L^\infty(\Omega)} \leq M_0$;
3. $\Gamma_0 \subseteq B(\Sigma; \varrho_0)$, $\quad \|h_0\|_{W^{3-\frac{2}{q}}(\Sigma)} \leq M_0$, $\quad \|h_0\|_{C^2(\Sigma)} < \varepsilon_0$;
4. $\mathrm{div} u_0 = 0$ in $\Omega \setminus \Gamma_0$, $\quad \mathrm{div} B_0 = 0$ in Ω;
5. $u_0 = B_0 = 0$ on $\partial\Omega$;
6. $\mathcal{P}_{\Gamma_0} \left[\!\!\left[\nu \tilde{D} u_0\right]\!\!\right] n_{\Gamma_0} = 0$.

Using the argument in Section 3.3, we obtain the initial condition of the transformed problem

$$(\overline{u}_0, \overline{B}_0, h_0) = (u_0 \circ \Theta_{h_0}, B_0 \circ \Theta_{h_0}, h_0).$$

Let $T_0 > 0$ be a fixed, sufficiently large number. Using again the same argument as in [15, Theorem 2], we obtain the auxiliary term b in (3.3.87) by extending $\overline{u}_0$ to $\overline{u}_b \in \mathcal{W}_1^{T_0}$ and letting

$$b := \overline{u}_b|_{[0,T_0]\times\Sigma}.$$

Using [15, Proposition 2], we can extend

$$\mathrm{div}\overline{u}_0 \quad \text{and} \quad 2\mathcal{P}_\Sigma \left[\!\!\left[\nu \tilde{D} \overline{u}_0\right]\!\!\right] n_\Sigma$$

to two auxiliary functions $\alpha_3 \in \mathcal{S}_3^{T_0}$ and $\alpha_4 \in \mathcal{S}_4^{T_0}$. Using the solution operator S from Section 3.4, we obtain an auxiliary solution

$$z_\alpha := (u_\alpha, B_\alpha, p_\alpha, \varpi_\alpha, h_\alpha) := S_{(\overline{u}_0, \overline{B}_0, h_0, b)}(0, 0, \alpha_3, \alpha_4, 0). \tag{3.6.33}$$

Similarly as in [15], we consider the equation

$$L(z + z_\alpha) = G(z + z_\alpha), \quad z(0) = 0 \tag{3.6.34}$$

with $z \in \mathring{\mathcal{W}}$, which can be rewritten as

$$Lz = G(z + z_\alpha) - Lz_\alpha.$$

The compatibility conditions for $G(z_0) - L(z_0)$ are exactly the transformation (via Θ_{h_0}) of the initial conditions

$$\mathrm{div} u_0 = 0 \quad \text{and} \quad \mathcal{P}_{\Gamma_0} \left[\!\!\left[2\nu \tilde{D} u_0 \right]\!\!\right] n_{\Gamma_0} = 0.$$

Using the solution operator $S_{(0,0,0,b)}$, we rewrite the equation as

$$z = S_{(0,0,0,b)} \left(G(z + z_\alpha) - Lz_\alpha \right) =: K(z). \tag{3.6.35}$$

Similarly as in Section 3.6.1, we fix $r_0 > 0$ and define for all $r \in (0, r_0]$ and $T \in (0, T_0]$ that

$$\mathcal{B}_r^T := \{ w \in \mathring{\mathcal{W}}^T : \|w\|_{\mathcal{W}^T} \leq r \}. \tag{3.6.36}$$

It remains to find suitable r and T such that K is a contraction mapping on $\mathcal{B}_r^T$. Since $h_0 \neq 0$, we need to slightly modify the estimates in Section 3.6.1.

Given any $z_1, z_2 \in \mathcal{B}_r^T$, we consider the same estimate as stated in Section 3.6.1:

$$\begin{aligned} \|K(z_1) - K(z_2)\|_{\mathcal{W}^T} &\leq \|S\| \, \|G(z_1 + z_\alpha) - G(z_2 + z_\alpha)\|_{\mathring{\mathcal{S}}^T} \\ &\leq C \, \|S\| \sup_{0 \leq c \leq 1} \|DG\left[cz_1 + (1-c)z_2 + z_\alpha\right]\| \, \|z_1 - z_2\|_{\mathring{\mathcal{W}}^T} . \end{aligned} \tag{3.6.37}$$

Letting $z_\xi := cz_1 + (1-c)z_2$ and $\varphi := z_1 - z_2$, we need to estimate (3.5.210) using similar ideas as in [15, 28]. We rewrite the inequality for convenience. Suppose $\|h\|_{\mathcal{C}_2^{T_0}} \le M_h$, then for all $T \in (0, T_0]$ we have

$$\begin{aligned}\|DG[z_\alpha + z_\xi]\varphi\|_{\mathring{\mathcal{S}}^T} \le C(\Sigma, T_0, M_h)\Big(1 &+ \|u_\alpha + u_\xi\|_{L^\infty([0,T]\times\Omega)} + \|B_\alpha + B_\xi\|_{L^\infty([0,T]\times\Omega)} \\ &+ \|h_\alpha + h_\xi\|_{\mathcal{C}_2^T} + \|z_\alpha + z_\xi\|_{\mathcal{W}^T}\Big)\left(\|h_\alpha + h_\xi\|_{\mathcal{C}_4^T} + \|z_\alpha + z_\xi\|_{\mathcal{W}^T}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T} \\ + C\left\|\mathrm{tr}L_\Sigma^2\varphi_h\right\|_{\mathring{\mathcal{S}}_4^T} + C(T_0, \Sigma)&\left(\|b - u_\alpha - u_\xi\|_{\mathcal{S}_5^T} + \|b - u_\alpha - u_\xi\|_{L^\infty([0,T]\times\Sigma)}\right)\|\varphi\|_{\mathring{\mathcal{W}}^T}.\end{aligned} \tag{3.6.38}$$

The estimates in Proposition 3.6.3 and Proposition 3.6.4 can be obtained without any change. We slightly modify the arguments in Proposition 3.6.1 and Proposition 3.6.2.

Proposition 3.6.5 (Modification of Proposition 3.6.1)**.** *For M_0 and ε_0 defined in the beginning of Section 3.6.2. For all $T \in (0, T_0]$ and all $z_\xi \in \mathcal{B}_{r_0}^T$, we have*

$$1 + \|u_\alpha + u_\xi\|_\infty + \|B_\alpha + B_\xi\|_\infty + \|h_\alpha + h_\xi\|_{\mathcal{C}_2^T} + \|z_\alpha + z_\xi\|_{\mathcal{W}^T} \le C(\Sigma, T_0, r_0, M_0). \tag{3.6.39}$$

The constant $C(\Sigma, T_0, r_0, M_0)$ is independent of h_0 or ε_0.

Proof. We recall that the solution operator S in Section 3.4 is continuous with respect to the initial value. Since $\|u_0\|_{W^{2-\frac{2}{q},q}(\Omega\setminus\Gamma_0)} \le M_0$, $\|B_0\|_{W^{2-\frac{2}{q},q}(\Omega)} \le M_0$ and $\|h_0\|_{W^{3-\frac{2}{q},q}(\Sigma)} \le M_0$, from the derivation of z_α, we have for all $T \in (0, T_0]$ that

$$\|z_\alpha\|_{\mathcal{W}^T} \le \|z_\alpha\|_{\mathcal{W}^{T_0}} \le C(M_0, T_0, \Sigma).$$

From the embedding theory in [28, Proposition 5.1 (d)], we have for all $T \in (0, T_0]$ that

$$\|h_\alpha\|_{\mathcal{C}_2^T} \le \|h_\alpha\|_{\mathcal{C}_2^{T_0}} \le C(T_0, \Sigma)\|h_\alpha\|_{\mathcal{W}_5^{T_0}} \le C(M_0, T_0, \Sigma).$$

The rest of the proof can be carried out using similar arguments as in Proposition 3.6.1, which implies (3.6.39). $\square$

We also modify Proposition 3.6.2 since $h_0 \neq 0$ in the current case.

Proposition 3.6.6 (Modification of Proposition 3.6.2)**.** *There exists a constant C_1, such that for any $\varepsilon_1 > 0$, there exists $T(\varepsilon_1) > 0$ and $r(\varepsilon_1) > 0$, such that for all $z_\xi \in \mathcal{B}_r^T$ we have*

$$\|h_\xi + h_\alpha\|_{\mathcal{C}_4^T} + \|z_\xi + z_\alpha\|_{\mathcal{W}^T} < \varepsilon_0 + C_1\varepsilon_1. \tag{3.6.40}$$

Here the number ε_0 is the upper bound of $\|h_0\|_{C^2(\Sigma)}$ as stated at the beginning of this section.

Proof. We only need to modify the estimate of h_α since $h_0 \neq 0$ in the current case. Notice that $\|h_\alpha(0)\|_{C^2} = \|h_0\|_{C^2} < \varepsilon_0$ by the assumption of the initial conditions. Since $h_\alpha \in C([0, T_0]; C^2(\Sigma)) =: \mathcal{C}_4^{T_0}$, we can let $T(\varepsilon)$ be sufficiently small, such that for all $t \in [0, T]$ we have

$$\|h_\alpha(t)\|_{C^2} \leq \|h_0\|_{C^2} + \|h_\alpha(t) - h_0\|_{C^2} < \varepsilon_0 + \varepsilon_1, \tag{3.6.41}$$

which implies

$$\|h_\alpha\|_{\mathcal{C}_4^T} < \varepsilon_0 + \varepsilon_1. \tag{3.6.42}$$

This completes the proof.

□

We can now prove Theorem 3.1.2 using similar ideas as in [15, 28].

Proof of Theorem 3.1.2. First, we verify the condition

$$\|h\|_{\mathcal{C}_4^T} := \|h\|_{C([0,T];C^2(\Sigma))} < \delta_0$$

as stated in Proposition 3.5.5, which enables us to use the estimate in (3.5.210). Notice that δ_0 only depends on Σ and thus is a fixed number. Without loss of generality, we assume $\varepsilon_0 < \delta_0/4$ and $C_1\varepsilon_1 < \delta_0/4$. From Proposition 3.6.6, there exists $T_1 \in (0, T_0]$, such that $\|h\|_{\mathcal{C}_4^{T_1}} < \delta_0$.

Given any ε_0 and ε_1, from Proposition 3.6.3, Proposition 3.6.4, Proposition 3.6.5 and Proposition 3.6.6, we can find $T \in (0, T_1]$ and $r \in (0, r_0]$ sufficiently small, such that

$$\begin{aligned} \|DG\,[z_\alpha + z_\xi]\,\varphi\|_{\mathring{\mathcal{S}}^T} \leq{}& C_2(M_0, T_0, r_0)\,(\varepsilon_0 + C_1\varepsilon_1)\,\|\varphi\|_{\mathring{\mathcal{W}}^T} + C_3(T_0)\varepsilon_1\,\|\varphi_h\|_{\mathring{\mathcal{W}}_5^T} \\ &+ C_4(T_0)\varepsilon_1\,\|\varphi\|_{\mathring{\mathcal{W}}^T}\,. \end{aligned} \tag{3.6.43}$$

Thus, for sufficiently small ε_0 and ε_1, we can obtain a contraction mapping. The rest of the proof can be proceeded using the same arguments as in the proof of Theorem 3.1.1.

□

4.0 Weak-Strong Uniqueness of the Two-Phase MHD Equations

In this chapter, we consider the two-phase MHD equations in the whole space $\mathbb{R}^3$, i.e.

$$\partial_t u + (u\cdot\nabla)u - (\nabla\times B)\times B - \nu^{\pm}\triangle u + \nabla p = 0 \quad \text{in } \mathbb{R}^3\setminus\Gamma(t), \tag{4.0.1}$$

$$\partial_t B - \nabla\times(u\times B) + \nabla\times(\sigma\nabla\times B) = 0 \quad \text{in } \mathbb{R}^3, \tag{4.0.2}$$

$$\mathrm{div} u = 0 \quad \text{in } \mathbb{R}^3\setminus\Gamma(t), \tag{4.0.3}$$

$$\mathrm{div} B = 0 \quad \text{in } \mathbb{R}^3, \tag{4.0.4}$$

$$-\left[\!\left[2\nu(\chi)Du - pI\right]\!\right] n = \kappa H n \quad \text{on } \Gamma(t), \tag{4.0.5}$$

$$V_\Gamma = u\cdot n \quad \text{on } \Gamma(t), \tag{4.0.6}$$

$$u|_{t=0} = u_0,\ B|_{t=0} = B_0,\ \Gamma(0) = \Gamma_0. \tag{4.0.7}$$

The two fluids occupy $\mathbb{R}^3$ and are separated by a closed interface $\Gamma(t)$. We denote the interior and exterior fluids by open sets $\Omega^+(t)$ and $\Omega^-(t)$ respectively. Then we have that $\Omega^+(t)$, $\Omega^-(t)$ and $\Gamma(t)$ are disjoint and $\Omega^+(t)\cup\Gamma(t)\cup\Omega^-(t) = \mathbb{R}^3$. In our equations, the term u denotes the fluid velocity; B denotes the magnetic field; p the pressure; H the mean curvature of $\Gamma(t)$; V_Γ is the speed of the interface; n is the normal vector of $\Gamma(t)$. The viscosity coefficient, magnetic diffusion coefficient and surface tension coefficient are denoted by $\nu^\pm$, σ and κ. Here $\nu^\pm$ takes different values in $\Omega^\pm(t)$ and σ remains a constant in $\mathbb{R}^3$. The initial interface Γ_0 is a compact C^3 surface. The notation $Du := \left(\nabla u + \nabla u^\top\right)/2$ is the strain rate tensor.

4.1 Preliminary and Main Result

In this section, we introduce some basic background knowledge and the main result. We first give the definition of strong solutions in the sense of [10].

Definition 4.1.1 (Strong solution)**.** Let $q > 5$ be a fixed number. Let (u_0, B_0, χ_0) be such that:

1. $\Gamma_0 := \partial\{x \in \mathbb{R}^3 : \chi_0(x) = 1\}$ is a compact C^3 surface;
2. $u_0 \in W^{2-\frac{2}{q},q}(\mathbb{R}^3 \setminus \Gamma_0)$, $\mathrm{div} u_0 = 0$ in $\mathbb{R}^3 \setminus \Gamma_0$, $[\![u_0]\!] = 0$ on Γ_0, $\mathcal{P}_{\Gamma_0} [\![\nu^\pm Du_0]\!] n_{\Gamma_0} = 0$ on Γ_0;
3. $B_0 \in W^{2-\frac{2}{q},q}(\mathbb{R}^3)$, $\mathrm{div} B_0 = 0$ in $\mathbb{R}^3$.

Here $[\![\cdot]\!]$ denotes the jump of functions; $\mathcal{P}_{\Gamma_0} := I - n_{\Gamma_0} \otimes n_{\Gamma_0}$ is the projection mapping; $Du_0 := \left(\nabla u_0 + \nabla u_0^\top\right)/2$ is the strain rate tensor.

A triple (u, B, χ) is called a strong solution to (4.0.1) - (4.0.7) with initial value (u_0, B_0, χ_0) if the following conditions are satisfied.

1. $u \in H^1([0,T_0]; L^2(\mathbb{R}^3)) \cap L^\infty([0,T_0]; H^1(\mathbb{R}^3))$. $\nabla u \in L^1([0,T_0]; BV(\mathbb{R}^3))$; div u =0;
2. $B \in H^1([0,T_0]; L^2(\mathbb{R}^3)) \cap L^\infty([0,T_0]; H^1(\mathbb{R}^3))$. $\nabla B \in L^1([0,T_0]; BV(\mathbb{R}^3))$.
3. $\chi \in L^\infty([0,T_0]; BV(\mathbb{R}^3))$ is an indicator function.
4. For all $\varphi \in C_c^\infty([0,T_0) \times \mathbb{R}^3)$ with $\mathrm{div}\varphi = 0$ and almost all $t \in [0, T_0)$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} u(t) \cdot \varphi(t) dx - \int_{\mathbb{R}^3} u_0 \cdot \varphi(0) dx = \int_0^t \int_{\mathbb{R}^3} u \cdot \partial_t \varphi dx d\tau + \int_0^t \int_{\mathbb{R}^3} u \otimes u : \nabla\varphi dx d\tau \\
&\quad - \int_0^t \int_{\mathbb{R}^3} B \otimes B : \nabla\varphi dx d\tau - \int_0^t \int_{\mathbb{R}^3} \nu(\chi) \left(\nabla u + \nabla u^\top\right) : \nabla\varphi dx d\tau \\
&\quad + \kappa \int_0^t \int_{\Gamma(\tau)} Hn \cdot \varphi dS d\tau.
\end{aligned} \tag{4.1.1}$$

5. For almost all $\varphi \in C_c^\infty([0,T_0) \times \mathbb{R}^3)$ with $\mathrm{div}\varphi = 0$ and all $t \in [0, T_0)$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} B(t) \cdot \varphi(t) dx - \int_{\mathbb{R}^3} B_0 \cdot \varphi(0) dx = \int_0^t \int_{\mathbb{R}^3} B \cdot \partial_t \varphi dx d\tau + \int_0^t \int_{\mathbb{R}^3} u \otimes B : \nabla\varphi dx d\tau \\
&- \int_0^t \int_{\mathbb{R}^3} B \otimes u : \nabla\varphi dx d\tau - \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B : \nabla\varphi dx d\tau.
\end{aligned} \tag{4.1.2}$$

6. For all $\varphi \in C_c^\infty([0,T_0) \times \mathbb{R}^3)$ and almost all $t \in [0, T_0)$,

$$\int_{\mathbb{R}^3} \chi(t)\varphi(t) dx - \int_{\mathbb{R}^3} \chi_0 \varphi(0) dx = \int_0^t \int_{\mathbb{R}^3} \chi \partial_t \varphi dx d\tau + \int_0^t \int_{\mathbb{R}^3} \chi u \cdot \nabla\varphi dx d\tau. \tag{4.1.3}$$

7. The terms $\nabla^i u$ and $\nabla^i B$ for $i = 0, 1, 2, 3$, and terms $\partial_t \nabla^k u$ and $\partial_t \nabla^k B$ for $k = 0, 1$, are all bounded on the set

$$\bigcup_{t \in [0,T_0)} \left(\mathbb{R}^3 \setminus \Gamma(t)\right) \times \{t\}.$$

8. There exists $\Theta(t, x) : [0, T_0) \times \mathbb{R}^3 \to \mathbb{R}^3$, which satisfies the following conditions.

a. For each $t \in [0, T_0)$, $\Theta(t, \cdot)$ is a C^3 diffeomorphism.

b. For $t = 0$, $\Theta(0, x) = x$.

c. The norm $\|\Theta\|_{L^\infty([0,T_0);W^{3,\infty}(\mathbb{R}^3))}$ is finite.

d. The time derivative $\partial_t \Theta \in C([0, T_0); C^2(\mathbb{R}^3))$.

e. There exists $\varrho_0 > 0$ such that for all $t \in [0, T_0)$ and $x \in \Gamma(t)$, the surface $\Gamma(t) \cap B(x; 2\varrho_0)$ can be represented as the graph of a function on the tangent plane $T_x\Gamma(t)$.

The sets that represent two fluids and the interface can be represented using Θ by $\Omega^+(t) = \Theta(t, \Omega_0^+)$, $\Omega^-(t) = \Theta(t, \Omega_0^-)$ and $\Gamma(t) = \Theta(t, \Gamma_0)$.

We now give the definition of varifold solutions in the whole space $\mathbb{R}^3$, which is based on the definition in [10, 14].

Definition 4.1.2 (Varifold solution)**.** Let $u_0, B_0 \in L^2(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ weakly. Let $\Omega_0^+ \subseteq \mathbb{R}^3$ be a bounded domain such that $\chi_0 = \chi_{\Omega_0^+}$ has finite perimeter. A quadruple (u, B, χ, V) with

$$
\begin{aligned}
&u \in L^2([0,T]; H^1(\mathbb{R}^3)) \cap L^\infty([0,T]; L^2(\mathbb{R}^3)),\\
&B \in L^2([0,T]; H^1(\mathbb{R}^3)) \cap L^\infty([0,T]; L^2(\mathbb{R}^3)),\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&\chi \in L^\infty([0,T]; BV(\mathbb{R}^3; \{0,1\})),\\
&V \in L^\infty([0,T]; \mathcal{M}(\mathbb{R}^3 \times \mathbb{S}^2)),
\end{aligned}
$$

is called a varifold solution to the two-phase MHD equations (2.1.1)-(2.1.8) with initial value (u_0, B_0, χ_0) if the following conditions are satisfied.

1. For almost every $t \in [0, T_0)$,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} u(t)\varphi(t)dx - \int_{\mathbb{R}^3} u_0\varphi(0)dx - \int_0^t\int_{\mathbb{R}^3} u\partial_t\varphi dxd\tau - \int_0^t\int_{\mathbb{R}^3} u\otimes u : \nabla\varphi dxd\tau\\
&+ \int_0^t\int_{\mathbb{R}^3} B\otimes B : \nabla\varphi dxd\tau + 2\int_0^t\int_{\mathbb{R}^3} \nu(\chi)Du : D\varphi dxd\tau\\
&+ \kappa\int_0^t\int_{\mathbb{R}^3\times\mathbb{S}^2} (I - s\otimes s) : \nabla\varphi dVd\tau = 0
\end{aligned}
\tag{4.1.4}
$$

   is satisfied for all $\varphi \in C_c^\infty([0, T_0) \times \mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$.

2. For almost every $t \in [0, T_0)$,

$$\int_{\mathbb{R}^3} B(t)\varphi(t)dx - \int_{\mathbb{R}^3} B_0\varphi(0)dx - \int_0^t \int_{\mathbb{R}^3} B\partial_t\varphi dxd\tau - \int_0^t \int_{\mathbb{R}^3} u \otimes B : \nabla\varphi dxd\tau$$
$$+ \int_0^t \int_{\mathbb{R}^3} B \otimes u : \nabla\varphi dxd\tau + \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B : \nabla\varphi dxd\tau = 0 \tag{4.1.5}$$

is satisfied for all $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^3)$ with $\mathrm{div}\varphi = 0$.

3. For almost every $t \in [0, T_0)$,

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} s \cdot \psi dV(t) = -\int_{\mathbb{R}^3} \psi d\nabla\chi(t) \tag{4.1.6}$$

is satisfied for all $\psi \in C_0(\mathbb{R}^3)$.

4. For almost every $t \in [0, T_0)$,

$$\int_{\mathbb{R}^3} \chi(t)\varphi(t)dx - \int_{\mathbb{R}^3} \chi_0\varphi(0)dx - \int_0^t \int_{\mathbb{R}^3} \chi\partial_t\varphi dxd\tau - \int_0^t \int_{\mathbb{R}^3} \chi v \cdot \nabla\varphi dxd\tau = 0 \tag{4.1.7}$$

for all $\varphi \in C_c^\infty([0, T_0) \times \mathbb{R}^3)$.

5. The generalized energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{1}{2}\|B(t)\|_{L^2}^2 + \kappa\|V(t)\|_{\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)} + 2\int_0^t \int_{\mathbb{R}^3} \nu(\chi)|Du|^2 dxd\tau$$
$$+\sigma \int_0^t \|\nabla B(\tau)\|_{L^2}^2 d\tau \le \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{2}\|B_0\|_{L^2}^2 + \kappa\|\nabla\chi_0\|_{\mathcal{M}(\mathbb{R}^3)} \tag{4.1.8}$$

holds for almost every $t \in [0, T_0)$.

6. The energy

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{1}{2}\|B(t)\|_{L^2}^2 + \kappa\|V(t)\|_{\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)} \tag{4.1.9}$$

is a nonincreasing function of t.

To measure the scale of the error between a strong solution and a varifold solution, we introduce the concept of relative entropy, which is constructed based on the structure of the relative entropy for two-phase Navier-Stokes equations in [10].

Definition 4.1.3 (Relative entropy)**.** Given a strong solution (v, B_v, Γ_v) and a varifold solution (u, B_u, χ_u, V_u) to the two-phase MHD equations (3.1.1)-(3.1.8). Given an auxiliary function $w \in L^2([0,T_0]; H^1(\mathbb{R}^3)) \cap H^1([0,T_0]; L^{4/3}(\mathbb{R}^3) + L^2(\mathbb{R}^3))$. Let $\varrho_0 > 0$ be such that the nearest point projection (see Section 3.2.4) is valid in $B(\Gamma(t); \varrho_0)$ for all $t \in [0, T_0)$. We define the relative entropy by

$$\begin{aligned} E(t) := \kappa \int_{\mathbb{R}^3} 1 - \xi \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|} d\left|\nabla \chi_u\right| + \int_{\mathbb{R}^3} \frac{1}{2} \left|u - v - w\right|^2 dx \\ + \int_{\mathbb{R}^3} \frac{1}{2} \left|B_u - B_v\right|^2 dx + \kappa \int_{\mathbb{R}^3} 1 - \theta(t) dV_* + \int_{\mathbb{R}^3} |\chi_u - \chi_v| \left|\beta\left(\frac{d(x)}{\varrho_0}\right)\right| dx. \end{aligned} \tag{4.1.10}$$

The term ξ is the extension of the normal vector n_v defined as

$$\xi(x) := \eta(d(x))(1 - d^2(x)) n_v(\Pi(x)), \tag{4.1.11}$$

where d and Π are the signed distance function and the projection mapping in the theory of nearest point projection (see Section 3.2.4 for details); the function η is a fixed cut-off function such that $\eta = 1$ on $[-\varrho_0/2, \varrho_0/2]$ and $\eta = 0$ on $(-\infty, -\varrho_0] \cup [\varrho_0, +\infty)$; the function $\theta(t)$ is the Radon-Nikodym derivative

$$\theta(t) := \frac{d\left|\nabla \chi_u(t)\right|}{dV_*(t)} \tag{4.1.12}$$

between the total variation measure $|\nabla \chi_u(t)|$ and the measure $V_*(t)$, which is defined as $(V_*(t))(A) := |V(t)|(A \times \mathbb{S}^2)$ for $A \subseteq \mathbb{R}^3$ such that $A \times \mathbb{S}^2$ is measurable.

We now state our main result.

Theorem 4.1.1. *Let (v, B_v, Γ_v) and (u, B_u, χ_u, V_u) be the strong solution (in the sense of Definition 4.1.1) and a varifold solution (in the sense of Definition 4.1.2) to* (3.1.1)-(3.1.8) *with the same initial value (u_0, B_0, χ_0). Then*

$$u = v \text{ and } B_u = B_v \text{ in } L^2([0,T_0]; H^1(\mathbb{R}^3)); \quad \chi_u = \chi_v \text{ in } L^\infty([0,T_0] \times \mathbb{R}^3);$$

and for almost every $t \in [0, T_0]$ and all $\varphi \in C_0(\mathbb{R}^3 \times \mathbb{S}^2)$ we have

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} \varphi(x, s) dV(t) = \int_{\mathbb{R}^3} \varphi(x, n_v(x)) d\left|\nabla \chi_v\right|.$$

Here n_v is the Radon-Nikodym derivative $\nabla \chi_v / \left|\nabla \chi_v\right|$.

4.2 Relative Entropy Inequality

In this section, we derive an inequality of the relative entropy E. The terms in this inequality will be estimated in the next section, which allows us to control $E(t)$ by utilizing the Gronwall's inequality.

Let (v, B_v, χ_v) be a strong solution and (u, B_u, χ_u, V) be a varifold solution. Suppose that the strong solution exists on $[0, T_0)$.

The terms without the magnetic field can be treated using similar arguments as in [10], while the magnetic terms will be treated differently. We include some important formulas and steps from [10] for completeness and convenience. First, in the equations of the strong solution, we test (4.0.1) with $\varphi \in C_c^\infty([0, T_0) \times \mathbb{R}^3)$, then for all $t \in [0, T_0)$ we have

$$\begin{aligned}
&-\int_{\mathbb{R}^3} v \cdot \varphi(t) + \int_{\mathbb{R}^3} v \cdot \varphi(0) = -\int_0^t \int_{\mathbb{R}^3} v \cdot \partial_t \varphi - \int_0^t \int_{\mathbb{R}^3} v \otimes v : \nabla \varphi + \int_0^t \int_{\mathbb{R}^3} B_v \otimes B_v : \nabla \varphi \\
&+ \int_0^t \int_{\mathbb{R}^3} \nu(\chi_v)(2Dv) : D\varphi - \kappa \int_0^t \int_{\Gamma_v(\tau)} Hn \cdot \varphi.
\end{aligned} \tag{4.2.1}$$

Similarly as in [10, (185)], we obtain from the transport equation of quantity $v\varphi$ that

$$\begin{aligned}
&\int_{\mathbb{R}^3} v(t) \cdot \varphi(t) - \int_{\mathbb{R}^3} v_0 \cdot \varphi(0) = \int_0^t \int_{\mathbb{R}^3} \partial_t v \cdot \varphi + \int_0^t \int_{\mathbb{R}^3} v \cdot \partial_t \varphi \\
&+ \int_0^t \int_{\mathbb{R}^3} v \nabla \varphi v + \int_0^t \int_{\mathbb{R}^3} v \nabla v \varphi.
\end{aligned} \tag{4.2.2}$$

Remark 4.2.1. When φ is a scalar function, for all $1 \le k \le n$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} v_k(t)\varphi(t) - \int_{\mathbb{R}^3} (v_0)_k \varphi(0) = \int_0^t \int_{\mathbb{R}^3} \partial_t (v_k \varphi) + \int_0^t \int_{\mathbb{R}^3} v \cdot \nabla(v_k \varphi) \\
&= \int_0^t \int_{\mathbb{R}^3} \partial_t v_k \varphi + \int_0^t \int_{\mathbb{R}^3} \sum_j v_k \partial_t \varphi + \int_0^t \int_{\mathbb{R}^3} \sum_i v_i \partial_i v_k \varphi + \int_0^t \int_{\mathbb{R}^3} \sum_i v_i v_k \partial_i \varphi,
\end{aligned} \tag{4.2.3}$$

which implies

$$\begin{aligned}
&\int_{\mathbb{R}^3} v(t)\varphi(t) - \int_{\mathbb{R}^3} v_0 \varphi(0) = \int_0^t \int_{\mathbb{R}^3} \partial_t v \varphi + \int_0^t \int_{\mathbb{R}^3} v \partial_t \varphi \\
&+ \int_0^t \int_{\mathbb{R}^3} v \nabla v \varphi + \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla \varphi) v.
\end{aligned} \tag{4.2.4}$$

When φ is a vector-valued function, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} v(t)\cdot\varphi(t) - \int_{\mathbb{R}^3} v_0\cdot\varphi(0) = \int_0^t\int_{\mathbb{R}^3}\partial_t(v\cdot\varphi) + \int_0^t\int_{\mathbb{R}^3} v\cdot\nabla(v\cdot\varphi)\\
&= \int_0^t\int_{\mathbb{R}^3}\partial_t\sum_i v_i\varphi_i + \int_0^t\int_{\mathbb{R}^3}\sum_j v_j\left(\partial_j\sum_i v_i\varphi_i\right)\\
&= \int_0^t\int_{\mathbb{R}^3}\sum_i\partial_t v_i\varphi_i + \int_0^t\int_{\mathbb{R}^3}\sum_i v_i\partial_t\varphi_i + \int_0^t\int_{\mathbb{R}^3}\sum_j v_j\left(\sum_i\partial_j v_i\varphi_i\right)\\
&\qquad + \int_0^t\int_{\mathbb{R}^3}\sum_j v_j\left(\sum_i v_i\partial_j\varphi_i\right)\\
&= \int_0^t\int_{\mathbb{R}^3}\partial_t v\cdot\varphi + \int_0^t\int_{\mathbb{R}^3} v\cdot\partial_t\varphi + \int_0^t\int_{\mathbb{R}^3}\sum_j\sum_i v_j\partial_j v_i\varphi_i + \int_0^t\int_{\mathbb{R}^3}\sum_j\sum_i v_j\partial_j\varphi_i v_i\\
&= \int_0^t\int_{\mathbb{R}^3}\partial_t v\cdot\varphi + \int_0^t\int_{\mathbb{R}^3} v\cdot\partial_t\varphi + \int_0^t\int_{\mathbb{R}^3} v\nabla v\varphi + \int_0^t\int_{\mathbb{R}^3} v\nabla\varphi v.
\end{aligned}
\tag{4.2.5}
$$

Let w be an auxiliary function as defined in [10]. Adding (4.2.1) and (4.2.2) and then let $\varphi = u - v - w$, we obtain

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\partial_t v\cdot(u-v-w) + \int_0^t\int_{\mathbb{R}^3} v\cdot((\nabla v)\cdot(u-v-w)) + \int_0^t\int_{\mathbb{R}^3} B_v\otimes B_v : \nabla(u-v-w)\\
&+ \int_0^t\int_{\mathbb{R}^3}\nu(\chi_v)(2Dv) : \nabla(u-v-w) - \kappa\int_0^t\int_{\Gamma_v(\tau)} Hn_v\cdot(u-v-w)dSd\tau = 0.
\end{aligned}
\tag{4.2.6}
$$

Next, we consider the equations of varifold solutions. Similarly as in [10, (184)], we consider the transport equation of the quantity $|v+w|^2/2$, which implies

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|v+w|^2(t) - \frac{1}{2}\int_{\mathbb{R}^3}|v+w|^2(0)\\
&= \int_0^t\int_{\mathbb{R}^3}(v+w)\cdot\partial_t(v+w) + u\cdot(\nabla(v+w)(v+w)).
\end{aligned}
\tag{4.2.7}
$$

We recall the energy inequality of the varifold solution:

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|u|^2(t) + \frac{1}{2}\int_{\mathbb{R}^3}|B_u|^2(t) + \kappa\|V(t)\|_{\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)} + 2\int_0^t\int_{\mathbb{R}^3}\nu(\chi_u)|Du|^2 + \sigma\int_0^t\int_{\mathbb{R}^3}|\nabla B_u|^2\\
\leq&\frac{1}{2}\int_{\mathbb{R}^3}|u|^2(0) + \frac{1}{2}\int_{\mathbb{R}^3}|B_u|^2(0) + \kappa|\nabla\chi_0|(\mathbb{R}^3).
\end{aligned}
\tag{4.2.8}
$$

Similarly as in [10], we test the momentum equation (4.0.1) of the varifold solution with $\varphi = v + w$, which implies

$$\begin{aligned}
&-\int_{\mathbb{R}^3} u(t)\cdot(v+w)(t) + \int_{\mathbb{R}^3} u(0)\cdot(v+w)(0)\\
&= -\int_0^t\int_{\mathbb{R}^3} u\cdot\partial_t(v+w) - \int_0^t\int_{\mathbb{R}^3} u\otimes u : \nabla(v+w) + \int_0^t\int_{\mathbb{R}^3} B_u\otimes B_u : \nabla(v+w) \qquad (4.2.9)\\
&+ \int_0^t\int_{\mathbb{R}^3} \nu(\chi_u)(2Du) : \nabla(v+w) + \kappa\int_0^t\int_{\mathbb{R}^3\times\mathbb{S}^2} (I - s\otimes s) : \nabla(v+w)dV(\tau)d\tau.
\end{aligned}$$

Now we consider the magnetic equation in the varifold solution. Similarly as in (4.2.7), we consider the transport equation (along velocity field u) of the quantity $\frac{1}{2}|B_v|^2$:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} |B_v|^2(t) - \frac{1}{2}\int_{\mathbb{R}^3} |B_v|^2(0) = \frac{1}{2}\int_0^t\int_{\mathbb{R}^3} \partial_t|B_v|^2 + \frac{1}{2}\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla(|B_v|^2)\\
&= \int_0^t\int_{\mathbb{R}^3} B_v\cdot\partial_t B_v + \int_0^t\int_{\mathbb{R}^3} u\otimes B_v : \nabla B_v.
\end{aligned} \qquad (4.2.10)$$

In the magnetic equation of the varifold solution, we let $\varphi = B_v$ and obtain

$$\begin{aligned}
&-\int_{\mathbb{R}^3} B_u(t)\cdot B_v(t) + \int_{\mathbb{R}^3} B_u(0)\cdot B_v(0) = -\int_0^t\int_{\mathbb{R}^3} B_u\cdot\partial_t B_v - \int_0^t\int_{\mathbb{R}^3} u\otimes B_u : \nabla B_v\\
&+ \int_0^t\int_{\mathbb{R}^3} B_u\otimes u : \nabla B_v + \sigma\int_0^t\int_{\mathbb{R}^3} \nabla B_u : \nabla B_v.
\end{aligned} \qquad (4.2.11)$$

Letting (4.2.6) + (4.2.7) + (4.2.8) + (4.2.9), we obtain for almost all $t \in [0, T_0)$ that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} |v+w|^2(t) - \frac{1}{2}\int_{\mathbb{R}^3} |v+w|^2(0) + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2(t) + \frac{1}{2}\int_{\mathbb{R}^3} |B_u|^2(t) \\
&\quad + \kappa \|V(t)\|_{\mathcal{M}(\mathbb{R}^3\times\mathbb{S}^2)} + 2\int_0^t\int_{\mathbb{R}^3} \nu(\chi_u)|Du|^2 + \sigma\int_0^t\int_{\mathbb{R}^3} |\nabla B_u|^2 \\
&\quad - \int_{\mathbb{R}^3} u(t)\cdot(v+w)(t) + \int_{\mathbb{R}^3} u(0)\cdot(v+w)(0) \\
&\le \int_0^t\int_{\mathbb{R}^3} \partial_t v\cdot(u-v-w) + \int_0^t\int_{\mathbb{R}^3} v\cdot(\nabla v(u-v-w)) \\
&\quad + \int_0^t\int_{\mathbb{R}^3} \nu(\chi_v)(2Dv) : \nabla(u-v-w) - \kappa\int_0^t\int_{\Gamma_v(\tau)} Hn_v\cdot(u-v-w)dSd\tau \\
&\quad + \int_0^t\int_{\mathbb{R}^3} B_v\otimes B_v : \nabla(u-v-w) + \int_0^t\int_{\mathbb{R}^3} (v+w)\cdot\partial_t(v+w) \\
&\quad + \int_0^t\int_{\mathbb{R}^3} u\cdot(\nabla(v+w)(v+w)) + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2(0) + \frac{1}{2}\int_{\mathbb{R}^3} |B_u|^2(0) + \kappa|\nabla\chi_{u,0}|(\mathbb{R}^3) \\
&\quad - \int_0^t\int_{\mathbb{R}^3} u\cdot\partial_t(v+w) - \int_0^t\int_{\mathbb{R}^3} u\otimes u : \nabla(v+w) + \int_0^t\int_{\mathbb{R}^3} B_u\otimes B_u : \nabla(v+w) \\
&\quad + 2\int_0^t\int_{\mathbb{R}^3} \nu(\chi_u)Du : \nabla(v+w) + \kappa\int_0^t\int_{\mathbb{R}^3\times\mathbb{S}^2} (I - s\otimes s) : \nabla(v+w)dV(t),
\end{aligned}
\tag{4.2.12}
$$

which is abbreviated to

$$I_1 + \cdots + I_9 \le J_1 + \cdots + J_{15}. \tag{4.2.13}$$

Now we reorder these terms and collect all the component terms of the relative entropy E. The terms without magnetic field can be treated in the same way as in [10, Section 7]. We include these terms in our arguments for convenience and completeness. First, the velocity term of the relative entropy E can be obtained by

$$
\begin{aligned}
I_1 + I_3 + I_8 &= \frac{1}{2}\int_{\mathbb{R}^3} |u-v-w|^2(t)dx, \\
I_2 + I_9 - J_8 &= -\frac{1}{2}\int_{\mathbb{R}^3} |u-v-w|^2(0)dx.
\end{aligned}
\tag{4.2.14}
$$

From [10, (194)], we obtain the varifold term in E:

$$
\begin{aligned}
I_5 &= \kappa|\nabla\chi_u(t)|(\mathbb{R}^3) + \kappa\int_{\mathbb{R}^3} 1-\theta(t)dV_*(t), \\
J_{10} &= \kappa|\nabla\chi_0|(\mathbb{R}^3).
\end{aligned}
\tag{4.2.15}
$$

Since the two fluids have the same density $\rho = 1$, the term R_{dt} in [10] vanishes in our case. Thus, the terms R_{adv}, A_{adv}, R_{dt} and A_{dt} in [10] satisfies

$$\begin{aligned}
&(J_2 + J_7 + J_{12}) + (J_1 + J_6 + J_{11}) \\
&= -\int_0^t \int_{\mathbb{R}^3} v\nabla w(u - v - w) - \int_0^t \int_{\mathbb{R}^3} (u - v - w)\nabla v(u - v - w) \\
&- \int_0^t \int_{\mathbb{R}^3} w\nabla(v + w)(u - v - w) - \int_0^t \int_{\mathbb{R}^3} (u - v - w)\nabla w(u - v - w) \\
&- \int_0^t \int_{\mathbb{R}^3} (u - v - w) \cdot \partial_t w = R_{adv} + A_{adv} + R_{dt} + A_{dt}.
\end{aligned} \tag{4.2.16}$$

The terms that contain mean curvature are

$$\begin{aligned}
J_4 + J_{15} &= \kappa \int_0^t \int_{\mathbb{R}^3\times\mathbb{S}^2} (I - s \otimes s) : \nabla v dV(\tau) - \kappa \int_0^t \int_{\Gamma_v(\tau)} Hn_v \cdot (u - v) dSd\tau \\
&+ \kappa \int_0^t \int_{\mathbb{R}^3\times\mathbb{S}^2} (I - s \otimes s) : \nabla w dV(\tau) + \kappa \int_0^t \int_{\Gamma_v(\tau)} Hn_v \cdot w dSd\tau.
\end{aligned} \tag{4.2.17}$$

The viscosity terms R_{visc} and A_{visc} in [10] are treated by

$$J_3 - I_6 + J_{14} = R_{visc} + A_{visc} - 2 \int_0^t \int_{\mathbb{R}^3} \nu(\chi_u)|D(u - v - w)|^2. \tag{4.2.18}$$

Therefore, we obtain:

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^3} |u - v - w|^2(t)dx + \kappa|\nabla\chi_u(t)|(\mathbb{R}^3) + \kappa \int_{\mathbb{R}^3} 1 - \theta(t)dV_*(t) \\
&+ \frac{1}{2} \int_{\mathbb{R}^3} |B_u|^2(t) + \sigma \int_0^t \int_{\mathbb{R}^3} |\nabla B_u|^2 + 2 \int_0^t \int_{\mathbb{R}^3} \nu(\chi_u)|D(u - v - w)|^2 \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} |u - v - w|^2(0)dx + \kappa|\nabla\chi_{u,0}|(\mathbb{R}^3) + \frac{1}{2} \int_{\mathbb{R}^3} |B_u|^2(0) \\
&+ \int_0^t \int_{\mathbb{R}^3} B_u \otimes B_u : \nabla(v + w) + \int_0^t \int_{\mathbb{R}^3} B_v \otimes B_v : \nabla(u - v - w) \\
&+ R_{adv} + A_{adv} + R_{dt} + A_{dt} + R_{visc} + A_{visc} + J_4 + J_{15}.
\end{aligned} \tag{4.2.19}$$

By (202), (208) and (210) in [10], we obtain

$$
\begin{aligned}
&-\kappa\int_{\mathbb{R}^3} n_u(t)\cdot\xi(t)d\left|\nabla\chi_u\right| + \int_{\mathbb{R}^3}|\chi_u(t)-\chi_v(t)|\left|\beta\left(\frac{d(x)}{\varrho_0}\right)\right| \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3}|u-v-w|^2(t)dx + \kappa|\nabla\chi_u(t)|(\mathbb{R}^3) + \kappa\int_{\mathbb{R}^3} 1-\theta(t)dV_*(t) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3}|B_u|^2(t) + \sigma\int_0^t\int_{\mathbb{R}^3}|\nabla B_u|^2 + 2\int_0^t\int_{\mathbb{R}^3}\nu(\chi_u)|D(u-v-w)|^2 \\
&\leq -\kappa\int_{\mathbb{R}^3} n_{u,0}\xi(0)d\left|\nabla\chi_{u,0}\right| + \frac{1}{2}\int_{\mathbb{R}^3}|u-v-w|^2(0)dx + \kappa|\nabla\chi_{u,0}|(\mathbb{R}^3) + \frac{1}{2}\int_{\mathbb{R}^3}|B_u|^2(0) \\
&\quad + \int_0^t\int_{\mathbb{R}^3} B_u\otimes B_u : \nabla(v+w) + \int_0^t\int_{\mathbb{R}^3} B_v\otimes B_v : \nabla(u-v-w) \\
&\quad + R_{adv} + A_{adv} + R_{dt} + A_{dt} + R_{visc} + A_{visc} + R_{surTen} + A_{surTen} + R_{weightVol} + A_{weightVol} \\
&\quad + \int_{\mathbb{R}^3}|\chi_{u,0}-\chi_{v,0}|\left|\beta\left(\frac{d(x)}{\varrho_0}\right)\right|.
\end{aligned}
\tag{4.2.20}
$$

The definitions of R_{surTen}, A_{surTen}, $R_{weightVol}$, $A_{weightVol}$ and β can be found in [10, Proposition 10]. It remains to obtain the term $|B_u - B_v|^2$ to complete the relative entropy. In order to do this, we add (4.2.10) and (4.2.11) to (4.2.20), which implies

$$
\begin{aligned}
&-\kappa \int_{\mathbb{R}^3} n_u(t) \cdot \xi(t) d\,|\nabla \chi_u| + \int_{\mathbb{R}^3} |\chi_u(t) - \chi_v(t)| \left| \beta \left(\frac{d(x)}{\varrho_0} \right) \right| \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} |u - v - w|^2(t) dx + \kappa |\nabla \chi_u(t)|(\mathbb{R}^3) + \kappa \int_{\mathbb{R}^3} 1 - \theta(t) dV_*(t) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} |B_u|^2(t) - \int_{\mathbb{R}^3} B_u(t) \cdot B_v(t) + \int_{\mathbb{R}^3} \frac{1}{2} |B_v|^2(t) \\
&\quad + \sigma \int_0^t \int_{\mathbb{R}^3} |\nabla B_u|^2 + 2 \int_0^t \int_{\mathbb{R}^3} \nu(\chi_u) |D(u - v - w)|^2 \\
&\leq -\kappa \int_{\mathbb{R}^3} n_{u,0} \xi(0) d\,|\nabla \chi_{u,0}| + \frac{1}{2} \int_{\mathbb{R}^3} |u - v - w|^2(0) dx + \kappa |\nabla \chi_{u,0}|(\Omega) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} |B_u|^2(0) - \int_{\mathbb{R}^3} B_u(0) \cdot B_v(0) + \int_{\mathbb{R}^3} \frac{1}{2} |B_v|^2(0) \\
&\quad + \int_0^t \int_{\mathbb{R}^3} B_u \otimes B_u : \nabla(v + w) + \int_0^t \int_{\mathbb{R}^3} B_v \otimes B_v : \nabla(u - v - w) \\
&\quad + R_{adv} + A_{adv} + R_{dt} + A_{dt} + R_{visc} + A_{visc} + R_{surTen} + A_{surTen} + R_{weightVol} + A_{weightVol} \\
&\quad + \int_{\mathbb{R}^3} |\chi_{u,0} - \chi_{v,0}| \left| \beta \left(\frac{d(x)}{\varrho_0} \right) \right| \\
&\quad - \int_0^t \int_{\mathbb{R}^3} B_u \cdot \partial_t B_v - \int_0^t \int_{\mathbb{R}^3} u \otimes B_u : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} B_u \otimes u : \nabla B_v \\
&\quad + \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B_u : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} B_v \cdot \partial_t B_v + \int_0^t \int_{\mathbb{R}^3} u \otimes B_v : \nabla B_v.
\end{aligned}
\tag{4.2.21}
$$

We recall that

$$
\kappa |\nabla \chi_u(t)|(\mathbb{R}^3) - \kappa \int_{\mathbb{R}^3} n_u \cdot \xi(t) d\,|\nabla \chi_u| = \int_{\mathbb{R}^3} 1 - n_u \cdot \xi(t) d\,|\nabla \chi_u(t)| \tag{4.2.22}
$$

and

$$
\int_{\mathbb{R}^3} \frac{1}{2} |B_u|^2(t) - \int_{\mathbb{R}^3} B_u(t) \cdot B_v(t) + \int_{\mathbb{R}^3} \frac{1}{2} |B_v|^2(t) = \int_{\mathbb{R}^3} \frac{1}{2} |B_u - B_v|^2(t). \tag{4.2.23}
$$

Using the same argument, we combine corresponding terms in (4.2.21) and obtain terms

$$
\int_{\mathbb{R}^3} 1 - n_{u,0} \cdot \xi(0) d\,|\nabla \chi_u(0)| \quad \text{and} \quad \int_{\mathbb{R}^3} \frac{1}{2} |B_u - B_v|^2(0).
$$

We can now replace the corresponding terms with $E(t)$ and $E(0)$ and obtain

$$
\begin{aligned}
&E(t)+\sigma\int_0^t\int_{\mathbb{R}^3}|\nabla B_u|^2+2\int_0^t\int_{\mathbb{R}^3}\nu(\chi_u)|D(u-v-w)|^2\\
&\le E(0)+R_{adv}+A_{adv}+R_{dt}+A_{dt}+R_{visc}+A_{visc}+R_{surTen}+A_{surTen}+R_{weightVol}+A_{weightVol}\\
&+\int_0^t\int_{\mathbb{R}^3}B_u\otimes B_u:\nabla(v+w)+\int_0^t\int_{\mathbb{R}^3}B_v\otimes B_v:\nabla(u-v-w)\\
&-\int_0^t\int_{\mathbb{R}^3}(B_u-B_v)\cdot\partial_t B_v-\int_0^t\int_{\mathbb{R}^3}u\otimes(B_u-B_v):\nabla B_v\\
&+\int_0^t\int_{\mathbb{R}^3}B_u\otimes u:\nabla B_v+\sigma\int_0^t\int_{\mathbb{R}^3}\nabla B_u:\nabla B_v.
\end{aligned}
\tag{4.2.24}
$$

Combining the second term and the last term in (4.2.24), we have

$$
\begin{aligned}
&\sigma\int_0^t\int_{\mathbb{R}^3}|\nabla B_u|^2-\sigma\int_0^t\int_{\mathbb{R}^3}\nabla B_u:\nabla B_v\\
&=\sigma\int_0^t\int_{\mathbb{R}^3}|\nabla(B_u-B_v)|^2+\sigma\int_0^t\int_{\mathbb{R}^3}\nabla B_v:\nabla(B_u-B_v).
\end{aligned}
\tag{4.2.25}
$$

Thus, we can rewrite (4.2.24) as

$$
\begin{aligned}
&E(t)+2\int_0^t\int_{\mathbb{R}^3}\nu(\chi_u)|D(u-v-w)|^2+\sigma\int_0^t\int_{\mathbb{R}^3}|\nabla(B_u-B_v)|^2\\
&\le E(0)+R_{adv}+A_{adv}+R_{dt}+A_{dt}+R_{visc}+A_{visc}+R_{surTen}+A_{surTen}\\
&+R_{weightVol}+A_{weightVol}+I_1+\cdots+I_6,
\end{aligned}
\tag{4.2.26}
$$

where $t\in[0,T_0)$ and

$$
\begin{aligned}
&I_1+\cdots+I_6\\
&:=\int_0^t\int_{\mathbb{R}^3}B_u\otimes B_u:\nabla(v+w)+\int_0^t\int_{\mathbb{R}^3}B_v\otimes B_v:\nabla(u-v-w)\\
&-\int_0^t\int_{\mathbb{R}^3}(B_u-B_v)\cdot\partial_t B_v-\int_0^t\int_{\mathbb{R}^3}u\otimes(B_u-B_v):\nabla B_v\\
&+\int_0^t\int_{\mathbb{R}^3}B_u\otimes u:\nabla B_v-\sigma\int_0^t\int_{\mathbb{R}^3}\nabla B_v:\nabla(B_u-B_v).
\end{aligned}
\tag{4.2.27}
$$

Now we rewrite terms I_1, I_4 and I_5. For I_1, we replace B_u with $B_u - B_v + B_v$ and obtain

$$\begin{aligned} I_1 &= \int_0^t \int_{\mathbb{R}^3} B_u \otimes B_u : \nabla(v+w) \\ &= \int_0^t \int_{\mathbb{R}^3} (B_u - B_v + B_v) \otimes (B_u - B_v + B_v) : \nabla(v+w) \\ &= \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (B_u - B_v) : \nabla(v+w) + \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes B_v : \nabla(v+w) \\ &+ \int_0^t \int_{\mathbb{R}^3} B_v \otimes (B_u - B_v) : \nabla(v+w) + \int_0^t \int_{\mathbb{R}^3} B_v \otimes B_v : \nabla(v+w). \end{aligned} \tag{4.2.28}$$

Using the same idea as for I_1, we obtain

$$\begin{aligned} I_4 &= -\int_0^t \int_{\mathbb{R}^3} u \otimes (B_u - B_v) : \nabla B_v \\ &= -\int_0^t \int_{\mathbb{R}^3} (u - (v+w) + (v+w)) \otimes (B_u - B_v) : \nabla B_v \\ &= -\int_0^t \int_{\mathbb{R}^3} (u - v - w) \otimes (B_u - B_v) : \nabla B_v \\ &- \int_0^t \int_{\mathbb{R}^3} (v+w) \otimes (B_u - B_v) : \nabla B_v \end{aligned} \tag{4.2.29}$$

and

$$\begin{aligned} I_5 &= \int_0^t \int_{\mathbb{R}^3} B_u \otimes u : \nabla B_v \\ &= \int_0^t \int_{\mathbb{R}^3} (B_u - B_v + B_v) \otimes (u - (v+w) + (v+w)) : \nabla B_v \\ &= \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (u - v - w) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (v+w) : \nabla B_v \\ &+ \int_0^t \int_{\mathbb{R}^3} B_v \otimes (u - v - w) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} B_v \otimes (v+w) : \nabla B_v. \end{aligned} \tag{4.2.30}$$

Thus, we have

$$
\begin{aligned}
I_1 + \cdots + I_6 &:= \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (B_u - B_v) : \nabla(v+w) \\
&+ \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes B_v : \nabla(v+w) + \int_0^t \int_{\mathbb{R}^3} B_v \otimes (B_u - B_v) : \nabla(v+w) \\
&+ \int_0^t \int_{\mathbb{R}^3} B_v \otimes B_v : \nabla(v+w) + \int_0^t \int_{\mathbb{R}^3} B_v \otimes B_v : \nabla(u-v-w) \\
&- \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \cdot \partial_t B_v - \int_0^t \int_{\mathbb{R}^3} (u-v-w) \otimes (B_u - B_v) : \nabla B_v \\
&- \int_0^t \int_{\mathbb{R}^3} (v+w) \otimes (B_u - B_v) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (u-v-w) : \nabla B_v \\
&+ \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (v+w) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} B_v \otimes (u-v-w) : \nabla B_v \\
&+ \int_0^t \int_{\mathbb{R}^3} B_v \otimes (v+w) : \nabla B_v - \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B_v : \nabla(B_u - B_v).
\end{aligned}
\tag{4.2.31}
$$

Notice that term 4 cancels term 12 and term 5 cancels term 11. Thus, we finally obtain the relative entropy inequality:

$$
\begin{aligned}
&E(t) + 2\int_0^t \int_{\mathbb{R}^3} \nu(\chi_u)|D(u-v-w)|^2 + \sigma \int_0^t \int_{\mathbb{R}^3} |\nabla(B_u - B_v)|^2 \\
&\le E(0) + R_{adv} + A_{adv} + R_{dt} + A_{dt} + R_{visc} + A_{visc} + R_{surTen} + A_{surTen} \\
&+ R_{weightVol} + A_{weightVol} + I_{\text{good}} + I_{\text{bad}},
\end{aligned}
\tag{4.2.32}
$$

where $t \in [0, T_0)$. The terms R and A are defined in [10]. The two terms I_{good} and I_{bad} are defined as:

$$
\begin{aligned}
I_{\text{good}} &:= \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (B_u - B_v) : \nabla(v+w) \\
&- \int_0^t \int_{\mathbb{R}^3} (u-v-w) \otimes (B_u - B_v) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (u-v-w) : \nabla B_v
\end{aligned}
\tag{4.2.33}
$$

and

$$
\begin{aligned}
I_{\text{bad}} := & \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes B_v : \nabla(v+w) + \int_0^t \int_{\mathbb{R}^3} B_v \otimes (B_u - B_v) : \nabla(v+w) \\
& - \int_0^t \int_{\mathbb{R}^3} (v+w) \otimes (B_u - B_v) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (v+w) : \nabla B_v \\
& - \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \cdot \partial_t B_v - \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B_v : \nabla(B_u - B_v) \\
= & \int_0^t \int_{\mathbb{R}^3} B_v \otimes (B_u - B_v) : \nabla(v+w) \\
& - \int_0^t \int_{\mathbb{R}^3} (v+w) \otimes (B_u - B_v) : \nabla B_v - \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \cdot \partial_t B_v - \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B_v : \nabla(B_u - B_v).
\end{aligned}
\tag{4.2.34}
$$

The term I_{bad} can only be controlled using $\int_0^t E(t)^{\frac{1}{2}}$ rather than $\int_0^t E(t)$. In order to utilize the Gronwall's inequality, we still need to treat I_{bad} using the equations of the strong solution. By the transport theorem we have

$$
\begin{aligned}
& \int_{\mathbb{R}^3} (B_v \cdot \varphi)(t) - \int_{\mathbb{R}^3} (B_0 \cdot \varphi)(0) = \int_0^t \int_{\mathbb{R}^3} \partial_t (B_v \cdot \varphi) + \int_0^t \int_{\mathbb{R}^3} v \cdot \nabla (B_v \cdot \varphi) \\
& = \int_0^t \int_{\mathbb{R}^3} \partial_t B_v \cdot \varphi + \int_0^t \int_{\mathbb{R}^3} B_v \cdot \partial_t \varphi + \int_0^t \int_{\mathbb{R}^3} v \otimes \varphi : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} v \otimes B_v : \nabla \varphi.
\end{aligned}
\tag{4.2.35}
$$

Next, we consider the magnetic equation (4.0.2) of the strong solution. Testing the equation with $\varphi \in C_c^\infty([0,T_0) \times \mathbb{R}^3)$, $\operatorname{div}\varphi = 0$, we obtain for all $t \in [0, T_0)$ that

$$
\begin{aligned}
& \int_{\mathbb{R}^3} B_v(t) \cdot \varphi(t) - \int_{\mathbb{R}^3} B_0 \cdot \varphi(0) = \int_0^t \int_{\mathbb{R}^3} B_v \cdot \partial_t \varphi + \int_0^t \int_{\mathbb{R}^3} v \otimes B_v : \nabla \varphi \\
& - \int_0^t \int_{\mathbb{R}^3} B_v \otimes v : \nabla \varphi - \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B_v : \nabla \varphi.
\end{aligned}
\tag{4.2.36}
$$

Subtracting (4.2.36) from (4.2.35), we have

$$
\int_0^t \int_{\mathbb{R}^3} \partial_t B_v \cdot \varphi + \int_0^t \int_{\mathbb{R}^3} v \otimes \varphi : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} B_v \otimes v : \nabla \varphi + \sigma \int_0^t \int_{\mathbb{R}^3} \nabla B_v : \nabla \varphi = 0.
\tag{4.2.37}
$$

Letting $\varphi = B_u - B_v$ in (4.2.37) and combining it with I_{bad}, we finally obtain

$$
I_{\text{bad}} = - \int_0^t \int_{\mathbb{R}^3} B_v \otimes w : \nabla(B_u - B_v) - \int_0^t \int_{\mathbb{R}^3} w \otimes (B_u - B_v) : \nabla B_v.
\tag{4.2.38}
$$

4.3 Control of Relative Entropy

Now we can estimate I_{good} and I_{bad} in terms of the relative entropy E. We let

$$
\begin{aligned}
& I_{\text{good}} + I_{\text{bad}} = \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (B_u - B_v) : \nabla(v+w) \\
& - \int_0^t \int_{\mathbb{R}^3} (u - v - w) \otimes (B_u - B_v) : \nabla B_v + \int_0^t \int_{\mathbb{R}^3} (B_u - B_v) \otimes (u - v - w) : \nabla B_v \\
& - \int_0^t \int_{\mathbb{R}^3} B_v \otimes w : \nabla(B_u - B_v) - \int_0^t \int_{\mathbb{R}^3} w \otimes (B_u - B_v) : \nabla B_v \\
& =: I_1 - I_2 + I_3 - I_4 - I_5.
\end{aligned}
\tag{4.3.1}
$$

We assume in advance that $E(t) \leq \epsilon^2(t)$ for an auxiliary function $\epsilon(t) \in C^1([0, T_0); [0, \varrho_0))$. The auxiliary function w has been carefully studied in [10, Proposition 28]. In the estimation of magnetic terms, we will use the following two properties of w in [10]:

$$
\begin{aligned}
& \|\nabla w(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \frac{1}{\varrho_0^4} |\log \epsilon(t)| \, \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(t))} + C \frac{1}{\varrho_0^3} \left\|\nabla^3 v\right\|_{L^\infty(\mathbb{R}^3 \setminus \Gamma_v(t))} \\
& + C \frac{1}{\varrho_0^9} \left(1 + \mathcal{H}^2(\Gamma_v(t))\right) \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(t))}
\end{aligned}
\tag{4.3.2}
$$

and

$$
\begin{aligned}
& \int_{\mathbb{R}^3} |w(t)|^2 \, dx \leq C \left(\frac{M^2}{\varrho_0^4} \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(t))} + 1 \right) \\
& \cdot \left(\int_{\Gamma_v(t)} \left|h^+_{\epsilon(t)}\right|^2 + \left|\nabla h^+_{\epsilon(t)}\right|^2 + \left|h^-_{\epsilon(t)}\right|^2 + \left|\nabla h^-_{\epsilon(t)}\right|^2 dS \right) \\
& \leq C(v, \varrho_0) E(t)
\end{aligned}
\tag{4.3.3}
$$

where $M > 0$ is a fixed number such that $B(\Gamma_v(t); \varrho_0) \subseteq B(0; M)$ and the functions $h^{\pm}_{\epsilon(t)}$ and $\nabla h^{\pm}_{\epsilon(t)}$ are defined in [10].

Now we estimate I_1 to I_5 in (4.3.1) term by term. Using (4.3.2), we obtain

$$
|I_1| \lesssim \int_0^t \|\nabla(v+w)\|_{L^\infty} \|B_u - B_v\|_{L^2}^2 \lesssim \int_0^t C(\tau) E(\tau) d\tau \tag{4.3.4}
$$

where

$$
\begin{aligned}
& C(\tau) = \|\nabla v\|_{L^\infty(\mathbb{R}^3 \setminus \Gamma_v(\tau))} + \frac{1}{\varrho_0^4} |\log \epsilon(\tau)| \, \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(\tau))} \\
& + \frac{1}{\varrho_0^3} \left\|\nabla^3 v\right\|_{L^\infty(\mathbb{R}^3 \setminus \Gamma_v(\tau))} + \frac{1}{\varrho_0^9} \left(1 + \mathcal{H}^2(\Gamma_v(\tau))\right) \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(\tau))}.
\end{aligned}
\tag{4.3.5}
$$

Using Hölder's inequality, we have

$$\begin{aligned}
|I_2| + |I_3| &\lesssim \int_0^t \int_{\mathbb{R}^3} |u - v - w| \, |B_u - B_v| \, |\nabla B_v| \\
&\lesssim \int_0^t \|\nabla B_v(\tau)\|_{L^\infty(\mathbb{R}^3)} \|u - v - w\|_{L^2(\mathbb{R}^3)} \|B_u - B_v\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \int_0^t \|\nabla B_v(\tau)\|_{L^\infty(\mathbb{R}^3)} \left(\|u - v - w\|^2_{L^2(\mathbb{R}^3)} + \|B_u - B_v\|^2_{L^2(\mathbb{R}^3)} \right) \\
&\lesssim \int_0^t \|\nabla B_v(\tau)\|_{L^\infty(\mathbb{R}^3)} E(\tau) d\tau.
\end{aligned} \tag{4.3.6}$$

Without loss of generality, we assume $\varrho_0 < 1$, then for all $\delta > 0$ we have

$$\begin{aligned}
|I_4| &\lesssim \int_0^t \|B_v\|_{L^\infty(\mathbb{R}^3)} \|w\|_{L^2(\mathbb{R}^3)} \|\nabla (B_u - B_v)\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \delta \int_0^t \|B_v\|_{L^\infty(\mathbb{R}^3)} \|\nabla (B_u - B_v)\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\delta} \int_0^t \|B_v\|_{L^\infty(\mathbb{R}^3)} \|w\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim \delta \int_0^t \|B_v\|_{L^\infty(\mathbb{R}^3)} \|\nabla (B_u - B_v)\|^2_{L^2(\mathbb{R}^3)} \\
&+ \frac{1}{\delta} \int_0^t \frac{C \|B_v\|_{L^\infty}}{\varrho_0^4} \left(\frac{1}{\varrho_0^4} \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(\tau))} + 1 \right) E(\tau).
\end{aligned} \tag{4.3.7}$$

Finally, using Hölder's inequality, we have

$$\begin{aligned}
|I_5| &\lesssim \int_0^t \|\nabla B_v\|_{L^\infty(\mathbb{R}^3)} \|w\|_{L^2(\mathbb{R}^3)} \|B_u - B_v\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \int_0^t \|\nabla B_v\|_{L^\infty(\mathbb{R}^3)} \|B_u - B_v\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla B_v\|_{L^\infty(\mathbb{R}^3)} \|w\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim \int_0^t \|\nabla B_v\|_{L^\infty(\mathbb{R}^3)} E(\tau) + \int_0^t \frac{\|\nabla B_v\|_{L^\infty}}{\varrho_0^4} \left(\frac{1}{\varrho_0^4} \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(\tau))} + 1 \right) E(\tau) \\
&\lesssim \int_0^t \frac{\|\nabla B_v\|_{L^\infty}}{\varrho_0^4} \left(\frac{1}{\varrho_0^4} \|v\|_{W^{2,\infty}(\mathbb{R}^3 \setminus \Gamma_v(\tau))} + 1 \right) E(\tau).
\end{aligned} \tag{4.3.8}$$

Thus, we obtain

$$\begin{aligned}
|I_1| + \cdots + |I_5| \leq C(v, B_v, \Gamma_v, \varrho_0, \delta) \int_0^t (1 + |\log \epsilon(\tau)|) \, E(\tau) d\tau \\
+ C(B_v) \delta \int_0^t \|\nabla (B_u - B_v)\|^2_{L^2} \, d\tau.
\end{aligned} \tag{4.3.9}$$

The estimates in the right-hand side of (4.3.9) do not change the structure of the relative entropy inequality in [10, (176)]. Thus, when the assumption $E(t) \leq \epsilon^2(t)$ holds, by choosing δ sufficiently small, we can find $c_1, c_2 > 0$, such that

$$
\begin{aligned}
& E(t) + c_1 \int_0^t \int_{\mathbb{R}^3} |\nabla (u - v - w)|^2 + c_2 \int_0^t \int_{\mathbb{R}^3} |\nabla (B_u - B_v)|^2 \\
& \leq E(0) + C(v, B_v, \Gamma_v, \varrho_0, \delta) \int_0^t (1 + |\log \epsilon(\tau)|) \, E(\tau) d\tau \\
& \quad + C(v, B_v, \Gamma_v, \varrho_0, \delta) \int_0^t (1 + |\log \epsilon(t)|) \, \epsilon(\tau) E^{\frac{1}{2}}(\tau) d\tau \\
& \quad + C(v, B_v, \Gamma_v, \varrho_0, \delta) \int_0^t \epsilon'(\tau) E(\tau) d\tau.
\end{aligned} \tag{4.3.10}
$$

The right-hand side of (4.3.10) is the same as in [10, Section 6.9]. Therefore, the arguments in [10] can be carried out directly to obtain $E(t) \equiv 0$, which completes the proof of the weak-strong uniqueness.

Appendix Basic Theorems and Details of Proofs

We include some useful details in the appendix for the readers' reference.

We state the Aubin-Lions lemma, which is used for obtaining stronger convergence in the study of varifold solutions.

Theorem A.0.1 ([23], Theorem 1.71)**.** *Given Banach spaces X, Y and Z such that $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$. Let $q \in (1, \infty]$ be a fixed number. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $L^q([0,T];Y) \cap L^1([0,T];X)$. Suppose that $\{\partial_t f_n\}_{n=1}^{\infty}$ is bounded in $L^1([0,T];Z)$. Then for all $p \in [1, q)$, the sequence $\{f_n\}_{n=1}^{\infty}$ is relatively compact in $L^p([0,T];Y)$.*

We write the estimate of the product of two functions from different spaces. The dependency of constant terms on parameters are carefully studied.

Proposition A.0.1. *Let Ω be a bounded open set. Let $T_0 > 0$, $s \in (0,1)$, $r \in (0,1)$ and $q \geq 1$ be fixed numbers. For all $T \in (0, T_0]$, suppose that $f \in C^1([0,T];C(\Omega)) \cap C([0,T];C^1(\Omega))$ and $g \in W^{s,q}([0,T];L^q(\Omega)) \cap L^q([0,T];W^{r,q}(\Omega))$. Then*

$$\|fg\|_{W^{s,q}L^q \cap L^q W^{r,q}} \leq C \|f\|_{C^1 C \cap C C^1} \|g\|_{W^{s,q}L^q \cap L^q W^{r,q}} .$$

Proof. Step 1:
For all $t \in [0,T]$ we have

$$\|fg\|_{W^{r,q}(\Omega)}(t) := \|fg\|_{L^q}(t) + [fg]_{W^{r,q}}(t).$$

We ignore the variable t when there is no confusion. For the L^q norm we have

$$\|fg\|_{L^q(\Omega)} \leq \|f\|_{C(\Omega)} \|g\|_{L^q(\Omega)} .$$

For the seminorm we have

$$\begin{aligned}
[fg]_{W^{r,q}(\Omega)} &:= \left(\int_\Omega\int_\Omega \frac{|f(x)g(x)-f(y)g(y)|^q}{|x-y|^{n+rq}}dydx\right)^{\frac{1}{q}} \\
&\leq \left(C(q)\int_\Omega\int_\Omega \frac{|f(x)|^q\,|g(x)-g(y)|^q+|f(x)-f(y)|^q\,|g(y)|^q}{|x-y|^{n+rq}}dydx\right)^{\frac{1}{q}} \\
&\leq \left(C(q)\int_\Omega\int_\Omega \frac{\|f\|_{C^0}^q\,|g(x)-g(y)|^q+\|f\|_{C^1}^q\,|x-y|^q\,|g(y)|^q}{|x-y|^{n+rq}}dydx\right)^{\frac{1}{q}} \\
&\leq C(q)\,\|f\|_{C^0}\,[g]_{W^{r,q}}+C(q)\,\|f\|_{C^1}\left(\int_\Omega |g(y)|^q\left(\int_\Omega |x-y|^{-n-rq+q}\,dx\right)dy\right)^{\frac{1}{q}} \\
&\leq C(q)\,\|f\|_{C^0}\,[g]_{W^{r,q}}+C(q,\mathrm{diam}(\Omega))\,\|f\|_{C^1}\,\|g\|_{L^q} \\
&\leq C(q,\mathrm{diam}(\Omega))\,\|f\|_{C^1}\,\|g\|_{W^{r,q}}
\end{aligned} \tag{A.0.1}$$

where $C(q,\mathrm{diam}(\Omega))$ is an increasing function of $\mathrm{diam}(\Omega)$. Thus, we have

$$\begin{aligned}
\|fg\|_{L^qW^{r,q}} &:= \left(\int_0^T \|fg\|_{W^{r,q}(\Omega)}^q\,(t)dt\right)^{\frac{1}{q}} \\
&\leq C(q,\mathrm{diam}(\Omega))\left(\int_0^T \|f\|_{C^1(\Omega)}^q\,(t)\,\|g\|_{W^{r,q}(\Omega)}^q\,(t)dt\right)^{\frac{1}{q}} \\
&\leq C(q,\mathrm{diam}(\Omega))\,\|f\|_{C^0C^1}\,\|g\|_{L^qW^{r,q}}\,,
\end{aligned} \tag{A.0.2}$$

where the constant is still an increasing function of $\mathrm{diam}(\Omega)$ and it is independent of T_0 or T.

Step 2: We recall that

$$\|fg\|_{W^{s,q}L^q} := \|fg\|_{L^qL^q} + [fg]_{W^{s,q}L^q}$$

with the seminorm defined as

$$[f]_{W^{s,q}L^q} := \left(\int_0^T\int_0^T \frac{|f(t)-f(\tau)|^q}{|x-y|^{1+sq}}dtd\tau\right)^{\frac{1}{q}}.$$

We have the estimate

$$
\begin{aligned}
[fg]_{W^{s,q}L^q} &:= \left(\int_0^T \int_0^T \frac{\|f(t)g(t)-f(\tau)g(\tau)\|^q_{L^q(\Omega)}}{|t-\tau|^{1+sq}} dtd\tau\right)^{\frac{1}{q}} \\
&\leq C(q)\left(\int_0^T \int_0^T \frac{\|f(t)\left(g(t)-g(\tau)\right)\|^q_{L^q(\Omega)} + \|\left(f(t)-f(\tau)\right)g(\tau)\|^q_{L^q(\Omega)}}{|t-\tau|^{1+sq}} dtd\tau\right)^{\frac{1}{q}} \\
&\leq C(q)\left(\int_0^T \int_0^T \frac{\|f(t)\|^q_{C^0(\Omega)}\|g(t)-g(\tau)\|^q_{L^q(\Omega)} + \|f(t)-f(\tau)\|^q_{C^0(\Omega)}\|g(\tau)\|^q_{L^q}}{|t-\tau|^{1+sq}} dtd\tau\right)^{\frac{1}{q}} \\
&\leq C(q)\left(\int_0^T \int_0^T \frac{\|f(t)\|^q_{C^0(\Omega)}\|g(t)-g(\tau)\|^q_{L^q(\Omega)} + \|f\|^q_{C^1C^0}|t-\tau|^q\|g(\tau)\|^q_{L^q}}{|t-\tau|^{1+sq}} dtd\tau\right)^{\frac{1}{q}} \\
&\leq C(q)\|f\|_{C^0C^0}[g]_{W^{s,q}L^q} + C(q)\|f\|_{C^1C^0}\left(\int_0^T \|g(\tau)\|^q_{L^q}\left(\int_0^T |t-\tau|^{-1-sq+q}\,dt\right)d\tau\right)^{\frac{1}{q}} \\
&\leq C(q)\|f\|_{C^0C^0}[g]_{W^{s,q}L^q} + C(q,T)\|f\|_{C^1C^0}\|g\|_{L^qL^q} \\
&\leq C(q,T)\|f\|_{C^1C^0}\|g\|_{W^{s,q}L^q}\,,
\end{aligned}
\tag{A.0.3}
$$

where $C(q,T)$ is an increasing function of T.

Consequently, since Ω and T_0 are fixed and $T \leq T_0$, we obtain for all $T \in (0, T_0]$ that

$$
\begin{aligned}
&\|fg\|_{W^{s,q}([0,T];L^q(\Omega))\cap L^q([0,T];W^{r,q}(\Omega))} := \|fg\|_{L^qW^{r,q}} + \|fg\|_{W^{s,q}L^q} \\
&\leq C(q,\mathrm{diam}(\Omega))\|f\|_{C^0C^1}\|g\|_{L^qW^{r,q}} + C(q,T_0)\|f\|_{C^1C^0}\|g\|_{W^{s,q}L^q} \\
&\leq C(q,\mathrm{diam}(\Omega),T_0)\|f\|_{C^1([0,T];C(\Omega))\cap C([0,T];C^1(\Omega))}\|g\|_{W^{s,q}([0,T];L^q(\Omega))\cap L^q([0,T];W^{r,q}(\Omega))}\,.
\end{aligned}
\tag{A.0.4}
$$

$\square$

Bibliography

[1] Helmut Abels. On generalized solutions of two-phase flows for viscous incompressible fluids. *Interfaces Free Bound.*, 9(1):31–65, 2007.

[2] Helmut Abels and Mathias Wilke. Well-posedness and qualitative behaviour of solutions for a two-phase Navier-Stokes-Mullins-Sekerka system. *Interfaces Free Bound.*, 15(1):39–75, 2013.

[3] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[4] Luigi Ambrosio, Gianluca Crippa, Camillo De Lellis, Felix Otto, and Michael Westdickenberg. *Transport equations and multi-D hyperbolic conservation laws*, volume 5 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin; UMI, Bologna, 2008. Lecture notes from the Winter School held in Bologna, January 2005, Edited by Fabio Ancona, Stefano Bianchini, Rinaldo M. Colombo, De Lellis, Andrea Marson and Annamaria Montanari.

[5] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[6] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[7] Lawrence C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[8] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[9] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[10] Julian Fischer and Sebastian Hensel. Weak-strong uniqueness for the Navier-Stokes equation for two fluids with surface tension. *Arch. Ration. Mech. Anal.*, 236(2):967–1087, 2020.

[11] Xumin Gu and Yanjin Wang. On the construction of solutions to the free-surface incompressible ideal magnetohydrodynamic equations. *J. Math. Pures Appl. (9)*, 128:1–41, 2019.

[12] Xianpeng Hu and Dehua Wang. Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. *Arch. Ration. Mech. Anal.*, 197(1):203–238, 2010.

[13] Song Jiang and Quansen Jiu. Existence theory of global large solutions to the high-dimensional compressible isentropic Navier-Stokes equations. Third version, 2009.

[14] Tian Jing. Varifold solutions of the two-phase three-dimensional magnetohydrodynamic equations. *J. Differential Equations*, 315:57–89, 2022.

[15] Matthias Köhne, Jan Prüss, and Mathias Wilke. Qualitative behaviour of solutions for the two-phase Navier-Stokes equations with surface tension. *Math. Ann.*, 356(2):737–792, 2013.

[16] N. V. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces*, volume 96 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[17] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968. Translated from the Russian by S. Smith.

[18] Giovanni Leoni. *A first course in Sobolev spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017.

[19] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[20] Frank Morgan. *Geometric measure theory*. Elsevier/Academic Press, Amsterdam, fifth edition, 2016. A beginner's guide, Illustrated by James F. Bredt.

[21] Josef Málek, Jindřich Nečas, Mirko Rokyta, and Michael Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[22] A. Nouri and F. Poupaud. An existence theorem for the multifluid Navier-Stokes problem. *J. Differential Equations*, 122(1):71–88, 1995.

[23] A. Novotný and I. Straškraba. *Introduction to the mathematical theory of compressible flow*, volume 27 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[24] M. Padula and V. A. Solonnikov. On the free boundary problem of magnetohydrodynamics. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 385(Kraevye Zadachi Matematicheskoĭ Fiziki i Smezhnye Voprosy Teorii Funktsiĭ. 41):135–186, 236, 2010.

[25] P. I. Plotnikov. Generalized solutions of a problem on the motion of a non-Newtonian fluid with a free boundary. *Sibirsk. Mat. Zh.*, 34(4):127–141, iii, ix, 1993.

[26] Jan Prüss, Jürgen Saal, and Gieri Simonett. Existence of analytic solutions for the classical Stefan problem. *Math. Ann.*, 338(3):703–755, 2007.

[27] Jan Prüss and Gieri Simonett. On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.*, 12(3):311–345, 2010.

[28] Jan Prüss and Gieri Simonett. Analytic solutions for the two-phase Navier-Stokes equations with surface tension and gravity. In *Parabolic problems*, volume 80 of *Progr. Nonlinear Differential Equations Appl.*, pages 507–540. Birkhäuser/Springer Basel AG, Basel, 2011.

[29] Jan Prüss and Gieri Simonett. *Moving interfaces and quasilinear parabolic evolution equations*, volume 105 of *Monographs in Mathematics*. Birkhäuser/Springer, [Cham], 2016.

[30] James C. Robinson, José L. Rodrigo, and Witold Sadowski. *The three-dimensional Navier-Stokes equations*, volume 157 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016. Classical theory.

[31] Rodolfo Salvi. On the existence of free surface problem for viscous incompressible flow. In *Topics in mathematical fluid mechanics*, volume 10 of *Quad. Mat.*, pages 247–275. Dept. Math., Seconda Univ. Napoli, Caserta, 2002.

[32] Paolo Secchi and Yuri Trakhinin. Well-posedness of the plasma-vacuum interface problem. *Nonlinearity*, 27(1):105–169, 2014.

[33] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[34] Leon Simon. *Theorems on regularity and singularity of energy minimizing maps*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Based on lecture notes by Norbert Hungerbühler.

[35] Roger Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[36] Roger Temam. *Navier-Stokes equations and nonlinear functional analysis*, volume 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1995.

[37] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[38] Yuri Trakhinin and Tao Wang. Well-posedness for the free-boundary ideal compressible magnetohydrodynamic equations with surface tension. *Math. Ann.*, 383(1-2):761–808, 2022.

[39] Yanjin Wang and Zhouping Xin. Global well-posedness of free interface problems for the incompressible inviscid resistive MHD. *Comm. Math. Phys.*, 388(3):1323–1401, 2021.

[40] Karen Yeressian. On varifold solutions of two-phase incompressible viscous flow with surface tension. *J. Math. Fluid Mech.*, 17(3):463–494, 2015.

[41] Eberhard Zeidler. *Nonlinear functional analysis and its applications. I.* Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.

[42] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/A.* Springer-Verlag, New York, 1990. Linear monotone operators, Translated from the German by the author and Leo F. Boron.

[43] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/B.* Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.