Three Essays on Microeconomic Theory

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This dissertation consists of three essays on microeconomic theory.

In the first chapter, I model the adjustment process in Chinese graduate school admission as a two-period decentralized matching game. Four cases are considered and they are different in the information structure and whether cutoffs are flexible. I find that: all students applying to the better school first is an equilibrium for all cases; it is the unique equilibrium when cutoffs are fixed; perfect sorting is the unique equilibrium outcome when students are uncertain about their abilities. The results suggest that when students have the opportunity to apply to every school, they can proceed in the order of their preference. However, when there are more schools than periods, students need to be strategic.

In the second chapter, I analyze censorship using a cheap-talk model. In the model, there is an external binary signal which can be censored by the sender. I characterize three partition equilibria - the full-censorship equilibrium, the no-censorship equilibrium, and the censor-by-lows equilibrium. I compare the ex-ante expected payoffs and find that partial censorship is always inferior to no censorship for both agents and the welfare comparison between partial and full censorship depends on the bias. The results suggest that it is in the government's (and citizens') interest to choose full or no censorship, but not something in between.

In the third chapter, I study a two-period model with a multi-product firm and a single consumer. The consumer's type is unknown to the firm, but her choice reveals some information about it. Two privacy settings are considered. In one setting, the consumer cannot hide her purchase history whereas she can in the other setting. I characterize the firm-optimal equilibria in both settings and show that when the opt-out choice is added, the ex-ante producer surplus increases while the ex-ante consumer surplus decreases. The results suggest that sometimes privacy protection tools can harm consumers and help firms, and the strategic interaction between the two sides need to be considered by the regulator.

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Preface

First and foremost, I would like to express my sincere gratitude to my advisor, Professor Richard Van Weelden, for his invaluable advice and continuous support during my PhD study. His insightful feedback and expertise have guided me at every stage of my academic research.

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Life as a PhD student is challenging. Obstacles came from not only research and but also life. I would like to express my gratitude to all my friends and everyone who helped me in my difficult times.

Last but now least, I am indebted to my parents Lingli Hao and Zhongyong Chen for their unconditional support and endless love. I know there are someone who will always back me up.

1.0 Decentralized Matching in Chinese Graduate School Admission

Graduate school admission in China has both centralized and decentralized features. In the initial assignment process, each student applies to one school and takes a nationwide exam. The score of the exam affects students' eligibility for school-specific exams. Students unadmitted after the initial assignment proceed to the adjustment process (*tiaoji* in Chinese) where they can apply up to three schools at the same time, and schools independently make admission decisions. In this study, we model the adjustment process as a two-period decentralized matching game and analyze the equilibrium strategies and outcomes. We consider four cases that are different in which side is uncertain about student ability, and whether cutoffs are fixed for the two periods. We find that 1) all students applying to the better school first is an equilibrium for all cases; 2) it is the unique equilibrium when cutoffs are fixed; 3) perfect sorting is the unique equilibrium outcome when students are uncertain about their abilities. Some comparative statics results are provided, and we relate them to a supply-demand framework. Finally, we discuss how the equilibrium may change when the number of schools is larger than the number of periods.

1.1 Introduction

Due to thick labor market, more and more students in China apply to graduate schools - in 2023, about 4.74 million applied to graduate schools, a 3.7% increase from 2022.[1] Graduate school admission has become increasingly important for students, schools, policy makers, and the society as a whole. Therefore it is beneficial to understand the admission process and the behaviors of students and schools.

Graduate school admission starts in December every year and ends in April in the following year, and it evolves important decisions from thousands of schools and millions of students. Unlike many countries where college graduates directly apply to graduate schools,

[1]Data source: `https://www.eol.cn/e_ky/zt/report/2023/content01.html#sc_1_1`

applicants in China must take the centralized nationwide exam and school-specific exam for admission. The admission process is divided into two stages:

1. (*Initial assignment process*) Students apply to one school, and take the nationwide exam. After the exam, the Ministry of Education sets the national minimum score line for different majors in the two regions.[2] 34 of the top schools then set their school minimum score line (which are always higher than the national line). All applicants whose score is higher or equal to the national line (or school line, if it is higher) can take the school-specific exams[3]. Applicants not admitted after school-specific exams proceed to the adjustment process (*tiaoji* in Chinese). Applicants whose score is lower than the national line are rejected and cannot apply to any other schools.
2. (*Adjustment process*) Applications and admission decisions are made via an online system. In the beginning, schools post the number of vacancies and requirements online. Students can apply to at most three schools at the same time. Each application will be temporarily locked in the system for schools to review. Once an application is unlocked, if it was rejected or not reviewed, the student can withdraw it and apply to another school. After reviewing the applications, schools invite some students to take the school-specific exams. After school-specific exams, students are either admitted or rejected and among the offers received, the applicant can accept at most one.

Apparently, the whole admission process has both centralized and decentralized features. In particular, the adjustment process is decentralized with an exogenous limit on the number of applications. Students must decide which schools to apply to and in which order. Schools consider who to enroll. In this chapter, we study the adjustment process and focus on the order of application. We construct a two-period model where there are two ranked schools, and students are restricted to apply to one of them in each period.[4] Two scenarios are considered. The first one is "uncertainty on the student side". Students face uncertainty about their ability, and only receive a signal of it. Alternatively, as in Nagypal (2004), we can

[2]Provinces are grouped into Regions A and B, and the national score line is higher for schools in Region A (which is composed of more developed provinces).

[3]School-specific exams usually composed of written tests and interviews

[4]Allowing multiple applications in each period complicates the analysis. Thus we abstract away from portfolio choice problems and focus on the order of application.

interpreted it as uncertainty about their ranking among the applicant pool. Schools observe applicant ability in this scenario. In the example of Chinese graduate school admission, scores from the nationwide exam act as signals. After conducting school specific-exams, schools know every applicant's ability (or ranking). On the contrary, the second scenario is "uncertainty on the school side". When students apply to schools, schools observe only signals of applicant ability because of noises (or measurement errors) in school-specific exams. In this scenario, students know their own ability. For both scenarios, we consider the cases where schools have fixed or flexible cutoffs across the two periods.

The main goal of this paper is equilibrium analysis. What application strategies will students use? Will they apply in the order of their preference? How will schools set their cutoffs? Can the outcome in the complete information benchmark be attained in equilibrium? By analyzing the application game we have the following findings.

First, all students applying to the better school first is an equilibrium for all cases. Since there are two schools and two periods, every students have the opportunity to apply to both schools. Therefore, they have the incentive to apply to the better school first. Moreover, it is the unique equilibrium when cutoffs are fixed. If students apply to the worse school first, they miss the chance of being admitted by the better school, and the probability of being admitted by the worse school does not increase. Thus their expected utilities decrease. Second, perfect sorting is the unique equilibrium outcome when students are uncertain about their abilities. The driving force behind this result is that schools observe applicants' abilities, and they will set their cutoffs accordingly. Some comparative statics results are provided at the end of the main part, and we relate them to a supply-demand framework.

From above results, the equilibrium strategy is straightforward for students - they can apply in the order of their preferences. However, the result may not apply in a more general setting. When there are more schools than application periods, students may be strategic and apply to a "safer" schools first. We discuss this extension and give an example.

We view this study as a first step to understand the mechanism of the adjustment process (*tiaoji*) in the Chinese graduate school admission. Empirical evidences are needed to draw any conclusion about the efficiency of the process in practice. For example, interviews with students and school officials (how they say) and students' application behavior and schools' cutoffs (how they do).

1.1.1 Related Literature

Since the seminal paper by Gale and Shapley (1962), centralized matching in school choice and college admission has been extensively studied (see Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu, Pathak, and Roth (2005), Ergin and Sönmez (2006)). In centralized college admissions, there is often a centralized exam and an admission mechanism.[5] Literature in this strand analyzes present mechanisms and proposes better[6] mechanisms. On the contrary, the research on decentralized matching is more recent. In decentralized college admissions, there is no central clearing house and students send applications to multiple schools (for example, college application in the US), and schools make admission decisions independently. Literature in this strand usually uses game theoretical model or general equilibrium model for the application and admission process. This paper is closely related to Nagypal (2004) and Chade, Gregory Lewis, and Smith (2014). In both of their models, students rank colleges in the same way, one side of the market is uncertain about student abilities, and students face a portfolio choice problem. We abstract away from the portfolio choice problem and in our model, students apply to one school at a time and choose the order of application. Che and Koh (2016) study a college admission model where student preferences are uncertain and schools strategically avoid head-on competitions with others. In Hafalir et al. (2018), students with different abilities choose effort level and the college to apply to. They show that students with lower abilities prefer the decentralized admission mechanism and students with higher abilities prefer the centralized admission mechanism. In all the decentralized models mentioned above, the setting is static, while in this paper we study a two-period application game.

School choice and college admission problem in China is relatively less studied theoretically. Wei (2009) analyzes the then mechanisms of college admission and doctorate program admission in China, and proposes improvements. Yan Chen and Kesten (2017) study the

[5] Examples include the National College Entrance Examination in China and the Student Selection and Placement System in Turkey.

[6] "Better" is in terms of stability, efficiency, and strategyproofness.

transition from “sequential” to “parallel” mechanism in the centralized college admission system, and show that the “parallel” the mechanism is less manipulable as defined in Pathak and Sönmez (2013) and more stable in their own definition, and their results are supported by empirical evidences (Yan Chen, Jiang, and Kesten (2020), Ha, Kang, and Song (2020)). Most papers in this literature focus on the Chinese college admission (*gaokao*). One reason could be that *gaokao* is a canonical centralized matching problem, and there have been various reforms of the mechanism in recent years. On the contrary, the Chinese graduate school admission has both centralized and decentralized features and received much less attention from economists.

The rest of the chapter is organized as follows: Sections 1.2 and 1.3 are for model analysis. In particular, we analyze the scenario “uncertainty on the student side” in Section 1.2, and “uncertainty on the school side” in Section 1.3. In both sections, we first study the fixed cutoff setting and then the flexible cutoff setting. In Section 1.4, some comparative statics results are provided. We discuss an extension in Section 1.5 and conclude the study in Section 1.6. Omitted proofs are in the Appendix.

1.2 Model - Uncertainty on the Student Side

1.2.1 Setup and the complete information benchmark

There is a unit mass of students whose *ability* (or the ranking in the applicants pool, as in Nagypal (2004)) are distributed according to $N(0, \sigma_x^2)$. There are two schools s_1 and s_2 with capacity q_1 and q_2, respectively. All students prefer s_1 to s_2, and throughout the paper, we sometimes refer s_1 (s_2) as the better (worse) school. I normalize the utility from remaining unmatched as 0. Denote u_j as the utility student gets from being admitted to s_j, and we have $u_1 > u_2 > 0$. For either school, enrolling a student with ability x gives it a value $v(x)$, where the function $v : \mathbb{R} \to \mathbb{R}^{++}$ satisfies:

1. $v(\cdot)$ is strictly increasing: $v(x) > v(x'), \forall x > x'$;
2. $lim_{x \to -\infty} v(x) = 0$;

3. The payoff a school can get from enrolling students is bounded: $\int_{-\infty}^{\infty} v(x)g(\frac{x}{\sigma_x})dx \leq M$ for some $M > 0$, where $g(\cdot)$ is the p.d.f. of standard normal.

Denote D_j as the set of students enrolled by school s_j, its payoff is

$$\int_{D_j} v(x)f(x)dx$$

where $f(\cdot)$ is the p.d.f. of $N(0, \sigma_x^2)$.

We can see that school preferences have two features: (a) they prefer students with higher abilities; (b) they prefer enrolling a student to keeping the empty seat. Payoff functions for students and schools remain the same for the following sections. Throughout the paper, we assume there is no application cost, but students can only send one application in each period. Furthermore, students and schools only care about their final match, so discount factor $\delta = 1$ for both sides. To avoid trivialization, we assume $q_1 + q_2 < 1$.

Since it is a continuum economy, technically, q_1 and q_2 are the measure of seats, instead of the "number of seats". However, to ease exposition, we use the two words interchangeably.

Remark 1.2.1 (Measure 0)**.** *Each student is of measure zero. Therefore, the cutoff or the remaining capacity is not affected by a single student's application behavior.*

When information is complete, every student's ability is observed by herself and the schools she applies to. It is obvious to see that there is a unique outcome: the "top-q_1" students are admitted by s_1; the "top-q_2" of the *remaining* students are admitted by s_2. Students perfectly sort in this scenario. We denote the cutoffs in s_1 and s_2 as c_1^* and c_2^*, respectively. Then they satisfy:

$$\int_{c_1^*}^{\infty} f(x)dx = q_1,$$

$$\int_{c_2^*}^{c_1^*} f(x)dx = q_2.$$

That is,

$$1 - G(\frac{c_1^*}{\sigma_x}) = q_1,$$

$$G(\frac{c_1^*}{\sigma_x}) - G(\frac{c_2^*}{\sigma_x}) = q_2,$$

where $G(\cdot)$ is the CDF of the standard normal distribution. Therefore, $c_1^* = \sigma_x G^{-1}(1-q_1)$ and $c_2^* = \sigma_x G^{-1}(1-q_1-q_2)$. We will use c_1^* and c_2^* later in the chapter.

1.2.2 Fixed cutoffs across periods

In this section, students cannot observe their own ability and only receive a signal about their ability prior to application. For student i, the signal she received is $\hat{x}_i = x_i + \epsilon$ where the noise term $\epsilon \sim N(0, \sigma_\epsilon^2)$. This distribution is common knowledge. When a student applies to a school, the school can observe her ability. As in Nagypal (2004), the interpretation is, students do not know their absolute ranking in the applicants pool, but after reviewing applicants' exam scores and conducting interviews, each school is able to rank its applicants. The timeline of this two-period admission game (denoted as G_{fixed}) is:

1. At the beginning of the game, each student observes a private signal $\hat{x}$ about own ability.
2. In period 1, after receiving the signal, each student applies to one school. After receiving applications, schools set their cutoff abilities c_j ($j \in \{1, 2\}$). School j admits applicants whose abilities are above or equal to c_j.[7] At the end of period 1, admitted students exit the game and schools announce the number of remaining seats and the cutoff c_j ($j \in \{1, 2\}$).
3. In period 2, unmatched students apply to the other school if that school still has seats left. Applicants whose abilities are above or equal to c_j ($j \in \{1, 2\}$) are admitted. After period 2, the game ends.

In this section, student's strategy is a function $S : \mathbb{R} \to \{s_1, s_2\}$, and it specifies which school to apply to first given the signal. For this game, school j's strategy is the cutoff it sets, c_j. An equilibrium is defined as follows:

Definition 1.2.1. *The triple $(S^*(\hat{x}), c_1^*, c_2^*)$ is an equilibrium in G_{fixed} if*

(a) *Given (c_1^*, c_2^*), $S^*(\hat{x})$ maximizes student's payoff, $\forall \hat{x}$;*

(b) *Given $(S^*(\cdot), c_{3-j}^*)$, c_j^* maximizes s_j's payoff ($j \in \{1, 2\}$) subject to capacity constraint.*

[7] Therefore schools need to consider their capacity constraint when setting cutoffs.

Before diving into the equilibrium analysis, let us first look at how a student's belief updates.

Belief updating after receiving the signal

The signal is $\hat{x} = x + \epsilon$ where $x \sim N(0, \sigma_x^2)$ and $\epsilon \sim N(0, \sigma_\epsilon^2)$, so we have densities $f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} exp\{-\frac{x^2}{2\sigma_x^2}\}$ and $f(\hat{x}|x) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} exp\{-\frac{(\hat{x}-x)^2}{2\sigma_\epsilon^2}\}$.[8] The posterior of x conditional on $\hat{x}$ is:

$$
\begin{aligned}
f(x|\hat{x}) =& \frac{f(\hat{x}|x)f(x)}{f(\hat{x})} \\
\propto& f(\hat{x}|x)f(x) \\
=& \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_\epsilon^2}} exp\{-\frac{x^2}{2\sigma_x^2} - \frac{(\hat{x}-x)^2}{2\sigma_\epsilon^2}\} \\
=& \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_\epsilon^2}} exp\{-\frac{x^2 - \frac{2\sigma_X^2}{\sigma_x^2+\sigma_\epsilon^2}\hat{x}x + \frac{\sigma_x^2\hat{x}^2}{\sigma_x^2+\sigma_\epsilon^2}}{\frac{2\sigma_x^2\sigma_\epsilon^2}{\sigma_x^2+\sigma_\epsilon^2}}\} \\
\propto& \frac{1}{\sqrt{2\pi\frac{\sigma_x^2\sigma_\epsilon^2}{\sigma_x^2+\sigma_\epsilon^2}}} exp\{-\frac{(x - \frac{\sigma_x^2}{\sigma_x^2+\sigma_\epsilon^2}\hat{x})^2}{2\frac{\sigma_x^2\sigma_\epsilon^2}{\sigma_x^2+\sigma_\epsilon^2}}\}
\end{aligned}
$$

Since the density integrates to 1 for $x \in (-\infty, \infty)$, we have $x|\hat{x} \sim N(\tilde{x}, \tilde{\sigma}^2)$ where $\tilde{x} = \frac{\sigma_x^2}{\sigma_x^2+\sigma_\epsilon^2}\hat{x}$ and $\tilde{\sigma}^2 = \frac{\sigma_x^2\sigma_\epsilon^2}{\sigma_x^2+\sigma_\epsilon^2}$.

Now we check if the unique outcome in the complete information benchmark can be attained as an equilibrium outcome. Consider the following strategy profile:

$$S(\hat{x}) = s_1, \forall \hat{x} \in \mathbb{R}$$

$$c_1 = c_1^*,$$

$$c_2 = c_2^*.$$

[8] To keep notations simple, when there is no risk of confusion, we denote $f_X, f_{\hat{X}|X}$ as f henceforth.

That is, all students apply to s_1 in period 1, and s_1 fills all its seats in period 1. s_2 fills all its seats with top students who were rejected by s_1. It is clear that both schools are best responding.

Lemma 1.2.1. *For game G_{fixed}, in equilibrium, all students apply to s_1 first.*

We prove this lemma by considering different cases. In equilibrium, the cutoff at s_1 can be equal to, higher than, or lower than the cutoff at s_2. It can be verified that in all cases, every student has higher expected utility if she applies to s_1 first. The intuition is as follows. Since the cutoffs are fixed and schools know applicant's ability, students know that they will not be rejected because they apply at the "wrong time" or because the evaluation is noisy. Therefore, if a student applies to s_2 first, she forgoes the chance of getting into the preferred school s_1, and it is never optimal.

Since in equilibrium all students apply to s_1 first, s_1 will set a cutoff so that it fills all seats with top students in period 1. Expecting this, s_2 will set a cutoff to fill all its seats with the top remaining students in period 2. The unique equilibrium is the one in the complete information benchmark.

Proposition 1.2.1. *For game G_{fixed}, there exists a unique equilibrium in which all students apply to s_1 in the first period, and schools set cutoffs*

$$c_1 = \sigma_x G^{-1}(1 - q_1), c_2 = \sigma_x G^{-1}(1 - q_1 - q_2).$$

1.2.3 Flexible cutoffs across periods

In the previous analysis, the assumption seems restrictive - schools cannot change their cutoffs across periods. In the actual adjustment process, as its name suggests, schools constantly adjust their admission standards: If initially, a school receives more applications from good students than expected, it may increase their standard for later rounds, and vice versa.

Now we relax this assumption. Strategy space for students is the same as before. But school j ($j \in \{1, 2\}$) chooses c_j^1 in period 1 and c_j^2 in period 2. The timeline of the game (denoted as G_{flex}) is:

1. At the beginning of the game, each student observes a private signal $\hat{x}$ about own ability.

2. In period 1, after receiving the signal, each student applies to one school. After receiving appications, schools set their cutoff abilities c_j^1 ($j \in \{1,2\}$). School j admits applicants whose abilities are above or equal to c_j^1. At the end of period 1, admitted students exit the game and schools announce the number of remaining seats the cutoff c_j^1 ($j \in \{1,2\}$).
3. In period 2, unmatched students apply to the other school if that school still has seats left. After receiving appications, schools set their cutoff abilities c_j^2 ($j \in \{1,2\}$). Applicants whose abilities are above or equal to c_j^2 ($j \in \{1,2\}$) are admitted. After period 2, the game ends.

The equilibrium can be defined analogously.

Definition 1.2.2. *The quintuple $(S^*(\hat{x}), c_1^{1*}, c_2^{1*}, c_1^{2*}, c_2^{2*})$ is an equilibrium in G_{flex} if*

(a) *Given $(c_1^{1*}, c_2^{1*}, c_1^{2*}, c_2^{2*})$, $S^*(\hat{x})$ maximizes student's payoff, $\forall \hat{x}$;*

(b) *Given $(S^*(\cdot), c_{3-j}^{1*}, c_{3-j}^{2*})$, (c_j^{1*}, c_j^{2*}) maximizes s_j's payoff ($j \in \{1,2\}$) subject to capacity constraint.*

We can easily check that the unique equilibrium in G_{fixed} is still an equilibrium here.

Now schools can change their cutoff across periods. After period 1, given the information in the market, each school can infer who will apply to it in period 2 (if any), and can thus compute the cutoff needed in period 2 so that all its seats are filled. Since both schools prefer filling a seat to leaving it empty, in equilibrium all seats are filled, and we have the following lemma.

Lemma 1.2.2. *For game G_{flex}, in equilibrium, both schools fill up their seats after two periods.*

By Lemma 1.2.2, in equilibrium, the total measure of admitted students is $q_1 + q_2$. Then a natural question arises: are all admitted students of higher ability than those who remain unmatched? The answer is "yes". The key is that each school observes its applicant's ability and knows the other school's capacity, hence it does not need to lower the cutoffs too much to fill all its seats. We formalize the result in the following lemma.

Lemma 1.2.3. *For game G_{flex}, in equilibrium, c_1^1, c_1^2, c_2^1 and c_2^2 are all higher than or equal to $c_2^* = \sigma_x G^{-1}(1 - q_1 - q_2)$.*

By Lemma 1.2.2 and Lemma 1.2.3, the following corollary follows immediately.

Corollary 1.2.1. *For game G_{flex}, in equilibrium, all students with ability higher than or equal to $c_2^* = \sigma_x G^{-1}(1 - q_1 - q_2)$ are admitted.*

Recall that in the complete information benchmark, cutoffs are $c_1^* = \sigma_x G^{-1}(1 - q_1)$ and $c_2^* = \sigma_x G^{-1}(1 - q_1 - q_2)$. Students sort perfectly between the two schools. It also holds in this setting.

Proposition 1.2.2. *For game G_{flex}, there exists a unique equilibrium outcome in which*

(i) *the set of students with ability in $[c_1^*, \infty)$ that are admitted by s_1 has measure q_1;*

(ii) *the set of students with ability in $[c_2^*, c_1^*)$ that are admitted by s_2 has measure q_2;*

(iii) *students with ability in $(-\infty, c_2^*)$ are unadmitted.*

We prove this proposition by contradiction. If in equilibrium, a positive measure of students with higher ability end up in s_2, then there is a set of lower-ability students with the same measure who end up in s_1. But then those higher-ability students have an incentive to change their application order and get into s_1.

The driving force behind this result is that although students do not know their true abilities, they know that schools can observe them. Since both schools share the same ranking of students and there is no noisy evaluation process, it is weakly undominated for students to apply to s_1 first.

In the next section, we will consider the case where students know their abilities, but the evaluation process is noisy.

1.3 Model - Uncertainty on the School Side

In this model, students know their own abilities, but schools only receive noisy signals for each application. We first analyze the game where schools cannot change their cutoffs and then relax this assumption. The payoff functions of students and schools are the same as in the previous section. But now student i knows ability x_i before application, and each time she applies, the target school receives a *conditionally independent signal* $\hat{x}_i = x_i + \epsilon$ where $\epsilon \sim N(0, \sigma_\epsilon^2)$. Equilibrium definitions are similar to those in the previous section.

In the previous section, schools know their applicant ability. We can interpret it as the two schools receiving perfectly correlated "signals". Hence, in that case, when a student is rejected by the school with a lower cutoff, she will also be rejected by the school with a higher cutoff. Whereas in this section, since signals are conditionally independent, when a student is rejected by a school (with either a lower or higher cutoff), there is still a positive possibility that she gets into the other school.

1.3.1 Fixed cutoffs across periods

The timeline of the game $\hat{G}_{fixed}$ where schools cannot change cutoffs are:

1. At the beginning of the game, each student observes their own ability.[9]
2. In period 1, each student applies to one school, and a signal is realized. After observing their applicant's signals, schools set their *cutoff signals* c_j ($j \in \{1, 2\}$) and admit applicants whose signals are above or equal to c_j. At the end of period 1, admitted students exit the game and schools announce the number of remaining seats and the cutoff c_j ($j \in \{1, 2\}$).
3. In period 2, unmatched students apply to the other school if that school still has seats left. For each applicant, a second independent signal is realized. Schools observe signals and make admission decisions. After period 2, the game ends.

This game has a unique equilibrium.

Proposition 1.3.1. *For game $\hat{G}_{fixed}$, there exists a unique equilibrium in which all students apply to s_1 first.*

Though with fixed cutoffs, applying to the better school first remains an optimal choice for all students, the intuition is different from the previous section: compared to applying to s_2 first, by applying to s_1 first, the probability of being admitted by s_1 increases, whereas the probability of being admitted by s_2 decreases, and the change in the two probabilities

[9] Therefore student's strategy $S(\cdot) : \mathbb{R} \to \{s_1, s_2\}$ is a function of the ability to school.

are the same. However, since being admitted by s_1 gives students higher payoff ($u_1 > u_2$), applying to s_1 first is optimal.

Next, we consider the game with flexible cutoffs.

1.3.2 Flexible cutoffs across periods

We denote the game as $\hat{G}_{flex}$. The timeline of $\hat{G}_{flex}$ is the same as $\hat{G}_{fixed}$, except that now school j set cutoffs c_j^1 and c_j^2 in period 1 and 2, respectively. Using the same reasoning as in $\hat{G}_{fixed}$, we establish the following result:

Proposition 1.3.2. *For game $\hat{G}_{flex}$, there exists an equilibrium in which all students apply to s_1 first.*

This equilibrium can be constructed by letting all students apply to s_1 first, and schools set c_1^1 and c_2^2 as in Proposition 1.3.1. As we emphasized at the beginning of this section, when a student is rejected by a school (with either a lower or higher cutoff), there is still a positive possibility that she gets into the other school. So a student may be accepted or rejected by some school by "good luck" or "bad luck".

Conjecture 1.3.1. *For game $\hat{G}_{flex}$, there exists an equilibrium in which some students apply to s_2 first.*

For a student with ability x, when deciding which school to apply first, she compares the following expected utilities. If she applies to s_1 first

$$
\begin{aligned}
&Pr(x+\epsilon \geq c_1^1|x)u_1 + Pr(c_1^1 > x+\epsilon|x) \times Pr(x+\epsilon' \geq c_2^2|x)u_2 \\
=&[1-G(\frac{c_1^1-x}{\sigma_\epsilon})]u_1 + G(\frac{c_1^1-x}{\sigma_\epsilon})[1-G(\frac{c_2^2-x}{\sigma_\epsilon})]u_2
\end{aligned}
$$

If she applies to s_2 first

$$
\begin{aligned}
&Pr(x+\epsilon \geq c_2^1|x)u_2 + Pr(c_2^1 > x+\epsilon|x) \times Pr(x+\epsilon' \geq c_1^2|x)u_1 \\
=&[1-G(\frac{c_2^1-x}{\sigma_\epsilon})]u_2 + G(\frac{c_2^1-x}{\sigma_\epsilon})[1-G(\frac{c_1^2-x}{\sigma_\epsilon})]u_1
\end{aligned}
$$

where $\epsilon, \epsilon' \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$.

Our conjecture is, when c_1^1 is high enough, only students with extremely high ability will apply to s_1 first. And other cutoffs are set so that the two schools fill all the seats after period 2. Constructing such an equilibrium is difficult because when both schools receive applications in period 1, their cutoffs depend on the applications, moreover, students have different signal realizations each period.

Another point worth mentioning is whether perfect sorting arises in equilibrium.

Remark 1.3.1 (No perfect sorting). *When the evaluation process is noisy, even in the equilibrium where all students apply to s_1 first, perfect sorting does not happen. This is because schools enroll applicants based on their signals, not their true abilities.*

1.4 Comparative Statics

In this section, we provide some comparative statics results and relate them to a supply-demand framework. We focus on the equilibrium in G_{fixed} and G_{flex} in which all students apply to s_1 first.[10] In particular, we explore how cutoffs change when some of the parameters change. In this section, when we say "equilibrium cutoffs", we refer to the equilibrium cutoffs c_1 and c_2 in G_{fixed} or the equilibrium cutoffs c_1^1 and c_2^2 in G_{flex}. For simplicity, we refer to them as c_1 and c_2.

When schools observe applicants' abilities, we have closed-form solutions for equilibrium cutoffs:

$$c_1 = \sigma_x G^{-1}(1 - q_1),$$

$$c_2 = \sigma_x G^{-1}(1 - q_1 - q_2).$$

Since $G(\cdot)$ is increasing, $G^{-1}(\cdot)$ is also increasing. We have the following result:

Proposition 1.4.1. *In games G_{fixed} and G_{flex},*

[10]We do not study comparative statics for "uncertainty on school side" because equilibrium cutoffs are for signals, not for abilities. Also, equilibrium cutoffs change as signal realizations change.

(a) *When σ_x increases, equilibrium cutoffs c_1 and c_2 both increase.*

(b) *When q_1 increases, equilibrium cutoffs c_1 and c_2 both decrease.*

(c) *When q_2 increases, equilibrium cutoff c_1 remains unchanged, whereas c_2 decreases.*

For the first part of the proposition, note that when σ_x increases, the distribution of student ability is "flatter", and the proportion of students with extremely high ability is larger, resulting in higher cutoffs. For parts (b) and (c), we can interpret them in a supply-demand framework.[11] Schools supply seats and students demand seats. The cutoffs act like "prices". Thus "price" at school j (c_j) is decreasing in its supply (q_j). Seats from the two schools are substitutable, but all students value s_1's seat more. Moreover, in the equilibrium considered, all students "buy" from s_1 first. Therefore, s_1's "price" is insulated from the change in s_2's supply: when q_2 increases, c_1 remains unchanged. But s_2's "price" is affected by the supply of s_1: when q_1 increases, c_2 decreases.

Apart from the previous results, we also have the following result.

Proposition 1.4.2. *As long as $u_1 > u_2 > 0$ holds, the equilibrium cutoffs do not depend on u_1 or u_2.*

Essentially, the role of u_1 and u_2 is to tell which school students prefer. As our goal is to study the application order in equilibrium, ordinal preferences of students are sufficient.

1.5 Extension and Discussion

Throughout the paper, we study a model with two schools and a unit mass of students. There is uncertainty on one side of the matching market and school seats are limited. At first glance, with uncertainty about ability (whether it's from schools or students) and limited seats, some students may "play it safe" and apply to a less preferred school in the first period. That is, some are expected not to apply in the order of their preference. However, in most cases, all students applying to the better school first is the unique equilibrium. The coordination among students is reached for the following reason. There are two periods, so

[11] Many papers in the matching literature use a supply-demand framework. See, for example, Chade, Gregory Lewis, and Smith (2014) and Azevedo and Leshno (2016).

every student has the opportunity to apply to both schools. Schools also know that and they set cutoffs such that seats are "reserved" for students with higher ability. As a result, every student applies to the better school first.

What if there are more schools than application periods? In this case, students do not have the opportunity to apply to every school, and must decide which schools to apply to. Students who are confident in their ability (or who received a good signal) may apply to better schools and those who are unconfident (or who received a bad signal) may apply to worse schools.

A complete analysis is complex and beyond the scope of this paper. Instead, we give a simple example to illustrate the point above.

Example 1.5.1. *Assume there are 4 students i_1, i_2, i_3, i_4 and 3 schools s_1, s_2, s_3. Similarly to our main model, there are two periods. All students prefer s_1 to s_2, prefer s_2 to s_3, and prefer being admitted to remaining unmatched. Let us denote the utility of being admitted by s_j as u_j, then for all students, $u_1 > u_2 > u_3 > 0$. Each school has one seat, and they rank students in the same way by their abilities. Consider a single student i_4 who is uncertain about her ability x_4 but knows that $x_4 > x_1 > x_2 > x_3$ with probability p and $x_1 > x_2 > x_4 > x_3$ with probability $1 - p$. Assume i_1, i_2, i_3 apply to s_1, s_2, s_3 in period 1, respectively.*[12] *Then the expected utility for i_4 is*

$$pu_1 \text{ if she applies to } s_1 \text{ first, and}$$

$$u_3 \text{ if she applies to } s_3 \text{ first.}$$

Therefore i_4 will apply to s_1 first when $p \geq \frac{u_3}{u_1}$. That is, she will apply to the best school s_1 if she is positive that her ability is high or if the utility of attending s_1 is much higher than the utility of attending the worse school s_3.

[12] Here we can also treat i_1, i_2, i_3 as continuum of students who will apply to s_1, s_2, s_3 in period 1, respectively. Although for each student, her ability is uncertain, when the population of students is large (or modeled as a continuum), there is no aggregate uncertainty about other students' ability for her. Hence it is reasonable to assume that i_4 knows $x_1 > x_2 > x_3$ for this example.

1.6 Conclusion

In the study, we model the adjustment process in the Chinese graduate admission as a two-period decentralized matching game and consider different scenarios in terms of whether schools or students are uncertain about ability, and whether schools can adjust cutoffs across periods. We find that all students applying to the better school first is an equilibrium for all cases and it is the unique equilibrium when cutoffs are fixed. Moreover, perfect sorting is the unique equilibrium outcome when students are uncertain about their abilities.

The results suggest that when there are two schools and two periods, it is in students' interest to apply to their preferred school first, and they do not need to strategize on their applications.

In the model we assume students have the same preference for different schools. Another extension is to make student preferences heterogeneous. For example, if some students prefer s_2 and schools cannot learn applicant preferences throughout the game, there will be additional uncertainty for schools when they decide who to enroll and in which period.

In a two-school and two-period setup, this paper suggests that the adjustment process works efficiently because students have the opportunity to apply to every schools they want. However, as discussed in the extension, equilibrium behavior and equilibrium outcome may change where there are more schools. To make any conclusion or policy recommendation on the adjustment process in practice, empirical evidences from students and schools are needed.

2.0 Cheap Talk with Censorship

In this chapter, we analyze censorship using a cheap-talk model. In our model, besides the message by the sender, there is an external binary signal. The sender does not learn the realization of the signal but can censor it. Apart from the message by the sender, the receiver learns the signal realization if the sender does not censor it. We focus on two-interval partition equilibria and characterize three of them - the full-censorship equilibrium, the censor-by-lows equilibrium, and the no-censorship equilibrium. We compare the *ex-ante* expected payoffs and find that 1) partial censorship is always inferior to no censorship for both agents; 2) the welfare comparison between partial censorship and full censorship depends on the bias.

2.1 Introduction

In the pioneering work by Crawford and Sobel (1982) (CS henceforth), a better-informed agent (the sender) communicates with an uninformed receiver who then chooses an action that affects the payoff for both agents. A natural extension is to endow the receiver with some private information. For example, a buyer has some information about the product prior to talking to the seller, or a patient may search online about his illness before visiting a doctor. Contrary to intuition, this additional information can sometimes make the receiver worse off (see De Barreda (2010) and Lai (2014), among others).

We go one step further and ask: What would happen if the sender can block the receiver from external information? This may not fit into the buyer-seller or the patient-doctor relationships. Here we present two applications, and both are related to censorship decisions by a government. For the first application, imagine that some event (it could be good or bad) happens, and the government asks the state media to report the event in a way it wants. At the same time, the government blocks reports (good or bad) from foreign media. If it does so, citizens have no information beyond the report by the state media. For the second application, imagine that the government announces a new and controversial policy on social media. Fearing criticism from the public, and to further guide public opinion, the government account may choose to close the comment section of the post.

In the two examples described above, the government (sender) knows the state of the world: it knows whether a good or bad event happened and whether the policy is good or bad. We assume that foreign reports and comments under an official account provide some external information about the state but they do not observe the state perfectly. Thus, if the government did not censor them, citizens (receiver) will have additional information. After receiving some information about the state, citizens take some action[1] that will affect their own payoff and the payoff of the government.

In this study, we intend to answer the following questions: Will the sender (government) always censor the signal (foreign reports or comments), or does the censorship decision depend on the state? How will censorship affect the welfare of both sides?

This paper is mostly related to Ying Chen (2009). In our model, the sender, after observing the state and without knowing the signal realization, can censor the signal. If the sender decides to censor the signal, then the receiver has no private information. As in Ying Chen (2009), the signal is affiliated with the state and it has two realizations h (high) and l (low). For tractability, we assume that the probability of high realization is linear in the state. We find three two-interval partition equilibria that are different in their level of censorship. First, the two-interval partition equilibrium in CS is still an equilibrium in our model. It can be regarded as a full-censorship equilibrium since the receiver has no private information in any state. There is also a no-censorship equilibrium - no sender-type censors the signal. Finally, there is an equilibrium in which the sender censors the signal when the state is below some cutoff. That is, the sender only censors the signal when the state is bad (or low) enough. We call it censor-by-lows equilibrium.

We measure welfare with the *ex-ante* expected payoff. It can be shown that given the bias, the difference between sender welfare and receiver welfare is constant. Thus, equilibria can be Pareto-ranked. We find that the censor-by-lows equilibrium is always inferior to the

[1]Examples of such action include revolt and forming a specific public opinion. We abstract from details of the receiver's action.

no-censorship equilibrium in terms of welfare. Comparing the welfare of the no-censorship equilibrium and the CS (full-censorship) equilibrium, when the bias is large (small), full (no) censorship is better.

The results above suggest that the government should never use partial censorship, and whether full or no censorship is better depends on the difference in its and the citizens' preferences.

2.1.1 Related literature

Since CS, cheap talk games have been extensively studied in various settings and applications. Most related to our paper is the literature on cheap talk with a privately-informed receiver. Ying Chen (2009) studies a model where the decision maker (the DM or the receiver) privately observes a binary signal. She finds that in one-way communication, non-monotone equilibria may arise, and the DM can only tell whether the state is extreme or intermediate. We follow the binary signal setting in her model and assume the probability of a high signal is linear in the state. In the model of De Barreda (2010), the DM privately observes an unbiased and symmetric (around the state) signal, and both the expert and DM have symmetric (around their bliss points) preferences. She shows that for quadratic-loss preferences, DM's private information sometimes reduces the information transmitted by the expert. In some cases, both agents have lower welfare. In Lai (2014), the DM has a private threshold and knows whether the state is above or below this threshold. Lai (2014) finds that for some types of DM, private information makes them worse off. Ishida and Shimizu (2019) departs from previous papers in that the receiver also faces a higher-order uncertainty - she does not know whether her private signal reveals the true state or is just randomly drawn. They find a case where the receiver's private information makes the sender's communication more informative.

Another related strand of literature is censorship. Censorship has been studied by many political scientists and economists (see Lorentzen (2014), Shadmehr and Bernhardt (2015), and Ananyev et al. (2019), among others). In these papers (and many others in the literature), the government censors media reports or online posts by citizens, and citizens decide whether to revolt. While in those papers, the government makes censorship decision *after* learning the signal realization, the censorship decision in our model is made *before or without* learning the signal realization.[2] Our model also abstracts from details of the receiver's action and assumes that the sender can communicate directly with the receiver (i.e., sending cheap-talk messages).

In Section 2.2 we build the model and characterize the three equilibria. We make the equilibrium analysis in Section 2.3. At the end of this section, some results of boundary types are provided. Section 2.4 is for welfare comparison. We conclude the chapter and discuss some potential extensions in Section 2.5. Omitted proofs are in the Appendix.

2.2 Model

There are two players in the game, the sender (S or she) and the receiver (R or he). State θ is drawn from $\Theta = [0, 1]$ according to a uniform distribution. We denote the CDF and PDF of uniform$(0, 1)$ as F and f. We also call the state sender's type. S can send a cheap talk message m from $M = [0, 1]$. R's action x is chosen from $X = \mathbb{R}$.

In our model, R has a private signal π with two realizations $\Pi = \{h, l\}$ (high and low). For $\theta \in \Theta$, we denote $Pr(\pi = h|\theta) = s(\theta)$ where $s(\theta) = \lambda\theta + \frac{1-\lambda}{2}$. We assume $\lambda > 0$, thus for $\theta < \theta'$, $Pr(\pi = h|\theta) < Pr(\pi = h|\theta')$. That is, the signal is increasing in the state. We also assume $\lambda < 1$ so that $s(\theta) \in (0, 1)$ for all θ. Moreover, when $\theta = \frac{1}{2}$, $Pr(\pi = h|\theta) = Pr(\pi = l|\theta)$. That is, when the state is right at the middle, the probability of high and low realization is equal. S does not know the realization of the signal but can censor it. If S censors it, R does not know the realization, either.

The timing of the game is as follows:

1. Nature draws the state θ and the realization of signal π.
2. Sender observes the state and chooses the message rule $q : \Theta \to \triangle(M)$ and the censorship rule $r : \Theta \to \{0, 1\}$ where 1 means to censor and 0 otherwise.

[2] An example of the former: the government account deletes adversarial comments. An example of the latter: the government account closes the comment section from the beginning.

3. Receiver gets the message, and in the case of no-censorship, observes the realization of π.
4. Receiver updates her belief about θ and chooses an action x.

The sender and the receiver have the standard quadratic utility functions:

$$U^S(x, \theta, b) := -(x - b - \theta)^2$$

where $b > 0$ is the bias, and

$$U^R(x, \theta) := -(x - \theta)^2.$$

Given the concavity of U^R, in equilibrium, R never mixes between actions, so we focus on pure strategies for the receiver. Denote R's action as $\alpha : M \times \{h, l, \phi\} \to X$ where ϕ means the signal is censored by S.

Upon receiving the message and the private signal (if not censored), R updates his belief using Bayes Rule:

For all $\theta \in \{\theta | r(\theta) = 1\}$,

$$\mu(\theta | m, \phi) = \frac{f(\theta) q(m|\theta)}{\int_0^1 f(t) q(m|t) dt}.$$

For all $\theta \in \{\theta | r(\theta) = 0\}$,

$$\mu(\theta | m, h) = \frac{f(\theta) q(m|\theta) s(\theta)}{\int_0^1 f(t) q(m|t) s(t) dt} \text{ after observing a high signal;}$$

$$\mu(\theta | m, l) = \frac{f(\theta) q(m|\theta)(1 - s(\theta))}{\int_0^1 f(t) q(m|t)(1 - s(t)) dt} \text{ after observing a low signal.}$$

The solution concept is perfect Bayesian equilibrium (equilibrium hereafter):

Definition 2.2.1. *The strategies (q^*, r^*, α^*) constitute an equilibrium if*

1. *For $\pi \in \{h, l, \emptyset\}$ and for each m received, $\alpha^*(m, \pi)$ solves*

$$max_{\alpha(m,\pi) \in X} \int_0^1 U^R(\alpha(m, \pi), \theta) \mu(\theta | m, \pi) d\theta;$$

2. For each $\theta \in [0,1]$, if m^* is in the support of $q^*(\cdot|\theta)$, then $(m^*, r^*(\theta))$ solves

$$max_{m \in M, r}(1-r)[s(\theta)U^S(\alpha^*(m,h),\theta,b) + (1-s(\theta))U^S(\alpha^*(m,l),\theta,b)] + rU^S(\alpha^*(m,\phi),\theta,b);$$

3. The receiver's belief is consistent with (q^*, r^*, α^*).

In other words, in equilibrium, given the state, S chooses the message and the censorship decision to maximize her expected utility. Based on his updated belief about the state, R chooses an action to maximize his expected utility.

Throughout the paper, we focus on pure-strategy equilibria. For trackability, we focus on two-interval partition equilibria.

2.3 Equilibrium Analysis

In this section, we will characterize three two-interval equilibria, which differ in their level of censorship.

2.3.1 Benchmark - Two-interval partition equilibrium in CS

In CS, for $0 < b < \frac{1}{4}$, the two-interval equilibrium exists and the boundary type is $a_{CS} = \frac{1-4b}{2}$.[3] The *ex-ante* expected utilities are:

$$\text{For sender: } -\int_0^{a_{CS}} (x_1(a_{CS}) - \theta - b)^2 d\theta - \int_{a_{CS}}^1 (x_2(a_{CS}) - \theta - b)^2 d\theta$$

$$\text{For receiver: } -\int_0^{a_{CS}} (x_1(a_{CS}) - \theta)^2 d\theta - \int_{a_{CS}}^1 (x_2(a_{CS}) - \theta)^2 d\theta$$

where $x_1 : (0,1) \to \mathbb{R}$ and $x_2 : (0,1) \to \mathbb{R}$ are defined as $x_1(a) := \frac{a}{2}$ and $x_2(a) := \frac{a+1}{2}$. In other words, x_1 and x_2 are the receiver's optimal actions when he believes that S is from the first and the second interval, respectively.

[3]Note that, when $b \in [\frac{1}{12}, \frac{1}{4})$, the maximum number of intervals in equilibrium is two, and both S and R prefer the two-interval equilibrium to other equilibria ex ante.

It can be verified that the two-interval equilibrium in CS is also an equilibrium in our model. Framing this equilibrium in our context, it is the equilibrium where all sender types censor the signal, and we will refer to it as the *full-censorship equilibrium* throughout the paper.

Proposition 2.3.1 (Full censorship). *For every $\lambda \in (0,1)$, when $b < \frac{1}{4}$, there exists a two-interval partition equilibrium with boundary type a_{CS} in which*

(a) *sender types in $[0, a_{CS})$ sends m_1, and sender types in $[a_{CS}, 1]$ sends m_2 for some $m_1 \neq m_2$;*

(b) *every sender type censors the signal.*

We specify R's off-path belief as follows: when R observes any off-path deviation on message or censorship decision, he believes that $\theta \in [0, a_{CS})$. We postpone the discussion of off-path belief and equilibrium selection after all three equilibria are introduced.

Full censorship corresponds to the scenario where citizens never have access to external information sources (e.g., foreign media), or the scenario where the comment section of an online post is closed from the beginning. Real-life examples of full censorship include a country banning foreign media from broadcasting in its territory. In 2022, as a result of the conflict between Russia and Ukraine, some Russian media were banned by EU countries.[4]

2.3.2 No-censorship equilibrium

To evaluate the effect of censorship, it is natural to first consider the scenario where no sender-type censors. We define four functions from $(0,1)$ to $\mathbb{R}$ as

$$x_{1h}(a) := \frac{\int_0^a s(\theta)\theta d\theta}{\int_0^a s(\theta) d\theta},$$

$$x_{1l}(a) := \frac{\int_0^a (1-s(\theta))\theta d\theta}{\int_0^a (1-s(\theta)) d\theta},$$

$$x_{2h}(a) := \frac{\int_a^1 s(\theta)\theta d\theta}{\int_a^1 s(\theta) d\theta},$$

[4]Source: https://www.consilium.europa.eu/en/press/press-releases/2022/03/02/eu-imposes-sanctions-on-state-owned-outlets-rt-russia-today-and-sputnik-s-broadcasting-in-the-eu/

$$x_{2l}(a) := \frac{\int_a^1 (1-s(\theta))\theta d\theta}{\int_a^1 (1-s(\theta))d\theta}.$$

Similar to $x_1(a)$ and $x_2(a)$, the four functions represent R's optimal actions: $x_{1h}(a)$ ($x_{1l}(a)$) is R's optimal action when he observes the high (low) signal and believes the state is in $[0, a)$. $x_{2h}(a)$ ($x_{2l}(a)$) is R's optimal action when he observes the high (low) signal and believes the state is in $[a, 1]$.

Given an interval, we expect R's optimal action to be higher when he observes the high signal than that of the low signal, and the optimal action without any signal is in between. Moreover, $x_1(a)$ ($x_2(a)$) is a convex combination of $x_{1l}(a)$ and $x_{1h}(a)$ ($x_{2l}(a)$ and $x_{2h}(a)$). The following lemma formalizes the results.

Lemma 2.3.1. $\forall \lambda \in (0, 1)$, $\forall a \in (0, 1)$, *the following hold:*

(a) $[\int_0^a (1-s(\theta))d\theta]x_{1l}(a) + [\int_0^a s(\theta)d\theta]x_{1h}(a) = [\int_0^a 1d\theta]x_1(a)$;

(b) $x_{1l}(a) < x_1(a) < x_{1h}(a)$;

(c) $[\int_a^1 (1-s(\theta))d\theta]x_{2l}(a) + [\int_a^1 s(\theta)d\theta]x_{2h}(a) = [\int_a^1 1d\theta]x_2(a)$;

(d) $x_{2l}(a) < x_2(a) < x_{2h}(a)$.

In the proposed two-interval partition equilibrium in which no sender type censors, the boundary type $a \in (0, 1)$ satisfies the following arbitrage condition (A):

$$-s(a)(x_{1h}(a) - a - b)^2 - (1-s(a))(x_{1l}(a) - a - b)^2 =$$

$$-s(a)(x_{2h}(a) - a - b)^2 - (1-s(a))(x_{2l}(a) - a - b)^2.$$

This condition means that in equilibrium, the boundary sender type is indifferent between sending the message in the first and the second interval.

Condition (A) can be rewritten as

$$(\lambda a + \frac{1-\lambda}{2})(x_{1h}(a) + x_{2h}(a) - 2a - 2b)(x_{2h}(a) - x_{1h}(a))$$

$$+(-\lambda a + \frac{1+\lambda}{2})(x_{1l}(a) + x_{2l}(a) - 2a - 2b)(x_{2l}(a) - x_{1l}(a)) = 0.$$

Define $g : (0,1) \to \mathbb{R}$ as

$$\begin{aligned} g(a) =& (\lambda a + \frac{1-\lambda}{2})(x_{1h}(a) + x_{2h}(a) - 2a - 2b)(x_{2h}(a) - x_{1h}(a)) \\ &+ (-\lambda a + \frac{1+\lambda}{2})(x_{1l}(a) + x_{2l}(a) - 2a - 2b)(x_{2l}(a) - x_{1l}(a)). \end{aligned}$$

Then $g(a)$ is the difference in payoffs for type-a sender when she follows the message rule in the first interval and the second interval. Furthermore, $g(a) = 0$ if and only if (A) holds for a. This function g will be useful in subsequent proofs.

It is clear that condition (A) is necessary for the existence of the proposed no-censorship equilibrium. Next, we prove that when the bias is small enough, such boundary type exists and is unique.

Proposition 2.3.2. *For every $\lambda \in (0,1)$, when $b < \frac{3-\lambda}{12}$, there exists a unique $a \in (0,1)$ such that the arbitrage condition (A) holds.*

The proof of Proposition 2.3.2 proceeds in two steps. We first use the Intermediate Value Theorem to prove the existence of a solution, and then prove uniqueness by contradiction.

In CS, a similar arbitrage condition is necessary and sufficient for the sender's message rule to be optimal. When the receiver has private information, in general, such a condition may not be sufficient.[5] However, with our linear signal structure, the arbitrage condition (A) is both necessary and sufficient, and therefore a *no-censorship equilibrium* exists.

Proposition 2.3.3 (No censorship)**.** *For every $\lambda \in (0,1)$, when $b < \frac{3-\lambda}{12}$, there exists a two-interval partition equilibrium with boundary type a_{NC} in which*

(a) *sender types in $[0, a_{NC})$ sends m_1, and sender types in $[a_{NC}, 1]$ sends m_2 for some $m_1 \neq m_2$;*

(b) *no sender type censors the signal.*

This equilibrium can be supported by the following off-path belief of R: whenever R observes any off-path deviation, he believes that $\theta \in [0, a_{NC})$. Then to prove Proposition 2.3.3, it is sufficient to prove that no sender type wants to mimic sender types in the other interval.

[5] See Proposition 1 in Ying Chen (2009) for an example.

This equilibrium corresponds to the following scenarios. The first scenario is that the government uses state media to report some events but also allows citizens access to foreign media. The second scenario is that an official account posts a thread, and allows all comments.

2.3.3 Equilibrium where low sender types censor the signal

Now we know that there exist a full-censorship equilibrium and a no-censorship equilibrium. One might postulate the existence of an equilibrium in which only the sender types below some threshold censor the signal. In such an equilibrium candidate, the arbitrage condition (A') is

$$-(x_1(a) - a - b)^2 =$$

$$-s(a)(x_{2h}(a) - a - b)^2 - (1 - s(a))(x_{2l}(a) - a - b)^2.$$

Condition (A') can be rewritten as

$$\begin{aligned}(\lambda a + \frac{1-\lambda}{2})(x_1(a) + x_{2h}(a) - 2a - 2b)(x_{2h}(a) - x_1(a))\\ +(-\lambda a + \frac{1+\lambda}{2})(x_1(a) + x_{2l}(a) - 2a - 2b)(x_{2l}(a) - x_1(a)) = 0.\end{aligned}$$

Similarly to g, we define $h : (0, 1) \to \mathbb{R}$ as

$$\begin{aligned}h(a) =&(\lambda a + \frac{1-\lambda}{2})(x_1(a) + x_{2h}(a) - 2a - 2b)(x_{2h}(a) - x_1(a))\\ &+ (-\lambda a + \frac{1+\lambda}{2})(x_1(a) + x_{2l}(a) - 2a - 2b)(x_{2l}(a) - x_1(a)).\end{aligned}$$

$h(a)$ is the difference in payoffs for the boundary type sender (type a) when she follows the message rule in the first interval and the second interval. And $h(a) = 0$ if and only if (A') holds for a. Same as g, $h(a) = 0$ has a unique solution.

Proposition 2.3.4. *For every $\lambda \in (0, 1)$, when $b < \frac{3-\lambda}{12}$, there exists a unique $a \in (0, 1)$ such that the arbitrage condition (A') holds.*

The proof of Proposition 2.3.4 is similar to that of Proposition 2.3.2.

Similarly, condition (A') is also sufficient for equilibrium existence:

Proposition 2.3.5 (Censorship by low types). *For every $\lambda \in (0, 1)$, when $b < \frac{3-\lambda}{12}$, there exists a two-interval partition equilibrium with boundary type a_{CL} in which*

(a) *sender types in $[0, a_{CL})$ sends m_1, sender types in $[a_{CL}, 1]$ sends m_2 for some $m_1 \neq m_2$;*

(b) *the sender censors the signal if the state is in $[0, a_{CL})$.*

We will call this equilibrium the *censor-by-lows equilibrium* henceforth. It can be supported by the following off-path belief of R: whenever R observes any off-path deviation, he believes that $\theta \in [0, a_{CL})$.

This equilibrium corresponds to the following scenarios. The first scenario is that the government uses state media to propaganda and allows citizens access to foreign media only when the state is good. The second scenario is that an official account post online, and close the comment section if it is known that the state is bad.

It is a kind of partial censorship. That is, the government's censorship decision depends on the state of the world. For example, Uganda blocks all social media platforms and messaging apps ahead of presidential election.[6] Here the government regards the potential opposition as a low state.

Before discussing boundary types in the next subsection, we would like to make some remarks on equilibria.

Remark 2.3.1 (The condition on bias). *For propositions in this and the previous subsections, we restrict $b < \frac{3-\lambda}{12}$. This is sufficient and not necessary. For example, when $\lambda = \frac{1}{2}$ and $b = 0.21$, this condition is violated, but we can find a no-censorship equilibrium and a censor-by-lows equilibrium.*

Remark 2.3.2 (Off-path beliefs and equilibrium selection). *For all three equilibria, we specify R's off-path belief to be the following: when he observes off-path deviations, he believes the sender type is in the first interval. Readers may wonder if the off-path beliefs survive standard equilibrium-selection criteria. Our answer to this question is twofold. First, the*

[6]Source: https://www.reuters.com/world/uganda-bans-social-media-ahead-presidential-election-2021-01-12/

focus of the paper is to compare the welfare of both players to CS and to see whether the right to censorship always reduces the receiver's ex-ante expected utility. Our objective is not equilibrium selection here. Second, as pointed out in the literature, standard refinement tools (e.g., Intuitive Criterion and D1) have no power in restricting equilibria in cheap talk games. The reason is that any equilibrium outcome can be supported with an equilibrium in which there is no unused message.[7] *We thus choose the NITS (no incentive to separate) condition proposed by Ying Chen, Kartik, and Sobel (2008).*[8] *It can be verified that the three equilibria (full-censorship, no-censorship, and censor-by-lows) satisfy the NITS condition in some cases. (see Appendix A.2.8 for details)*

2.3.4 About boundary types

Before presenting the welfare results, we analyze the boundary types.

Recall that in CS, the boundary type of the two-interval equilibrium is $a_{CS}(b) = \frac{1-4b}{2}$. Fix b, we can compare a_{CS} with the boundary type in the no-censorship equilibrium (a_{NC}) and in the censor-by-lows equilibrium (a_{CL}).

Proposition 2.3.6 (Boundary types)**.** *For every $0 < b < \frac{3-\lambda}{12}$, the boundary types satisfy $0 < a_{CL} < a_{NC} < a_{CS} < \frac{1}{2}$.*

In previous analysis, we have shown that given b, a_{CL} and a_{NC} are determined by functions h and g, respectively. Then $a_{CL} < a_{NC}$ can be proved by showing $(g - h)(a) > 0$ for $a \in (0, 1)$. Since a_{CS} has a simple closed form, $a_{NC} < a_{CS}$ can be proved by showing $g(a_{CS}) < 0$.

Since for every $0 < b < \frac{3-\lambda}{12}$, there exists a unique a_{NC} such that (A) holds, and there exists a unique a_{CL} such that (A') holds. With slight abuse of notation, we define boundary types in the no-censorship equilibrium and the censor-by-lows equilibrium as functions as bias: $a_{NC}(b)$ and $a_{CL}(b)$. Since $a_{CS}(b) = \frac{1-4b}{2}$, we know that as bias increases, the boundary

[7] See Farrell (1993) and Ying Chen, Kartik, and Sobel (2008) for detailed discussions. This problem also leads them to propose new refinement tools.

[8] As discussed in Ying Chen, Kartik, and Sobel (2008), the neologism-proof refinement by Farrell (1993) rules out all equilibria in the quadratic example in CS. Announcement-proofness by Matthews, Okuno-Fujiwara, and Postlewaite (1991) also eliminate all equilibria in the quadratic example while credible rationalizability by Rabin (1990) eliminate none.

type in CS moves to the left. Similar results hold for a_{NC} and a_{CL}.

Lemma 2.3.2. *For $0 < b < \frac{3-\lambda}{12}$, $a_{CS}(b)$, $a_{NC}(b)$, and $a_{CL}(b)$ are strictly decreasing in b.*

To prove this lemma, we can write b as a function of a_{NC} (or a_{CL}) from condition (A) (or (A')). Thus $a_{NC}(b)$ and $a_{CL}(b)$ must be monotone, and then ruling out the case of increasing in b is straightforward.

Why the boundary types are decreasing in the bias? Let's first consider the extreme case where $b = 0$. In this case, the interests of S and R are perfectly aligned. When censorship decisions of the two intervals are the same, the boundary type is exactly at the middle point of $[0, 1]$. That is, if $b = 0$, then $a_{CS} = a_{NC} = \frac{1}{2}$. When censorship decisions of the two intervals are different, the boundary type is close (but not at) the middle point. Hence we have $a_{CL} < \frac{1}{2}$ when $b = 0$.

As the bias increases, previous boundary types strictly prefer sending the message from the second interval, and as a result, new boundary types move to the left. The result above can be clearly seen in Figure 1.[9]

2.4 Welfare

From the previous section, we prove the existence of equilibria where all types, no type, or some types censor the signal. A natural question is, does the power of censorship benefit or hurt the sender and the receiver? The short answer is, it depends. Next, we will answer this question in more details.

For our study, we use the *ex-ante* expected utility to measure welfare. Then the welfare of the sender and the receiver are:

In the CS (full-censorship) equilibrium,

$$EU^S_{CS} = -\int_0^{a_{CS}} (x_1(a_{CS}) - \theta - b)^2 d\theta - \int_{a_{CS}}^1 (x_2(a_{CS}) - \theta - b)^2 d\theta,$$

[9] We plot the figure for $\lambda = 0.99$, figures of other λ have similar patterns.

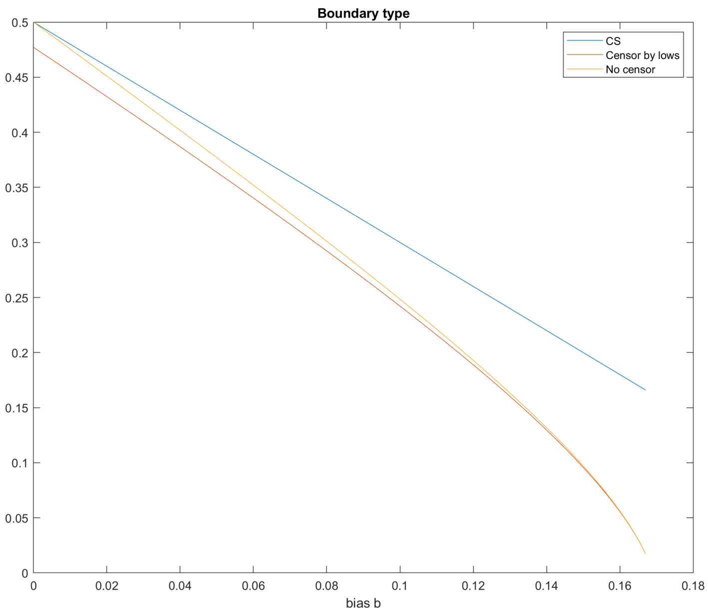

Figure 1: Boundary types in the three equilibria when $\lambda = 0.99$

$$EU^R_{CS} = -\int_0^{a_{CS}} (x_1(a_{CS}) - \theta)^2 d\theta - \int_{a_{CS}}^1 (x_2(a_{CS}) - \theta)^2 d\theta.$$

In the no-censorship equilibrium,

$$\begin{aligned} EU^S_{NC} = &-\int_0^{a_{NC}} s(\theta)(x_{1h}(a_{NC}) - \theta - b)^2 d\theta - \int_0^{a_{NC}} (1 - s(\theta))(x_{1l}(a_{NC}) - \theta - b)^2 d\theta \\ &- \int_{a_{NC}}^1 s(\theta)(x_{2h}(a_{NC}) - \theta - b)^2 d\theta - \int_{a_{NC}}^1 (1 - s(\theta))(x_{2l}(a_{NC}) - \theta - b)^2 d\theta, \end{aligned}$$

$$\begin{aligned} EU^R_{NC} = &-\int_0^{a_{NC}} s(\theta)(x_{1h}(a_{NC}) - \theta)^2 d\theta - \int_0^{a_{NC}} (1 - s(\theta))(x_{1l}(a_{NC}) - \theta)^2 d\theta \\ &- \int_{a_{NC}}^1 s(\theta)(x_{2h}(a_{NC}) - \theta)^2 d\theta - \int_{a_{NC}}^1 (1 - s(\theta))(x_{2l}(a_{NC}) - \theta)^2 d\theta. \end{aligned}$$

In the censor-by-lows equilibrium,

$$
\begin{aligned}
EU^S_{CL} =& -\int_0^{a_{CL}} (x_1(a_{CL}) - \theta - b)^2 d\theta \\
& - \int_{a_{CL}}^1 s(\theta)(x_{2h}(a_{CL}) - \theta - b)^2 d\theta - \int_{a_{CL}}^1 (1 - s(\theta))(x_{2l}(a_{CL}) - \theta - b)^2 d\theta,
\end{aligned}
$$

$$
\begin{aligned}
EU^R_{CL} =& -\int_0^{a_{CL}} (x_1(a_{CL}) - \theta)^2 d\theta \\
& - \int_{a_{CL}}^1 s(\theta)(x_{2h}(a_{CL}) - \theta)^2 d\theta - \int_{a_{CL}}^1 (1 - s(\theta))(x_{2l}(a_{CL}) - \theta)^2 d\theta.
\end{aligned}
$$

It can be verified that for all three equilibria, $EU^S = EU^R - b^2$, so equilibria can be Pareto ranked. In other words, if the sender's welfare is higher in one equilibrium than the other, then the receiver's welfare is also higher in the former equilibrium than the latter.

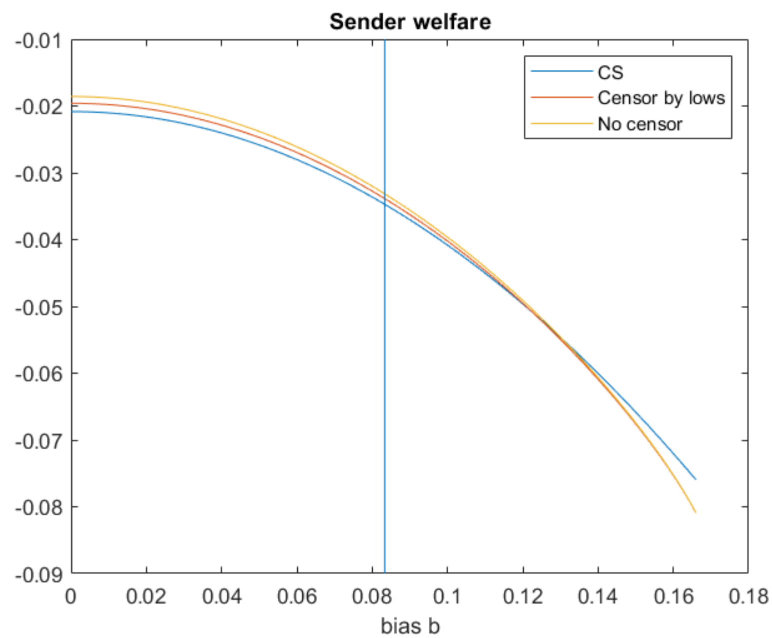

Figure 2: Sender welfare in the three equilibria when $\lambda = 0.99$

We plot the welfare of the sender and receiver in Figure 2 and Figure 3 for $\lambda = 0.99$.[10] To the right of the blue vertical line (i.e., $b \geq \frac{1}{12}$), in the original CS paper, the maximum

[10] Figures of other λ have similar patterns.

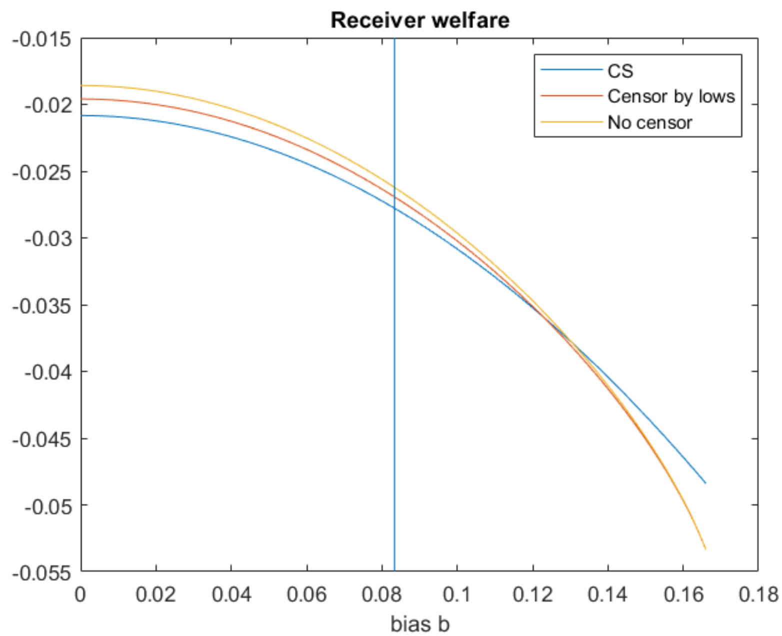

Figure 3: Receiver welfare in the three equilibria when $\lambda = 0.99$

number of intervals in equilibrium is two, and the welfare of both S and R are maximized in the two-interval equilibrium. Hence we compare welfare in that region.

Several observations can be made from the figures. First, for all of the three equilibria, sender and receiver welfare are decreasing in bias. Second, given the bias, sender and receiver welfare in the censor-by-lows equilibrium are lower than in the no-censorship equilibrium. Third, when the bias is large enough, the CS equilibrium is the best of the three in terms of welfare.

In fact, these observations can be generalized.

Proposition 2.4.1. *For the three equilibria, the ex-ante expected payoffs of the sender and receiver are strictly decreasing in bias.*

When preferences of S and R are more aligned, the boundary type is closer to $\frac{1}{2}$, and more information is transmitted. Therefore, both sides are better off.

Proposition 2.4.2. *For every $b < \frac{3-\lambda}{12}$, the censor-by-lows equilibrium is inferior to the* *no-censorship equilibrium in terms of the welfare of the sender and the receiver.*

Now let us discuss the proof and the economic intuition for this result. In the proof, we define $W_{CL} : (0,1) \to \mathbb{R}$ and $W_{NC} : (0,1) \to \mathbb{R}$ as

$$\begin{aligned} W_{CL}(a) = &-\int_0^a (x_1(a) - \theta - b)^2 d\theta \\ &- \int_a^1 s(\theta)(x_{2h}(a) - \theta - b)^2 d\theta - \int_a^1 (1 - s(\theta))(x_{2l}(a) - \theta - b)^2 d\theta, \end{aligned}$$

$$\begin{aligned} W_{NC}(a) = &-\int_0^a s(\theta)(x_{1h}(a) - \theta - b)^2 d\theta - \int_0^a (1 - s(\theta))(x_{1l}(a) - \theta - b)^2 d\theta \\ &- \int_a^1 s(\theta)(x_{2h}(a) - \theta - b)^2 d\theta - \int_a^1 (1 - s(\theta))(x_{2l}(a) - \theta - b)^2 d\theta. \end{aligned}$$

Fix b, we have $W_i(a_i(b)) = EU_i^S(b)$ for $i \in \{CL, NC\}$. That is, the function values are equal to the sender welfare at corresponding boundary types.

The difference of sender welfare in the two equilibrium can be decomposed into two parts:

$$\begin{aligned} W_{CL}(a_{CL}) - W_{NC}(a_{NC}) = &(W_{CL}(a_{CL}) - W_{NC}(a_{CL})) \\ &+ (W_{NC}(a_{CL}) - W_{NC}(a_{NC})). \end{aligned}$$

We prove Proposition 2.4.2 by showing that the two parts $(W_{CL}(a_{CL}) - W_{NC}(a_{CL}))$ and $(W_{NC}(a_{CL}) - W_{NC}(a_{NC}))$ are negative. In particular, for $a \in (0,1)$,

$$\begin{aligned} W_{CL}(a) - W_{NC}(a) = &\int_0^a s(\theta)(x_{1h}(a) - \theta - b)^2 d\theta + \int_0^a (1 - s(\theta))(x_{1l}(a) - \theta - b)^2 d\theta \\ &- \int_0^a (x_1(a) - \theta - b)^2 d\theta \\ &< 0. \end{aligned}$$

Given an interval, with additional information (i.e., the private signal), the average quadratic distance from receiver's optimal action to sender's bliss point is smaller, therefore $W_{CL}(a_{CL}) - W_{NC}(a_{CL}) < 0$.

Moreover, since $0 < a_{CL} < a_{NC} < \frac{1}{2}$, the intervals in the no-censorship equilibrium are more evenly distributed than in the censor-by-lows equilibrium and more information is transmitted in the former. As a result, $W_{NC}(a_{CL}) - W_{NC}(a_{NC}) < 0$.

For the comparison between the no-censorship equilibrium and the CS (full-censorship) equilibrium, it depends on the bias and can go either way. We give the following two examples.

Example 2.4.1 (No censorship is better)**.** *Suppose* $\lambda = 0.99$ *and* $b = 0.1$[11]*. In this case,* $a_{CS} = 0.3$*,* $a_{NC} = 0.24847$*, and* $a_{CL} = 0.24217$*. The ex-ante expected utilities are:*

$$EU^S_{CS} = -0.0408333, EU^S_{NC} = -0.0396450, EU^S_{CL} = -0.0402321.$$

$$EU^R_{CS} = -0.0308333, EU^R_{NC} = -0.0296450, EU^R_{CL} = -0.0302321.$$

In Example 2.4.1, the no-censorship equilibrium is better than the CS (full-censorship) equilibrium. In the following example, we increase the bias from 0.1 to 0.15, and the relation is reversed.

Example 2.4.2 (Full censorship is better)**.** *Suppose* $\lambda = 0.99$ *and* $b = 0.15$*. In this case,* $a_{CS} = 0.2$*,* $a_{NC} = 0.09703$*, and* $a_{CL} = 0.09573$*. The ex-ante expected utilities are:*

$$EU^S_{CS} = -0.0658333, EU^S_{NC} = -0.0674327, EU^S_{CL} = -0.0675813.$$

$$EU^R_{CS} = -0.0433333, EU^R_{NC} = -0.0449327, EU^R_{CL} = -0.0450813.$$

Though we are unable to prove a general result, we provide some intuitions below. Similar to the proof of Proposition 2.4.2, now we define $W_{CS} : (0,1) \to \mathbb{R}$ as

$$W_{CS}(a) = -\int_0^a (x_1(a) - \theta - b)^2 d\theta - \int_a^1 (x_2(a) - \theta - b)^2 d\theta$$

[11]Note that for the values of b in Example 2.4.1 and 2.4.2, the maximum number of intervals in the CS equilibrium is 2, hence the comparison is meaningful.

Fix b, we have $W_i(a_i(b)) = EU_i^S(b)$ for $i \in \{NC, CS\}$. That is, the function values are equal to the sender welfare at corresponding boundary types. Comparing W_{CS} and W_{NC}, we have the following results.

Claim 2.4.1. *For $a \in (0,1)$, $W_{CS}(a) - W_{NC}(a) < 0$.*

In other words, when the CS "equilibrium" and the no-censorship "equilibrium" have the same "boundary types"[12], the sender welfare is lower in the CS equilibrium. The driving force is similar to that of $W_{CL}(a) - W_{NC}(a) < 0$.

Claim 2.4.2. *For $a < \frac{1}{2}$, $W'_{CS}(a) > 0$.*

Since $0 < a_{NC} < a_{CS} < \frac{1}{2}$, the intervals in the CS (full-censorship) equilibrium are more evenly distributed than in the no-censorship equilibrium, more information is transmitted in the former, and $W_{CS}(a_{CS}) - W_{CS}(a_{NC}) > 0$.

Again, the difference in sender welfare can be decomposed as

$$\begin{aligned} W_{CS}(a_{CS}) - W_{NC}(a_{NC}) =& (W_{CS}(a_{NC}) - W_{NC}(a_{NC})) \\ &+ (W_{CS}(a_{CS}) - W_{CS}(a_{NC})). \end{aligned}$$

By Claim 2.4.1, $W_{CS}(a_{NC}) - W_{NC}(a_{NC}) < 0$. From Proposition 2.3.6, we have $a_{NC} < a_{CS} < \frac{1}{2}$. Then by Claim 2.4.2, we have $W_{CS}(a_{CS}) - W_{CS}(a_{NC}) > 0$. Therefore, the sign of $W_{CS}(a_{CS}) - W_{NC}(a_{NC})$ depends on the magnitude of the two parts.

From Figure 1, we can see that when b is small, a_{CS} and a_{NC} are close, and $(W_{CS}(a_{CS}) - W_{CS}(a_{NC}))$ is close to 0. As a result, the first part $(W_{CS}(a_{NC}) - W_{NC}(a_{NC}))$ dominates, and $W_{CS}(a_{CS}) - W_{NC}(a_{NC})$ is negative. As b increases, a_{CS} and a_{NC} move farther apart. After some point, the second part $(W_{CS}(a_{CS}) - W_{CS}(a_{NC}))$ dominates and $W_{CS}(a_{CS}) - W_{NC}(a_{NC})$ becomes positive.

To summarize this section, in our model with a binary signal, given the bias, the difference between sender welfare and receiver welfare is constant in all three equilibria, and both agents always have higher welfare in the no-censorship equilibrium than the censor-by-lows equilibrium. For moderate biases, the welfare in the no-censorship equilibrium is also higher

[12] Strictly speaking, they are not equilibria or boundary types.

than in the CS (full-censorship) equilibrium. On the contrary, when the bias is large enough, full censorship dominates the other two.

To interpret the results in real-life examples, censorship only in bad/low states are never the best option for the government. Whether to censor external information sources at all depends on the bias between the government and the citizens: it is better for the government (and for the citizens as well) to censor external information sources only when the bias in their preferences is large enough.

2.5 Conclusion

We study censorship using a cheap talk model. In our model, there is an external binary signal affiliated with the state. The sender does not observe the realization of the signal but can censor it. If the signal is censored, the only information for the receiver is the message transmitted by the sender. If the signal is not censored, the receiver will learn its realization. We characterize two-interval equilibria where there is no censorship, partial censorship, or full censorship. We compare sender and receiver welfare across the three equilibria and have two main findings. First, partial censorship is always inferior to no censorship for both agents. Second, the welfare comparison between no censorship and full censorship depends on the bias. We provide two examples and conjecture that when bias is large enough, full censorship is the best for both agents.

Regarding real-life applications, our results suggest that it is not in the interest of the government to base censorship decisions on the state of the world. Whether to censor external information depends on the difference in the government's and citizens' preferences.

Our study focuses on two-interval partition equilibria and hence the structure of equilibria is limited. In particular, in the censor-by-lows equilibrium, messages essentially play no role in changing the receiver's belief - he can update belief solely based on the censorship decision. Generalizing to equilibria with more than two intervals will lead to more interesting structures, though the analysis would be much more complicated.

In our model, the sender makes censorship decisions *before* learning the signal realization.

This corresponds to the type of censorship where the government bans foreign media from broadcasting in the country or reporting a certain event. Hence the government do not know how the foreign report would be. An interesting direction of future work is to explore censorship decisions *after* learning the signal realization and to compare the two cases. The latter corresponds to the type of censorship where the government notices the foreign report and removes it.

Another extension regards the timing of learning the state. In the paper, we assume the sender (government) learns the state before taking actions on message and censorship. Alternatively, we can assume the sender (government) does not know the state but has to act. This would better fit into the following scenario: Some emergent public event happens (e.g., disease outbreak), and the government does not know how severe it will be. But it needs to report the event to the citizens and decide whether coverage from foreign media is allowed.

3.0 Privacy and Personalization in a Dynamic Model

In this chapter, we study a two-period model with a monopoly firm and a consumer. The multi-product firm offers two contracts (product-price pairs) in the first period and offers one contract in the second period. The single consumer's type is unknown to the firm, and she chooses at most one contract each period. Two privacy settings are considered. In one setting, the consumer cannot hide her purchase history (i.e., cannot opt out) whereas she can in the other setting. We characterize the firm-optimal equilibria in both settings and show that when the opt-out choice is added, the *ex-ante* producer surplus increases while the *ex-ante* consumer surplus decreases. How the *interim* consumer surplus changes depends on how close the type is to the middle point of the type space. Our results suggest that sometimes privacy protection tools can harm consumers and help firms, and the strategic interaction between the two sides need to be considered by the regulator.

3.1 Introduction

With the development of information technology, people nowadays rely heavily on the internet to shop, entertain, and socialize with friends. Activities like buying a coffeemaker on Amazon, searching for movies on Netflix, and liking a post on Instagram all carry important information for internet firms. On the one hand, those consumer information can be used for personalization, e.g. recommendations and targeted ads. Such personalization may provide consumers with more relevant products and content. On the other hand, the consumer may have privacy concerns and firms sometimes use those information to practice price discrimination.[1]

Researchers in different disciplines have their own definition of privacy, and the one used by Acquisti, C. Taylor, and Wagman (2016) are widely adopted by economists: "protecting

[1] See Hannak et al. (2014), Stucke and Ezrachi (2018), and Gautier, Ittoo, and Van Cleynenbreugel (2020) for empirical evidences.

or sharing of personal data". Specifically, in the consumer and online retailer scenario, privacy means the consumer's ability to opt out of being tracked by the retailer.

In response to privacy concerns, several data privacy laws were enacted across the world in recent years. The General Data Protection Regulation (GDPR) came into effect in the European Union in 2018, and China introduced the Personal Information Protection Law (PIPL) in 2021. There is no single comprehensive data privacy law in the US, but there are sector-specific laws (e.g., Health Insurance Portability and Accountability Act (HIPAA)) and some states have or will have state privacy laws (e.g., California).[2] These data privacy laws are aimed to make consumers aware of the use of their data and give them control over it.

Some firms also provide consumers with the choice to opt out. For example, under "Interest-based ad preferences" on Amazon, the consumer can choose to "Show me interest-based ads provided by Amazon" or "Do not show me interest-based ads provided by Amazon". Consumers themselves have alternative ways to "hide" from firms including deleting cookies, using different accounts, and guest checkout. Particularly in the online retail setting, when a consumer determines whether to opt out of personalization, the trade-off is the benefit of better-matched products and the cost of potential price discrimination.

In this paper, we study this trade-off by a two-period model with a multi-product firm and a consumer. The consumer type (i.e., her favorite product) is unknown to the firm and does not change between periods. In the first period, the firm offers two contracts (product-price pairs) to the consumer, and it offers one contract in the second period. Our model features the following scenario. When an online retailer and a consumer meet for the first time, little is known by the retailer, and thus it provides a range of products hoping to get some information from the consumer's choice. After some interactions, the retailer then can target a particular product to consumers via emailed offers or recommendations.

Our model is based on Hidir and Vellodi (2021), and some of their results in the static setting are used. In particular, when the firm and the consumer interact only once, there exist the optimal pair and the optimal triple of contracts. We first study the case where

[2] See a commentary from Reuters for an introduction of the current US data privacy laws and the comparison with GDPR: `https://www.reuters.com/legal/legalindustry/us-data-privacy-laws-enter-new-era-2023-2023-01-12/`

the consumer cannot hide her purchase history (i.e., no opt-out choice is available), and characterize the firm-optimal equilibrium. Intuitively, in the firm-optimal equilibrium, the firm offers the optimal pair in the first period, and conditioning on the purchase history, the firm offers one of the contracts from the optimal pair in the second period. Then we consider the case where the consumer can choose to opt out at the beginning of the second period. We find an equilibrium that increases the firm's *ex-ante* expected surplus and prove that it is firm-optimal in this setting. Comparing the two firm-optimal equilibria, the consumer is worse off *ex ante* when the opt-out choice is available. The reason is as follows. When there is an opt-out choice and some consumer-types use it, the firm can segment the whole market into more submarkets and increase prices. For the *interim* consumer surplus, whether opt-out benefits the consumer depends on her type.

Our paper suggests that tools that seemingly help with consumer privacy protection can actually have negative effects on them. One possible reason is that the choice of opt-out itself reveals some information to the firm. Therefore, when evaluating the effect of a specific privacy protection tool, regulators need to ponder the interactions between firms and consumers in a dynamic and strategic environment.

3.1.1 Related literature

Price discrimination is a classical economic question dating back to Pigou (1920). Recently, Bergemann, Brooks, and Morris (2015) study the limit of price discrimination where a monopolist sells a single good to a continuum of consumers. In particular, they characterize the "surplus triangle" for all possible segmentations of the market. In the same vein, but for a multi-product seller setting, Haghpanah and Siegel (2022) study the achievability of first-best consumer surplus and Haghpanah and Siegel (2023) focus on Pareto improvement.

Consumer privacy has received increasing attention from researchers. In an early study of C. R. Taylor (2004), consumers interact sequentially with two firms, and depending on the privacy setting, the first firm may or may not sell consumer information to the second firm. It is shown that when consumers anticipate the sale of their information, the firm actually prefers to commit to keeping consumer information private. Acquisti and Varian

(2005) consider a two-period model where consumers have privacy protection options. They provide conditions under which conditioning prices on purchase history are profitable for the seller. In the two-period model of Conitzer, C. R. Taylor, and Wagman (2012), increasing the cost of anonymity can sometimes benefit consumers. But when the firm or a third party controls the cost of anonymity, consumers are usually harmed. In the papers discussed above, consumers only have limited ability to protect their privacy: they can only choose to hide their purchase history. On the contrary, the consumer in Ali, Greg Lewis, and Vasserman (2023) is more proactive: she can disclose hard information about her value for the product. The authors find that whether consumer's control over their data improves welfare depends on what they can disclose and whether there is a competitor firm in the market.

Recent studies also focus on multi-product sellers. Apart from Haghpanah and Siegel (2022) and Haghpanah and Siegel (2023), Ichihashi (2020) models consumer information disclosure in a Bayesian persuasion framework. The main finding is that the seller prefers to commit to not using consumer information for price discrimination and the consumer is harmed by this commitment. The paper that is closest to ours is Hidir and Vellodi (2021). They introduce the notion of incentive-compatible market segmentation (IC-MS), and characterize the buyer-optimal IC-MS. Our paper extends their model to a two-period setting and examines the effect of "opt-out" on the welfare of the firm and the consumer.

In Section 3.2, we present the model. In Section 3.3, we characterize the optimal contract, the optimal pair, and the optimal triple in the static setting. Some results in Hidir and Vellodi (2021) are used. We analyze the two-period model in Sections 3.3 and 3.4. Particularly, we construct the firm-optimal equilibrium *without* opt-out in Section 3.3 and the firm-optimal equilibrium *with* opt-out in Section 3.4. Welfare comparisons of the two equilibria are also done in Section 3.4. We conclude the chapter in Section 3.5. Omitted proofs are in the Appendix.

3.2 Model

There are two periods, t_1 and t_2. The firm is a multi-product monopoly with a range of products $v \in V \triangleq [0,1]$. We assume there is no production cost, and the firm maximizes expected revenue from the two periods. There is a single consumer whose type θ follows a uniform distribution on $[0,1]$, and we denote the type space as $\Theta \triangleq [0,1]$. The consumer's utility of buying product v at price p is

$$u(v,\theta,p) \triangleq \bar{u} - a(v-\theta)^2 - p.$$

Here $a > 0$ measures the degree of "substitutability" of products - a larger a means that the products are less substitutable. The consumer prefers products closer to her type. Assume that there is a common discount factor $\delta \in (0,1]$, and both agents maximize the discounted expected utility from the two periods. Following the notation of Hidir and Vellodi (2021), any $x \in X \triangleq \Delta(\Theta)$ is a market. In this paper, we will work with markets of the form $[c,d]$ for some $0 \leq c \leq d \leq 1$, i.e., the market where type is uniformally distributed on $[c,d]$.

A contract $(v,p) \in [0,1] \times \mathbb{R}_+$ is a product-price pair. In period 1, the firm offers two contracts (v_l,p_l) and (v_r,p_r) where $v_l \leq v_r$, and the consumer can choose at most one contract. Depending on the setting studied, at the beginning of period 2, the consumer may be able to hide her purchase history by a "opt-out" choice. Then, after observing the consumer's purchase history (or possibly the opt-out choice), the firm offers one contract (v_2,p_2).

Below is the timeline of the game with the opt-out choice, and the timeline without opt-out is similar:

1. At the beginning of the game, the consumer observes her type θ;
2. In period 1, the firm offers two product-price contracts (v_l,p_l) and (v_r,p_r);
3. The consumer accepts one or rejects all contracts;
4. At the beginning of period 2, the consumer chooses to opt in or opt out. In case she chooses to opt out, the firm cannot observe her purchase history from period 1 but can observe the opt-out choice itself;
5. Then the firm offer one contract (v_2,p_2);

6. The consumer accepts or rejects it and the game ends.

The model is constructed to capture the trade-off of better-matched products and the possibility of price discrimination. In practice, when a new customer visits an online retailer's website for the first time, the retailer knows little about her preference, thus it provides a range of products hoping to learn the customer's taste from her choice. This is modeled as the first period of the game. When the customer already has some interactions with the online retailer, some information is revealed by the customer's browsing activity and purchase history. Now the retailer can target a particular product to the customer via interest-based ads, recommendations, or personalized offers in an email. This is modeled as the second period. The customer can circumvent such personalizations in a variety of ways such as guest checkout and using different accounts. We refer to them as "opt-out" in our model. Notice that from the firm's perspective, a consumer with an opt-out choice is different from a consumer who didn't buy anything in period 1 and chose to opt in.[3]

Denote $\Theta(v,p) \triangleq \{\theta \in \Theta | u(v,\theta) \geq p\}$. In Hidir and Vellodi (2021), $\Theta(v,p)$ is the set of consumer types that would buy product v at price p. However, in our dynamic environment (and particularly in period 1), types in $\Theta(v,p)$ may not buy the product while types outside $\Theta(v,p)$ may buy the product. This is because when the consumer makes a purchase decision in the first period, she takes into account the effect on the second-period contract. Throughout the paper, we refer to $\Theta(v,p)$ as the set of types *served* or *covered* by (v,p).

In this paper, we focus on perfect Bayesian equilibria where in each period, consumer types with the same purchase (or opt-out) behavior constitute an interval.

3.3 Optimal contracts for the firm in a static setting

In this section, we consider the firm's optimal contract(s) in a static setting. Specifically, we show the optimal contract, the optimal pair, and the optimal triple of contracts for the

[3]This is different from Acquisti and Varian (2005) and Conitzer, C. R. Taylor, and Wagman (2012). In their papers, the seller cannot distinguish the consumer who didn't purchase the product and the consumer who anonymized.

firm if it interacts with the consumer for only one period. Some results from Hidir and Vellodi (2021) are also presented.

3.3.1 Optimal contract

First, we study the firm's optimal contract when facing a consumer whose type $\theta \in [\underline{\theta}, \bar{\theta}] \subseteq [0,1]$. From Proposition 1 of Hidir and Vellodi (2021), a market $[\underline{\theta},\bar{\theta}]$ is *clearing* if and only if $|\bar{\theta} - \underline{\theta}| \leq 2\sqrt{\frac{\bar{u}}{3a}}$, and clearing means that any optimal offer rule results in all types in the market accepting the contract. Moreover, for such a market, the optimal offer rule is unique.

We rephrase their result in our notations below.

Lemma 3.3.1. (From Hidir and Vellodi (2021)) *When consumer type $\theta \in [\underline{\theta}, \bar{\theta}]$ and the firm and the consumer only interact for one period, there in an optimal contract $(\frac{\underline{\theta}+\bar{\theta}}{2}, p^*)$ where*

(a) *if $\bar{u} \geq \frac{3a}{4}(\bar{\theta} - \underline{\theta})^2$, all consumer types accept the offer, and $p^* = \bar{u} - \frac{a}{4}(\bar{\theta} - \underline{\theta})^2$. The firm's expected revenue is $\bar{u}(\bar{\theta} - \underline{\theta}) - \frac{a}{4}(\bar{\theta} - \underline{\theta})^3$;*

(b) *if $\bar{u} < \frac{3a}{4}(\bar{\theta} - \underline{\theta})^2$, not all consumer types accept the offer, and $p^* = \frac{2}{3}\bar{u}$. The firm's expected revenue is $\frac{4\bar{u}}{3}\sqrt{\frac{\bar{u}}{3a}}$.*

When products are more substitutable ($\bar{u} \geq \frac{3a}{4}(\bar{\theta} - \underline{\theta})^2$), it's optimal for the firm to include all consumer types and the price is determined by leaving zero surplus for the boundary types of the market ($\bar{\theta}$ and $\underline{\theta}$). When products are less substitutable ($\bar{u} < \frac{3a}{4}(\bar{\theta} - \underline{\theta})^2$), it's optimal for the firm to only serve some of the consumer types and the monopoly price is charged.

3.3.2 Optimal pair

Now we look at the scenario in which the firm can provide two contracts. In the Online Appendix of Hidir and Vellodi (2021), they consider menus of offers. That is, the seller can offer menus of k goods in each segment. They show that the seller's optimal offer is *balanced*, i.e., the optimal offer partitions each segment into groups with equal width. In the remainder of this section, we will prove similar results using different methods and different interpretations.

We denote the two contracts as (v_l, p_l) and (v_r, p_r) with $v_l \leq v_r$. They serve $[\underline{\theta}_l, \bar{\theta}_l]$ and $[\underline{\theta}_r, \bar{\theta}_r]$, respectively. Thus (v_l, p_l) and (v_r, p_r) satisfy

$$\bar{u} - a(v_l - \underline{\theta}_l)^2 - p_l = 0,$$

$$\bar{u} - a(v_l - \bar{\theta}_l)^2 - p_l = 0,$$

$$\bar{u} - a(v_r - \underline{\theta}_r)^2 - p_r = 0,$$

$$\bar{u} - a(v_r - \bar{\theta}_r)^2 - p_r = 0.$$

First note that in any optimal pair, $[\underline{\theta}_l, \bar{\theta}_l]$ and $[\underline{\theta}_r, \bar{\theta}_r]$ are subsets of $[0, 1]$. Otherwise, the firm can relocate the products, increase prices, and have higher expected revenue. Therefore, $0 \leq \underline{\theta}_l \leq \underline{\theta}_r$ and $\bar{\theta}_l, \bar{\theta}_r \leq 1$. Second, a longer covered interval corresponds to a lower price. To solve for the optimal pair, let's first prove the following lemma.

Lemma 3.3.2. *Suppose the firm and the consumer interact for one period, then for any optimal pair of contracts (v_l, p_l) and (v_r, p_r), the intersection of the intervals covered is empty or a singleton, that is, $\bar{\theta}_l \leq \underline{\theta}_r$.*

In other words, the submarkets served by any optimal two contracts do not overlap. Otherwise, the firm can shift one product slightly away from the other, increase its price, and extracts higher revenue.

With the previous lemma, we now solve for the optimal pair. By offering a pair (v_l, p_l) and (v_r, p_r), consumer types in $[\underline{\theta}_l, \bar{\theta}_l]$ and $[\underline{\theta}_r, \bar{\theta}_r]$ buy v_l and v_r, respectively. The firm's expected revenue is

$$\begin{aligned}&[\bar{u} - a(\underline{\theta}_l - \frac{\underline{\theta}_l + \bar{\theta}_l}{2})^2](\bar{\theta}_l - \underline{\theta}_l)\\ &+[\bar{u} - a(\underline{\theta}_r - \frac{\underline{\theta}_r + \bar{\theta}_r}{2})^2](\bar{\theta}_r - \underline{\theta}_r).\end{aligned}$$

Maximizing the expected revenue is equivalent to maximizing

$$[\bar{u} - a(\underline{\theta}_l - \frac{\underline{\theta}_l + \bar{\theta}_l}{2})^2](\bar{\theta}_l - \underline{\theta}_l)$$

and

$$[\bar{u} - a(\underline{\theta}_r - \frac{\underline{\theta}_r + \bar{\theta}_r}{2})^2](\bar{\theta}_r - \underline{\theta}_r)$$

separately. And in any optimal pair, (v_l, p_l) and (v_r, p_r) will cover the same width of intervals at the same price.

Dropping the subscripts (l and r), the problem can be rewritten as

$$\max_{\bar{\theta}, \underline{\theta}} \bar{u}(\bar{\theta} - \underline{\theta}) - \frac{a(\bar{\theta} - \underline{\theta})^3}{4}.$$

We ignore the boundaries of type space for now. The expected revenue is solely determined by the width of the covered interval. Denote the width $|\bar{\theta} - \underline{\theta}|$ as w.

The problem is now

$$\max_{w} \bar{u}w - \frac{a}{4}w^3.$$

The maximizer is $w^* = \sqrt{\frac{4\bar{u}}{3a}}$. When $w^* < \frac{1}{2}$, the optimal pair only covers a subset of consumer types. When $w^* \geq \frac{1}{2}$, all consumer types are covered, and the price is determined by making boundary types indifferent between buying and not buying the designed product.

Formalizing the above result, we have the following lemma.

Lemma 3.3.3 (Optimal pair)**.** *Suppose the firm and the consumer interact for one period,*

(a) *if $\bar{u} \geq \frac{3a}{16}$, there exists a unique optimal pair: $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ and $(\frac{3}{4}, \bar{u} - \frac{a}{16})$;*

(b) *if $\bar{u} < \frac{3a}{16}$, any pair $(v_l, \frac{2}{3}\bar{u})$ and $(v_r, \frac{2}{3}\bar{u})$ $(v_l < v_r)$ is optimal for the firm if they satisfy the following conditions*

$$v_l \geq \frac{w^*}{2},$$
$$1 - v_r \geq \frac{w^*}{2},$$

$$v_r - v_l \geq w^*,$$

where $w^* = \sqrt{\frac{4\bar{u}}{3a}}$.

Similarly to the optimal contract, when a is small, the degree of "substitutability" between products is small, and for each consumer type, the difference in utility obtained from the products are small. In this case, it is in the firm's interest to serve the whole market. When a is large, the degree of "substitutability" is large, and it is better to only serve part of the market. The conditions in the second case are that the intervals covered don't overlap and they don't extend beyond the boundaries of the market (0 and 1).

3.3.3 Optimal three contracts

Though in our model, the firm cannot offer three contracts in any period, it does not mean that the firm cannot segment the markets into three parts. In later sections, we will show it is possible. For now, let us answer the following question: when the firm is able to provide three contracts, what will the optimal triple look like? For the case of optimal two contracts, we have proved that the contracts do not overlap. Moreover, whether the optimal pair of contracts cover the whole market depends on parameters $\bar{u}$ and a. We have similar results for the optimal three contracts.

We denote the triple as (v_l, p_l), (v_m, v_m) and (v_r, p_r) with $v_l \leq v_m \leq v_r$, and they cover $[\underline{\theta}_l, \bar{\theta}_l]$, $[\underline{\theta}_m, \bar{\theta}_m]$ and $[\underline{\theta}_r, \bar{\theta}_r]$, respectively. For $i \in \{l, m, r\}$,

$$\bar{u} - a(v_i - \underline{\theta}_i)^2 - p_i = 0,$$

$$\bar{u} - a(v_i - \bar{\theta}_i)^2 - p_i = 0.$$

Similarly, for any optimal triple, $[\underline{\theta}_l, \bar{\theta}_l]$, $[\underline{\theta}_m, \bar{\theta}_m]$ and $[\underline{\theta}_r, \bar{\theta}_r]$ are subsets of $[0, 1]$. Thus $0 \leq \underline{\theta}_l \leq \underline{\theta}_m \leq \underline{\theta}_r$ and $\bar{\theta}_l, \bar{\theta}_m, \bar{\theta}_r \leq 1$. We have the following result.

Lemma 3.3.4. *Suppose the firm and the consumer interact for one period, then for any optimal triple (v_l, p_l), (v_m, p_m), and (v_r, p_r), the intersection of any two of the intervals covered is empty or a singleton, that is, $\bar{\theta}_l \leq \underline{\theta}_m$ and $\bar{\theta}_m \leq \underline{\theta}_r$.*

Intuitively, the submarkets served by any optimal three contracts do not overlap. We can now solve for the optimal triple. By offering (v_l, p_l), (v_m, p_m), and (v_r, p_r), the firm's expected revenue is

$$\begin{aligned}
&[\bar{u} - a(\underline{\theta}_l - \frac{\underline{\theta}_l + \bar{\theta}_l}{2})^2](\bar{\theta}_l - \underline{\theta}_l) \\
+&[\bar{u} - a(\underline{\theta}_m - \frac{\underline{\theta}_m + \bar{\theta}_m}{2})^2](\bar{\theta}_m - \underline{\theta}_m) \\
+&[\bar{u} - a(\underline{\theta}_r - \frac{\underline{\theta}_r + \bar{\theta}_r}{2})^2](\bar{\theta}_r - \underline{\theta}_r).
\end{aligned}$$

Again, when we ignore the boundaries of type space, the problem is reduced to

$$\max_w \bar{u}w - \frac{a}{4}w^3$$

and $w^* = \sqrt{\frac{4\bar{u}}{3a}}$ is the maximizer.

We have the optimal three contracts in different cases depending on the values of $\bar{u}$ and a.

Lemma 3.3.5 (Optimal triple)**.** *Suppose the firm and the consumer interact for one period,*

(a) *if $\bar{u} \geq \frac{a}{12}$, there exists a unique optimal triple: $(\frac{1}{6}, \bar{u} - \frac{a}{36})$, $(\frac{1}{2}, \bar{u} - \frac{a}{36})$, and $(\frac{5}{6}, \bar{u} - \frac{a}{36})$;*

(b) *if $\bar{u} < \frac{a}{12}$, any triple $(v_l, \frac{2}{3}\bar{u})$, $(v_m, \frac{2}{3}\bar{u})$, and $(v_r, \frac{2}{3}\bar{u})$ ($v_l < v_m < v_r$) is optimal for the firm if they satisfy the following conditions*

$$v_l \geq \frac{w^*}{2},$$

$$1 - v_r \geq \frac{w^*}{2},$$

$$v_m - v_l \geq w^*,$$

$$v_r - v_m \geq w^*,$$

where $w^* = \sqrt{\frac{4\bar{u}}{3a}}$.

Similarly, when a is small, the optimal triple covers the entire market, and when a is large, only parts of the market are covered.

Before moving to the two-period game, we would like to highlight some differences between the current paper and Hidir and Vellodi (2021).

First, the setting in Hidir and Vellodi (2021) is static. They are interested in incentive-compatible market segmentation, and segmentations can be implemented using a cheap talk game. On the contrary, this paper studies the repeated interaction of the firm and the consumer. Specifically, we are interested in the effect of the "opt-out" option. Second, although Hidir and Vellodi (2021) also characterize the seller-optimal segmentation, their focus is on the buyer-optimal segmentation. Our paper explores the seller-optimal equilibrium with and without "opt-out" for the following reason. Currently, many privacy protection tools are provided by online retailers themselves. For example, they decide whether to allow customers to use guest checkout or to turn off personalized recommendations. We believe studying the welfare effect in different seller-optimal equilibria improves our understanding of those privacy protection tools.

3.4 Firm and consumer interact for two periods

Throughout the paper, we assume that $\bar{u} \geq \frac{3a}{4}$. Within this range, when the firm and the consumer interact only once, the optimal contract, the optimal pair, and the optimal triple all cover the whole market and they are unique. We make this assumption for clearer exposition.

In this section, we first define and characterize the firm-optimal equilibrium when there is no opt-out choice. Then we explore the welfare effect after adding the opt-out choice.

3.4.1 Firm-optimal equilibrium where there is no opt-out choice

When there is no opt-out choice, at the beginning of period 2, the firm observes the consumer's purchase history from period 1. In the first period, the firm provides two products.

Let us call them the left product (v_l) and the right product (v_r). There are three possible purchase histories for the consumer: she bought v_l, she bought v_r, or she bought nothing in the first period. Observing different purchase histories, the firm gets some information about the consumer type, and it can base the second product offer on its updated belief. One equilibrium is straightforward: in period 1, the firm offers the optimal pair; in period 2, the firm offers the same contract if the consumer chose some contract in t_1 , and offers $(\frac{1}{2}, \bar{u})$ if nothing was bought in the first period.[4] In this equilibrium, all consumer types buy some product in both periods.

Proposition 3.4.1. *In the two-period game without opt-out choice, there exists an equilibrium as follows:*

(i) *In the first period, the firm offers* $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ *and* $(\frac{3}{4}, \bar{u} - \frac{a}{16})$*;*

(ii) *In the second period, observing a purchase history of* $\frac{1}{4}$ *(or* $\frac{3}{4}$*), the firm offers* $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ *(or* $(\frac{3}{4}, \bar{u} - \frac{a}{16})$*). If the firm observes that there is no purchase in the first period, it offers* $(\frac{1}{2}, \bar{u})$*;*

(iii) *In the first period, the consumer chooses the contract which gives her highest surplus in that period as long as it is non-negative;*

(iv) *In the second period, the consumer accepts the contract as long as it gives her a non-negative surplus in that period.*

In the equilibrium described above, the firm is essentially providing the optimal pair twice. Given the constraint on the number of contracts in each period, one may wonder whether this is the best the firm can do. The answer is "yes". Before presenting this result, let us introduce the welfare criterion and formally define firm-optimal equilibrium.

Consider an equilibrium where the firm offers (v_l, p_l) and (v_r, p_r) in the first period, and offers (v_2, p_2) in the second period. For $i \in \{l, r, 2\}$, let $\hat{\Theta}(v_i, p_i)$ denote the set of types who accept (v_i, p_i) in this equilibrium. Denote vectors of products and prices offered as $\mathbf{v}$ and $\mathbf{p}$ respectively, then the *ex-ante* surplus for the firm is:

$$PS(\mathbf{v}, \mathbf{p}) = \int_{\hat{\Theta}(v_l, p_l)} p_l d\theta + \int_{\hat{\Theta}(v_r, p_r)} p_r d\theta + \delta \int_{\hat{\Theta}(v_2, p_2)} p_2 d\theta.$$

[4] Off the equilibrium path, when the firm offers product $\frac{1}{2}$ for non-buying types, it assigns probability 1 to $\theta = \frac{1}{2}$.

Definition 3.4.1 (Firm-optimal equilibrium)**.** *A Perfect Bayesian Equilibrium is firm-optimal if it gives the firm the highest ex-ante surplus among all equilibria.*

Throughout the paper, we also refer to the *ex-ante* PS as welfare for the firm.

Recall that for the equilibrium in Proposition 3.4.1, in period 1 the the firm provides the optimal pair and all types buy some products. In particular, $\hat{\Theta}(\frac{1}{4}, \bar{u} - \frac{a}{16}) = [0, \frac{1}{2}]$ and $\hat{\Theta}(\frac{3}{4}, \bar{u} - \frac{a}{16}) = [\frac{1}{2}, 1]$. In period 2, if the firm observes a purchase history of product $\frac{1}{4}$ (or $\frac{3}{4}$), it offers $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ (or $(\frac{3}{4}, \bar{u} - \frac{a}{16})$), and in both cases, the consumer will accept the contract.

Therefore, the welfare of the firm in this equilibrium is

$$\begin{aligned} PS &= \bar{u} - \frac{a}{16} + \delta(\bar{u} - \frac{a}{16}) \\ &= (1 + \delta)(\bar{u} - \frac{a}{16}). \end{aligned}$$

We now show it is the maximum welfare the firm can get in equilibrium.

Proposition 3.4.2. *When there is no opt-out choice, the equilibrium in Proposition 3.4.1 is firm-optimal.*

Intuitively, when $\bar{u} \geq \frac{3a}{4}$, the products are close substitutes (or different consumer types have similar preferences), so it is in the interest of the firm to sell to all types in period 1, instead of excluding some types. Thus offering the optimal pair twice gives the firm the highest surplus.

3.4.2 An improvement for the firm when the opt-out choice is available

Let us refer to the equilibrium in Proposition 3.4.1 as *the firm-optimal equilibrium without opt-out*. In this equilibrium, the firm segments the whole market into two groups - the left and the right half of $[0, 1]$. In Hidir and Vellodi (2021) where the seller can segment the market as finely as possible, full segmentation is optimal for the seller. In general, more segments are better for the seller as it can charge higher prices in each segment. In our model, the firm is restricted to two contracts in period 1 and one contract in period 2. When all types buy some product in period 1, the best the firm can do is to segment the market into two. However, when the opt-out choice is available, there exists an equilibrium in which the market is segmented into three sub-markets in period 2. Moreover, the *ex-ante* PS is higher in this equilibrium.

Proposition 3.4.3. *In the two-period game with opt-in/out choice, there exists an equilibrium as follows:*

(i) *In the first period, the firm offers $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ and $(\frac{3}{4}, \bar{u} - \frac{a}{16})$;*

(ii) *In the second period, observing a purchase history of $\frac{1}{4}$ (or $\frac{3}{4}$), the firm offers $(\frac{1}{6}, \bar{u} - \frac{a}{36})$ (or $(\frac{5}{6}, \bar{u} - \frac{a}{36})$). If the firm observes that there is no purchase in the first period, it offers $(\frac{1}{2}, \bar{u})$; if the firm observes that the consumer chose to opt out, it offers $(\frac{1}{2}, \bar{u} - \frac{a}{36})$;*

(iii) *In the first period, the consumer chooses the contract which gives her highest surplus in that period as long as it is non-negative;*

(iv) *In the second period, the consumer chooses to opt out if her type is in $[\frac{1}{3}, \frac{2}{3}]$. The consumer accepts the contract as long as it gives her a non-negative surplus in that period.*

In the proof of Proposition 3.4.3, we list the purchase decisions and opt-out choices for different consumer types. In particular, all types buy some product at the same price in each period. Thus the *ex-ante* PS is

$$(\bar{u} - \frac{a}{16}) + \delta(\bar{u} - \frac{a}{36}).$$

The first term is the firm's expected revenue from period 1 and the second term is the expected revenue from period 2. Recall that the *ex-ante* PS in the firm-optimal equilibrium without opt-out is $(1+\delta)(\bar{u} - \frac{a}{36})$, so the firm's surplus is improved. Firm-optimal equilibrium when opt-out is available can be defined in a similar way as in the previous subsection. Now we show that the equilibrium in Proposition 3.4.3 is firm-optimal.

Proposition 3.4.4. *When there is an opt-in/out choice, the equilibrium in Proposition 3.4.3 is firm-optimal.*

We refer to the equilibrium in Proposition 3.4.3 as *the firm-optimal equilibrium with opt-out*. For the firm, the key to welfare improvement lies in the second period. Now with the opt-out choice, the firm can segment the market more finely and charge higher prices. In general, when the consumer chooses whether to opt out, she reveals additional information which is used to the advantage of the firm. In particular, in the firm-optimal equilibrium with opt-out, consumer types in $[\frac{1}{3}, \frac{2}{3}]$ choose to opt out. Observing such a choice, the firm expects the consumer to be types in the middle of the interval $[0, 1]$ and provide the product in the middle.

Comparing the two equilibria, the firm is better off with the opt-out choice. What about the consumer? At first glance, in the equilibrium with opt-out, all types pay higher prices in period 2 but some types get products closer to her ideal product, so the answer is not obvious. In the next subsection we will compare the *ex-ante* and *interim* consumer surpluses across the two equilibria.

3.4.3 Comparing consumer surpluses

Similar to previous subsections, the *ex-ante* surplus for the consumer is defined as:

$$CS(\mathbf{v}, \mathbf{p}) = \int_{\hat{\Theta}(v_l, p_l)} [\bar{u} - a(v_l - \theta)^2 - p_l] d\theta + \int_{\hat{\Theta}(v_r, p_r)} [\bar{u} - a(v_r - \theta)^2 - p_r] d\theta$$

$$+\delta \int_{\hat{\Theta}(v_2, p_2)} [\bar{u} - a(v_2 - \theta)^2 - p_2] d\theta.$$

For any particular type θ, if she chooses (v_i, p_i) $(i \in \{l, r\})$ and (v_2, p_2), her *interim* CS is:

$$CS(\mathbf{v}, \mathbf{p}; \theta) = \bar{u} - a(v_i - \theta)^2 - p_i + \delta[\bar{u} - a(v_2 - \theta)^2 - p_2].$$

First, we compare the **ex-ante** CS. All calculations are relegated to the Appendix.

Proposition 3.4.5 (Ex-ante CS). *The ex-ante CS in the firm-optimal equilibrium without opt-out is higher than in the firm-optimal equilibrium with opt-out.*

From an *ex-ante* view, the consumer is worse off when the opt-out choice is available. Our interpretation is that the expected cost of higher prices exceeds the expected benefit of better-matched products. When we calculate the *ex-ante* CS, we aggregate the effect of opt-out on each type, it is interesting to see how the welfare of each consumer type changes after adding the opt-out choice.

Now we compare the welfare after the consumer learns her type.

Proposition 3.4.6 (Interim CS). *For $\theta \in (0, \frac{4}{9}) \cup (\frac{5}{9}, 1)$, the interim CS in the firm-optimal equilibrium without opt-out is higher. For $\theta \in (\frac{4}{9}, \frac{5}{9})$, the interim CS in the firm-optimal equilibrium with opt-out is higher. For boundary types $\theta \in \{0, \frac{4}{9}, \frac{5}{9}, 1\}$, the interim CS is the same in the two equilibria.*

We illustrate the result above in Figure 4. In the firm-optimal equilibrium with opt-out, the whole market is represented by the interval $[0, 1]$, and the three submarkets $[0, \frac{1}{3})$, $[\frac{1}{3}, \frac{2}{3}]$, and $(\frac{2}{3}, 1]$ are represented in different colors.

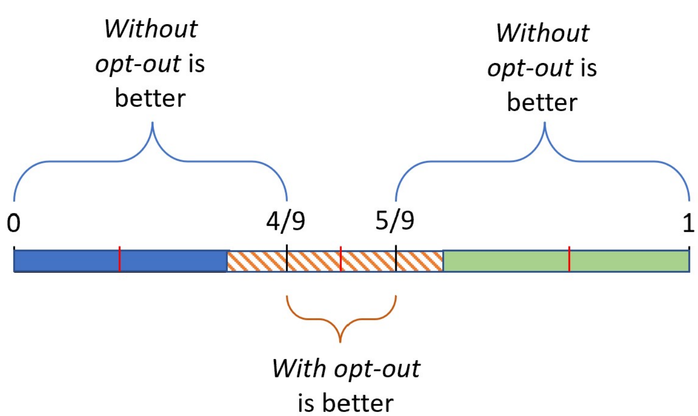

Figure 4: Interim CS comparison

Generally, the shorter the covered sub-market is, the higher the price for that sub-market will be. In any particular sub-market, as the firm offers the product at the middle, consumer types closer to the middle are better off compared to types farther away from the middle. Therefore, for the entire market, only types close to $\frac{1}{2}$ benefit from the opt-out choice: they buy product $\frac{1}{4}$ or $\frac{3}{4}$ in the equilibrium without opt-out and buy product $\frac{1}{2}$ when opt-out is available. Given the contracts offered, the consumer cannot commit not to opt out if her type is in the middle.

3.5 Conclusion

This paper studies a two-period with a multi-product monopoly firm and a consumer. The consumer's type is unknown to the firm but her choice reveals some information. In the first period, the firm is able to offer two contracts (product-price pairs) and in the second period, it is only able to offer one contract. We study the setting where the consumer cannot hide her purchase history (i.e., cannot opt out) and the setting where she can. Our model is constructed to study the following scenario: when a new customer visits an online retailer's website, little is known about her preference, and thus the retailer provides a range of products, hoping to learn from the customer's choice. After the initial interaction, some information is revealed, and now the retailer targets one product to the customer. Our main research question is, what is the welfare implication when the consumer can opt out of personalization?

We first characterize the firm-optimal equilibrium without opt-out and calculate the surpluses for the firm and the consumer, and then do the same thing after adding the opt-out choice. We find that from an *ex-ante* view, with the opt-out choice, the firm is better off while the consumer is worse off. The main reason is that with the opt-out choice, the firm can segment the market more finely and increase prices in each segment. Though some types now have products closer to their ideal point, this benefit is smaller than the cost of higher prices when we consider all types together. But from an *interim* view, those types closer to the middle point of the type space are better off when they can choose to opt out.

As consumers have more privacy concerns, regulators and internet companies give them more control over their data. For example, consumers can now opt out of personalization on many websites and apps. However, this option comes at a cost, and sometimes consumers are worse off. The mere choice to hide one's purchase history, browsing history, or other activities online can reveal valuable information to the internet companies and the latter may use that information to increase revenue at the expense of consumers. Our seemingly counter-intuitive results suggest that regulators need to consider the privacy protection tools in a dynamic and strategic setting.

Appendix A Omitted Proofs

A.1 Omitted Proofs in Chapter 1

A.1.1 Proof of Lemma 1.2.1

Proof. Denote the equilibrium cutoffs as $\underline{c}_1$ and $\underline{c}_2$. There are three cases to consider.

Case 1: $\underline{c}_1 = \underline{c}_2$.

For a student with signal $\hat{x}$, her expected utility

$$\text{if she applies to } s_1 \text{ first: } Pr(x \geq \underline{c}_1|\hat{x})u_1$$

$$\text{if she applies to } s_2 \text{ first: } Pr(x \geq \underline{c}_2|\hat{x})u_2$$

Since $Pr(x \geq \underline{c}_1|\hat{x})u_1 > Pr(x \geq \underline{c}_2|\hat{x})u_2$ for all $\hat{x}$, all students apply to s_1 first in this case.

Case 2: $\underline{c}_1 > \underline{c}_2$.

For a student with signal $\hat{x}$, her expected utility

$$\text{if she applies to } s_1 \text{ first: } Pr(x \geq \underline{c}_1|\hat{x})u_1 + Pr(\underline{c}_1 > x \geq \underline{c}_2|\hat{x})u_2$$

$$\text{if she applies to } s_2 \text{ first: } Pr(x \geq \underline{c}_2|\hat{x})u_2$$

Since for all $\hat{x}$,

$$\begin{aligned} Pr(x \geq \underline{c}_1|\hat{x})u_1 + Pr(\underline{c}_1 > x \geq \underline{c}_2|\hat{x})u_2 &> Pr(x \geq \underline{c}_1|\hat{x})u_2 + Pr(\underline{c}_1 > x \geq \underline{c}_2|\hat{x})u_2 \\ &= Pr(x \geq \underline{c}_2|\hat{x})u_2, \end{aligned}$$

all students apply to s_1 first in this case.

Case 3: $\underline{c}_1 < \underline{c}_2$.

For a student with signal $\hat{x}$, her expected utility

$$\text{if she applies to } s_1 \text{ first: } Pr(x \geq \underline{c}_1|\hat{x})u_1$$

$$\text{if she applies to } s_2 \text{ first: } Pr(x \geq \underline{c}_2|\hat{x})u_2 + Pr(\underline{c}_2 > x \geq \underline{c}_1|\hat{x})u_1$$

Since for all $\hat{x}$,

$$\begin{aligned} Pr(x \geq \underline{c}_1|\hat{x})u_1 &= Pr(x \geq \underline{c}_2|\hat{x})u_1 + Pr(\underline{c}_2 > x \geq \underline{c}_1|\hat{x})u_1 \\ &> Pr(x \geq \underline{c}_2|\hat{x})u_2 + Pr(\underline{c}_2 > x \geq \underline{c}_1|\hat{x})u_1, \end{aligned}$$

all students apply to s_1 first in this case.

Thus proof is complete. □

A.1.2 Proof of Lemma 1.2.2

Proof. Suppose not. Then there is an equilibrium such that some school s_k still has seats left after period 2.

First, we prove that s_k rejected some applicants in some period. Suppose s_k reject no applicant in either period. Denote the measure of students who apply to s_k first as m. Then the measure of students who apply to the other school (denote as s_j) first is $1-m$. In period 1, s_j admitted as most q_j students, so at least $\max\{0, 1-m-q_j\}$ students apply to s_k in period 2. Since s_k rejected no students, it admitted at least $m + (1-m-q_j) = 1-q_j$ students. But $1-q_j > q_k$, contradicting the fact that s_k still has seat left after period 2.

Since s_k rejected some applicants at some period, it has an incentive to lower its cutoff at that period (and it will not affect the students enrolled by s_k in the other period). But this contradicts the fact it is an equilibrium. □

A.1.3 Proof of Lemma 1.2.3

Proof. The total measure of students whose ability is no less than c_2^* is $q_1 + q_2$. Schools know that they can fill all the seats with those students, so in equilibrium no school will set a cutoff lower than c_2^*. □

A.1.4 Proof of Proposition 1.2.2

Proof. Part (iii) follows from Corollary 1.2.1. Now we prove (i) and (ii) by contradiction.

Part (i): Suppose in equilibrium there is a set of students $A = \{i|x_i \geq c_1^*\}$ with a positive measure that is admitted by s_2. Then by Lemma 1.2.2, there is a set of students $B = \{i|c_2^* \leq x_i < c_1^*\}$ with the same measure that is admitted by s_1. Students in A have the incentive to follow the same application order as students in B and will be admitted by s_1. This is a contradiction to the equilibrium.

Part (ii): Suppose in equilibrium there is a set of students $C = \{i|c_2^* \leq x_i < c_1^*\}$ with a positive measure that is admitted by s_1. Then by Lemma 1.2.2, there is a set of students $D = \{i|x_i \geq c_1^*\}$ with the same measure that is admitted by s_2. Students in D have the incentive to follow the same application order as students in C and will be admitted by s_1. This is a contradiction to the equilibrium. □

A.1.5 Proof of Proposition 1.3.1

Proof. We first show that in equilibrium, all students apply to s_1 first.

Denote the equilibrium cutoffs as $\hat{c}_1$ and $\hat{c}_2$.

For a student with ability x, if she applies to s_1 first, her expected utility is

$$\begin{aligned} &Pr(x+\epsilon \geq \hat{c}_1|x)u_1 + Pr(\hat{c}_1 > x+\epsilon|x) \times Pr(x+\epsilon' \geq \hat{c}_2|x)u_2 \\ =&[1-G(\frac{\hat{c}_1 - x}{\sigma_\epsilon})]u_1 + G(\frac{\hat{c}_1 - x}{\sigma_\epsilon})[1-G(\frac{\hat{c}_2 - x}{\sigma_\epsilon})]u_2 \end{aligned}$$

where $\epsilon, \epsilon' \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$.

If she applies to s_2 first, her expected utility is

$$\begin{aligned} &Pr(x+\epsilon \geq \hat{c}_2|x)u_2 + Pr(\hat{c}_2 > x+\epsilon|x) \times Pr(x+\epsilon' \geq \hat{c}_1|x)u_1 \\ =&[1-G(\frac{\hat{c}_2 - x}{\sigma_\epsilon})]u_2 + G(\frac{\hat{c}_2 - x}{\sigma_\epsilon})[1-G(\frac{\hat{c}_1 - x}{\sigma_\epsilon})]u_1. \end{aligned}$$

Subtracting the latter from the former, we get the difference:

$$[1-G(\frac{\hat{c}_1 - x}{\sigma_\epsilon})][1-G(\frac{\hat{c}_2 - x}{\sigma_\epsilon})](u_1 - u_2) > 0 \text{ for all } x$$

Hence in equilibrium, all students apply to s_1 in period 1, and s_1 sets a cutoff such that the top-q_1 are admitted. In period 2, rejected students apply to s_2, and the cutoff is set such that top-q_2 of those students are admitted.

$\square$

A.2 Omitted Proofs in Chapter 2

A.2.1 Proof of Proposition 2.3.1

Proof. Given that all sender types censor the signal, when we check potential deviations for the receiver, we are back to the case of CS and indeed there is no profitable deviation.

We specify R's off-path belief as follows: when observing no censorship or any off-path message, he believes the sender's type is from the first interval $[0, a_{CS})$. Then the sender has no profitable deviation.

$\square$

A.2.2 Useful results for other proofs

We first present some results that will be used in other proofs.

In the main part, we define R's optimal actions, and they are functions of boundary type a: $x_{1h}(a) = \frac{4\lambda a^3+3(1-\lambda)a^2}{6\lambda a^2+6(1-\lambda)a}$, $x_{1l}(a) = \frac{-4\lambda a^3+3(1+\lambda)a^2}{-6\lambda a^2+6(1+\lambda)a}$, $x_{2h}(a) = \frac{3+\lambda-4\lambda a^3-3(1-\lambda)a^2}{6-6\lambda a^2-6(1-\lambda)a}$, and $x_{2l}(a) = \frac{3-\lambda+4\lambda a^3-3(1+\lambda)a^2}{6+6\lambda a^2-6(1+\lambda)a}$. Taking derivatives, we have the following fact.

Fact A.2.1. *For $a \in (0, 1)$,*

$$\frac{\partial x_{1h}}{\partial a} = \frac{2}{3} - \frac{(1-\lambda)^2}{6(\lambda a + 1 - \lambda)^2} \in (\frac{1}{2}, \frac{2}{3} - \frac{(1-\lambda)^2}{6}) \subset (\frac{1}{2}, \frac{2}{3}),$$
$$\frac{\partial x_{1l}}{\partial a} = \frac{2}{3} - \frac{(1+\lambda)^2}{6(-\lambda a + 1 + \lambda)^2} \in (\frac{2}{3} - \frac{(1+\lambda)^2}{6}, \frac{1}{2}) \subset (0, \frac{1}{2}),$$

$$\frac{\partial x_{2h}}{\partial a} = \frac{2}{3} - \frac{(1+\lambda)^2}{6(\lambda a + 1)^2} \in (\frac{2}{3} - \frac{(1+\lambda)^2}{6}, \frac{1}{2}) \subset (0, \frac{1}{2}),$$
$$\frac{\partial x_{2l}}{\partial a} = \frac{2}{3} - \frac{(1-\lambda)^2}{6(1 - \lambda a)^2} \in (\frac{1}{2}, \frac{2}{3} - \frac{(1-\lambda)^2}{6}) \subset (\frac{1}{2}, \frac{2}{3}).$$

Lemma A.2.1. *$x_{2h}(a) - x_{1h}(a) > x_{2l}(a) - x_{1l}(a)$ for $a \in (0, \frac{1}{2})$, $x_{2h}(a) - x_{1h}(a) < x_{2l}(a) - x_{1l}(a)$ for $a \in (\frac{1}{2}, 1)$, and $x_{2h}(\frac{1}{2}) - x_{1h}(\frac{1}{2}) = x_{2l}(\frac{1}{2}) - x_{1l}(\frac{1}{2})$.*

Proof. With some calculations,

$$x_{2h}(a) - x_{2l}(a) = \frac{\lambda(1-a)^2}{3 - 3\lambda^2 a^2},$$

$$x_{1h}(a) - x_{1l}(a) = \frac{\lambda a^2}{3 - 3\lambda^2(1-a)^2}.$$

Define $\beta : (0,1) \to \mathbb{R}$ as $\beta(t) = \frac{\lambda(1-t)^2}{3-3\lambda^2 t^2}$. Differentiating we get $\beta'(t) = \frac{6\lambda(1-t)(\lambda^2 t-1)}{(3-3\lambda^2 t^2)^2} < 0$.

Note that $x_{2h}(a) - x_{2l}(a) = \beta(a)$ and $x_{1h}(a) - x_{1l}(a) = \beta(1-a)$. Therefore, $a < \frac{1}{2} \Leftrightarrow a < 1 - a \Rightarrow \beta(a) > \beta(1-a) \Leftrightarrow x_{2h}(a) - x_{2l}(a) > x_{1h}(a) - x_{1l}(a)$. Similarly we have $a > \frac{1}{2} \Rightarrow x_{2h}(a) - x_{2l}(a) < x_{1h}(a) - x_{1l}(a)$ and $x_{2h}(\frac{1}{2}) - x_{2l}(\frac{1}{2}) = x_{1h}(\frac{1}{2}) - x_{1l}(\frac{1}{2})$.

Rearrange the terms and the proof is complete. □

Claim A.2.1. *For $a \in (0, \frac{1}{2})$, $2x_{2l}(a) > x_{1h}(a) + x_{2h}(a)$.*

Proof. From Fact A.2.1, $\frac{\partial(x_{2l} - x_{1h})}{\partial a} = \frac{(1-\lambda)^2}{6(\lambda a + 1 - \lambda)^2} - \frac{(1-\lambda)^2}{6(1-\lambda a)^2} > 0$ and $\frac{\partial(x_{2l} - x_{2h})}{\partial a} > 0$. Therefore,

$$\begin{aligned}
2x_{2l}(a) - x_{1h}(a) - x_{2h}(a) \geq& \lim_{a \to 0}[2x_{2l}(a) - x_{1h}(a) - x_{2h}(a)] \\
=& \frac{3-\lambda}{3} - \frac{3+\lambda}{6} \\
=& \frac{1-\lambda}{2} > 0.
\end{aligned}$$

□

Claim A.2.2. *For $a \in (\frac{1}{2}, 1)$, $2x_{1h}(a) < x_{1l}(a) + x_{2l}(a)$.*

Proof. From Fact A.2.1, $\frac{\partial(x_{1h} - x_{1l})}{\partial a} > 0$, and $\frac{\partial(x_{1h} - x_{2l})}{\partial a} = \frac{(1-\lambda)^2}{6(1-\lambda a)^2} - \frac{(1-\lambda)^2}{6(\lambda a + 1 - \lambda)^2} > 0$.

Then

$$\begin{aligned}
2x_{1h}(a) - x_{1l}(a) - x_{2l}(a) \leq& \lim_{a \to 1}[2x_{1h}(a) - x_{1l}(a) - x_{2l}(a)] \\
=& \frac{3+\lambda}{3} - \frac{3-\lambda}{6} - 1 \\
=& \frac{\lambda - 1}{2} < 0.
\end{aligned}$$

□

The above results are useful in the proof of Proposition 2.3.2.

A.2.3 Proof of Lemma 2.3.1

Proof. Since

$$x_1(a) = \frac{a}{2} = \frac{\int_0^a \theta d\theta}{\int_0^a 1 d\theta},$$

$$x_{1h}(a) = \frac{\int_0^a s(\theta)\theta d\theta}{\int_0^a s(\theta) d\theta},$$

$$x_{1l}(a) = \frac{\int_0^a (1 - s(\theta))\theta d\theta}{\int_0^a (1 - s(\theta)) d\theta},$$

part (a) is straightforward.

For part (b), with some calculations, we have

$$x_{1h}(a) = \frac{4\lambda a^3 + 3(1-\lambda)a^2}{6\lambda a^2 + 6(1-\lambda)a},$$

$$x_{1l}(a) = \frac{-4\lambda a^3 + 3(1+\lambda)a^2}{-6\lambda a^2 + 6(1+\lambda)a}.$$

Thus

$$x_{1h}(a) - x_1(a) = \frac{\lambda a^2}{6(\lambda a - \lambda + 1)} > 0,$$

$$x_{1l}(a) - x_1(a) = \frac{\lambda a^2}{6(\lambda a - \lambda - 1)} < 0.$$

Similarly,

$$x_2(a) = \frac{a+1}{2} = \frac{\int_a^1 \theta d\theta}{\int_a^1 1 d\theta},$$

$$x_{2h}(a) = \frac{\int_a^1 s(\theta)\theta d\theta}{\int_a^1 s(\theta) d\theta},$$

$$x_{2l}(a) = \frac{\int_a^1 (1 - s(\theta))\theta d\theta}{\int_a^1 (1 - s(\theta)) d\theta},$$

and part (c) is apparent.

For part (d), we have

$$x_{2h}(a) = \frac{3 + \lambda - 4\lambda a^3 - 3(1-\lambda)a^2}{6 - 6\lambda a^2 - 6(1-\lambda)a},$$

$$x_{2l}(a) = \frac{3-\lambda+4\lambda a^3 - 3(1+\lambda)a^2}{6+6\lambda a^2 - 6(1+\lambda)a}.$$

Thus

$$x_{2h}(a) - x_2(a) = \frac{\lambda(a-1)^2}{6(\lambda a+1)} > 0,$$

$$x_{2l}(a) - x_2(a) = \frac{\lambda(a-1)^2}{6(\lambda a-1)} < 0.$$

□

A.2.4 Proof of Proposition 2.3.2

Proof. Part 1: We prove that for $b < \frac{3-\lambda}{12}$, the arbitrage condition holds for some $a \in (0, \frac{1}{2})$.

We want to show $g(a) = 0$ for some $a \in (0, \frac{1}{2})$.

With some calculations, we get $x_{1h}(a) = \frac{4\lambda a^3+3(1-\lambda)a^2}{6\lambda a^2+6(1-\lambda)a}$, $x_{1l}(a) = \frac{-4\lambda a^3+3(1+\lambda)a^2}{-6\lambda a^2+6(1+\lambda)a}$, $x_{2h}(a) = \frac{3+\lambda-4\lambda a^3-3(1-\lambda)a^2}{6-6\lambda a^2-6(1-\lambda)a}$, and $x_{2l}(a) = \frac{3-\lambda+4\lambda a^3-3(1+\lambda)a^2}{6+6\lambda a^2-6(1+\lambda)a}$. Note that

$\lim_{a\to 0} x_{1h}(a) = \lim_{a\to 0} x_{1l}(a) = 0$ and $\lim_{a\to 1} x_{2h}(a) = \lim_{a\to 1} x_{2l}(a) = 1$ by the L'Hospital's Rule.

Define $x_{1h}(0) = x_{1l}(0) = 0$ and $x_{2h}(1) = x_{2l}(1) = 1$. We have extended g to a continuous function on $[0,1]$.

$$\begin{aligned} g(0) &= \frac{1-\lambda}{2}(\frac{3+\lambda}{6} - 2b)\frac{3+\lambda}{6} + \frac{1+\lambda}{2}(\frac{3-\lambda}{6} - 2b)\frac{3-\lambda}{6} \\ &= \frac{-5\lambda^2+9}{36} - \frac{3-\lambda^2}{3}b \end{aligned}$$

Thus $g(0) > 0$ iff $b < \frac{9-5\lambda^2}{12(3-\lambda^2)}$. It can be verified that for $0 < \lambda < 1$, $\frac{3-\lambda}{12} < \frac{9-5\lambda^2}{12(3-\lambda^2)}$, so $g(0) > 0$.

$$\begin{aligned} g(\frac{1}{2}) =& \frac{1}{2}(\frac{3-\lambda}{12-6\lambda} + \frac{9+5\lambda}{12+6\lambda} - 1 - 2b)(\frac{9+5\lambda}{12+6\lambda} - \frac{3-\lambda}{12-6\lambda}) \\ &+ \frac{1}{2}(\frac{3+\lambda}{12+6\lambda} + \frac{9-5\lambda}{12-6\lambda} - 1 - 2b)(\frac{9-5\lambda}{12-6\lambda} - \frac{3+\lambda}{12+6\lambda}) \\ =& -b(\frac{9+5\lambda}{12+6\lambda} - \frac{3-\lambda}{12-6\lambda}) - b(\frac{9-5\lambda}{12-6\lambda} - \frac{3+\lambda}{12+6\lambda}) \\ &< 0 \end{aligned}$$

By the Intermediate Value Theorem, there is a $a \in (0, \frac{1}{2})$ such that (A) holds.

Part 2: We prove that the solution is unique in $[0, \frac{1}{2}]$.

Clearly, 0 and $\frac{1}{2}$ are not solutions to (A). Suppose there are at least two solutions in $(0, \frac{1}{2})$. Since $g(0) > 0$, $g(\frac{1}{2}) < 0$, and g is differentiable on $(0, 1)$, there is some $a^* \in (0, \frac{1}{2})$ such that

$$g(a^*) = 0 \text{ and } g'(a^*) \geq 0.$$

We have

$$\begin{aligned} g(a^*) =& s(a^*)(x_{1h}(a^*) + x_{2h}(a^*) - 2a^* - 2b)(x_{2h}(a^*) - x_{1h}(a^*)) \\ &+ (1 - s(a^*))(x_{1l}(a^*) + x_{2l}(a^*) - 2a^* - 2b)(x_{2l}(a^*) - x_{1l}(a^*)) \\ =& 0. \end{aligned}$$

Since $s(a^*), x_{2h}(a^*) - x_{1h}(a^*), 1 - s(a^*),$ and $x_{2l}(a^*) - x_{1l}(a^*)$ are all positive, and $x_{1h}(a^*) + x_{2h}(a^*) - 2a^* - 2b > x_{1l}(a^*) + x_{2l}(a^*) - 2a^* - 2b$, we must have

$$x_{1h}(a^*) + x_{2h}(a^*) - 2a^* - 2b > 0$$

and

$$x_{1l}(a^*) + x_{2l}(a^*) - 2a^* - 2b < 0.$$

$$\begin{aligned} g'(a) =& \lambda(x_{1h}(a) + x_{2h}(a) - 2a - 2b)(x_{2h}(a) - x_{1h}(a)) \\ &+ s(a)(\frac{\partial x_{1h}}{\partial a} + \frac{\partial x_{2h}}{\partial a} - 2)(x_{2h}(a) - x_{1h}(a)) \\ &+ s(a)(x_{1h}(a) + x_{2h}(a) - 2a - 2b)(\frac{\partial x_{2h}}{\partial a} - \frac{\partial x_{1h}}{\partial a}) \\ &- \lambda(x_{1l}(a) + x_{2l}(a) - 2a - 2b)(x_{2l}(a) - x_{1l}(a)) \\ &+ (1 - s(a))(\frac{\partial x_{1l}}{\partial a} + \frac{\partial x_{2l}}{\partial a} - 2)(x_{2l}(a) - x_{1l}(a)) \\ &+ (1 - s(a))(x_{1l}(a) + x_{2l}(a) - 2a - 2b)(\frac{\partial x_{2l}}{\partial a} - \frac{\partial x_{1l}}{\partial a}) \end{aligned}$$

From Lemma A.2.1, we know that for $a \in (0, \frac{1}{2})$, $x_{2h}(a) - x_{1h}(a) > x_{2l}(a) - x_{1l}(a)$. From Fact A.2.1, $\frac{\partial x_{1h}}{\partial a} \in (\frac{1}{2}, \frac{2}{3})$, $\frac{\partial x_{2h}}{\partial a} \in (0, \frac{1}{2})$, $\frac{\partial x_{1l}}{\partial a} \in (0, \frac{1}{2})$, and $\frac{\partial x_{2l}}{\partial a} \in (\frac{1}{2}, \frac{2}{3})$.

Thus,

$$\begin{aligned}
g'(a^*) <&\lambda(x_{2h}(a^*) - x_{1h}(a^*))(x_{1h}(a^*) + x_{2h}(a^*) - x_{1l}(a^*) - x_{2l}(a^*)) \\
&+ s(a^*)(\frac{\partial x_{1h}}{\partial a}(a^*) + \frac{\partial x_{2h}}{\partial a}(a^*) - 2)(x_{2h}(a^*) - x_{1h}(a^*)) \\
&+ (1 - s(a^*))(\frac{\partial x_{1l}}{\partial a}(a^*) + \frac{\partial x_{2l}}{\partial a}(a^*) - 2)(x_{2l}(a^*) - x_{1l}(a^*)) \\
<&\lambda(x_{2h}(a^*) - x_{1h}(a^*))(x_{1h}(a^*) + x_{2h}(a^*) - x_{1l}(a^*) - x_{2l}(a^*)) \\
&- \frac{5}{6}s(a^*)(x_{2h}(a^*) - x_{1h}(a^*)) \\
&- \frac{5}{6}(1 - s(a^*))(x_{2l}(a^*) - x_{1l}(a^*)) \\
<&\lambda(x_{2h}(a^*) - x_{1h}(a^*))(x_{1h}(a^*) + x_{2h}(a^*) - x_{1l}(a^*) - x_{2l}(a^*)) \\
&- \frac{5}{6}(x_{2l}(a^*) - x_{1l}(a^*)).
\end{aligned}$$

Then from Claim A.2.1,

$$\begin{aligned}
g'(a^*) <&\lambda(x_{2h}(a^*) - x_{1h}(a^*))(x_{2l}(a^*) - x_{1l}(a^*)) \\
&- \frac{5}{6}(x_{2l}(a^*) - x_{1l}(a^*)) \\
=&[\lambda(x_{2h}(a^*) - x_{1h}(a^*)) - \frac{5}{6}](x_{2l}(a^*) - x_{1l}(a^*)).
\end{aligned}$$

Since $\frac{\partial(x_{2h} - x_{1h})}{\partial a} < 0$, $x_{2h}(a^*) - x_{1h}(a^*) \leq \lim_{a \to 0}[x_{2h}(a) - x_{1h}(a)] = \frac{3+\lambda}{6}$. Then $g'(a^*) < (\frac{3\lambda+\lambda^2}{6} - \frac{5}{6})(x_{2l}(a^*) - x_{1l}(a^*)) < 0$, which contradicts $g'(a^*) \geq 0$.

Part 3: We prove that there is no solution in $(\frac{1}{2}, 1]$.

Suppose $g(a) = 0$ has a solution in $(\frac{1}{2}, 1]$.

$$\begin{aligned}
g(1) =&\frac{1+\lambda}{2}(\frac{3+\lambda}{6} + 1 - 2 - 2b)(1 - \frac{3+\lambda}{6}) \\
&+ \frac{1-\lambda}{2}(\frac{3-\lambda}{6} + 1 - 2 - 2b)(1 - \frac{3-\lambda}{6}) \\
=&\frac{1+\lambda}{2}(\frac{\lambda-3}{6} - 2b)\frac{3-\lambda}{6} + \frac{1-\lambda}{2}(\frac{-\lambda-3}{6} - 2b)\frac{3+\lambda}{6} \\
<&0
\end{aligned}$$

Then since $g(\frac{1}{2}) < 0$, there is some $\hat{a} \in (\frac{1}{2}, 1)$ such that

$$g(\hat{a}) = 0 \text{ and } g'(\hat{a}) \geq 0.$$

Again, by the same reasoning, we must have

$$x_{1h}(\hat{a}) + x_{2h}(\hat{a}) - 2\hat{a} - 2b > 0$$

and

$$x_{1l}(\hat{a}) + x_{2l}(\hat{a}) - 2\hat{a} - 2b < 0.$$

From Lemma A.2.1, for $a \in (\frac{1}{2}, 1)$, $x_{2h}(a) - x_{1h}(a) < x_{2l}(a) - x_{1l}(a)$. Thus,

$$\begin{aligned}
g'(\hat{a}) <& \lambda(x_{2l}(\hat{a}) - x_{1l}(\hat{a}))(x_{1h}(\hat{a}) + x_{2h}(\hat{a}) - x_{1l}(\hat{a}) - x_{2l}(\hat{a})) \\
& + s(\hat{a})(\frac{\partial x_{1h}}{\partial a}(\hat{a}) + \frac{\partial x_{2h}}{\partial a}(\hat{a}) - 2)(x_{2h}(\hat{a}) - x_{1h}(\hat{a})) \\
& + (1 - s(\hat{a}))(\frac{\partial x_{1l}}{\partial a}(\hat{a}) + \frac{\partial x_{2l}}{\partial a}(\hat{a}) - 2)(x_{2l}(\hat{a}) - x_{1l}(\hat{a})) \\
<& \lambda(x_{2l}(\hat{a}) - x_{1l}(\hat{a}))(x_{1h}(\hat{a}) + x_{2h}(\hat{a}) - x_{1l}(\hat{a}) - x_{2l}(\hat{a})) \\
& - \frac{5}{6} s(\hat{a})(x_{2h}(\hat{a}) - x_{1h}(\hat{a})) \\
& - \frac{5}{6}(1 - s(\hat{a}))(x_{2l}(\hat{a}) - x_{1l}(\hat{a})) \\
\leq& \lambda(x_{2l}(\hat{a}) - x_{1l}(\hat{a}))(x_{1h}(\hat{a}) + x_{2h}(\hat{a}) - x_{1l}(\hat{a}) - x_{2l}(\hat{a})) \\
& - \frac{5}{6}(x_{2h}(\hat{a}) - x_{1h}(\hat{a})).
\end{aligned}$$

From Claim A.2.2,

$$\begin{aligned}
g'(\hat{a}) <& \lambda(x_{2l}(\hat{a}) - x_{1l}(\hat{a}))(x_{2h}(\hat{a}) - x_{1h}(\hat{a})) \\
& - \frac{5}{6}(x_{2h}(\hat{a}) - x_{1h}(\hat{a})) \\
=& [\lambda(x_{2l}(\hat{a}) - x_{1l}(\hat{a})) - \frac{5}{6}](x_{2h}(\hat{a}) - x_{1h}(\hat{a})).
\end{aligned}$$

Since $\frac{\partial(x_{2l} - x_{1l})}{\partial a} > 0$, $x_{2l}(\hat{a}) - x_{1l}(\hat{a}) \leq \lim_{a \to 1}[x_{2l}(a) - x_{1l}(a)] = \frac{3+\lambda}{6}$. Then $g'(\hat{a}) < (\frac{3\lambda+\lambda^2}{6} - \frac{5}{6})(x_{2h}(\hat{a}) - x_{1h}(\hat{a})) < 0$, which contradicts $g'(\hat{a}) \geq 0$.

Thus proof is complete. □

A.2.5 Proof of Proposition 2.3.3

Proof. Let $a \in (0, \frac{1}{2})$ denote the unique solution to $g = 0$. For this particular a, define

$$L(\theta) := -s(\theta)(x_{1h}(a) - \theta - b)^2 - (1 - s(\theta))(x_{1l}(a) - \theta - b)^2$$

$$R(\theta) := -s(\theta)(x_{2h}(a) - \theta - b)^2 - (1 - s(\theta))(x_{2l}(a) - \theta - b)^2$$

Since a is fixed, for simplicity, we omit the dependence of x's on a. Clearly, $L(a) - R(a) = 0$.

$L(\theta)$ ($R(\theta)$) is the *ex-ante* expected payoff for sender when the state is θ and the receiver believes that the state is in $[0, a)$ ($[a, 1]$). For every $\theta \in [0, 1]$, denote $L(\theta) - R(\theta)$ as $(L - R)_\theta$, $L'(\theta) - R'(\theta)$ as $(L - R)'_\theta$, and $L''(\theta) - R''(\theta)$ as $(L - R)''_\theta$. Then

$$\begin{aligned}(L - R)_\theta =&(\lambda\theta + \frac{1 - \lambda}{2})(x_{1h} + x_{2h} - 2\theta - 2b)(x_{2h} - x_{1h})\\ &+ (-\lambda\theta + \frac{1 + \lambda}{2})(x_{1l} + x_{2l} - 2\theta - 2b)(x_{2l} - x_{1l})\end{aligned}$$

$$\begin{aligned}(L - R)'_\theta =&\lambda(x_{1h} + x_{2h} - 2\theta - 2b)(x_{2h} - x_{1h})\\ &- 2(\lambda\theta + \frac{1 - \lambda}{2})(x_{2h} - x_{1h})\\ &- \lambda(x_{1l} + x_{2l} - 2\theta - 2b)(x_{2l} - x_{1l})\\ &- 2(-\lambda\theta + \frac{1 + \lambda}{2})(x_{2l} - x_{1l})\end{aligned}$$

$$(L - R)''_\theta = -4\lambda[(x_{2h} - x_{1h}) - (x_{2l} - x_{1l})]$$

By Lemma A.2.1, since $a < \frac{1}{2}$, $(L - R)''_\theta < 0$, that is, $L - R$ is concave. Then if $(L - R)_0 > 0$, by concavity, $(L - R)_\theta > 0$ for $\theta \in [0, a)$ and $(L - R)_\theta < 0$ for $\theta \in (a, 1]$. Therefore, it is sufficient to prove $(L - R)_0 > 0$.

$$(L-R)_0 = \frac{1-\lambda}{2}(x_{1h}+x_{2h}-2b)(x_{2h}-x_{1h}) + \frac{1+\lambda}{2}(x_{1l}+x_{2l}-2b)(x_{2l}-x_{1l}) > 0$$

$$\iff b < \frac{(1-\lambda)(x_{2h}+x_{1h})(x_{2h}-x_{1h}) + (1+\lambda)(x_{2l}+x_{1l})(x_{2l}-x_{1l})}{2(1-\lambda)(x_{2h}-x_{1h}) + 2(1+\lambda)(x_{2l}-x_{1l})}$$

Now we prove that for each λ, $\frac{3-\lambda}{12} < \frac{(1-\lambda)(x_{2h}+x_{1h})(x_{2h}-x_{1h})+(1+\lambda)(x_{2l}+x_{1l})(x_{2l}-x_{1l})}{2(1-\lambda)(x_{2h}-x_{1h})+2(1+\lambda)(x_{2l}-x_{1l})}$. Denote $\triangle = \frac{(1-\lambda)(x_{2h}+x_{1h})(x_{2h}-x_{1h})+(1+\lambda)(x_{2l}+x_{1l})(x_{2l}-x_{1l})}{2(1-\lambda)(x_{2h}-x_{1h})+2(1+\lambda)(x_{2l}-x_{1l})}$.

First, note that

$$\triangle - \frac{x_{2l}+x_{1l}}{2}$$

$$= \frac{(1-\lambda)(x_{2h}+x_{1h})(x_{2h}-x_{1h}) - (1-\lambda)(x_{2l}+x_{1l})(x_{2h}-x_{1h})}{2(1-\lambda)(x_{2h}-x_{1h}) + 2(1+\lambda)(x_{2l}-x_{1l})} > 0.$$

Denote $x_{1l} = \frac{C1}{E1} = \frac{-4\lambda a^3+3(1+\lambda)a^2}{-6\lambda a^2+6(1+\lambda)a}$ and $x_{2l} = \frac{C2}{E2} = \frac{3-\lambda+4\lambda a^3-3(1+\lambda)a^2}{6+6\lambda a^2-6(1+\lambda)a}$,

$$\frac{x_{2l}+x_{1l}}{2} \geq \min\{\frac{C1+C2}{2\times E1}, \frac{C1+C2}{2\times E2}\}$$

Simplifying we get

$$\frac{x_{2l}+x_{1l}}{2} \geq \min\{\frac{3-\lambda}{2[-6\lambda a^2+6(1+\lambda)a]}, \frac{3-\lambda}{2[6+6\lambda a^2-6(1+\lambda)a]}\}.$$

For $a \in (0, \frac{1}{2})$, $-6\lambda a^2 + 6(1+\lambda)a \in (0, 3+\frac{3\lambda}{2})$ and $6+6\lambda a^2 - 6(1+\lambda)a \in (3-\frac{3\lambda}{2}, 6)$, so $\triangle > \frac{x_{2l}+x_{1l}}{2} > \frac{3-\lambda}{12}$.

Therefore, when $b < \frac{3-\lambda}{12}$, we have $(L-R)_0 > 0$. It implies that $(L-R)_\theta > 0$ for $\theta \in [0, a)$ and $(L-R)_\theta < 0$ for $\theta \in (a, 1]$.

To constitute an equilibrium, we specify that when R observes off-path strategy, he believes the state is in $[0, a)$.

Thus proof is complete. □

Claim A.2.3. *For $a \in (0,1)$, $x_{2h}(a) - x_{2l}(a) < x_{2l}(a) - x_1(a)$.*

Proof. From Fact 1, $\frac{\partial(x_{2h}-x_{2l})}{\partial a} < 0$ and $\frac{\partial(x_{2l}-x_1)}{\partial a} > 0$.

Then

$$\begin{aligned} x_{2h}(a) - x_{2l}(a) - (x_{2l}(a) - x_1(a)) \leq & x_{2h}(0) - 2x_{2l}(0) + x_1(0) \\ = & \frac{3+\lambda}{6} - \frac{3-\lambda}{3} \\ = & \frac{\lambda - 1}{2} < 0. \end{aligned}$$

□

A.2.6 Proof of Proposition 2.3.4

Proof. The method of proof is the same as that of Proposition 2.3.2.

Part 1: We prove that for $b < \frac{3-\lambda}{12}$, the arbitrage condition holds for some $a \in (0, \frac{1}{2})$.

We want to show $h(a) = 0$ for some $a \in (0, \frac{1}{2})$.

Define $x_{2h}(1) = x_{2l}(1) = 1$. We have extended h to a continuous function on $[0, 1]$.

$$\begin{aligned} h(0) &= \frac{1-\lambda}{2}(\frac{3+\lambda}{6} - 2b)\frac{3+\lambda}{6} + \frac{1+\lambda}{2}(\frac{3-\lambda}{6} - 2b)\frac{3-\lambda}{6} \\ &= \frac{-5\lambda^2+9}{36} - \frac{3-\lambda^2}{3}b \end{aligned}$$

Note that $g(0) = h(0)$, so for $b < \frac{3-\lambda}{12}$, $h(0) > 0$.

$$\begin{aligned} h(\frac{1}{2}) = & \frac{1}{2}(\frac{1}{4} + \frac{9+5\lambda}{12+6\lambda} - 1 - 2b)(\frac{9+5\lambda}{12+6\lambda} - \frac{1}{4}) \\ & + \frac{1}{2}(\frac{1}{4} + \frac{9-5\lambda}{12-6\lambda} - 1 - 2b)(\frac{9-5\lambda}{12-6\lambda} - \frac{1}{4}) \\ = & \frac{\lambda^2(7\lambda^2-20)}{144(\lambda^2-4)^2} \\ & - b(\frac{9+5\lambda}{12+6\lambda} - \frac{1}{4}) - b(\frac{9-5\lambda}{12-6\lambda} - \frac{1}{4}) \\ < & 0 \end{aligned}$$

By the Intermediate Value Theorem, there is a $a \in (0, \frac{1}{2})$ such that (A') holds.

Part 2: We prove that the solution is unique in $[0, \frac{1}{2}]$.

Clearly, 0 and $\frac{1}{2}$ are not solutions to (A). Suppose there are at least two solutions in $(0, \frac{1}{2})$. Since $h(0) > 0$, $h(\frac{1}{2}) < 0$, and h is differentiable on $(0, 1)$, there is some $a^* \in (0, \frac{1}{2})$ such that

$$h(a^*) = 0 \text{ and } h'(a^*) \geq 0.$$

We have

$$\begin{aligned} h(a^*) =& s(a^*)(x_1(a^*) + x_{2h}(a^*) - 2a^* - 2b)(x_{2h}(a^*) - x_1(a^*)) \\ &+ (1 - s(a^*))(x_1(a^*) + x_{2l}(a^*) - 2a^* - 2b)(x_{2l}(a^*) - x_1(a^*)) \\ =& 0. \end{aligned}$$

Since $s(a^*), x_{2h}(a^*) - x_1(a^*), 1 - s(a^*),$ and $x_{2l}(a^*) - x_1(a^*)$ are all positive, and $x_1(a^*) + x_{2h}(a^*) - 2a^* - 2b > x_1(a^*) + x_{2l}(a^*) - 2a^* - 2b$, we must have

$$x_1(a^*) + x_{2h}(a^*) - 2a^* - 2b > 0$$

and

$$x_1(a^*) + x_{2l}(a^*) - 2a^* - 2b < 0.$$

$$\begin{aligned} h'(a) =& \lambda(x_1(a) + x_{2h}(a) - 2a - 2b)(x_{2h}(a) - x_1(a)) \\ &+ s(a)(\frac{\partial x_{2h}}{\partial a} - \frac{3}{2})(x_{2h}(a) - x_1(a)) \\ &+ s(a)(x_1(a) + x_{2h}(a) - 2a - 2b)(\frac{\partial x_{2h}}{\partial a} - \frac{1}{2}) \\ &- \lambda(x_1(a) + x_{2l}(a) - 2a - 2b)(x_{2l}(a) - x_1(a)) \\ &+ (1 - s(a))(\frac{\partial x_{2l}}{\partial a} - \frac{3}{2})(x_{2l}(a) - x_1(a)) \\ &+ (1 - s(a))(x_1(a) + x_{2l}(a) - 2a - 2b)(\frac{\partial x_{2l}}{\partial a} - \frac{1}{2}) \end{aligned}$$

Since $x_{2h}(a) - x_1(a) > x_{2l}(a) - x_1(a)$ and from Fact A.2.1, $\frac{\partial x_{1h}}{\partial a} \in (\frac{1}{2}, \frac{2}{3})$, $\frac{\partial x_{2h}}{\partial a} \in (0, \frac{1}{2})$, $\frac{\partial x_{1l}}{\partial a} \in (0, \frac{1}{2})$, and $\frac{\partial x_{2l}}{\partial a} \in (\frac{1}{2}, \frac{2}{3})$

$$\begin{aligned}
h'(a^*) <& \lambda(x_{2h}(a^*) - x_1(a^*))(x_{2h}(a^*) - x_{2l}(a^*)) \\
&+ s(a^*)(\frac{\partial x_{2h}}{\partial a}(a^*) - \frac{3}{2})(x_{2h}(a^*) - x_1(a^*)) \\
&+ (1 - s(a^*))(\frac{\partial x_{2l}}{\partial a}(a^*) - \frac{3}{2})(x_{2l}(a^*) - x_1(a^*)) \\
<& \lambda(x_{2h}(a^*) - x_1(a^*))(x_{2h}(a^*) - x_{2l}(a^*)) \\
&- s(a^*)(x_{2h}(a^*) - x_1(a^*)) \\
&- \frac{5}{6}(1 - s(a^*))(x_{2l}(a^*) - x_1(a^*)) \\
<& \lambda(x_{2h}(a^*) - x_1(a^*))(x_{2h}(a^*) - x_{2l}(a^*)) \\
&- \frac{5}{6}(x_{2l}(a^*) - x_1(a^*))
\end{aligned}$$

Then from Claim A.2.3,

$$\begin{aligned}
h'(a^*) <& \lambda(x_{2h}(a^*) - x_1(a^*))(x_{2l}(a^*) - x_1(a^*)) \\
&- \frac{5}{6}(x_{2l}(a^*) - x_1(a^*)) \\
=& [\lambda(x_{2h}(a^*) - x_1(a^*)) - \frac{5}{6}](x_{2l}(a^*) - x_1(a^*)).
\end{aligned}$$

Since $\frac{\partial(x_{2h} - x_1)}{\partial a} < 0$, $x_{2h}(a^*) - x_1(a^*) \leq x_{2h}(0) - x_1(0) = \frac{3+\lambda}{6}$. Then $h'(a^*) < (\frac{3\lambda+\lambda^2}{6} - \frac{5}{6})(x_{2l}(a^*) - x_1(a^*)) < 0$, which contradicts $h'(a^*) \geq 0$.

Part 3: We prove that there is no solution in $(\frac{1}{2}, 1]$.

Suppose $h(a) = 0$ has a solution in $(\frac{1}{2}, 1]$.

$$\begin{aligned}
h(1) =& \frac{1+\lambda}{2}(\frac{1}{2} + 1 - 2 - 2b)(1 - \frac{1}{2}) \\
&+ \frac{1-\lambda}{2}(\frac{1}{2} + 1 - 2 - 2b)(1 - \frac{1}{2}) \\
=& \frac{1}{2}(-\frac{1}{2} - 2b) \\
<& 0
\end{aligned}$$

Then since $h(\frac{1}{2}) < 0$, there is some $\hat{a} \in (\frac{1}{2}, 1)$ such that

$$h(\hat{a}) = 0 \text{ and } h'(\hat{a}) \geq 0.$$

Again, by the same reasoning, we must have

$$x_1(\hat{a}) + x_{2h}(\hat{a}) - 2\hat{a} - 2b > 0$$

and

$$x_1(\hat{a}) + x_{2l}(\hat{a}) - 2\hat{a} - 2b < 0.$$

Thus,

$$\begin{aligned}
h'(\hat{a}) <& \lambda(x_{2h}(\hat{a}) - x_1(\hat{a}))(x_{2h}(\hat{a}) - x_{2l}(\hat{a})) \\
&+ s(\hat{a})(\frac{\partial x_{2h}}{\partial a}(\hat{a}) - \frac{3}{2})(x_{2h}(\hat{a}) - x_1(\hat{a})) \\
&+ (1 - s(\hat{a}))(\frac{\partial x_{2l}}{\partial a}(\hat{a}) - \frac{3}{2})(x_{2l}(\hat{a}) - x_1(\hat{a})) \\
<& \lambda(x_{2h}(\hat{a}) - x_1(\hat{a}))(x_{2h}(\hat{a}) - x_{2l}(\hat{a})) \\
&- s(\hat{a})(x_{2h}(\hat{a}) - x_1(\hat{a})) \\
&- \frac{5}{6}(1 - s(\hat{a}))(x_{2l}(\hat{a}) - x_1(\hat{a})) \\
<& \lambda(x_{2h}(\hat{a}) - x_1(\hat{a}))(x_{2h}(\hat{a}) - x_{2l}(\hat{a})) \\
&- \frac{5}{6}(x_{2l}(\hat{a}) - x_1(\hat{a})).
\end{aligned}$$

From Claim A.2.3,

$$\begin{aligned}
h'(\hat{a}) <& \lambda(x_{2h}(\hat{a}) - x_1(\hat{a}))(x_{2l}(\hat{a}) - x_1(\hat{a})) \\
&- \frac{5}{6}(x_{2l}(\hat{a}) - x_1(\hat{a})) \\
=& [\lambda(x_{2h}(\hat{a}) - x_1(\hat{a})) - \frac{5}{6}](x_{2l}(\hat{a}) - x_1(\hat{a})) \\
<& 0.
\end{aligned}$$

This contradicts $h'(\hat{a}) \geq 0$.

Thus proof is complete. $\square$

A.2.7 Proof of Proposition 2.3.5

Proof. Let $a \in (0, \frac{1}{2})$ be the unique solution to $h = 0$. For this particular a, we define

$$l(\theta) := -(x_1(a) - \theta - b)^2$$

$$r(\theta) := -s(\theta)(x_{2h}(a) - \theta - b)^2 - (1 - s(\theta))(x_{2l}(a) - \theta - b)^2$$

Since a is fixed, for simplicity, we omit the dependence of x's on a. Clearly, $l(a) - r(a) = 0$.

$l(\theta)$ ($r(\theta)$) is the *ex-ante* expected payoff for sender when the state is θ and the receiver believes that the state is in $[0, a)$ ($[a, 1]$). For every $\theta \in [0, 1]$, denote $l(\theta) - r(\theta)$ as $(l - r)_\theta$, $l'(\theta) - r'(\theta)$ as $(l - r)'_\theta$, and $l''(\theta) - r''(\theta)$ as $(l - r)''_\theta$. Then

$$\begin{aligned}(l - r)_\theta =& (\lambda\theta + \frac{1 - \lambda}{2})(x_1 + x_{2h} - 2\theta - 2b)(x_{2h} - x_1) \\ &+ (-\lambda\theta + \frac{1 + \lambda}{2})(x_1 + x_{2l} - 2\theta - 2b)(x_{2l} - x_1)\end{aligned}$$

$$\begin{aligned}(l - r)'_\theta =& \lambda(x_1 + x_{2h} - 2\theta - 2b)(x_{2h} - x_1) \\ &- 2(\lambda\theta + \frac{1 - \lambda}{2})(x_{2h} - x_1) \\ &- \lambda(x_1 + x_{2l} - 2\theta - 2b)(x_{2l} - x_1) \\ &- 2(-\lambda\theta + \frac{1 + \lambda}{2})(x_{2l} - x_1)\end{aligned}$$

$$(l - r)''_\theta = -4\lambda(x_{2h} - x_{2l}) < 0$$

Then if $(l - r)_0 > 0$, by concavity, $(l - r)_\theta > 0$ for $\theta \in [0, a)$ and $(l - r)_\theta < 0$ for $\theta \in (a, 1]$. Therefore, it is sufficient to prove $(l - r)_0 > 0$.

$$(l - r)_0 = \frac{1 - \lambda}{2}(x_1 + x_{2h} - 2b)(x_{2h} - x_1) + \frac{1 + \lambda}{2}(x_1 + x_{2l} - 2b)(x_{2l} - x_1) > 0$$

$$\Longleftrightarrow b < \frac{(1-\lambda)(x_{2h}+x_1)(x_{2h}-x_1)+(1+\lambda)(x_{2l}+x_1)(x_{2l}-x_1)}{2(1-\lambda)(x_{2h}-x_1)+2(1+\lambda)(x_{2l}-x_1)}$$

Now we prove that for each λ, $\frac{3-\lambda}{12} < \frac{(1-\lambda)(x_{2h}+x_1)(x_{2h}-x_1)+(1+\lambda)(x_{2l}+x_1)(x_{2l}-x_1)}{2(1-\lambda)(x_{2h}-x_1)+2(1+\lambda)(x_{2l}-x_1)}$. Denote $\nabla = \frac{(1-\lambda)(x_{2h}+x_1)(x_{2h}-x_1)+(1+\lambda)(x_{2l}+x_1)(x_{2l}-x_1)}{2(1-\lambda)(x_{2h}-x_1)+2(1+\lambda)(x_{2l}-x_1)}$.

First, note that

$$\nabla - \frac{x_1+x_{2l}}{2}$$

$$= \frac{(1-\lambda)(x_{2h}+x_1)(x_{2h}-x_1)-(1-\lambda)(x_{2l}+x_1)(x_{2h}-x_1)}{2(1-\lambda)(x_{2h}-x_1)+2(1+\lambda)(x_{2l}-x_1)} > 0.$$

Since $\frac{d(x_1+x_{2l})}{da} > 0$, for each $\lambda \in (0,1)$, $\nabla > \frac{x_1+x_{2l}}{2} > \frac{x_1(0)+x_{2l}(0)}{2} = \frac{3-\lambda}{12}$.

Therefore, when $b < \frac{3-\lambda}{12}$, we have $(l-r)_0 > 0$. It implies that $(l-r)_\theta > 0$ for $\theta \in [0,a)$ and $(l-r)_\theta < 0$ for $\theta \in (a,1]$.

To constitute an equilibrium, we specify that when R observes off-path strategy, he believes the state is in $[0,a)$.

Thus proof is complete. $\square$

A.2.8 Checking for the NITS condition

In our setting, the NITS condition by Ying Chen, Kartik, and Sobel (2008) has the following definition.

Definition A.2.1. *An equilibrium (q^*, r^*, α^*) satisfies the no incentive to separate (NITS) condition if*

(a) $s(0)U^S(\alpha^*(q^*(0),h),0,b) + (1-s(0))U^S(\alpha^*(q^*(0),l),0,b) \geq U^S(0,0,b)$ *if* $r^*(0) = 0$ *or*

(b) $U^S(\alpha^*(q^*(0),\phi),0,b) \geq U^S(0,0,b)$ *if* $r^*(0) = 1$.

That is, an equilibrium satisfies the NITS condition if the lowest type of sender (type-0 sender) has a weakly higher payoff in equilibrium than the payoff if she can credibly reveal her type. Note that $U^S(0,0,b) = -b^2$.

Now we check the NITS condition for the three equilibria.

CS equilibrium:

Equilibrium payoff for the type-0 sender:

$$
\begin{aligned}
-(x_1(a_{CS}) - 0 - b)^2 &= -(\frac{a_{CS}}{2} - b)^2 \\
&= -(\frac{1-4b}{4} - b)^2 \\
&= -(\frac{1}{4} - 2b)^2
\end{aligned}
$$

The NITS condition is satisfied when $b \geq \frac{1}{12}$.

No-censorship equilibrium:

Equilibrium payoff for the type-0 sender:

$$
\begin{aligned}
&- s(0)(x_{1h}(a_{NC}) - 0 - b)^2 - (1 - s(0))(x_{1l}(a_{NC}) - 0 - b)^2 \\
= &- s(0)(x_{1h}^2(a_{NC}) - 2bx_{1h}(a_{NC})) - (1 - s(0))(x_{1h}^2(a_{NC}) - 2bx_{1h}(a_{NC})) - b^2
\end{aligned}
$$

Thus the NITS condition is satisfied if

$$
b \geq \frac{(1-\lambda)x_{1h}^2(a_{NC}) + (1+\lambda)x_{1h}^2(a_{NC})}{2[(1-\lambda)x_{1h}(a_{NC}) + (1+\lambda)x_{1l}(a_{NC})]}
$$

It can be verified that when b is sufficiently large, the above inequality holds.

Censor-by-lows equilibrium:

Equilibrium payoff for the type-0 sender:

$$
-(x_1(a_{CL}) - 0 - b)^2 = -(\frac{a_{CL}}{2} - b)^2
$$

The NITS condition is satisfied when $b \geq \frac{a_{CL}}{4}$. It can be verified that the inequality holds when b is sufficiently large.

A.2.9 Proof of Proposition 2.3.6

Proof. Part 1: We prove that $0 < a_{CL} < a_{NC} < \frac{1}{2}$.

Recall that condition (A) holds if and only if $g(a) = 0$ and condition (A') if and only if $h(a) = 0$.

We have

$$(g-h)(a) = (x_1(a) - a - b)^2 - s(a)(x_{1h}(a) - a - b)^2 - (1 - s(a))(x_{1l}(a) - a - b)^2.$$

Substituting $x_1(a) = \frac{a}{2}$ and rewrite:

$$\begin{aligned}(g-h)(a) =& \frac{a^2}{4} - s(a)(x_{1h}(a) - a)^2 - (1 - s(a))(x_{1l}(a) - a)^2 \\ &+ b[2s(a)x_{1h}(a) + 2(1 - s(a))x_{1l}(a) - a]\end{aligned}$$

Since

$$x_{1h}(a) \int_0^a s(\theta)d\theta + x_{1l}(a) \int_0^a (1 - s(\theta))d\theta = x_1(a) \int_0^a 1d\theta$$

and

$$x_{1h}(a) > x_{1l}(a),$$

we have

$$s(a)x_{1h}(a) + (1 - s(a))x_{1l}(a) > x_1(a) = \frac{a}{2}.$$

Thus $b[2s(a)x_{1h} + 2(1 - s(a))x_{1l} - a] > 0$.

Denote $A(a) = \frac{a^2}{4} - s(a)(x_{1h}(a) - a)^2 - (1 - s(a))(x_{1l}(a) - a)^2$. With some calculations, we have

$$A(a) = \frac{\lambda^2 a^4[(-3a^2 + 8a - 5)\lambda^2 + 5]}{36[(a-1)^2\lambda^2 - 1]^2}.$$

It can be verified that $A(a) > 0$ for $a \in (0, 1), \lambda \in (0, 1)$.

Therefore $(g-h)(a) > 0$ for all $a \in (0, 1), \lambda \in (0, 1)$. Note that $g(0) = h(0) > 0$. Given that the unique solution of $g(a) = 0$ and the unique solution of $h(a) = 0$ exist in $(0, \frac{1}{2})$, we thus have $0 < a_{CL} < a_{NC} < \frac{1}{2}$.

Part 2: We prove that $0 < a_{NC} < a_{CS} < \frac{1}{2}$.

Since $g(a) = 0$ has a unique solution in $(0, 1)$, $g(0) > 0$, and $g(1) < 0$, we have $g(a) > 0$ for $a \in [0, a_{NC})$ and $g(a) < 0$ for $a \in (a_{NC}, 1]$.

So we prove $a_{NC} < a_{CS}$ by proving $g(a_{CS}) < 0$. Substitute $b = \frac{1-2a_{CS}}{4}$ into $g(a_{CS})$ (omit x's dependence on a_{CS}),

$$\begin{aligned}
g(a_{CS}) =& s(a_{CS})[(x_{2h} - a_{CS} - \frac{1-2a_{CS}}{4})^2 - (x_{1h} - a_{CS} - \frac{1-2a_{CS}}{4})^2] \\
&+ (1 - s(a_{CS}))[(x_{2l} - a_{CS} - \frac{1-2a_{CS}}{4})^2 - (x_{1l} - a_{CS} - \frac{1-2a_{CS}}{4})^2] \\
=& s(a_{CS})(x_{1h} + x_{2h} - a_{CS} - \frac{1}{2})(x_{2h} - x_{1h}) \\
&+ (1 - s(a_{CS}))(x_{1l} + x_{2l} - a_{CS} - \frac{1}{2})(x_{2l} - x_{1l})
\end{aligned}$$

Note that $x_{1h}(a_{CS}) + x_{2h}(a_{CS}) - a_{CS} - \frac{1}{2} = x_{1h}(a_{CS}) + x_{2h}(a_{CS}) - (x_1(a_{CS}) + x_2(a_{CS})) > 0$. Also note that for $a \in (0, \frac{1}{2})$,

$$s(a)(x_{2h}(a) - x_{1h}(a)) - (1 - s(a))(x_{2l}(a) - x_{1l}(a))$$

$$= -\frac{(2a-1)\lambda[\lambda^2((a-1)a(\lambda^2+3)+2)-2]}{6[(a-1)\lambda-1][(a-1)\lambda+1](a\lambda-1)(a\lambda+1)} < 0$$

Therefore, (omit x's dependence on a_{CS})

$$\begin{aligned}
g(a_{CS}) <& (1 - s(a_{CS}))(x_{2l} - x_{1l})(x_{1l} + x_{2l} + x_{1h} + x_{2h} - 2a_{CS} - 1) \\
=& (1 - s(a_{CS}))(x_{2l} - x_{1l}) \times \frac{(a-1)a(2a-1)\lambda^2[((a-1)a+1)\lambda^2 - 1]}{3[(a-1)\lambda-1][(a-1)\lambda+1](a\lambda-1)(a\lambda+1)} \\
<& 0.
\end{aligned}$$

Thus $0 < a_{CL} < a_{NC} < a_{CS} < \frac{1}{2}$. $\square$

A.2.10 Proof of Lemma 2.3.2

Proof. Since $a_{CS}(b) = \frac{1-4b}{2}$, it's straightforward for a_{CS}. We now prove the result for a_{NC} and a_{CL}.

Part 1: We first prove the result for $a_{NC}(b)$.

Note that $a_{NC}(b)$ is implicitly defined as (omitting the subscript of a)

$$
\begin{aligned}
&s(a)(x_{2h}(a)-a-b)^2 + (1-s(a))(x_{2l}(a)-a-b)^2 \\
&-s(a)(x_{1h}(a)-a-b)^2 - (1-s(a))(x_{1l}(a)-a-b)^2 = 0
\end{aligned}
$$

Rewriting:

$$
b = \frac{NUM_1(a)}{DEN_1(a)}
$$

where

$$
\begin{aligned}
NUM_1(a) =& s(a)(x_{1h}(a)-a)^2 + (1-s(a))(x_{1l}(a)-a)^2 \\
&- s(a)(x_{2h}(a)-a)^2 - (1-s(a))(x_{2l}(a)-a)^2
\end{aligned}
$$

and

$$
\begin{aligned}
DEN_1(a) =& 2[s(a)(x_{1h}(a)-a) + (1-s(a))(x_{1l}(a)-a)] \\
&- 2[s(a)(x_{2h}(a)-a) + (1-s(a))(x_{2l}(a)-a)].
\end{aligned}
$$

Since a_{NC} is a function of b and b is a function of a_{NC}, we claim that $a_{NC}(b)$ is strictly monotone. Suppose not, then there exists $b < b'$ such that $a_{NC}(b) = a_{NC}(b')$. But this implies $b = b'$ which is a contradiction.

Now we prove $a_{NC}(b)$ is strictly decreasing. Suppose not, then $a_{NC}(b)$ is strictly increasing. Note that $a_{NC}(\frac{1}{6}) < a_{CS}(\frac{1}{6}) = \frac{1}{6}$. Define $a_{NC}(0^+)$ as $\lim_{b\to 0^+} a_{NC}(b)$, then $a_{NC}(0^+) < \frac{1}{6}$ and $NUM_1(a_{NC}(0^+)) = 0$.

Rewriting NUM_1 we have (omitting x's dependence on $a_{NC}(0^+)$)

$$\begin{aligned}
&s(a_{NC}(0^+))(x_{1h}+x_{2h}-2a_{NC}(0^+))(x_{1h}-x_{2h})\\
+&[1-s(a_{NC}(0^+))](x_{1l}+x_{2l}-2a_{NC}(0^+))(x_{1l}-x_{2l})=0
\end{aligned}$$

It implies that

$$x_{1l}(a_{NC}(0^+))+x_{2l}(a_{NC}(0^+))-2a_{NC}(0^+)<0.$$

Since $\frac{\partial(x_{1l}(a)+x_{2l}(a)-2a)}{\partial a}<0$, we have

$$\begin{aligned}
&x_{1l}(a_{NC}(0^+))+x_{2l}(a_{NC}(0^+))-2a_{NC}(0^+)\\
>&x_{1l}(\frac{1}{6})+x_{2l}(\frac{1}{6})-2\times\frac{1}{6}\\
=&\frac{1}{30(5\lambda+6)}+\frac{25}{6(\lambda-6)}+\frac{46}{45}\\
>&0
\end{aligned}$$

which contradicts $x_{1l}(a_{NC}(0^+))+x_{2l}(a_{NC}(0^+))-2a_{NC}(0^+)<0$. Therefore $a_{NC}(b)$ is strictly decreasing.

<u>Part 2: Now we prove $a_{CL}(b)$ is strictly decreasing.</u> It follows the same logic as in Part 1.

$a_{CL}(b)$ is implicitly defined as (omitting the subscript of a)

$$s(a)(x_{2h}(a)-a-b)^2+(1-s(a))(x_{2l}(a)-a-b)^2-(x_1(a)-a-b)^2=0$$

Rewriting:

$$b=\frac{NUM_2(a)}{DEN_2(a)}$$

where

$$NUM_2(a)=(x_1(a)-a)^2-s(a)(x_{2h}(a)-a)^2-(1-s(a))(x_{2l}(a)-a)^2$$

and

$$DEN_2(a)=2[(x_1(a)-a)-s(a)(x_{2h}(a)-a)-(1-s(a))(x_{2l}(a)-a)].$$

By the same argument, we know that $a_{CL}(b)$ is strictly monotone. Suppose $a_{NC}(b)$ is strictly increasing. Similarly, we have $a_{CL}(\frac{1}{6}) < a_{CS}(\frac{1}{6}) = \frac{1}{6}$. Define $a_{CL}(0^+)$ as $\lim_{b\to 0^+} a_{CL}(b)$, then $a_{CL}(0^+) < \frac{1}{6}$ and $NUM_2(a_{CL}(0^+)) = 0$.

Rewriting NUM_2 we have (omitting x's dependence on $a_{CL}(0^+)$)

$$\begin{aligned}
&s(a_{CL}(0^+))(x_1 + x_{2h} - 2a_{CL}(0^+))(x_1 - x_{2h}) \\
+&[1 - s(a_{CL}(0^+))](x_1 + x_{2l} - 2a_{CL}(0^+))(x_1 - x_{2l}) = 0
\end{aligned}$$

It implies that

$$x_1(a_{CL}(0^+)) + x_{2l}(a_{CL}(0^+)) - 2a_{CL}(0^+) < 0.$$

Since $\frac{\partial(x_1(a)+x_{2l}(a)-2a)}{\partial a} < 0$, we have

$$\begin{aligned}
&x_1(a_{CL}(0^+)) + x_{2l}(a_{CL}(0^+)) - 2a_{CL}(0^+) \\
>&x_1(\frac{1}{6}) + x_{2l}(\frac{1}{6}) - 2 \times \frac{1}{6} \\
=&\frac{37\lambda - 72}{36(\lambda - 6)} \\
>&0
\end{aligned}$$

which contradicts $x_1(a_{CL}(0^+))+x_{2l}(a_{CL}(0^+))-2a_{CL}(0^+) < 0$. Therefore $a_{CL}(b)$ is strictly decreasing. □

A.2.11 Proof of Proposition 2.4.1

Before proving Proposition 2.4.1, we first define two functions and prove a lemma.

For each $b \in (0, \frac{3-\lambda}{12})$, define $\gamma : (0,1) \to \mathbb{R}$ and $\rho : (0,1) \to \mathbb{R}$ as

$$\gamma(a) = -\int_0^a s(\theta)(x_{1h}(a) - \theta - b)^2 d\theta - \int_a^1 s(\theta)(x_{2h}(a) - \theta - b)^2 d\theta$$

$$\rho(a) = -\int_0^a (1 - s(\theta))(x_{1l}(a) - \theta - b)^2 d\theta - \int_a^1 (1 - s(\theta))(x_{2l}(a) - \theta - b)^2 d\theta$$

Lemma A.2.2. *Given $b \in (0, \frac{3-\lambda}{12})$, $\gamma + \rho$ is strictly increasing in $[a_{CL}(b), a_{NC}(b)]$.*

Proof. Fix $b \in (0, \frac{3-\lambda}{12})$, we omit the dependence of a_{CL} and a_{NC} on b. Rewrite γ we have:

$$\begin{aligned}\gamma(a) =& -\int_0^a s(\theta)(x_{1h}(a)-\theta)^2 d\theta - \int_a^1 s(\theta)(x_{2h}(a)-\theta)^2 d\theta \\ &+ 2b\int_0^a s(\theta)(x_{1h}(a)-\theta)d\theta + 2b\int_a^1 s(\theta)(x_{2h}(a)-\theta)d\theta \\ &- b^2\int_0^1 s(\theta)d\theta \\ =& -\int_0^a s(\theta)(x_{1h}(a)-\theta)^2 d\theta - \int_a^1 s(\theta)(x_{2h}(a)-\theta)^2 d\theta \\ &- b^2\int_0^1 s(\theta)d\theta\end{aligned}$$

Applying the Leibniz integral rule:

$$\begin{aligned}\gamma'(a) =& - s(a)(x_{1h}(a)-a)^2 - 2\int_0^a s(\theta)(x_{1h}(a)-\theta)\frac{\partial x_{1h}}{\partial a}d\theta \\ &+ s(a)(x_{2h}(a)-a)^2 - 2\int_a^1 s(\theta)(x_{2h}(a)-\theta)\frac{\partial x_{2h}}{\partial a}d\theta \\ =& - s(a)(x_{1h}(a)-a)^2 + s(a)(x_{2h}(a)-a)^2\end{aligned}$$

Similarly we rewrite ρ:

$$\rho(a) = -\int_0^a (1-s(\theta))(x_{1l}(a)-\theta)^2 d\theta - \int_a^1 (1-s(\theta))(x_{2l}(a)-\theta)^2 d\theta - b^2\int_0^1 (1-s(\theta))d\theta.$$

Applying the Leibniz integral rule:

$$\begin{aligned}\rho'(a) =& - (1-s(a))(x_{1l}(a)-a)^2 - 2\int_0^a (1-s(\theta))(x_{1l}(a)-\theta)\frac{\partial x_{1l}}{\partial a}d\theta \\ &+ (1-s(a))(x_{2l}(a)-a)^2 - 2\int_a^1 (1-s(\theta))(x_{2l}(a)-\theta)\frac{\partial x_{2l}}{\partial a}d\theta \\ =& - (1-s(a))(x_{1l}(a)-a)^2 + (1-s(a))(x_{2l}(a)-a)^2\end{aligned}$$

We can infer from the proof of Proposition 2.3.2 that $g(a) \geq 0$ for $a \in [a_{CL}, a_{NC}]$. That is,

$$-s(a)(x_{1h}(a)-a-b)^2 - (1-s(a))(x_{1l}(a)-a-b)^2 \geq$$

$$-s(a)(x_{2h}(a)-a-b)^2-(1-s(a))(x_{2l}(a)-a-b)^2$$

Rewriting the above inequality (omit the dependence of x's on a):

$$-s(a)(x_{1h}-a)^2-(1-s(a))(x_{1l}-a)^2+2s(a)b(x_{1h}-a)+2(1-s(a))b(x_{1l}-a)\geq$$

$$-s(a)(x_{2h}-a)^2-(1-s(a))(x_{2l}-a)^2+2s(a)b(x_{2h}-a)+2(1-s(a))b(x_{2l}-a)$$

Since $x_{1l}<x_{1h}<a<x_{2l}<x_{2h}$, we have

$$-s(a)(x_{1h}-a)^2-(1-s(a))(x_{1l}-a)^2>$$

$$-s(a)(x_{2h}-a)^2-(1-s(a))(x_{2l}-a)^2$$

which is equivalent to $\gamma'+\rho'>0$. □

Now we proceed to prove Proposition 2.4.1.

Proof. We first prove the result for the CS equilibrium.

Since $a_{CS}(b)=\frac{1-4b}{2}$, the expected payoffs for both agents can be written as functions of b:

$$EU^S_{CS}(b)=-\int_0^{a_{CS}(b)}(x_1(a_{CS}(b))-\theta-b)^2d\theta-\int_{a_{CS}(b)}^1(x_2(a_{CS}(b))-\theta-b)^2d\theta,$$

$$EU^R_{CS}(b)=-\int_0^{a_{CS}(b)}(x_1(a_{CS}(b))-\theta)^2d\theta-\int_{a_{CS}(b)}^1(x_2(a_{CS}(b))-\theta)^2d\theta.$$

Simplify:

$$\begin{aligned}EU^S_{CS}(b)=&\frac{1}{3}[(-\frac{a_{CS}(b)}{2}-b)^3-(\frac{a_{CS}(b)}{2}-b)^3]\\&+\frac{1}{3}[(\frac{a_{CS}(b)-1}{2}-b)^3-(\frac{1-a_{CS}(b)}{2}-b)^3]\end{aligned}$$

Take derivative and simplify:

$$(EU^S_{CS})'=-4b$$

Since $EU^S_{CS}(b)=EU^R_{CS}(b)-b^2$, we have $(EU^R_{CS})'=-4b+2b=-2b$.

From Lemma 2.3.2, for $b \in (0, \frac{3-\lambda}{12})$, a_{NC} and a_{CL} are functions of b. From the proof of Lemma 2.3.2, b can be written as a function of a_{NC} and a_{CL}, respectively, and moreover, the two functions are differentiable and the derivatives are negative. Therefore, $a'_{NC}(b)$ and $a'_{CL}(b)$ exists and are negative.

We then prove the result for the no-censorship equilibrium.

The ex-ante expected payoff for the sender is a function of b:

$$\begin{aligned} EU^S_{NC}(b) = &- \int_0^{a_{NC}(b)} s(\theta)(x_{1h}(a_{NC}(b)) - \theta - b)^2 d\theta \\ &- \int_0^{a_{NC}(b)} (1 - s(\theta))(x_{1l}(a_{NC}(b)) - \theta - b)^2 d\theta \\ &- \int_{a_{NC}(b)}^1 s(\theta)(x_{2h}(a_{NC}(b)) - \theta - b)^2 d\theta \\ &- \int_{a_{NC}(b)}^1 (1 - s(\theta))(x_{2l}(a_{NC}(b)) - \theta - b)^2 d\theta \end{aligned}$$

Similarly for the receiver:

$$\begin{aligned} EU^R_{NC}(b) = &- \int_0^{a_{NC}(b)} s(\theta)(x_{1h}(a_{NC}(b)) - \theta)^2 d\theta \\ &- \int_0^{a_{NC}(b)} (1 - s(\theta))(x_{1l}(a_{NC}(b)) - \theta)^2 d\theta \\ &- \int_{a_{NC}(b)}^1 s(\theta)(x_{2h}(a_{NC}(b)) - \theta)^2 d\theta \\ &- \int_{a_{NC}(b)}^1 (1 - s(\theta))(x_{2l}(a_{NC}(b)) - \theta)^2 d\theta \end{aligned}$$

In particular, $EU^S_{NC}(b) = EU^R_{NC}(b) - b^2$.

Note that the derivatives exists and $(EU^S_{NC})' = (EU^R_{NC})' - 2b$. Applying the Leibniz integral rule and simplifying:

$$\begin{aligned}(EU^R_{NC})' =& a'_{NC}(b)s(a_{NC}(b))(x_{2h}(a_{NC}(b)) - a_{NC}(b))^2\\ &- a'_{NC}(b)s(a_{NC}(b))(x_{1h}(a_{NC}(b)) - a_{NC}(b))^2\\ &+ a'_{NC}(b)(1 - s(a_{NC}(b)))(x_{2l}(a_{NC}(b)) - a_{NC}(b))^2\\ &- a'_{NC}(b)(1 - s(a_{NC}(b)))(x_{1l}(a_{NC}(b)) - a_{NC}(b))^2\\ =& a'_{NC}(b)(\gamma'(a_{NC}(b)) + \rho'(a_{NC}(b)))\end{aligned}$$

From Lemma A.2.2, $\gamma'(a_{NC}(b)) + \rho'(a_{NC}(b)) > 0$. Thus $(EU^R_{NC})' < 0$ and $(EU^S_{NC})' = (EU^R_{NC})' - 2b < 0$.

Finally we prove the result for the censor-by-lows equilibrium.

Similarly, we have:

$$\begin{aligned}EU^S_{CL}(b) =& -\int_0^{a_{CL}(b)} (x_1(a_{CL}(b)) - \theta - b)^2 d\theta\\ &- \int_{a_{CL}(b)}^1 s(\theta)(x_{2h}(a_{CL}(b)) - \theta - b)^2 d\theta\\ &- \int_{a_{CL}(b)}^1 (1 - s(\theta))(x_{2l}(a_{CL}(b)) - \theta - b)^2 d\theta\end{aligned}$$

$$\begin{aligned}EU^R_{CL}(b) =& -\int_0^{a_{CL}(b)} (x_1(a_{CL}(b)) - \theta)^2 d\theta\\ &- \int_{a_{CL}(b)}^1 s(\theta)(x_{2h}(a_{CL}(b)) - \theta)^2 d\theta\\ &- \int_{a_{CL}(b)}^1 (1 - s(\theta))(x_{2l}(a_{CL}(b)) - \theta)^2 d\theta\end{aligned}$$

Also, $EU^S_{CL}(b) = EU^R_{CL}(b) - b^2$ and $(EU^S_{CL})' = (EU^R_{CL})' - 2b$.

Applying the Leibniz integral rule and simplifying:

$$\begin{aligned}(EU^R_{CL})' =& a'_{CL}(b)s(a_{CL}(b))(x_{2h}(a_{CL}(b)) - a_{CL}(b))^2\\ &- a'_{CL}(b)s(a_{CL}(b))(x_1(a_{CL}(b)) - a_{CL}(b))^2\\ &+ a'_{CL}(b)(1 - s(a_{CL}(b)))(x_{2l}(a_{CL}(b)) - a_{CL}(b))^2\\ &- a'_{CL}(b)(1 - s(a_{CL}(b)))(x_1(a_{CL}(b)) - a_{CL}(b))^2\end{aligned}$$

Note that condition (A') implies (omitting the dependence of a_{CL} on b)

$$\begin{aligned}(x_1(a_{CL}) - a_{CL} - b)^2 =& s(a_{CL})(x_{2h}(a_{CL}) - a_{CL} - b)^2 \\ &+ (1 - s(a_{CL}))(x_{2l}(a_{CL}) - a_{CL} - b)^2\end{aligned}$$

Rewriting:

$$\begin{aligned}&(x_1(a_{CL}) - a_{CL})^2 - 2b(x_1(a_{CL}) - a_{CL}) \\ =& s(a_{CL})(x_{2h}(a_{CL}) - a_{CL})^2 \\ &- 2bs(a_{CL})(x_{2h}(a_{CL}) - a_{CL}) \\ &+ (1 - s(a_{CL}))(x_{2l}(a_{CL}) - a_{CL})^2 \\ &- 2b(1 - s(a_{CL}))(x_{2l}(a_{CL}) - a_{CL})\end{aligned}$$

which is equivalent to

$$\begin{aligned}(x_1(a_{CL}) - a_{CL})^2 - s(a_{CL})(x_{2h}(a_{CL}) - a_{CL})^2 \\ - (1 - s(a_{CL}))(x_{2l}(a_{CL}) - a_{CL})^2 \\ =& 2b[x_1(a_{CL}) - s(a_{CL})x_{2h}(a_{CL}) - (1 - s(a_{CL}))x_{2l}(a_{CL})]\end{aligned}$$

Since $x_1(a_{CL}) < a_{CL} < x_{2l}(a_{CL}) < x_{2h}(a_{CL})$, we have

$$\begin{aligned}(x_1(a_{CL}) - a_{CL})^2 <& s(a_{CL})(x_{2h}(a_{CL}) - a_{CL})^2 \\ &+ (1 - s(a_{CL}))(x_{2l}(a_{CL}) - a_{CL})^2\end{aligned}$$

Since $a'_{CL}(b) < 0$, we have $(EU^R_{CL})' < 0$, and $(EU^S_{CL})' = (EU^R_{CL})' - 2b < 0$. $\square$

A.2.12 Proof of Proposition 2.4.2

Proof. We proceed in two steps. In step 1 we show that if the censor-by-lows "equilibrium" and the no-censorship "equilibrium" (technically, they are not equilibria) have the same boundary type, then S and R's welfare is higher in the no-censorship "equilibrium".

In step 2, we show that as a moves from a_{CL} to a_{NC}, S and R's welfare in the no-censorship "equilibrium" increases.

Step 1: Define $W_{CL} : (0,1) \to \mathbb{R}$ and $W_{NC} : (0,1) \to \mathbb{R}$ as

$$\begin{aligned} W_{CL}(a) = &- \int_0^a (x_1(a) - \theta - b)^2 d\theta \\ &- \int_a^1 s(\theta)(x_{2h}(a) - \theta - b)^2 d\theta - \int_a^1 (1 - s(\theta))(x_{2l}(a) - \theta - b)^2 d\theta \end{aligned}$$

$$\begin{aligned} W_{NC}(a) = &- \int_0^a s(\theta)(x_{1h}(a) - \theta - b)^2 d\theta - \int_0^a (1 - s(\theta))(x_{1l}(a) - \theta - b)^2 d\theta \\ &- \int_a^1 s(\theta)(x_{2h}(a) - \theta - b)^2 d\theta - \int_a^1 (1 - s(\theta))(x_{2l}(a) - \theta - b)^2 d\theta \end{aligned}$$

Particularly, $W_{CL}(a_{CL})$ and $W_{NC}(a_{NC})$ are the sender's welfare in the censor-by-lows equilibrium and the no-censorship equilibrium, respectively.

$$\begin{aligned} &W_{CL}(a) - W_{NC}(a) \\ =& \int_0^a s(\theta)(x_{1h}(a) - \theta - b)^2 d\theta + \int_0^a (1 - s(\theta))(x_{1l}(a) - \theta - b)^2 d\theta \\ &- \int_0^a (x_1(a) - \theta - b)^2 d\theta \\ =& \int_0^a s(\theta)[(x_{1h}(a) - \theta)^2 - 2b(x_{1h}(a) - \theta)] d\theta \\ &+ \int_0^a (1 - s(\theta))[(x_{1l}(a) - \theta)^2 - 2b(x_{1l}(a) - \theta)] d\theta \\ &- \int_0^a [(x_1(a) - \theta)^2 - 2b(x_1(a) - \theta)] d\theta \\ =& \int_0^a s(\theta)(x_{1h}(a) - \theta)^2 d\theta + \int_0^a (1 - s(\theta))(x_{1l}(a) - \theta)^2 d\theta - \int_0^a (x_1(a) - \theta)^2 d\theta \\ =& - x_{1h}(a) \int_0^a s(\theta)\theta d\theta - x_{1l}(a) \int_0^a (1 - s(\theta))\theta d\theta + x_1(a) \int_0^a \theta d\theta \end{aligned}$$

Note that by Lemma 2.3.1, $x_1(a)$ can be expressed as a convex combination of $x_{1h}(a)$ and $x_{1l}(a)$:

$$x_{1h}(a)\int_0^a s(\theta)d\theta + x_{1l}(a)\int_0^a (1-s(\theta))d\theta = x_1(a)\int_0^a 1d\theta.$$

Also, note that

$$\frac{\int_0^a s(\theta)d\theta}{\int_0^a 1d\theta} \div \frac{\int_0^a s(\theta)\theta d\theta}{\int_0^a \theta d\theta} = \frac{x_1(a)}{x_{1h}(a)} < 1$$

and

$$\frac{\int_0^a (1-s(\theta))d\theta}{\int_0^a 1d\theta} \div \frac{\int_0^a (1-s(\theta))\theta d\theta}{\int_0^a \theta d\theta} = \frac{x_1(a)}{x_{1l}(a)} > 1.$$

Hence for all $a \in (0,1)$,

$$\begin{aligned}
& W_{CL}(a) - W_{NC}(a) \\
=& -x_{1h}(a)\int_0^a s(\theta)\theta d\theta - x_{1l}(a)\int_0^a (1-s(\theta))\theta d\theta + x_1(a)\int_0^a \theta d\theta \\
=& [x_1(a)\int_0^a 1d\theta - x_{1h}(a)\int_0^a s(\theta)\theta d\theta \cdot \frac{\int_0^a 1d\theta}{\int_0^a \theta d\theta} - x_{1l}(a)\int_0^a (1-s(\theta))\theta d\theta \cdot \frac{\int_0^a 1d\theta}{\int_0^a \theta d\theta}]\frac{\int_0^a \theta d\theta}{\int_0^a 1d\theta} \\
<& [x_1(a)\int_0^a 1d\theta - x_{1h}(a)\int_0^a s(\theta)d\theta - x_{1l}(a)\int_0^a (1-s(\theta))d\theta]\frac{\int_0^a \theta d\theta}{\int_0^a 1d\theta} \\
=& 0.
\end{aligned}$$

The inequality above is because the "weight" of the larger number decreases (i.e., $\int_0^a s(\theta)\theta d\theta \cdot \frac{\int_0^a 1d\theta}{\int_0^a \theta d\theta} > \int_0^a s(\theta)d\theta$) and the "weight" of the smaller number increases (i.e., $\int_0^a (1-s(\theta))\theta d\theta \cdot \frac{\int_0^a 1d\theta}{\int_0^a \theta d\theta} < \int_0^a (1-s(\theta))d\theta$).

Step 2: We want to show: For every a and a' such that $a_{CL} \leq a < a' \leq a_{NC}$, $W_{NC}(a) < W_{NC}(a')$.

Note that $W_{NC}(a) = \gamma(a)+\rho(a)$. From Lemma A.2.2, we have $\gamma+\rho$ is strictly increasing in $[a_{CL}, a_{NC}]$.

Therefore, as a increases from a_{CL} to a_{NC}, $W_{NC}(a)$ increases.

Thus $W_{NC}(a_{NC}) > W_{NC}(a_{CL}) > W_{CL}(a_{CL})$. Since $EU^S_{CL} = EU^R_{CL} - b^2$ and $EU^S_{NC} = EU^R_{NC} - b^2$, the proof is complete.

□

A.2.13 Proof of Claim 2.4.1

Proof. For simple notation, sometimes we omit the dependence of x on a.

$$\begin{aligned}
&W_{CS}(a) - W_{NC}(a) \\
=&\int_0^a s(\theta)(x_{1h} - \theta - b)^2 d\theta + \int_0^a (1 - s(\theta))(x_{1l} - \theta - b)^2 d\theta \\
&+ \int_a^1 s(\theta)(x_{2h} - \theta - b)^2 d\theta + \int_a^1 (1 - s(\theta))(x_{2l} - \theta - b)^2 d\theta \\
&- \int_0^a (x_1 - \theta - b)^2 d\theta - \int_a^1 (x_2 - \theta - b)^2 d\theta \\
=&\int_0^a s(\theta)(x_{1h} - \theta)^2 d\theta + \int_0^a (1 - s(\theta))(x_{1l} - \theta)^2 d\theta \\
&+ \int_a^1 s(\theta)(x_{2h} - \theta)^2 d\theta + \int_a^1 (1 - s(\theta))(x_{2l} - \theta)^2 d\theta \\
&- \int_0^a (x_1 - \theta)^2 d\theta - \int_a^1 (x_2 - \theta)^2 d\theta \\
=&- x_{1h}(a) \int_0^a s(\theta)\theta d\theta - x_{1l}(a) \int_0^a (1 - s(\theta))\theta d\theta + x_1(a) \int_0^a \theta d\theta \qquad (1) \\
&- x_{2h}(a) \int_a^1 s(\theta)\theta d\theta - x_{2l}(a) \int_a^1 (1 - s(\theta))\theta d\theta + x_2(a) \int_a^1 \theta d\theta \qquad (2)
\end{aligned}$$

In the proof of Proposition 2.4.2, we already proved that the part on line (1) is negative. Using the same reasoning, we can prove that the part on line (2) is negative.

Thus $W_{CS}(a) - W_{NC}(a) < 0$. □

A.2.14 Proof of Claim 2.4.2

Proof. With some calculation

$$\begin{aligned}
W_{CS}(a) =&\frac{1}{3}[(-\frac{a}{2} - b)^3 - (\frac{a}{2} - b)^3] \\
&+ \frac{1}{3}[(\frac{a-1}{2} - b)^3 - (\frac{1-a}{2} - b)^3],
\end{aligned}$$

hence

$$\begin{aligned}
W'_{CS}(a) =& - \frac{1}{2}(-\frac{a}{2} - b)^2 - \frac{1}{2}(\frac{a}{2} - b)^2 \\
&+ \frac{1}{2}(\frac{a-1}{2} - b)^2 + \frac{1}{2}(\frac{1-a}{2} - b)^2 \\
=&\frac{1}{2}(\frac{1}{2} - a)
\end{aligned}$$

which is positive for $a < \frac{1}{2}$. $\square$

A.3 Omitted Proofs in Chapter 3

A.3.1 Proof of Lemma 3.3.1

Proof. First ignoring the upper and lower bounds ($\bar{\theta}$ and $\underline{\theta}$). Without loss of generality, we solve the monopoly pricing problem with a product located at $\frac{\underline{\theta}+\bar{\theta}}{2}$.

In a one-period setting, the buyer will accept the offer (v, p) if and only if $\bar{u} - a(v-\theta)^2 - p \geq 0$. Rewriting:

$$v - \sqrt{\frac{\bar{u}-p}{a}} \leq \theta \leq v + \sqrt{\frac{\bar{u}-p}{a}}.$$

Therefore the firm's problem is

$$\max_p 2p\sqrt{\frac{\bar{u}-p}{a}}$$

Define $\Pi(p) = 2p\sqrt{\frac{\bar{u}-p}{a}}$.

Taking derivative:

$$\Pi'(p) = -\frac{1}{a}(\frac{\bar{u}-p}{a})^{-\frac{1}{2}}p + 2(\frac{\bar{u}-p}{a})^{\frac{1}{2}}$$

Second-order derivative:

$$\Pi''(p) = -\frac{2}{a}(\frac{\bar{u}-p}{a})^{-\frac{1}{2}} - \frac{1}{2a^2}(\frac{\bar{u}-p}{a})^{-\frac{3}{2}}p < 0$$

The first order condition is satisfied at $p = \frac{2}{3}\bar{u}$.

This is the solution to the original problem when the interval of accepting consumer types doesn't exceed $[\underline{\theta}, \bar{\theta}]$:

$$2\sqrt{\frac{\bar{u}-\frac{2}{3}\bar{u}}{a}} < \bar{\theta} - \underline{\theta}$$

$$\Leftrightarrow \bar{u} < \frac{3a}{4}(\bar{\theta} - \underline{\theta})^2$$

In this case, not all types in $[\underline{\theta}, \bar{\theta}]$ accepts the offer, and the firm's expected revenue is $\Pi(\frac{2}{3}\bar{u}) = \frac{4\bar{u}}{3}\sqrt{\frac{\bar{u}}{3a}}$.

If $\bar{u} \geq \frac{3a}{4}(\bar{\theta} - \underline{\theta})^2$, the contract $(\frac{\underline{\theta}+\bar{\theta}}{2}, \frac{2}{3}\bar{u})$ will be accepted by all types. The firm can actually increase the price as long as the surplus of type $\underline{\theta}$ and $\bar{\theta}$ is not negative. We have a corner solution in this case, and the price satisfies

$$\bar{u} - a(\frac{\underline{\theta}+\bar{\theta}}{2} - \underline{\theta})^2 - p = 0$$

$$\Leftrightarrow p = \bar{u} - \frac{a}{4}(\bar{\theta} - \underline{\theta})^2.$$

The expected revenue in this case is $\bar{u}(\bar{\theta} - \underline{\theta}) - \frac{a}{4}(\bar{\theta} - \underline{\theta})^3$. □

A.3.2 Proof of Lemma 3.3.2

Proof. Suppose not, then we have some optimal pair (v_l, p_l) and (v_r, p_r) with $\bar{\theta}_l > \underline{\theta}_r$.

We now show that there exists a new pair that gives the firm a higher expected revenue.

Case 1: $\bar{\theta}_l \geq \bar{\theta}_r$.

In this case $p_l \leq p_r$. Note that in the original pair (v_l, p_l) and (v_r, p_r), consumer types in $[\underline{\theta}_l, \underline{\theta}_r)$ and $(\bar{\theta}_r, \bar{\theta}_l]$ choose (v_l, p_l), and types in $[\underline{\theta}_r, \bar{\theta}_r]$ choose whichever gives them higher payoff.

Since $p_r \geq p_l$, the revenue is bounded above by $(width_l - width_r)p_l + width_r \cdot p_r$ where $width_l = \bar{\theta}_l - \underline{\theta}_l$ and $width_r = \bar{\theta}_r - \underline{\theta}_r$. Consider a new pair (v_l', p_l') and (v_r', p_r) such that 1) (v_r', p_r) covers $[\underline{\theta}_r', \bar{\theta}_l]$ which has the same width as $[\underline{\theta}_r, \bar{\theta}_r]$; 2) (v_l', p_l') covers $[\underline{\theta}_l, \underline{\theta}_r']$. Then the revenue with the new pair is

$$\begin{aligned}&(width_l - width_r)p_l' + width_r \cdot p_r \\ >&(width_l - width_r)p_l + width_r \cdot p_r\end{aligned}$$

where the inequality is because (v_l', p_l') covers a smaller range of types than (v_l, p_l).

Case 2: $\bar{\theta}_l < \bar{\theta}_r$.

In the original pair (v_l, p_l) and (v_r, p_r), consumer types in $[\underline{\theta}_l, \underline{\theta}_r)$ choose (v_l, p_l), types in $(\bar{\theta}_l, \bar{\theta}_r]$ choose (v_r, p_r), and types in $[\underline{\theta}_r, \bar{\theta}_l]$ choose whichever gives them a higher payoff.

Again, define $width_l$ and $width_r$ in the same way as in Case 1. Define $Olap$ as $\bar{\theta}_l - \underline{\theta}_r$. That is, $Olap$ is the width of the overlapping sub-interval. Then the revenue is bounded above by

$$\begin{aligned} \max\{width_l \cdot p_l + (width_r - Olap) \cdot p_r, \quad (3) \\ (width_l - Olap) \cdot p_l + width_r \cdot p_r\} \quad (4) \end{aligned}$$

When $p_l \leq p_r$, the term in Line (2) is larger. Consider a new pair $(\hat{v}_l, \hat{p}_l)$ and (v_r, p_r) such that $(\hat{v}_l, \hat{p}_l)$ covers $[\underline{\theta}_l, \underline{\theta}_r]$. Since it covers a smaller interval, $\hat{p}_l > p_l$. This new pair gives a revenue of $(width_l - Olap) \cdot \hat{p}_l + width_r \cdot p_r$, which is higher than the upper bound of the original revenue.

When $p_l > p_r$, the term in Line (1) is larger. Similarly, consider a new pair (v_l, p_l) and $(\hat{v}_r, \hat{p}_r)$ such that $(\hat{v}_r, \hat{p}_r)$ covers $[\bar{\theta}_l, \bar{\theta}_r]$. Again, we have $\hat{p}_r > p_r$. This new pair gives a revenue of $width_l \cdot p_l + (width_r - Olap) \cdot \hat{p}_r$, which is higher than the upper bound of the original revenue.

Therefore, in any optimal pair (v_l, p_l) and (v_r, p_r), we have $\bar{\theta}_l \leq \underline{\theta}_r$. □

A.3.3 Proof of Lemma 3.3.3

Proof. Note that $\bar{u} \geq \frac{3a}{16}$ is equivalent to $w^* \geq \frac{1}{2}$. In this case, all types are covered, and the firm offers product $\frac{1}{4}$ and $\frac{3}{4}$. The price of the two products satisfy

$$\bar{u} - a(0 - \frac{1}{4})^2 = p$$

$$\Rightarrow p = \bar{u} - \frac{a}{16}.$$

$\bar{u} < \frac{3a}{16}$ is equivalent to $w^* < \frac{1}{2}$. In this case, only a subset of types are covered by some contract, and the width of each interval of types is w^*. Suppose product v_l covers $[\underline{\theta}_l, \bar{\theta}_l]$ and v_r covers $[\underline{\theta}_r, \bar{\theta}_r]$.

By Lemma 3.3.2, $0 \leq \underline{\theta}_l < \bar{\theta}_l \leq \underline{\theta}_r < \bar{\theta}_r \leq 1$, which are exactly the conditions listed in the lemma.

In this case,

$$
\begin{aligned}
&\bar{u} - a(\underline{\theta}_l - \frac{\bar{\theta}_l - \underline{\theta}_l}{2})^2 = p \\
\Rightarrow &\bar{u} - \frac{a{w^*}^2}{4} = p \\
\Rightarrow &p = \frac{2}{3}\bar{u}.
\end{aligned}
$$

□

A.3.4 Proof of Lemma 3.3.4

Proof. Similar to that of Lemma 3.3.2, and is done by contradiction. □

A.3.5 Proof of Lemma 3.3.5

Proof. The proof is similar to that of Lemma 3.3.3. $\bar{u} \geq \frac{a}{12}$ corresponds to the case that all consumer types are covered whereas $\bar{u} < \frac{a}{12}$ is the case where some types are left out. □

A.3.6 Proof of Proposition 3.4.1

Proof. Given the consumer's strategy, it is clear that the firm has no profitable deviation as it's already providing the optimal pair. Given the firm's strategy, if the consumer deviates in the first period, the highest surplus she can get in period 2 is zero. She also has no incentive to deviate in t_2 as it's the last period. □

A.3.7 Proof of Proposition 3.4.2

Proof. It's straightforward to see that in any equilibria where all types buy some product in t_1, the equilibrium in Proposition3.4.1 indeed gives the firm the highest ex-ante surplus. The reason is as follows. In all such equilibria, consumer types are pooled into two groups in t_2: those who bought v_l and those who bought v_r. In this case, the best the firm can do is to provide the optimal pair in both periods.

On the contrary, in equilibria where some types don't buy any product in t_1, consumer types are pooled into three segments in t_2: those who bought v_l, those who bought v_r, and those who didn't buy anything. Recall that $\Theta(v, p)$ is the set of types served by (v, p). We denote $\Theta(v_l, p_l) \equiv [A, B]$ and $\Theta(v_r, p_r) \equiv [C, D]$ for some $0 \leq A < B < C < D \leq 1$. Notice that if this equilibrium is firm-optimal, then $A = 0$ and $D = 1$. This is because, if $A > 0$ or $D < 1$, then those types who didn't buy anything does not constitute an interval, and thus the expected revenue from those types can be increased by shifting $[A, B]$ towards 0 or shifting $[C, D]$ towards 1.

Formally, consider the following kind of equilibrium. In t_1, the firm provides (v_l, p_l) and (v_r, p_r) with $\Theta(v_l, p_l) = [0, B]$ and $\Theta(v_r, p_r) = [C, 1]$. B and C can be expressed in terms of v_i ($i \in \{l, r\}$):

$$B = 2v_l, \ C = 2v_r - 1.$$

Prices can also expressed in terms of v_i ($i \in \{l, r\}$):

$$\bar{u} - a(0 - v_l)^2 - p_l = 0,$$

$$\bar{u} - a(1 - v_r)^2 - p_r = 0.$$

That is, $p_l = \bar{u} - av_l^2$ and $p_r = \bar{u} - a(1 - v_r)^2$. To maximize its expected revenue among all equilibria of this kind, in t_2, the firm offers $(v_l, \bar{u} - av_l^2)$ (or $(v_r, \bar{u} - a(1 - v_r)^2)$) if the consumer bought v_l (or v_r) in t_1.

If the consumer didn't buy anything, the firm offers $(v_l + v_r - \frac{1}{2}, p_m)$ such that $\bar{u} - a[2v_l - (v_l + v_r - \frac{1}{2})]^2 - p_m = 0$. Therefore, $p_m = \bar{u} - a(v_l - v_r + \frac{1}{2})^2$. In other words, if the firm observes that the consumer didn't buy any product, it believes that the consumer type is between $2v_l$ and $2v_r - 1$, and offers the optimal contract for that interval. It's easy to confirm that no type wants to mimic any type outside her group.

Thus the firm's ex-ante surplus is

$$2v_l(\bar{u} - av_l^2)(1 + \delta) + (2v_r - 1 - 2v_l)[\bar{u} - a(v_l - v_r + \frac{1}{2})^2]\delta + (2 - 2v_r)[\bar{u} - a(1 - v_r)^2](1 + \delta),$$

which can be rewritten as

$$wid_l(\bar{u} - \frac{a}{4}wid_l^2)(1+\delta) + wid_m(\bar{u} - \frac{a}{4}wid_m^2)\delta + wid_r(\bar{u} - \frac{a}{4}wid_r^2)(1+\delta) \quad (5)$$

where wid_l, wid_m, and wid_r are the width of the left interval, the middle interval, and the right interval, respectively.

We want to find the maximum value of (3) and then compare it to $(1+\delta)(\bar{u} - \frac{a}{16})$.

To this end, we first prove the following claim.

Claim A.3.1. *For each given width of the middle interval wid_m, the value of (3) is maximized when $wid_l = wid_r$.*

Proof. Denote $wid_m = 1 - \triangle$. Then $wid_l + wid_r = \triangle$. Maximizing (3) is equivalent to maximizing

$$wid_l(\bar{u} - \frac{a}{4}wid_l^2) + (\triangle - wid_l)[\bar{u} - \frac{a}{4}(\triangle - wid_l)^2].$$

The first-order derivative w.r.p. to wid_l is $-\frac{3a}{4}wid_l^2 + \frac{3a}{4}(\triangle - wid_l)^2$, which is equal to 0 when $wid_l = \triangle - wid_l$. The second-order derivative $-\frac{3a}{2}wid_l - \frac{3a}{2}(\triangle - wid_l)$ is negative.

□

Now we want to find the maximum value of:

$$wid_l(\bar{u} - \frac{a}{4}wid_l^2)(1+\delta) + (1-2wid_l)[\bar{u} - \frac{a}{4}(1-2wid_l)^2]\delta + wid_l(\bar{u} - \frac{a}{4}wid_l^2)(1+\delta) \quad (6)$$

The first-order derivative w.r.p. to wid_l is $2\bar{u} + \frac{3a}{2}\delta(1-2wid_l)^2 - \frac{3a}{2}(1+\delta)wid_l^2$. Since $\bar{u} \geq \frac{3a}{4}$, $0 < \delta \leq 1$, and $wid_l \leq \frac{1}{2}$, the derivative is no less than $2 \times \frac{3a}{4} - \frac{3a}{2} \times 2 \times \frac{1}{4} = \frac{3a}{4} > 0$. The second-order derivative is negative.

Thus for $wid_l \in (0, \frac{1}{2})$, (4) is bounded above by $\frac{1}{2}(\bar{u} - \frac{a}{4} \times \frac{1}{4})(1+\delta) + \frac{1}{2}(\bar{u} - \frac{a}{4} \times \frac{1}{4})(1+\delta) = (1+\delta)(\bar{u} - \frac{a}{16})$.

Therefore, the equilibrium is firm-optimal. □

A.3.8 Proof of Proposition 3.4.3

Proof. In the proposed equilibrium, given the consumer's strategy, the firm offers the optimal pair in period 1 and the optimal triple in period 2. Thus the firm has no incentive to deviate in either period.

For the consumer, depending on the product purchased and the opt-out choice, her type space can be partitioned into 4 intervals:

(a) When $\theta \in [0, \frac{1}{3})$, the consumer accepts $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ in period 1 and accepts $(\frac{1}{6}, \bar{u} - \frac{a}{36})$ in period 2;

(b) When $\theta \in [\frac{1}{3}, \frac{1}{2}]$, the consumer accepts $(\frac{1}{4}, \bar{u} - \frac{a}{16})$ in period 1, chooses to opt out, and accepts $(\frac{1}{2}, \bar{u} - \frac{a}{36})$ in period 2;

(c) When $\theta \in (\frac{1}{2}, \frac{2}{3}]$, the consumer accepts $(\frac{3}{4}, \bar{u} - \frac{a}{16})$ in period 1, chooses to opt out, and accepts $(\frac{1}{2}, \bar{u} - \frac{a}{36})$ in period 2;

(d) When $\theta \in (\frac{2}{3}, 1]$, the consumer accepts $(\frac{3}{4}, \bar{u} - \frac{a}{16})$ in period 1 and accepts $(\frac{5}{6}, \bar{u} - \frac{a}{36})$ in period 2.

Given the firm's strategy, if the consumer deviates from her opt-in/out choice, her expected payoff in period 2 will decrease. If she deviates from her period-1 purchase decision, her expected payoffs will decrease in both periods. Moreover, since period 2 is the last period, she has no incentive to deviate from the purchase decision in that period.

Therefore, it is indeed an equilibrium. □

A.3.9 Proof of Proposition 3.4.4

Proof. First, consider equilibria where all types buy some product in period 1. In such equilibria, at the beginning of period 2, the firm can segment the market into at most three sub-markets - those who bought v_l, those who bought v_r, and those who chose to opt out. In period 1, the firm can segment the market into at most two sub-market since all types buy some product. Therefore, the expected revenue from period 1 is bounded above by the revenue of offering the optimal pair (which gives $\bar{u} - \frac{a}{16}$). The expected revenue from period 2 is bounded above by the revenue of offering the optimal triple (which gives $\bar{u} - \frac{a}{36}$). Thus for any equilibria where all types buy some product in period 1, the ex-ante PS is bounded above by $\bar{u} - \frac{a}{16} + \delta(\bar{u} - \frac{a}{36})$.

Next, we consider equilibria where some types don't buy anything in period 1. For equilibria of this kind, the potential firm-optimal equilibria induce three segments in t_1 (buy v_l, buy v_r, and don't buy anything) and four segments in t_2 (bought v_l, bought v_r, didn't buy anything, and opted out). We first rule out some candidate equilibria by the following claim.

Claim A.3.2. *Equilibria where a positive measure of types reject both contracts in t_1 and opt out in t_2 are not firm-optimal.*

Proof. Denote the set of types who don't buy anything in t_1 as $\Theta_{1\emptyset}$. In t_2, $\Theta_{1\emptyset}$ can be partitioned into two (possibly empty) sets - $\Theta_{1\emptyset}^{in}$ and $\Theta_{1\emptyset}^{out}$. The two sets correspond to types in $\Theta_{1\emptyset}$ who choose opt-in and opt-out, respectively.

If $\Theta_{1\emptyset}^{in} = \emptyset$, then consumer types are pooled into three segments in t_2 - those bought v_l,those bought v_r, and those who opted out. The equilibrium is not firm-optimal since equilibria with four segments in t_2 is superior.

If $\Theta_{1\emptyset}^{in} \neq \emptyset$, the consumer types are pooled into four segments in t_2. Consider the sub-market $[0,1]\backslash\Theta_{1\emptyset}^{in}$. In this sub-market, at least types in $\Theta_{1\emptyset}\backslash\Theta_{1\emptyset}^{in}$ reject both contracts in t_1. We can construct a new pair of period-1 contracts to increase the firm's revenue in period 1 while keeping its period-2 revenue unchanged. Therefore the candidate equilibrium is not firm-optimal.

□

Now we consider equilibria where every type who didn't buy anything in t_1 chooses opt-in. $\Theta_{1\emptyset}$ is an interval and we denote its width as $wid_\emptyset$. We fixed $wid_\emptyset$ and calculate the maximum ex-ante PS, and then argue that the highest of those maximum values are attained at $wid_\emptyset = 0$.

Note that when $wid_\emptyset$ is fixed, the sub-market $[0,1]\backslash\Theta_{1\emptyset}$ can be treated as a market where all types bought some product in t_1. According to the first case of this proof, in the firm-optimal "equilibrium" of the sub-market, the firm offers the optimal pair of contracts on $[0, 1-wid_\emptyset]$ in t_1 and offers the optimal triple on $[0, 1-wid_\emptyset]$ in t_2. Therefore the maximum ex-ante PS with given $wid_\emptyset$ is

$$\bar{u}(1 - wid_\emptyset) - \frac{a}{16}(1 - wid_\emptyset)^3 + \delta[\bar{u}(1 - wid_\emptyset) - \frac{a}{36}(1 - wid_\emptyset)^3]$$
$$+\delta(\bar{u} \cdot wid_\emptyset - \frac{a}{4} \cdot wid_\emptyset^3).$$

The above expression is a function of $wid_\emptyset$. Taking derivative we get

$$-\bar{u} + \frac{3a}{16}(1 - wid_\emptyset)^2 + \delta\frac{a}{12}(1 - wid_\emptyset)^2 - \delta\frac{3a}{4}wid_\emptyset^2.$$

Since $\bar{u} \geq \frac{3a}{4}$, we have

$$\begin{aligned}
& -\bar{u} + \frac{3a}{16}(1 - wid_\emptyset)^2 + \delta\frac{a}{12}(1 - wid_\emptyset)^2 - \delta\frac{3a}{4}wid_\emptyset^2 \\
\leq & -\frac{3a}{4} + \frac{3a}{16}(1 - wid_\emptyset)^2 + \delta\frac{a}{12}(1 - wid_\emptyset)^2 - \delta\frac{3a}{4}wid_\emptyset^2 \\
< & -\frac{3a}{4} + \frac{3a}{16} + \frac{a}{12} \\
= & -\frac{23}{48}a < 0.
\end{aligned}$$

Thus highest of the maximum values are attained at $wid_\emptyset = 0$, which corresponds to the equilibrium in Proposition 3.4.4. Therefore this equilibrium is firm-optimal. □

A.3.10 Proof of Proposition 3.4.5

Proof. In the equilibrium without opt-out, the type space is divided into two intervals.

For $\theta \in [0, \frac{1}{2}]$, the expected utility is

$$\begin{aligned}
& (1 + \delta)[\bar{u} - a(\theta - \frac{1}{4})^2 - (\bar{u} - \frac{a}{16})] \\
= & a(1 + \delta)(-\theta^2 + \frac{\theta}{2}).
\end{aligned}$$

For $\theta \in (\frac{1}{2}, 1]$, the expected utility is

$$\begin{aligned}
& (1 + \delta)[\bar{u} - a(\theta - \frac{3}{4})^2 - (\bar{u} - \frac{a}{16})] \\
= & a(1 + \delta)(-\theta^2 + \frac{3}{2}\theta - \frac{1}{2}).
\end{aligned}$$

Thus the *ex-ante* CS is

$$\begin{aligned}
&\int_0^{\frac{1}{2}} a(1+\delta)(-\theta^2+\frac{\theta}{2})d\theta + \int_{\frac{1}{2}}^1 a(1+\delta)(-\theta^2+\frac{3}{2}\theta-\frac{1}{2})d\theta \\
=&a(1+\delta)[-\frac{1}{3}\theta^3|_0^1+\frac{1}{4}\theta^2|_0^{\frac{1}{2}}+(\frac{3}{4}\theta^2-\frac{1}{2}\theta)|_{\frac{1}{2}}^1] \\
=&\frac{a}{24}(1+\delta).
\end{aligned}$$

In the equilibrium with opt-out, as previously shown, the type space is divided into four intervals.

For $\theta \in [0, \frac{1}{3})$, the expected utility is

$$\begin{aligned}
&\bar{u}-a(\theta-\frac{1}{4})^2-(\bar{u}-\frac{a}{16}) \\
&+\delta[\bar{u}-a(\theta-\frac{1}{6})^2-(\bar{u}-\frac{a}{36})] \\
=&a(-\theta^2+\frac{\theta}{2})+a\delta(-\theta^2+\frac{\theta}{3}).
\end{aligned}$$

For $\theta \in [\frac{1}{3}, \frac{1}{2}]$,the expected utility is

$$\begin{aligned}
&\bar{u}-a(\theta-\frac{1}{4})^2-(\bar{u}-\frac{a}{16}) \\
&+\delta[\bar{u}-a(\theta-\frac{1}{2})^2-(\bar{u}-\frac{a}{36})] \\
=&a(-\theta^2+\frac{\theta}{2})+a\delta(-\theta^2+\theta-\frac{2}{9}).
\end{aligned}$$

For $\theta \in (\frac{1}{2}, \frac{2}{3}]$, the expected utility is

$$\begin{aligned}
&\bar{u}-a(\theta-\frac{3}{4})^2-(\bar{u}-\frac{a}{16}) \\
&+\delta[\bar{u}-a(\theta-\frac{1}{2})^2-(\bar{u}-\frac{a}{36})] \\
=&a(-\theta^2+\frac{3}{2}\theta-\frac{1}{2})+a\delta(-\theta^2+\theta-\frac{2}{9}).
\end{aligned}$$

For $\theta \in (\frac{2}{3}, 1]$, the expected utility is

$$\begin{aligned}
&\bar{u}-a(\theta-\frac{3}{4})^2-(\bar{u}-\frac{a}{16}) \\
&+\delta[\bar{u}-a(\theta-\frac{5}{6})^2-(\bar{u}-\frac{a}{36})] \\
=&a(-\theta^2+\frac{3}{2}\theta-\frac{1}{2})+a\delta(-\theta^2+\frac{5}{3}\theta-\frac{2}{3}).
\end{aligned}$$

Thus the *ex-ante* CS is

$$
\begin{aligned}
&\int_0^{\frac{1}{3}}[a(-\theta^2+\frac{\theta}{2})+a\delta(-\theta^2+\frac{\theta}{3})]d\theta\\
&+\int_{\frac{1}{3}}^{\frac{1}{2}}[a(-\theta^2+\frac{\theta}{2})+a\delta(-\theta^2+\theta-\frac{2}{9})]d\theta\\
&+\int_{\frac{1}{2}}^{\frac{2}{3}}[a(-\theta^2+\frac{3}{2}\theta-\frac{1}{2})+a\delta(-\theta^2+\theta-\frac{2}{9})]d\theta\\
&+\int_{\frac{2}{3}}^{1}[a(-\theta^2+\frac{3}{2}\theta-\frac{1}{2})+a\delta(-\theta^2+\frac{5}{3}\theta-\frac{2}{3})]d\theta\\
=&\frac{a}{24}+\frac{a\delta}{54}\\
<&\frac{a}{24}(1+\delta).
\end{aligned}
$$

□

A.3.11 Proof of Proposition 3.4.6

Proof. For all types, the difference is in the second period. We calculate the difference in expected utilities below (*with* - *without*).

For $\theta \in [0, \frac{1}{3})$, the difference is

$$
\begin{aligned}
&a\delta(-\theta^2+\frac{\theta}{3})-a\delta(-\theta^2+\frac{\theta}{2})\\
=&-\frac{\theta}{6}a\delta.
\end{aligned}
$$

For $\theta \in [\frac{1}{3}, \frac{1}{2}]$,the difference is

$$
\begin{aligned}
&a\delta(-\theta^2+\theta-\frac{2}{9})-a\delta(-\theta^2+\frac{\theta}{2})\\
=&(\frac{\theta}{2}-\frac{2}{9})a\delta.
\end{aligned}
$$

For $\theta \in (\frac{1}{2}, \frac{2}{3}]$, the difference is

$$
\begin{aligned}
&a\delta(-\theta^2+\theta-\frac{2}{9})-a\delta(-\theta^2+\frac{3}{2}\theta-\frac{1}{2})\\
=&(-\frac{\theta}{2}+\frac{5}{18})a\delta.
\end{aligned}
$$

For $\theta \in (\frac{2}{3}, 1]$, the difference is

$$
\begin{aligned}
&a\delta(-\theta^2 + \frac{5}{3}\theta - \frac{2}{3}) - a\delta(-\theta^2 + \frac{3}{2}\theta - \frac{1}{2}) \\
=&(\frac{\theta}{6} - \frac{1}{6})a\delta.
\end{aligned}
$$

Clearly, the difference is positive for $\theta \in (\frac{4}{9}, \frac{5}{9})$, negative for $\theta \in (0, \frac{4}{9}) \cup (\frac{5}{9}, 1)$, and zero for $\theta \in \{0, \frac{4}{9}, \frac{5}{9}, 1\}$. $\square$

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