PROBLEM SOLVING, SCAFFOLDING AND LEARNING

by

Shih-Yin Lin

B.S. Physics, National Taiwan University, 2006

M.S. Physics, National Taiwan University, 2008

Submitted to the Graduate Faculty of

the Department of Physics and Astronomy in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2012

UNIVERSITY OF PITTSBURGH

SCHOOL OF ARTS AND SCIENCES

DEPARTMENT OF PHYSICS AND ASTRONOMY

This dissertation was presented

by

Shih-Yin Lin

It was defended on

Nov 30, 2011

and approved by

Dr. Robert P. Devaty, Associate Professor, Department of Physics and Astronomy

Dr. Andrew Zentner, Associate Professor, Department of Physics and Astronomy

Dr. Russell Clark, Lecturer, Department of Physics and Astronomy

Dr. Larry Shuman, Professor, Department of Industrial Engineering

Dissertation Advisor: Dr. Chandralekha Singh, Professor, Department of Physics and Astronomy

PROBLEM SOLVING, SCAFFOLDING, AND LEARNING

Shih-Yin Lin, PhD

University of Pittsburgh, 2012

Helping students to construct robust understanding of physics concepts and develop good problem solving skills is a central goal in many physics classrooms. This thesis examines students' problem solving abilities from different perspectives and explores strategies to scaffold students' learning. In studies involving analogical problem solving between isomorphic problems, we evaluate introductory physics students' abilities to learn from the solved problems provided and transfer their learning to solve the corresponding quiz problems which involve the same physics principles but different surface features. Findings suggest that postponing the providing of the solved problems until students have attempted to solve the quiz problems first without help is a good way to scaffold students' analogical problem solving. Categorization of problems based upon similarity of solution provides another angle to evaluate and scaffold students' ability to reflect on the deep features of the problems. A study on categorization of quantum mechanics problems reveals that the faculty overall perform better categorization than the students. However, unlike the categorization of introductory mechanics problems, in which the categories created by the faculty are uniform and based on the fundamental physics principles, the categorization in quantum mechanics is based on the concepts and procedures, and is more diverse. In addition to investigating strategies that may guide students to develop a better knowledge structure in physics, from the learners' perspective, we also explore possible strategies to help instructors improve their teaching of problem solving and to assess student difficulties more efficiently. Investigating how teaching assistants (TAs) design problem

solutions in view of the recommendations from research literature, we find that the TAs don't necessarily notice all components in a problem solution that are valued by the educational researchers. There is much room for improvement when it comes to actual practice. Another study involving comparison between different assessment tools reveals that carefully designed multiple-choice questions can reflect the relative performance on the free-response problems while maintaining the benefit of ease of grading, especially if the different choices in the multiple-choice questions are weighted to reflect the different levels of understanding that students display.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

PREFACE-ACKNOWLEDGMENTS

First of all, I would like to thank my advisor, Dr. Chandralekha Singh, for all her help and support these years. I've learned a lot about the cognitive theories, the way to conduct research, as well as the important attitudes from her. Her guidance and support enable me to complete my study and step further to the next stage in my career.

I would like to thank my informal co-advisor, Dr. Edit Yerushalmi for her advice and constant feedback and support on the TA study. I am grateful to her substantial support and encouragement throughout this period. Her help is a key element in the completion of the study related to TA training.

I would also like to thank my committee members: Dr. Russell Clark, Dr. Robert Devaty, Dr. Larry Shuman, and Dr. Andrew Zentner for their insight and critiques on my research. I would like to express my additional thanks to Dr. Robert Devaty for his valuable help and suggestions on the presentation of our studies.

In addition, I would like to thank Dr. Charles Henderson and William Mamudi for their ideas and contributions as the collaborators in the TA study.

I would also like to thank Dr. Jeremy Levy for his encouragement as well as his insight in the categorization study and the study involving comparison of different assessment tools.

I am grateful to all the faculty, TAs and students who participated in our studies. Their willingness to participate as well as their efforts and time made these studies possible.

I would like thank many of my friends for their support and encouragement through all these years. I would like to express my special thanks to Tze-Wei Liu and Chien-Lin Liu for their substantial support in my life.

Finally, I would like to thank my parents, Ji-Cheng Lin and Shu-Nu Chen, and my brother, Tsung-Hsien Lin, for their endless love and continuous support throughout my whole life. I am incredibly grateful to them for encouraging and accompanying me through this journey. I would like to dedicate this thesis to them.

1.0 INTRODUCTION

Improving students' understanding of physics concepts and enhancing their problem solving skills are two central goals in many physics classrooms. Problem solving is an important component in many introductory- and advanced-level physics courses. Many instructors make use of the problem solving activities to clarify and emphasize the physics concepts and principles. In addition, students are often provided with opportunities to practice applying the knowledge they acquired when solving their homework problems. Problem solving is also commonly used as a tool to assess students' learning in physics.

According to Reif's definition, a problem is "a task which requires one to devise a sequence of actions leading from some initial situation to some specified goal" (Reif 1995). There are several mental tasks involved in a problem solving process, such as creating representations for the problem, recalling relevant information in order to solve the problem, and monitoring and assessing the solution process (Larkin 1979; Larkin et al. 1980; Larkin 1981; Chi et al. 1982; Eylon and Reif 1984; Bagno and Eylon 1997; Hsu et al. 2004). If a person is a good problem solver, he or she has the ability to develop strategies and employ relevant knowledge to reach the targeted goal when presented with a novel situation (Larkin and Reif 1979; Reif 1981; Eylon and Reif 1984; Heller and Reif 1984; Reif 1986; Reif and Larkin 1991; Reif and Allen 1992; Reif 1995).

Problem solving in physics, however, is typically challenging for students. Physics is a subject which contains only a few fundamental principles that are condensed into compact mathematical form. To learn physics effectively, it is essential to unpack the meaning of the abstract principles, and understand their applicability in diverse situations (Larkin and Reif 1979; Chi et al. 1981; Reif 1981; Eylon and Reif 1984; Maloney 1994; Reif 1995; Redish et al. 1998; Hammer 2000; Redish et al. 2006). Research on the difference between physics experts and novices indicates that experts usually see a physics problem at a more abstract level. Unlike experts who focus on the deep features of the problems in terms of the physics principles involved, novices are more likely to be distracted by the surface features (Chi et al. 1981; Hardiman et al. 1989). Research suggests that experts in physics have a highly hierarchical knowledge structure, which helps them apply their knowledge in novel or complex situations and approach the problems in a systematic way (Johnson-Laird 1972; Bobrow and Norman 1975; Larkin 1980; Larkin 1980; Chi et al. 1981; Larkin 1981; Reif and Heller 1982; Schoenfeld and Herrmann 1982; Eylon and Reif 1984; Cheng and Holyoak 1985; Marshall 1995; Johnson and Mervis 1997; Dufresne et al. 2005). Novices, on the other hand, have a less organized knowledge structure and they may only be able to apply what they learned to similar situations following routine procedures.

A lot of efforts have been devoted to investigating possible strategies to help students acquire the content knowledge, to think like a physics expert and to perform an expert-like problem solving (Heller and Reif 1984; Van Heuvelen 1991; Dufresne et al. 1992; Mestre et al. 1993; Leonard et al. 1996). Cognitive research suggests that effective teaching and learning is not simply a process of pouring knowledge into students' brains. For learning to be meaningful, students need to be actively engaged to construct a robust knowledge structure. My research in

this thesis examines different aspects to assess and improve students' problem solving abilities. The topics cover both the introductory physics course and the upper level undergraduate quantum mechanics course. While some studies examine strategies from the learners' perspective that may guide students to develop a better knowledge structure in physics, other studies focus on possible ways for instructors to identify student difficulties more efficiently and to teach problem solving more effectively.

In particular, the studies in chapters 2 to 4 examine the effect of using analogical problem solving to help students learn introductory physics. Students in these studies were explicitly asked to learn from a solved problem provided to them and take advantage of what they learned to solve another isomorphic problem (which we call the "quiz problem") that involves the same underlying physics principle(s) but has different surface features. According to Hayes and Simon, problems are isomorphic if they can be mapped to each other in a one-to-one relation in terms of their problem solving trajectories (Hayes and Simon 1977). In our studies, we call problems isomorphic if they can be solved using the same physics principle(s). Different scaffolding supports are designed to help students process through the analogy between isomorphic problems deeply. The effects of different scaffolding are evaluated and compared. In addition, interviews with several students were conducted in order to get an in-depth account of their reasoning and difficulties. The study in chapter 2 evaluates students' transfer from a 2-step problem to another 2-step problem which is typically known to be difficult for students. Both of the problems involve the principles of conservation of energy and Newton's 2nd Law with centripetal acceleration. The study in chapter 3 examines the extent of transfer (with different scaffolding strategies) when the transfer problem (the quiz problem) involves static friction, for which many students often have a misconception. The study in chapter 4 examines whether students are able

to transfer their learning from a 2-step problem to a 3-step problem in which the same physics principles of conservation of momentum and conservation of mechanical energy come into play. These analogical problem solving activities are designed to guide students to focus not on the surface features in the problems but on the deep physics principle(s), a process which can be beneficial to the construction of a robust knowledge structure.

In chapter 5, I describe a study which investigates the categorization of quantum mechanics problems by physics professors and students. A group of undergraduate students in the upper-level quantum mechanics courses and several physics faculty members were asked to categorize 20 quantum mechanics problems based upon similarity of solution. The way different people categorize the problems provides insight into the way knowledge is structured in their minds. In addition, I'll discuss the interesting result found by comparing the categorization of quantum mechanics problems in this study to the categorization of introductory level mechanics presented in the research literature.

To scaffold students' learning and problem solving skills in a more comprehensive way, a study which provides implications on possible strategies to improve the teaching of problem solving in introductory physics classrooms is discussed in chapter 6. This study builds on a former line of research which investigated physics faculty beliefs about the teaching and learning of problem solving and examines graduate teaching assistants' views about how example problem solutions should be designed and the role that example problem solving should play. In many institutions, graduate teaching assistants lead recitations in introductory physics courses and therefore play a central role in the teaching of problem solving. According to the recommendation from research literature, the modeling of expert thinking is an important component in the teaching of problem solving. If we wish to help instructors make problem

solving approaches explicit on example problem solutions they provide students, it is necessary to first understand how these instructors currently perceive and value the different components of the design of example problem solutions. This study in chapter 6 investigates the goals a group of TAs expressed for the use of instructor solutions and explores how the goals were materialized into a concrete solution through different solution features. TAs' actual practices are compared to their self-reported beliefs and compared further with the recommendations from research literature.

In chapter 7, I evaluate the extent to which performance on carefully designed multiple-choice questions can reflect students' relative performance on the corresponding free-response problems. When it comes to assessing students' learning in physics, there appears to be a trade-off between multiple-choice questions and free-response questions. A test in multiple-choice format is appreciated by many instructors because it provides an efficient tool to evaluate students' learning, especially when there are a lot of students enrolled in the course and there is a severe time constraint. On the other hand, many of the instructors also believe that a test in a free-response format may facilitate a more accurate understanding of student difficulties. In addition, it allows students to get partial credits for displaying different extents of understanding of the subject matter tested, which is appreciated by many instructors and students. With an attempt to incorporate the advantages of both assessment tools and to evaluate whether carefully designed multiple-choice questions can mirror the relative student performance on the free-response questions while maintaining the ease of grading and quantitative analysis, two research-based multiple-choice questions were designed and implemented in an introductory physics course. Common student difficulties found via research were incorporated when designing the alternative choices in these questions. The multiple-choice questions were also transformed into

a free-response format and administered in another introductory physics course which is equivalent. Students' performance in the two courses is compared. In addition, we develop a "weighted" scheme (which is commensurate with the rubric used for grading the free-response questions) so that the different partial credits assigned to different choices in the multiple-choice questions can reflect the different levels of understanding students have.

Before I describe each study in detail in the later chapters, I'll first discuss some findings from cognitive theories and physics education research (PER) which shed light on possible ways to improve students' learning in the remaining paragraphs of chapter 1.

1.1 THEORETICAL FRAMEWORKS OF LEARNING FROM COGNITIVE SCIENCE

Cognitive science is an important resource for the physics education research. Since learning and problem solving are cognitive processes, theories from cognitive science provide basic frameworks for interpreting students' learning processes and performance in physics. Cognitive principles also provide guidelines for physics education researchers to develop effective instructional strategies and materials to enhance students' learning. My studies in this thesis are informed by the learning theories and models proposed by many cognitive scientists, in particular Piaget's theory which includes notions of "optimal mismatch" and "assimilation, and accommodation", Vygotsky's model of "zone of proximal development", and Schwartz, Bransford and Sears's framework of "efficiency and innovation in transfer" and preparation for future learning. I'll highlight these models in the following sections.

1.1.1 Assimilation, Accommodation, and Optimal Mismatch

Piaget describes two important processes – assimilation and accommodation - to explain how a person internalizes the new information from the outside world into his/her pre-existing knowledge structure (Ripple and Rockcastle 1964; Ginsberg and Opper 1969). Through assimilation, the new information that conforms to the original mental structure is incorporated into the pre-existing knowledge structure. If the new information doesn't fit with the pre-existing cognitive structure, the knowledge schema is modified in order to accommodate the new information. Piaget argues that when a person encounters a new event, he/she strives through the process of assimilation and accommodation until equilibrium between the information from the outside world and his/her own mental structure is established.

In Piaget's theory, a positive cognitive developmental progress is likely to occur if there is an optimal mismatch between the demand from the outside world and the students' internal knowledge structure. It is suggested that with an optimal level of fit and an optimal level of mismatch, the learning can be productive. For example, if common student difficulties or misconceptions are elicited by the instructor, and students themselves realize that there is a conflict between their predictions and observations, such conflict can encourage the students to reflect and revise their original knowledge structure so that the conflict can be resolved. Through this process, students can make progress in learning, repairing and organizing their knowledge structure until the conflict is resolved and a new equilibrium between the outside world and the students' internal mental structure is achieved. It is suggested that at the conflict stage, an appropriate level of guidance and support can be provided to the students to help them assimilate and accommodate the appropriate concepts in order to build a robust knowledge structure.

In one of my studies (chapter 3) on a topic which involves common student misconceptions, the idea of providing optimal mismatch is employed in the design of one of the instructional materials. Students are explicitly guided to reflect on whether there is a conflict between the observations from their daily experience and their conceptual understanding. We expected that the explicit awareness of the conflict can challenge students' initial knowledge schema and then providing appropriate guidance can improve their learning.

1.1.2 Zone of Proximal Development

In the early twentieth century, Vygotsky, a Soviet psychologist, proposed the idea of the zone of proximal development (ZPD). According to Vygotsky, the zone of proximal development is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (Vygotsky 1978). He suggested that there is a difference between what a student can do on his/her own and what he or she can do under guidance. A central idea in ZPD is to put students in this zone and provide scaffolding supports to them in order to gradually stretch their ability beyond their initial state. With an understanding of students' initial knowledge, the instructor can design instructional activity that is targeted at a level which is slightly above the students' current knowledge state so that the scaffolding provided can help the students overcome the gap between the new knowledge state desired and their initial knowledge state. When the first targeted state is reached, additional scaffolding support can be provided to stretch students' learning further. With these repeated processes, students can gradually make progress and can be weaned slowly to help them develop independence. It is important that the instruction in each step is designed to be commensurate

with students' prior knowledge and not too far beyond what the students are currently able to accomplish, so that students can build a connection between the new knowledge they learn and their prior knowledge, which facilitates effective learning.

In my studies, the concept of ZPD is used to evaluate the effect of different scaffolding supports to help students take advantage of what they learn to solve a novel problem which is typically known to be difficult for them. The interviews conducted in these studies (chapters 2 to 4) aim at examining the difficulties students have in this process and investigating the additional scaffolding supports required to gradually bridge students to the targeted goal. In the study about how TAs design problem solutions for their students, it's also interesting to examine whether the TAs are aware of the fact that the problem solutions should be written in ways which provide enough scaffolding to students and fit with students' prior knowledge, so that the solution provided is not beyond students' zone of proximal development and the students can learn from the problem solutions effectively.

1.1.3 Innovation and Efficiency in Transfer

There has been much research effort devoted to the study of transfer of learning. In 1990s, Bransford and Schwartz proposed a perspective to think of transfer in terms of the "preparation for future learning." (Bransford and Schwartz 1999) They also proposed a two dimensional learning and performance space, which involve two elements – efficiency and innovation - as two orthogonal coordinates (Schwartz et al. 2005). Using this two dimensional space, they discussed an optimal learning trajectory which moves along the diagonal direction. They argued that both the innovation task and efficiency task play an important role in students' learning, and the preparation for future learning or transfer is optimal if the instructional activity does not go

toward the extreme of any one of these two coordinates but rather involves both components of efficiency and innovation.

For example, although instructions focusing on efficiency (e.g., solving problems through much practice) can shorten the time needed for the problem solver to retrieve information from the long term memory (Anderson 2000) and improve students' abilities to break a problem into several routine sub-problems that they can easily handle (Schwartz et al. 2005), over-emphasizing the efficiency dimension may reduce the amount of cognitive engagement by the students. If students do not go through the cognitive processing at a deep enough level, they may not develop a functional understanding which is helpful for a farther transfer. They may not be able to solve other problems which go beyond the routine procedures although they can solve some similar routine problems correctly and quickly. If, on the other hand, the instruction goes to another extreme dimension which is far beyond the students' prior knowledge and is too innovative, they may get frustrated and may not be able to make sense of the new information. In this case, a meaningful learning and transfer are again impeded.

This framework of efficiency and innovation underlies the rationale of many instructional activities in my studies. The instructional activities in these studies (such as the categorization task in chapter 5 or the analogical problem solving activity in chapters 2 to 4) aim at helping students construct a good knowledge structure which allows them to transfer their learning to new contexts and to develop better problem solving skills. Although the direct show and tell is often considered as an efficient way to convey factual knowledge, if students are not actively engaged in deep processing and simply watch and mindlessly practice the routine procedures, they may only memorize the algorithms without constructing a solid knowledge structure that facilitates a far transfer. On the other hand, it is also important to provide guidelines to students

so that they can make sense of the instructional activities and take advantage of such activities to expand their learning. The activities in these studies are therefore designed by incorporating both elements of efficiency and innovation, with the expectation to engage students in deeper thinking, and to help them develop a deeper understanding.

1.2 BRIEF REVIEW OF RELEVANT STUDIES FROM PHYSICS EDUCATION RESEARCH AND OTHER COGNITIVE SCIENCE STUDIES

There is a close relationship between cognitive research and physics education research (PER). While cognitive studies provide the broad frameworks for interpreting and analyzing students' learning, physics education research adds to the cognitive research because many of the topics that are of interest to researchers in both fields can be studied in the domain of physics. Although physics education research is a new field which became established only in the late twentieth century, research from this field has produced significant findings that help to deepen our understanding of student difficulties and broaden our views about physics education from different perspectives. Based on this understanding, different teaching strategies and instructional materials have been developed to help students overcome their difficulties and construct robust understanding. A general overview of physics education research can be found in McDermott and Redish's article (McDermott and Redish 1999). Hsu et al. summarize information about problem solving in particular in their articles in 2004 (Hsu et al. 2004). My research in this thesis is inspired by many of the former studies from both PER and cognitive science. I'll review some of them in the following sections.

1.2.1 Research on Expert-Novice Difference

As mentioned earlier, research on expert-novice difference indicates that when solving problems, experts tend to focus on deep features while novices are more likely to be distracted by the surface features (Chi et al. 1981; Hardiman et al. 1989). Prior studies on the categorization of introductory mechanics problems (Chi et al. 1981) indicates that while novices may group two problems together because both of them involve an inclined plane, experts are likely to notice that one of the problems involves the principle of conservation of energy but the other problem involves a different principle (such as Newton's 2nd Law) so that they place these problems in two different categories. The findings suggest that experts usually group problems based upon the physics principles but novices usually group problems based on the surface features (such as the inclined plane or pulley.) The different ways experts and novices categorize problems reflect the different ways knowledge is organized in their minds (Johnson-Laird 1972; Bobrow and Norman 1975; Chi et al. 1981; Larkin 1981; Reif and Heller 1982; Schoenfeld and Herrmann 1982; Eylon and Reif 1984; Cheng and Holyoak 1985; Marshall 1995; Johnson and Mervis 1997; Dufresne et al. 2005). Experts have a pyramid-like knowledge hierarchy in which the most fundamental principles are placed at the top, followed by layers of subsidiary details. This organized knowledge structure allows the experts to focus on the fundamental physics principles when solving problems and it also allows them to transfer better between various contexts (Chi et al. 1981; Novick 1988; Bassok and Holyoak 1989; Brown 1989; Detterman and Sternberg 1993; Dufresne et al. 2005; Ozimek et al. 2005).

The research on expert-novice differences inspires my study on the categorization of quantum mechanics problems (chapter 5). In order to investigate if similar expert-novice difference also exists in the context of undergraduate level quantum mechanics, several faculty

members and students were asked to categorize 20 quantum mechanics problems based upon similarity of solution. Moreover, since the importance of looking beyond the surface features and building an organized knowledge structure is emphasized in these studies, the analogical problem solving activities (chapters 2 to 4) implemented in introductory physics courses were designed with an objective to guide students to look for deep similarities of problems.

1.2.2 Research on Analogical Reasoning and Transfer

Developing students' ability to apply the knowledge acquired in one context to a different situation is an important goal in education, and issues about transfer of knowledge have been widely discussed from different perspectives (Duncker 1945; Holyoak 1985; Genter and Toupin 1986; Novick 1988; Bassok and Holyoak 1989; Brown 1989; Adey and Shayer 1993; Detterman and Sternberg 1993; Holyoak and Thagard 1995; Kurz and Tweney 1998; Bransford and Schwartz 1999; Klahr et al. 2001; Mestre 2001; Mestre 2002; Lobato 2003; Dufresne et al. 2005; Gray and Rebello 2005; Ozimek et al. 2005; Rebello and Zollman 2005; Schwartz et al. 2005; Lobato 2006; Rebello et al. 2007). For example, the type of knowledge that students transfer (Dufresne et al. 2005; Mestre 2005; Singh 2005), factors that facilitate or hinder transfer (Hammer et al. 2005; Singh 2008), and the possible mechanisms for transfer (Rebello and Zollman 2005) are discussed in different contexts in prior studies. It is pointed out that the amount of knowledge a person has, the knowledge structure that the person constructs, and the context in which the knowledge is learned can all affect the person's ability to transfer knowledge acquired from one situation to another (Dufresne et al. 2005). Some recent studies on transfer also emphasize the view of considering transfer as a process of (re)constructing knowledge in the new context and suggest a dynamic assessment to measure transfer (Bransford

and Schwartz 1999; Lobato 2003; diSessa and Wagner 2005; Hammer et al. 2005; Rebello and Zollman 2005).

In particular, research on transfer involving analogical reasoning (Gentner 1983; Holyoak 1985; Holyoak and Koh 1987; Ross 1987; Gentner 1989; Ross 1989; Reeves and Weisberg 1994) provides great insight to my studies in chapters 2 to 4. Analogy to familiar situations is a good strategy to scaffold learning because it can help people understand an unfamiliar phenomenon more easily by creating a connection between the new information and the existing knowledge structure (Shapiro 1988; Duit 1991). Similar to Piaget's idea of accommodation process, new schema can be created by transferring the existing cognitive structure from the source domain to the target domain in which analogy comes into play (Shapiro 1988; Duit 1991). Studies have shown that using analogy can improve students' learning and reasoning in many domains (Reed et al. 1974; Novick 1988; Shapiro 1988; Ross 1989; Duit 1991). It is also a common practice for students to solve new problems by first looking for similar problems that they already know how to solve and applying similar strategies from one problem to another.

Theory suggests that in an analogical problem solving process, the analogous problems can involve similarity at two different levels (Holyoak and Koh 1987). Although the surface similarity may help people recall the analogy better, the structural similarity is essential in order to apply the analogy in the new situation appropriately (Holyoak and Koh 1987). In our studies in chapters 2 to 4, students are explicitly guided to perform analogical problem solving by first learning from one solved problem provided and transfering what they learn to solve another new problem which is analogous. Although some of the analogous problem pairs may look distinct to the students in terms of the surface features, these problem pairs involve deep structural similarity in terms of physics principles applied to the solution.

1.2.3 Research on Learning from Examples

My studies in this thesis are also greatly informed by research on learning from examples. Research suggests that at the initial stages of skill acquisition, learning can be more effective through the studying of worked out examples than the actual practice of problem solving (Ward and Sweller 1990). Because the cognitive overload is less when studying worked examples than when actually solving problems, more spaces in short term memory can become available for students to extract useful strategies and to develop knowledge schemas (Paas 1992; Sweller et al. 1998; Atkinson et al. 2000). Research also suggests that there is a difference between how good students and poor students study worked-out examples (Chi et al. 1989; Ferguson-Hessler and Jong 1990). Good students typically engage in deeper processing than the poor students (Chi et al. 1989; Ferguson-Hessler and Jong 1990). It is also pointed out that students who "self-explain" the worked-out example more are able to benefit more from reading the worked-out example. In view of the suggestion from these studies, different scaffolding supports in chapters 2 to 4 are designed with the attempt to help students process through the worked-out example deeply.

1.2.4 Research on Expert Problem Solving Approaches

The studies on how experts approach problem solving suggest another perspective to help enhance students' problem solving performance by adopting a systematic problem solving approach (Polya 1945; Van Heuvelen 1991; Reif 1995; Heller and Heller 2000). For example, Reif proposes a prescribed problem solving strategy which describes that effective problem solving involves three major steps: the initial problem analysis, the solution construction and the checking of the solution (Reif 1995). Research on expert problem solving strategies indicates

that experts typically start with a re-description of the problem information and then use the relevant information to plan the solution before executing it (Larkin 1979; Larkin et al. 1980; Larkin 1981; Chi et al. 1982; Eylon and Reif 1984; Bagno and Eylon 1997). A recent study on physics experts' problem solving approaches when their intuition fails also observes that expert problem solvers typically adopt a systematic problem-solving heuristic (such as first visualizing the problem, considering different conservation principles, and examining limiting cases) when they are presented with a novel situation (Singh 2002). Research indicates that by explicitly modeling and encouraging students to follow a set of problem solving procedures, students are likely to achieve a better performance (Van Weeren et al. 1982; Heller and Reif 1984; Wright and Williams 1986; Huffman 1997). In view of these studies, when we provided solved problems to students in the analogical problem solving activities described in chapters 2 to 4, the solutions were presented in a way which follows these systematic solution strategies.

On the other hand, if we wish to help instructors to teach these problem solving strategies effectively in their classrooms, it is necessary to first investigate their current views about example problem solutions. In chapter 5, we investigate teaching assistants' beliefs about the role example problem solution should play and how the example problem solutions should be designed. In particular, we focus on different solution features that pertain to the three stages of the prescribed problem solving strategy proposed by Reif (Reif 1995). TAs' notions and values related to these features were examined and compared to the recommendations from research literature. The goal of this comparison is to provide implications for how we can improve the teaching practices of these graduate teaching assistants and help them develop teaching strategies that they will hopefully carry with them when they become faculty members.

1.2.5 Common Student Difficulties and the Multiple-Choice Assessment

Research on student difficulties is one of the earliest focuses in physics education research. As student difficulties were discovered and analyzed in various topics, these findings provide a basis for further physics education research for developing effective teaching strategies, instructional materials, and assessment tools. One example of a multiple-choice test that was developed based on this understanding of student difficulties is the Force Concept Inventory (FCI) created by Hestenes et al. (Hestenes et al. 1992) This multiple-choice test examines students' conceptual understanding of motion and force. It is one of the well-known standardized tests to evaluate students' learning in introductory physics.

There has been research effort devoted to investigating the extent to which students' performance on multiple-choice tests agrees with the performance on equivalent free-response tests. In Hudson and Hudson's study (Hudson and Hudson 1981), a high correlation between students' performance on multiple-choice questions and free-response questions was found. Scott, Stelzer, and Gladding also reported a high consistency when students' ranking based on the multiple-choice exam scores was compared to the ranking based on students' written explanations graded by the instructors in an introductory electricity and magnetism course (Scott et al. 2006). Although the multiple-choice exam is easier to grade, its quality depends highly on the careful design of each question and each response. In chapter 7, I discuss a study along this line of research using two research-based multiple-choice questions that we designed. Students' performance on the research-based multiple-choice questions is compared to the performance on the corresponding free-response problems. A new grading scheme which allows us to assign different partial scores to students who display different levels of understanding of the concepts tested is also discussed in this study.

1.3 CHAPTER REFERENCES

Adey, P. and M. Shayer (1993). "An exploration of long-term far-transfer effects following an extended intervention program in the high school science curricula." *Cogn. Instruct.* **11**, 1-29.

Anderson, J. (2000). *Learning and memory: An integrative approach*. New York, Wiley.

Atkinson, R. K., S. J. Derry, A. Renkl and D. Wortham (2000). "Learning from examples: Instructional principles from the worked examples research." *Rev. Educ. Res.* **70**(2), 181-214.

Bagno, E. and B. Eylon (1997). "From problem solving to knowledge structure: An example from the domain of electromagnetism." *Am. J. Phys.* **65**(8), 726-736.

Bassok, M. and K. Holyoak (1989). "Interdomain transfer between isomorphic topics in algebra and physics." *J. Experimental Psychology: Memory, Learning and Cognition* **15**(1), 153-166.

Bobrow, D. and D. Norman (1975). "Some principles of memory schemata". In *Representation and understanding: Studies in cognitive science*. D. Bobrow and A. Collins. New York, Academic Press**:** 131-149.

Bransford, J. D. and D. Schwartz (1999). "Rethinking transfer: A simple proposal with multiple implications." *Rev. Res. Educ.* **24**, 61-100.

Brown, A. (1989). "Analogical learning and transfer: What develops?". In *Similarity and analogical reasoning*. S. V. a. A. O. Eds. NY, Cambridge University Press**:** 369-412.

Cheng, P. W. and K. J. Holyoak (1985). "Pragmatic reasoning schema." *Cognitive Psychology* **17**, 391-416.

Chi, M. T. H., M. Bassok, M. W. Lewis, P. Reimann and R. Glaser (1989). "Self-explanations: How students study and use examples in learning to solve problems." *Cognitive Science* **13**(2), 145-182.

Chi, M. T. H., P. J. Feltovich and R. Glaser (1981). "Categorization and representation of physics knowledge by experts and novices." *Cog. Sci.* **5**, 121-152

Chi, M. T. H., R. Glaser and E. Rees (1982). "Expertise in problem solving". In *Advances in the psychology of human intelligence*. R. J. Sternberg. Hillsdale, NJ, Lawrence Erlbaum Associates. **1:** 7-75.

Detterman, D. K. and R. J. Sternberg, Eds. (1993). *Transfer on trial: Intelligence, cognition and instruction*. Ablex, Norwood, NJ.

diSessa, A. and J. Wagner (2005). "What coordination has to say about transfer". In *Transfer of learning from a modern multidisciplinary perspective*. J. Mestre. Greenwich, CT, Information Age Publishing.

Dufresne, R., W. Gerace, P. Hardiman and J. Mestre (1992). "Constraining novices to perform expert-like problem analyses: Effects on schema acquisition." *Journal of the Learning Sciences* **2**(3), 307-331.

Dufresne, R., J. Mestre, T. Thaden-Koch, W. Gerace and W. Leonard (2005). "Knowledge representation and coordination in the transfer process". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Information Age Publishing**:** 155-215.

Duit, R. (1991). "On the role of analogies and metaphors in learning science." *Science Education* **75**, 649-672.

Duncker, K. (1945). "On problem solving." *Psychol. Monogr.* **58**, 1.

Eylon, B. and F. Reif (1984). "Effect of knowledge organization on task performance." *Cognition and Instruction* **1**, 5.

Ferguson-Hessler, M. G. M. and T. d. Jong (1990). "Studying physics texts: Differences in study processes between good and poor performers." *Cognition and Instruction* **7**(1), 41-54.

Genter, D. and C. Toupin (1986). "Systematicity and surface similarity in the development of analogy." *Cognitive Science* **10**, 277-300.

Gentner, D. (1983). "Structure-mapping: A theoretical framework for analogy." *Cog. Sci.* **7**, 155-170.

Gentner, D. (1989). "Mechanisms of analogical learning.". In *Similarity and analogical reasoning*. S. Vosniadou and A. Ortony. London, Cambridge University press**:** 199-241.

Ginsberg, H. and S. Opper (1969). *Piaget's theory of intellectual development*. Englewood Cliffs, NJ, Prentice Hall.

Gray, K. E. and N. S. Rebello (2005). "Transfer between paired problems in an interview." *AIP Conf. Proc.* **790**, 157.

Hammer, D. (2000). "Student resources for learning introductory physics." *Am. J. Phys.* **68**(7), S52-S59.

Hammer, D., A. Elby, R. Scherr and E. F. Redish (2005). "Resources, framing and transfer". In *Transfer of learning from a modern multidisciplinary perspective*. J. Mestre. Greenwich, CT, Information Age Publishing.

Hardiman, P., R. Dufresne and J. Mestre (1989). "The relationship between problem categorization and problem solving among experts and novices." *Memory and Cognition* **17**(5), 627-638.

Hayes, J. R. and H. A. Simon (1977). "Psychological differences among problem isomorphs". In *Cognitive theory*. N. J. Castellan, D. B. Pisoni and G. R. Potts. Hillsdale, NJ, Lawrence Erlbaum.

Heller, J. and F. Reif (1984). "Prescribing effective human problem-solving processes: Problem description in physics." *Cognition and Instruction* **1**, 177-216.

Heller, K. and P. Heller (2000). *The competent problem solver for introductory physics*. New York, McGraw-Hill.

Hestenes, D., M. Wells and G. Swackhamer (1992). "Force concept inventory." *The Phys. Teach.* **30**, 141-158.

Holyoak, K. (1985). "The pragmatics of analogical transfer". In *The psychology of learning and motivation*. G. Bower. New York, Academic Press. **19**.

Holyoak, K., Ed. (1985). *The psychology of learning and motivation*. New York, Academic.

Holyoak, K. and K. Koh (1987). "Surface and structural similarity in analogical transfer." *Memory and Cognition* **15**, 332-340.

Holyoak, K. and P. Thagard (1995). *Mental leaps: Analogy in creative thought*. Cambridge, MA, MIT Press.

Hsu, L., E. Brewe, T. Foster and K. Harper (2004). "Resource letter rps-1: Research in problem solving." *American Journal of Physics* **72**(9), 1147-1156.

Hudson, H. and C. Hudson (1981). "Suggestions on the construction of multiple-choice tests." *Am. J. Phys.* **49**, 838-841.

Huffman, D. (1997). "Effect of explicit problem solving instruction on high school students' problem-solving performance and conceptual understanding of physics." *Journal of Research on Science Teaching* **34**(6), 551-570.

Johnson-Laird, P. (1972). *Psychology of reasoning, structure and content*. Cambridge, MA, Harvard University Press.

Johnson, K. and C. Mervis (1997). "Effects of varying the levels of expertise on the basic level of categorization." *J. Exp. Psychol. Gen.* **126**(3), 248.

Klahr, D., Z. Chen and E. Toth (2001). "From cognition to instruction to cognition: A case study in elementary school science instruction". In *Designing for science, implications from everyday, classroom and professional settings*. C. D. Schuun, T. Okada and K. Crowley. Mahwah, NJ, Lawrence Erlbaum.

Kurz, E. M. and R. D. Tweney (1998). "The practice of mathematics and science: From calculus to the clothesline problem". In *Rational models of cognition*. M. Oalesfard and N. Chater. London, Oxford University Press**:** 415-438.

Larkin, J. (1979). "Processing information for effective problem solving." *Engineering Education* **70**(3), 285-288.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning approach." *J. Struct. Learn.* **6**, 121-130.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning model." *J. Struct. Learn.* **6**, 271-297.

Larkin, J. (1981). "Cognition of learning physics." *Am. J. Phys.* **49**(6), 534-541.

Larkin, J. H., J. McDermott, D. P. Simon and H. A. Simon (1980). "Expert and novice performance in solving physics problems." *Sciences* **208**(4450), 1335-1342.

Larkin, J. H. and F. Reif (1979). "Understanding and teaching problem solving in physics." *Eur. J. Sci. Ed.* **1**(2), 191-203.

Leonard, W., R. Dufresne and J. Mestre (1996). "Using qualitative problem-solving strategies to highlight the role of conceptual knowledge in solving problems." *Am. J. Phys.* **64**, 1495-1503.

Lobato, J. (2003). "How design experiments can inform a rethinking of transfer and vice versa." *Educ. Res.* **32**, 17.

Lobato, J. (2006). "Transfer strand: Alternative perspectives on the transfer of learning: History, issues and challenges for future research." *J. Learn. Sci.* **15**, 431.

Maloney, D. (1994). "Research in problem solving: Physics". In *Handbook of research on the teaching and learning of science*. D. Gabel. New York, MacMillan.

Marshall, S. (1995). *Schemas in problem solving*. New York, NY, Cambridge University Press.

McDermott, L. and E. Redish (1999). "Resource letter: Per-1: Physics education research." *Am. J. Phys.* **67**(9), 755.

Mestre, J. (2001). "Implications of research on learning for the education of prospective science and physics teachers." *Phys. Educ.* **36**, 44.

Mestre, J. (2002). "Probing adults' conceptual understanding and transfer of learning via problem solving." *J. of Applied Developmental Psychology* **23**(1), 9-50.

Mestre, J., Ed. (2005). *Transfer of learning from a modern multidisciplinary perspective*. Greenwich, CT, Information Age Publishing.

Mestre, J., R. Dufresne, W. Gerace, P. Hardiman and J. Touger (1993). "Promoting skilled problem solving behavior among beginning physics students." *Journal of Research in Science Teaching* **30**, 303-317.

Novick, L. (1988). "Analogical transfer, problem similarity and expertise." *J. Exp. Psychol. Learn. Mem. Cogn.* **14**, 510.

Ozimek, D. J., P. V. Engelhardt, A. G. Bennett and N. S. Rebello (2005). "Retention and transfer from trigonometry to physics." *Proceedings of the 2004 Physics Education Research Conference* **790**, 173-176.

Paas, F. (1992). "Training strategies for attaining transfer of problem-solving skill in statistics: A cognitive-load approach." *Journal of Educational Psychology* **84**, 429-434.

Polya, G. (1945). *How to solve it*. Princeton, NJ, Princeton University Press.

Rebello, N. S., L. Cui, A. G. Bennett, D. A. Zollman and D. J. Ozimek (2007). "Transfer of learning in problem solving in the context of mathematics and physics". In *Learning to solve complex scientific problems*. D. Jonassen. New York, Lawrence Earlbaum Associates.

Rebello, N. S. and D. A. Zollman (2005). "Dynamic transfer: A perspective from physics education research". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Infromation Age Publishing.

Redish, E. F., J. M. Saul and R. N. Steinberg (1998). "Student expectations in introductory physics." *Am. J. Phys.* **66**, 212-224.

Redish, E. F., R. E. Scherr and J. Tuminaro (2006). "Reverse-engineering the solution of a "simple" physics problem: Why learning physics is harder than it looks." *The Phys. Teach.* **44**, 293-300.

Reed, S. K., G. W. Ernst and R. Bannerji (1974). "The role of analogy in transfer between similar problem states." *Cogn. Psychol.* **6**, 436.

Reeves, L. M. and R. W. Weisberg (1994). "The role of content and abstract information in analogical transfer." *Psychological Bulletin* **115**(3), 381-400.

Reif, F. (1981). "Teaching problem solving- a scientific approach." *The Phys. Teach.* **33**, 310.

Reif, F. (1986). "Scientific approaches to science education." *Physics Today* **39**(11), 48-54.

Reif, F. (1995). "Millikan lecture 1994: Understanding and teaching important scientific thought processes." *Am. J. Phys.* **63**(1), 17-32.

Reif, F. (1995). *Understanding basic mechanics*. New York, Wiley.

Reif, F. and S. Allen (1992). "Cognition for interpreting scientific concepts: A study of acceleration." *Cognit. Instruct.* **9**(1), 1-44.

Reif, F. and J. I. Heller (1982). "Knowledge structure and problem solving in physics." *Educ. Psychologist* **17**, 102.

Reif, F. and J. Larkin (1991). "Cognition in scientific and everyday domains: Comparison and learning implications." *Journal of Research in Science Teaching* **28**, 733-760.

Ripple, R. and V. Rockcastle, Eds. (1964). *Piaget rediscovered*. New York, Cornell University Press.

Ross, B. (1989). "Distinguishing types of superficial similarities: Different effects on the access and use of earlier problems." *J. Experimental Psychology: Learning, memory and cognition* **15**, 456-468.

Ross, B. H. (1987). "This is like that: The use of earlier problems and the separation of similarity effects." *Journal of Experimental Psychology: Learning, Memory, and Cognition* **13**, 629-639.

Ross, B. H. (1989). "Distinguishing types of superficial similarities: Different effects on the access and use of earlier problems." *Journal of Experimental Psychology: Learning, Memory, and Cognition* **15**, 456-468.

Schoenfeld, A. and D. J. Herrmann (1982). "Problem perception and knowledge structure in expert novice mathematical problem solvers." *J. Exp. Psych: Learning, Memory, and Cognition* **8**, 484-494.

Schwartz, D., J. Bransford and D. Sears (2005). "Efficiency and innovation in transfer". In *Transfer of learning : Research and perspectives*. J. Mestre. Greenwich, CT, Information Age Publishing.

Scott, M., T. Stelzer and G. Gladding (2006). "Evaluating multiple-choice exams in large introductory physics courses." *Phys. Rev. ST Phys. Ed. Res.* **2**, 020102.

Shapiro, M. A. (1988). "Analogies, visualization and mental processing of science stories". In *Communication yearbook*. M. McLaughlin. Beverly Hills, CA, Sage. **9**.

Singh, C. (2002). "When physical intuition fails." *Am. J. Phys.* **70**(11), 1103-1109.

Singh, C. (2005). "Transfer of learning in quantum mechanics." *AIP Conf. Proc.* **790**, 23-26.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. II. Effect of some potential factors on problem solving and transfer." *Phys. Rev. ST Phys. Educ. Res.* **4**, 010105.

Sweller, J., J. J. G. V. Merrienboer and F. G. W. C. Paas (1998). "Cognitive architecture and instructional design." *Educational Psychology Review* **10**(3), 251-296.

Van Heuvelen, A. (1991). "Learning to think like a physicist: A review of research-based strategies." *Am. J. Phys.* **59**(10), 891-897.

Van Heuvelen, A. (1991). "Overview, case study physics." *Am. J. Phys.* **59**, 898-907.

Van Weeren, J. H. P., F. F. M. de Mul, M. J. Peters, H. Kramers-Pals and H. J. Roossink (1982). "Teaching problem-solving in physics: A course in electromagnetism." *Am. J. Phys.* **50**, 725-732.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA, Harvard University Press.

Ward, M. and J. Sweller (1990). "Structuring effective worked examples." *Cognition and Instruction* **7**(1), 1-39.

Wright, D. S. and C. D. Williams (1986). "A wise strategy for introductory physics." *The Phys. Teach.* **24**, 211-216.

2.0 USING ISOMORPHIC PROBLEMS TO LEARN INTRODUCTORY PHYSICS: CHALLENGES IN APPLYING NEWTON'S 2ND LAW IN A NON-EQULIBRIUM SITUATION

2.1 ABSTRACT

In this study, we examine introductory physics students' ability to perform analogical reasoning between two isomorphic problems which employ the same underlying physics principles but have different surface features. Three hundred sixty-two students from a calculus-based and an algebra-based introductory physics course were given a quiz in the recitation in which they had to first learn from a solved problem provided and take advantage of what they learned from it to solve another problem (which we call the quiz problem) which was isomorphic. Previous research suggests that the multiple-concept quiz problem is challenging for introductory students. Students in different recitation classes received different interventions in order to help them discern and exploit the underlying similarities of the isomorphic solved and quiz problems. We also conducted think-aloud interviews with four introductory students in order to understand in depth the difficulties they had and explore strategies to provide better scaffolding. We found that most students were able to learn from the solved problem to some extent with the scaffolding provided and invoke the relevant principles in the quiz problem. However, they were not necessarily able to apply the principles correctly. Research suggests that more scaffolding is

needed to help students in applying these principles appropriately. We outline a few possible strategies for future investigation.

2.2 INTRODUCTION

Learning physics is challenging. Physics is a subject in which diverse physical phenomena can be explained by just a few basic physics principles. Learning physics requires unpacking these principles and understanding their applicability in a variety of contexts that share deep features (Chi et al. 1981; Eylon and Reif 1984). A major goal of most calculus-based and algebra-based introductory physics courses is to help students learn to recognize the applicability of a physics principle in diverse situations and discern the deep similarities between the problems that share the same underlying physics principles but have different surface features.

It is well known that two physics problems that look very similar to a physics expert because both involve the same physics principle don't necessary look similar to the beginning students (Chi et al. 1981). Research has shown that when physics experts and novices are given several introductory physics problems and asked to categorize the problems based upon similarity of solution, experts tend to categorize them based upon the fundamental physics principles (e.g., conservation of mechanical energy, Newton's 2nd Law, etc.) while novices tend to group them based upon the surface features such as pulley or inclined plane (Chi et al. 1981). Similarly, when a group of introductory physics students and physics faculty were asked to rate the similarities between different pairs of problems, it was found that for problem pairs which only involve surface similarity but employ different principles, students were more likely to rate them as similar compared to the faculty members (Mateycik et al. 2009). The different patterns

that experts and novices discern in these problems reflect the difference between the ways in which the knowledge structure of experts and novices is structured and how they exploit it to solve problems. The fact that experts in physics have a well-organized knowledge hierarchy where the most fundamental physics principles are placed at the top, followed by layers of subsidiary knowledge and details facilitates their problem solving process, allowing them to approach the problems in a more effective and systematic way (Johnson-Laird 1972; Bobrow and Norman 1975; Larkin 1980; Larkin 1980; Chi et al. 1981; Larkin 1981; Reif and Heller 1982; Schoenfeld and Herrmann 1982; Eylon and Reif 1984; Cheng and Holyoak 1985; Marshall 1995; Johnson and Mervis 1997). It also guides the experts to see the problems beyond the surface features, and makes the transfer of knowledge between different contexts easier.

There has been much research effort devoted to investigating and improving transfer of learning (Duncker 1945; Holyoak 1985; Genter and Toupin 1986; Bassok and Holyoak 1989; Brown 1989; Adey and Shayer 1993; Detterman and Sternberg 1993; Holyoak and Thagard 1995; Kurz and Tweney 1998; Klahr et al. 2001; Mestre 2002; Gray and Rebello 2004; Ozimek et al. 2005). In these investigations, issues about transfer of knowledge from one context to another have been discussed from different perspectives (Sternberg 1977; Novick 1988; Bransford and Schwartz 1999; Mestre 2001; Lobato 2003; Dufresne et al. 2005; Rebello and Zollman 2005; Schwartz et al. 2005; Lobato 2006; Rebello et al. 2007). The amount of knowledge a person has, the knowledge structure that the person constructs, and the context in which the knowledge is learned could all affect the person's ability to transfer knowledge acquired in one situation to another (Dufresne et al. 2005).

One way to help students learn physics is via analogical reasoning (Chi et al. 1981; Eylon and Reif 1984). Students can be explicitly taught to make an analogy between a solved problem

and a new problem, even if the surface features of the problems are different. In doing so, students may develop an important skill shared by experts: the ability to transfer from one context to another, based upon shared deep features. Here, we examine introductory physics students' ability to perform analogical problem solving. In this investigation, students were explicitly asked to point out the similarities between a solved problem and a quiz problem and then use the analogy to solve the quiz problem. In particular, students were asked in a recitation quiz to browse through and learn from a solved problem and then solve a quiz problem that has different surface features but the same underlying physics. Different types of scaffolding were provided in different intervention groups (recitation sections). The goal is to investigate what students are able to do with the analogy provided, and to understand if students could discern the similarities between the solved and the quiz problems, take advantage of them and transfer their learning from the solved problem to solve the quiz problem.

Our investigation also has overlap with prior investigations involving isomorphic problems since we focus on the effect of using an isomorphic problem pair to help students learn introductory physics. In particular, students were explicitly asked to learn from a solved problem and then solve another problem which is isomorphic. According to Hayes and Simon (Hayes and Simon 1977), isomorphic problems are defined as problems that can be mapped to each other in a one-to-one relation in terms of their solutions and the moves in the problem solving trajectories. For example, the “tower of Hanoi problem” and the “cannibal and the missionary problem” are isomorphic to each other and have the same structure if they are reduced to the abstract mathematical form (Hayes and Simon 1977). In this investigation, we call problems isomorphic if they can be solved using the same physics principles. The ballerina problem in which the ballerina’s rotational speed changes when she pulls her arms closer to or farther away

from her body is isomorphic to a neutron star problem in which the collapse due to gravity makes the neutron star spin faster. Both these problems require the conservation of angular momentum principle to solve them, but the contexts are very different.

Cognitive theory suggests that, depending on a person's expertise in the field, different contexts and representations may trigger the recall of a relevant principle more in one problem than another, and two problems which are isomorphic are not necessarily perceived as being at the same level of difficulty especially by a beginning learner (Simon and Hayes 1976; Kotovsky et al. 1985). Changing the context of the problem, making one problem in the isomorphic pair conceptual and the other quantitative, or introducing distracting features into one of the problems can to different extent raise the difficulty in discerning the similarity and make the transfer of learning between the two problems more challenging (Singh 2008). A previous study on transfer in which isomorphic problem pairs in introductory physics were given back to back to the students suggests that those who were given both the quantitative and conceptual problems in the isomorphic pairs were often able to perform better on the conceptual problem (which was typically more challenging for them) than the students who were given the conceptual problem alone (Singh 2008). For problem pairs that didn't involve a conceptual and a quantitative one but one problem provided a hint for the other, students typically were able to discern the similarity between the two problems and took advantage of what they learned from one problem to solve the other. However, for those problems in which the context triggered an alternative approach (which was not necessarily correct) to solve the problem (for example, in problems involving friction), the alternative view prevented the students from making a connection between the two problems. This study suggests that isomorphic problem pairs may be a useful tool to help students learn physics, but in some cases, more scaffolding may be needed (Lin and Singh 2011).

As noted earlier, the study here could also be viewed from a broader perspective of learning and reasoning by analogy. Analogy is often useful in helping people understand an unfamiliar phenomenon. Theories suggest that analogy can make the mental processing of new information more efficient by modifying the existing knowledge schemata (Shapiro 1988; Duit 1991). Similar to Piaget's idea of accommodation process, new schema can be created by transferring the existing cognitive structure from the source domain to the target domain in which analogy comes into play (Shapiro 1988; Duit 1991). As pointed out in the literature (Shapiro 1988), a good analogy not only creates an efficient connection between the new and existing information, but can also make the new information more concrete and easier to comprehend. Analogy can also be made by drawing a connection between different contexts involving similar reasoning strategies, e.g., in problems where the same physics principles are applicable, which is what we aim at here. The view of how analogy plays a role in the learning process which involves connecting the new material with the existing structure and modifying the existing cognitive structure to accommodate the new information is consonant with the view which describes learning as a construction process, emphasizing the importance of prior knowledge as a basis of learning. Studies have shown that using analogy could help improve students' learning and reasoning in many domains (Reed et al. 1974; Novick 1988; Shapiro 1988; Ross 1989; Duit 1991), and it has long been an effective strategy adopted by many instructors in the practical classrooms.

Another important thread of research related to the study discussed here is that of learning from examples. Examples can serve a goal similar to that served by analogy because they can be used to draw connections between different materials and make the unfamiliar familiar (Duit 1991). Presenting students with examples to demonstrate the meaning and

application of a physics concept is a very common pedagogical tool in physics. Research on learning from worked-out examples (Chi et al. 1989; Aleven et al. 1999; Atkinson et al. 2000; Chi 2000; Yerushalmi et al. 2008) (such as those in a textbook) has shown that students who self-explain the underlying reasoning in the example extensively learn more than those who don't self-explain even if the self-explanations given by the students are sometimes fragmented or incorrect. It is suggested that the largest learning gain can be achieved if students are actively engaged in the process of sense making while learning from examples (Chi et al. 1989; Aleven et al. 1999; Chi 2000; Yerushalmi et al. 2008).

2.3 METHODOLOGY

In this study, students from a calculus-based and an algebra-based introductory physics course were given two isomorphic problems in the recitation quiz. The solution to one of the problems (which we call the "solved problem") was provided. Students were explicitly asked to learn from the solution to the solved problem, point out the similarities between the two problems, explain whether they can use the solved problem to solve the other problem (which we call the "quiz problem"), and then they were asked to solve the quiz problem. The solution provided was presented in a detailed and systematic way. It started with a description of the problem with the knowns, unknowns, and target quantity listed, followed by a plan for solving the problem in which the reasons why each principle was applicable were explicated. After the plan was executed in the mathematical representation, the last part of the solution provided a check for the answer by examining the limiting cases. A full solution to the solved problems can be found in the Appendix.

In the quiz, the solved problem was about a girl riding on a rollercoaster car on a smooth track. The problem asked for the apparent weight of the girl when the car went over the top of a hump around which the track was part of a circle. Conservation of mechanical energy can be used to find the speed at the point of interest, followed by the application of Newton's 2nd Law in the non-equilibrium situation with a centripetal acceleration to solve for the normal force, which is related to the target variable. This problem was isomorphic to the quiz problem, which was about a boy swinging on a tire swing created by a rope tied to a branch. Students were told that the rated maximum value of tension that the rope could hold was 2500 N. They were asked to evaluate whether the ride was safe by solving for the maximum tension in the rope during the ride, assuming the boy initially started at rest at a certain height. Again, the problem can be solved using the principles of conservation of mechanical energy and Newton's 2nd Law as well as the concept of centripetal acceleration. The same problems have been used in another study, which examines the effect of students' self-diagnosing of their own solutions to quiz problems on subsequent problem solving and transfer (Yerushalmi et al. ; Yerushalmi et al. 2008). In that study, students were asked in the quiz to solve the rollercoaster problem first, and then diagnose their own mistakes with different types of scaffolding provided to aid the self-diagnosis process. The swing problem was later given in the midterm exam. Although the solution to the swing problem can be mapped to that of the rollercoaster problem in an almost one-to-one fashion, many students didn't necessarily recall and transfer what they learned from the self-diagnosing task and didn't perform well on the swing problem (Yerushalmi et al. ; Yerushalmi et al. 2008). It is possible that the time separation between the quiz and the midterm exam as well as the different contexts of the two problems made it difficult for students to discern the deep connection between the two problems. By explicitly placing the two problems in a pair,

providing students with a detailed solution to one problem and asking them to point out the similarities between the two problems before solving the quiz problem, our goal in this study is to examine whether such explicit hints can help them make better connections between the two problems and help them solve the quiz problem by learning from the solved problem.

Three hundred and sixty two students from an algebra-based and a calculus-based introductory physics course were involved in this study (181 students in each, respectively). In each course, students were randomly divided into one comparison group and three intervention groups based on the different recitation classes. Students in the comparison group were given only the quiz problem in the recitation quiz. Similar to a traditional quiz, students in this comparison group were asked to solve the quiz problem on their own with no scaffolding support provided. The performance of this group of students could help us understand what students were able to do without being explicitly provided a solved isomorphic problem to learn from.

Students in the other three intervention groups, on the other hand, were given an opportunity to learn from the solved isomorphic problem during the quiz. Our previous research (Lin and Singh) indicates that simply providing students with a similar solved problem doesn't necessarily help them because students may simply follow the procedures in the solution without thinking carefully about the deep similarity of the problems. In order to help students process through the analogy more deeply and contemplate issues which they often have difficulty with, different kinds of scaffolding were provided in addition to the solved problem to the students in different intervention groups.

In particular, students in the intervention group 1 were asked to take the first few minutes in the quiz to learn from the solution to the solved problem. They were explicitly told at the beginning of the quiz that after 10 minutes, they had to turn in the solution, and then solve two

problems in the quiz: one of them would be exactly the same as the one they just browsed over (the rollercoaster problem), and the other one would be similar (the swing problem). In order to help students discern the connection between the two problems, students were also explicitly asked to identify the similarities between the two problems and explain whether they could use the similarities to solve the quiz problem before actually solving it. We hypothesized that since they had to solve the same problem whose solution they browsed over and another isomorphic problem in the quiz, students would try hard to get the most out of the solution in the allocated learning period. In order to apply what they learned from the solution to solve exactly the same problem on their own as well as an isomorphic problem, they had to not only focus on what principles are useful, but also understand why and how each principle is applicable in different circumstances. We hypothesized that an advantage could be achieved over the comparison group if students in the intervention group 1 went through a deep reasoning while browsing over the solved problem. Students' performance on both problems was later analyzed and compared with the comparison group.

The scaffolding in the 2nd intervention group was designed based on a different framework. Students in this group were first asked to solve the quiz problem on their own. After a designated period of time, they turned in their solution, and were given the isomorphic solved problem to learn from. Then, with the solved problem and its solution in their possession, they were asked to redo the quiz problem a second time after pointing out the similarities between the two problems and explicitly asked to discuss the implication of these similarities in constructing their solution to the quiz problem. We hypothesized that postponing the browsing over the solved isomorphic problem until the students have actually tried to solve the quiz problem on their own could be beneficial to them because in this way, students would have already searched

through their knowledge base of physics and attempted to organize the information given in the quiz problem. We hypothesized that having tried the quiz problem on their own may make the browsing over the solved problem for relevant information more structured and productive before students attempted the quiz problem a second time. Students had the opportunity to display what they learned from the solved isomorphic problem when they solved the quiz problem a second time. The fact that the solution we provided had made explicit the consideration for using the principles but was not directly the solution to the quiz problem was inspired by Schwartz, Bransford and Sears' theory of transfer (Schwartz et al. 2005), which states that two components, efficiency and innovation, are both important in the learning process.

Unlike the students in the intervention groups 1 and 2 who had to figure out the similarities between the two problems themselves, students in the 3rd intervention group were given both the quiz problem and the solved problem at the same time and were explicitly told that "Similar to the solved problem, the quiz problem can be solved using conservation of energy and Newton's 2nd Law (with centripetal acceleration)". We hypothesized that deliberately pointing out the principles that are useful in solving both problems may guide students to focus more on the deep physics instead of the surface features while browsing over the solved problem. In addition to the instruction which asked them to first learn from the solved problem and then exploit the similarity to solve the quiz problem, students in this group also received extra hints to help them deal with the common difficulties in solving this problem found in previous research (Lin and Singh ; Singh and Rosengrant 2003; Singh 2009).

Research suggests that introductory physics students have great difficulty dealing with the non-equilibrium situation and they usually think of a non-equilibrium situation which involves the centripetal acceleration as an equilibrium situation by treating the centripetal force

as an additional force (Singh 2009). In the swing problem, the correct use of the centripetal acceleration and Newton's 2nd Law should yield $T - mg = \frac{mv^2}{r}$. However, students who treat it as an equilibrium problem and believe that "the centripetal force is an additional force" obtain an answer of the type $T - mg + \frac{mv^2}{r} = 0 \Rightarrow T - mg = -\frac{mv^2}{r}$, which has a wrong sign. To help students with these issues, we presented to students in the intervention group 3 a dialogue between two people discussing whether the centripetal force is an additional force or whether it is simply a name given to the net force in a circular motion. (See the Appendix.) Students were asked to explain which person they agreed with and why before solving the quiz problem. To assist students in correctly analyzing the dialogue, a practical situation similar to the rollercoaster cart which went over the top of a circle was discussed. Free-body diagrams as well as mathematical equations were presented with the dialogue. We hypothesized that if students did not know how to assess which person is correct in the dialogue, they could always go back to the solution of the rollercoaster problem provided and figure out the correct answer by comparing either the free-body diagrams or the mathematical equations. We hypothesized that after students contemplated the issues discussed in the dialogue and acquired a better understanding of the centripetal acceleration and centripetal force, they may perform better on the quiz problem.

Table 2-1. Summary of the rubric for the quiz problem. The rubric for the solved problem is almost identical.

Description	*Correct answer*	*Common mistakes*		*Points taken off*
Invoking and applying the principle of conservation of mechanical energy to find the speed (3 points)	$mg\Delta h = \frac{1}{2}mv^2$	use 1-D kinematics equations to find v		2
		wrong Δh		1
Identifying the centripetal acceleration and using Newton's 2nd Law to find the tension (7 points)	$a = a_c = \frac{v^2}{r}$ $T - mg = ma_c = m\frac{v^2}{r}$ $\Rightarrow T = mg + m\frac{v^2}{r}$	$a = 0, \quad T = mg$		5
		$a \neq 0$ but wrong formula for a	$2500 = ma$	3
			$a = \frac{mv^2}{r}$	1
		$T = m\frac{v^2}{r}$		3
		$T = mg - m\frac{v^2}{r}$	Using $\sum F = 0$ (centripetal force as an additional force)	2
			Using $\sum F = T - mg$ $= ma = -\frac{mv^2}{r}$ (didn't pay attention to the direction of a)	1

Students' performance on the quiz was graded by two researchers using a rubric. Summary of the "physics part of the rubric highlights" for the quiz problem is shown in Table 2-1. The rubric for the solved problem is not listed here because the solutions to the two problems can be mapped directly to each other and the rubrics are almost identical except for the problem specific details involving the application of physics principles. As shown in Table 2-1, the rubric had a full score of 10 points, divided into two parts based upon the two principles

involved. Three points were devoted to using the principle of conservation of mechanical energy (CME) to find the speed at the point where Newton's 2nd Law was applied; seven points were devoted to identifying the centripetal acceleration, recognizing all relevant forces and applying Newton's 2nd law correctly to obtain the final answer. Students' common mistakes and the corresponding points taken off are also listed. In the case of intervention 3, which included an additional dialogue problem, the same rubric was used for grading their answer for finding the tension. If the students didn't answer the dialogue problem correctly, an additional 2 points were taken off from the score they received for solving for the tension force if it didn't result in a negative score. The minimum score was zero. An inter-rater reliability of more than 80 percent was achieved when two researchers scored independently a sample of 20 students. When five researchers scored independently a sample of five students, the inter-rater reliability was more than 95 percent.

Students' performance in different intervention groups was later compared to each other. In order to examine the effects of interventions on students with different expertise and to evaluate whether the interventions were more successful in helping students at a particular level of expertise, we further classified the students in each course as top, middle, and bottom based on their scores on the final exam. Students in the whole course (no distinction between different recitation classes) were first ranked by their scores on the final exam. About 1/3 of the students were assigned to the top, middle, and bottom groups, respectively. The overall performance of each intervention group is represented by an unweighted mean of students' performance from the three different levels of expertise. To compare how similar the students in different intervention groups were, their performance on the Force Concept Inventory (FCI) (Hestenes et al. 1992) administered at the beginning of the semester was investigated. There was no statistically

significant difference between the different intervention groups in terms of the FCI score. Moreover, in order to take into account the possible difference which may develop as the semester progresses between different recitation classes, the effects of different interventions on the quiz were compared based on the unweighted means described earlier.

In addition to the comparison between the different intervention groups, we also compared the students' performance in these algebra-based and calculus-based introductory physics courses with the performance of a group of first-year physics graduate students who were asked to solve the tire swing problem on their own without any solved problem provided. The performance of the graduate students can serve as a benchmark for how well the undergraduate students can perform. Moreover, we also conducted think-aloud interviews with four introductory physics students (who were selected from other introductory physics classes) to get an in-depth account of their difficulties with the scaffolding provided and examine additional ways to help them. The details of the interviews will be discussed later.

2.4 RESULTS AND DISCUSSION

2.4.1 Quantitative data from the two introductory physics courses

We found that the similarities between the solved and quiz problems that the students described in the first part of their quiz solution had no correlation with their ability to actually solve the quiz problem. Many students described the similarities based on the details of the problems (e.g., the initial speeds in both problems are zero, both problem are asking for a force, etc.) whether or not they could solve the problem correctly. For example, one student who correctly solved the

quiz problem described the following three similarities: "*<1> going around a circle with m (30kg) and radius (15m) <2> need to solve for velocity at a point <3> start from rest*". However, the student did not mention the deep similarities regarding the physics principles involved. In particular, without looking at his actual solution to the quiz problem, it is not possible to tell whether this student would be able to solve the quiz problem correctly. On the other hand, the fact that some students described the similarities in terms of the physics principles involved didn't necessarily mean that they knew how to apply the principles correctly, and sometimes they didn't even make use of the principles they mentioned as similar (for the solved and quiz problems) when solving the quiz problem. For example, one student described the similarities as follows: "*The initial velocity of both is 0 m/s. The theory of conservation of energy is used in both. The tension is going to be* $T = mg + m\frac{{v_B}^2}{R_B}$ *instead of* $N_B = mg - m\frac{{v_B}^2}{R_B}$. *This is because in problem 1, the cart is moving up whereas in problem 2, the swing is moving downwards in the arc, so the forces are acting as one combined force.*" Although these statements about the similarity seem to indicate that this student was capable of solving the quiz problem correctly, examination of his actual work shows that he didn't make use of the principle of conservation of energy at all in his actual attempt to solve the quiz problem. Instead, he tried to find the speed at the bottom of the ride by connecting the centripetal acceleration to the acceleration due to gravity and set $a_c = \frac{v^2}{r} = g = 10\mathrm{m/s}$. Because of such inconsistencies, in the following discussion, we will only focus on students' *solutions* to the quiz problem (and not focus on their response to the question asking for the similarities between the two problems).

Table 2-2 and Table 2-3 present students' average scores on the tire swing problem (the quiz problem) in the calculus-based and algebra-based courses, respectively. For the intervention group 2, students' performance when they solved the problem the 2nd time is presented. Due to

the instructor's time constraint in the recitation classes, the allotted time for students in intervention group 2 to try the quiz problem on their own before learning from the solved problem was slightly less than the time given to those in the comparison group. Therefore, instead of examining how intervention 2 students' pre-scaffolding performance compares to that of the comparison group, we only focus on the performance of students in intervention group 2 after the scaffolding support was provided. Moreover, as noted earlier, the initial FCI scores were comparable for the comparison group and all intervention groups.

The p-values presented in Table 2-4 show that all three intervention (Intv) groups in the algebra-based course and the intervention group 2 in the calculus-based course significantly outperformed the comparison group, indicating that these students, to a moderate extent, could reason about the similarities between the two problems and take advantage of the solved problem provided to solve the quiz problem. On the other hand, while the score of the intervention group 1 in the calculus-based course was higher than the comparison group in the same course, the difference is not statistically significant. The performance of intervention 3 students in the same course was comparable to that of the comparison group. It is possible that many students in these groups failed to process the analogy between the solutions to the solved and quiz problems deeply the way we had hypothesized. We'll describe the possible reasons for the difficulty in analogical reasoning in the later paragraphs.

The algebra-based students benefited more from the interventions overall in the sense that students in all three intervention groups in general performed significantly better than the comparison group students. However, comparison of the absolute scores of students in the same intervention group from the two courses indicates that the calculus-based students on average scored higher than the algebra-based students whether or not the scaffolding was provided. We

note that how well a student performed may depend not only on the scaffolding provided, but also on their initial knowledge relevant for the problem. An improvement would easily be seen if the students who initially had no clue about how the solution should be constructed were able to invoke an appropriate concept or principle by learning from the isomorphic problem provided. The fact that 26% of students in the algebra-based comparison group received a score of zero because they incorrectly connected the tension force directly to the energy (for example, with the equation $T = mgh$) suggests that there was plenty of room for improvement in invoking the principles correctly. A noticeable progress would be made if the students were able to recognize the similarity between the solved and quiz problems and identify correctly the principles to be used. However, in order to apply the physics principles successfully, more understanding and mathematical competence is required and students must also be able to understand the nuances between the solved and quiz problems.

Table 2-2. Students' average scores out of 10 on the tire swing problem (the quiz problem) in the calculus-based course. The number of students in each case is shown in parentheses. The performance of the whole group taken together is represented by an unweighted mean of students' average scores from the top, middle and bottom categories.

	Comparison (38)	Intervention 1 (35)	Intervention 2 (34)	Intervention 3 (74)
Top	8.6 (14)	9.3 (15)	9.2 (13)	7.6 (19)
Middle	7.6 (10)	8.7 (9)	9.4 (12)	7.5 (35)
Bottom	4.2 (14)	4.6 (11)	8.7 (9)	5.1 (20)
Average	6.8	7.5	9.1	6.7

Table 2-3. Students' average scores out of 10 on the tire swing problem in the algebra-based course. The number of students in each case is shown in parentheses. The performance of the whole group taken together is represented by an unweighted mean of students' average scores from the top, middle and bottom categories.

	Comparison (54)	Intervention 1 (46)	Intervention 2 (33)	Intervention 3 (48)
Top	6.0 (19)	8.0 (10)	6.8 (12)	7.2 (27)
Middle	2.7 (15)	7.3 (20)	6.7 (10)	3.5 (11)
Bottom	2.0 (20)	6.6 (16)	4.8 (11)	6.2 (10)
Average	3.5	7.3	6.1	5.6

Table 2-4. The p values for the comparison of students' performance between different groups in the calculus-based and algebra-based courses. The "c" stands for the comparison group.

	c vs. 1	c vs. 2	c vs. 3	1 vs. 2	1 vs. 3	2 vs. 3
Calculus	0.200	0.000	0.829	0.091	0.417	0.000
Algebra	0.000	0.000	0.001	0.417	0.371	1.000

For comparison, Table 2-5 lists the different answers graduate students provided to the tire swing quiz problem on which they achieved an average score of 8.4 out of 10. Out of the 26 graduate students, 21 students successfully figured out the correct answer. The most common mistakes the graduate students made were ignoring the fact that there was an acceleration involved and treating the problem as an equilibrium problem. A similar result has been reported (Reif and Allen 1992) when the same problem was given to a group of physics professors.

In our study with the introductory physics students here, not recognizing the existence of the acceleration was one of the common mistakes, but this was not the only difficulty introductory students had. Without the interventions, some students (especially in the algebra-

based course) simply had no clue about how to solve the problem and they tried to associate the tension force with some quantity that didn't even have the same dimension. Some students realized that they should apply Newton's 2nd law in the non-equilibrium situation but they didn't know how to find the acceleration. Even if some of them knew the expression for the magnitude of the acceleration as $a_c = \frac{v^2}{r}$, they didn't necessarily know how to find the speed of the object. These difficulties, as well as the mistake of neglecting the gravitational force term in the solution, were reduced after the students were provided with the solved problem. With the scaffolding, more students were able to identify the existence of both the gravitational force and the centripetal acceleration, and most students could apply the principle of CME to find the speed correctly.

Table 2-5. Graduate students' answers to the tire swing problem.

Answers	Number of people
$T = mg + \frac{mv^2}{r}$ (correct)	21
$T = mg$	4
$T\cos\theta = mg,\ T\sin\theta = \frac{mv^2}{r}$	1

Examining intervention 2 students' performance shows that students did improve significantly by learning from the isomorphic solved problem provided after struggling with the quiz problem first. In particular, this intervention worked very well for the calculus-based students. With the solved problem in their possession to learn from, the calculus-based students achieved an average score of 9.1 (out of 10) the second time they solved the quiz problem, which was a higher score than the benchmark (8.4) set by the graduate students. Even the bottom students in this group earned an average score of 8.7 out of 10. Table 2-6 provides insight on how the pre and post performance of this group of students evolved by binning the students into

different categories based on their solutions. A comparison of the number of students who had difficulty figuring out the acceleration and the speed correctly before and after the scaffolding was provided is shown in Table 2-7. These tables suggest that most calculus-based students were able to correctly invoke the necessary knowledge which they lacked initially. They corrected at least part of their mistakes after browsing over the solution, and a significant improvement in the scores was found.

Table 2-6. Different answers calculus-based intervention 2 students provided for the tire swing problem before and after the scaffolding was provided. The corresponding number of students in each case is listed. The correct answer is indicated by the shaded background.

	Before	After
$T = mg + \frac{mv^2}{r}$	13 (38.2%)	26 (76.5%)
$T = mg - m\frac{v^2}{r}$ or $T = -mg + \frac{mv^2}{r}$	3 (8.8%)	4 (11.8%)
$T = \frac{mv^2}{r}$	3 (8.8%)	3 (8.8%)
$T - mg = ma$ but didn't know how to find a	3 (8.8%)	0
$T = mg$	4 (11.8%)	0
Other (e.g., $T = mv_f$, $T = mv_f$)	8 (23.5%)	1 (2.9%) (This person thought T_{max} occured when $\theta = 45^{\circ}$ and said $T_{max}\cos\theta - mg = m\frac{v^2}{r}$)

Table 2-7. Comparison of the number of students who had difficulty figuring out the acceleration and the speed correctly before and after the scaffolding was provided in the calculus-based intervention group 2.

	Before	After
Mentioned a but had no idea how to find a or used incorrect method to find a (e.g., used $ma = 2500\text{N}$ to find a)	4	0
Used incorrect method to find v (e.g., $v = 0$, $v = 9.8\ \text{m/s}$, $\frac{v^2}{r} = g$, using 1-D kinematics equations)	12	1 (used 1-D kinematics equations)

Table 2-8 presents intervention 1 students' performance on the rollercoaster problem right after learning from and returning its solution to the instructor. It shows that many students in both the calculus-based and algebra-based courses were capable of reproducing the solved problem immediately. The average scores on the solved problem reproduced from students with different levels of expertise were 8.5 (calculus) and 9.0 (algebra); even the scores of the "bottom" students in both courses were high. The fact that students were immediately able to reproduce the problem they browsed over, however, doesn't necessarily mean that they could transfer their learning to a new isomorphic problem. An average drop of 1.0 and 1.7 points were found for the calculus-based and algebra-based students for the transfer problem. In fact, the "bottom" calculus-based students' average score on the quiz problem dropped to 4.6. One possible reason for this low score is that this group of students might not have as strong a motivation to perform well as the algebra-based students, and they didn't process through the solutions provided as deeply as we had hypothesized. The fact that these "bottom" students in the calculus-based course didn't perform well on the quiz problem as compared to other students who received the same intervention could be a possible reason for why on average the score of

the intervention 1 students in the calculus-based course was not significantly better than the comparison group students.

Although the solved problem provided was useful in helping students construct an appropriate solution plan for the quiz problem by invoking the relevant principles and correcting the terms they might have missed before browsing over the solved problem, students weren't necessarily able to apply the principles correctly when a change in the details of application was required in order to solve the transfer problem in the new situation. One common incorrect answer intervention 1 and 2 students provided (after learning from the solved problem) for the swing problem was $T = mg - m\frac{v^2}{r}$(or sometimes $T = -mg + m\frac{v^2}{r}$ if the students noticed that the former answer would result in a negative value) instead of the correct answer of $T = mg + m\frac{v^2}{r}$. One possible reason for this mistake may be that the vector nature involved in Newton's 2nd law was challenging for the students. To apply the principle correctly, students need to realize that when applying Newton's 2nd law, not only do they have to take into account the direction of the forces, they also must remember the fact that the acceleration is also a vector in which a positive or negative sign based on the direction should be considered and assigned accordingly. If the students didn't realize that the centripetal accelerations were pointing in the opposite directions in these two problems (because in one problem the object was at the top and in the other, it was at the bottom) and they simply copied down the equations from the solved problem, they were likely to make the mistake.

Table 2-8. Average scores out of 10 on the roller coaster problem (solved problem) and the tire swing problem (quiz problem) for intervention 1 in the algebra-based and calculus-based courses. The performance of the whole group is represented by an unweighted mean of students' average scores from the top, middle and bottom categories.

	Solved Problem		Quiz Problem	
	Calculus	Algebra	Calculus	Algebra
Top	9.0	9.6	9.3	8.0
Middle	8.7	9.0	8.7	7.3
Bottom	7.9	8.5	4.6	6.6
Average	8.5	9.0	7.5	7.3

Another possible reason for why students came up with a wrong sign for the centripetal acceleration term may be that they interpreted the quantity $\frac{mv^2}{r}$ as an additional force acting on the object undergoing a circular motion and they treated the situation as an equilibrium problem in which all the forces should sum up to zero. Intervention 3 students' answers to the additional dialogue question show that 30% and 35% of the calculus-based and algebra-based students, respectively, agreed with the first person who argued that "If an object is undergoing a circular motion, then there's an extra centripetal force acting on it" and that "If an object is traveling on a track of a vertical circle, using Newton's 2nd law in equilibrium situation, at the top we have $\sum F = 0 \Rightarrow N = mg + \frac{mv^2}{r}$." (See the Appendix.) However, examination of students' work indicates that students were not always consistent between the answers they chose for the dialogue question and the actual solution they provided for the tire swing problem. The answers "agreeing with person 1" and "$T = mg - m\frac{v^2}{r}$" should be correlated if the students were

consistent. Another consistent answer pair would be "agreeing with person 2" and "$T = mg + m\frac{v^2}{r}$".

Table 2-9 lists the intervention 3 students' answers to the dialogue question and the tire swing problem; the consistent answer pairs are indicated by the shaded background. The table suggests that a large fraction of the students were not consistent in their answers in both the algebra-based and calculus-based courses. It appears that some students didn't understand the key points in the two arguments and incorrectly agreed with one person based on some subsidiary factor. A student who correctly proceduralized Newton's 2nd Law in the non-equilibrium situation and came up with a correct answer agreed with person 1 "because centripetal force points into the center of the circle" (despite the fact that person 2 had a similar statement of "centripetal acceleration's direction is pointing from the object to the center of the circle."). It is also likely that some students chose the inconsistent answer pairs because they expected the dialogue question to be directly applicable to the tire swing problem to be solved and they didn't recognize that these two cases involved different situations and different application details (since in the dialogue, the object was at the top but in the swing problem it was at the bottom). They either directly copied the final answer from the person they agreed with in the dialogue as their answer to the tire swing problem without thinking through it in the new situation, or they first solved for the tension in the tire swing problem and argued that whichever person had the same equation as theirs (if the normal force in the dialogue situation was substituted by the tension force in the quiz problem) would be the one they agreed with. In either of these cases, students lost 1~2 points because the person they agreed with reflected gaps in their knowledge structure (listed in the last item of the rubric) or because the equation they used from the dialogue had a wrong sign (which would not have happened if they used correct

concepts and derived the equation in the new situation themselves). The fact that some students lost additional points for the answer they gave to the dialogue question is one of the reasons why students in the intervention group 3 didn't perform as well as students in the other intervention groups. Another reason may be that providing students with more hints, e.g., by directly telling them the principles involved, may have reduced the amount of cognitive engagement and students may not be as actively involved in the reasoning in intervention 3.

Table 2-9. Intervention 3 students' answers to the dialogue question and the tire swing problem and the corresponding number of students in each case. The consistent answer pairs are indicated by the shaded backgrounds. In the calculus-based course, there were only 73 students in total because one student who answered that he "agreed with either student 1 or 2" was not included in this table.

	Calculus		Algebra	
	Person 1	Person 2	Person 1	Person 2
$T = mg + \frac{mv^2}{r}$	17	19	7	10
$T = mg - m\frac{v^2}{r}$ or $T = -mg + \frac{mv^2}{r}$	2	19	4	12
$T = \frac{mv^2}{r}$	0	3	0	1
$T = mg$	0	1	1	3
Other	3	9	5	5

Comparing the performance of different intervention groups, we found that all three intervention groups were significantly better than the comparison group in the algebra-based course and there was no significant difference between any of the intervention groups. In the calculus-based course, intervention 2 was the only group which statistically significantly outperformed the comparison group. It was also statistically significantly better than intervention

group 3. As described earlier, the interventions would be useful if the scaffolding supports provided matched well with students' abilities and if the students were actively engaged in the thinking process as hypothesized during the design of each intervention. We found that to begin with, many algebra-based students had no clue about how to construct the quiz problem. Providing them with the solved problem (regardless of the different interventions) did help them invoke the relevant principles and an improvement was observed. As for the calculus-based students, whose initial performance was better, the intervention which let them struggle first before any scaffolding was provided benefited them the most. It is likely that this intervention was the one which made students think through the analogy between the solved and quiz problems with the greatest depth because the struggling experience can make students aware of their initial knowledge explicitly. Comparing what they learned from the solved problem with what they had initially thought, they had a good probability of detecting any discrepancy between them and were more likely to be forced to think about how to modify their initial knowledge and incorporate the new information to their existing knowledge structure in a coherent way. It is possible that students in the other two intervention groups were not forced to go through the analogy in great depth and some of them didn't think through the analogy between the solutions the way we had hypothesized. We'll describe the students' responses to interventions 1 and 3 further in the interview section.

2.4.2 Interview

2.4.2.1 General description

In addition to the students from the previously discussed calculus-based and algebra-based courses who took the quiz, four students from several other introductory physics classes were

recruited for one-on-one interviews to get an in-depth account of their reasoning while they solved the problems. Two of the four students we interviewed were enrolled in an algebra-based introductory mechanics course at the time of the interview; the other two were enrolled in two different calculus-based mechanics courses. The interviews were conducted after all the relevant topics had been covered in the lectures. All four students recruited had a midterm score which fell in the middle of their own introductory physics course, ranging from +3 to -9 points above or below the class averages (which fell between 70% and 76% for different sections of the courses). The audio-recorded interviews were typically 0.5-1 hour long.

During the interviews, students were asked to learn from the solved problem provided and solve the isomorphic quiz problem given. Different students received different kinds of interventions in the interviews, which are listed in Table 2-10. Most of the interventions were the same as the previous interventions used in the quantitative data discussed in the earlier section. One of them (what student A received) was new in the sense that a slight modification was made to the interventions used earlier. Instead of letting student A read the rollercoaster problem on his own and reproduce the rollercoaster problem again, the researcher outlined the solution to the solved problem to the student. After the student understood how to solve the rollercoaster problem, the researcher then asked him to solve the tire swing problem (quiz problem).

The interviews were conducted using a think-aloud protocol to follow and record the students' thinking processes. Students were asked to perform the task (whether they were reading the solved problem or trying to solve the quiz problem) while thinking aloud; they were not disturbed during the task. After the students completed the quiz, the researcher would first ask clarification questions in order to understand what they did not make explicit earlier and what their difficulties were. Based on this understanding, the researcher then provided some guidance

(sometimes including the physics knowledge required) to the students in order to help them solve the quiz problem correctly if they had not done so. After helping students learn how to solve the quiz problem correctly, the researcher invited them to reflect on the learning process they just went through (for example, by asking explicitly what was the thing that helped them figure out how to solve the problem) and provide some suggestion from the student's own perspective on how to improve students' performance on the problem. The goal of the students' reflection was to help us identify the possible helpful scaffoldings not only based upon what the researchers observed but also based upon students' reflection of their own learning.

Table 2-10. The interventions students received in the interview.

	Intervention used in the interview
Student A	Modified intervention 1
Student B	Intervention 1
Student C	Intervention 3
Student D	Intervention 3
* Modified intervention 1: (1) The researcher first discussed with the student how to solve the rollercoaster problem using Newton's 2nd law and the reason why there is a minus sign in the centripetal acceleration term (2) The student looked at the solution to the solved problem for a short period of time (3) The student attempted to solve the quiz problem.	

2.4.2.2 Interview results

We found that many of the student difficulties observed in the quantitative data were observed in the interviews as well. In the following section, we will discuss some findings from the interviews which provided more in-depth understanding of students' thinking processes. Some

check points that are likely to provide guidance to the students in successfully solving the tire swing problem will be summarized at the end.

First of all, we found that some students didn't take advantage of the solved problem to think through the analogy in great depth as we had hoped. When designing intervention 1, we hoped that students will not only learn from the solved example regarding what principles should be invoked and why but also how the principles should be applied. We also hoped that requesting students to reproduce the solved problem could give them an opportunity to practice applying the principles before applying them to the quiz problem. When student B, who was given intervention 1, was instructed to solve the rollercoaster problem he just browsed over, he tried to reproduce the solved problem by simply recalling the equation he had just read. He didn't start from the fundamental principles to derive the equation, but rather simply wrote down the equations he remembered for the speed at the point of interest and the final targeted variable (which were both incorrect). His answers for the solved and quiz problems, which are displayed in Figure 2-1 and Figure 2-2, indicate that he superficially mapped the two problems together without carefully examining the differences. As this interview suggests, if the students didn't carefully think through the problems as we had hoped, it's less likely that they would benefit significantly from the interventions.

$M = 120$ kg $\quad h_A = 15$m $\quad h_B = 5$m $\quad R = 30$m

$m = 55$kg

$$v_B = \sqrt{2gh_A + 2gh_B}$$

$$v_B = \sqrt{2(10)(15\text{m}) + 2(10\ \text{m/s}^2)(5\text{m})}$$

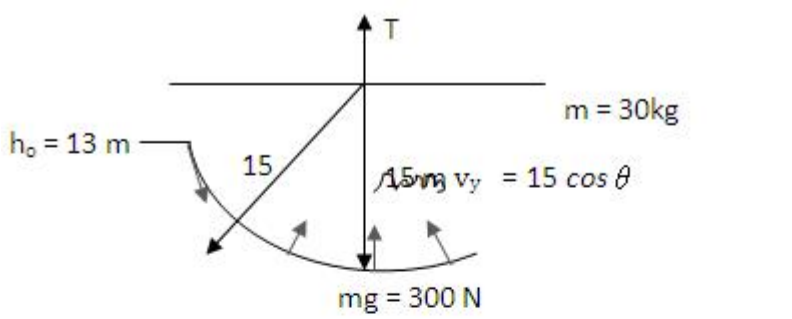
$$v_B = 20\ \text{m/s}$$

$$N_B = mg + \frac{(M+m){v_B}^2}{R^2}$$

$$N_B = 55kg\left(10\frac{\text{m}}{\text{s}^2}\right) + \frac{175kg\ (20\frac{\text{m}}{\text{s}})^2}{30\text{m}^{\ 2}}$$

$$\boxed{N_B = 96.11\text{N}}$$

Figure 2-1. Student B's answer to the solved problem.



$h_A = 13$m $\qquad v_B = \sqrt{2gh_A}$

$h_B = 0$ m $\qquad v_B = \sqrt{2(10)(13)} = 16.12$ m/s

$$N_B = mg + \frac{mv^2}{r^2}$$

$$= 30\ \text{kg}\left(10\frac{\text{m}}{\text{s}^2}\right) + \frac{30\ \text{kg}\left(16.12\frac{\text{m}}{\text{s}}\right)^2}{15^2}$$

$$= 300 + 34.65$$

$$\boxed{= 334.65\text{N}}$$

The ride will be safe for Ryan

Figure 2-2. Student B's answer to the quiz problem.

We also found that students didn't necessarily think of Newton's 2nd Law as a vector equation. In addition, even if students knew that both the solved and quiz problems were dealing with centripetal acceleration, which is a vector, they didn't necessarily notice the difference between the two (one is at the top of the circle; the other is at the bottom) on their own. When the researcher asked student A to explain how he got the minus sign in his final answer of $T = -\frac{mv^2}{r} + mg$ in the tire swing problem, he answered:

Student A: Isn't that the same as this [pointing to the solved problem] ... wait…'cause the centripetal acceleration is going… Wait… No… No, I was wrong. Wait a second. This time [in the solved problem] I'm on the top, not the bottom… so [in the quiz problem] instead of negative, the centripetal acceleration will be positive, correct?

When the researcher later asked him to reflect on his learning, he also mentioned that:

Student A: At first I thought they were just the same situation. I just kind of assumed that they were. I forgot that this one is at the bottom. So I just used whatever I knew from here. It wasn't right.

The conversation above suggests that even though the student may have all the physics knowledge required to answer the quiz problem, the knowledge might not be structured in a well-organized manner to allow him to quickly detect the difference between the two situations (quiz problem and solved problem). More specifically, it's possible that the connection between Newton's 2nd Law and its vector nature (which implies that the direction of the net force and the acceleration should be contemplated carefully) was not strong enough in the student's mind. Therefore, the student didn't realize that a modification in the application detail should be made

in the new situation until additional guidance which directed his attention to this issue was explicitly provided by the researcher.

As pointed out in the section on written quantitative data, some interviewed students were also not consistent while answering different parts of the quiz. The following conversation with student D is an example. Although student D's answer to the dialogue question in intervention 3 was correct and he didn't think there would be an extra force in a circular motion, he later said that he was thinking about what the 1st person in the dialogue question said (which was wrong) when he was asked to explain how he obtained his answer ($T = \frac{mv^2}{r}$) for the tire swing problem.

Student D: [reading the dialogue problem]… I'd agree with person two just because I don't think that… uh… I don't' think it's an extra force. I know that centripetal force is what keeps it going in the circle… but I don't think it's an extra…or is it? Uh… mg… N-mg equal… No. I agree with person two. I don't think… I think… I don't think it's an extra force at point A.

Researcher: So… when you wrote down this one [$T = F_c = \frac{mv^2}{r}$ *, his answer to the quiz problem], can you tell me which principle you were using?*

Student D: Tension is equal to the centripetal force if there's … No I think it's almost wrong… but… I think maybe I was thinking about centripetal force… no I was not thinking about centripetal force at all……

[Student D tried to solve the quiz problem again, this time using $F = ma$ *,* $N - W = ma$ *. (He later noted that what he had as N was in fact the tension.) After he came up with the correct answer for tension, he noted the following]*

Student D: This [his original work] is wrong. I was just thinking about the centripetal force just because… because of the part A [pointing to what the 1st person in the dialogue question said, which is incorrect.]

We can invoke the knowledge in pieces (diSessa 1988; diSessa 1993) framework to understand the student's response. The conversation above suggests that Student D had some relevant knowledge but the student's knowledge was not organized in a knowledge structure and he didn't notice the inconsistency between different knowledge elements he referred to unless explicitly guided.

Although the dialogue in intervention 3 didn't necessarily help all students, the interview with student A suggests that the dialogue could be useful for helping students learn the concept of centripetal force if the student tries to incorporate the newly acquired knowledge into his original knowledge structure and is made aware of the conflicts between the knowledge he acquired from the quiz activity and his prior knowledge. In the interview with student A, we found that the notion of associating the centripetal force as an additional force coming from a single physical object was strong. The student could correct his own mistake regarding the incorrect sign for the centripetal acceleration term (after realizing that the direction of the acceleration in the quiz problem was not the same as in the solved problem) and came up with the correct equation by following the procedure in the solved problem (first drawing a correct free body diagram (FBD) and then applying Newton's 2nd Law correctly). However, when he later explained why the tension was maximum at the bottom during the ride, the diagram he drew still suggested that he had a tendency to consider the centripetal force as an additional force coming from a physical object. Figure 2-3 and Figure 2-4 show the different diagrams he drew to

solve for the tension force and to explain why tension would be maximum at the bottom of the ride, respectively. When he later compared his new figure (Figure 2-4) to his final answer for tension ($T = mg + \frac{mv^2}{r}$), he became confused because in his diagram, ma_c and mg pointed in different directions but in the equation they were added together.

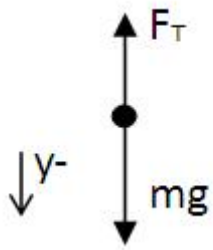


Figure 2-3. The diagram student A drew from which he came up with the correct answer

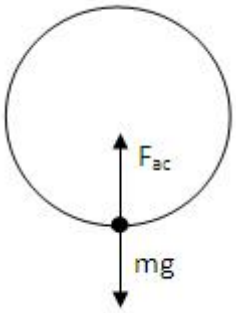


Figure 2-4. The diagram student A later drew which implied that he was thinking of centripetal force as an additional force.

In order to help student A, the researcher discussed with the student the implications of considering the centripetal force as an additional force versus the net force in the radial direction. This discussion was very similar to the information presented in the dialogue question in intervention group 3 except that the case the researcher discussed was for an object at the bottom

of the circle instead of at the top of the circle. After the discussion about the two different diagrams and the corresponding equations (similar to those presented for intervention 3 in the Appendix) the student realized why he had difficulty. He changed his labeling of F_{ac} on Figure 2-4 to F_T , and had the following conversation with the researcher:

Student A: I see what I was doing wrong. I was confused about that. Now it makes more sense.

Researcher: Yeah, but still I don't understand how I helped you. So, can you explain more?

Student A: Yeah, you helped me because I guess I was thinking of this [F_{ac}] as a force, like as a physical force, so I put it up this way [pointing to his new diagram of Figure 2-4]. And then I'm really confused because they are acting in two different directions.

Researcher: Yeah but still you use…

Student A: Yeah. But when I originally did it, I just wrote this [pointing to his original diagram of Figure 2-3], which makes more sense, because my F_T minus mg equals this. So whenever you make the equation, you end up you're adding them

Researcher: OK

Student A: 'cause there… [sigh…] why… or another word is [that they are] acting in the same direction… I… I just got confused by thinking of the ma part as… not the net force but as the… like force acting on that [the object] like that. So whenever you put ma_c equals that [the net force, F_{ac}] and then use Newton's 2nd Law, it makes a lot more sense to me.

Researcher: Yeah, so I think that's another reason why I prefer to draw the acceleration… I mean, beside, not on the…

Student A: Yeah, not like direct on that because it confuses [me]

The discussion above suggests that the dialogue problem and the related concepts presented in intervention 3 can be used as a tool to help students understand the centripetal force. Moreover, it would also be helpful to explicitly require students to draw the acceleration on the side of a FBD (but not directly with other forces). Overall, based on the interviews, we found that if students were actively engaged in the thinking process and if sufficient scaffolding support was provided to help them contemplate the following issues, they were very likely to solve the quiz problem correctly: (1) They realized that the centripetal force is just a name given to one component of the "net force" in a circular motion. It is not always associated with a single physical force unless only one force is present in that direction. (2) They knew how to find the acceleration and its direction. They also discovered that the positions of the objects (relative to the circles) are different in the two problems since one object is at the top and the other is at the bottom. (3) They realized how to use Newton's 2nd law correctly as a vector equation instead of as a scalar equation. (4) They were required to draw an arrow indicating the direction of the acceleration not on the FBD but on the side of it. Follow-up studies including interviews with students from all levels of expertise could be conducted in the future to thoroughly explore the specific effects different scaffolding supports could have on each of these issues.

2.5 SUMMARY AND FUTURE OUTLOOK

In this study, we found that students in both the calculus-based and algebra-based courses were able to recognize the similarities between the isomorphic problems in terms of the relevant physics principles involved when they were asked to learn from a solved problem and transfer

what they learned from the example problem to solve another isomorphic quiz problem. The algebra-based students in all three intervention groups on average outperformed the comparison group students in the same course because many of them had no clue about how to approach the quiz problem if no support was provided. Providing algebra-based students with a solved isomorphic problem to learn from (regardless of the types of additional scaffolding supports involved in the three different intervention groups) improved their performance by helping them invoke the relevant principles in the quiz problem. On the other hand, students in the calculus-based course were better than the algebra-based students in the sense that even without the solved problem provided, they already had some idea about the structure of the problem, although they may not have been able to proceduralize the principles correctly. Therefore, a significant improvement would be observed if the students were not only able to identify the similar principles involved in the two problems, but were also capable of applying what they learned from the solved example in an appropriate way to the new situation presented in the quiz problem. Among all three interventions, we found that intervention 2, in which students were asked to try the quiz problem on their own before the solved problem was provided, was the best intervention in helping the calculus-based students. The findings suggest that postponing the scaffolding support until students have attempted to solve the quiz problem without help is consistently beneficial for students in both courses because the clear targeted goal and the thinking process students went through in their first attempt facilitates better transfer to the other problem.

As noted earlier, the greatest difficulty students had in the analogical reasoning activity discussed was in the correct application of the principles in the new context. One common difficulty observed, for example, was that many students failed to differentiate between the

situations in which an object is going over the top versus the bottom of a circle and they didn't contemplate the direction of the corresponding centripetal acceleration and its sign in the corresponding equation. In general, calculus-based students performed better than the algebra-based students on the transfer problems.

In order to help students perform better on the transfer problem, more scaffolding may be required. Deliberately guiding students to think more about the relations between the isomorphic problems by helping them discern not only the similarities, but also the differences between the isomorphic problems and asking them to discuss the implications of both the similarities and differences before actually solving the transfer problem may be a useful strategy. It is possible that by performing a systematic and thorough comparison of the two problems, students may think through the analogy more comprehensively and carefully. If students are new to such activities and they have difficulty identifying the differences they should be looking for in the isomorphic problems, other strategies that are helpful for learning such as instructor modeling, peer discussion, etc. may be combined to assist students (at least in the beginning). It is likely that with more practice and feedback on such analogical reasoning activity, students will gradually develop expertise. The scaffolding support can be reduced as the students develop self-reliance.

A similar strategy to assist students in discerning the differences between the problems and contemplating the application details is to provide them with more than one solved problem to learn from. If two isomorphic solved problems which contain different contexts and different application details are provided to them, students can no longer simply match the quiz problem to either one of them without thinking. They will have to carefully examine the similarities and differences between the three problems and combine what they learned from both solved

problems to come up with a new solution that is suitable for the quiz problem. The different application details presented in the two solutions could also serve as a model and/or a hint for how different situations may require the application of the same principles but the application details must be adjusted in each situation.

Some additional scaffolding supports could be designed (and may be combined with the previous strategies) to help students with specific difficulties. For example, one common difficulty found in students' work on the quiz problem was that they didn't draw a free-body diagram when solving the quiz problem. It is possible that mistakes related to missing the gravitational force or having an incorrect sign for the acceleration term (as described in the results section) could be reduced if, in addition to the current intervention, students are explicitly asked to draw a free body diagram before solving the problem, and a comparison between the free body diagrams for the tire swing problem and the roller coaster problem is explicitly enforced. It is also useful to help students develop the habit of drawing the acceleration on the side of the FBD as discussed in the interview. The acceleration vector drawn on the side may help remind students about the fact that they have to consider the vector nature of both forces and accelerations when applying Newton's 2nd law. At the same time, it avoids the difficulty of students confusing the centripetal force as an additional force if the arrow signifying the acceleration is drawn together with all the forces.

In summary, deliberately using isomorphic worked out examples to help students transfer what they learned from one context to another can be a useful tool to help students understand the applicability of physics principles in diverse situations and develop a coherent knowledge structure of physics. For introductory students, such well-thought out activity could provide a model for effective physics learning since the idea of looking at deep similarities beyond the

surface features is enforced throughout the activity. It is possible that students will become more facile at the analogical reasoning processes if practice and feedback are constantly provided to them. The greatest benefit may be achieved if similar activities are sustained throughout the course over different topics and the coherence of physics as well as the importance of looking at the deep features of the problems is consistently explained, emphasized, demonstrated and rewarded by the instructors.

2.6 CHAPTER REFERENCES

Adey, P. and M. Shayer (1993). "An exploration of long-term far-transfer effects following an extended intervention program in the high school science curricula." *Cogn. Instruct.* **11**, 1-29.

Aleven, V., K. R. Koedinger and K. Cross (1999). "Tutoring answer explanation fosters learning with understanding". the 9th International Conference on Artificial Intelligence in Education, Le Mans, France, IOS Press, Amsterdam.

Atkinson, R. K., S. J. Derry, A. Renkl and D. Wortham (2000). "Learning from examples: Instructional principles from the worked examples research." *Rev. Educ. Res.* **70**(2), 181-214.

Bassok, M. and K. Holyoak (1989). "Interdomain transfer between isomorphic topics in algebra and physics." *J. Experimental Psychology: Memory, Learning and Cognition* **15**(1), 153-166.

Bobrow, D. and D. Norman (1975). "Some principles of memory schemata". In *Representation and understanding: Studies in cognitive science*. D. Bobrow and A. Collins. New York, Academic Press**:** 131-149.

Bransford, J. D. and D. Schwartz (1999). "Rethinking transfer: A simple proposal with multiple implications." *Rev. Res. Educ.* **24**, 61-100.

Brown, A. (1989). "Analogical learning and transfer: What develops?". In *Similarity and analogical reasoning*. S. V. a. A. O. Eds. NY, Cambridge University Press**:** 369-412.

Cheng, P. W. and K. J. Holyoak (1985). "Pragmatic reasoning schema." *Cognitive Psychology* **17**, 391-416.

Chi, M. (2000). "Self-explaining expository texts: The dual processes of generating inferences and repairing mental models.". In *Advances in instructional psychology*. R. Glaser. Hillsdale, NJ, Erlbaum Associates**:** 161-238.

Chi, M., M. Bassok, M. W. Lewis, P. Remann and R. Glaser (1989). "Self-explanations: How students study and use examples in learning to solve problems." *Cog. Sci.* **13**, 145.

Chi, M. T. H., P. J. Feltovich and R. Glaser (1981). "Categorization and representation of physics knowledge by experts and novices." *Cog. Sci.* **5**, 121-152

Detterman, D. K. and R. J. Sternberg, Eds. (1993). *Transfer on trial: Intelligence, cognition and instruction*. Ablex, Norwood, NJ.

diSessa, A. (1988). "Knowledge in pieces". In *Constructivism in the computer age.* G. Forman and P. B. Pufall. Hillsdale, NJ, Lawrence Erlbaum Associates**:** 49-70.

diSessa, A. (1993). "Toward an epistemology of physics." *Cogn. Instr.* **10**, 105-225.

Dufresne, R., J. Mestre, T. Thaden-Koch, W. Gerace and W. Leonard (2005). "Knowledge representation and coordination in the transfer process". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Information Age Publishing**:** 155-215.

Duit, R. (1991). "On the role of analogies and metaphors in learning science." *Science Education* **75**, 649-672.

Duncker, K. (1945). "On problem solving." *Psychol. Monogr.* **58**, 1.

Eylon, B. and F. Reif (1984). "Effect of knowledge organization on task performance." *Cognition and Instruction* **1**, 5.

Genter, D. and C. Toupin (1986). "Systematicity and surface similarity in the development of analogy." *Cognitive Science* **10**, 277-300.

Gray, K. E. and N. S. Rebello (2004). "Transfer between paired problems in an interview." *Proceedings of the 2004 Physics Education Research Conference*.

Hayes, J. R. and H. A. Simon (1977). "Psychological differences among problem isomorphs". In *Cognitive theory*. N. J. Castellan, D. B. Pisoni and G. R. Potts. Hillsdale, NJ, Lawrence Erlbaum.

Hestenes, D., M. Wells and G. Swackhamer (1992). "Force concept inventory." *The Phys. Teach.* **30**, 141-158.

Holyoak, K. (1985). "The pragmatics of analogical transfer". In *The psychology of learning and motivation*. G. Bower. New York, Academic Press. **19**.

Holyoak, K. and P. Thagard (1995). *Mental leaps: Analogy in creative thought*. Cambridge, MA, MIT Press.

Johnson-Laird, P. (1972). *Psychology of reasoning, structure and content*. Cambridge, MA, Harvard University Press.

Johnson, K. and C. Mervis (1997). "Effects of varying the levels of expertise on the basic level of categorization." *J. Exp. Psychol. Gen.* **126**(3), 248.

Klahr, D., Z. Chen and E. Toth (2001). "From cognition to instruction to cognition: A case study in elementary school science instruction". In *Designing for science, implications from everyday, classroom and professional settings*. C. D. Schuun, T. Okada and K. Crowley. Mahwah, NJ, Lawrence Erlbaum.

Kotovsky, K., J. R. Hayes and H. A. Simon (1985). "Why are some problems hard? Evidence from the tower of hanoi." *Cogn. Psychol.* **17**, 248.

Kurz, E. M. and R. D. Tweney (1998). "The practice of mathematics and science: From calculus to the clothesline problem". In *Rational models of cognition*. M. Oalesfard and N. Chater. London, Oxford University Press**:** 415-438.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning approach." *J. Struct. Learn.* **6**, 121-130.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning model." *J. Struct. Learn.* **6**, 271-297.

Larkin, J. (1981). "Cognition of learning physics." *Am. J. Phys.* **49**(6), 534-541.

Lin, S. and C. Singh "Learning introductory physics by analogical reasoning " *unpublished*.

Lin, S. and C. Singh (2011). "Challenges in using analogies." *Phys. Teach.* **49**, 512.

Lobato, J. (2003). "How design experiments can inform a rethinking of transfer and vice versa." *Educ. Res.* **32**, 17.

Lobato, J. (2006). "Transfer strand: Alternative perspectives on the transfer of learning: History, issues and challenges for future research." *J. Learn. Sci.* **15**, 431.

Marshall, S. (1995). *Schemas in problem solving*. New York, NY, Cambridge University Press.

Mateycik, F., D. Jonassen and N. S. Rebello (2009). "Using similarity rating tasks to assess case reuse in problem solving." *AIP Conf. Proc.* **1179**, 201-204.

Mestre, J. (2001). "Implications of research on learning for the education of prospective science and physics teachers." *Phys. Educ.* **36**, 44.

Mestre, J. (2002). "Probing adults' conceptual understanding and transfer of learning via problem solving." *J. of Applied Developmental Psychology* **23**(1), 9-50.

Novick, L. (1988). "Analogical transfer, problem similarity and expertise." *J. Exp. Psychol. Learn. Mem. Cogn.* **14**, 510.

Novick, L. (1988). "Analogical transfer, problem similarity and expertise." *J. Experimental Psychology: Learning, memory and cognition* **14**(3), 510-520.

Ozimek, D. J., P. V. Engelhardt, A. G. Bennett and N. S. Rebello (2005). "Retention and transfer from trigonometry to physics." *Proceedings of the 2004 Physics Education Research Conference* **790**, 173-176.

Rebello, N. S., L. Cui, A. G. Bennett, D. A. Zollman and D. J. Ozimek (2007). "Transfer of learning in problem solving in the context of mathematics and physics". In *Learning to solve complex scientific problems*. D. Jonassen. New York, Lawrence Earlbaum Associates.

Rebello, N. S. and D. A. Zollman (2005). "Dynamic transfer: A perspective from physics education research". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Infromation Age Publishing.

Reed, S. K., G. W. Ernst and R. Bannerji (1974). "The role of analogy in transfer between similar problem states." *Cogn. Psychol.* **6**, 436.

Reif, F. and S. Allen (1992). "Cognition for interpreting scientific concepts: A study of acceleration." *Cognit. Instruct.* **9**(1), 1-44.

Reif, F. and J. I. Heller (1982). "Knowledge structure and problem solving in physics." *Educ. Psychologist* **17**, 102.

Ross, B. (1989). "Distinguishing types of superficial similarities: Different effects on the access and use of earlier problems." *J. Experimental Psychology: Learning, memory and cognition* **15**, 456-468.

Schoenfeld, A. and D. J. Herrmann (1982). "Problem perception and knowledge structure in expert novice mathematical problem solvers." *J. Exp. Psych: Learning, Memory, and Cognition* **8**, 484-494.

Schwartz, D., J. Bransford and D. Sears (2005). "Efficiency and innovation in transfer". In *Transfer of learning: Research and perspectives*. J. Mestre. Greenwich, CT, Information Age Publishing**:** 1-51.

Shapiro, M. A. (1988). "Analogies, visualization and mental processing of science stories". In *Communication yearbook*. M. McLaughlin. Beverly Hills, CA, Sage. **9**.

Simon, H. A. and J. R. Hayes (1976). "The understanding process: Problem isomorphs." *Cogn. Psychol.* **8**, 165.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. I. Performance on nonintuitive problem pair from introductory physics." *Phys. Rev. ST Phys. Educ. Res.* **4**(1), 010104.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. II. Effect of some potential factors on problem solving and transfer." *Physical Review Special Topic Physics Education Research* **4**(1), 010105.

Singh, C. (2009). "Centripetal acceleration: Often forgotten or misinterpreted." *Phys. Educ.* **44**(5), 464-468.

Singh, C. and D. Rosengrant (2003). "Multiple-choice test of energy and momentum concepts." *Am. J. Phys.* **71**, 607.

Sternberg, R. J. (1977). "Component processes in analogical reasoning." *Psychol. Rev.* **84**, 353.

Yerushalmi, E., E. Cohen, A. Mason and C. Singh "What do students do when asked to diagnose their mistakes? Does it help them? I: A non-typical quiz context." unpublished.

Yerushalmi, E., A. Mason, E. Cohen and C. Singh (2008). "Effect of self-diagnosis on subsequent problem solving performance." *Proceedings of the Phys. Ed. Res. Conference, AIP Conf. Proc.* **1064**, 53-36.

3.0 USING ISOMORPHIC PROBLEMS WITH ADDITIONAL SCAFFOLDING SUPPORTS TO HELP STUDENTS LEARN ABOUT FRICTION

3.1 ABSTRACT

Research suggests many students have the incorrect notion that the magnitude of the static frictional force is always equal to its maximum value. In this study, we examine introductory students' ability to learn from analogical problem solving between two isomorphic problems that are similar in the application of a physics principle (Newton's 2nd Law) but one problem involves static friction, which often triggers the misleading notion. Students from algebra- and calculus-based introductory physics courses were asked in a quiz to take advantage of what they learned from a solved problem provided, which was about tension in a rope, to solve another problem involving friction. To help students process through the analogy between the isomorphic problems deeply and contemplate whether the static frictional force was at its maximum value, students in different recitation classrooms received different scaffolding. We find that one difficulty students had with the static friction was that they often focused on the fact that the static friction cannot be greater than the coeffeicient of static friction times the normal force, but neglected the fact that it can also be smaller than this value. We also find that if students were guided to contemplate issues related to static frictional force and asked to solve the friction problem on their own before learning from the solved problem, they were more likely to avoid using their incorrect notion to solve for friction. The think-aloud interviews suggest that a well-

designed post-activity discussion can be beneficial in helping students organize their learning to build a better understanding of static friction. We believe the analogical reasoning and transfer activity with isomorphic problems in this study provide a good starting point to help students learn about static friction, especially if it is followed by a well-designed post-activity discussion.

3.2 INTRODUCTION

Physics is a subject which contains only a few fundamental principles that are condensed into compact mathematical form. Learning physics requires unpacking these principles and understanding their applicability in different contexts which have distinct surface features but involve the same physics (Larkin and Reif 1979; Chi et al. 1981; Reif 1981; Eylon and Reif 1984; Maloney 1994; Reif 1995; Redish et al. 1998; Hammer 2000; Redish et al. 2006). Research suggests that experts in physics have a hierarchical knowledge structure where the most fundamental physics principles are placed at the top, followed by layers of subsidiary knowledge and details. This well-organized knowledge structure facilitates their problem solving processes and allows them to approach the problems in a systematic way (Johnson-Laird 1972; Bobrow and Norman 1975; Larkin 1980; Larkin 1980; Chi et al. 1981; Larkin 1981; Reif and Heller 1982; Schoenfeld and Herrmann 1982; Eylon and Reif 1984; Cheng and Holyoak 1985; Marshall 1995; Johnson and Mervis 1997). It also guides the experts to see the problems beyond the surface features and makes the transfer of knowledge from one context to another easier (Chi et al. 1981; Novick 1988; Bassok and Holyoak 1989; Brown 1989; Detterman and Sternberg 1993; Bransford and Schwartz 1999; Lobato 2003; Dufresne et al. 2005; Ozimek et al. 2005).

Issues about transfer of learning have been widely discussed from different perspectives (Holyoak 1985; Novick 1988; Bassok and Holyoak 1989; Brown 1989; Detterman and Sternberg 1993; Holyoak and Thagard 1995; Kurz and Tweney 1998; Bransford and Schwartz 1999; Klahr et al. 2001; Lobato 2003; Dufresne et al. 2005; Gray and Rebello 2005; Ozimek et al. 2005; Rebello and Zollman 2005). Different factors that affect the transfer of knowledge have been pointed out in the research literature (Duncker 1945; Genter and Toupin 1986; Adey and Shayer 1993; Mestre 2001; Mestre 2002; Schwartz et al. 2005; Lobato 2006; Rebello et al. 2007). For example, the way the learned material is organized and the context in which the knowledge is learned can both affect a person's ability to apply the knowledge flexibly (Mestre 2002). In order to assist students in better learning and help them recognize the similarities between different contexts in which the same physics principle is applicable, various scaffolding mechanisms can be used. For example, students can be taught to perform analogical reasoning between problems that share deep features (Sternberg 1977; Chi et al. 1981; Gick and Holyoak 1983; Eylon and Reif 1984; Holyoak 1985; Genter and Toupin 1986; Novick 1988; Brown 1989; Bransford and Schwartz 1999; Lin and Singh 2010). Studies have shown that using analogy can help improve students' learning and reasoning in many domains (Reed et al. 1974; Novick 1988; Shapiro 1988; Ross 1989; Duit 1991). A good analogy can create an efficient connection between the new and existing information, making the mental processing of new information more efficient by modifying the existing knowledge schemata (Shapiro 1988; Duit 1991). It can also make the new information more concrete and easier to comprehend (Shapiro 1988). To help students learn physics by performing analogical problem solving between problems that involve similar reasoning strategy (e.g. the same physics principles), students can be explicitly guided to point out the similarities between two problems (which may not look similar on the surface but involve

the same physics principles) and take advantage of what they learned from one problem to solve the other. Our previous research shows that if sufficient scaffolding is provided to help students process through the analogy between two problems deeply, they are likely to invoke the relevant principles correctly in the new context even if the problems involve multiple concepts and are known to be difficult for students (Lin and Singh 2010).

In this study, we examine introductory students' ability to learn from worked out examples (Chi et al. 1989; Aleven et al. 1999; Atkinson et al. 2000; Chi 2000; Yerushalmi et al. 2008) and perform analogical problem solving between two isomorphic problems that are similar in the application of a physics principle, but one problem often triggers a misleading notion which is not applicable in that particular case. According to Hayes and Simon's definition, problems are isomorphic if they can be mapped to each other in a one-to-one relation in terms of their solutions and the moves in the problem solving trajectories (Hayes and Simon 1977). For example, the "tower of Hanoi problem" and the "cannibal and the missionary problem" have the same structure if they are reduced to the abstract mathematical form and are isomorphic to each other. Here, we call problems isomorphic if they can be solved using the same physics principle. Cognitive theory (Simon and Hayes 1976; Kotovsky et al. 1985) suggests that isomorphic problems are not necessarily perceived as being at the same level of difficulty because different contexts and representations may trigger the recall of a relevant principle more in one problem than another, especially for a beginning learner. A previous study on transfer between isomorphic problem pairs in introductory physics (Singh 2008; Singh 2008) indicates that if students were given both the conceptual and quantitative problems in the isomorphic problem pairs, student were often able to perform better on the conceptual problem (which was typically more challenging for them) than students who were given the conceptual problem alone.

However, for those problems in which one context triggered an alternative approach which was not necessarily correct, the alternative view may deter the analogical reasoning and transfer of knowledge between the two problems. The study suggests that isomorphic problem pairs may be a useful tool to help students learn physics, but in some cases, more scaffolding may be needed.

In this study, students were asked in a mandatory recitation quiz to learn from a solved problem provided and take advantage of what they learned from it to solve another problem (called the quiz problem) that is isomorphic. Before solving the quiz problem, students were also explicitly guided to point out the similarities between the two problems. Both problems were about a car in equilibrium on an inclined plane with an inclination of 30 degrees. In the solved problem, the inclined plane was frictionless and the car was held at rest by a rope. The problem asked for the tension in the rope. In the quiz problem, there was no rope present and the car was held at rest by friction. Students were asked to find the frictional force acting on the car. We note that the two problems are similar because in both problems, the car is at rest on an incline, so the net force acting on it is zero. In addition, since the weight of the car and normal force exerted on the car by the inclined plane in both problems are the same, the only other force acting on the car (the tension in one problem and the friction in the other problem) must be the same. However, prior research suggests that many students have the notion that the magnitude of the static frictional force (f_s) is always equal to its maximum value, the coefficient of static friction (μ_s) times the normal force (F_N) (Singh 2007; Singh 2008). This notion is not valid for our quiz problem because the maximum value of static friction exceeds the actual frictional force needed to hold the car at rest.

In a previous study (Singh 2008), when the same friction problem was given to a group of introductory physics students in the multiple-choice format, only 20% of the students chose

the correct answer. The most common incorrect response was $f_s = \mu_s F_N$ (about 40%) . Preceding the friction problem with the isomorphic tension problem didn't significantly improve student performance on the friction problem and students did not fully discern the similarity between the problems. A related study was conducted in which another group of introductory physics students was explicitly instructed to focus on the similarity between these two problems (Lin and Singh 2011). They were given the tension problem (for which the solution was provided) and friction problem at the same time and were asked to exploit the similarity between the problems to solve the friction problem. Thirty five percent of the students got the correct answer, but the incorrect response, $f_s = \mu_s F_N$, was still common. It is likely that the strong alternative notion about static friction prevents the students from drawing a connection between the two problems and deters the transfer of knowledge from one context to another (in particular, from a problem not involving static friction to another problem involving static friction). The prior study (Lin and Singh 2011) also showed that a few students used Newton's 2nd law to solve for static friction on the car while simultaneously believing that f_s should equal $\mu_s F_N$ (for example, they may first solve for static friction using Newton's 2nd law correctly and then incorrectly calculate F_N by using the equation $f_s = \mu_s F_N$). This result suggests that even for students who are able to discern the similarities between the two problems and employ similar reasoning (i.e., the same physics principles), there may still be an unnoticed deficiency in their knowledge structure, which corresponds to a conflict between their notions of $f_s = \mu_s F_N$ and Newton's 2nd law. Therefore, the goal of our current study is to investigate strategies to help students build a coherent knowledge structure by (1) discerning the deep similarity between the two problems and understanding its implication for the use of the same physics principle, and (2) overcoming the conflict between Newton's 2nd Law and their misleading notion about static friction (if any).

In particular, we designed additional scaffolding supports for the students based on some pedagogical hypotheses and investigated how these different scaffolding strategies affect students' performance on the friction problem. Since the prior studies suggest that students may use Newton's 2nd Law to solve for friction while simultaneously believing that $f_s = \mu_s F_N$ is also applicable, in order to better explore students' thought processes about static frictional force, students in our current study were asked to solve for both (1) the static friction and (2) the normal force in the quiz problem. We note that although the solved problem didn't explicitly ask for the normal force, the answer for the normal force can be found in the solution provided.

3.3 METHODOLOGY

Four hundred and ten students from a calculus-based and an algebra-based introductory physics course were recruited in this study (183 and 227 students from each course, respectively). They were divided into four groups – one comparison group and three intervention groups - based on different recitation classes. Students in the comparison group were asked to solve the friction problem in a 15-minute long quiz on their own. No scaffolding support was provided. Examining the performance of this group of students can help us understand what students in this population can accomplish when no scaffolding support was provided.

Students in the three intervention groups, on the other hand, received the tension problem (which we call the "solved problem") to help them solve the friction problem (which we call the "quiz problem"). During the quiz, students in the three intervention groups were instructed to learn from the solution of the tension problem provided to them, draw an analogy between the solved and quiz problems by pointing out the similarities between them, and discuss how they

can take advantage of the solved problem provided to solve the friction problem. Moreover, different scaffolding supports were implemented in different intervention groups to help students process through the analogy deeply and/or to contemplate the applicability of associating the static frictional force with its maximum value. Depending on the different support provided, different amounts of time were given to students in different groups in order to complete the quiz. The amount of time given to each group was carefully determined so that students in all the groups would be able to complete the task.

In particular, students in the intervention group 1 were asked to take a few minutes to learn from the solution to the tension problem provided to them before they received the friction problem. They were explicitly told at the beginning of the quiz that after ten minutes, they had to return the solution to the instructor, and then they would be given two problems to solve: one of them would be the exact same problem they just browsed over (the tension problem), and the other one (involving static friction) would be similar. We hypothesized that by asking students to display how to solve the tension problem again on their own, students will process through the concepts they learned from the tension problem in more depth than if they are simply asked to browse over it. If the students adopt the problem solving approach used in the solved problem and set up both problems by drawing the free-body diagrams first, they are likely to realize that the two problems have the same free-body diagram and that the same concept (Newton's 2nd Law in the equilibrium situation) is applicable in the problem involving static friction as well.

The scaffolding supports provided to the intervention groups 2 and 3 were designed based on different frameworks to tackle students' alternative notions about the static frictional force. As we found in a previous study (Singh 2008), many students incorrectly believed that the friction problem should be solved differently from the tension problem because there is a special

formula for the static frictional force, i.e., its magnitude is always the coefficient of static friction times the normal force. They asserted that since there was a special formula associated with friction but not with tension, the special formula and the free-body diagram should be used in the corresponding problems, respectively. Not many of them appeared concerned that the two problems share the same free-body diagram but they were NOT solving them in the same way.

In order to help them, intervention 2 students were asked to make a qualitative prediction about the magnitude of the static frictional force (whether it's larger or smaller) when the same car is at rest on a steeper inclined plane (with the same static coefficient of friction) based on their daily experience. They were also explicitly instructed to quantitatively calculate the magnitudes of the frictional force acting on the car with two different degrees of inclination and compare their quantitative result with their qualitative prediction to check for consistency. We hypothesized that students could reason from their daily experience that it's more difficult to stand still on a steeper inclined plane; therefore, a larger frictional force would be required in order for the same car to stay at rest on a steeper incline. However, if they used $f_s = \mu_s F_N = \mu_s mg\, cos\, \theta$ (where θ is the angle of inclination) to calculate the magnitude of the frictional force, there would be a conflict because as the degree of inclination increases, the normal force decreases, making the frictional force calculated in this manner smaller. We hypothesized that if students are provided with the solution to the tension problem after noticing this conflict, they are more likely to notice the deficiency in their original argument. They may pay more attention to the similarities and benefit more from the solved tension problem provided. Therefore, students in the intervention group 2 were asked to take the first 10 minutes to do the quiz (which in addition to the original friction problem includes extra sub-problems asking for a qualitative prediction and a quantitative calculation of the magnitude of f_s on a steeper incline as well as a

consistency check) on their own before the solved tension problem was provided as a scaffolding tool. After they completed the quiz the first time, they turned in their first solution, and then they were given the tension problem with its solution. With the solved tension problem in their possession, they were given 20 minutes to learn from the solved problem and take the quiz (again the extra sub-problems are included) a second time. The design of this intervention was inspired by cognitive theory (Ginsberg and Opper 1969; Gorman 1972) which indicates that cognitive conflict can be useful for helping students learn concepts and build a better understanding.

A different scaffolding support which aimed at guiding students to examine the applicability of the equation $f_s = \mu_s F_N$ was implemented in the intervention group 3. Students in this group were provided with the solved problem and the quiz problem at the same time in a 25-minute quiz. In addition to the instruction which asked them to discuss the similarity between the two problems before solving for the frictional force, they were also asked to explain the meaning of the inequality in $f_s \leq \mu_s F_N$ and discuss whether they can find the frictional force on the car in the quiz problem without knowing μ_s. We intended that this additional questioning provide a direct hint to students to resolve the "conflict" if they are able to recognize the similar roles played by the tension and the friction in the two problems (the solved and quiz problems are similar from the physics point of view except that the tension in the former case is substituted by the friction in the latter case) but are concerned about the fact that the equation $f_s = \mu_s F_N$ doesn't yield an answer which has the same magnitude as the tension. In order to increase the possibility of students discerning this discrepancy between the free-body diagram and the special equation they used for static friction, after solving for the frictional force, the last part of the quiz also explicitly asked students to solve for the magnitude of the normal force "using the

component of force perpendicular to the inclined plane" (and check that the calculated normal force is consistent with what they obtained previously if the normal force was previously involved in solving for friction). We hypothesized that if the students used a convoluted reasoning as discussed earlier in the introduction section and set $mg\,sin\,\theta - f_s = mg\,sin\,\theta - \mu_s F_N = 0$ when solving for friction, this additional hint and instruction provided may help them.

In order to examine whether students in different groups were comparable, their performances on the Force Concept Inventory (FCI) (Hestenes et al. 1992) administered at the beginning of the semester and their scores on the final exam were investigated. There was no statistically significant difference between different groups in terms of the FCI score or the score on the final exam.

Student performance on the quiz was graded using a rubric. For the intervention group 2, in which students were asked to calculate the friction with different angles of inclination, in the few cases for which there was a discrepancy between the two calculations, the score was assigned based on the 30 degree case, which is the same as the grading of the students in the other intervention groups. When two researchers scored independently a sample of 10% of the students using the rubric, an inter-rater reliability of more than 80% was achieved. Table 3-1 summarizes the rubric for the calculation of friction, which had a full score of 10 points. It was constructed based on students' different problem solving approaches and their common mistakes. Different approaches were assigned different maximum scores. For example, the maximum score a student could receive if she/he correctly used Newton's 2nd Law in the equilibrium situation ($\sum F = 0$) was 10, while a student who used $f_s = \mu_s F_N$ to solve for tension could earn a maximum score of only 5. If the students used $\sum F = 0$ and came up with the correct value ($f_s = mg\,sin\,\theta$) for the calculation of friction, but their answers to the other sub-problems

indicated that they still related the static friction with its maximum value (for example, first finding $f_s = mg\,sin\,\theta$ correctly but then incorrectly using $f_s = \mu_s F_N$ to solve for the normal force in the next sub-problem), they were classified as having the same notion as students using the $f_s = \mu_s F_N$ approach and the maximum score they could receive was 5. Under each approach, the common mistakes students made and the corresponding points taken off are listed in the rubric. For example, students lost point(s) for decomposing the force incorrectly or for confusing weight with mass. Intervention 1 students' ability to reproduce the solution to the tension problem was graded using a similar rubric (the part associated with the "using $\sum F = 0$" approach in Table 3-1).

Table 3-1. Summary of the rubric for the calculation of frictional force.

Problem solving approach	Maximum score	Common mistakes (Points taken off)
Using $\sum F = 0$	10	(1) Decomposed the force incorrectly: $f_s = mg\,cos\,\theta$ (-1) (2) Decomposed the force incorrectly: $f_s = mg/\ sin\,\theta$ (-2) (3) Confused weight with mass and multiplied the weight by an additional g=9.8m/s^2 (-1)
$f_s = \mu_s F_N$	5	(1) Decomposed the normal force incorrectly (-1) (2) N = mg (-2) (3) Confused weight with mass and multiplied the weight by an additional g=9.8m/s^2 (-1)
$f_k = \mu_k F_N$	3 or 4	
Combined μ_s and μ_k (e.g. $f_{friction} = \mu_s F_N + \mu_k F_N$)	2	
$f_s = \mu_s F_N - mg\,sin\,\theta$	2	

In order to examine the effects of the interventions and to evaluate whether the interventions were more successful in helping students at a particular level of expertise, students in each course were further classified as top, middle, bottom, and "none" by listing them in order based on their scores on the final exam and then splitting them into thirds (students in the category "none" didn't take the final exam). We analyzed how the top, middle and bottom students within the same intervention group performed with the same scaffolding provided. We also compared introductory physics students' performance in the algebra- and calculus-based courses with the performance of a group of first-year physics graduate students who were asked to solve for the friction in the quiz problem on their own without any scaffolding provided. Moreover, in order to get an in-depth account of students' reasoning and examine the possible ways to help them with their difficulties, three introductory physics students from other introductory physics classes were recruited for interviews. The interviews were conducted using a think aloud protocol to allow the researchers to follow and record students' thinking process. The details of the interviews will be discussed later.

3.4 RESULTS FROM TWO INTRODUCTORY PHYSICS COURSES

The performance of 26 first-year graduate students on the friction problem is shown in Table 3-2, which bins the students into categories based on their problem solving approach. The number of students in each case and the mistakes students made are presented. We found that 31% of the graduate students had the notion that the magnitude of the static frictional force is always equal to its maximum value. The average score of this group of graduate students was 8.6 out of 10.

Table 3-2 Graduate students' performance on the calculation of friction.

Problem solving approach	Number of students	Types of Mistakes and the corresponding number of students who made a particular mistake
$f_s = mg\, sin\, \theta$	18 (69.2%)	Had an extra g factor (1 person)
$f_s = \mu_s F_N$	6 (23.1%)	Used $F_N = \mu_s f_s$ instead of using $f_s = \mu_s F_N$ (1 person)
Both $f_s = mg\, sin\, \theta$ and $f_s = \mu_s F_N$ (thinking $\mu_s F_N - mg \sin\theta = 0$)	2 (7.7%)	--

Before discussing the results in the introductory physics courses, we note that students in all groups had adequate time to work on the quiz. Table 3-3 and Table 3-4 present students' average scores on the calculation of friction in the calculus-based and algebra-based courses, respectively. The p-values listed in Table 3-5 indicate that in both courses, students in all intervention groups significantly outperformed the comparison group students (with p-values less than 0.05), except for the calculus-based intervention group 1 students whose performance is marginally better (with p-value of 0.053) than the corresponding comparison group. Comparing the effects of different interventions, we found that in the calculus-based course, students in the intervention groups 2 and 3 on average achieved the score of 7.2 or 7.1 out of 10, respectively, followed by the students in the intervention group 1, whose average score was 5.9. All of these scores were better than that for the comparison group, and there was no significant difference between any of the intervention groups. For the algebra-based course, on the other hand, all three intervention groups significantly outperformed the comparison group, and among them, intervention 2 was the best. The p-values show that intervention group 2 students, who achieved an average score of 6.8, performed significantly better than students in intervention groups 1 and 3, whose average scores were 5.1 and 4.9, respectively. There was no significant difference

between the latter two groups. Comparing the data from the two courses, we found that intervention 2 was always one of the best interventions for both the algebra- and calculus-based students.

Table 3-3. Students' average scores out of 10 on the calculation of friction (the quiz problem) in the calculus-based course. The numbers of students in the comparison (comp) group and each of the 3 intervention (intv) groups are shown in parentheses. For students in the intervention group 2, their performance before and after they received the scaffolding was examined. The normalized gain is defined by the change in score over the maximum possible score for improvement.

	Comp	Intv 1	Intv 2			Intv 3
			Before	After	Normalized gain (%)	
Top	4.7 (13)	7.4 (13)	5.7	8.5 (22)	65	9.0 (10)
Middle	4.4 (12)	5.7 (9)	4.3	6.6 (25)	40	8.3 (18)
Bottom	5.4 (9)	4.4 (12)	2.8	6.9 (22)	57	3.7 (9)
None	1.0 (4)		2.3	5.7 (3)	43	2.5 (2)
All	4.4 (38)	5.9 (34)	4.2	7.2 (72)	52	7.1 (39)

Table 3-4. Students' average scores out of 10 on the calculation of friction (the quiz problem) in the algebra-based course. For students in the intervention group 2, their performance before and after they received the scaffolding was examined. The numbers of students in the comparison (comp) group and each of the 3 intervention (intv) groups are shown in parentheses.

	Comp	Intv 1	Intv 2			Intv 3
			Before	After	Normalized gain (%)	
Top	4.1 (14)	7.2 (20)	5.4	7.4 (27)	45	6.0 (14)
Middle	2.9 (17)	4.6 (18)	3.1	6.2 (11)	45	5.0 (22)
Bottom	2.5 (13)	3.4 (20)	1.6	5.2 (10)	43	4.5 (24)
None	2.7 (3)	5.4 (5)	3.3	8.7 (3)	80	3.7 (6)
All	3.1 (47)	5.1 (63)	4.0	6.8 (51)	47	4.9 (66)

Table 3-5. The p values (from ANOVA) for the comparison of students' performance between different groups in the calculus-based and algebra-based courses. The algebra-based course is indicated by the shaded background.

	Comparison	Intervention 1	Intervention 2	Intervention 3
comparison	--	0.053	0.000	0.001
Intervention 1	0.003	--	0.058	0.129
Intervention 2	0.000	0.016	--	0.906
Intervention 3	0.005	0.820	0.008	--

Examining introductory physics students' performance on the friction problem by binning students into categories based on their problem solving approach provides another angle to look into how students were able to extract information and benefit from the scaffolding provided. Table 3-6 and Table 3-7 list the students' different approaches for finding friction and the corresponding percentage of students in each group in the calculus- and algebra-based course, respectively. The p-values, which compare the difference between the number of students in the intervention groups and comparison groups who adopted different problem solving approaches, are presented in Table 3-8. As discussed previously, one common mistake students made was to first find the normal force by using Newton's law in the equilibrium situation and then using $f_s = \mu_s F_N$ to solve for friction. We note that if the students' values of the frictional force were correct but the overall performance for the whole quiz indicated that they were still connecting the static friction to its maximum value (for example, by using $f_s = \mu_s F_N$ to solve for the normal force in the next sub-problem after finding f_s correctly), they were classified in the 2nd category of "$f_s = \mu_s F_N$". Unlike graduate students whose problem solving approaches always fell into either the first or second category, introductory students had additional difficulties with the friction problem and the alternative approaches to the friction problem were not limited exclusively to $f_s = \mu_s F_N$. For example, some students failed to realize

the vector nature of Newton's 2nd Law and treated it as a scalar equation. Some students multiplied μ_s with a quantity other than the normal force such as the component of the weight parallel to the incline surface. Some students found both the $mg\,sin\,\theta$ and $\mu_s F_N$ terms and set f_s as a combination of them by either adding or subtracting one term to/from the other. There were also students who confused the static friction with the kinetic friction and used μ_k instead of μ_s to solve the problem. Some students erroneously combined μ_s and μ_k together and came up with an answer such as $f_s = \mu_s F_N + \mu_k F_N$. All these different approaches were placed in the "other" category.

Table 3-6 and Table 3-7 show that in all the intervention groups, the percentages of students who correctly used Newton's 2nd Law in the equilibrium situation to solve for static friction without connecting it to its maximum value were higher than those in the comparison groups in both the calculus- and algebra-based courses. Among them, interventions 2 and 3 both provided excellent scaffolding in helping calculus-based students solve the static friction problem correctly, while the best intervention in the algebra-based course was intervention 2. The percentages of students who correctly used Newton's 2nd Law in these three groups were more than two times higher than that for the comparison group in the corresponding course. This result is similar to what we found by looking at the average scores in Table 3-3 and Table 3-4. We note that the increase in the percentage of students who solved the friction problem correctly would be accompanied by the decrease of the number of students who used either $f_s = \mu_s F_N$ or other approaches. Comparing the percentages of students in the "$f_s = \mu_s F_N$" category in particular, however, we found that only intervention group 2 in the algebra-based course showed a significant decrease. Although the percentages in the calculus-based intervention groups 2 and 3 also decreased, the differences from the comparison group (especially that of the intervention

group 3) were not great enough to be statistically significant. The finding suggests that although providing students with the solved isomorphic problem gave them more clues about how to construct the problem solution (and therefore the percentages of students in the "other" group and sometimes the "$f_s = \mu_s F_N$" group were reduced), overall, the notion of $f_s = \mu_s F_N$ was still common. We not only found that different interventions had different effects in helping students adopt a suitable problem solving strategy and avoid common mistakes, but also observed that calculus and algebra-based students didn't benefit equally from the same intervention (e.g., intervention 3). In the following paragraphs, we'll discuss the students' responses to the different additional tasks/scaffoldings contained in different interventions in more detail and investigate the possible reasons why some interventions were more beneficial to the students than others.

Table 3-6. Percentage of students in each group based on their problem solving approaches in the calculus-based course.

	Percentage of Students					Percentage changed with respect to the comparison group			
	comp	Intv 1	Intv 2		Intv 3	Intv 1	Intv 2		Intv 3
			pre	post			pre	post	
Correct use of Newton's 2nd Law	21.1	38.2	22.2	56.9	56.4	17.2	1.1	35.8	35.3
$f_s = \mu_s F_N$	42.1	38.2	43.1	25.0	30.8	-3.9	1.0	-17.1	-11.3
Other	36.8	23.5	34.7	18.1	12.8	-13.3	-2.1	-18.7	-24.0

Table 3-7. Percentage of students in each group based on their problem solving approaches in the algebra-based course.

	Percentage of Students					Percentage changed with respect to the comparison group			
	comp	Intv 1	Intv 2		Intv 3	Intv 1	Intv 2		Intv 3
			Pre	post			Pre	post	
Correct use of Newton's 2nd Law	14.9	30.2	19.6	60.8	28.8	15.3	4.7	45.9	13.9
$f_s = \mu_s F_N$	34.0	36.5	41.2	15.7	37.9	2.5	7.2	-18.3	3.9
Other	51.1	33.3	39.2	23.5	33.3	-17.8	-11.9	-27.6	-17.8

Table 3-8. P values (using the Chi-square tests) for the comparison of the number of students who adopted different problem solving approaches in different groups. The differences that are significant are indicated by the asterisk (*). The pound symbol ($^{\#}$) indicates a marginally significant difference with a p-value between 0.05 and 0.10.

		Comparison vs. intervention 1	Comparison vs. intervention 2	Comparison vs. intervention 3
Calculus	Correct use of Newton's 2nd Law	0.109	0.000*	0.001*
	$f_s = \mu_s F_N$	0.738	0.065$^{\#}$	0.301
	Other	0.221	0.029*	0.015*
Algebra	Correct use of Newton's 2nd Law	0.062$^{\#}$	0.000*	0.084$^{\#}$
	$f_s = \mu_s F_N$	0.789	0.035*	0.676
	Other	0.061$^{\#}$	0.005*	0.059$^{\#}$

As we mentioned earlier, intervention 2 was always one of the best interventions for students in both the calculus- and algebra-based courses. Students in this group were asked to try the quiz problem on their own before learning from the solved example. Moreover, they were advised to make a qualitative prediction about the magnitude of the static frictional force on a steeper incline based on their daily experience and compare their prediction with their calculated

result. Examining students' answers to these additional questions about the steeper incline when they tried the problem for the first time, we found that not all students could make a correct prediction, and there was not much difference between the answers provided by students from the calculus- and algebra-based courses. Table 3-9 compares the percentage of students who predicted/calculated that the static friction should be larger or smaller before and after learning from the solved example. In both courses, most students' reasoning behind their first predictions could be classified into one of three categories: (1) daily experience and correct interpretation/prediction, (2) daily experience and incorrect interpretation/prediction, and (3) answer based on the calculated result. There were students from both the calculus-based and algebra-based courses who were able to connect the problem with their daily experience and make a correct prediction. For example, a calculus-based student correctly stated that "Based on my daily experience, I would predict that the magnitude should be larger because the steeper angle makes objects want to move more than the slight angle". Similar statements such as "If the inclined plane is steeper, the frictional force between the object and the surface will be larger because the frictional force is equal to the magnitude of the force pulling you down the incline (just in the opposite direction) and from daily experience it feels like more force is trying to pull you down a plane when the plane is steeper" were made by some algebra-based students as well.

Table 3-9. Percentage of students who predicted (pred)/calculated (cal) that the static friction should be larger or smaller on a steeper incline before and after learning from the solved example.

	Calculus				Algebra			
	Before		After		Before		After	
	Pred	Cal	Pred	Cal	Pred	Cal	Pred	Cal
Larger	54.2	30.6	75.0	73.6	56.9	33.3	68.7	68.7
Smaller	33.3	31.9	23.6	20.8	27.5	21.6	23.5	23.5
The same	1.4				2.0	3.9		
None/ not complete	11.1	37.5	1.4	5.6	13.7	41.2	7.8	7.8

However, although students knew from their daily experiences that it is less likely for an object to stay at rest on a steeper incline, some of them had an alternative explanation so that their prediction was opposite to their intuition. For example, one student said "Based on my daily experience, the frictional force should be less on a larger incline because it's harder to stay at rest on a steeper incline." Such alternative explanations were found in both the algebra- and calculus-based courses. We note that the purpose of this prediction question was to help students who originally adopted the $f_s = \mu_s F_N$ approach to discover the conflict between the qualitative trend suggested by the daily experience (static friction should be larger on a steeper incline) and their quantitative answer (showing that the static friction calculated using $f_s = \mu_s F_N$ is smaller). We hypothesized that such questioning will provide incentive to re-examine their problem solving approach. Not all students, however, were able to discover the inconsistency in their responses. Some students provided alternative explanations about their daily experience as described above, others made a prediction not based on their daily experience but based on a quantitative calculation, and some made a mistake in the subsequent calculation (for example, switching the $sin\,\theta$ and $cos\,\theta$) and therefore their calculated results accidentally coincided with their qualitative prediction. In summary, we found that these additional questions work in the way we had intended for some students, but not for all of them. Despite this fact, Table 3-6 and Table 3-7 still suggest that students benefited overall from intervention 2. It is likely that the fact that students in this group had the opportunity to try solving the problem on their own before the solved example was provided is beneficial to them because the clear targeted goal and the thinking process they went through in their first attempt facilitates better transfer to the other problem. Similar findings showing an advantage in postponing scaffolding until students have

attempted to solve the problem without help have also been discussed in the literature (Lin and Singh 2010; Lin and Singh 2010).

As for intervention 3, which not only provided students with the solved tension problem but also exposed them to the correct inequality $f_s \leq \mu_s F_N$ and asked them to explicitly discuss whether μ_s is needed to solve the quiz problem by thinking about the meaning of the inequality, the percentage of students who explicitly answered whether μ_s is needed/not needed is listed in Table 3-10. As we found previously by looking at the percentages of students using different problem solving approaches, this scaffolding support was more beneficial to the calculus-based students than the algebra-based students. Even though students were advised to identify the similarity between two problems and were also explicitly shown that the correct expression for the static friction was not $f_s = \mu_s F_N$ but $f_s \leq \mu_s F_N$, more algebra-based students had difficulty in making sense of the inequality and its implication for the static friction problem. As Table 3-10 shows, fifty percent of the algebra-based students explicitly said that in order to find the frictional force on the car, μ_s needs to be given. Examining students' explanations of the inequality, we found that many algebra-based students weren't able to take advantage of the scaffolding provided because they focused only on one aspect of the inequality and failed to see its full implication. Instead of realizing that "f_s can be any value from zero to the maximum value, which is $\mu_s F_N$, depending on how strong the opposing force is", they only focused on the fact that static friction can't be larger than $\mu_s F_N$. They explained that if this maximum amount is exceeded, the object could no longer be stationary; however, since the car in the problem was at rest, the coefficient of static friction must be used. The similarities between the two problems and explicitly asking them to explain the inequality sign didn't help them realize that the static friction in the quiz problem was not equal to its maximum value.

We also note that some students incorrectly interpreted the inequality and the maximum static friction. For example, one student stated that "$f_s \leq \mu_s F_N$ means that the normal force multiplied by μ_s must be greater than f_s in order for the car to overcome the frictional force. If it is not greater, then the car will not move." Another student noted "It is an inequality because if $\mu_s F_N$ were any smaller than f_s, it would cause the force to be too small and the car would move." Such difficulty in understanding the relationship between the static frictional force and its maximum value was more commonly found in the algebra-based course than in the calculus-based course. It is likely that the scaffolding support provided in intervention 3 requires an ability to interpret inequalities at a level which is suitable for calculus-based students but too innovative for many algebra-based students in the framework of preparation for future learning by Schwartz et al. (Schwartz et al. 2005). Therefore, intervention 3 may be more commensurate with calculus-based students' prior skills but may be beyond the zone of proximal development (Vygotsky 1978) of many algebra-based students. Accordingly, more calculus-based students benefited from it than the algebra-based students.

Table 3-10. Percentage of students in intervention group 3 who answered that μ_s is needed/not needed in the quiz problem after they attempted to explain the meaning of the inequality $f_s \leq \mu_s F_N$.

	Calculus	Algebra
μ_s not needed	69.2 %	45.5 %
μ_s needed	28.2 %	50.0 %
Irrelevant answer or no answer	2.6 %	4.5 %

Table 3-11 shows intervention 1 students' average scores on the solved problem (tension problem) reproduced after returning its solution to the instructor. Students' performance on the friction problem is also listed for comparison. We found that almost every student, whether in a calculus-based or an algebra-based course, was able to solve for the tension in the solved

problem correctly except for some minor mistake (if any) such as confusing the weight and the mass and therefore multiplying the weight by an additional factor of g. The average score on the tension problem was 9.7 out of 10. When it comes to transfer to the friction problem, however, the average score dropped significantly to 5.9 and 5.1 in the calculus- and algebra-based courses, respectively. As Table 3-6 and Table 3-7 show, only 38% and 30% of the students, respectively, adopted Newton's 2nd law in equilibrium to the transfer problem involving friction; $f_s = \mu_s F_N$ was still common in both courses after the scaffolding. It is likely that the scaffolding support, which included asking students to identify the similarities between the two problems and reproduce the solved problem, was not meaningful enough to engage many students in the analogical reasoning, especially if they had a strong belief about being able to solve the friction problem using an alternative approach. Therefore, their improvement with intervention 1 was not as great as for some other intervention(s) in which students were provided with more direct hints to help them contemplate the applicability of the equation $f_s = \mu_s F_N$ carefully in the problem given.

Table 3-11. Average scores out of 10 on the tension problem (solved problem) and the friction problem (quiz problem) for intervention 1 in the algebra-based and calculus-based courses.

	Tension Problem		Friction Problem	
	Calculus	Algebra	Calculus	Algebra
Top	10	9.8	7.4	7.2
Middle	9.9	9.3	5.7	4.6
Bottom	9.3	9.8	4.4	3.4
None		10		5.4
All	9.7	9.7	5.9	5.1

3.5 INTERVIEWS

3.5.1 General Description

As the quantitative data suggest, the notion that "static friction always equals $\mu_s F_N$" was strong in many introductory physics students' minds and providing them with the solved isomorphic problem didn't necessarily help all students. In particular, even though students in intervention groups 2 and 3 received extra hints to help them deal with their misleading notion, overall, the notion of $f_s = \mu_s F_N$ was still common. In order to get a better understanding of the rationale behind students' responses, we conducted a few in-depth interviews with individual students to explore their reasoning while they solved the problem with scaffolding. Another goal of the interviews is to further explore strategies to help students overcome their common incorrect responses. The interviews were not conducted for quantitative purposes because the quantitative data already exists from a large number of students described in the previous sections. Rather, the interviews were conducted with the intention to get an in-depth account of students' thought processes in order to help us understand in more depth what students were able to learn from the scaffolding provided and what additional support students might need.

Three student volunteers from other introductory physics classes which didn't participate in the quiz were recruited for one-on-one interviews. Two of them were enrolled in a calculus-based introductory physics course at the time of the interview; the third was enrolled in an algebra-based introductory physics course. The interviews were conducted after all the relevant topics had been covered in the lectures. All three students had midterm scores which fell in the middle of their own introductory physics class performance.

During the interviews, students were asked to learn from the solved problem provided and solve the isomorphic quiz problem. The algebra-based student and one of the calculus-based students were provided with the scaffolding of intervention 2. The other calculus-based student was given the scaffolding of intervention 3. Students were asked to perform the task while thinking aloud; they were not disturbed during the task. After the students completed the quiz, the researcher would first ask clarification questions in order to understand what they did not make explicit earlier and what their difficulties were. Based on this understanding, the researcher then provided additional support to the students in order to help them solve the quiz problem correctly if they had not done so. At the end of the interviews, students were invited to reflect on the learning process they just went through and provide some suggestion from their own perspective on how to improve students' performance on the quiz problem. The goal of the students' reflection was to help us identify possible helpful scaffoldings not only based upon what the researchers observed but also based upon students' reflection of their own learning. The two researchers later looked at the interview data and discussed the interpretation of the data with each other. The agreement between the researchers was very good.

3.5.2 Interview Results

The interviews suggest that although all three students were able to discern the similarities between the tension in one problem and the friction in the other, they struggled with the fact that the static friction isn't necessarily equal to its maximum value. For example, when a student (student A) who was provided with intervention 3 in the interview tried to use both Newton's 2nd Law in the equilibrium situation and the equation $f_s = \mu_s F_N$ to solve for friction, he was confused when he saw that different methods yielded different answers. He was not sure if he

should say yes or no to the question related to "whether μ_s is needed in order to solve for friction." The student continued "*Here, the question is: 'can you find the frictional force on the car in the problem without knowing the coefficient of static friction?' I would say no, but my equation says yes. That doesn't make sense. Because, judging by my free body diagram,* $mgsin\theta$ *would actually…would have to equal force done by friction. But I thought the definition of the force done by static friction was the coefficient of friction times the normal force.*" This student also explicitly noted that he didn't know how to explain the meaning of the inequality. He knew that the inequality stated the static friction is smaller than or equal to the coefficient of static friction times the normal force, and he knew that the static friction couldn't be larger, but he didn't know how to interpret the inequality. In particular, he didn't know why f_s can also be smaller, not just equal to $\mu_s F_N$.

Moreover, the interviews suggest that even if students calculated the static frictional force and normal force correctly without using $f_s = \mu_s F_N$, it does not necessarily indicate that they were completely devoid of this incorrect notion. For example, while one student who was given intervention 2 was able to write down correct answers to all of the questions the first time she tried the problem, her remarks during the "thinking aloud" process revealed she also believed that the static friction and the normal force she calculated would be connected by the formula $f_s = \mu_s F_N$ even though she didn't explicitly write it down. When the researcher later asked her to check this relation by plugging in the numbers she got, she found a conflict and didn't know what to do about it. Although learning from the solved problem made her confident that her original answer $f_s = mg\sin\theta$ was correct, she didn't understand why the static friction doesn't have to equal $\mu_s F_N$.

These findings suggest that the quantitative data presented previously could be considered as an upper limit for how well the interventions can help students overcome their misleading notions by reasoning about the similarities between the quiz problem and the solved problem. Some students may require more help in order to re-organize their knowledge about static friction and interpret the associated inequality correctly. From the interview, we identified some post-activity discussion that was useful in helping students build a better understanding of static friction. We'll discuss it in the following paragraphs.

We found it helpful to improve students' understanding of the inequality $f_s \leq \mu_s F_N$ to quantitatively discuss with them how the static friction acting on an object (such as a heavy desk) placed on a horizontal surface keeps increasing when we push the object harder and harder until the maximum static friction is reached and the static frictional force is no longer able to hold the object in place so that the object starts to move. We note that although the information contained here is similar to that in intervention 2 when the angle of inclination is increased, the example of an object on a horizontal surface may be easier for students to comprehend because it doesn't require a decomposition of forces and the normal force stays the same. We also note that although a similar example is often used by many instructors while lecturing, it is likely that students pay more attention to one aspect of the inequality (that the static friction cannot be larger than $\mu_s F_N$, otherwise the object will start to move) than the other (the situations in which f_s was not equal to but smaller than $\mu_s F_N$) as we discussed in the quantitative results section. It would therefore be beneficial in the post-activity (after the interventions) discussion to guide students' attention to this latter aspect by starting with some cueing questions. For example, students could be asked: "Since the inequality $f_s \leq \mu_s F_N$ implies that the static friction can be smaller than (not equal to) $\mu_s F_N$, can you think of any situation in which f_s is indeed smaller?" If

the students struggle with this question, we could guide them to think about the static friction acting on an object resting on a horizontal surface when no horizontal force is applied, which many students are able to answer immediately, so that students can build their understanding on a solid base in a familiar situation. We found from the interviews that although simply asking students to consider the case in which there is no static friction while the normal force is nonzero may not be enough to totally clear up student' confusion about whether it is correct to solve for the friction without using μ_s, it provides a good starting point. After successive follow up questions in which student A was asked to calculate the static frictional force on an object when 1 N, 2 N…. of external horizontal force is applied until the value of $\mu_s F_N$ was exceeded and the object starts to move, he gradually understood the full implication of the inequality and he was no longer perplexed by the fact that his answer to the quiz problem didn't involve the coefficient of static friction. When the researcher later asked him to reflect on his learning during the whole activity and identify the support which he found most helpful, he said "*I think it was the…the analogy of the desk [that really helped me]. I understand that now. Because…like….at first it [$f_s \leq \mu_s F_N$] just looks like an equation to me. But after I understand that it's gonna be less than or equal to until that point where you exceed it, and it starts moving the other direction that's gonna be greater than, that makes sense to me.*"

Similarly, when another student was asked at the end of the interview to provide some suggestions on how to help students learn that f_s is not always equal to $\mu_s F_N$ by reflecting on her learning process during the activity, she pointed out that the example of an object on a horizontal surface was very helpful, especially the part in which the researcher guided her to examine what will happen if static friction was always equal to $\mu_s F_N$. By drawing the free body diagram, this student was able to reason that if the static frictional force had a fixed value of $\mu_s F_N$, giving the

object a small push toward the left (with a magnitude smaller than $\mu_s F_N$) would result in a freak phenomenon of the object moving toward the right (since a static friction acting toward the right should exist to resist the tendency to move and its larger magnitude suggests that the object would move in a direction opposite to the direction in which it was pushed), which was contradictory to her daily experience. She claimed that this example helped her the most in realizing that $f_s = \mu_s F_N$ is only true in special situations and she was able to reason about it using the free body diagram.

The student also suggested that writing both $f_s^{max} = \mu_s F_N$ and $f_s \leq \mu_s F_N$ together would help students understand the concept of static friction better. It is likely that listing both the equation and the inequality together would help students focus on both aspects of the inequality when the maximum static friction is/ is not reached. She pointed out that "*It's easy if you put that 'f_s max would equal' equation and the inequality, and then you were to explain why. That would help a lot as to why $\mu_s F_N$ doesn't equal [the static friction]*." She reflected on her own learning of the subject and said: "*Because when my professor first taught it to us, he wrote maximum and I was like, 'hey…what does that mean?' Like… you know…it was just max, whatever. Now that we went over this, I do understand why he would put that, and I've grasped the concept better.*" Her remarks also suggested that if the analogical reasoning activity as well as the post-activity discussion were designed and implemented in a way that is commensurate with students' ability, they are likely to benefit from it.

3.6 SUMMARY AND DISCUSSION

In summary, we found that introductory physics students to some extents were able to take advantage of the analogical reasoning activity and transfer their learning from the solved problem provided to solve the quiz problem involving friction. After learning from the scaffolding provided, many students were able to identify the relevant concepts involved in solving the quiz problem, and the score on average improved in all the intervention groups. However, we also found that a large portion of the improvement came from the fact that the number of students who had no clue about how to construct the problem solution was reduced after learning from the solved problem. The notion of "static friction is always equal to its maximum value $\mu_s F_N$" was still prevalent. Although the percentage of students who used $f_s = \mu_s F_N$ to solve the quiz problem was reduced in some of the intervention groups, in most cases the decrease was not large enough to make a statistical difference. Among all the different scaffoldings provided, intervention 2 in which students had to think before the solved problem was provided was consistently the best in helping students refrain from using $f_s = \mu_s F_N$ in both the algebra- and calculus-based courses. This result suggests that providing the solved problem to students only after they have tried to solve the quiz problem on their own was the most beneficial to students in both the calculus- and algebra-based courses. The additional questioning about the change in the magnitude of static friction when the inclination changes may also provide some advantage. It is likely that the cumulative effect of all these scaffolding supports in intervention 2 together helped students engage in a deeper thinking when constructing their solution to the friction problem.

Although the various scaffolding supports provided didn't produce a large enough effect to make the introductory students' performance comparable with that of the graduate students'

(who didn't receive any scaffolding), and many introductory students struggled with the fact that the static friction is not always equal to its maximum value, in general, we believe this analogical reasoning activity, especially interventions 2 and 3, can serve as a good starting point to help students contemplate their understanding of friction and when $f_s = \mu_s F_N$ is applicable. Since the improvement on the raw score indicates introductory students' ability to recognize the similarity between the isomorphic problems, instructors can provide similar guidance to help students and explicitly ask them to compare what they learned with their existing knowledge structure, assess whether the different approaches are consistent with each other, and re-construct a better knowledge structure as needed. Instead of solving the quiz problem, students can be asked to evaluate different solutions of the same problem (e.g., the static friction problem) created by the instructors and discuss the mistakes they find in the incorrect solutions. If the students realize that one solution, e.g., using $f_s = \mu_s F_N$ yields an answer different from the other solution using Newton's 2nd Law in equilibrium, they can be encouraged to think about the difference between these approaches in order to assess which solution is correct, and they can be guided to draw better analogy with the isomorphic solved problem. Various interventions that are likely to help students, such as interventions 2 and 3, can be combined to provide a scaffolding support which is most commensurate with students' current knowledge structure.

Based on the findings from individual interviews, students are likely to benefit most from analogical reasoning activities if post-activity discussion is carried out by the instructor. If any of the three interventions discussed are adopted by an instructor, after the analogical reasoning task is performed by students, it will be advantageous to discuss with students why the static friction should not always equal its maximum value with the quiz as an example. Reviewing the similarity between the quiz and solved problem after the discussion may help students repair

their knowledge structure. Issues about the meaning of the inequality sign, the change in the magnitude of static friction when a larger and larger force is applied to an object while the object remains at rest, and what would have happened if f_s is always equal to $\mu_s F_N$, could be discussed to help students consolidate their understanding about friction. Our research suggests that it would be important to keep in mind that one difficulty students have in learning about the inequality related to static friction is that they often focus on static friction not being greater than $\mu_s F_N$, ignoring the fact that f_s can be smaller than this value. Special effort should therefore be made to address related issues in discussions.

In summary, analogical reasoning tasks can provide a good opportunity to help students not only learn about friction, a very challenging topic even at the introductory level, but can also help them build a better knowledge structure. If similar activities and post activity discussions are sustained throughout an introductory physics course, students are likely to develop expertise in physics and become better problem solvers.

3.7 CHAPTER REFERENCES

Adey, P. and M. Shayer (1993). "An exploration of long-term far-transfer effects following an extended intervention program in the high school science curricula." *Cogn. Instruct.* **11**, 1-29.

Aleven, V., K. R. Koedinger and K. Cross (1999). "Tutoring answer explanation fosters learning with understanding". the 9th International Conference on Artificial Intelligence in Education, Le Mans, France, IOS Press, Amsterdam.

Atkinson, R. K., S. J. Derry, A. Renkl and D. Wortham (2000). "Learning from examples: Instructional principles from the worked examples research." *Rev. Educ. Res.* **70**(2), 181-214.

Bassok, M. and K. Holyoak (1989). "Interdomain transfer between isomorphic topics in algebra and physics." *J. Experimental Psychology: Memory, Learning and Cognition* **15**(1), 153-166.

Bobrow, D. and D. Norman (1975). "Some principles of memory schemata". In *Representation and understanding: Studies in cognitive science*. D. Bobrow and A. Collins. New York, Academic Press**:** 131-149.

Bransford, J. D. and D. Schwartz (1999). "Rethinking transfer: A simple proposal with multiple implications." *Rev. Res. Educ.* **24**, 61-100.

Brown, A. (1989). "Analogical learning and transfer: What develops?". In *Similarity and analogical reasoning*. S. V. a. A. O. Eds. NY, Cambridge University Press**:** 369-412.

Cheng, P. W. and K. J. Holyoak (1985). "Pragmatic reasoning schema." *Cognitive Psychology* **17**, 391-416.

Chi, M. (2000). "Self-explaining expository texts: The dual processes of generating inferences and repairing mental models.". In *Advances in instructional psychology*. R. Glaser. Hillsdale, NJ, Erlbaum Associates**:** 161-238.

Chi, M., M. Bassok, M. W. Lewis, P. Remann and R. Glaser (1989). "Self-explanations: How students study and use examples in learning to solve problems." *Cog. Sci.* **13**, 145.

Chi, M. T. H., P. J. Feltovich and R. Glaser (1981). "Categorization and representation of physics knowledge by experts and novices." *Cog. Sci.* **5**, 121-152

Detterman, D. K. and R. J. Sternberg, Eds. (1993). *Transfer on trial: Intelligence, cognition and instruction*. Ablex, Norwood, NJ.

Dufresne, R., J. Mestre, T. Thaden-Koch, W. Gerace and W. Leonard (2005). "Knowledge representation and coordination in the transfer process". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Information Age Publishing**:** 155-215.

Duit, R. (1991). "On the role of analogies and metaphors in learning science." *Science Education* **75**, 649-672.

Duncker, K. (1945). "On problem solving." *Psychol. Monogr.* **58**, 1.

Eylon, B. and F. Reif (1984). "Effect of knowledge organization on task performance." *Cognition and Instruction* **1**, 5.

Genter, D. and C. Toupin (1986). "Systematicity and surface similarity in the development of analogy." *Cognitive Science* **10**, 277-300.

Gick, M. and K. Holyoak (1983). "Schema induction and analogical transfer." *Cognitive Psychology* **15**, 1-38.

Ginsberg, H. and S. Opper (1969). *Piaget's theory of intellectual development*. Englewood Cliffs, NJ, Prentice Hall.

Gorman, R. (1972). *Discovering piaget: A guide for teachers*. Columbus, OH, Merrill.

Gray, K. E. and N. S. Rebello (2005). "Transfer between paired problems in an interview." *AIP Conf. Proc.* **790**, 157.

Hammer, D. (2000). "Student resources for learning introductory physics." *Am. J. Phys.* **68**(7), S52-S59.

Hayes, J. R. and H. A. Simon (1977). "Psychological differences among problem isomorphs". In *Cognitive theory*. N. J. Castellan, D. B. Pisoni and G. R. Potts. Hillsdale, NJ, Lawrence Erlbaum.

Hestenes, D., M. Wells and G. Swackhamer (1992). "Force concept inventory." *The Phys. Teach.* **30**, 141-158.

Holyoak, K. (1985). "The pragmatics of analogical transfer". In *The psychology of learning and motivation*. G. Bower. New York, Academic Press. **19**.

Holyoak, K., Ed. (1985). *The psychology of learning and motivation*. New York, Academic.

Holyoak, K. and P. Thagard (1995). *Mental leaps: Analogy in creative thought*. Cambridge, MA, MIT Press.

Johnson-Laird, P. (1972). *Psychology of reasoning, structure and content*. Cambridge, MA, Harvard University Press.

Johnson, K. and C. Mervis (1997). "Effects of varying the levels of expertise on the basic level of categorization." *J. Exp. Psychol. Gen.* **126**(3), 248.

Klahr, D., Z. Chen and E. Toth (2001). "From cognition to instruction to cognition: A case study in elementary school science instruction". In *Designing for science, implications from everyday, classroom and professional settings*. C. D. Schuun, T. Okada and K. Crowley. Mahwah, NJ, Lawrence Erlbaum.

Kotovsky, K., J. R. Hayes and H. A. Simon (1985). "Why are some problems hard? Evidence from the tower of hanoi." *Cogn. Psychol.* **17**, 248.

Kurz, E. M. and R. D. Tweney (1998). "The practice of mathematics and science: From calculus to the clothesline problem". In *Rational models of cognition*. M. Oalesfard and N. Chater. London, Oxford University Press**:** 415-438.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning approach." *J. Struct. Learn.* **6**, 121-130.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning model." *J. Struct. Learn.* **6**, 271-297.

Larkin, J. (1981). "Cognition of learning physics." *Am. J. Phys.* **49**(6), 534-541.

Larkin, J. H. and F. Reif (1979). "Understanding and teaching problem solving in physics." *Eur. J. Sci. Ed.* **1**(2), 191-203.

Lin, S. and C. Singh (2010). "Using analogy to solve a three-step physics problem." *AIP Conf. Proc.* **1289**, 29-32.

Lin, S. and C. Singh (2011). "Challenges in using analogies." *Phys. Teach.* **49**, 512.

Lin, S. Y. and C. Singh (2010). "Using analogies to learn introductory physics." *AIP Conf. Proc.* **1289**, 209-212.

Lobato, J. (2003). "How design experiments can inform a rethinking of transfer and vice versa." *Educ. Res.* **32**, 17.

Lobato, J. (2006). "Transfer strand: Alternative perspectives on the transfer of learning: History, issues and challenges for future research." *J. Learn. Sci.* **15**, 431.

Maloney, D. (1994). "Research in problem solving: Physics". In *Handbook of research on the teaching and learning of science*. D. Gabel. New York, MacMillan.

Marshall, S. (1995). *Schemas in problem solving*. New York, NY, Cambridge University Press.

Mestre, J. (2001). "Implications of research on learning for the education of prospective science and physics teachers." *Phys. Educ.* **36**, 44.

Mestre, J. (2002). "Probing adults' conceptual understanding and transfer of learning via problem solving." *J. of Applied Developmental Psychology* **23**(1), 9-50.

Novick, L. (1988). "Analogical transfer, problem similarity and expertise." *J. Exp. Psychol. Learn. Mem. Cogn.* **14**, 510.

Ozimek, D. J., P. V. Engelhardt, A. G. Bennett and N. S. Rebello (2005). "Retention and transfer from trigonometry to physics." *Proceedings of the 2004 Physics Education Research Conference* **790**, 173-176.

Rebello, N. S., L. Cui, A. G. Bennett, D. A. Zollman and D. J. Ozimek (2007). "Transfer of learning in problem solving in the context of mathematics and physics". In *Learning to solve complex scientific problems*. D. Jonassen. New York, Lawrence Earlbaum Associates.

Rebello, N. S. and D. A. Zollman (2005). "Dynamic transfer: A perspective from physics education research". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Infromation Age Publishing.

Redish, E. F., J. M. Saul and R. N. Steinberg (1998). "Student expectations in introductory physics." *Am. J. Phys.* **66**, 212-224.

Redish, E. F., R. E. Scherr and J. Tuminaro (2006). "Reverse-engineering the solution of a "simple" physics problem: Why learning physics is harder than it looks." *The Phys. Teach.* **44**, 293-300.

Reed, S. K., G. W. Ernst and R. Bannerji (1974). "The role of analogy in transfer between similar problem states." *Cogn. Psychol.* **6**, 436.

Reif, F. (1981). "Teaching problem solving- a scientific approach." *The Phys. Teach.* **33**, 310.

Reif, F. (1995). "Millikan lecture 1994: Understanding and teaching important scientific thought processes." *Am. J. Phys.* **63**(17).

Reif, F. and J. I. Heller (1982). "Knowledge structure and problem solving in physics." *Educ. Psychologist* **17**, 102.

Ross, B. (1989). "Distinguishing types of superficial similarities: Different effects on the access and use of earlier problems." *J. Experimental Psychology: Learning, memory and cognition* **15**, 456-468.

Schoenfeld, A. and D. J. Herrmann (1982). "Problem perception and knowledge structure in expert novice mathematical problem solvers." *J. Exp. Psych: Learning, Memory, and Cognition* **8**, 484-494.

Schwartz, D., J. Bransford and D. Sears (2005). "Efficiency and innovation in transfer". In *Transfer of learning : Research and perspectives*. J. Mestre. Greenwich, CT, Information Age Publishing.

Shapiro, M. A. (1988). "Analogies, visualization and mental processing of science stories". In *Communication yearbook*. M. McLaughlin. Beverly Hills, CA, Sage. **9**.

Simon, H. A. and J. R. Hayes (1976). "The understanding process: Problem isomorphs." *Cogn. Psychol.* **8**, 165.

Singh, C. (2007). "Effect of misconceptions on transfer in problem solving." *AIP Conf. Proc.* **951**, 196-199.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. I. Performance on nonintuitive problem pair from introductory physics." *Phys. Rev. ST Phys. Educ. Res.* **4**(1), 010104.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. II. Effect of some potential factors on problem solving and transfer." *Phys. Rev. ST Phys. Educ. Res.* **4**, 010105.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. Ii. Effect of some potential factors on problem solving and transfer." *Physical Review Special Topic Physics Education Research* **4**(1).

Sternberg, R. J. (1977). "Component processes in analogical reasoning." *Psychol. Rev.* **84**, 353.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA, Harvard University Press.

Yerushalmi, E., A. Mason, E. Cohen and C. Singh (2008). "Effect of self-diagnosis on subsequent problem solving performance." *Proceedings of the Phys. Ed. Res. Conference, AIP Conf. Proc.* **1064**, 53-36.

4.0 USING AN ISOMORPHIC PROBLEM PAIR TO LEARN INTRODUCTORY PHYSICS: TRANSFERRING FROM A TWO-STEP PROBLEM TO A THREE-STEP PROBLEM

4.1 ABSTRACT

In this study, we examine introductory physics students' ability to perform analogical reasoning between two isomorphic problems which employ the same underlying physics principles but have different surface features. Three hundred and eighty two students from a calculus-based and an algebra-based introductory physics course were given a quiz in the recitation in which they had to first learn from a solved problem provided and take advantage of what they learned from it to solve another problem (which we called the quiz problem) which was isomorphic. The solved problem provided has two sub-problems while the quiz problem has three sub-problems, which is known to be challenging for introductory students from previous research. Students in different recitation classes received different interventions in order to help them discern and exploit the underlying similarities of the isomorphic solved and quiz problems. We also conducted think-aloud interviews with six introductory students in order to understand in-depth the difficulties they had and explore strategies to provide better scaffolding. We found that students had difficulty in transferring what they learned from a 2-step problem to a 3-step problem. Although most students were able to learn from the solved problem to some extent with

the scaffolding provided and invoke the relevant principles in the quiz problem, they were not necessarily able to apply the principles correctly. The interviews suggest that students often superficially mapped the principles employed in the solved problem to the quiz problem without necessarily understanding the governing conditions underlying each principle and examining the applicability of the principle in the new situation in an in-depth manner. Findings suggest that more scaffolding is needed to help students in applying these principles appropriately. We outline a few possible strategies for future investigation.

4.2 INTRODUCTION

Learning physics is challenging. Physics is a subject in which diverse physical phenomena can be explained by just a few basic physics principles. Learning physics requires unpacking these principles and understanding their applicability in a variety of contexts that share deep features (Chi et al. 1981; Eylon and Reif 1984). A major goal of most calculus-based and algebra-based introductory physics courses is to help students learn to recognize the applicability of a physics principle in diverse situations and discern the deep similarities between the problems that share the same underlying physics principles but have different surface features.

It is well known that two physics problems that look very similar to a physics expert because both involve the same physics principle don't necessary look similar to the beginning students (Chi et al. 1981). Research has shown that when physics experts and novices are given several introductory physics problems and asked to categorize the problems based upon similarity of solution, experts tend to categorize them based upon the fundamental physics principles (e.g., conservation of mechanical energy, Newton's 2nd Law, etc.) while novices tend

to group them based upon the surface features such as pulley or inclined plane (Chi et al. 1981). Similarly, when a group of introductory physics students and physics faculty are asked to rate the problem similarities between different pairs of problems, it is found that for problem pairs which involve facial similarity but principle difference, students' rating of similarity is higher than that from the faculty members (Mateycik et al. 2009). The different patterns that experts and novices discern in these problems reflect the difference between the ways in which the knowledge structure of experts and novices is structured and how they exploit it to solve problems. The fact that experts in physics have a well-organized knowledge hierarchy where the most fundamental physics principles are placed at the top, followed by layers of subsidiary knowledge and details facilitates their problem solving process, allowing them to approach the problems in a more effective and systematic way (Johnson-Laird 1972; Bobrow and Norman 1975; Larkin 1980; Larkin 1980; Chi et al. 1981; Larkin 1981; Reif and Heller 1982; Schoenfeld and Herrmann 1982; Eylon and Reif 1984; Cheng and Holyoak 1985; Marshall 1995; Johnson and Mervis 1997). It also guides the experts to see the problems beyond the surface features, and makes the transfer of knowledge between different contexts easier.

There also has been much research effort devoted to investigating and improving transfer of learning (Duncker 1945; Sternberg 1977; Holyoak 1985; Genter and Toupin 1986; Genter and Toupin 1986; Novick 1988; Bassok and Holyoak 1989; Brown 1989; Adey and Shayer 1993; Detterman and Sternberg 1993; Holyoak and Thagard 1995; Kunz and Tweney 1998; Kurz and Tweney 1998; Bransford and Schwartz 1999; Klahr et al. 2001; Mestre 2001; Mestre 2002; Lobato 2003; Gray and Rebello 2004; Dufresne et al. 2005; Ozimek et al. 2005; Rebello and Zollman 2005; Schwartz et al. 2005; Lobato 2006; Rebello et al. 2007). In these investigations, issues about transfer of knowledge from one context to another have been discussed from

different perspectives (Duncker 1945; Sternberg 1977; Holyoak 1985; Genter and Toupin 1986; Genter and Toupin 1986; Novick 1988; Bassok and Holyoak 1989; Brown 1989; Adey and Shayer 1993; Detterman and Sternberg 1993; Holyoak and Thagard 1995; Kunz and Tweney 1998; Kurz and Tweney 1998; Bransford and Schwartz 1999; Klahr et al. 2001; Mestre 2001; Mestre 2002; Lobato 2003; Gray and Rebello 2004; Dufresne et al. 2005; Ozimek et al. 2005; Rebello and Zollman 2005; Schwartz et al. 2005; Lobato 2006; Rebello et al. 2007). The amount of knowledge a person has, the knowledge structure that the person constructs, and the context in which the knowledge is learned could all affect the person's ability to transfer knowledge acquired in one situation to another (Mestre 2001) .

One way to help students learn physics is via analogical reasoning (Chi et al. 1981; Eylon and Reif 1984). Students can be explicitly taught to make an analogy between a solved problem and a new problem, even if the surface features of the problems are different. In doing so, students may develop an important skill shared by experts: the ability to transfer from one context to another, based upon shared deep features. Here, we examine introductory physics students' ability to perform analogical problem solving. In this investigation, students were explicitly asked to focus on the similarities between a solved problem and a quiz problem and then use the analogy to solve the quiz problem. In particular, students were asked in a recitation quiz to browse through and learn from a solved problem and then solve a quiz problem that has different surface features but the same underlying physics. Different types of scaffolding were provided in different intervention groups (recitation sections). The goal is to investigate what students are able to do with the analogy provided, and to understand if students could discern the similarities between the solved and the quiz problems, take advantage of them and transfer their learning to solve the quiz problem. In a previous study (Lin and Singh 2011) in which a group of

students were asked to take advantage of what they learned from the solution provided to a solved problem (which was a 2-step problem that involves the principles of conservation of mechanical energy and Newton's 2nd Law) to solve another 2-step quiz problem in which the same physics principles come into play, we found that if suitable scaffolding was provided, students were able to reason through the analogy between two problems and performed significantly better on the quiz problem than students who were not provided with the isomorphic solved problem to learn from. In this study, the goal is to investigate if students are able to transfer what they learned from a 2-step problem to solve a 3-step problem and examine the possible scaffolding supports to help the students.

Our investigation also has overlap with prior investigations involving isomorphic problems since we focus on the effect of using isomorphic problem pairs to help students learn introductory physics. In particular, students were explicitly asked to learn from a solved problem and then solve another problem which is isomorphic. According to Hayes and Simon (Hayes and Simon 1977), isomorphic problems are defined as problems that can be mapped to each other in a one-to-one relation in terms of their solutions and the moves in the problem solving trajectories. For example, the "tower of Hanoi problem" and the "cannibal and the missionary problem" are isomorphic to each other and have the same structure if they are reduced to the abstract mathematical form (Hayes and Simon 1977). In this investigation, we call problems isomorphic if they can be solved using the same physics principles. For example, the ballerina problem in which the ballerina's rotational speed changes when she pulls her arm closer to or farther away from her body is isomorphic to a neutron star problem in which the collapse due to gravity makes the neutron star spin faster. Both these problems require the conservation of angular momentum principle to solve them, but the contexts are very different.

Cognitive theory suggests that, depending on a person's expertise in the field, different contexts and representations may trigger the recall of a relevant principle more in one problem than another, and two problems which are isomorphic are not necessarily perceived as being at the same level of difficulty especially by a beginning learner (Simon and Hayes 1976; Kotovsky et al. 1985). Changing the context of the problem, making one problem in the isomorphic pair conceptual and the other quantitative, or introducing distracting features into one of the problems can to different extent raise the difficulty in discerning the similarity and make the transfer of learning between the two problems more challenging (Singh 2008). A previous study on transfer in which isomorphic problem pairs in introductory physics were given back to back to the students suggests that those who were given both the quantitative and conceptual problems in the isomorphic pairs were often able to perform better on the conceptual problem (which was typically more challenging for them) than the students who were given only the conceptual problem alone (Singh 2008). For problem pairs that didn't involve a conceptual and a quantitative one but one problem provided a hint for the other, students typically were able to discern the similarity between the two problems and took advantage of what they learned from one problem to solve the other. However, for those problems in which the context triggered an alternative approach (which was not necessarily correct) to solve the problem (for example, in problems involving friction), the alternative view prevented the students from making a connection between the two problems. This study suggests that isomorphic problem pairs may be a useful tool to help students learn physics, but in some cases, more scaffolding may be needed (Lin and Singh 2011).

As noted earlier, the study here could also be viewed from a broader perspective of learning and reasoning by analogy. Analogy to familiar situations is a good strategy to scaffold

learning because it can help people understand an unfamiliar phenomenon more easily by creating a connection between the new information and the existing knowledge structure (Shapiro 1988; Duit 1991). Similar to Piaget's idea of accommodation process, new schema can be created by transferring the existing cognitive structure from the source domain to the target domain in which analogy comes into play (Shapiro 1988; Duit 1991). Studies have shown that using analogy can improve students' learning and reasoning in many domains (Reed et al. 1974; Novick 1988; Shapiro 1988; Ross 1989; Duit 1991). It is also a common practice for students to solve new problems by first looking for similar problems that they already know how to solve and applying similar strategies from one problem to another. As pointed out in the literature (Shapiro 1985), a good analogy not only creates an efficient connection between the new and existing information, but can also make the new information more concrete and easier to comprehend. Analogy can also be made by drawing a connection between different contexts involving similar reasoning strategies, e.g., in problems where the same physics principles are applicable, which is what we aim at here. The view of how analogy plays a role in the learning process which involves connecting the new material with the existing structure and modifying the existing cognitive structure to accommodate the new information is consonant with the view of learning which describes learning as a construction process, emphasizing the importance of prior knowledge as a basis of learning. Studies have shown that using analogy could help improve students' learning and reasoning in many domains (Reed et al. 1974; Shapiro 1985; Novick 1988; Ross 1989; Duit 1991), and it has long been an effective strategy adopted by many teachers in the practical classrooms.

Another important thread of research related to the study discussed here is that of learning from examples. Examples can serve a goal similar to that served by analogy because

they can be used to draw connection between different materials and make the unfamiliar familiar (Duit 1991). Presenting students with examples to demonstrate the meaning and application of a physics concept is a very common pedagogical tool in physics. Research suggests that at the initial stages of skill acquisition, learning can be more effective through the studying of worked out examples than the actual practice of problem solving (Ward and Sweller 1990). Because the cognitive overload is less when studying worked examples than actually solving problems, more spaces in short term memory can become available for students to extract useful strategies and to develop knowledge schemas (Paas 1992; Sweller et al. 1998; Atkinson et al. 2000). Research on learning from worked-out examples (Chi et al. 1989; Aleven et al. 1999; Atkinson et al. 2000; Chi 2000; Yerushalmi et al. 2008) (such as those in a textbook) has shown that students who self-explain the underlying reasoning in the example extensively learn more than those who don't self-explain even if the self-explanations given by the students are sometimes fragmented or incorrect. It is suggested that the largest learning gain can be achieved if students are actively engaged in the process of learning from examples (Chi et al. 1989; Aleven et al. 1999; Chi 2000; Yerushalmi et al. 2008).

4.3 METHODOLOGY

In this study, students from a calculus-based and an algebra-based introductory physics course were given two isomorphic problems in the recitation quiz. The solution to one of the problems (which we call the "solved problem") was provided. Students were explicitly asked to learn from the solution to the solved problem, point out the similarities between the two problems, explain whether they can use the solved problem to solve the other problem (which we call the "quiz

problem”), and then they were asked to solve the quiz problem. The solution provided was presented in a detailed and systematic way. It started with a description of the problem with the knowns, unknowns, and target quantity listed, followed by a plan for solving the problem in which the reasons why each principle was applicable were explicated. After the plan was executed in the mathematical representation, the last part of the solution provided a check for the answer by examining the limiting cases. A full solution to the solved problem can be found in the Appendix.

The solved problem was about a boy who took a running start, jumped onto a stationary snowboard and then went up a hill with the snowboard. The problem asked for the minimum speed at which the boy should run (right before jumping onto the snowboard) in order to go up to a certain height assuming the frictional force can be neglected. The quiz problem, on the other hand, was about two putty spheres hanging on massless strings of equal length. Sphere A was raised to a height h_o while keeping the string straight. After it was released, it collided with the other sphere B, which has the same mass; the two spheres then stuck and swung together to a maximum height h_f. Students were asked to find h_f in terms of h_o. Both the solved and quiz problems involve an inelastic collision and process(es) in which something goes up or down while there’s no work done by the non-conservative forces. Both problems can be solved using the principles of conservation of momentum (CM) and conservation of mechanical energy (CME). However, the snowboard problem can be solved by decomposing it into two steps (first the inelastic collision process, which involves the CM principle, followed by the process of the person and snowboard together going up the hill, which requires the CME principle) while the putty problem involves a three-step solution (with the CME, CM, and CME principles applicable to the processes of putty A going down, inelastic collision, and putties A and B together going up

to a maximum height, respectively.) Unlike the study in chapter 2, in which both the solved and quiz problems are two-step problems and the solutions can be mapped directly to each other, in this study, only the last two steps of the quiz problem and not the whole problem can be mapped directly to the solution of the solved problem. We note that even though the two problems may look very similar to a physics expert and both are relatively easy for them, our previous research indicates that the three-step putty problem is typically very challenging for the introductory students (C. Singh and Rosengrant 2003). The investigation in this study was designed with the expectation that providing different types of scaffolding support to students to think about the similarities between the solved problem and the quiz problem may facilitate transfer of what they learned in the two-step solved problem to solve the three-step quiz problem.

One hundred and eighty students from a calculus-based introductory physics course and 202 students from an algebra-based introductory physics course were involved in this study. In each of the courses, students were randomly divided into one comparison group and three intervention groups based on different recitation classes. There was no significant difference between any of the group in each course in terms of students' force concept inventory (FCI) score conducted at the beginning of the semester.

Students in the comparison group were given only the quiz problem in the recitation quiz. Similar to a traditional quiz, students in this comparison group were asked to solve the quiz problem on their own; no scaffolding support was provided. The performance of this group of students could help us understand what students were able to do without being explicitly provided a solved isomorphic problem to learn from.

Students in the three intervention groups, on the other hand, were given an opportunity to learn from the solved isomorphic problem during the quiz. Our previous research (Lin and

Singh) indicates that simply providing students with a similar solved problem doesn't necessary help them because students may simply follow the procedures in the solution without thinking carefully about the deep similarity of the problems. In order to help students process through the analogy more deeply and contemplate issues which they often have difficulty with, different kinds of scaffolding were provided in addition to the solved problem to the students in different intervention groups.

In particular, students in the intervention group 1 were asked to take the first 10 minutes in the quiz to learn from the solution to the solved problem. They were explicitly told at the beginning of the quiz that after 10 minutes, they had to turn in the solution, and then solve two problems in the quiz: one of them would be exactly the same as the one they just browsed over (the snowboard problem), and the other one would be similar (the putty problem.) In order to help students discern the connection between the two problems, students were also explicitly asked to identify the similarities between the two problems and explain whether they could use the similarities to solve the quiz problem before actually solving it. We hypothesized that since they had to solve the same problem they browsed over and an isomorphic problem in the quiz, students would try hard to get the most out of the solution in the allocated learning period. In order to apply what they learned from the solution to solve exactly the same problem on their own as well as the isomorphic problem, they had to not only figure out what principles to use, but also understand why and how each principle is applicable in different circumstances. We hypothesized that an advantage could be achieved over the comparison group if students in the intervention group 1 went through a deep reasoning while browsing over the solved problem as we intended. Students' performance on both problems was later analyzed.

The scaffolding in the intervention group 2 was designed based on a different framework. Students in this group were first asked to solve the quiz problem on their own. After a designated period of time, they turned in their solutions, and were given the isomorphic solved problem to learn from. Then, with the solved problem and its solution in their possession, they were asked to redo the quiz problem a second time after pointing out the similarities between the two problems and explicitly asked to discuss the implication of these similarities in constructing their solution to the quiz problem. We hypothesized that postponing the browsing over the solved isomorphic problem until the students have actually tried to solve the quiz problem on their own could be beneficial to them because in this way, students would have already searched through their knowledge base of physics and attempted to organize the information given in the quiz problem. We hypothesized that having tried the quiz problem on their own may make the browsing over the solved problem for relevant information more structured and productive before students attempted the quiz problem a second time. Even if their initial method of solution was incorrect or couldn't lead them very far, the thinking processes involved may still provide a useful framework for interpreting, incorporating and accommodating the material that they later learned from the solved problem. We hypothesized that if they got stuck in the first trial without scaffolding, this initial struggle and then browsing over the solved isomorphic problem would give them some perspective on why they were stuck and they may become more deliberate and directed in terms of what to look for in the solution. If they failed to recall a certain principle or forgot to take into consideration a certain part in the problem, the similarity between the two problems may trigger the recall of the previously inaccessible knowledge resource. Moreover, if students were not sure whether their solution was correct, the comparison between the two solutions (one provided, one their own) could also serve as a basis for examining the correctness

of their answers. The fact that the solution we provided made explicit the consideration for using the principles but was not directly the solution to the quiz problem was inspired by Schwartz, Bransford and Sears' theory of transfer (Schwartz et al. 2005), which states that two components -efficiency and innovation- are both important in the learning process. Students had the opportunity to display what they learned from the solved isomorphic problem when they solved the quiz problem a second time.

Unlike the students in the intervention groups 1 and 2 who had to figure out the similarities between the two problems themselves, students in the intervention group 3 were given a different type of hint in the quiz. They were given both the quiz problem and the solved problem at the same time. In addition to the instruction which asked them to first learn from the solved problem and then exploit the similarity to solve the quiz problem, students were explicitly told that "Similar to the solved problem, the quiz problem can be solved using conservation of momentum and conservation of mechanical energy." We hypothesized that deliberately pointing out the similar principles that should be used in both problems may guide students to focus more on the deep physics principles. Moreover, students in this group were explicitly told that they may have to use the conservation of energy twice because our previous research indicates that it's challenging for students to recognize the three-step nature of the putty problem (Lin and Singh ; Singh and Rosengrant 2003).

Students' performance was graded by two researchers using the rubrics. Summaries of the rubrics for the solved problem and the quiz problem are shown in Table 4-1 and Table 4-2, respectively. The rubrics were constructed based on the common student difficulties. Each of them consists of 2 parts based upon the principles required. Different scores were assigned in the solved problem than in the quiz problem because the former involves a 2-step solution and the

latter involves 3 steps. An inter-rater reliability of more than 80 percents was achieved when two researchers scored independently a sample of 20 students.

Students' performance in different intervention groups was later compared to each other. Moreover, in order to examine the effects of interventions on students with different expertise and to evaluate whether the interventions were more successful in helping students at a particular level of expertise, we further classify the students in each course as top, middle or bottom based on their scores on the final exam. Students in the whole course (not distinguished between different recitation classrooms) were first ranked by their scores on the final exam. About 1/3 of the students were assigned to the top, middle, and bottom groups, respectively. As noted earlier, there was no significant difference between any of the groups in each course in terms of students' force concept inventory (FCI) score conducted at the beginning of the semester. In order to take into account the possible difference which may develop as the semester progresses between different recitation classes, the overall performance of each intervention group is represented by an unweighted mean of students' performance from the three different levels of expertise. We also compared the students' performance in these algebra-based and calculus-based introductory physics courses with the performance of a group of first-year physics graduate students who were asked to solve the quiz problem on their own without any solved problem provided. The performance of the graduate students can serve as a benchmark for how well the undergraduate students can achieve as an upper limit. In addition, we also conducted think-aloud interviews with six introductory physics students (who were selected from other introductory physics classes) to get an in-depth account of their difficulties and examine the possible ways to help students. The details of the interviews will be discussed later.

Table 4-1. Summary of the rubric for the solved problem.

Description	*Scores*
Conservation of Momentum in the 1st sub-problem (5 points)	Invoking physics principle: 3 points
	Applying physics principle: 2 points
Conservation of Mechanical Energy in the 2nd sub-problem (5 points)	Invoking physics principle: 3 points
	Applying physics principle: 2 points

Table 4-2. Summary of the rubric for the quiz problem.

Description	*Scores*
Conservation of Mechanical Energy in the 1st and 3rd sub-problems (6 points)	Invoking physics principle: 2 points (1 point for each sub-problem)
	Applying physics principle: 4 points (2 points for each sub-problem)
Conservation of Momentum in the 2nd sub-problem (4 points)	Invoking physics principle: 1 point
	Applying physics principle: 1 point
	Showed relevance of work to the final answer: 2 points

4.4 RESULTS AND DISCUSSION

4.4.1 Quantitative data from the two introductory physics courses

We found that the similarities between the solved and quiz problems that the students described in the first part of their quiz solution didn't provide much information about their ability to actually solve the quiz problem. Common similarities that the students recognized include: that both problems involve an inelastic collision, that the principle of conservation of mechanical energy can be used. However, the students didn't necessarily point out how the quiz problem can be broken into different sub-problems and where should each principle be applied. Therefore, in the following discussion, we will only focus on their *solution* to the quiz problem.

Table 4-3 and Table 4-4 present students' average scores on the quiz problem in the calculus-based and algebra-based courses. Due to the instructor's time constraint in the recitation classes, the allotted time for students in intervention group 2 to try the quiz problem on their own before learning from the solved problem was slightly less than the time given to those in the comparison group. Therefore, instead of examining how intervention 2 students' pre-scaffolding performance compares to that of the comparison group, in these tables we only focus on the performance of students in intervention group 2 after the scaffolding support. Table 4-3 and Table 4-4 show that even though students in the three intervention groups received the solved problem and other scaffoldings to help them solve the quiz problem, their performance didn't show great improvement. In the calculus-based course, the comparison group students who solved the quiz problem on their own received an average score of 6.3 out of 10. The average scores of the three intervention groups were similar. Analysis of variance (ANOVA) indicates that none of the intervention groups in the calculus-based course show a statistically different

performance from that of the comparison group. In the algebra-based course, even though the scores went up significantly ($p < 0.05$) from 2.5 (in the comparison group) to 4.4, 5.4, and 5.2 in the three intervention groups, respectively, these absolute scores are not very good and there is still much room for improvement. It turns out that this problem was challenging for the calculus-based students and even more difficult for the algebra-based students. The p-values, which compared the performance of the comparison group students with various intervention group students, are listed in Table 4-5.

Table 4-3. Students' average scores out of 10 on the quiz problem in the calculus-based course. The number of students in each case is shown in parentheses. The performance of the whole group taken together is represented by an unweighted mean of students' average scores from the top, middle and bottom categories.

	Comparison (34)	Intervention 1 (37)	Intervention 2 (35)	Intervention 3 (74)
Top	8.2 (13)	9.2 (13)	8.4 (13)	8.2 (19)
Middle	6.8 (12)	6.1 (10)	8.4 (10)	6.9 (35)
Bottom	3.9 (9)	3.8 (14)	5.2 (12)	5.4 (20)
Average	6.3	6.4	7.3	6.8

Table 4-4. Students' average scores out of 10 on the quiz problem in the algebra-based course. The number of students in each case is shown in parentheses. The performance of the whole group taken together is represented by an unweighted mean of students' average scores from the top, middle and bottom categories.

	Comparison (46)	Intervention 1 (46)	Intervention 2 (62)	Intervention 3 (48)
Top	3.8 (10)	5.3 (27)	7.3 (21)	6.2 (15)
Middle	1.9 (19)	3.3 (11)	4.2 (17)	5.3 (17)
Bottom	1.9 (17)	4.5 (8)	4.6 (24)	4.2 (16)
Average	2.5	4.4	5.4	5.2

Table 4-5. The p values for the comparison of students' performance between the control group and different intervention groups in the calculus-based and algebra-based courses.

	Intervention 1	Intervention 2	Intervention 3
Calculus	0.880	0.146	0.382
Algebra	0.003	0.000	0.000

Table 4-6 shows the different answers graduate students provided when they were asked to solve the quiz problem on their own without scaffolding. The frequencies of each type of answer are listed. The 26 graduate students on average scored 9.2 out of 10 on the quiz problem, which was significantly better than students from both introductory physics courses whether or not the scaffolding with the solved isomorphic problem was provided to the introductory students. Twenty three graduate students were able to figure out the 3-step nature of the solution even though some of them erroneously used $\frac{1}{2}mv$ instead of $\frac{1}{2}mv^2$ to calculate the kinetic

energy or made mistakes related to the masses on the two sides of the equation in the 3rd step. Two graduate students believed that the total mechanical energy was conserved throughout (including all the processes), forgetting about the fact that there was an inelastic collision involved in which some mechanical energy will be transformed into other forms of energy when two objects stick together. The principle of CM was not invoked in their solution. All these mistakes that the graduate students made were present in introductory students' solutions as well.

Examination of introductory students' work indicates that forgetting to invoke the principle of CM is one of the most common mistakes introductory students made when no scaffolding was provided. Some of them simply related the initial potential energy of putty A (when it is raised to the initial height h_o) to the final potential energy of putty A and B (when both of them reach the maximum height h_f) and came up with an expression $m_A g h_o = (m_A + m_B) g h_f$ without considering the process in between. Other students took into account the intermediate process but still came up with a similar answer $m_A g h_o = 1/2\ m v^2 = (m_A + m_B) g h_f$. (Depending on the student, m and v here could stand for the mass and the speed of putty A right before the collision, or the mass and the speed of both putties together right after the collision.) Even though some students recognized that the CM principle is applicable to the collision process after learning from the solved problem, they didn't necessarily make use of it. Some of them successfully found that the speed of two putty spheres together immediately after the collision would be half of the speed of putty A right before the collision by using CM principle, but they just left it aside after that and did not make use of it later. They resorted to their original idea (e.g., $m_A g h_o = (m_A + m_B) g h_f$) to come up with the final answer. An example of the students' work is shown in Figure 4-1.

Table 4-6. Graduate students' answers to the putty problem.

Descriptions of Graduate Students' Answers	Number of students
Correct 3-step solution: $$m_A g h_o = \frac{1}{2} m_A {v_A}^2 \Rightarrow v_A = \sqrt{2gh_o}$$ $$m_A v_A = (m_A + m_B) v_{A+B} \Rightarrow v_{A+B} = \frac{\sqrt{2gh_o}}{2}$$ $$\frac{1}{2}(m_A + m_B){v_{A+B}}^2 = (m_A + m_B) g h_f \Rightarrow h_f = \frac{1}{4} h_o$$	20
Correct except that in the 3rd step, the student used $mgh = \frac{1}{2}mv$	1
Correct except that in both the 1st and 3rd step, the student used $mgh = \frac{1}{2}mv$	1
Correct except that in the 3rd step, the masses on the two sides of the equation are not consistent $mgh_f = \frac{1}{2}(2m){v_{A+B}}^2$	1
$$m_A g h_o = \frac{1}{2} m v^2 = (m + m) g h_f$$	1
$$mgh_o = 2mgh_f$$	1
Both $m_A g h_o = \frac{1}{2} m_A v^2 = (m_A + m_B) g h_f$ and 3-step solution (but in the 3rd step the student used $\frac{1}{2}(m_A + m_B) v_{A+B} = (m_A + m_B) g h_f$)	1

$$mv_A = 2mv_{AB}$$
$$v_{AB} = \frac{1}{2}v_A$$
$$KE_{fA} = KE_{iAB}$$
$$PE_{iA} = PE_{fAB}$$
$$mgh_o = 2mgh_f$$
$$h_f = \frac{1}{2}h_o$$

Figure 4-1. An example of an introductory student's answer to the putty problem. Even though the student invoked the CM principle, he didn't use this principle to find the final answer.

There were also other difficulties many introductory students shared in common. Even though they were in general able to recognize the similarity between the isomorphic problems in terms of the principles involved, many of them didn't understand the circumstances in which each principle is applicable. Many of them incorrectly combined the various sub-problems into one and applied the principles in incorrect situations. Such difficulties were commonly found in both the calculus-based and algebra-based courses even though the latter group had more difficulty. Figure 4-2 shows an example of a student's work. Instead of a correct 3-step solution, there were only 2 steps involved in the solution; the first step involved the CM principle and the other involved the CME principle. Why the student applied these two principles in the manner he did, however, is not clear. One way of interpreting the student's work is to assume that v_A and v_f stand for the speed of putty A right before the collision and the speed for both putty A and B together immediately after the collision. If this assumption is correct, the student would have applied the principle of CM correctly to the collision process but made a mistake with the CME part because the student erroneously combined the initial potential energy of putty A (when it was released) with the kinetic energy at a later instance (when putty A reached the bottom) and

set it equal to the kinetic energy of putty A+B together right after the collision plus the final potential energy of putty A+B when they reached the maximum height. The mistake of summing up potential energy and kinetic energy from different instances on one side of the CME equation indicates that the student didn't fully understand the meaning of the CME principle and he didn't know how to apply it correctly. Figure 4-3 is another example of a student's work who made a similar mistake of mixing up several processes into one and applying the CME principle to an incorrect situation.

$$\Delta\vec{P} = 0 \qquad \vec{P}_f = \vec{P}_i \qquad 2mV_f = mV_A + \cancel{mV_B} \qquad V_B = 0 \qquad V_f = \frac{1}{2}V_A$$

$$\Delta E_{mec} = \frac{1}{2}mV_A^{\ 2} + mgh_o = \frac{1}{2}2mV_f^{\ 2} + 2mgh_f$$

$$h_f = \frac{\frac{1}{2}mV_A^{\ 2} + mgh_o - mV_f^{\ 2}}{2mg}$$

$$h_f = \frac{\frac{1}{2}V_A^{\ 2} + gh_o - \left(\frac{1}{2}V_A\right)^2}{2g}$$

$$h_f = \frac{\frac{1}{2}V_A^{\ 2} + gh_o - \frac{1}{4}V_A^{\ 2}}{2g}$$

$$h_f = \frac{gh_o - \frac{1}{2}V_A^{\ 2}}{2g}$$

Figure 4-2. An example of a student's answer. The situations in which the CM and CME principles were applied were not clear.

$$PE_1 = m_A g h_o \qquad PE_f = m_{A+B} g h_f$$

$$m_A g h_o + \frac{1}{2} m_{A+B} {v_{A+B}}^2 = m_{A+B} g h_f$$

$$P_1 = P_f$$

$$m_A v_A + \cancel{m_B v_B} = (m_A + m_B) v_{A+B}$$

$$v_{A+B} = \frac{m_A v_A}{m_A + m_B}$$

$$m_A g h_o + \frac{1}{2} m_{A+B} \left(\frac{m_A v_A}{m_A + m_B} \right) = m_{A+B} g h_f$$

$$h_f = \frac{m_A g h_o + \frac{1}{2} m_{A+B} \left(\frac{m_A v_A}{m_A + m_B} \right)}{m_{A+B} \cdot g}$$

Figure 4-3. Another example of a student's work which mixed up several processes into one and applied the CME principle to an incorrect situation.

Another possible way to interpret the student's work in Figure 4-2 is to postulate that the student realized he should only combine the potential energy and kinetic energy of a system at the same moment on one side of the equation. In this case, v_A and v_f would stand for the speed of putty A when it was released and the speed of putty AB together when they momentarily reached the maximum height h_f, which would mean that v_A and v_f should both be zero. The student may then be thinking about the mechanical energy being conserved during the whole process, which can be reduced to the previously described common mistake of setting $mgh_o = mgh_f$. The student, however, would have invoked the CM principle in an incorrect situation. It is possible that the concept of the "infinitesimal" time before and after the collision involved in the CM principle was very challenging for the students. If students didn't realize that momentum of the two putty system was conserved only immediately before and after the collision, they were likely to make a mistake. Figure 4-4 shows an example of the work of another student who explicitly said that the initial momentum of the system equals the final momentum of the system where all the speeds involved were zero. This mistake suggests that the student didn't realize that the CM principle is applicable only during the collision process and not during the entire process.

$$m_A v_{Ai} + m_B v_{Bi} = v_{f\,A+B}(m_A + m_B)$$
$$m_A(0) + m_B(0) = 0(m_A + m_B)$$
$$0 = 0$$

$$\cancel{KE_i} + PE_i = \cancel{KE_f} + PE_f$$
$$PE_i = PE_f$$
$$m_A g h_i + \cancel{m_B g h_i} = (m_{A+B})(g)(h) \qquad m_A = m_B \quad \therefore m_A + m_B = 2m$$
$$(m)_A(\cancel{g})(h_i) = (2m)(\cancel{g})(h_f)$$
$$\boxed{\frac{1}{2} h_i = h_f}$$

Figure 4-4. Example work by a student who applied the CM principle to an incorrect situation.

The great difficulty many students had with this putty problem may be due to the fact that decomposing a problem appropriately into several temporally separated sub-problems as well as figuring out how the different sub-problems should be connected are extremely challenging for the students. To solve the problem correctly, students have to not only realize the 3-step structure of the problem solution, but also carefully think through the fact that the final speed of putty A in the 1st sub-problem when it reaches the bottom will become the initial speed for the collision process in the 2nd sub-problem. Similarly, the final speed of putties A and B together right after the collision in the 2nd sub-problem will become the new initial speed in the 3rd sub-problem when the two putties swing together to their maximum height. If students don't have a holistic picture of the entire process of how the speeds in the different sub-problems connect and if they don't use appropriate notation for the various speeds involved, they are likely to make mistakes. Figure 4-5 shows the work of a student who would have solved the problem correctly if the roles of v_i and v_f were switched. The common mistakes students made are summarized in Table 4-7. Overall, the data suggest that providing students with the solution to

the snowboard problem doesn't necessarily help them figure out the structure of the three part putty problem and apply the principles correctly.

$m_1 = m = m_2$

$mv_{1i} + mv_{2i} = mv_f + mv_f$

$v_{1i} = 2v_f$

$$V_f^{\,2} = \left(\tfrac{1}{2}v_{1i}\right)^2 = \left(\tfrac{1}{4}v_{1i}^{\,2}\right)$$

Before	After
$mgh_o = \frac{1}{2}m\left(\frac{1}{4}v_{1i}^{\,2}\right)$	$\frac{1}{2}mv_{1i}^{\,2} = mgh_f$
$8gh_o = v_{1i}^{\,2}$	$\frac{1}{2}(8gh_o) = gh_f$
	$\boxed{4h_o = h_f}$?

Figure 4-5. An example of a student's work which shows that the student didn't have a holistic picture of the entire process of how the speeds in different sub-problems are connected.

Table 4-7. Summary of students' common mistakes on the putty problem.

Description of Students' common mistakes	Example of students' answers illustrating the mistakes
Mechanical Energy is conserved during the whole process	$m_A gh_o = (m_A+m_B)gh_f$
	$m_A gh_o = ½\,(m_A+m_B)v^2$, $½\,(m_A+m_B)v^2 = (m_A+m_B)gh_f$
Velocity is the same before and after the collision	$m_A gh_o = ½\,m_A v^2 \Rightarrow v^2 = 2g\,h_o$, $½\,(m_A+m_B)v^2 = (m_A+m_B)gh_f$ $\Rightarrow h_f = v^2/2g = h_o$
Combining several processes into one (regardless of whether the CM principle was invoked)	$m_A gh_o + ½\,m_A v_A^{\,2} = (m_A+m_B)gh_f + ½\,(m_A+m_B)v_f^{\,2}$
	$v_{A+B} = m_A v_A/(m_A+m_B)$, $m_A gh_o + ½\,(m_A+m_B)v_{A+B}^{\,2} = (m_A+m_B)\,g\,h_f$

Table 4-8 presents intervention 1 students' scores on the isomorphic snowboard problem reproduced immediately after browsing over and returning its solution. The scores on the putty problem (the quiz problem) are listed for comparison. Similar to the findings in chapters 2 and 3, both algebra-based and calculus based students were good at reproducing the solved problem they just learned from, but the high score they achieved on the solved problem reproduced didn't imply their ability to transfer their learning to the isomorphic quiz problem. On average, the quiz problem score dropped by 3.2 and 3.9 points out of 10 in the calculus-based and algebra-based courses, respectively.

Table 4-8. Average scores out of 10 on the snowboard problem (solved problem) and the putty problem (quiz problem) for intervention 1 in the algebra-based and calculus-based courses.

	Solved Problem		Quiz Problem	
	Calculus	Algebra	Calculus	Algebra
Top	9.9	8.8	9.2	5.3
Middle	9.9	6.8	6.1	3.3
Bottom	8.9	9.4	3.8	4.5
All	9.6	8.3	6.4	4.4

Comparing intervention 2 students' work before and after browsing over the isomorphic snowboard problem indicates that many algebra-based students simply had no clue about how to solve the putty problem at the beginning. Some of them invoked the 1-D kinematics equations and weren't able to go far after that. Some of them simply wrote down some potential-energy-like or kinetic-energy-like terms separately without writing any equation. The fact that after learning from the solved problem, most of them were able to invoke either one or both of the

correct principles (which they were not able to do when asked to solve the problem on their own) is the main reason why their scores went up significantly the second time they solved the same quiz problem. The same reason explains why all three intervention groups performed significantly better than the comparison group in the algebra-based courses. However, the not very high absolute scores (around 44% to 54%) after the scaffolding also reflects the fact that the algebra-based students weren't necessarily able to apply the relevant principle correctly.

For the calculus-based course, on the other hand, students typically were able to invoke the relevant principle(s) even without being provided the solved problem. The main difficulty for the calculus-based students therefore lay in how to proceduralize these principles in an appropriate manner. As pointed out earlier, such difficulty still remained after they received the scaffolding. Although some improvement was seen among students who were able to take advantage of the snowboard problem and successfully map the last two sub-problems of the putty problem to it, many of them didn't know what to do with the 1st sub-problem that was not included in the solved problem and some just left it unattempted. Other students who struggled more weren't able to discern the three-step nature of the quiz problem or the correspondence between the quiz and solved problems. They often mistakenly thought that the mechanical energy of the system was conserved during the whole process or incorrectly combined several processes into one after browsing over the solved problem.

4.4.2 Interviews

4.4.2.1 General Description

In addition to the students from the previously discussed calculus-based and algebra-based courses who took the quiz, six students from several other introductory physics classes were

recruited for one-on-one interviews to get an in-depth account of their reasoning while they solved the problems. We also explored possible strategies to help the students during the interviews. Three of the six students we interviewed were enrolled in an algebra-based introductory mechanics course at the time of the interview; the other 3 were enrolled in 2 different calculus-based mechanics courses. The interviews were conducted in the middle of the semester, after all the relevant topics had been covered in the lectures. All the students recruited for the interviews had a midterm score which fell in the middle of their own introductory physics course, ranging from +6 to -15 points above or below the class averages (which fell between 70% and 76% for different sections of the courses). The audio-recorded interviews which were typically 0.5-1 hour long were carried out using a think-aloud protocol.

During the interviews, students were asked to learn from the solved problem provided and solve the isomorphic quiz problem given. Similar to the previously discussed quiz situation, different students in different interviews received different kinds of interventions. Some of the interventions were the same as the previous interventions used in the quantitative data discussed in section 4.4.1. Some of them were new in the sense that a slight modification was made to the interventions used earlier. For example, in the interviews with students E and F, we examined the effect of a modified intervention which added another problem (the “two-block problem” shown in the Appendix) as a bridging problem to help students solve the putty problem involving three parts. This new two-block problem consists of only two steps: an object going down, colliding, sticking and moving together with another object on the horizontal part of the track. This bridging problem is a 2-step problem which is very similar to the solved problem except that the processes are reversed. After students realized how to solve the new bridging problem correctly, we then asked them to take advantage of what they learned from these two problems to solve the

3-step putty problem. We hypothesized that after the students understand how to solve the snowboard problem and the two-block bridging problem, they will have a better idea of the three processes involved in the putty problem and they may be able to construct a holistic picture of how the different sub-problems should be connected. The different interventions students received are listed in Table 4-9.

As noted earlier, the interviews were conducted using a think-aloud protocol, which allowed the researchers to follow and record their thinking process. Students were asked to perform the task (whether they were reading the solved problem or trying to solve the quiz problem) while thinking aloud; they were not disturbed during the task. All the questions were asked to the students after they were completely done with the problem solving to the best of their abilities.

The interviews focused not just on understanding the difficulties students had, but also on examining the scaffoldings that may be helpful for the students. In the interviews, after the students completed the quiz while thinking aloud, the researcher would first ask clarification questions in order to understand what they did not make explicit earlier and what their difficulties were. Based on this understanding, the researcher then provided some guidance (sometimes including the physics knowledge required) to the students in order to help them solve the quiz problem correctly if they had not done so. The researcher also outlined or even demonstrated part of the solutions to the students if needed. After helping students learn how to solve the quiz problem correctly, the researcher invited them to reflect on the learning process they just went through (for example, by asking explicitly what was the thing that helped them figure out how to solve the problem) and provide some suggestion from the student's own perspective on how to improve students' performance on the quiz problem. The goal of the

students' reflection was to help us identify the possible helpful scaffoldings not only based upon what the researchers observed but also based upon students' reflection of their own learning.

Table 4-9. The interventions students received in the interview.

Student A	Intervention 3
Student B	Intervention 3
Student C	Intervention 2
Student D	Intervention 2
Student E	Two quiz problems (version 1)
Student F	Two quiz problems (version 2)

* Two quiz problems (version1): (1) The student first learned from the solved snowboard problem provided and then solved another problem about "two blocks colliding" (with the solved snowboard problem in his hand) (2) The researcher discussed with the student how to solve the "two-block problem" correctly (3) The student was asked to take advantage of what he learned from the previous two problems to solve the putty problem.

* Two quiz problems (version 2): (1) The student first learned from the solved snowboard problem provided and then solved the two quiz problems (the two-block bridging problem and the putty problem) with the solved problem in his hand (2) The researcher discussed with the student how to solve the two block problem correctly (3) The student was asked to take advantage of what he learned from the previous two problems and attempted to solve the putty problem the second time.

4.4.2.2 Interview Results

Similar to the quantitative data presented in the previous section, we found that the putty problem was very difficult for the students interviewed. The difficulties that these interviewed students had fell into the same categories discussed previously from the in-class administration of the quiz. In the following section, we will discuss the difficulties interviewed students had when solving the putty problem by focusing on two aspects: (1) the general problem solving approach and (2) the specific physics knowledge related to inelastic versus elastic collision as well as the CM principle that the students had. Some quotations from the interviews will be presented. We will also discuss the effect of different kinds of scaffolding support we provided to these students during the interview, not only from the researcher's point of view, but also from the students' own perspective as much as students were able to articulate it.

General Problem Solving Approach

From the interviews, we found that the students in general didn't systematically come up with a plan for solving the problem before implementing the plan. Many of them first wrote down the principle they "believed" should be used (because the same principle was shown in the given solved problem) and then tried to plug in some variables from the new situation (the quiz problem) in order to solve for the target variable. They didn't carefully examine whether the same principle could be used again in the new problem (and if it can, in which situation should the principle be applied.) This tendency of blindly mimicking the solved problem was prevalent when they had no clue how to solve the quiz problem. Because the solution to the quiz problem had many differences compared to the solved problem, such strategies didn't get them too far; they soon encountered problems and were confused. For example, by looking at the solved problem provided, some students quickly wrote down the conservation of momentum principle

as their first equation to solve the quiz problem. However, after that they didn't know what to do with that equation and didn't know how to connect it to the target variable. They just left it aside and started working on the conservation of mechanical energy equation without coming back to their original work with the momentum principle, which is similar to the tendencies we've observed in the quantitative data.

A similar situation involving not knowing how to exploit the principle in the solved problem and apply it to the quiz problem occurred in the CME part as well. If the potential energy is chosen to be zero at the lowest point in both problems, the fact that the solved problem starts with some kinetic energy and no potential energy confused the students because the quiz problem started with a different situation involving non-zero potential energy but no kinetic energy. Although the students tried to manipulate the equation to come up with an answer, (see Figure 4-6 for an example, in which one student crossed out the PE_i and KE_f in his CME equation similar to what the solved problem did), one interviewed student explicitly said "*I think this is weird. Uh….I don't think this would be correct*" while writing down the equation. The following is a quote from one of the interviewed students when the researcher asked him to talk about whether the two problems look similar to him and whether the solved problem was useful to solve the quiz problem after the student had completed attempting to solve the quiz problem. It suggests that the student couldn't discern the deep similarities between the two problems.

Student B: I think they are sort of similar. They both use the conservation of energy equations. But this one here...um, you are solving more for, at least in my mind, you are solving more for…um...something. It was happening with… It seems like in this one [solved problem] you had everything start off at zero [potential energy] whereas in this [quiz problem], the thing does not start off at zero because that [putty A] was released, and this one [solved problem] thing starts

off at zero in the sample problem since he[the person in the snowboard problem] starts off at zero. I don't know. They look…they are similar… because the same theorem they use, but it just seems to me that it didn't really help. But it did a little bit by letting me see the equation that I need to use. But as far as like plugging in things, it doesn't help me too much in that regard.

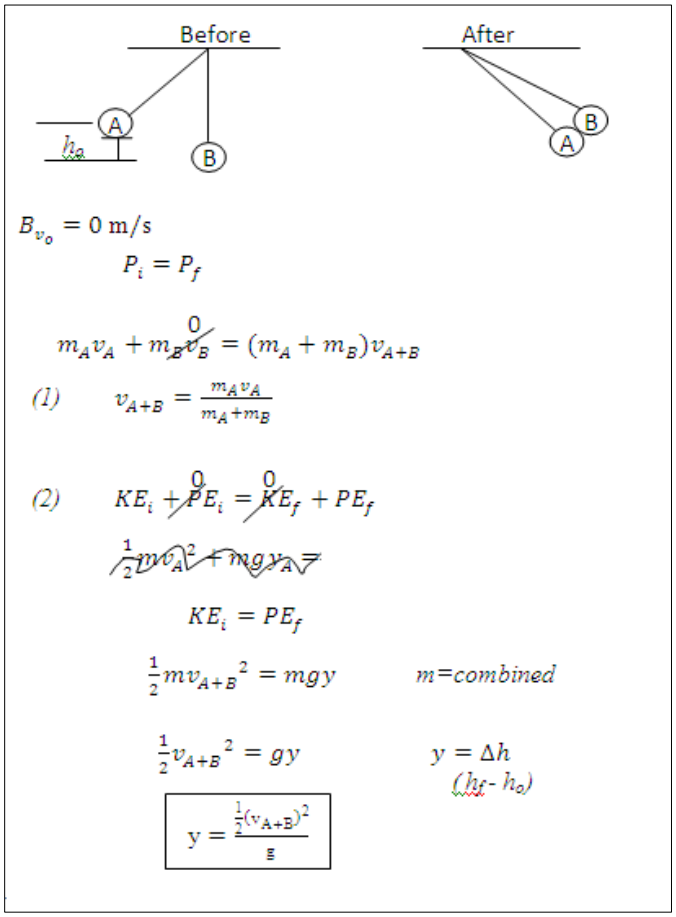


Figure 4-6. Students A's answer to the putty problem.

In addition, we found that students in general failed to describe the meaning of each variable they used in the solution in a precise way. This difficulty might have to do with the fact that students didn't have a clear picture in mind about what was going on in the quiz problem and they didn't necessarily think through the problem in sufficient depth. When the researcher asked students to explicate what each variable, especially the various "*v*"s (for example, in the

equation of $m_A v_A + \cancel{m_B v_B} = (m_A + m_B) v_{A+B}$) mean, they often answered that v_A was the velocity "before collision", and v_{A+B} was the velocity "after collision". It was not easy for them to articulate by themselves which velocity before collision they were talking about (e.g., whether it was the velocity of sphere A "right before the collision", or the velocity of sphere A "at the very beginning when it was released"). It was possible that the students initially didn't recognize that in both "before the collision" and "after the collision", there were processes involved in which the speed of the sphere(s) varied with the height. Such difficulties in articulating what each variable in their answer meant if no specific guidance was provided were commonly observed in the interviews. Sometimes it took the researcher some effort to explain to the students that "velocity before/after the collision" could mean many different things since at different heights the velocities were different. As suggested by one student during the reflection, the idea of "snapshots of the putty at different points" was very helpful in solving the problem. However, the student was able to articulate this idea only after the discussion with the researcher.

Specific knowledge related to the physics principles and collision process

Another possible reason why the students couldn't articulate their variables as precisely as the researcher would have hoped may be that the students didn't fully understand the applicability of each principle. As pointed out in the previous discussion of the quantitative data, many students didn't realize that the CM principle should be applicable only at times right before and after the collision. All students during the interview were asked explicitly to identify what their variables v_i and v_f in the CM equation refer to. They were sometimes given the following choices to help them articulate the meaning of the different "v"s and the situation in which their CM principle was applied: (1) the very beginning to the very end (2) right before the collision to right after the collision (3) somewhere in between (1) and (2). One student responded in the following manner:

Student B : Wouldn't they...Wouldn't the total momentum for the system be the same throughout? Or would [it] not be?

Researcher: What do you think?

Student B: I think it would be.

Researcher: OK. So…why?

Student B: Um…just because from what I heard for the conservation of momentum, from what I've been told about, momentum is conserved throughout…Uh…yea, no matter what time it is, whether it's t [equals] zero or t [equals] infinity, the momentum should be conserved throughout.

Researcher: OK. Do you remember…when your professor taught you about conservation of momentum, did he or she say when should that principle be applicable?

Student B: Um…I believe…it may have been not applicable for inelastic collision but I don't really remember. And I believe it [the solution to the snowboard problem] says that this was inelastic… Um…I believe you would use it [conservation of momentum] more in elastic collision than you would in inelastic collision.

The dialogue above suggests that when the student exploited the CM principle to solve the quiz problem, he didn't carefully examine the applicability of the CM principle by considering whether or not there was an external force acting on the system. In fact, the interaction between the researcher and the student suggests that it is very likely that the student didn't know how to do so. When the researcher asked him about the situations in which the CM principle could be applicable, the student discussed the applicability incorrectly based upon whether the collision was elastic or inelastic.

Another situation in which students focused more on the "surface features" rather than the "governing conditions" to determine whether the CM principle should be applicable was observed when the researcher asked students to identify whether the collision involved was an elastic or inelastic collision and describe the difference between the two. Instead of mentioning the definition of an elastic (inelastic) collision as a collision in which mechanical energy is (is not) conserved, most students discussed the difference between two collisions by some surface feature such as (i) whether the object keeps its original shape and/or (ii) whether the two objects move apart or become one after the collision, but not the different implication for mechanical energy conservation. The followings are two examples:

Researcher: Can you tell me…uh…is it [the collision in the quiz problem] an elastic collision or inelastic collision?

Student A: What's the difference? I forget. Elastic…I remember in the class it was like… it bounces back… and I guess inelastic…it would be… it doesn't bounce back…it stick together?

Researcher: Do you remember if this collision is an elastic or inelastic collision?

Student C: Uh. Yea. Elastic means that they come together and then they can retain their shape, right?

Researcher: Retain their shape?

Student C: Yea…like… when they hit together, then when they come ap[apart]…they can come apart, and so be the same shape as they were when they came into collision.

Researcher: Ok.

Student C: And then… and… inelastic forces they hit and they are one object.

Researcher: do you mean they stick…

Student C: Yea they stick together and they stick together throughout the system.

Researcher: Ok. So do you still remember which principle is applicable during the collision process and which is not? I mean, is there any difference between these two kinds of collision?

Student C: Uh…I mean… there is. I just...I just don't know.

Researcher: How about… let's think about the momentum?

Student C: Yea, it would be the difference between…like…. conserves momentum and [does] not conserves momentum? Would that be it?

Researcher: What do you think?

Student C: I… I… I actually don't know.

Researcher: Ok. So.… momentum is conserved in both kinds of collisions.

Student C: Oh, really?

Researcher: Yea.

Student C: Ok.

Researcher: So do you still remember anything about energy?

Student C: I do. I… I remember like… kinetic energy equ[equation]… uh... conservation of energy equation, just 'cause that was what we were taught a lot about. And then, we are taught a little bit about momentum, but… it shoots off very quickly. And I haven't taken physics before, so it's all new to me. So…

Researcher: Yea, I understand that. Don't feel bad… I mean… we know the [CM] principle is very difficult for lots of students.

Student C: OK.

Researcher: So…um… you are not quite sure whether energy is conserved during…

Student C: between elastic and inelastic?

Researcher: Uh huh. Do you remember anything about that?

Student C: Uh...energy is conserved in a…inelastic collision? Or? Uh….Let me think about this.

Researcher: Yea, take your time.

Student C: Uh….Uh….I guess energy is… conserved in inelastic collision? Yeah, I think so.

Researcher: So is energy conserved in an elastic collision?

Student C: Uh…is energy conserved in an elastic collision…

[silence]

Student C: Uh...I attempt to say no. But at the same time I don't think so. I don't think it's yes. I'm going to go with no because once you hit it, like you… like for this object, when the one ball hits the other ball, and if it were inelastic, they come together, and all the energies is gonna fall in with them. Whereas if the one hits, this one is still on the, like going up a little bit…but this one is also going up, so…uh…I guess energy is … I'm going to go with energy is conserved in both.

The example above shows that when the researcher asked student C about the specific principles that were applicable or not applicable in the elastic and inelastic collisions, student C didn't know the correct answer and was unable to make up his mind. Such difficulties were found among other interviewed students as well. A quote from another student is listed below. Out of the six students we interviewed, only one could correctly answer that both CM and CME principles are valid in an elastic collision but only the CM (not the CME) principle is valid in an inelastic collision.

Student F: There's one where you can use conservation of momentum and energy. And there's another one where you can use one of those. But I don't remember which one [elastic or inelastic] it goes to and which one [CM or CME] works.

Scaffoldings that may be helpful for the students

As mentioned before, one important goal of the interviews was to examine the additional possible scaffoldings that could help the students solve the quiz problem correctly. The different scaffoldings we tried in the interviews included:

< i> Telling students explicitly that the problem can be decomposed into 3 parts.

<ii> Directing students' attention to the fact that energy is not conserved in the inelastic collision.

<iii> Directing students' attention to the fact that the CM principle is only valid right before to right after the collision.

<iv> Helping students learn how to solve a simpler 2-step problem (the bridging two-block problem) first.

Depending on the prior knowledge and difficulties students had, different students received different levels of scaffolding in the interviews to help them solve the quiz problem correctly. As we discussed in the previous sections, even though students could recognize the inelastic collision process involved, they didn't necessarily understand that during an inelastic collision, some mechanical energy would be transformed into heat or other forms of energy and therefore the mechanical energy of the system was not conserved. For some students, just asking them to identify the type of the collision (elastic or inelastic) and think about whether the CME principle was valid or not in an inelastic collision was enough to help them recognize their mistakes related to applying the CME principle during the collision process. For other students,

we had to provide more direct guidance related to the physics knowledge they lacked so that they could understand what they did incorrectly.

However, understanding that the mechanical energy was not conserved during the inelastic collision process didn't guarantee success. If the students didn't have a holistic picture of the complete problem, they were still easily lost. In the quantitative data discussed in the earlier section, we found that the additional hint about "using the conservation of mechanical energy twice" in intervention 3 didn't help students much. The same phenomenon was observed in the interview as well. When one student was told by the researcher (after he had tried the putty problem on his own) that the snowboard problem was a 2-part problem and the putty problem was a 3-part problem in which we had to use the CME principle twice, he thought that the 1st step to solve the putty problem was to use the CM principle similar to what he did in his original work. He felt that the instruction meant that the putty problem must be solved by using CM first, followed by the use of CME twice. He still incorrectly interpreted that it was the last CME part which would lead him to express h_f in terms of h_o that he missed in his original work. The original work done by this student is shown in Figure 4-6. This interview suggests that students might interpret the instruction of "using CME twice" in a different way than what was intended, which could be one possible reason why providing the additional instruction that may be considered a huge hint by the experts in intervention group 3 didn't work very well for the students.

Another scaffolding support required to help students solve the putty problem correctly involves helping them realize that the CM principle can be applied for only a short period of time during the collision since this was one of the most difficult hurdles students had. Directing students to go back to the solved problem and figure out the situation in which the CM principle

was applicable didn't help much because they didn't necessarily ask themselves why this principle was applicable here, and more importantly, why wasn't it applicable to some broader situation which contains non-zero net force on the system. As an earlier quote suggests, the student drew from his previous experience that the CM principle was always valid from t=0 to t=∞. It is possible that the student didn't examine the condition for the applicability of the CM principle because he didn't know how to do so. However, if we use the solved problem to help students learn that when the snowboard goes up the ramp, the net external force acting on the system is not zero and the velocity of the snowboard keeps changing, so the momentum couldn't be conserved, they are likely to realize that in the quiz problem, the momentum principle should be applicable only for right before and right after the collision.

In the interviews with students A to D, in which we didn't use the solved problem to discuss the issues related to the applicability of the CM principle, unless the students received an explicit instruction in "applying the CM principle only right before and right after the collision in the putty problem", they were not able to solve the putty problem correctly. Some of them required significant help from the researcher, e.g., in breaking the whole problem into sub-problems in which the target in each sub-problem was specified after they attempted to solve the problem to the best of their abilities. In the interviews with students E and F, however, we found that after enough discussion and explanation about why and how each principle was applied in the way shown in the solution to the snowboard problem, and after the students understood how to solve a bridging problem (i.e. the two-block problem) correctly following the discussion, they could take advantage of what they learned from the two problems (snowboard problem and the two-block problem) and correctly solved the putty problem on their own. The critical scaffolding provided was the help in recognizing the similarities between the snowboard problem and the

two-block problem and understanding how to solve the two-block problem correctly. One student required more help in understanding that something going down in the two-block problem is in principle the same as something going up in the solved snowboard problem. However, after students E and F recognized this similarity, understood that the mechanical energy of the system is not conserved during the inelastic collision, and realized why CM is not valid throughout the whole process in the snowboard problem or the two-block problem, the putty problem was not as difficult for them as for other students. After the scaffolding, without the help from the researcher, they themselves recognized that the putty problem should be split into 3 sub-problems and CME, CM, and CME should be applied to the three consecutive sub-problems, respectively. Although the fact that the final velocity in one sub-problem becomes the initial velocity in the next sub-problem was somewhat frustrating for them, they had a clear picture of the whole solution process, and the issue of how the different velocities in the sub-problems should be connected didn't seem to be difficult for them.

When students were asked to reflect upon what they considered to be beneficial in their physics learning so that they would be able to solve the putty problem or other physics problems, some students believed that going through the problem with them would be helpful:

Student F: I think like…just going through a specific example like this in class… like doing it together on the board. 'Cause my professor never does that…he kinds of just explains… like… in the fine stuff. He never like... goes through like a specific problem [when] he has really solved it. He does like... he does go through… he goes through like deriving equation for us… But I just get really confused like when to use those equations… that kind of stuff…

Similar to what student F said, most students pointed out in the interview that they were more interested in understanding how to apply the principles than in the derivation of the principle itself. They explicitly said that they wanted to learn something more "practical" (for example, they would appreciate if the instructor could tell them explicitly how to use a principle in different situations) rather than learning some theoretical details. This preference toward the practical applications is manifested in the following quotation.

Researcher: So…is there anything else that you think is important? If I somehow provided some help for you, then what was that thing? I mean…

Student C: Uh…what else could help me solve this problem?

Researcher: Yea

Student C: Uh…

Researcher: For example, from your initial work to…

Student C: Yea. The whole explanation of uh… like in elasticity and inelasticity… just kind of very simply… uh… by saying that… like when you have an elastic collision that momentum is conserved… and that energy is conserved. And then when you have inelastic collision, your momentum is conserved. Just… like… just saying that… that would help me. But I think he did that in…. uh… my professor is [XXX], I think he did that. But uh, he went into it in much more detail about it. And sometimes some of the detail is good for… theory? But I mean… most of the time when we do homework, I don't really think about the theory behind them. I kind of think of the application. So that's… I think… just more simple explanations like…

Researcher: So you mean sometimes (he) gives you too much detail….

Student C: Yea.

Researcher: What kind of detail are you talking about? Can you give me an example?

[Student C is trying to think]

Student C: I guess I can give you an example. I can give you… uh… like how do you derive all the formalism things… I mean… I think it is very good for our math perspective, and I guess that's part of the reason why it's considered physics for science and engineering. But at the same time, they never ask you to derive, you know, derive the equation for force or Newton's 2nd Law. So, you kind of, it's… it's information that really isn't nece[necessary]. It's necessary for if you're like curious about it, but it's not necessary for practice. And I think the more you kind of want using it more practically so that… it makes it less confusing but at the same time, it's directly to the point. And then you know "yea to use this when it's this."

In addition to wanting more "practical applications of the principles" in the class, a student added that having multiple-part problems in the class would be beneficial to their physics learning. He reflected on his experience of solving the putty problem and identified the multiple parts of the problem as the main source of difficulty for him.

Student D: I have… like I understand the concepts, and I understand like when we learn it in class. I just don't understand how do I... I guess like thinking about it in terms of separate steps, I might be able to understand the problem. I think… I think I tried to think of it as a whole without breaking it down, which is why I can't solve it.

Researcher: OK. So you mean, when I tried to break the problem into several parts [for you]

Student D: Yeah, breaking them into parts, and then… yea… because… like I understand everything. I understand conservation of momentum. I understand mechanical

energy. But I don't understand it when it's all thrown on me. I get lost. I think that's the problem.

Researcher: OK. That [point] sounds good. And… oh, I mean… just add on what you said… because we know from research that knowing when to apply the principles is a very difficult part for students, but…

Student D: And I think that professors have such… like a mind set of getting completely through all the chapters instead of understanding like how…. like instead of explaining how you know when to apply each concept... sort of.

Researcher: OK. So… um… did he or she try to explain when a principle will be applicable?

Student D: Sort of, but not really. Like, like we did sample… uh examples in class

Researcher: Can you give me an example?

Student D: Like…he didn't tell anything like this where we have to combine everything. Like… we did a sample… uh an example where we have to use conservation of momentum. We did a sample, or example where we have to use conservation of mechanical energy. But we didn't do, or at least I don't think we did… we may have... but I don't think we did an example where we needed to use the conservation of momentum and…

Researcher: OK

Student D: You get it, whatever.

Researcher: So you mean because there's only one principle involved in…

Student D: Right.

Researcher: So when you see the problem you know that's the thing you need to apply?

Student D: Yea.

Researcher: And then... so… the problem doesn't really tell you to examine whether these principles should be used.

Student D: right.

As student D pointed out, a multi-step problem in which several principles were combined was very difficult for him even if he understood each individual principle separately, especially because the instructor did not go over multiple-step problems in class. Moreover, not having any multiple-step problem in the class may be disadvantageous to his learning of physics principles because he didn't have an opportunity to see the instructor demonstrating how to examine the applicability of a physics principle in a multiple-step problem or got an opportunity to practice it himself.

Based on the students' own reflection and the observation from the interviews mentioned above, we found that there was a need for teaching students the applicability of principles in a more effective way in the physics classrooms. As most students suggested, they would like to receive more practical guidance from the lectures so that they could learn when and how to apply the principles. However, instead of spoon-feeding the students by listing all the situations in which a certain principle would be applicable, one strategy that may allow for better transfer is to constantly demonstrate to the students how to examine the applicability of a certain principle. For example, after showing that the net external force on the system $\vec{F}_{\text{external}} = \vec{0}$ implies momentum conservation, students can be given several examples and asked to discuss conceptually why the principle they just learned can or cannot be applied in those situations. The instructor can coach the students to examine whether the governing condition (such as $\vec{F}_{\text{external}} = \vec{0}$) is met in each example which can help students develop the habit of verifying the applicability of a principle based upon the deep features. It will be useful to have examples of

situations in which the principle is valid and also those in which the same principle is not valid. The putty problem, for instance, could be used as a good example for discussing the applicability of the CM principle.

In addition to emphasizing the importance of examining the applicability of a principle in class, instructors could also explicitly ask students to write down their reasoning for applying a certain principle in the homework problems so that the students have an opportunity for further practice. Moreover, if the instructors adopt the analogical problem solving activity discussed in this study (e.g., to help students transfer their learning from the snowboard problem to the putty problem), it can be useful for them to guide students to think about certain issues related to the solved problem before students start solving the quiz problem. A list of important issues for discussion related to the solution of the snowboard problem which may be beneficial for the students include: (a) When is the CM principle applicable? (The instructors may give students several choices including “from immediately before to immediately after the collision”, “from the very beginning to the very end when the person reaches the maximum height”, etc.) (b) Why isn’t the CM principle applicable elsewhere in the problem? Could it be applicable elsewhere? Why or why not? (c) From where to where in the problem is the CME principle applicable? Why do we apply CME only in this situation? Can we go beyond that and apply it throughout the whole process? (d) What is an appropriate system for applying each of these principles? Such questions could keep students more actively involved in the learning process from the solved example and help them benefit more from the self-explanation process as discussed in Chi’s study (Chi et al. 1989). In summary, if the classroom is designed to focus on contemplating the applicability of the physics principles, and not simply on how to execute them, students may benefit more.

4.5 SUMMARY AND FUTURE OUTLOOK

In this study, we find that it is challenging for students to transfer what they learned in the 2-step snowboard problem to solve the 3-step putty problem. When students were asked to learn from the solved snowboard problem provided and take advantage of what they learned from the snowboard problem to solve the putty problem which is isomorphic, only students in the algebra-based course benefitted from scaffolding supports provided. However, examination of their absolute scores on the putty problem suggests that there is still much room for improvement. Findings revealed that the greatest difficulty students had in transfer was in applying what they learned from the solved problem in an appropriate way to the new situation presented in the quiz problem. Even though the solved problem could help students invoke the relevant principles in the quiz problem (which is the main reason why in the algebra-based course, students who received the scaffolding of the solved problem outperformed students in the comparison group), many students didn't have a clear plan for how to solve the quiz problem. They didn't realize how to decompose the quiz problem into suitable sub-problems and they sometimes combined several processes into one, applied the principles in inappropriate situations, or applied the principles correctly but didn't discern their relevance to the final answer (target variable). For calculus-based students, many of them were able to invoke the relevant principles even without learning from the solved problem and the greatest difficulty was in applying the principles correctly. The scaffolding supports provided didn't help them much in this regard.

A previous study (Yerushalmi et al. 2008) suggests that if the target problem was also a 2-step problem, students who self-diagnosed their own mistakes in the snowboard problem were capable of transferring their learning to solve an isomorphic 2-step problem even after a time delay of 1-2 weeks. In the study presented here, we focused on the effect of immediate transfer

by providing students with a solved solution to learn from and found that transferring from a 2-step problem to a 3-step problem was not easy. Similarly, comparing students' ability to transfer in this study to that in chapter 2 (in which both the solved and quiz problems were 2-step problems), we find that even though the problems in chapter 2 required the application of Newton's 2nd Law in the non-equilibrium situation, which is typically challenging for students, on average students displayed better transfer for the case discussed in chapter 2. The fact that in this study, the solved problem provided was a two-step problem whereas the targeted problem was a three-step problem made the transfer very challenging. With the existence of an additional step in the quiz problem, students could no longer map the solved problem directly to the quiz problem. They had to learn from the solved example and understand the circumstances for which each principle is applicable, so as to be able to systematically decompose the problem into several sub-problems (that can be dealt with one at a time with a single principle). The interviews suggest that students often superficially mapped the principles employed in the solved problem to the quiz problem without necessarily understanding the governing conditions underlying each principle and examining the applicability of the principle in the new situation in an in-depth manner.

Findings reveal that figuring out the 3-step structure of the quiz problem was an extremely difficult part for most students. Even intervention 3 students in the calculus-based course who received an explicit hint about "applying the conservation of mechanical energy twice" had great difficulty figuring out the correct process to solve the quiz problem. Prior research suggests that the perceived complexity of a problem depends not only on its inherent complexity but also on the experience, familiarity, and intuition the problem solver has built about a certain class of problems (Singh 2002). Two problems which look very similar for a

physics expert may not look similar to the students. Moreover, the fact that experts (graduate students) performed better on one quiz problem than the other (e.g., the graduate students performed better on the putty problem in this study than on the tire swing problem in chapter 2) doesn't necessarily mean that the introductory students will perceive the problem complexity in the same way.

Our research suggests that in order to help students perform better on the transfer problem, more scaffolding may be required. However, the idea is not to spoon-feed them; rather, the dimensions of efficiency and innovation as described in Schwartz, Bransford and Sears's model are both important for transfer (Bransford and Schwartz 1999; Schwartz et al. 2005). Students should be actively engaged in the analogical reasoning process themselves and in reconstructing, organizing and extending their knowledge structure. It is possible that if students are guided to think about the solution in more depth and contemplate the applicability of various principles in the solution, they are more likely to benefit from the solved problem provided. One possible way to guide students' self-explanation toward this goal is suggested in the interview part of this study and can be investigated in-depth in the future. In particular, if suitable questions are designed about the applicability of the principles used in the solved problem and students are asked to justify why some physics principles are applicable in certain situations before they solve the quiz problem, they may learn better from the solved problem provided.

For students who couldn't figure out the 3 step structure of the solution and combined several processes into one when solving the putty problem, one possible strategy to help students with this specific difficulty is to add a bridging problem. After students learn from the solved snowboard problem, we can ask them to solve a bridging problem (such as the two-block problem discussed in the interview section) first before they solve the putty problem. As we

found in the interviews, if the students understand why and how the CM and CME principles are applicable in the snowboard problem and the two-block problem, why the CM principle isn't valid for other sub-problems before or after the collision, and why the CME principle isn't applicable throughout the whole process in the two problems, they are likely to solve the putty problem correctly on their own.

Additional scaffoldings may also be designed to help students with specific difficulties. For students who believe that the mechanical energy of the system is conserved throughout the whole process in the putty problem, it may be helpful to explicitly ask them to discuss the kind of collision that is involved and the implication it has on the total linear momentum as well as total mechanical energy of the system if the students know (or are guided to) the correct answers to those questions. However, as found in the interview, not all students realized that the CM principle is applicable for both elastic and inelastic collisions and CME is valid only for the elastic collision. If the students understand that they can use the CM principle for the inelastic collision process involved but they are not sure about the CME principle, instead of simply telling them the answer, an intervention could deliberately direct students to think about both the momentum and mechanical energy right before and right after the collision (for example, by asking them to compute the speeds and kinetic energies at these two instances and compare whether their results are consistent with the predictions they made for both conservation laws). It is possible that by doing so, they are more likely to recall the fact that some mechanical energy will be transformed into other forms of energy and they cannot simply set the initial potential energy of one putty sphere equal to the final potential energy of both putty spheres together without contemplating the collision process in between.

For students who have difficulties understanding that the CM principle is applicable only from right before to right after the collision, the putty problem itself could be used as an effective tool to draw students' attention to this issue. Students could be taught to consider the velocities (both the direction and magnitude) of the putty spheres and the forces acting on them at various points conceptually. As discussed in the interview section, it is important to help students learn and develop the ability to examine the applicability of the physics principles whenever they encounter a new problem. Demonstrating how to examine the applicability of the principles by using the putty problem as an example and discussing with students why a certain principle may or may not be applicable to each of the sub-problems could be a useful strategy for helping them understand the applicability of each principle. It may also be useful to investigate if students are able to transfer what they learned from the 3-step solved example to solve another 2-step problem by making the "putty problem" the solved problem and the "snowboard problem" or the "two-block problem" the quiz problem in future studies.

In summary, deliberately using an isomorphic worked out example to help students transfer what they learned from one context to another can be a useful tool to help students understand the applicability of physics principles in diverse situations and develop a coherent knowledge structure of physics. For introductory students, such well-thought out activities could provide a model for effective physics learning since the idea of looking at deep similarities beyond the surface features is enforced throughout these activities. However, it can be challenging for students to correctly apply what they learned from a 2-step problem to solve a 3-step problem. More scaffolding supports that are commensurate with students' prior knowledge may be required to help them realize the structure of the solution and to learn from the solved example effectively. It can be beneficial if the importance of looking for governing conditions

underlying each principle and examining the applicability of the physics principles in the new situation in an in-depth manner are consistently explained, emphasized, demonstrated and rewarded by the instructors. It is possible that students will become more facile at the analogical problem solving processes if practice and feedback are constantly provided to them throughout the whole course.

4.6 CHAPTER REFERENCES

Adey, P. and M. Shayer (1993). "An exploration of long-term far-transfer effects following an extended intervention program in the high school science curricula." *Cogn. Instruct.* **11**, 1-29.

Aleven, V., K. R. Koedinger and K. Cross (1999). "Tutoring answer explanation fosters learning with understanding." *Proceedings of AIED* **99**, 199-206.

Atkinson, R. K., S. J. Derry, A. Renkl and D. Wortham (2000). "Learning from examples: Instructional principles from the worked examples research." *Review of Educational Research* **70**(2), 181-214.

Atkinson, R. K., S. J. Derry, A. Renkl and D. Wortham (2000). "Learning from examples: Instructional principles from the worked examples research." *Rev. Educ. Res.* **70**(2), 181-214.

Bassok, M. and K. Holyoak (1989). "Interdomain transfer between isomorphic topics in algebra and physics." *J. Experimental Psychology: Memory, Learning and Cognition* **15**(1), 153-166.

Bobrow, D. and D. Norman (1975). "Some principles of memory schemata". In *Representation and understanding: Studies in cognitive science*. D. Bobrow and A. Collins. New York, Academic Press**:** 131-149.

Bransford, J. D. and D. Schwartz (1999). "Rethinking transfer: A simple proposal with multiple implications." *Rev. Res. Educ.* **24**, 61-100.

Brown, A. (1989). "Analogical learning and transfer: What develops?". In *Similarity and analogical reasoning*. S. V. a. A. O. Eds. NY, Cambridge University Press**:** 369-412.

C. Singh and D. Rosengrant (2003). "Multiple-choice test of energy and momdntum concepts." *Am. J. Phys.* **71**, 607.

Cheng, P. W. and K. J. Holyoak (1985). "Pragmatic reasoning schema." *Cognitive Psychology* **17**, 391-416.

Chi, M. (2000). "Self-explaining expository texts: The dual processes of generating inferences and repairing mental models.". In *Advances in instructional psychology*. R. Glaser. Mahwah, NJ, Erlbaum Associates**:** 161-238.

Chi, M., M. Bassok, M. W. Lewis, P. Remann and R. Glaser (1989). "Self-explanations: How students study and use examples in learning to solve problems." *Cognitive Science* **13**, 145.

Chi, M. T. H., P. J. Feltovich and R. Glaser (1981). "Categorization and representation of physics knowledge by experts and novices." *Cog. Sci.* **5**, 121-152

Detterman, D. K. and R. J. Sternberg, Eds. (1993). *Transfer on trial: Intelligence, cognition and instruction*. Ablex, Norwood, NJ.

Dufresne, R., J. Mestre, T. Thaden-Koch, W. Gerace and W. Leonard (2005). "Knowledge representation and coordination in the transfer process". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Information Age Publishing**:** 155-215.

Duit, R. (1991). "On the role of analogies and metaphors in learning science." *Science Education* **75**, 649-672.

Duncker, K. (1945). "On problem solving." *Psychological Monographs*, 58.

Eylon, B. and F. Reif (1984). "Effect of knowledge organization on task performance." *Cognition and Instruction* **1**, 5.

Genter, D. and C. Toupin (1986). "Systematicity and surface similarity in the development of analogy." *Cognitive Science* **10**, 277-300.

Genter, D. and C. Toupin (1986). "Systematicity and surface similarity in the development of analogy." *Cogn. Sci.* **10**, 277.

Gray, K. E. and N. S. Rebello (2004). "Transfer between paired problems in an interview." *Proceedings of the 2004 Physics Education Research Conference*.

Hayes, J. R. and H. A. Simon (1977). "Psychological differences among problem isomorphs". In *Cognitive theory*. N. J. Castellan, D. B. Pisoni and G. R. Potts. Hillsdale, NJ, Lawrence Erlbaum.

Holyoak, K. (1985). "The pragmatics of analogical transfer". In *The psychology of learning and motivation*. G. Bower. New York, Academic Press. **19**.

Holyoak, K. and P. Thagard (1995). *Mental leaps: Analogy in creative thought*. Cambridge, MA, MIT Press.

Johnson-Laird, P. (1972). *Psychology of reasoning, structure and content*. Cambridge, MA, Harvard University Press.

Johnson, K. and C. Mervis (1997). "Effects of varying the levels of expertise on the basic level of categorization." *J. Exp. Psychol. Gen.* **126**(3), 248.

Klahr, D., Z. Chen and E. Toth (2001). "From cognition to instruction to cognition: A case study in elementary school science instruction". In *Designing for science, implications from everyday, classroom and professional settings*. C. D. Schuun, T. Okada and K. Crowley. Mahwah, NJ, Lawrence Erlbaum.

Kotovsky, K., J. R. Hayes and H. A. Simon (1985). "Why are some problems hard? Evidence from the tower of hanoi." *Cogn. Psychol.* **17**, 248.

Kunz, E. and R. Tweney (1998). "The practice of math and science: From calculations to the clothesline problems". In *Rational models of cognition*. M. O. a. N. Chater**:** 415-438.

Kurz, E. M. and R. D. Tweney (1998). "The practice of mathematics and science: From calculus to the clothesline problem". In *Rational models of cognition*. M. Oalesfard and N. Chater. London, Oxford University Press**:** 415-438.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning approach." *J. of Structured Learn.* **6**, 121-130.

Larkin, J. (1980). "Skilled problem solving in physics: A hierarchical planning model." *J. Structural Learning* **6**, 271-297.

Larkin, J. (1981). "Cognition of learning physics." *Am. J. Phys.* **49**(6), 534-541.

Lin, S. and C. Singh "Learning introductory physics by analogical reasoning " *unpublished*.

Lin, S. and C. Singh (2011). "Challenges in using analogies." *Phys. Teach.* **49**, 512.

Lin, S. and C. Singh (2011). "Using isomorphic problems to learn introductory physics." *Phys. Rev. ST Phys. Ed. Res.* **7**, 020104.

Lobato, J. (2003). "How design experiments can inform a rethinking of transfer and vice versa." *Educ. Res.* **32**, 17.

Lobato, J. (2006). "Transfer strand: Alternative perspectives on the transfer of learning: History, issues and challenges for future research." *J. Learn. Sci.* **15**, 431.

Marshall, S. (1995). *Schemas in problem solving*. New York, NY, Cambridge University Press.

Mateycik, F., D. Jonassen and N. S. Rebello (2009). "Using similarity rating tasks to assess case reuse in problem solving." *AIP Conf. Proc.* **1179**, 201-204.

Mestre, J. (2001). "Implications of research on learning for the education of prospective science and physics teachers." *Phys. Educ.* **36**, 44.

Mestre, J. (2002). "Probing adults' conceptual understanding and transfer of learning via problem solving." *J. of Applied Developmental Psychology* **23**(1), 9-50.

Novick, L. (1988). "Analogical transfer, problem similarity and expertise." *J. Experimental Psychology: Learning, memory and cognition* **14**(3), 510-520.

Novick, L. (1988). "Analogical transfer, problem similarity and expertise." *J. Exp. Psychol. Learn. Mem. Cogn.* **14**, 510.

Ozimek, D. J., P. V. Engelhardt, A. G. Bennett and N. S. Rebello (2005). "Retention and transfer from trigonometry to physics." *Proceedings of the 2004 Physics Education Research Conference* **790**, 173-176.

Paas, F. (1992). "Training strategies for attaining transfer of problem-solving skill in statistics: A cognitive-load approach." *Journal of Educational Psychology* **84**, 429-434.

Rebello, N. S., L. Cui, A. G. Bennett, D. A. Zollman and D. J. Ozimek (2007). "Transfer of learning in problem solving in the context of mathematics and physics". In *Learning to solve complex scientific problems*. D. Jonassen. New York, Lawrence Earlbaum Associates.

Rebello, N. S. and D. A. Zollman (2005). "Dynamic transfer: A perspective from physics education research". In *Transfer of learning from a modern multidisciplinary perspective*. J. P. Mestre. Greenwich, CT, Infromation Age Publishing.

Reed, S. K., G. W. Ernst and R. Bannerji (1974). "The role of analogy in transfer between similar problem states." *Cogn. Psychol.* **6**, 436.

Reif, F. and J. I. Heller (1982). "Knowledge structure and problem solving in physics." *Educ. Psychology* **17**, 102.

Ross, B. (1989). "Distinguishing types of superficial similarities: Different effects on the access and use of earlier problems." *J. Experimental Psychology: Learning, memory and cognition* **15**, 456-468.

Schoenfeld, A. and D. J. Herrmann (1982). "Problem perception and knowledge structure in expert novice mathematical problem solvers." *J. Exp. Psych: Learning, Memory, and Cognition* **8**, 484-494.

Schwartz, D., J. Bransford and D. Sears (2005). "Efficiency and innovation in transfer". In *Transfer of learning: Research and perspectives*. J. Mestre. Greenwich, CT, Information Age Publishing**:** 1-51.

Shapiro, M. A. (1985). "Analogies, visualization and mentalprocessing of science stories". Information Systems Division of the International Communication Association.

Shapiro, M. A. (1988). "Analogies, visualization and mental processing of science stories". In *Communication yearbook*. M. McLaughlin. Beverly Hills, CA, Sage. **9**.

Simon, H. A. and J. R. Hayes (1976). "The understanding process: Problem isomorphs." *Cogn. Psychol.* **8**, 165.

Singh, C. (2002). "When intuition fails." *Am. J. Phys.* **70**(11), 1103-1109.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. I. Performance on nonintuitive problem pair from introductory physics." *Phys. Rev. ST Phys. Educ. Res.* **4**(1), 010104.

Singh, C. (2008). "Assessing student expertise in introductory physics with isomorphic problems. II. Effect of some potential factors on problem solving and transfer." *Phys. Rev. ST Phys. Educ. Res.* **4**, 010105.

Singh, C. and D. Rosengrant (2003). "Multiple-choice test of energy and momentum concepts." *Am. J. Phys.* **71**, 607.

Sternberg, R. J. (1977). "Component processes in analogical reasoning." *Psychol. Rev.* **84**, 353.

Sweller, J., J. J. G. V. Merrienboer and F. G. W. C. Paas (1998). "Cognitive architecture and instructional design." *Educational Psychology Review* **10**(3), 251-296.

Ward, M. and J. Sweller (1990). "Structuring effective worked examples." *Cognition and Instruction* **7**(1), 1-39.

Yerushalmi, E., A. Mason, E. Cohen and C. Singh (2008). "Effect of self-diagnosis on subsequent problem solving performance". Phys. Ed. Res. Conference, Edmonton, CA, AIP Conf. Proc.

5.0 CATEGORIZATION OF QUANTUM MECHANICS PROBLEMS BY PROFESSORS AND STUDENTS

5.1 ABSTRACT

We discuss the categorization of 20 quantum mechanics problems by physics professors and undergraduate students from two honors-level quantum mechanics courses. Professors and students were asked to categorize the problems based upon similarity of solution. We also had individual discussions with professors who categorized the problems. Faculty members' categorizations were overall rated higher than those of students by three faculty members who evaluated all of the categorizations. The categories created by faculty members were more diverse compared to the categories they created for a set of introductory mechanics problems. Some faculty members noted that the categorization of introductory physics problems often involves identifying fundamental principles relevant for the problem, whereas in upper-level undergraduate quantum mechanics problems, it mainly involves identifying concepts and procedures required to solve the problem. Moreover, physics faculty members who evaluated others' categorizations expressed that the task was very challenging and they sometimes found another person's categorization to be better than their own. They also rated some concrete categories such as 'hydrogen atom' or 'simple harmonic oscillator' higher than other concrete categories such as 'infinite square well' or 'free particle'.

5.2 INTRODUCTION

A crucial difference between the problem-solving strategies used by experts in physics and beginning students lies in the interplay between how their knowledge is organized and how it is retrieved to solve problems (Larkin and Reif 1979; Chi et al. 1981; Reif 1981; Hardiman et al. 1989; Singh 2009; Tabor-Morris et al. 2009). Categorizing or grouping together problems based upon similarity of solution can give a glimpse of the 'pattern' an individual sees in a problem while contemplating how to solve it (Chi et al. 1981). In a classic study by Chi *et al.* (Chi et al. 1981), a categorization task was used to assess introductory physics students' level of expertise in physics. In Chi's study (Chi et al. 1981), eight introductory physics students were asked to group together introductory mechanics problems into categories based upon similarity of solution. They found that, unlike experts (physics graduate students in their study) who categorized them based on the physical principles required to solve them, introductory students categorized problems involving inclined planes in one category and pulleys in a separate category (Chi et al. 1981). Previously, we conducted a categorization study in which 7 professors, 21 physics graduate students and more than a hundred introductory physics students in a classroom environment were asked to group together introductory physics problems based upon similarity of solution (Singh 2009). We found that the professors significantly outperformed both the graduate students and introductory physics students in grouping together problems based upon the physics principles involved rather than basing the grouping of the problems on the

surface features of the problems and they created very similar categories (Singh 2009). The graduate students performed better than the introductory physics students in the categorization task. However, there is a large overlap in the performance of graduate students and introductory students in the calculus-based courses on the categorization of introductory physics problems into groups based upon the fundamental principles of physics required to solve the problems (Singh 2009).

While learning introductory physics is challenging, learning quantum mechanics is perhaps even more so (Jolly et al. 1998; Singh 2001; Mannila et al. 2002; Singh 2005; Magalhaes and Vasconcelos 2006; Singh 2006; Singh et al. 2006; Singh 2007; Singh 2007; Singh 2008; Singh 2008; Matteucci et al. 2009). Unlike classical mechanics, we do not have direct experience with the microscopic quantum world. Also, quantum mechanics has an abstract theoretical framework in which the most fundamental equation, the time-dependent Schrödinger equation (TDSE), describes the time evolution of the wavefunction or the state of a quantum system according to the Hamiltonian of the system. This wavefunction is in general complex and does not directly represent a physical entity. However, the wavefunction at a given time can be used to calculate the probability of measuring a particular value for a given physical observable associated with the system. For example, the absolute square of the wavefunction in position space gives the probability density. Since the TDSE does not describe the evolution or motion of a physical entity, unlike Newton's second law, the modeling of the microscopic world in quantum mechanics is generally more abstract than the modeling of the macroscopic world in classical mechanics.

The conceptual framework of quantum mechanics is often counterintuitive to our everyday experiences. According to quantum theory, the position, momentum, energy and other

observables for a quantum mechanical entity are in general not well defined. We can only predict the probability of measuring different values based upon the wavefunction when a measurement is performed. This probabilistic interpretation of quantum mechanics, which even Einstein found disconcerting, is challenging for students. Moreover, according to the Copenhagen interpretation of quantum mechanics, which is widely taught to students, measurement of a physical observable 'collapses' the wavefunction into an eigenstate of the operator corresponding to the observable measured. Thus, the usual time evolution of the system according to the TDSE is treated differently from measurement processes. Students often have difficulty with this notion of an instantaneous change or 'collapse' of the wavefunction during the measurement (Singh 2005). The proper way to interpret quantum mechanics is still the subject of debate, making the subject even more challenging for physics instructors.

Here, we discuss a study in which 22 physics juniors and seniors in two undergraduate quantum mechanics courses and six physics faculty members (professors) were asked to categorize 20 quantum mechanics problems based upon similarity of solution. We also interviewed some faculty members concerning issues related to categorization of quantum mechanics problems. All but one faculty member had taught an upper-level undergraduate or graduate level quantum mechanics course. The faculty member who had not taught quantum mechanics regularly teaches other physics graduate 'core' courses including electricity and magnetism and statistical mechanics. All undergraduate students in the upper-level quantum mechanics classes (12 and 10 students in the two classes who were present on the day the categorization task was given as a quiz) participated. The students were given 35–40 min to perform the categorization. The faculty members performed the categorization at a time convenient to them. Except for the faculty member who had not taught quantum mechanics and

took longer to categorize the problems, other professors noted that it took them less than 30 min to perform the categorizations.

The 20 problems to be categorized (given in the appendix) were adapted from the problems found among the end of the chapter exercises in commonly used upper-level undergraduate quantum mechanics textbooks. All those who performed the categorization were provided with the instructions given at the beginning of the appendix. The sheet on which individuals were asked to perform the categorization of problems had three columns. In the first column, they were asked to place their own 'category name' for each category (in other words, they had to come up with their own category names); in the second column, they had to place a description of the category that explains why those problems can be grouped together; in the third column, they had to list the problem numbers for the problems that should be placed in that category. We note that for solving a problem, more than one approach may be useful. The instruction for the categorization explicitly noted that a problem could be placed in more than one category.

The goal was to investigate differences in categorization by faculty members and students and whether there are major differences in the ways in which individuals in each group categorize quantum mechanics problems. This study was partly inspired by the fact that a physics faculty member who was teaching advanced undergraduate quantum mechanics in a previous semester had given a take-home exam in which one problem asked students to find the wavefunction of a free particle after a time t given the initial wavefunction (which was a Gaussian). Two students approached the faculty member complaining that this material was not covered in the class. The faculty member pointed out to them that he had discussed in the class how to find the wavefunction after a time t given an initial wavefunction in the context of a

problem involving an infinite square well. But the students insisted that, while the time development of the wavefunction may have been discussed in the context of an infinite square well, it was not discussed in the context of a free particle. It appears that the two students did not categorize the time-development issues for the infinite square well and the free particle in the same category. They did not realize that a solution procedure very similar to what they had learned in the context of the time development of the wavefunction for an infinite square well should be applicable to the free particle case except they must use the energy eigenstates and eigenvalues corresponding to the free particle and replace the discrete sum over energy levels for an infinite square well by an integral since the energy levels for a free particle are continuous. This difficulty in discerning that the same concepts and procedures should be applicable in both contexts is similar to the difficulty introductory students have in discerning that the same principle is applicable in two problems that have different contexts.

5.3 SCORING OF CATEGORIZATION

We note that each individual who categorized the problems had to come up with his/her own category names and justify why each problem should be placed in a particular category. The 20 questions for categorization were such that the 'context' in four of them was the hydrogen atom, the harmonic oscillator, the infinite square well and the free particle (see the appendix). Three of the problems were related to the spin angular momentum and one was about the Dirac delta function. Within these different contexts, there were questions about the time evolution of the wavefunction, time dependence of expectation value, measurement of physical observables, expectation value including uncertainty, commutation relations between different components of

the spin angular momentum, etc. As noted earlier, we wanted to investigate if the questions were grouped together based upon the physics concepts and procedures required for solving them or the 'surface features' of the problems such as the contexts used. For problems related to the time dependence of a wavefunction or time dependence of an expectation value, we wanted to investigate if the faculty members and students categorized problems involving the stationary states differently from those involving the non-stationary states.

We find that the categorizations of a problem performed by the students were diverse and they seldom placed a problem in more than one category although they were explicitly told that they could do so if they wish. Moreover, the faculty members often used a diverse set of categories unlike the highly uniform categorization by faculty members for introductory physics problems (Singh 2009).

To analyze the quality of categories created by the professors and the students quantitatively, we placed each category created by each individual into a matrix which consisted of problem numbers along the columns and categories along the rows. A '1' was assigned if the problem appeared in the given category and a '0' was assigned if the opposite was true. Categories that were very similar were combined, e.g., 'time dependence of wavefunction', 'time development of wavefunction' or 'dynamics of wavefunction' were combined into a single category. In order to score the categorizations by students and faculty members, three faculty members (a subset of those who had categorized the problems themselves) were recruited. They were given the categorizations by students and faculty in the matrix form we had created (without identifiers and with the categorizations by the faculty and students jumbled up). For example, all the different categories created by different individuals for problem (1) were placed one after the other to aid faculty members who were scoring the categorizations.

For each question, the three faculty members doing the scoring were advised to read the question, think about how they would categorize it and then evaluate and score everybody's categorization. They were asked to evaluate whether each of the categories created by an individual should be considered 'good' (assigned a score of 2), 'moderate' (assigned a score of 1) or 'poor' (assigned a score of zero). We note that if all three faculty members scored a particular problem for an individual as 'good', the score of that individual on that problem will be 6 (maximum possible). If one faculty member scored it as 'good' but the other two scored it as 'medium', the score of that individual on that problem will be 4.

5.4 RESULTS

Each of the 22 students and 6 faculty members categorized 20 problems. Faculty members often placed a problem in more than one category. As noted earlier, some of the categories created for a problem by more than one individual were the same or similar. Several categories that were similar were combined into a single category. All three faculty members noted that evaluating and scoring other people's categorization was a very challenging task and required intense focus. One faculty member noted that it took him several hours to complete the scoring. Moreover, two of the faculty members who evaluated everybody's categorization noted that they would prefer not to use the terms 'good' or 'poor' for judging the categories although some categories were better than others. The faculty members who scored others' categorizations also noted that sometimes they liked the categorizations of a problem by others much more than their own. Interestingly, in our earlier studies with introductory physics categorization, we had asked three faculty members to evaluate the categorizations of a subset of randomly selected individuals (in

that case we did not ask them to score all categorizations because the introductory physics classes had several hundred students) (Singh 2009). In scoring introductory physics categorizations, faculty members were not hesitant in calling the categories good/poor and they did not say that the task was challenging (Singh 2009). They also never said that they preferred others' categorizations of a problem more than their own perhaps because there was great conformity in faculty categorizations (which were based upon physics principles such as the conservation of mechanical energy, conservation of momentum, conservation of angular momentum, Newton's second law, etc.) (Singh 2009).

Table 5-1 shows examples of category names for each question divided into three groups with a score of '5 or 6', '3 or 4' or 'less than 3'. With each category name, many faculty members and students provided an explanation justifying why certain problems should be placed in that category. Inspection of Table 5-1 shows that the categories that obtained a total score of less than 3 (out of 6) included both concrete and abstract categories. For example, 'change in basis' for problem (1), 'commutation relation' for problem (12), 'matrix element' for problem (17), 'rotation group' for problem (19), etc. are abstract categories that received a score of less than 3. On the other hand, 'infinite square well' for problems (4), (12), (16) and (18) and 'free particle' for problems (3), (7) and (10) etc are examples of concrete categories that received a score of less than 3.

Table 5-1. Examples of categories created for each question (Q) divided into three groups with a score of ‘5 or 6’, ‘3 or 4’, or ‘less than 3’. ‘EV’ is an abbreviation for ‘expectation value’ and ‘FT’ is an abbreviation for ‘Fourier transform’.

Q	Score of 5 or 6	Score of 3 or 4	Scores less than 3
1/2	Time dependence of EV / Stationary State	Eigenvalue and function / angular momentum / Larmor precession	Stern-Gerlach / change in basis / charged particle in magnetic field
3	Time evolution of wavefunction	Time dependency, evolution	Superposition / free particle
4	EV / EV and uncertainty	Measurement, observables and uncertainty relations	Infinite square well / $\Psi(x,t)$ manipulations
5	EV / eigenstate / time dependence of EV	Simple harmonic oscillator / operator properties / $\Psi(x,t)$	Math / little concept
6	Time dependence of EV / symmetry argument	Simple harmonic oscillator/ eigenstates	Matrix element / $\Psi(x,t)$ / EV and uncertainty
7	Time evolution of wavefunction	--	Free particle / math / FT
8	EV / EV and uncertainty	Hydrogen atom / matrix element / eigenstates	Energy and momentum / math / $\Psi(x,t)$
9	FT / Dirac delta function	Math	Graphing
10	EV / EV and uncertainty	Probability and EV	Free particle / uncertainty principle
11	Time dependence of EV	Superposition / time dependent Schrödinger equation / EV	Energy and time / math / hydrogen atom
12	Measurement / expansion in eigenfunctions	Collapsed wavefunction / scalar product / FT	Infinite square well / commutation relation
13	Measurement / collapsed wavefunction	Scalar product / eigenvalue	Superposition / stationary state
14	Time evolution of wavefunction	Hydrogen atom / time dependence	Math / time
15	--	EV / stationary state / selection rules /symmetry (even/odd)	Time/ time dependent Schrödinger equation
16	Expansion in eigenfunctions / time evolution of wavefunction	Stationary state / time dependent function	Infinite square well / math
17	EV / EV and uncertainty	Symmetry / probability and EV	Free particle / matrix element / uncertainty relation
18	Time dependence of EV	EV / superposition	Infinite square well
19	Spin	Commutation / uncertainty	Math / rotation group
20	Collapsed wavefunction	Hydrogen atom	Stationary state/ EV

Figure 5-1 shows a histogram of the percentage of people (students or faculty) versus percentage of problems with a score of 50% or better (at least 3 out of 6), and Figure 5-2 shows a histogram of the percentage of people versus average score on the categorization task out of a maximum of 6 (averaged over all problems). We note that what one faculty member scored as 'good' was often scored as 'medium' by another. While three of the six faculty members who categorized the problems were recruited to score all of the categories by all faculty members and students, the average score of the three faculty members who scored all problems was lower than those of the other three faculty members who did not score the categories. Also, faculty members who scored the categorizations explicitly noted that they sometimes preferred other's categorizations more than their own. Thus, we do not believe that the faculty members who scored everybody's categorizations were partial to their own categories. It is interesting to note that the faculty member who had never taught quantum mechanics (but had taught statistical mechanics and electricity and magnetism at the graduate level) performed slightly better on average (though not statistically significant) than the faculty members who scored the categorizations. In fact, the faculty member who had never taught quantum mechanics but performed the categorization commented that he would like to teach quantum mechanics but was not assigned that course despite asking for it. He added that the main reason was that many other faculty members wanted to teach quantum mechanics but they did not want to teach the other graduate level courses that he was assigned.

Figure 5-1 shows that the categorizations by faculty members were rated higher overall than those by students, despite the diversity in faculty responses. We find that the faculty members were more likely to categorize the problems based upon the procedures and concepts

required to solve the problems rather than the contexts involved. But Figure 5-2 shows that none of the faculty members had an average score of 5–6 on the categorization task, implying that none of the faculty member placed all 20 problems in categories that were considered uniformly excellent (although their categories were on average better than those of the students). Faculty members sometimes categorized problems based upon the contexts used, e.g., hydrogen atom, simple harmonic oscillator, angular momentum, etc. However, most of the time when they did such categorizations, they also categorized the same problems in other categories which were based upon the procedures for solving the problems. They were also more likely than students to make use of the nuances in the questions to group problems, e.g., whether the system was in a stationary state in order to categorize problems involving the time dependence of wavefunction or the time dependence of expectation value.

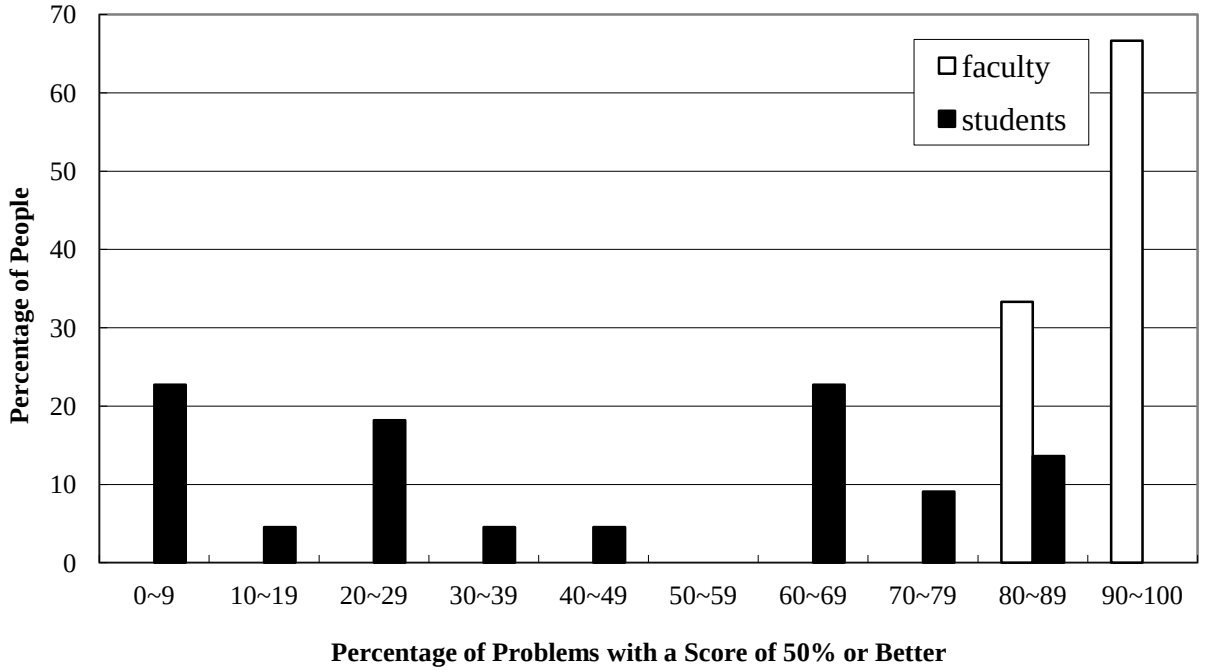


Figure 5-1. Percentage of people versus percentage of problems with a score of 50% or better (at least 3 out of 6).

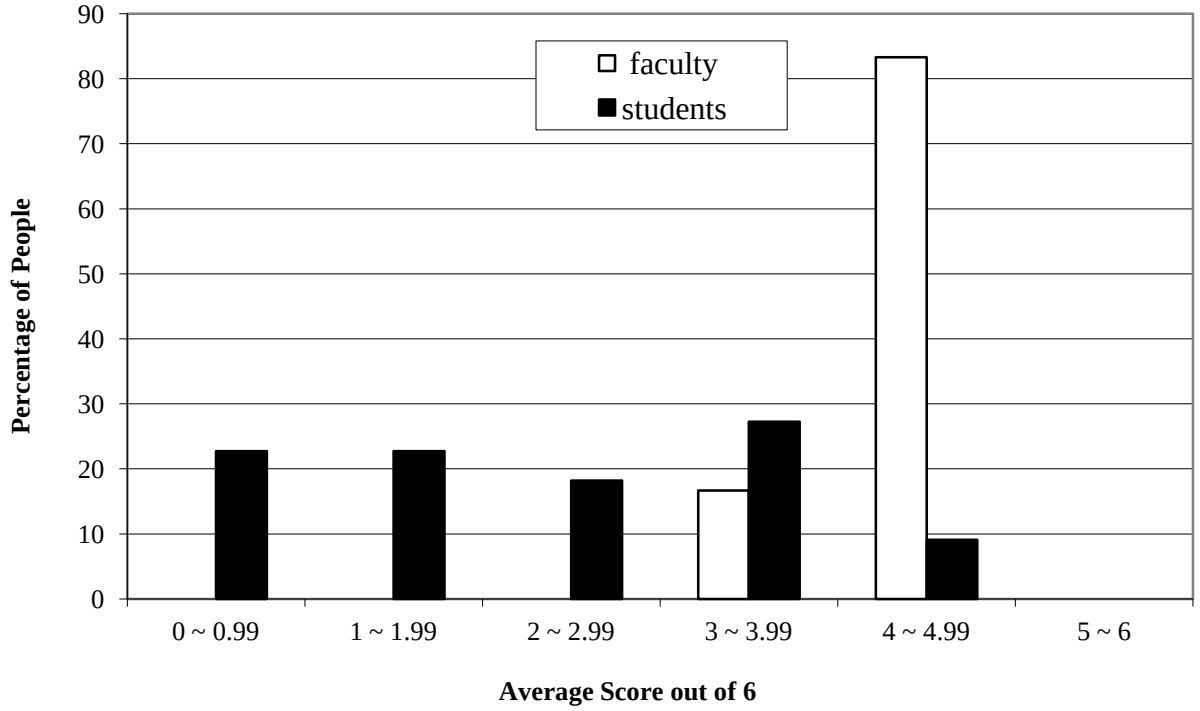


Figure 5-2. Percentage of people versus average score out of 6.

The same category was sometimes assigned different scores for different questions depending upon whether the faculty members who scored them felt they were appropriate categories for those questions. For example, for question (15), the category 'stationary state' obtained a score of at least 3 because the faculty members felt that it was relevant for determining the expectation value of momentum and for explaining whether it should depend on time. On the other hand, for question (20) (see the appendix), the category 'stationary state' obtained a score less than 3 because it was not considered relevant for finding the possible values of energy after the measurement of the distance of the electron from the nucleus.

Faculty members who scored the categories were careful to distinguish between the categories 'uncertainty' and 'uncertainty principle' (or 'uncertainty relation'). For example, in questions (10) and (17), the category 'expectation value and uncertainty' obtained an average score of 5 or 6 whereas 'uncertainty principle' or 'uncertainty relation' obtained a score of less

than 3. Individual discussions with the faculty who scored the categorization suggest that they saw a clear distinction between these categories. In particular, they asserted that calculating the standard deviation σ_x was about calculating the uncertainty in position but it was not about 'uncertainty principle' or 'uncertainty relation'. The question did not ask whether the product of the uncertainties in position and momentum is greater than or equal to $\hbar/2$.

The overall scores (by the three faculty members who evaluated all of the categorizations) on concrete or context-based categories such as 'hydrogen atom' or 'harmonic oscillator' were higher than other concrete categories such as 'infinite square well' or 'free particle' (where four questions out of 20 given in the categorization task belong to each of these four systems as noted earlier). Discussions with individual faculty suggest that they have a notion of a canonical quantum system that they use for thinking about concepts and to help clarify ideas about quantum mechanics. 'Hydrogen atom' and 'harmonic oscillator' fit their notion of canonical quantum systems. One faculty member explicitly noted that the hydrogen atom and harmonic oscillator are quintessential in quantum mechanics. He added that the hydrogen atom embodies many essential features of other complex quantum systems but is exactly soluble and widely applicable. Similarly, the harmonic oscillator is used as a model to understand diverse quantum systems such as molecular excitations and quantum optics. Such explanations about why the average score for 'hydrogen atom' as a category was at least 50% (3 out of 6 including the scores of all the three faculty members who evaluated the responses) for three of the four questions that related to the hydrogen atom but the average score for 'infinite square well' as a category was not 50% for any of the four questions related to the infinite square well shed some light on why the faculty do not view all 'concrete' categories on the same footing.

As noted earlier, most of the time when faculty members placed problems in a category involving context, such as ‘hydrogen atom’ or ‘simple harmonic oscillator’, they also placed the same problem in another category based on the procedure involved in solving the problem. But sometimes they placed some of the problems only in concrete categories. For example, one faculty member grouped some problems about the hydrogen atom in the ‘hydrogen atom’ category or in a category based upon the procedure for solving the problems, e.g., ‘measurement’ or ‘time evolution of wavefunction’ or in both these types of categories. During individual discussions, these faculty members were asked why their choices were more context based in some of their groupings and more focused on the procedures and concepts to solve the problems for creating other categories. In response, some faculty members reasoned that they were perhaps using lenses with different ‘zoom factors’ for categorizing different problems. They noted that the categorization task was challenging and they sometimes zoomed in and out while categorizing different problems focusing on the contexts or the procedures for solving them. Faculty members who scored the categorizations also noted that while scoring others’ categorizations they realized that there were many different ways to categorize the problems and sometimes others’ categorizations were better than their own.

The faculty members were reminded during the individual discussions that while categorizing introductory physics problems, faculty always scored ‘inclined plane category’, ‘cliff category’ or ‘spring category’ as poor categories explaining that they were based on the ‘surface features’ of the problems rather than the ‘deep’ features (fundamental principles of physics required to solve them). They were asked to comment on whether making categories such as ‘angular momentum’ or ‘hydrogen atom’ was also based on the ‘surface features’ of the problems rather than the procedures relevant for solving the problems. In response to such

questions, faculty members often noted that while these categories were less directly related to the procedure for solving the problems, they were hesitant to call them ‘poor’ categories. They noted, e.g., that the knowledge about the hydrogen atom is relevant for solving the problems involving hydrogen atom even though that knowledge alone may not be the central component of how to set up the solution of the problem. For example, questions (11) and (14) in the appendix are about the hydrogen atom in a linear superposition of stationary states. In question (11), the knowledge that the expectation value of an operator corresponding to a physical observable which does not commute with the Hamiltonian depends on time in a non-stationary state is relevant to solve the problem. Similarly, in question (14), knowledge about the time dependence of the wavefunction in a non-stationary state is relevant for solving the problem. Simply categorizing these problems in the ‘hydrogen atom category’ does not indicate whether the individual knows the procedure for solving the problem. While the faculty members agreed that some of these concrete categories may not be the best way to categorize the problems, they sometimes scored some of these context-based categories (even if they did not give an indication of the procedures for solving the problems) as ‘1’ instead of ‘0’ (but rarely gave them a score of ‘2’). As shown in Table 5-1, ‘angular momentum’ for questions (1) and (2), harmonic oscillator for questions (5) and (6), hydrogen atom for questions (8), (14) and (20) are examples of such context-based categories that were judged favorably.

Individual discussions with faculty members suggest that some felt that the structure of knowledge in quantum mechanics is more complex than that in introductory physics. Moreover, the complexity of knowledge structure in quantum mechanics is due to both the requisite conceptual and mathematical knowledge. This complexity may make it difficult for everybody to focus on the same aspects of solution when asked to categorize (although there are often

underlying relations in faculty categorizations). One possible implication is that the way concepts are emphasized in a quantum mechanics course may differ based upon the 'patterns' that appear to be most central to the faculty member teaching the course. For example, one faculty member may emphasize the conceptual aspects while another may emphasize the mathematical aspects.

During individual discussions, faculty members were asked if they were surprised that the categories in which a problem was placed by different faculty members were not always similar and some faculty came up with categories that were more abstract than others. They were also asked to comment on the fact that the faculty members who scored the categorizations gave low scores not only to the concrete categories but also to some abstract categories. For example, as noted earlier, 'matrix element' for question (17) and 'rotation group' for question (19) received a cumulative score of less than 3. In response to these questions, faculty members asserted that they were not very surprised about these because they felt that how one teaches quantum mechanics and how abstractly or concretely one presents the material depends strongly on the instructor. During discussions, several faculty members pointed out that if one takes a look at the quantum mechanics textbooks, he/she will realize that the textbooks are laid out very differently and emphasize different things. Some faculty members mentioned that some undergraduate textbooks do not emphasize the postulates of quantum mechanics. Also, the postulates in different textbooks are not identical (e.g., only some of the textbooks list the time-dependent Schrödinger equation as a postulate). Some textbooks are hesitant to mention the 'collapse' of the wavefunction during measurement while others discuss these issues in detail. They also mentioned that some textbooks start with the infinite-dimensional vector space while others start with the quantum mechanics of a spin-half particle. The proponents of the spin-half

first believe that it provides a simple two-dimensional vector space to teach the foundations of quantum mechanics whereas those who discuss, e.g., the infinite square well, first believe that spin is too abstract and continuity with the topics covered in the earlier courses is important. The extent to which symmetry ideas are emphasized and the conservation laws derived from them also varies in the undergraduate textbooks. Discussions suggest that most faculty members believed that if there is no agreement on the basic issues about teaching undergraduate quantum mechanics, the differences in how the faculty members categorize problems, teach their courses and what they emphasize are perhaps expected.

Another common theme that emerged is that categorization of introductory physics problems involves identifying fundamental principles relevant for the problems, whereas in the upper-level undergraduate quantum mechanics problems, it mainly involves identifying concepts and procedures, because problem solving in such a course is tied to conceptual and procedural knowledge. Some faculty members asserted that the fundamental principles of physics such as conservation of energy and conservation of momentum are important even for understanding quantum processes. However, the application of fundamental principles to quantum processes is not typically the focus of an upper-level undergraduate course. For example, one faculty member noted that for understanding the properties of a solid using neutron scattering, one will have to carefully account for the conservation of energy and momentum but questions involving these topics are typically not common in an undergraduate quantum mechanics course. He added that if such questions were given in the categorization task, there may be more uniformity in the faculty responses.

5.5 SUMMARY

The categorization of problems by students in a quantum mechanics course can be a useful tool for understanding the patterns students see in a problem when contemplating how to solve it. Even in the context of quantum mechanics problems, professors overall scored higher than students in grouping together problems based on similarity of solutions.

However, unlike the categorization of introductory physics problems, in which professors' categorizations are generally uniform, their categorizations were more varied in the context of quantum mechanics. The diversity of categories created for quantum mechanics may partly be due to the fact that the solution to a typical quantum mechanics problem in an upper-level quantum mechanics course typically requires knowledge of requisite concepts and procedures. On the other hand, categorization in introductory physics is typically based on the fundamental principles of physics. Faculty members noted that the fundamental principles, e.g., conservation laws, are also important in understanding quantum processes but they are not the focus of an upper-level undergraduate quantum mechanics course. Some faculty members created more abstract categories than others. It will be useful to investigate how different is the teaching emphasis of faculty members in a quantum mechanics course depending upon the types of categories they created.

5.6 CHAPTER REFERENCES

Chi, M. T. H., P. J. Feltovich and R. Glaser (1981). "Categorization and representation of physics knowledge by experts and novices." Cog. Sci. 5, 121-152

Hardiman, P., R. Dufresne and J. Mestre (1989). "The relationship between problem categorization and problem solving among experts and novices." Memory and Cognition 17(5), 627-638.

Jolly, P., D. Zollman, N. S. Rebello and A. Dimitrova (1998). "Visualizing potential energy diagrams." Am. J. Phys. 66, 57.

Larkin, J. H. and F. Reif (1979). "Understanding and teaching problem solving in physics." Eur. J. Sci. Ed. 1(2), 191-203.

Magalhaes, A. and V. P. S. Vasconcelos (2006). "Particle in a box: Software for computer-assisted learning in introductory quantum mechanics courses." Eur. J. Phys. 27(1425-1435).

Mannila, K., I. T. Koponen and J. Niskanen (2002). "Building a picture of students' conceptions of wave- and particle-like properties of quantum entities." Eur. J. Phys. 23, 45-53.

Matteucci, G., A. Migliori, F. Medina and R. Castaneda (2009). "An experiment on the particle-wave nature of electrons." Eur. J. Phys. 30, 217-226.

Reif, F. (1981). "Teaching problem solving- a scientific approach." The Phys. Teach. 33, 310.

Singh, C. (2001). "Student understanding of quantum mechanics." Am. J. Phys. 69, 885-896.

Singh, C. (2005). "Transfer of learning in quantum mechanics." AIP Conf. Proc. 790, 23-26.

Singh, C. (2006). "Improvinig student understanding of quantum mechanics." AIP Conf. Proc. 818, 69-72.

Singh, C. (2007). "Helping students learn quantum mechanics for quantum computing." AIP Conf. Proc. 883, 42-45.

Singh, C. (2007). "Student difficulties with quantum mechanics formalism." AIP Conf. Proc. 883, 185-188.

Singh, C. (2008). "Interactive learning tutorials on quantum mechanics." Am. J. Phys. 76, 400-405.

Singh, C. (2008). "Student understanding of quantum mechanics at the beginning of graduate instruction." Am. J. Phys. 76, 277-287.

Singh, C. (2009). "Categorization of problems to assess and improve proficiency as teacher and learner." Am. J. Phys. 77, 73-80.

Singh, C., M. Belloni and W. Christian (2006). "Improving student's understanding of quantum mechanics." Phys. Today 59, 43-49.

Tabor-Morris, A. E., K. A. Froriep, T. M. Briles and C. M. McGuire (2009). "City of physics-analogies to increase cognitive coherence in physics learning." Phys. Educ. 44, 195-198.

6.0 TEACHING ASSISTANTS' BELIEFS ABOUT THE DESIGN OF PROBLEM SOLUTIONS FOR INTRODUCTORY PHYSICS

6.1 ABSTRACT

We investigated how graduate teaching assistants (TAs) believe the worked-out example problem solutions should be designed. TAs are an important population to understand; they often provide significant instruction and they also represent the pool of future physics faculty. Twenty-four first-year graduate TAs enrolled in a training course were provided with different example problem solutions for the same physics problem. They were asked to discuss their preferences for prominent solution features that they noticed as well as their reasons (goals). Their beliefs are compared to the recommendations from the research literature and their practices. Findings suggest that although "modeling expert-like problem solving and decision making", a goal aligned with the recommendation from the research literature, was mentioned by most of the TAs, they did not necessarily notice all features that help with this goal. Moreover, there is a discrepancy between the self-reported preferences and the actual practices. A challenge in materializing all the goals coherently was also observed in this study. We believe the activity in this study can serve as a starting point for TAs' professional development.

6.2 INTRODUCTION

Cognitive apprenticeship approach (Collins et al. 1991) underlies many pedagogical techniques that have been shown to promote expert-like problem solving. In this approach, a prescribed problem-solving framework is made explicit through "modeling" it in instructors' solutions to problems. The framework involves: 1) initial problem analysis, 2) solution construction (choice of sub-problems), and 3) checking of solution (Reif 1995).

If we wish to help instructors make problem solving approaches explicit on problem solutions they provide students, it is necessary to understand how these instructors currently perceive and value the design features of solutions to problems. A former line of research (Henderson et al. 2007; Yerushalmi et al. 2007) investigated physics faculty beliefs and values about the teaching and learning of problem solving. This study builds on the former line of research to investigate graduate teaching assistants' beliefs about the role that worked examples should play in introductory physics instruction. At many institutions, graduate teaching assistants play a central role in the teaching of problem solving. Many teaching assistants lead recitations in which they present students with worked-out examples for physics problems, guide students in solving problems and assess students' solutions. They also represent the pool of future physics faculty. Their practices may depend on factors such as the individual characteristics of the TAs, the context that they are teaching in, and their beliefs about the role that problem solutions should play in physics instruction. This study particularly aims to find out TAs' beliefs regarding example problem solutions in educating introductory physics students how to approach physics problems. The main research questions are:

(1) What goals (and/or concerns) do TAs express when designing instructor solutions to problems?

(2) How do TAs externalize these goals through different features when designing a solution to a problem?

In particular, we study whether the TAs notice, value, and make use of features that (a) help to explicate the expert decision-making process along the prescribed problem-solving stages suggested in the research literature (Reif 1995) (b) relate to other aspects of a solution. We also examine the extent to which valued design features and goals cohere with each other and the recommendation from research literature.

6.3 METHODOLOGY

In designing the data collection tool for this study, we aimed at:

1) Uncovering TAs' beliefs that drive their decision making in-vivo (in classroom) regarding how to design example problem solutions
2) Encouraging TAs' introspection and articulation related to the issue above
3) Reliability – minimal distortion of data by researchers' personal bias
4) Comparison with educational research-based pedagogies

To achieve the goals above we adapted a methodology that was used in a former study of the considerations that shape instructional choices regarding worked-out examples of physics problems (Henderson et al. 2007). This data collection tool made use of the "artifact comparison" technique and semi-structured individual interviews. However, there are several concerns regarding the method of interviews. First, from the practical perspective, it requires significant time for both data collection and analysis. Second, the interviewer interventions

required to clarify respondents' answers endanger reliability. Finally, as the data collected is extremely rich, there is ambiguity in categorization of the data, which endangers validity.

In this study we made use of an alternative data collection tool: the Group Administered Interactive Questionnaire (GAIQ). This tool was designed to respond to the aforementioned concerns by taking advantage of the opportunity to conduct the study in a TA training course at the University of Pittsburgh. The “artifact comparison” technique used in the former study (Henderson et al. 2007) is retained in the GAIQ approach. A detailed comparison between the interview and the GAIQ approach can be found in the article by Yerushalmi et al. (Yerushalmi et al. 2011).

6.3.1 The artifact comparison technique

The GAIQ made use of the "artifact comparison" technique that was previously used in the former study (Henderson et al. 2007). Respondents were asked to make judgments about instructional artifacts that were carefully designed to activate, in an imaginary classroom setting, the instructional decision-making that takes place in an authentic classroom. Through making and justifying instructional decisions, research subjects expose the beliefs and values that underlie these decisions in a way that is not possible through observational studies. The technique also allows us to standardize data collection across participants (Henderson et al. 2007).

The artifacts were three example instructor solutions for a single problem selected to be one that could reasonably be given in most calculus-based introductory physics courses. It was important that the problem be considered difficult enough by an instructor to require an average student to use an exploratory decision making process as opposed to an algorithmic procedure.

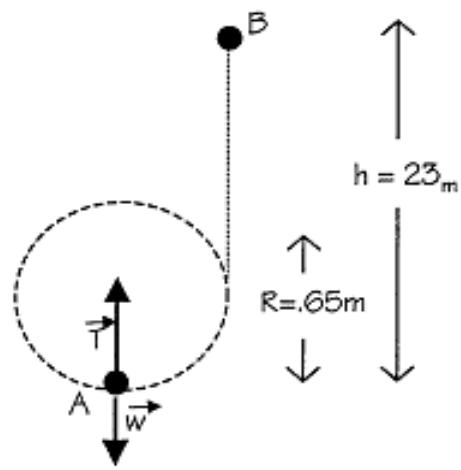
The problem is presented in Figure 6-1. The example instructor solutions presented in Figure 6-2, Figure 6-3, and Figure 6-4 reflect various instructional styles.

Instructor solution 1 (Figure 6-2) was a “bare-bones” solution that left many of the minor steps to be filled in by the reader. Instructor solution 2 (Figure 6-3) explicates some of the details of the solution process. Instructor solution 3 (Figure 6-4) is designed to reflect a systematic decision making process characteristic of expert problem solvers. It begins with the problem goal and attempts to relate it to the known information. The reasoning behind each step is explicated.

You are whirling a stone tied to the end of a string around in a vertical circle having a radius of 65 cm. You wish to whirl the stone fast enough so that when it is released at the point where the stone is moving directly upward it will rise to a maximum height of 23 meters above the lowest point in the circle. In order to do this, what force will you have to exert on the string when the stone passes through its lowest point one-quarter turn before release? Assume that by the time that you have gotten the stone going and it makes its final turn around the circle, you are holding the end of the string at a fixed position. Assume also that air resistance can be neglected. The stone weighs 18 N.

Figure 6-1. Problem used in the artifact comparison technique.

Instructor solution I



The tension does no work

Conservation of energy between point A and B

$$Mv_A^2/2 = mgh$$

$$V_A^2 = 2gh$$

At point A, Newton's 2nd Law gives us:

$$\vec{T} - \vec{w} = m\vec{a}$$

$$T - w = mv_A^2/R$$

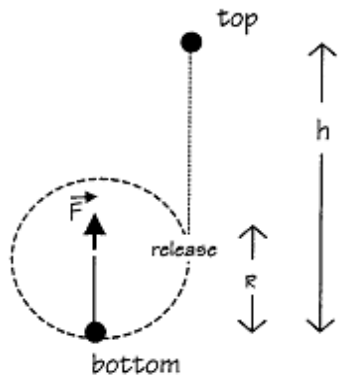
$$T = 18_N + 2 \cdot 18_N \cdot 23_m/.65_m = \boxed{1292N}$$

Figure 6-2. Example instructor solution 1.

Instructor solution II

Known
- $w = 18N$ = weight of stone
- $R = 0.65m$
- $h = 23m$
- $v_t = 0$ = velocity at top

Unknown
- $v_r = ?$ = velocity at release
- $v_b = ?$ = velocity at bottom
- force my hand exerts = F=?



Step 1) Find v_r needed to reach h

$E_i = E_f$

$E_{release} = E_{top}$

$PE_{release} + KE_{release} = PE_{top} + KE_{top}$

$\cancel{m}gR + \cancel{m}v_r^2/2 = \cancel{m}gh + \cancel{m}v_t^2/2$ (0)

$v_r^2 = 2g(h - R)$

Conservation of energy for the stone earth system, since no external forces.

Note: you could also choose other systems.

KE of earth estimated to be 0

You could also use kinematics to find v_r

Step 2) Find v_b needed to have v_r at release

$E_{bottom} = E_{release}$

$PE_{bottom} + KE_{bottom} = PE_{release} + KE_{release}$

$mgO + \cancel{m}v_b^2/2 = \cancel{m}gR + \cancel{m}v_r^2/2$

Using v_r from above:

$V_b = [2gh]^{1/2}$

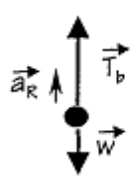
Conservation of energy for the stone earth system. Since $T \perp v$ in circular path, T does no work.

Step 3) Find T_b, tension at bottom, needed for stone to have v_b at bottom

$\sum \vec{F} = m\vec{a}$

$\sum F_R = ma_R$

$T_b - w = m\, v_b^2/R$



Free body diagram

To relate the forces to velocity we can look at the radial component, and use $a_R = v^2/R$.

Using v_b from above:

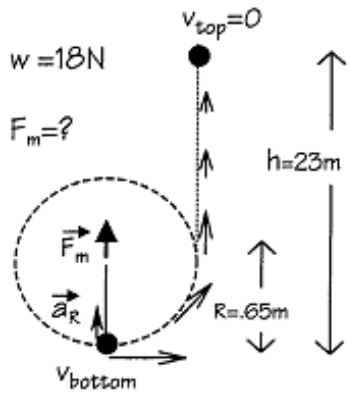
$T_b - w = 2\, mgh/R$

$T_b = w + 2\, w\, h/R = 18 + 2\ 18\ 23/.65 = \boxed{1292N}$

T_b equals F, the force my hand exerts, for a massless string

Figure 6-3. Example instructor solution 2.

Instructor solution III



Approach:

I need to find F_m, force exerted by me. I know the path, h (height at top) and v_t (velocity at top)

A) For a massless string $F_m = T_b$ (T_b-Tension at bottom)

B) I can relate T_b to v_b (velocity at bottom) using the radial component of $\sum \vec{F}=m\vec{a}$, and radial acceleration $a_R=v^2/R$, since stone is in circular path

C) I can relate v_b to v_t using either i) energy ii) Dynamics and kinematics

ii) Messy since forces/accelerations change through the circular path

i) I can apply work-energy theorm for stone. Path has 2 parts:

first - circular, earth and rope interact with stone,

second - vertical, earth interacts with stone

In both parts the only force that does work is weight, since in first part hand is not moving $\Rightarrow \vec{T}\perp\vec{v} \Rightarrow \vec{T}$ does no work.

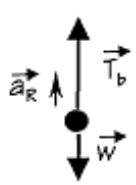
Execution:

B) Relate T_b to v_b

$\sum \vec{F}= m\vec{a}$

$\sum F_R= ma_R$

$T_b - w = m\, v_b^2/R$



C) Relate v_b to v_t

Work = ΔKE

For constant force

$\vec{F}\cdot\vec{d} = KE_f - KE_i$

$F_y d_y = KE_{top} - KE_{bottom}$

Substituting C) into B)

$T_b - w = 2\, w\, h/R$

$F_m = T_b = w + 2\, w\, h/R$

$= 18 + 2\ 18\ 23/.65$

$= \boxed{1292N}$

N=N m/m
units O.K.

Large compared to weight, but stone needs to travel up large distance

Check limits: $T_b\uparrow$ as $R\downarrow$, for smaller circle I'll need bigger force, reasonable

Figure 6-4. Example instructor solution 3.

6.3.2 The GAIQ (Group-Administered Interactive Questionnaire) approach

In the former interview studies (Henderson et al. 2007), interviewee's considerations about the design of instructor solutions are clarified through discussion between the interviewer and the interviewee. In this study, the GAIQ took advantage of a methods course for physics graduate assistants at the University of Pittsburgh. The GAIQ replaced the one-on-one discussion that takes place in an interview with a sequence of activities that took place during the first three weeks of the course. Twenty four graduate students were involved in this study. Table 6-1 summarizes the sequence.

Table 6-1. GAIQ sequence of activities.

Time	**Activity**
Pre	Individually, TAs wrote a solution to the target problem (Figure 6-1). After their own solutions were turned in, the TAs answered questions in pre-discussion worksheet (Table 6-2) that are related to the three example instructor solutions (Figure 6-2, Figure 6-3, and Figure 6-4).
Discussion	In groups of three, TAs answered the same questions in group worksheets. Then, a whole class discussion took place in which each group shared their work.
Post	Individually, TAs answered the same questions in post-discussion worksheet (Table 6-3).

In the pre-discussion stage, as part of their homework, TAs were asked to write a solution (to the problem presented in Figure 6-1) that they would hand out to their students. The TAs were later provided with three instructor solutions (shown in Figure 6-2, Figure 6-3, and Figure 6-4) for the problem and were asked to fill in a pre-discussion individual worksheet (Table 6-2) where they identified prominent features of the solutions, ranked the solutions based on i) which solution has more of each feature and ii) their preference for including each feature in their own solutions, and explained their reasons.

Table 6-2. Pre-discussion worksheet. TAs were asked to identify prominent features in the solutions, ranked the solutions (sol.) based on i) which solution has more of each feature and ii) their preference for including each features in their own solutions, and explained their reasons.

Attached are several instructor solutions for the problem you solved that were designed to be posted or distributed to students. They are based on actual instructor solutions. Take a look at each of these instructor solutions and describe the prominent features of those solutions. Which features of these solutions would you like to include in solutions you are writing for your students? Please explain your reasons.							
Solution features	Rank the solutions based on which solution has more of this feature. (You could also mark + for the solutions in which this feature exists.)			Rank the solutions based on your **preference** for this feature (A - for the one you like the most in how it represents this feature to C-for the one you like the least)			Why do you like/ not like this feature?
	Sol. I	Sol. II	Sol. III	Sol. I	Sol. II	Sol. III	

In the discussion stage, the TAs interacted in small groups to share their ideas regarding different features in the example problem solutions. After the small-group discussion, a whole class discussion took place in which each group shared their work.

Table 6-3. Post-discussion worksheet.

Write down the features' numbers that you originally noticed (using the feature list attached (Table 6-4). You might have termed them somewhat differently). For each of your features, please write down how you originally named this feature. Describe how and why, if at all, your preference towards it changed following the class discussions.					
Feature number	Your original feature name	Rate the solutions based on your current preference for this feature			In case your preference towards it changed following the class discussion, elaborate your final preferences: Why do you like or dislike this feature?
		Sol. I	Sol.II	Sol. III	

Table 6-4. Pre-defined feature list (from pilot study).

1. Providing a schematic visualization of the problem (a diagram)
2. Providing a list of knowns/unknowns
3. Providing a "separate" overview of how the problem will be tackled (Explaining premise and concepts -- big picture -- prior to presenting solution details)
4. Explicit sub-problems are identified (Explicitly identifying intermediate variables and procedures to solve for them)
5. Reasoning is explained in explicit words (Description/justification of why principles and/or sub-problems are appropriate/useful in this situation)
6. The principles/concepts used are explicitly written using words and/or basic mathematical representations (e.g., F=ma or Newton's 2nd Law)
7. Thorough derivation (Detailed/verbose vs. Concise/short/simplified/skips lots of derivation)
8. Long physical length (Long/verbose vs. Short/concise vs. Balanced/not too long, not too short)
9. Including details that are not necessary for explaining the problem solution (The solution is technically correct and complete without these 'unnecessary' details)
10. Providing an alternative approach
11. Solution is presented in an organized and clear manner
12. Direction for the progress of the solution: Backward vs. forward
13. Symbolic solution (Numbers are plugged-in only at the end)
14. Providing a check of the final result (e.g. if the unit is correct, or if the answer makes sense by examining the limits)

Finally, the TAs were provided with the opportunity to explain whether (and why) their preference changed by filling in a similar post-discussion worksheet (Table 6-3). On this post-discussion worksheet they were also asked to match the features they identified on the pre-discussion worksheet to a list of pre-defined features (presented in Table 6-4) corresponding to different aspects of the solution presentation. The list represents categories of features identified in a pilot study with another group of TAs. Some of these categories relate to the key stages in an expert problem solving and decision making process (Reif 1995).

Both the pre- and post-discussion worksheets as well as TAs' own solutions were collected for analysis. (All 24 TAs submitted their pre- and post-discussion worksheets and 23 TAs provided their own solutions). Features on the pre-worksheet that were not matched to Table 6-4 by the TAs in the post-worksheet were categorized as additional features by the researchers.

To represent the goals that TAs expressed when designing the example problem solution, TAs' reasons for including or not including specific features in an instructor solution were analyzed. Open coding (Strauss and Corbin 1990) was used to generate initial categories that were constantly compared to the new data and refined by the entire research team of 5 researchers together to arrive at a final set of categories. After developing coding categories, coding was done by one researcher (SL), with approximately 1/3 of the codes checked by other researchers. Any disagreements were discussed by 4 researchers until full agreement was established.

6.4 RESULTS

6.4.1 TAs' goals when designing instructor solutions to problems

By analyzing TAs' reasons underlying their preferences for different solution features, we identify the goals that the TAs have when contemplating how to design instructor solutions to problems:

Goal 1: Keeping students cognitively involved (expressed by 21 TAs): An instructor solution should be communicated in a manner that allows students to follow it. Thus, solutions should be "*easy to understand*" and avoid the situation where "*someone who is lost could not follow this*".

Goal 2: Modeling expert-like problem solving and decision making (expressed by 21 TAs): An instructor solution should externalize internal decision-making and representations that could aid a solver searching for a solution to a problem. For example, by looking at a solution "*the students should understand where the thought process comes from.*" In addition, there are several tools frequently found in expert problem solving that the TAs believe should be included in the example solutions because these tools facilitate their thinking. For example, a diagram "*allows students to visualize the problem*", and doing a unit check at the end "*allows students to evaluate their final answer - does it make sense.*"

Goal 3: Setting the basic requirement of an adequate solution (expressed by 14 TAs): There are some features that the TAs prefer because they are considered as the basic requirement of an adequate solution. For example, the solution should include the full process (e.g., "*showing how you arrived at an answer*"), be efficient (e.g. "*physics is straight, it should be solved in the most simple way*"), and orderly presented.

Goal 4: Promoting conceptual understanding (expressed by 14 TAs): An instructor solution should help students understand the physics concepts in a way that facilitates transfers to other situations. For example, the solution should "*make students think, not spoon feed them*". It should "*guide students to pay more attention to concepts rather than equations*" and "*not promoting mindless plug and chug*".

Goal 5: Keeping students emotionally involved (expressed by 7 TAs): An instructor solution should maintain students' attention and interest. For example, some TAs explained that they don't like to have a detailed solution "*because students won't have patience to finish it*." Another TA explained that the solution should "*easily explain concepts without scary math*".

Goal 6: Saving time (expressed by 5 TAs): Some TAs preferred a concise solution because a short solution "*saves time*". However, not many of them point out explicitly whose time is saved. It could mean that a short solution takes the TAs, who are busy with their own coursework in the graduate school, less time to write. But it could also mean that writing a solution in the most concise way would help students to save time in an exam, a situation in which time is an important issue.

Goal 7: Preventing exposure of mistakes (expressed by 2 TAs): Some TAs believe that a concise solution lowers the possibility of exposing oneself to critique. Students should refrain from length to avoid lowering their grades (e.g. "*fewer steps, less mistake*", "*more simple, less mistake*").

In general, goals 1 and 5 reflect TAs' concerns that in order for a solution to be effective, the message needs to be successfully conveyed. Goals 2, 3, and 4 focus on the conveyed content of the solution. Goal 2 is aligned with the cognitive apprenticeship research literature (Collins et al. 1991) which suggests that moving students further along the novice-expert continuum

requires modeling. In addition, we find that there are also some TAs who are concerned about the “practical” issues in problem solving – Goals 6 and 7.

As Figure 6-5 shows, most TAs (N=21) expressed the goals (Gs) of “keeping students cognitively involved (G1)” and “modeling expert-like problem solving and decision making (G2)”, followed by goals of “setting the basic requirement of an adequate solution (G3)” and “promoting conceptual understanding (G4)” (N=14). Goals such as “keeping students emotionally involved (G5)”, “saving time (G6)”, and “preventing exposure of mistakes (G7)” are mentioned by fewer than one-third of the TAs. It is promising to see that “modeling expert-like problem solving and decision making” is one of the most prevalent goals expressed by the TAs.

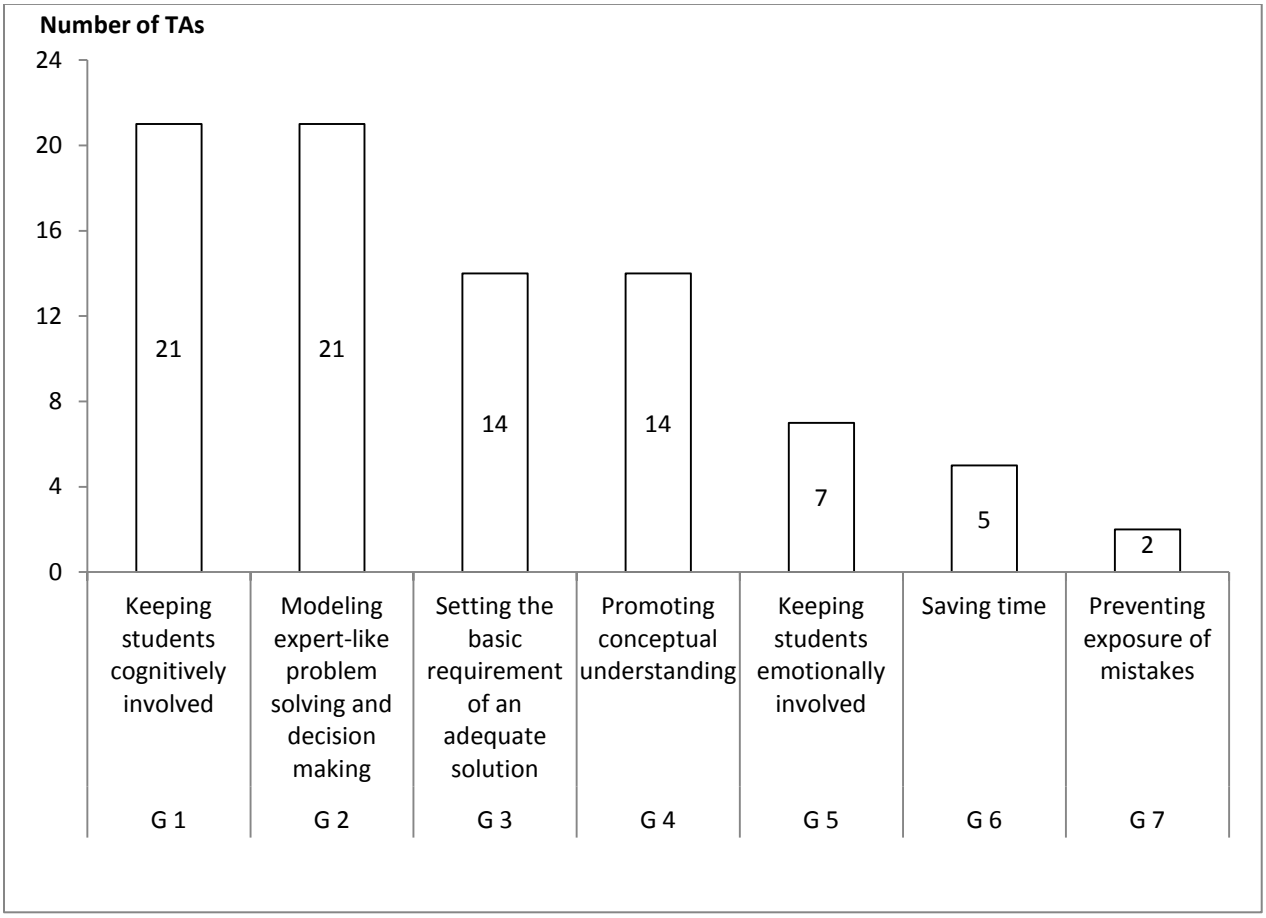


Figure 6-5. Number of TAs who mentioned each goal (G).

6.4.2 How do TAs believe a concrete instructor solution should be designed through different features in order to externalize these goals?

In the following sections, we'll first discuss the features that the TAs noticed from the 3 example solution artifacts and their preferences for those features. For features that correspond to the key stages in an expert-like problem solving and decision making process, TAs' self-reported preferences are compared to their actual practices as well as the recommendation from research literature. We'll also discuss the relationship between the goals and the features as well as whether the valued features and goals cohere with each other.

6.4.2.1 Features that the TAs noticed, valued, and used

In addition to the 14 pre-defined features given in Table 6-4, there were 3 additional features ("solution boxed", "meaning of symbols" and "in first-person narrative") that the TAs noticed. Because each was mentioned by only 1 or 2 TAs, we will focus only on the pre-defined features. Figure 6-6 presents the number of TAs who noticed each of the pre-defined features, and whether or not they liked it or were conflicted about it. If the TAs' preference for the feature changed after the discussion, or if the TAs explained both the pros and cons of a feature, they are placed in the "conflict" category.

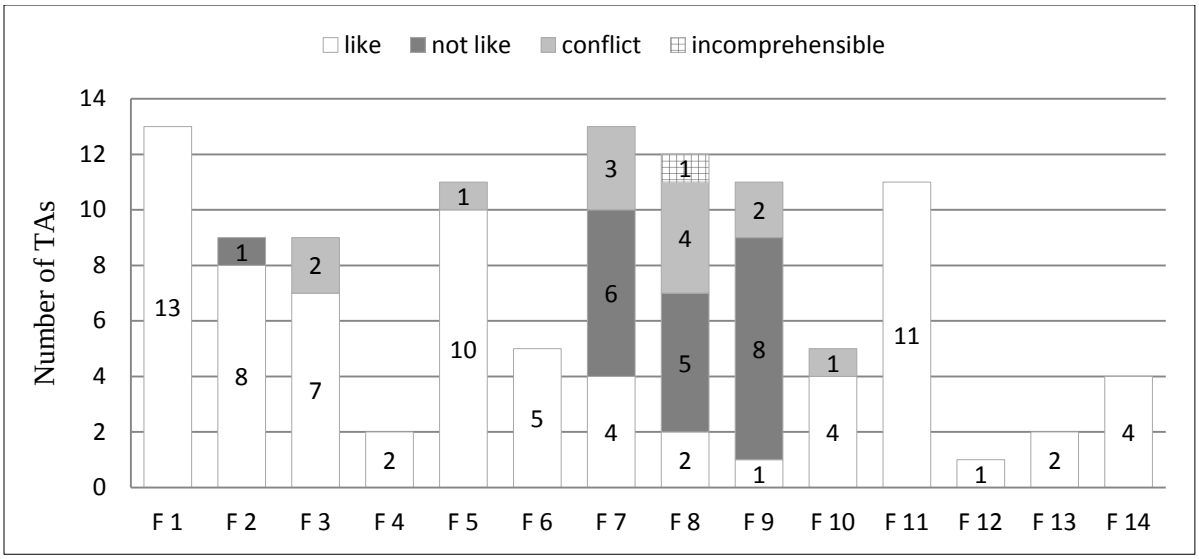


Figure 6-6. Number of TAs who mentioned each of the features (Fs). If the TAs' preference for the feature changed after the discussion, or if the TAs explained both the pros and cons of a feature, they are placed in the "conflict" category.

The 14 pre-defined features can be classified into two groups, one relates to the features that help to explicate an expert decision-making process within different key stages in a prescribed problem solving process (Reif 1995) and the other relates to the communication of the solutions. Each of these two groups consists of 3 clusters of features as shown in Table 6-5.

Table 6-5. Clusters (Cs) of Features (Fs).

Key stages in an expert-like problem solving and decision making process
C1-Initial problem analysis: F1) Providing a schematic visualization of the problem (a diagram) F2) Providing a list of knowns/unknowns
C2-Solution construction: Choices made (major solution steps): F4) Explicit sub-problems are identified F6) Principles/concepts used are explicitly written Additional explanations - Reasons for choices: F3) Providing a separate overview F5) Reasoning is explained in explicit words Framework within which choices are made: F10) Providing alternative approach F12) Backward vs. forward solution
C3-Checking solution: F14) Providing a check of the final result
Aspects related to the communication of the solution
C4-Extended details: F7) Thorough derivation F8) Long physical length F9) Including details that are not necessary for explaining the problem solution (i.e., the solution is correct and complete without these 'unnecessary' details)
C5-Organization and clarity F11) Organized and clear solution presentation
C6-Symbolic solution F13) Symbolic solution

In the following sections, we will separate our discussion based on the two groups.

Features related to the key stages in an expert-like problem solving and decision making process

Features Related to Initial Problem Analysis (C1)

Providing a schematic visualization of the problem (F1) and providing a list of knowns/unknowns (F2) are the features that relate to the explication of the initial problem analysis stage in an expert-like problem solving process (Reif 1995). F1 was one of the most

mentioned features (13 out of 24 TAs). F2 was mentioned by 9 TAs (the median for all features). These features were valued by almost all TAs who mentioned them. Only one TA expressed that he didn't like to provide a list of knowns/unknowns because it encourages students to solve the problem via mindless plug and chug. Other TAs valued the list of knowns/unknowns because it "*gives an idea of what you have and what you need*." Examination of TAs' own solutions (which 23 TAs provided) indicates that all TA solutions included a diagram. The list of knowns (and sometimes with the unknown targeted variable included) was found in the solutions of 12 TAs.

Although all TAs valued F1 (visualization), different TAs had different ideas about the preferred visualization shown in Figure 6-7. Table 6-6 shows that initially 9/13 TAs distinguished between the quality of diagrams, with 6 of them preferring a detailed drawing as presented in solution 3. Most of the TAs did not articulate why the detailed diagram was better than the others. TAs who chose the less detailed diagrams in solution 1 and/or 2 explained, for example, that they didn't like diagram 3 because "*complicated diagrams can be confusing*". Some TAs worried that the arrows in diagram 3 could be confusing to the students because they are used to represent both acceleration and velocity. It is likely that this concern was spread during the peer discussion stage, and therefore on the post discussion worksheet, the number of TAs who did not distinguish between solutions decreased and the number of TAs preferring solution 1 increased.

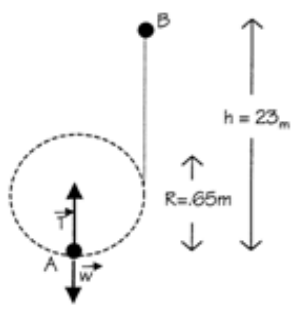
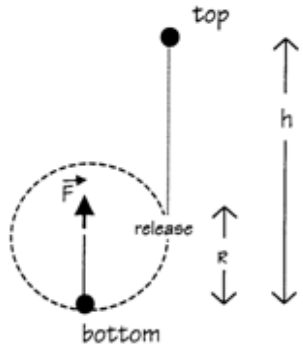
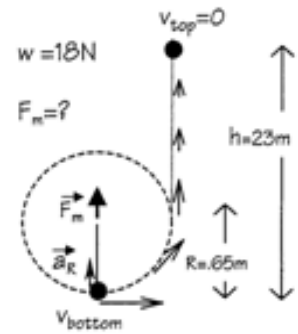
Solution 1 (S1)	Solution 2 (S2)	Solution 3 (S3)
		

Figure 6-7. Diagram used in each of the 3 example solution artifacts.

Table 6-6. TAs' preferences for each type of diagram.

Solution	Number of TAs (pre discussion)	Number of TAs (post discussion)
S1	1	3
S2	1	1
S3	6	5
S1=S2	1	1
S2=S3	0	1
S1=S2=S3	4	2

Features Related to Solution Construction (C2)

Six of the features (F3, F4, F5, F6, F10, F12) relate to the solution construction stage in an expert-like problem solving and decision making process. They can be further classified into 3 groups shown below:

Choices made (major solution steps):

F4) Explicit sub-problems are identified

F6) Principles/concepts used are explicitly written

Reasons for choices (additional explanations):

F3) Providing a "separate" overview

F5) Reasoning is explained in explicit words

Framework within which choices are made:

F10) Providing alternative approach

F12) Backward vs. forward solution

Based on Reif's suggestion (Reif 1995) to represent the process of solving a problem as a decision-making process, the major choices a person makes in a solution process involve defining sub-problems: intermediate variables and principles to find them. Underlying these choices is the solver's reasoning. While F4 and F6 present the major choices one makes, F3 and F5 provide additional explanations regarding the reasons underlying these choices. We note that this reasoning is guided by the solver's general perception of the framework within which choices are made (e.g., as a process that involves choosing between alternatives, or arriving at identified goal in a backward manner) represented in F10 and F12. Figure 6-6 shows that features related to reasons for choices were the most noticed ones.

Table 6-7 shows the solutions TAs believed best represent features related to reasons for choices. Most of the TAs who noticed these features thought that they were best represented in solution 2 or 3. However, as shown in Figure 6-3 and Figure 6-4, these solutions present reasoning in different ways. Solution 2 identifies the goal of each sub-problem and provides justification for the principles separately along the progress of the solution. Solution 3 describes a complete overview of how the problem should be broken into sub-problems and explains the principles applicable in each of the sub-problems at the very beginning. In general, solution 3 was slightly preferred by TAs for its enactment of F3 while solution 2 was generally preferred as

the best enactment of F5. Although most TAs did not explicate why one presentation is better than the other in the worksheets, in the whole-class discussion several TAs raised their concerns that students may not have the patience to read the whole chunk of text at the beginning of solution 3. Students may simply ignore all the explanations in the first part and jump directly into the second part with equations. They believed that reasoning that is presented beside the equations, as in solution 2, makes it easier to reference and students are more likely to process the information better.

Table 6-7. TAs' preferences for F3 and F5.

	F3 (pre)	F3 (post)	F5 (pre)	F5 (post)
S1	0	1	0	0
S2	2	3	7	6
S3	5	4	2	2
S1=S3	1	1	0	0
S2=S3	0	0	2	2
S1=S2=S3	1	0	0	1

In general, we find that F3 and F5 were valued by most TAs who noticed them. The TAs believed that these features play an important role in instructor solutions because they make the solution process clear and make the solution easier to follow. The TAs also believed that these features help students understand the internal thinking process that the instructor went through when solving the problem and facilitate better transfer to other problems. Except for minor concerns, such as "*overdoing the motivations can lead to undesired chunks of text*", which was the major reason why a few of the TAs expressed a conflicted preference, these features were generally valued by TAs. However, examination of TAs' own solutions indicates a discrepancy between their self-reported preferences and their actual practice. In total, only 3 out of 23 TAs

provided some outline of the sub-problems (F3) either at the very beginning or along the solution progression, and only 6 of the 23 TAs provided any justification for the principles used (F5).

Features 4 and 6, which explicate the choices made, were less noticed (2 and 5 TAs, respectively), although they were valued by all TAs who noticed them. One TA explained that "*I enjoy this feature [F4] because it helps set up a logical progression of the problem*"; other TAs explained their preference towards F6 in that "*the concepts may be more important than the answer*" or "*if we can use less math, I think we should do that, so students focus on physics*". Examination of TAs' own solutions indicates that no TA presented a solution in which the goals for each sub-problem were clearly stated. On the other hand, the concepts of "conservation of energy" and "Newton's 2nd Law" were explicitly written in words or the basic mathematical forms by 18 and 8 TAs, respectively.

Regarding the framework within which choices are made, 4 of the 5 TAs who noticed F10 (providing alternative approach) preferred this feature, explaining, for example, that "*this [feature] demonstrates how to develop an expert knowledge structure and how it makes the problem much simpler*." One TA was conflicted about this feature, as presenting an alternative approach "*could possibly confuse students*." However, no TA provided an alternative approach in their own solutions. As for F12 (backward vs. forward solution), most TAs did not notice it as an important consideration in the design of a solution. One difference between experts and novices is that experts (teachers) commonly regard introductory physics problems as exercises while they are actually problems for novices (students). As a result, experts may present problem solutions in a forward manner (such as solutions 1 and 2, which start with the knowns), reflecting their knowledge of the problem solution in an algorithmic way. Yet, to explicate the decision making process of an expert when solving a real problem, as suggested by instructional strategies

aligned with cognitive apprenticeship framework (Collins et al. 1991), one has to present the solution in a backward manner (such as solution 3, which starts with the targeted variable). Only one TA mentioned this feature. Although this TA pointed out that he preferred the backward solution as in solution 3, this TA presented his own solution in a forward manner. On the other hand, there were 8 TAs who originally presented a backward solution, even though they did not mention F12 in the worksheets. It is likely that many of the TAs considered the backward and forward solutions as interchangeable.

Features Related to Checking of Solution (C3)

F14, providing a check of the final result, is the feature which is related to the third stage of an expert problem solving process: checking of solution. We expected this feature to stand out in the artifact comparison technique since only 1 of the 3 solutions included it. However, only 4 TAs noticed this feature. In addition, examination of TAs' own solutions indicates that only one TA performed an answer check (this TA performed the unit check but didn't examine the limits) in the solutions they prepared for the introductory students. Although this feature was valued by all the TAs who noticed it, the findings suggest that this feature was underrated or ignored by most of the TAs.

Features related to the communication of solution

Features Related to Extended Details (C4)

Among all the features, F7 (thorough derivation), F8 (long physical length) and F9 (details that are not necessary for explaining the problem solution, i.e., the solution is technically correct and complete without these "unnecessary" details) were the ones that have the most TAs who didn't like them. They are all related to the "long/detailed" aspect of a solution.

For F7 and F8, there was no consensus on the preferences among the TAs. While some TAs consistently preferred a detailed solution before and after the class discussion, some TAs consistently preferred a concise one; other TAs (represented in the conflict category in Figure 6-6) expressed both the pros and cons of a detailed solution or changed their preferences after the group discussion. The TAs explained the value of a concise solution from various perspectives such as: "*physics is straight, it should be solved in most simple way*", "*long solution can be confusing*", "*concise [solution]: makes the students think and write, verbose: students more likely to doze off*", or "*[concise solution:] save time, fastest way*", etc. On the other hand, TAs who valued a detailed solution in general focused on the fact that a concise solution can be difficult for students to understand and an appropriate length and details will help students follow the steps better.

Feature 9, which is about details that are not necessary for explaining the problem solution in the sense that the solution is technically correct and complete without them, is associated to either solution 2, solution 3 or both solutions by the TAs who mentioned this feature. Except for one TA who originally named this feature as "more instruction" and explained that he liked solution 3 for this feature because "*that's what students need; more instruction is good for an example but may not [be] necessary for a solution of homework*", the majority of TAs didn't like this feature. The original feature names given by the TAs (such as "verbose", "way too complicated", "sensory overload", "lengthy statement", etc.) and the reasons they expressed for their preferences suggest that they were worried that this feature may confuse the students and lose students' attention.

Features Related to Organization and Clarity (C5)

F11, solution presented in an organized and clear manner, is valued by all TAs who mentioned it. However, there is no consensus on which solution best represents this feature. The different feature names that the TAs originally used reveal the different elements which constitute a clear and organized solution in their thoughts. Table 6-8 presents some examples of the feature names given by the TAs based on the different solutions they preferred.

Table 6-8. Examples of TAs' original feature names for F11 based on the different solutions they preferred.

Solution 1	Solution 2	Solution 3
• Simple mathematics • Clear • Elegant conceptual approach	• Self-explanatory • Step by step • Solution with steps • Presentation of the problem	• Self-explanatory • Steps/procedure • Clear view of the problem • Concrete procedure in solution • Presentation of the problem

Features Related to Symbolic Solution (C6)

Feature 13, symbolic solution, is an important feature in the teaching of physics problem solving because many students tend to plug in the numbers at the beginning of their solutions. Examination of TAs' own solutions indicates that all TAs' solutions were symbolic. However, only 2 TAs noticed this feature. It is likely that many TAs didn't notice this feature because solving problems symbolically has become a natural practice for them and because this feature was presented in all 3 solution artifacts provided. On the other hand, the fact that many more TAs noticed F1 (providing a schematic visualization) than F13 even though both features can be found in all 3 solutions suggests that F13 is a deeper feature that may require a deep familiarity with the teaching of physics problem solving in order to be able to notice it.

6.4.2.2 Relationship between the goals and the features

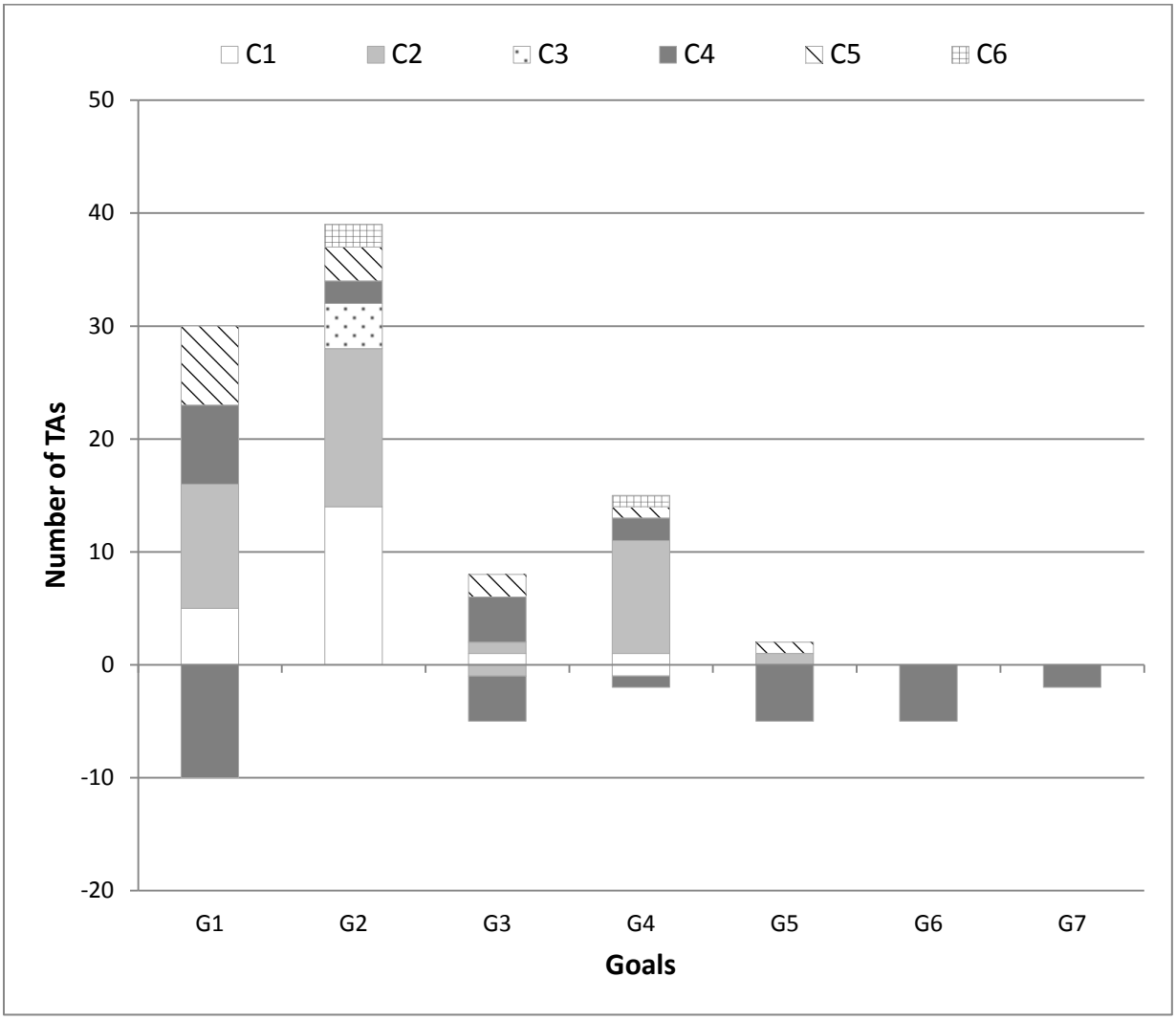


Figure 6-8. Number of TAs who mentioned features as supportive or contradictive to the goals (Gs). To get a somewhat more global picture, the 14 features are compressed into 6 clusters (Cs) as described in the previous section. The length of each bar indicates the number of the TAs who noticed at least one feature in that particular cluster and believed that the features support (positive) or contradict (negative) the goals displayed on the horizontal axis.

In order to investigate the relationship between the goals and the features, the number of TAs who perceive different features as supportive or contradictive to the different goals are plotted in

Figure 6-8. To get a somewhat more global picture, in Figure 6-8, the 14 features (Fs) are compressed into 6 clusters (Cs) as described in Table 6-5.

In Figure 6-8, the length of each bar indicates the number of the TAs who noticed at least one feature in that particular cluster and believed that the features support (positive) or contradict (negative) the goals displayed on the horizontal axis. We note that *initial problem analysis* (C1) and *solution construction* (C2) are prominent clusters that TAs perceive as supportive to the goals of *keeping students cognitively involved* (G1), *modeling expert-like problem solving and decision making* (G2), or *promoting conceptual understanding* (G4). The other prominent cluster is *extended details* (C4), which is considered as disadvantageous in regard to most goals (such as G5 - *keeping students emotionally involved*, G6 - *time saving* and G7 – *preventing exposure of mistakes*), and in some cases both positive and negative (such as G3-*requirement of an adequate solution* and G1-*cognitively involved*). For example, although a detailed solution may make it easier for students to follow, it could also work in the opposite way and make the students lose the thread more easily.

6.4.2.3 The extent to which valued design features and goals cohere with each other: Challenges in materializing the goals coherently

Figure 6-8 suggests a conflict between features supporting different goals. In particular, the negative occurrence of cluster C4- *extended details* conflicts with the positive occurrence of cluster C2- *demonstrating solution construction*. While C2 is perceived as one of the prominent clusters supporting goals such as G1, G2, and G4, it usually requires a longer length and more details, which is represented by the cluster of *extended details* (C4). However, C4 is considered as disadvantageous in regard to many goals such as G5, G6, G7, and sometimes G1 and G3 as

well. Such finding suggests a challenge in materializing all the goals coherently between different features.

Above all, the two features that most require extended details in the C2 cluster are *F5 - Reasoning is explained in explicit words, and F3 - Providing a separate overview*. To get more insight into the nature of the conflict we examined how TAs weighed these features as compared to features in the C4 cluster*: F7- Thorough derivation* and *F8- Long physical length* (we ignored F9, which is stated in a negative manner from the beginning). We will discuss this conflict along with the TAs' backgrounds since we observe that TAs' backgrounds may play a role in the nature of this conflict.

Table 6-9 shows that before the peer discussion, non-American TAs (N=13), most of whom had their secondary and post-secondary education in China or India, were more likely than American-educated TAs (N=11) to dislike F7 and/or F8 even though both groups were likely to prefer F5 and/or F3. TAs from foreign countries may have different expectations about what an introductory physics student is able or expected to do, that relate to their own learning experiences in the past. As one TA who was formerly educated in China explicitly pointed out after the activity: "*TA solution should be clearer than just a few key steps. That's what I really learned. In the class, all of the native students [TAs] tended to avoid using a simple key step solution. That's surprising because in my own country I have only seen such solutions. I used to avoid using many words explaining what is going on and why we have to apply these theorems, because that's the situation in my own country, where students have to think all by themselves*."

Table 6-9. Comparison of the number of TAs who (1) noticed either F3 and/or F5 vs. F7 and/or F8 (either one of them) (2) expressed positive (+) or negative (-) preference for the feature(s) in the pre-discussion worksheet**.** If the TAs explained both the pros and cons of the same feature or displayed a somewhat conflicting preference, they are placed in the +/- category.

	Former education: USA (N=11)		Former education: Other (N=13)	
	Notice	Preference	Notice	Preference
F3 or F5	9/11 (82%)	+: 9/9 (100%)	7/13 (54%)	+: 6/7 (86%) +/-: 1/7 (14%) (*)
F7 or F8	8/11 (73%)	+: 4/8 (50%) - : 4/8 (50%)	12/13 (92%)	+: 1/12 (8.3%) -: 9/12 (75%) +/-: 2/12 (17%) (**)
(*): This TA noticed F5, which he originally named as "marginal notes" and in general valued it. He explained that this feature "*give notes for some procedures*". However, he also added a comment saying that "*but it's not good for too many notes*". (**): One TA noticed F8 and indicated that there are pros and cons for a concise solution. He explained that a concise solution "*saves time, but could also cause confusion*". Overall speaking, this TA preferred solution 1 (the concise solution) the best. The other TA expressed a somewhat conflicting preference between F7 and F8. He valued F7, which he originally named as "sufficient details", but preferred a brief demonstration when discussing F8.				

The above statement demonstrates a process where, following the discussion with peers - a process embedded in the GAIQ tool, the TAs re-considered their former preferences regarding the design of problem solutions. In particular, two non-American TAs and one American TA explicitly changed their preferences from concise solution to thorough presentation after the group discussion. Reasons why they initially preferred a concise solution include "*less*

exhaustive, more efficient", "*use the best solution with least steps*", etc. After the group discussion, they focused on a different goal and explained that they preferred a longer solution because "*appropriate physical length will help student follow the steps*" or that "*if it's too concise, people may be confused*".

In general, the difficulty of materializing all the goals coherently in a single solution can be observed through shifts in TAs' preferences between the pre and the post worksheets, as well as the different perspectives regarding a single feature that were raised in either one of the worksheets themselves. For example, although one TA consistently preferred a concise solution (the opposite of F8) in the pre and post worksheets, he raised his concern about the disadvantage of this feature in the post-discussion worksheet, noting that "*Solution 1 is short and sweet, hard to understand for a layman though.*" Another TA expressed that "*although a brief demonstration may cause confusion for some students, it will make more students feel comfortable.*" Two other TAs expressed that there's a need to find a middle ground that people should aim for.

6.5 CONCLUSIONS

In summary, the most prominent goals that the TAs expressed when contemplating how to design problem solutions are *keeping students cognitively involved* (G1) and *modeling expert-like problem solving and decision making* (G2), followed by *demonstrating the standard for adequate solution* (G3) and *promoting conceptual understanding* (G4). It is promising to see that Goal G2, which is aligned with the research literature about how to move students further along the novice-expert continuum (Collins et al. 1991) is mentioned by many of the TAs.

However, findings indicate that the TAs didn't necessarily notice all features that help with this goal. In general, most features that the TAs noticed were "surface features" such as F1 (drawing), F5 (explaining reasoning in explicit words), F8 (length) that one is likely to be aware of even if s/he doesn't know much about the teaching of physics problem solving. This is compared to features such as F6 (principles used are explicitly written), F12 (backward vs. forward solution) or F13 (symbolic solution) that are deeper features of the solution and were less commonly identified by the TAs.

Moreover, we find that the self-reported preferences didn't match well with the solutions the TAs wrote on their own before seeing the 3 example solution artifacts. Although features in all 3 clusters that are aligned with the key stages in an expert-like problem solving and decision making process (clusters C1, C2 and C3) were in general valued by the TAs once they were noticed, only features related to problem re-description (especially F1) were generally found in their own solutions. The majority of the TA solutions contained little or no reasoning to explicate the underlying thought processes. The answer check was found in only one TA's solution.

A challenge in materializing the goals coherently in a concrete solution was also observed in this study. For example, feature cluster C2- *solution construction*, a prominent cluster that TAs believe to support G1 and G2, usually requires a longer length and more detail, which contradict goals such as G1 and G5 - *keeping students cognitively and emotionally involved* and G6 - *time saving*. It is likely that this conflict may be one of the reasons why the TAs didn't use some of the features that they valued (such as features F3 and F5 in C2) in their own solutions.

We note that this study is conducted at the beginning of the TA training course, when the TAs had just entered graduate school and started their TA jobs. Many TAs might not be well

acquainted with students' prior knowledge, and they might not be familiar with the specific scaffolding supports – especially those deep features that are aligned with the recommendation from research literature – that can help students learn from the problem solutions effectively. It is likely that this activity, which helps to elicit TAs' initial ideas about the design of problem solutions in physics teaching, can provide a starting point for TAs' professional development and influence their practices in the future. It can be beneficial if ideas from the research literature are explicitly imported in the follow-up activities, and the TAs are explicitly guided to reflect on their practice in light of these new ideas. The follow-up activities can also invite the TAs to explore the possible ways to deal with the conflicts between different features and goals. For example, students may have different needs at different stages of the learning process (e.g., during lecture, after homework or after a test). Solutions may be designed differently to reflect the most important goal in various situations, so that each of the solutions is commensurate with the students' need, and all the different solutions combined together would cover most of the goals that the TAs aim for.

6.6 CHAPTER REFERENCES

Collins, A., J. S. Brown and A. Holum (1991). "Cognitive apprenticeship: Making thinking visible." American Educator 6(11), 38-46.

Henderson, C., E. Yerushalmi, V. Kuo, K. Heller and P. Heller (2007). "Physics faculty beliefs and values about the teaching and learning of problem solving. II. Procedures for measurement and analysis." Phys. Rev. ST Phys. Ed. Res. 3(2), 020110.

Reif, F. (1995). "Millikan lecture 1994: Understanding and teaching important scientific thought processes." Am. J. Phys. 63(1), 17-32.

Strauss, A. and J. M. Corbin (1990). Basics of qualitative research: Grounded theory procedures and techniques. Thousand Oaks, CA, Sage Publications.

Yerushalmi, E., C. Henderson, K. Heller, P. Heller and V. Kuo (2007). "Physics faculty beliefs and values about the teaching and learning of problem solving. I. Mapping the common core." Phys. Rev. ST Phys. Ed. Res. 3(2), 020109.

Yerushalmi, E., C. Henderson, W. Mamudi, C. Singh and S. Lin (2011). "The group administered interactive questionnaire: An alternative to individual interviews". Physics Education Research Conference, Omaha, NE.

7.0 CAN MULTIPLE-CHOICE QUESTIONS SIMULATE FREE-RESPONSE QUESTIONS?

7.1 ABSTRACT

We discuss a study to evaluate the extent to which free-response questions could be approximated by multiple-choice equivalents. Two carefully designed research-based multiple-choice questions were transformed into a free-response format and administered on the final exam in a calculus-based introductory physics course. The original multiple-choice questions were administered in another similar introductory physics course on the final exam. Findings suggest that carefully designed multiple-choice questions can reflect the relative performance on the free-response questions while maintaining the benefits of ease of grading and quantitative analysis, especially if the different choices in the multiple-choice questions are weighted to reflect the different levels of understanding that students display.

7.2 INTRODUCTION

When it comes to assessing students' learning in physics, there is always concern about the format of the assessment tool. While a multiple-choice (MC) test provides an efficient tool for assessment because it is easy to grade, instructors are often concerned when using it because a

test in a free-response format facilitates a more accurate understanding of students' thought processes. In addition, free-response questions allow students to get partial credit for displaying different extent of understanding of the subject matter tested, which is appreciated by many instructors and students. Thus, there appears to be a trade-off between the two assessment tools. If the instructors choose to implement a multiple choice test, they often feel that they are completely sacrificing the benefits that the free-response questions could provide.

Research indicates that the difficulties students have related to a given topic can be classified into relatively few categories. If the choices in the MC questions are designed carefully to reflect the common difficulties students have, it is possible that the multiple choice questions will faithfully reflect the performance on the free-response questions while maintaining their benefits of ease of grading and comparison of classes taught using different instructional approaches. Here, we present a study designed to investigate the relation between students' performance on quantitative free-response and MC questions. We converted two research-based MC questions into free-response format and administered them on the final exam in a calculus-based introductory physics course (course A). The original MC questions were administered in an equivalent introductory physics course (course B) on the final exam. Students' performance in two different courses is compared. Moreover, we investigate the correlation between students' actual performance on the free-response questions and a "simulated" multiple-choice performance had the problems been given in the MC format in course A.

7.3 METHODOLOGY

An object which weighs 20,000 N is at rest on a 25° incline as shown. The coefficient of static friction between the object and the inclined plane is 0.90, and the coefficient of kinetic friction is 0.80. Find the magnitude of the frictional force on the object.
Note: These trigonometric results might be useful: sin 25°=0.423, cos 25°=0.906

(a) 8460 N (b) 14496N (c) 16308N
(d) 18120 N (e) none of the above

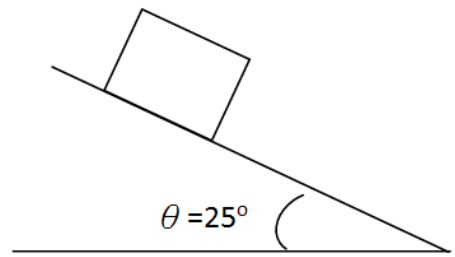
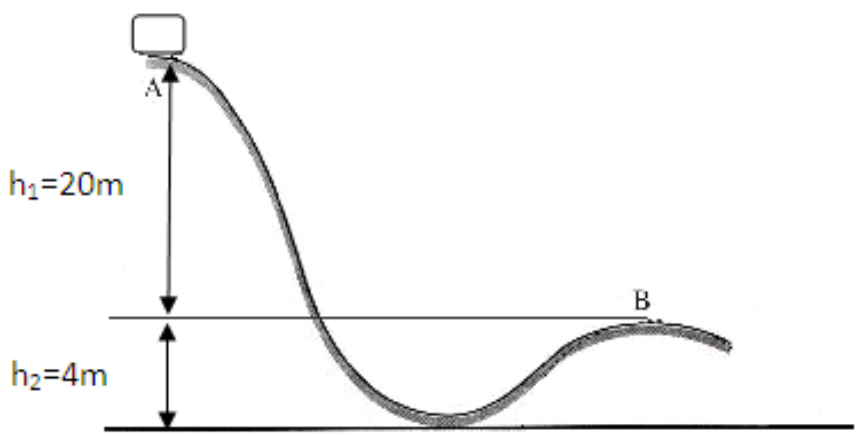


Figure 7-1. Problem Statement for Question 1



A cart of mass 40kg is sliding down a frictionless track as shown in the figure. Close to point B, the track is part of a circle with a radius of 50 m. Assuming the cart starts from rest at point A, find the magnitude of the normal force on the car at point B.

(a) 392N (b) 705.6N (c) 313.6N (d) 15.68N (e) 78.4N

Figure 7-2. Problem Statement for Question 2.

Figure 7-1 and Figure 7-2 present the MC questions that were administered in this study. Question 1 concerns an object at rest on an inclined plane. Students were asked to find the

magnitude of the static friction acting on the object, which is equal to $mg\,sin\,\theta$ according to Newton's 2nd Law. Research (Singh 2007) suggests that many students struggle with this question because they believe that the magnitude of static friction (f_s) is always equal to its maximum value, the coefficient of static friction (μ_s) times the normal force (F_N). This notion is not valid for this question because the maximum value of static friction exceeds the actual frictional force needed to hold the object at rest. Other student difficulties found from a pilot study include the confusion between static and kinetic friction, the challenge in decomposing the force correctly, etc. The alternative choices in the MC question are designed to reflect these difficulties. Table 7-1 presents the choices in algebraic form where all symbols have their standard meaning (actual choices were numerical).

Table 7-1. The algebraic form for the choices in question 1 and the different scores assigned in the "weighted multiple-choice" simulation. The correct answer is indicated by the shaded background.

Choice	Question 1	Score
(a)	$mgsin\theta$	1.0
(b)	$\mu_k\, mgcos\theta$	0.3
(c)	$\mu_s\, mgcos\theta$	0.5
(d)	$mgcos\theta$	0.2
(e)	none of the above	0.0

Question 2 concerns a roller coaster cart on a frictionless track. The question asks for the normal force acting on the cart when it goes over a hump, which can be solved by using the principles of the conservation of mechanical energy and Newton's 2nd law in the non-equilibrium situation (with centripetal acceleration involved). Previous research (Singh 2009) indicates that a common difficulty introductory physics students have is that they think of a non-equilibrium situation which involves the centripetal acceleration as an equilibrium situation by treating the centripetal force as an additional force. The correct use of the centripetal acceleration and Newton's 2nd Law in this question should yield $N - mg = -\frac{mv^2}{r}$. However, students who treat it

as an equilibrium question and believe that the centripetal force is an additional force obtain an answer of the type $N - mg - \frac{mv^2}{r} = 0 \Rightarrow N = mg + \frac{mv^2}{r}$, which has an incorrect sign. In addition to this common mistake, a pilot study indicates that some students incorrectly believe that the normal force is equal to the gravitational force ($N = mg$) without contemplating the centripetal acceleration. On the other hand, there are also students who completely skip the gravitational force and claim that $N = \frac{mv^2}{r}$. Moreover, some students have difficulty figuring out the speed of the cart at point B because they are confused by the two different heights provided. These common difficulties are incorporated in the design of the multiple-choice questions. Table 7-2 presents the choices in algebraic form where all symbols have their standard meaning.

Table 7-2. The algebraic form for the choices in question 2 and the different scores assigned in the "weighted multiple-choice" simulation. The correct answer is indicated by the shaded background. Except for choice (d), the speed at point B (v_B) is calculated correctly using the square root of $2gh_1$ in choices (b), (c) and (e).

Choice	Question 2	Score
(a)	$N=mg$	0.2
(b)	$N = mg + m\frac{{v_B}^2}{r}$	0.8
(c)	$N = m\frac{{v_B}^2}{r}$	0.7
(d)	$N = mg - m\frac{{v_B}^2}{r}$ v calculated using $\sqrt{2g(h_1 + h_2)}$	0.9
(e)	$N = mg - m\frac{{v_B}^2}{r}$	1.0

The MC questions and the corresponding free-response questions were administered on an exam in two introductory physics courses (with 185 and 153 students.) The Force Concept Inventory scores (pre-/post-instruction) indicate that students in these two courses are

comparable (no statistically significant difference). The free-response questions were the same as their counterparts in the MC format except that there were no choices provided. Students' performance on the free-response questions was graded using rubrics, with a full score of 1 for each question. The rubrics incorporated students' common difficulties. Different partial scores were assigned based on the problem solving approach and the principles used. An example of a rubric can be found in (Lin and Singh 2010).

To construct two types of "simulated" MC score from the answers students provided for the free-response questions, student responses were first binned into different categories by comparing and matching their answers to the different choices in the MC questions. For the dichotomous MC simulation, a score of 1 (correct choice) or 0 (incorrect choice) was then assigned for the various categories. For the weighted MC simulation, to simulate the partial credit which is usually awarded for a free-response question, we assigned partial credit to different binned responses based upon approaches students used for the free-response questions. The scores assigned to each of the choices in this "weighted" MC simulation are shown in Table 7-1 and Table 7-2. The different weights for the choices reflect the different levels of understanding students display. The weights are commensurate with the rubrics used to grade the free-response questions.

Table 7-3. Summary of grading methods in the two courses.

	Course A (given free-response questions)	Course B (given multiple-choice questions)
Graded using a rubric	Yes	--
Multiple-choice (dichotomous)	simulated (*)	Yes
Multiple-choice (weighted)	simulated (*)	Yes
(*): Student responses were first binned into different categories by comparing and matching their answers to the different choices in the multiple-choice format and then assigning a score as discussed in the text.		

To summarize, students who were given the free-response questions were graded using three different methods: using a rubric, using a dichotomous MC simulation, and using a weighted MC simulation. On the other hand, students who were given the MC questions were graded using two methods involving dichotomous or weighted scoring. Table 7-3 summarizes the different methods used to analyze student performance in the two courses. In order to compare student performance in the two courses, in each course, students were divided into groups 1 to 5 based on their overall performance on the final exam (with group 5 representing the group of students performing the best on the final exam, followed by those in group 4, etc.). Students in the whole course were first ranked by their scores on the final exam. About 1/5 of the students were assigned to groups 5, 4, 3, 2, and 1, respectively. For each group, students' average scores on each question were plotted. We compared the trends in student performance in the two courses.

7.4 FINDINGS

Table 7-4 presents the percentage of students who were binned into different categories by matching their free-response answers to the choices in the corresponding MC questions. We find that out of the 153 students involved, 84% and 88% of students' free-response answers could be matched to the a priori choices in the multiple-choice questions 1 and 2, respectively. Except for the mistake of using 1-D kinematics equations instead of the principle of conservation of mechanical energy to find the speed at point B in question 2 (which cannot be detected in the multiple-choice question because both methods yield the same numerical value for an option in

the MC question), the findings suggest that a carefully designed research-based multiple choice question can reasonably reflect the distribution of common difficulties students have (as detected in their free-response answers).

Table 7-4. Percentage of students binned into different categories for simulated MC by comparing their free-response answers to the choices in both MC questions. The correct answer for each question is indicated by the shaded background.

Choice	Question 1	%	Question 2	%
(a)	$mgsin\theta$	28	$N=mg$	9
(b)	$\mu_k F_N$	7	$N = mg + m\frac{{v_B}^2}{r}$	31
(c)	$\mu_s F_N$	46	$N = m\frac{{v_B}^2}{r}$	28
(d)	$mgcos\theta$	3	$N = mg - m\frac{{v_B}^2}{r}$ (v calculated using $\sqrt{2g(h_1 + h_2)}$)	6
(e)	none of the above	16 (*)	$N = mg - m\frac{{v_B}^2}{r}$	14
--	--	--	Other	12
(*): Both choice (b) and choice (c) were found in one students' free-response answer in this category.				

Students' average performance on questions 1 and 2 is presented in Figure 7-3. The black and white labels are used to distinguish students in course B, who were given the multiple-choice questions and students in course A, who were given the free-response questions. In general, we find that the trends for students' performance in the two courses are similar regardless of the question format they were given. For example, comparing students' rubric-graded free-response performance in course A to students' dichotomous multiple-choice performance in course B, we find that in both courses, students who displayed a higher level of expertise on the final exam

(e.g., students in groups 4 and 5) on average typically scored higher on questions 1 and 2 in both formats than those who did not perform as well on the final exam.

Comparing students' performance in course A to course B, we also find that there is a better correspondence between students' performance on free-response questions in one class and the multiple-choice questions in another class (shown in Figure 7-3) if partial credits are awarded for both types of questions. The reason free-response performance for students in one class has a better match with the weighted multiple-choice performance in the other class than the dichotomous multiple-choice performance is that the weights for the weighted MC performance were similar to those used in the rubric to score the free-response questions.

Table 7-5 presents the correlation coefficients between students' free-response performance (graded using the rubrics) and the simulated multiple-choice performance (both dichotomous and weighted) in course A. It shows that the correlation coefficient is always higher for weighted multiple-choice simulation. The correlation between free-response and simulated dichotomous MC performance is higher for the question with a single very stronge distracter choice (question 1) in MC compared to the question with several distracter choices (question 2) each of which represent a different level of understanding.

Table 7-5. Correlation (N=153) between the free-response performance graded using the rubrics (FR) vs. the simulated multiple-choice performance for questions (Q) 1 and 2.

	FR vs. simulated multiple-choice (dichotomous)		FR vs. simulated multiple-choice (weighted)	
	Q 1	Q 2	Q 1	Q 2
Correlation coefficient (r)	0.890	0.483	0.928	0.945
p-value	0.000	0.000	0.000	0.000

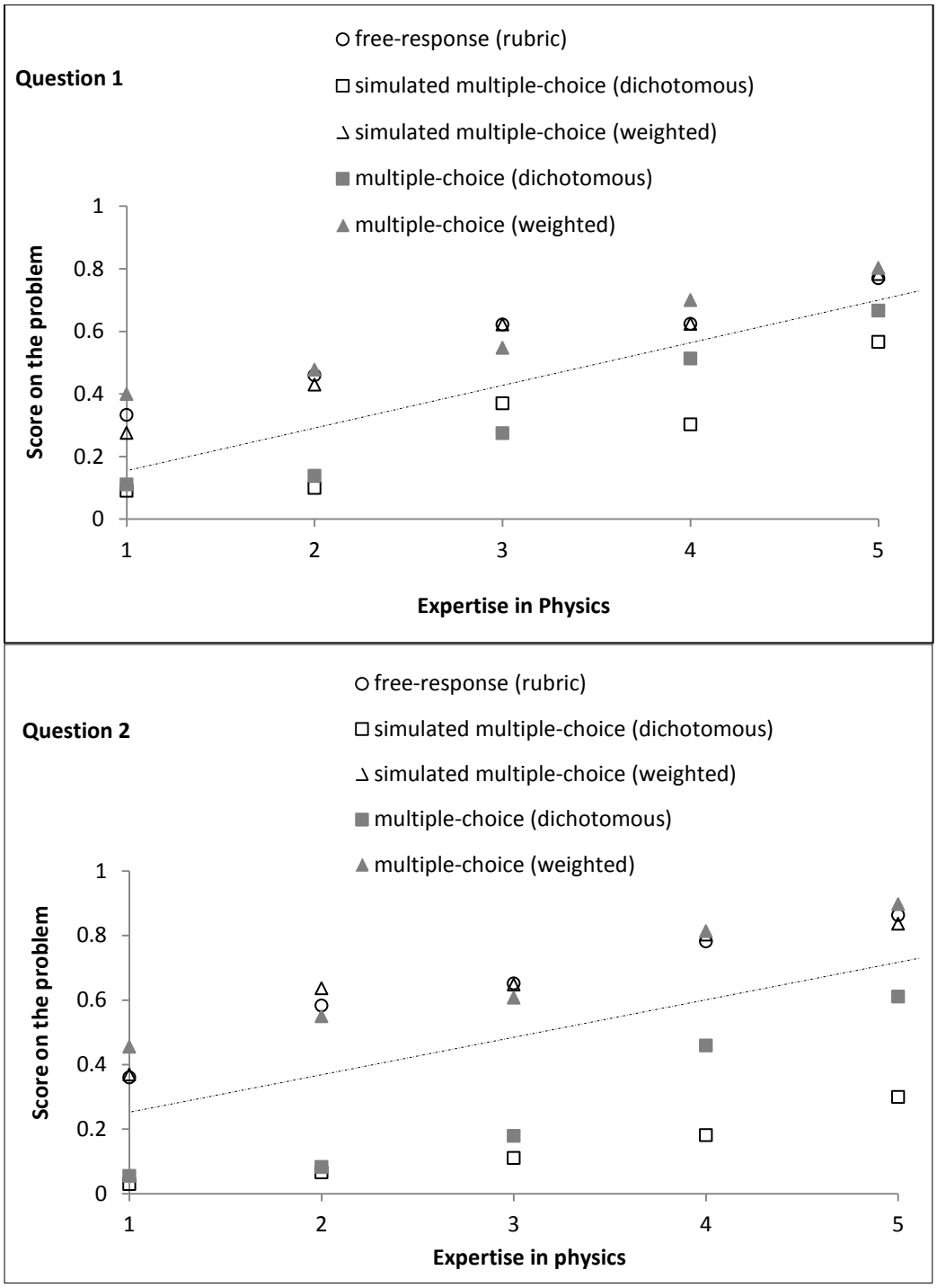


Figure 7-3. Students' average performance on questions 1 and 2. The white and black data labels are used to indicate students in course A (who were given the free-response question) and students in course B (who were given the multiple-choice questions), respectively. A dashed line is included on the figure to separate the data for the dichotomous case vs. the case where partial credits are assigned to the students.

7.5 DISCUSSION

We find that the trends in student performance on the research-based multiple-choice questions given to one class (in which the distracter choices correspond to students' common difficulties) and the free-response questions given to another equivalent class are similar in that those who displayed a higher level of expertise on the final exam in each of the classes performed better on the questions than those who displayed a lower level of expertise regardless of the format of questions provided to them. Moreover, there is a good match between students' free-response answers in one class and the a priori choices in the MC questions administered to another class.

The findings suggest that research-based MC questions can reasonably reflect the relative performance of students on the free-response questions, especially if the answers for the MC questions are graded in a weighted manner by assigning partial credit to the different choices students selected. We note that (similar to the rubrics for the free-response questions) the weightings for the different alternative choices in the MC questions reflect the fact that some mistakes are not as "bad" as others despite the fact that they lead to students selecting an incorrect choice. In summary, if different scores are assigned to the different choices in the MC questions in the weighted model to reflect the different levels of understanding students display, there is a good overlap between students' MC performance in one class and the free-response performance in another class.

We re-emphasize that the fidelity of a MC question to a free-response performance depends strongly on the incorrect choices given (Aubrecht and Aubrecht 1983). If students' common difficulties found via research are incorporated, instructors can utilize MC questions without sacrificing accuracy in assessment of students' thinking processes. Free-response questions are useful only if they are graded carefully based on a good rubric. When they are

graded leniently without a good rubric, the resulting scores will not typically reflect the appropriate level of understanding students' have. Weighted MC questions can be graded by a computer with weights corresponding to a good rubric for each distracter choice. Once the weights for the choices have been determined via research, MC questions can be as accurate for assessment purposes as rubric-based free-response questions without the time constraint.

7.6 CHAPTER REFERENCES

Aubrecht, G. and J. Aubrecht (1983). "Constructing objective tests." Am. J. Phys. 51(7), 613-620.

Lin, S. Y. and C. Singh (2010). "Using analogies to learn introductory physics." AIP Conf. Proc. 1289, 209-212.

Singh, C. (2007). "Effect of misconceptions on transfer in problem solving." AIP Conf. Proc. 951, 196-199.

Singh, C. (2009). "Centripetal acceleration: Often forgotten or misinterpreted." Phys. Educ. 44(5), 464-468.

8.0 CONCLUSIONS AND FUTURE CONSIDERATIONS

In this thesis, I discuss several investigations designed to enhance and/or assess students' problem solving skills from different perspectives. Problem solving is a central component in many physics courses. A major goal of many physics courses is to help students overcome the difficulties they have and help them develop good problem solving skills as well as good knowledge structure so that they can transfer what they learned from one context to another. In order to achieve this goal, it is important to be able to effectively identify the difficulties students have and employ strategies from both the learner's and the instructor's perspectives to assist students in effective learning. The studies in this thesis, which cover investigation on issues mentioned above in both the introductory physics courses and upper-level undergraduate quantum mechanics course, have implications for improving the classroom practice on problem solving.

8.1 USING ISOMORPHIC PROBLEMS TO LEARN INTRODUCTORY PHYSICS

In these studies, we investigate students' abilities to perform analogical problem solving between isomorphic problems. Students were explicitly asked in the quizzes to learn from the solved problem provided to them and take advantage of what they learned from the solved problem to solve another quiz problem which involves the same physics principles but has different surface

features. Different scaffolding supports were provided to help students process through the analogy deeply. The findings show that the solved problems provided were typically useful for helping students invoke the relevant principles in the corresponding quiz problems, but the greatest challenge was in applying the principles they learned appropriately to the new situation presented in the transfer problem. For example, one common difficulty students had in the first study was that they failed to differentiate between the situations of an object going over the top versus the bottom of a circle and they didn't contemplate the direction of the corresponding centripetal acceleration and its sign in the corresponding equation. Another example from the third study is that many students didn't have a clear plan for how to solve the three-step quiz problem. They didn't realize how to decompose the quiz problem into suitable sub-problems and they sometimes combined several processes into one, applied the principles in inappropriate situations, or applied the principles correctly but didn't discern their relevance to the target variable. In general, we found that it was more challenging for students to transfer their learning from a two-step problem to a three-step problem than transferring from a two-step problem to another two-step problem.

If the transfer problem involves context which often triggers a common student misconception, such as the notion that "static friction should always equal its maximum value" as discussed in the second study, the misconception can hinder a full transfer of learning to the quiz problem. Although many students were able to identify some of the relevant concepts involved in the solved problem and employ this learning in the solution to the quiz problem, the answers they provided suggest that students might not necessarily realize their misconception and they might still hold the misconception when solving the quiz problem unless the scaffolding support provided explicitly guided them to contemplate issues which are directly relevant to the

validity of their misconceptions. We also found in this study that the scaffolding support which was good for the calculus-based students may be beyond the zone of proximal development of the algebra-based students. The finding in this study re-emphasizes the importance of building scaffolding supports based on students' prior knowledge and skills.

Combining the findings from three studies across different topics, we find that out of the different interventions provided, the one which postpones the providing of the solved problem until students have attempted to solve the quiz problem without help first is typically one of the best interventions for students in both the algebra-based and calculus-based courses. Having tried the quiz problem on their own first may make the browsing over the solved problem for relevant information more structured and productive because students have already searched through their knowledge base of physics and attempted to organize the information given in the quiz problem. Moreover, many students were found to be able to fix at least part of their initial mistakes when they attempted the quiz problem a second time after learning from the solved problem.

Further strategies that may assist students in the analogical problem solving activity to apply the principles they learned from the solved problem correctly to the new context in the quiz problem involve providing more than one solved problem for students to learn from. If two isomorphic solved problems which contain different contexts and different application details are provided to students, the different application details presented in the two solutions may serve as a model and/or a hint for how different situations may require the application of the same principles but the application details must be adjusted in each situation. The additional solved problem provided may also be used as a bridging problem to help students transfer their learning from the two-step problems to a three-step problem. Moreover, it may be useful to deliberately guide students to think about not only the similarities, but also the differences between the

isomorphic problems and discuss how the similarities as well as the differences provide implications to the solution before actually solving the transfer problem. Future studies can focus on examining the effects of these general strategies as well as other possible scaffolding supports that are designed to improve students' learning for specific physics concepts as discussed in each study. It is also useful to conduct think-aloud interviews with students from all levels of expertise to examine the effect of these strategies thoroughly and deeply.

8.2 CATEGORIZATION IN QUANTUM MECHANICS

The categorization of problems can be a useful tool to investigate how knowledge is structured in a person's mind and examine the patterns that the person sees in a problem when contemplating how to solve it. In the study on categorization of quantum mechanics problems by faculty and students, we find that unlike the categorization of introductory mechanics problems, in which the categories created by the faculty members are uniform and there is strong agreement on what is considered as a "good category" or "poor category", the categorization in quantum mechanics is more diverse. Moreover, the faculty members who are recruited to evaluate the quality of categorization indicate that it is challenging to evaluate other people's categorization in quantum mechanics. Although in the categorization of introductory mechanics problems, categories based on surface features (such as "incline" or "pulley") are never considered as good categories, some faculty members consider categories such as "hydrogen atom" as good categories in the quantum categorization. The interviews with faculty members suggest that the categorization of introductory physics problems typically involves identifying

the fundamental physics principles while the categorization in quantum mechanics is based on the concepts and procedures.

Although the categorization of quantum mechanics problems by the faculty is more diverse, the faculty still overall scored higher in grouping together problems based on similarity of solutions as compared to the students. Future studies can focus on comparing an individual's performance on the categorization task to their abilities in actually solving the problems, and examining the effect of using categorization tasks as a learning activity to help students organize their thoughts in quantum mechanics. It will be useful to investigate whether students who participate in the categorization task can learn from this activity and achieve a better problem solving performance than students who are not involved in the categorization task. It will also be useful to investigate if there is a different teaching emphasis in the quantum mechanics courses among faculty members depending on the different types of categories they create.

8.3 TA TRAINING

In the study on first-year graduate teaching assistants' beliefs about the use and design of example problem solutions, we found that the goals that the teaching assistants expressed for the use of example problem solutions involve modeling expert-like problem solving and decision making, which is aligned with the recommendations from research literature. However, when asked to compare different example solution artifacts and express the preferences for the different features that they noticed in the solutions, the TAs didn't necessarily notice all features that help to demonstrate the expert problem solving approaches. Moreover, there was a potential challenge to coherently materialize all the goals and the considerations that the TAs expressed in

a concrete solution. For example, the modeling of expert-like problem solving usually requires a longer solution and more details, which contradict the consideration to present the solution in a concise manner because a short solution can save time or because a solution presented in a more concise way can keep students cognitively and emotionally involved without distracting them or losing their attention. A difference between TAs' conceptions of different considerations and the corresponding features based on their former education was also observed. In general, TAs with foreign background (in our study most of them had their secondary and post-secondary education in China or India) were more likely to value concise solutions as compared to the American TAs.

Comparing the features in the TAs' own solutions to their self-reported preferences and the recommendations from research literature about the modeling of expert-like problem solving, we found that there was much room for improvement regarding TAs' actual practices. Since the study was implemented when the graduate teaching assistants just entered graduate school and started their teaching jobs, the activity discussed in this study may provide a starting point for TAs' professional development. It is likely that this activity in which TAs were explicitly asked to contemplate and discuss their views about the design of problem solutions in physics teaching can influence their future practices. Future studies can focus on examining the influence of such activity on TAs' future practices and identifying the scaffolding support needed to enhance their expertise in teaching physics. It is also useful to compare the TAs' views with the views from the faculty members who have much experience in teaching. This comparison can provide implications for physics education researchers to develop strategies to enhance both the classroom practices as well as TAs' professional development.

8.4 MULTIPLE-CHOICE ASSESSMENT AND FREE-RESPONSE ASSESMENT

The study in this thesis suggests that carefully designed multiple-choice questions can reasonably reflect the relative performance on the corresponding free-response questions if common student difficulties found via research are incorporated when designing the alternative choices. By comparing students' performance on two research-based multiple-choice questions to students' performance on the corresponding free-response questions, we found that more than 80 percent of students' free-response answers in one class could be matched to the a priori choices in the multiple-choice questions administered to another class. The trend in students' performance on the research-based multiple-choice questions in one class was similar to the trend in students' performance on the free-response questions given to another equivalent class. In addition, in this study, we developed a "weighted" scheme to grade the multiple-choice questions by assigning different partial credits to different choices in the multiple-choice question so that the score assigned could reflect the different levels of understanding students have. Findings suggest that the research-based multiple-choice questions could to good extent mirror the relative performance on the free-response questions, especially if students' performance on the multiple-choice questions was graded using the weighted scheme.

We note that the free-response questions are useful if they are graded based on good rubrics. If they are graded leniently without a good rubric, the free-response questions lose much of their value. On the other hand, if the multiple-choice questions are designed with alternative choices to reflect the different common difficulties students have, and if the appropriate weights for each choice is determined, the computer can grade the multiple-choice questions efficiently and effectively, which is similar to grading using a rubric. Future studies can focus on developing research-based multiple-choice questions (with an appropriate partial score assigned

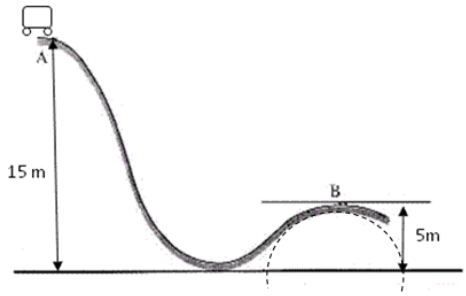
to each alternative choice) in different topics. It will be useful if a collection of high quality research-based multiple-choice questions is available to the instructors so that they can make use of this efficient assessment tool to investigate the difficulties their students have and to improve learning.

APPENDIX A

MATERIALS GIVEN TO STUDENTS IN DIFFERENT INTERVENTION GROUPS (CHAPTER 2)

A.1 THE SOLVED PROBLEM (ROLLERCOASTER PROBLEM)

A friend told a girl that he had heard that if you sit on a scale while riding a roller coaster, the dial on the scale changes all the time. The girl decides to check the story and takes a bathroom scale to the amusement park. There she receives an illustration, depicting the riding track of a roller coaster car along with information on the track (the illustration scale is not accurate). The operator of the ride informs her that the rail track is smooth, the mass of the car is 120 kg, and that the car sets in motion from a rest position at the height of 15 m. He adds that point B is at 5m height and that close to point B the track is part of a circle with a radius of 30 m. Before leaving the house, the girl stepped on the scale which indicated 55 kg (the scale is designed to be used on earth and displays the mass of the object placed on it). In the rollercoaster car the girl sits on the scale. According to your calculation, what will the scale show at point B?



<Solution>

1. Description of the problem

Knowns:

The height from which the car was released: $h_A = 15$ m

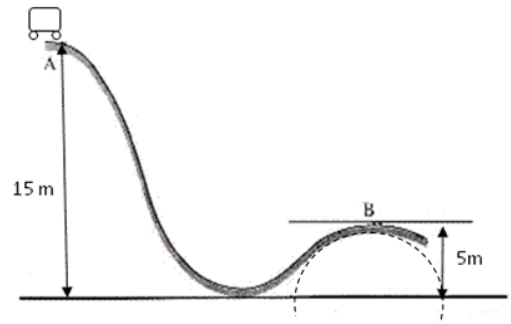
The velocity of the car at point A: $v_A = 0$

The height of point B: $h_B = 5$ m

The radius at point B : $R_B = 30$ m

The mass of the car: M = 120 kg

The mass of the girl: m = 55 kg



Target quantity:

N_B = the normal force (as indicated by the scale) at point B.

Assumptions:

The friction with the track is negligible; the acceleration of gravity g =10 m/s^2

2. Constructing the solution

Step A: Plan

During the motion of the girl along the curved track, the magnitude of her velocity as well as the radial acceleration change from point to point. If the radial acceleration changes, from Newton's 2nd Law ($\sum \vec{F} = m\vec{a}$) we infer that the net force acting in the radial direction on the girl changes as well. The net force on the girl is the sum of two forces acting on her: the force of gravity and the normal force.

To calculate the normal force at point B, we can use Newton's 2nd Law: $\sum \vec{F} = m\vec{a}$; however we will need to know the acceleration at this point.

Since close to point B the track is part of a circle, we can think of the motion near point B as a circular motion. Therefore, the radial acceleration at point B is given by the centripetal

acceleration: $a_B = \frac{v^2}{R}$. In order to calculate the centripetal acceleration at B, we need to know the speed at point B (v_B) and the radius of the track at that point (R_B).

We will calculate the speed of the girl at point B using the law of conservation of mechanical energy between point of departure A and point B (the mechanical energy is conserved since the only force that does work is the force of gravity which is a conservative force. The normal force does no work because it is perpendicular to the velocity at every point on the curve.)

Step B: Execution of the plan

<u>Sub-problem 1</u> <u>– calculating the speed at point B</u>

We will set ground as the reference level for gravitational potential energy and compare the total mechanical energies of the girl and car at points A and B: $E_A = E_B$

Since the speed is zero at point A, the kinetic energy at that point is zero. Therefore we get:

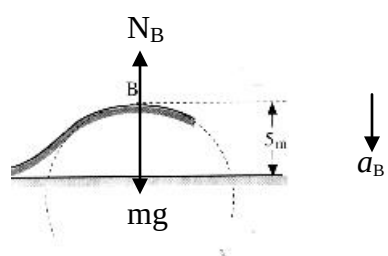
$$\mathrm{PE_A} = \mathrm{PE_B} + \mathrm{KE_B} \quad \Rightarrow \quad (\mathrm{M+m})\mathrm{gh_A} = (\mathrm{M+m})\mathrm{gh_B} + \frac{(\mathrm{M+m})\mathrm{v_B^2}}{2} \quad \Rightarrow \quad \mathrm{gh_A} = \mathrm{gh_B} + \frac{\mathrm{v_B^2}}{2}$$

We can calculate the speed at point B in the following manner:

$$\mathrm{v_B^2} = 2\ \mathrm{g}\ \mathrm{h_A} - 2\ \mathrm{g}\ \mathrm{h_B} = 2(10\ \mathrm{m/s^2})(15\ \mathrm{m} - 5\ \mathrm{m}) = 200\ \mathrm{m^2/s^2}$$

<u>Sub problem 2</u> <u>– calculating the normal force at point B</u>

Using Newton's 2nd Law:



$$\sum \vec{F}_B = m\vec{a}_B$$

$$N_B - mg = -ma_B$$

$$N_B = mg - m\frac{v_B^{\ 2}}{\mathrm{R_B}}$$

$$N_B = mg - m\frac{v_B^{\ 2}}{\mathrm{R_B}} \Rightarrow N_B = (55\ \mathrm{kg})(10\ \mathrm{m/s^2}) - (55\ \mathrm{kg})\frac{200\ \mathrm{m^2/s^2}}{30\ \mathrm{m}} = 183.3\mathrm{N}$$

Final Result: When the car crosses Point B on the track, the scale indicates 18 kg (the calibration of the scale is in kilograms, per 10N →1kg since we assume the acceleration of gravity g =10 m/s^2)

3. Reasonability check of the solution:

Reasonability check of the parametric solution $N_B = mg - m\frac{v_B^{\ 2}}{\mathrm{R_B}}$

- limiting case 1 : At rest (v = 0): $N_B = mg$

- limiting case 2 : On a horizontal surface ($R_B \to \infty$): $N_B = mg$

A.2 INTERVENTION 1

A.2.1 Intervention 1: step 1

In this quiz, first browse over and learn from the solved problem. After 10 minutes, please turn in the solved problem, and then solve two quiz problems. One of the quiz problems will be exactly the same as the solved problem that you browsed over and the other problem will be similar.

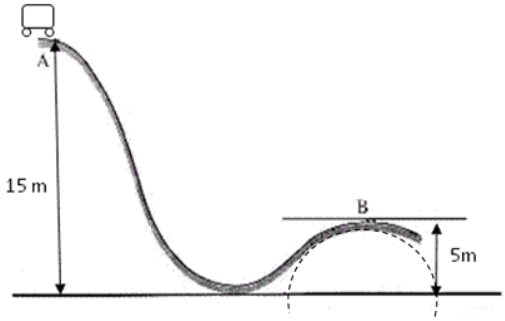
A.2.2 Intervention 1: step 2

Now, first solve the problem you just browsed over (problem 1) and then solve the other problem (problem 2) which is similar.

Problem 1:

A friend told a girl that he had heard that if you sit on a scale while riding a roller coaster, the dial on the scale changes all the time. The girl decides to check the story and takes a bathroom

scale to the amusement park. There she receives an illustration, depicting the riding track of a roller coaster car along with information on the track (the illustration scale is not accurate). The operator of the ride informs her that the rail track is smooth, the mass of the car is 120 kg, and that the car sets in motion from a rest position at the height of 15 m. He adds that point B is at 5m height and that close to point B the track is part of a circle with a radius of 30 m.



Before leaving the house, the girl stepped on the scale which indicated 55 kg (the scale is designed to be used on earth and displays the mass of the object placed on it). In the rollercoaster car the girl sits on the scale. According to your calculation, what will the scale show at point B?

Before you solve problem 2, identify the similarities between problem 1 and problem 2, and answer the questions below.

Problem 2:

A family decides to create a tire swing in their backyard for their son Ryan. They tie a 15 m nylon rope to a branch. To make the ride more exciting, they construct a launch point that is 13 m above the lowest point in the ride so that they don't have to push Ryan all the time. You are their neighbor, and you are concerned that the ride might not be safe, so you calculate the maximum tension in the rope to see if it will hold. Calculate the maximum tension in the rope, assuming that Ryan (mass 30 kg) starts from rest from his launch pad. Is it greater than the maximum rated value of 2500 N?

(a) Write down explicitly the similarities between problem 1 and problem 2 in detail and whether you can use the similarities to solve problem 2.

(b) Explain where the tension is maximum in the swing's trajectory and why?

(c) Now solve for the maximum tension in the rope. Is the maximum tension greater than 2500 N?

A.3 INTERVENTION 2

A.3.1 Intervention 2: step 1

First, solve the quiz problem below. After 10 minutes, please turn in the answer sheet, and you'll be provided with another solved problem to browse over and learn from. Then, you will have to redo this quiz problem.

<u>Quiz Problem:</u>

A family decides to create a tire swing in their backyard for their son Ryan. They tie a 15 m nylon rope to a branch. To make the ride more exciting, they construct a launch point that is 13 m above the lowest point in the ride so that they don't have to push Ryan all the time. You are their neighbor, and you are concerned that the ride might not be safe, so you calculate the maximum tension in the rope to see if it will hold. Calculate the maximum tension in the rope, assuming that Ryan (mass 30 kg) starts from rest from his launch pad. Is it greater than the maximum rated value of 2500 N?

(a) Explain where the tension is maximum in the swing's trajectory and why?

(b) Now solve for the maximum tension in the rope. Is the maximum tension greater than 2500 N?

A.3.2 Intervention 2: step 2

Before you solve the same quiz problem again, go over the solved problem first. Identify the similarities between the solved problem and the quiz problem you just solved. Then answer the questions below and redo the quiz problem.

Quiz Problem:

A family decides to create a tire swing in their backyard for their son Ryan. They tie a 15 m nylon rope to a branch. To make the ride more exciting, they construct a launch point that is 13 m above the lowest point in the ride so that they don't have to push Ryan all the time. You are their neighbor, and you are concerned that the ride might not be safe, so you calculate the maximum tension in the rope to see if it will hold. Calculate the maximum tension in the rope, assuming that Ryan (mass 30 kg) starts from rest from his launch pad. Is it greater than the maximum rated value of 2500 N?

(a) Write down explicitly the similarities between problem 1 and problem 2 in detail and whether you can use the similarities to solve problem 2.

(b) Explain where the tension is maximum in the swing's trajectory and why?

(c) Now solve for the maximum tension in the rope. Is the maximum tension greater than 2500 N?

A.4 INTERVENTION 3

In this quiz, first browse over and learn from the solved problem, then solve the quiz problem below. Similar to the solved problem, this quiz problem below could be solved using

conservation of energy and Newton's 2nd law (with centripetal acceleration). Before you solve the quiz problem, go over the solved problem and then answer the following questions.

Quiz Problem:

A family decides to create a tire swing in their backyard for their son Ryan. They tie a 15 m nylon rope to a branch. To make the ride more exciting, they construct a launch point that is 13 m above the lowest point in the ride so that they don't have to push Ryan all the time. You are their neighbor, and you are concerned that the ride might not be safe, so you calculate the maximum tension in the rope to see if it will hold. Calculate the maximum tension in the rope, assuming that Ryan (mass 30 kg) starts from rest from his launch pad. Is it greater than the maximum rated value of 2500 N?

(a) Consider the following discussion between two students about circular motion.

Student 1: If an object is undergoing a circular motion, then there's an extra centripetal force acting on it. The magnitude of this centripetal force is $\frac{mv^2}{r}$, and the direction is pointing from the object to the center of the circle. So, if an object is traveling on a track of a vertical circle, the free body diagram at the top is as shown in fig 1. Using Newton's 2nd law in equilibrium situation, we have

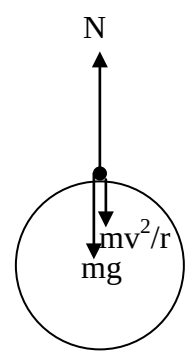


fig 1

$$\sum F = 0 \Rightarrow N = mg + \frac{mv^2}{r}$$

Student 2: No, centripetal force is not a physical force. It is just a name given to the net force for circular motion. If an object is undergoing a circular motion, it has a centripetal

acceleration (a_c) with magnitude $\frac{v^2}{r}$ and its direction is pointing from the object to the center of the circle. Therefore, the free body diagram is as shown in fig 2. Using Newton's 2nd law in the non-equilibrium situation, we have

$$\sum F = ma_c \Rightarrow N - mg = -\frac{mv^2}{r} \Rightarrow N = mg - \frac{mv^2}{r}$$

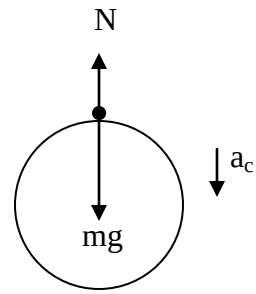


fig 2

Which one do you agree with? Why?

(b) Explain where the tension is maximum in the swing's trajectory and why?

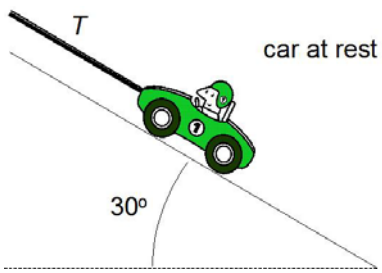
(c) Now solve for the maximum tension in the rope. Is the maximum tension greater than 2500 N?

APPENDIX B

MATERIALS GIVEN TO STUDENTS IN DIFFERENT INTERVENTION GROUPS (CHAPTER 3)

B.1 THE SOLVED PROBLEM (TENSION PROBLEM)

A car which weighs 15,000 N is at rest on a frictionless 30° incline as shown. The car is held in place by a light strong cable parallel to the incline. Find the magnitude of the tension force in the cable. Note: These trigonometric results might be useful:
sin 30°=0.500, cos 30°=0.866 to three places.



<Solution>

1. Description of the problem

Knowns:

weight of the car: W = mg = 15000 N

angle of the incline: $\theta = 30^o$

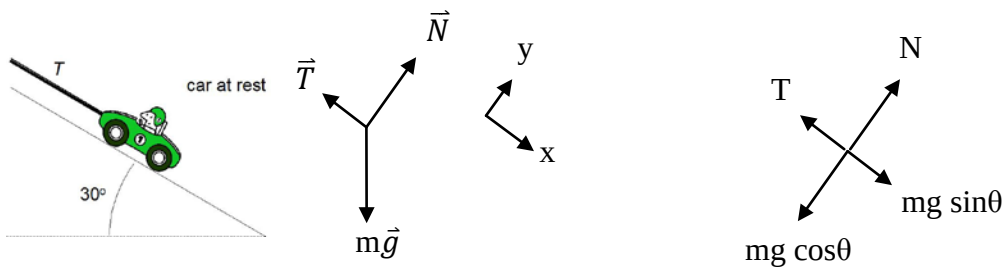
The incline is frictionless.

Target Quantity:

the magnitude of the tension force (T)

2. Constructing the solution

Free body diagram:



Since the incline is frictionless, there are only 3 forces acting on the car: the gravitational force (mg), the normal force (N), and the tension (T). Because the car is stationary, the velocity of the car, which is 0, doesn't change with time; therefore, the acceleration ($\vec{a}$) of the car is zero, i.e., $\vec{a} = \vec{0}$. From Newton's 2nd Law:

$$\vec{F}_{net} = m\vec{a}$$

we know the net force ($\vec{F}_{net}$), which is defined as the vector sum of all the forces acting on the car, should be $\vec{0}$ as well:

$$\vec{F}_{net} = \vec{0}.$$

If we decompose the force into x and y components as indicated above, both

$$\vec{F}_{net,x} = \vec{0} \text{ and } \vec{F}_{net,y} = \vec{0}.$$

From $\vec{F}_{net,x} = \vec{0}$, we have $T - mg\sin\theta = 0$

$$\therefore T = mg\sin\theta$$
$$= 15000 \text{ (N)} \cdot \sin 30^{o}$$
$$= 7500 \text{ (N)}$$

(If we want to find the magnitude of normal force, then from $\vec{F}_{net,y} = \vec{0}$, we have

$$N - mg\cos\theta = 0$$

$$\therefore N = mg\cos\theta$$
$$= 15000\ (\text{N}) \cdot \cos 30^{o} \qquad)$$
$$= 12990\ (\text{N})$$

3. **Reasonability check of the final result:**

- limiting case 1: θ= 0, we expect T to be zero, which agrees with our result

  $T = mg\sin\theta = mg \cdot \sin(0) = 0$

- limiting case 2: θ= 90°, we expect T to be equal to the weight of the car, which agrees with our result $T = mg\sin\theta = mg \cdot \sin(90^{o}) = mg$

B.2 INTERVENTION 1

B.2.1 Intervention 1: step 1

In this quiz, first browse over and learn from the solution to a problem. After 10 minutes, please turn in the solution, and then solve two quiz problems. One of the quiz problems will be exactly the same as the solved problem that you browsed over and the other problem will be similar.

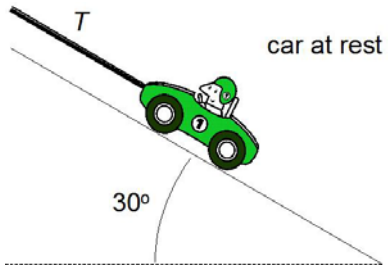
B.2.2 Intervention 1: step 2

Now, first solve the problem you just browsed over (problem 1) and then solve the other problem (problem 2) which is similar.

Problem 1:

A car which weighs 15,000 N is at rest on a frictionless 30° incline as shown. The car is held in place by a light strong cable parallel to the incline. Find the magnitude of the tension force in the

cable. Note: These trigonometric results might be useful: sin 30°=0.5, cos 30°=0.866 to three places.



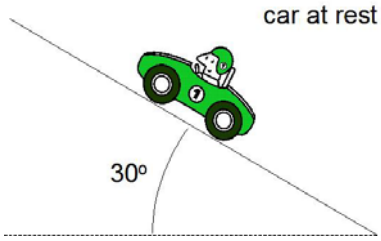
Before you solve problem 2, identify the similarities between problem 1 and problem 2, and answer the questions below.

Problem 2:

A car which weighs 15,000 N is at rest on a 30° incline, as shown below. The coefficient of static friction between the car's tires and the road is 0.90, and the coefficient of kinetic friction is 0.80. Find the magnitude of the frictional force on the car.

Note: These trigonometric results might be useful:

sin 30°=0.500, cos 30°=0.866 to three places.



a) Write down explicitly the similarities between the quiz problem and the solved problem provided in detail and whether you can use the similarities to solve this quiz problem.
b) Now solve for the magnitude of the frictional force on the car.
c) Solve for the magnitude of the normal force.

B.3 INTERVENTION 2

B.3.1 Intervention 2: step 1

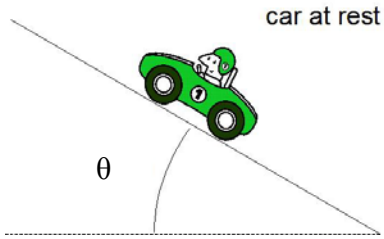
First, solve the quiz problem below. After 10 minutes, please turn in the answer sheet, and you'll be provided with another solved problem to browse over and learn from. Then, you will have to redo this quiz problem.

Quiz Problem:

A car which weighs 15,000 N is at rest on a 30° incline as shown ($\theta = 30^o$). The coefficient of static friction between the car's tires and the road is 0.90, and the coefficient of kinetic friction is 0.80. Find the magnitude of the frictional force on the car.

Note: These trigonometric results might be useful:

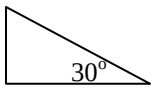
$\sin 30^o = 0.500$, $\cos 30^o = 0.866$ to three places.

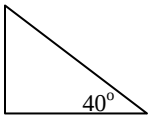


a) Solve for the magnitude of the frictional force on the car.

b) Solve for the magnitude of the normal force on the car.

c) An object is on a surface whose inclination can be changed. The object on this inclined plane is at rest for two separate angles of inclination 30° and 40°. <u>Based on your daily experience</u>, if the inclined plane is steeper (corresponding to $\theta = 40^o$ in our case), should the magnitude of the frictional force between the object and the surface be larger or smaller? Please explain.





d) Now solve for the magnitude of the frictional force on the car if the angle of the inclined plane (θ) is 40°. Note: These trigonometric results might be useful:

sin 40°=0.643, cos 40°=0.766.

e) Is your answer in d) consistent with what you predicted in c)?

B.3.2 Intervention 2: step 2

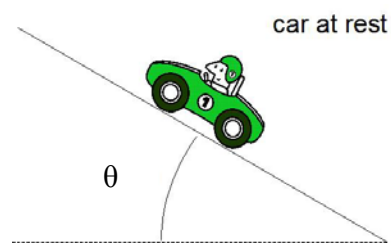
Before you solve the same quiz problem again, go over the solved problem first. Identify the similarities between the solved problem provided to you and the quiz problem you just solved, and then answer the questions below.

Quiz Problem:

A car which weighs 15,000 N is at rest on a 30° incline as shown ($\theta = 30^o$). The coefficient of static friction between the car's tires and the road is 0.90, and the coefficient of kinetic friction is 0.80. Find the magnitude of the frictional force on the car.

Note: These trigonometric results might be useful:

sin 30°=0.500, cos 30°=0.866 to three places.



a) Write down explicitly the similarities between the quiz problem and the solved problem provided in detail and whether you can use the similarities to solve this quiz problem.

b) Solve for the magnitude of the frictional force on the car.

c) Solve for the magnitude of the normal force on the car.

d) An object is on a surface whose inclination can be changed. The object on this inclined plane is at rest for two separate angles of inclination 30° and 40°. <u>Based on your daily experience</u>, if the inclined plane is steeper (corresponding to $\theta = 40^o$ in our case), should the magnitude of the frictional force between the object and the surface be larger or smaller? Please explain.

e) Now solve for the magnitude of the frictional force on the car if the angle of the inclined plane (θ) is 40°. Note: These trigonometric results might be useful:

$\sin 40^o = 0.643$, $\cos 40^o = 0.766$.

f) Is your answer in e) consistent with what you predicted in d)?

B.4 INTERVENTION 3

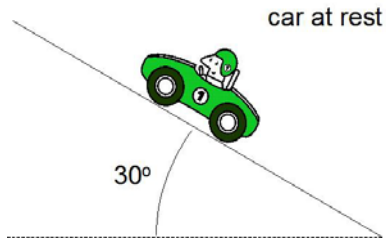
Before you solve the quiz problem below, browse over and learn from the solution to a solved problem. Identify the similarities between the solved problem provided to you and your quiz problem, and then answer the questions below.

Quiz Problem:

A car which weighs 15,000 N is at rest on a 30° incline, as shown below. The coefficient of static friction between the car's tires and the road is 0.90, and the coefficient of kinetic friction is 0.80. Find the magnitude of the frictional force on the car.

Note: These trigonometric results might be useful:

sin 30°=0.500, cos 30°=0.866 to three places.



a) Write down explicitly the similarities between the quiz problem and the solved problem provided in detail and whether you can use the similarities to solve this quiz problem.

b) Explain the meaning of the <u>inequality</u> in $f_s \leq \mu_s N$ where f_s stands for the static frictional force, μ_s is the coefficient of static friction, and N is the normal force. Can you find the frictional force on the car in this quiz problem without knowing μ_s? If yes, discuss why μ_s is not needed to solve this problem. If no, discuss why μ_s is needed to solve this problem.

c) Now solve for the magnitude of the frictional force on the car.

d) Solve for the magnitude of the normal force using the component of force perpendicular to the inclined plane. Also, if the normal force was involved in solving part c) above, please check that the normal force in that case is consistent with what you calculated here. If you did not use the normal force to solve for part c), you need not check for consistency between parts c) and d).

APPENDIX C

MATERIALS GIVEN TO STUDENTS IN DIFFERENT INTERVENTION GROUPS (CHAPTER 4)

C.1 THE SOLVED PROBLEM (SNOWBOARD PROBLEM)

Your friend Dan, who is in a ski resort, competes with his twin brother Sam on who can glide higher with the snowboard. Sam, whose mass is 60 kg, puts his 15 kg snowboard on a level section of the track, 5 meters from a slope (inclined plane). Then, Sam takes a running start and jumps onto the stationary snowboard. Sam and the snowboard glide together till they come to rest at a height of 1.8 m above the starting level. What is the minimum speed at which Dan should run to glide higher than his brother to win the competition? (Dan has the same weight as Sam and his snowboard weighs the same as Sam's snowboard.)

<Solution>

1. Description of the problem

Knowns:

Dan's mass : $m_D = 60$ kg

The mass of Dan's snowboard : $m_B = 15$ kg

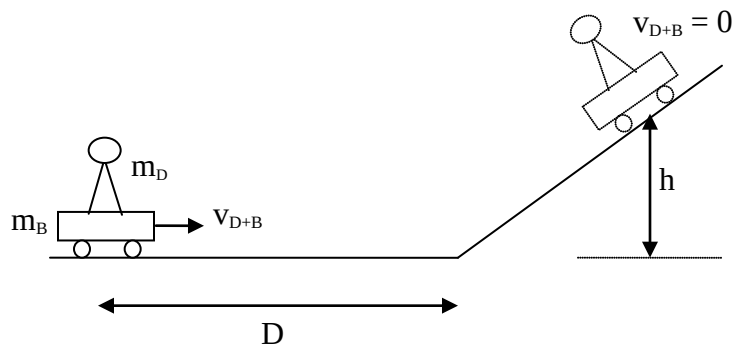
Desired minimum height : $h_{min} = 1.8$m

The distance between the initial position of the snowboard and the inclined plane: D = 5 m

Target quantity:

The minimum speed that Dan should run: $v_{D,min}$ (If $v_D \geq v_{D,\ min} \Rightarrow h \geq h_{min}$)

Diagram:



Assumptions:

Ignore retarding effects of friction and air resistance.

2. Constructing the solution

Plan:

Suppose Dan runs with a speed v_D, and the height he reaches is h. If we can find v_D in terms of h, then we can solve for $v_{D,\ min}$ for a desired h_{min} given.

We notice that the problem has two distinct components:

* Dan jumping over the snowboard and coming to rest with respect to the snowboard is completely inelastic collision. We must find the speed of the snowboard with Dan in it after collision (assuming Dan's running speed is v_D).
* the system consisting of Dan and the snowboard go up the inclined plane and then stop at height h when the kinetic energy is zero.
* We note that we can use conservation of momentum for the first part to find the speed of the snowboard and Dan together.
* Then we can use conservation of mechanical energy for second part to find the height h at which the snowboard stops.

Step B: Execution of the plan

Sub-problem 1 – calculating speed v_{D+B} of Dan and the snowboard after inelastic collision

Since the momentum of the system consisting of Dan and Snowboard is conserved

$$p_i = p_f$$

$$m_D v_D + m_B v_B = (m_D + m_B) v_{D+B}$$

Since the initial speed of the snowboard is zero,

$$m_D v_D = (m_D + m_B) v_{D+B}$$

$$v_{D+B} = \frac{m_D v_D}{(m_D + m_B)}$$

Sub problem 2 – calculating the height reached before coming to a stop momentarily

From conservation of mechanical energy

$$\underbrace{KE_i + PE_i}_{\text{mechanical energy on horizontal surface}} = \underbrace{KE_f + PE_f}_{\text{mechanical energy on the incline when the snowboard stops momentarily}}$$

. $PE_i = 0$ if we choose the reference height to be on the horizontal surface

. $KE_f = 0$ at highest point since the speed there is zero

$$\therefore \frac{1}{2} m v_{D+B}{}^2 = mgh \quad \text{where } m = m_D + m_B$$

$$v_{D+B}{}^2 = 2gh$$

$$v_{D+B} = \sqrt{2gh}$$

$$\frac{m_D v_D}{(m_D + m_B)} = \sqrt{2gh}$$

$$v_D = \frac{m_D + m_B}{m_D}\sqrt{2gh}$$

$$v_{D,\ \min} = \frac{m_D + m_B}{m_D}\sqrt{2gh_{\min}}$$

$$v_{D,\ \min} = \frac{60kg + 15kg}{60kg}\sqrt{2 \times 9.8\frac{m}{s^2} \times 1.8m}$$

$$= 7.4\ m/s$$

<u>Final Result</u>: To win the competition, the minimun speed that Dan should run is 7.4 m/s.

<u>3. Reasonability check of the solution:</u>

. unit is correct for v

. In the limiting case $m_B = 0$, we expect $v_D = \sqrt{2gh}$, which agrees with our result $v_D = \frac{m_D + 0}{m_D}\sqrt{2gh} = \sqrt{2gh}$

C.2 INTERVENTION 1

C.2.1 Intervention 1: step 1

In this quiz, first browse over and learn from a solved problem. After 10 minutes, please turn in the solved problem, and then solve two quiz problems. One of the quiz problems will be exactly the same as the solved problem that you browsed over and the other problem will be similar.

C.2.2 Intervention 1: step 2

Now, first solve the problem you just browsed over (problem 1) and then solve the other problem (problem 2) which is similar.

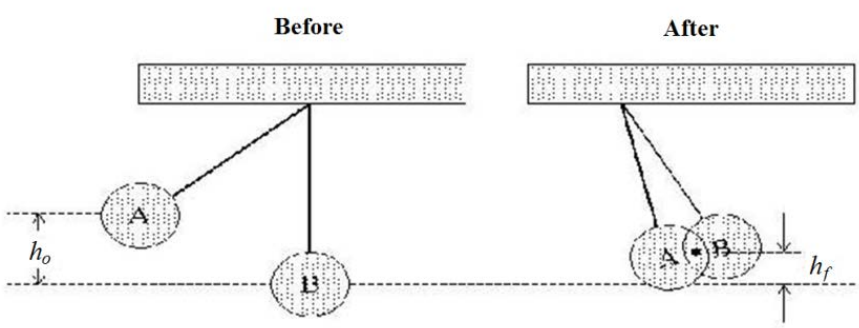
Problem 1:

Your friend Dan, who is in a ski resort, competes with his twin brother Sam on who can glide higher with the snowboard. Sam, who's mass is 60 kg, puts his 15 kg snowboard on a level section of the track, 5 meters from a slope (inclined plane). Then, Sam takes a running start and

jumps onto the stationary snowboard. Sam and the snowboard glide together till they come to rest at a height of 1.8 m above the starting level. What is the minimum speed at which Dan should run to glide higher than his brother to win the competition? (Dan has the same weight as Sam and his snowboard weighs the same as Sam's snowboard.)

Problem 2:

Two small spheres of putty, A and B, of equal mass, hang from the ceiling on massless strings of equal length. Sphere A is raised to a height h_0 as shown below and released. It collides with sphere B (which is initially at rest); they stick and swing together to a maximum height h_f. Find the height h_f <u>in terms of h_0</u>.



a) Write down explicitly the similarities between problem 1 and problem 2 in detail and whether you can use the similarities to solve problem 2.

b) Now solve for the maximum height h_f <u>in terms of h_0</u>.

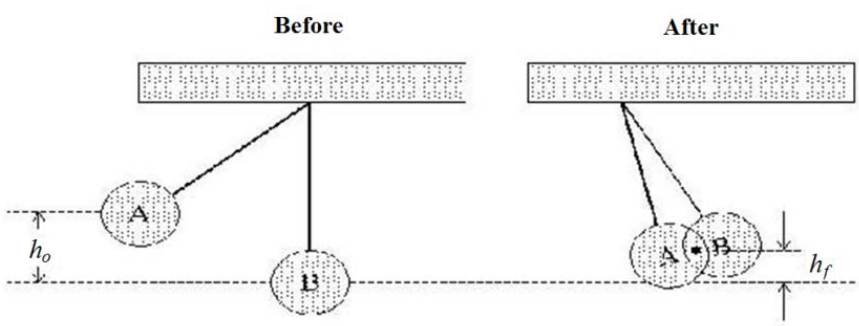
C.3 INTERVENTION 2

C.3.1 Intervention 2: step 1

First, solve the quiz problem below. After 10 minutes, please turn in the answer sheet, and you'll be provided with another solved problem to browse over and learn from. Then, you will have to redo this quiz problem.

<u>Quiz Problem:</u>

Two small spheres of putty, A and B, of equal mass, hang from the ceiling on massless strings of equal length. Sphere A is raised to a height h_0 as shown below and released. It collides with sphere B (which is initially at rest); they stick and swing together to a maximum height h_f. Find the height h_f <u>in terms of h_0</u>.

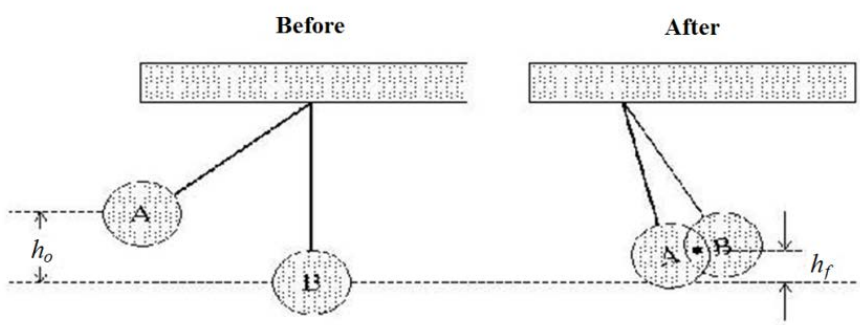


C.3.2 Intervention 2: step 2

Before you solve the same quiz problem again, go over the solved problem first. Identify the similarities between the solved problem and the quiz problem you just solved. Then answer the questions below and redo the quiz problem.

<u>Quiz Problem:</u>

Two small spheres of putty, A and B, of equal mass, hang from the ceiling on massless strings of equal length. Sphere A is raised to a height h_0 as shown below and released. It collides with sphere B (which is initially at rest); they stick and swing together to a maximum height h_f. Find the height h_f <u>in terms of h_0.</u>



a) Write down explicitly the similarities between the quiz problem and the solved problem provided in detail and whether you can use the similarities to solve this quiz problem.

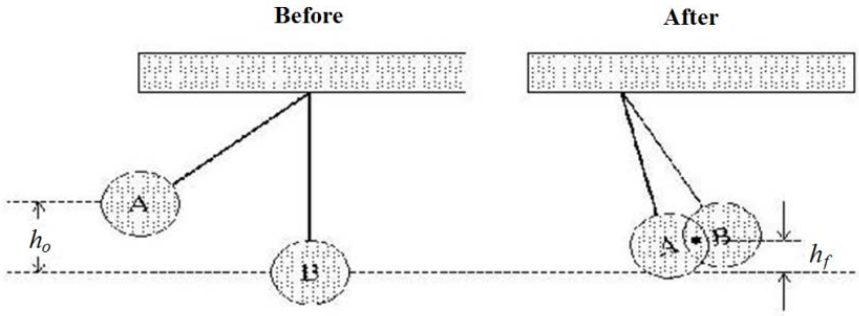
b) Now solve for the maximum height h_f <u>in terms of h_0.</u>

C.4 INTERVENTION 3

In this quiz, first browse over and learn from a solved problem, then solve the quiz problem below. Similar to the solved problem, this quiz problem below could be solved using conservation of energy and conservation of momentum. (You might have to use conservation of energy twice to find the height h_f in terms of h_0.) Before you solve the quiz problem, go over the solved problem and then answer the following questions.

Quiz Problem:

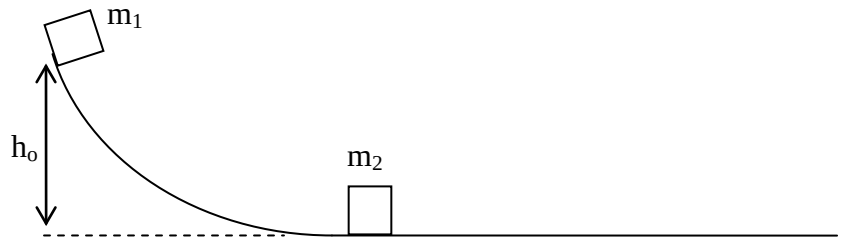
Two small spheres of putty, A and B, of equal mass, hang from the ceiling on massless strings of equal length. Sphere A is raised to a height h_0 as shown below and released. It collides with sphere B (which is initially at rest); they stick and swing together to a maximum height h_f. Find the height h_f in terms of h_0.



C.5 THE "TWO-BLOCK" PROBLEM USED IN THE INTERVIEW WITH STUDENTS E AND F

A block of mass m_1, initially at rest at a height of h_o, slides down a frictionless track (with an elevated end and a horizontal part as shown in the figure) and collides with another block of

mass m_2. Then, the blocks stick and slide together on the horizontal surface. Find the speed of the blocks sliding together.



APPENDIX D

CATEGORIZATION QUESTIONS (CHAPTER 5)

- Your task is to group the 20 problems below into various groups based upon similarity of solution on the sheet of paper provided. You can create as many categories as you wish. The grouping of problems should *not* be in terms of 'easy problems', 'medium difficulty problems' and 'difficult problems' but rather it should be based upon the features and characteristics of the problems that make them similar. A problem can be placed in more than one group created by you. Please provide a brief explanation for why you placed a set of questions in a particular group. You need *not* solve any problems.

The first *two* questions refer to the following system: an electron is in an external magnetic field B which is pointing in the z-direction. The Hamiltonian for the electron spin is given by $\hat{H} = -\gamma B \hat{S}_z$ where γ is the gyromagnetic ratio and $\hat{S}_z$ is the z-component of the spin angular momentum operator.

(1) If the electron is initially in an eigenstate of $\hat{S}_x$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

(2) If the electron is initially in an eigenstate of $\hat{S}_z$, does the expectation value of $\hat{S}_x$ depend on time? Justify your answer.

(3) A free particle has the initial wavefunction $\Psi(x, t = 0) = Ae^{-ax^2}e^{ik_0x}$ where A, a and k_0 are constants (a and k_0 are real and positive). Find $|\Psi(x, t)|^2$.

(4) A particle in an infinite square well ($0 \leq x \leq a$) has the initial wavefunction $\psi(x, 0) = Ax(a - x)$. Find the uncertainty in position and momentum.

(5) In the ground state of the harmonic oscillator, what are the expectation values of position, momentum and energy? Do these expectation values depend on time?

(6) A particle is in the first excited state of a harmonic oscillator potential. Without any calculations, explain what the expectation value of momentum is and whether it should depend on time.

(7) A free particle has the initial wavefunction $\Psi(x, t = 0) = Ae^{ik_0x}$ where A and k_0 are constants (k_0 is real and positive). Find $|\Psi(x, t)|^2$.

(8) An electron is in the ground state of a hydrogen atom. Find the uncertainty in the energy and the z-component of angular momentum.

(9) Make a qualitative sketch of a Dirac delta function $\delta(x)$. Then, make a qualitative sketch of the absolute value of the Fourier transform of $\delta(x)$. Label the axes appropriately for each plot.

(10) A free particle has the initial wavefunction $\Psi(x, t = 0) = Ae^{-ax^2}e^{ik_0x}$ where A, a and k_0 are constants (a and k_0 are real and positive). Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$.

(11) An electron in a hydrogen atom is in a linear superposition of the first and third excited states. Does the expectation value of its kinetic energy depend on time?

(12) Suppose that the measurement of the position of a particle in an infinite square well ($0 \leq x \leq a$) yields the value $x = a/2$ at the centre of the well. Show that if energy is measured immediately after the position measurement, it is equally probable to find the particle in any odd-energy stationary state.

(13) An electron is in a linear combination of the ground and fourth excited states in a harmonic oscillator potential. A measurement of energy is performed and then followed by a measurement of position. What can you say about the possible results for the energy and position measurements?

(14) An electron in a hydrogen atom is in a linear superposition of the first and third excited states. Find the wavefunction after time *t*.

(15) A particle is in the third excited state of a harmonic oscillator potential. Without any calculations, explain what the expectation value of momentum is and whether it should depend on time.

(16) A particle in an infinite square well ($0 \leq x \leq a$) has the initial wavefunction $\psi(x, 0) = Ax(a - x)$. Without normalizing the wavefunction, find $\psi(x, t)$.

(17) A free particle has the initial wavefunction wavefunction $\Psi(x, t = 0) = Ae^{ik_0 x}$ where A and k_0 are constants (k_0 is real and positive). Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$.

(18) A particle is initially in a linear combination of the ground state and the first excited state of an infinite square well. Without any calculations, explain whether the expectation value of position should depend on time.

(19) What is the commutation relation $[\hat{S}_x , \hat{S}_y]$?

(20) A hydrogen atom is in the first excited state. You measure the distance of the electron from the nucleus first and then measure energy. Describe the possible values of energy you may measure.