A STUDY ON FUNCTION SPACES

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Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2023

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

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It was defended on

March 28th, 2023

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A STUDY ON FUNCTION SPACES

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University of Pittsburgh, 2023

In this thesis, we investigate the properties of homogeneous function spaces. We study related basic definitions and prerequisite lemmas. We state and prove a complex interpolation theorem for the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ when $1 < p, q < \infty$ and $s \in \mathbb{R}$. We prove a Fourier multiplier theorem for sequences of functions and deduce another for homogeneous Triebel-Lizorkin spaces. This thesis provides improved results of function spaces found in the famous literature [93] and [94] by H. Triebel. The first pair of results improves the restrictions of characterizations by maximal functions of the iterated differences $\Delta^L_h f$ to $0 < p < \infty$, $0 < q \leq \infty$, $\frac{n}{\min\{p,q\}} < s < L$ for the homogeneous spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and to $0 < p, q \leq \infty$, $\frac{n}{p} < s < L$ for the homogeneous spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$. The following inequalities are also proven. Denote $\sigma_{pq} = \max\{0, n(\frac{1}{\min\{p,q\}} - 1)\}$, $\tilde{\sigma}_{pq} = \max\{0, n(\frac{1}{p} - \frac{1}{q})\}$, $\tilde{\sigma}^1_{pq} = \max\{0, \frac{1}{p} - \frac{1}{q}\}$, $\sigma_p = \max\{0, n(\frac{1}{p} - 1)\}$. Let $L \in \mathbb{N}$, $0 < p < \infty$, $0 < q \leq \infty$, $s, t \in \mathbb{R}$, $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ and $h, e_j \in \mathbb{R}^n$, e_j is the elementary unit vector for $1 \leq j \leq n$. And $\Delta^L_h f$ is the iterated difference, $\Delta^L_{t,j} f = \Delta^L_{te_j} f$. If $0 < p, q < \infty$, $\tilde{\sigma}_{pq} < s < L$, then

$$\|(\int_{\mathbb{R}^n} |h|^{-sq} |\Delta^L_h f|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{1}$$

If $0 < p < \infty$, $0 < q < 1$ and $\sigma_{pq} + \tilde{\sigma}_{pq} < s < \infty$, or if $0 < p < \infty$, $1 \leq q < \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|(\int_{\mathbb{R}^n} |h|^{-sq} |(\Delta^L_h f)(\cdot)|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}. \tag{2}$$

If $0 < p < \infty$, $q = \infty$ and $\frac{n}{p} < s < L$, then

$$\|\operatorname*{ess\,sup}_{h \in \mathbb{R}^n} \frac{|\Delta^L_h f|}{|h|^s}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{3}$$

If $0 < p < \infty$, $q = \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \|\operatorname*{ess\,sup}_{h \in \mathbb{R}^n} \frac{|(\Delta^L_h f)(\cdot)|}{|h|^s}\|_{L^p(\mathbb{R}^n)}. \tag{4}$$

If $0 < p, q < \infty$, $\tilde{\sigma}^1_{pq} < s < L$, then for each $j \in \{1, \cdots, n\}$

$$\|(\int_0^\infty t^{-sq}|\Delta^L_{t,j}f|^q \frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{5}$$

If $1 < \min\{p, q\}$, $q < \infty$ and $s \in \mathbb{R}$, or if $\min\{p, q\} \leq 1$, $q < \infty$ and $\sigma_{pq} + \tilde{\sigma}^1_{pq} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n} \|(\int_0^\infty t^{-sq}|\Delta^L_{t,j}f(\cdot)|^q \frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}. \tag{6}$$

If $0 < p < \infty$, $q = \infty$ and $\frac{1}{p} < s < L$, then for each $j \in \{1, \cdots, n\}$

$$\|\operatorname*{ess\,sup}_{t>0} \frac{|\Delta^L_{t,j}f|}{t^s}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{7}$$

If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \leq 1$, $q = \infty$ and $\sigma_p + \frac{1}{p} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n} \|\operatorname*{ess\,sup}_{t>0} \frac{|\Delta^L_{t,j}f(\cdot)|}{t^s}\|_{L^p(\mathbb{R}^n)}. \tag{8}$$

This thesis also provides the counterparts of the above inequalities for the homogeneous $\dot{B}^s_{p,q}(\mathbb{R}^n)$ spaces.

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Preface

The author expresses his earnest gratitude to his advisor, Professor Armin Schikorra, for the numerous suggestions he counseled, for the insightful and inspirational communications that took place, and most importantly, for suggesting a suitable and fruitful research project that enlightens many of the author's works afterward.

The author also thanks the dissertation committee for their time and advice.

The author sincerely appreciates Professors Armin Schikorra, Piotr Hajlasz, Jason DeBlois, and Juan J. Manfredi for the many recommending efforts made during the author's job application.

The author is very grateful to the Department of Mathematics at the University of Pittsburgh for all the financial and academic support during the author's Ph.D. study, for the friendly and collaborative colleagues, and for the amiable environment.

1.0 Introduction

1.1 General Introduction

Historically the theory of function spaces constituted an indispensable and significant part in the development of classical and modern mathematics. Function spaces, consisting of continuous or differentiable or p-integrable elements, were of interest not just on their own but also in conjunction with the theoretical development of ordinary and partial differential equations. These spaces were called the classical basic spaces with examples like $L^p(\mathbb{R}^n)$ spaces and C^m spaces of $m \in \mathbb{N}$ times continuously differentiable functions. In the meanwhile the Hölder spaces C^s with $0 < s \notin \mathbb{N}$ and the Hardy spaces H^p with $0 < p < \infty$ were also carefully and thoroughly studied. Subsequently, along with the introduction of Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ with $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, the theory of distributions was successfully established and new techniques and results such as embedding theorems were widely utilized for the continued investigation into partial differential equations. During this period, many new spaces were constructed with explicit norms or quasinorms, which were usually considered direct descendants of the aforementioned classical spaces. To extend the Hölder spaces C^s with $0 < s \notin \mathbb{N}$ to values $s = 1, 2, 3, \cdots$, the Zygmund spaces $\mathcal{C}^s$ were defined, revealing the advantages of the second-order difference of functions over the first-order difference. As an attempt to fill the gaps between Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ with $m \in \mathbb{N}$, the Slobodeckij spaces $W^{s,p}(\mathbb{R}^n)$ with $0 < s \notin \mathbb{N}$ were introduced. Merging the above two ideas, the use of the second-order difference of functions instead of the first-order difference and the replacement of the supremum norm in the Hölder spaces by the $L^p(\mathbb{R}^n)$ norm in the Slobodeckij spaces, yielded the Besov spaces $\Lambda^s_{p,q}(\mathbb{R}^n)$. Another attempt to fill the gaps between Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ was the defining of Bessel-potential spaces $L^p_s(\mathbb{R}^n)$ for $s \in \mathbb{R}$, also known as fractional Sobolev spaces nowadays, via Fourier transforms. Of course, this was due to the important role Fourier transform plays in the theory of distributions. As the theory of function spaces continued to flourish and the investigation into differential equations continued to deepen, many other spaces were treated extensively, such as the space of bounded mean oscillation, Lorentz spaces, Campanato-Morrey spaces, Orlicz spaces, and Orlicz-Sobolev spaces. The number of function spaces grew plethoric and this gave rise to the need for deep and profound theories that can characterize diversified spaces from a few unified perspectives. To name a few, the abstract interpolation theory, Fourier analysis, and the theory of maximal inequalities emerged as far-reaching and powerful tools to achieve this goal. Many of the aforementioned classical and constructive spaces fell into the two categories of inhomogeneous function spaces $B^s_{p,q}$ and $F^s_{p,q}$, with their homogeneous counterparts $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, all of which can be defined in the framework of Fourier analysis. Encouraged by this spirit, it is the goal of the present thesis to study and investigate the classical properties of function spaces of the above types, practicing knowledge and modern techniques from interpolation theory, Fourier analysis, and maximal inequalities.

The arrangement of this thesis is described below. In the present chapter 1, we give a general introduction to the historical background of function spaces along with the notations and basic definitions widely adopted throughout the thesis. In chapter 2, powerful and technical lemmas and remarks are introduced and they are the devices that will be extensively used in the proofs of ensuing results. In chapter 3, we look into the interpolation property of homogeneous spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ while providing the reader with modern and detailed proof. In chapter 4, the Fourier multiplier property is studied and the argument given there is of Hörmander type. Chapter 5 is dedicated to the characterization of homogeneous function spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by maximal functions of iterated differences. In chapter 6, we present inequalities in function spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ in terms of iterated differences. And chapter 7 furnishes the reader with inequalities in function spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ in terms of iterated differences along coordinate axes.

1.2 Notations And Definitions

Here in this section, we introduce notations and definitions widely used throughout the thesis. Let $\mathbb{N}$ denote the set of positive integers. Let $C_c^\infty(\mathbb{R}^n)$ be the set of smooth functions on $\mathbb{R}^n$ with compact supports and if a function f is in $C_c^\infty(\mathbb{R}^n)$, we use $spt.f$ to denote the support set of this function. Also, let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions, and the notation "$X \lesssim Y$" means X is dominated by a constant multiple of Y and the constant is determined by some fixed parameters, and when we want to emphasize the constant is 1 we still use the usual notation "$X \leq Y$". If $X \lesssim Y$ and $Y \lesssim X$, then we consider X and Y are equivalent and write $X \sim Y$. For a sufficiently smooth function f and a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote the derivative by

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

For a function $f \in L^1(\mathbb{R}^n)$, we denote its n-dimensional Fourier transform by

$$\mathcal{F}_n f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx,$$

and the n-dimensional inverse Fourier transform is denoted by

$$\mathcal{F}_n^{-1} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx,$$

where for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, $x \cdot \xi$ is the inner product. If f is a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$, we use the same notation to denote n-dimensional distributional Fourier transform and its inverse. We also give the definition of iterated differences. Let $L \in \mathbb{N}$, for a function f defined on $\mathbb{R}^n$ and $x, h \in \mathbb{R}^n$ we define

$$\Delta_h^1 f(x) = f(x+h) - f(x), \qquad (\Delta_h^L f)(x) = \Delta_h^1 (\Delta_h^{L-1} f)(x). \tag{9}$$

It is not hard to prove by induction on L that

$$(-1)^{L+1} (\Delta_h^L f)(x) = \sum_{j=1}^{L} d_j f(x + jh) - f(x), \tag{10}$$

where

$$\sum_{j=1}^{L} d_j = 1 \text{ and } d_j \in \mathbb{Z} \text{ for } 1 \leq j \leq L. \tag{11}$$

Assuming the existence of Fourier transform, we have

$$(\Delta_h^1 \mathcal{F}_n f)(x) = \mathcal{F}_n((e^{-2\pi i h \cdot \xi} - 1) f(\xi))(x), \tag{12}$$

and by iteration, we can obtain

$$(\Delta_h^L \mathcal{F}_n f)(x) = \mathcal{F}_n((e^{-2\pi i h\cdot\xi} - 1)^L f(\xi))(x). \tag{13}$$

In a similar way, we also have

$$(\Delta_h^L \mathcal{F}_n^{-1} f)(x) = \mathcal{F}_n^{-1}((e^{2\pi i h\cdot\xi} - 1)^L f(\xi))(x), \tag{14}$$

and therefore the following is true

$$(\Delta_h^L f)(x) = \mathcal{F}_n^{-1}((e^{2\pi i h\cdot\xi} - 1)^L \mathcal{F}_n f(\xi))(x). \tag{15}$$

Let $\mathcal{H}^{n-1}$ denote the $(n-1)$-dimensional Hausdorff measure on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and A denote the annulus $A := \{z \in \mathbb{R}^n : 1 \le |z| < 2\}$. For $k \in \mathbb{Z}$, we also use the notation

$$A_k := \{x \in \mathbb{R}^n : 2^{-k} \le |x| < 2^{1-k}\}.$$

And the n-dimensional ball is denoted by $B^n(x,t) := \{y \in \mathbb{R}^n : |y - x| < t\}$.

In this paper we denote the Lebesgue measure and integrals with respect to the Lebesgue measure in the usual way, then $|\mathbb{S}^{n-1}| = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ and $|A|$ represent the corresponding surface measure and volume respectively. Also "$\fint$" is the mean value integral. Given a sequence $\{f_k(x)\}_{k\in\mathbb{Z}}$ of functions defined on $\mathbb{R}^n$ and $0 < p, q \le \infty$, we use the following notations

$$\|\{f_k\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} := (\int_{\mathbb{R}^n} (\sum_{k\in\mathbb{Z}} |f_k(x)|^q)^{\frac{p}{q}} dx)^{\frac{1}{p}}$$

and

$$\|\{f_k\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} := (\sum_{k\in\mathbb{Z}} \|f_k\|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}}$$

with modifications if $p = \infty$ and/or $q = \infty$, and on the left side, we omit the domain of $L^p(\mathbb{R}^n)$-quasinorms since for most cases in this paper the domain is $\mathbb{R}^n$ by default. Sometimes the range of k may not be all of $\mathbb{Z}$ then we make some modifications such as

$$\|\{f_k\}_{k\ge 0}\|_{L^p(l^q)} = (\int_{\mathbb{R}^n} (\sum_{k=0}^{\infty} |f_k(x)|^q)^{\frac{p}{q}} dx)^{\frac{1}{p}}$$

and

$$\|\{f_k\}_{k\geq 0}\|_{l^q(L^p)} = (\sum_{k=0}^{\infty} \|f_k\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}}.$$

Furthermore, $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f_k(x)|$ denotes the essential supremum of the function $|f_k(x)|$ over $\mathbb{R}^n$, that is, the least upper bound of $|f_k(x)|$ over $\mathbb{R}^n$ except on a subset of $\mathbb{R}^n$ of Lebesgue measure zero. Moreover, $\operatorname{ess\,sup}_{k\in\mathbb{Z}} |f_k(x)|$ denotes the essential supremum of the sequence $\{|f_k(x)|\}_{k\in\mathbb{Z}}$ at $x \in \mathbb{R}^n$, that is, the least upper bound of $\{|f_k(x)|\}_{k\in\mathbb{Z}}$ except on a subset of $\mathbb{Z}$ of counting measure zero, and in this sense $\operatorname{ess\,sup}_{k\in\mathbb{Z}} |f_k(x)| = \sup_{k\in\mathbb{Z}} |f_k(x)|$.

For the purpose of the complex interpolation theorem, we cite the definition of $\mathcal{S}'(\mathbb{R}^n)$-analytic functions from section 2.4.4 of [93]. Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ denote the open strip on the complex plane and $\bar{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ be its closure.

Definition 1.2.1. We say that f_z is a $\mathcal{S}'(\mathbb{R}^n)$-analytic function in S if the following properties are satisfied:

(1) for every $z \in \bar{S}$, f_z is a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$;

(2) for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is compactly supported in $\mathbb{R}^n$, the convolution $\varphi * f_z(x)$ is a uniformly continuous and bounded function in $\mathbb{R}^n \times \bar{S}$;

(3) for every $x \in \mathbb{R}^n$ and every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is compactly supported in $\mathbb{R}^n$, the convolution $\varphi * f_z(x)$ is also analytic in S.

Now we introduce the definitions of related function spaces and maximal functions. We fix throughout this paper $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$0 \leq \mathcal{F}_n\psi(\xi) \leq 1 \qquad \text{and} \qquad spt.\mathcal{F}_n\psi \subseteq \{\frac{1}{2} \leq |\xi| < 2\} \tag{16}$$

and also

$$\sum_{j\in\mathbb{Z}} \mathcal{F}_n\psi(2^{-j}\xi) = 1 \qquad \text{if} \qquad \xi \neq 0, \tag{17}$$

then the function ϕ is defined in a way so that

$$\mathcal{F}_n\phi(\xi) = \begin{cases} \sum_{j\leq 0} \mathcal{F}_n\psi(2^{-j}\xi) & \text{if } \xi \neq 0, \\ 1 & \text{if } \xi = 0, \end{cases} \tag{18}$$

then

$$spt.\mathcal{F}_n\phi(\xi) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \text{ and } \mathcal{F}_n\phi(\xi) = 1 \text{ if } |\xi| \leq 1. \tag{19}$$

Furthermore, we have the equality

$$\mathcal{F}_n\phi(\xi) + \sum_{j=1}^{\infty} \mathcal{F}_n\psi(2^{-j}\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n. \tag{20}$$

Define for $f \in \mathcal{S}'(\mathbb{R}^n)$, $j \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n$, the function $f_j(x) := \psi_{2^{-j}} * f(x)$ where $\psi_{2^{-j}}(y) = 2^{jn}\psi(2^j y)$ and thus we have the following decompositions:

$$f = \sum_{j\in\mathbb{Z}} f_j, \tag{21}$$

where the sum in (21) converges in $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ is the space of tempered distributions modulo polynomials (cf. section 1.1.1 of [42]), and

$$f = f * \phi + \sum_{j=1}^{\infty} f_j, \tag{22}$$

where the sum in (22) converges in $\mathcal{S}'(\mathbb{R}^n)$. Also due to the support condition of $\mathcal{F}_n\psi$, we have the following

$$f_j(x) = \sum_{l=j-1}^{j+1} (f_j)_l(x) \qquad \text{for almost every } x \in \mathbb{R}^n, \tag{23}$$

where $(f_j)_l = \psi_{2^{-l}} * f_j = \psi_{2^{-l}} * \psi_{2^{-j}} * f$.

Definition 1.2.2. For $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, the homogeneous function space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ as a subspace of the space $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ is

$$\dot{F}^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) : \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} := \|\{2^{ks} f_k\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} < \infty\}. \tag{24}$$

For $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, the homogeneous function space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as a subspace of the space $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ is

$$\dot{B}^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \|\{2^{ks} f_k\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} < \infty\}. \tag{25}$$

It is a well-known fact that the space $\mathcal{S}_0(\mathbb{R}^n)$ of Schwartz functions that satisfy the condition

$$\int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0 \text{ for all multi-indices } \alpha$$

is dense in $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ when $0 < p, q < \infty$ and $s \in \mathbb{R}$. The above equation is also equivalent to the condition that all the derivatives of the Fourier transform $\mathcal{F}_n\varphi$ equal 0 at the origin.

Definition 1.2.3. For $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, the inhomogeneous function space $F^s_{p,q}(\mathbb{R}^n)$ as a subspace of the space $\mathcal{S}'(\mathbb{R}^n)$ is

$$F^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \|\phi * f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} f_k\}_{k>0}\|_{L^p(l^q)} < \infty\}. \tag{26}$$

For $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, the inhomogeneous function space $B^s_{p,q}(\mathbb{R}^n)$ as a subspace of the space $\mathcal{S}'(\mathbb{R}^n)$ is

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|\phi * f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} f_k\}_{k>0}\|_{l^q(L^p)} < \infty\}. \tag{27}$$

Given $L \in \mathbb{N}$, $h \in \mathbb{R}^n$, $f \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, observe the facts that $< \Delta^L_h f, \varphi >=< f, \Delta^L_{-h}\varphi >$ and both spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_0(\mathbb{R}^n)$ are closed under the operation Δ^L_{-h}, then (21) and (22) also suggest that

$$\Delta^L_h f = \sum_{j\in\mathbb{Z}} \Delta^L_h f_j \qquad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n), \tag{28}$$

and

$$\Delta^L_h f = \Delta^L_h(f * \phi) + \sum_{j=1}^{\infty} \Delta^L_h f_j \qquad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n). \tag{29}$$

Furthermore if $0 < p, q < \infty$, $s \in \mathbb{R}$ and f is a function, we define the generalized Gagliardo seminorm of f (see section 2 on page 524 of [31] for the usual Gagliardo seminorm) is

$$[f]_{W^s_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{n+sq}} dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}. \tag{30}$$

And for $1 < p < \infty$, $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the inhomogeneous Sobolev norm of f (see section 1.3.1 of [42] for details) is

$$\|f\|_{L^p_s(\mathbb{R}^n)} := \|\mathcal{F}^{-1}_n((1 + |\xi|^2)^{s/2}\mathcal{F}_n f)\|_{L^p(\mathbb{R}^n)}, \tag{31}$$

where $\|\cdot\|_{L^p(\mathbb{R}^n)}$ is considered via duality. In particular when $p = 2$ we use Plancherel's identity and get

$$\|f\|_{L^2_s(\mathbb{R}^n)} = \|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_n f\|_{L^2(\mathbb{R}^n)}. \tag{32}$$

We recall the definition of the famous Hardy-Littlewood maximal function.

Definition 1.2.4. If a function f is locally integrable on $\mathbb{R}^n$, then

$$\mathcal{M}_n(f)(x) := \operatorname*{ess\,sup}_{\delta>0} ⨍_{B^n(x,\delta)} |f(y)|dy$$

is the n-dimensional Hardy-Littlewood maximal function of f at x.

We also define the Peetre-Fefferman-Stein maximal function for functions whose Fourier transforms have compact supports in $\mathbb{R}^n$.

Definition 1.2.5. If f is a function defined on $\mathbb{R}^n$ whose distributional Fourier transform is compactly supported in the ball $B^n(0,t) \subseteq \mathbb{R}^n$ centered at origin with radius $t > 0$, then the associated n-dimensional Peetre-Fefferman-Stein maximal function of f at x is given by

$$\mathcal{P}_n f(x) = \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|f(x-z)|}{(1+t|z|)^{\frac{n}{r}}}$$

where in most cases of this paper we pick r to be a positive number satisfying either $0 < r < \min\{p,q\}$ or $0 < r < p$.

In general, the n-dimensional Peetre-Fefferman-Stein maximal function can be defined as

$$\mathcal{P}_n f(x) = \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|f(x-z)|}{(1+t|z|)^{a}}$$

for any positive real number a but for the convenience of notations in this paper, we choose $a = \frac{n}{r}$ for the specified r.

Remark 1.2.1. For the Fourier transform, Hardy-Littlewood maximal function and Peetre-Fefferman-Stein maximal function, when we want to apply these operations only to some specific coordinates, we use a subscript number different from the dimension of the ambient space $\mathbb{R}^n$. For example if $f(x) \in \mathcal{S}(\mathbb{R}^n)$ for $n > 1$, let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and we denote $x_1' = (x_2, \cdots, x_n)$ then $x = (x_1, x_1')$ and $f(x) = f(x_1, x_1')$. If the 1-dimensional Fourier transform is done with respect to x_1 then we use the notation

$$\mathcal{F}_1 f(\cdot, x_1')(y_1) := \int_{\mathbb{R}} f(x_1, x_1') e^{-2\pi i x_1 y_1} dx_1,$$

and if the $(n-1)$-dimensional Fourier transform is done with respect to x_1' then we use the notation

$$\mathcal{F}_{n-1} f(x_1, \cdot)(y_1') := \int_{\mathbb{R}^{n-1}} f(x_1, x_1') e^{-2\pi i x_1' \cdot y_1'} dx_1',$$

where $y_1' \in \mathbb{R}^{n-1}$ and $x_1' \cdot y_1'$ is the inner product in $\mathbb{R}^{n-1}$, $dx_1' = dx_2 \cdots dx_n$. Similar notations are used for the inverse Fourier transforms. If we fix $x_1' \in \mathbb{R}^{n-1}$ then the 1-dimensional Hardy-Littlewood maximal function of f, with respect to the first coordinate, centered at $u \in \mathbb{R}$ is given by

$$\mathcal{M}_1(f(\cdot, x_1'))(u) := \operatorname*{ess\,sup}_{\delta > 0} ⨍_{-\delta < t < \delta} |f(u + t, x_1')| dt,$$

and if furthermore the 1-dimensional Fourier transform $\mathcal{F}_1 f(\cdot, x_1')(u)$ is supported in the interval $\{u \in \mathbb{R} : |u| < t\}$, $t \in \mathbb{R}$, then we can also define the associated 1-dimensional Peetre-Fefferman-Stein maximal function of f at u, with respect to the first coordinate, as follows

$$\mathcal{P}_1 f(\cdot, x_1')(u) = \operatorname*{ess\,sup}_{z \in \mathbb{R}} \frac{|f(u - z, x_1')|}{(1 + t|z|)^{\frac{1}{r}}}.$$

We continue introducing more maximal functions below.

Definition 1.2.6. For a function f defined on $\mathbb{R}^n$, let $t > 0$, $r > 0$, $x \in \mathbb{R}^n$, $0 \neq h \in \mathbb{R}^n$, $L \in \mathbb{N}$, and A is the annulus $\{z \in \mathbb{R}^n : 1 \leq |z| < 2\}$, then the following maximal functions are defined

$$(S_t^L f)(x) = \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} |⨍_{\mathbb{S}^{n-1}} (\Delta_{tz}^L f)(x - y) d\mathcal{H}^{n-1}(z)| \cdot (1 + t^{-1}|y|)^{\frac{-n}{r}}, \tag{33}$$

$$(V_t^L f)(x) = \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} |⨍_A (\Delta_{tz}^L f)(x - y) dz| \cdot (1 + t^{-1}|y|)^{\frac{-n}{r}}, \tag{34}$$

$$(D_h^L f)(x) = \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} |(\Delta_h^L f)(x - y)| \cdot (1 + \frac{|y|}{|h|})^{\frac{-n}{r}}. \tag{35}$$

It should be noted that the number r here could be a general positive number, but for most cases in this thesis, we just consider this number r coincides with the number r given in Definition 1.2.5.

2.0 Lemmas And Remarks

In this chapter, we collect some useful results and lemmas. In order to write this thesis in a more self-included fashion, we cite these useful results directly from the literature and the proofs of cited results can be found in their respective source. We also provide succinct proofs for those interesting ones. And then we deduce frequently used remarks right after the closely related citations.

The following lemma is cited from section 1.3.3 of [41] and serves as the main tool we will use to prove the interpolation theorem in chapter 3.

Lemma 2.0.1 (cf. Lemma 1.3.8 of [41])**.** Let F be analytic on the open strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on its closure such that for some $A < \infty$ and $0 \leq \tau_0 < \pi$ we have

$$\log|F(z)| \leq Ae^{\tau_0|\operatorname{Im} z|} \tag{36}$$

for all $z \in \bar{S}$. Then

$$|F(x+iy)| \leq \exp\{\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}[\frac{\log|F(it+iy)|}{\cosh(\pi t)-\cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)}]dt\} \tag{37}$$

whenever $0 < x < 1$, and y is real.

Remark 2.0.1. When $z = x + iy \in S$ and $F(z)$ satisfies the conditions of Lemma 2.0.1, then by writing $\cosh(\pi t) = \frac{1}{2}(e^{\pi t} + e^{-\pi t})$ and using the change of variable $\xi = e^{\pi t}$, we have

$$\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t)-\cos(\pi x)}dt = 1 - x, \tag{38}$$

$$\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t)+\cos(\pi x)}dt = x. \tag{39}$$

Hence using concavity of logarithmic function with respect to measures

$$\frac{\sin(\pi x)}{2}\cdot\frac{1}{\cosh(\pi t)-\cos(\pi x)}dt \qquad \text{and} \qquad \frac{\sin(\pi x)}{2}\cdot\frac{1}{\cosh(\pi t)+\cos(\pi x)}dt,$$

we deduce from (37) that

$$\log|F(x+iy)|\lesssim(1-x)\cdot\log\frac{\sin(\pi x)}{2(1-x)}\int_{-\infty}^{\infty}\frac{|F(it+iy)|}{\cosh(\pi t)-\cos(\pi x)}dt$$
$$+x\cdot\log\frac{\sin(\pi x)}{2x}\int_{-\infty}^{\infty}\frac{|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)}dt. \tag{40}$$

Therefore if we denote for $0 < x < 1$

$$G_0(x,t)=\frac{\sin(\pi x)}{2(1-x)}\cdot\frac{1}{\cosh(\pi t)-\cos(\pi x)} \tag{41}$$

and

$$G_1(x,t)=\frac{\sin(\pi x)}{2x}\cdot\frac{1}{\cosh(\pi t)+\cos(\pi x)}, \tag{42}$$

then we obtain

$$\int_{-\infty}^{\infty}G_0(x,t)dt=\int_{-\infty}^{\infty}G_1(x,t)dt=1 \tag{43}$$

and

$$|F(x+iy)|\lesssim\Big(\int_{-\infty}^{\infty}|F(it+iy)|G_0(x,t)dt\Big)^{1-x}\cdot\Big(\int_{-\infty}^{\infty}|F(1+it+iy)|G_1(x,t)dt\Big)^{x} \tag{44}$$

whenever $F(z)$ satisfies the conditions of Lemma 2.0.1.

To prove the Fourier multiplier theorem in chapter 4, we would like to cite the following fundamental theorem for Banach-valued integral operators found in H. Triebel's book [90]. Let A be a Banach space and $L_0(A)$ denote the set of all A-measurable and A-bounded functions with compact support defined on $\mathbb{R}^n$. Furthermore $L(A_0,A_1)$ is the space of bounded linear operators from A_0 into A_1 equipped with the usual operator norm $\|\cdot\|_{L(A_0,A_1)}$.

Lemma 2.0.2 (cf. Theorem in section 2.2.2 of [90])**.** Let A_0 and A_1 be two reflexive Banach spaces. Let $K(x)$ be a function with values in $L(A_0,A_1)$ defined for almost all $x\in\mathbb{R}^n$. Let $K(x)$ be locally $L(A_0,A_1)$-integrable. Further, it is assumed that there exist numbers $\infty>q\geq 1$, $B>0$, and $C>0$ such that for all $t>0$ and for all y with $|y|\leq B^{-1}$

$$\left(\int_{|x|\geq B}\|K(t(x-y))-K(tx)\|^q_{L(A_0,A_1)}dx\right)^{1/q}\leq C\cdot t^{-n/q}. \tag{45}$$

Let

$$(\mathscr{K}f)(x)=\int_{\mathbb{R}^n}<K(x-y),f(y)>dy,\quad f\in L_0(A_0). \tag{46}$$

Further, it is assumed that there exist numbers p and r with

$$\infty > p > 1, \quad \infty > r > 1, \quad \frac{1}{p} - \frac{1}{r} = 1 - \frac{1}{q}, \tag{47}$$

such that

$$\|\mathscr{K} f\|_{L_r(A_1)} \leq C\|f\|_{L_p(A_0)}, \quad f \in L_0(A_0), \tag{48}$$

where C is the same number as in (45). If

$$1 < s \leq \sigma < \infty, \quad \frac{1}{s} - \frac{1}{\sigma} = 1 - \frac{1}{q}, \tag{49}$$

then the mapping $\mathscr{K}$ (after a uniquely determined extension by continuity) belongs to $L(L_s(A_0), L_\sigma(A_1))$, and the operator norm

$$\|\mathscr{K}\|_{L(L_s(A_0), L_\sigma(A_1))} \leq \alpha C, \tag{50}$$

where C has the same meaning as in (45) and (48), and α depends only on n, B, q, r, p, s, and σ.

Remark 2.0.2. A careful inspection of the proof of the above Theorem in section 2.2.2 of [90] reveals that condition (45) can be replaced by the following Hörmander type assumption: there exists $1 \leq q < \infty$ and $C > 0$ such that

$$\Big(\int_{|x| \geq 2|y|} \|K(x-y) - K(x)\|^q_{L(A_0,A_1)} dx \Big)^{\frac{1}{q}} \lesssim C, \text{ uniformly in } y \in \mathbb{R}^n. \tag{51}$$

And the constant C in (51) has the same meaning as in (45), (48) and (50). Since (51) is trivially true when $y = 0$, without loss of generality we can assume that K given in (51) is defined on $\mathbb{R}^n \setminus \{0\}$.

Remark 2.0.3. Directly from Definition 1.2.5 and the use of some basic inequalities we can obtain useful inequalities for the Peetre-Fefferman-Stein maximal function. If $x, y \in \mathbb{R}^n$ then

$$\begin{aligned}\mathcal{P}_n f(x-y) &= \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|f(x-y-z)|}{(1+t|z+y|)^{n/r}} \cdot \frac{(1+t|y+z|)^{n/r}}{(1+t|z|)^{n/r}} \\ &\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|f(x-y-z)|}{(1+t|z+y|)^{n/r}} \cdot \frac{(1+t|y|)^{n/r}\cdot(1+t|z|)^{n/r}}{(1+t|z|)^{n/r}} \\ &= \mathcal{P}_n f(x)\cdot(1+t|y|)^{n/r}.\end{aligned}$$

And also, using the above inequality gives that

$$\mathcal{P}_n f(x) = \mathcal{P}_n f(x-y+y) \lesssim \mathcal{P}_n f(x-y)\cdot(1+t|y|)^{n/r}.$$

Therefore we reach the conclusion that for $x, y \in \mathbb{R}^n$

$$\mathcal{P}_n f(x)\cdot(1+t|y|)^{-n/r} \lesssim \mathcal{P}_n f(x-y) \lesssim \mathcal{P}_n f(x)\cdot(1+t|y|)^{n/r}, \tag{52}$$

where the constants involved are independent of t and y. We also infer from (52) that if $\mathcal{P}_n f(x)$ vanishes at some point $x \in \mathbb{R}^n$ then the Peetre-Fefferman-Stein maximal function $\mathcal{P}_n f$ vanishes on all of $\mathbb{R}^n$ and hence f vanishes on all of $\mathbb{R}^n$. So for a nonzero function f, the associated Peetre-Fefferman-Stein maximal function $\mathcal{P}_n f$ is positive everywhere. And it would be obvious that if $\mathcal{P}_n f(x) = \infty$ for some $x \in \mathbb{R}^n$ then $\mathcal{P}_n f = \infty$ on all of $\mathbb{R}^n$.

Remark 2.0.4. If f is a function defined on $\mathbb{R}^n$ and its distributional Fourier transform satisfies $spt.\mathcal{F}_n f \subseteq B^n(0,t)$ for $t > 0$ and φ is a Schwartz function whose Fourier transform is compactly supported in $B^n(0,1) \subseteq \mathbb{R}^n$ and $\varphi_{1/t}(x) = t^n\varphi(tx)$ then the distributional Fourier transform of the convolution $\varphi_{1/t} * f$ is also compactly supported in $B^n(0,t) \subseteq \mathbb{R}^n$, and we have for

$$\begin{aligned}\mathcal{P}_n(\varphi_{1/t} * f)(x) &= \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|\varphi_{1/t} * f(x-z)|}{(1+t|z|)^{n/r}} \\ &\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\varphi_{1/t}(y) f(x-z-y)|\cdot(1+t|z+y|)^{n/r}}{(1+t|z|)^{n/r}(1+t|z+y|)^{n/r}} dy \\ &\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\varphi_{1/t}(y) f(x-z-y)|\cdot(1+t|y|)^{n/r}}{(1+t|z+y|)^{n/r}} dy \\ &\lesssim \mathcal{P}_n f(x)\cdot\int_{\mathbb{R}^n} |\varphi_{1/t}(y)|\cdot(1+t|y|)^{n/r} dy,\end{aligned}$$

that is,

$$\mathcal{P}_n(\varphi_{1/t} * f)(x) \lesssim \mathcal{P}_n f(x) \tag{53}$$

for $t > 0, x \in \mathbb{R}^n$ and the constant is independent of t.

For the reader's convenience, we also would like to cite some useful results from the well-known literature [41] and [42] below.

Lemma 2.0.3 (cf. Lemma 2.2.3 of [42])**.** Let $0 < r < \infty$. Then there exist constants C_1 and C_2 such that for all $t > 0$ and for all $\mathscr{C}^1$ functions u on $\mathbb{R}^n$ whose distributional Fourier transform is supported in the ball $|\xi| \leq t$ we have

$$\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \leq C_1 \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}, \tag{54}$$

$$\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \leq C_2 \mathcal{M}_n(|u|^r)(x)^{\frac{1}{r}}, \tag{55}$$

where $\mathcal{M}_n$ denotes the Hardy-Littlewood maximal operator. The constants C_1 and C_2 depend only on the dimension n and r; in particular, they are independent of t.

Remark 2.0.5. The above Lemma 2.0.3 is significant in the sense that it provides a pointwise estimate by the famous Hardy-Littlewood maximal function to a function u whose distributional Fourier transform has compact support in the ball $B^n(0,t)$ of center 0 and radius t and we have the following

$$|u(x)| \lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} = \mathcal{P}_n u(x) \lesssim \mathcal{M}_n(|u|^r)(x)^{\frac{1}{r}}$$

where r can be a positive finite number chosen to satisfy particular needs.

Remark 2.0.6. Assuming sufficient smoothness of the function u as a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$ and iterating (54) of Lemma 2.0.3 repeatedly, since for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, the distributional Fourier transform of $\partial^\alpha u$ is also supported in $B^n(0,t) \subseteq \mathbb{R}^n$, we obtain

$$\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{1}{t^{|\alpha|}}\frac{|\partial^\alpha u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \tag{56}$$

and the constant is independent of t.

Lemma 2.0.4 (cf. Corollary 2.2.4 of [42])**.** Let $0 < p \leq \infty$ and α a multi-index. Then there are constants $C = C(\alpha, n, p)$ and $C' = C(\alpha, n, p)$ such that for all Schwartz functions u on $\mathbb{R}^n$ whose Fourier transform is supported in the ball $B^n(0, t)$, for some $t > 0$, we have

$$\|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \leq C t^{|\alpha|} \|u\|_{L^p(\mathbb{R}^n)} \tag{57}$$

and

$$\|\partial^\alpha u\|_{L^\infty(\mathbb{R}^n)} \leq C' t^{|\alpha| + \frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)}. \tag{58}$$

Remark 2.0.7. Let p, u, t, α be given as in Lemma 2.0.4 and $q \in \mathbb{R}$ satisfies $0 < p \leq q \leq \infty$ then a simple interpolation with (57) and (58) reveals that

$$\|\partial^\alpha u\|_{L^q(\mathbb{R}^n)} \lesssim t^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p(\mathbb{R}^n)}, \tag{59}$$

where u is a Schwartz function on $\mathbb{R}^n$ and $\mathcal{F}_n u$ is supported in the ball $B^n(0, t)$. This inequality is also known as the Plancherel-Polya-Nikol'skij inequality and the constant on the right side of (59) only depends on α, n, p, q. For a more general introduction to the Plancherel-Polya-Nikol'skij inequality, we would like to refer the interested reader to section 1.3 of [93].

The Plancherel-Polya-Nikol'skij inequality can be generalized to the class of sufficiently smooth functions that are also tempered distributions.

Lemma 2.0.5. Suppose $u(x)$ defined on $\mathbb{R}^n$ is a sufficiently smooth function as a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$, and its n-dimensional distributional Fourier transform is supported in the ball $\{\xi \in \mathbb{R}^n : |\xi| \leq t\}$ for some $t > 0$. Assume $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index and $0 < p \leq q \leq \infty$, then

$$\|\partial^\alpha u\|_{L^q(\mathbb{R}^n)} \lesssim t^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p(\mathbb{R}^n)}, \tag{60}$$

and the constant on the right side of (60) only depends on α, n, p, q.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the following conditions

$$0 \le \mathcal{F}_n\varphi \le 1,\ spt.\mathcal{F}_n\varphi \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \text{ and } \mathcal{F}_n\varphi = 1 \text{ if } |\xi| \le 1.$$

Then $\mathcal{F}_n\varphi_{\frac{1}{t}}(\xi) = \mathcal{F}_n\varphi(\frac{\xi}{t})$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \le 2t\}$ and equals 1 if $|\xi| \le t$. Let g be an arbitrary element in $\mathcal{S}(\mathbb{R}^n)$, then

$$< u, g >=< \mathcal{F}_n u, \mathcal{F}_n^{-1} g >=< \mathcal{F}_n u, \mathcal{F}_n\varphi_{\frac{1}{t}} \cdot \mathcal{F}_n^{-1} g > + < \mathcal{F}_n u, (1 - \mathcal{F}_n\varphi_{\frac{1}{t}}) \cdot \mathcal{F}_n^{-1} g > .$$

The Schwartz function $(1 - \mathcal{F}_n\varphi_{\frac{1}{t}}) \cdot \mathcal{F}_n^{-1} g$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| > t\}$ and the distributional Fourier transform $\mathcal{F}_n u$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \le t\}$, then

$$< \mathcal{F}_n u, (1 - \mathcal{F}_n\varphi_{\frac{1}{t}}) \cdot \mathcal{F}_n^{-1} g >= 0.$$

Hence we obtain the following

$$< u, g >=< \mathcal{F}_n u, \mathcal{F}_n^{-1}(\tilde{\varphi}_{\frac{1}{t}} * g) >=< \varphi_{\frac{1}{t}} * u, g >, \tag{61}$$

where $\tilde{\varphi}(x) = \varphi(-x)$. Equation (61) shows that $u(x) = \varphi_{\frac{1}{t}} * u(x)$ for almost every $x \in \mathbb{R}^n$. If $1 \le p \le \infty$, using Hölder's inequality, we have

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |u(y)| \cdot t^n |\varphi(tx - ty)| dy \le t^{\frac{n}{p}} \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \cdot \|u\|_{L^p(\mathbb{R}^n)}, \tag{62}$$

where p' is the Hölder's conjugate of p. If $0 < p < 1$, then we have

$$\begin{aligned} \|u\|_{L^\infty(\mathbb{R}^n)} &\le t^n \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |u(y)|^p \cdot |u(y)|^{1-p} dy \\ &\le t^n \|\varphi\|_{L^\infty(\mathbb{R}^n)} \cdot \|u\|_{L^\infty(\mathbb{R}^n)}^{1-p} \cdot \|u\|_{L^p(\mathbb{R}^n)}^p. \end{aligned} \tag{63}$$

If $0 < p \le q \le \infty$, we then use (62) and (63) to get

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^q dx &= \int_{\mathbb{R}^n} |u(x)|^{q-p} \cdot |u(x)|^p dx \\ &\lesssim t^{\frac{n}{p}(q-p)} \|u\|_{L^p(\mathbb{R}^n)}^{q-p} \cdot \|u\|_{L^p(\mathbb{R}^n)}^p \\ &= t^{\frac{n}{p}(q-p)} \|u\|_{L^p(\mathbb{R}^n)}^q. \end{aligned} \tag{64}$$

Thus we can obtain (60) if $|\alpha| = 0$. When u is sufficiently smooth and $|\alpha| > 0$, we use Remark 2.0.6 and Lemma 2.0.3 to obtain

$$|\partial^\alpha u(x)| \lesssim \mathcal{P}_n(\partial^\alpha u)(x) \lesssim t^{|\alpha|}\mathcal{P}_n(u)(x) \lesssim t^{|\alpha|}\mathcal{M}_n(|u|^r)(x)^{\frac{1}{r}}, \tag{65}$$

where the constants depend on n and r. By picking $r < q$ and invoking the mapping property of the Hardy-Littlewood maximal function, we have

$$\|\partial^\alpha u\|_{L^q(\mathbb{R}^n)} \lesssim t^{|\alpha|}\|\mathcal{M}_n(|u|^r)(x)^{\frac{1}{r}}\|_{L^q(\mathbb{R}^n)} \lesssim t^{|\alpha|}\|u\|_{L^q(\mathbb{R}^n)} \lesssim t^{|\alpha|+n(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\mathbb{R}^n)}, \tag{66}$$

where constants depend only on n, r, p, q and are independent of t. □

An example that shows the advantage of Lemma 2.0.5 over Remark 2.0.7 can be given below. Consider the function $f_j = f * \psi_{2^{-j}}$ where $f \in \mathcal{S}'(\mathbb{R}^n)$, $j \in \mathbb{Z}$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (16), (17). Then for each $j \in \mathbb{Z}$, the distributional Fourier transform of f_j is supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| < 2^{j+1}\}$ and by Theorem 2.3.20 of [41], f_j is a smooth function which has at most polynomial growth at infinity. Then f_j is a smooth tempered distribution and applying Lemma 2.0.5, we have

$$f_j(x) \lesssim \|f_j\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{jn/p}\|f_j\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 0 < p < \infty.$$

Lemma 2.0.6 (cf. Theorem 5.6.6 of [41])**.** For $1 < p, r < \infty$ the Hardy-Littlewood maximal function $\mathcal{M}$ satisfies the vector-valued inequalities

$$\|(\sum_j |\mathcal{M}(f_j)|^r)^{\frac{1}{r}}\|_{L^{1,\infty}} \le C_n'(1 + (r-1)^{-1})\|(\sum_j |f_j|^r)^{\frac{1}{r}}\|_{L^1}, \tag{67}$$

$$\|(\sum_j |\mathcal{M}(f_j)|^r)^{\frac{1}{r}}\|_{L^p} \le C_n c(p,r)\|(\sum_j |f_j|^r)^{\frac{1}{r}}\|_{L^p}, \tag{68}$$

where $c(p,r) = (1 + (r-1)^{-1})(p + (p-1)^{-1})$.

Remark 2.0.8. Recall that for each $l \in \mathbb{Z}$ the distributional Fourier transform of $f_l := \psi_{2^{-l}} * f$ is supported in the compact annulus $2^{l-1} \leq |\xi| < 2^{l+1}$ and

$$\mathcal{P}_n f_l(x) = \operatorname*{ess\,sup}_{z \in \mathbb{R}^n} \frac{|\psi_{2^{-l}} * f(x-z)|}{(1+2^{l+1}|z|)^{\frac{n}{r}}}.$$

Applying Lemma 2.0.3 first and then Lemma 2.0.6 and assuming that $s \in \mathbb{R}, 0 < p, q < \infty, 0 < r < \min\{p, q\}$ we obtain

$$\begin{aligned}
&(\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} \mathcal{P}_n f_l(y)|^q)^{\frac{p}{q}} dy)^{\frac{1}{p}} \\
\lesssim &(\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} \mathcal{M}_n(|f_l|^r)(y)^{\frac{1}{r}}|^q)^{\frac{p/r}{q/r}} dy)^{\frac{1/r}{p/r}} \\
\lesssim &(\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} f_l(y)|^q)^{\frac{p}{q}} dy)^{\frac{1}{p}}. \qquad (69)
\end{aligned}$$

And by definition of the Peetre-Fefferman-Stein maximal function, $f_l(y) \lesssim \mathcal{P}_n f_l(y)$ for every $y \in \mathbb{R}^n$ and $l \in \mathbb{Z}$ thus

$$(\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} f_l(y)|^q)^{\frac{p}{q}} dy)^{\frac{1}{p}} \lesssim (\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} \mathcal{P}_n f_l(y)|^q)^{\frac{p}{q}} dy)^{\frac{1}{p}}. \qquad (70)$$

Therefore we have reached the conclusion that for $s \in \mathbb{R}, 0 < p, q < \infty$, the following is true

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} f_l(y)|^q)^{\frac{p}{q}} dy)^{\frac{1}{p}} \sim (\int_{\mathbb{R}^n} (\sum_{l=-\infty}^{\infty} |2^{ls} \mathcal{P}_n f_l(y)|^q)^{\frac{p}{q}} dy)^{\frac{1}{p}}. \qquad (71)$$

Next, we provide useful lemmas in the proofs of our main theorems.

Lemma 2.0.7. Let $x, y \in \mathbb{R}^n$, $L \in \mathbb{N}$, $r > 0$ and for each $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$, if $h \in \mathbb{R}^n$ and $|h| \sim 2^{-k}$ and $f_j = \psi_{2^{-j}} * f$, then we have the following two estimates

$$|(\varDelta_h^L f_j)(x-y)| \cdot (1 + \frac{|y|}{|h|})^{\frac{-n}{r}} \lesssim 2^{(j-k)L} (1 + 2^{j-k})^{\frac{n}{r}} \mathcal{P}_n f_j(x), \qquad (72)$$

$$|(\varDelta_h^L f_j)(x-y)| \cdot (1 + \frac{|y|}{|h|})^{\frac{-n}{r}} \lesssim (1 + 2^{j-k})^{\frac{n}{r}} \mathcal{P}_n f_j(x), \qquad (73)$$

and constants are independent of y and h.

Proof. To prove (72), we use the mean value theorem and the iteration formula (9) consecutively and obtain

$$|(\Delta_h^L f_j)(x-y)| \lesssim \sum_{|\alpha|=L} |\partial^\alpha f_j(x-y+\sum_{l=1}^{L} t_{\alpha,l}h)| \cdot |h|^L, \tag{74}$$

where α represents a multi-index and each $t_{\alpha,l}$ is in the interval $(0,1)$. For each multi-index α with $|\alpha| = L$ and since $\mathcal{F}_n \partial^\alpha f_j(\xi)$ is supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| < 2^{j+1}\}$, we use the basic inequality

$$(1+2^{j+1}|y-\sum_{l=1}^{L} t_{\alpha,l}h|)^{\frac{n}{r}} \lesssim (1+2^{j-k}\cdot 2^k|y|+2^j|\sum_{l=1}^{L} t_{\alpha,l}h|)^{\frac{n}{r}} \lesssim (1+2^k|y|)^{\frac{n}{r}} \cdot (1+2^{j-k})^{\frac{n}{r}} \tag{75}$$

and obtain that

$$\begin{aligned}
&|\partial^\alpha f_j(x-y+\sum_{l=1}^{L} t_{\alpha,l}h)| \\
=&\frac{|\partial^\alpha f_j(x-y+\sum_{l=1}^{L} t_{\alpha,l}h)|}{(1+2^{j+1}|y-\sum_{l=1}^{L} t_{\alpha,l}h|)^{\frac{n}{r}}} \cdot (1+2^{j+1}|y-\sum_{l=1}^{L} t_{\alpha,l}h|)^{\frac{n}{r}} \\
\lesssim&(1+2^k|y|)^{\frac{n}{r}} \cdot (1+2^{j-k})^{\frac{n}{r}} \mathcal{P}_n(\partial^\alpha f_j)(x).
\end{aligned} \tag{76}$$

We also use Remark 2.0.6 to get $\mathcal{P}_n(\partial^\alpha f_j)(x) \lesssim 2^{jL}\mathcal{P}_n f_j(x)$. Since $|h| \sim 2^{-k}$, we also have

$$(1+2^k|y|)^{\frac{n}{r}} \lesssim (1+\frac{|y|}{|h|})^{\frac{n}{r}}.$$

We insert these estimates and (76) into (74) and obtain (72). To prove (73), we use (10), (11) and Remark 2.0.3 to get

$$|(\Delta_h^L f_j)(x-y)| \lesssim \sum_{l=0}^{L} |f_j(x-y+lh)| \lesssim \sum_{l=0}^{L} (1+2^j|y-lh|)^{\frac{n}{r}} \mathcal{P}_n f_j(x). \tag{77}$$

Since $|h| \sim 2^{-k}$, then

$$(1+2^j|y-lh|)^{\frac{n}{r}} \lesssim (1+2^k|y|)^{\frac{n}{r}} \cdot (1+2^{j-k})^{\frac{n}{r}} \lesssim (1+\frac{|y|}{|h|})^{\frac{n}{r}} \cdot (1+2^{j-k})^{\frac{n}{r}}. \tag{78}$$

Inserting (78) into (77) yields (73). □

Lemma 2.0.8. Let $f \in C_c^\infty(\mathbb{R}^n)$, $\tau \in [1,2]$ and $\theta \in \mathbb{S}^{n-1}$ satisfy the condition that for all $\xi \in spt.f$, there are positive real numbers a and b, which can be sufficiently small and are independent of τ and θ, such that $a \le |\theta \cdot \xi| \le b$. Assume L is a positive integer. Then for any positive integer N, there exists a positive constant C such that

$$|\mathcal{F}_n(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L})(x)| \le \frac{C}{(1+|x|)^N} \qquad \text{for all } x \in \mathbb{R}^n. \tag{79}$$

Most importantly, the constant C may depend on f, L, N, a, b but is independent of $\tau \in [1,2]$ and $\theta \in \mathbb{S}^{n-1}$.

Proof. If $|x| < 1$, we use Taylor expansion and obtain

$$(e^{2\pi i\tau\theta\cdot\xi}-1)^L = (2\pi i\tau\theta\cdot\xi)^L \cdot (1 + O(2\pi i\tau\theta\cdot\xi)). \tag{80}$$

Since $\tau \in [1,2]$ and if $\xi \in spt.f$, a and b are sufficiently small, then we have

$$|(e^{2\pi i\tau\theta\cdot\xi}-1)^L| \ge C_1 a^L(1 - C_2 b) > 0, \tag{81}$$

where C_1 and C_2 are constants independent of τ and θ. Hence we have

$$|\mathcal{F}_n(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L})(x)| \le \int_{\mathbb{R}^n} \frac{|f(\xi)|}{|(e^{2\pi i\tau\theta\cdot\xi}-1)^L|} d\xi \lesssim \|f\|_{L^1(\mathbb{R}^n)} \lesssim \frac{C}{(1+|x|)^N} \tag{82}$$

if $|x| < 1$ and the constant C is independent of $\tau \in [1,2]$ and $\theta \in \mathbb{S}^{n-1}$.

If $|x| \ge 1$, without loss of generality we can assume that $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $|x_1| = \max\{|x_1|, |x_2|, \cdots, |x_n|\} > 0$. Using integration by parts with respect to ξ_1, the condition that $f \in C_c^\infty(\mathbb{R}^n)$ and the basic formula

$$e^{-2\pi i x_1\xi_1} = \frac{\partial}{\partial\xi_1}(\frac{e^{-2\pi i x_1\xi_1}}{-2\pi i x_1}),$$

we can obtain

$$\mathcal{F}_n(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L})(x) = \frac{1}{2\pi i x_1}\int_{\mathbb{R}^n} \frac{\partial}{\partial\xi_1}(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L}) \cdot e^{-2\pi i x\cdot\xi} d\xi.$$

We can iterate the integration by parts with respect to ξ_1 for N times and obtain

$$\mathcal{F}_n(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L})(x) = \frac{1}{(2\pi i x_1)^N}\int_{\mathbb{R}^n} \frac{\partial^N}{\partial\xi_1^N}(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L}) \cdot e^{-2\pi i x\cdot\xi} d\xi. \tag{83}$$

By direct calculation, we have

$$|\frac{\partial^k}{\partial \xi_1^k}(e^{2\pi i\tau\theta\cdot\xi}-1)^{-L}| \lesssim |\theta_1|^k \sum_{j=0}^{k} |(e^{2\pi i\tau\theta\cdot\xi}-1)^{-L-j}| \lesssim \sum_{j=0}^{k} |(e^{2\pi i\tau\theta\cdot\xi}-1)^{-L-j}| \tag{84}$$

for every nonnegative integer k and the constants are independent of $\tau \in [1,2]$ and $\theta \in \mathbb{S}^{n-1}$. Furthermore, using (80) and (81), if $\xi \in spt.f$ we can choose sufficiently small positive numbers a and b so that $a \leq |\theta \cdot \xi| \leq b$ implies

$$|(e^{2\pi i\tau\theta\cdot\xi}-1)^{L+j}| \geq C_3 > 0 \qquad \text{for all } 0 \leq j \leq N, \tag{85}$$

and the constant C_3 is independent of $\tau \in [1,2]$ and $\theta \in \mathbb{S}^{n-1}$. Therefore Leibniz rule, (83), (84) and (85) tell us that

$$|\mathcal{F}_n(\frac{f(\xi)}{(e^{2\pi i\tau\theta\cdot\xi}-1)^L})(x)| \lesssim \frac{1}{|x|^N}\sum_{k=0}^{N}\|\frac{\partial^k f}{\partial \xi_1^k}\|_{L^1(\mathbb{R}^n)} \lesssim \frac{C}{(1+|x|)^N} \tag{86}$$

if $|x| \geq 1$ and the constants are independent of $\tau \in [1,2]$ and $\theta \in \mathbb{S}^{n-1}$. The proof of Lemma 2.0.8 is now complete. □

Lemma 2.0.9. Suppose f is a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$. Recall that $f_l(x) = \psi_{2^{-l}} * f(x)$ for every $l \in \mathbb{Z}$, $x = (x_1, x_1') \in \mathbb{R}^n$. Then for every fixed $x_1' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$, the smooth function $x_1 \in \mathbb{R} \mapsto f_l(x_1, x_1')$ is an element in $\mathcal{S}'(\mathbb{R})$ and its distributional Fourier transform $\mathcal{F}_1 f_l(\cdot, x_1')$ is supported in the set $\{u \in \mathbb{R} : |u| \leq 2^{l+1}\}$ and hence the associated 1-dimensional Peetre-Fefferman-Stein maximal function can be defined as

$$\mathcal{P}_1 f_l(\cdot, x_1')(u) = \operatorname*{ess\,sup}_{z\in\mathbb{R}} \frac{|f_l(u-z, x_1')|}{(1+2^{l+1}|z|)^{\frac{1}{r}}}. \tag{87}$$

Proof. Since $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi_{2^{-l}} \in \mathcal{S}(\mathbb{R}^n)$, by Theorem 2.3.20 of [41], f_l is a smooth function and there exist positive constants a and b such that

$$|f_l(x_1, x_1')| \le a(1 + |x_1| + |x_1'|)^b \le a(1 + |x_1'|)^b(1 + |x_1|)^b.$$

This inequality shows for fixed $x_1' \in \mathbb{R}^{n-1}$, the smooth function $x_1 \in \mathbb{R} \mapsto f_l(x_1, x_1')$ is in $\mathcal{S}'(\mathbb{R})$. To prove the distributional Fourier transform $\mathcal{F}_1 f_l(\cdot, x_1')$ is supported in the set $\{u \in \mathbb{R} : |u| \le 2^{l+1}\}$, we find a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ so that $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to f in the sense of $\mathcal{S}'(\mathbb{R}^n)$. Next, we establish the equality

$$< \mathcal{F}_1 f_l(\cdot, x_1'), g >= \lim_{k\to\infty} < \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1')), g > \tag{88}$$

for every $g \in \mathcal{S}(\mathbb{R})$. With an argument like above, the smooth function $x_1 \in \mathbb{R} \to \psi_{2^{-l}} * \varphi_k(x_1, x_1')$ is an element of $\mathcal{S}'(\mathbb{R})$, thus we have the following

$$\begin{aligned} & < \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1')), g > \\ =& \int_{\mathbb{R}} \psi_{2^{-l}} * \varphi_k(u, x_1') \cdot \mathcal{F}_1 g(u) du \\ =& \int_{\mathbb{R}^n} \varphi_k(y_1, y_1') \int_{\mathbb{R}} \psi_{2^{-l}}(u - y_1, x_1' - y_1') \mathcal{F}_1 g(u) du dy_1 dy_1'. \end{aligned} \tag{89}$$

It is not hard to check by direct calculations that for fixed $x_1' \in \mathbb{R}^{n-1}$, the function $(y_1, y_1') \in \mathbb{R}^n \mapsto \int_{\mathbb{R}} \psi_{2^{-l}}(u - y_1, x_1' - y_1') \mathcal{F}_1 g(u) du$ is an element of $\mathcal{S}(\mathbb{R}^n)$. Therefore we have

$$\begin{aligned} & \lim_{k\to\infty} < \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1')), g > \\ =& \lim_{k\to\infty} < \varphi_k, \int_{\mathbb{R}} \psi_{2^{-l}}(u - \cdot, x_1' - \cdot) \mathcal{F}_1 g(u) du > \\ =& < f, \int_{\mathbb{R}} \psi_{2^{-l}}(u - \cdot, x_1' - \cdot) \mathcal{F}_1 g(u) du > . \end{aligned} \tag{90}$$

We now justify the equation

$$< f, \int_{\mathbb{R}} \psi_{2^{-l}}(u - \cdot, x_1' - \cdot) \mathcal{F}_1 g(u) du >= \int_{\mathbb{R}} < f, \psi_{2^{-l}}(u - \cdot, x_1' - \cdot) > \mathcal{F}_1 g(u) du. \tag{91}$$

Since f is a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$, (91) requires the Riemann sums of the integral $\int_{\mathbb{R}} \psi_{2^{-l}}(u - y_1, x_1' - y_1') \mathcal{F}_1 g(u) du$ converges to that integral in Schwartz seminorms with respect to $y = (y_1, y_1') \in \mathbb{R}^n$. For a sufficiently large $N \in \mathbb{N}$, we consider the interval $[-N, N]$ and decompose it into $2N^2$ disjoint subintervals $\{I_m\}_{m=1}^{2N^2}$ of equal length $\frac{1}{N}$ and pick $u_m \in I_m$, then $[-N,N]=\bigcup_{m=1}^{2N^2} I_m$. The difference between the integral and its N-th Riemann sum is given by

$$\int_{\mathbb{R}} \psi_{2^{-l}}(u-y_1,x_1'-y_1')\mathcal{F}_1 g(u)du - \sum_{m=1}^{2N^2} |I_m|\cdot \psi_{2^{-l}}(u_m-y_1,x_1'-y_1')\mathcal{F}_1 g(u_m) \tag{92}$$

and can be written as a sum of the following two terms,

$$\int_{|u|>N} \psi_{2^{-l}}(u-y_1,x_1'-y_1')\mathcal{F}_1 g(u)du \tag{93}$$

and

$$\sum_{m=1}^{2N^2}\int_{I_m} \psi_{2^{-l}}(u-y_1,x_1'-y_1')\mathcal{F}_1 g(u) - \psi_{2^{-l}}(u_m-y_1,x_1'-y_1')\mathcal{F}_1 g(u_m)du. \tag{94}$$

It is sufficient to show both (93) and (94) converge to zero in Schwartz seminorms with respect to y as $N\to\infty$. Let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_1')$ and $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$ denote multi-indices. To estimate the Schwartz seminorms of (93), we compute as follows

$$\begin{aligned}
&|y^\alpha\cdot\partial_y^\beta(\int_{|u|>N}\psi_{2^{-l}}(u-y_1,x_1'-y_1')\mathcal{F}_1 g(u)du)|\\
\lesssim&|y_1|^{\alpha_1}\cdot|y_1'|^{|\alpha_1'|}\cdot\int_{|u|>N}|(\partial^\beta\psi)(2^l u-2^l y_1,2^l x_1'-2^l y_1')\mathcal{F}_1 g(u)|du\\
\lesssim&|y_1|^{\alpha_1}\cdot|y_1'|^{|\alpha_1'|}\cdot\int_{|u|>N}\frac{1}{(1+4^l|u-y_1|^2+4^l|x_1'-y_1'|^2)^{2M}}\cdot\frac{1}{(1+u^2)^M}du\\
\lesssim&|y_1|^{\alpha_1}\cdot|y_1'|^{|\alpha_1'|}\cdot\int_{|u|>N}\frac{1}{(1+|u-y_1|^2)^M(1+|x_1'-y_1'|^2)^M(1+u^2)^M}du,
\end{aligned}\tag{95}$$

for some arbitrarily large positive integer M. We apply the following estimates

$$(1+y_1^2)^{\frac{M}{2}}\le 2^{\frac{M}{2}}(1+|u-y_1|^2)^{\frac{M}{2}}\cdot(1+u^2)^{\frac{M}{2}}, \tag{96}$$

$$\frac{1}{(1+|u-y_1|^2)^{\frac{M}{2}}}\le 1,\ \text{and}\ |y_1'|^{|\alpha_1'|}\le(|y_1'-x_1'|^2+|x_1'|^2)^{\frac{|\alpha_1'|}{2}}, \tag{97}$$

then we can estimate (95) from above by

$$\frac{|y_1|^{\alpha_1}\cdot|y_1'|^{|\alpha_1'|}}{(1+y_1^2)^{\frac{M}{2}}(1+|x_1'-y_1'|^2)^M}\cdot\int_{|u|>N}\frac{1}{(1+u^2)^{\frac{M}{2}}}du, \tag{98}$$

where the constant may depend on $\alpha,\beta,l,\psi,g,M,x_1'$. And (98) tends to zero as $N\to\infty$ uniformly in $y\in\mathbb{R}^n$ if we pick $M\in\mathbb{N}$ to be sufficiently large. This shows the Schwartz seminorms of (93) with respect to y converge to zero as $N\to\infty$. To estimate the Schwartz seminorms of each term in the summation of (94), we use Mean Value Theorem with respect to u and obtain

$$\begin{aligned}
&|y^\alpha \cdot \partial_y^\beta (\int_{I_m} \psi_{2^{-l}}(u-y_1, x_1'-y_1')\mathcal{F}_1 g(u) - \psi_{2^{-l}}(u_m - y_1, x_1'-y_1')\mathcal{F}_1 g(u_m) du)| \\
\lesssim & |y^\alpha| \int_{I_m} |\frac{d}{du}[(\partial^\beta \psi)(2^l u - 2^l y_1, 2^l x_1' - 2^l y_1')\mathcal{F}_1 g(u)]\Big|_{u=t_m} | \cdot |u-u_m| du \\
\lesssim & \frac{|y_1|^{\alpha_1}\cdot |y_1'|^{|\alpha_1'|}}{N} \int_{I_m} \frac{1}{(1+4^l|t_m - y_1|^2 + 4^l |x_1'-y_1'|^2)^{2M}} \cdot \frac{1}{(1+t_m^2)^M} du \\
\lesssim & \frac{|y_1|^{\alpha_1}\cdot |y_1'|^{|\alpha_1'|}}{N} \int_{I_m} \frac{1}{(1+|x_1'-y_1'|^2)^M (1+|t_m-y_1|^2)^M (1+t_m^2)^M} du,
\end{aligned} \tag{99}$$

for some $t_m \in I_m$ between u and u_m, and the constants do not rely on m. When N is large, we have $|u - t_m| \le \frac{1}{N} < \frac{1}{2}$ and $u^2 \le 2t_m^2 + \frac{1}{2}$. We also use the inequality $(1+y_1^2)^{\frac{M}{2}} \le 2^{\frac{M}{2}}(1+|t_m - y_1|^2)^{\frac{M}{2}} \cdot (1+t_m^2)^{\frac{M}{2}}$ and then we can estimate (99) from above by

$$\frac{|y_1|^{\alpha_1}\cdot |y_1'|^{|\alpha_1'|}}{N(1+y_1^2)^{\frac{M}{2}}(1+|x_1'-y_1'|^2)^M} \cdot \int_{I_m} \frac{1}{(1+u^2)^{\frac{M}{2}}} du. \tag{100}$$

Summing over $m = 1, \cdots, 2N^2$ and taking supremum over $y \in \mathbb{R}^n$ yield that the Schwartz seminorms of (94) with respect to y can be estimated from above by

$$\frac{1}{N}\int_{[-N,N]} \frac{1}{(1+u^2)^{\frac{M}{2}}} du, \tag{101}$$

where M is a sufficiently large positive integer and the constant may depend on α, β, l, ψ, g, M, x_1'. And (101) tends to zero as $N \to \infty$. Therefore the validity of equation (91) has been proved. Furthermore from equation (2.3.21) on page 127 of [41], we know $< f, \psi_{2^{-l}}(u - \cdot, x_1' - \cdot) >$ can be identified with the function $f_l(u, x_1')$. By inserting this identification into (91) and combining the result with (90), we have obtained

$$< \mathcal{F}_1 f_l(\cdot, x_1'), g >=< f_l(\cdot, x_1'), \mathcal{F}_1 g >= \lim_{k\to\infty} < \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1')), g > . \tag{102}$$

Since $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi_k \in C_c^\infty(\mathbb{R}^n)$, we have

$$\psi_{2^{-l}} * \varphi_k(x) = \mathcal{F}_1^{-1}[\mathcal{F}_{n-1}^{-1}[\mathcal{F}_n\psi(2^{-l}\xi)\mathcal{F}_n\varphi_k(\xi)](x_1')](x_1), \tag{103}$$

where $\xi = (\xi_1, \xi_1') \in \mathbb{R}^n$ and the $(n-1)$-dimensional inverse Fourier transform is done with respect to $\xi_1' \in \mathbb{R}^{n-1}$ and the 1-dimensional inverse Fourier transform is done with respect to $\xi_1 \in \mathbb{R}$. Hence for every $k \in \mathbb{N}$, we have

$$
\begin{aligned}
& < \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1')), g > \\
=& \int_{\mathbb{R}} \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1'))(u) \cdot g(u) du \\
=& \int_{\mathbb{R}} \mathcal{F}_{n-1}^{-1}[\mathcal{F}_n \psi(2^{-l}u, 2^{-l}\xi_1') \mathcal{F}_n \varphi_k(u, \xi_1')](x_1') \cdot g(u) du \\
=& \mathcal{F}_{n-1}^{-1}[\int_{\mathbb{R}} \mathcal{F}_n \psi(2^{-l}u, 2^{-l}\xi_1') \mathcal{F}_n \varphi_k(u, \xi_1') g(u) du](x_1'),
\end{aligned}
\tag{104}
$$

and from (104) we also see that if $spt.g$ is contained in the complement of the set $\{u \in \mathbb{R} : |u| \leq 2^{l+1}\}$, then $< \mathcal{F}_1(\psi_{2^{-l}} * \varphi_k(\cdot, x_1')), g >= 0$ for every $k \in \mathbb{N}$. Therefore the distributional Fourier transform $\mathcal{F}_1 f_l(\cdot, x_1')$ is supported in the set $\{u \in \mathbb{R} : |u| \leq 2^{l+1}\}$ and the proof of Lemma 2.0.9 is now concluded. □

3.0 A Complex Interpolation Theorem For Homogeneous Triebel-Lizorkin Spaces

3.1 Chapter Introduction

In section 2.4.7 of [93], H. Triebel proved an interpolation space theorem for inhomogeneous Triebel-Lizorkin spaces where the parameters are in their full ranges. In this section, we state an interpolation theorem for homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ where $s \in \mathbb{R}, 1 < p, q < \infty$ and we provide a direct proof of this interpolation theorem using the complex method and duality. It has been shown that when restricted to the boundary of the open strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, the growth of bound of the analytic family $\{T_z\}_{z\in\mathbb{C}}$ of linear operators can be exponential in terms of $\operatorname{Im} z$. We limit ourselves to the case when $1 < p, q < \infty$ because the duality of the homogeneous space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ becomes complicated when one of p, q is less than or equal to 1 or becomes ∞. The main result is the Theorem 3.1.1 below and Theorem 3.1.1 is new in the sense that it provides modern and clearer conditions on the family of linear operators $\{T_z\}_{z\in\mathbb{C}}$ while considering the homogeneous Triebel-Lizorkin spaces.

Theorem 3.1.1. Assume that $\{T_z\}_{z\in\mathbb{C}}$ is a family of linear operators defined on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$ and taking values in the set of $\mathcal{S}'(\mathbb{R}^n)$-analytic functions in the open strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and we also assume the family $\{T_z\}_{z\in\mathbb{C}}$ satisfy the condition: for every $\psi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform has a compact support in $\mathbb{R}^n$ and for all u and v in $\mathcal{S}'(\mathbb{R}^n)$, the map

$$z \longmapsto \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \psi_{2^{-k}} * T_z(u)(x) \cdot \overline{\psi_{2^{-k}} * v(x)} dx$$

is analytic in the open strip S and continuous on its closure $\bar{S}$. Let $s_0, s_1 \in \mathbb{R}$, $1 < p_0, p_1, q_0, q_1 < \infty$ and suppose that M_0 and M_1 are positive functions defined on $\mathbb{R}$ such that for some $0 \le A < \infty$ and $0 \le B < \pi$ we have

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}} e^{-B|t|} \cdot M_j(t) \le A < \infty \qquad \text{for} \qquad j = 0, 1. \tag{105}$$

Let $0 < \theta < 1$ and define s, p, q by the equations

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{106}$$

Suppose $f \in \mathcal{S}(\mathbb{R}^n)$ is an arbitrary Schwartz function and we have

$$\|T_{it}(f)\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)} \lesssim M_0(t)\|f\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)}, \tag{107}$$

$$\|T_{1+it}(f)\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim M_1(t)\|f\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}, \tag{108}$$

then we have

$$\|T_\theta(f)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \lesssim M(\theta)\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \tag{109}$$

where

$$M(\theta) = \exp\Big\{\frac{\sin(\pi\theta)}{2}\int_{-\infty}^{\infty}\Big[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi\theta)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi\theta)}\Big]dt\Big\}. \tag{110}$$

Thus by density T_θ has a unique bounded extension from $\dot{F}^s_{p,q}(\mathbb{R}^n)$ to $\dot{F}^s_{p,q}(\mathbb{R}^n)$ when s, p, q are as in (106).

Remark 3.1.1. If $0 < \theta < 1$, we can be certain that $M(\theta)$ given in (110) is a positive finite number. Using (38), (39) and assumption (105), the exponent in (110) is dominated by

$$\log A + \sin(\pi\theta)\int_0^{\infty} \frac{2Bt}{e^{\pi t} + e^{-\pi t} - 2\cos(\pi\theta)} + \frac{2Bt}{e^{\pi t} + e^{-\pi t} + 2\cos(\pi\theta)}dt, \tag{111}$$

since the integrand in (110) can be dominated by an even function in t. Let

$$m(t) := \frac{2Bt}{e^{\pi t} + e^{-\pi t} - 2\cos(\pi\theta)} + \frac{2Bt}{e^{\pi t} + e^{-\pi t} + 2\cos(\pi\theta)}$$

then $m(t)$ is positive for all $t > 0$ and we choose $t_0 > 0$ so that $t > t_0$ implies $e^{-\pi t} < \frac{1}{4}$. We split the integral in (111) into two parts $\int_0^{t_0} m(t)dt$ and $\int_{t_0}^{\infty} m(t)dt$. Since $0 < \theta < 1$ and $m(t)$ is continuous and bounded on $[0, t_0]$, the first part is convergent. As for the second part we have

$$\int_{t_0}^{\infty} m(t)dt \lesssim \int_{t_0}^{\infty} \frac{2Bte^{-\pi t}}{1 - 2\cos(\pi\theta)e^{-\pi t}} + \frac{2Bte^{-\pi t}}{1 + 2\cos(\pi\theta)e^{-\pi t}}dt \lesssim \int_{t_0}^{\infty} 8Bt \cdot e^{-\pi t}dt < \infty,$$

due to the choice of t_0 and the assumption that $0 \le B < \pi$. And the requirement $B < \pi$ can be seen from (190) and (193) below.

We also introduce other mathematicians' results related to the interpolation of function spaces. In [58], P.C. Kunstmann introduced the l^q-interpolation method which came from the modification of the real interpolation method for Banach spaces and the interpolation theory presented in that paper was related to the more abstract homogeneous generalized Triebel-Lizorkin spaces $\dot{X}^{\theta}_{q,A}$ where $0 < \theta < 1$. In [107], W. Yuan introduced the inhomogeneous Hausdorff type Besov space $\mathcal{B}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $1 \leq q < \infty$ and Hausdorff type Triebel-Lizorkin space $\mathcal{F}H^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $1 < q < \infty$, which are the predual spaces of Besov-Morrey space $\mathcal{B}M^{s,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin-Morrey space $\mathcal{F}M^{s,\tau}_{p,q}(\mathbb{R}^n)$, and the complex interpolation of these spaces were also obtained. In [71], T. Noi and Y. Sawano introduced the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, the variable exponent inhomogeneous Triebel-Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and the variable exponent inhomogeneous Besov-Lipschitz space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, furthermore they also give the generalized Hölder's inequality and Minkowski's inequality for variable exponent Lebesgue spaces, moreover complex interpolation spaces for $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are also considered as the main results of [71]. In [46], D. I. Hakim, T. Nogayama, and Y. Sawano gave the definition of the Lizorkin-Triebel-Morrey space $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ and they also proved the interpolation space theorem that says $[\mathcal{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^n), \mathcal{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n)]_\theta = \mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ in the sense of equivalent norms under the given conditions in [46]. In [110], C. Zhuo, M. Hovemann, and W. Sickel defined the Lizorkin-Triebel-Morrey space $\mathcal{E}^s_{u,p,q}(\Omega)$ on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ for $1 \leq p \leq u < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, and the authors also proved there exists a linear and bounded extension operator from $\mathcal{E}^s_{u,p,q}(\Omega)$ into the space $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ under the given conditions, finally a complex interpolation space theorem, which states that $[\mathcal{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathcal{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\theta = \mathring{\mathcal{E}}^s_{u,p,q}(\Omega)$, was formulated and proven with the Lemarié-Rieusset condition $p_0u_1 = p_1u_0$ and $\mathring{\mathcal{E}}^s_{u,p,q}(\Omega)$ denotes the closure with respect to $\mathcal{E}^s_{u,p,q}(\Omega)$ of the set of all smooth functions f such that $\partial^\alpha f \in \mathcal{E}^s_{u,p,q}(\Omega)$ for all multi-indices α. In [87], W. Sickel, L. Skrzypczak, and J. Vybíral studied the complex interpolation of weighted Besov and Lizorkin-Triebel spaces equipped with local Muckenhoupt weights w_0 and w_1, and the authors also obtained results on complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces on $\mathbb{R}^d$. In [11], M. Bownik introduced the definition of the anisotropic Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n, A, \mu)$, where A is an $n \times n$ real matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$ and is often called an expansive dilation, and μ is a doubling measure respecting the action of A, furthermore the author also identified the dual space of the anisotropic space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n, A)$ and proved real interpolation and complex interpolation results of anisotropic Triebel-Lizorkin spaces with the help of Calderón products. In [56], V. L. Krepkogorskiĭ showed the spaces $BL^{s,k}_{p,q}$ can be obtained from the Besov spaces $B^s_p(\mathbb{R}^n)$ and from the Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$ upon interpolation along a straight line with slope k, and the author also gave a counterexample showing that the interpolation spaces $(B^{s_0}_{p_0,1}, B^{s_1}_{p_1,1})_{\theta,q}$ and $(B^{s_0}_{p_0,\infty}, B^{s_1}_{p_1,\infty})_{\theta,q}$ are different in general. In a recent paper [15], J. Byeon, H. Kim, and J. Oh established several results on sufficient and necessary conditions for the interpolation inequality of the type $\|f\|_X \lesssim \|f\|^{1-\theta}_{X_1}\|f\|^{\theta}_{X_2}$ $(0 < \theta < 1)$ where X, X_1, and X_2 can be inhomogeneous and homogeneous Triebel-Lizorkin-Lorentz spaces, and inhomogeneous and homogeneous Besov-Lorentz spaces. The Triebel-Lizorkin-Lorentz quasinorm is the generalization of the Triebel-Lizorkin quasinorm obtained by replacing the underlining Lebesgue quasinorm by the Lorentz quasinorm, and the Besov-Lorentz quasinorm can be obtained from the Besov quasinorm in the same way. The authors of [15] also extended the Gagliardo-Nirenberg inequalities to the setting of Lorentz spaces, including the limiting case when some exponent equals 1 or ∞, and afterward various interpolation inequalities were derived, such as the famous Ladyzhenskaya inequality and Nash's inequality. In [108], W. Yuan, W. Sickel, and D. Yang systematically studied numerous interpolation space theorems of Besov-type spaces, Triebel-Lizorkin-type spaces, Besov-Morrey spaces, Triebel-Lizorkin-Morrey spaces, and Morrey-Campanato spaces via different interpolation methods such as the $\pm$-method of Gustavsson and Peetre, the Peetre-Gagliardo interpolation method, the complex interpolation method, the second complex interpolation method of Calderón, and the real interpolation method. The authors of [108] also studied the interpolation of Morrey spaces on a bounded domain and the interpolation of Besov-Morrey spaces on a Lipschitz domain (either a special or a bounded Lipschitz domain) by the Peetre-Gagliardo interpolation method. Moreover, the interpolation properties of linear operators on some of the smoothness function spaces built on Morrey spaces were obtained in their article. In [51], X. Jiang, D. Yang, and W. Yuan introduced the grand Besov spaces $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and the grand Triebel-Lizorkin spaces $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$ on an RD-space $\mathscr{X}$ via the grand Littlewood-Paley g-function, where an RD-space $\mathscr{X}$ is a metric space having both doubling and reverse doubling properties. The paper also established some real interpolation conclusions of the spaces $\mathscr{A}\dot{B}^s_{p,q}(\mathscr{X})$ and $\mathscr{A}\dot{F}^s_{p,q}(\mathscr{X})$, which generalized the real interpolation theorems of Besov and Triebel-Lizorkin spaces on Ahlfors n-regular metric spaces and RD-spaces. In [104], D. Yang, W. Yuan, and C. Zhuo established the complex interpolation theorems on Triebel–Lizorkin-type spaces, Besov-type spaces, and Besov-Morrey spaces. Furthermore, as a corollary, the authors obtained the complex interpolation for Morrey spaces. In [33], D. Drihem presented the Fourier analytical definition of Herz-type Triebel-Lizorkin spaces $\dot{K}^{\alpha,q}_p F^s_\beta$ and studied the complex interpolation of Herz-type Triebel–Lizorkin spaces using Calderón products. As some applications of the main theorems, D. Drihem also obtained results concerning the complex interpolation between bmo (or h_p) spaces and Herz spaces, and the complex interpolation of Triebel-Lizorkin spaces equipped with Muckenhoupt weights. In [34], the same author also studied the complex interpolation of variable Triebel–Lizorkin spaces and considered some limiting cases.

3.2 Proof of Theorem 3.1.1

Proof. Let $1 < p, p', q, q' < \infty$ where p', q' are Hölder's conjugates of p, q respectively. Since $T_\theta(f)$ is in $\dot{F}^s_{p,q}(\mathbb{R}^n)$ if and only if the sequence $\{2^{ks}\psi_{2^{-k}} * T_\theta(f)\}_{k\in\mathbb{Z}}$ is in $L^p(\mathbb{R}^n, l^q)$, and g is in $\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)$ if and only if the sequence $\{2^{-ks}\psi_{2^{-k}} * g\}_{k\in\mathbb{Z}}$ is in $L^{p'}(\mathbb{R}^n, l^{q'})$, and since $L^p(\mathbb{R}^n, l^q)$ is the dual space of $L^{p'}(\mathbb{R}^n, l^{q'})$ when $1 < p, q < \infty$, therefore $\dot{F}^s_{p,q}(\mathbb{R}^n)$ is the dual space of $\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)$. And in order to prove (109), it suffices to show

$$|\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \psi_{2^{-k}} * T_\theta(f)(x) \cdot \overline{\psi_{2^{-k}} * g(x)} dx| \lesssim M(\theta) \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)} \tag{112}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, where ψ is the chosen Schwartz function in the definition of Triebel-Lizorkin spaces which also satisfies (16), (17), (21), (23) and $\psi_{2^{-k}}(y) = 2^{kn}\psi(2^k y)$. We will use Lemma 2.0.1 to prove (112) so first, we will construct an analytic extension of the left side of (112). Recall Definition 1.2.5 and we let $f_j = \psi_{2^{-j}} * f$ and $g_j = \psi_{2^{-j}} * g$ for $j \in \mathbb{Z}$ then the n-dimensional Fourier transforms $\mathcal{F}_n f_j$ and $\mathcal{F}_n g_j$ are both supported in the annulus

$2^{j-1} \leq |\xi| < 2^{j+1}$ and thus the Peetre-Fefferman-Stein maximal functions of f_j and g_j are well-defined and

$$\mathcal{P}_n f_j(x) = \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|f_j(x-z)|}{(1+2^{j+1}|z|)^{n/r}} \qquad\qquad \mathcal{P}_n g_j(x) = \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|g_j(x-z)|}{(1+2^{j+1}|z|)^{n/r}}.$$

Since $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ and $g \in \dot{F}^{-s}_{p',q'}(\mathbb{R}^n)$, we infer from Remark 2.0.8 that for almost every $y \in \mathbb{R}^n$, $(\sum_{l=-\infty}^{\infty} |2^{ls}\mathcal{P}_n f_l(y)|^q)^{\frac{1}{q}}$ and $(\sum_{l=-\infty}^{\infty} |2^{-ls}\mathcal{P}_n g_l(y)|^{q'})^{\frac{1}{q'}}$ are finite real numbers. And if there exists y_0 such that $(\sum_{l=-\infty}^{\infty} |2^{ls}\mathcal{P}_n f_l(y_0)|^q)^{\frac{1}{q}} = 0$ then from the discussion in Remark 2.0.3, $f_l(x) = 0$ for all $l \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, thus by (21) the Schwartz function f is identically zero and we have a trivial discussion. The same conclusion can be drawn for $(\sum_{l=-\infty}^{\infty} |2^{-ls}\mathcal{P}_n g_l(y)|^{q'})^{\frac{1}{q'}}$. Now assuming both

$$(\sum_{l=-\infty}^{\infty} |2^{ls}\mathcal{P}_n f_l(y)|^q)^{\frac{1}{q}}$$

and

$$(\sum_{l=-\infty}^{\infty} |2^{-ls}\mathcal{P}_n g_l(y)|^{q'})^{\frac{1}{q'}}$$

are positive, we denote for $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$

$$\mathit{\Gamma}^s_q(f)(y) = (\sum_{l=-\infty}^{\infty} |2^{ls}\mathcal{P}_n f_l(y)|^q)^{\frac{1}{q}} \qquad \mathit{\Gamma}^s_{q,k}(f)(y) = (\sum_{l=-\infty}^{k} |2^{ls}\mathcal{P}_n f_l(y)|^q)^{\frac{1}{q}}, \tag{113}$$

$$\mathit{\Gamma}^{-s}_{q'}(g)(y) = (\sum_{l=-\infty}^{\infty} |2^{-ls}\mathcal{P}_n g_l(y)|^{q'})^{\frac{1}{q'}} \qquad \mathit{\Gamma}^{-s}_{q',k}(g)(y) = (\sum_{l=-\infty}^{k} |2^{-ls}\mathcal{P}_n g_l(y)|^{q'})^{\frac{1}{q'}}, \tag{114}$$

and let $\eta(y) = 2^{-n}\psi(\frac{y}{2}) + \psi(y) + 2^n\psi(2y)$ so that

$$\mathcal{F}_n\eta(2^{-k}\xi) = \mathcal{F}_n\psi(2^{-k+1}\xi) + \mathcal{F}_n\psi(2^{-k}\xi) + \mathcal{F}_n\psi(2^{-k-1}\xi) \tag{115}$$

and

$$\mathcal{F}_n\eta(2^{-k}\xi) = 1 \qquad \text{on the support of } \mathcal{F}_n\psi(2^{-k}\xi) \tag{116}$$

due to the support condition of $\mathcal{F}_n\psi$ and (17). For $z \in \mathbb{C}$ we denote

$$\rho_1(z) = sq(\frac{1-z}{q_0} + \frac{z}{q_1}) - (1-z)s_0 - zs_1, \tag{117}$$

$$\rho_2(z) = p(\frac{1-z}{p_0} + \frac{z}{p_1}) - q(\frac{1-z}{q_0} + \frac{z}{q_1}), \tag{118}$$

$$\rho_3(z) = 1 - p(\frac{1-z}{p_0} + \frac{z}{p_1}), \qquad \rho_4(z) = q(\frac{1-z}{q_0} + \frac{z}{q_1}), \tag{119}$$

and in accordance with relation (106) we also have

$$-s = -(1-\theta)s_0 - \theta s_1, \qquad \frac{1}{p'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}, \qquad \frac{1}{q'} = \frac{1-\theta}{q_0'} + \frac{\theta}{q_1'}, \tag{120}$$

and we denote

$$\rho_5(z) = -sq'(\frac{1-z}{q_0'} + \frac{z}{q_1'}) + (1-z)s_0 + zs_1, \tag{121}$$

$$\rho_6(z) = p'(\frac{1-z}{p_0'} + \frac{z}{p_1'}) - q'(\frac{1-z}{q_0'} + \frac{z}{q_1'}), \tag{122}$$

$$\rho_7(z) = 1 - p'(\frac{1-z}{p_0'} + \frac{z}{p_1'}), \qquad \rho_8(z) = q'(\frac{1-z}{q_0'} + \frac{z}{q_1'}). \tag{123}$$

Then we also define for $z \in \mathbb{C}$ with $0 \le \operatorname{Re} z \le 1$, $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$,

$$U_{z,k}(y) := 2^{\rho_1(z)k} \mathit{\Gamma}_{q,k}^s(f)(y)^{\rho_2(z)} \|\mathit{\Gamma}_q^s(f)\|_{L^p(\mathbb{R}^n)}^{\rho_3(z)} f_k(y)^{\rho_4(z)}, \tag{124}$$

and

$$V_{z,k}(y) := 2^{\rho_5(z)k} \mathit{\Gamma}_{q',k}^{-s}(g)(y)^{\rho_6(z)} \|\mathit{\Gamma}_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}^{\rho_7(z)} g_k(y)^{\rho_8(z)}. \tag{125}$$

Finally for Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^n)$ we define their analytic extensions by

$$f_z = \sum_{k\in\mathbb{Z}} \eta_{2^{-k}} * U_{z,k} \qquad \text{and} \qquad g_z = \sum_{k\in\mathbb{Z}} \eta_{2^{-k}} * V_{z,k}. \tag{126}$$

The convergence in (126) is in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$. We claim that f_z and g_z are tempered distributions when $0 \le \operatorname{Re} z \le 1$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ denote an arbitrary Schwartz function, then we can use (117), (118), (119) and deduce the following

$$|< f_z, \varphi >| = \lim_{N\to\infty} |\sum_{k=-N}^{N} < \eta_{2^{-k}} * U_{z,k}, \varphi >| \lesssim \lim_{N\to\infty} \sum_{k=-N}^{N} |< U_{z,k}, \tilde{\eta}_{2^{-k}} * \varphi >|$$

$$\lesssim \lim_{N\to\infty} \sum_{k=-N}^{N} \int_{\mathbb{R}^n} |U_{z,k}(x)| \cdot |\tilde{\eta}_{2^{-k}} * \varphi(x)| dx \le C \|\varphi\|_{L^\infty(\mathbb{R}^n)},$$

where $\tilde{\eta}(x) = \eta(-x)$ and the positive finite constant C in the last inequality above may depend on η and certain homogeneous Triebel-Lizorkin and/or Besov-Lipschitz quasinorms of the Schwartz function f, and the indices of these quasinorms depend on $\operatorname{Re} z$. This shows f_z is a tempered distribution. In a similar way, we can also show g_z is a tempered distribution.

Next, we show that

$$f_\theta = f, \qquad g_\theta = g, \qquad \text{for } z = \theta \in (0,1), \tag{127}$$

$$\|f_{it}\|_{\dot F^{s_0}_{p_0,q_0}(\mathbb{R}^n)} \lesssim \|f\|_{\dot F^{s}_{p,q}(\mathbb{R}^n)}, \qquad \|f_{1+it}\|_{\dot F^{s_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim \|f\|_{\dot F^{s}_{p,q}(\mathbb{R}^n)}, \tag{128}$$

$$\|g_{it}\|_{\dot F^{-s_0}_{p_0',q_0'}(\mathbb{R}^n)} \lesssim \|g\|_{\dot F^{-s}_{p',q'}(\mathbb{R}^n)}, \qquad \|g_{1+it}\|_{\dot F^{-s_1}_{p_1',q_1'}(\mathbb{R}^n)} \lesssim \|g\|_{\dot F^{-s}_{p',q'}(\mathbb{R}^n)}, \tag{129}$$

and the constants in (128) and (129) are independent of t. Using relations (106), (120) and (116), it becomes clear that when $z = \theta \in (0,1)$

$$f_\theta = \sum_{k\in\mathbb{Z}} \eta_{2^{-k}} * f_k = \sum_{k\in\mathbb{Z}} \eta_{2^{-k}} * \psi_{2^{-k}} * f = f,$$
$$g_\theta = \sum_{k\in\mathbb{Z}} \eta_{2^{-k}} * g_k = \sum_{k\in\mathbb{Z}} \eta_{2^{-k}} * \psi_{2^{-k}} * g = g,$$

and so (127) is proved. To prove (128), we first notice that for $z \in \bar{S} = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$, $\operatorname{Re}\rho_4(z) = \rho_4(\operatorname{Re} z) = q(\frac{1-\operatorname{Re} z}{q_0} + \frac{\operatorname{Re} z}{q_1}) \ge 0$, and applying inequality (52) of Remark 2.0.3 to f_k where $\mathcal{F}_n f_k(\xi)$ is supported in the annulus $2^{k-1} \le |\xi| < 2^{k+1}$, we get

$$\begin{aligned} |f_k(x-y)^{\rho_4(z)}| &\lesssim |f_k(x-y)|^{\operatorname{Re}\rho_4(z)} \lesssim |\mathcal{P}_n f_k(x-y)|^{\operatorname{Re}\rho_4(z)} \\ &\lesssim |\mathcal{P}_n f_k(x)|^{\operatorname{Re}\rho_4(z)} (1+2^{k+1}|y|)^{\frac{n}{r}\operatorname{Re}\rho_4(z)}. \end{aligned} \tag{130}$$

Also we notice that when $\operatorname{Re}\rho_2(z) \ge 0$, using the right side of inequality (52) of Remark 2.0.3 we can obtain

$$\begin{aligned} |\mathit{\Gamma}^s_{q,k}(f)(x-y)^{\rho_2(z)}| &\lesssim \Big(\sum_{l=-\infty}^{k} 2^{lsq}\mathcal{P}_n f_l(x-y)^q\Big)^{\frac{\operatorname{Re}\rho_2(z)}{q}} \\ &\lesssim \Big(\sum_{l=-\infty}^{k} 2^{lsq}\mathcal{P}_n f_l(x)^q (1+2^{l+1}|y|)^{\frac{nq}{r}}\Big)^{\frac{\operatorname{Re}\rho_2(z)}{q}} \\ &\lesssim \Big(\sum_{l=-\infty}^{k} 2^{lsq}\mathcal{P}_n f_l(x)^q\Big)^{\frac{\operatorname{Re}\rho_2(z)}{q}} (1+2^{k+1}|y|)^{\frac{n}{r}\operatorname{Re}\rho_2(z)}, \end{aligned}$$

and when $\operatorname{Re}\rho_2(z) < 0$, using the left side of inequality (52) of Remark 2.0.3 we can obtain

$$\begin{aligned}|\mathit{\Gamma}^s_{q,k}(f)(x-y)^{\rho_2(z)}| &\lesssim \Big(\sum_{l=-\infty}^{k} 2^{lsq}\mathcal{P}_n f_l(x-y)^q\Big)^{\frac{\operatorname{Re}\rho_2(z)}{q}} \\ &\lesssim \Big(\sum_{l=-\infty}^{k} 2^{lsq}\mathcal{P}_n f_l(x)^q(1+2^{l+1}|y|)^{-\frac{nq}{r}}\Big)^{\frac{\operatorname{Re}\rho_2(z)}{q}} \\ &\lesssim \Big(\sum_{l=-\infty}^{k} 2^{lsq}\mathcal{P}_n f_l(x)^q\Big)^{\frac{\operatorname{Re}\rho_2(z)}{q}}(1+2^{k+1}|y|)^{-\frac{n}{r}\operatorname{Re}\rho_2(z)},\end{aligned}$$

therefore combining the two cases together gives

$$|\mathit{\Gamma}^s_{q,k}(f)(x-y)^{\rho_2(z)}| \lesssim \mathit{\Gamma}^s_{q,k}(f)(x)^{\operatorname{Re}\rho_2(z)}(1+2^{k+1}|y|)^{\frac{n}{r}|\operatorname{Re}\rho_2(z)|}. \tag{131}$$

Using (130) and (131) we estimate

$$\begin{aligned}|\eta_{2^{-k}} * U_{z,k}(x)| &\lesssim \int_{\mathbb{R}^n} |\eta_{2^{-k}}(y)|\cdot|U_{z,k}(x-y)|dy \\ &\lesssim \int_{\mathbb{R}^n} |\eta_{2^{-k}}(y)|\cdot 2^{\operatorname{Re}\rho_1(z)k}\mathit{\Gamma}^s_{q,k}(f)(x-y)^{\operatorname{Re}\rho_2(z)} \\ &\quad \cdot\|\mathit{\Gamma}^s_q(f)\|^{\operatorname{Re}\rho_3(z)}_{L^p(\mathbb{R}^n)}|f_k(x-y)|^{\operatorname{Re}\rho_4(z)}dy \\ &\lesssim 2^{\operatorname{Re}\rho_1(z)k}\mathit{\Gamma}^s_{q,k}(f)(x)^{\operatorname{Re}\rho_2(z)}\|\mathit{\Gamma}^s_q(f)\|^{\operatorname{Re}\rho_3(z)}_{L^p(\mathbb{R}^n)}|\mathcal{P}_n f_k(x)|^{\operatorname{Re}\rho_4(z)} \\ &\quad \cdot\int_{\mathbb{R}^n} |\eta_{2^{-k}}(y)|(1+2^{k+1}|y|)^{\frac{n}{r}(|\operatorname{Re}\rho_2(z)|+\operatorname{Re}\rho_4(z))}dy,\end{aligned}$$

and since the last integral above is independent of $k\in\mathbb{Z}$ and

$$|\operatorname{Re}\rho_2(z)|+\operatorname{Re}\rho_4(z) \lesssim \frac{p}{p_0}+\frac{p}{p_1}+\frac{2q}{q_0}+\frac{2q}{q_1} \qquad \text{for } z\in\bar{S},$$

we obtain

$$|\eta_{2^{-k}} * U_{z,k}(x)| \lesssim 2^{\operatorname{Re}\rho_1(z)k}\mathit{\Gamma}^s_{q,k}(f)(x)^{\operatorname{Re}\rho_2(z)}\|\mathit{\Gamma}^s_q(f)\|^{\operatorname{Re}\rho_3(z)}_{L^p(\mathbb{R}^n)}|\mathcal{P}_n f_k(x)|^{\operatorname{Re}\rho_4(z)}, \tag{132}$$

and the constant is independent of $k\in\mathbb{Z}$ and $z\in\bar{S}$. When $z=it$ for $t\in\mathbb{R}$, since $\mathcal{F}_n\psi(2^{-j}\xi)$ is supported in $2^{j-1}\le|\xi|<2^{j+1}$ and by (115) $\mathcal{F}_n\eta(2^{-k}\xi)$ is supported in $2^{k-2}\le|\xi|<2^{k+2}$, we use (126) with $z=it$ and obtain

$$\|f_{it}\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{l=-2}^{2} 2^{js_0}\psi_{2^{-j}} * \eta_{2^{-j-l}} * U_{it,j+l}(x)\Big|^{q_0}\Big)^{p_0/q_0}dx\Big)^{1/p_0}. \tag{133}$$

When $1 < p_0, q_0 < \infty$, $\|\cdot\|_{l^{q_0}}$ and $\|\cdot\|_{L^{p_0}(\mathbb{R}^n)}$ are norms of Banach spaces, then we can dominate (133) by

$$\begin{aligned}
&\sum_{l=-2}^{2}\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|2^{js_0}\psi_{2^{-j}}*\eta_{2^{-j-l}}*U_{it,j+l}(x)|^{q_0}\Big)^{p_0/q_0}dx\Big)^{1/p_0}\\
=&\sum_{l=-2}^{2}\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|2^{(j+l)s_0}\cdot 2^{-ls_0}\psi_{2^{-j-l+l}}*\eta_{2^{-j-l}}*U_{it,j+l}(x)|^{q_0}\Big)^{p_0/q_0}dx\Big)^{1/p_0}\\
=&\sum_{l=-2}^{2}2^{-ls_0}\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|2^{js_0}\psi_{2^{-j+l}}*\eta_{2^{-j}}*U_{it,j}(x)|^{q_0}\Big)^{p_0/q_0}dx\Big)^{1/p_0}. \qquad (134)
\end{aligned}$$

The Fourier transform of $\eta_{2^{-j}}*U_{it,j}(x)$ is supported in the ball $B(0,2^{j+2})\subseteq\mathbb{R}^n$, so using an argument like that of Remark 2.0.4, we obtain for $l\in\{-2,-1,0,1,2\}$

$$\begin{aligned}
|\psi_{2^{-j+l}}*\eta_{2^{-j}}*U_{it,j}(x)|&\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n}\frac{|\psi_{2^{-j+l}}*\eta_{2^{-j}}*U_{it,j}(x-z)|}{(1+2^{j+2}|z|)^{n/r}}\\
&\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\psi_{2^{-j+l}}(y)|(1+2^{j+2}|y|)^{n/r}\cdot\frac{|\eta_{2^{-j}}*U_{it,j}(x-z-y)|}{(1+2^{j+2}|z+y|)^{n/r}}dy\\
&\lesssim \mathcal{P}_n(\eta_{2^{-j}}*U_{it,j})(x)\cdot\int_{\mathbb{R}^n}2^{jn-ln}|\psi(2^{j-l}y)|(1+2^{j+2}|y|)^{n/r}dy,
\end{aligned}$$

where the last integral only depends on l,n,r,ψ and is independent of $j\in\mathbb{Z}$, thus we have

$$|\psi_{2^{-j+l}}*\eta_{2^{-j}}*U_{it,j}(x)|\lesssim \mathcal{P}_n(\eta_{2^{-j}}*U_{it,j})(x) \qquad (135)$$

for every $x\in\mathbb{R}^n$ and the constant is independent of $j\in\mathbb{Z}$. Put (135) into (134) and recall that in Definition 1.2.5 we can chose $r>0$ so that

$$0<r<\min\{p,p_0,p_1,q,q_0,q_1,p',p_0',p_1',q',q_0',q_1'\} \qquad (136)$$

and by an application of Lemma 2.0.3 and Lemma 2.0.6, we have

$$\begin{aligned}
(134)&\lesssim\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}2^{js_0q_0}|\mathcal{P}_n(\eta_{2^{-j}}*U_{it,j})(x)|^{q_0}\Big)^{p_0/q_0}dx\Big)^{1/p_0}\\
&\lesssim\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}\mathcal{M}_n(2^{js_0r}|\eta_{2^{-j}}*U_{it,j}|^r)(x)|^{q_0/r}\Big)^{\frac{p_0/r}{q_0/r}}dx\Big)^{\frac{1/r}{p_0/r}}\\
&\lesssim\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}2^{js_0q_0}|\eta_{2^{-j}}*U_{it,j}(x)|^{q_0}\Big)^{p_0/q_0}dx\Big)^{1/p_0}. \qquad (137)
\end{aligned}$$

Inserting estimate (132) with $z = it$ into (137) yields

$$(137) \lesssim \|\mathit{\Gamma}_q^s(f)\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{p_0}} \cdot (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q (\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0 p}{q p_0}-1})^{\frac{p_0}{q_0}} dx)^{\frac{1}{p_0}}. \tag{138}$$

We discuss two cases: $\frac{q_0 p}{q p_0} - 1 \geq 0$ or $\frac{q_0 p}{q p_0} - 1 < 0$. If $\frac{q_0 p}{q p_0} - 1 \geq 0$, we use

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q (\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0 p}{q p_0}-1} &\lesssim (\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q) \cdot (\sum_{l\in\mathbb{Z}} |2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0 p}{q p_0}-1} \\ &= (\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q)^{\frac{q_0 p}{q p_0}},\end{aligned}$$

and thus

$$\begin{aligned}(138) &\lesssim \|\mathit{\Gamma}_q^s(f)\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{p_0}} \cdot (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q)^{\frac{p}{q}} dx)^{\frac{1}{p}\cdot\frac{p}{p_0}} \\ &= (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q)^{\frac{p}{q}} dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)},\end{aligned} \tag{139}$$

where the last inequality is due to Remark 2.0.8. If $\frac{q_0 p}{q p_0} - 1 < 0$, since

$$0 < \lambda := \mathit{\Gamma}_q^s(f)(x)^q = \sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q < \infty \tag{140}$$

for almost every $x \in \mathbb{R}^n$ and $\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q$ increases as j increases, so if we assume $J_0 = \infty$ and pick J_1 as the least integer so that

$$\frac{1}{2}\lambda \leq \sum_{l=-\infty}^{J_1} |2^{ls}\mathcal{P}_n f_l(x)|^q < \lambda \qquad \text{and} \qquad \sum_{l=-\infty}^{J_1-1} |2^{ls}\mathcal{P}_n f_l(x)|^q < \frac{1}{2}\lambda, \tag{141}$$

and for a natural number $K \in \mathbb{N}$ we pick J_K as the least integer so that

$$2^{-K}\lambda \leq \sum_{l=-\infty}^{J_K} |2^{ls}\mathcal{P}_n f_l(x)|^q < 2^{-K+1}\lambda \qquad \text{and} \qquad \sum_{l=-\infty}^{J_K-1} |2^{ls}\mathcal{P}_n f_l(x)|^q < 2^{-K}\lambda \tag{142}$$

then $\{J_K\}_{K\geq 0}$ is a decreasing sequence of integers and we get

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}|2^{js}\mathcal{P}_n f_j(x)|^q(\sum_{l=-\infty}^{j}|2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0p}{qp_0}-1}\\
=&\sum_{K=1}^{\infty}\sum_{j=J_K}^{J_{K-1}-1}|2^{js}\mathcal{P}_n f_j(x)|^q(\sum_{l=-\infty}^{j}|2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0p}{qp_0}-1}\\
\lesssim&\sum_{K=1}^{\infty}(\sum_{j=J_K}^{J_{K-1}-1}|2^{js}\mathcal{P}_n f_j(x)|^q)(\sum_{l=-\infty}^{J_K}|2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0p}{qp_0}-1}\\
\lesssim&\sum_{K=1}^{\infty}(\sum_{j=-\infty}^{J_{K-1}-1}|2^{js}\mathcal{P}_n f_j(x)|^q)(\sum_{l=-\infty}^{J_K}|2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_0p}{qp_0}-1}\\
\lesssim&2\sum_{K=1}^{\infty}2^{-K\cdot\frac{q_0p}{qp_0}}\lambda^{\frac{q_0p}{qp_0}}\lesssim(\sum_{j\in\mathbb{Z}}|2^{js}\mathcal{P}_n f_j(x)|^q)^{\frac{q_0p}{qp_0}}.
\end{aligned}\tag{143}$$

Inserting (143) into (138) and invoking (71) of Remark 2.0.8 yield (139). Finally we combine (133), (134), (137), (138) and (139) together and obtain

$$\|f_{it}\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)}\lesssim\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)},\tag{144}$$

where the constant above is determined by fixed parameters and is independent of $t\in\mathbb{R}$.

Now we prove $\|f_{1+it}\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}\lesssim\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$, which is the second part of (128). By definition (126) and the support conditions of $\mathcal{F}_n\psi(2^{-j}\xi)$ and $\mathcal{F}_n\eta(2^{-k}\xi)$ we have

$$\|f_{1+it}\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}=(\int_{\mathbb{R}^n}(\sum_{j\in\mathbb{Z}}|\sum_{l=-2}^{2}2^{js_1}\psi_{2^{-j}}*\eta_{2^{-j-l}}*U_{1+it,j+l}(x)|^{q_1})^{p_1/q_1}dx)^{1/p_1}.\tag{145}$$

When $1<p_1,q_1<\infty$, $\|\cdot\|_{l^{q_1}}$ and $\|\cdot\|_{L^{p_1}(\mathbb{R}^n)}$ are norms of Banach spaces, then we can dominate (145) by

$$\begin{aligned}
&\sum_{l=-2}^{2}(\int_{\mathbb{R}^n}(\sum_{j\in\mathbb{Z}}|2^{js_1}\psi_{2^{-j}}*\eta_{2^{-j-l}}*U_{1+it,j+l}(x)|^{q_1})^{p_1/q_1}dx)^{1/p_1}\\
=&\sum_{l=-2}^{2}(\int_{\mathbb{R}^n}(\sum_{j\in\mathbb{Z}}|2^{(j+l)s_1}\cdot 2^{-ls_1}\psi_{2^{-j-l+l}}*\eta_{2^{-j-l}}*U_{1+it,j+l}(x)|^{q_1})^{p_1/q_1}dx)^{1/p_1}\\
=&\sum_{l=-2}^{2}2^{-ls_1}(\int_{\mathbb{R}^n}(\sum_{j\in\mathbb{Z}}|2^{js_1}\psi_{2^{-j+l}}*\eta_{2^{-j}}*U_{1+it,j}(x)|^{q_1})^{p_1/q_1}dx)^{1/p_1}.
\end{aligned}\tag{146}$$

The Fourier transform of $\eta_{2^{-j}} * U_{1+it,j}(x)$ is supported in the ball $B(0,2^{j+2}) \subseteq \mathbb{R}^n$, so using an argument like that of Remark 2.0.4, we obtain for $l \in \{-2,-1,0,1,2\}$

$$\begin{aligned}
&|\psi_{2^{-j+l}} * \eta_{2^{-j}} * U_{1+it,j}(x)| \\
\lesssim &\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|\psi_{2^{-j+l}} * \eta_{2^{-j}} * U_{1+it,j}(x-z)|}{(1+2^{j+2}|z|)^{n/r}} \\
\lesssim &\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\psi_{2^{-j+l}}(y)|(1+2^{j+2}|y|)^{n/r} \cdot \frac{|\eta_{2^{-j}} * U_{1+it,j}(x-z-y)|}{(1+2^{j+2}|z+y|)^{n/r}} dy \\
\lesssim &\mathcal{P}_n(\eta_{2^{-j}} * U_{1+it,j})(x) \cdot \int_{\mathbb{R}^n} 2^{jn-ln}|\psi(2^{j-l}y)|(1+2^{j+2}|y|)^{n/r} dy,
\end{aligned}$$

where the last integral only depends on l,n,r,ψ and is independent of $j \in \mathbb{Z}$, thus we have

$$|\psi_{2^{-j+l}} * \eta_{2^{-j}} * U_{1+it,j}(x)| \lesssim \mathcal{P}_n(\eta_{2^{-j}} * U_{1+it,j})(x) \tag{147}$$

for every $x \in \mathbb{R}^n$ and the constant is independent of $j \in \mathbb{Z}$. Put (147) back into (146) and recall the condition (136) and by an application of Lemma 2.0.3 and Lemma 2.0.6, we have

$$\begin{aligned}
(146) \lesssim &(\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} 2^{js_1q_1}|\mathcal{P}_n(\eta_{2^{-j}} * U_{1+it,j})(x)|^{q_1})^{p_1/q_1} dx)^{1/p_1} \\
\lesssim &(\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} \mathcal{M}_n(2^{js_1r}|\eta_{2^{-j}} * U_{1+it,j}|^r)(x)|^{q_1/r})^{\frac{p_1/r}{q_1/r}} dx)^{\frac{1/r}{p_1/r}} \\
\lesssim &(\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} 2^{js_1q_1}|\eta_{2^{-j}} * U_{1+it,j}(x)|^{q_1})^{p_1/q_1} dx)^{1/p_1}.
\end{aligned} \tag{148}$$

Inserting estimate (132) with $z = 1+it$ into (148) yields

$$(148) \lesssim \|\varGamma_q^s(f)\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{p_1}} \cdot (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q (\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_1p}{qp_1}-1})^{\frac{p_1}{q_1}} dx)^{\frac{1}{p_1}}. \tag{149}$$

We discuss two cases like before: $\frac{q_1p}{qp_1} - 1 \geq 0$ or $\frac{q_1p}{qp_1} - 1 < 0$. If $\frac{q_1p}{qp_1} - 1 \geq 0$, we use

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q (\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_1p}{qp_1}-1} \lesssim &(\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q) \cdot (\sum_{l\in\mathbb{Z}} |2^{ls}\mathcal{P}_n f_l(x)|^q)^{\frac{q_1p}{qp_1}-1} \\
= &(\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q)^{\frac{q_1p}{qp_1}},
\end{aligned}$$

and thus

$$
\begin{aligned}
(149) &\lesssim \|\mathit{\Gamma}_q^s(f)\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{p_1}} \cdot \Big(\int_{\mathbb{R}^n} \big(\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q\big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}\cdot\frac{p}{p_1}} \\
&= \Big(\int_{\mathbb{R}^n} \big(\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q\big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \qquad (150)
\end{aligned}
$$

where the last inequality is due to (71) of Remark 2.0.8. If $\frac{q_1 p}{q p_1} - 1 < 0$, we use the same notations λ, J_0, J_1, J_K for $K \in \mathbb{N}$ as given in (140), (141), (142) and estimate

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q \Big(\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q\Big)^{\frac{q_1 p}{q p_1}-1} \\
=&\sum_{K=1}^{\infty} \sum_{j=J_K}^{J_{K-1}-1} |2^{js}\mathcal{P}_n f_j(x)|^q \Big(\sum_{l=-\infty}^{j} |2^{ls}\mathcal{P}_n f_l(x)|^q\Big)^{\frac{q_1 p}{q p_1}-1} \\
\lesssim&\sum_{K=1}^{\infty} \Big(\sum_{j=J_K}^{J_{K-1}-1} |2^{js}\mathcal{P}_n f_j(x)|^q\Big) \Big(\sum_{l=-\infty}^{J_K} |2^{ls}\mathcal{P}_n f_l(x)|^q\Big)^{\frac{q_1 p}{q p_1}-1} \\
\lesssim&\sum_{K=1}^{\infty} \Big(\sum_{j=-\infty}^{J_{K-1}-1} |2^{js}\mathcal{P}_n f_j(x)|^q\Big) \Big(\sum_{l=-\infty}^{J_K} |2^{ls}\mathcal{P}_n f_l(x)|^q\Big)^{\frac{q_1 p}{q p_1}-1} \\
\lesssim&2\sum_{K=1}^{\infty} 2^{-K\cdot\frac{q_1 p}{q p_1}} \lambda^{\frac{q_1 p}{q p_1}} \lesssim \Big(\sum_{j\in\mathbb{Z}} |2^{js}\mathcal{P}_n f_j(x)|^q\Big)^{\frac{q_1 p}{q p_1}}. \qquad (151)
\end{aligned}
$$

Inserting (151) into (149) and invoking (71) of Remark 2.0.8 yield (150). Finally we combine (145), (146), (148), (149) and (150) together and obtain

$$
\|f_{1+it}\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \qquad (152)
$$

where the constant above is determined by fixed parameters and is independent of $t \in \mathbb{R}$. With (144) and (152), we prove (128).

Next, we move on to the proof of (129). The proof of (129) is very much alike to the proof of (128) so we will only sketch the main steps and leave the details to the reader. Still notice that for $z \in \bar{S} = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$, $\operatorname{Re}\rho_8(z) = \rho_8(\operatorname{Re} z) = q'(\frac{1-\operatorname{Re} z}{q_0'} + \frac{\operatorname{Re} z}{q_1'}) \ge 0$ and applying inequality (52) of Remark 2.0.3 to $g_k = \psi_{2^{-k}} * g$ whose Fourier transform is supported in $B(0, 2^{k+1}) \subseteq \mathbb{R}^n$, we get

$$
|g_k(x-y)^{\rho_8(z)}| \lesssim |\mathcal{P}_n g_k(x-y)|^{\operatorname{Re}\rho_8(z)} \lesssim |\mathcal{P}_n g_k(x)|^{\operatorname{Re}\rho_8(z)} (1 + 2^{k+1}|y|)^{\frac{n}{r}\cdot\operatorname{Re}\rho_8(z)}. \qquad (153)
$$

By using the right side of (52) when $\operatorname{Re}\rho_6(z) \geq 0$ and using the left side of (52) when $\operatorname{Re}\rho_6(z) < 0$, we obtain

$$\begin{aligned} |\Gamma_{q',k}^{-s}(g)(x-y)^{\rho_6(z)}| &\lesssim |\Gamma_{q',k}^{-s}(g)(x-y)|^{\operatorname{Re}\rho_6(z)} \\ &\lesssim \Gamma_{q',k}^{-s}(g)(x)^{\operatorname{Re}\rho_6(z)}(1+2^{k+1}|y|)^{\frac{n}{r}\cdot|\operatorname{Re}\rho_6(z)|}. \end{aligned} \tag{154}$$

Therefore using (153), (154), an argument like the one to deduce (132) and the fact that $|\operatorname{Re}\rho_6(z)| + \operatorname{Re}\rho_8(z) \leq \frac{p'}{p_0'} + \frac{p'}{p_1'} + \frac{2q'}{q_0'} + \frac{2q'}{q_1'}$ for $z \in \bar{S}$, we obtain

$$|\eta_{2^{-k}} * V_{z,k}(x)| \lesssim 2^{\operatorname{Re}\rho_5(z)k}\Gamma_{q',k}^{-s}(g)(x)^{\operatorname{Re}\rho_6(z)}\|\Gamma_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}^{\operatorname{Re}\rho_7(z)}|\mathcal{P}_n g_k(x)|^{\operatorname{Re}\rho_8(z)}, \tag{155}$$

and the constant is independent of $k \in \mathbb{Z}$ and $z \in \bar{S}$. When $z = it$ for $t \in \mathbb{R}$, using support conditions of $\mathcal{F}_n\psi(2^{-j}\xi)$ and $\mathcal{F}_n\eta(2^{-k}\xi)$ and an argument like that to deduce (134) we have

$$\|g_{it}\|_{\dot{F}^{-s_0}_{p_0',q_0'}(\mathbb{R}^n)} \lesssim \sum_{l=-2}^{2} 2^{ls_0}\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|2^{-js_0}\psi_{2^{-j+l}} * \eta_{2^{-j}} * V_{it,j}(x)|^{q_0'}\Big)^{p_0'/q_0'}dx\Big)^{1/p_0'}. \tag{156}$$

With a similar argument for (135) we also have

$$|\psi_{2^{-j+l}} * \eta_{2^{-j}} * V_{it,j}(x)| \lesssim \mathcal{P}_n(\eta_{2^{-j}} * V_{it,j})(x) \tag{157}$$

for $l \in \{-2,-1,0,1,2\}$ and every $x \in \mathbb{R}^n$ with a constant independent of $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Insert (157) into (156), recall the condition (136) and apply Lemma 2.0.3 and Lemma 2.0.6 and use (155) with $z = it$ in the final step, we get

$$(156) \lesssim \|\Gamma_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}^{1-\frac{p'}{p_0'}}\cdot\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|2^{-sj}\mathcal{P}_n g_j(x)|^{q'}\Big(\sum_{l=-\infty}^{j}|2^{-ls}\mathcal{P}_n g_l(x)|^{q'}\Big)^{\frac{p'q_0'}{p_0'q'}-1}\Big)^{\frac{p_0'}{q_0'}}dx\Big)^{\frac{1}{p_0'}}. \tag{158}$$

When $\frac{p'q_0'}{p_0'q'} - 1 \geq 0$, we use $\sum_{l=-\infty}^{j}|2^{-ls}\mathcal{P}_n g_l(x)|^{q'} \leq \sum_{l=-\infty}^{\infty}|2^{-ls}\mathcal{P}_n g_l(x)|^{q'}$ and (71) of Remark 2.0.8 to deduce that

$$\|g_{it}\|_{\dot{F}^{-s_0}_{p_0',q_0'}(\mathbb{R}^n)} \lesssim \Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|2^{-sj}\mathcal{P}_n g_j(x)|^{q'}\Big)^{p'/q'}dx\Big)^{1/p'} \lesssim \|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}. \tag{159}$$

When $\frac{p'q_0'}{p_0'q'} - 1 < 0$, we use an argument like the one in (140), (141), (142) and (143). Since

$$0 < \omega := \Gamma_{q'}^{-s}(g)(x)^{q'} = \sum_{l\in\mathbb{Z}}|2^{-sl}\mathcal{P}_n g_l(x)|^{q'} < \infty \tag{160}$$

for almost every $x \in \mathbb{R}^n$, we let $I_0 = \infty$ and pick I_1 as the least integer so that

$$\frac{1}{2}\omega \leq \sum_{l=-\infty}^{I_1} |2^{-sl}\mathcal{P}_n g_l(x)|^{q'} < \omega \qquad \text{and} \qquad \sum_{l=-\infty}^{I_1-1} |2^{-sl}\mathcal{P}_n g_l(x)|^{q'} < \frac{1}{2}\omega, \tag{161}$$

and for a natural number $K \in \mathbb{N}$, we pick I_K as the least integer so that

$$2^{-K}\omega \leq \sum_{l=-\infty}^{I_K} |2^{-sl}\mathcal{P}_n g_l(x)|^{q'} < 2^{-K+1}\omega \qquad \text{and} \qquad \sum_{l=-\infty}^{I_K-1} |2^{-sl}\mathcal{P}_n g_l(x)|^{q'} < 2^{-K}\omega, \tag{162}$$

then $\{I_K\}_{K\geq 0}$ is a decreasing sequence of integers and we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'} \Big(\sum_{l=-\infty}^{j} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'}\Big)^{\frac{p'q_0'}{p_0'q'}-1} \\
=&\sum_{K=1}^{\infty} \sum_{j=I_K}^{I_{K-1}-1} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'} \Big(\sum_{l=-\infty}^{j} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'}\Big)^{\frac{p'q_0'}{p_0'q'}-1} \\
\lesssim&\sum_{K=1}^{\infty} \Big(\sum_{j=-\infty}^{I_{K-1}-1} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'}\Big) \Big(\sum_{l=-\infty}^{I_K} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'}\Big)^{\frac{p'q_0'}{p_0'q'}-1} \\
\lesssim&2\sum_{K=1}^{\infty} 2^{-K\frac{p'q_0'}{p_0'q'}} \omega^{\frac{p'q_0'}{p_0'q'}} \lesssim \Big(\sum_{j\in\mathbb{Z}} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'}\Big)^{\frac{p'q_0'}{p_0'q'}}.
\end{aligned} \tag{163}$$

Inserting (163) into (158) yields (159). When $z = 1 + it$ for $t \in \mathbb{R}$, the counterparts of (156) and (157) are given respectively by

$$\|g_{1+it}\|_{\dot{F}^{-s_1}_{p_1',q_1'}(\mathbb{R}^n)} \lesssim \sum_{l=-2}^{2} 2^{ls_1} \Big(\int_{\mathbb{R}^n} \Big(\sum_{j\in\mathbb{Z}} |2^{-js_1}\psi_{2^{-j+l}} * \eta_{2^{-j}} * V_{1+it,j}(x)|^{q_1'}\Big)^{p_1'/q_1'} dx\Big)^{1/p_1'}, \tag{164}$$

and

$$|\psi_{2^{-j+l}} * \eta_{2^{-j}} * V_{1+it,j}(x)| \lesssim \mathcal{P}_n(\eta_{2^{-j}} * V_{1+it,j})(x) \tag{165}$$

for $l \in \{-2, -1, 0, 1, 2\}$ and every $x \in \mathbb{R}^n$ with a constant independent of $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Insert (165) into (164), recall the condition (136) and apply Lemma 2.0.3 and Lemma 2.0.6 and use (155) with $z = 1 + it$ in the final step, we can dominate $\|g_{1+it}\|_{\dot{F}^{-s_1}_{p_1',q_1'}(\mathbb{R}^n)}$ by

$$\|\mathit{\Gamma}^{-s}_{q'}(g)\|_{L^{p'}(\mathbb{R}^n)}^{1-\frac{p'}{p_1'}} \cdot \Big(\int_{\mathbb{R}^n} \Big(\sum_{j\in\mathbb{Z}} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'} \Big(\sum_{l=-\infty}^{j} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'}\Big)^{\frac{p'q_1'}{p_1'q'}-1}\Big)^{\frac{p_1'}{q_1'}} dx\Big)^{\frac{1}{p_1'}}. \tag{166}$$

If $\frac{p'q_1'}{p_1'q'} - 1 \geq 0$, we use $\sum_{l=-\infty}^{j} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'} \leq \sum_{l=-\infty}^{\infty} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'}$ and (71) of Remark 2.0.8 to deduce that

$$\|g_{1+it}\|_{\dot{F}^{-s_1}_{p_1',q_1'}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'})^{p'/q'} dx)^{1/p'} \lesssim \|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}. \tag{167}$$

If $\frac{p'q_1'}{p_1'q'} - 1 < 0$, we use the notations in (160), (161), (162) and argue as in (163) to get

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'} (\sum_{l=-\infty}^{j} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'})^{\frac{p'q_1'}{p_1'q'}-1} \\
=&\sum_{K=1}^{\infty} \sum_{j=I_K}^{I_{K-1}-1} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'} (\sum_{l=-\infty}^{j} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'})^{\frac{p'q_1'}{p_1'q'}-1} \\
\lesssim&\sum_{K=1}^{\infty} (\sum_{j=-\infty}^{I_{K-1}-1} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'}) (\sum_{l=-\infty}^{I_K} |2^{-ls}\mathcal{P}_n g_l(x)|^{q'})^{\frac{p'q_1'}{p_1'q'}-1} \\
\lesssim&2\sum_{K=1}^{\infty} 2^{-K\frac{p'q_1'}{p_1'q'}} \omega^{\frac{p'q_1'}{p_1'q'}} \lesssim (\sum_{j\in\mathbb{Z}} |2^{-sj}\mathcal{P}_n g_j(x)|^{q'})^{\frac{p'q_1'}{p_1'q'}}.
\end{aligned} \tag{168}$$

Inserting (168) into (166) yields (167). And hence we complete the proof of (129).

Upon the proof of both (128) and (129), we want to use Lemma 2.0.1 to prove Theorem 3.1.1. First, we define the complex extension of the left side of (112). Let f_z and g_z be as given in (126) and

$$F(z) = \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \psi_{2^{-k}} * T_z(f_z)(x) \cdot \overline{\psi_{2^{-k}} * g_z(x)} dx, \tag{169}$$

then by our assumption on the family $\{T_z\}_{z\in\mathbb{C}}$ of linear operators and the constructions of f_z and g_z, $F(z)$ is analytic on the open strip S and continuous on the closure $\bar{S}$. We assume for now that $F(z)$ satisfies the condition (36) of Lemma 2.0.1. Then by Hölder's inequalities

$$\begin{aligned}
|F(it)| \lesssim& \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |2^{ks_0}\psi_{2^{-k}} * T_{it}(f_{it})(x)| \cdot |2^{-ks_0}\psi_{2^{-k}} * g_{it}(x)| dx \\
\lesssim& \int_{\mathbb{R}^n} (\sum_{k\in\mathbb{Z}} |2^{ks_0}\psi_{2^{-k}} * T_{it}(f_{it})(x)|^{q_0})^{\frac{1}{q_0}} \cdot (\sum_{k\in\mathbb{Z}} |2^{-ks_0}\psi_{2^{-k}} * g_{it}(x)|^{q_0'})^{\frac{1}{q_0'}} dx \\
\lesssim& \|(\sum_{k\in\mathbb{Z}} |2^{ks_0}\psi_{2^{-k}} * T_{it}(f_{it})|^{q_0})^{\frac{1}{q_0}}\|_{L^{p_0}(\mathbb{R}^n)} \cdot \|(\sum_{k\in\mathbb{Z}} |2^{-ks_0}\psi_{2^{-k}} * g_{it}|^{q_0'})^{\frac{1}{q_0'}}\|_{L^{p_0'}(\mathbb{R}^n)} \\
\lesssim& M_0(t)\|f_{it}\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)} \|g_{it}\|_{\dot{F}^{-s_0}_{p_0',q_0'}(\mathbb{R}^n)} \qquad (170) \\
\lesssim& M_0(t)\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}, \qquad (171)
\end{aligned}$$

where (170) is due to assumption (107) and (171) is because of (128) and (129). Likewise, we also have

$$
\begin{aligned}
|F(1+it)| &\lesssim \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |2^{ks_1}\psi_{2^{-k}} * T_{1+it}(f_{1+it})(x)| \cdot |2^{-ks_1}\psi_{2^{-k}} * g_{1+it}(x)| dx \\
&\lesssim \int_{\mathbb{R}^n} (\sum_{k\in\mathbb{Z}} |2^{ks_1}\psi_{2^{-k}} * T_{1+it}(f_{1+it})(x)|^{q_1})^{\frac{1}{q_1}} \cdot (\sum_{k\in\mathbb{Z}} |2^{-ks_1}\psi_{2^{-k}} * g_{1+it}(x)|^{q_1'})^{\frac{1}{q_1'}} dx \\
&\lesssim \|(\sum_{k\in\mathbb{Z}} |2^{ks_1}\psi_{2^{-k}} * T_{1+it}(f_{1+it})|^{q_1})^{\frac{1}{q_1}}\|_{L^{p_1}(\mathbb{R}^n)} \cdot \|(\sum_{k\in\mathbb{Z}} |2^{-ks_1}\psi_{2^{-k}} * g_{1+it}|^{q_1'})^{\frac{1}{q_1'}}\|_{L^{p_1'}(\mathbb{R}^n)} \\
&\lesssim M_1(t) \|f_{1+it}\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)} \|g_{1+it}\|_{\dot{F}^{-s_1}_{p_1',q_1'}(\mathbb{R}^n)} \qquad (172) \\
&\lesssim M_1(t) \|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}, \qquad (173)
\end{aligned}
$$

where (172) is due to assumption (108) and (173) is because of (128) and (129). We also note that constants in (171) and (173) are independent of $t \in \mathbb{R}$. Therefore applying Lemma 2.0.1 to $F(z)$ along with (171), (173) and (38), (39) from Remark 2.0.1 yields for $0 < \theta < 1$

$$
\begin{aligned}
|F(\theta)| &\lesssim \exp\{\frac{\sin(\pi\theta)}{2}\int_{-\infty}^{\infty}[\frac{\log|F(it)|}{\cosh(\pi t)-\cos(\pi\theta)} + \frac{\log|F(1+it)|}{\cosh(\pi t)+\cos(\pi\theta)}]dt\} \\
&\lesssim \exp\{\frac{\sin(\pi\theta)}{2}\int_{-\infty}^{\infty}[\frac{\log M_0(t)}{\cosh(\pi t)-\cos(\pi\theta)} + \frac{\log M_1(t)}{\cosh(\pi t)+\cos(\pi\theta)}]dt \\
&\quad + \log(\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}\|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)})\} \\
&\lesssim M(\theta)\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}\|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}. \qquad (174)
\end{aligned}
$$

Recall (127), then by (174) we have proven (112). Then by the fact that the dual space of $\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)$ is $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$, (109) is true for all Schwartz functions $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$ is a Banach space when $1 < p, q < \infty, s \in \mathbb{R}$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$, we can pick a sequence of Schwartz functions $\{h_l\}_{l\in\mathbb{N}}$ that converges to $h \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)$ in $\|\cdot\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}$, then the linearity of T_θ says that

$$\|T_\theta(h_m) - T_\theta(h_l)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \lesssim M(\theta)\|h_m - h_l\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)},$$

thus $\{T_\theta(h_l)\}_{l\in\mathbb{N}}$ is Cauchy in $\|\cdot\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}$ and converges to a unique element in $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$. By defining $T_\theta(h) = \lim_{l\to\infty} T_\theta(h_l)$, we can obtain a unique bounded extension of T_θ to all of $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$ that also satisfies (109) and hence prove the theorem.

Last but not least, we prove that $F(z)$ defined in (169) does satisfy the condition (36) of Lemma 2.0.1. Let $z = \alpha + i\beta$ represent an arbitrary element in $\mathbb{C}$. Since $\{T_z\}_{z\in\mathbb{C}}$ is a family of linear operators taking values in the set of $\mathcal{S}'(\mathbb{R}^n)$-analytic functions in the open strip S, then by Definition 1.2.1, $\psi_{2^{-j}} * T_z(f_z)(x)$ is uniformly continuous and bounded in $\mathbb{R}^n \times \bar{S}$ for every $j \in \mathbb{Z}$ and the mapping $z \in \mathbb{C} \mapsto \psi_{2^{-j}} * T_z(f_z)(x)$ is analytic in S for every $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, thus this mapping also satisfies the conditions of Lemma 2.0.1. We invoke (44) of Remark 2.0.1 and obtain

$$\begin{aligned} |\psi_{2^{-j}} * T_z(f_z)(x)| \lesssim & \Big(\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{it+i\beta}(f_{it+i\beta})(x)| G_0(\alpha,t)dt \Big)^{1-\alpha} \\ & \cdot \Big(\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{1+it+i\beta}(f_{1+it+i\beta})(x)| G_1(\alpha,t)dt \Big)^{\alpha}, \end{aligned} \tag{175}$$

for every $j \in \mathbb{Z}$ and every $x \in \mathbb{R}^n$ and G_0, G_1 are given by (41), (42) and satisfy

$$\int_{-\infty}^{\infty} G_0(\alpha,t)dt = \int_{-\infty}^{\infty} G_1(\alpha,t)dt = 1$$

for $\alpha \in (0,1)$. We also want to prove the analytic function $z \in \mathbb{C} \mapsto \psi_{2^{-j}} * g_z(x)$ satisfies the condition (36) of Lemma 2.0.1. Recalling the definition of g_z given in (126), the support conditions of $\mathcal{F}_n\psi(2^{-j}\xi)$ and $\mathcal{F}_n\eta(2^{-k}\xi)$ as well as (155), we have

$$\begin{aligned} |\psi_{2^{-j}} * g_z(x)| \lesssim & \sum_{l=-2}^{2} \int_{\mathbb{R}^n} |\psi_{2^{-j}}(y)| \cdot |\eta_{2^{-j-l}} * V_{z,j+l}(x-y)| dy \\ \lesssim & \sum_{l=-2}^{2} \int_{\mathbb{R}^n} |\psi_{2^{-j}}(y)| \cdot 2^{\rho_5(\alpha)(j+l)} \mathit{\Gamma}_{q',j+l}^{-s}(g)(x-y)^{\rho_6(\alpha)} \\ & \cdot \|\mathit{\Gamma}_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}^{\rho_7(\alpha)} |\mathcal{P}_n g_{j+l}(x-y)|^{\rho_8(\alpha)} dy, \end{aligned} \tag{176}$$

where in (176) the parameters s, p, p', q, q' are determined by $\theta \in (0,1)$, s, p, q satisfy (106) and p', q' satisfy (120), $\rho_5, \rho_6, \rho_7, \rho_8$ are given by (121), (122), (123). Invoking (153), (154) and the fact that $0 \leq \operatorname{Re} z = \alpha \leq 1$ yields

$$\begin{aligned} |\psi_{2^{-j}} * g_z(x)| \lesssim & \sum_{l=-2}^{2} 2^{\rho_5(\alpha)(j+l)} \mathit{\Gamma}_{q',j+l}^{-s}(g)(x)^{\rho_6(\alpha)} \|\mathit{\Gamma}_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}^{\rho_7(\alpha)} |\mathcal{P}_n g_{j+l}(x)|^{\rho_8(\alpha)} \\ & \cdot \int_{\mathbb{R}^n} |\psi_{2^{-j}}(y)| (1 + 2^{j+l+1}|y|)^{\frac{n}{r}(|\rho_6(\alpha)| + \rho_8(\alpha))} dy \\ \lesssim & \sum_{l=-2}^{2} 2^{|j+l|(\frac{|s|q'}{q_0'} + \frac{|s|q'}{q_1'} + |s_0| + |s_1|)} \mathit{\Gamma}_{q',j+l}^{-s}(g)(x)^{\frac{p'}{p_0'} + \frac{p'}{p_1'} + \frac{q'}{q_0'} + \frac{q'}{q_1'}} \\ & \cdot \|\mathit{\Gamma}_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}^{1 + \frac{p'}{p_0'} + \frac{p'}{p_1'}} |\mathcal{P}_n g_{j+l}(x)|^{\frac{q'}{q_0'} + \frac{q'}{q_1'}}. \end{aligned} \tag{177}$$

We notice that $\varGamma_{q',j+l}^{-s}(g)(x)$, $\|\varGamma_{q'}^{-s}(g)\|_{L^{p'}(\mathbb{R}^n)}$ and $|\mathcal{P}_n g_{j+l}(x)|$ are finite for almost every $x \in \mathbb{R}^n$ when $g \in \mathcal{S}(\mathbb{R}^n)$ is in $\dot{F}_{p',q'}^{-s}(\mathbb{R}^n)$, thus (177) tells us that the mapping $z \in \mathbb{C} \mapsto \psi_{2^{-j}} * g_z(x)$ is uniformly bounded on the set $\bar{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ for every $j \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, and satisfies the condition (36) of Lemma 2.0.1. Using (44) of Remark 2.0.1, we obtain

$$|\psi_{2^{-j}} * g_z(x)| \lesssim \Big(\int_{-\infty}^{\infty} |\psi_{2^{-j}} * g_{it+i\beta}(x)| G_0(\alpha, t) dt \Big)^{1-\alpha} \cdot \Big(\int_{-\infty}^{\infty} |\psi_{2^{-j}} * g_{1+it+i\beta}(x)| G_1(\alpha, t) dt \Big)^{\alpha}, \tag{178}$$

for almost every $x \in \mathbb{R}^n$, every $j \in \mathbb{Z}$ and $z = \alpha + i\beta \in S$. Now we denote by $s_\alpha, p_\alpha, p'_\alpha, q_\alpha, q'_\alpha$ interpolation indices determined by $\alpha \in (0,1)$, that is, they satisfy the following relations

$$s_\alpha = (1-\alpha)s_0 + \alpha s_1, \tag{179}$$

$$\frac{1}{p_\alpha} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1} \qquad \frac{1}{q_\alpha} = \frac{1-\alpha}{q_0} + \frac{\alpha}{q_1}, \tag{180}$$

$$\frac{1}{p'_\alpha} = \frac{1-\alpha}{p'_0} + \frac{\alpha}{p'_1} \qquad \frac{1}{q'_\alpha} = \frac{1-\alpha}{q'_0} + \frac{\alpha}{q'_1}. \tag{181}$$

For $z = \alpha + i\beta \in S$, since $1 < p_\alpha, q_\alpha < \infty$ we estimate

$$\begin{aligned}
&(\sum_{j\in\mathbb{Z}} |2^{js_\alpha} \psi_{2^{-j}} * T_z(f_z)(x)|^{q_\alpha})^{1/q_\alpha} \\
\lesssim &(\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)s_0 q_\alpha} (\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{it+i\beta}(f_{it+i\beta})(x)| G_0(\alpha,t) dt)^{(1-\alpha)q_\alpha} \\
&\cdot 2^{j\alpha s_1 q_\alpha} (\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{1+it+i\beta}(f_{1+it+i\beta})(x)| G_1(\alpha,t) dt)^{\alpha q_\alpha})^{1/q_\alpha} &(182)\\
\lesssim &(\sum_{j\in\mathbb{Z}} 2^{js_0 q_0} (\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{it+i\beta}(f_{it+i\beta})(x)| G_0(\alpha,t) dt)^{q_0})^{\frac{1-\alpha}{q_0}} \\
&\cdot (\sum_{j\in\mathbb{Z}} 2^{js_1 q_1} (\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{1+it+i\beta}(f_{1+it+i\beta})(x)| G_1(\alpha,t) dt)^{q_1})^{\frac{\alpha}{q_1}}, &(183)
\end{aligned}$$

where (182) is due to (175) and (179) while (183) is because of Hölder's inequality and relation (180). We apply $\|\cdot\|_{L^{p_\alpha}(\mathbb{R}^n)}$ norm to (183) and use Hölder's inequality again with relation (180) then we can dominate $\|T_z(f_z)\|_{\dot{F}^{s_\alpha}_{p_\alpha,q_\alpha}(\mathbb{R}^n)}$ by the product of

$$\|(\sum_{j\in\mathbb{Z}} 2^{js_0 q_0} (\int_{-\infty}^{\infty} |\psi_{2^{-j}} * T_{it+i\beta}(f_{it+i\beta})(x)| G_0(\alpha,t) dt)^{q_0})^{\frac{1}{q_0}}\|_{L^{p_0}(\mathbb{R}^n)}^{1-\alpha}, \tag{184}$$

and

$$\|(\sum_{j\in\mathbb{Z}} 2^{js_1q_1}(\int_{-\infty}^{\infty}|\psi_{2^{-j}} * T_{1+it+i\beta}(f_{1+it+i\beta})(x)|G_1(\alpha,t)dt)^{q_1})^{\frac{1}{q_1}}\|_{L^{p_1}(\mathbb{R}^n)}^{\alpha}. \tag{185}$$

We use Minkowski's integral inequalities, assumption (107), assumption (105) and (128) in a sequence to obtain that

$$\begin{aligned}
(184)\lesssim&(\int_{-\infty}^{\infty}\|(\sum_{j\in\mathbb{Z}} 2^{js_0q_0}|\psi_{2^{-j}} * T_{it+i\beta}(f_{it+i\beta})|^{q_0})^{\frac{1}{q_0}}\|_{L^{p_0}(\mathbb{R}^n)}G_0(\alpha,t)dt)^{1-\alpha}\\
=&(\int_{-\infty}^{\infty}\|T_{it+i\beta}(f_{it+i\beta})\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)}G_0(\alpha,t)dt)^{1-\alpha}\\
\lesssim&(\int_{-\infty}^{\infty}M_0(t+\beta)\|f_{it+i\beta}\|_{\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)}G_0(\alpha,t)dt)^{1-\alpha}\\
\lesssim& e^{B|\beta|(1-\alpha)}\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}^{1-\alpha}(\int_{-\infty}^{\infty}e^{B|t|}G_0(\alpha,t)dt)^{1-\alpha},
\end{aligned}\tag{186}$$

where $B\in[0,\pi)$ is given by assumption (105). Likewise, we use Minkowski's integral inequalities, assumption (108), assumption (105), and (128) in a sequence to obtain that

$$\begin{aligned}
(185)\lesssim&(\int_{-\infty}^{\infty}\|(\sum_{j\in\mathbb{Z}} 2^{js_1q_1}|\psi_{2^{-j}} * T_{1+it+i\beta}(f_{1+it+i\beta})|^{q_1})^{\frac{1}{q_1}}\|_{L^{p_1}(\mathbb{R}^n)}G_1(\alpha,t)dt)^{\alpha}\\
=&(\int_{-\infty}^{\infty}\|T_{1+it+i\beta}(f_{1+it+i\beta})\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}G_1(\alpha,t)dt)^{\alpha}\\
\lesssim&(\int_{-\infty}^{\infty}M_1(t+\beta)\|f_{1+it+i\beta}\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}G_1(\alpha,t)dt)^{\alpha}\\
\lesssim& e^{B|\beta|\cdot\alpha}\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}^{\alpha}(\int_{-\infty}^{\infty}e^{B|t|}G_1(\alpha,t)dt)^{\alpha},
\end{aligned}\tag{187}$$

where $B\in[0,\pi)$ is given by assumption (105). Now we show that the integrals

$$\int_{-\infty}^{\infty}e^{B|t|}G_j(\alpha,t)dt \quad \text{for} \quad j=0,1$$

can be dominated by a positive finite constant A' and A' is independent of $\operatorname{Re} z=\alpha\in(0,1)$. First we pick $t_0>0$ so that $t>t_0$ implies $e^{-\pi t}<\frac{1}{4}$. Also, we notice the basic facts that $\sin\pi\alpha=\sin\pi(1-\alpha)$, $\lim_{t\to 0}\frac{\sin\pi t}{t}=\pi$ and $\lim_{t\to 1}\frac{\sin\pi t}{t}=0$, and thus both $\frac{\sin\pi\alpha}{1-\alpha}$ and

$\frac{\sin \pi\alpha}{\alpha}$ are uniformly bounded for $\alpha \in [0,1]$ and the upper bound is a positive finite constant independent of α. Then recalling the defining expression (41) of G_0, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{B|t|} G_0(\alpha,t)dt \\
\lesssim & e^{Bt_0} \int_0^{t_0} \frac{\sin \pi\alpha}{(1-\alpha)(\cosh \pi t - \cos \pi\alpha)} dt + \int_{t_0}^{\infty} \frac{e^{Bt}}{\cosh \pi t - \cos \pi\alpha} dt \qquad (188)\\
\lesssim & \frac{2e^{Bt_0} \sin \pi\alpha}{1-\alpha} \int_{-\infty}^{\infty} \frac{1}{e^{\pi t} + e^{-\pi t} - 2\cos \pi\alpha} dt + \int_{t_0}^{\infty} \frac{2e^{Bt}}{e^{\pi t} + e^{-\pi t} - 2\cos \pi\alpha} dt, \qquad (189)
\end{aligned}$$

where in the second integral of (188) we used the uniform boundedness of $\frac{\sin \pi\alpha}{1-\alpha}$. Using the change of variable $y = e^{\pi t}$, we get

$$\int \frac{1}{e^{\pi t} + e^{-\pi t} - 2\cos \pi\alpha} dt = \frac{1}{\pi \sin \pi\alpha} \cdot \arctan\left(\frac{e^{\pi t}}{\sin \pi\alpha} - \tan\left(\frac{\pi}{2} - \pi\alpha\right)\right).$$

We also use the estimate

$$\frac{2e^{Bt}}{e^{\pi t} + e^{-\pi t} - 2\cos \pi\alpha} = \frac{2e^{(B-\pi)t}}{1 + e^{-2\pi t} - 2e^{-\pi t}\cos \pi\alpha} \lesssim \frac{2e^{(B-\pi)t}}{1 - 2e^{-\pi t}} \lesssim 4e^{(B-\pi)t}$$

when $t > t_0$. Therefore putting back the above estimates into (189) yields

$$\int_{-\infty}^{\infty} e^{B|t|} G_0(\alpha,t)dt \lesssim 2e^{Bt_0} + \int_{t_0}^{\infty} 4e^{(B-\pi)t} dt =: A' < \infty \qquad (190)$$

where $B \in [0,\pi)$ is given by assumption (105) and A' is a positive finite constant that is independent of $\alpha \in (0,1)$. Likewise by (42) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{B|t|} G_1(\alpha,t)dt \\
\lesssim & e^{Bt_0} \int_0^{t_0} \frac{\sin \pi\alpha}{\alpha(\cosh \pi t + \cos \pi\alpha)} dt + \int_{t_0}^{\infty} \frac{e^{Bt}}{\cosh \pi t + \cos \pi\alpha} dt \qquad (191)\\
\lesssim & \frac{2e^{Bt_0} \sin \pi\alpha}{\alpha} \int_{-\infty}^{\infty} \frac{1}{e^{\pi t} + e^{-\pi t} + 2\cos \pi\alpha} dt + \int_{t_0}^{\infty} \frac{2e^{Bt}}{e^{\pi t} + e^{-\pi t} + 2\cos \pi\alpha} dt, \qquad (192)
\end{aligned}$$

where in the second integral of (191) we used the uniform boundedness of $\frac{\sin \pi\alpha}{\alpha}$. Using the change of variable $y = e^{\pi t}$, we get

$$\int \frac{1}{e^{\pi t} + e^{-\pi t} + 2\cos \pi\alpha} dt = \frac{1}{\pi \sin \pi\alpha} \cdot \arctan\left(\frac{e^{\pi t}}{\sin \pi\alpha} + \tan\left(\frac{\pi}{2} - \pi\alpha\right)\right).$$

We also use the estimate

$$\frac{2e^{Bt}}{e^{\pi t}+e^{-\pi t}+2\cos\pi\alpha}=\frac{2e^{(B-\pi)t}}{1+e^{-2\pi t}+2e^{-\pi t}\cos\pi\alpha}\lesssim\frac{2e^{(B-\pi)t}}{1-2e^{-\pi t}}\lesssim 4e^{(B-\pi)t}$$

when $t>t_0$. Thus putting back these estimates into (192) yields

$$\int_{-\infty}^{\infty}e^{B|t|}G_1(\alpha,t)dt\lesssim 2e^{Bt_0}+\int_{t_0}^{\infty}4e^{(B-\pi)t}dt=A'<\infty. \tag{193}$$

Now we infer from (184), (185), (186), (187), (190) and (193) that

$$\|T_z(f_z)\|_{\dot F^{s_\alpha}_{p_\alpha,q_\alpha}(\mathbb{R}^n)}\lesssim A'e^{B|\beta|}\|f\|_{\dot F^{s}_{p,q}(\mathbb{R}^n)} \tag{194}$$

where $\operatorname{Im}z=\beta\in\mathbb{R}$ and the constants on the right side of (194) are independent of $\operatorname{Re}z=\alpha\in(0,1)$. Next we show that $\|g_z\|_{\dot F^{-s_\alpha}_{p'_\alpha,q'_\alpha}(\mathbb{R}^n)}\lesssim\|g\|_{\dot F^{-s}_{p',q'}(\mathbb{R}^n)}$. Using (178) and (179), we have

$$\begin{aligned}\|g_z\|_{\dot F^{-s_\alpha}_{p'_\alpha,q'_\alpha}(\mathbb{R}^n)}\lesssim&\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}2^{-js_0(1-\alpha)q'_\alpha}\Big(\int_{-\infty}^{\infty}|\psi_{2^{-j}}*g_{it+i\beta}(x)|G_0(\alpha,t)dt\Big)^{(1-\alpha)q'_\alpha}\\ &\cdot2^{-js_1\alpha q'_\alpha}\Big(\int_{-\infty}^{\infty}|\psi_{2^{-j}}*g_{1+it+i\beta}(x)|G_1(\alpha,t)dt\Big)^{\alpha q'_\alpha}\Big)^{\frac{p'_\alpha}{q'_\alpha}}dx\Big)^{\frac{1}{p'_\alpha}}.\end{aligned} \tag{195}$$

Using Hölder's inequalities with (181) yields

$$\begin{aligned}(195)\lesssim&\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}2^{-js_0q'_0}\Big(\int_{-\infty}^{\infty}|\psi_{2^{-j}}*g_{it+i\beta}(x)|G_0(\alpha,t)dt\Big)^{q'_0}\Big)^{\frac{p'_0}{q'_0}}dx\Big)^{\frac{1-\alpha}{p'_0}}\\ &\cdot\Big(\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}2^{-js_1q'_1}\Big(\int_{-\infty}^{\infty}|\psi_{2^{-j}}*g_{1+it+i\beta}(x)|G_1(\alpha,t)dt\Big)^{q'_1}\Big)^{\frac{p'_1}{q'_1}}dx\Big)^{\frac{\alpha}{p'_1}}.\end{aligned} \tag{196}$$

Since $1<p'_0,q'_0,p'_1,q'_1<\infty$, by using Minkowski's integral inequalities, we can dominate (196) by

$$\Big(\int_{-\infty}^{\infty}\|g_{it+i\beta}\|_{\dot F^{-s_0}_{p'_0,q'_0}(\mathbb{R}^n)}G_0(\alpha,t)dt\Big)^{1-\alpha}\cdot\Big(\int_{-\infty}^{\infty}\|g_{1+it+i\beta}\|_{\dot F^{-s_1}_{p'_1,q'_1}(\mathbb{R}^n)}G_1(\alpha,t)dt\Big)^{\alpha}. \tag{197}$$

Applying (129) and (43) of Remark 2.0.1 to (197) yields

$$\|g_z\|_{\dot F^{-s_\alpha}_{p'_\alpha,q'_\alpha}(\mathbb{R}^n)}\lesssim\|g\|_{\dot F^{-s}_{p',q'}(\mathbb{R}^n)}, \tag{198}$$

where $s_\alpha, p'_\alpha, q'_\alpha$ satisfy (179) and (181) while s, p', q' are determined by (120), and the constant is independent of $\operatorname{Re} z = \alpha \in (0,1)$ and $\operatorname{Im} z = \beta \in \mathbb{R}$. Therefore by (194) and (198) we have obtained that

$$\begin{aligned}\log |F(z)| \lesssim & \log \|T_z(f_z)\|_{\dot{F}^{s_\alpha}_{p_\alpha,q_\alpha}(\mathbb{R}^n)} + \log \|g_z\|_{\dot{F}^{-s_\alpha}_{p'_\alpha,q'_\alpha}(\mathbb{R}^n)} \\ \lesssim & B|\beta| + \log(A'\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \|g\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}) \lesssim A'' e^{B|\beta|} \end{aligned} \tag{199}$$

for $z = \alpha + i\beta$ in the open strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, and the constant A'' relies on f, g and is independent of $\operatorname{Re} z = \alpha$, also the constant $B \in [0, \pi)$ is given by assumption (105). Recall (171) and (173) then we know $F(z)$ defined in (169) satisfies the condition (36) of Lemma 2.0.1 for all of $z \in \bar{S}$. Hereby we conclude the proof of Theorem 3.1.1. □

Remark 3.2.1. A careful inspection into (186) and (187) tells us that we can extend the exponential growth condition (105) of $M_j, j = 0, 1$ to an exponential growth condition of $\log M_j, j = 0, 1$ by assuming

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}} e^{-B|t|} \cdot \log M_j(t) \leq A < \infty \qquad \text{for} \qquad j = 0, 1. \tag{200}$$

for some $0 \leq A < \infty$ and $0 \leq B < \pi$, if we add in the statement of Theorem 3.1.1 the assumption that the family $\{T_z\}_{z \in \mathbb{C}}$ of linear operators is of admissible growth (cf. section 1.3.3 of [41]). That is, by assuming that $F(z)$ defined in (169) satisfies the condition (36) of Lemma 2.0.1 for all of $z \in \bar{S}$, we can omit the last part of the above proof and replace condition (105) by the assumption (200). With the new assumption (200) on $M_j, j = 0, 1$, the function $M(\theta)$ given in (110) is still finite for every $\theta \in (0, 1)$.

Remark 3.2.2. In sections 2.1.3 and 2.1.5 of [80], it has been revealed that when $1 \leq p, q \leq \infty$ and $-\infty < s < \infty$, the inhomogeneous space $F^s_{p,q}(\mathbb{R}^n)$ is the dual space of $f^{-s}_{p',q'}(\mathbb{R}^n)$ where $f^{-s}_{p',q'}(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{F^{-s}_{p',q'}(\mathbb{R}^n)}$. We believe the counterpart for the homogeneous space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ also exists and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ can be seen as the dual space of $\dot{f}^{-s}_{p',q'}(\mathbb{R}^n)$ where $\dot{f}^{-s}_{p',q'}(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{\dot{F}^{-s}_{p',q'}(\mathbb{R}^n)}$ in the space of tempered distributions modulo polynomials. But whether we can generalize Theorem 3.1.1 to the case where $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $-\infty < s_0, s_1 < \infty$ is still unknown.

4.0 A Fourier Multiplier Theorem For Sequences Of Functions

4.1 Chapter Introduction

In this chapter, we prove a Fourier multiplier theorem for sequences of functions defined on $\mathbb{R}^n$ when $1 < p, q < \infty$. Results in this chapter are known in the esteemed literature such as [74], [21], [103], but the author would like to derive these results independently as part of his study for the Ph.D. degree. Recall (31) and (32), then the statement of the theorem is below.

Theorem 4.1.1. If $1 < p, q < \infty$, $\tau > \frac{n}{2}$, then for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ and $\{m_k\}_{k\in\mathbb{Z}}$ of functions defined on $\mathbb{R}^n$ that satisfy the following conditions:

$$\Big(\int_{\mathbb{R}^n}(\sum_{k\in\mathbb{Z}}|f_k(x)|^q)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}} < \infty, \tag{201}$$

and the n-dimensional distributional Fourier transform $\mathcal{F}_n f_k$ is supported in the annulus $\{\xi \in \mathbb{R}^n : 2^{k-1} \leq |\xi| < 2^{k+1}\}$ for every $k \in \mathbb{Z}$, and each m_k is a function in the inhomogeneous Sobolev space $L^2_\tau(\mathbb{R}^n)$ and

$$\operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} < \infty, \tag{202}$$

we have

$$\Big(\int_{\mathbb{R}^n}(\sum_{k\in\mathbb{Z}}|\mathcal{F}_n^{-1}(m_k\mathcal{F}_n f_k)(x)|^q)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}} \lesssim \operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \cdot \Big(\int_{\mathbb{R}^n}(\sum_{k\in\mathbb{Z}}|f_k(x)|^q)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}. \tag{203}$$

The statement of the theorem originates from Theorem 2 in section 2.4.9 of [93] but the original literature did not provide proof of the result nor explained why the factor $\operatorname{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}$ would appear on the right side of the inequality. Since this factor $\operatorname{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}$ plays a crucial role in the theorem's other applications, it is interesting to give an independent proof of the Fourier multiplier theorem here in this paper after studying related materials in [41], [90] and [93]. We use an argument of Hörmander type to show that the Fourier multiplier theorem is valid when the constant on the right has a factor $\operatorname{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}$ and when $1 < p, q < \infty$. The proof of Theorem 4.1.1 is given in section 4.2.

Since we consider the homogeneous Triebel-Lizorkin norm $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ is the $\|\cdot\|_{L^p(\mathbb{R}^n)}$-norm of the $\|\cdot\|_{l^q}$-norm of the sequence $\{2^{js}\psi_{2^{-j}} * f\}_{j\in\mathbb{Z}}$, we state in the following corollary that the classical Hörmander's condition is sufficient for a function to be not only a $L^p(\mathbb{R}^n)$-multiplier (see Theorem 6.2.7 of [41]) but also a $\dot{F}^s_{p,q}(\mathbb{R}^n)$-multiplier.

Corollary 4.1.1. Let $1 < p, q < \infty$, $s \in \mathbb{R}$. And let $m(\xi)$ be a function that is continuously differentiable up to order $[\frac{n}{2}] + 1$ and satisfy the classical Hörmander's condition:

$$\operatorname*{ess\,sup}_{R>0} R^{-n+2|\alpha|} \cdot \int_{\frac{R}{4}<|\xi|<4R} |(\partial^\alpha m)(\xi)|^2 d\xi \lesssim A^2 < \infty \tag{204}$$

for all multi-indices α with $|\alpha| \leq [\frac{n}{2}] + 1$, and A is a positive finite constant. Then m is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$-multiplier, that is, for all $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ we have

$$\|\mathcal{F}_n^{-1}(m\mathcal{F}_n f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim A \cdot \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{205}$$

The proof of the above corollary can be found in section 4.3. We also note that in most applications, the condition on the decay of derivatives of the multiplier

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} \tag{206}$$

implies the condition (204) and is easier to verify.

We also introduce mathematicians' results related to Fourier multiplier theorems. In [16], A. Bényi, L. Grafakos, K. Gröchenig, and K. Okoudjou used Gabor frames and methods from time-frequency analysis to study the boundedness of a general class of Fourier multipliers, in particular of the Hilbert transform, on modulation spaces $\mathcal{M}^{p,q}$ for $1 < p < \infty$ and $1 \leq q \leq \infty$. In general, the Fourier multipliers in this class fail to be bounded on L^p spaces. In [24], Y.-K. Cho and D. Kim studied Fourier multiplier operators whose symbols satisfy a generalization of Hörmander's condition on the homogeneous Besov-Lipschitz spaces and by applying their result to the symbol $|\xi|^{-\alpha}$, they also obtained a Sobolev imbedding result. In [53], A. Karlovich and E. Shargorodsky considered the abstract Lorentz spaces $\Lambda_q(X)$ where $0 < q \leq \infty$ and X is a Banach function space satisfying the weak doubling property, and proved that the space of Fourier multipliers acting from $\Lambda_q(X)$ to $\Lambda_\infty(X)$ is continuously embedded into L^∞ for every q in the full range above. In [4], N. Asmar, F. Newberger, and S. Watson defined a new type of multiplier operator on $L^p(\mathbb{T}^N)$ for $1 < p < \infty$ where $\mathbb{T}^N$ is the N-dimensional torus, and their main theorem is known to be the first application of the tangent sequences from probability theory to harmonic analysis and it proves that the operator norms of these multipliers are independent of the dimension N. In [39], A. Figà-Talamanca and J. F. Price applied the theory of random Fourier series to construct a type of Rudin-Shapiro sequence and then used this sequence to obtain slightly more restricted versions of several known families of strict inclusions for Fourier multipliers over infinite compact groups and over infinite compact Lie groups. In [14], T. A. Bui and X. T. Duong developed a theory of homogeneous and inhomogeneous Besov and Triebel–Lizorkin spaces associated with the Hermite operator $H = -\Delta + |x|^2$ on the Euclidean space $\mathbb{R}^n$ and proved the boundedness of negative powers and spectral multipliers of the Hermite operators on some appropriate Besov and Triebel–Lizorkin spaces. In [22], L. Chen, G. Lu, and X. Luo proved that under the limited smoothness conditions, multi-parameter Fourier multiplier operators are bounded on multi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces, and they also proved the boundedness of multi-parameter Fourier multiplier operators on weighted multi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces when the Fourier multiplier is only assumed with limited smoothness. In [28], M. Congo and M. F. Ouedraogo studied the boundedness of nonregular pseudo-differential operators on variable exponent Besov-Morrey spaces, the x-regularity of whose symbols is measured in Hölder-Zygmund spaces. In [6], S. Baron, E. Liflyand, and U. Stadtmüller investigated a notion of complementary space for double Fourier series of functions of bounded variation and gave sufficient conditions for when a double sequence is a multiplier of a class. In [23], Y.-K. Cho gave a set of continuous characterizations for the homogeneous Triebel-Lizorkin spaces and used them to deduce mapping properties of Fourier multiplier operators on Triebel-Lizorkin and Besov-Lipschitz spaces, and the symbols of these Fourier multiplier operators satisfy a generalization of Hörmander's condition. In [18], A. Carbery proved an extension of the Marcinkiewicz multiplier theorem for $L^p(\mathbb{R}^n)$ $(1 < p)$ with the help of the so-called "differentiation in lacunary directions" operator and the usual argument with Rademacher functions. In [60], H.-G. Leopold stated and proved a vector-valued multiplier theorem for pseudo-differential operators, which generalized the corresponding results in section 1.6.3 of [93]. In [88], T. Steenstrup provided a closed expression for the completely bounded Fourier multiplier norm of the spherical functions on the generalized Lorentz groups and proved that there is no uniform bound on the completely bounded Fourier multiplier norm of the spherical functions on the generalized Lorentz groups. In [74], B. J. Park studied sharp generalizations of $\dot{F}^0_{p,q}(\mathbb{R}^n)$ multiplier theorems of Mikhlin-Hörmander type, whose sufficient conditions involve the Herz spaces $K^s_{u,t}$, and these results definitely improved and generalized Triebel's results in [91] and [93]. In [21], D. Cardona and M. Ruzhansky proved the boundedness of Fourier multipliers on a compact Lie group when acting on Triebel-Lizorkin spaces with Hörmander-Mihlin-Marcinkiewicz type conditions, and their results covered the sharp Hörmander-Mihlin theorem on Lebesgue spaces and also other historical results on this subject. In [38], H. G. Feichtinger and G. Narimani applied techniques concerning pointwise multipliers for generalized Wiener amalgam spaces and provided a complete characterization of the Fourier multipliers of modulation spaces, and they also showed that any function with $([d/2]+1)$-times bounded derivatives is a Fourier multiplier for all modulation spaces $M^{p,q}(\mathbb{R}^d)$ for $1 < p < \infty$ and $1 \leq q \leq \infty$. In [25], G. Cleanthous, A. G. Georgiadis, and M. Nielsen derived a boundedness result for Fourier multipliers on anisotropic decomposition spaces of modulation and Triebel-Lizorkin type. In [103], D. Yang, W. Yuan, and C. Zhuo obtained the boundedness of Fourier multipliers on Triebel-Lizorkin-type spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, Besov-Hausdorff spaces $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, and Triebel-Lizorkin-Hausdorff spaces $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, with symbols satisfying some generalized Hörmander's condition, and their results covered the corresponding existing results for the classical Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ when $\tau = 0$. In [3], W. Arendt and S. Bu considered the Fourier series of functions in $L^p(0, 2\pi; X)$ where X is a Banach space and presented the Marcinkiewicz theorem for operator-valued multipliers and gave applications to differential equations. In [81], M. Ruzhansky and J. Wirth proved L^p Fourier multiplier theorems for invariant and noninvariant operators on compact Lie groups and gave applications to a-priori estimates for non-hypoelliptic operators. In [57], V. Kumar and M. Ruzhansky proved the $L^p - L^q$ boundedness of (k, a)-generalised Fourier multipliers by establishing Paley inequality and Hausdorff-Young-Paley inequality for (k, a)-generalised Fourier transform. In [8], E. Berkson proved that for continuous bounded functions having uniformly bounded r-variations on $\mathbb{T}$, the associated Fourier series of the operator ergodic Stieltjes convolution converges at each point in $\mathbb{T}$ with respect to the strong operator topology, and the results also encompassed the Fourier multiplier actions of these functions in the setting of A_p-weighted sequence spaces. In [5], R. Bañuelos and A. Osekowski identified the L^p-norms of certain Fourier multipliers such as the second order Riesz transforms and some Lévy multipliers, and they used the argument of Geiss, Montgomery-Smith, and Saksman, and a new martingale inequality in the proofs of their main results. In [50], Petr Honzík studied the associated maximal function of a type of bilinear operator whose Fourier multiplier is defined on $\mathbb{R}^{2d}$ and satisfies a certain decay condition and proved that such a maximal function maps $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with the norm at most a constant multiple $\sqrt{\log(N+2)}$, where p_1, p_2, p satisfy $1 < p_1, p_2 < \infty$, $\frac{1}{2} < p < \infty$, and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, furthermore the author also provided an example to indicate the sharpness of this result. In [26], G. Cleanthous, A. G. Georgiadis, and M. Nielsen constructed smooth molecular decompositions for holomorphic Besov and Triebel-Lizorkin spaces on the unit disk of the complex plane, obtained a boundedness result for Fourier multipliers, and furthermore provided equivalent norms for the spaces under consideration. The implications of the results of [26] on Hardy and Hardy-Sobolev spaces and the boundedness of coefficient multipliers were also studied by the authors. In [75], L-E. Persson, L. Sarybekova, and N. Tleukhanova proved a generalization and sharpening of the Lizorkin theorem concerning Fourier multipliers between L^p and L^q, the proof of which used some multidimensional Lorentz spaces and an interpolation technique of Sparr type as crucial tools. In [20], D. Cardona and M. Ruzhansky proved spectral and Fourier multiplier theorems in the setting of graded Lie groups and presented a Nikolskii-type inequality and the Littlewood–Paley theorem. In [99], C. Watari investigated a class of multiplier transformations of Walsh Fourier series, which shares most of the properties with fractional integration. In [105], D. Yang, W. Yuan, and C. Zhuo introduced the Musielak–Orlicz Besov-type spaces and the Musielak-Orlicz Triebel–Lizorkin-type spaces and obtained the boundedness on these spaces of Fourier multipliers with symbols satisfying some generalized Hörmander condition. The spaces considered in [105] included Musielak–Orlicz Hardy spaces, unweighted and weighted Besov(-type) and Triebel-Lizorkin(-type) spaces as special cases. In [63], T. R. McConnell obtained analogues of the Mihlin multiplier theorem and Littlewood-Paley inequalities for functions with values in a Banach space having the unconditionality property for martingale difference sequences. In [36], D. E. Edmunds, V. Kokilashvili, and M. Alexander studied two-weighted estimates for multipliers of Fourier transforms and derived conditions for the pairs of weights ensuring two-weight estimates for several classes of multipliers in Triebel-Lizorkin spaces. Furthermore, the authors of [36] presented examples of pairs of weights governing two-weighted estimates for Fourier multipliers. In [73], B. P. Osilenker studied multipliers for Fourier series in polynomials orthogonal in continuous-discrete Sobolev spaces and obtained existence results and norm estimates for the multiplier operator. In [49], Y. Heo, F. Nazarov, and A. Seeger investigated connections between radial Fourier multipliers on $\mathbb{R}^d$ and certain conical Fourier multipliers on $\mathbb{R}^{d+1}$ and obtained a new weak type endpoint bound for the Bochner-Riesz multipliers associated with the light cone in $\mathbb{R}^{d+1}$, where $d \geq 4$. In [29], E. Curcă proved that if $d \geq 2$, every Fourier multiplier on $\dot{W}^{l,1}(\mathbb{R}^d)$ or on $\dot{W}^{l,\infty}(\mathbb{R}^d)$ is a bounded continuous function on $\mathbb{R}^d$ for every integer $l \geq 1$ and this result is a generalization of the corresponding result proven by Kazaniecki and Wojciechowski in 2013. In [54], K. Kazaniecki and M. Wojciechowski proved that every Fourier multiplier on the homogeneous Sobolev space is a continuous function. In [111], F. Zimmermann generalized the classical Fourier multiplier theorems of Littlewood-Paley, Marcinkiewicz, and Mikhlin to the vector-valued setting in d dimension using a tensor product approach. In [1], H. Amann extended and complemented the theory of vector-valued Besov spaces by proving that translation-invariant operators with operator-valued symbols act continuously on Besov spaces of Banach-space-valued distributions and gave applications to a variety of problems from elliptic and parabolic differential and integrodifferential equations. In [86], V. B. Shakhmurov studied the operator-valued Fourier multiplier theorems in E-valued weighted Lebesgue and Besov spaces, proved the embedding theorems in weighted Besov-Lions type spaces, and established the Ehrling-Nirenberg-Gagliardo type sharp estimates. In [2], W. Arendt and S. Bu proved that the analogue of Marcinkiewicz's Fourier multiplier theorem on $L^p(\mathbb{T})$ is true for the Besov space $B^s_{p,q}(\mathbb{T}; X)$ if and only if $1 < p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$, and X is a UMD-space, furthermore the authors also obtained a periodic Fourier multiplier theorem by imposing stronger conditions and then used their results to characterize maximal regularity of periodic Cauchy problems. In [7],

B. Barraza Martinez, I. Gonzalez Martinez, and J. Hernandez Monzon proved that a sequence $M : \mathbb{Z}^n \to \mathcal{L}(E)$ of bounded variation is a Fourier multiplier of the Besov space $B^s_{p,q}(\mathbb{T}^n, E)$ for $1 < p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$, and E is a Banach space if and only if E is a UMD-space, and then studied the solvability of two abstract Cauchy problems with periodic boundary conditions. In [100], R. Xia and X. Xiong developed some Fourier multiplier theorems for square functions and then studied the operator-valued Triebel-Lizorkin spaces on $\mathbb{R}^d$. In [19], A. Carbery, G. Gasper, and W. Trebels gave the best possible sufficient conditions, in terms of differentiability and growth properties, for a radial function to be an $L^p(\mathbb{R}^2)$ Fourier multiplier and established a multiplier theorem for a class of functions of which the Bochner-Riesz multipliers are prototypical members. In [109], G. Zhao, J. Chen, and W. Guo studied the boundedness properties of the Fourier multiplier operator $e^{i\mu(D)}$ on α-modulation spaces and Besov spaces and improved the conditions for the boundedness of Fourier multipliers with compact supports and for the boundedness of $e^{i\mu(D)}$ on α-modulation spaces. In [79], J. Rozendaal and M. Veraar developed the theory of Fourier multiplier operators $T_m : L^p(\mathbb{R}^d; X) \to L^q(\mathbb{R}^d; Y)$ for Banach spaces X and Y, $1 \leq p \leq q \leq \infty$ and $m : \mathbb{R}^d \to \mathcal{L}(X, Y)$ is an operator-valued symbol, furthermore the authors also showed that when $p < q$, other geometric conditions on X and Y, such as the notions of type and cotype, can be used to study Fourier multipliers, moreover they also obtained boundedness results for T_m without any smoothness properties of m. In [64], C. Muscalu, T. Tao, and C. Thiele unified previous results by C. Calderon, by Coifman and Meyer, and by Lacey and Thiele and proved the boundedness of the multi-linear operator T where the associated multiplier belongs to a class of functions that are singular on a subspace of the $(n-1)$-dimensional vector space $\Gamma := \{\xi \in \mathbb{R}^n : \xi_1 + \cdots + \xi_n = 0\}$. Their result can be viewed as a generalization of Hölder's inequality and also includes the bilinear Hilbert transform as a special case. In [44], L. Grafakos and B. J. Park proved an improvement of Calderón and Torchinsky's version of the Hörmander multiplier theorem on Hardy spaces H^p $(0 < p < \infty)$, substituting the Sobolev space by the Lorentz-Sobolev space, and their result is sharp in the sense that the preceding Lorentz-Sobolev space cannot be replaced by a larger Lorentz-Sobolev space. In [72], A. Osekowski established a related estimate for a large class of Fourier multipliers in the more general setting of continuous-time martingales. In [52], M. Junge, T. Mei, and J.

Parcet investigated Fourier multipliers on the compact dual of arbitrary discrete groups and proved an Hörmander-Mihlin multiplier theorem for finite-dimensional cocycles with optimal smoothness conditions, furthermore the authors also found the Littlewood-Paley type inequalities in group von Neumann algebras and characterize $L^\infty \to BMO$ boundedness for radial Fourier multipliers. In [43], L. Grafakos, D. He, P. Honzik, and H. V. Nguyen discussed $L^p(\mathbb{R}^n)$ boundedness for Fourier multiplier operators that satisfy the hypotheses of the Hörmander multiplier theorem in terms of an optimal condition that relates the distance $|\frac{1}{p} - \frac{1}{2}|$ to the smoothness s of the associated multiplier measured in some Sobolev norm and provided new counterexamples to justify the optimality of the condition $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$, furthermore the authors also discussed the endpoint case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$. In [37], C. Fefferman proved that the Fourier multiplier operator whose multiplier is the characteristic function of the unit ball is bounded only on L^2 and disproved the L^p-boundedness of such an operator for $p \neq 2$. In [106], A. Ydyrys, L. Sarybekova, and N. Tleukhanova studied the multipliers of multiple Fourier series for a regular system on anisotropic Lorentz spaces and gave the sufficient conditions for a sequence of complex numbers to be a multiplier of multiple trigonometric Fourier series from $L^p[0;1]^n$ to $L^q[0;1]^n$, $p < q$. In [65], S. Neuwirth and É. Ricard inspected the relationship between relative Fourier multipliers on noncommutative Lebesgue-Orlicz spaces of a discrete group and relative Toeplitz-Schur multipliers on Schatten-von-Neumann-Orlicz classes. In [17], A. Bényi, K. Gröchenig, K. A. Okoudjou, and L. G. Rogers investigated the boundedness of unimodular Fourier multipliers on modulation spaces and proved that the multipliers with general symbol $e^{i|\xi|^\alpha}$ $(0 \leq \alpha \leq 2)$ are bounded on all modulation spaces and deduced that the phase-space concentration of the solutions to the free Schrödinger and wave equations are preserved, furthermore the authors also obtained boundedness results on modulation spaces for singular multipliers $|\xi|^{-\delta} \sin(|\xi|^\alpha)$ for $0 \leq \delta \leq \alpha$. In [61], Y. Liu proved the boundedness of bilinear Fourier multiplier operators on the variable exponent Besov spaces using Fourier transform, inverse Fourier transform, and the Littlewood-Paley decomposition technique. In [35], D. Drihem and W. Hebbache studied the boundedness of nonregular pseudodifferential operators, with symbols belonging to some vector-valued Besov spaces, on Besov spaces with variable smoothness and integrability, and these symbols include the classical Hörmander type. In [82], L. O. Sarybekova, T. V. Tararykova, and N. T.

Tleukhanova proved a generalization of the Lizorkin theorem on Fourier multipliers using the so-called net spaces and interpolation theorems, and the authors also gave an example of a Fourier multiplier which satisfies the assumptions of the generalized theorem but does not satisfy the assumptions of the Lizorkin theorem. In [92], H. Triebel stated the natural Fourier multipliers for the spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. In [76], L.-E. Persson, L. Sarybekova, and N. Tleukhanova proved a new Fourier series multiplier theorem of Lizorkin type for the case $1 < q < p < \infty$ in the setting of a general strong regular system, and if it is a trigonometric system, their result implies an analogy of the original Lizorkin theorem. In [96], R. M. Trigub proved new statements regarding multipliers of trigonometric Fourier series in the space C of continuous periodic functions. In [40], M. Girardi and L. Weis proved a general Fourier multiplier theorem for operator-valued multiplier functions on vector-valued Besov spaces where the required smoothness of the multiplier functions depends on the geometry of the underlying Banach space, and their main result covers many classical multiplier conditions, such as Mihlin and Hörmander conditions. In [32], P. Dintelmann presented a discrete characterization of Besov and Triebel spaces which is used to determine various classes of Fourier multipliers for these spaces and recovered results of R. Johnson. In [70], T. Noi proved Fourier multiplier theorems on Besov and Triebel-Lizorkin spaces with variable exponents, and as the consequences of the main results, the author also obtained Fourier multiplier theorems on variable Bessel potential spaces, variable Sobolev spaces, and variable Lebesgue spaces. In [27], G. Cleanthous, A. G. Georgiadis, and M. Nielsen introduced a new general Hörmander type condition involving anisotropies and mixed norms, and the authors also obtained boundedness results for Fourier multipliers on anisotropic Besov and Triebel-Lizorkin spaces of distributions with mixed Lebesgue norms. In [78], T. S. Quek obtained a sufficient condition for a bounded measurable function on $\mathbb{R}^n$ to be a Fourier multiplier on $H^p_\alpha(\mathbb{R}^n)$ for $0 < p < 1$ and $-n < \alpha \leq 0$ using Herz spaces and generalized a recent result obtained by Baernstein and Sawyer. In [13], H.-Q. Bui, T. A. Bui, and X. T. Duong developed the theory of weighted Besov spaces and weighted Triebel–Lizorkin spaces built upon a homogeneous space X associated with a nonnegative self-adjoint operator L on $L^2(X)$. The operator L satisfies the Gaussian upper bounds on its heat kernels, the parameters take value in the full range, and the weight function is in the Muckenhoupt weight class A_∞. The authors of [13] also proved that their new spaces satisfy important features such as continuous characterizations in terms of square functions, atomic decompositions, and identifications with some well-known function spaces such as Hardy-type spaces and Sobolev-type spaces, furthermore they applied their results to prove the boundedness of the fractional power of the operator L, the spectral multiplier of L in these new function spaces. In [62], Y. Liu and J. Zhao proved the boundedness of bilinear Fourier multiplier operators on variable exponent Triebel-Lizorkin spaces.

4.2 Proof of Theorem 4.1.1

Proof. We want to use Lemma 2.0.2 to prove the theorem. We let $A_0 = A_1 = l^q$ be the reflexive Banach spaces as in Lemma 2.0.2. We fix a nonnegative Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $spt.\varphi \subseteq \{\frac{1}{4} \le |\xi| < 4\}$, $\varphi = 1$ on $\{\frac{1}{2} \le |\xi| < 2\}$ and thus $\varphi(2^{-k}\xi) = 1$ on $spt.\mathcal{F}_n f_k$. For a sequence $f(x) = \{f_k(x)\}_{k\in\mathbb{Z}}$ satisfying (201), we consider the operator

$$\mathscr{K} f(x) := \int_{\mathbb{R}^n} \langle K(x-y), f(y)\rangle dy, \tag{207}$$

where for every $x \in \mathbb{R}^n$, $K(x)$ is an infinite diagonal matrix that maps from l^q to l^q with diagonal elements $\{K_k(x)\}_{k\in\mathbb{Z}} = \{\mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(x)\}_{k\in\mathbb{Z}}$. By using Cauchy-Schwartz inequality, Plancherel's identity and condition (202), one can verify that $m_k(\xi)\varphi(2^{-k}\xi)$ is an integrable function on $\mathbb{R}^n$ and hence each K_k in the sequence is well-defined. Therefore componentwisely $\mathscr{K} f(x)$ can be written as

$$\mathscr{K} f(x) = \{K_k * f_k(x)\}_{k\in\mathbb{Z}}.$$

And due to the support condition of φ, it suffices to prove that the operator $\mathscr{K}$ satisfies all the conditions of Lemma 2.0.2 and the conclusion of Lemma 2.0.2 will tell us that

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} \big(\sum_{k\in\mathbb{Z}} |\mathcal{F}_n^{-1}(m_k \mathcal{F}_n f_k)(x)|^q\big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \\
=&\Big(\int_{\mathbb{R}^n} \big(\sum_{k\in\mathbb{Z}} |K_k * f_k(x)|^q\big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \\
\lesssim&\operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \cdot \Big(\int_{\mathbb{R}^n} \big(\sum_{k\in\mathbb{Z}} |f_k(x)|^q\big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}.
\end{aligned} \tag{208}$$

Notice that

$$\|K(x-y)-K(x)\|_{L(l^q,l^q)} \lesssim \operatorname*{ess\,sup}_{k\in\mathbb{Z}} |K_k(x-y)-K_k(x)| \lesssim \sum_{k\in\mathbb{Z}} |K_k(x-y)-K_k(x)| \quad (209)$$

where $K_k(x) = \mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(x)$ for each $k \in \mathbb{Z}$. We want to use the condition (51) given in Remark 2.0.2 instead of condition (45). First, we give two estimates of $\int_{|x|\geq 2|y|} |K_k(x-y)-K_k(x)|dx$. Since $\tau > \frac{n}{2}$, we pick t and t' so that

$$0 < t < \min\{1, \tau - \frac{n}{2}\} \qquad \text{and} \qquad \tau + \frac{n}{2} < t'. \quad (210)$$

Then

$$\int_{|x|\geq 2|y|} |K_k(x-y)-K_k(x)|dx \lesssim \int_{|x|\geq 2|y|} |K_k(x-y)|dx + \int_{|x|\geq 2|y|} |K_k(x)|dx \lesssim \int_{|x|\geq |y|} |K_k(x)|dx. \quad (211)$$

And by Hölder's inequality, we have

$$\begin{aligned}\int_{|x|\geq|y|} |K_k(x)|dx \lesssim& (1+|2^k y|^2)^{-\frac{t}{2}} \cdot \int_{|x|\geq|y|} (1+|2^k x|^2)^{\frac{t-\tau}{2}} \cdot (1+|2^k x|^2)^{\frac{\tau}{2}} |K_k(x)|dx \\ \lesssim& (1+|2^k y|^2)^{-\frac{t}{2}} \cdot (\int_{\mathbb{R}^n} (1+|2^k x|^2)^{t-\tau} dx)^{\frac{1}{2}} \\ &\cdot (\int_{\mathbb{R}^n} (1+|2^k x|^2)^{\tau} |\mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(x)|^2 dx)^{\frac{1}{2}}. \quad (212)\end{aligned}$$

We apply the change of variable $z = 2^k x$ and the property of Fourier transform that $\mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(2^{-k}z) = \mathcal{F}_n^{-1}(2^{kn} m_k(2^k\xi)\varphi(\xi))(z)$ then (212) is dominated by

$$\begin{aligned}&(1+|2^k y|^2)^{-\frac{t}{2}} \cdot (\int_{\mathbb{R}^n} (1+|z|^2)^{t-\tau} dz)^{\frac{1}{2}} \cdot (\int_{\mathbb{R}^n} (1+|z|^2)^{\tau} |\mathcal{F}_n^{-1}(m_k(2^k\xi)\varphi(\xi))(z)|^2 dz)^{\frac{1}{2}} \\ \lesssim& (1+|2^k y|^2)^{-\frac{t}{2}} \cdot (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} (1+|z|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(m_k(2^k\xi))(w)| \cdot |\mathcal{F}_n^{-1}\varphi(z-w)|dw)^2 dz)^{\frac{1}{2}} \quad (213) \\ \lesssim& (1+|2^k y|^2)^{-\frac{t}{2}} \cdot (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} (1+|w|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(m_k(2^k\xi))(w)| \\ &\cdot (1+|z-w|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}\varphi(z-w)|dw)^2 dz)^{\frac{1}{2}}, \quad (214)\end{aligned}$$

where (213) is because by the choice of t the integral $\int_{\mathbb{R}^n}(1+|z|^2)^{t-\tau}dz$ converges. We use Young's inequality for convolutions and then Hölder's inequality to obtain

$$
\begin{aligned}
(214)\lesssim&(1+|2^k y|^2)^{-\frac{t}{2}}\|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\cdot\int_{\mathbb{R}^n}(1+|z|^2)^{\frac{\tau}{2}}|\mathcal{F}_n^{-1}\varphi(z)|dz\\
=&(1+|2^k y|^2)^{-\frac{t}{2}}\|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\cdot\int_{\mathbb{R}^n}(1+|z|^2)^{\frac{\tau-t'}{2}}\cdot(1+|z|^2)^{\frac{t'}{2}}|\mathcal{F}_n^{-1}\varphi(z)|dz\\
\lesssim&(1+|2^k y|^2)^{-\frac{t}{2}}\|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\|\varphi\|_{L^2_{t'}(\mathbb{R}^n)}\cdot(\int_{\mathbb{R}^n}(1+|z|^2)^{\tau-t'}dz)^{\frac{1}{2}}\\
\lesssim&(1+|2^k y|^2)^{-\frac{t}{2}}\cdot\operatorname*{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)},
\end{aligned}
\tag{215}
$$

where due to the choice of t' in (210), $\int_{\mathbb{R}^n}(1+|z|^2)^{\tau-t'}dz$ is convergent and the constant in (215) is independent of $k\in\mathbb{Z}$. Combining (211), (212), (214) and (215) yields the first estimate

$$
\int_{|x|\geq 2|y|}|K_k(x-y)-K_k(x)|dx\lesssim(2^k|y|)^{-t}\cdot\operatorname*{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)},\tag{216}
$$

where $\tau>\frac{n}{2}$, t satisfies (210) and the constant in (216) is independent of $k\in\mathbb{Z}$ and $0\neq y\in\mathbb{R}^n$. For the second estimate, we notice that

$$
|K_k(x-y)-K_k(x)|\lesssim\int_0^1|\nabla K_k(x-ty)|\cdot|y|dt
$$

and $|x-ty|\geq|y|$ if $|x|\geq 2|y|$ and $t\in(0,1)$ hence

$$
\begin{aligned}
&\int_{|x|\geq 2|y|}|K_k(x-y)-K_k(x)|dx\\
\lesssim&\int_{|x|\geq 2|y|}\int_0^1|\nabla K_k(x-ty)|\cdot|y|dtdx\\
\lesssim&\int_0^1\int_{|x|\geq|y|}|\nabla K_k(x)|\cdot|y|dxdt\\
=&|y|\cdot\int_{|x|\geq|y|}|\nabla K_k(x)|dx.
\end{aligned}
\tag{217}
$$

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}^n$, $\alpha_i \geq 0$ for $1 \leq i \leq n$, denote a multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, then by the basic property of Fourier transform we have

$$\begin{aligned}\int_{|x|\geq|y|} |\nabla K_k(x)| \cdot |y| dx \lesssim & \sum_{|\alpha|=1} |y| \cdot \int_{|x|\geq|y|} |\mathcal{F}_n^{-1}(\xi^\alpha m_k(\xi)\varphi(2^{-k}\xi))(x)| dx \\ \lesssim & \sum_{|\alpha|=1} \frac{|y|}{(1+|2^k y|^2)^{\frac{t}{2}}} \cdot \int_{|x|\geq|y|} (1+|2^k x|^2)^{\frac{t-\tau}{2}} \\ & \cdot (1+|2^k x|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(\xi^\alpha m_k(\xi)\varphi(2^{-k}\xi))(x)| dx. \qquad (218)\end{aligned}$$

We apply in a sequence Hölder's inequality, the change of variable $z = 2^k x$ and the property of Fourier transform that $\mathcal{F}_n^{-1}(\xi^\alpha m_k(\xi)\varphi(2^{-k}\xi))(2^{-k}z) = \mathcal{F}_n^{-1}(2^{k(n+1)}\xi^\alpha m_k(2^k\xi)\varphi(\xi))(z)$ for $|\alpha| = 1$, then we can estimate (218) from above by

$$\begin{aligned} & \sum_{|\alpha|=1} \frac{|y|}{(1+|2^k y|^2)^{\frac{t}{2}}} \cdot \left(\int_{\mathbb{R}^n} (1+|2^k x|^2)^{t-\tau} dx\right)^{\frac{1}{2}} \\ & \cdot \left(\int_{\mathbb{R}^n} (1+|2^k x|^2)^{\tau} |\mathcal{F}_n^{-1}(\xi^\alpha m_k(\xi)\varphi(2^{-k}\xi))(x)|^2 dx\right)^{\frac{1}{2}} \\ = & \sum_{|\alpha|=1} \frac{2^k |y|}{(1+|2^k y|^2)^{\frac{t}{2}}} \cdot \left(\int_{\mathbb{R}^n} (1+|z|^2)^{t-\tau} dz\right)^{\frac{1}{2}} \\ & \cdot \left(\int_{\mathbb{R}^n} (1+|z|^2)^{\tau} |\mathcal{F}_n^{-1}(\xi^\alpha m_k(2^k\xi)\varphi(\xi))(z)|^2 dz\right)^{\frac{1}{2}}. \qquad (219)\end{aligned}$$

Recall (210) and the integral $\int_{\mathbb{R}^n} (1+|z|^2)^{t-\tau} dz$ converges, thus we can obtain the following inequality

$$\begin{aligned}(219) \lesssim & \sum_{|\alpha|=1} \frac{2^k |y|}{(1+|2^k y|^2)^{\frac{t}{2}}} \cdot \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (1+|z|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(m_k(2^k\xi))(w)| \right.\right. \\ & \left.\left. \cdot |\mathcal{F}_n^{-1}(\xi^\alpha \varphi(\xi))(z-w)| dw\right)^2 dz\right)^{\frac{1}{2}} \\ \lesssim & \sum_{|\alpha|=1} \frac{2^k |y|}{(1+|2^k y|^2)^{\frac{t}{2}}} \cdot \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (1+|w|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(m_k(2^k\xi))(w)| \right.\right. \\ & \left.\left. \cdot (1+|z-w|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(\xi^\alpha \varphi(\xi))(z-w)| dw\right)^2 dz\right)^{\frac{1}{2}}. \qquad (220)\end{aligned}$$

Using Young's inequality for convolutions, we have

$$
\begin{aligned}
(220)\lesssim&\sum_{|\alpha|=1}\frac{2^k|y|}{(1+|2^ky|^2)^{\frac{t}{2}}}\cdot\|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\cdot\int_{\mathbb{R}^n}(1+|z|^2)^{\frac{\tau}{2}}|\mathcal{F}_n^{-1}(\xi^\alpha\varphi(\xi))(z)|dz\\
=&\sum_{|\alpha|=1}\frac{2^k|y|}{(1+|2^ky|^2)^{\frac{t}{2}}}\cdot\|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\cdot\int_{\mathbb{R}^n}(1+|z|^2)^{\frac{\tau-t'}{2}}(1+|z|^2)^{\frac{t'}{2}}|\mathcal{F}_n^{-1}(\xi^\alpha\varphi(\xi))(z)|dz\\
\lesssim&\sum_{|\alpha|=1}\frac{2^k|y|}{(1+|2^ky|^2)^{\frac{t}{2}}}\cdot\|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\cdot\|\xi^\alpha\varphi(\xi)\|_{L^2_{t'}(\mathbb{R}^n)}\cdot(\int_{\mathbb{R}^n}(1+|z|^2)^{\tau-t'}dz)^{\frac{1}{2}}\\
\lesssim&\frac{2^k|y|}{(1+|2^ky|^2)^{\frac{t}{2}}}\cdot\operatorname*{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)},
\end{aligned}\tag{221}
$$

where the last inequality is because of (210) and the integral $\int_{\mathbb{R}^n}(1+|z|^2)^{\tau-t'}dz$ converges. Combining (217), (218), (219), (220) and (221) together yields the second estimate

$$
\int_{|x|\geq 2|y|}|K_k(x-y)-K_k(x)|dx\lesssim(2^k|y|)^{1-t}\cdot\operatorname*{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)},\tag{222}
$$

where $\tau>\frac{n}{2}$, t satisfies (210) and the constant in (222) is independent of $k\in\mathbb{Z}$ and $y\in\mathbb{R}^n$. Recall (209). We use (216) when $k\in\mathbb{Z}$ satisfies $2^k|y|\geq 1$ and use (222) when $k\in\mathbb{Z}$ satisfies $2^k|y|\leq 1$, then we get

$$
\begin{aligned}
&\int_{|x|\geq 2|y|}\|K(x-y)-K(x)\|_{L(l^q,l^q)}dx\\
\lesssim&\sum_{k\in\mathbb{Z}}\int_{|x|\geq 2|y|}|K_k(x-y)-K_k(x)|dx\\
\lesssim&\Big(\sum_{\substack{k\in\mathbb{Z}\\2^k|y|\geq 1}}(2^k|y|)^{-t}+\sum_{\substack{k\in\mathbb{Z}\\2^k|y|\leq 1}}(2^k|y|)^{1-t}\Big)\cdot\operatorname*{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}\\
\lesssim&\operatorname*{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)},
\end{aligned}\tag{223}
$$

and the constant in (223) is independent of $k\in\mathbb{Z}$, $y\in\mathbb{R}^n$ and $y\neq 0$. It is trivial to see that (223) still holds true for $y=0$. This shows the infinite diagonal matrix $K(\cdot)$ satisfies condition (51) of Remark 2.0.2 and the constant C on the right side of (51) contains the factor $\operatorname{ess\,sup}_{j\in\mathbb{Z}}\|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}$.

Next we prove that the operator $\mathscr{K}$ satisfies assumptions (47) and (48) of Lemma 2.0.2. For every $k \in \mathbb{Z}$ we recall $K_k(x) = \mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(x)$ and deduce the following

$$
\begin{aligned}
\int_{\mathbb{R}^n} |K_k(x)|dx =& \int_{\mathbb{R}^n} (1+|2^k x|^2)^{\frac{-\tau}{2}} \cdot (1+|2^k x|^2)^{\frac{\tau}{2}} |K_k(x)|dx \\
\lesssim& \Big(\int_{\mathbb{R}^n} (1+|2^k x|^2)^{-\tau} dx\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}^n} (1+|2^k x|^2)^{\tau} |K_k(x)|^2 dx\Big)^{\frac{1}{2}} \qquad (224)\\
=& \Big(\int_{\mathbb{R}^n} (1+|z|^2)^{-\tau} dz\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}^n} (1+|z|^2)^{\tau} |\mathcal{F}_n^{-1}(m_k(2^k\xi)\varphi(\xi))(z)|^2 dz\Big)^{\frac{1}{2}} \qquad (225)\\
\lesssim& \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} (1+|z|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(m_k(2^k\xi))(w)| \cdot |\mathcal{F}_n^{-1}\varphi(z-w)|dw\Big)^2 dz\Big)^{\frac{1}{2}}, \qquad (226)
\end{aligned}
$$

where (224) is due to Hölder's inequality, (225) is by the change of variable $z = 2^k x$ and the property of Fourier transform that $\mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(2^{-k}z) = \mathcal{F}_n^{-1}(2^{kn}m_k(2^k\xi)\varphi(\xi))(z)$, and (226) is because $\tau > \frac{n}{2}$ and thus the integral in the first factor of (225) converges. Using Young's inequality for convolutions, Hölder's inequality and the definition of t' in (210) sequentially, we can estimate (226) from above by

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} (1+|w|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}(m_k(2^k\xi))(w)| \cdot (1+|z-w|^2)^{\frac{\tau}{2}} |\mathcal{F}_n^{-1}\varphi(z-w)|dw\Big)^2 dz\Big)^{\frac{1}{2}} \\
\lesssim& \|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \cdot \int_{\mathbb{R}^n} (1+|z|^2)^{\frac{\tau-t'}{2}} \cdot (1+|z|^2)^{\frac{t'}{2}} |\mathcal{F}_n^{-1}\varphi(z)|dz \\
\lesssim& \|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \cdot \|\varphi\|_{L^2_{t'}(\mathbb{R}^n)} \cdot \Big(\int_{\mathbb{R}^n} (1+|z|^2)^{\tau-t'} dz\Big)^{\frac{1}{2}}. \qquad (227)
\end{aligned}
$$

From (226) and (227), we deduce that

$$
\|K_k\|_{L^1(\mathbb{R}^n)} \lesssim \|m_k(2^k\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \qquad (228)
$$

and the constant is independent of $k \in \mathbb{Z}$. Therefore using Young's inequality for convolutions again, we obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |K_k * f_k(x)|^q dx \\
=& \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |K_k * f_k(x)|^q dx \\
\lesssim& \sum_{k\in\mathbb{Z}} \|K_k\|^q_{L^1(\mathbb{R}^n)} \cdot \|f_k\|^q_{L^q(\mathbb{R}^n)} \\
\lesssim& (\operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)})^q \cdot \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |f_k(x)|^q dx, \qquad (229)
\end{aligned}
$$

and this inequality implies that the operator $\mathscr{K}$ satisfies assumptions (47) and (48) of Lemma 2.0.2 and the constant C on the right side of (48) contains the factor

$$\operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}.$$

Finally, we check that the infinite diagonal matrix $K(x)$ with values in the space of linear operators from l^q to l^q is defined for almost every $x \in \mathbb{R}^n$ and $\|K(x)\|_{L(l^q,l^q)}$ is locally integrable in the domain of $K(x)$. Let $\delta > 0$ be a positive number. In (212), (213), (214) and (215), we replace $|y|$ by δ then for each $k \in \mathbb{Z}$ we have

$$\int_{|x|\geq\delta} |K_k(x)|dx \lesssim (1+2^{2k}\delta^2)^{\frac{-t}{2}} \cdot \operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)}, \tag{230}$$

and hence

$$\int_{|x|\geq\delta} \sum_{\substack{k\in\mathbb{Z}\\k>0}} |K_k(x)|dx \lesssim \big(\sum_{\substack{k\in\mathbb{Z}\\k>0}} 2^{-kt}\big) \cdot \delta^{-t} \cdot \operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} < \infty. \tag{231}$$

Since δ can be any positive number, then $\sum_{\substack{k\in\mathbb{Z}\\k>0}} |K_k(x)|$ is finite for almost every $x \in \mathbb{R}^n$ and locally integrable away from zero. Also for each $k \in \mathbb{Z}$, we have

$$\begin{aligned}
|K_k(x)| &= |\mathcal{F}_n^{-1}(m_k(\xi)\varphi(2^{-k}\xi))(2^{-k}\cdot 2^k x)| \\
&= 2^{kn}|\mathcal{F}_n^{-1}(m_k(2^k\cdot)) * \mathcal{F}_n^{-1}\varphi(2^k x)| \\
&\lesssim 2^{kn}\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}(m_k(2^k\cdot))(y)| \cdot |\mathcal{F}_n^{-1}\varphi(2^k x - y)|dy \\
&\lesssim 2^{kn}\|m_k(2^k\cdot)\|_{L^2(\mathbb{R}^n)} \cdot \big(\int_{\mathbb{R}^n} |\mathcal{F}_n\varphi(y-2^k x)|^2 dy\big)^{\frac{1}{2}} \\
&\lesssim 2^{kn} \operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \cdot \|\varphi\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{232}$$

and hence

$$\sum_{\substack{k\in\mathbb{Z}\\k\leq0}} |K_k(x)| \lesssim \sum_{\substack{k\in\mathbb{Z}\\k\leq0}} 2^{kn} \operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|m_j(2^j\cdot)\|_{L^2_\tau(\mathbb{R}^n)} < \infty \tag{233}$$

for every $x \in \mathbb{R}^n$ and is locally integrable away from zero. Recall the following inequality

$$\|K(x)\|_{L(l^q,l^q)} \lesssim \operatorname*{ess\,sup}_{k\in\mathbb{Z}} |K_k(x)| \lesssim \sum_{k>0} |K_k(x)| + \sum_{k\leq0} |K_k(x)|$$

and then the proof of Theorem 4.1.1 is complete. □

4.3 Proof of Corollary 4.1.1

Proof. Let ψ be as given in (16) and (17) and recall $f_j = \psi_{2^{-j}} * f$ for $j \in \mathbb{Z}$. We also fix a nonnegative Schwartz function φ such that $spt.\varphi \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{4} \le |\xi| < 4\}$, $\varphi = 1$ on $\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| < 2\}$ and thus $\varphi(2^{-j}\xi) = 1$ on $\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| < 2^{j+1}\}$. By definition of $\|\cdot\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$, we have

$$\begin{aligned}\|\mathcal{F}_n^{-1}(m\mathcal{F}_n f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} =& (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} 2^{jsq} |\psi_{2^{-j}} * (\mathcal{F}_n^{-1} m) * f(x)|^q)^{p/q} dx)^{1/p} \\ =& (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} 2^{jsq} |\mathcal{F}_n^{-1}(m(\xi)\mathcal{F}_n f_j(\xi))(x)|^q)^{p/q} dx)^{1/p} \\ =& (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} 2^{jsq} |\mathcal{F}_n^{-1}(m(\xi)\varphi(2^{-j}\xi)\mathcal{F}_n f_j(\xi))(x)|^q)^{p/q} dx)^{1/p}. \end{aligned} \quad (234)$$

Since $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$, the sequence $\{2^{js} f_j\}_{j\in\mathbb{Z}}$ satisfies condition (201) of Theorem 4.1.1 and $2^{js}\mathcal{F}_n f_j(\xi)$ is supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| < 2^{j+1}\}$. To prove (205), it suffices to show that the Hörmander's condition (204) implies the sequence $\{g_j(\xi)\}_{j\in\mathbb{Z}}$, where $g_j(\xi) := m(\xi) \cdot \varphi(2^{-j}\xi)$, satisfies

$$\operatorname*{ess\,sup}_{j\in\mathbb{Z}} \|g_j(2^j\cdot)\|_{L^2_{[\frac{n}{2}]+1}(\mathbb{R}^n)} \lesssim A < \infty. \quad (235)$$

First, by a change of variable $y = \xi/R$, we see condition (204) is equivalent to

$$\operatorname*{ess\,sup}_{R>0} \int_{\frac{1}{4}<|y|<4} |\partial_y^\alpha (m(Ry))|^2 dy \lesssim A^2 < \infty \quad (236)$$

for all multi-indices α with $|\alpha| \le [\frac{n}{2}] + 1$ and $\partial_y^\alpha(m(Ry))$ means the partial derivative of the function $y \mapsto m(Ry)$ with respect to y. Since $[\frac{n}{2}] + 1$ is an integer, then we have for $j \in \mathbb{Z}$

$$\|g_j(2^j\cdot)\|_{L^2_{[\frac{n}{2}]+1}(\mathbb{R}^n)} = \|g_j(2^j\cdot)\|_{W^{[\frac{n}{2}]+1,2}(\mathbb{R}^n)} \lesssim \sum_{|\alpha|\le[\frac{n}{2}]+1} \|\partial^\alpha(g_j(2^j\cdot))\|_{L^2(\mathbb{R}^n)}. \quad (237)$$

Using the Leibniz rule, we have

$$\partial_y^\alpha(g_j(2^j y)) = \partial_y^\alpha(m(2^j y)\varphi(y)) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial_y^\beta(m(2^j y)) \partial^{\alpha-\beta}\varphi(y)$$

where multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ satisfy $\beta \le \alpha$, that is, $0 \le \beta_i \le \alpha_i$ for all $i = 1, \cdots, n$, and $\alpha - \beta$ is the multi-index $(\alpha_1 - \beta_1, \cdots, \alpha_n - \beta_n)$, and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n}.$$

Notice that $|\beta| \le |\alpha| \le [\frac{n}{2}] + 1$ and $\partial^{\alpha-\beta}\varphi$ is supported in $\{y \in \mathbb{R}^n : \frac{1}{4} \le |y| < 4\}$, then we obtain

$$\begin{aligned} \|g_j(2^j\cdot)\|_{L^2_{[\frac{n}{2}]+1}(\mathbb{R}^n)} &\lesssim \sum_{\substack{|\alpha|\le[\frac{n}{2}]+1\\ \beta\le\alpha}} \Big(\int_{\mathbb{R}^n} |\partial_y^\beta(m(2^j y))\partial^{\alpha-\beta}\varphi(y)|^2 dy\Big)^{1/2} \\ &\lesssim \sum_{\substack{|\alpha|\le[\frac{n}{2}]+1\\ \beta\le\alpha}} \|\partial^{\alpha-\beta}\varphi\|_{L^\infty(\mathbb{R}^n)}\Big(\int_{\frac{1}{4}<|y|<4} |\partial_y^\beta(m(2^j y))|^2 dy\Big)^{1/2} \\ &\lesssim \sum_{\substack{|\alpha|\le[\frac{n}{2}]+1\\ \beta\le\alpha}} \|\partial^{\alpha-\beta}\varphi\|_{L^\infty(\mathbb{R}^n)} \cdot A, \end{aligned} \tag{238}$$

where (238) is due to condition (236), and constants involved are independent of $j \in \mathbb{Z}$. Taking essential supremum over $j \in \mathbb{Z}$ in (238) gives us (235) and henth (205) is proved by invoking Theorem 4.1.1. $\square$

5.0 Characterization Of Function Spaces By Maximal Functions Of Iterated Differences

5.1 Chapter Introduction

In section 2.5.9 of [93], H. Triebel proposed an equivalence characterization theorem of the inhomogeneous function spaces $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ by maximal functions given in Definition 1.2.6 and we cite this theorem below with notations adjusted to the notations used in this paper. Let $G_p = n + 3 + \frac{3n}{p}$ and $G_{pq} = n + 3 + \frac{3n}{\min\{p,q\}}$.

Theorem 5.1.1. (i) Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \geq G_p$. If M is an integer with $M > 2G_p + s$ and if $r < p$ in (33)-(35), then the following five quasinorms are equivalent quasinorms in $B^s_{p,q}(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} S^M_{2^{-k}} f\}_{k\geq 0}\|_{l^q(L^p)}, \tag{239}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} \operatorname*{ess\,sup}_{1\leq\tau\leq 2} S^M_{\tau 2^{-k}} f\}_{k\geq 0}\|_{l^q(L^p)}, \tag{240}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} V^M_{2^{-k}} f\}_{k\geq 0}\|_{l^q(L^p)}, \tag{241}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} \operatorname*{ess\,sup}_{1\leq\tau\leq 2} V^M_{\tau 2^{-k}} f\}_{k\geq 0}\|_{l^q(L^p)}, \tag{242}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} \operatorname*{ess\,sup}_{1\leq|h|\leq 2} D^M_{2^{-k}h} f\}_{k\geq 0}\|_{l^q(L^p)}. \tag{243}$$

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \geq G_{pq}$. If M is an integer with $M > 2G_{pq} + s$ and if $r < \min\{p, q\}$ in (33)-(35), then the following five quasinorms are equivalent quasinorms in $F^s_{p,q}(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} S^M_{2^{-k}} f\}_{k\geq 0}\|_{L^p(l^q)}, \tag{244}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} \operatorname*{ess\,sup}_{1\leq\tau\leq 2} S^M_{\tau 2^{-k}} f\}_{k\geq 0}\|_{L^p(l^q)}, \tag{245}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} V^M_{2^{-k}} f\}_{k\geq 0}\|_{L^p(l^q)}, \tag{246}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} \operatorname*{ess\,sup}_{1\leq\tau\leq 2} V^M_{\tau 2^{-k}} f\}_{k\geq 0}\|_{L^p(l^q)}, \tag{247}$$

$$\|f\|_{L^p(\mathbb{R}^n)} + \|\{2^{ks} \operatorname*{ess\,sup}_{1\leq|h|\leq 2} D^M_{2^{-k}h} f\}_{k\geq 0}\|_{L^p(l^q)}. \tag{248}$$

It seems that the restrictions $s \geq G_p$, $M > 2G_p + s$, $s \geq G_{pq}$, $M > 2G_{pq} + s$ are unnatural and the ranges of τ and $|h|$ under the supremums in Theorem 5.1.1 can be extended. Below we would like to propose the improved versions of the above theorem for homogeneous function spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$. The results below are published in the author's paper [98].

Theorem 5.1.2. Let $n \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ is a function and assume $L \in \mathbb{N}$, $s \in \mathbb{R}$ satisfy $\frac{n}{\min\{p,q\}} < s < L$, then for every r as in Definition 1.2.5 and Definition 1.2.6 satisfying $\frac{n}{s} < r < \min\{p, q\}$, the following five quasinorms are equivalent quasinorms in $\dot{F}^s_{p,q}(\mathbb{R}^n)$,

$$\|\{2^{ks} S^L_{2^{-k}} f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}, \tag{249}$$

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}, \tag{250}$$

$$\|\{2^{ks} V^L_{2^{-k}} f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}, \tag{251}$$

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} V^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}, \tag{252}$$

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<|h|<2} D^L_{2^{-k}h} f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}. \tag{253}$$

Theorem 5.1.3. Let $n \geq 2$, $0 < p \leq \infty$, $0 < q \leq \infty$, $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function and assume $L \in \mathbb{N}$, $s \in \mathbb{R}$ satisfy $\frac{n}{p} < s < L$, then for every r as in Definition 1.2.5 and Definition 1.2.6 satisfying $\frac{n}{s} < r < p$, the following five quasinorms are equivalent quasinorms in $\dot{B}^s_{p,q}(\mathbb{R}^n)$,

$$\|\{2^{ks} S^L_{2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}, \tag{254}$$

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}, \tag{255}$$

$$\|\{2^{ks} V^L_{2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}, \tag{256}$$

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} V^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}, \tag{257}$$

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<|h|<2} D^L_{2^{-k}h} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}. \tag{258}$$

The proof of Theorem 5.1.2 can be found in section 5.2 and the proof of Theorem 5.1.3 is given in section 5.3.

5.2 Proof Of Theorem 5.1.2

Proof. We first prove that

$$\|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \tag{259}$$

when $0<p<\infty$, $0<q\le\infty$, $\frac{n}{\min\{p,q\}}<s<L$ and for every $r\in\mathbb{R}$ with $\frac{n}{s}<r<\min\{p,q\}$. Recall $f_j = f*\psi_{2^{-j}}$. We denote

$$S^*_k f(x) = \sum_{j\in\mathbb{Z}} \operatorname*{ess\,sup}_{\substack{1\le\tau<2\\ y\in\mathbb{R}^n}} \fint_{\mathbb{S}^{n-1}} |\Delta^L_{\tau 2^{-k}z} f_j(x-y)|\cdot(1+2^k|y|)^{\frac{-n}{r}}\,d\mathcal{H}^{n-1}(z). \tag{260}$$

Since $|\tau 2^{-k}z|\sim 2^{-k}$ if $1\le\tau<2$ and $z\in\mathbb{S}^{n-1}$, then we have

$$\begin{aligned}&\fint_{\mathbb{S}^{n-1}} |\Delta^L_{\tau 2^{-k}z} f_j(x-y)|\cdot(1+2^k|y|)^{\frac{-n}{r}}\,d\mathcal{H}^{n-1}(z)\\ \lesssim&\fint_{\mathbb{S}^{n-1}} |\Delta^L_{\tau 2^{-k}z} f_j(x-y)|\cdot\Big(1+\frac{|y|}{|\tau 2^{-k}z|}\Big)^{\frac{-n}{r}}\,d\mathcal{H}^{n-1}(z).\end{aligned}$$

We use (72) for $j\le k$ and (73) for $j>k$, and obtain

$$\begin{aligned}S^*_k f(x) &\lesssim \sum_{j\le k} 2^{(j-k)L}(1+2^{j-k})^{\frac{n}{r}}\mathcal{P}_n f_j(x) + \sum_{j>k}(1+2^{j-k})^{\frac{n}{r}}\mathcal{P}_n f_j(x)\\ &\lesssim \sum_{j\le k} 2^{(j-k)L}\mathcal{P}_n f_j(x) + \sum_{j>k} 2^{(j-k)\frac{n}{r}}\mathcal{P}_n f_j(x).\end{aligned} \tag{261}$$

For $0<q<\infty$, we pick $0<\varepsilon<\min\{L-s, s-\frac{n}{r}\}$ and deduce from (261) the following

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} 2^{ksq}(S^*_k f(x))^q\\ \lesssim&\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\sum_{j\le k} 2^{j\varepsilon}\cdot 2^{-j\varepsilon+(j-k)L}\mathcal{P}_n f_j(x)\Big)^q\\ &+\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\sum_{j>k} 2^{-j\varepsilon}\cdot 2^{j\varepsilon+(j-k)\frac{n}{r}}\mathcal{P}_n f_j(x)\Big)^q\\ \lesssim&\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\sum_{j\le k} 2^{j\varepsilon}\Big)^q\cdot \operatorname*{ess\,sup}_{l\le k}\, 2^{-lq\varepsilon+(l-k)qL}\mathcal{P}_n f_l(x)^q\\ &+\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\sum_{j>k} 2^{-j\varepsilon}\Big)^q\cdot \operatorname*{ess\,sup}_{l>k}\, 2^{lq\varepsilon+(l-k)q\frac{n}{r}}\mathcal{P}_n f_l(x)^q\\ \lesssim&\sum_{k\in\mathbb{Z}} 2^{kq(s+\varepsilon)}\sum_{l\le k} 2^{-lq\varepsilon+(l-k)qL}\mathcal{P}_n f_l(x)^q\\ &+\sum_{k\in\mathbb{Z}} 2^{kq(s-\varepsilon)}\sum_{l>k} 2^{lq\varepsilon+(l-k)q\frac{n}{r}}\mathcal{P}_n f_l(x)^q\\ \lesssim&\sum_{l\in\mathbb{Z}} 2^{lsq}\mathcal{P}_n f_l(x)^q,\end{aligned} \tag{262}$$

where in the last inequality we switched the order of summation. Then we raise the power to $\frac{1}{q}$, apply $\|\cdot\|_{L^p(\mathbb{R}^n)}$-quasinorm to both sides of (262), use Remark 2.0.8 and we can obtain the estimate

$$\|\{2^{ks}S_k^*f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \lesssim (\int_{\mathbb{R}^n} (\sum_{j\in\mathbb{Z}} 2^{jsq}\mathcal{P}_n f_j(x)^q)^{\frac{p}{q}} dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{263}$$

If $q = \infty$, we use (261) and the same ε as above to obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}S_k^*f(x)\\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\sum_{j\leq k} 2^{j\varepsilon} \operatorname*{ess\,sup}_{l\leq k} 2^{-l\varepsilon+(l-k)L}\mathcal{P}_n f_l(x) + \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\sum_{j>k} 2^{-j\varepsilon} \operatorname*{ess\,sup}_{l>k} 2^{l\varepsilon+(l-k)\frac{n}{r}}\mathcal{P}_n f_l(x)\\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s+\varepsilon)} \operatorname*{ess\,sup}_{l\leq k} 2^{-l\varepsilon+(l-k)L}\mathcal{P}_n f_l(x) + \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s-\varepsilon)} \operatorname*{ess\,sup}_{l>k} 2^{l\varepsilon+(l-k)\frac{n}{r}}\mathcal{P}_n f_l(x)\\
=&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k\geq l} 2^{k(s+\varepsilon-L)}\cdot 2^{l(L-\varepsilon)}\mathcal{P}_n f_l(x) + \operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k<l} 2^{k(s-\varepsilon-\frac{n}{r})}\cdot 2^{l(\frac{n}{r}+\varepsilon)}\mathcal{P}_n f_l(x)\\
\lesssim&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls}\mathcal{P}_n f_l(x).
\end{aligned} \tag{264}$$

We apply $\|\cdot\|_{L^p(\mathbb{R}^n)}$-quasinorm to both sides of (264), use Remark 2.0.5 and the mapping property of Hardy-Littlewood maximal function to get

$$\begin{aligned}
&\|\{2^{ks}S_k^*f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^\infty)}\\
\lesssim&\|\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls}\mathcal{M}_n(|f_l|^r)(x)^{\frac{1}{r}}\|_{L^p(\mathbb{R}^n)}\\
\lesssim&\|\mathcal{M}_n(\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{lsr}|f_l|^r)(x)\|^{\frac{1}{r}}_{L^{\frac{p}{r}}(\mathbb{R}^n)}\\
\lesssim&\|\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls}|f_l|\|_{L^p(\mathbb{R}^n)} = \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}.
\end{aligned} \tag{265}$$

The above proof also shows for every $k\in\mathbb{Z}$,

$$\|\operatorname*{ess\,sup}_{\substack{1\leq\tau<2\\ y\in\mathbb{R}^n}} \fint_{\mathbb{S}^{n-1}} \sum_{j\in\mathbb{Z}} |\varDelta^L_{\tau 2^{-k}z} f_j(x-y)| d\mathcal{H}^{n-1}(z)\cdot(1+2^k|y|)^{\frac{-n}{r}}\|_{L^p(\mathbb{R}^n)} < \infty, \tag{266}$$

and thus $\sum_{j\in\mathbb{Z}} |\varDelta^L_{\tau 2^{-k}z} f_j(x-y)| < \infty$ for almost every $1\leq\tau<2$, $z\in\mathbb{S}^{n-1}$, $x,y\in\mathbb{R}^n$. Therefore we can infer from (28) that

$$\varDelta^L_{\tau 2^{-k}z} f(x-y) = \sum_{j\in\mathbb{Z}} \varDelta^L_{\tau 2^{-k}z} f_j(x-y) \tag{267}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ for every $k \in \mathbb{Z}$, and almost every $1 \le \tau < 2$, $z \in \mathbb{S}^{n-1}$, $x, y \in \mathbb{R}^n$. The above justification of decomposition also tells us that

$$\operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f(x) = \operatorname*{ess\,sup}_{\substack{1\le\tau<2\\ y\in\mathbb{R}^n}} |\fint_{\mathbb{S}^{n-1}} \Delta^L_{\tau 2^{-k} z} f(x-y) d\mathcal{H}^{n-1}(z)| \cdot (1+\tau^{-1}2^k|y|)^{\frac{-n}{r}} \lesssim S^*_k f(x),$$

and this estimate, combined with (263) and (265), finishes the proof of (259). We also observe that for $0 < q < \infty$

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{ksq}\big(\operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f(x)\big)^q \\
\lesssim&\sum_{k\in\mathbb{Z}}\sum_{j\ge 0} 2^{ksq}\big(\operatorname*{ess\,sup}_{2^{-j}\le\tau<2^{1-j}} S^L_{\tau 2^{-k}} f(x)\big)^q \\
=&\sum_{j\ge 0} 2^{-jsq}\sum_{k\in\mathbb{Z}} 2^{(k+j)sq}\big(\operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f(x)\big)^q \\
\lesssim&\sum_{k\in\mathbb{Z}} 2^{ksq}\big(\operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f(x)\big)^q,
\end{aligned} \tag{268}$$

and for $q = \infty$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f(x) \\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j\ge 0} 2^{-js}\cdot 2^{(k+j)s} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f(x) \\
\lesssim&\big(\sum_{j\ge 0} 2^{-js}\big) \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f(x),
\end{aligned} \tag{269}$$

therefore (250) can be estimated from above by $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$. Using the same method, we can also estimate (252) from above by $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ under the conditions of Theorem 5.1.2. As for (253), by using the same method as (259) we can show that

$$\|\{2^{ks} \operatorname*{ess\,sup}_{1\le|h|<2} D^L_{2^{-k}h} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \tag{270}$$

when p, q, s, r, L satisfy conditions of Theorem 5.1.2. And then we use the arguments in (268) and (269) to prove that

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<|h|<2} D^L_{2^{-k}h} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \lesssim \|\{2^{ks} \operatorname*{ess\,sup}_{1\le|h|<2} D^L_{2^{-k}h} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \tag{271}$$

for all $0 < p < \infty$, $0 < q \leq \infty$, since we have the decomposition

$$\{h \in \mathbb{R}^n : 0 < |h| < 2\} = \bigcup_{j=0}^{\infty} \{h \in \mathbb{R}^n : 2^{-j} \leq |h| < 2^{1-j}\}.$$

To prove the reverse directions, we first show that for any $0 < \tau < 2$, $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, we have the estimate

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|\{2^{ks} S^L_{\tau 2^{-k}} f(x)\}_{k \in \mathbb{Z}}\|_{L^p(l^q)}. \tag{272}$$

Let $\tau \in (0, 2)$ be fixed for now, and let a denote the tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$ whose distributional Fourier transform is the function below

$$\mathcal{F}_n a(\xi) = \fint_{\mathbb{S}^{n-1}} (e^{2\pi i \tau \xi \cdot z} - 1)^L d\mathcal{H}^{n-1}(z). \tag{273}$$

For example, we can choose $a = \sum_{m=0}^{L} \binom{L}{m} (-1)^{L-m} \fint_{\mathbb{S}^{n-1}} \delta_{-m\tau z} d\mathcal{H}^{n-1}(z)$, where $\delta_{-m\tau z}$ is the Dirac mass at $-m\tau z$. Then from (15) and Definition 1.2.6 we deduce the following equality

$$S^L_{\tau 2^{-k}} f(x) = \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|\mathcal{F}_n^{-1}(\mathcal{F}_n a(2^{-k}\xi) \cdot \mathcal{F}_n f(\xi))(x-y)|}{(1 + \tau^{-1} 2^k |y|)^{\frac{n}{r}}}. \tag{274}$$

Using the formula given in Appendix D.3 of [41], we have

$$\mathcal{F}_n a(\xi) = C_n \cdot \int_{-1}^{1} (e^{2\pi i t \tau |\xi|} - 1)^L (1 - t^2)^{\frac{n-3}{2}} dt, \tag{275}$$

where C_n is a positive constant depending on n. By using Taylor expansion, we can write

$$(e^{2\pi i t \tau |\xi|} - 1)^L = \sum_{k=0}^{\infty} A_{L+k} (t\tau|\xi|)^{L+k}, \tag{276}$$

and each A_{L+k} is a complex number whose value is independent of t, τ, ξ and satisfies $|A_{L+k}| > 0$. Hence we have the expression

$$\mathcal{F}_n a(\xi) = C_n \cdot \sum_{k=0}^{\infty} A_{L+k} B_{L+k} |\tau\xi|^{L+k} \text{ for every } \xi \in \mathbb{R}^n \text{ when } |\xi| \text{ is small}, \tag{277}$$

where

$$B_{L+k} = \int_{-1}^{1} t^{L+k} (1 - t^2)^{\frac{n-3}{2}} dt \qquad \text{for } k \geq 0, \tag{278}$$

and $B_{L+k} = 0$ if $L + k$ is an odd integer, $B_{L+k} > 0$ if $L + k$ is an even integer. If L is a positive even integer, then

$$\mathcal{F}_n a(\xi) = C_n A_L B_L |\tau\xi|^L (1 + O(|\tau\xi|^2))$$

and $|\mathcal{F}_n a(\xi)| \sim |\xi|^L > 0$ if $|\xi| > 0$ is sufficiently small. If L is a positive odd integer, then

$$\mathcal{F}_n a(\xi) = C_n A_{L+1} B_{L+1} |\tau\xi|^{L+1} (1 + O(|\tau\xi|^2))$$

and $|\mathcal{F}_n a(\xi)| \sim |\xi|^{L+1} > 0$ if $|\xi| > 0$ is sufficiently small. Therefore we can pick a sufficiently large positive integer m_1 so that $|\mathcal{F}_n a(\xi)| > 0$ if $0 < |\xi| < 2^{1-m_1}$, and hence $\frac{\mathcal{F}_n \psi(2^{m_1}\xi)}{\mathcal{F}_n a(\xi)}$ is a well-defined function in $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{F}_n^{-1}(\frac{\mathcal{F}_n \psi(2^{m_1}\xi)}{\mathcal{F}_n a(\xi)})(\cdot) \in \mathcal{S}(\mathbb{R}^n)$. Furthermore by using (274) we have for each $k \in \mathbb{Z}$

$$\begin{aligned}
&|\mathcal{F}_n^{-1}(\mathcal{F}_n \psi(2^{m_1-k}\xi)\mathcal{F}_n f(\xi))(x)| \\
\lesssim &\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}(\frac{\mathcal{F}_n \psi(2^{m_1-k}\xi)}{\mathcal{F}_n a(2^{-k}\xi)})(y)| \cdot |\mathcal{F}_n^{-1}(\mathcal{F}_n a(2^{-k}\xi)\mathcal{F}_n f(\xi))(x-y)| dy \\
\lesssim &\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}(\frac{\mathcal{F}_n \psi(2^{m_1}\xi)}{\mathcal{F}_n a(\xi)})(2^k y)| \cdot 2^{kn}(1 + \tau^{-1}2^k|y|)^{\frac{n}{r}} dy \cdot S^L_{\tau 2^{-k}} f(x) \\
\lesssim &S^L_{\tau 2^{-k}} f(x),
\end{aligned} \tag{279}$$

and the constants are independent of $k \in \mathbb{Z}$. By using (279) above, we reach the conclusion that for $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$

$$\begin{aligned}
\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} =& 2^{-sm_1} \|\{2^{ks}\mathcal{F}_n^{-1}(\mathcal{F}_n \psi(2^{m_1-k}\xi)\mathcal{F}_n f(\xi))(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\
\lesssim& \|\{2^{ks} S^L_{\tau 2^{-k}} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}.
\end{aligned} \tag{280}$$

We let $\tau = 1$ in (280) and get that $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ can be estimated from above by (249). Therefore we have shown that (249) and (250) are equivalent quasinorms in $\dot{F}^s_{p,q}(\mathbb{R}^n)$ when parameters p, q, s, r, L satisfy the conditions of Theorem 5.1.2.

To show that when $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, the quasinorm $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ can be estimated from above by (251), we consider the tempered distribution $b \in \mathcal{S}'(\mathbb{R}^n)$ whose distributional Fourier transform is the function below

$$\mathcal{F}_n b(\xi) = \fint_A (e^{2\pi i \xi\cdot z} - 1)^L dz = \frac{1}{|A|}\int_1^2 \tau^{n-1} \int_{\mathbb{S}^{n-1}} (e^{2\pi i\tau\xi\cdot z} - 1)^L d\mathcal{H}^{n-1}(z) d\tau, \tag{281}$$

where A is the annulus $\{z \in \mathbb{R}^n : 1 \le |z| < 2\}$. For example, we can choose $b = \sum_{m=0}^{L} \binom{L}{m} (-1)^{L-m} \fint_A \delta_{-mz} dz$, where δ_{-mz} is the Dirac mass at $-mz$. Using (273), (275), (276) and (277), we obtain that

$$\begin{aligned}\mathcal{F}_n b(\xi) =& C'_n \sum_{k=0}^{\infty} \int_1^2 \tau^{n+k+L-1} d\tau \cdot A_{L+k} B_{L+k} |\xi|^{L+k} \\ =& C'_n \sum_{k=0}^{\infty} A'_{L+k} B_{L+k} |\xi|^{L+k} \end{aligned} \tag{282}$$

for every $\xi \in \mathbb{R}^n$ when $|\xi|$ is small, where C'_n is a positive constant depending on n, each A'_{L+k} is a complex number satisfying $|A'_{L+k}| > 0$, and each B_{L+k} is defined by (278). Therefore using a similar analysis like the one for $\mathcal{F}_n a(\xi)$, we can find a sufficiently large positive integer m_2 so that $|\mathcal{F}_n b(\xi)| > 0$ if $0 < |\xi| < 2^{1-m_2}$. Hence $\frac{\mathcal{F}_n \psi(2^{m_2}\xi)}{\mathcal{F}_n b(\xi)}$ is a well-defined function in $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{F}_n^{-1}(\frac{\mathcal{F}_n \psi(2^{m_2}\xi)}{\mathcal{F}_n b(\xi)})(\cdot) \in \mathcal{S}(\mathbb{R}^n)$. Using a similar argument like the one to deduce (279) and the estimate

$$|\mathcal{F}_n^{-1}(\mathcal{F}_n b(2^{-k}\xi) \mathcal{F}_n f(\xi))(x-y)| = |\fint_A \varDelta^L_{2^{-k}z} f(x-y) dz| \le (1 + 2^k |y|)^{\frac{n}{r}} \cdot V^L_{2^{-k}} f(x), \tag{283}$$

which can be obtained by invoking (15), (34), and (281), we can obtain

$$|\mathcal{F}_n^{-1}(\mathcal{F}_n \psi(2^{m_2-k}\xi) \mathcal{F}_n f(\xi))(x)| \lesssim V^L_{2^{-k}} f(x) \lesssim \operatorname*{ess\,sup}_{0<\tau<2} V^L_{\tau 2^{-k}} f(x) \qquad \text{for every } k \in \mathbb{Z}, \tag{284}$$

and the constant is independent of k. By using (284) above, we reach the conclusion that for $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$

$$\begin{aligned}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} =& 2^{-sm_2} \|\{2^{ks} \mathcal{F}_n^{-1}(\mathcal{F}_n \psi(2^{m_2-k}\xi) \mathcal{F}_n f(\xi))(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\ \lesssim& \|\{2^{ks} V^L_{2^{-k}} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \lesssim \|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} V^L_{\tau 2^{-k}} f(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}. \end{aligned} \tag{285}$$

By using their defining expressions in Definition 1.2.6, it is easy to see that $V^L_{2^{-k}} f(x) \lesssim \operatorname{ess\,sup}_{0<|h|<2} D^L_{2^{-k}h} f(x)$ for every $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, thus $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ can be estimated from above by (253) for all $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Hereby we conclude the proof of Theorem 5.1.2. □

5.3 Proof Of Theorem 5.1.3

Proof. The proof of Theorem 5.1.3 is alike to the proof of Theorem 5.1.2 and thus we will just sketch it. We first prove the counterpart of (259), that is,

$$\|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \tag{286}$$

when $0 < p \le \infty$, $0 < q \le \infty$, $\frac{n}{p} < s < L$ and for every $r \in \mathbb{R}$ with $\frac{n}{s} < r < p$. By using Lemma 2.0.7, we still have (261) with $S^*_k f(x)$ given in (260). If $1 \le p \le \infty$, we use Minkowski's inequality for $L^p(\mathbb{R}^n)$-norms, Remark 2.0.5 and the mapping property of Hardy-Littlewood maximal function in a sequence and obtain the following

$$\|S^*_k f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j\le k} 2^{(j-k)L}\|f_j\|_{L^p(\mathbb{R}^n)} + \sum_{j>k} 2^{(j-k)\frac{n}{r}}\|f_j\|_{L^p(\mathbb{R}^n)}. \tag{287}$$

With (287), we use the calculation method of (262) when $0 < q < \infty$ and the calculation method of (264) when $q = \infty$ and justify the decomposition in a similar way like (267), then we can obtain (286) for the case $1 \le p \le \infty$. If $0 < p < 1$, we raise the power of both sides of (261) to p and integrate over $\mathbb{R}^n$ with respect to x, use Remark 2.0.5 and the mapping property of Hardy-Littlewood maximal function in a sequence and obtain the following

$$\|S^*_k f\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_{j\le k} 2^{(j-k)Lp}\|f_j\|^p_{L^p(\mathbb{R}^n)} + \sum_{j>k} 2^{(j-k)\frac{np}{r}}\|f_j\|^p_{L^p(\mathbb{R}^n)}. \tag{288}$$

With (288), we use the calculation method of (262) when $0 < \frac{q}{p} < \infty$ and the calculation method of (264) when $\frac{q}{p} = \infty$ and justify the decomposition then we can obtain (286) for the case $0 < p < 1$. Next we show that

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \lesssim \|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \tag{289}$$

for $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. We have the following pointwise estimate

$$\operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f(x) \lesssim \sum_{j=0}^{\infty} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f(x) \qquad \text{for every } x \in \mathbb{R}^n. \tag{290}$$

If $1 \le p \le \infty$, then we use Minkowski's inequality for $L^p(\mathbb{R}^n)$-norms and get

$$\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \lesssim \|\{2^{ks} \sum_{j=0}^{\infty} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|_{L^p(\mathbb{R}^n)}\}_{k\in\mathbb{Z}}\|_{l^q}. \tag{291}$$

When $0 < q < 1$, we can switch the order of summation and obtain

$$\begin{aligned}
(291) &\lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \sum_{j=0}^{\infty} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{j=0}^{\infty} 2^{-jsq} \sum_{k\in\mathbb{Z}} 2^{(k+j)sq} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}} \\
&\lesssim \|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}.
\end{aligned} \tag{292}$$

When $1 \le q \le \infty$, we use Minkowski's inequality for l^q-norms and obtain

$$\begin{aligned}
(291) &\lesssim \sum_{j=0}^{\infty} 2^{-js} \|\{2^{(k+j)s} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|_{L^p(\mathbb{R}^n)}\}_{k\in\mathbb{Z}}\|_{l^q} \\
&\lesssim \|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}.
\end{aligned} \tag{293}$$

If $0 < p < 1$, then we raise the power of both sides of (290) to p and integrate over $\mathbb{R}^n$ with respect to x to obtain

$$\| \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=0}^{\infty} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^p_{L^p(\mathbb{R}^n)}. \tag{294}$$

When $0 < \frac{q}{p} < 1$, we use (294) to obtain

$$\begin{aligned}
&\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \\
\lesssim& \Big(\sum_{k\in\mathbb{Z}} \Big(\sum_{j=0}^{\infty} 2^{ksp} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^p_{L^p(\mathbb{R}^n)}\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \\
\lesssim& \Big(\sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{ksq} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}} \\
=& \Big(\sum_{j=0}^{\infty} 2^{-jsq} \sum_{k\in\mathbb{Z}} 2^{(k+j)sq} \| \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}} \\
\lesssim& \|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}.
\end{aligned} \tag{295}$$

When $1 \le \frac{q}{p} \le \infty$, we use (294) and Minkowski's inequality for $l^{\frac{q}{p}}$-norms and obtain

$$
\begin{aligned}
&\|\{2^{ks} \operatorname*{ess\,sup}_{0<\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \\
\lesssim&\|\{\sum_{j=0}^{\infty} 2^{ksp} \|\operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|^p_{L^p(\mathbb{R}^n)}\}_{k\in\mathbb{Z}}\|^{\frac{1}{p}}_{l^{\frac{q}{p}}} \\
\lesssim&(\sum_{j=0}^{\infty} 2^{-jsp} \|\{2^{(k+j)s} \|\operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k-j}} f\|_{L^p(\mathbb{R}^n)}\}_{k\in\mathbb{Z}}\|^p_{l^q})^{\frac{1}{p}} \\
\lesssim&\|\{2^{ks} \operatorname*{ess\,sup}_{1\le\tau<2} S^L_{\tau 2^{-k}} f\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}.
\end{aligned}
\tag{296}
$$

From (291), (292), (293), (295) and (296), we see that (289) has been proved. Combining (286) and (289) gives that (255) can be estimated from above by $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ when conditions of Theorem 5.1.3 are satisfied. Using the same method, we also prove that (257) and (258) can be estimated from above by $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ under the conditions of Theorem 5.1.3.

To prove the reverse directions, we just notice that (279) and (284) are pointwise estimates for every $x \in \mathbb{R}^n$ and then we use the same method given in (280) and (285) to prove that $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ can be estimated from above by (254) and (256) for all $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Furthermore (256) can be estimated from above by (258) by using Definition 1.2.6. The proof of Theorem 5.1.3 is complete. □

6.0 Inequalities In Function Spaces In Terms Of Iterated Differences

6.1 Chapter Introduction

In section 2.5.10 of [93], H. Triebel gave an equivalence characterization theorem of the inhomogeneous function space $F^s_{p,q}(\mathbb{R}^n)$ by iterated differences and we would cite this theorem below with adjusted notations.

Theorem 6.1.1. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \frac{n}{\min(p,q)}$. If M is an integer such that $M > s$, then

$$\|f|F^s_{p,q}(R_n)\|^{(1)}_M = \|f|L_p(R_n)\| + \left\|\left(\int_{R_n} |h|^{-sq} \operatorname*{ess\,sup}_{\substack{|\rho|\leq|h| \\ \rho\in R_n}} |(\varDelta^M_\rho f)(\cdot)|^q \frac{dh}{|h|^n}\right)^{\frac{1}{q}} \Big| L_p(R_n)\right\| \tag{297}$$

and

$$\|f|F^s_{p,q}(R_n)\|^{(2)}_M = \|f|L_p(R_n)\| + \left\|\left(\int_{R_n} |h|^{-sq}|(\varDelta^M_h f)(\cdot)|^q \frac{dh}{|h|^n}\right)^{\frac{1}{q}} \Big| L_p(R_n)\right\| \tag{298}$$

are equivalent quasi-norms in $F^s_{p,q}(R_n)$ (modification if $q = \infty$).

In Theorem 1 on page 102 of [89], E. M. Stein gave the equivalence characterization

$$[f]_{W^\alpha_{p,2}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p_\alpha(\mathbb{R}^n)}$$

where the restrictions $0 < \alpha < 1$, $1 < p < \infty$ and $\frac{2n}{n+2\alpha} < p < \infty$ were considered essentially sharp. Since the inhomogeneous spaces satisfy $L^p_\alpha(\mathbb{R}^n) \sim F^\alpha_{p,2}(\mathbb{R}^n)$ if $1 < p < \infty$ and since

$$[f]_{W^\alpha_{p,2}(\mathbb{R}^n)} = \|(\int_{\mathbb{R}^n} |h|^{-2\alpha}|(\varDelta^1_h f)(\cdot)|^2 \frac{dh}{|h|^n})^{\frac{1}{2}}\|_{L^p(\mathbb{R}^n)},$$

we would consider the above result in [89] is a better result in the special case. Furthermore in Theorem 1 on page 393 of [85], A. Seeger provided another improvement and generalization for the homogeneous space

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \sim \|S^\alpha_{q,r,m} f\|_{L^p(\mathbb{R}^n)}$$

where $0 < p < \infty$, $0 < q \le \infty$, $m > \alpha/a_0$, $r \ge 1$ with

$$\alpha > \max\{0, \nu(\frac{1}{p} - \frac{1}{r}), \nu(\frac{1}{q} - \frac{1}{r})\},$$

and

$$S^{\alpha}_{q,r,m} f(x) = (\int_0^{\infty} [\fint_{\varrho(h)\le t} |(\Delta_h^m f)(x)|^r dh]^{q/r} \frac{dt}{t^{1+\alpha q}})^{1/q}.$$

If we consider the isotropic spaces in which $\varrho(h)$ above can be deemed as $|h|$ and a_0 can be deemed as 1, by letting $r = q$ and changing the order of integration, then we can obtain

$$\begin{aligned}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} &\sim \|(\int_0^{\infty} \int_{|h|\le t} t^{-1-n-q\alpha} \cdot |(\Delta_h^m f)(\cdot)|^q dh dt)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \\ &\sim \|(\int_{\mathbb{R}^n} |h|^{-q\alpha} \cdot |(\Delta_h^m f)(\cdot)|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

for $0 < p < \infty$, $1 \le q \le \infty$ and $\max\{0, \nu(\frac{1}{p} - \frac{1}{q})\} < \alpha < m$, and this is the homogeneous counterpart of (298). Recently in Theorem 1.2 on page 693 of [77] M. Prats also proves an equivalence characterization theorem of the inhomogeneous norm $\|f\|_{F^s_{p,q}(\Omega)}$ in terms of the sum of $\|f\|_{W^{k,p}(\Omega)}$ and

$$\sum_{|\alpha|=k} \Big(\int_{\Omega} \Big(\int_{\Omega} \frac{|D^{\alpha} f(x) - D^{\alpha} f(y)|^q}{|x-y|^{\sigma q + d}} dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \tag{299}$$

when parameters satisfy $1 \le p < \infty, 1 \le q \le \infty, s = k + \sigma$, $\max\{0, d(\frac{1}{p} - \frac{1}{q})\} < \sigma < 1$ and Ω is a uniform domain in $\mathbb{R}^d$. Furthermore, M. Prats also shows under the same conditions on parameters, the equivalence relation stands if (299) is replaced by

$$\sum_{|\alpha|=k} \Big(\int_{\Omega} \Big(\int_{\mathbf{Sh}(x)} \frac{|D^{\alpha} f(x) - D^{\alpha} f(y)|^q}{|x-y|^{\sigma q + d}} dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \tag{300}$$

where $\mathbf{Sh}(x) := \{y \in \Omega : |y - x| \le c_{\Omega}\delta(x)\}$ is the Carleson box centered at x, $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and $c_{\Omega} > 1$ is a constant. Moreover when $1 \le q \le p < \infty$, the set $\mathbf{Sh}(x)$ in (300) can be improved and replaced by the Whitney ball $B(x, \rho\delta(x))$ for $0 < \rho < 1$.

In this paper, we would like to furnish the reader with a further improvement of Theorem 6.1.1 for the homogeneous space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ which includes the case $0 < q < 1$. We use Fourier analytic techniques to prove the improved inequality for $0 < q < 1$ and also provide an independent proof for $1 \le q \le \infty$. We now state this further improvement below. Let

$$\sigma_{pq} = \max\{0, n(\frac{1}{\min\{p,q\}} - 1)\}, \quad \tilde{\sigma}_{pq} = \max\{0, n(\frac{1}{p} - \frac{1}{q})\}, \quad \sigma_p = \max\{0, n(\frac{1}{p} - 1)\}. \tag{301}$$

Theorem 6.1.2. Let $L \in \mathbb{N}$, $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$.
(i)If $0 < p, q < \infty$, $\tilde{\sigma}_{pq} < s < L$, then

$$\|(\int_{\mathbb{R}^n} |h|^{-sq}|\Delta^L_h f|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{302}$$

(ii)Suppose f is a function. If $0 < p < \infty$, $0 < q < 1$ and $\sigma_{pq} + \tilde{\sigma}_{pq} < s < \infty$, or if $0 < p < \infty$, $1 \le q < \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|(\int_{\mathbb{R}^n} |h|^{-sq}|(\Delta^L_h f)(\cdot)|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}. \tag{303}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $\frac{n}{p} < s < L$, then

$$\| \operatorname*{ess\,sup}_{h \in \mathbb{R}^n} \frac{|\Delta^L_h f|}{|h|^s}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{304}$$

(iv)Suppose f is a function. If $0 < p < \infty$, $q = \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \| \operatorname*{ess\,sup}_{h \in \mathbb{R}^n} \frac{|(\Delta^L_h f)(\cdot)|}{|h|^s}\|_{L^p(\mathbb{R}^n)}. \tag{305}$$

The proof of Theorem 6.1.2 can be found in section 6.2. Theorem 6.1.2 (i) shows the term

$$\|(\int_{\mathbb{R}^n} |h|^{-sq}|\Delta^L_h f|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \tag{306}$$

may not be independently defined for tempered distributions, since the iterated difference $\Delta^L_h f$ may not have a function representative if f is a member of $\mathcal{S}'(\mathbb{R}^n)$. Another example is that if $P(x) = x^\alpha$ is a polynomial function and we put it into (306) then the resulting term may not have finite value. However if we consider P as a tempered distribution in $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and the conditions of Theorem 6.1.2 (i) are met, then (400) designates the function representative of $\Delta^L_h P$ is given by $\sum_{j \in \mathbb{Z}} \Delta^L_h(\psi_{2^{-j}} * P)(x) = 0$ for all $x \in \mathbb{R}^n$. This is because $(\psi_{2^{-j}} * P)(x)$ can be expressed as a linear combination, with coefficients depending on x, of derivatives of the Fourier transform $\mathcal{F}_n\psi$ evaluated at 0 and these evaluations are identically zero due to the support condition of $\mathcal{F}_n\psi$. We believe Theorem 6.1.2 (i) extends the definition of the term (306). The same discussion is also true for Theorem 6.1.2 (iii).

Comparing Theorem 6.1.2 with Theorem 6.1.1, we find that if $0 < q < 1$ and $\frac{q}{q+1} \leq p < \infty$, then the restriction $\sigma_{pq} + \tilde{\sigma}_{pq} < s < L$ is better than the restriction of s in Theorem 6.1.1. However if $0 < p < \frac{q}{q+1} < q < 1$ then we have

$$\frac{n}{\min\{p, q\}} < \sigma_{pq} + \tilde{\sigma}_{pq}$$

and the restrictions of Theorem 6.1.1 remain better. If in addition the number s also satisfies the condition

$$s \leq n + \frac{n}{q}, \tag{307}$$

then $\sigma_{pq} + \tilde{\sigma}_{pq} < s$ implies $\frac{q}{q+1} < p$ and hence the restrictions in Theorem 6.1.2 are better than the restrictions in Theorem 6.1.1. This happens for sure when we pick $L = 1$ or $L = 2$ since $n + \frac{n}{q} > 2$ for $0 < q < 1$. Therefore we formulate these two cases as corollaries below.

Corollary 6.1.1. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p, q < \infty$, $\tilde{\sigma}_{pq} < s < 1$, then

$$\Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x - y|^{n+sq}} dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{308}$$

(ii)If $0 < p < \infty$, $0 < q < 1$ and $\sigma_{pq} + \tilde{\sigma}_{pq} < s < \infty$, or if $0 < p < \infty$, $1 \leq q < \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x - y|^{n+sq}} dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}. \tag{309}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $\frac{n}{p} < s < 1$, then

$$\Big(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} dx\Big)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{310}$$

(iv)If $0 < p < \infty$, $q = \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \Big(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{sp}} dx\Big)^{\frac{1}{p}}. \tag{311}$$

Proof of Corollary 6.1.1. Apply inequalities (302), (303), (304) and (305) with $L = 1$ and use appropriate change of variable. We also note that the quantity

$$[f]_{W^s_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x - y|^{n+sq}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

is usually called the generalized Gagliardo seminorm. In the case $1 \leq q \leq \infty$, inequalities (309) and (311) are still true for $s = 0$ or $s = 1$. In particular, if we let $0 < p \leq 1, q = 2, s = 0$ and apply the equivalence relation $\| \cdot \|_{\dot{F}^0_{p,2}(\mathbb{R}^n)} \sim \| \cdot \|_{H^p(\mathbb{R}^n)}$ in (309), then we have

$$\|f\|_{H^p(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^n} dy)^{\frac{p}{2}} dx)^{\frac{1}{p}}, \tag{312}$$

where $\| \cdot \|_{H^p(\mathbb{R}^n)}$ represents the Hardy quasinorm. □

Corollary 6.1.2. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ is a function.

(i)If $0 < p, q < \infty$, $\tilde{\sigma}_{pq} < s < 2$, then

$$(\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|^q}{|x - y|^{n+sq}} dy)^{\frac{p}{q}} dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{313}$$

(ii)If $0 < p < \infty$, $0 < q < 1$ and $\sigma_{pq} + \tilde{\sigma}_{pq} < s < \infty$, or if $0 < p < \infty$, $1 \leq q < \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|^q}{|x - y|^{n+sq}} dy)^{\frac{p}{q}} dx)^{\frac{1}{p}}. \tag{314}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $\frac{n}{p} < s < 2$, then

$$(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|^p}{|x - y|^{sp}} dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{315}$$

(iv)If $0 < p < \infty$, $q = \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|^p}{|x - y|^{sp}} dx)^{\frac{1}{p}}. \tag{316}$$

Proof of Corollary 6.1.2. Apply inequalities (302), (303), (304) and (305) with $L = 2$ and use appropriate change of variable. In the case $1 \le q \le \infty$, inequalities (314) and (316) are still true for $s = 0, 1, 2$. In particular, if we let $0 < p \le 1, q = 2, s = 0$ and apply the equivalence relation $\|\cdot\|_{\dot{F}^0_{p,2}(\mathbb{R}^n)} \sim \|\cdot\|_{H^p(\mathbb{R}^n)}$ in (314), then we have

$$\|f\|_{H^p(\mathbb{R}^n)} \lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\frac{|f(x)+f(y)-2f(\frac{x+y}{2})|^2}{|x-y|^n}dy\Big)^{\frac{p}{2}}dx\Big)^{\frac{1}{p}}, \tag{317}$$

where $\|\cdot\|_{H^p(\mathbb{R}^n)}$ represents the Hardy quasinorm. □

The following Theorem 6.1.3 is the counterpart of Theorem 6.1.2 for $\dot{B}^s_{p,q}(\mathbb{R}^n)$ spaces.

Theorem 6.1.3. Let $L \in \mathbb{N}$, $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$.
(i)If $0 < p \le \infty$, $0 < q < \infty$ and $0 < s < L$, then

$$\Big(\int_{\mathbb{R}^n}|h|^{-sq}\|\Delta^L_h f\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{318}$$

(ii)Suppose f is a function. If $1 < p \le \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$, or if $0 < p < 1$, $0 < q < \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \Big(\int_{\mathbb{R}^n}|h|^{-sq}\|\Delta^L_h f\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}. \tag{319}$$

(iii)If $0 < p \le \infty$, $q = \infty$ and $0 < s < L$, then

$$\operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s}\cdot\|\Delta^L_h f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{320}$$

(iv)Suppose f is a function. If $1 < p \le \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p < 1$, $q = \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s}\cdot\|\Delta^L_h f\|_{L^p(\mathbb{R}^n)}. \tag{321}$$

(v)Suppose f is a function. If $p = 1$, $1 \le q < \infty$ and $-n < s < \infty$, or if $p = 1$, $0 < q < 1$ and $0 < s < \infty$, then

$$\|f\|_{\dot{B}^s_{1,q}(\mathbb{R}^n)} \lesssim \Big(\int_{\mathbb{R}^n}|h|^{-sq}\|\Delta^L_h f\|^q_{L^1(\mathbb{R}^n)}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}. \tag{322}$$

If $p = 1$, $q = \infty$ and $-n < s < \infty$, then

$$\|f\|_{\dot{B}^s_{1,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s}\cdot\|\Delta^L_h f\|_{L^1(\mathbb{R}^n)}. \tag{323}$$

In [47], D. D. Haroske and H. Triebel provided a characterization of the inhomogeneous space $B^s_{p,q}(\mathbb{R}^n)$ in the sense of equivalent quasinorms via the following expression

$$\|f\|_{L^p(\mathbb{R}^n)} + \Big(\int_0^1 t^{-sq} \omega_k(f,t)_p^q \frac{dt}{t} \Big)^{\frac{1}{q}}, \tag{324}$$

where $\omega_k(f,t)_p = \sup_{0<|h|\le t} \|\Delta^k_h f\|_{L^p(\mathbb{R}^n)}$ is the k-th modulus of smoothness of the function f, and also via the following expression

$$\|f\|_{L^p(\mathbb{R}^n)} + \Big(\int_{0<|h|\le 1} \frac{\|\Delta^k_h f\|^q_{L^p(\mathbb{R}^n)}}{|h|^{sq}} \frac{dh}{|h|^n} \Big)^{\frac{1}{q}}, \tag{325}$$

under the conditions that $0 < p, q \le \infty$, $\sigma_p < s < k$. However Theorem 6.1.3 (ii), (iv), and (v) above achieve better conditions on parameters when $p = 1$, $1 \le q \le \infty$, and $-n < s < \infty$, and when $1 < p \le \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. The authors of [47] also proved that the inhomogeneous space $B^s_{p,q}(\mathbb{R}^n)$ can be continuously embedded into $L^r(\mathbb{R}^n)$ if and only if

$$\|f\|_{L^r(\mathbb{R}^n)} + \sup_{0<|h|\le 1} \frac{\|\Delta^m_h f\|_{L^r(\mathbb{R}^n)}}{|h|^m} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \Big(\int_{0<|h|\le 1} \frac{\|\Delta^M_h f\|^q_{L^p(\mathbb{R}^n)}}{|h|^{(s+m)q}} \frac{dh}{|h|^n} \Big)^{\frac{1}{q}}, \tag{326}$$

and the parameters satisfy $0 < p < \infty$, $0 < q \le \infty$, $1 < r < \infty$, $m \in \mathbb{N}_0$, $M \in \mathbb{N}$, and $0 < s < M - m$ with $s - \frac{n}{p} = -\frac{n}{r}$. Another embedding result in terms of moduli of smoothness was derived in [47] as a corollary. In [90, Theorem 2.5.1], H. Triebel gave the following characterization in the sense of equivalent quasinorms,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \sum_{j=1}^n \Big(\int_{Q_\delta} \Big\| |h|^{-(s-k)} \Delta^l_h \frac{\partial^k f}{\partial x_j^k} \Big\|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^n} \Big)^{\frac{1}{q}}, \tag{327}$$

where $Q_\delta = \{y | y = (y_1, \cdots, y_n); 0 < y_j < \delta\}$ (notation cf. [90, Section 1.13.4]), under the conditions that $1 < p < \infty$, $1 \le q \le \infty$, k and l are integers such that $0 \le k < s < l + k$, and $0 < \delta \le \infty$. But Theorem 6.1.3 (ii) and (iv) still achieve better conditions on parameters in case $k = 0$. Inequalities (319), (321), (322), and (323) were also given in [10, Proposition 10 (i)] by the authors G. Bourdaud, M. Moussai, and W. Sickel under the rough conditions that $1 \le p \le \infty$, $0 < q \le \infty$, $0 < s < m$, and m is the iteration number (see also [10, Section 4.4] for the definition of $M^{s,m}_{p,q} f$). The reverse inequality was also given under the same conditions in [10, Proposition 10 (ii)]. The proof of Theorem 6.1.3 can be found in section 6.3. The corollaries of Theorem 6.1.3 are formulated below.

Corollary 6.1.3. Let $0 < p, q < \infty$, $s \in \mathbb{R}$ and $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p < \infty$, $0 < q < \infty$ and $0 < s < 1$, then

$$(\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{328}$$

(ii)If $1 < p < \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$, or if $0 < p < 1$, $0 < q < \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}. \tag{329}$$

(iii)If $p = 1$, $1 \leq q < \infty$ and $-n < s < \infty$, or if $p = 1$, $0 < q < 1$ and $0 < s < \infty$, then

$$\|f\|_{\dot{B}^s_{1,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+h) - f(x)| dx)^q \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}. \tag{330}$$

Proof of Corollary 6.1.3. Apply inequalities (318), (319) and (322) with $L = 1$. Inequalities (329) and (330) also indicate

$$2^{ks} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}, \tag{331}$$

for every $k \in \mathbb{Z}$, and hence $\lim_{k \to +\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right sides of (329) and (330) are finite. □

We can also pick some special values for p, q, s in the above inequalities and then deduce some other interesting inequalities. For example, let $1 \leq q \leq p < \frac{n}{s}$, then $0 < s < 1$ and $s < \frac{n}{p}$. By Lemma 2.0.5, we have $\|f_k\|_{L^p(\mathbb{R}^n)} \lesssim 2^{kn(\frac{1}{q} - \frac{1}{p})} \|f_k\|_{L^q(\mathbb{R}^n)}$ and hence

$$\begin{aligned} \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} &\lesssim \Big(\sum_{k \leq 0} 2^{kq(s + \frac{n}{q} - \frac{n}{p})} \|f_k\|^q_{L^q(\mathbb{R}^n)} + \sum_{k > 0} 2^{kq(s + \frac{n}{q} - \frac{n}{p})} \|f_k\|^q_{L^q(\mathbb{R}^n)} \Big)^{\frac{1}{q}} \\ &\lesssim \Big(\sum_{k \leq 0} \|f_k\|^q_{L^q(\mathbb{R}^n)} \Big)^{\frac{1}{q}} + \Big(\sum_{k > 0} 2^{kn} \|f_k\|^q_{L^q(\mathbb{R}^n)} \Big)^{\frac{1}{q}} \\ &\lesssim \|f\|_{\dot{B}^0_{q,q}(\mathbb{R}^n)} + \|f\|_{\dot{B}^{\frac{n}{q}}_{q,q}(\mathbb{R}^n)}. \end{aligned} \tag{332}$$

By the inequalities given in Corollary 6.1.3 and Fubini's theorem, we can further deduce the following inequality

$$\begin{aligned} &(\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \\ \lesssim\ &\Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^n} dx dy\Big)^{\frac{1}{q}} + \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{2n}} dx dy\Big)^{\frac{1}{q}}, \end{aligned} \tag{333}$$

when the corresponding conditions on the parameters are satisfied.

Corollary 6.1.4. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and at least one of p and q is infinity. Assume $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.

(i)If $p = \infty$, $0 < q < \infty$ and $0 < s < 1$, then

$$(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{|f(x+h) - f(x)|^q}{|h|^{n+sq}} dh)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)}. \tag{334}$$

(ii)If $p = \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$, then

$$\|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{|f(x+h) - f(x)|^q}{|h|^{n+sq}} dh)^{\frac{1}{q}}. \tag{335}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $0 < s < 1$, then

$$\operatorname*{ess\,sup}_{h \in \mathbb{R}^n} |h|^{-s} \cdot (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{336}$$

(iv)If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $p = 1$, $q = \infty$ and $-n < s < \infty$, or if $0 < p < 1$, $q = \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h \in \mathbb{R}^n} |h|^{-s} \cdot (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{1}{p}}. \tag{337}$$

(v)If $p = q = \infty$ and $0 < s < 1$, then

$$\operatorname*{ess\,sup}_{x,y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^s} \lesssim \|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)}. \tag{338}$$

(vi)If $p = q = \infty$ and $s \in \mathbb{R}$, then

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{x,y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^s}. \tag{339}$$

Proof of Corollary 6.1.4. Apply Theorem 6.1.3 with $L = 1$. From (334) we can see that

$$(\int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^q}{|h|^{n+sq}} dh)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)}, \tag{340}$$

for almost every $x \in \mathbb{R}^n$ when conditions of Corollary 6.1.4 (i) are satisfied. From (336) and (338), we also deduce the following inequality

$$\|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim |h|^s \cdot \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \tag{341}$$

for almost every $h \in \mathbb{R}^n$ when $0 < p \leq \infty$, $q = \infty$ and $0 < s < 1$. From (337) with a proper change of variable, we can obtain

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{sp}} dx)^{\frac{1}{p}}, \tag{342}$$

when conditions of Corollary 6.1.4 (iv) are satisfied. Furthermore from (335) we have

$$2^{ks}|\psi_{2^{-k}} * f(x)| \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{|f(x+h) - f(x)|^q}{|h|^{n+sq}} dh)^{\frac{1}{q}} \tag{343}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, and from (339) we have

$$2^{ks}|\psi_{2^{-k}} * f(x)| \lesssim \operatorname*{ess\,sup}_{x,y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^s} \tag{344}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, therefore $\lim_{k \to +\infty} |\psi_{2^{-k}} * f(x)| = 0$ when $s > 0$ and the right sides of (335) and (339) are finite. Moreover from (337) we have

$$2^{ks}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h \in \mathbb{R}^n} |h|^{-s} \cdot (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{\frac{1}{p}} \tag{345}$$

for every $k \in \mathbb{Z}$, hence $\lim_{k \to +\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right side of (337) is finite. □

If $0 < \alpha \leq p < \infty$, $0 < s < 1$ and $0 < \beta < \infty$, then by Lemma 2.0.5, $f_k = \psi_{2^{-k}} * f$ satisfies $\|f_k\|_{L^p(\mathbb{R}^n)} \lesssim 2^{kn(\frac{1}{\alpha}-\frac{1}{p})}\|f_k\|_{L^\alpha(\mathbb{R}^n)}$, and we have

$$\begin{aligned}\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} &\lesssim \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s+\frac{n}{\alpha}-\frac{n}{p})}\|f_k\|_{L^\alpha(\mathbb{R}^n)} \\ &\lesssim \Big(\int_{\mathbb{R}^n} (\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s+\frac{n}{\alpha}-\frac{n}{p})}|f_k(x)|)^\alpha dx\Big)^{\frac{1}{\alpha}} \\ &\lesssim \Big(\int_{\mathbb{R}^n} (\sum_{k\in\mathbb{Z}} 2^{k\beta(s+\frac{n}{\alpha}-\frac{n}{p})}|f_k(x)|^\beta)^{\frac{\alpha}{\beta}} dx\Big)^{\frac{1}{\alpha}} = \|f\|_{\dot{F}^{s+\frac{n}{\alpha}-\frac{n}{p}}_{\alpha,\beta}(\mathbb{R}^n)}. \end{aligned} \tag{346}$$

By combining (336), (346), and (309) altogether, we can obtain

$$\operatorname*{ess\,sup}_{h\in\mathbb{R}^n} \frac{\|f(\cdot+h)-f(\cdot)\|_{L^p(\mathbb{R}^n)}}{|h|^s} \lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^\beta}{|x-y|^{n+\beta(s+\frac{n}{\alpha}-\frac{n}{p})}}dy\Big)^{\frac{\alpha}{\beta}}dx\Big)^{\frac{1}{\alpha}}, \tag{347}$$

where the parameters satisfy

$$\max\{0, n(\frac{1}{\min\{\alpha,\beta\}}-1)\} + \max\{0, n(\frac{1}{\alpha}-\frac{1}{\beta})\} < s+\frac{n}{\alpha}-\frac{n}{p}$$

if $0 < \beta < 1$, and there are no extra conditions for parameters if $1 \leq \beta < \infty$ since $-n < s+\frac{n}{\alpha}-\frac{n}{p} < \infty$ is always true for $0 < \alpha \leq p < \infty$ and $0 < s < 1$. In particular, letting $\alpha = p$ in (347) yields

$$\operatorname*{ess\,sup}_{h\in\mathbb{R}^n} \frac{\|f(\cdot+h)-f(\cdot)\|_{L^p(\mathbb{R}^n)}}{|h|^s} \lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^\beta}{|x-y|^{n+s\beta}}dy\Big)^{\frac{p}{\beta}}dx\Big)^{\frac{1}{p}} = [f]_{W^s_{p,\beta}(\mathbb{R}^n)}, \tag{348}$$

when the above conditions are met. If $0 < \alpha < p = \infty$, $0 < s < 1$, and $0 < \beta < \infty$, then by Lemma 2.0.5 we have $\|f_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{\frac{kn}{\alpha}}\|f_k\|_{L^\alpha(\mathbb{R}^n)}$ and

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s+\frac{n}{\alpha})}\|f_k\|_{L^\alpha(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^{s+\frac{n}{\alpha}}_{\alpha,\infty}(\mathbb{R}^n)}. \tag{349}$$

By (338), (349), and (311), we have

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^s} \lesssim \Big(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \frac{|f(x)-f(y)|^\alpha}{|x-y|^{n+s\alpha}}dx\Big)^{\frac{1}{\alpha}} \tag{350}$$

for all $0 < \alpha < \infty$ and $0 < s < 1$. Because $\|f\|_{\dot{F}^{s+\frac{n}{\alpha}}_{\alpha,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^{s+\frac{n}{\alpha}}_{\alpha,\beta}(\mathbb{R}^n)}$ for all $0 < \beta < \infty$, then (338), (349) and (309) combined together give us the following inequality

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^s} \lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^\beta}{|x-y|^{n+(s+\frac{n}{\alpha})\beta}}dy\Big)^{\frac{\alpha}{\beta}}dx\Big)^{\frac{1}{\alpha}}, \tag{351}$$

where the parameters satisfy

$$\max\{0, n(\frac{1}{\min\{\alpha,\beta\}} - 1)\} + \max\{0, n(\frac{1}{\alpha} - \frac{1}{\beta})\} < s + \frac{n}{\alpha}$$

if $0 < \beta < 1$, and there are no extra conditions for parameters if $1 \leq \beta < \infty$. In particular, when α and β are related by the equation $\beta = \alpha \cdot \gamma$ for some $\gamma > 0$, then (351) becomes

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^s} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^{\alpha\cdot\gamma}}{|x-y|^{n+n\gamma+s\alpha\cdot\gamma}} dy)^{\frac{1}{\gamma}} dx)^{\frac{\gamma}{\beta}}, \tag{352}$$

when the corresponding conditions are satisfied.

Corollary 6.1.5. Let $0 < p, q < \infty$, $s \in \mathbb{R}$ and $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p < \infty$, $0 < q < \infty$ and $0 < s < 2$, then

$$(\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{353}$$

(ii)If $1 < p < \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$, or if $0 < p < 1$, $0 < q < \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}. \tag{354}$$

(iii)If $p = 1$, $1 \leq q < \infty$ and $-n < s < \infty$, or if $p = 1$, $0 < q < 1$ and $0 < s < \infty$, then

$$\|f\|_{\dot{B}^s_{1,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)| dx)^q \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}. \tag{355}$$

Proof of Corollary 6.1.5. Apply inequalities (318), (319) and (322) with $L = 2$. Inequalities (354) and (355) also indicate

$$2^{ks}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^p dx)^{\frac{q}{p}} \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}, \tag{356}$$

for every $k \in \mathbb{Z}$, and hence $\lim_{k\to+\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right sides of (354) and (355) are finite. □

Corollary 6.1.6. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and at least one of p and q is infinity. Assume $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.

(i)If $p = \infty$, $0 < q < \infty$ and $0 < s < 2$, then

$$(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^q \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)}. \tag{357}$$

(ii)If $p = \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$, then

$$\|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^q \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}}. \tag{358}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $0 < s < 2$, then

$$\operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s} \cdot (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^p dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{359}$$

(iv)If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $p = 1$, $q = \infty$ and $-n < s < \infty$, or if $0 < p < 1$, $q = \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s} \cdot (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^p dx)^{\frac{1}{p}}. \tag{360}$$

(v)If $p = q = \infty$ and $0 < s < 2$, then

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|}{|x-y|^s} \lesssim \|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)}. \tag{361}$$

(vi)If $p = q = \infty$ and $s \in \mathbb{R}$, then

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|}{|x-y|^s}. \tag{362}$$

Proof of Corollary 6.1.6. Apply Theorem 6.1.3 with $L = 2$. From (357) we can see that

$$(\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^q \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)}, \tag{363}$$

for almost every $x \in \mathbb{R}^n$ when conditions of Corollary 6.1.6 (i) are satisfied. From (359) and (361), we also deduce the following inequality

$$\|f(\cdot + 2h) - 2f(\cdot + h) + f(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim |h|^s \cdot \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \tag{364}$$

for almost every $h \in \mathbb{R}^n$ when $0 < p \leq \infty$, $q = \infty$ and $0 < s < 2$. From (360) with a proper change of variable, we can obtain

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|^p}{|x-y|^{sp}} dx)^{\frac{1}{p}}, \tag{365}$$

when conditions of Corollary 6.1.6 (iv) are satisfied. Furthermore from (358) we have

$$2^{ks}|\psi_{2^{-k}} * f(x)| \lesssim (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^q \frac{dh}{|h|^{n+sq}})^{\frac{1}{q}} \tag{366}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, and from (362) we have

$$2^{ks}|\psi_{2^{-k}} * f(x)| \lesssim \operatorname*{ess\,sup}_{x,y \in \mathbb{R}^n} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|}{|x-y|^s} \tag{367}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, therefore $\lim_{k\to+\infty} |\psi_{2^{-k}} * f(x)| = 0$ when $s > 0$ and the right sides of (358) and (362) are finite. Moreover from (360) we have

$$2^{ks}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h \in \mathbb{R}^n} |h|^{-s} \cdot (\int_{\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|^p dx)^{\frac{1}{p}} \tag{368}$$

for every $k \in \mathbb{Z}$, hence $\lim_{k\to+\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right side of (360) is finite. □

Theorem 6.1.2, Theorem 6.1.3, and their corresponding corollaries are newly published results in the author's paper [98]. And we also introduced other mathematicians' results related to iterated differences below. In [97], F. Wang, Z. He, D. Yang, and W. Yuan introduced the spaces of Lipschitz type on spaces of homogeneous type in the sense of Coifman and Weiss, and discussed their relations with Besov and Triebel–Lizorkin spaces, furthermore the authors also established the difference characterization of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type without the dependence on the reverse doubling assumption of the considered measure of the underlying space. This major novelty is achieved by using the geometrical property of the underlying space in terms of its dyadic reference points, dyadic cubes, and the (local) lower bound. In [102], D. Yang, W. Yuan, and Y. Zhou provided the characterization of homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, in the sense of equivalent quasinorms, via a new square function, and they proved the equivalence relation

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \sim \|\{2^{k\alpha} \fint_{B(\cdot,2^{-k})} [f(\cdot) - f(y)]dy\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}, \tag{369}$$

under the condition that $f \in L^1_{loc}(\mathbb{R}^n) \bigcap \mathcal{S}'(\mathbb{R}^n)$, $0 < \alpha < 2$, $1 < p < \infty$, and $1 < q \leq \infty$. The authors of [102] also considered the case when $p = \infty$ and extended this result to higher order Sobolev space for $\alpha \in (2N, 2N + 2)$ and N is a positive integer. The corresponding results for inhomogeneous spaces are also included in this paper. In [12], H.-Q. Bui, M. Paluszyński, and M. Taibleson gave continuous characterizations of the weighted homogeneous Triebel-Lizorkin $\dot{F}^{\alpha,w}_{p,q}$ and Besov-Lipschitz $\dot{B}^{\alpha,w}_{p,q}$ spaces by using Schwartz functions satisfying the moment condition and the Tauberian condition, and their result reads as follows,

$$\|(\int_0^\infty (t^{-\alpha}\mu_t^* f(x))^q \frac{dt}{t})^{\frac{1}{q}}\|_{p,w} \lesssim \|f\|_{\dot{F}^{\alpha,w}_{p,q}} \lesssim \|(\int_0^\infty (t^{-\alpha}\nu_t^* f(x))^q \frac{dt}{t})^{\frac{1}{q}}\|_{p,w}, \tag{370}$$

where $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, w is a function in the Muckenhoupt weight class A_∞, $r_0 = \inf\{r : w \in A_r\}$, and $\lambda > \max\{nr_0/p, n/q\}$. Here in the characterization (370), μ and ν are Schwartz functions, μ satisfies the moment condition (that is, $\int_{\mathbb{R}^n} x^\kappa \mu(x)dx = 0$ for all $|\kappa| \leq [\alpha]$), and ν satisfies the Tauberian condition (that is, for all $\xi \neq 0$ there exists $t > 0$ such that $\mathcal{F}_n\nu(t\xi) \neq 0$), and $\mu_t^* f(\cdot)$ and $\nu_t^* f(\cdot)$ are the associated Peetre-Fefferman-Stein maximal function, the name of which was (firstly) introduced at the beginning of section 3 of [12]. Under the same conditions except that $\lambda > nr_0/p$, the authors of [12] also proved the following characterization

$$(\int_0^\infty (t^{-\alpha}\|\mu_t * f\|_{H^p_w})^q \frac{dt}{t})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^{\alpha,w}_{p,q}} \lesssim (\int_0^\infty (t^{-\alpha}\|\nu_t^* f\|_{p,w})^q \frac{dt}{t})^{\frac{1}{q}} \tag{371}$$

for the weighted homogeneous Besov-Lipschitz spaces $\dot{B}^{\alpha,w}_{p,q}$. In [59], L. Liu proved the $L^p(\mathbb{R}^n) \to \dot{F}^{\beta}_{q,\infty}(\mathbb{R}^n)$ boundedness, the $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ boundedness, and the $L^1(\mathbb{R}^n) \to L^{\frac{n}{n-\delta-\beta},\infty}(\mathbb{R}^n)$ weak type boundedness of the multilinear Littlewood-Paley operator defined by

$$g^A_\mu(f)(x) = [\iint_{\mathbb{R}^{n+1}_+} (\frac{t}{t+|x-y|})^{n\mu} |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+1}}]^{\frac{1}{2}}, \tag{372}$$

where

$$F^A_t(f)(x,y) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz, \tag{373}$$

$$R_{m+1}(A;x,z) = A(x) - \sum_{|\alpha|\le m} \frac{D^\alpha A(z)}{\alpha!}(x-z)^\alpha, \tag{374}$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$, and A is a function such that $D^\alpha A$ is in the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ for $|\alpha| = m$. In [69], V. K. Nguyen, M. Ullrich, and T. Ullrich defined the Besov space of dominating mixed smoothness, the Triebel–Lizorkin space of dominating mixed smoothness, and the mixed iterated differences of a multivariate function, furthermore the authors provided equivalence characterizations of the above spaces via rectangular means of mixed iterated differences. Their results are considered as the counterpart of the characterization by the ball means of iterated differences for isotropic Triebel-Lizorkin spaces (see [93, Theorem 2.5.11]). In [30], F. Dai, A. Gogatishvili, D. Yang, and W. Yuan characterized homogeneous Besov and Triebel-Lizorkin spaces via sequences consisting of the differences between f and the ball average $B_{l,2^{-k}}f$. Namely, they characterized $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ by the following expression

$$\Big(\sum_{k\in\mathbb{Z}} 2^{kq\alpha}\|f - B_{l,2^{-k}}f\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}}, \tag{375}$$

when $p \in (1,\infty]$, $q \in (0,\infty]$, $l \in \mathbb{N}$, and $\alpha \in (0, 2l)$, and

$$B_{l,2^{-k}}f(x) = \frac{-2}{\binom{2l}{l}} \sum_{j=1}^{l} (-1)^j \binom{2l}{l-j} B_{j2^{-k}}f(x), \tag{376}$$

where $B_{j2^{-k}}f(x)$ denotes the integral average of f on the ball $B(x, j2^{-k})$. The authors of [30] also characterized $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ by the following expression

$$\|\big(\sum_{k\in\mathbb{Z}} 2^{kq\alpha}|f - B_{l,2^{-k}}f|^q\big)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}, \tag{377}$$

when $p \in (1, \infty)$, $q \in (1, \infty]$, $l \in \mathbb{N}$, and $\alpha \in (0, 2l)$, and by the expression

$$\sup_{x\in\mathbb{R}^n}\sup_{m\in\mathbb{Z}}\big(\fint_{B(x,2^{-m})}\sum_{k=m}^{\infty} 2^{kq\alpha}|f(y) - B_{l,2^{-k}}f(y)|^q dy\big)^{\frac{1}{q}}, \tag{378}$$

when $p = \infty$, $q \in (1, \infty]$, $l \in \mathbb{N}$, and $\alpha \in (0, 2l)$. (With obvious modifications if $q = \infty$.) The corresponding results for inhomogeneous Besov and Triebel–Lizorkin spaces are also obtained in [30]. In [83], C. Schneider and J. Vybíral proved the homogeneity property

$$\|f(\lambda\cdot)\|_{\mathbf{B}^s_{p,q}(\mathbb{R}^n)} \sim \lambda^{s-n/p}\|f\|_{\mathbf{B}^s_{p,q}(\mathbb{R}^n)} \quad \text{for } 0 < \lambda \leq 1,\ f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n) \text{ and } spt.f \subset B_\lambda, \tag{379}$$

where $\|f\|_{\mathbf{B}^s_{p,q}(\mathbb{R}^n)}$ is defined to be the following expression

$$\|f\|_{L^p(\mathbb{R}^n)} + \big(\int_0^1 t^{-sq}\omega_r(f,t)_p^q\frac{dt}{t}\big)^{\frac{1}{q}}, \tag{380}$$

(with the usual modification if $q = \infty$) and $0 < p, q \leq \infty$, $0 < s < r \in \mathbb{N}$, and $\omega_r(f,t)_p = \sup_{|h|\leq t}\|\Delta^r_h f\|_{L^p(\mathbb{R}^n)}$ is the r-th modulus of smoothness. The authors of [83] also defined the space $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ in terms of the ball means of the iterated difference $\Delta^r_h f$ (see [83, Definition 2.1 (ii)]) and derived its corresponding homogeneity property. The spaces in [83, Definition 2.1] are independent of r, meaning that different values of $r > s$ result in norms that are equivalent. In [101], D. Yang and W. Yuan introduced the α-order Hajłasz type gradient sequence of a locally integrable function on $\mathbb{R}^n$ and gave the definitions of homogeneous Hajłasz Besov spaces $\dot{\mathcal{B}}^{\alpha}_{p,q}(\mathbb{R}^n)$ and homogeneous Hajłasz Triebel-Lizorkin spaces $\dot{\mathcal{F}}^{\alpha}_{p,q}(\mathbb{R}^n)$, furthermore their main result showed that homogeneous Hajłasz Besov spaces $\dot{\mathcal{B}}^{\alpha}_{p,q}(\mathbb{R}^n)$ coincide with the classical homogeneous Besov spaces $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ in the sense of equivalent quasinorms when $0 < \alpha < 2$, $1 < p \leq \infty$, and $0 < q \leq \infty$, and that homogeneous Hajłasz Triebel-Lizorkin spaces $\dot{\mathcal{F}}^{\alpha}_{p,q}(\mathbb{R}^n)$ coincide with the classical homogeneous Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ in the sense of equivalent quasinorms when $0 < \alpha < 2$,

$1 < p \leq \infty$, and $1 < q \leq \infty$. The authors of [101] also derived the higher order variant of their main result, and these results provided a possible way to introduce Besov and Triebel-Lizorkin spaces with arbitrary positive smoothness order on metric measure spaces. In [66, Section 3.4], V. K. Nguyen and W. Sickel defined the spaces $\mathcal{Z}^s_{mix}((0,1)^d)$ of Hölder-Zygmund type via mixed iterated differences and identified $\mathcal{Z}^s_{mix}((0,1)^d)$ with the d-fold tensor product $S^s_{\infty,\infty}B((0,1)^d)$ of the univariate Besov space $B^s_{\infty,\infty}(0,1)$. The authors of [66] also investigated the asymptotic behavior of the n-th Weyl number of the identity map $id: S^t_{p_1,p_1}B((0,1)^d) \to \mathcal{Z}^s_{mix}((0,1)^d)$ under the conditions that $s > 0$, $t > s + \frac{1}{p_1}$, $n \geq 2$, and $S^t_{p_1,p_1}B((0,1)^d)$ denotes the d-fold tensor product of the univariate Besov space $B^t_{p_1,p_1}(0,1)$. The asymptotic behavior of the n-th approximation number of the same identity map was studied in [66, Theorem 3.13]. In [45], P. Hajłasz proved that in the case Ω is a bounded domain with the extension property or in the case $\Omega = \mathbb{R}^n$, the sufficient and necessary condition for the gradient ∇f of a measurable function f to belong to $L^p(\Omega)$ $(1 < p \leq \infty)$ is that the inequality $|f(x) - f(y)| \leq |x-y|(g(x)+g(y))$ holds true almost everywhere for some nonnegative function $g \in L^p(\Omega)$, and the author also showed that this condition can be generalized to define Sobolev spaces on metric measure spaces with a finite diameter and a finite positive Borel measure. The above condition has been generalized to the case of higher-order iterated differences in [95] by H. Triebel. The main result of [95] shows that when $1 < p < \infty$ and $k \in \mathbb{N}$, the classical Sobolev space $W^k_p(\mathbb{R}^n)$ can be identified with the space of $L^p(\mathbb{R}^n)$-functions f, for every f there exists a function $0 \leq g \in L^p(\mathbb{R}^n)$ such that for all $h \in \mathbb{R}^n$ with $0 < |h| \leq 1$, we have $|h|^{-k}|\varDelta^k_h f(x)| \leq \sum_{l=0}^k g(x+lh)$ for almost every $h, x \in \mathbb{R}^n$. In [9], B. Bojarski proved that such an identification still holds if one replaces the above inequality by $|\varDelta^k_h f(x)| \leq |x-y|^k(g(x)+g(y))$ for almost every $x, h \in \mathbb{R}^n$, and $y = x + kh$. In [48], D. D. Haroske and H. Triebel surveyed some recent developments of distributional Sobolev-Besov spaces and Sobolev-Besov spaces of measurable functions of positive smoothness which can be characterized in terms of differences. In [112], Ó. Domínguez, A. Seeger, B. Street, J. Van Schaftingen, and P.-L. Yung proved that when $0 < s < M$, $1 < p < \infty$, $1 \leq r \leq \infty$, and $\gamma \in \mathbb{R}$, the Lorentz norm $\|\mathcal{Q}_{M,s+\frac{\gamma}{p}} f_\circ\|_{L^{p,r}(\nu_\gamma)}$ of the function $\mathcal{Q}_{M,s+\frac{\gamma}{p}} f_\circ(x,h) := \frac{\varDelta^M_h f_\circ(x)}{|h|^{s+\frac{\gamma}{p}}}$ with respect to the measure $\nu_\gamma(E) := \iint_E \frac{dxdh}{|h|^{d-\gamma}}$ for $E \subseteq \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ can be controlled by $\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}$, where $f_\circ$ is the unique function representative of $f \in \mathcal{S}'_\infty(\mathbb{R}^d)$ and the term

$\|f\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}$ is defined in (1.4) and (1.5) of [112], and the authors also established the equivalence between Fourier analytic definitions and definitions via difference operators acting on measurable functions. In [68] and [67], V. K. Nguyen and W. Sickel provided the definition of Sobolev and Besov spaces of dominating mixed smoothness in terms of the mixed difference operator and its associated modulus of smoothness and then gave necessary and sufficient conditions for these spaces to form algebras with respect to pointwise multiplication and the description of the space of all pointwise multipliers for $S^r_{p,q}B(\mathbb{R}^d)$ in case $p \leq q$. In [84], C. Schneider and J. Vybíral defined the Besov space $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and the Besov space $\mathbf{B}^s_{p,q}(\Omega)$ on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with a Lipschitz boundary Γ via iterated differences and moduli of smoothness, and they also studied the (σ, p)-atomic decomposition, the Lipschitz atomic decomposition of $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, and the atomic decomposition for the space $\mathbf{B}^s_{p,q}(\Gamma)$ introduced via the resolution of unity and the local Lipschitz diffeomorphisms. The authors of [84] also proved the boundedness of the linear trace operator $Tr : \mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega) \to \mathbf{B}^s_{p,q}(\Gamma)$ when $n \geq 2$, $0 < p, q \leq \infty$, and $0 < s < 1$, as well as the existence of a bounded nonlinear extension operator $Ext : \mathbf{B}^s_{p,q}(\Gamma) \to \mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega)$ when the parameters satisfy the same conditions. In [55], H. Kempka and J. Vybíral proved that the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ of Besov and Triebel-Lizorkin types of variable exponents allow a characterization in the time-domain with the help of classical ball means of differences.

6.2 Proof Of Theorem 6.1.2

Proof. We first prove Theorem 6.1.2 (i). Let $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ be an element of $\mathcal{S}'(\mathbb{R}^n)$. We note that $\tilde{\sigma}_{pq} < s$ implies $\frac{nq}{n+sq} < p$. And we recall the notation $A_k = \{h \in \mathbb{R}^n : 2^{-k} \leq |h| < 2^{1-k}\}$ for $k \in \mathbb{Z}$. For $|h| \lesssim 2^{-k}$ and $f_j = \psi_{2^{-j}} * f$, we deduce two estimates for $|(\varDelta^L_h f_j)(x)|$. Using mean value theorem and the iteration formula (9) consecutively, we get

$$|(\varDelta^L_h f_j)(x)| \lesssim \sum_{|\alpha|=L} |\partial^\alpha f_j(x + \sum_{l=1}^{L} t_{\alpha,l} h)| \cdot |h|^L, \tag{381}$$

where α represents a multi-index and each $t_{\alpha,l}$ is in $(0, 1)$. Since the n-dimensional distributional Fourier transform $\mathcal{F}_n f_j$ is supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| < 2^{j+1}\}$, we use Remark

2.0.6 to get

$$|\partial^\alpha f_j(x+\sum_{l=1}^L t_{\alpha,l}h)| \lesssim \mathcal{P}_n(\partial^\alpha f_j)(x+\sum_{l=1}^L t_{\alpha,l}h) \lesssim 2^{jL}\mathcal{P}_n f_j(x+\sum_{l=1}^L t_{\alpha,l}h). \tag{382}$$

Since $|\sum_{l=1}^L t_{\alpha,l}h| \lesssim L2^{-k}$, by (52) of Remark 2.0.3 we have

$$\mathcal{P}_n f_j(x+\sum_{l=1}^L t_{\alpha,l}h) \lesssim \mathcal{P}_n f_j(x)\cdot(1+L2^{j-k})^{n/r}, \tag{383}$$

where r is the chosen positive number in Definition 1.2.5 and satisfies $0<r<\min\{p,q\}$. We infer from (381), (382) and (383) the first estimate

$$|(\mathit{\Delta}_h^L f_j)(x)| \lesssim 2^{(j-k)L}(1+L2^{j-k})^{n/r}\mathcal{P}_n f_j(x) \quad \text{for} \quad |h| \lesssim 2^{-k}, \tag{384}$$

and the constant is independent of $h\in\mathbb{R}^n$, $j,k\in\mathbb{Z}$. Also by using (10), we get

$$|(\mathit{\Delta}_h^L f_j)(x)| \lesssim \sum_{l=0}^L |f_j(x+lh)|. \tag{385}$$

If $0\le l\le L$, $|h|\lesssim 2^{-k}$ and $j>k$, we recall that $0<r<\min\{p,q\}$ and the value of r will be determined later, then using Remark 2.0.3, Lemma 2.0.3 and the definition of Peetre-Fefferman-Stein maximal function, we obtain

$$\mathcal{P}_n f_j(x+lh) \lesssim (1+2^{j+1}l|h|)^{n/r}\mathcal{P}_n f_j(x) \lesssim 2^{(j-k)n/r}\mathcal{M}_n(|f_j|^r)(x)^{1/r}, \tag{386}$$

and the constant in (386) is independent of $h\in\mathbb{R}^n$, $0\le l\le L$ and $j,k\in\mathbb{Z}$. Using a proper change of variable, we also have

$$2^{kn}\int_{A_k}|f_j(x+lh)|^r dh \lesssim ⨍_{l2^{-k}\le|y|<l2^{-k+1}}|f_j(x+y)|^r dy \lesssim \mathcal{M}_n(|f_j|^r)(x) \tag{387}$$

for $0 < l \le L$, and $|f_j(x)|^r \le \mathcal{M}_n(|f_j|^r)(x)$ by Lebesgue's differentiation theorem. Applying (386) and (387), we can obtain the second estimate

$$\begin{aligned}
&2^{kn}\int_{A_k} |(\Delta_h^L f_j)(x)|^q dh \\
\lesssim&\sum_{l=0}^{L} 2^{kn}\int_{A_k} |f_j(x+lh)|^r \cdot |f_j(x+lh)|^{q-r} dh \\
\lesssim&\sum_{l=0}^{L} 2^{kn}\int_{A_k} |f_j(x+lh)|^r dh \cdot \mathcal{P}_n f_j(x+lh)^{q-r} \\
\lesssim&2^{(j-k)n(\frac{q}{r}-1)}\mathcal{M}_n(|f_j|^r)(x)^{\frac{q}{r}}. \qquad (388)
\end{aligned}$$

And estimate (388) is true for $0 < q < \infty$ and $j > k$. Now we consider the following estimate

$$\begin{aligned}
&(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)}\int_{A_k} (\sum_{j\in\mathbb{Z}} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}} \qquad (389)\\
\lesssim&(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)}\int_{A_k} (\sum_{j\le k} |(\Delta_h^L f_j)(x)|)^q + (\sum_{j>k} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}} \\
\lesssim&(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)}\int_{A_k} (\sum_{j\le k} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}} \qquad (390)\\
&+(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)}\int_{A_k} (\sum_{j>k} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}}. \qquad (391)
\end{aligned}$$

For $0 < q < \infty$, we pick $0 < \varepsilon < \min\{s, L-s\}$ and the value of ε will be determined later. Then we have

$$\begin{aligned}
(\sum_{j\le k} |(\Delta_h^L f_j)(x)|)^q =& (\sum_{j\le k} 2^{j\varepsilon}\cdot 2^{-j\varepsilon}|(\Delta_h^L f_j)(x)|)^q \\
\lesssim& (\sum_{j\le k} 2^{j\varepsilon})^q \cdot \operatorname*{ess\,sup}_{l\le k} 2^{-lq\varepsilon}|(\Delta_h^L f_l)(x)|^q \\
\lesssim& 2^{kq\varepsilon}\cdot \sum_{j\le k} 2^{-jq\varepsilon}|(\Delta_h^L f_j)(x)|^q, \qquad (392)
\end{aligned}$$

and

$$\begin{aligned}
(\sum_{j>k} |(\Delta_h^L f_j)(x)|)^q =& (\sum_{j>k} 2^{-j\varepsilon}\cdot 2^{j\varepsilon}|(\Delta_h^L f_j)(x)|)^q \\
\lesssim& (\sum_{j>k} 2^{-j\varepsilon})^q \cdot \operatorname*{ess\,sup}_{l>k} 2^{lq\varepsilon}|(\Delta_h^L f_l)(x)|^q \\
\lesssim& 2^{-kq\varepsilon}\cdot \sum_{j>k} 2^{jq\varepsilon}|(\Delta_h^L f_j)(x)|^q. \qquad (393)
\end{aligned}$$

Using (392) and (384), we can estimate (390) from above by

$$\begin{aligned}&(\sum_{k\in\mathbb{Z}}\sum_{j\le k}2^{k(n+qs+q\varepsilon)}2^{-jq\varepsilon}\int_{A_k}|(\Delta_h^L f_j)(x)|^q dh)^{\frac{1}{q}}\\ \lesssim&(\sum_{k\in\mathbb{Z}}\sum_{j\le k}2^{k(qs+q\varepsilon)}2^{-jq\varepsilon}2^{(j-k)Lq}(1+L2^{j-k})^{\frac{nq}{r}}\mathcal{P}_n f_j(x)^q)^{\frac{1}{q}}. \qquad (394)\end{aligned}$$

We notice that $(1+L2^{j-k})^{nq/r}\lesssim C$ if $j\le k$ and C is a constant determined by n,q,r,L and we switch the order of summation to obtain

$$\begin{aligned}(394)\lesssim&(\sum_{j\in\mathbb{Z}}\sum_{k\ge j}2^{kq(s+\varepsilon-L)}2^{-jq\varepsilon+jqL}\mathcal{P}_n f_j(x)^q)^{\frac{1}{q}}\\ =&(\sum_{j\in\mathbb{Z}}2^{jqs}\mathcal{P}_n f_j(x)^q)^{\frac{1}{q}}\\ \lesssim&(\sum_{j\in\mathbb{Z}}2^{jqs}\mathcal{M}_n(|f_j|^r)(x)^{\frac{q}{r}})^{\frac{1}{q}}, \qquad (395)\end{aligned}$$

where we also used Remark 2.0.5 and the condition that $\varepsilon<L-s$. Using (393) and (388) and switching the order of summation, we can estimate (391) from above by

$$\begin{aligned}&(\sum_{k\in\mathbb{Z}}\sum_{j>k}2^{k(n+qs-q\varepsilon)}2^{jq\varepsilon}\int_{A_k}|(\Delta_h^L f_j)(x)|^q dh)^{\frac{1}{q}}\\ \lesssim&(\sum_{k\in\mathbb{Z}}\sum_{j>k}2^{kq(s-\varepsilon)+jq\varepsilon}\cdot 2^{(j-k)n(\frac{q}{r}-1)}\mathcal{M}_n(|f_j|^r)(x)^{\frac{q}{r}})^{\frac{1}{q}}\\ \lesssim&(\sum_{j\in\mathbb{Z}}\sum_{k<j}2^{kq[s-\varepsilon-n(\frac{1}{r}-\frac{1}{q})]}\cdot 2^{jq[\varepsilon+n(\frac{1}{r}-\frac{1}{q})]}\mathcal{M}_n(|f_j|^r)(x)^{\frac{q}{r}})^{\frac{1}{q}}. \qquad (396)\end{aligned}$$

If $q\le p<\infty$, we have

$$\lim_{\varepsilon\to 0,r\to q}s-\varepsilon-n(\frac{1}{r}-\frac{1}{q})=s>0.$$

If $\frac{nq}{n+sq}<p<q$, we have

$$\lim_{\varepsilon\to 0,r\to p}s-\varepsilon-n(\frac{1}{r}-\frac{1}{q})=s-n(\frac{1}{p}-\frac{1}{q})>0.$$

Therefore if we pick ε sufficiently small and r sufficiently close to $\min\{p,q\}$, then we can make $s-\varepsilon-n(\frac{1}{r}-\frac{1}{q})$ a positive finite number and hence

$$\sum_{k<j}2^{kq[s-\varepsilon-n(\frac{1}{r}-\frac{1}{q})]}\lesssim 2^{jq[s-\varepsilon-n(\frac{1}{r}-\frac{1}{q})]}. \qquad (397)$$

Inserting (397) into (396) yields

$$(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} (\sum_{j>k} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}} \lesssim (\sum_{j\in\mathbb{Z}} 2^{jqs}\mathcal{M}_n(|f_j|^r)(x)^{\frac{q}{r}})^{\frac{1}{q}}. \tag{398}$$

Combining (390), (394), (395), (391) and (398) and also invoking Lemma 2.0.6, we can obtain

$$\|(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} (\sum_{j\in\mathbb{Z}} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \tag{399}$$

when $0 < p, q < \infty$ and $\tilde{\sigma}_{pq} < s < L$. From the assumption $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$, we know inequality (399) also shows $\sum_{j\in\mathbb{Z}} |(\Delta_h^L f_j)(x)| < \infty$ for every $k \in \mathbb{Z}$ and for almost every $x \in \mathbb{R}^n, h \in A_k$. Together with (28), we have reached the conclusion that

$$\Delta_h^L f = \sum_{j\in\mathbb{Z}} \Delta_h^L f_j(x) \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \tag{400}$$

for every $k \in \mathbb{Z}$ and almost every $h \in A_k, x \in \mathbb{R}^n$, and the tempered distribution $\Delta_h^L f$ has a function representative which is the pointwise limit of the series $\sum_{j\in\mathbb{Z}} \Delta_h^L f_j(x)$. Furthermore, integration of $\Delta_h^L f$ with respect to the Lebesgue measure is justified, and the inequality

$$\|(\int_{\mathbb{R}^n} |h|^{-sq}|\Delta_h^L f|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} (\sum_{j\in\mathbb{Z}} |(\Delta_h^L f_j)(x)|)^q dh)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \tag{401}$$

is also validated. Therefore the proof of Theorem 6.1.2 (i) is now complete.

Now we prove Theorem 6.1.2 (ii) when f is a function, $0 < p < \infty$, $0 < q < 1$ and $\sigma_{pq} + \tilde{\sigma}_{pq} < s < \infty$. Without loss of generality, we also assume the right side of (303) is finite, otherwise, inequality (303) is trivial. To do this, recall that $spt.\mathcal{F}_n\psi \subseteq A' = \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| < 2\}$ and by Taylor expansion of $e^{2\pi it}$, we have

$$(e^{2\pi it} - 1)^L = (2\pi it)^L(1 + O(2\pi it)) \tag{402}$$

and there exists a sufficiently large positive integer m_0 such that

$$0 < |t| < 2^{2-m_0} \text{ implies } |(e^{2\pi it} - 1)^L| > 0. \tag{403}$$

For a unit vector $\theta \in \mathbb{S}^{n-1}$, we can find $\delta > 0$ so small that if $\xi \in A' \subseteq \mathbb{R}^n$ and $\frac{1}{4} \leq |\theta\cdot\xi| < 2$, then for all other θ' in the spherical cap $C_\theta := \{\theta' \in \mathbb{S}^{n-1} : |\theta' - \theta| < \delta\}$, we also have $\frac{1}{4} \leq |\theta'\cdot\xi| < 2$. We choose properly distributed unit vectors $\theta_1, \theta_2, \cdots, \theta_M$ where $M \in \mathbb{N}$

is sufficiently large so that the spherical caps $C_1, C_2, \cdots, C_M$, respectively associated with $\theta_1, \theta_2, \cdots, \theta_M$ in the above way, cover the unit sphere $\mathbb{S}^{n-1}$. For each cap C_l, $1 \le l \le M$, we consider the set

$$P_l := \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| < 2, \frac{\xi}{|\xi|} \in C_l\}, \tag{404}$$

then from the construction of $\{C_l\}_{l=1}^M$, we have that

$$\frac{1}{4} \le |\theta \cdot \xi| < 2 \qquad \text{for all } \xi \in P_l \text{ and } \theta \in C_l \tag{405}$$

and

$$\bigcup_{l=1}^{M} P_l = A'. \tag{406}$$

We use a partition of unity associated with $\{P_l\}_{l=1}^M$ by smooth functions $\{\rho_l\}_{l=1}^M$ with compact supports and $\{\rho_l\}_{l=1}^M$ also satisfy

$$\sum_{l=1}^{M} \rho_l(\xi) = 1 \text{ if } \xi \in A' \qquad \text{and} \qquad spt.\rho_l \bigcap A' \subseteq P_l \text{ for each } l. \tag{407}$$

Recall the definition of $\mathcal{F}_n\phi$ given in (18), we pick a large positive integer $J > m_0$ and the value of J will be determined later, then we have for each $k \in \mathbb{Z}$

$$\mathcal{F}_n\phi(2^{m_0-J-k}\xi) = 1 \qquad \text{if} \qquad |\xi| \le 2^{k+J-m_0}, \tag{408}$$

and by (20),

$$\mathcal{F}_n\phi(2^{m_0-J-k}\xi) = 1 - \sum_{j=1}^{\infty} \mathcal{F}_n\psi(2^{m_0-J-k-j}\xi) = 1 - \sum_{j=J+1}^{\infty} \mathcal{F}_n\psi(2^{m_0-k-j}\xi). \tag{409}$$

Furthermore if $\tau \in [1,2]$, $\theta \in C_l$, $2^{m_0-k}\xi \in spt.\mathcal{F}_n\psi \bigcap spt.\rho_l \subseteq P_l$, $1 \le l \le M$ then (403) and (405) tell us that

$$0 < 2^{-m_0-2} \le 2^{-k}\tau|\xi \cdot \theta| < 2^{2-m_0}, \tag{410}$$

and

$$|(e^{2\pi i 2^{-k}\tau\theta\cdot\xi} - 1)^L| > 0. \tag{411}$$

Hence if we let

$$\lambda_{l,\tau\theta}(\xi) := \frac{\mathcal{F}_n\psi(2^{m_0}\xi)\rho_l(2^{m_0}\xi)}{(e^{2\pi i\tau\theta\cdot\xi} - 1)^L}, \tag{412}$$

then $\lambda_{l,\tau\theta}(2^{-k}\xi)$ is a well-defined function in $C_c^\infty(\mathbb{R}^n)$ for every $k \in \mathbb{Z}$. Using formula (15), we have

$$\begin{aligned} &|\mathcal{F}_n^{-1}(\lambda_{l,\tau\theta}(2^{-k}\xi)) * [\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)](x)| \\ \lesssim &\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}(\lambda_{l,\tau\theta}(2^{-k}\xi))(y) \cdot \Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|dy. \end{aligned} \tag{413}$$

The Fourier transform of the Schwartz function

$$y \mapsto \mathcal{F}_n^{-1}(\lambda_{l,\tau\theta}(2^{-k}\xi))(y) \cdot \Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)$$

is supported in $\{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^{k+J-m_0}\}$. Since $0 < r < \min\{p, q\} < 1$ as mentioned in Definition 1.2.5, we use (408), observe the simple fact that both $\int_1^2 \frac{d\tau}{\tau}$ and $\mathcal{H}^{n-1}(C_l)$ are fixed positive finite constants, and then apply Lemma 2.0.5 to (413) and obtain

$$\begin{aligned} &|\mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f)(x)|^r \\ =&\fint_{[1,2]}\fint_{C_l} |\mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_n f)(x)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} \\ =&\fint_{[1,2]}\fint_{C_l} |\mathcal{F}_n^{-1}(\frac{\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)}{(e^{2\pi i 2^{-k}\tau\theta\cdot\xi}-1)^L} \cdot \\ &(e^{2\pi i 2^{-k}\tau\theta\cdot\xi}-1)^L \mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_n f)(x)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} \\ =&\fint_{[1,2]}\fint_{C_l} |\mathcal{F}_n^{-1}(\lambda_{l,\tau\theta}(2^{-k}\xi)) * [\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)](x)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} \\ \lesssim&2^{(J+k-m_0)n(1-r)}\int_1^2\int_{C_l}\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}(\lambda_{l,\tau\theta}(2^{-k}\xi))(y)|^r \cdot \\ &|\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau}. \end{aligned} \tag{414}$$

We let $k = 0$ in (410) and pick m_0 so large that conditions of Lemma 2.0.8 are satisfied. Applying Lemma 2.0.8 to the smooth function $\mathcal{F}_n\psi(2^{m_0}\xi)\rho_l(2^{m_0}\xi)$ whose support set is compactly contained in P_l yields that for a sufficiently large positive integer N, whose value will be determined later, we can find a constant C such that

$$|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(x)| \le \frac{C}{(1+|x|)^N} \qquad \text{for all } x \in \mathbb{R}^n, \tag{415}$$

and the constant C may depend on ψ, ρ_l, m_0, L, N but it is independent of $\tau \in [1,2]$ and $\theta \in C_l$. Recall that A_{k-m} denotes the annulus $\{y \in \mathbb{R}^n : 2^m \leq 2^k|y| < 2^{m+1}\}$ for integers k, m. With (415), we can estimate the most inside integral of (414) as follows,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}(\lambda_{l,\tau\theta}(2^{-k}\xi))(y)|^r \cdot |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy \\
=&\sum_{m\in\mathbb{Z}} \int_{A_{k-m}} 2^{knr}|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^k y)|^r \cdot |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy \\
\lesssim&\sum_{m<0} 2^{knr} \int_{A_{k-m}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy \\
&+\sum_{m\geq 0} 2^{knr-mNr} \int_{A_{k-m}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy \\
\lesssim&\sum_{m<0} 2^{kn(r-1)+mn} ⨍_{A_{k-m}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy \\
&+\sum_{m\geq 0} 2^{kn(r-1)+m(n-Nr)} ⨍_{A_{k-m}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy. \qquad (416)
\end{aligned}$$

We insert (416) into (414), apply Fubini's Theorem to switch the order of integration, use the following simple estimate

$$\begin{aligned}
&⨍_{A_{k-m}} \int_1^2 \int_{C_l} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} dy \\
\lesssim&⨍_{|y|\leq 2^{m+1-k}} \int_1^2 \int_{\mathbb{S}^{n-1}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} dy \\
\lesssim&\mathcal{M}_n(\int_1^2 \int_{\mathbb{S}^{n-1}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})(x), \qquad (417)
\end{aligned}$$

and also pick N so that $n - Nr < 0$, then we obtain the estimate

$$\begin{aligned}
&|\mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f)(x)|^r \\
\lesssim&2^{(J-m_0)n(1-r)}(\sum_{m<0} 2^{mn} + \sum_{m\geq 0} 2^{m(n-Nr)}) \\
&\cdot\mathcal{M}_n(\int_1^2 \int_{\mathbb{S}^{n-1}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})(x) \\
\lesssim&2^{(J-m_0)n(1-r)}\mathcal{M}_n(\int_1^2 \int_{\mathbb{S}^{n-1}} |\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})(x), \qquad (418)
\end{aligned}$$

where $\mathcal{M}_n$ is the Hardy-Littlewood maximal function, and we can obtain all these inequalities above because the constant C in (415) does not rely on $\tau \in [1,2]$ and $\theta \in C_l \subseteq \mathbb{S}^{n-1}$. Recall (407) and the fact that $0 < r < \min\{p,q\} < 1$, then we have

$$\begin{aligned}
&|\psi_{2^{m_0-k}} * f(x)| \\
=&|\sum_{l=1}^{M} \mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f)(x)| \\
\lesssim&(\sum_{l=1}^{M} |\mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f)(x)|^r)^{\frac{1}{r}} \\
\lesssim&2^{(J-m_0)n(\frac{1}{r}-1)}\mathcal{M}_n(\int_1^2\int_{\mathbb{S}^{n-1}} |\varDelta_{2^{-k}\tau\theta}^L(\phi_{2^{m_0-k-J}} * f)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})(x)^{\frac{1}{r}}. \qquad (419)
\end{aligned}$$

We insert (419) into $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ below, incorporate those coefficients that contain m_0 into constants since m_0 will be fixed, apply Lemma 2.0.6 and then we can obtain

$$\begin{aligned}
&\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \\
=&2^{-sm_0}\|\{2^{ks}|\psi_{2^{m_0-k}} * f|\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\
\lesssim&2^{Jn(\frac{1}{r}-1)}\|\{2^{ks}(\int_1^2\int_{\mathbb{S}^{n-1}} |\varDelta_{2^{-k}\tau\theta}^L(\phi_{2^{m_0-k-J}} * f)(\cdot)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})^{\frac{1}{r}}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}. \qquad (420)
\end{aligned}$$

We use Hölder's inequality to obtain

$$\begin{aligned}
&(\int_1^2\int_{\mathbb{S}^{n-1}} |\varDelta_{2^{-k}\tau\theta}^L(\phi_{2^{m_0-k-J}} * f)(\cdot)|^r d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})^{\frac{1}{r}} \\
\lesssim&(\int_1^2\int_{\mathbb{S}^{n-1}} |\varDelta_{2^{-k}\tau\theta}^L(\phi_{2^{m_0-k-J}} * f)(\cdot)|^q d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})^{\frac{1}{q}}. \qquad (421)
\end{aligned}$$

Inserting (421) into (420) yields $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ can be estimated from above by

$$2^{Jn(\frac{1}{r}-1)}\|\{2^{ks}(\int_1^2\int_{\mathbb{S}^{n-1}} |\varDelta_{2^{-k}\tau\theta}^L(\phi_{2^{m_0-k-J}} * f)(\cdot)|^q d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})^{\frac{1}{q}}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}. \qquad (422)$$

Recall (409) and the notation $f_j = \psi_{2^{-j}} * f$ then we have

$$\phi_{2^{m_0-k-J}} * f = f - \sum_{j=J+1}^{\infty} f_{k+j-m_0} \qquad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n). \qquad (423)$$

We can use an argument like the one for deducing (29) to obtain

$$\varDelta_{2^{-k}\tau\theta}^L(\phi_{2^{m_0-k-J}} * f) = \varDelta_{2^{-k}\tau\theta}^L f - \sum_{j=J+1}^{\infty} \varDelta_{2^{-k}\tau\theta}^L f_{k+j-m_0} \qquad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n). \qquad (424)$$

Inferring from (423) and assuming the validity of decomposition, then (422) can be estimated from above by the sum of the following two terms,

$$2^{Jn(\frac{1}{r}-1)}\|\{2^{ks}(\int_1^2\int_{\mathbb{S}^{n-1}}|\Delta^L_{2^{-k}\tau\theta}f(\cdot)|^q d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})^{\frac{1}{q}}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}, \tag{425}$$

and

$$2^{Jn(\frac{1}{r}-1)}\|\{2^{ks}(\int_1^2\int_{\mathbb{S}^{n-1}}|\Delta^L_{2^{-k}\tau\theta}(\sum_{j=J+1}^{\infty}f_{k+j-m_0})(\cdot)|^q d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau})^{\frac{1}{q}}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}. \tag{426}$$

For the first term, we use the change of variable formulas $t=2^{-k}\tau$ for $\tau\in[1,2]$ and $h=t\theta$ for $\theta\in\mathbb{S}^{n-1}$ in a sequence and we can get

$$\int_1^2\int_{\mathbb{S}^{n-1}}|\Delta^L_{2^{-k}\tau\theta}f(x)|^q d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau}=\int_{A_k}\frac{|\Delta^L_h f(x)|^q}{|h|^n}dh, \tag{427}$$

where A_k is the annulus $\{h\in\mathbb{R}^n: 2^{-k}\le|h|<2^{1-k}\}$, and hence

$$(425)\lesssim 2^{Jn(\frac{1}{r}-1)}\cdot\|(\int_{\mathbb{R}^n}\frac{|\Delta^L_h f(\cdot)|^q}{|h|^{n+sq}}dh)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}<\infty, \tag{428}$$

and the value of the large positive integer J will be determined later. For the second term (426), we begin with the same change of variable as in (427) and obtain

$$\begin{aligned}
&\int_1^2\int_{\mathbb{S}^{n-1}}(\sum_{j=J+1}^{\infty}|\Delta^L_{2^{-k}\tau\theta}f_{k+j-m_0}(x)|)^q d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau}\\
\lesssim&2^{kn}\int_{A_k}(\sum_{j=J+1}^{\infty}|\Delta^L_h f_{k+j-m_0}(x)|)^q dh\\
\lesssim&\sum_{j=J+1}^{\infty}2^{kn}\int_{A_k}|\Delta^L_h f_{k+j-m_0}(x)|^q dh \qquad (429)\\
\lesssim&\sum_{j=J+1}^{\infty}2^{(j-m_0)n(\frac{q}{r}-1)}\mathcal{M}_n(|f_{k+j-m_0}|^r)(x)^{\frac{q}{r}}, \qquad (430)
\end{aligned}$$

where in (429) we used the condition $0 < q < 1$, and in (430) we used estimate (388) since $k + j - m_0 > k + J - m_0 > k$. Therefore we have the estimate

$$2^{Jn(\frac{1}{r}-1)} \|\{2^{ks}(\int_1^2 \int_{\mathbb{S}^{n-1}} (\sum_{j=J+1}^{\infty} |\Delta^L_{2^{-k}\tau\theta} f_{k+j-m_0}(\cdot)|)^q d\mathcal{H}^{n-1}(\theta) \frac{d\tau}{\tau})^{\frac{1}{q}}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \tag{431}$$

$$\lesssim 2^{Jn(\frac{1}{r}-1)} \|(\sum_{k\in\mathbb{Z}} \sum_{j=J+1}^{\infty} 2^{ksq} \cdot 2^{jqn(\frac{1}{r}-\frac{1}{q})} \mathcal{M}_n(|f_{k+j-m_0}|^r)(\cdot)^{\frac{q}{r}})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim 2^{Jn(\frac{1}{r}-1)} \|(\sum_{j=J+1}^{\infty} 2^{jq[n(\frac{1}{r}-\frac{1}{q})-s]} \cdot \sum_{k\in\mathbb{Z}} 2^{ksq} \mathcal{M}_n(|f_k|^r)(\cdot)^{\frac{q}{r}})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim 2^{J[n(\frac{1}{r}-1)+n(\frac{1}{r}-\frac{1}{q})-s]} \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \tag{432}$$

where in the above calculation we incorporate coefficients containing m_0 into constants since m_0 is fixed. Recall the conditions $0 < q < 1$, $\sigma_{pq} + \tilde{\sigma}_{pq} < s$ and $0 < r < \min\{p, q\}$. If $\min\{p, q\} = q$, then the condition $\sigma_{pq} + \tilde{\sigma}_{pq} < s$ means $s > n(\frac{1}{q} - 1)$ and we can pick r sufficiently close to q so that

$$s > n(\frac{1}{r} - 1) + n(\frac{1}{r} - \frac{1}{q}) > n(\frac{1}{r} - \frac{1}{q}). \tag{433}$$

If $\min\{p, q\} = p$, then the condition $\sigma_{pq} + \tilde{\sigma}_{pq} < s$ means $s > n(\frac{1}{p} - 1) + n(\frac{1}{p} - \frac{1}{q})$ and we can pick r sufficiently close to p so that (433) still holds true. Hence by invoking Lemma 2.0.6, the last inequality (432) is justified. We also infer from the assumption $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ and inequality (432) that $\sum_{j=J+1}^{\infty} |\Delta^L_{2^{-k}\tau\theta} f_{k+j-m_0}(x)| < \infty$ for every $k \in \mathbb{Z}$, almost every $\tau \in [1, 2], \theta \in \mathbb{S}^{n-1}, x \in \mathbb{R}^n$. Therefore (424), (428), the above inference and the supposition of f being a function validate the decomposition

$$\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x) = \Delta^L_{2^{-k}\tau\theta} f(x) - \sum_{j=J+1}^{\infty} \Delta^L_{2^{-k}\tau\theta} f_{k+j-m_0}(x) \tag{434}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)$ for every $k \in \mathbb{Z}$, almost every $\tau \in [1, 2], \theta \in \mathbb{S}^{n-1}, x \in \mathbb{R}^n$ when $0 < p < \infty$, $0 < q < 1$ and $\sigma_{pq} + \tilde{\sigma}_{pq} < s < \infty$, furthermore estimating (422) from above by the sum of (425) and (426) is justified, moreover (426) can be estimated from above by (431) and hence by (432). We have reached the conclusion

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \le C' 2^{Jn(\frac{1}{r}-1)} \cdot \|(\int_{\mathbb{R}^n} \frac{|\Delta^L_h f(\cdot)|^q}{|h|^{n+sq}} dh)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}$$

$$+ C' 2^{J[n(\frac{1}{r}-1)+n(\frac{1}{r}-\frac{1}{q})-s]} \cdot \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \tag{435}$$

where the constant C' is independent of J. From (433) we see that if we pick J sufficiently large so that the coefficient $C'2^{J[n(\frac{1}{r}-1)+n(\frac{1}{r}-\frac{1}{q})-s]}$ is less than $\frac{1}{2}$ and then shift the second term on the right side of (435) to the left side of (435), then we can finish the proof of the first part of Theorem 6.1.2 (ii).

Next we prove the second part of Theorem 6.1.2 (ii) when f is a function, $0 < p < \infty$, $1 \le q < \infty$ and $-n < s < \infty$. It seems that the same method as in the proof of the first part of Theorem 6.1.2 (ii) produces a worse result in the case $0 < p < 1 \le q < \infty$, therefore we use a different method to prove the second part. Still, we assume the right side of (303) is finite. We use equalities (10) and (11) and integrate $[(-1)^{L+1}\Delta^L_{2^{-k}z}f(x)]$ against a Schwartz function $g(z)$ of chosen properties. We let g be a radial Schwartz function whose radial Fourier transform $\mathcal{F}_n g$ satisfies

$$0 \le \mathcal{F}_n g \le 1, \quad \mathcal{F}_n g \text{ is supported in } \{\xi \in \mathbb{R}^n : \frac{1}{4} \le |\xi| < 4L\} \tag{436}$$

and

$$\mathcal{F}_n g(\xi) = 1 \text{ on } \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| < 2L\}. \tag{437}$$

Since $\mathcal{F}_n g(0) = 0$, then $\int_{\mathbb{R}^n} g(z)dz = 0$ and we obtain the equality

$$\int_{\mathbb{R}^n} g(z)[(-1)^{L+1}(\Delta^L_{2^{-k}z}f)(x)]dz = \int_{\mathbb{R}^n} g(z)[\sum_{j=1}^{L} d_j f(x+2^{-k}jz)]dz = \sum_{j=1}^{L} d_j g_{2^{-k}j} * f(x), \tag{438}$$

where the kernel $G_k(z) := \sum_{j=1}^{L} d_j g_{2^{-k}j}(z)$ satisfies

$$spt.\mathcal{F}_n G_k \subseteq \{\xi \in \mathbb{R}^n : 2^{k-2}/L \le |\xi| \le 2^{k+2}L\},\ \mathcal{F}_n G_k(\xi) = 1 \text{ if } 2^{k-1} \le |\xi| < 2^{k+1}. \tag{439}$$

We first estimate the term $\|\{2^{ks}G_k * f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}$. Since $g(z)$ is a bounded Schwartz function, then $|g(z)| \lesssim |z|^{-N'}$ for $0 \ne z \in \mathbb{R}^n$ and N' can be a sufficiently large positive integer whose value will be determined later. Recall that $A_l = \{z \in \mathbb{R}^n : 2^{-l} \leq |z| < 2^{1-l}\}$ and $A_0 = \{h \in \mathbb{R}^n : 1 \leq |h| < 2\}$ and by (438), we have for every $x \in \mathbb{R}^n$

$$
\begin{aligned}
2^{ks}|(G_k * f)(x)| &\lesssim 2^{ks} \int_{\mathbb{R}^n} |g(z)| \cdot |(\Delta^L_{2^{-k}z} f)(x)| dz && (440)\\
&= 2^{ks} \sum_{l \in \mathbb{Z}} \int_{A_l} |g(z)| \cdot |(\Delta^L_{2^{-k}z} f)(x)| dz \\
&= \sum_{l \in \mathbb{Z}} 2^{ks-ln} \int_{A_0} |g(2^{-l}h)| \cdot |(\Delta^L_{2^{-k-l}h} f)(x)| dh \\
&\lesssim \sum_{l \geq 0} 2^{ks-ln} \int_{A_0} |(\Delta^L_{2^{-k-l}h} f)(x)| dh && (441)\\
&\quad + \sum_{l < 0} 2^{ks+l(N'-n)} \int_{A_0} |(\Delta^L_{2^{-k-l}h} f)(x)| dh. && (442)
\end{aligned}
$$

Applying Minkowski's inequality for $\|\cdot\|_{l^q}$-norm for $1 \leq q < \infty$ to the above inequality yields

$$
\begin{aligned}
\|\{2^{ks}(G_k * f)(x)\}_{k\in\mathbb{Z}}\|_{l^q} &\lesssim \sum_{l \geq 0} 2^{-l(n+s)} (\sum_{k \in \mathbb{Z}} 2^{(k+l)sq} (\int_{A_0} |(\Delta^L_{2^{-k-l}h} f)(x)| dh)^q)^{\frac{1}{q}} \\
&\quad + \sum_{l < 0} 2^{l(N'-n-s)} (\sum_{k \in \mathbb{Z}} 2^{(k+l)sq} (\int_{A_0} |(\Delta^L_{2^{-k-l}h} f)(x)| dh)^q)^{\frac{1}{q}} \\
&\lesssim (\sum_{k \in \mathbb{Z}} 2^{ksq} (\int_{A_0} |(\Delta^L_{2^{-k}h} f)(x)| dh)^q)^{\frac{1}{q}}, && (443)
\end{aligned}
$$

if N' is chosen so that $N' > n + s > 0$. Using Hölder's inequality for $1 \leq q < \infty$, we have

$$
(\int_{A_0} |(\Delta^L_{2^{-k}h} f)(x)| dh)^q \lesssim \int_{A_0} |(\Delta^L_{2^{-k}h} f)(x)|^q dh. \qquad (444)
$$

Inserting (444) into (443) and applying the appropriate change of variable $z = 2^{-k}h$ and then inserting the resulting inequality into $\|\cdot\|_{L^p(\mathbb{R}^n)}$ quasinorm yield

$$
\begin{aligned}
&\|\{2^{ks} G_k * f\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\
&\lesssim \|(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} |(\Delta^L_z f)(\cdot)|^q dz)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \|(\int_{\mathbb{R}^n} |z|^{-sq} \cdot |(\Delta^L_z f)(\cdot)|^q \frac{dz}{|z|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}. && (445)
\end{aligned}
$$

From inequalities (440) and (445), we also deduce for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, the integral on the left end of (438) is absolutely convergent and hence well-defined. Thus for each $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$

$$f_k(x) = \mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{-k}\xi)\mathcal{F}_n f)(x) = \psi_{2^{-k}} * G_k * f(x) \tag{446}$$

and we argue as in Remark 2.0.4 to obtain

$$\begin{aligned}
|\psi_{2^{-k}} * G_k * f(x)| &\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \frac{|\psi_{2^{-k}} * G_k * f(x-z)|}{(1+2^{k+2}L|z|)^{n/r}} \\
&\lesssim \operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\psi_{2^{-k}}(y)|(1+2^{k+2}L|y|)^{n/r} \cdot \frac{|G_k * f(x-z-y)|}{(1+2^{k+2}L|z+y|)^{n/r}} dy \\
&\lesssim \mathcal{P}_n(G_k * f)(x) \cdot \int_{\mathbb{R}^n} |\psi_{2^{-k}}(y)|(1+2^{k+2}L|y|)^{n/r} dy \\
&\lesssim \mathcal{P}_n(G_k * f)(x),
\end{aligned} \tag{447}$$

and we recall that $\psi_{2^{-k}}(y) = 2^{kn}\psi(2^k y)$ thus the constant in (447) is independent of $k \in \mathbb{Z}$. From (446), (447) and (71) of Remark 2.0.8, we deduce that

$$\begin{aligned}
\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} |f_k(x)|^q\Big)^{p/q} dx\Big)^{1/p} \\
&\lesssim \Big(\int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} |\mathcal{P}_n(G_k * f)(x)|^q\Big)^{p/q} dx\Big)^{1/p} \\
&\sim \Big(\int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} |(G_k * f)(x)|^q\Big)^{p/q} dx\Big)^{1/p}.
\end{aligned} \tag{448}$$

Combining (448) and (445), we conclude the proof of the second part of Theorem 6.1.2 (ii).

For the case $q = \infty$, we first prove Theorem 6.1.2 (iii). We begin with estimating the term $\operatorname{ess\,sup}_{k\in\mathbb{Z}} \operatorname{ess\,sup}_{h\in A_k} 2^{ks} \sum_{j\in\mathbb{Z}} |(\Delta_h^L f_j)(x)|$ from above by the following

$$\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j\le k} 2^{ks} \operatorname*{ess\,sup}_{h\in A_k} |(\Delta_h^L f_j)(x)| + \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j>k} 2^{ks} \operatorname*{ess\,sup}_{h\in A_k} |(\Delta_h^L f_j)(x)|. \tag{449}$$

We pick $0 < \varepsilon < \min\{s, L-s\}$ and estimate the first term of (449) as follows

$$\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j\le k} 2^{ks} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_j)(x)| \\
=&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j\le k} 2^{j\varepsilon} 2^{-j\varepsilon+ks} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_j)(x)| \\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j\le k} 2^{j\varepsilon} \operatorname*{ess\,sup}_{l\le k} 2^{-l\varepsilon+ks} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_l)(x)| \\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{l\le k} 2^{-l\varepsilon+k(s+\varepsilon)} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_l)(x)|. \qquad (450)
\end{aligned}$$

We use estimate (384) and Remark 2.0.5 to get

$$\begin{aligned}
(450)\lesssim&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k\ge l} 2^{k(s+\varepsilon-L)} 2^{l(L-\varepsilon)} \mathcal{P}_n f_l(x) \\
\lesssim&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls} \mathcal{P}_n f_l(x) \\
\lesssim&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls} \mathcal{M}_n(|f_l|^r)(x)^{\frac{1}{r}} \\
\lesssim&\mathcal{M}_n(\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{lsr}|f_l|^r)(x)^{\frac{1}{r}}. \qquad (451)
\end{aligned}$$

We estimate the second term of (449) from above by

$$\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j> k} 2^{ks} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_j)(x)| \\
=&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j> k} 2^{-j\varepsilon} \cdot 2^{j\varepsilon+ks} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_j)(x)| \\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \sum_{j> k} 2^{-j\varepsilon} \operatorname*{ess\,sup}_{l>k} 2^{l\varepsilon+ks} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_l)(x)| \\
\lesssim&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k<l} 2^{k(s-\varepsilon)} 2^{l\varepsilon} \operatorname*{ess\,sup}_{h\in A_k} |(\varDelta_h^L f_l)(x)|. \qquad (452)
\end{aligned}$$

From Lebesgue's differentiation theorem, we have $|f_l(x)| \lesssim \mathcal{M}_n(|f_l|^r)(x)^{\frac{1}{r}}$ for almost every $x \in \mathbb{R}^n$. Putting (386) back into (385) yields the estimate

$$|(\varDelta_h^L f_l)(x)| \lesssim 2^{(l-k)n/r} \mathcal{M}_n(|f_l|^r)(x)^{1/r} \quad \text{for } |h| \lesssim 2^{-k}, l > k, \qquad (453)$$

where the constant in (453) is independent of $h \in \mathbb{R}^n$, $l, k \in \mathbb{Z}$. The value of $r \in (0, p)$ will be determined later. Inserting (453) into (452) yields

$$(452) \lesssim \operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k<l} 2^{k(s-\varepsilon-\frac{n}{r})} 2^{l(\varepsilon+\frac{n}{r})} \mathcal{M}_n(|f_l|^r)(x)^{\frac{1}{r}}. \qquad (454)$$

Since $\frac{n}{s} < p$ and $s - \varepsilon - \frac{n}{r} \to s - \frac{n}{p} > 0$ as $\varepsilon \to 0$ and $r \to p$, we can pick ε sufficiently small and r sufficiently close to p so that $s - \varepsilon - \frac{n}{r}$ is a positive number and hence

$$(454) \lesssim \operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls}\mathcal{M}_n(|f_l|^r)(x)^{\frac{1}{r}} \lesssim \mathcal{M}_n(\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{lsr}|f_l|^r)(x)^{\frac{1}{r}}. \tag{455}$$

From the above discussion and the $L^{\frac{p}{r}}(\mathbb{R}^n)$-boundedness of the Hardy-Littlewood maximal function, we have proven that

$$\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{h\in A_k} 2^{ks}\sum_{j\in\mathbb{Z}}|(\Delta_h^L f_j)(\cdot)|\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{M}_n(\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{lsr}|f_l|^r)(\cdot)^{\frac{1}{r}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \tag{456}$$

for all $0 < p < \infty$ and $\frac{n}{p} < s < L$. The above inequality and the assumption $f \in \dot{F}^s_{p,\infty}(\mathbb{R}^n)$ have shown $\sum_{j\in\mathbb{Z}}|(\Delta_h^L f_j)(x)| < \infty$ for every $k \in \mathbb{Z}$, almost every $h \in A_k, x \in \mathbb{R}^n$. In conjunction with (28), we have justified the claim that

$$\Delta_h^L f = \sum_{j\in\mathbb{Z}} \Delta_h^L f_j(x) \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \tag{457}$$

for every $k \in \mathbb{Z}$, almost every $h \in A_k, x \in \mathbb{R}^n$, and hence also the inequality

$$\|\operatorname*{ess\,sup}_{h\in\mathbb{R}^n} \frac{|\Delta_h^L f|}{|h|^s}\|_{L^p(\mathbb{R}^n)} \lesssim \|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{h\in A_k} 2^{ks}\sum_{j\in\mathbb{Z}}|(\Delta_h^L f_j)(\cdot)|\|_{L^p(\mathbb{R}^n)}. \tag{458}$$

Now (458) and (456) conclude the proof of Theorem 6.1.2 (iii).

To prove Theorem 6.1.2 (iv), we assume the right side of (305) is finite and use (441) and (442) with $N' > n + s > 0$, and then we can deduce that

$$\begin{aligned}
&\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}|G_k * f(\cdot)|\|_{L^p(\mathbb{R}^n)} \\
\lesssim&(\sum_{l\geq 0} 2^{-l(n+s)} + \sum_{l<0} 2^{l(N'-n-s)})\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\int_{A_0}|(\Delta^L_{2^{-k}h} f)(\cdot)|dh\|_{L^p(\mathbb{R}^n)} \\
\lesssim&\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{h\in A_k} 2^{ks}|(\Delta_h^L f)(\cdot)|\|_{L^p(\mathbb{R}^n)} \\
\lesssim&\|\operatorname*{ess\,sup}_{h\in\mathbb{R}^n} \frac{|(\Delta_h^L f)(\cdot)|}{|h|^s}\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{459}$$

And the above estimate, in conjunction with (440), shows the absolute convergence of the integral on the left end of (438) for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. Using equality (446), estimate (447), Remark 2.0.5 and the $L^{\frac{p}{r}}(\mathbb{R}^n)$-boundedness of Hardy-Littlewood maximal function in a sequence, we can also obtain

$$
\begin{aligned}
\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} =& \| \operatorname*{ess\,sup}_{k\in\mathbb{Z}} |2^{ks} f_k(\cdot)| \|_{L^p(\mathbb{R}^n)} \\
\lesssim& \| \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \mathcal{M}_n(|G_k * f|^r)(\cdot)^{\frac{1}{r}} \|_{L^p(\mathbb{R}^n)} \\
\lesssim& \| \mathcal{M}_n(\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ksr} |G_k * f|^r)(\cdot)^{\frac{1}{r}} \|_{L^p(\mathbb{R}^n)} \\
\lesssim& \| \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} |G_k * f(\cdot)| \|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{460}
$$

Inequalities (459) and (460) finish the proof of Theorem 6.1.2 (iv). The proof of Theorem 6.1.2 is now complete. □

6.3 Proof Of Theorem 6.1.3

Proof. We first prove Theorem 6.1.3 (i). We continue using the notation $A_k = \{h \in \mathbb{R}^n : 2^{-k} \leq |h| < 2^{1-k}\}$ for $k \in \mathbb{Z}$ and thus $\mathbb{R}^n \setminus \{0\} = \bigcup_{k\in\mathbb{Z}} A_k$. We also pick the number r in the definition of the Peetre-Fefferman-Stein maximal function so that $0 < r < p$. We begin with estimating $(\int_{\mathbb{R}^n} |h|^{-sq} \| \sum_{j\in\mathbb{Z}} |\Delta^L_h f_j| \|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^n})^{\frac{1}{q}}$ from above by the following

$$
(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} \| \sum_{j\leq k} |\Delta^L_h f_j| \|^q_{L^p(\mathbb{R}^n)} dh)^{\frac{1}{q}} + (\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} \| \sum_{j>k} |\Delta^L_h f_j| \|^q_{L^p(\mathbb{R}^n)} dh)^{\frac{1}{q}}. \tag{461}
$$

We can pick $0 < \varepsilon < \min\{s, L-s\}$ and use the same calculation method as in (617), (618), (619), (620) to obtain

$$
\| \sum_{j\leq k} |\Delta^L_h f_j| \|^q_{L^p(\mathbb{R}^n)} \lesssim 2^{kq\varepsilon} \sum_{j\leq k} 2^{-jq\varepsilon} \|\Delta^L_h f_j\|^q_{L^p(\mathbb{R}^n)}, \tag{462}
$$

$$
\| \sum_{j>k} |\Delta^L_h f_j| \|^q_{L^p(\mathbb{R}^n)} \lesssim 2^{-kq\varepsilon} \sum_{j>k} 2^{jq\varepsilon} \|\Delta^L_h f_j\|^q_{L^p(\mathbb{R}^n)}. \tag{463}
$$

And (462), (463) are true for $0 < p \leq \infty$, $0 < q < \infty$. To estimate the first term in (461), we use (384), Remark 2.0.5 and the mapping property of Hardy-Littlewood maximal function for $\|\cdot\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}$-norm to obtain

$$\|\Delta_h^L f_j\|_{L^p(\mathbb{R}^n)} \lesssim 2^{(j-k)L}(1+2^{j-k})^{\frac{n}{r}}\|\mathcal{P}_n f_j\|_{L^p(\mathbb{R}^n)} \lesssim 2^{(j-k)L}\|f_j\|_{L^p(\mathbb{R}^n)}, \tag{464}$$

for $j \leq k$, $|h| \lesssim 2^{-k}$ and constants are independent of $h \in \mathbb{R}^n$ and $j, k \in \mathbb{Z}$. We put (462) and (464) into the first term of (461) then we have

$$\begin{aligned}
&(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} \|\sum_{j\leq k} |\Delta_h^L f_j|\|^q_{L^p(\mathbb{R}^n)} dh)^{\frac{1}{q}} \\
\lesssim&(\sum_{k\in\mathbb{Z}}\sum_{j\leq k} 2^{kq(s+\varepsilon-L)} \cdot 2^{jq(L-\varepsilon)}\|f_j\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}} \\
\lesssim&(\sum_{j\in\mathbb{Z}}\sum_{k\geq j} 2^{kq(s+\varepsilon-L)} \cdot 2^{jq(L-\varepsilon)}\|f_j\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.
\end{aligned} \tag{465}$$

To estimate the second term of (461), we can use (10) and proper change of variable to obtain

$$\|\Delta_h^L f_j\|_{L^p(\mathbb{R}^n)} \lesssim \|f_j\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } j \in \mathbb{Z}, \tag{466}$$

and the constant is independent of $j \in \mathbb{Z}$ and $h \in \mathbb{R}^n$. We put (463) and (466) into the second term of (461) then we have

$$\begin{aligned}
&(\sum_{k\in\mathbb{Z}} 2^{k(sq+n)} \int_{A_k} \|\sum_{j>k} |\Delta_h^L f_j|\|^q_{L^p(\mathbb{R}^n)} dh)^{\frac{1}{q}} \\
\lesssim&(\sum_{k\in\mathbb{Z}}\sum_{j>k} 2^{kq(s-\varepsilon)} \cdot 2^{jq\varepsilon}\|f_j\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}} \\
\lesssim&(\sum_{j\in\mathbb{Z}}\sum_{k<j} 2^{kq(s-\varepsilon)} \cdot 2^{jq\varepsilon}\|f_j\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.
\end{aligned} \tag{467}$$

Combining (461), (465) and (467), we have proven

$$(\int_{\mathbb{R}^n} |h|^{-sq}\|\sum_{j\in\mathbb{Z}} |\Delta_h^L f_j|\|^q_{L^p(\mathbb{R}^n)} \frac{dh}{|h|^n})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{468}$$

The assumption $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ implies $\sum_{j\in\mathbb{Z}} |\Delta_h^L f_j(x)| < \infty$ for almost every $h \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. In conjunction with (28), we have shown

$$\Delta_h^L f = \sum_{j\in\mathbb{Z}} \Delta_h^L f_j(x) \tag{469}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ for almost every $h \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ when $0 < p \leq \infty$, $0 < q < \infty$ and $0 < s < L$. Therefore we obtain

$$(\int_{\mathbb{R}^n} |h|^{-sq} \|\varDelta_h^L f\|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|^n})^{\frac{1}{q}} \lesssim (\int_{\mathbb{R}^n} |h|^{-sq} \| \sum_{j\in\mathbb{Z}} |\varDelta_h^L f_j| \|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|^n})^{\frac{1}{q}}. \tag{470}$$

Inequalities (468) and (470) conclude the proof of Theorem 6.1.3 (i).

To prove Theorem 6.1.3 (ii), we assume the right side of inequality (319) is finite otherwise the inequality is trivial. We recall that $spt.\mathcal{F}_n\psi \subseteq A' = \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| < 2\}$ and use the positive integer m_0 satisfying (403). We also continue using the spherical caps $\{C_l\}_{l=1}^M$ constructed right after (403), the corresponding sets $\{P_l\}_{l=1}^M$ given in (404), (405), (406), and the associated smooth partition of unity $\{\rho_l\}_{l=1}^M$ satisfying (407). We have the apparent estimate

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \|\{2^{k-m_0} f_{k-m_0}\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \lesssim (\sum_{k\in\mathbb{Z}} 2^{ksq} \|\psi_{2^{m_0-k}} * f\|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}}. \tag{471}$$

When $1 < p \leq \infty$, we can obtain from (407) the following

$$\psi_{2^{m_0-k}} * f(x) = \sum_{l=1}^M \mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f](x). \tag{472}$$

For each $l \in \{1, \cdots, M\}$, $\theta \in C_l$ and $1 \leq \tau \leq 2$, we can infer from (410), (411) and (412) the following

$$\begin{aligned} &\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f](x) \\ =&\mathcal{F}_n^{-1}[\frac{\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}\tau\theta\cdot\xi} - 1)^L} \cdot (e^{2\pi i\cdot 2^{-k}\tau\theta\cdot\xi} - 1)^L \mathcal{F}_n f](x) \\ =&\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^{-k}\xi)(\cdot) * (\varDelta_{2^{-k}\tau\theta}^L f)(x). \end{aligned} \tag{473}$$

Due to Lemma 2.0.8 and hence (415), we have

$$\begin{aligned}
&|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^{-k}\xi)(\cdot) * (\Delta^L_{2^{-k}\tau\theta}f)(x)| \\
\lesssim&\int_{2^k|y|<1} 2^{kn}|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^k y)|\cdot|(\Delta^L_{2^{-k}\tau\theta}f)(x-y)|dy \\
&+\sum_{l=0}^{\infty}\int_{A_{k-l}} 2^{kn}|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^k y)|\cdot|(\Delta^L_{2^{-k}\tau\theta}f)(x-y)|dy \\
\lesssim&\fint_{2^k|y|<1}|(\Delta^L_{2^{-k}\tau\theta}f)(x-y)|dy+\sum_{l=0}^{\infty}2^{l(n-N)}\fint_{A_{k-l}}|(\Delta^L_{2^{-k}\tau\theta}f)(x-y)|dy \\
\lesssim&(1+\sum_{l=0}^{\infty}2^{l(n-N)})\mathcal{M}_n(|\Delta^L_{2^{-k}\tau\theta}f|)(x)\lesssim\mathcal{M}_n(|\Delta^L_{2^{-k}\tau\theta}f|)(x),
\end{aligned} \tag{474}$$

if in the last step above we pick $N > n$. Since $1 < p \le \infty$, we invoke the mapping property of Hardy-Littlewood maximal function and obtain for $0 < q < \infty$,

$$\begin{aligned}
\|\psi_{2^{m_0-k}} * f\|^q_{L^p(\mathbb{R}^n)}\lesssim&\sum_{l=1}^{M}\fint_{[1,2]}\fint_{C_l}\|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_nf]\|^q_{L^p(\mathbb{R}^n)}d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} \\
\lesssim&\sum_{l=1}^{M}\int_1^2\int_{C_l}\|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^{-k}\xi)(\cdot)*(\Delta^L_{2^{-k}\tau\theta}f)\|^q_{L^p(\mathbb{R}^n)}d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} \\
\lesssim&\int_1^2\int_{\mathbb{S}^{n-1}}\|\Delta^L_{2^{-k}\tau\theta}f\|^q_{L^p(\mathbb{R}^n)}d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau} \\
\lesssim&\int_{A_k}\|\Delta^L_hf\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n},
\end{aligned} \tag{475}$$

for every $k \in \mathbb{Z}$. Inserting (475) into (471) proves (319) when $1 < p \le \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$. When $0 < p < 1$, $0 < q < \infty$ and $\sigma_p < s < \infty$, we use the function ϕ satisfying conditions (19), (408), (409), and $J > m_0$ is a large positive integer whose value will be determined later. Then we have

$$\psi_{2^{m_0-k}} * f(x) = \sum_{l=1}^{M}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_nf](x). \tag{476}$$

Furthermore for each $l \in \{1,\cdots,M\}$, $\theta \in C_l$ and $1 \le \tau \le 2$, we can infer from (410), (411) and (412) the following

$$\begin{aligned}
&\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_nf](x) \\
=&\mathcal{F}_n^{-1}[\frac{\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)}{(e^{2\pi i\cdot2^{-k}\tau\theta\cdot\xi}-1)^L}\cdot(e^{2\pi i\cdot2^{-k}\tau\theta\cdot\xi}-1)^L\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_nf](x) \\
=&\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^{-k}\xi)(\cdot)*(\Delta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-J-k}}*f))(x).
\end{aligned} \tag{477}$$

The Fourier transform of the Schwartz function

$$y \mapsto 2^{kn}\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^k y)\cdot \varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-J-k}} * f)(x-y)$$

is compactly supported in a ball centered at the origin of radius about 2^{k+J-m_0} in $\mathbb{R}^n$, therefore by invoking Remark 2.0.7 or the more general Lemma 2.0.5 of Plancherel-Polya-Nikol'skij inequality, we can estimate (477) from above by

$$\begin{aligned}&\|2^{kn}\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^k\cdot)\cdot \varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-J-k}} * f)(x-\cdot)\|_{L^1(\mathbb{R}^n)}\\ \lesssim&2^{(k+J-m_0)n(\frac{1}{r}-1)}\|2^{kn}\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^k\cdot)\cdot \varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-J-k}} * f)(x-\cdot)\|_{L^r(\mathbb{R}^n)},\end{aligned}\tag{478}$$

for $0<r<p<1$. We insert (416) with $n-Nr<0$ into (478) and use the following inequality

$$⨍_{A_{k-m}}|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)(x-y)|^r dy \lesssim \mathcal{M}_n(|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r)(x),\tag{479}$$

and then combine the result with (477) to obtain

$$\begin{aligned}&|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_n f](x)|\\ \lesssim&2^{Jn(\frac{1}{r}-1)}\mathcal{M}_n(|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r)(x)^{\frac{1}{r}},\end{aligned}\tag{480}$$

for every $l\in\{1,\cdots,M\}$, $\theta\in C_l$ and $\tau\in[1,2]$. Then we use (476), (480), the calculation method displayed in (475) and the mapping property of Hardy-Littlewood maximal function for $\|\cdot\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}$-norm to obtain

$$\begin{aligned}\|\psi_{2^{m_0-k}} * f\|^q_{L^p(\mathbb{R}^n)}\lesssim&2^{Jnq(\frac{1}{r}-1)}\sum_{l=1}^M\int_1^2\int_{C_l}\|\mathcal{M}_n(|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r)^{\frac{1}{r}}\|^q_{L^p(\mathbb{R}^n)}d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau}\\ \lesssim&2^{Jnq(\frac{1}{r}-1)}\int_1^2\int_{\mathbb{S}^{n-1}}\|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)\|^q_{L^p(\mathbb{R}^n)}d\mathcal{H}^{n-1}(\theta)\frac{d\tau}{\tau}\\ \lesssim&2^{Jnq(\frac{1}{r}-1)}\int_{A_k}\|\varDelta^L_h(\phi_{2^{m_0-k-J}} * f)\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n},\end{aligned}\tag{481}$$

and (481) is true for $0<q<\infty$. We insert (481) into (471), recall (409), (423), and (424), and we also assume the validity of decomposition, then $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ can be estimated from above by the sum of the following two terms,

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_{A_k}\|\varDelta^L_h f\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n})^{\frac{1}{q}}\sim 2^{Jn(\frac{1}{r}-1)}(\int_{\mathbb{R}^n}|h|^{-sq}\|\varDelta^L_h f\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n})^{\frac{1}{q}}<\infty\tag{482}$$

and

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_{A_k}\|\Delta_h^L(\sum_{j=J+1}^{\infty}f_{k+j-m_0})\|_{L^p(\mathbb{R}^n)}^q\frac{dh}{|h|^n})^{\frac{1}{q}}. \tag{483}$$

To estimate (483), we begin with

$$\begin{aligned}&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_{A_k}\|\sum_{j=J+1}^{\infty}|\Delta_h^L f_{k+j-m_0}|\|_{L^p(\mathbb{R}^n)}^q\frac{dh}{|h|^n})^{\frac{1}{q}}\\ \lesssim&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{k(sq+n)}\int_{A_k}(\sum_{j=J+1}^{\infty}\|\Delta_h^L f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^p)^{\frac{q}{p}}dh)^{\frac{1}{q}},\end{aligned} \tag{484}$$

since $0<p<1$. When $0<q\le p$, we use (466) and the following inequality

$$(\sum_{j=J+1}^{\infty}\|\Delta_h^L f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^p)^{\frac{q}{p}}\lesssim\sum_{j=J+1}^{\infty}\|\Delta_h^L f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^q\lesssim\sum_{j=J+1}^{\infty}\|f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^q, \tag{485}$$

where the constants are independent of $j,k\in\mathbb{Z}$ and $h\in\mathbb{R}^n$. When $p<q<\infty$, we pick $0<\varepsilon<s$, use (466) and the following inequality

$$\begin{aligned}&(\sum_{j=J+1}^{\infty}2^{-jp\varepsilon}\cdot 2^{jp\varepsilon}\|\Delta_h^L f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^p)^{\frac{q}{p}}\\ \lesssim&(\sum_{l=J+1}^{\infty}2^{-lp\varepsilon})^{\frac{q}{p}}\operatorname*{ess\,sup}_{j>J}2^{jq\varepsilon}\|\Delta_h^L f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^q\\ \lesssim&2^{-Jq\varepsilon}\sum_{j=J+1}^{\infty}2^{jq\varepsilon}\|f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^q,\end{aligned} \tag{486}$$

where the constants are independent of $j,k\in\mathbb{Z}$ and $h\in\mathbb{R}^n$. Insert (485) and (486) into (484) and exchange the order of summation, and then we can estimate (484) from above by $2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$. The assumption $f\in\dot{B}^s_{p,q}(\mathbb{R}^n)$ implies $\sum_{j=J+1}^{\infty}|\Delta_h^L f_{k+j-m_0}(x)|<\infty$ for every $k\in\mathbb{Z}$ and almost every $h\in A_k, x\in\mathbb{R}^n$. In conjunction with (29), (423) and (482), we have shown

$$\Delta_h^L(\phi_{2^{m_0-k-J}}*f)(x)=\Delta_h^L f(x)-\sum_{j=J+1}^{\infty}\Delta_h^L f_{k+j-m_0}(x) \tag{487}$$

is true not only in the sense of $\mathcal{S}'(\mathbb{R}^n)$ but also for every $k\in\mathbb{Z}$ and almost every $h\in A_k, x\in\mathbb{R}^n$ when $0<p<1$, $0<q<\infty$ and $\sigma_p<s<\infty$. Furthermore estimating $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ from above by the sum of (482) and (483) is justified, moreover (483) can be estimated from above by (484) and hence by $2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$. We have obtained the inequality

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leq C'2^{Jn(\frac{1}{r}-1)}(\int_{\mathbb{R}^n}|h|^{-sq}\|\Delta_h^L f\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n})^{\frac{1}{q}} + C''2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{488}$$

The condition $\sigma_p < s < \infty$ implies $n(\frac{1}{r}-1)-s<0$ when r is sufficiently close to p. Hence when the positive integer J is sufficiently large, the coefficient $C''2^{J[n(\frac{1}{r}-1)-s]}$ is less than $\frac{1}{2}$ and then we can shift the second term on the right side of (488) to its left side and prove the inequality (319) when $0<p<1$, $0<q<\infty$ and $\sigma_p<s<\infty$. The proof for Theorem 6.1.3 (ii) is now concluded.

Next, we prove Theorem 6.1.3 (iii). We pick $0<\varepsilon<\min\{s, L-s\}$ and begin with estimating $\operatorname{ess\,sup}_{h\in\mathbb{R}^n}|h|^{-s}\|\sum_{j\in\mathbb{Z}}|\Delta_h^L f_j|\|_{L^p(\mathbb{R}^n)}$ from above by

$$\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{h\in A_k} 2^{ks}\|\sum_{j\leq k}|\Delta_h^L f_j|\|_{L^p(\mathbb{R}^n)} + \operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{h\in A_k} 2^{ks}\|\sum_{j>k}|\Delta_h^L f_j|\|_{L^p(\mathbb{R}^n)}. \tag{489}$$

If $1\leq p\leq\infty$, by using Minkowski's inequality, (384) and the calculation method displayed in (660), we can estimate the first term in (489) from above by $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}$. And by using Minkowski's inequality, (466) and the calculation method displayed in (661), we can estimate the second term in (489) from above by $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}$. If $0<p<1$, by applying (384) and the calculation method given in (662) to the first term of (489), and by applying (466) and the calculation method given in (663) to the second term of (489), we can still estimate (489) from above by $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}$. Thus we can obtain the inequality

$$\operatorname*{ess\,sup}_{h\in\mathbb{R}^n}|h|^{-s}\|\sum_{j\in\mathbb{Z}}|\Delta_h^L f_j|\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{490}$$

The assumption $f\in\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ shows $\sum_{j\in\mathbb{Z}}|\Delta_h^L f_j(x)|<\infty$ for almost every $h\in\mathbb{R}^n, x\in\mathbb{R}^n$. In conjunction with (28), we have shown (469) is true not only in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ but also for almost every $h\in\mathbb{R}^n, x\in\mathbb{R}^n$ when $0<p\leq\infty$, $q=\infty$ and $0<s<L$. Therefore we have

$$\operatorname*{ess\,sup}_{h\in\mathbb{R}^n}|h|^{-s}\|\Delta_h^L f\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h\in\mathbb{R}^n}|h|^{-s}\|\sum_{j\in\mathbb{Z}}|\Delta_h^L f_j|\|_{L^p(\mathbb{R}^n)}. \tag{491}$$

Inequalities (490) and (491) conclude the proof of Theorem 6.1.3 (iii).

Now we prove Theorem 6.1.3 (iv). We assume the right side of (321) is finite, otherwise, the inequality is trivial. We still use the positive integer m_0 satisfying (403), the spherical caps $\{C_l\}_{l=1}^M$ constructed right after (403), the corresponding sets $\{P_l\}_{l=1}^M$ given in (404), (405), (406), and the associated smooth partition of unity $\{\rho_l\}_{l=1}^M$ satisfying (407). When $1 < p \leq \infty$, we use (472), (473) and (474) to obtain

$$\begin{aligned}\|\psi_{2^{m_0-k}} * f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{l=1}^{M} \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in C_l} \|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{l=1}^{M} \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in C_l} \|\mathcal{F}_n^{-1}\lambda_{l,\tau\theta}(2^{-k}\xi)(\cdot) * (\varDelta^L_{2^{-k}\tau\theta} f)\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{l=1}^{M} \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in C_l} \|\mathcal{M}_n(|\varDelta^L_{2^{-k}\tau\theta} f|)\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in\mathbb{S}^{n-1}} \|\varDelta^L_{2^{-k}\tau\theta} f\|_{L^p(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{h\in A_k} \|\varDelta^L_h f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{492}$$

Therefore we have the inequality

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\|\psi_{2^{m_0-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s}\|\varDelta^L_h f\|_{L^p(\mathbb{R}^n)}, \tag{493}$$

and inequality (493) is true for $1 < p \leq \infty$, $q = \infty$ and $s \in \mathbb{R}$. When $0 < p < 1$, we also use the function ϕ satisfying conditions (19), (408), (409), and $J > m_0$ is a large positive integer whose value will be determined later. Then we can use (476) and (480) with $0 < r < p < 1$ to obtain

$$\begin{aligned}&\|\psi_{2^{m_0-k}} * f\|_{L^p(\mathbb{R}^n)} \\ \lesssim&\sum_{l=1}^{M} \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in C_l} \|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_l(2^{m_0-k}\xi)\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)} \\ \lesssim&2^{Jn(\frac{1}{r}-1)} \sum_{l=1}^{M} \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in C_l} \|\mathcal{M}_n(|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)|^r)^{\frac{1}{r}}\|_{L^p(\mathbb{R}^n)} \\ \lesssim&2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{\tau\in[1,2]} \operatorname*{ess\,sup}_{\theta\in\mathbb{S}^{n-1}} \|\varDelta^L_{2^{-k}\tau\theta}(\phi_{2^{m_0-k-J}} * f)\|_{L^p(\mathbb{R}^n)} \\ =&2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{h\in A_k} \|\varDelta^L_h(\phi_{2^{m_0-k-J}} * f)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{494}$$

Insert (494) into $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}$, recall (423) and assume the validity of decomposition, then we can estimate $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}$ from above by the sum of the following two terms,

$$2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s}\|\varDelta^L_h f\|_{L^p(\mathbb{R}^n)} < \infty \tag{495}$$

and

$$2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{h\in A_k} 2^{ks} \|\Delta_h^L(\sum_{j=J+1}^{\infty} f_{k+j-m_0})\|_{L^p(\mathbb{R}^n)}. \tag{496}$$

To estimate (496), we use (466) and begin with the following

$$\begin{aligned}
&2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{h\in A_k} 2^{ks} \|\sum_{j=J+1}^{\infty} |\Delta_h^L f_{k+j-m_0}|\|_{L^p(\mathbb{R}^n)} &(497)\\
\lesssim &2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{h\in A_k} 2^{ks} (\sum_{j=J+1}^{\infty} \|\Delta_h^L f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^p)^{\frac{1}{p}}\\
\lesssim &2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{k\in\mathbb{Z}} (\sum_{j=J+1}^{\infty} 2^{(m_0-j)sp} \cdot 2^{(k+j-m_0)sp} \|f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}^p)^{\frac{1}{p}}\\
\lesssim &2^{J[n(\frac{1}{r}-1)-s]} \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. &(498)
\end{aligned}$$

The assumption $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ implies $\sum_{j=J+1}^{\infty} |\Delta_h^L f_{k+j-m_0}(x)| < \infty$ for every $k \in \mathbb{Z}$ and almost every $h \in A_k, x \in \mathbb{R}^n$. In conjunction with (29), (423) and (495), we have shown (487) is true not only in the sense of $\mathcal{S}'(\mathbb{R}^n)$ but also for every $k \in \mathbb{Z}$ and almost every $h \in A_k, x \in \mathbb{R}^n$ when $0 < p < 1$, $q = \infty$ and $\sigma_p < s < \infty$. Furthermore estimating $\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}$ from above by the sum of (495) and (496) is justified, moreover (496) can be estimated from above by (497) and hence by (498). We have obtained the inequality

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \leq C' 2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{h\in\mathbb{R}^n} |h|^{-s} \|\Delta_h^L f\|_{L^p(\mathbb{R}^n)} + C'' 2^{J[n(\frac{1}{r}-1)-s]} \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{499}$$

The condition $\sigma_p < s < \infty$ indicates $n(\frac{1}{r} - 1) - s < 0$ when r is sufficiently close to p. Thus the coefficient $C'' 2^{J[n(\frac{1}{r}-1)-s]}$ is less than $\frac{1}{2}$ when the positive integer J is sufficiently large, and then we can shift the second term on the right side of (499) to its left side and prove the desired inequality (321). The proof of Theorem 6.1.3 (iv) is now complete.

Finally, we come to the proof of Theorem 6.1.3 (v). By using a different method, some better conditions can be obtained in the case $p = 1$. We assume the right sides of (322) and (323) are finite. If $1 \leq q \leq \infty$ and $-n < s < \infty$, we use the radial Schwartz function g satisfying (436), (437), (438), and the kernel $G_k(\cdot) = \sum_{j=1}^{L} d_j g_{2^{-k}j}(\cdot)$ satisfying (439), (446), (447) with $0 < r < p = 1$. Then from (440), (441) and (442), we deduce

$$\begin{aligned}
2^{ks}\|G_k * f\|_{L^1(\mathbb{R}^n)} \lesssim &\sum_{l\geq 0} 2^{-l(n+s)} \cdot 2^{(k+l)s} \int_{A_0} \|\Delta_{2^{-k-l}h}^L f\|_{L^1(\mathbb{R}^n)} dh\\
&+ \sum_{l<0} 2^{l(N'-n-s)} \cdot 2^{(k+l)s} \int_{A_0} \|\Delta_{2^{-k-l}h}^L f\|_{L^1(\mathbb{R}^n)} dh. &(500)
\end{aligned}$$

When $1 \le q < \infty$, we use Minkowski's inequality for $\|\cdot\|_{l^q}$-norm, Hölder's inequality and compute as in (443) and (444) with $N' > n+s > 0$ then we can obtain

$$\begin{aligned}
&\|\{2^{ks}G_k * f\}_{k\in\mathbb{Z}}\|_{l^q(L^1)} \\
\lesssim&(\sum_{k\in\mathbb{Z}} 2^{ksq}\int_{A_0}\|\Delta^L_{2^{-k}h}f\|^q_{L^1(\mathbb{R}^n)}dh)^{\frac{1}{q}} \\
\lesssim&(\int_{\mathbb{R}^n}|h|^{-sq}\|\Delta^L_h f\|^q_{L^1(\mathbb{R}^n)}\frac{dh}{|h|^n})^{\frac{1}{q}}.
\end{aligned} \tag{501}$$

When $q = \infty$, we use Minkowski's inequality for $\|\cdot\|_{l^\infty}$-norm and the inequality (500) with $N' > n+s > 0$ to obtain

$$\begin{aligned}
&\|\{2^{ks}G_k * f\}_{k\in\mathbb{Z}}\|_{l^\infty(L^1)} \\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\int_{A_0}\|\Delta^L_{2^{-k}h}f\|_{L^1(\mathbb{R}^n)}dh \\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\operatorname*{ess\,sup}_{h\in A_0}\|\Delta^L_{2^{-k}h}f\|_{L^1(\mathbb{R}^n)} \\
\lesssim&\operatorname*{ess\,sup}_{h\in\mathbb{R}^n}|h|^{-s}\|\Delta^L_h f\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{502}$$

Indicated by (440), (500), (501) and (502), we know that the integral on the left end of (438) is absolutely convergent and hence well-defined for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. By (446), (447), Remark 2.0.5 and the mapping property of Hardy-Littlewood maximal function, we have $\|f\|_{\dot{B}^s_{1,q}(\mathbb{R}^n)} \lesssim \|\{2^{ks}G_k * f\}_{k\in\mathbb{Z}}\|_{l^q(L^1)}$ for $1 \le q \le \infty$ and any $s \in \mathbb{R}$. In conjunction with (501) and (502), we have proven the inequality (323), and the inequality (322) when $p = 1$, $1 \le q < \infty$ and $-n < s < \infty$. To prove (322) is true for $p = 1$, $0 < q < 1$ and $0 < s < \infty$, we notice that by picking $0 < r < p = 1$, the method given for the proof of the second part of Theorem 6.1.3 (ii) still applies and $\sigma_1 = 0$. The proof of Theorem 6.1.3 is now complete. □

7.0 Inequalities In Function Spaces In Terms Of Iterated Differences Along Coordinate Axes

7.1 Chapter Introduction

In Theorem 6.1.2, it was shown that the quasinorm $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ is equivalent to

$$\|(\int_{\mathbb{R}^n} |h|^{-sq}|\Delta_h^L f|^q \frac{dh}{|h|^n})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} = \|(\int_{\mathbb{S}^{n-1}} \int_0^\infty t^{-sq}|\Delta_{t\theta}^L f|^q \frac{dt}{t} d\mathcal{H}^{n-1}(\theta))^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)},$$

and the expression inside the parenthesis on the right side is an integral of the term

$$\int_0^\infty t^{-sq}|\Delta_{t\theta}^L f|^q \frac{dt}{t}$$

over the set of all the unit directions $\theta \in \mathbb{S}^{n-1}$. Therefore it is natural to ask: is there a similar equivalence relation if we replace the integral over $\mathbb{S}^{n-1}$ by a finite sum of unit vectors? It seems this question can be answered when θ takes value in the set of elementary unit vectors $\{e_j\}_{j=1}^n$, where each $e_j \in \mathbb{R}^n$ has its j-th coordinate equal to 1 and all the other $(n-1)$ coordinates equal to 0. We use the notation $\Delta_{t,j}^L f = \Delta_{te_j}^L f$ for $j \in \{1, \cdots, n\}$, then for example, when f is a function defined on $\mathbb{R}^n$, $\Delta_{t,1}^1 f(x) = f(x_1 + t, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)$. Denote

$$\sigma_{pq} = \max\{0, n(\frac{1}{\min\{p,q\}} - 1)\}, \quad \tilde{\sigma}_{pq}^1 = \max\{0, \frac{1}{p} - \frac{1}{q}\}, \quad \sigma_p = \max\{0, n(\frac{1}{p} - 1)\}. \quad (503)$$

Theorem 7.1.1. Let $L \in \mathbb{N}$, $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$.
(i)If $0 < p, q < \infty$, $\tilde{\sigma}_{pq}^1 < s < L$, then for each $j \in \{1, \cdots, n\}$

$$\|(\int_0^\infty t^{-sq}|\Delta_{t,j}^L f|^q \frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \quad (504)$$

(ii)Suppose f is a function. If $1 < \min\{p, q\}$, $q < \infty$ and $s \in \mathbb{R}$, or if $\min\{p, q\} \leq 1$, $q < \infty$ and $\sigma_{pq} + \tilde{\sigma}_{pq}^1 < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \|(\int_0^\infty t^{-sq}|\Delta_{t,j}^L f(\cdot)|^q \frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}. \quad (505)$$

(iii)If $0 < p < \infty$, $q = \infty$ and $\frac{1}{p} < s < L$, then for each $j \in \{1, \cdots, n\}$

$$\| \operatorname*{ess\,sup}_{t>0} \frac{|\Delta_{t,j}^L f|}{t^s} \|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{506}$$

(iv)Suppose f is a function. If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \leq 1$, $q = \infty$ and $\sigma_p + \frac{1}{p} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \| \operatorname*{ess\,sup}_{t>0} \frac{|\Delta_{t,j}^L f(\cdot)|}{t^s} \|_{L^p(\mathbb{R}^n)}. \tag{507}$$

It is worth noting that the condition in Theorem 7.1.1 (i) and the condition in Theorem 7.1.1 (ii) when $1 < \min\{p, q\}$ are completely independent of the dimension of the ambient space $\mathbb{R}^n$ while the restriction of s in Theorem 7.1.1 (ii) when $\min\{p, q\} \leq 1$ is only partially dependent on the dimension n. H. Triebel formulated the counterpart of Theorem 7.1.1 for the inhomogeneous $F^s_{p,q}(\mathbb{R}^n)$ space in [94, section 2.6.2] with rough conditions $0 < p < \infty$, $0 < q \leq \infty$ and $\frac{n}{\min\{p,q\}} < s < M$. Theorem 7.1.1 is a newly published result in the author's paper [98]. The proof of Theorem 7.1.1 can be found in section 7.2. The corollaries of Theorem 7.1.1 are given below.

Corollary 7.1.1. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p, q < \infty$, $\tilde{\sigma}^1_{pq} < s < 1$, then for each $j \in \{1, \cdots, n\}$

$$\Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty |f(x + te_j) - f(x)|^q \frac{dt}{t^{1+sq}}\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{508}$$

(ii)If $1 < \min\{p, q\}$, $q < \infty$ and $s \in \mathbb{R}$, or if $\min\{p, q\} \leq 1$, $q < \infty$ and $\sigma_{pq} + \tilde{\sigma}^1_{pq} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty |f(x + te_j) - f(x)|^q \frac{dt}{t^{1+sq}}\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}. \tag{509}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $\frac{1}{p} < s < 1$, then for each $j \in \{1, \cdots, n\}$

$$\Big(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x + te_j) - f(x)|^p}{t^{sp}} dx\Big)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{510}$$

(iv)If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \leq 1$, $q = \infty$ and $\sigma_p + \frac{1}{p} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \Big(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x + te_j) - f(x)|^p}{t^{sp}} dx\Big)^{\frac{1}{p}}. \tag{511}$$

Proof of Corollary 7.1.1. Apply inequalities (504), (505), (506) and (507) with $L = 1$. $\square$

Corollary 7.1.2. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p, q < \infty$, $\tilde{\sigma}^1_{pq} < s < 2$, then for each $j \in \{1, \cdots, n\}$

$$(\int_{\mathbb{R}^n} (\int_0^\infty |f(x+2te_j) - 2f(x+te_j) + f(x)|^q \frac{dt}{t^{1+sq}})^{\frac{p}{q}} dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{512}$$

(ii)If $1 < \min\{p, q\}$, $q < \infty$ and $s \in \mathbb{R}$, or if $\min\{p, q\} \leq 1$, $q < \infty$ and $\sigma_{pq} + \tilde{\sigma}^1_{pq} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_{\mathbb{R}^n} (\int_0^\infty |f(x+2te_j) - 2f(x+te_j) + f(x)|^q \frac{dt}{t^{1+sq}})^{\frac{p}{q}} dx)^{\frac{1}{p}}. \tag{513}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $\frac{1}{p} < s < 2$, then for each $j \in \{1, \cdots, n\}$

$$(\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x+2te_j) - 2f(x+te_j) + f(x)|^p}{t^{sp}} dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{514}$$

(iv)If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \leq 1$, $q = \infty$ and $\sigma_p + \frac{1}{p} < s < \infty$, then

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x+2te_j) - 2f(x+te_j) + f(x)|^p}{t^{sp}} dx)^{\frac{1}{p}}. \tag{515}$$

Proof of Corollary 7.1.2. Apply inequalities (504), (505), (506) and (507) with $L = 2$. $\square$

Finally, as part of a systematic study, we also state and prove the counterpart of Theorem 7.1.1 and the corresponding corollaries for $\dot{B}^s_{p,q}(\mathbb{R}^n)$ spaces.

Theorem 7.1.2. Let $L \in \mathbb{N}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$.
(i)If $0 < p \leq \infty$, $0 < q < \infty$ and $0 < s < L$, then for each $j \in \{1, \cdots, n\}$

$$(\int_0^\infty t^{-sq} \|\Delta^L_{t,j} f\|^q_{L^p(\mathbb{R}^n)} \frac{dt}{t})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{516}$$

(ii)Suppose f is a function. If $1 < p \leq \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$, or if $1 < p \leq \infty$, $0 < q < 1$ and $0 < s < L$, or if $0 < p \leq 1$, $0 < q < \infty$ and $\sigma_p < s < L$, then

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_0^\infty t^{-sq} \|\Delta^L_{t,j} f\|^q_{L^p(\mathbb{R}^n)} \frac{dt}{t})^{\frac{1}{q}}. \tag{517}$$

(iii)If $0 < p \le \infty$, $q = \infty$ and $0 < s < L$, then for each $j \in \{1, \cdots, n\}$

$$\operatorname*{ess\,sup}_{t>0} t^{-s} \|\Delta_{t,j}^L f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{518}$$

(iv)Suppose f is a function. If $1 < p \le \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \le 1$, $q = \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n} \operatorname*{ess\,sup}_{t>0} t^{-s} \|\Delta_{t,j}^L f\|_{L^p(\mathbb{R}^n)}. \tag{519}$$

Theorem 7.1.2 is a newly published result in the author's paper [98]. The proof of Theorem 7.1.2 is given in section 7.3. The counterpart of Theorem 7.1.2 for the inhomogeneous $B^s_{p,q}(\mathbb{R}^n)$ space was obtained by H. Triebel in [94, section 2.6.1] with rough conditions $0 < p, q \le \infty$ and $\sigma_p < s < M$. The corollaries of Theorem 7.1.2 are formulated below.

Corollary 7.1.3. Let $0 < p, q < \infty$, $s \in \mathbb{R}$ and $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p < \infty$, $0 < q < \infty$ and $0 < s < 1$, then for each $j \in \{1, \cdots, n\}$

$$\Big(\int_0^\infty \Big(\int_{\mathbb{R}^n} |f(x + te_j) - f(x)|^p dx\Big)^{\frac{q}{p}} \frac{dt}{t^{1+sq}}\Big)^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{520}$$

(ii)If $1 < p < \infty$, $1 \le q < \infty$ and $s \in \mathbb{R}$, or if $1 < p < \infty$, $0 < q < 1$ and $0 < s < 1$, or if $0 < p \le 1$, $0 < q < \infty$ and $\sigma_p < s < 1$, then

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n} \Big(\int_0^\infty \Big(\int_{\mathbb{R}^n} |f(x + te_j) - f(x)|^p dx\Big)^{\frac{q}{p}} \frac{dt}{t^{1+sq}}\Big)^{\frac{1}{q}}. \tag{521}$$

Proof of Corollary 7.1.3. Apply Theorem 7.1.2 (i) and (ii) with $L = 1$. And (521) also indicates the following inequality

$$2^{ks} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n} \Big(\int_0^\infty \Big(\int_{\mathbb{R}^n} |f(x + te_j) - f(x)|^p dx\Big)^{\frac{q}{p}} \frac{dt}{t^{1+sq}}\Big)^{\frac{1}{q}} \tag{522}$$

for every $k \in \mathbb{Z}$, and hence $\lim_{k \to +\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right side of (521) is finite. □

Corollary 7.1.4. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and at least one of p and q is infinity. Assume $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.

(i)If $p = \infty$, $0 < q < \infty$ and $0 < s < 1$, then for each $j \in \{1, \cdots, n\}$

$$(\int_0^\infty \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x+te_j) - f(x)|^q \frac{dt}{t^{1+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)}. \tag{523}$$

(ii)If $p = \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$, or if $p = \infty$, $0 < q < 1$ and $0 < s < 1$, then

$$\|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_0^\infty \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x+te_j) - f(x)|^q \frac{dt}{t^{1+sq}})^{\frac{1}{q}}. \tag{524}$$

(iii)If $0 < p < \infty$, $q = \infty$ and $0 < s < 1$, then for each $j \in \{1, \cdots, n\}$

$$\operatorname*{ess\,sup}_{t>0} t^{-s} (\int_{\mathbb{R}^n} |f(x+te_j) - f(x)|^p dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \tag{525}$$

(iv)If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \leq 1$, $q = \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \operatorname*{ess\,sup}_{t>0} t^{-s} (\int_{\mathbb{R}^n} |f(x+te_j) - f(x)|^p dx)^{\frac{1}{p}}. \tag{526}$$

(v)If $p = q = \infty$ and $0 < s < 1$, then for each $j \in \{1, \cdots, n\}$

$$\operatorname*{ess\,sup}_{t>0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{|f(x+te_j) - f(x)|}{t^s} \lesssim \|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)}. \tag{527}$$

(vi)If $p = q = \infty$ and $s \in \mathbb{R}$, then

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \operatorname*{ess\,sup}_{t>0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{|f(x+te_j) - f(x)|}{t^s}. \tag{528}$$

Proof of Corollary 7.1.4. Apply Theorem 7.1.2 with $L = 1$. We also deduce from (523) that for almost every $x \in \mathbb{R}^n$,

$$(\int_0^\infty |f(x+te_j) - f(x)|^q \frac{dt}{t^{1+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)} \tag{529}$$

when conditions of Corollary 7.1.4 (i) are satisfied. Furthermore if $0 < p \leq \infty$, $q = \infty$ and $0 < s < 1$, then we can infer from (525) and (527) the following inequality

$$\|f(\cdot + te_j) - f(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim t^s \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \qquad \text{for } t > 0. \tag{530}$$

Moreover when conditions of Corollary 7.1.4 (iv) are satisfied, the inequality (526) indicates

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x+te_j) - f(x)|^p}{t^{sp}} dx)^{\frac{1}{p}}. \tag{531}$$

From (524) and (528) we see that for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, $2^{ks}|\psi_{2^{-k}} * f(x)|$ can be estimated from above by the right sides of (524) and (528) respectively, and hence we deduce $\lim_{k\to+\infty} \psi_{2^{-k}} * f(x) = 0$ when $s > 0$ and the right sides of (524) and (528) are finite. From (526) we see that for every $k \in \mathbb{Z}$

$$\begin{aligned} 2^{ks}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{j=1}^n \operatorname*{ess\,sup}_{t>0} t^{-s} (\int_{\mathbb{R}^n} |f(x+te_j) - f(x)|^p dx)^{\frac{1}{p}} \\ &\lesssim \sum_{j=1}^n (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x+te_j) - f(x)|^p}{t^{sp}} dx)^{\frac{1}{p}}, \end{aligned} \tag{532}$$

and hence we deduce $\lim_{k\to+\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right side of (532) is finite. □

Corollary 7.1.5. Let $0 < p, q < \infty$, $s \in \mathbb{R}$ and $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.
(i)If $0 < p < \infty$, $0 < q < \infty$ and $0 < s < 2$, then for each $j \in \{1, \cdots, n\}$

$$(\int_0^\infty (\int_{\mathbb{R}^n} |f(x+2te_j) - 2f(x+te_j) + f(x)|^p dx)^{\frac{q}{p}} \frac{dt}{t^{1+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{533}$$

(ii)If $1 < p < \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$, or if $1 < p < \infty$, $0 < q < 1$ and $0 < s < 2$, or if $0 < p \leq 1$, $0 < q < \infty$ and $\sigma_p < s < 2$, then

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_0^\infty (\int_{\mathbb{R}^n} |f(x+2te_j) - 2f(x+te_j) + f(x)|^p dx)^{\frac{q}{p}} \frac{dt}{t^{1+sq}})^{\frac{1}{q}}. \tag{534}$$

Proof of Corollary 7.1.5. Apply Theorem 7.1.2 (i) and (ii) with $L = 2$. And (534) also indicates the following inequality

$$2^{ks}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n}(\int_0^\infty(\int_{\mathbb{R}^n}|f(x+2te_j)-2f(x+te_j)+f(x)|^p dx)^{\frac{q}{p}}\frac{dt}{t^{1+sq}})^{\frac{1}{q}} \quad (535)$$

for every $k \in \mathbb{Z}$, and hence $\lim_{k\to+\infty}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right side of (534) is finite. □

Corollary 7.1.6. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and at least one of p and q is infinity. Assume $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a function.

(i)If $p = \infty$, $0 < q < \infty$ and $0 < s < 2$, then for each $j \in \{1, \cdots, n\}$

$$(\int_0^\infty \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x+2te_j)-2f(x+te_j)+f(x)|^q\frac{dt}{t^{1+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)}. \quad (536)$$

(ii)If $p = \infty$, $1 \le q < \infty$ and $s \in \mathbb{R}$, or if $p = \infty$, $0 < q < 1$ and $0 < s < 2$, then

$$\|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n}(\int_0^\infty \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x+2te_j)-2f(x+te_j)+f(x)|^q\frac{dt}{t^{1+sq}})^{\frac{1}{q}}. \quad (537)$$

(iii)If $0 < p < \infty$, $q = \infty$ and $0 < s < 2$, then for each $j \in \{1, \cdots, n\}$

$$\operatorname*{ess\,sup}_{t>0} t^{-s}(\int_{\mathbb{R}^n}|f(x+2te_j)-2f(x+te_j)+f(x)|^p dx)^{\frac{1}{p}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \quad (538)$$

(iv)If $1 < p < \infty$, $q = \infty$ and $s \in \mathbb{R}$, or if $0 < p \le 1$, $q = \infty$ and $\sigma_p < s < \infty$, then

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n}\operatorname*{ess\,sup}_{t>0} t^{-s}(\int_{\mathbb{R}^n}|f(x+2te_j)-2f(x+te_j)+f(x)|^p dx)^{\frac{1}{p}}. \quad (539)$$

(v)If $p = q = \infty$ and $0 < s < 2$, then for each $j \in \{1, \cdots, n\}$

$$\operatorname*{ess\,sup}_{t>0}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} t^{-s}|f(x+2te_j)-2f(x+te_j)+f(x)| \lesssim \|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)}. \quad (540)$$

(vi)If $p = q = \infty$ and $s \in \mathbb{R}$, then

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n}\operatorname*{ess\,sup}_{t>0}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} t^{-s}|f(x+2te_j)-2f(x+te_j)+f(x)|. \quad (541)$$

Proof of Corollary 7.1.6. Apply Theorem 7.1.2 with $L = 2$. We also deduce from (536) that for almost every $x \in \mathbb{R}^n$,

$$(\int_0^\infty |f(x+2te_j) - 2f(x+te_j) + f(x)|^q \frac{dt}{t^{1+sq}})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{\infty,q}(\mathbb{R}^n)} \tag{542}$$

when conditions of Corollary 7.1.6 (i) are satisfied. Furthermore if $0 < p \leq \infty$, $q = \infty$ and $0 < s < 2$, then we can infer from (538) and (540) the following inequality

$$\|f(\cdot + 2te_j) - 2f(\cdot + te_j) + f(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim t^s \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \qquad \text{for } t > 0. \tag{543}$$

Moreover when conditions of Corollary 7.1.6 (iv) are satisfied, the inequality (539) indicates

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x+2te_j) - 2f(x+te_j) + f(x)|^p}{t^{sp}} dx)^{\frac{1}{p}}. \tag{544}$$

From (537) and (541) we see that for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$, $2^{ks}|\psi_{2^{-k}} * f(x)|$ can be estimated from above by the right sides of (537) and (541) respectively, and hence we deduce $\lim_{k\to+\infty} \psi_{2^{-k}} * f(x) = 0$ when $s > 0$ and the right sides of (537) and (541) are finite. From (539) we see that for every $k \in \mathbb{Z}$

$$\begin{aligned} 2^{ks}\|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{j=1}^n \operatorname*{ess\,sup}_{t>0} t^{-s} (\int_{\mathbb{R}^n} |f(x+2te_j) - 2f(x+te_j) + f(x)|^p dx)^{\frac{1}{p}} \\ &\lesssim \sum_{j=1}^n (\int_{\mathbb{R}^n} \operatorname*{ess\,sup}_{t>0} \frac{|f(x+2te_j) - 2f(x+te_j) + f(x)|^p}{t^{sp}} dx)^{\frac{1}{p}}, \end{aligned} \tag{545}$$

and hence we deduce $\lim_{k\to+\infty} \|\psi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)} = 0$ when $s > 0$ and the right side of (545) is finite. □

7.2 Proof Of Theorem 7.1.1

Proof. We first prove inequality (504) when $0 < p, q < \infty$, $\tilde{\sigma}^1_{pq} < s < L$ and f is a tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$. Without loss of generality, we only need to prove the inequality for $j = 1$, the cases for $j = 2, \cdots, n$ can be proved in the same way. Recall that for $x \in \mathbb{R}^n$, $x = (x_1, x_1')$ and $x_1' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$. We still denote $f_l = \psi_{2^{-l}} * f$ and begin with estimating $\sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} (\sum_{l\in\mathbb{Z}} |\Delta^L_{t,1} f_l(x)|)^q dt$ from above by the following

$$\sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} (\sum_{l\le k} |\Delta^L_{t,1} f_l(x)|)^q dt + \sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} (\sum_{l>k} |\Delta^L_{t,1} f_l(x)|)^q dt. \tag{546}$$

Using the same calculation technique exhibited in (392) and (393), if $0 < \varepsilon < \min\{s, L - s\}$, we can obtain

$$(\sum_{l\le k} |\Delta^L_{t,1} f_l(x)|)^q \lesssim 2^{kq\varepsilon} \cdot \sum_{l\le k} 2^{-lq\varepsilon} |\Delta^L_{t,1} f_l(x)|^q, \tag{547}$$

$$(\sum_{l>k} |\Delta^L_{t,1} f_l(x)|)^q \lesssim 2^{-kq\varepsilon} \cdot \sum_{l>k} 2^{lq\varepsilon} |\Delta^L_{t,1} f_l(x)|^q. \tag{548}$$

Inserting these estimates into (546), we can estimate (546) from above by

$$\sum_{k\in\mathbb{Z}} 2^{k(sq+\varepsilon q+1)} \sum_{l\le k} 2^{-lq\varepsilon} \int_{2^{-k}}^{2^{1-k}} |\Delta^L_{t,1} f_l(x)|^q dt + \sum_{k\in\mathbb{Z}} 2^{k(sq-\varepsilon q+1)} \sum_{l>k} 2^{lq\varepsilon} \int_{2^{-k}}^{2^{1-k}} |\Delta^L_{t,1} f_l(x)|^q dt. \tag{549}$$

Now we give an important estimate for $\Delta^L_{t,1} f_l(x)$. By using Mean Value Theorem consecutively with respect to the first coordinate, we obtain

$$\Delta^L_{t,1} f_l(x) = \partial^\alpha f_l(x_1 + \lambda t, x_1') \cdot t^L$$

for some λ between 0 and L and $\alpha = (L, 0, \cdots, 0)$ is a multi-index. From Lemma 2.0.9, we know for fixed $x_1' \in \mathbb{R}^{n-1}$, the 1-dimensional Peetre-Fefferman-Stein maximal function of $f_l(\cdot, x_1')$ is well-defined. Using the 1-dimensional version of Remark 2.0.6, we have

$$|\partial^\alpha f_l(x_1 + \lambda t, x_1')| \lesssim \mathcal{P}_1 \partial^\alpha f_l(\cdot, x_1')(x_1 + \lambda t) \lesssim 2^{lL} \mathcal{P}_1 f_l(\cdot, x_1')(x_1 + \lambda t).$$

Using 1-dimensional version of Remark 2.0.3 and assuming $t \lesssim 2^{-k}$, we can further obtain

$$\mathcal{P}_1 f_l(\cdot, x_1')(x_1 + \lambda t) \lesssim (1 + 2^{l-k})^{\frac{1}{r}} \mathcal{P}_1 f_l(\cdot, x_1')(x_1),$$

where $0 < r < \min\{p, q\}$. Therefore the estimate is given as follows

$$|\Delta_{t,1}^L f_l(x)| \lesssim 2^{(l-k)L}(1+2^{l-k})^{\frac{1}{r}}\mathcal{P}_1 f_l(\cdot, x_1')(x_1) \quad \text{for} \quad |t| \lesssim 2^{-k}. \tag{550}$$

For the first term in (549), we use the above estimate (550) and 1-dimensional version of Lemma 2.0.3 to obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{k(sq+\varepsilon q+1)} \sum_{l\le k} 2^{-lq\varepsilon} \int_{2^{-k}}^{2^{1-k}} |\Delta_{t,1}^L f_l(x)|^q dt \\
\lesssim &\sum_{l\in\mathbb{Z}} (\sum_{k\ge l} 2^{kq(s+\varepsilon-L)}) \cdot 2^{lq(L-\varepsilon)} \mathcal{P}_1 f_l(\cdot, x_1')(x_1)^q \\
\lesssim &\sum_{l\in\mathbb{Z}} 2^{lqs} \mathcal{P}_1 f_l(\cdot, x_1')(x_1)^q \\
\lesssim &\sum_{l\in\mathbb{Z}} 2^{lqs} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{q}{r}},
\end{aligned} \tag{551}$$

since $(1+2^{l-k})^{\frac{q}{r}}$ is bounded from above by a constant when $l \le k$, and $\mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)$ is the 1-dimensional Hardy-Littlewood maximal function of $|f_l(\cdot, x_1')|^r$ centered at x_1. For the second term in (549), we use (10) to get

$$|\Delta_{t,1}^L f_l(x)| \lesssim \sum_{m=0}^{L} |f_l(x_1 + mt, x_1')|. \tag{552}$$

When $0 \le m \le L$, $|t| \lesssim 2^{-k}$ and $l > k$, we also have

$$|f_l(x_1 + mt, x_1')| \lesssim (1+2^{l-k})^{\frac{1}{r}} \mathcal{P}_1 f_l(\cdot, x_1')(x_1) \lesssim 2^{\frac{l-k}{r}} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}}, \tag{553}$$

by using 1-dimensional versions of Remark 2.0.5, Remark 2.0.3 and Lemma 2.0.3, and constants are independent of l, k, m, t. And the following inequality is true for $0 \le m \le L$

$$\fint_{[2^{-k}, 2^{1-k}]} |f_l(x_1 + mt, x_1')|^r dt \lesssim \fint_{|t| \le 2^{1-k}} |f_l(x_1 + mt, x_1')|^r dt \lesssim \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1). \tag{554}$$

Applying estimates (552), (553) and (554) yields the following

$$\begin{aligned}
&2^k \int_{2^{-k}}^{2^{1-k}} |\Delta_{t,1}^L f_l(x)|^q dt \\
\lesssim &\sum_{m=0}^{L} 2^k \int_{2^{-k}}^{2^{1-k}} |f_l(x_1+mt,x_1')|^r \cdot |f_l(x_1+mt,x_1')|^{q-r} dt \\
\lesssim &\sum_{m=0}^{L} \fint_{[2^{-k},2^{1-k}]} |f_l(x_1+mt,x_1')|^r dt \cdot 2^{(l-k)(\frac{q}{r}-1)} \mathcal{M}_1(|f_l(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}-1} \\
\lesssim &2^{(l-k)(\frac{q}{r}-1)} \mathcal{M}_1(|f_l(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}}. \qquad (555)
\end{aligned}$$

And estimate (555) is true for $0 < q < \infty$ and $l > k$. Therefore we can estimate the second term in (549) as follows

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{k(sq-\varepsilon q+1)} \sum_{l>k} 2^{lq\varepsilon} \int_{2^{-k}}^{2^{1-k}} |\Delta_{t,1}^L f_l(x)|^q dt \\
\lesssim &\sum_{k\in\mathbb{Z}} 2^{kq(s-\varepsilon)} \sum_{l>k} 2^{lq\varepsilon} \cdot 2^{(l-k)(\frac{q}{r}-1)} \mathcal{M}_1(|f_l(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}} \\
\lesssim &\sum_{l\in\mathbb{Z}} \sum_{k<l} 2^{kq(s-\varepsilon-\frac{1}{r}+\frac{1}{q})} \cdot 2^{lq(\varepsilon+\frac{1}{r}-\frac{1}{q})} \mathcal{M}_1(|f_l(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}} \\
\lesssim &\sum_{l\in\mathbb{Z}} 2^{lqs} \mathcal{M}_1(|f_l(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}}, \qquad (556)
\end{aligned}$$

where the last step is because the assumption $\tilde{\sigma}_{pq}^1 < s$ indicates that $s - \varepsilon - \frac{1}{r} + \frac{1}{q} > 0$ if we pick ε sufficiently close to 0 and r sufficiently close to $\min\{p, q\}$. Combining (546), (549), (551) and (556) altogether, raising the power to $\frac{1}{q}$ and inserting the result into the $\|\cdot\|_{L^p(\mathbb{R}^n)}$-quasinorm yield

$$\begin{aligned}
&\|(\sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} (\sum_{l\in\mathbb{Z}} |\Delta_{t,1}^L f_l(x)|)^q dt)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \\
\lesssim &(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\sum_{l\in\mathbb{Z}} 2^{lqs} \mathcal{M}_1(|f_l(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}})^{\frac{p/r}{q/r}} dx_1 dx_1')^{\frac{1}{p}} \\
\lesssim &(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\sum_{l\in\mathbb{Z}} 2^{lqs} |f_l(x_1,x_1')|^q)^{\frac{p}{q}} dx_1 dx_1')^{\frac{1}{p}} = \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \qquad (557)
\end{aligned}$$

where we also used the 1-dimensional version of Lemma 2.0.6, and inequality (557) is true for $0 < p, q < \infty$, $\tilde{\sigma}_{pq}^1 < s < L$. The assumption $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ and inequality (557) also tell the absolute convergence of the series $\sum_{l\in\mathbb{Z}} \Delta^L_{t,1} f_l(x)$ for every $k \in \mathbb{Z}$ and almost every $t \in [2^{-k}, 2^{1-k}], x \in \mathbb{R}^n$. In conjunction with (28), we have proven the following claim that

$$\Delta^L_{t,1} f = \sum_{l\in\mathbb{Z}} \Delta^L_{t,1} f_l(x) \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \tag{558}$$

for every $k \in \mathbb{Z}$ and almost every $t \in [2^{-k}, 2^{1-k}], x \in \mathbb{R}^n$ when $0 < p, q < \infty$, $\tilde{\sigma}^1_{pq} < s < L$. Therefore the tempered distribution $\Delta^L_{t,1} f$ has a function representative which is the pointwise limit of the series $\sum_{l\in\mathbb{Z}} \Delta^L_{t,1} f_l(x)$ and integration of $\Delta^L_{t,1} f$ with respect to Lebesgue measure is justified. Furthermore, we have obtained the following inequality

$$\|(\int_0^\infty t^{-sq} |\Delta^L_{t,1} f|^q \frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \lesssim \|(\sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} (\sum_{l\in\mathbb{Z}} |\Delta^L_{t,1} f_l(x)|)^q dt)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}, \tag{559}$$

and then (557) and (559) conclude the proof of Theorem 7.1.1 (i).

Next, we show that inequality (505) is true under the conditions of Theorem 7.1.1 (ii). We assume the right side of (505) is finite, otherwise, the inequality is trivial. We still use the sufficiently large positive integer m_0 given in (403). Observe that if $\xi \in spt.\mathcal{F}_n\psi \subseteq A' = \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| < 2\}$, then

$$\frac{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2}{n} \geq \frac{1}{4n}.$$

This means given a sufficiently small positive number δ, there exists at least one ξ_j such that $\delta \leq |\xi_j| < 2$. Therefore we obtain the decomposition $A' = \bigcup_{j=1}^n A'_j$, where $A'_j = \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| < 2, \delta \leq |\xi_j| < 2\}$. Let $\{\rho_j\}_{j=1}^n$ be the partition of unity associated with this decomposition, that is, each ρ_j is a smooth function with a compact support in $\mathbb{R}^n$, and $spt.\rho_j$ is contained in a small neighborhood of A'_j, furthermore

$$\sum_{j=1}^n \rho_j(\xi) = 1 \text{ if } \xi \in A'. \tag{560}$$

Without loss of generality, we can assume that

$$\frac{1}{2} \leq |\xi| < 2 \text{ and } \delta \leq |\xi_j| < 2 \text{ for } \xi \in spt.\rho_j. \tag{561}$$

Then we have

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n 2^{-sm_0} \|\{2^{ks}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f](x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}. \tag{562}$$

Thus to prove (505), it is sufficient to prove

$$\begin{aligned}&\|\{2^{ks}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_nf](x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}\\ \lesssim& C_0\|(\int_0^\infty t^{-sq}|\Delta_{t,j}^L f(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}+C_{00}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\end{aligned} \tag{563}$$

for every $j\in\{1,2,\cdots,n\}$ and C_{00} is a positive constant that can be arbitrarily small. We only need to prove (563) for $j=1$, and the cases for $j=2,\cdots,n$ can be proved in the same way. Notice that both $\mathcal{F}_n\psi(2^{m_0-k}\xi)$ and $\rho_1(2^{m_0-k}\xi)$ are supported in a ball centered at 0 of radius 2^{k+1-m_0} in $\mathbb{R}^n$, thus using the argument of Remark 2.0.4, we have

$$\begin{aligned}&|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](x)|\\ \lesssim&\mathcal{P}_n\{\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf]\}(x)\\ \lesssim&\mathcal{P}_n\{\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf]\}(x).\end{aligned} \tag{564}$$

By Lemma 2.0.3, Remark 2.0.5 and Lemma 2.0.6, we have

$$\begin{aligned}&\|\{2^{ks}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}\\ \lesssim&\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}.\end{aligned} \tag{565}$$

If $1<\min\{p,q\}$ and $s\in\mathbb{R}$, then we have

$$\begin{aligned}&\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](x)\\ =&\fint_{[1,2]}\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}\cdot(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L\mathcal{F}_nf](x)\frac{dt}{t}\\ =&\fint_{[1,2]}\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}]*\Delta^L_{2^{-k}t,1}f(x)\frac{dt}{t}.\end{aligned} \tag{566}$$

According to the support condition of ρ_1, when $\rho_1(2^{m_0}\xi)\neq 0$ and $t\in[1,2]$, we have $0<2^{-m_0}\delta\leq t|\xi_1|<2^{2-m_0}$ and thus $|(e^{2\pi i\cdot t\xi_1}-1)^L|\geq c>0$ for some constant c independent of t and ξ_1. Using the same method as in Lemma 2.0.8, in particular since a similar condition like (81) is satisfied because of the assumption on ρ_1, we can obtain

$$|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0}\xi)}{(e^{2\pi i\cdot t\xi_1}-1)^L}](y)|\lesssim\frac{1}{(1+|y|)^N} \tag{567}$$

for arbitrarily large positive integer N, and the constant is independent of t. We still use the notation $A_{k-l} = \{y \in \mathbb{R}^n : 2^{l-k} \le |y| < 2^{1+l-k}\}$ and hence

$$\begin{aligned}
&|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}] * \Delta^L_{2^{-k}t,1}f(x)| \\
\lesssim &\sum_{l\in\mathbb{Z}} \int_{A_{k-l}} 2^{kn}|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0}\xi)}{(e^{2\pi i\cdot t\xi_1}-1)^L}](2^k y)| \cdot |\Delta^L_{2^{-k}t,1}f(x-y)|dy \\
\lesssim &\sum_{l\le 0} 2^{ln} \fint_{A_{k-l}} |\Delta^L_{2^{-k}t,1}f(x-y)|dy + \sum_{l>0} 2^{l(n-N)} \fint_{A_{k-l}} |\Delta^L_{2^{-k}t,1}f(x-y)|dy. \qquad (568)
\end{aligned}$$

Insert (568) into (566), exchange the order of integration, and use the inequality

$$\fint_{A_{k-l}} \int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(x-y)|\frac{dt}{t}dy \lesssim \mathcal{M}_n(\int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(\cdot)|\frac{dt}{t})(x),$$

then we can obtain

$$\begin{aligned}
&|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](x)| \\
\lesssim &(\sum_{l\le 0} 2^{ln} + \sum_{l>0} 2^{l(n-N)})\mathcal{M}_n(\int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(\cdot)|\frac{dt}{t})(x) \\
\lesssim &\mathcal{M}_n(\int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(\cdot)|\frac{dt}{t})(x), \qquad (569)
\end{aligned}$$

if we pick $N > n$. Inserting (569) into (565), applying Lemma 2.0.6 which requires the condition $1 < \min\{p, q\}$, and also using Hölder's inequality for $1 < q$ yield the following

$$\begin{aligned}
&\|\{2^{ks}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\
\lesssim &\|\{2^{ks}\mathcal{M}_n(\int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(\cdot)|\frac{dt}{t})(x)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\
\lesssim &\|\{2^{ks}\int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(x)|\frac{dt}{t}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)} \\
\lesssim &\|(\sum_{k\in\mathbb{Z}} 2^{ksq}\int_{[1,2]} |\Delta^L_{2^{-k}t,1}f(x)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \\
\lesssim &\|(\int_0^\infty t^{-sq}|\Delta^L_{t,1}f(x)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}, \qquad (570)
\end{aligned}$$

and this inequality is true for any $s \in \mathbb{R}$. By now we have proven (563) for $1 \le j \le n$ and $C_{00} = 0$, and inserting these inequalities back into (562) proves (505) under the conditions of Theorem 7.1.1 (ii) when $1 < \min\{p, q\}$, $q < \infty$ and $s \in \mathbb{R}$. Now we show that (505) is still true under the conditions of Theorem 7.1.1 (ii) when $\min\{p, q\} \le 1$, $q < \infty$ and $\sigma_{pq} + \tilde{\sigma}^1_{pq} < s <$

∞, we will have to use the n-dimensional Plancherel-Polya-Nikol'skij inequality and hence introduce σ_{pq}, a number depending on the dimension n, into the restriction of s. We use the function ϕ satisfying conditions (19), (408) and (409), and J is still a large positive integer whose value will be determined later. Because $spt.\rho_1(2^{m_0-k}\xi) \subseteq \{\xi \in \mathbb{R}^n : |\xi| < 2^{k+1-m_0}\}$, the n-dimensional Fourier transform of the Schwartz function

$$y \mapsto \mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}](y) \cdot \varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x-y)$$

is supported in a ball of radius about 2^{J+k-m_0}, centered at the origin in $\mathbb{R}^n$. Therefore by using Plancherel-Polya-Nikol'skij inequality or the more general Lemma 2.0.5 and the condition $0 < r < \min\{p, q\} \leq 1$, we have

$$\begin{aligned} &|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}] * [\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(\cdot)](x)|^r \\ \lesssim&(\int_{\mathbb{R}^n} |\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}](y) \cdot \varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x-y)|dy)^r \\ \lesssim&2^{(J+k-m_0)n(1-r)} \int_{\mathbb{R}^n} 2^{knr} |\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0}\xi)}{(e^{2\pi i\cdot t\xi_1}-1)^L}](2^k y)|^r \\ &\cdot|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x-y)|^r dy. \end{aligned} \tag{571}$$

Recall that $\mathbb{R}^n = \bigcup_{l\in\mathbb{Z}} A_{k-l}$ where A_{k-l} is the annulus $\{y \in \mathbb{R}^n : 2^{l-k} \leq |y| < 2^{1+l-k}\}$ and use (567) with a sufficiently large positive integer $N' > \frac{n}{r}$, then we can estimate (571) from above by

$$\begin{aligned} &2^{Jn(1-r)} \cdot \{\sum_{l\leq 0} 2^{ln} \fint_{A_{k-l}} |\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x-y)|^r dy \\ &+ \sum_{l>0} 2^{l(n-N'r)} \fint_{A_{k-l}} |\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x-y)|^r dy\}. \end{aligned} \tag{572}$$

Therefore from (403) we know that when $\rho_1(2^{m_0-k}\xi)\neq 0$ and $t\in[1,2]$, $|(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L|>0$ and we can obtain

$$\begin{aligned}
&|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](x)|^r\\
=&\fint_{[1,2]}|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}\cdot(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L\mathcal{F}_n\phi(2^{m_0-J-k}\xi)\mathcal{F}_nf](x)|^r\frac{dt}{t}\\
=&\fint_{[1,2]}|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}]*[\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(\cdot)](x)|^r\frac{dt}{t}\\
\lesssim&2^{Jn(1-r)}\cdot\{\sum_{l\leq 0}2^{ln}\fint_{A_{k-l}}\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(x-y)|^r\frac{dt}{t}dy\\
&+\sum_{l>0}2^{l(n-N'r)}\fint_{A_{k-l}}\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(x-y)|^r\frac{dt}{t}dy\}.
\end{aligned}\tag{573}$$

Using the fact that $\fint_{A_{k-l}}\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(x-y)|^r\frac{dt}{t}dy$ can be dominated by

$$\mathcal{M}_n(\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(\cdot)|^r\frac{dt}{t})(x),$$

we obtain

$$\begin{aligned}
&|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](x)|\\
\lesssim&2^{Jn(\frac{1}{r}-1)}\mathcal{M}_n(\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(\cdot)|^r\frac{dt}{t})(x)^{\frac{1}{r}}.
\end{aligned}\tag{574}$$

Inserting (574) into (565) and using Lemma 2.0.6 and Hölder's inequality since $0<r<\min\{p,q\}$ yield

$$\begin{aligned}
&\|\{2^{ks}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](\cdot)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}\\
\lesssim&2^{Jn(\frac{1}{r}-1)}\cdot\|\{2^{ks}(\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(\cdot)|^r\frac{dt}{t})^{\frac{1}{r}}\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}\\
\lesssim&2^{Jn(\frac{1}{r}-1)}\cdot\|(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}.
\end{aligned}\tag{575}$$

Inferring from (423), (29) and assuming the validity of decomposition, then (575) can be estimated from above by the sum of the two terms

$$\begin{aligned}
&2^{Jn(\frac{1}{r}-1)}\cdot\|(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2|\varDelta^L_{2^{-k}t,1}f(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}\\
\sim&2^{Jn(\frac{1}{r}-1)}\cdot\|(\int_0^\infty t^{-sq}|\varDelta^L_{t,1}f(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}<\infty,
\end{aligned}\tag{576}$$

and

$$2^{Jn(\frac{1}{r}-1)}\cdot\|(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2|\Delta^L_{2^{-k}t,1}(\sum_{l=J+1}^{\infty}f_{k+l-m_0})(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}. \tag{577}$$

To estimate (577), we begin by applying the calculation method used for obtaining the second term of (549) to the following term below and obtain

$$2^{Jn(\frac{1}{r}-1)}\|(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2(\sum_{l=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+l-m_0}(\cdot)|)^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)} \tag{578}$$

$$\lesssim 2^{Jn(\frac{1}{r}-1)-J\varepsilon}\|(\sum_{k\in\mathbb{Z}}\sum_{l>J}2^{ksq+k+lq\varepsilon}\int_{2^{-k}}^{2^{1-k}}|\Delta^L_{t,1}f_{k+l-m_0}(\cdot)|^q dt)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}, \tag{579}$$

where we only need $\varepsilon>0$. Considering $x_1'=(x_2,\cdots,x_n)\in\mathbb{R}^{n-1}$ is fixed for now, we use Lemma 2.0.9 and estimate (555) since $k+l-m_0>k+J-m_0>k$ and then we can obtain the following

$$2^k\int_{2^{-k}}^{2^{1-k}}|\Delta^L_{t,1}f_{k+l-m_0}(x)|^q dt\lesssim 2^{l(\frac{q}{r}-1)}\mathcal{M}_1(|f_{k+l-m_0}(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}}. \tag{580}$$

Inserting (580) into (579), because the assumption $\sigma_{pq}+\tilde{\sigma}^1_{pq}<s<\infty$ implies

$$n(\frac{1}{r}-1)+(\frac{1}{r}-\frac{1}{q})<s\qquad\text{and}\qquad\varepsilon+\frac{1}{r}-\frac{1}{q}<s \tag{581}$$

when r is sufficiently close to $\min\{p,q\}$ and ε is sufficiently close to 0, we can estimate (579) from above by the following

$$2^{Jn(\frac{1}{r}-1)-J\varepsilon}\cdot(\int_{\mathbb{R}^n}(\sum_{l>J}2^{lq(\varepsilon+\frac{1}{r}-\frac{1}{q}-s)}\cdot\sum_{k\in\mathbb{Z}}2^{ksq}\mathcal{M}_1(|f_k(\cdot,x_1')|^r)(x_1)^{\frac{q}{r}})^{\frac{p}{q}}dx)^{\frac{1}{p}}, \tag{582}$$

and this term can be further estimated from above by

$$2^{J[n(\frac{1}{r}-1)+(\frac{1}{r}-\frac{1}{q})-s]}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}, \tag{583}$$

due to the 1-dimensional version of Lemma 2.0.6. Putting together (579), (582) and (583) yields

$$2^{Jn(\frac{1}{r}-1)}\|(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2(\sum_{l=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+l-m_0}(\cdot)|)^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim 2^{J[n(\frac{1}{r}-1)+(\frac{1}{r}-\frac{1}{q})-s]}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{584}$$

Inequality (584) and the assumption of f being a member of $\dot{F}^s_{p,q}(\mathbb{R}^n)$ also suggest that $\sum_{l=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+l-m_0}(x)|<\infty$ for every $k\in\mathbb{Z}$ and almost every $t\in[1,2], x\in\mathbb{R}^n$. From (29), (576), the above inference and the supposition of f being a function, we can deduce that

$$\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(x)=\Delta^L_{2^{-k}t,1}f(x)-\sum_{l=J+1}^{\infty}\Delta^L_{2^{-k}t,1}f_{k+l-m_0}(x) \tag{585}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)$ for every $k\in\mathbb{Z}$ and almost every $t\in[1,2], x\in\mathbb{R}^n$ when $\min\{p,q\}\le 1$, $q<\infty$ and $\sigma_{pq}+\tilde{\sigma}^1_{pq}<s<\infty$, furthermore estimating (575) from above by the sum of (576) and (577) is justified, moreover (577) can be estimated from above by (578) and hence by (584). We have reached the conclusion

$$\begin{aligned}&\|\{2^{ks}\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](\cdot)\}_{k\in\mathbb{Z}}\|_{L^p(l^q)}\\ \le& C_12^{Jn(\frac{1}{r}-1)}\|(\int_0^{\infty}t^{-sq}|\Delta^L_{t,1}f(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}+C_1'2^{J[n(\frac{1}{r}-1)+(\frac{1}{r}-\frac{1}{q})-s]}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.\end{aligned} \tag{586}$$

In a similar way, we can also prove (586) if we replace ρ_1, $\Delta^L_{t,1}$, C_1, C_1' by ρ_j, $\Delta^L_{t,j}$, C_j, C_j' respectively for $j=2,\cdots,n$. From (562) we have obtained

$$\begin{aligned}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\le& C'2^{Jn(\frac{1}{r}-1)}\sum_{j=1}^{n}\|(\int_0^{\infty}t^{-sq}|\Delta^L_{t,j}f(\cdot)|^q\frac{dt}{t})^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n)}\\ &+C''2^{J[n(\frac{1}{r}-1)+(\frac{1}{r}-\frac{1}{q})-s]}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.\end{aligned} \tag{587}$$

By (581) we can pick a sufficiently large positive integer J so that the coefficient

$$C''2^{J[n(\frac{1}{r}-1)+(\frac{1}{r}-\frac{1}{q})-s]}<\frac{1}{2},$$

and shift the second term on the right side of (587) to its left side and hence complete the proof of (505) when $\min\{p,q\}\le 1$, $q<\infty$ and $\sigma_{pq}+\tilde{\sigma}^1_{pq}<s<\infty$.

Now we prove Theorem 7.1.1 (iii). We only need to prove inequality (506) when $j=1$ and the other cases when $j=2,\cdots,n$ can be proved in the same way. We begin with estimating $\|\operatorname{ess\,sup}_{t>0}t^{-s}\sum_{l\in\mathbb{Z}}|\Delta^L_{t,1}f_l(\cdot)|\|_{L^p(\mathbb{R}^n)}$ from above by the sum

$$\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})}2^{ks}\sum_{l\le k}|\Delta^L_{t,1}f_l(\cdot)|\|_{L^p(\mathbb{R}^n)}+\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})}2^{ks}\sum_{l>k}|\Delta^L_{t,1}f_l(\cdot)|\|_{L^p(\mathbb{R}^n)}. \tag{588}$$

Pick $0 < \varepsilon < \min\{s - \frac{1}{p}, L - s\}$ and let $x_1' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$ be fixed for now. When $l \le k$, we use Lemma 2.0.9 and (550) and calculate as follows

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \sum_{l\le k} \operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})} |\Delta^L_{t,1} f_l(x_1, x_1')| \\
\lesssim &\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s-L)} \sum_{l\le k} 2^{l\varepsilon} \cdot 2^{l(L-\varepsilon)} \mathcal{P}_1 f_l(\cdot, x_1')(x_1) \\
\lesssim &\operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k\ge l} 2^{k(s+\varepsilon-L)} \cdot 2^{l(L-\varepsilon)} \mathcal{P}_1 f_l(\cdot, x_1')(x_1) \\
\lesssim &\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}},
\end{aligned} \tag{589}
$$

where we also used the 1-dimensional version of Lemma 2.0.3 and $0 < r < p$. Inserting (589) into the $\|\cdot\|_{L^p(\mathbb{R}^n)}$-quasinorm and invoking the mapping property of the 1-dimensional Hardy-Littlewood maximal function, we can estimate the first term in (588) from above by the following

$$
\begin{aligned}
&\|\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}}\|_{L^p(\mathbb{R}^n)} \\
\lesssim &\Big(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathcal{M}_1(\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{lsr} |f_l(\cdot, x_1')|^r)(x_1)^{\frac{p}{r}} dx_1 dx_1'\Big)^{\frac{1}{p}} \\
\lesssim &\Big(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{lsp} |f_l(x_1, x_1')|^p dx_1 dx_1'\Big)^{\frac{1}{p}} = \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}.
\end{aligned} \tag{590}
$$

When $l > k$ and $|t| \lesssim 2^{-k}$, we use (552) and (553) with temporarily fixed $x_1' \in \mathbb{R}^{n-1}$ to obtain

$$
\operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})} |\Delta^L_{t,1} f_l(x_1, x_1')| \lesssim 2^{\frac{l-k}{r}} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}}. \tag{591}
$$

Since $\frac{1}{p} < s < L$ by the assumption, we can pick $\varepsilon > 0$ to be sufficiently close to 0 and r to be sufficiently close to p so that $s - \frac{1}{r} - \varepsilon$ is a positive finite number, and then we can obtain

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \sum_{l> k} \operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})} |\Delta^L_{t,1} f_l(x_1, x_1')| \\
\lesssim &\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s-\frac{1}{r})} \sum_{l> k} 2^{-l\varepsilon} \cdot 2^{l(\varepsilon+\frac{1}{r})} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}} \\
\lesssim &\operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k<l} 2^{k(s-\frac{1}{r}-\varepsilon)} \cdot 2^{l(\varepsilon+\frac{1}{r})} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}} \\
\lesssim &\operatorname*{ess\,sup}_{l\in\mathbb{Z}} 2^{ls} \mathcal{M}_1(|f_l(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}}.
\end{aligned} \tag{592}
$$

Inserting (592) into the $\|\cdot\|_{L^p(\mathbb{R}^n)}$-quasinorm and proceeding as in (590), we can estimate the second term in (588) from above by $\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}$. Therefore we have obtained the inequality

$$\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})} 2^{ks}\sum_{l\in\mathbb{Z}} |\varDelta^L_{t,1}f_l(\cdot)|\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}, \tag{593}$$

when $0<p<\infty$, $q=\infty$ and $\frac{1}{p}<s<L$. The assumption $f\in\dot{F}^s_{p,\infty}(\mathbb{R}^n)$ and the above inequality also show $\sum_{l\in\mathbb{Z}}|\varDelta^L_{t,1}f_l(x)|<\infty$ for every $k\in\mathbb{Z}$ and almost every $t\in[2^{-k},2^{1-k}), x\in\mathbb{R}^n$. In conjunction with (28), we have reached the conclusion that

$$\varDelta^L_{t,1}f = \sum_{l\in\mathbb{Z}} \varDelta^L_{t,1}f_l(x) \tag{594}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ for every $k\in\mathbb{Z}$ and almost every $t\in[2^{-k},2^{1-k}), x\in\mathbb{R}^n$ when $0<p<\infty$, $q=\infty$ and $\frac{1}{p}<s<L$, furthermore we also obtain

$$\|\operatorname*{ess\,sup}_{t>0} \frac{|\varDelta^L_{t,1}f|}{t^s}\|_{L^p(\mathbb{R}^n)} \lesssim \|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{t\in[2^{-k},2^{1-k})} 2^{ks}\sum_{l\in\mathbb{Z}} |\varDelta^L_{t,1}f_l(\cdot)|\|_{L^p(\mathbb{R}^n)}. \tag{595}$$

Inequalities (593) and (595) conclude the proof of Theorem 7.1.1 (iii).

Finally, we come to the proof of Theorem 7.1.1 (iv). We assume the right side of (507) is finite, otherwise, the inequality is trivial. We also use the sufficiently large positive integer m_0 given in (403) and let $0<r<p$. We continue using the partition of unity $\{\rho_j\}_{j=1}^n$ associated with the set $A'=\{\xi\in\mathbb{R}^n : \frac{1}{2}\le|\xi|<2\}$ introduced at the beginning of the proof of Theorem 7.1.1 (ii), and also continue assuming $\frac{1}{2}\le|\xi|<2$ and $\delta\le|\xi_j|<2$ for $\xi\in spt.\rho_j$ and δ being a sufficiently small positive number. Then by using Remark 2.0.4, Lemma 2.0.3, and the mapping property of the Hardy-Littlewood maximal function, we can obtain the estimate

$$\begin{aligned}
\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} &\lesssim 2^{-m_0 s}\sum_{j=1}^n \|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}|\mathcal{F}_n^{-1}(\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_nf)(\cdot)|\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{j=1}^n \|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\mathcal{P}_n(\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_nf])(\cdot)\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{j=1}^n \|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\mathcal{M}_n(|\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_nf]|^r)(\cdot)^{\frac{1}{r}}\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{j=1}^n \|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}|\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_nf](\cdot)|\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{596}$$

We now estimate each term in the sum of (596). It suffices to provide estimate for the term $\|\operatorname{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)|\|_{L^p(\mathbb{R}^n)}$ when $j=1$, and estimates for the terms when $j=2,\cdots,n$ can be obtained in the same way. When $1<p<\infty$, we have

$$\begin{aligned}
&|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](x)| \\
=&\operatorname*{ess\,sup}_{t\in[1,2]} |\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](x)| \\
=&\operatorname*{ess\,sup}_{t\in[1,2]} |\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}\cdot(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L\mathcal{F}_n f](x)| \\
=&\operatorname*{ess\,sup}_{t\in[1,2]} |\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}]*\Delta^L_{2^{-k}t,1}f(x)|.
\end{aligned} \tag{597}$$

We use (568) and the following estimate

$$\fint_{A_{k-l}} \operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}f(x-y)|dy \lesssim \mathcal{M}_n(\operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}f(\cdot)|)(x), \tag{598}$$

and then we can estimate $|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](x)|$ from above by the following

$$(\sum_{l\le 0}2^{ln}+\sum_{l>0}2^{l(n-N)})\mathcal{M}_n(\operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}f|)(x) \lesssim \mathcal{M}_n(\operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}f|)(x) \tag{599}$$

for every $x\in\mathbb{R}^n$ if we pick $N>n$. Therefore when $1<p<\infty$, $q=\infty$ and $s\in\mathbb{R}$, we invoke the mapping property of Hardy-Littlewood maximal function and obtain

$$\begin{aligned}
&\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f]|\|_{L^p(\mathbb{R}^n)} \\
\lesssim&\|\mathcal{M}_n(\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}|\Delta^L_{2^{-k}t,1}f|)\|_{L^p(\mathbb{R}^n)} \lesssim \|\operatorname*{ess\,sup}_{t>0}\frac{|\Delta^L_{t,1}f(\cdot)|}{t^s}\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{600}$$

Inequality (600) is still true if we replace $\rho_1, \Delta^L_{2^{-k}t,1}, \Delta^L_{t,1}$ by $\rho_j, \Delta^L_{2^{-k}t,j}, \Delta^L_{t,j}$ respectively for $j=2,\cdots,n$. Inserting these inequalities back into (596) proves the first part of Theorem 7.1.1 (iv). To prove the second part of Theorem 7.1.1 (iv), we use the function ϕ satisfying conditions (19), (408), (409), and the same m_0 as in (403), and the large positive integer $J>m_0$ whose value will be determined later. From (571), (572) and the inequality

$$\operatorname*{ess\,sup}_{t\in[1,2]}\fint_{A_{k-l}} |\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(x-y)|^r dy \lesssim \mathcal{M}_n(\operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)|^r)(x), \tag{601}$$

we can obtain

$$
\begin{aligned}
&|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](x)|^r \\
=&\operatorname*{ess\,sup}_{t\in[1,2]} |\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}] * [\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)](x)|^r \qquad (602)\\
\lesssim& 2^{Jn(1-r)}\cdot(\sum_{l\le 0} 2^{ln} + \sum_{l>0} 2^{l(n-N'r)})\mathcal{M}_n(\operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)|^r)(x) \\
\lesssim& 2^{Jn(1-r)}\mathcal{M}_n(\operatorname*{ess\,sup}_{t\in[1,2]} |\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)|^r)(x), \qquad (603)
\end{aligned}
$$

if we pick $N' > \frac{n}{r}$. From this inequality and the mapping property of the Hardy-Littlewood maximal function, we deduce the estimate

$$
\begin{aligned}
&\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)|\|_{L^p(\mathbb{R}^n)} \\
\lesssim& 2^{Jn(\frac{1}{r}-1)}\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(\cdot)|\|_{L^p(\mathbb{R}^n)}. \qquad (604)
\end{aligned}
$$

Assuming the validity of decomposition, then (604) can be estimated from above by the sum of the following two terms,

$$
2^{Jn(\frac{1}{r}-1)}\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}|\Delta^L_{2^{-k}t,1} f(\cdot)|\|_{L^p(\mathbb{R}^n)} \sim 2^{Jn(\frac{1}{r}-1)}\|\operatorname*{ess\,sup}_{t>0} \frac{|\Delta^L_{t,1} f(\cdot)|}{t^s}\|_{L^p(\mathbb{R}^n)} < \infty, \qquad (605)
$$

and

$$
2^{Jn(\frac{1}{r}-1)}\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}|\Delta^L_{2^{-k}t,1}(\sum_{l=J+1}^{\infty} f_{k+l-m_0})(\cdot)|\|_{L^p(\mathbb{R}^n)}. \qquad (606)
$$

To estimate (606), we use (552), (553), Lebesgue's differentiation theorem and the fact that $k+l-m_0 > k+J-m_0 > k$ to obtain for $k\in\mathbb{Z}$, $l>J$ and $t\in[1,2]$,

$$
|\Delta^L_{2^{-k}t,1} f_{k+l-m_0}(x)| \lesssim 2^{\frac{l}{r}}\mathcal{M}_1(|f_{k+l-m_0}(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}}, \qquad (607)
$$

where the constant is independent of l,k,t. Therefore we have

$$
\begin{aligned}
&2^{Jn(\frac{1}{r}-1)}\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}\sum_{l=J+1}^{\infty}|\Delta^L_{2^{-k}t,1} f_{k+l-m_0}(\cdot)|\|_{L^p(\mathbb{R}^n)} \qquad (608)\\
=&2^{Jn(\frac{1}{r}-1)}\|\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}\sum_{l=J+1}^{\infty} 2^{-l\varepsilon}\cdot 2^{l\varepsilon}|\Delta^L_{2^{-k}t,1} f_{k+l-m_0}(\cdot)|\|_{L^p(\mathbb{R}^n)} \\
\lesssim& 2^{Jn(\frac{1}{r}-1)-J\varepsilon}\|\operatorname*{ess\,sup}_{l>J}\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks+l\varepsilon}|\Delta^L_{2^{-k}t,1} f_{k+l-m_0}(\cdot)|\|_{L^p(\mathbb{R}^n)} \\
\lesssim& 2^{Jn(\frac{1}{r}-1)-J\varepsilon}\|\operatorname*{ess\,sup}_{l>J} 2^{l(\varepsilon+\frac{1}{r}-s)}\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{(k+l-m_0)s}\mathcal{M}_1(|f_{k+l-m_0}(\cdot, x_1')|^r)(x_1)^{\frac{1}{r}}\|_{L^p(\mathbb{R}^n)}. \qquad (609)
\end{aligned}
$$

The assumption $\sigma_p + \frac{1}{p} < s < \infty$ indicates

$$\varepsilon + \frac{1}{r} < n(\frac{1}{r} - 1) + \frac{1}{r} < s \tag{610}$$

if we pick $\varepsilon > 0$ to be sufficiently small and r to be sufficiently close to p when $0 < r < p \leq 1$. Thus by the mapping property of the 1-dimensional Hardy-Littlewood maximal function, we can continue from (609) and obtain the following estimate

$$\begin{aligned}
&2^{Jn(\frac{1}{r}-1)} \| \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks} \sum_{l=J+1}^{\infty} |\Delta^L_{2^{-k}t,1} f_{k+l-m_0}(\cdot)| \|_{L^p(\mathbb{R}^n)} \\
\lesssim & 2^{J[n(\frac{1}{r}-1)+\frac{1}{r}-s]} (\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathcal{M}_1(\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ksr} |f_k(\cdot, x_1')|^r)(x_1)^{\frac{p}{r}} dx_1 dx_1')^{\frac{1}{p}} \\
\lesssim & 2^{J[n(\frac{1}{r}-1)+\frac{1}{r}-s]} \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}.
\end{aligned} \tag{611}$$

The assumption $f \in \dot{F}^s_{p,\infty}(\mathbb{R}^n)$ implies $\sum_{l=J+1}^{\infty} |\Delta^L_{2^{-k}t,1} f_{k+l-m_0}(x)| < \infty$ for every $k \in \mathbb{Z}$ and almost every $t \in [1,2], x \in \mathbb{R}^n$. From this implication, (29) and (605), we deduce

$$\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x) = \Delta^L_{2^{-k}t,1} f(x) - \sum_{l=J+1}^{\infty} \Delta^L_{2^{-k}t,1} f_{k+l-m_0}(x) \tag{612}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)$ for every $k \in \mathbb{Z}$ and almost every $t \in [1,2], x \in \mathbb{R}^n$ when $0 < p \leq 1$, $q = \infty$ and $\sigma_p + \frac{1}{p} < s < \infty$, furthermore estimating (604) from above by the sum of (605) and (606) is justified, moreover (606) can be estimated from above by (608). We have obtained

$$\begin{aligned}
&\| \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} |\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)| \|_{L^p(\mathbb{R}^n)} \\
\leq & C_1' 2^{Jn(\frac{1}{r}-1)} \| \operatorname*{ess\,sup}_{t>0} \frac{|\Delta^L_{t,1} f(\cdot)|}{t^s} \|_{L^p(\mathbb{R}^n)} + C_1'' 2^{J[n(\frac{1}{r}-1)+\frac{1}{r}-s]} \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}.
\end{aligned} \tag{613}$$

Inequality (613) is still true if we replace $\rho_1, \Delta^L_{t,1}, C_1', C_1''$ by $\rho_j, \Delta^L_{t,j}, C_j', C_j''$ respectively for $j = 2, \cdots, n$. Inserting these inequalities back into (596) yields

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} \leq C' 2^{Jn(\frac{1}{r}-1)} \sum_{j=1}^{n} \| \operatorname*{ess\,sup}_{t>0} \frac{|\Delta^L_{t,j} f(\cdot)|}{t^s} \|_{L^p(\mathbb{R}^n)} + C'' 2^{J[n(\frac{1}{r}-1)+\frac{1}{r}-s]} \|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)}. \tag{614}$$

Due to (610), we can pick J to be sufficiently large so that the coefficient $C'' 2^{J[n(\frac{1}{r}-1)+\frac{1}{r}-s]} < \frac{1}{2}$ and shift the second term on the right side of (614) to its left side to complete the proof of Theorem 7.1.1 (iv). Now we conclude the proof of Theorem 7.1.1. □

7.3 Proof Of Theorem 7.1.2

Proof. To prove Theorem 7.1.2 (i), it suffices to prove inequality (516) for $j = 1$, and the other cases for $j = 2, \cdots, n$ can be proven in the same way. We begin with estimating the following two terms

$$
\begin{aligned}
&(\sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} \| \sum_{l\le k} |\Delta_{t,1}^L f_l| \|_{L^p(\mathbb{R}^n)}^q dt)^{\frac{1}{q}} \\
\lesssim &(\sum_{k\in\mathbb{Z}} 2^{ksq} \operatorname*{ess\,sup}_{t\in[1,2]} \| \sum_{l\le k} |\Delta_{2^{-k}t,1}^L f_l| \|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}},
\end{aligned}
\tag{615}
$$

and

$$
\begin{aligned}
&(\sum_{k\in\mathbb{Z}} 2^{k(sq+1)} \int_{2^{-k}}^{2^{1-k}} \| \sum_{l> k} |\Delta_{t,1}^L f_l| \|_{L^p(\mathbb{R}^n)}^q dt)^{\frac{1}{q}} \\
\lesssim &(\sum_{k\in\mathbb{Z}} 2^{ksq} \operatorname*{ess\,sup}_{t\in[1,2]} \| \sum_{l> k} |\Delta_{2^{-k}t,1}^L f_l| \|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}}.
\end{aligned}
\tag{616}
$$

We pick $0 < \varepsilon < \min\{s, L-s\}$. If $1 \le p \le \infty$, we use Minkowski's inequality for $\| \cdot \|_{L^p(\mathbb{R}^n)}$-norm and obtain

$$
\begin{aligned}
&\| \sum_{l\le k} |\Delta_{2^{-k}t,1}^L f_l| \|_{L^p(\mathbb{R}^n)}^q \\
\lesssim &(\sum_{l\le k} 2^{l\varepsilon} \cdot 2^{-l\varepsilon} \| \Delta_{2^{-k}t,1}^L f_l \|_{L^p(\mathbb{R}^n)})^q \\
\lesssim &(\sum_{l\le k} 2^{l\varepsilon})^q \operatorname*{ess\,sup}_{j\le k} 2^{-jq\varepsilon} \| \Delta_{2^{-k}t,1}^L f_j \|_{L^p(\mathbb{R}^n)}^q \\
\lesssim &2^{kq\varepsilon} \sum_{l\le k} 2^{-lq\varepsilon} \| \Delta_{2^{-k}t,1}^L f_l \|_{L^p(\mathbb{R}^n)}^q,
\end{aligned}
\tag{617}
$$

and

$$
\begin{aligned}
&\| \sum_{l> k} |\Delta_{2^{-k}t,1}^L f_l| \|_{L^p(\mathbb{R}^n)}^q \\
\lesssim &(\sum_{l> k} 2^{-l\varepsilon} \cdot 2^{l\varepsilon} \| \Delta_{2^{-k}t,1}^L f_l \|_{L^p(\mathbb{R}^n)})^q \\
\lesssim &(\sum_{l> k} 2^{-l\varepsilon})^q \operatorname*{ess\,sup}_{j> k} 2^{jq\varepsilon} \| \Delta_{2^{-k}t,1}^L f_j \|_{L^p(\mathbb{R}^n)}^q \\
\lesssim &2^{-kq\varepsilon} \sum_{l> k} 2^{lq\varepsilon} \| \Delta_{2^{-k}t,1}^L f_l \|_{L^p(\mathbb{R}^n)}^q.
\end{aligned}
\tag{618}
$$

If $0 < p < 1$, then we have

$$\begin{aligned}
&\|\sum_{l\leq k}|\varDelta^L_{2^{-k}t,1}f_l|\|^q_{L^p(\mathbb{R}^n)}\\
\lesssim&(\sum_{l\leq k}2^{lp\varepsilon}\cdot 2^{-lp\varepsilon}\|\varDelta^L_{2^{-k}t,1}f_l\|^p_{L^p(\mathbb{R}^n)})^{\frac{q}{p}}\\
\lesssim&2^{kq\varepsilon}\sum_{l\leq k}2^{-lq\varepsilon}\|\varDelta^L_{2^{-k}t,1}f_l\|^q_{L^p(\mathbb{R}^n)},
\end{aligned}\tag{619}$$

and

$$\begin{aligned}
&\|\sum_{l>k}|\varDelta^L_{2^{-k}t,1}f_l|\|^q_{L^p(\mathbb{R}^n)}\\
\lesssim&(\sum_{l>k}2^{-lp\varepsilon}\cdot 2^{lp\varepsilon}\|\varDelta^L_{2^{-k}t,1}f_l\|^p_{L^p(\mathbb{R}^n)})^{\frac{q}{p}}\\
\lesssim&2^{-kq\varepsilon}\sum_{l>k}2^{lq\varepsilon}\|\varDelta^L_{2^{-k}t,1}f_l\|^q_{L^p(\mathbb{R}^n)}.
\end{aligned}\tag{620}$$

When $t \in [1,2]$ and $l \leq k$, we use estimate (550) with $0 < r < p$, the 1-dimensional version of Lemma 2.0.3 and the mapping property of the 1-dimensional Hardy-Littlewood maximal function to obtain

$$\|\varDelta^L_{2^{-k}t,1}f_l\|_{L^p(\mathbb{R}^n)} \lesssim 2^{(l-k)L}\|f_l\|_{L^p(\mathbb{R}^n)} \text{ for } 0 < p \leq \infty, \tag{621}$$

where the constant is independent of t, l, k. Inserting (617), (619) and (621) into (615) yields

$$(615) \lesssim (\sum_{l\in\mathbb{Z}}\sum_{k\geq l}2^{kq(s+\varepsilon-L)}\cdot 2^{lq(L-\varepsilon)}\|f_l\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{622}$$

We can also use estimate (552) and proper change of variable to obtain

$$\|\varDelta^L_{2^{-k}t,1}f_l\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{m=0}^{L}\|f_l(\cdot + 2^{-k}mte_1)\|_{L^p(\mathbb{R}^n)} \lesssim \|f_l\|_{L^p(\mathbb{R}^n)}, \tag{623}$$

where constants are independent of t, k, l, and (623) is true for all $0 < p \leq \infty$. Inserting (618), (620) and (623) into (616) yields

$$(616) \lesssim (\sum_{l\in\mathbb{Z}}\sum_{k<l}2^{kq(s-\varepsilon)}\cdot 2^{lq\varepsilon}\|f_l\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{624}$$

Combining (615), (616), (622) and (624), we have proven

$$\begin{aligned}
&(\int_0^\infty t^{-sq}\|\sum_{l\in\mathbb{Z}}|\varDelta_{t,1}^L f_l|\|_{L^p(\mathbb{R}^n)}^q\frac{dt}{t})^{\frac{1}{q}}\\
\lesssim&(\sum_{k\in\mathbb{Z}}2^{ksq}\operatorname*{ess\,sup}_{t\in[1,2]}\|\sum_{l\in\mathbb{Z}}|\varDelta_{2^{-k}t,1}^L f_l|\|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}}\\
\lesssim&\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)},
\end{aligned} \tag{625}$$

if $0<p\le\infty$, $0<q<\infty$ and $0<s<L$. The assumption $f\in\dot{B}^s_{p,q}(\mathbb{R}^n)$ implies $\sum_{l\in\mathbb{Z}}|\varDelta_{t,1}^L f_l(x)|<\infty$ for almost every $t\in(0,\infty)$ and $x\in\mathbb{R}^n$. In conjunction with (28), we have reached the conclusion that

$$\varDelta_{t,1}^L f=\sum_{l\in\mathbb{Z}}\varDelta_{t,1}^L f_l(x) \tag{626}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ for almost every $t\in(0,\infty)$ and $x\in\mathbb{R}^n$ when $0<p\le\infty$, $0<q<\infty$ and $0<s<L$. Hence we also have the estimate

$$(\int_0^\infty t^{-sq}\|\varDelta_{t,1}^L f\|_{L^p(\mathbb{R}^n)}^q\frac{dt}{t})^{\frac{1}{q}}\lesssim(\int_0^\infty t^{-sq}\|\sum_{l\in\mathbb{Z}}|\varDelta_{t,1}^L f_l|\|_{L^p(\mathbb{R}^n)}^q\frac{dt}{t})^{\frac{1}{q}}. \tag{627}$$

Inequalities (625) and (627) conclude the proof of Theorem 7.1.2 (i).

Now we prove the first and the second parts of Theorem 7.1.2 (ii). We can assume the right side of (517) is finite and the left side of (517) is positive, otherwise, inequality (517) will be trivial. In the definition of the Peetre-Fefferman-Stein maximal function, we pick the number r so that $0<r<p$. We also use the positive integer m_0 given in (403) and $\{\rho_j\}_{j=1}^n$ is the partition of unity given in (560) and (561). Then we have

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}=&(\sum_{k\in\mathbb{Z}}2^{(k-m_0)sq}\|\psi_{2^{m_0-k}}*f\|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}}\\
\lesssim&\sum_{j=1}^n(\sum_{k\in\mathbb{Z}}2^{ksq}\|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)}^q)^{\frac{1}{q}}.
\end{aligned} \tag{628}$$

By Remark 2.0.4 and Remark 2.0.5, we can obtain the following pointwise estimate

$$\begin{aligned}
&\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f](x)\\
\lesssim&\mathcal{P}_n\{\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f]\}(x)\\
\lesssim&\mathcal{P}_n\{\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f]\}(x)\\
\lesssim&\mathcal{M}_n(|\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f]|^r)(x)^{\frac{1}{r}},
\end{aligned} \tag{629}$$

for $j \in \{1, \cdots, n\}$ and $x \in \mathbb{R}^n$. Inserting (629) into (628) and invoking the mapping property of Hardy-Littlewood maximal function yield

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^{n} \|\{2^{ks}\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}. \tag{630}$$

We estimate the term with $j = 1$ and estimates for other terms with $j = 2, \cdots, n$ can be obtained in the same way. If $1 < p \leq \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$, we use (566), (568), (569), the mapping property of Hardy-Littlewood maximal function, Minkowski's inequality for $\|\cdot\|_{L^p(\mathbb{R}^n)}$-norm and Hölder's inequality for $1 \leq q < \infty$ in a sequence and then we have

$$\begin{aligned}
&\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \\
\lesssim&\|\{2^{ks}\mathcal{M}_n(\int_{[1,2]} |\Delta^L_{2^{-k}t,1} f(\cdot)|\frac{dt}{t})(\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \\
\lesssim&(\sum_{k\in\mathbb{Z}} 2^{ksq}(\int_{[1,2]} \|\Delta^L_{2^{-k}t,1} f\|_{L^p(\mathbb{R}^n)}\frac{dt}{t})^q)^{\frac{1}{q}} \qquad (631)\\
\lesssim&(\sum_{k\in\mathbb{Z}} 2^{ksq}\int_{[1,2]} \|\Delta^L_{2^{-k}t,1} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}} \\
\lesssim&(\int_0^\infty t^{-sq}\|\Delta^L_{t,1} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}. \qquad (632)
\end{aligned}$$

Inequality (632) is also true if we replace $\rho_1, \Delta^L_{2^{-k}t,1}, \Delta^L_{t,1}$ by $\rho_j, \Delta^L_{2^{-k}t,j}, \Delta^L_{t,j}$ respectively for $j = 2, \cdots, n$. Inserting these inequalities into (630) proves inequality (517) when $1 < p \leq \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$. If $1 < p \leq \infty$, $0 < q < 1$ and $0 < s < L$, we can still obtain estimate (631). We continue from there and estimate each term in the summation of (631) as below

$$\begin{aligned}
&2^{ksq}(\int_{[1,2]} \|\Delta^L_{2^{-k}t,1} f\|_{L^p(\mathbb{R}^n)}\frac{dt}{t})^q \\
=&(\int_{[1,2]} 2^{ksq}\|\Delta^L_{2^{-k}t,1} f\|^q_{L^p(\mathbb{R}^n)} \cdot 2^{ks(1-q)}\|\Delta^L_{2^{-k}t,1} f\|^{1-q}_{L^p(\mathbb{R}^n)}\frac{dt}{t})^q \\
\lesssim&(\int_{[1,2]} 2^{ksq}\|\Delta^L_{2^{-k}t,1} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^q \cdot 2^{ks(1-q)q} \operatorname*{ess\,sup}_{t\in[1,2]} \|\Delta^L_{2^{-k}t,1} f\|^{(1-q)q}_{L^p(\mathbb{R}^n)}, \qquad (633)
\end{aligned}$$

and then by using Hölder's inequality with conjugates $\frac{1}{q}$ and $\frac{1}{1-q}$, we can estimate (631) from above by the product of the following two terms,

$$\sum_{k\in\mathbb{Z}} \int_{[1,2]} 2^{ksq}\|\Delta^L_{2^{-k}t,1} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t} \lesssim \int_0^\infty t^{-sq}\|\Delta^L_{t,1} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t} \tag{634}$$

and

$$(\sum_{k\in\mathbb{Z}} 2^{ksq} \operatorname*{ess\,sup}_{t\in[1,2]} \|\Delta^L_{2^{-k}t,1} f\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}-1}. \tag{635}$$

Furthermore (635) can be estimated from above by $\|f\|^{1-q}_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ if we apply (625) and the argument for justifying the decomposition afterward. Combining these estimates together, we have shown

$$\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \lesssim \int_0^\infty t^{-sq}\|\Delta^L_{t,1} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t}\cdot\|f\|^{1-q}_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{636}$$

Inequality (636) is still true if we replace $\rho_1, \Delta^L_{t,1}$ by $\rho_j, \Delta^L_{t,j}$ respectively for $j = 2, \cdots, n$. Inserting these inequalities into (630) yields

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \int_0^\infty t^{-sq}\|\Delta^L_{t,j} f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t}\cdot\|f\|^{1-q}_{\dot{B}^s_{p,q}(\mathbb{R}^n)}, \tag{637}$$

when $1 < p \leq \infty$, $0 < q < 1$ and $0 < s < L$. Then inequality (637) indicates the desired inequality (517). Next, we come to the proof of the third part of Theorem 7.1.2 (ii). We prove (517) when $0 < p \leq 1$, $0 < q < \infty$ and $\sigma_p < s < L$, we pick $0 < r < p$ and estimate the term $\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}$. We still use the sufficiently large positive integer m_0 given in (403), the function ϕ satisfying conditions (19), (408), (409), and $J > m_0$ is a large positive integer whose value will be determined later. From (574), the mapping property of Hardy-Littlewood maximal function and Minkowski's inequality for $\|\cdot\|_{L^{p/r}(\mathbb{R}^n)}$-norm, we can deduce the following

$$\begin{aligned}
&\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)} \\
\lesssim& 2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\int_1^2 |\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(\cdot)|^r\frac{dt}{t}\|^{q/r}_{L^{p/r}(\mathbb{R}^n)})^{\frac{1}{q}} \\
\lesssim& 2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}} 2^{ksq}(\int_1^2 \|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)\|^r_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{q/r})^{\frac{1}{q}}.
\end{aligned} \tag{638}$$

If furthermore q and r satisfy $q \geq r$, then by using Hölder's inequality we have

$$(\int_1^2 \|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)\|^r_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{q/r} \lesssim \int_1^2 \|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t}. \tag{639}$$

Hence we can estimate (638) from above by

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2\|\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}. \tag{640}$$

From (423) and (29), we can infer

$$\varDelta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}}*f)(x)=\varDelta^L_{2^{-k}t,1}f(x)-\sum_{j=J+1}^{\infty}\varDelta^L_{2^{-k}t,1}f_{k+j-m_0}(x) \tag{641}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)$. Assuming the validity of decomposition, we can estimate (640) from above by the sum of the following two terms,

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2\|\varDelta^L_{2^{-k}t,1}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}\sim 2^{Jn(\frac{1}{r}-1)}(\int_0^\infty t^{-sq}\|\varDelta^L_{t,1}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}<\infty, \tag{642}$$

and

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2\|\varDelta^L_{2^{-k}t,1}(\sum_{j=J+1}^{\infty}f_{k+j-m_0})\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}. \tag{643}$$

To estimate (643), we use (623) and the condition $0<p\le 1$ and begin with the following

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\int_1^2\|\sum_{j=J+1}^{\infty}|\varDelta^L_{2^{-k}t,1}f_{k+j-m_0}|\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}} \tag{644}$$

$$\lesssim 2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\sum_{j=J+1}^{\infty}\|f_{k+j-m_0}\|^p_{L^p(\mathbb{R}^n)})^{\frac{q}{p}})^{\frac{1}{q}}. \tag{645}$$

In case $q\le p$, since $0<\sigma_p<s$, we have

$$\begin{aligned}(645)&\lesssim 2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}\sum_{j=J+1}^{\infty}\|f_{k+j-m_0}\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}}\\ &\lesssim 2^{Jn(\frac{1}{r}-1)}(\sum_{j=J+1}^{\infty}2^{(m_0-j)sq}\sum_{k\in\mathbb{Z}}2^{(k+j-m_0)sq}\|f_{k+j-m_0}\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}}\\ &\lesssim 2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.\end{aligned} \tag{646}$$

In case $p<q$, we use $0<\varepsilon<\min\{s,L-s\}$ and estimate as follows

$$\begin{aligned}
(645)=&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\sum_{j=J+1}^{\infty}2^{-jp\varepsilon}\cdot 2^{jp\varepsilon}\|f_{k+j-m_0}\|^p_{L^p(\mathbb{R}^n)})^{\frac{q}{p}})^{\frac{1}{q}}\\
\lesssim&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\sum_{j=J+1}^{\infty}2^{-jp\varepsilon})^{\frac{q}{p}}\cdot\operatorname*{ess\,sup}_{l>J}2^{lq\varepsilon}\|f_{k+l-m_0}\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}}\\
\lesssim&2^{J[n(\frac{1}{r}-1)-\varepsilon]}(\sum_{k\in\mathbb{Z}}2^{ksq}\sum_{l=J+1}^{\infty}2^{lq\varepsilon}\|f_{k+l-m_0}\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}}\\
\lesssim&2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.
\end{aligned}\tag{647}$$

Combining (644), (645), (646), (647) and the assumption $f\in\dot{B}^s_{p,q}(\mathbb{R}^n)$ altogether, we find $\sum_{j=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+j-m_0}(x)|<\infty$ for every $k\in\mathbb{Z}$ and almost every $t\in[1,2],x\in\mathbb{R}^n$. In conjunction with (641) and (642), we have proven (641) is true not only in the sense of $\mathcal{S}'(\mathbb{R}^n)$ but also for every $k\in\mathbb{Z}$ and almost every $t\in[1,2],x\in\mathbb{R}^n$ when $0<r<p\le 1$, $r\le q<\infty$ and $\sigma_p<s<L$. Furthermore estimating (640) from above by the sum of (642) and (643) is justified, moreover (643) can be estimated from above by (644) and hence by the term $2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$. Recall (638) and (640), then we have obtained

$$\begin{aligned}
&\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_nf](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}\\
\le&C_1'2^{Jn(\frac{1}{r}-1)}(\int_0^{\infty}t^{-sq}\|\Delta^L_{t,1}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}+C_1''2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.
\end{aligned}\tag{648}$$

Inequality (648) is also true if we replace $\rho_1,\Delta^L_{t,1},C_1',C_1''$ by $\rho_j,\Delta^L_{t,j},C_j',C_j''$ respectively for $j=2,\cdots,n$. Inserting these inequalities into (630) yields

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}\le C'2^{Jn(\frac{1}{r}-1)}\sum_{j=1}^{n}(\int_0^{\infty}t^{-sq}\|\Delta^L_{t,j}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{q}}+C''2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.\tag{649}$$

The condition $\sigma_p<s<L$ implies $n(\frac{1}{r}-1)-s<0$ if r is sufficiently close to p, and hence the coefficient $C''2^{J[n(\frac{1}{r}-1)-s]}$ is less than $\frac{1}{2}$ when J is a sufficiently large positive integer. Shifting the second term on the right side of (649) to its left side proves the desired inequality (517) when $0<r<p\le 1$, $r\le q<\infty$ and $\sigma_p<s<L$. If $0<q<r<p\le 1$ and $\sigma_p<s<L$, then we continue from (638). Applying (641) and assuming the validity of decomposition, then (638) can be estimated from above by the sum of the following two terms,

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\int_1^2\|\Delta^L_{2^{-k}t,1}f\|^r_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{q}{r}})^{\frac{1}{q}}\tag{650}$$

and

$$2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\int_1^2\|\Delta^L_{2^{-k}t,1}(\sum_{j=J+1}^{\infty}f_{k+j-m_0})\|^r_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{q}{r}})^{\frac{1}{q}}. \tag{651}$$

To estimate (650), we first rewrite each term in the summation of (650) as follows

$$\begin{aligned}
&2^{ksq}(\int_1^2\|\Delta^L_{2^{-k}t,1}f\|^r_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{q}{r}}\\
=&(\int_1^2 2^{ksq}\|\Delta^L_{2^{-k}t,1}f\|^q_{L^p(\mathbb{R}^n)}\cdot 2^{ks(r-q)}\|\Delta^L_{2^{-k}t,1}f\|^{r-q}_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{q}{r}}\\
\lesssim&(\int_1^2 2^{ksq}\|\Delta^L_{2^{-k}t,1}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{q}{r}}\cdot(2^{ks}\operatorname*{ess\,sup}_{t\in[1,2]}\|\Delta^L_{2^{-k}t,1}f\|_{L^p(\mathbb{R}^n)})^{(r-q)\frac{q}{r}}, \qquad (652)
\end{aligned}$$

and then we apply Hölder's inequality with conjugates $\frac{r}{q}$ and $\frac{r}{r-q}$ to obtain

$$\begin{aligned}
(650)\lesssim&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}\int_1^2 2^{ksq}\|\Delta^L_{2^{-k}t,1}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{r}}\cdot(\sum_{k\in\mathbb{Z}}2^{ksq}\operatorname*{ess\,sup}_{t\in[1,2]}\|\Delta^L_{2^{-k}t,1}f\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}-\frac{1}{r}}\\
\lesssim&2^{Jn(\frac{1}{r}-1)}(\int_0^{\infty}t^{-sq}\|\Delta^L_{t,1}f\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{1}{r}}\cdot\|f\|^{1-\frac{q}{r}}_{\dot{B}^s_{p,q}(\mathbb{R}^n)}<\infty, \qquad (653)
\end{aligned}$$

where in (653) we used estimate (625) and the argument afterward to justify the decomposition. And (625) requires $0<s<L$. To estimate (651), we use (623) and the condition $0<q<r<p\le 1$ to get

$$\|\sum_{j=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+j-m_0}|\|^r_{L^p(\mathbb{R}^n)}\lesssim\sum_{j=J+1}^{\infty}\|f_{k+j-m_0}\|^r_{L^p(\mathbb{R}^n)}, \tag{654}$$

and then we can insert (654) into the following term and obtain

$$\begin{aligned}
&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\int_1^2\|\sum_{j=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+j-m_0}|\|^r_{L^p(\mathbb{R}^n)}\frac{dt}{t})^{\frac{q}{r}})^{\frac{1}{q}} \qquad (655)\\
\lesssim&2^{Jn(\frac{1}{r}-1)}(\sum_{k\in\mathbb{Z}}2^{ksq}(\sum_{j=J+1}^{\infty}\|f_{k+j-m_0}\|^r_{L^p(\mathbb{R}^n)})^{\frac{q}{r}})^{\frac{1}{q}}\\
\lesssim&2^{Jn(\frac{1}{r}-1)}(\sum_{j=J+1}^{\infty}2^{(m_0-j)sq}\sum_{k\in\mathbb{Z}}2^{(k+j-m_0)sq}\|f_{k+j-m_0}\|^q_{L^p(\mathbb{R}^n)})^{\frac{1}{q}}\\
\lesssim&2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \qquad (656)
\end{aligned}$$

The assumption $f\in\dot{B}^s_{p,q}(\mathbb{R}^n)$ tells us $\sum_{j=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+j-m_0}(x)|<\infty$ for every $k\in\mathbb{Z}$ and almost every $t\in[1,2],x\in\mathbb{R}^n$. In conjunction with (653) and (641), we have proven (641) is true not only in the sense of $\mathcal{S}'(\mathbb{R}^n)$ but also for every $k\in\mathbb{Z}$ and almost every

$t \in [1,2], x \in \mathbb{R}^n$ when $0 < q < r < p \leq 1$ and $\sigma_p < s < L$. Furthermore estimating (638) from above by the sum of (650) and (651) is justified, moreover (651) can be estimated from above by (655) and hence by (656). Combining (638), (650), (651), (653), (655) and (656) altogether, we have obtained

$$\begin{aligned}&\|\{2^{ks}\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f](\cdot)\}_{k\in\mathbb{Z}}\|_{l^q(L^p)}\\ \leq& C_1' 2^{Jn(\frac{1}{r}-1)}(\int_0^\infty t^{-sq}\|\Delta_{t,1}^L f\|_{L^p(\mathbb{R}^n)}^q \frac{dt}{t})^{\frac{1}{r}} \cdot \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}^{1-\frac{q}{r}} + C_1'' 2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \quad (657)\end{aligned}$$

Inequality (657) is also true if we replace $\rho_1, \Delta_{t,1}^L, C_1', C_1''$ by $\rho_j, \Delta_{t,j}^L, C_j', C_j''$ respectively for $j = 2, \cdots, n$. Inserting these inequalities into (630) yields

$$\begin{aligned}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leq& C' 2^{Jn(\frac{1}{r}-1)} \sum_{j=1}^n (\int_0^\infty t^{-sq}\|\Delta_{t,j}^L f\|_{L^p(\mathbb{R}^n)}^q \frac{dt}{t})^{\frac{1}{r}} \cdot \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}^{1-\frac{q}{r}}\\ &+ C'' 2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \quad (658)\end{aligned}$$

The condition $\sigma_p < s < L$ implies $n(\frac{1}{r}-1)-s < 0$ if r is sufficiently close to p, and hence the coefficient $C'' 2^{J[n(\frac{1}{r}-1)-s]}$ is less than $\frac{1}{2}$ when J is a sufficiently large positive integer. And then we can shift the second term on the right side of (658) to its left side, divide both sides of the resulting inequality by $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}^{1-\frac{q}{r}}$ and raise the power to $\frac{r}{q}$, finally, we can reach the desired inequality (517) when $0 < q < r < p \leq 1$ and $\sigma_p < s < L$. We have finished the proof for the third part of Theorem 7.1.2 (ii).

Now we prove Theorem 7.1.2 (iii). We only need to prove inequality (518) for $j = 1$, and the other cases for $j = 2, \cdots, n$ can be proved in the same way. We pick $0 < \varepsilon < \min\{s, L-s\}$ and begin with estimating the following term

$$\operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{2^{-k}\leq t<2^{1-k}} 2^{ks}\|\sum_{l\leq k}|\Delta_{t,1}^L f_l|\|_{L^p(\mathbb{R}^n)} + \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{2^{-k}\leq t<2^{1-k}} 2^{ks}\|\sum_{l>k}|\Delta_{t,1}^L f_l|\|_{L^p(\mathbb{R}^n)}. \quad (659)$$

If $1 \leq p \leq \infty$, by using Minkowski's inequality and (621), we can estimate the first term of (659) from above by

$$\begin{aligned}&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\sum_{l\leq k} 2^{l\varepsilon}\cdot 2^{-l\varepsilon} \operatorname*{ess\,sup}_{2^{-k}\leq t<2^{1-k}} \|\Delta_{t,1}^L f_l\|_{L^p(\mathbb{R}^n)}\\ \lesssim& \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s+\varepsilon-L)} \operatorname*{ess\,sup}_{l\leq k} 2^{l(L-\varepsilon)}\|f_l\|_{L^p(\mathbb{R}^n)}\\ \lesssim& \operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k\geq l} 2^{k(s+\varepsilon-L)}\cdot 2^{l(L-\varepsilon)}\|f_l\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}, \quad (660)\end{aligned}$$

and by using Minkowski's inequality and (623), we can estimate the second term of (659) from above by

$$\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\sum_{l>k} 2^{-l\varepsilon}\cdot 2^{l\varepsilon} \operatorname*{ess\,sup}_{2^{-k}\le t<2^{1-k}} \|\Delta_{t,1}^{L} f_l\|_{L^p(\mathbb{R}^n)}\\
\lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{k(s-\varepsilon)} \operatorname*{ess\,sup}_{l>k} 2^{l\varepsilon}\|f_l\|_{L^p(\mathbb{R}^n)}\\
\lesssim&\operatorname*{ess\,sup}_{l\in\mathbb{Z}} \operatorname*{ess\,sup}_{k<l} 2^{k(s-\varepsilon)}\cdot 2^{l\varepsilon}\|f_l\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}.
\end{aligned}\tag{661}$$

If $0<p<1$, by using (621) for $l\le k$, we can estimate the first term of (659) from above by

$$\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \operatorname*{ess\,sup}_{2^{-k}\le t<2^{1-k}} (\sum_{l\le k} 2^{lp\varepsilon}\cdot 2^{-lp\varepsilon}\|\Delta_{t,1}^{L} f_l\|^p_{L^p(\mathbb{R}^n)})^{\frac{1}{p}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)},\tag{662}$$

and by using (623) for $l>k$, we can estimate the second term of (659) from above by

$$\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks} \operatorname*{ess\,sup}_{2^{-k}\le t<2^{1-k}} (\sum_{l>k} 2^{-lp\varepsilon}\cdot 2^{lp\varepsilon}\|\Delta_{t,1}^{L} f_l\|^p_{L^p(\mathbb{R}^n)})^{\frac{1}{p}} \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}.\tag{663}$$

From (659), (660), (661), (662) and (663), we have proven

$$\operatorname*{ess\,sup}_{t>0} t^{-s}\|\sum_{l\in\mathbb{Z}}|\Delta_{t,1}^{L} f_l|\|_{L^p(\mathbb{R}^n)} \lesssim (659) \lesssim \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)},\tag{664}$$

when $0<p\le\infty$, $q=\infty$ and $0<s<L$. The assumption $f\in\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ implies $\sum_{l\in\mathbb{Z}}|\Delta_{t,1}^{L} f_l(x)|<\infty$ for almost every $t>0, x\in\mathbb{R}^n$. In conjunction with (28), we have shown

$$\Delta_{t,1}^{L} f = \sum_{l\in\mathbb{Z}} \Delta_{t,1}^{L} f_l(x)\tag{665}$$

in the sense of $\mathcal{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ for almost every $t>0, x\in\mathbb{R}^n$ when $0<p\le\infty$, $q=\infty$ and $0<s<L$. Therefore we have obtained the inequality

$$\operatorname*{ess\,sup}_{t>0} t^{-s}\|\Delta_{t,1}^{L} f\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{t>0} t^{-s}\|\sum_{l\in\mathbb{Z}}|\Delta_{t,1}^{L} f_l|\|_{L^p(\mathbb{R}^n)}.\tag{666}$$

Inequalities (664) and (666) complete the proof of Theorem 7.1.2 (iii).

Finally, we come to the proof of Theorem 7.1.2 (iv). We assume the right side of (519) is finite, otherwise, inequality (519) is trivial. In the definition of the Peetre-Fefferman-Stein maximal function, we pick the number r so that $0<r<p$. We also use the positive integer

m_0 given in (403) and $\{\rho_j\}_{j=1}^n$ is the partition of unity given in (560) and (561). Then we have

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \lesssim \sum_{j=1}^n \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\|\mathcal{F}_n^{-1}[\mathcal{F}_n\psi(2^{m_0-k}\xi)\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)}$$
$$\lesssim \sum_{j=1}^n \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\|\mathcal{F}_n^{-1}[\rho_j(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)}, \tag{667}$$

where (667) is a consequence by Remark 2.0.4, Remark 2.0.5 and the mapping property of the Hardy-Littlewood maximal function. It suffices to estimate the term for $j=1$ in (667), and estimates for the other terms in (667) when $j=2,\cdots,n$ can be obtained in the same way. If $1<p\le\infty$, $q=\infty$ and $s\in\mathbb{R}$, we can infer from the calculation method displayed in (597) that

$$\|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)}$$
$$= \operatorname*{ess\,sup}_{t\in[1,2]} \|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)}$$
$$= \operatorname*{ess\,sup}_{t\in[1,2]} \|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}] * \Delta^L_{2^{-k}t,1} f\|_{L^p(\mathbb{R}^n)}. \tag{668}$$

We use (568), Minkowski's inequality for $\|\cdot\|_{L^p(\mathbb{R}^n)}$-norm, the mapping property of the Hardy-Littlewood maximal function and the following estimate

$$\|\fint_{A_{k-l}} |\Delta^L_{2^{-k}t,1} f(\cdot-y)|dy\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{M}_n(|\Delta^L_{2^{-k}t,1} f|)(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \|\Delta^L_{2^{-k}t,1} f\|_{L^p(\mathbb{R}^n)}, \tag{669}$$

where A_{k-l} denotes the annulus $\{y\in\mathbb{R}^n : 2^{l-k}\le|y|<2^{1+l-k}\}$, and then we can estimate (668) from above by

$$\operatorname*{ess\,sup}_{t\in[1,2]} (\sum_{l\le 0} 2^{ln} + \sum_{l>0} 2^{l(n-N)})\|\Delta^L_{2^{-k}t,1} f\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{t\in[1,2]} \|\Delta^L_{2^{-k}t,1} f\|_{L^p(\mathbb{R}^n)}, \tag{670}$$

if we pick $N>n$. Combining (668) and (670), we have proven

$$\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)} \lesssim \operatorname*{ess\,sup}_{t>0} t^{-s}\|\Delta^L_{t,1} f\|_{L^p(\mathbb{R}^n)} \tag{671}$$

for any $s\in\mathbb{R}$. And inequality (671) is true if we replace $\rho_1, \Delta^L_{t,1}$ by $\rho_j, \Delta^L_{t,j}$ respectively for $j=2,\cdots,n$. Inserting these inequalities into (667) yields the desired inequality (519). If $0<p\le 1$, $q=\infty$ and $\sigma_p<s<\infty$, then we use the function ϕ satisfying conditions (19), (408), (409), and $J > m_0$ is a large positive integer whose value will be determined later. Since $0 < r < p \le 1$, we use the following inequality

$$\fint_{A_{k-l}} |\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(x-y)|^r dy \lesssim \mathcal{M}_n(|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)|^r)(x), \tag{672}$$

and deduce from (571) and (572) the estimate below

$$\begin{aligned} &|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}] * [\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)(\cdot)](x)| \\ \lesssim & 2^{Jn(\frac{1}{r}-1)}\mathcal{M}_n(|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)|^r)(x)^{\frac{1}{r}}, \end{aligned} \tag{673}$$

where we let $N' > \frac{n}{r}$ in (572). From (403) we know that when $\rho_1(2^{m_0-k}\xi) \neq 0$ and $t \in [1,2]$, $|(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L| > 0$ and we can infer from the calculation method displayed in (602) that

$$\begin{aligned} &\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\ 2^{ks}\|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)} \\ =&\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\ 2^{ks}\operatorname*{ess\,sup}_{t\in[1,2]}\ \|\mathcal{F}_n^{-1}[\frac{\rho_1(2^{m_0-k}\xi)}{(e^{2\pi i\cdot 2^{-k}t\xi_1}-1)^L}] * [\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)]\|_{L^p(\mathbb{R}^n)} \\ \lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\ 2^{ks+Jn(\frac{1}{r}-1)}\operatorname*{ess\,sup}_{t\in[1,2]}\ \|\mathcal{M}_n(|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)|^r)\|^{\frac{1}{r}}_{L^{\frac{p}{r}}(\mathbb{R}^n)} \\ \lesssim&\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\ 2^{ks+Jn(\frac{1}{r}-1)}\operatorname*{ess\,sup}_{t\in[1,2]}\ \|\Delta^L_{2^{-k}t,1}(\phi_{2^{m_0-J-k}} * f)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{674}$$

Recall (641) and assume the validity of decomposition, then we can estimate (674) from above by the sum of the following two terms,

$$2^{Jn(\frac{1}{r}-1)}\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\ \operatorname*{ess\,sup}_{t\in[1,2]}\ 2^{ks}\|\Delta^L_{2^{-k}t,1}f\|_{L^p(\mathbb{R}^n)} \sim 2^{Jn(\frac{1}{r}-1)}\operatorname*{ess\,sup}_{t>0}\ t^{-s}\|\Delta^L_{t,1}f\|_{L^p(\mathbb{R}^n)} < \infty, \tag{675}$$

and

$$2^{Jn(\frac{1}{r}-1)}\operatorname*{ess\,sup}_{k\in\mathbb{Z}}\ \operatorname*{ess\,sup}_{t\in[1,2]}\ 2^{ks}\|\Delta^L_{2^{-k}t,1}(\sum_{j=J+1}^{\infty} f_{k+j-m_0})\|_{L^p(\mathbb{R}^n)}. \tag{676}$$

To estimate (676), we pick $0 < \varepsilon < s$ and use (623) to obtain

$$\begin{aligned} &\|\sum_{j=J+1}^{\infty}|\Delta^L_{2^{-k}t,1}f_{k+j-m_0}|\|_{L^p(\mathbb{R}^n)} \\ \lesssim&(\sum_{j=J+1}^{\infty}\|\Delta^L_{2^{-k}t,1}f_{k+j-m_0}\|^p_{L^p(\mathbb{R}^n)})^{\frac{1}{p}} \\ \lesssim&(\sum_{j=J+1}^{\infty}2^{-jp\varepsilon}\cdot 2^{jp\varepsilon}\|f_{k+j-m_0}\|^p_{L^p(\mathbb{R}^n)})^{\frac{1}{p}} \\ \lesssim&2^{-J\varepsilon}\operatorname*{ess\,sup}_{j>J}\ 2^{j\varepsilon}\|f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}, \end{aligned} \tag{677}$$

where the constants are independent of t. And then we can have the following estimate

$$
\begin{aligned}
&2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{t\in[1,2]} 2^{ks}\| \sum_{j=J+1}^{\infty} |\Delta^L_{2^{-k}t,1} f_{k+j-m_0}| \|_{L^p(\mathbb{R}^n)} &(678)\\
\lesssim &2^{J[n(\frac{1}{r}-1)-\varepsilon]} \operatorname*{ess\,sup}_{j>J} 2^{j(\varepsilon-s)+m_0 s} \operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{(k+j-m_0)s}\|f_{k+j-m_0}\|_{L^p(\mathbb{R}^n)}\\
\lesssim &2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. &(679)
\end{aligned}
$$

The assumption $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ implies $\sum_{j=J+1}^{\infty} |\Delta^L_{2^{-k}t,1} f_{k+j-m_0}(x)| < \infty$ for every $k \in \mathbb{Z}$ and almost every $t \in [1,2], x \in \mathbb{R}^n$. In conjunction with (29) and (675), we have shown (641) is true not only in the sense of $\mathcal{S}'(\mathbb{R}^n)$ but also for every $k \in \mathbb{Z}$ and almost every $t \in [1,2], x \in \mathbb{R}^n$ when $0 < p \leq 1$, $q = \infty$ and $\sigma_p < s < \infty$. Furthermore estimating (674) from above by the sum of (675) and (676) is justified, moreover (676) can be estimated from above by (678) and hence by (679). We have obtained the inequality

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{k\in\mathbb{Z}} 2^{ks}\|\mathcal{F}_n^{-1}[\rho_1(2^{m_0-k}\xi)\mathcal{F}_n f]\|_{L^p(\mathbb{R}^n)}\\
\leq &C_1' 2^{Jn(\frac{1}{r}-1)} \operatorname*{ess\,sup}_{t>0} t^{-s}\|\Delta^L_{t,1} f\|_{L^p(\mathbb{R}^n)} + C_1'' 2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. &(680)
\end{aligned}
$$

Inequality (680) is also true if we replace $\rho_1, \Delta^L_{t,1}, C_1', C_1''$ by $\rho_j, \Delta^L_{t,j}, C_j', C_j''$ respectively for $j = 2, \cdots, n$. Inserting these inequalities into (667) yields

$$
\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \leq C' 2^{Jn(\frac{1}{r}-1)} \sum_{j=1}^{n} \operatorname*{ess\,sup}_{t>0} t^{-s}\|\Delta^L_{t,j} f\|_{L^p(\mathbb{R}^n)} + C'' 2^{J[n(\frac{1}{r}-1)-s]}\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}. \quad (681)
$$

The assumption $\sigma_p < s < \infty$ allows $n(\frac{1}{r}-1) - s < 0$ when r is sufficiently close to p. Thus when J is a sufficiently large positive integer, the coefficient $C'' 2^{J[n(\frac{1}{r}-1)-s]}$ is less than $\frac{1}{2}$, and we can shift the second term on the right side of (681) to its left side and then the desired inequality (519) is proved. Now the proof of Theorem 7.1.2 is complete. □

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