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Extension of Preferences to an Ordered Set

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Abstract If a decision maker prefers x to y to z, would he choose orderd set $[x, z]$ or $[y, x]$? This article studies extension of preferences over individual alternatives to an ordered set which is prevalent in closed ballot elections with proportional representation and other real life problems where the decision maker is to choose from groups with an associated hierarchy inside. I introduce five ordinal decision rules: highest-position, top-q, lexicographic order, max-best, highest-of-best rules and provide axiomatic characterization of them. I also investigate the relationship between ordinal decision rules and the expected utility rule. In particular, whether some ordinal rules induce the same (weak) ranking of ordered sets as the expected utility rule.

Keywords Extension of preferences · Ordered Set · List · Expected utility

1 Introduction

Decision makers often need to choose among different sets of alternatives. In some circumstances, there is an order inside each set and the decision maker has to form a preference over sets of ordered alternatives based on his preference over individual alternatives. Thus the problem is extension of preference on a set to its ordered power set. I restrict attention to situations where all sets to be compared are finite and have the same cardinality. Then the analysis becomes extension of preference over individual alternatives to their finite permutations of the same size. A prominent example is voter choice in closed ballot elections with proportional representation. In this system, utilized in many European democracies, every constituency has a prespecified number of congressmen to represent itself in the parliament. Each political party proposes its ordered list of candidates. The number of votes to parties and the election rule

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determine how many seats each party obtains from that constituency. Then elected candidates corresponding to those seats of a party are picked from its list starting from the highest position. Therefore candidates at the higher order of a list have higher chance of becoming a congressman. Note that since voters are voting for political parties, they are essentially voting for list of candidates. Therefore a voter must establish a preference over lists of candidates. As the order of candidates in the list is critical, voter must extend his preference over individual candidates to ordered lists of candidates.[1] Instances of preference extension to ordered sets exist in other contexts as well. Consider for instance a principal deciding which consultant firm to hire. Consultant firms typically have many experts arranged according to an organizational structure. Suppose the principal knows qualification of all experts in all consultant firms. His problem is then choice among groups of experts with an order in each group. In this case the order reflects rank of experts. Using similar logic, every problem where the principal must choose among groups of people with an associated hierarchy, is basically extension of preferences to ordered sets.

As another example consider a committee evaluating applicants for a fellowship grant. The sole criterion is educational background. The committee must take into account applicant's success in all educational levels like baccalaureate, high school, secondary school; however the committee will give more emphasis to the recent educational institutions. Thus an applicant is a set of ordered scholastic records.

In this setting, which decision rules can a decision maker use to choose among sets of ordered alternatives, and how are these decision rules rationalized based on behavioral choice axioms? Assuming ordinal preferences, I present some decision rules and provide axiomatic characterization of them. They are as I name, highest-position, lexicographic-order, max-best and highest-of-best. In the highest-position rule, the decision maker considers the alternative at the top of each set and prefers the set with the best alternative. He is indifferent among sets that have equivalent alternatives at their highest position. In the lexicographic-order rule, the decision maker first looks at the alternatives at the highest position of each set and prefers the set with a better alternative. If he is indifferent among highest position alternatives of some sets, he compares alternatives at the second highest position and prefers the set with a better one. If tie is still not broken, he compares alternatives at the third highest position and so on. In the max-best rule, the decision maker identifies the best alternative(s) of each set. Inside a set, the best alternative at the highest position is its max-best and the decision maker chooses the set whose max-best is superior to max-best of others, regardless of location. If he is indifferent among max-best of some sets, he prefers the one with max-best at higher position. The last rule, highest-of-best, considers the max-best of each set with its position. The decision maker prefers a set if its max-best is both more qualified and at a higher position than max-best of other sets. In the case of quality-order tradeoff among two sets, if one set weakly Pareto dominates the other set in those two positions, the decision maker chooses this set. Otherwise he prefers the set whose max-best is inferior but at a higher order.

These decision rules are practical in situations where the decision maker has only ordinal preference ranking over individual alternatives but cannot attribute numerical values to them; or the decision maker attributes numerical values to alternatives but he cannot specify numerical values to measure importance of different positions in the

[1] Whether a voter votes sincerely or strategically, he must first identify his true preference over lists.

hierarchy. Even when the decision maker quantifies all aspects of the problem, he may still follow one of the above rules if he is boundedly rational. That is he is not capable of going through complex operations to calculate the expected utility but rather employs simpler methods and concentrates on focal points like highest position or most qualified alternative in a set.

Still, one may investigate the expected utility approach and its relationship to the ordinal approach in this setup. Suppose the decision maker has cardinal utility over individual elements and also attributes numerical weight to each position in the list. This way he can compute expected utility of an ordered set and rank the sets. Now is the expected utility rule, under some conditions, equivalent to an ordinal decision rule? That is, do they generate the same ranking over sets for any preference over alternatives? First note that there are infinitely many cardinal utility functions that represent the same ordinal preference over individual alternatives. Yet these various cardinal utility functions may result different rankings over sets although ordinal ranking over alternatives remains the same. Then the question is whether there are ordinal decision rules for which expected utility counterpart yields the same ranking over sets independent of variation of cardinal utility. I find such ordinal rules to be the degenerate ones: Rules that solely consider the element at a single and specific position in the list. Among degenerate rules, the decision maker would sensibly consider the highest position in the list as it is the most important one. Thus the only rational cardinal-proof ordinal rule is the highest-position rule. This somewhat comes as an impossibility result so I relax equality condition between ranking under ordinal and cardinal rule. I just require the equality between weak ranking over sets under two rules. Then the group of admissible ordinal rules expands. In addition to degenerate rules, pairwise dominance (top-q) rules also satisfy weakened cardinality proofness. Inside this group, the Pareto dominance rule is the rule whose induced weak ranking is the same as expected utility rule even when we vary numerical values of weights. Thus Pareto-dominance rule is the only ordinal rule which is weakly consistent with the expected utility rule under both cardinal proofness and weight proofness.

The article is organized as follows: Section 2 reviews the literature, Section 3 presents the basic setup and notation. Section 4 explains the main axioms used to characterize various extension rules. Section 5 demonstrates ordinal extension rules and provides their axiomatic characterization. Section 6 examines the connection between ordinal decision rules and the expected utility approach. Section 7 concludes.

2 Related Literature

Previous literature has studied extension of preferences to the power set and choice under complete uncertainty. In these models, the decision maker is to form preference over sets of alternatives but he treats all the alternatives in a set equally. Hence there is no order or hierarchy among the elements of a set. In the analogous problem choice under complete uncertainty with the set based approach, the decision maker needs to choose among actions where each action generates a stochastic outcome. The decision maker knows the set of possible outcomes for each action but he doesn't know the probabilities or even likelihood comparison of the outcomes. Therefore each action is equivalent to a set of outcomes and the decision maker has to choose among sets of outcomes based on his preference over individual outcomes. Then the problem becomes extension of preferences over a set to its power set. The literature has established impossibility and possibility results regarding extension rules. See Arrow and Hurwicz (1972), Kannai and Peleg (1984), Pattanaik and Peleg (1984), Bossert, Pattanaik and Xu (2000) and others for these contributions and axiomatic characterization of decision rules.

Another subfield of research, choice under complete uncertainty with the vector approach, is also related to my research. In the vector approach, the decision maker knows the possible states of the nature and the outcome that an action yields in each state. So each action can be interpreted as a vector of outcomes where a specific position in vectors of different actions corresponds to the same state. This model has been first introduced by Pattanaik and Peleg (1984) and the decision rules investigated by . Although there exists some sort of order inside the outcome set of an action, the order is up to the reordering of states. Since the decision maker does not know the probabilities of states, all states are equally treated. Being located at a particular position in the outcome vector does not offer any information about likelihood or importance of that alternative. This is the difference from choice over ordered sets. Yet when the decision maker can differentiate likelihood of possible states, choice among alternative actions with the vector method becomes equivalent to choice over ordered sets.

3 The Model

Let X be the finite and nonempty set of all alternatives. I define an ordered subset $\underline{\text{A}}$ of X, or equivalently a list in X, as a finite vector of elements in X: $\underline{A} = [a_1, ..., a_n]$ where $a_i \in X, \forall i = 1, ..., n$ and $n = |\underline{A}|$ is the length or size of ordered set $\underline{A}$. i denotes the position or index of a_i in $\underline{A}$. Elements with smaller indicies are said to be at relatively higher positions in the list and elements with greater indicies are said to be at relatively lower positions in the list. Therefore a_1 is at the top of $\underline{A}$ and a_n is at the bottom of $\underline{A}$. Higher positions in a list are more probable or more important depending on context. If $\underline{A}$ is an ordered subset of X of size n, then $\underline{B} = [\underline{A}|x_{n+1}]$ is the (n+1) size ordered subset of X constructed by augmenting $x_{n+1} \in X$ to the end (bottom) of $\underline{A}$; likewise $\underline{C} = [x_0|\underline{A}]$ is the (n+1) size ordered subset of X constructed by augmenting $x_0 \in X$ to the beginning (top) of $\underline{A}$.

$\underline{X}^n \underline{X}^n$ denotes the set of all n-size ordered subsets of X. $\underline{X}^n = \{\underline{A} : |\underline{A}| = n, a_i \in X\}$. $\underline{X}$ is the set of all nonempty and finite ordered subsets of X, i.e. $\underline{X} =_{n^+} \bigcup X^n$. I name $\underline{X}^n$ as the n-ordered power set of X and $\underline{X}$ as the ordered power set of X.

Let Rbe a complete, linear preference order[2] over X. I write $\boldsymbol{R}$for the set of all linear preference orders on X. xRy means x is at least as good as y. Let $P$and Istand for antisymmetric and symmetric parts of Rrespectively. Namely, $xPy \Leftrightarrow xRy \wedge \neg yRx$ and $xIy \Leftrightarrow xRy \wedge yRx$. Observe that some elements in Xcan be indifferent to each other and I don't impose strict order among elements in an ordered set $\underline{A}$ and allow indifferences. $\mathbf{U}_R$ denotes the set of real-valued utility functions $u : \Re \to \Re$ that represent $R \in \mathbf{R}$ i.e. $\forall x, y \in X, xPy \Leftrightarrow u(x) > u(y) \wedge xIy \Leftrightarrow u(x) = u(y)$ and $\mathbf{U}$denotes the set of all real-valued utility functions (cardinal preferences) on X.

An extension of $R \in \boldsymbol{R}$ to $\underline{X}^n$ is a linear order $\succeq^n_R$ on elements of set $\underline{X}^n$ that satisfies the *Fundamental* axiom below. $\succ^n_R$ and $\sim^n_R$ stand for asymmetric and symmetric parts of $\succeq^n_R$ respectively.

[2] A linear preference order is a reflexive,transitive and antisymmetric binary relation

Definition 1 $\succeq_R^n$ satisfies Fundamental axiom if $\forall \underline{A}, \underline{B} \in \underline{X}^n, \underline{A} = [a_1, ..., a_n], \underline{B} = [b_1, ..., b_n]$

1. If $x_i P y_i \forall i = 1, ..., n \Rightarrow \underline{A} \succ_R^n \underline{B}$
2. *If* $x_i I y_i \forall i = 1, ..., n \Rightarrow \underline{A} \sim_R^n \underline{B}$

An extension of $R \in \boldsymbol{R}$ to $\underline{\boldsymbol{X}}$ is a collection of linear orders $\succeq_R \equiv \left\{\succeq_R^1, \succeq_R^2, ...\right\} = \left\{\succeq_R^i\right\}_i$. Therefore if $\underline{A}, \underline{B} \in \underline{X}^m, \underline{A} \succeq_R \underline{B}$ is equivalent to $\underline{A} \succeq_R^m \underline{B}$, where $\succeq_R^m$ is the restriction of $\succeq_R$ to $\underline{X}^n$. Similarly $\underline{A} \succ_R \underline{B}$ (or $\underline{A} \sim_R \underline{B}$) means $\underline{A} \succ_R^m \underline{B}$ (or $\underline{A} \sim_R^m \underline{B}$) where $\succ_R$ ($\sim_R$) is the asymmetric (symmetric) part of $\succeq_R$. Observe that $\succeq$ is a complete order over each $\underline{X}^n, n \in N$, but not a complete order over $\underline{\boldsymbol{X}}$. So I call $\succeq$ a *lateral preference order.*

Let Σ be the set of all $\succeq$ induced by all $R \in \boldsymbol{R}$. An extension rule π is a mapping from $\boldsymbol{R}$ to $\underline{\boldsymbol{X}}$. Namely an extension rule maps a preference order R on X to a lateral preference order $\pi(R) \equiv \succeq_R$ on $\underline{\boldsymbol{X}}$.

4 Ordinal Decision Rules

The decision maker has a known linear preference $R \in \boldsymbol{R}$ over individual alternatives in X and he needs to form a lateral preference over ordered subsets of X. I restrict attention to the cases where the decision maker is to choose from ordered sets of equal size, as in the examples of introduction.[3] The elements at higher positions in an ordered set have greater likelihood (as in political party list in closed ballot elections) or more importance (as in selection of applicant for fellowship grant).

An ordinal decision rule specifies a lateral preference order $\succeq_R$ over $\underline{\boldsymbol{X}}$ for an ordinal preference R over X. In this sense, an ordinal decision rule induces an extension rule $\pi_{\succeq}$ from X to $\underline{\boldsymbol{X}}$.

I will present and characterize five different ordinal decision rules in the paper. They are highest-position, lexicographic-order, Pareto dominance, max-best and highest-of-best rules.

Definition 2 Let $\underline{A}, \underline{B} \in \underline{X}, \underline{A} = [a_1, ..., a_n], \underline{B} = [b_1, ..., b_n]$. Then,

1. $\succeq_h$ is defined as $\underline{A} \succ_h^n \underline{B} \Leftrightarrow a_1 P b_1$ and $\underline{A} \sim_h^n \underline{B} \Leftrightarrow a_1 I b_1$
2. $\succeq_{tq}$ is defined as $\left(\underline{A} \sim_{tq}^n \underline{B} \Leftrightarrow a_i I b_i, \forall i = 1, ..., q\right)$ and $\left(\underline{A} \succ_{tq}^n \underline{B} \Leftrightarrow \left(a_i R b_i, \forall i = 1, ..., q; a_j P b_j, \exists j \in [1, q]\right)\right)$
3. $\succeq_{lx}$ is defined as $(\underline{A} \sim_{lx}^n \underline{B} \Leftrightarrow a_i I b_i, \forall i = 1, ..., n)$ and $(\underline{A} \succ_{lx}^n \underline{B} \Leftrightarrow (a_1 P b_1 \vee \exists k \in [1, n]\, a_k P b_k, a_i I b_i \forall i < k))$

Under the highest position rule, when comparing two lists $\underline{A}$ and $\underline{B}$, the decision maker looks at the elements at the highest position of these two sets. If the element at the highest position of one list is better than the element at the highest position of the other, he prefers the former list. If the elements at the highest order of $\underline{A}$ and $\underline{B}$ are indifferent, then the decision maker is indifferent between the two lists. Thus the highest position rule is a degenerate decision rule that considers merely one position in the list. The top-q rule widens the scope of the decision maker and considers positions 1 through q in the list: In order for the decision maker to weakly prefer list $\underline{A}$ to $\underline{B}$, it must be that the first element in list $\underline{A}$ is at least as good as the first element in list $\underline{B}$, the second element in $\underline{A}$ is at least as good as the second element in $\underline{B}$, and so

[3] Ranking among lists of different size would also involve size and reference effect in addition to preference. I do not study this problem in the paper.

on until position q. And at least one relation among these q must be strict for strict preference of $\underline{A}$ to $\underline{B}$. The special top-q rule with $q = N$ is the Pareto dominance rule. The decision maker strictly prefers list $\underline{A}$ over list $\underline{B}$ under the Pareto dominance rule if and only if every element in list $\underline{A}$ is at least as good as element at corresponding position in $\underline{B}$, with at least one relation strict. Next, I introduce the lexicographic order rule. To rank two lists $\underline{A}$ and $\underline{B}$, a decison maker with the lexicographic order decision rule first compares the elements at the highest position of $\underline{A}$ and $\underline{B}$ and prefers the list with with the better element. Only if the elements at the highest position are indifferent, he then compares elements at the second highest position of $\underline{A}$ and $\underline{B}$ and prefers the list with a better element in its second position. If tie is still not broken, he compares elements at the third highest position and so on. The decision maker is indifferent between $\underline{A}$ and $\underline{B}$ if he is indifferent between elements at the same locations across the two lists.

The decision rules so far share a common feature: They rank the lists by specifying some position(s) in the list and assessing elements pairwise in those position(s) across lists. They don't consider an element independent of its position in the list. The following two ordinal decision rules focus on a special element within the list: The best one(s)

Definition 3 The set of best elements in a list $\underline{A}$ is $B(\underline{A}) = \{x : xRa,\ \forall a \in \underline{A}\}$. The max-best element $\underline{A}^\circ$ of list $\underline{A}$ is the one among best elements that is located at the highest position $\underline{A}^m = \left\{a_j \in \underline{A} : j \leq i,\ \forall i\, a_i \in B(\underline{A})\right\}$ and $\bot_{\underline{A}}$ is the index of max-best element in the list.

When there are more than one best elements in different positions within a list, the decision maker is likely to concentrate on the max-best element, the one among best elements located at the highest. Then, depending on the priority of the decision maker between quality and position of the max-best across lists, there are two types of max-best element based decision rule:

Definition 4 Let $\underline{A}, \underline{B} \in \underline{X}$. Then,

1. $\succeq_{mb}$ is defined as $\underline{A} \succ^n_{mb} \underline{B} \Rightarrow [\underline{A}^\circ P \underline{B}^\circ]$ or $\left[\underline{A}^\circ I \underline{B}^\circ,\ \bot_{\underline{A}} < \bot_{\underline{B}}\right]$
2. $\succeq_{hb}$ is defined as $\underline{A} \succ^n_{hb} \underline{B} \Rightarrow \left[\underline{A}^\circ P \underline{B}^\circ, \bot_{\underline{A}} \leq \bot_{\underline{B}}\right]$ or $\left[\underline{A}^\circ P \underline{B}^\circ,\ \bot_{\underline{A}} > \bot_{\underline{B}}, a_{\bot_{\underline{B}}} R b_{\bot_{\underline{B}}}\right]$ or $\left[\underline{B}^\circ P \underline{A}^\circ,\ \bot_{\underline{A}} < \bot_{\underline{B}}, a_{\bot_{\underline{A}}} P b_{\bot_{\underline{A}}}\right]$

Definition 5 Let $\underline{A}, \underline{B} \in \underline{X}$. Then,

1. $\succeq_{mb}$ is defined as $\underline{A} \succ^n_{mb} \underline{B} \Rightarrow [\underline{A}^\circ P \underline{B}^\circ]$ or $\left[\underline{A}^\circ I \underline{B}^\circ,\ \bot_{\underline{A}} < \bot_{\underline{B}}\right]$
2. $\succeq_{hb}$ is defined as $\underline{A} \succ^n_{hb} \underline{B} \Rightarrow \left[\underline{A}^\circ P \underline{B}^\circ, \bot_{\underline{A}} \leq \bot_{\underline{B}}\right]$ or $\left[\underline{A}^\circ P \underline{B}^\circ,\ \bot_{\underline{A}} > \bot_{\underline{B}}, a_{\bot_{\underline{B}}} R \underline{B}^\circ\right]$ or $\left[\underline{B}^\circ P \underline{A}^\circ,\ \bot_{\underline{A}} < \bot_{\underline{B}}, \underline{A}^\circ P b_{\bot_{\underline{A}}}\right]$

Among two lists $\underline{A}$ and $\underline{B}$, if there is a list whose max-best is superior and at a higher (or same) position compared to the max-best of the other list, both the max-best and the highest-of-best rules choose the former list. Besides, if the max-best elements of $\underline{A}$ and $\underline{B}$ are indifferent to each other, both rules choose the list whose max-best element is located at a higher position. In case positions of max-best elements are also identical, then the decision maker is indifferent between $\underline{A}$ and $\underline{B}$. A harder decision problem, where these two rules may differ, is for instance max-best element of $\underline{A}$ is better than max-best element of $\underline{B}$ but located at a higher position i.e. $\underline{A}^{\circ} P \underline{B}^{\circ}$, $\perp_{\underline{A}} > \perp_{\underline{B}}$. Here the max-best rule ranks list $\underline{A}$ over list $\underline{B}$, while the highest-of-best rule ranks list $\underline{B}$ over list $\underline{A}$ as long as max-best of $\underline{B}$ is better than corresponding element in $\underline{A}$ at $\perp_{\underline{B}}$ position of $\underline{B}$'s max-best. Otherwise, if $\underline{B}$ has an element in $\perp_{\underline{A}}$ the position of $\underline{A}$'s max-best which is at least as good as $\underline{A}$'s max-best, that means list $\underline{B}$ weakly dominates list $\underline{A}$ in these two positions $\perp_{\underline{A}}$, $\perp_{\underline{B}}$. Thus the decision maker with the highest-of-best rule would now strictly prefer list $\underline{B}$.

As the next proposition shows, all these rules generate a linear preference ordering on sets of equal size in $\underline{\mathbf{X}}$. In other words, for every $R \in \mathbf{R}$ each of the above rules yield a reflexive, transitive and antisymmetric binary relation over $\underline{X}^n$ for all $n \in \mathbf{N}$. Hence the rules qualify for an ordinal decision rule and preference extension rule from X to $\underline{\mathbf{X}}$ defined for the setup.

Proposition 1 *Let $R \in \boldsymbol{R}$. The preference relation over $\underline{X}^n$ induced by $\succeq^n_{R,h}$, $\succeq^n_{R,tq}$, $\succeq^n_{R,lx}$, $\succeq^n_{R,mb}$, $\succeq^n_{R,hb}$ is complete, reflexive, transitive and antisymmetric for $\forall n \in N$.*

Ordinal decision rules illustrated in this section are convenient options for the decision maker to utilize when he cannot form expected utility of ordered sets. Even when the decision maker can form an expected utility, he may still utilize an ordinal decision rule rather than expected utility if he is boundedly rational. The decision environment might require too many or complex computations to figure out expected utility of each set. A completely rational decision maker may also utilize an ordinal decision rule just because he is in short of time and expected utility calculation takes long time. For example, in a large constituency like Istanbul where party lists include 35 candidates or in graduate applications with more than 1000 candidates, the decision maker can employ simple and time saving methods to choose from lists.

A particular ordinal rule(s) may be more intuitive to employ in some circumstances depending on the characteristics of the decision problem. In closed ballot elections, voters tend to apply max-best, highest-of-best or highest-position rule. are Lexicographic decision rule would fit to recruitment of academic job market candidates, hiring a consultant firm, choice among alternative products with multiple criteria. Medical organ transplants, blood transfers, tree inoculation and manure selection require a decision consistent with the top-q or Pareto dominance rule.

5 Main Axioms

In this section I introduce the main choice axioms that are frequently refered throughout the paper. These axioms, together with other choice axioms will be used to characterize the ordinal decision rules presented in the last section. The first axiom is Independence (from lower Augmentation). It states that the preference relation between two lists $\underline{A}$ and $\underline{B}$ will remain the same after augmenting any alternative x to the bottom of list $\underline{A}$ and any alternative y to the bottom of list $\underline{B}$.

(IND B) $\succeq$ satisfies Independence From Bottom Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;

1. $\underline{A} \succ^n \underline{B} \Rightarrow [\underline{A}|x] \succ^{n+1} [\underline{B}|y]$
2. $\underline{A} \sim^n \underline{B} \Rightarrow [\underline{A}|x] \sim^{n+1} [\underline{B}|y]$

At first sight, this independence axiom seems natural. Since lower positions are less important, the preference order among two lists, strict or indifference, should be independent from appending one more element to the bottom of each list. However if this condition is applied for higher and higher positions in the list, the inductive process reveals that it is solely the highest element that determines the preference ordering of lists. A list is preferred to another as long as its top element is better. Inserting another element to $2^{nd}, 3^{rd}$, ... positions never changes the preference ordering between the two lists, regardless of the quality of the appended elements. Thus $2^{nd}, 3^{rd}$ and lower positions have no effect on preference formation. In fact, this IND_B axiom characterizes the highest-position rule.

Theorem 1 *A lateral preference order* $\succeq$ *satisfies* IND_B *if and only if* $\succeq = \succeq_h$

So the highest-position rule is the only rule that satisfies Independence From Lower Augmentation axiom. This somehow comes as an impossibility result as the IND_B axiom does not allow any other nondegenerate decision rule. Then to be able to characterize other decision rules, I relax the IND_B axiom and obtain weaker forms of it. One imminent way is requiring independence from lower augmentation only for strict preference between lists.

(WIND B) $\succeq$ satisfies Weak Independence From Bottom Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \succ^n \underline{B} \Rightarrow [\underline{A}|x] \succ^{n+1} [\underline{B}|y]$

WIND B states that if the decision maker strictly prefers list $\underline{A}$ over $\underline{B}$, then appending elements x and y to the bottom of $\underline{A}$ and $\underline{B}$ respectively will not change the strict preference. But the axiom does not state anything about ranking of augmented lists $[\underline{A}|x]$, $[\underline{B}|y]$ when the decision maker is indifferent between $\underline{A}$ and $\underline{B}$. Thus WIND B weakens IND B axiom by removing its second part.

The IND B and WIND B axioms can be further relaxed. One can require strict preference between lists to remain after augmenting under certain properties of appended elements.

(WIND BI) $\succeq$ satisfies Weak Independence From Bottom Augmentation of Indifferent Elements if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \succ^n \underline{B}$ and $xIy \Rightarrow [\underline{A}|x] \succ^{n+1} [\underline{B}|y]$

(IMM B) $\succeq$ satisfies Immunity to Bottom Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \succ^n \underline{B}$ and $\exists s \in \underline{A}, \exists t \in \underline{B}$ such that sRx, tRy implies $[\underline{A}|x] \succ^{n+1} [\underline{B}|y]$

(IMM I) $\succeq \in \Sigma$ satisfies Immunity to Interim Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^m, \underline{W}, \underline{V} \in \underline{X}^k$ and $x, y \in X$;

$[\underline{A}|\underline{W}] \succ [\underline{B}|\underline{V}]$ and $\exists s \in [\underline{A}|\underline{W}], \exists t \in [\underline{B}|\underline{V}]$ such that sRx, tRy implies $[\underline{A}|x|\underline{W}] \succ [\underline{B}|y|\underline{V}]$

WIND IB argues that when there is a strict preference between list $\underline{A}$ and list $\underline{B}$, augmenting indifferent elements, x to the bottom of A and y to the bottom of B will keep the strict preference between the two list. IMM B states that if a list $\underline{A}$ is strictly preferred to list $\underline{B}$ and if x is added to the bottom of $\underline{A}$ and element y is added to the bottom of $\underline{B}$, strict preference relation will continue to hold between augmented lists $[\underline{A}|x]$ and $[y|\underline{B}]$, provided that x is not better than $\underline{A}$'s best element(s) and y is not better than $\underline{B}$'s best element(s). Therefore IMM B is a weaker version of WIND B in the sense that strict preference order between two lists remains the same under IMM B if, for each list, an inferior or indifferent element (with respect to the best element(s) of that list) is added to the bottom of it.

IMM I applies similar notion for intermediate augmentation. To explain Immunity to Interim Augmentation, consider two composite lists $[\underline{A}|\underline{W}]$ and $[\underline{B}|\underline{V}]$ with the former strictly preferred. Suppose x is not better than the best element(s) of $[\underline{A}|\underline{W}]$ and y is not better than the best element(s) of $[\underline{B}|\underline{V}]$. If x is added in between $\underline{A}$ and $\underline{W}$ and y is added in between $\underline{B}$ and $\underline{V}$, then the decision maker will still strictly prefer $[\underline{A}|x|\underline{W}]$ over $[\underline{B}|y|\underline{V}]$ in augmented form. Namely, strict preference between two lists is immune from adding element to an intermediate position of each list provided that the element is not better than the best element(s) of the augmented list.

Note that IND B and WIND B imply both IMM B and IMM I but not the other direction. Another axiom that will be used in characterization of decision rules is Immunity from Top Augmentation. It is somehow dual of IMM B in terms of augmentation direction, yet it is independent of IND B and WIND B.

(IMM T) $\succeq$ satisfies Immunity to Top Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \succ^n \underline{B}$ and $\exists s \in \underline{A}, \exists t \in \underline{B}$ such that sPx, tPy implies $[x|\underline{A}] \succ^{n+1} [y|\underline{B}]$

IMM T argues that if a list $\underline{A}$ is strictly preferred to list $\underline{B}$, then affixing an element x, worse than $\underline{A}$'s best element(s), to the top of $\underline{A}$ and affixing an element y, worse than $\underline{B}$'s best element(s), to the top of $\underline{B}$ will not change the preference ranking among the two lists. According to IMM T, adding poor elements to the top of lists will not affect the strict preference relation among them. This choice axiom makes sense when the decision maker directs attention to best elements in the list as we shall see later in characterization theorems.

Until now, I have examined conditions on strict preference relation among lists by working on the first part of IND B axiom. One can also deal with the indifference situation among lists as mentioned by the second part of IND B. Defining it as an axiom,

(I IND B) $\succeq$ satisfies Independence of Indifference From Bottom Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \sim^n \underline{B} \Rightarrow [\underline{A}|x] \sim^{n+1} [\underline{B}|y]$

However one may normally call for strict preference after appending different elements to two indifferent lists.

(LMON) $\succeq$ satisfies Lower Monotonicity if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \sim^n \underline{B}$ and xPy implies $\Rightarrow [\underline{A}|x] \succ^{n+1} [\underline{B}|y]$

(W LMON) $\succeq$ satisfies Weak Lower Monotonicity if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \sim^n \underline{B}$ and xRy implies $\Rightarrow [\underline{A}|x] \succeq^{n+1} [\underline{B}|y]$

LMON stipulates that if a list $\underline{A}$ is strictly preferred to another list $\underline{B}$ and element x is better than y, then the augmented list $[\underline{A}|x]$ is strictly preferred to augmented list $[\underline{B}|y]$. W LMON weakens LMON by requiring only that $[\underline{B}|y]$ will not be strictly preferred to augmented list $[\underline{A}|x]$ when x is at least as good as y.

Before proceeding to axiomatic characterizations, I summarize the relationship between the axioms in this section.

Remark 1 1. IND B $\Rightarrow$ WIND B $\Rightarrow$ WIND BI, IMM B, IMM I, IMM T
2. IND B $\Rightarrow$ I IND B $\Rightarrow$ W LMON
3. LMON $\Rightarrow$ W LMON; I IND B and LMON are independent.
4. WIND B, WIND BI and I IND B are an independent set of axioms.
5. WIND BI, IMM B, IMM I and IMM T are an independent set of axioms.

6 Axiomatic Characterization of Ordinal Decision Rules

6.1 Lexicographic Rule

I first provide axiomatic characterization of lexicographic rule. It turns out that LMON with WIND B fully characterize the lexicographic ordinal decision rule. To see this, first observe that these two axioms imply a third axiom Neutrality:

(NEUTR) $\succeq$ satisfies Neutrality if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \sim^n \underline{B}$ and xIy implies $\Rightarrow [\underline{A}|x] \sim^{n+1} [\underline{B}|y]$

Under WIND B and LMON, the only situation list $\underline{A}$ is indifferent to list $\underline{B}$ is when each element of $\underline{A}$ is indifferent to the corresponding element of list $\underline{B}$ at the same position. This propoerty will continue to hold when indifferent elements x and y are appended to $\underline{A}$ and $\underline{B}$, respectively. Then, in accordance with the FUND axiom,[4] the decision maker must be indifferent between $[\underline{A}|x]$ and $[\underline{B}|y]$. Now it is straightforward to see that the three axioms WIND B, LMON, NEUTR are necessary and sufficient for the lexicographic rule:

Theorem 2 *A lateral preference order $\succeq$ satisfies WIND B, LMON and NEUTR if and only if $\succeq = \succeq_{lx}$*

To understand the proof, if the highest element of one list is better than that of the other, then FUND and WIND B axioms assign a strict preference relation. If highest elements of the two lists are indifferent or the indifference persists in the $2_n d, 3_r d$ ... positions, then NEUTR implies continued indifference among lists. In case the tie is broken at a position, LMON identifies the ranking among the lists and after that lower elements are not considered due to WIND B. Note that I have used WIND B axiom but dropped the second part of IND B. Instead, I used LMON and NEUTR axioms that restrict the prevalence of indifference in IND B. With LMON and NEUTR, the indifference among original lists $\underline{A}$ and $\underline{B}$ will carry on to augmented lists $[\underline{A}|x]$ and $[\underline{B}|y]$ only when x and y are indifferent. This manner we could escape from the highest-position rule and achieved the lexicographic rule.

As I require independent axioms for characterization, I drop the NEUTR axiom and get the main theorem of this section:

Theorem 3 *A lateral preference order $\succeq$ satisfies WIND B and LMON if and only if $\succeq = \succeq_{lx}$. Moreover the axioms are independent from each other.*

6.2 Max-Best Rule

To characterize the max-best rule, I define four new axioms as below:

(I-IMM B) $\succeq$ satisfies Indifference Immunity to Bottom Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \sim^n \underline{B}$ and $\exists s \in \underline{A}, \exists t \in \underline{B}$ such that sRx, tRy implies $[\underline{A}|x] \sim^{n+1} [\underline{B}|y]$

(I-IMM T) $\succeq$ satisfies Indifference Immunity to Top Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and $x, y \in X$;
$\underline{A} \sim^n \underline{B}$ and $\exists s \in \underline{A}, \exists t \in \underline{B}$ such that sPx, tPy implies $[x|\underline{A}] \sim^{n+1} [y|\underline{B}]$

[4] Recall that ordinal decision rules, by definition, satisfy FUND

(ORD MON) $\succeq$ satisfies Order Monotonicity if for $x, m, n \in X$, xPm and xPn implies $[x, m] \succ [n, x]$

(BF) $\succeq$ satisfies Best Focal Choice if for $x, y, m, n \in X$, xPy, xPm and yPn implies $[m, x] \succ [y, n]$

Order Monotonicity requires that if the decision maker strictly prefers individual alternative x to m and n, then he strictly prefers ordered pair $[m, x]$ over $[y, n]$. That is he distinguishes x as the best element and strictly prefers the pair in which x is in the first position rather than the second position. The next axiom Best Focal Choice states that when x and y are the better element inside their own pair $[m, x]$ and $[y, n]$ and x is better than y, then the decision maker strictly prefers $[m, x]$ over $[y, n]$. Though y is better than m and y is in the first position, he strictly prefers ordered pair $[m, x]$ possibly because it has element x which is superior to y. So under BF, the decision maker gives priority to quality of the best element in the pair rather than its position, which is related to the max-best rule.

Theorem 4 shows that the three axioms IMM B, IMM S, IMM T together with I-IMM B, I-IMM T, ORD MON, BF characterize the max-best rule. Besides the axioms are independent.

Theorem 4 *A lateral preference order $\succeq \in \Sigma$ satisfies IMM B, I-IMM B, IMM T, I-IMM T, IMM S, ORD MON, BF if and only if $\succeq = \succeq_{mb}$. The axioms are independent from each other.*

The following lemma will be helpful to understand this characterization. It states that when the first five axioms in Theorem 4, IMM B, I-IMM B, IMM T, I-IMM T and IMM I are imposed, the ranking of two lists (of any size) basically reduces to either ranking of two sublists of size two or ranking of two singleton sublists. The elements of sublists are those elements of the original lists at the two position index of the max-best elements of the lists. If the max-best elements of the two lists happen to be at identical position, the sublists will contain single element, the element of the original list in this position index. Note that in case a list has best elements in more than one position, in accordance with I-IMM B, the decision maker takes into account the max-best and disregards other best elements at lower positions.

Lemma 1 *Suppose $\succeq \in \Sigma$ satisfies IMM B, I-IMM B, IMM T, I-IMM T, IMM I. Then for $\underline{A}, \underline{B} \in \underline{X}$,*

1. *If $\perp_{\underline{A}} < \perp_{\underline{B}}$, $\underline{A} \succ \underline{B} \Leftrightarrow \left[\underline{A}^\circ, a_{\perp_{\underline{B}}}\right] \succ \left[b_{\perp_{\underline{A}}}, \underline{B}^\circ\right]$*
2. *If $\perp_{\underline{A}} > \perp_{\underline{B}}$, $\underline{A} \succ \underline{B} \Leftrightarrow \left[a_{\perp_{\underline{B}}}, \underline{A}^\circ\right] \succ \left[\underline{B}^\circ, b_{\perp_{\underline{A}}}\right]$*
3. *If $\perp_{\underline{A}} = \perp_{\underline{B}}$, $\underline{A} \succ \underline{B} \Leftrightarrow \underline{A}^\circ P \underline{B}^\circ$*

To see this lemma, assume without loss of generality that the max-best element of list $\underline{A}$ is at an identical or higher position than the max-best element of $\underline{B}$, i.e. $\perp_{\underline{A}} \leq \perp_{\underline{B}}$. Under IMM B, I-IMM B, IMM T, I-IMM T, elements located above index $\perp_{\underline{A}}$ and below index $\perp_{\underline{B}}$ in both lists do not affect the preference ranking of the lists. The comparison then becomes among sublist of $\underline{A}$ between indices $\perp_{\underline{A}}$, $\perp_{\underline{B}}$ and sublist of $\underline{B}$ between indices $\perp_{\underline{A}}$, $\perp_{\underline{B}}$. (If $\perp_{\underline{A}} = \perp_{\underline{B}}$, the comparison is simply among max-best elements of $\underline{A}$ and $\underline{B}$, $\underline{A}^\circ$ and $\underline{B}^\circ$). If, in addition, IMM I is imposed, the intermediate elements [5] located between $\perp_{\underline{A}}$ and $\perp_{\underline{B}}$ do not have impact on the ranking between two

[5] By construction of sublists, the intermediate elements of the sublists are not better than best element(s) of their own list

sublists either, and thus are eliminated. Thereafter the problem reduces to ranking two ordered pairs (or two singletons), as illustrated in the lemma, and one can use axioms ORD MON and BF to identify the preference.

6.3 Highest-of-Best Rule

I continue with the characterization of the highest-of-best decision rule. This entails two additional axioms:

(H FOC) $\succeq$ satisfies High Focal Choice if for $x, y, z \in X$, xPy and yPz implies $[y, z] \succ [z, x]$

(W PAR) $\succeq$ satisfies Weak Pairwise Pareto Dominance if for $x, y, z, m \in X$, xPy, yPz and mIy implies $[m, x] \succ [y, z]$

High Focal Choice states that if x is better than y and y is better than z, then the decision maker strictly prefers ordered pair $[y, z]$ over ordered pair $[z, x]$. Here in the first position, $[y, z]$ includes a better element y compared to $[z, x]$. Though $[z, x]$ has a better element x than y, it is in the secondary position.

Weak Pairwise Pareto Dominance axiom complements High Focal Choice axiom by arguing that in a similar situation, if there were an ordered pair $[y, x]$ (or $[m, x]$, mIy) instead of $[z, x]$, then the decision maker would reverse his pereference and now strictly prefer $[y, x]$ over $[y, z]$. The insight is both lists have the same or equivalent element in their first position however the second element of $[y, x]$ exceeds its first element while the second element of $[y, z]$ falls below its first element.

These two axioms H FOC, W PAR with IMM B, I-IMM B, IMM T, I-IMM T, IMM I defined before characterize the highest-of-best decision rule:

Theorem 5 *A lateral preference order $\succeq \in \Sigma$ satisfies IMM B, I-IMM B, IMM T, I-IMM T, IMM I, H FOC, W PAR if and only if $\succeq = \succeq_{hb}$. The axioms are independent from each other.*

Lemma 1 already instituted that when IMM B, I-IMM B, IMM T, I-IMM T, IMM I axioms are imposed on $\succeq$, ranking of two lists $\underline{A}$ and $\underline{B}$ becomes equivalent to ranking of two orderd pairs or ranking of two singletons (if max-best elements are at the same position). Thereafter one can use axioms IMM B, H FOC, W PAR to find the preference relation between the ordered pairs and thus the preference relation between the original lists.

7 Expected Utility

As an alternative to ordinal approach, the decision maker can also use expected utility to form his preference over ordered sets. To do so, he needs to have a cardinal preference over X, i.e. a cardinal utility function over X and attribute a numerical weight to each position in the ordered set. As the decision maker chooses among sets of equal size, I study the lateral preference order induced by the expected utility. With cardinal utility and position weight vector, the decision maker can compute the expected utility or weighted aggregate utility of each ordered set in $\underline{X}^n$ and choose the set(s) that yield highest expected utility.

Definition 6 A cardinal utility function $u : X \to \Re$ maps a real-valued utility $u(x) \in \Re$ to every $x \in X$. A weight vector for the ordered set is $\underline{w} = [w_1, ..., w_n]$, $0 \leq w_n \leq w_{n-1} \leq ... \leq w_1 < \infty$ where w_i represents the weight of position i in the set. Then, the expected utility decision rule $\succeq_{EU\langle u,\underline{w}\rangle} \in \Sigma$ is defined, for any $\underline{A}, \underline{B} \in \underline{X}^n, \underline{A} = [a_1, ..., a_n]$, $\underline{B} = [b_1, ..., b_n]$,

$\underline{A} \succ^n_{EU\langle u,\underline{w}\rangle} \underline{B}$ if and only if $\sum_n^{i=0} w_i u(a_i) > \sum_n^{i=0} w_i u(b_i)$ and $\underline{A} \sim^n_{EU\langle u,\underline{w}\rangle} \underline{B} \Rightarrow \sum_n^{i=0} w_i u(a_i) = \sum_n^{i=0} w_i u(b_i)$

The weights w_i are positive, finite and weakly decreasing downwards in the list. I also assume w_i, weight of a particular position in the list, is the same across all lists of the same size. Thus an expected utility rule has a unique weight vector $\underline{w}$ over $\underline{X}^n, n \in \mathbf{N}$. Depending on the context, the weight of a position may have different meaning. In the consultant firm, fellowship grant and academic recruitment examples, the weight reflects importance of an item being at that position. In the case of closed ballot elections the weight of a position denotes the ex-ante probability of a candidate in this position being elected.

Next I invesigate the relationship between ordinal and cardinal decision rules: Is the expected utility decision rule equivalent to an ordinal decision rule in terms of ranking of ordered sets? And if so, under what conditions? I note that there are infinitely many cardinal utility functions $u \in \mathbf{U}_R$ that represent the same ordinal preference R over X. However these cardinal utility functions may generate, for a fixed weight vector $\underline{w}$, different ranking of ordered sets in $\underline{X}^n$, while ordinal preference R and thus ranking of ordered sets under a particular ordinal decision rule stays the same. Therefore for potential equivalence between some ordinal and cardinal decision rules, first there must exist a specific expected utility decision rule, identified by its weight vector $\underline{w}$, which yields the same ranking of ordered sets in $\underline{X}^n, n \in \mathbf{N}$ for all cardinal preference that represent the same ordinal preference. I name these rules as cardinal-proof decision rules. If a cardinal-proof expected utility rule exists, there must be an ordinal decision rule that yields the same ranking of ordered sets with the expected utility rule for all ordinal preference $R \in \mathbf{R}$. Such an ordinal decision rule(s), if exists, is characterized solely by the weight vector $\underline{w}$ of its equivalent expected utility counterpart since the ranking of ordered sets under expected utility rule is cardinal-proof. I find the class of cardinal-proof expected utility decision rules are those that consider only the highest position in the list and their equivalent ordinal decision rule is the highest position rule. (As they have a degenerate weight vector that support the highest position)

Definition 7 An expected utility decision rule $\succeq^n_{EU\langle u,\underline{w}\rangle}$ is cardinal-proof if for $\underline{A}, \underline{B} \in \underline{X}^n$; $\underline{A} \succeq^n_{EU\langle u_1,\underline{w}\rangle} \underline{B} \Leftrightarrow \underline{A} \succeq^n_{EU\langle u_2,\underline{w}\rangle} \underline{B}$, $\forall R \in \mathbf{R}$ and $\forall u_1, u_2 \in \mathbf{U}_R$

Definition 8 An ordinal decision rule $\succeq \in \Sigma$ is cardinal-proof if there exists $\underline{w}$ such that $\succeq^n_{EU\langle u,\underline{w}\rangle} \equiv \succeq^n_R$ for $\forall R \in \mathbf{R}$ and $\succeq^n_{EU\langle u,\underline{w}\rangle}$ is cardinal-proof.

Theorem 6 *If $\succeq \in \Sigma$ is cardinal proof, then $\succeq = \succeq_h$ and its corresponding expected utility rule has a degenerate weight vector $\underline{w}$ i.e. $w_1 > 0$, $w_2 = ... = w_n = 0$ for $j \in [2, n]$*

Definition 9 An ordinal decision rule $\succeq \in \Sigma$ is cardinal-proof if there exists $\underline{w}$ such that $\succeq^n_{EU\langle u,\underline{w}\rangle} = \succeq^n_R$ for $\forall R \in \mathbf{R}$ and for $\forall u \in \mathbf{U}_R$.

Therefore the only cardinal-proof ordinal decision rule is the highest-position rule. This is somehow an impossibility result regarding the equivalence between ordinal and cardinal decision rules. So I relax my cardinal-proofness condition as follows: Instead of requiring eqality in both symmetric and asymmetric part between ranking of sets under ordinal and cardinal rules, now I just require equality in weak ranking of sets under two types of rules. If the weak ranking of an ordinal rule $\succeq \in \Sigma$ is the same of weak ranking of an expected utility rule $\succeq^n_{EU\langle u,\underline{w}\rangle}$ for all preferences then $\succeq$ and $\succeq^n_{EU\langle u,\underline{w}\rangle}$ are partially equal and in this case I name $\succeq$ as weakly cardinal-proof.

Definition 10 $\succeq \in \Sigma$ and $\succeq^n_{EU\langle u,\underline{w}\rangle}$ are partially equal if for all $\underline{A}, \underline{B} \in \underline{X}^n$, $\underline{A} \succeq^n_R \underline{B} \Leftrightarrow \underline{A} \succeq^n_{EU\langle u,\underline{w}\rangle} \underline{B}$, $\forall R \in \mathbf{R}$ and $\forall u \in \mathbf{U}_R$

Definition 11 An ordinal decision rule $\succeq \in \Sigma$ is weakly cardinal-proof if there exists $\underline{w}$ such that $\succeq$ and $\succeq^n_{EU\langle u,\underline{w}\rangle}$ are partially equal.

Thus weak cardinality-proofness entails equivalence in only weak ranking of ordinal and cardinal rules. If $\succeq$ is cardinal-proof then it is weakly cardinal-proof, but the other direction is not necessarily true. The reason is if $\succeq$ and $\succeq^n_{EU\langle u,\underline{w}\rangle}$ are partially equal, their strict ranking may be different and $\succeq^n_{EU\langle u,\underline{w}\rangle}$ need not be cardinal-proof and strict ranking may be different. Theorem 7 shows that the class of ordinal rules that are partially equivalent to the expected utility rule are the top-q rules defined in Section 3.

Theorem 7 *Suppose $\succeq \in \Sigma$ is weakly cardinal-proof. Then $\succeq = \succeq_{tq}$*

Highest-position rule and Pareto dominance rule are special instances of top-q rule with $q = 1$ and $q = N$ respectively. Top-q rules, by Theorem 7, are weakly cardinal-proof but except the highest-position rule, they are not cardinal-proof. So with the weakened cardinality-proofness principle, the set of admissible ordinal decision rules expands.

For weak cardinal-proofness, it suffices that the ordinal decision rule is partially equal to *some*expected utility rule. But among those weakly cardinal-proof ordinal rules, is there one that is partially equal to *all*expected utility rules? That is for which ordinal rules, the induced weak ranking is independent from variation of weight vectors? I define this concept as weak weight and cardinal-proofness (WWCP). An ordinal decision rule is WWCP if it yields the same weak ranking of ordered sets as the expected utility rule for all cardinal utility and weight vector combination.

Definition 12 $\succeq \in \Sigma$ is weakly weight and cardinal-proof (WWCP) if for all $\underline{A}, \underline{B} \in \underline{X}^n$, $\underline{A} \succeq^n_R \underline{B} \Leftrightarrow \underline{A} \succeq^n_{EU\langle u,\underline{w}\rangle} \underline{B}$, for all $\underline{\mathrm{w}}$, for all $R \in \mathbf{R}$ and for all $u \in \mathbf{U}_R$

Thus WWCP refines the set of weakly cardinal-proof ordinal decision rules. Actually, the only ordinal rule that respects WWCP criterion is the Pareto dominance rule.

Theorem 8 *A lateral preference order $\succeq \in \Sigma$ is weakly weight and cardinal-proof (WWCP) if and only if it is the Pareto dominance rule i.e. $\succeq = \succeq_{tq}$ with $q = N$.*

As Theorem 8 proves, Pareto dominance rule is the only ordinal decision rule which is partially equal to the expected utility rule under variation of both cardinal utility and numerical values of weights.[6] Recall that Pareto dominance rule yields strict preference between two lists only when every element in one list is at least as good as the corresponding element in the identical position in the other list, with at least one element strictly better. Otherwise the two lists are indifferent. Therefore the strict preference ranking by the Pareto dominance rule can never be overturned by an expected utility rule with any weight vector or any cardinal preference (representing the same ordinal preference). This is the basis of Pareto dominance rule being WWCP.

Exhibiting a special feature, I attempt to characterize the Pareto dominance rule. For this I will utilize three additional axioms:

(PD) $\succeq$ satisfies Perfectness for Dominance if for $x, y \in X$ and for $\underline{A}, \underline{B} \in \underline{X}^n$ with $\underline{A} \succ^n \underline{B}$;
$[\underline{A}|x] \sim^{n+1} [\underline{B}|y] \Leftrightarrow yPx$

(I IMM H) $\succeq$ satisfies Immunity of Indifference to Higher Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and for $x, y \in X$;
xPy and $[\underline{A}|x] \sim^{n+1} [\underline{B}|y] \Rightarrow [x|\underline{A}] \sim^{n+1} [y|\underline{B}]$

(I IMM L) $\succeq$ satisfies Immunity of Indifference to Further Lower Augmentation if for $\underline{A}, \underline{B} \in \underline{X}^n$ and for $x, y \in X$;
xPy and $[\underline{A}|x] \sim^{n+1} [\underline{B}|y] \Rightarrow [\underline{A}|x|\underline{W}] \sim [\underline{B}|y|\underline{V}]$ for all $\underline{W}, \underline{V} \in \underline{X}^k$, $k \in \mathbf{N}$

According to PD, when list $\underline{A}$ is strictly preferred to list $\underline{B}$ but x is worse than y, appending the worse element x to $\underline{A}$ makes this list indifferent to $[\underline{B}|y]$. PD also states that this is the only case where $[\underline{A}|x]$ is indifferent to $[\underline{B}|y]$. Namely, if $\underline{A} \succ^n \underline{B}$ yet in augmented form $[\underline{A}|x] \sim^{n+1} [\underline{B}|y]$, then x must be worse than y.

I IMM H contemplates that if individual element x is better than y and augmented list $[\underline{A}|x]$ is indifferent to $[\underline{B}|y]$, then appending x and y to top of their list instead of bottom cannot break the indifference between lists possibly in favor of $\underline{A}$. I IMM L fortifies I IMM H by stating that indifference will persist among these two lists upon further appending any lists $\underline{W}$ and $\underline{V}$ to the bottom of $[\underline{A}|x]$ and $[\underline{B}|y]$, respectively. The rationale is if $[\underline{A}|x]$ is indifferent to $[\underline{B}|y]$ even with an appended better element x, then appending additional lists to the bottom of them will not be able to break the tie.

Finally, I obtain

Theorem 9 *A lateral preference order $\succeq \in \Sigma$ satisfies WIND BI, W LMON, PD, I IMM H, I IMM L if and only if $\succeq$ is the Pareto dominance rule. Moreover the axioms are independent from each other.*

PD and I IMM L together imply that if a list $\underline{A}$, compared to another list $\underline{B}$, has better element(s) in some position(s) but at the same time has worse element(s) in other position(s), then the two lists are indifferent to each other. Thus in order for $\underline{A}$ to be strictly preffered to $\underline{B}$, it is necessary that $\underline{A}$ must possess better or indifferent element in all positions. From WIND BI and first part of FUND, only the highest position element being better and elements in all other locations indifferent is sufficient for strict preference. The two axioms also imply that elements at top-q positions $q \in [2, N]$ being better and lower elements being indifferent is sufficient for strict preference.

What if the highest position element of $\underline{A}$ is indifferent to highest position element of $\underline{B}$, or top-q position elements of $\underline{A}$ are indifferent to top-q elements in $\underline{B}$? In other

[6] Note here that by the definition of weight vector, there are restrictions on values of weights. In particular weights must be decreasing toward the bottom of the list.

words how to find the preference relation between $[\underline{A}|x]$ and $[\underline{B}|y]$, when $\underline{A} \sim \underline{B}$ and xPy? Here note that $\underline{A}$ can indifferent to $\underline{B}$ for two reasons and one needs to identify the two to conclude the preference. $\underline{A}$ can indifferent to $\underline{B}$ because of the symmetry: When elements in identical positions are indifferent across $\underline{A}$ and $\underline{B}$. In this case $[\underline{A}|x]$ should be strictly preferred to $[\underline{B}|y]$. However $\underline{A}$ can indifferent to $\underline{B}$ because of the asymmetry illustrated in PD axiom as well. In this case indifference between $\underline{A}$ and $\underline{B}$ should remain. The I IMM H axiom differentiates the two reasons for $\underline{A} \sim \underline{B}$. If $\underline{A}$ and $\underline{B}$ were to be indifferent due to the first reason, then $[\underline{A}|x]$ cannot be indifferent to $[\underline{B}|y]$; otherwise I IMM H would imply indifference between $[x|\underline{A}]$ and $\sim^{n+1}$ $[y|\underline{B}]$ violating FUND and WIND BI axioms. So if $[\underline{A}|x]$ is not be indifferent to $[\underline{B}|y]$, then it must be strictly preferred to $[\underline{B}|y]$ in accordance with WIND BI.

8 Conclusion

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