Production Networks Resilience: Cascading Failures, Power Laws and Optimal Interventions

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Problem definition: The pandemic crisis made apparent the need to take a holistic view of production networks to understand risk factors that arise due to their many interdependent units. We investigate the structural factors contributing to the severity of cascading failures in production networks, how to quantify the effect of these risk factors with resilience metrics, and how to design and evaluate interventions to improve resilience.

Methodology/Results: We propose a node percolation process on production networks that model product suppliers failing independently due to exogenous, systemic shocks and causing other products to fail as their production requirements are unmet. We first show that the size of the cascading failures follows a power law in random directed acyclic graphs, whose topology encodes the natural ordering of products from simple raw materials to complex products. This motivates the need for a resilience metric, which we define as the maximum magnitude shock the production network can withstand with only a small fraction of products failing. Next, we study the resilience of several architectures, such as trees, parallel products, and random trellis, and classify them as resilient or fragile depending on their topological attributes. In the next step, we give general lower bounds and optimal interventions for improving resilience as a function of Katz centralities. Finally, we empirically calculate the resilience metric and study interventions in various real-world networks to validate our theoretical results.

Managerial implications: We offer new thoughts and theories on reasoning about supply chain resilience based on topological attributes such as source dependency and product complexity and propose novel, theory-informed metrics that we evaluate empirically on real-world networks. Our theories also inform intervention designs for improving resilience, not just from a firm’s perspective but also on a national scale.

Key words: Production networks, Resilience, Cascading failures, Interventions

History: This version: March 23, 2023.
1. Introduction

The global economy consists of many interconnected entities which are responsible for the supply and sourcing of products (Carvalho and Tahbaz-Salehi, 2019). These production networks consist of products, each of which has some required inputs that can be sourced from a group of suppliers, and they play a critical role in day-to-day operations of the world economy (Long Jr and Plosser, 1983). Such interdependence between products yields cascades once parts of the supply chain are disrupted (Horvath, 1998; Carvalho and Tahbaz-Salehi, 2019; Lucas et al., 1995; Gabaix, 2011; Acemoglu et al., 2012; Hallegatte, 2008). For instance, the recent COVID-19 pandemic and the war in Ukraine disrupted parts of the global supply, whose failures then propagated to other parts of the supply chain network, causing bottlenecks and choke points (Ergun et al., 2023; Elliott et al., 2022; Guan et al., 2020; Walmsley et al., 2021). In light of such problems, there is an ever-emerging need to study the resilience of supply chain networks to identify vulnerabilities, such as bottlenecks and choke points, and take steps to mitigate their effects (Kleindorfer and Saad, 2005; Gurnani et al., 2012; Simchi-Levi et al., 2014). A motivating example is a tree production network where a variety of raw materials leads to the production of a specialized product. There, it is easy to observe that the failure to produce any of the raw materials, because all of the suppliers that make it failed, has a devastating effect on the network, and almost nothing can be produced.

There are various methods and approaches that can be used to study the resilience of supply chain networks (Perera et al., 2017). One direction is to use network analysis tools to identify critical components of the network and assess their vulnerability to disruption. This can include identifying key suppliers, products, and transportation routes and evaluating their importance to the overall network. Another approach is to use simulation modeling to analyze the impact of different disruptions on the network and assess the ability of the network to recover from these disruptions (Simchi-Levi et al., 2015). This can include analyzing the effect of different types of
disturbances (e.g., natural disasters, supply chain failures, etc.) and different scenarios in which the disruptions occur (e.g., simultaneous disruptions, sequential disruptions, etc.). In addition to these technical approaches, it is important to consider organizational and strategic approaches to building resilience in supply chain networks (Braunscheidel and Suresh, 2009; Simchi-Levi et al., 2018). This can include implementing contingency plans, developing relationships with alternative suppliers, and diversifying the network to reduce the impact of a single point of failure. Overall, building resilience in supply chain networks requires a combination of technical analysis, strategic planning, and organizational preparedness. By understanding the structure and dynamics of these networks, it is possible to build systems that can withstand disruptions and recover quickly when they occur.

1.1. Main Related Work

We model the production network as a graph that undergoes a percolation process on its nodes, representing products. Each product has some suppliers that, if they all fail, then the product cannot be produced. Subsequently, a cascade is caused because of such a failure. Our modeling decision to study the supply network as a graph that undergoes percolation has its roots in prior work (see the referenced works in Appendix F), the most closely related is the recent work of Elliott et al. (2022). They consider a hierarchical production network which undergoes a link percolation process. Specifically, firms are operational if all of their inputs have operating links to at least one firm producing the specific inputs. Elliott et al. (2022) study the reliability of the supply network as the probability that the root product is produced. They show that three important factors affect reliability: (i) the depth/size of the supply chain, (ii) the number of inputs that each product requires, and (iii) the number of suppliers (which corresponds to the ability of firms to multi-source their required inputs).

In our model, we investigate the effect of similar structural factors – i.e., topology, number of inputs, and number of suppliers – however, under a fundamentally expanded setup by (i) considering production shocks at the level of individual suppliers – rather than links; (ii) by considering
more general architectures than the hierarchical production networks of Elliott et al. (2022), and (iii) by studying a novel, theory-informed resilience metric that ensures almost all products can be produced in the event of a supply chain shock. One of our main results shows that production networks are either resilient or fragile. Elliott et al. (2022) show that when the systemic shock magnitude is above some critical value, then reliability goes to zero corresponding to fragile networks, and when the shocks’ magnitude is below this critical value, then reliability attains a strictly positive value corresponding to resilient networks. Briefly, they observe that if multi-sourcing (having a multitude of suppliers) increases, then reliability increases, and as interdependency (requiring a multitude of inputs) increases, reliability decreases. Our analysis differentiates from Elliott et al. (2022) and focuses on determining the largest possible shock that a network can withstand so that a significant fraction of the products are producible (survive) in the event of such a shock as the network size increases. Our networks are parametrized similarly by size, multi-sourcing, and interdependency parameters. In agreement with Elliott et al. (2022), we show that networks become more resilient when multi-sourcing increases, and when interdependencies among products increase, networks become less resilient. Additionally, we observe that certain types of networks are resilient, i.e., they can withstand sufficiently large shocks without experiencing catastrophic, cascading failures, and others are fragile – i.e., arbitrarily small shocks are enough to render a significant proportion of their products unproducible. Furthermore, we provide additional evidence in the form of the emergence of power laws for the size of cascading failures, which further motivates the study of resilience in complex production networks. Compared to existing indices that combine multitudes of temporal, financial, and economic risk factors (Gao et al., 2019), our resilience metric identifies topological risk factors and focuses on cascading failures to identify fragile and resilient supply chain architectures.

Perera et al. (2017) systematically reviews the literature on modeling the topology and robustness of production networks with applications to real-world data. Their first observation is that many real-world production networks follow a power law degree distribution with an exponent of
around two. Moreover, resilience (called “robustness” in their work) of such networks is defined as the size of the largest connected component or the average/max path length in presence of random failures. On the contrary, we propose a novel, theory-informed resilience metric that complements the existing measures in Perera et al. (2017), and references therein, and supplement their empirical works with novel theories about how topological attributes contribute to increased risk of cascading failures in production networks. We also test our proposed metric on real-world data and provide empirical insights comparable to Perera et al. (2017). Finally, we propose interventions for improving resilience with theoretical guarantees similar to optimal allocations in the presence of shocks and financial risk contagion (Eisenberg and Noe, 2001; Papachristou et al., 2022; Blume et al., 2013; Erol and Vohra, 2022; Bimpikis et al., 2019), and complementary to other supply chain interventions that, e.g., increase visibility and traceability (Blaettchen et al., 2021). An extended literature review is provided in Appendix F.

In the next Subsection, we use the above context to summarize our main contributions. Next, in Section 2, we introduce some preliminaries, which we use to motivate our proposed measure of resilience in Section 2.3. In Section 3, we present our new resilience metric formally and study it in various architectures – see also Table 1 in the next subsection, which lists our main results for different architectures. In Section 4, we present a general computational framework for lower-bounding the proposed metric in general graphs and make additional connections to financial networks, financial contagions, and centrality measures, as well as propose algorithms for optimal interventions in supply chains. In Section 5, we supply our theoretical analysis with experiments with real-world supply-chain data, where we numerically calculate resilience and design intervention policies. Finally, in Section 6, we summarize our main insights and conclusions and provide potential avenues for future work.

1.2. Main Contributions

To motivate our investigation of resiliency in supply chain networks, we first present a model of why supply chain networks are prone to catastrophic failures and why a measure of resilience is
needed to distinguish different supply chain architectures. Our model is based on a node percolation process in which the possible suppliers of a product fail independently with probability $x$, and a product can be produced when its requirements are met and its suppliers do not fail.

**Cascading Failures and Emergence of Power Laws (Section 2).** To start with, we state that the size of cascading failures in a supply chain follows a power law when the underlying supply graph is a random DAG — representing a natural topological hierarchy among products whereby the production of more complex products is contingent on the supply of simpler products and raw materials:

**Informal Theorem 1.** Consider the production network of $K$ products that is realized according to a random DAG model with edge probability $p$, and consider the node percolation model with failure probability $x$ for each of the $n$ suppliers of each product. Let $F$ be the total number of products that cannot be produced due to the unavailability of their supplier or other requisite products. Then $F$ follows a power-law distribution and admits a tail lower-bound of $\mathbb{P}[F \geq f] \geq C/f$ for some constant $C > 0$ is a constant dependent on $K$, $p$, and $x$.

**Study of Resilient Architectures (Section 3).** In the sequel, we study the resilience of many production networks and identify resilient and non-resilient architectures. Recall that we use $x$ to denote the probability of suppliers failing randomly. Consider a production network $\mathcal{G}$ with $K$ products. Our notion of resilience, denoted by $R_\mathcal{G}(\varepsilon)$, determines the maximum value of $x$ such that at least $(1 - \varepsilon)$ fraction of the products in the production network $\mathcal{G}$ are produced with probability at least $1 - 1/K$. In particular, we are interested in the limit of $R_\mathcal{G}(\varepsilon)$ for different structures as $K \to \infty$, and for resilient structures, this limit is bounded away from zero.

We study the resilience of the following production networks, whose parameters we state below:

1. **Random DAGs:** $K$ products ordered as $1, 2, \ldots, K$ and connected with directed edge probability $p$ that respects their ordering (see Section 2.3 and Figure 2(a)).

2. **Parallel products with dependencies:** A set of raw materials produces $K$ complex products, each complex product requires $m$ raw inputs (source dependencies), and each raw material is used by $d$ complex products ($d$ is called supply dependency) (see Section 3.1 and Figure 2(b)).
3. Hierarchical production networks: (i) Backward hierarchical production network: A tree production network with depth $D$ and fanout $m \geq 1$, i.e., each product has $m$ inputs independently of the other products. In this supply chain, the failures start from the raw materials we position at the tree’s leaves, and the cascades grow from the leaves to the root (see Section 3.2.1 and Figure 3(a)), and (ii) Forward hierarchical production network: A tree production network generated by a branching process (also known as the Galton-Watson process) with branching distribution $\mathcal{D}$ with mean $\mu$, which has (random) extinction time $\tau$, and extinction probability $\eta^*(\mathcal{D}) = \mathbb{P}[\tau < \infty]$. In this regime, the percolation starts from the root node (raw material), and proceeds to the leaves (see Section 3.2.2 and Figure 3(b)).

4. Random width-$w$ trellis: A trellis network with $D$ tiers, with each tier having width $w$. Random edges are generated independently with probability $p$ between (only) tier $d$ and $d + 1$. The network has $K = wD$ nodes (see Appendix D and Figure 5). It serves as a generalization of the parallel products and is a special instance of the random DAG.

Table 1 summarizes our results for the resilience of the above architectures and the corresponding theorems where the results are proved. Namely, we show:

INFORMAL THEOREM 2. The random DAG, backward production network, and random trellis with $pw \geq 1$ and constant width are always fragile with $R_G(\varepsilon) \to 0$ as $K \to \infty$ for all $\varepsilon$. In contrast, parallel products with dependencies and the random trellis with constant depth and $pw \leq 1$ are resilient. The forward production network is resilient with a positive probability.

In addition to identifying fragile architectures, our asymptotic analyses of $R_G(\varepsilon)$ reveal how various factors such as increased dependency ($d$ and $m$) affect how fast $R_G(\varepsilon) \to 0$ in case of fragile...
networks, or how far it is bounded away from zero, in case of resilient structures. These nuances are expanded upon in Section 3 as we present our results for each architecture in Table 1.

**Resilience of General Graphs & Optimal Interventions (Section 4).** Our results in Table 1 address the various instances of the DAG architecture. Our next set of results concentrates on generalizing the lower-bound analysis of the resilience metric, $R_G(\varepsilon)$, to any production network. We do this by upper-bounding the expected number of failures for a given graph. The latter is, in general, #P-hard to compute, but its upper-bound can be solved in polynomial time as an LP:

**Informal Theorem 3.** An upper bound to the expected number of failures for any production network $G$ can be found by solving the LP presented in Equation (9).

In Theorem 6, we give this LP for a more general percolation process than the one we study for DAG production networks. In particular, we allow links to also fail independently at random with probability $y \in (0, 1)$. Moreover, in Theorem 6, we show that under specific conditions on $y$, our problem is a special instance of the financial contagion model of Eisenberg and Noe (2001), thus establishing a promising connection between financial network contagion and our percolation analysis.

Finally, the formulation of Theorem 6 can be used to identify “vulnerable” products – i.e., products that are more likely to fail than others – and, subsequently, to design interventions in supply chains (see Section 4.1) based on Katz centrality. In Proposition 1, we show that under certain assumptions, the probability that each node is failing is related to its Katz centrality (Katz, 1953), which is in agreement with the existing literature in financial networks (Siebenbrunner, 2018; Bartesaghi et al., 2020). Additionally, Proposition 1 allows us to establish a generic lower bound on $R_G(\varepsilon)$ and an optimal intervention policy (under assumptions) as follows:

**Informal Theorem 4.** Consider a production network $G$ with adjacency matrix $A$ and max outdegree $\Delta \geq 1$, and its reverse network $G^R$ with maximum outdegree $\Delta_R \geq 1$. If $0 < y < \frac{1}{\max\{\Delta, \Delta_R\}}$ and $0 < x < (1 - y \max\{\Delta, \Delta_R\})^{1/n}$, then the following are true:

1. $R_G(\varepsilon) \geq \left(\frac{\varepsilon}{1 - y A^T} (I - y A^T)^{-1}\right)^{1/n}$, where $\gamma_{Katz}(G, y) = (I - y A^T)^{-1} 1$ is the Katz centrality.
2. If a planner can protect up to $T$ suppliers (which corresponds to protecting $T$ products), then the optimal intervention of the planner is to protect the $T$ products in decreasing order of Katz centrality in $G^R$, i.e.,

$$\gamma_{\text{Katz}}(G^R, y) = (I - yA)^{-1}.$$ 

Therefore, the Katz centrality of each node in the production network can inform the social planner’s effort to design contagion mitigation strategies to improve supply chain resiliency. Furthermore, we show that the resilience of any DAG is $\Omega\left(\left(\frac{1}{\varepsilon K}\right)^{1/n}\right)$ which is maximized when $y = 1/K$; thence, giving an $\Omega\left(\left(\frac{1}{K}\right)^{1/n}\right)$ lower bound on $R_G(\varepsilon)$ for any DAG; a result which is consistent with our earlier analysis of the proposed resilience metric in DAGs.

**Empirical Results (Section 5).** We experiment with real-world production networks from Willems (2008) and the World I-O database from Timmer et al. (2015). In the former case, the production networks are DAGs, whereas in the latter case, the networks correspond to countries’ economies and can also have cycles. We do Monte Carlo simulations to numerically determine the resilience of such real-world networks and show that overall more densely connected networks and those with more concentration of products in their early tiers are less resilient. Then, we design interventions based on the policy devised in Informal Theorem 4 and numerically show different resilience lower bounds between the different networks. In both cases, interventions make less resilient networks have a higher resilience lower bound.

2. Preliminary Results

2.1. The Production Network

We start by describing the production network. We consider the production of a set of products $\mathcal{K}$ with cardinality $|\mathcal{K}| = K$ where each product $i \in \mathcal{K}$ can be produced by a number of suppliers and
also requires certain inputs in order to be produced. Specifically, each product \( i \in \mathcal{K} \) has a set of requirements (inputs), denoted by \( \mathcal{N}(i) \) that it needs in order to be made. The products and the set of requirements for each product define the production network \( \mathcal{G} = \mathcal{G}(\mathcal{V}(\mathcal{G}) = \mathcal{K}, \mathcal{E}(\mathcal{G})) \).

The production network \( \mathcal{G} \) contains raw materials (or sources), which are materials that do not require any inputs, i.e., have \( |\mathcal{N}(i)| = 0 \) and are the “initial products” that are used in the production of others. We denote the set of raw materials as \( \mathcal{R} \subseteq \mathcal{K} \). Each product \( i \in \mathcal{K} \) can be sourced from a set of suppliers \( \mathcal{S}(i) \). We assume that a supplier of a product can source from any supplier of a product that the supplier depends on (one may argue here that it is impossible that every supplier of a product \( i \) can source from every supplier of, say, product \( j \), due to geographical constraints. The issue can be mitigated by expanding vertex \( j \) to multiple sub-nodes).

For simplicity, throughout the rest of the paper, we will assume that \( |\mathcal{S}(i)| = n \) for all \( i \in \mathcal{K} \).

**Structural Assumptions on \( \mathcal{G} \).** For our main results, we make the structural assumption that the production network \( \mathcal{G} \) is a Directed Acyclic Graph (DAG), i.e., a natural order exists among the products being produced, i.e., the production of more complex products depends on sourcing inputs from simpler products. Such an assumption has appeared in prior work, e.g., Elliott et al. (2022), Blaettchen et al. (2021), and Bimpikis et al. (2019). Later in the paper (Section 4), we show how we can relax this assumption and allow for cycles \( \mathcal{G} \).

**Additional Notation.** We use \([K]\) to denote the set \( \{1, \ldots, K\} \). For vectors (resp. matrices), we use \( \|x\|_p \) for the \( p \)-norm of \( x \) (resp. for the induced \( p \)-norm); for the Euclidean norm (i.e., \( p = 2 \)), we omit the subscript. \( \mathbf{0} \) (resp. \( \mathbf{1} \)) denotes all zeros (resp. all ones) column vector, and \( \mathbf{1}_S \) represents the indicator column vector of the set \( S \). We use \( x \wedge y \) (resp. \( x \vee y \)) as shorthand for the coordinate-wise minimum (resp. maximum) of vectors \( x \) and \( y \). Finally order relations \( \geq, \leq, >, < \) denote coordinate-wise ordering. \( \text{vec}(A) \) corresponds to the vectorization of matrix \( A \). We use \( \mathcal{G}^R \) to denote the reverse graph of \( \mathcal{G} \), i.e., the graph which has the same vertex set as \( \mathcal{G} \) and reversed edges. In our context, \( \mathcal{G} \) corresponds to the graph of supply relations, and \( \mathcal{G}^R \) corresponds to the graph of source relations. The notation \( \{y_n\}_{n \in \mathbb{N}} \) the notation \( x_n \asymp y_n \) means that \( \lim_{n \to \infty} \frac{x_n}{y_n} = 1 \).
2.2. Node Percolation

The supply chain graph $G$ undergoes a node percolation process at which each supplier fails independently at random with probability $x \in (0, 1)$. Each product can be produced if, and only if, (i) all of its requirements $j \in \mathcal{N}(i)$ can be produced, and, (ii) at least one of the suppliers $s \in \mathcal{S}(i)$ is operational. Upon the completion of the percolation process, a random number $F$ of products fails, and the remaining $S = K - F$ products survive. The number of surviving products can be expressed as $S = \sum_{i \in K} Z_i$ where $\{Z_i\}_{i \in K}$ are the indicator variables that equal to one if, and only if, product $i$ is produced and are zero otherwise. The sequence of variables $\{Z_i\}_{i \in K}$ obeys the following random system of equations for every $i \in K$:

$$Z_i = \prod_{j \in \mathcal{N}(i)} Z_j \left(1 - \prod_{s \in \mathcal{S}(i)} Y_{is}\right),$$

(1)

where $Y_{is} \sim \text{Be}(x)$ for all $i \in K$, $s \in \mathcal{S}(i)$. Our paper aims to study the random behavior of $F$ (resp. $S$). A planner is interested in finding the maximum probability value $x$ such that the number of failures is at most $\varepsilon K$ (e.g., sublinear) with high probability. We show that some very simple production networks can experience power-law cascades to motivate such a resilience metric.

2.3. Motivation for a Resilience Metric: Cascading failures and the emergence of power laws in random DAG structures

We start by motivating the need for the definition of a resilience measure for supply chain graphs. More specifically, similarly to large social networks (Leskovec et al., 2007; Wegrzycki et al., 2017) and power networks (Dobson et al., 2004; Dobson et al., 2005; Nesti et al., 2020), we show that a randomly generated supply chain with random DAG structure exhibits cascades that obey a power law, namely the average cascade size is dominated by a few very large scale cascades rather than the many smaller ones.

In a supply chain, we can assume that there is a “natural order” among the products being produced; namely, the production of more complex products is contingent on supplying simpler products. In a supply chain, raw materials and component parts are typically transformed into intermediate products and then into finished products through a series of production processes. The
production of more complex products often depends on the availability of simpler products, as the simpler products are used as inputs in the production of the more complex products. For example, in the production of a car, the production of the car’s engine may depend on the availability of simpler components such as computer chips. In its simplest form, this kind of behavior can be captured by a random DAG model, where a DAG is created by independently sampling edges via coin tosses.

To observe this, we start with the random DAG model \( \text{rdag}(K, p) \) described in the work of Wegrzycki et al. (2017). More specifically, we consider a supply chain with \( K \) products \( i_1, \ldots, i_K \) connected as follows: for every \( k \in [K] \) and for every \( 1 \leq l \leq K - 1 \) we add a directed edge \((i_l, i_k)\) independently with probability \( p \in (0, 1) \). Figure 2(a) shows the creation of a \( \text{rdag}(K, p) \) with probability \( p \) and \( K = 3 \) vertices.

The percolation process happens as described in Section 2.2. The following theorem determines the distribution of the failure cascade size \( F \) and shows that it grows at least as a power law with exponent one. Our proof follows similar arguments as those made by Wegrzycki et al. (2017).

**Theorem 1.** Let \( G \sim \text{rdag}(K, p) \) be the production network of \( K \) products that is realized according to a random DAG model, and consider the node percolation model with failure probability \( x \) on the supplier graph associated with the production network \( G \). Then \( \mathbb{P}[F = f] \propto \frac{x^n}{K(1-(1-x^n)(1-p)^f)} \geq \frac{C(K, p, x, n)}{f} \) where \( C(K, p, x, n) > 0 \) is a constant dependent on \( K, p, \) and \( x \).

**Proof Sketch.** We define \( P_{k,f} \) to be the probability that there are \( f \) distinct failures in the random DAG with \( k \) nodes, conditioned on a failure in node 1. Based on case analysis for node \( i_k \) – specifically, \( i_k \) can fail or not fail – we deduce a recurrence relation for \( P_{k,f} \) as a function of \( P_{k-1,f-1} \) and \( P_{k-1,f} \), then by symmetry arguments, the distribution of failures obeys \( \mathbb{P}[F = f] = \frac{x^n}{K} \sum_{k \in [K]} P_{k,f} \). By using the recurrence relation for \( P_{k,f} \) we devise a recurrence relation for \( Q_{k,f} = \sum_{k \in [K]} P_{k,f} \) and solve it at the regime of \( K \rightarrow \infty \) to get the asymptotic expression for the degree distribution, which implies the tail lower bound. \( Q.E.D. \)
The above result implies that $F$ has a tail lower bound, i.e., $\Pr[F \geq f] \geq C/f$. Having proven Theorem 1, the next question is: How can we calculate the probability that a fractional cascade emerges? Conceptually, if the probability of a fractional cascade emerging is $O(1/K)$ for some choice of the percolation probability $x$, then, with a high probability, we are going to have the majority of products surviving. In order to quantify this phenomenon, we can first calculate $\Pr[F \geq \varepsilon K]$ for a fixed fraction $\varepsilon \in (0, 1)$. A simple calculation shows that, for large enough $K$,

$$\Pr[F \geq \varepsilon K] \approx x^n \left[1 - \varepsilon + \frac{1}{K \log \left(\frac{1}{1-p} \right)} \log \left(\frac{1 - (1 - x^n)(1-p)^K}{1 - (1 - x^n)(1-p)^{\varepsilon K}}\right)\right] = g(x, K, \varepsilon, n). \quad (2)$$

We want to find values of $x$ such that for every $\varepsilon \in (0, 1)$, the probability that a cascade of size at least $\varepsilon K$ emerges goes to zero as $K \to \infty$. Note that as $K \to \infty$ we have that

$$\frac{1}{K \log \left(\frac{1}{1-p} \right)} \log \left(\frac{1 - (1 - x^n)(1-p)^K}{1 - (1 - x^n)(1-p)^{\varepsilon K}}\right) \to 0$$

and therefore $\Pr[F \geq \varepsilon K] \to x^n (1 - \varepsilon)$. In order to make this zero for every $\varepsilon \in (0, 1)$, we should set $x \to 0$. In Appendix A, we give an analytical bound on $x$ to ensure $\Pr[F \geq \varepsilon K] = O(1/K)$.

The above calculation shows that for a large random DAG, it is impossible to save a $(1 - \varepsilon)$-fraction of the products for any non-zero percolation probability $x$. Therefore, we could, on a high level, characterize the random DAG as a “fragile” architecture since even the tiniest shock can be devastating for the production network.

Identifying fragile supply chains is important, as it allows companies and organizations to identify potential vulnerabilities and risks in their supply chain operations. This can then be used to design more robust supply chains through targeted interventions. Thus, once fragile supply chains are identified, companies can take a number of steps to make them more robust, such as diversifying their supplier base, increasing inventory levels, and implementing contingency plans.

Finally, systematizing the analysis above for other supply chain graphs already yields the definition of the resiliency metric, which we formalize in the next Section.

3. A New Resilience Metric for Production Networks

The fact that power laws can arise in supply chain graphs, as we show in Theorem 1, yields the need to define a resilience metric. It is important to have a proper metric for resilience to understand
the behavior of complex systems and identify potential vulnerabilities. There are many different ways to define and measure resilience, and which metric is most appropriate will depend on the specific system being studied and the goals of the analysis. Some common approaches to defining resilience include looking at the system’s ability to recover from disturbances, absorb or adapt to change, and maintain function in the face of stress or disruption.

In our percolation model, ideally, a “more resilient” network is a network that can withstand larger productivity shocks, which are associated with larger percolation probabilities $x$. Therefore, it is natural to assume that in a resilient network, we want to find the maximum value that the percolation probability $x$ can get in order for a “large” fraction of the items to survive almost surely. Formally, for $\varepsilon \in (0, 1)$, the resilience of a (possibly random) product graph $G$ is defined as

$$R_G(\varepsilon) = \sup \left\{ x \in (0, 1) : \mathbb{P}_{G,x}[S \geq (1 - \varepsilon)K] \geq 1 - \frac{1}{\mathbb{E}_G[K]} \right\},$$

and corresponds to the maximum percolation probability for which at most $\varepsilon K$ products fail with high probability. The expectation $\mathbb{E}_G[\cdot]$ corresponds to the randomness of the graph generation process, and the probability $\mathbb{P}_{G,x}[\cdot]$ corresponds to the joint randomness of the percolation and the graph where the randomness of the graph can be over the nodes, the edges, or both. In the case of $\text{rdag}(K, p)$, our lower bound on $x$ in Appendix A for ensuring $\mathbb{P}[F \geq \varepsilon K] = O(1/K)$ is also a lower bound on the resilience for this particular architecture.

We are interested in the following questions: Do supply chains $G$ for which $R_G \to 0$ as $K \to \infty$ exist? Do supply chains $G$ for which $R_G \to R$ for some $R \in (0, 1]$ as $K \to \infty$ exist? The former type of architecture can be characterized as a fragile architecture since even the tiniest failure can devastate the network. The latter one can be characterized as a resilient architecture since it can withstand non-trivial shocks in the limit of $K \to \infty$.

In the sequel, we study a variety of network architectures and derive lower bounds for the resilience of such supply chain architectures. Table 1 summarizes our results. Briefly, in order to prove that a supply chain $G$ is resilient it suffices to choose some percolation probability $R_G(\varepsilon) \in (0, 1)$ such that $\mathbb{P}_{x=R_G(\varepsilon)}[F \geq \varepsilon K] = O(1/K)$ and prove that $\lim_{K \to \infty} R_G(\varepsilon) > 0$, which implies that
Figure 2 Production networks of Section 2.3 and Section 3.1. Failures are drawn in pink color.

As a warmup, the analysis of Section 2.3 shows that \textit{rdag}(K, p) is a fragile architecture, and therefore, we get our first theorem:

\textbf{Theorem 2.} Let $\mathcal{G} \sim \text{rdag}(K, p)$. Then, as $K \to \infty$, we have that $R_{\mathcal{G}}(\varepsilon) \to 0$.

In the next Sections, we study various architectures and derive upper and lower bounds of $R_{\mathcal{G}}(\varepsilon)$.

\section{3.1. Parallel Products with Dependencies}

The first architecture we study is parallel products. Here we aim to produce $K$ complex products, which we denote by $\mathcal{C}$, and each complex product requires $|\mathcal{N}(i)| = m$ inputs (raw materials; source dependencies) to be produced. The set of raw materials (denoted by $\mathcal{R}$) contains $|\mathcal{R}| = \rho$ raw materials. We further introduce supply dependencies among the raw materials by assuming that each raw material can supply $d$ products. Figure 2(b) shows an example of such a supply chain together with an instance of the percolation process (affected nodes are drawn in pink). Here it is interesting to study both the resilience of the whole graph, i.e., the graph with vertex set $\mathcal{C} \cup \mathcal{R}$, as well as the resilience of the complex products $\mathcal{C}$ alone. We show that if the source dependency $m$ and the supply dependency $d$ between the products is bounded, then the production network is resilient. The resilience metric is lower-bounded by $\left(\frac{\varepsilon}{2(d+1)m}\right)^{1/n}$ in both cases.
alone or together with raw materials), as the number of products goes to infinity. Moreover, if the number of inputs $m$ goes to infinity, the resilience of the whole graph goes to 0 at rate $O\left(e^{-\frac{1}{mn}}\right)$, and the resilience of the complex products goes to 0 at rate $O\left(e^{-c\frac{1}{m}}/n\right)$ for some constant $c \in (0, 1)$.

Formally, we prove the following for the resilience of the parallel products:

**Theorem 3.** Let $G$ consist of $K$ parallel products and assume that these products can be produced by $|R| = \rho$ raw materials, and each material is used by at most $d$ products. The resilience of the complex products $C$ satisfies:

$$R_C(\varepsilon) \leq \left(1 - \frac{\varepsilon^2}{2} \right)^{1/m}$$

Also, the resilience of all products satisfies:

$$R_G(\varepsilon) \leq \left(1 - \frac{\varepsilon^2}{2(m+1)} \right)^{1/n}$$

Subsequently, if $\varepsilon, d$ and $m$ are independent of $K$, then the resilience is $\Omega\left(\frac{\varepsilon^2}{d^2} / n\right)$ as $K \to \infty$.

**Proof Sketch.** For brevity, we give an analysis in expectation for $R_C(\varepsilon)$. The high-probability analysis has been deferred to Appendix [B.3], where the lower bound is improved by an additive factor of $\sqrt{\log K / 2mK}$ via Chernoff bounds. The analysis for $G$ is similar. To show the lower bound, we show that to have at least $\varepsilon K$ complex products fail in total, we need at least $\varepsilon K/d$ raw materials to fail. The expected number of raw materials that fail is $\rho x^n \leq mK x^n$, and therefore by equating $\varepsilon K$ with $mK x^n$, we get the desired answer. To derive the upper bounds, we simply upper bound the expected number of surviving complex products $E[S_C] = K(1 - x^n)^m$ and apply Markov’s inequality and Lemma [1].

**Additional Results.** In Appendix [D] we analyze the resilience of a random width-$w$ trellis graph width $D$ tiers (and $K = wD$ nodes) where edges between subsequent tiers are generated i.i.d. with probability $p$, which can be thought of as an extension to the parallel products’ architecture. In Theorem [7] we discover an interesting “duality phenomenon”; i.e., when $pw \leq 1$ a constant depth random trellis resilient, and when $pw \geq 1$ a constant width trellis is fragile.

### 3.2. Hierarchical Production Networks

A supply chain can be organized hierarchically, with different levels representing different stages of the production process. The raw materials or components that go into the production of a
product are at the bottom of the hierarchy, and the finished product is at the top. Each level of the hierarchy represents a stage of production where the materials or components are transformed into a more advanced or finished product (Elliott et al., 2022). This hierarchical structure helps to visualize the flow of materials and information through the supply chain and identify potential bottlenecks or inefficiencies. Moreover, another hierarchy that is possible is producing complex products by starting from a source of raw products. In this section, we study these two hierarchies, which we call backward and forward (referring to the directions of the percolation with respect to the network growth), production networks visualized in Figure 3. More specifically, we consider:

- The **backward production network** (Figure 3(a)) at which the tree grows from the root, and then the percolation starts from the leaves and proceeds to the root. For the scope of this paper, we study backward percolation in deterministic $m$-ary trees. In Section 3.2.1 we prove that, as someone would expect, such supply chains are, in fact, fragile and give lower bounds on resilience.

- The **forward production network** (Figure 3(b)) at which the tree grows from the root, and then the percolation starts from the root and proceeds to the leaves. Here, the production network is generated by a stochastic branching process; the Galton-Watson process. In Section 3.2.2 we prove that, under specific conditions, such supply chains are fragile with a non-negative probability and are otherwise resilient.

In the sequel, we present the two processes and the results on $R_G(\varepsilon)$, and in Section 3.2.3 we compare our conclusions with the ones of Elliott et al. (2022).

### 3.2.1. Backward Production Network

In the case of the backward production network, we consider an $m$-ary tree with height $D$ and fanout $m \geq 1$. The levels of the tree correspond to “tiers” with raw materials being positioned at tier $D$ and more complicated products positioned in higher tiers. Each product has $n$ potential suppliers, and each product at tier $d \in [D - 1]$ has exactly $|N(i)| = m$ inputs from tier $d + 1$. Tier $d = D$, which corresponds to the raw products, does not have any inputs. Figure 3(a) shows how the percolation process evolves in a tree with $m = 2$. 
and $D = 3$, where failures (drawn in pink) propagate from the two faulty raw materials up to the root.

In addition to the power law result (Theorem 1), the case of the $m$-ary tree is another example that motivates the resiliency measure $R_G(\varepsilon)$. Specifically, let’s think about the probability of a catastrophic failure in a tree — i.e., one that affects a substantial proportion of the suppliers in the production network. A raw material failing to be produced can cause its parent product not to be produced and inductively create a cascade up to the root. The complete cascade will start from the failed product at tier $D - 1$ since some products at tier $D - 1$ may be made if their corresponding raw materials are produced. However, no product can be produced from tier $D - 2$ and onward. As a result, only $o(K)$ products survive. The probability of such an event equals:

$$\mathbb{P}[S = o(K)] \geq \mathbb{P}[\geq 1 \text{ raw material malfunctions}] = 1 - (1 - x^n)^{m^{D-1}} \geq 1 - e^{-x^n m^{D-1}}. \quad (3)$$

It is easy to see that if $x = \Omega \left( (m \log K/K)^{1/n} \right)$, then a catastrophe happens with a high probability in the tree structure, meaning that failure probabilities as small as $(m \log K/K)^{1/n} + o(1)$ can cause catastrophes with probability approaching one (and therefore, the backward hierarchical production network is a fragile architecture). Therefore, it is interesting to study cases where such a scenario does not happen; on the contrary, we have many products surviving. The following Theorem formalizes lower bounds and provides an additional upper bound for the resilience of the backward production network.
Theorem 4. Let $G$ be a backward production network with fanout $m$ and depth $D$. Then,

$$
\left[1 - \left(1 - \frac{1}{K}\right)^{(1-\varepsilon)K}\right]^{1/n} \leq R_G(\varepsilon) \leq \begin{cases} 
\left(\frac{2}{K(1-\varepsilon)}\right)^{1/n}, & m = 1 \\
\left(\frac{(1-\varepsilon)\log m}{\log K}\right)^{1/n} \times \left(\frac{1-\varepsilon}{D}\right)^{1/n}, & m \geq 2.
\end{cases}
$$

(4)

Therefore, as $K \to \infty$ with $m$ finite, we have $R_G(\varepsilon) \to 0$.

Proof Sketch. To derive the lower bound (for both $m = 1$ and $m \geq 2$), we show that if a failure happens at tier $\tau$, then all of the products up to tier $\tau$ have to be operational, which implies a lower bound on the tail probability of $S$, i.e., $\mathbb{P}[S \geq (1-\varepsilon)K] \geq (1-x^n)(1-\varepsilon)^K$. To derive an upper bound, we show that when $m = 1$ then $\mathbb{E}[S] \leq 1/x^n$, and when $m \geq 2$ we show (see Appendix C) that $\mathbb{E}[S] \leq KDx^n/2$, and the rest follows by Lemma 1 and $D \geq \log K/\log m$ (for $m \geq 2$). Q.E.D.

Upper Bound Comparison. Since $\varepsilon \in (0,1)$ the above quantity behaves asymptotically as $O\left(\left(\frac{\log m}{\log K}\right)^{1/n}\right)$ for all values of $\varepsilon$. Therefore, the resilience goes to 0 with rate $\log K$. However, note that in Equation (3), we showed a better rate of $O\left(\left(\frac{m\log K}{K}\right)^{1/n}\right)$, and therefore we state that the resilience goes to 0 with a rate of $O\left(\left(\frac{m\log K}{K}\right)^{1/n}\right)$ (for $m \geq 2$).

3.2.2. Forward Production Network We consider a random hierarchical network where the products in each level $d$ are denoted by $\mathcal{K}_d$. Starting from one raw material $r$, i.e., $\mathcal{K}_1 = \mathcal{R} = \{r\}$, we branch out via a Galton-Watson (GW) process such that every product $i \in \mathcal{K}_d$ at level $d \geq 1$ creates $\xi_i^{(d)}$ supply dependencies, where $\{\xi_i^{(d)}\}_{i \in \mathcal{K}_d : d \geq 0}$ are generated i.i.d. from a distribution $\mathcal{D}$, with mean $\mathbb{E}_{\mathcal{D}}[\xi_i^{(d)}] = \mu > 0$. Subsequently, the number of products at each level obeys

$$
|\mathcal{K}_{d+1}| = \begin{cases} 
\sum_{i \in \mathcal{K}_d} \xi_i^{(d)}, & d \geq 2 \\
1 & d = 1
\end{cases}
$$

(5)

Adding the node percolation process, we start a percolation from the children of the root node $r$ and subsequently proceed to their children, and so on. The number of the surviving products $S$
in this case can be expressed as \( S = \sum_{d:|K_d| \geq 1} \sigma_d \), where \( \{\sigma_d\}_{d \geq 0} \) follow another branching process, namely
\[
\sigma_{d+1} = \begin{cases} 
\sum_{1 \leq i \leq \sigma_d} \xi^{(d)}_i \left( 1 - \prod_{s \in \mathcal{S}(i)} Y_{is} \right), & d \geq 2 \\
1 - \prod_{s \in \mathcal{S}(r)} Y_{rs} & d = 1 
\end{cases}.
\] (6)

In Figure 3(b) we show such an example where failures propagate from the
Under these scenarios, we prove the following Theorem

**Theorem 5.** Let \( \mathcal{G} \) be generated by a GW process where the number of children of each node is generated by a distribution \( \mathcal{D} \) with mean \( \mu > 0 \) and extinction time \( \tau \). Let \( G_{\mathcal{D}}(\eta) = \mathbb{E}_{\xi \sim \mathcal{D}}[e^{\eta \xi}] \) be the moment generating function of \( \mathcal{D} \), and let \( \mathbb{P}[\tau < \infty] = \eta^* = \inf\{\eta \in [0,1]: G_\mathcal{D}(\eta) = \eta\} \) be the extinction probability of the GW process. Then the following are true: (i) If \( \mu < 1 \), then the \( \mathcal{G} \) is always resilient, (ii) If \( \mu(1 - x^n) > 1 \), then with probability \( 1 - \eta^* \), \( \mathcal{G} \) is fragile.

Moreover, the expected upper bound on the resilience is, for \( \mu \in (0,1) \cup (e^2,\infty) \), given by
\[
\mathbb{E}_\mathcal{G}[\mathcal{R}(\varepsilon)] = \sum_{1 \leq k < \infty} \mathbb{P}[\tau = k] x(\mu,\tau,\varepsilon) \text{ with } 
\]
\[
x(\mu,\tau,\varepsilon) = \inf \left\{ x \in \left[ 0, 1 \{\mu < 1\} + \left( 1 - \frac{1}{\mu} \right)^{1/n} 1\{\mu > 1\} \right]: (1 - x^n) \frac{\mu^r(1 - x^n)^r - 1}{\mu(1 - x^n) - 1} \leq \frac{1 - \varepsilon}{2} \right\}.
\] (7)

The expected lower bound on the resilience is, for \( \mu \in (0,1) \cup (e,\infty) \), given by
\[
\mathbb{E}_\mathcal{G}[\mathcal{R}(\varepsilon)] = \sum_{1 \leq k < \infty} \mathbb{P}[\tau = k] \underline{x}(\mu,\tau,\varepsilon) \text{ with } 
\]
\[
\underline{x}(\mu,\tau,\varepsilon) = \sup \left\{ x \in \left[ 0, 1 \{\mu < 1\} + \left( 1 - \frac{1}{\mu} \right)^{1/n} 1\{\mu > 1\} \right]: \frac{\mu^r - 1}{\mu - 1} - (1 - x^n) \frac{\mu^r(1 - x^n)^r - 1}{\mu(1 - x^n) - 1} \leq \varepsilon \right\}.
\] (8)

*Proof Sketch.* If \( Z_r = 0 \), which happens with probability \( x^n \), then the number of surviving products is \( S = 0 \). If \( Z_r = 1 \), which happens with probability \( 1 - x^n \), the cascade behaves as a GW process with mean \( \mu_x = \mu(1 - x^n) \). Now, conditioned on the fact that \( Z_r = 1 \) we bound the percentage of the expected number of surviving products over the total expected number of products to devise the upper and lower bound. For the upper bound we require \( \mathbb{E}_{\mathcal{G},x}[S] \leq \frac{1 - \varepsilon}{2} \mathbb{E}_{\mathcal{G}}[K] \), and for the lower bound we require \( \mathbb{E}_{\mathcal{G},x}[F] = \mathbb{E}_{\mathcal{G}}[K] - \mathbb{E}_{\mathcal{G},x}[S] \leq \varepsilon \). We prove that if the process takes infinite time
to terminate, then the only possible solution is \( x = 0 \) which makes the supply chain fragile. In all of the other cases, we study the existence of an upper and a lower bound to the resilience by solving two inequalities, whose roots are studied in (the auxiliary) Lemma 2. Q.E.D.

Applying Theorem 5 for the case where \( D \) is a point-mass function that equals \( \mu \) with probability 1, yields the following corollary for deterministic structures (when \( \mu \in \mathbb{N}^* \) these are trees).

**Corollary 1.** Let \( D \) have \( \mathbb{P}_{\xi \sim D}[\xi = \mu] = 1 \) for \( \mu > 1 \). Then \( G \) is always fragile.

For the subcritical regime, we plot the expected resilience bounds in Figure 3(c) for a subcritical GW process with branching distribution \( D = \text{Bin}(k, p \in (0, 1/k)) \), as a function of \( \mu = kp \).

**3.2.3. Comparison with the results of Elliott et al. (2022).** The work of Elliott et al. (2022) considers a hierarchical supply network similar to the one presented in Section 3.2.1 – though by assuming a different percolation process. In Section II of their paper they observe that as the interdependency increases then their reliability metric decreases, and as the number of suppliers increases then the reliability increases. This is in agreement with the lower bound presented in Theorem 4 for the backward production network which increases as \( n \) increases and decreases as \( m \) increases, probability that there’s at least a raw material failure (Equation (3)) increases. Moreover, in their paper, if the shocks are below a value then for large depths, the reliability goes to zero, which is conceptually in agreement with the upper bounds presented in Theorem 4 which go to zero as \( D \) grows. Finally, for the forward production network – which is not studied by Elliott et al. (2022) – we again get a result that is in agreement with their results, since as the average interdependency \( \mu \) increases we observe that the upper and lower bounds in the resilience decrease, as empirically shown in Figure 3(c).

**4. Resilience of General Graphs and Optimal Interventions**

So far, we have focused our attention on the cases where \( G \) is a DAG and under the assumption that the network undergoes a node percolation process. It is certainly possible for a production network to have cycles, e.g., when complex products are used in the production of simpler products.
It is also possible that production networks have cycles which represent recycling or other forms of circular flows of materials or resources in modern economies; cf., circular economies (Geissdoerfer et al., 2017). In such cases, it is important to consider the impact of percolation on these cycles and the overall structure of the network. One way to analyze the effect of percolation on a network with cycles is to use tools from graph theory, such as connectivity and centrality measures, to study how the removal of nodes affects the overall structure of the network. It is also possible to extend percolation models to account for the presence of cycles in the network, although this can be more complex to analyze mathematically. Thus, the following questions arises naturally in this setting:

**Question 1.** How is the resiliency $R_G(\varepsilon)$ affected if we relax the DAG assumption?

**Question 2.** How do we identify the most vulnerable nodes?

**Question 3.** How can we design interventions to minimize the (expected) size of failure cascades?

As one would expect, the most vulnerable nodes are the ones that are most “central” to the network since lots of other materials would have (potentially higher-order) connections. As it turns out, such nodes can be identified by computing the Katz centrality of the network (Katz, 1953), which arise naturally under certain assumptions in our model. Moreover, minimizing the cascade size can be achieved (partly) by safeguarding the most vulnerable nodes, which is again related to the Katz centrality, as we discuss in Section 4.1.

A surprising way to identify such nodes involves extending the percolation process to account for some link percolation phenomena, creating a noisy version of the node percolation process introduced in Section 2.2. More specifically, for each edge $(i, j) \in \mathcal{E}(\mathcal{G})$ of the production network, we flip a coin of bias $y \in (0, 1)$, independently of the other edges and suppliers, and decide to keep the edge with probability $y$. This creates a subsampled graph $\mathcal{G}_y \subseteq \mathcal{G}$, which under reasonable assumptions for $x$ and $y$, can be used to identify “likely to fail” products. The above process can also be viewed as a *joint percolation* on both nodes and edges or a node percolation on the noisy subnetwork $\mathcal{G}_y$, where in order for a product to function, on average it only needs a $y$-fraction of its inputs to operate.
Algorithm 1 Solution to Equation (9) for DAGs

Input: DAG \( G \), node percolation probability \( x \), number of suppliers \( n \), edge sampling probability \( y \).
Output: The solution \( \beta^* \) to the linear program in Theorem 6 — Equation (9).

1. Use depth-first search to create a topological ordering of DAG \( G \), \( \pi: K \rightarrow K \), and let \( \beta^*_{\pi(1)} = x^n \).
2. For \( 2 \leq i \leq K \), set \( \beta^*_i = \min \left\{ 1, y \sum_{j \in N(\pi(i))} \beta^*_j + x^n \right\} \).

Below, we answer the questions raised above and provide a systematic way of treating general graphs that undergo a joint percolation process. More specifically, we devise a systematic way to bound the expected number of failures and subsequently derive bounds for the resilience metric. Finally, we show that our analysis has deep connections to financial networks. To put our analysis into a mathematical framework, Markov’s inequality states that \( P[F \geq \varepsilon K] \leq \frac{E[F]}{\epsilon K} \), so, limiting a failure of at least \( \varepsilon K \) products requires upper bounding \( E[F] = \sum_{i \in K} P[Z_i = 0] \).

At first glance, bounding \( E[F] \) looks to be instance-specific since different architectures behave differently under the percolation process; cf. Section 3. The following Theorem gives a way to characterize an upper bound on \( E[F] \) as the solution to a linear program.

**Theorem 6.** Let \( G \) be a graph of maximum outdegree \( \Delta \geq 1 \), that undergoes a joint percolation process with node failure probability \( x \in (0, 1) \) and edge survival probability \( y \in (0, 1) \). If \( A \) is the adjacency matrix of \( G \), and \( \beta = (\beta_i)_{i \in K} = (P[Z_i = 0])_{i \in K} \), then an upper bound to \( E[F] \) can be found via solving the following linear program

\[
E[F] \leq \text{OPT}_1 = \max_{\beta \in [0,1]} \sum_{i \in K} \beta_i \quad \text{s.t.} \quad \beta \leq yA^T \beta + x^n 1. \tag{9}
\]

Moreover, if \( y \leq \frac{1}{\Delta} \), then the solution to the LP can be found as the greatest fixed point \( \beta^* \in [0, 1] \) of the non-expansive operator \( \Phi(\beta) = 1 \wedge (yA^T \beta + x^n 1) \).

**Proof Sketch.** The proof uses a union bound on \( \beta_i = P[Z_i = 0] \) for every \( i \in K \) and maximizes \( \sum_{i \in K} P[Z_i = 0] \) under the union bound constraint and the fact that \( \{P[Z_i = 0]\}_{i \in K} \) are valid probabilities. The second part of the proof applies (Eisenberg and Noe, [2001] Lemma 4). Q.E.D.

Theorem 6 shows that there is a systematic way of bounding \( E[F] \) via the solution of a linear program or a fixed-point equation (if the edge survival probability is less than \( 1/\Delta \)). It is surprising
to note that an elegant upper bound on the expected number of failures becomes possible by introducing shocks at the edge level. However, Theorem 6 gives a polynomial time algorithm to compute an upper bound on $\mathbb{E}[F]$ by solving the aforementioned LP. Algorithm 1 solves this LP when $G$ is a DAG in $O(K + |E(G)|)$ time. When we allow cycles in $G$, the calculation becomes more complicated and Algorithm 1 does not produce a valid solution; this case is treated by the following proposition where we prove that under certain conditions, the fixed-point solution in Theorem 6 corresponds to the Katz centrality of $G$. Recall $\gamma_{\text{Katz}}(G, y) = (I - yA^T)^{-1}1$, where $A$ is the adjacency matrix of $G$.

**Proposition 1.** Consider the joint percolation process on graph $G$ with maximum outdegree $\Delta \geq 1$, adjacency matrix $A$, node failure probability $x \in (0, 1)$, and edge survival probability $y \in (0, 1)$, and let $\gamma_{\text{Katz}}(G, y)$ be its Katz centrality vector. If $0 < y < \frac{1}{\Delta}$ and $0 < x < (1 - y\Delta)^{1/n}$, then the fixed point solution $\beta^*$ to the problem of Theorem 6 is the Katz centrality measure of $G$ scaled by $x^n$: $\beta^* = x^n \gamma_{\text{Katz}}(G, y) = x^n(I - yA^T)^{-1}1$. Subsequently, $R_G(\varepsilon) \geq \left(\frac{\varepsilon}{\|x^n(I - yA^T)^{-1}1\|_1}\right)^{1/n} = \left(\frac{\varepsilon}{1^T\gamma_{\text{Katz}}(G, y)}\right)^{1/n}$.

**Proof Sketch.** We show that for these values of $x$ and $y$ the union bound constraints are enclosed in the box constraints $[0, 1]^K$, and since $\beta^*$ which solves the linear program in Equation (9) is also the unique solution to the fixed point problem in Theorem 6 (due to Banach’s theorem because $\Phi(\beta)$ is a contraction), we have $\beta^* = x^n \gamma_{\text{Katz}}(G, y)$. Q.E.D.

In the case of DAGs, we can get a special lower bound for $R_G(\varepsilon)$, in the large-shock regime for edges, i.e., when edges fail with high probability $1 - y$.

**Proposition 2.** Let $G$ be a DAG, and consider the joint percolation process with node percolation probability $x \in (0, 1)$, and edge survival probability $y \in (0, 1)$. The number of failures on $G$ obeys $\mathbb{E}[F] \leq \frac{\varepsilon x^nK_y}{y}$, and therefore the resilience of any DAG obeys $R_G(\varepsilon) \geq \left(\frac{\varepsilon}{x^nK_y}\right)^{1/n}$. When $y = \frac{1}{K}$, the lower bound on the resilience is maximized and equals $\left(\frac{\varepsilon}{xK}\right)^{1/n}$.

**Proof Sketch.** We consider a topological order of the DAG and prove that the solution $\beta^*$ to the linear program of Theorem 6 — Equation (9) — satisfies $\beta^*_i \leq (1 + y)^{i-1}x^n$ for all $i \in \mathcal{K}$. Therefore,
we can prove that the expected number of failures is at most \( x^n e^{Ky} / y \) which is minimized for 
\( y = 1/K \), yielding the maximized lower bound on \( R_G(\varepsilon) \).

\[ \text{Q.E.D.} \]

An immediate consequence of Proposition 2 is that whenever there is an over-diversification of suppliers, i.e., \( n \geq Ky \), then every DAG is resilient — in agreement with the results of Elliott et al. (2022). This idea of maximizing \( \mathbb{E}[F] \) has its roots in financial networks, specifically the Eisenberg-Noe model of financial contagion among agents with assets and liabilities (Eisenberg and Noe, 2001; Glasserman and Young, 2015). Similar connections between the Eisenberg-Noe clearing problem and network centrality have been made by Siebenbrunner (2018) and Bartesaghi et al. (2020) and references therein. Moreover, the solution to Equation (9) gives a measure of “vulnerability” of each node, i.e., how likely they are to be affected by cascading failures. In case of DAGs, we can relate this ranking of the node vulnerabilities to their topological ordering as follows:

**Corollary 2.** If \( G \) is a DAG, \( \pi : K \to K \) is a topological ordering of the nodes in \( G \), and \( \beta^* \) is the solution to the linear program in Equation (9), then \( \beta_{\pi(i)}^* \leq \beta_{\pi(j)}^* \) for all \( 1 \leq i \leq j \leq K \).

The preceding corollary, which is a direct consequence of Theorem 6 and Algorithm 1, states that when \( G \) is a DAG, the least vulnerable nodes are the raw materials that precede more complex products in their topological ordering. On the other hand, the failure of raw materials in a DAG will cause large cascading failures, affecting all the complex products that succeed the raw materials in their topological ordering. Hence, to prevent large cascading failures, it seems intuitive to intervene to protect nodes that rank highest in the reversed graph. Our results in the next section formalize this intuition, giving an explicit solution for optimal interventions in terms of the Katz centrality of the reversed graph (Proposition 3).

### 4.1. Intervention Design

Safeguarding supply chains is an ever-important issue in supply-chain management. Many potential risks can disrupt the supply chain and impact a business’s operations and bottom line. These risks can include natural disasters, transportation issues, supplier bankruptcy, cyber-attacks, geopolitical...
turmoils, etc. It is important for businesses to have contingency plans to address potential supply chain disruptions and minimize their impact. In our model, we start by building intuition behind designing interventions to safeguard the supply chain. Our problem involves a global planner that can cure a maximum of $T$ suppliers in the network. Their aim is to minimize failures, which can be achieved in many ways, such as diversifying the supplier base, building inventory buffers, and implementing robust risk management and monitoring systems. In mathematical terms, since each product can be produced if it has at least one functional supplier, this is equivalent to selecting a maximum of $T$ products to intervene. The decision variables are set to be $t_i \in \{0, 1\}$, which implies that the budget constraint is $\sum_{i \in K} t_i \leq T$ and a random number of failures $F = F(t)$. Also, every product’s failure probability $(x(1 - t_i))^n = x^n(1 - t_i)$, $t_i \in \{0, 1\}$. A “good” problem for the planner would be to try to minimize the upper bound on the worst-case expected damage, i.e.,

$$\min_{t \in \{0, 1\}^n} \mathbb{E}[F(t)] \leq \min_{t \in \{0, 1\}^n} \max_{\beta \in [0, 1]} 1^T \beta \quad \text{s.t.} \quad \beta \leq y A^T \beta + (1 - t)x^n, \ 1^T t \leq T. \quad (10)$$

Note that if $0 < y < \frac{1}{\Delta}$ and $0 < x < (1 - y\Delta)^{1/n}$, then Proposition 1 implies that the solution to the internal maximization in (10) is $\hat{\beta}(t) = x^n(I - y A^T)^{-1}(1 - t)$. This yields the following Proposition:

**Proposition 3.** Let $G$ be a graph of maximum outdegree $\Delta \geq 1$, let $G^R$ be the graph where the direction of edges in $G$ are reversed, and let $\Delta_R$ be the maximum outdegree of $G^R$. Let $0 < y < \frac{1}{\max(\Delta, \Delta_R)}$, $0 < x < (1 - y\Delta)^{1/n}$. Consider $\gamma_{\text{Katz}}(G^R, y) = (I - y A)^{-1}1$, the Katz centrality of $G^R$, and let $\pi : K \rightarrow K$ be a decreasing ordering on the entries of $\gamma_{\text{Katz}}(G^R, y)$. Then, the optimal policy $\hat{t}$ sets $\hat{t}_{\pi(i)} = 1$ for $i \in [T]$ and sets it to zero otherwise. Subsequently, the resilience is at least $\left(\frac{\gamma_{\text{Katz}}(G^R, y)(1 - t)}{\gamma_{\text{Katz}}(G^R, y)(1 - t)}\right)^{1/n}$.

So, one can think of the “riskiest” to be the ones with high Katz centrality in $G^R$ — the reversed graph representing the sourcing relationships between products. This agrees with our intuition on DAGs which says to intervene starting from the raw materials and progressing in the topological order of the DAG until the budget is exhausted.
5. Resilience of Empirical Production Networks

With the definition of resilience and the theoretical results developed in the previous sections, we study the resilience of production networks in practice. We examine the resilience of networks contained in the following two datasets of production networks (Tables 2 and 3):

1. **Multi-echelon Supply-chain Networks** from Willems (2008). The dataset contains 38 different multi-echelon (see also Figure 1) supply chain networks, from which we select three networks to run simulations on. The echelon statistics of these networks are presented in Figure 4(a). Similar supply chain networks have been used in prior literature, see, e.g., Blaettchen et al. (2021) and Perera et al. (2017).

2. **Country Economy Production Networks** derived from *World Input-Output Tables* taken from the World Input-Output Database (Timmer et al., 2015). We focus on the economies of six countries in 2014: the USA, Japan, China, Great Britain, Indonesia, and India. We consider any non-zero amount cell at the input-output tables as an edge between two industries in a country. For space considerations, the results have been deferred to Appendix G.

**Resilience Metrics.** First, we study the resilience of the above networks as a function of $\varepsilon$ for $\varepsilon$ ranging from 0 to 1 numerically. To achieve this, we run 1000 Monte Carlo (MC) simulations where we sample the (spontaneous) state of $n$ suppliers and then propagate the state of each product to the adjacent ones, based on Equation (1). To calculate resilience, we estimate the probability $\mathbb{P}[S \geq (1 - \varepsilon)K]$ for various values of $x \in (0, 1)$ with MC simulation and find the maximum value of $x$ for which the estimate is at least $1 - 1/K$. We plot the estimated resilience $\hat{R}_G(\varepsilon)$ as a function of $\varepsilon$ and present the results in Figures 4(b) and 6(a). As an additional resilience metric that’s independent of $\varepsilon$, we also report the Area Under the Curve (AUC), i.e., $\text{AUC} = \int_0^1 \hat{R}_G(\varepsilon) d\varepsilon$.

**Optimal Interventions.** To study the effect of targeted interventions in real-world supply chain networks, we apply the results of Proposition 3 for the networks from the two datasets. More specifically, for a value of $y = \frac{1}{10^{-y} + \Delta_R}$, and $\varepsilon = 0.2$, we plot the lower bound for the resilience devised by Proposition 3 as a function of the total intervention budget $T$. This involves calculating...
Table 2  Network Statistics and AUC for Willems (2008) dataset. The edge density is computed as $|E(G)|/K^2 - K$.

<table>
<thead>
<tr>
<th>Network ID</th>
<th>Size ($K$)</th>
<th>Avg. Degree</th>
<th>Density</th>
<th>Min/Max In-degree</th>
<th>Min/Max Out-degree</th>
<th>AUC</th>
</tr>
</thead>
<tbody>
<tr>
<td>#10</td>
<td>58</td>
<td>3.03</td>
<td>0.053</td>
<td>0–27</td>
<td>0–13</td>
<td>0.136</td>
</tr>
<tr>
<td>#20</td>
<td>156</td>
<td>1.08</td>
<td>0.006</td>
<td>0–29</td>
<td>0–3</td>
<td>0.117</td>
</tr>
<tr>
<td>#30</td>
<td>626</td>
<td>1.00</td>
<td>0.001</td>
<td>0–2</td>
<td>0–48</td>
<td>0.357</td>
</tr>
</tbody>
</table>

6. Managerial Insights and Concluding Remarks

Increased Risk from Interdependencies  Our insights so far show that more interconnected networks (with increased interdependencies among products) are less resilient. Specifically, in the case of tree networks, no more than a $(1/m)$-fraction of the products survive as we increase the node degrees ($m \to \infty$), making the structure fragile — see Equation (3) and the discussion in its preceding paragraph. Similarly, in the case of parallel product networks, we observe that resilience metrics converge to zero as the source dependency increases (in-degree $m \to \infty$; see Theorem 3). Furthermore, in the trellis case, the network becomes less resilient when the interdependency – which corresponds to the average in-degree $pw$ – is large. This agrees with our empirical observations. Firstly, in the case of Willems (2008) supply-chain networks, the networks with higher sourcing dependencies and higher average degrees were less resilient. Additionally, our results using
Input-Output networks of world economies show that networks that are denser (with higher average degrees) — which generally correspond to more developed economies — are less resilient since shocks can spread more easily in denser networks (see Table 3 and Figure 6). This agrees with our theoretical results and prior works that state networks with higher interdependence are less resilient (Elliott et al., 2022).

**Raw Materials vs. Complex Products.** A second potential point of discussion is whether for an economy to be resilient — through the lens of our model — should it focus on producing more raw materials or more complex products. First, Theorem 4 shows that as we increase the depth $D$ of the network, the network becomes less resilient. Similarly, in the case of trellis architectures, we show that denser and deeper, rather than sparser and wider, production networks are less resilient (see Theorem 7). Experimentally, this is also verified in Figure 4(b) since Network #30, which is more concentrated towards complex products (i.e., later tiers), is found to be less resilient experimentally. In contrast, Network #10 is mostly concentrated on raw materials and is found to be the most resilient network.

**Supplier Heterogeneity & Optimal Interventions.** Proposition 3 shows that targeting the nodes that have high Katz centrality in the reversed (source) graph is optimal — these nodes represent less complex products. This argument agrees with recent observations from the COVID-19 pandemic, where the failure of chip manufacturing industries (which are relatively close to the raw materials, compared, e.g., to more complicated electronic devices) caused very large cascades on a global scale. Another interesting insight that we can harvest from the above results is the effect of heterogeneity in multisourcing. Following our design of optimal interventions in Section 4.1, we can attempt to increase the number of suppliers of product $i$, from $n$ to $n + \nu_i$ for some $0 \leq \nu_i \leq \overline{\nu}_i$, subject to a budget $\sum_{i \in K} \nu_i \leq N$, to decrease its self-percolation probability from $x^n$ to $x^{n+\nu_i}$. The optimal intervention policy would be, again, given an ordering $\pi$ of the Katz centralities in the reverse production network as in Proposition 3 and under the same conditions, to set $\hat{\nu}_{\pi(i)} = \left( \overline{\nu}_{\pi(i)} \wedge \left( N - \sum_{j < i} \hat{\nu}_{\pi(j)} \right) \right)^+ ;$ see Appendix E.
**Future Directions.** There are various ways that our model can be extended. The first interesting extension is to fit our model of the production network to actual economic and financial networks. After that, our model can be extended to contain costs, quantities, and link capacities, and the desired objective would be to optimize the network total profits subject to production shocks, extending, thus, a long-standing line of work on economic networks (Hallegatte, 2008; Acemoglu et al., 2016; Acemoglu et al., 2012). Another promising avenue is to study networks with varying interdependency and multi-sourcing. So far, we have assumed that products have the same number \( n \) of suppliers. However, this can be extended to cases where each industry has random size \( n_i \sim D_{\text{industry}} \), where \( D_{\text{industry}} \) is a distribution (e.g., power law, as in Gabaix (2011)). It would be interesting to study this heterogeneity in the size of different industries and how it affects \( R_G(\varepsilon) \); see also Gabaix (2011). Yet another interesting roadmap would be to study \( R_G(\varepsilon) \) under heterogeneous hierarchical networks, where the (potentially random) fanout changes at each tier. Moreover, as we saw in Section 4, there is a direct connection between cascading failures in supply chains and systemic risk in financial networks. Therefore, we can readily adapt tools from the study of financial risk to supply chain resiliency and risk.

**Conclusions.** Through our paper, we aim to devise a systematic way to test which supply chain networks are resilient, i.e., can withstand “large enough” shocks while sourcing almost all their productions. Through this metric of resilience, we classify some common supply chain architectures – represented by DAGs – as resilient or fragile and identify structural factors that affect resiliency. We then generalize our analysis and provide tools for studying the resilience of general supply networks, as well as motivating methods that can be used for the design of optimal interventions.

**Acknowledgments**

MP was partially supported by a LinkedIn Ph.D. Fellowship, a grant from the A.G. Leventis Foundation, and a grant from the Gerondelis Foundation. MAR acknowledges support from a Pitt Momentum Funds Grant on Socially Responsible Data Collection and Network Intervention Designs. Data and code can be found at: [https://github.com/papachristoumarios/supply-chain-resilience](https://github.com/papachristoumarios/supply-chain-resilience)
References


Online Appendix

Appendix A: Analytical Bound on $x$ to ensure $\mathbb{P}[F \geq \varepsilon K] = O(1/K)$ for $\text{rdag}(K, p)$

We can convert the statement of Section 2.3 to a high-probability statement, if we require $\mathbb{P}[F \geq \varepsilon K]$ to be $O(1/K)$. Since $g(x, K, p, \varepsilon)$ is an increasing function of $x$, we are asking to find the largest possible $x$ such that $g(x, K, p, \varepsilon) \leq \frac{C}{K}$. In order to get an analytically tractable expression for $x$, we use the fact that $\log t \leq t$ for all $t > 0$ and get that Equation (2) becomes

$$g(x, K, p, \varepsilon) \leq x^n \left[ 1 - \varepsilon + \frac{1}{K \log \left( \frac{1}{1-p} \right)} \left( \frac{1 - (1-x^n)(1-p)^K}{1 - (1-x^n)(1-p)^{\varepsilon K}} \right) \right] \leq x^n \left[ 1 - \varepsilon + \frac{1}{K \log \left( \frac{1}{1-p} \right)} \left( \frac{1}{1 - (1-x^n)(1-p)^0} \right) \right]$$

$$= x^n (1 - \varepsilon) + \frac{1}{K \log \left( \frac{1}{1-p} \right)} = \bar{g}(x, K, p, \varepsilon)$$

If $p$ is constant, then, choosing $x = \left( \frac{1}{K \log \left( \frac{1}{1-p} \right)(1-\varepsilon)} \right)^{1/n}$, makes $\bar{g}(x, K, p, \varepsilon) \leq \frac{2}{\log \left( \frac{1}{1-p} \right)K} = O \left( \frac{1}{K} \right)$.

Appendix B: Omitted Proofs

B.1. Proof of Theorem 1

Let $G \sim \text{rdag}(K, p)$, with nodes $i_1, \ldots, i_K$ (in this order). Let $P_{i,f}$ be the probability of having $f$ distinct failures in the random DAG with $k$ nodes conditioned on a failure on node 1. We have that $P_{1,1} = 1$ and $P_{k,f} = 0$ for $f > i$ and $f < 1$. To devise a recurrence formula for $P_{i,f}$ note that for the $i$-th node we have the following:

1. $i_k$ is affected by the cascade. That happens if at least one connection to $f-1$ infected nodes up to node $i_{k-1}$, or if $i_k$ fails due to percolation. This happens with probability $\left\{ (1 - (1-p)^{f-1} + x^n - [1 - (1-p)^{f-1}]) P_{k-1,f-1} \right\} P_{k-1,f-1} = [1 - (1-p)^{f-1}(1-x^n)] P_{k-1,f-1}$.

2. $i_k$ is not affected by the cascade. That means that $i_k$ has $\geq 1$ functional supplier, and no connection exists from the $f$ infected nodes. That happens with probability $(1-p)^f(1-x^n)P_{k-1,f}$.

This yields the following recurrence,

$$P_{k,f} = [1 - (1-p)^{f-1}(1-x^n)] P_{k-1,f-1} + (1-p)^f(1-x^n)P_{k-1,f}.$$  \hspace{1cm} (11)

To determine the distribution of $F$ in $\text{rdag}(K, p)$, we assume that the cascade can start at any node with equal probability $1/K$ and that the probability of failure is conditioned on the choice of a node is $x^n$. Also, since a cascade in $\text{rdag}(K, p)$ starting from node 1 is the same as starting from node $i$ in $\text{rdag}(K + i - 1, p)$, the distribution obeys the following,

$$\mathbb{P}[F = f] = \frac{x^n}{K} \sum_{k \in [K]} P_{k,f}.$$  \hspace{1cm} (12)
We let $Q_{K,f} = \sum_{k \in [K]} P_{k,f}$, so that $\mathbb{P}[F = f] = \frac{x^n}{K}Q_{K,f}$. Summing Equation (11) for $k \in [K]$ and using the definition of $Q_{K,f}$ yields a recurrence relation for $Q_{K,f}$, i.e. $Q_{K,f} = (1-p)^f(1-x^n)Q_{K-1,f} + [1 - (1-p)^f](1-x^n)Q_{K-1,f-1}$. We take the limit for $K$ large, we let $q_f = \lim_{K \to \infty} Q_{K,f}$, and solve the recurrence
\[
q_f = (1-p)^f(1-x^n)q_f + [1 - (1-p)^f](1-x^n)q_{f-1}
\]
to get $q_f = \frac{1}{1-(1-p)(1-x^n)}$. Since $e^\varepsilon \geq x$, we have that $(1-p)^f \leq \log(1-p)f$ and subsequently $1 - (1-x^n)(1-p)^f \leq f\left(1 + (1-x^n)\log\left(\frac{1}{1-p}\right)\right)$. Therefore, for sufficiently large $K$,
\[
\mathbb{P}[F = f] \geq \frac{x^n q_f}{K(1-(1-x^n)(1-p)^f)} \geq \frac{x^n}{K\left(1 + (1-x^n)\log\left(\frac{1}{1-p}\right)\right)^f}
\]
\[C(K,p,x,n) > 0\]

### B.2. Proof of Lemma 1

If $S(x)$ is the number of surviving products for percolation probability $x$, and $x_1 \leq x_2$ are two percolation probabilities, a straightforward coupling argument shows that $S(x_1) \geq S(x_2)$, and subsequently for every $s \in [0,K]$ we have that $\mathbb{P}_{s=x_1}[S \geq s] \geq \mathbb{P}_{s=x_2}[S \geq s]$. Now, in order to arrive at a contradiction, let $\overline{R}_\varepsilon(\varepsilon) \leq R_\varepsilon$, and $s = (1-\varepsilon)K$. Then $1 - 1/K \leq \mathbb{P}_{s=R_\varepsilon(\varepsilon)}[S \geq (1-\varepsilon)K] \leq \mathbb{P}_{s=\overline{R}_\varepsilon(\varepsilon)}[S \geq (1-\varepsilon)K] \leq 1/2$ which yields a contradiction.

### B.3. Proof of Theorem 3

**Lower Bound.** For $C$, let $\varepsilon \in (0,1)$. If $F_R$ (resp. $F_C$) is the number of failed raw materials (resp. complex products), we have that $\{F_C \geq \varepsilon K\} \implies \{F_R \geq \varepsilon K/d\}$. Let $\delta = \frac{1}{\log K} \sqrt{\frac{\log K}{2R}}$ and let $\frac{\delta K}{d} = (1 + \delta)\mathbb{E}[F_R] = (1 + \delta)\rho x^n$. We apply the one-sided Chernoff bound and get $\mathbb{P}[F_C \geq \varepsilon K] \leq \mathbb{P}\left[F_R \geq \frac{\delta K}{d}\right] = \mathbb{P}\left[F_R \geq (1 + \delta)\mathbb{E}[F_R]\right] \leq e^{-2\delta^2\mathbb{E}[F_R]^2/R} = \frac{1}{K}$. Finally, by resolving the last equation $(1+\delta)\rho x^n = \frac{\varepsilon K}{d}$, we get that $x = \left(\frac{\varepsilon K}{d} + \sqrt{\frac{\log K}{2d\rho}}\right)^{1/n}$. Also, we have that $\rho \leq mK$ and therefore $R_C(\varepsilon) \geq \left(\frac{\varepsilon}{d^m} + \sqrt{\frac{\log K}{2mK}}\right)^{1/n}$. If $\varepsilon, m$ and $d$ are independent of $K$ then for $K \to \infty$ we have that $R_C(\varepsilon) \geq \left(\frac{\varepsilon}{d^m}\right)^{1/n} > 0$.

For $G$, the analysis is similar to the above. For brevity, we give the analysis in expectation (it is easy to extend it to an analysis in high probability): If on expectation $\mathbb{E}[F_R] = \rho x^n$ raw materials fail, that implies that at most $\mathbb{E}[F] = \mathbb{E}[F_R] + \mathbb{E}[F_C] \leq \rho x^n + d\rho x^n = (d+1)\rho x^n \leq mKx^n(d+1)$ total products fail in expectation. We want the fraction of failed products to be at least $\varepsilon(K + \rho) \geq \varepsilon K/2$. Therefore by solving the inequality, we get that the lower bound in the resilience is $\left(\frac{K}{2m(d+1)}\right)^{1/n}$. The high-probability analysis would be similar to the above case, yield and extra additive $\sqrt{\frac{\log(K/2)}{2mK}}$ factor.
**Upper Bound.** For $C$, to derive the upper bound, we first bound $E[S_C]$. It is easy to see that due to the linearity of expectation $E[S_C] = K(1-x^n)^m$. Thus by Markov’s inequality we have that $P[S_C \geq (1-\varepsilon)K] \leq \frac{E[S]}{(1-\varepsilon)K} \leq \frac{(1-x^n)^m}{1-\varepsilon}$. To make this probability $1/2$ it suffices to set $x = \left(1 - \frac{(1-\varepsilon)^{1/m}}{2}\right)^{1/n}$, thus from Lemma 1, this establishes an upper bound on $R_C(\varepsilon)$.

For $\mathcal{G}$, we proceed similarly by showing that the number of expected products is $E[S] = \rho(1-x^n) + K(1-x^n)^m \leq mK(1-x^n)^m \leq K(m+1)(1-x^n)$. Similarly to the above, from Lemma 1 we get that the upper bound on $R_G$ is $\left(1 - \frac{1-\varepsilon}{2(m+1)}\right)^{1/n}$.

**B.4. Proof of Theorem 4**

**Lower Bound.** Depending on the range of $m$ we have two choices

- **Case where $m = 1$.** For every $\tau \in D$ we have that $P[S \geq D - \tau] = P\left[\bigcap_{d > \tau} \{Z_i = 1\}\right] = P[Z_1 = 1]P[Z_2 = 1|Z_1 = 1]\cdots = \prod_{d > \tau} (1-x^n) = (1-x^n)^{D-\tau}$. We let $\tau = \varepsilon D$ for some $\varepsilon \in (0,1)$ and thus $P[S \geq (1-\varepsilon)D] = (1-x^n)^{(1-\varepsilon)D}$. We want to make this probability at least $1-1/D$, and therefore, the resilience of the path graph is $R_G(\varepsilon) \geq \left(1 - \frac{1}{D}\right)^{(1-\varepsilon)D/m}$ since $K = D$ we get the desired result.

- **Case where $m \geq 2$.** Let $\mathcal{K}_d$ be the products of tier $d$. We let $\tau = \sup\{d \in D: \exists i \in \mathcal{K}_d: Z_i = 0\}$ be the bottom-most tier for which a product failure happens. If at level $\tau$ a failure happens, all levels above $\tau$ are deactivated. The probability that all products up to tier $\tau$ operate is given by

$$P[\text{all products up to tier $\tau$ operate}] = P\left[\bigcap_{d > \tau} \bigcap_{i \in \mathcal{K}_d} \{Z_i = 1\}\right] = \prod_{d = D}^{\tau + 1} P\left[\bigcap_{i \in \mathcal{K}_d} \{Z_i = 1\}\bigg| \bigcap_{i \in \mathcal{K}_d} \big\{Z_i = 1\}\right]$$

$$= \prod_{d = D}^{\tau + 1} (1-x^n)^{m_d} = (1-x^n)^{\sum_{d = D}^{\tau + 1} m_d} = (1-x^n)^{\frac{mD - m^\tau}{m-1}}$$

Also, $\{\text{all products up to tier $\tau$ operate}\} \implies \{S \geq \frac{mD - m^\tau}{m-1}\}$. Therefore, the tail probability of $S$ for $\tau \in D$ is given by $P\left[S \geq \frac{mD - m^\tau}{m-1}\right] = P\left[S \geq \left(1 - \frac{m^\tau}{m^D}\right)\frac{mD}{m-1}\right] \geq (1-x^n)^{(1-\varepsilon)\frac{mD}{m-1}}$. For large enough $D$ we approach the continuous distribution and thus $P[S \geq (1-\varepsilon)K] \geq (1-x^n)^{(1-\varepsilon)K}$. Letting the above be at least $1-1/K$ we get that $R_G(\varepsilon) \geq \left[1 - \frac{1}{K}\right]^{\frac{1}{n}}$.

**Upper Bound.** To derive an upper bound, we have the following cases, depending on the value of $m$

- **Case $m = 1$.** We follow the same logic as the $m \geq 2$ case, and upper bound $E[S] \leq \sum_{d \geq 0} (1-x^n)^d = \frac{1}{x^n}$ which yields an upper bound $R_G(\varepsilon) < \left(\frac{2}{\rho(1-\varepsilon)}\right)^{1/n} \to 0$ as $D \to \infty$. 

In order to prove Theorem 5, we first prove this auxiliary lemma:

**Lemma 2.** For $\tau$ finite, $\frac{1}{\log \mu} < \alpha < \frac{1}{2}$, and $0 < \beta < 1\{\mu < 1\} + 1\{\mu > 1\} \frac{\log \mu - 1}{\mu}$, let

$$\phi(z) = z \frac{\mu^\tau z - 1}{\mu z - 1} - \alpha \frac{\mu^\tau - 1}{\mu - 1}, \text{ for } z \neq \frac{1}{\mu}, \quad \psi(z) = \frac{\mu^\tau - 1}{\mu - 1} - z \frac{\mu^\tau z - 1}{\mu z - 1} - \beta, \text{ for } z \neq \frac{1}{\mu}.$$ 

Then

1. If $\mu < 1$, then there exist $z_1, z_2 \in (0, 1)$ such that $\phi(z_1) = \psi(z_2) = 0$.
2. If $\mu > e^2$, then there exists $z_1 \in (1/\mu, 1)$ such that $\phi(z_1) = 0$.
3. If $\mu > e$, then there exists $z_2 \in (1/\mu, 1)$ such that $\phi(z_2) = 0$.

**Analysis for $\phi(z)$**. We do case analysis:

- If $\mu < 1$ then $\phi$ is defined everywhere in $[0, 1]$ and is also continuous. It is also easy to prove that $\phi$ is increasing in $[0, 1]$ since its the product of two non-negative increasing functions, $z$ and $\frac{(\mu z - 1)^\tau - 1}{\mu z - 1} = \sum_{i=0}^{\tau-1}(\mu z)^i$. Moreover, note that $\phi(0) < 0$ and $\phi(1) > 0$. Therefore, there exists a unique solution $z_1 \in (0, 1)$ such that $\phi(z_1) = 0$.

- If $\mu > e^2$, we study $\phi$ in $(1/\mu, 1]$. Again, $\phi$ is increasing (for the same reason as above), continuous in $(1/\mu, 1]$, and has $\phi(1) > 0$. We also have that, by using L'Hôpital’s rule,

$$\lim_{z \to 1/\mu} \frac{\mu^\tau z - 1}{\mu z - 1} = \lim_{z \to 1/\mu} \frac{(\mu^\tau z - 1)'}{(\mu z - 1)'} = \lim_{z \to 1/\mu} \frac{\mu^\tau z^\tau - 1}{\mu} = \tau \implies \lim_{z \to 1/\mu} \phi(z) = \tau - \alpha \frac{\mu^\tau - 1}{\mu - 1} \frac{\mu^\tau z^\tau - 1}{\mu} < 0 \text{ for } \alpha > \frac{1}{\log \mu}.$$ 

Therefore, for $\alpha \in (1/\log \mu, 1/2)$, there exists a unique solution $z_1 \in (1/\mu, 1]$ such that $\phi(z_1) = 0$.

**Analysis for $\psi(z)$**. Note that $\psi$ is a decreasing function of $z$. We do case analysis:

- If $\mu < 1$, then $\psi$ is defined everywhere in $[0, 1]$ and is continuous in $[0, 1]$. We have that $\psi(0) > 0$ and $\psi(1) < 0$ therefore there exists a unique solution $z_2$ such that $\psi(z_2) = 0$.

- If $\mu > e$, then $\psi$ is decreasing and continuous in $(1/\mu, 1]$, with $\psi(1) < 0$. We also have that

$$\lim_{z \to 1/\mu} \psi(z) = \frac{\mu^\tau - 1}{\mu} - \frac{z}{\mu} - \beta > \frac{\tau (\log \mu - 1 - \mu \beta)}{\mu} > 0 \text{ for } \beta < \frac{\log \mu - 1}{\mu}.$$ 

Subsequently we prove Theorem 5.
Proof of Theorem 3 Upper Bound. Let $\tau = \inf \{d \geq 1 : |K_d| = 0 \}$ be the extinction time of the GW process. In order to establish an upper bound on the resilience it suffices to set the expected number of surviving products to be at most $\frac{1-\varepsilon}{2} \mathbb{E}_\varnothing[K]$, since by Markov’s inequality the probability of a fraction of at least $(1-\varepsilon)$-fraction of products surviving would be at most $1/2$ and by Lemma 1 we would get an upper bound on the resilience $R_\varnothing(\varepsilon)$; namely, $\mathbb{P}_{\varnothing,x}[S \geq (1-\varepsilon)\mathbb{E}_\varnothing[K]] \leq \frac{\mathbb{E}_{\varnothing,x}[S]}{(1-\varepsilon)\mathbb{E}_\varnothing[K]} \leq \frac{1}{2}$. Conditioned on $Z = 1$, which happens with probability $1 - x^n$, the surviving products grow as a GW process with mean $\mu_x = (1 - x^n)\mu$. Therefore, the condition $\mathbb{E}_{\varnothing,x}[S] = \frac{1-\varepsilon}{2} \mathbb{E}_\varnothing[K]$, conditioned on the extinction time being $\tau$, is equivalent to

$$
(1-x^n)\frac{\mu^\tau}{\mu_x} - 1 \leq \frac{1-\varepsilon}{2} \mu^\tau - 1 = \frac{1-\varepsilon}{2} \mu - 1 \iff (1-x^n)\frac{\mu^\tau(1-x^n)^\tau - 1}{\mu(1-x^n) - 1} \leq \frac{1-\varepsilon}{2} \frac{\mu^\tau - 1}{\mu - 1}.
$$

(13)

We have the following cases

1. If $\mu < 1$ then $\mathbb{P}[\tau < \infty] = 1$ (i.e., the process goes extinct after a finite number of steps), then the upper bound on the resilience is always finite due to Lemma 2 which can be found by numerically solving Equation (13).

2. If $\mu(1 - x^n) > 1$, then $\mathbb{P}[\tau = \infty] > 0$ and in this case Equation (13) is only feasible if and only iff $x = 0$, at which case the upper bound on the resilience is 0, and the GW process is not resilient. If $\tau < \infty$, which happens with non-zero probability, the upper bound on the resilience is finite when $\mu > e^2$ due to Lemma 2

For a specific triplet $(\mu, \tau, \varepsilon)$, let $\pi(\mu, \tau, \varepsilon)$ be the smallest possible solution to Equation (13), which exists for $\mu \in (0, 1) \cup (e^2, \infty)$ due to Lemma 2. Then the expected upper bound on the resilience $\mathbb{E}_\varnothing[\overline{R}_\varnothing(\varepsilon)]$, can be expressed as $\mathbb{E}_\varnothing[\overline{R}_\varnothing(\varepsilon)] = \mathbb{E}_\tau[\overline{R}_\varnothing(\varepsilon)] = \sum_{1 \leq k < \infty} \mathbb{P}[\tau = k|\tau(\mu, \tau, \varepsilon) > 0].$

Lower Bound. Similarly to the upper bound, in order to devise a lower bound, it suffices to set $\mathbb{E}_\varnothing[K] - \mathbb{E}_\varnothing[S]$ to be at most $\varepsilon$, since, again, by Markov’s inequality, we are going to get that the probability that at least a $\varepsilon$-fraction of products fails is at most $\frac{1}{\mathbb{E}_\varnothing[K]}$; namely, $\mathbb{P}_{\varnothing,x}[F \geq \varepsilon \mathbb{E}_\varnothing[K]] \leq \frac{\mathbb{E}_{\varnothing,x}[F]}{\varepsilon \mathbb{E}_\varnothing[K]} \leq \frac{1}{\mathbb{E}_\varnothing[K]}$. This yields

$$
\frac{\mu^\tau - 1}{\mu - 1} - (1-x^n)\frac{\mu^\tau}{\mu_x} - 1 \leq \varepsilon \iff \frac{\mu^\tau - 1}{\mu - 1} - (1-x^n)\frac{\mu^\tau(1-x^n)^\tau - 1}{\mu(1-x^n) - 1} \leq \varepsilon.
$$

(14)

Similarly to the upper bound, we have the following cases,

1. In the subcritical regime $\mu < 1$, we can again prove that the lower bound is always finite due to Lemma 2.

2. In the supercritical regime $\mu(1 - x^n) > 1$, we have that when $\tau < \infty$, which happens with positive probability then for $\mu > e$ from Lemma 2 we get the existence of the resilience. When $\tau = \infty$, we have again that the only way Equation (14) can hold is iff $x = 0$. 


For a specific triplet \((\mu, \tau, \epsilon)\), let \(\varepsilon(\mu, \tau, \epsilon)\) be the largest possible solution to Equation (14), which exists for \(\mu \in (0, 1) \cup (\epsilon, \infty)\) due to Lemma 2. Then the expected lower bound on the resilience \(E_{\beta} [R_{\beta}(\varepsilon)]\), can be expressed as \(E_{\beta} [R_{\beta}(\varepsilon)] = E_{\gamma} [R_{\beta}(\varepsilon)] = \sum_{1 \leq k < \infty} \mathbb{P}[\tau = k] \varepsilon(\mu, \tau, \epsilon) > 0\).

**Determining \(P[\tau < \infty]\) when \(\mu(1-x^n) > 1\).** It is known from the analysis of GW processes (see, e.g., Srinivasan (2013)) that the extinction probability \(\mathbb{P}[\tau < \infty]\) can be found as the smallest solution \(\eta \in [0, 1]\) to the fixed-point equation \(\eta = G_D(\eta)\) where \(G_D(s) = E_{\xi \sim \mathcal{D}}[e^{\xi}]\) is the moment generating function of the branching distribution \(\mathcal{D}\).

**B.6. Proof of Theorem 6**

For \(i \in \mathcal{K}\) let \(\beta_i = \mathbb{P}[Z_i = 0] \in [0, 1]\). By the union bound, we have that

\[
\beta_i = \mathbb{P}[(\exists j \in \mathcal{N}(i) : (i, j)\) is operational \(\land Z_j = 0) \lor (\forall s \in \mathcal{S}(i)Y_{is} = 0)] \\
\leq \mathbb{P}[\exists j \in \mathcal{N}(i) : (i, j)\) is operational \(\land Z_j = 0] + \mathbb{P}[\forall s \in \mathcal{S}(i)Y_{is} = 0] \\
\leq \text{\(y\)} \sum_{j \in \mathcal{N}(i)} \beta_j + x^n.
\]

The number of failed products equals \(\mathbb{E}[F] = \sum_{i \in \mathcal{K}} \beta_i\). Thus, finding the upper bound on \(\mathbb{E}[F]\) corresponds to solving the following LP,

\[
\max_{\beta \in [0, 1]} \sum_{i \in \mathcal{K}} \beta_i \quad \text{s.t.} \quad \beta \leq yA^T \beta + x^n 1.
\]

When \(y \|A^T\|_1 \leq 1\), which is equivalent to \(y \leq \frac{1}{\Delta}\), this problem resembles the financial clearing problem of Eisenberg and Noe (2001), and from Lemma 4 of Eisenberg and Noe (2001), we know that we can also compute \(\beta\) by solving the fixed point equation \(\beta = 1 \land (yA^T \beta + x^n 1) = \Phi(\beta)\). The set of fixed points \(\text{FIX}(\Phi) = \{\beta \in [0, 1] : \beta = \Phi(\beta)\}\) has a greatest fixed point \(\overline{\beta}\) and a least fixed point \(\underline{\beta}\), due to the Knaster-Tarski Theorem.

**B.7. Proof of Proposition 1**

First of all, it is easy to see that \(\Phi(\beta)\) is a contraction under the assumption \(y < \frac{1}{\Delta}\), and, thus, by Banach’s fixed point theorem, \(\Phi(\beta)\) has a unique fixed point \(\beta^*\). Finally, note that, for any \(i \in \mathcal{K}\) we have that

\((yA^T \beta^* + x^n 1)_i = y \sum_{j \in \mathcal{N}(i)} \beta_j^* + x^n < y \Delta + (1 - y \Delta) < 1\).

Therefore, the fixed point equation simplifies to

\(\beta^* = yA^T \beta^* + x^n 1\), which has a solution \(\beta^* = x^n (I - yA^T)^{-1} 1\), which corresponds to the Katz centrality, since \(yA\) is substochastic, and subsequently, \(I - yA^T\) is invertible. Finally, from Markov’s inequality \(\mathbb{P}[F \geq \varepsilon K] \leq \frac{x^n 1((I - yA^T)^{-1})_1}{\varepsilon K} = \frac{x^n \|\varepsilon 1((I - yA^T)^{-1})\|_1}{\varepsilon K}\), where the last equality holds since \((I - yA^T)^{-1}\) has non-negative
elements (since $A^T$ is an adjacency matrix and $(I - yA^T)^{-1}$ can be expressed as Neumann series). To make the above $1/K$ is suffices to pick, $x = \left(\frac{1}{\text{vec}(I - yA^T)^{-1}1}1\right)^{1/n}$, hence the lower bound in $R_G(\varepsilon)$.

B.8. Proof of Proposition 3

Since $0 < y < \frac{1}{x}$ and $0 < x < (1 - y\Delta)^{1/n}$, the optimal solution of the internal maximization equals $\hat{\beta}(t) = x(I - yA^T)^{-1}(1 - t)$. Substituting that in the objective function of Equation (10), we get that

$$\hat{t} = \arg \min_{t \in \{0, 1\}^n} 1^T \hat{\beta}(t) = \arg \min_{t \in \{0, 1\}^n} 1^T (I - yA^T)^{-1}(1 - t) = \arg \min_{t \in \{0, 1\}^n} (1 - t)^T((I - yA^T)^{-1})^T 1$$

$$= \arg \min_{t \in \{0, 1\}^n} (1 - t)^T(I - yA)^{-1}1 = \arg \min_{t \in \{0, 1\}^n} \gamma_{\text{Katz}}^{\text{R}}(G^R, y)(1 - t)$$

where, we remind that $\gamma_{\text{Katz}}(G^R, y) = (I - yA)^{-1}1$ is the vector of Katz centralities for the reverse graph $G^R$. The third equality is true since $(I - yA^T)^{-T} = \left(\sum_{k \geq 0}(yA^T)^k\right)^T = \sum_{k \geq 0}(yA)^k y \preceq \frac{1}{y} (I - yA)^{-1}$. It is easy to observe that, by the rearrangement inequality, the optimal solution would be to intervene to the top-$T$ nodes in terms of Katz centrality in $G^R$.

Appendix C: Upper and Lower Bounds on $E[S]$ for the $m$-ary tree

Lemma 3. Let $q_d$ be the probability that a product in tier $d$ can be produced. Then

$$q_d = \begin{cases} 
(1 - x^n) \frac{m^{D-d+1} - 1}{m-1}, & m \geq 2 \\
(1 - x^n)^{D-d+1}, & m = 1
\end{cases}$$

Let $q_d = \mathbb{P}[\text{a product in tier } d \text{ can be produced}] = \mathbb{P}[\exists \text{a functional supplier at tier } d]$. To calculate $q_d$, note that all inputs for a product node at tier $d$ succeed with probability $q_{d+1}^n$ and then the probability that at least one supplier is functionally conditioned on the all the inputs working is $1 - x^n$. That yields the following recurrence relation $q_d = q_{d+1}^n(1 - x^n)$ with $q_{D+1} = 1$. Solving this recurrence relation, we get that for $d \in [D]$,

$$q_{D-d} = (1 - x^n)^{\sum_{i=0}^{d-1} m^i}.$$ This yields to $q_d = \begin{cases} 
(1 - x^n) \frac{m^{D-d+1} - 1}{m-1}, & m \geq 2 \\
(1 - x^n)^{D-d+1}, & m = 1
\end{cases}$.

C.1. Proof of Proposition 2

Let $G$ be a DAG, and let $\beta^*$ be an optimal solution to Theorem 6 and let the link operation probability $y$ be a value to be determined later. Assume that $v_1, \ldots, v_K$ is a topological ordering of the DAG from raw materials to more complex materials. We have that for the $i + 1$-th vertex: $\beta_i^* = y \sum_{j \in N(v_i)} \beta_j^* + x^n \leq y \sum_{j \leq i} \beta_j^*$.

It is easy to see (by induction) that the solution to the above recurrence is: $\beta_i^* \leq (1 + y)^{i-1} x^n$ which is tight when all the possible $\binom{K}{2}$ edges in the DAG exist. Therefore, $E[F] \leq \sum_{i \in K} \beta_i^* \leq x^n \sum_{i=1}^{K} (1 + y)^{i-1} \leq x^n \sum_{i=0}^{K} (1 + y)^i$.

Appendix C: Upper and Lower Bounds on $E[S]$ for the $m$-ary tree

Lemma 3. Let $q_d$ be the probability that a product in tier $d$ can be produced. Then

$$q_d = \begin{cases} 
(1 - x^n) \frac{m^{D-d+1} - 1}{m-1}, & m \geq 2 \\
(1 - x^n)^{D-d+1}, & m = 1
\end{cases}$$

Let $q_d = \mathbb{P}[\text{a product in tier } d \text{ can be produced}] = \mathbb{P}[\exists \text{a functional supplier at tier } d]$. To calculate $q_d$, note that all inputs for a product node at tier $d$ succeed with probability $q_{d+1}^n$ and then the probability that at least one supplier is functionally conditioned on the all the inputs working is $1 - x^n$. That yields the following recurrence relation $q_d = q_{d+1}^n(1 - x^n)$ with $q_{D+1} = 1$. Solving this recurrence relation, we get that for $d \in [D]$,

$$q_{D-d} = (1 - x^n)^{\sum_{i=0}^{d-1} m^i}.$$ This yields to $q_d = \begin{cases} 
(1 - x^n) \frac{m^{D-d+1} - 1}{m-1}, & m \geq 2 \\
(1 - x^n)^{D-d+1}, & m = 1
\end{cases}$.

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It is easy to see (by induction) that the solution to the above recurrence is: $\beta_i^* \leq (1 + y)^{i-1} x^n$ which is tight when all the possible $\binom{K}{2}$ edges in the DAG exist. Therefore, $E[F] \leq \sum_{i \in K} \beta_i^* \leq x^n \sum_{i=1}^{K} (1 + y)^{i-1} \leq x^n \sum_{i=0}^{K} (1 + y)^i$.
\[
\frac{x^n(1+y)^K}{y} \leq \frac{Kx^ne^{Ky}}{K^y}. \]
This yields the first part of the proof. For the second part we want to minimize the RHS so that we maximize the lower bound on the resilience. Hence, we choose \( z = Ky > 0 \) so that \( \frac{e^z}{z} \) is minimized, which happens at \( z = 1 \), and equivalently \( y = \frac{1}{K} \), at which case we have that \( E[F] \leq Kx^n \min_{z \geq 0} \frac{e^z}{z} = Kx^n e \).

By Markov’s inequality we can get, similarly to Theorem 6, that \( R_G(\varepsilon) \geq \left( \frac{1}{\varepsilon} \right)^{1/n} \).

**Lemma 4 (Upper Bound when \( m \geq 2 \)).** Under the tree structure and \( m \geq 2 \) the expected size obeys \( E[S] \leq \frac{Kx^n(D-1)}{2} = U(x) \).

From Lemma 3 we have that \( q_d = (1-x^n)^{\frac{mD-d+1}{m+1}} \). Using the inequality \( (1-t)^a \leq \frac{1}{1+at} \) for \( a > 0 \) and \( t \in (0,1) \) we get that

\[
E[S] = \sum_{d=1}^{D} m^{d-1} q_d \leq \sum_{d=1}^{D} m^{d-1} \frac{1}{1+x^n \frac{mD-d+1}{2(m-1)}} = \int_{t=1}^{D} m^{t-1} \frac{1}{1+x^n \frac{mD-t+1}{2(m-1)}}.
\]

By letting \( u = \frac{Dx^n}{2(m-1)} \) we get that the above integral equals

\[
\frac{m^Dx^n}{2 \log m(m-1)} \log \left( \frac{1-x^n(1-p)^K(1+(1-x^n)(1-p)^K)}{(1-x^n)(1-p)^K(1+(1-x^n)(1-p)^K)} \right) \bigg|_{u_1=x^n m/2(m-1)}^{u_2=x^n mD/2(m-1)} \leq \frac{m^Dx^n}{2 \log m(m-1)} \log (m^{D-1}) \leq \frac{Kx^n(D-1)}{2} = U(x).
\]

**Lemma 5 (Lower bound when \( m \geq 2 \)).** Under the tree structure and \( m \geq 2 \) the expected cascade size obeys \( E[S] \geq K(1-x^n(D-1)) = L(x) \) where \( K = m^D - 1 \) is the number of products.

From Lemma 3 we have that \( q_d = (1-x^n)^{\frac{mD-d+1}{m+1}} \). By Bernoulli’s inequality \( q_d \geq 1-x^n \left( \frac{mD-d+1}{m+1} \right) \).

Since every level has \( m^{d-1} \) nodes we have that

\[
E[S] = \sum_{d=1}^{D} m^{d-1} q_d \geq \sum_{d=1}^{D} m^{d-1} \left[ 1-x^n \left( \frac{mD-d+1}{m+1} - 1 \right) \right] = K(1+x^n) - x^n \frac{1}{m-1} \sum_{d=1}^{D} m^{d-1} m^{D-d+1} = K(1+x^n) - x^n \frac{1}{m-1} \sum_{d=1}^{D} m^{D} = K(1+x^n) - x^n D(K-1) \geq K(1-x^n(D-1)).
\]

**Appendix D: Random Width-\( w \) Trellis Architectures**

**Figure 5** Instance of random width-\( w \) trellis with \( w = 2, D = 3, K = 6 \). Failures are drawn in pink.

We analyze the resilience of the random width-\( w \) trellis depicted in Figure 5. The width-\( w \) trellis, \( rt(K,p,w) \) is a graph with \( K = wD \) nodes, that consists of \( D \) tiers of products \( K_1, \ldots, K_D \) and each tier has size/width
$|K_d| = w$ for all $d \in [D]$. The edges are generated randomly and independently with probability $p$ if $i \in K_d, j \in K_{d+1}, d \in [D-1]$ and do not exist otherwise. We prove the following,

**Theorem 7.** Let $G \sim \rt(K,p,w)$ be a random width-$w$ trellis with $K = wD$ nodes. Then the resiliency of $G$ satisfies,

$$R_G(\epsilon) \geq \begin{cases} 
\Omega \left( \left( \frac{e^{-\omega^2}}{\omega^2} \right)^{1/n} \right), & \text{for } pw \leq 1 \\
\Omega \left( \left( \frac{e^{-\omega^2}}{\omega^2} \right)^{1/n} \right), & \text{for } pw > 1 
\end{cases}$$

Therefore,

- If $pw \leq 1$, $K, w \to \infty$, and $K/w \to D < \infty$, then $G$ is resilient.
- If $pw \geq 1$, $K, D \to \infty$, and $K/D \to w < \infty$, then $G$ is fragile.

**Lower Bound.** To determine a lower bound in $R_G(\epsilon)$ we let $\gamma_d$ be the probability that a node at tier $d \in [D]$ fails (due to symmetry the probability is the same for each product of a given tier). The expected number of failures is $E[F] = D \sum_{d \in [D]} \gamma_d$. By applying the union bound – similarly to Theorem 6 – we get that $\gamma_{d+1} \leq pw \gamma_d + x^n$. Solving the recurrence, yields $\gamma_d \leq x^n \left( (pw)^{d-1} + \frac{1}{1-pw} \right)$ for $pw \neq 1$, and $\gamma_d \leq dx^n$ for $pw = 1$. Summing up everything,

$$E[F] = \frac{K}{w} \sum_{d \in [D]} \gamma_d \leq \begin{cases} 
\frac{K x^n (pw)^{K/w}}{pw-1}, & \text{for } pw > 1 \\
\frac{k^{2x^n}}{w^2} \frac{1}{1-pw^2}, & \text{for } pw < 1 \\
\frac{k^{2x^n}}{w^2}, & \text{for } pw = 1 
\end{cases}$$

Therefore, letting $E[F] = \epsilon$ yields the following lower bounds for the resilience: $R_G(\epsilon) = \begin{cases} 
\Omega \left( \left( \frac{e^{-\omega^2}}{\omega^2} \right)^{1/n} \right), & \text{for } pw \leq 1 \\
\Omega \left( \left( \frac{e^{-\omega^2}}{\omega^2} \right)^{1/n} \right), & \text{for } pw > 1 
\end{cases}$. If $w$ is constant, as $K \to \infty$ we have that $\lim_{K \to \infty} R_G(\epsilon) = 0$ for all $p \in (0,1]$, and $w \in \mathbb{N}$. Now, if $w, K \to \infty$ but $K/w \to D$ where $D$ is finite, then $G$ is resilient with resilience $\Omega \left( \left( \frac{e^{-\omega^2}}{\omega^2} \right)^{1/n} \right)$ for $pw \leq 1$. If $pw > 1$, then it is easy to prove that the lower bound goes to 0.

**Upper Bound.** To construct an upper bound, we need a lower bound on $E[F]$. Let $f_d$ be the expected number of failures conditioned on any node in tier $d$ failing solely (and spontaneously). As base case we have that $f_D = 1$. We have that a cascade starting from any node at tier $d$ causes at least $a_{d,k}$ failures on expectation at an offset $k$ from tier $d$. We have that for all $d$, $a_{d,0} = 1$, and that $a_{d,k+1} \geq \min \{ w, pw a_{d,k} \}$ since the failures are amplified by a factor of $pw$ at each level, up to being $w$, since the width of the trellis is $w$. Subsequently $f_d \geq \sum_{k=0}^{D-d} a_{d,k}$.
For $pw = 1$ we have that $a_{d,k+1} \geq \min\{w,a_{d,k}\}$, and since $a_{d,0} = 1$ we have that $a_{d,k} = 1$ for all $k$.

For $pw < 1$ we have that at every new offset the number of failures is decreasing by a factor of $pw$.

Therefore, we deduce that $a_{d,k+1} \geq (pw)a_{d,k}$, and since $a_{d,0} = 1$ we have that $a_{d,k} \geq (pw)^k$.

For $pw > 1$, for $k \leq \frac{w}{\log(pw)}$, we have that at least $(pw)^k \geq k\log(pw)$ nodes are influenced and are at most $w$. For $k > \frac{w}{\log(pw)}$, $w$ nodes are infected at every step on expectation.

$$f_d \geq \sum_{k=0}^{D-d} a_{d,k} \begin{cases} 1 - (pw)^{D-d+1}, & \text{for } pw < 1 \\ D - d + 1, & \text{for } pw = 1 \\ \sum_{k=0}^{\frac{w}{\log(pw)}} \log(pw)k + w \left(D - d - \frac{w}{\log(pw)}\right), & \text{for } pw > 1 \end{cases}$$

Therefore, we get that $f_d \geq \sum_{k=0}^{D-d} a_{d,k} \geq \begin{cases} 1 - (pw)^{D-d+1}, & \text{for } pw < 1 \\ D - d + 1, & \text{for } pw = 1 \\ \sum_{k=0}^{\frac{w}{\log(pw)}} \log(pw)k + w \left(D - d - \frac{w}{\log(pw)}\right), & \text{for } pw > 1 \end{cases}$

Therefore the lower bound on $\mathbb{E}[F]$ is $\mathbb{E}[F] \geq x^n \sum_{d=1}^{D} f_d$. We have the following three cases

1. If $pw = 1$ then by a change of summation indices $\mathbb{E}[F] \geq x^n \sum_{k=1}^{D} k = x^n \frac{D(D+1)}{2} \geq \frac{x^n D^2}{2} = \frac{x^n K^2}{2w}$.

2. If $pw < 1$ then, by change of summation indices we have that $\mathbb{E}[F] \geq x^n \sum_{d=1}^{D} (D-k+1)(pw)^{k-1} \geq \frac{x^n D}{1-pw}$.

3. If $pw > 1$, we have that for $w \geq 2$,

$$\mathbb{E}[F] \geq x^n \left\{ \log\left(\frac{w}{\log(pw)}\right) \sum_{k=0}^{\frac{w}{\log(pw)}} k + w \left(D - d - \frac{w}{\log(pw)}\right) \right\} \geq x^n \sum_{d=1}^{D} \left\{ \frac{w^2}{2} - \frac{w}{\log(pw)} + w(D - d) \right\} \geq x^n w \sum_{d=1}^{D} (D-d) \geq x^n w \frac{D^2}{2} = \frac{x^n K^2}{2w}$$

We want to make each of the above lower bounds equal to $\frac{1+\nu}{2} K$, since that would give an $\frac{1+\nu}{2} K$ upper bound on $\mathbb{E}[S]$, and due to Lemma [1] we get the upper bound

$$\mathcal{R}_G(\varepsilon) = \left\{ \begin{array}{ll} \left( \frac{(1+\epsilon)w(1-pw)}{2} \right)^{1/n}, & \text{for } pw < 1 \\ O\left(\left(\frac{w^2}{K}\right)^{1/n}\right), & \text{for } pw = 1 \end{array} \right.$$ Therefore, if $pw \geq 1$, $K, D \to \infty$ and $K/D \to w$ with $w$ finite, then $G$ is fragile.

### Appendix E: Supplier Heterogeneity & Optimal Interventions

Here we give a proof of the derivation of the optimal intervention mechanism through supplier heterogeneity presented in Section [6]. Namely, we increase the number of suppliers of product $i$, from $n$ to $n + \nu_i$ for some $0 \leq \nu_i \leq \nu$, subject to a budget $\sum_{i \in K} \nu_i \leq N$, to decrease its self-percolation probability from $x^n$ to $x^{n+\nu_i}$.

Under the Assumptions of Proposition [3] the optimization problem is:

$$\hat{\nu} = \arg \min_{0 \leq \nu \leq \nu} x^n \gamma_{\text{kat}}(G^R, \nu) x^{\hat{\nu}} = \arg \min_{0 \leq \nu \leq \nu} \gamma_{\text{kat}}(G^R, \nu) x^{\hat{\nu}}.$$
where $x^\odot \nu$ is the Hadamard power (elementwise power) of $x$ raised to the vector $\nu$, i.e. $x^{\odot \nu} = (x^{\nu_1}, \ldots, x^{\nu_K})^T$.

Since $x^{\nu}$ is a decreasing function of $\nu$, the optimal policy is to order the nodes in decreasing order $\pi$ in terms of their Katz centrality in $G^R$ and then try to put as many suppliers as possible in $\pi(1), \pi(2), \ldots$, while respecting the budget constraint $N$ at the same time. This yields the following optimal policy $\hat{\nu}_{\pi(i)} = \left( \nu_{\pi(i)} \land \left( N - \sum_{j<i} \hat{\nu}_{\pi(j)} \right) \right)^+$. where $(a)^+ = \max\{a, 0\}$.

Appendix F: Extended Related Work

Supply Chain Contagion. There have been multiple works on production networks in macroeconomics and how shocks in production networks propagate through the production network’s input-output relations, see a comprehensive survey by Carvalho and Tahbaz-Salehi (2019). One of the earliest works dates back to Horvath (1998) introduces a multi-sector model that extends the earlier Long-Plosser model Long Jr and Plosser, 1983 and argues that sector-specific shocks are, in fact, affected by the graph topology between producing sectors. Such arguments contrast the prior arguments of Lucas et al. (1995), which argue that small microeconomic shocks would significantly affect the economy. In the 2008 financial crisis, the ex-CEO of Ford, Alan Mulally, argued that the collapse of GM or Chrysler would significantly impact Ford’s production capabilities for non-trivial amounts of time. The works of Acemoglu et al. (2012) and Gabaix (2011) build on the above observation and shows that even small shocks lead to cascades that can have devastating effects on the economy. Specifically, Gabaix (2011) argues that firm-level idiosyncratic shocks can translate into large fluctuations in the production network when the firm sizes are heavy-tailed, and, in the sequel, Acemoglu et al. (2012) replaced Gabaix (2011)’s analysis on the firm size with the intersectoral network, building on the Long-Plosser model. Hallegatte (2008) introduces a supply-chain model in which when a firm cannot satisfy its orders, it rations its production to the firms that depend on it via the Input-Output matrix. The model of Hallegatte (2008) has been used to study supply chain effects of the COVID-19 pandemic (Guan et al., 2020, Walmsley et al., 2021, Inoue and Todo, 2020, Pichler et al., 2020). Finally, the recent work of Elliott et al., 2022 studies the propagation of shocks in a network (see Section I). Our work is related to the above works and attempts to study the propagation of individual shocks at the supplier level to the (aggregate) production network through the definition of a resilience metric.

Seeding Problems in Supply Chains. In a different flavor from the literature that we discussed above, the work of Blaettchen et al. (2021) study how to find the least costly set of firms to target as early adopters of a tractability technology in a supply chain network, where hyperedges model individual supply chains. The
connections to our paper involve the spread of information in a networked environment through interventions. However, in this paper, we do not focus on building algorithms for interventions; rather, we provide metrics to assess the vulnerability of nodes in a supply chain network, which can be informative for interventions.

*Node Percolation.* Goldschmidt et al. (1994) and Yu et al. (2010) study node percolation processes, wherein graph nodes fail independently with probability $p$. Their goal is to find the graphs – among graphs with a fixed number of nodes and edges – such that the probability that the induced subgraph (after percolation) is connected is maximized.

*Resilience and Risk Contagion in Financial Networks.* In a different context, similar models have been used to study financial networks, and optimal allocations in the presence of shocks (Eisenberg and Noe, 2001; Glasserman and Young, 2015; Papachristou and Kleinberg, 2022; Papachristou et al., 2022; Papp and Wattenhofer, 2021; Friedetzky et al., 2023; Demange, 2018; Ahn and Kim, 2019) and financial network formation and risk, see, e.g., (Blume et al., 2013; Jalan et al., 2022; Jackson and Pernoud, 2019; Aymanns and Georg, 2015; Babus, 2013; Erol, 2019; Erol and Vohra, 2022; Amelkin and Vohra, 2019; Talamas and Vohra, 2020; Bimpikis et al., 2019). Our work is related to the above works and attempts to study the propagation of individual shocks at the supplier level to the (aggregate) production network through the definition of a resilience metric. Acemoglu et al. (2015) study financial networks and state that contagion in financial networks has a phase transition: for small enough shocks, a densely connected financial network is more stable, and, on the contrary, for large enough shocks, a densely connected system is a more fragile financial system. In a similar spirit, Amini et al. (2012) and Battiston et al. (2012b) characterize the size of defaults under financial contagion in random graphs. They show that firms that contribute the most to network instability are highly connected and have highly contagious links. Moreover, it has also been shown – see, e.g., Siebenbrunner (2018), Battiston et al. (2012a), and Bartesaghi et al. (2020), and the references therein – that risky nodes in financial networks are connected to centrality measures.

*Cascading Failures and Emergence of Power Laws.* There has been a large body of literature on cascading failures in networks and how the cascade distributions behave as power laws in social networks (Leskovec et al., 2007; Wegzrycki et al., 2017), and power grids, see e.g., (Dobson et al., 2004; Nesti et al., 2020). We bring the perspective of supply chains and production networks to this literature and offer new insights on how complexity of products and their interdependence affect production network resilience.
Appendix G: Experiments Addendum: World I-O Tables

<table>
<thead>
<tr>
<th>Country</th>
<th>Size ($K$)</th>
<th>Avg. Degree</th>
<th>Density</th>
<th>Min/Max In-degree</th>
<th>Min/Max Out-degree</th>
<th>AUC</th>
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<td>1.000</td>
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Table 3 Network Statistics and AUC for the world economies. The edge density is computed as $\frac{|E(G)|}{K^2-K}$.

![Graph](image)

(a) Estimating $R_G(\varepsilon)$

(b) Optimal Interventions

Figure 6 World Economy Input-Output Networks. We set the number of suppliers for each product to $n = 1$.

References


Erol, Selman (2019). Network hazard and bailouts. *Available at SSRN 3034406*.


