APPROXIMATING FAST, VISCOUS FLUID FLOW IN COMPLICATED DOMAINS

by

Ross Ingram

B.S. in Mathematics, The Pennsylvania State University, 2002

M.S. in Mathematics, College of Charleston, 2005

Submitted to the Graduate Faculty of

the Department of Mathematics in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2011

UNIVERSITY OF PITTSBURGH

DEPARTMENT OF MATHEMATICS

This dissertation was presented

by

Ross Ingram

It was defended on

April 27, 2011

and approved by

W. Layton, Ph. D., Professor

I. Yotov, Ph. D., Professor

C. Trenchea, Ph. D., Professor

G. Galdi, Ph. D., Professor

Dissertation Director: W. Layton, Ph. D., Professor

APPROXIMATING FAST, VISCOUS FLUID FLOW IN COMPLICATED DOMAINS

Ross Ingram, PhD

University of Pittsburgh, 2011

Typical industrial and biological flows often occur in complicated domains that are either infeasible or impossible to resolve. Alternatives to solving the Navier-Stokes equations (NSE) for the fluid velocity in the pores of these problems *must* be considered. We propose and analyze a finite element discretization of the Brinkman equation for modeling non-Darcian fluid flow by allowing the Brinkman viscosity $\tilde{\nu} \to \infty$ and permeability $K \to 0$ in solid obstacles, and $K \to \infty$ in fluid domain. In this context, the Brinkman parameters are generally highly discontinuous. Furthermore, we consider inhomogeneous Dirichlet boundary conditions $\mathbf{u}|_{\partial\Omega} = \phi \neq 0$ and non-solenoidal velocity $\nabla \cdot \mathbf{u} = g \neq 0$ (to model sources/sinks). Coupling between these two conditions makes even existence of solutions subtle.

We establish conditions for the well-posedness of the continuous and discrete problem. We also establish convergence as $\tilde{\nu} \to \infty$ and $K \to 0$ in solid obstacles, as $K \to \infty$ in fluid region, and as the mesh width $h \to 0$. We prove similar results for time-dependent Brinkman equations for backward-Euler (BE) time-stepping. We provide numerical examples confirming theory including convergence of velocity, pressure, and drag/lift.

We also investigate the stability and convergence of the fully-implicit, linearly extrapolated Crank-Nicolson (CNLE) time-stepping for finite element spatial discretization of the Navier-Stokes equations. Although presented in 1976 by Baker and applied and analyzed in various contexts since then, all known convergence estimates of CNLE require a time-step restriction. We show herein that no such restriction is required. Moreover, we propose a *new* linear extrapolation of the convecting velocity for CNLE so that the approximating velocities converge without without time-step restriction in $l^\infty(H^1)$ along with the discrete time derivative of the velocity in $l^2(L^2)$. The new extrapolation ensures energetic *stability* of CNLE in the case of inhomogeneous boundary data. Such a result is unknown for conventional CNLE (usual techniques fail!). Numerical illustrations are provided showing that our new extrapolation clearly improves upon stability and accuracy from conventional CNLE.

keywords: Navier, Stokes, Brinkman, Darcy, Crank-Nicolson, backward, Euler, finite element, porous media, volume penalization, extrapolation, linearization, implicit, stability, error, convergence, analysis, very porous media, non-Darcy, inhomogeneous, non-solenoidal.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

PREFACE

Many people have shaped my work and life, but none more than my busy, curious, and charming daughter, Lucy, and my beautiful, patient, and loving wife, Ashley. Lucy, more than solving million dollar Clay prizes, I cherish time spent with you. Ashley, thank you so much for understanding that my 9am-5pm job is really more like a 5am-9pm job. Thank you for working your frustrations of my absence into batches and batches of chocolate chip cookies. It is possible that the cookies kept me working hard. It is also possible that the cookies kept me working hard to get you to bake more cookies. Regardless and independently, I love you dearly and forever.

Thank you Mom, Dad, Brian and Tammy. It is hard to imagine equaling your life achievements. Your continual and constant support are inspirations to my own life work. Thank you Granny, Grandpa and Grandma, Aunt Mary, Uncle Tom, Lydia, and Aunt Joan for your interest in my life. Thanks also to Sharon, Ken, Emily, and Rob. Beyond mathematics, I find comfort in my dynamic, impressive, and inspiring family.

Thank you to my advisor, Professor William Layton. Your passion for mathematics, your students, and your life are inspirations for me beyond my research. Thank you for your time and ideas. Thank you for giving me the freedom to follow my interests. I do not know how you know everything about everything, but somehow you always do. Fortunately, this generally works to my advantage. Thank you endlessly for all of your help.

Thank you to my committee members Professors Giovanni Galdi, Catalin Trenchea, and Ivan Yotov. I am incredibly fortunate to have such a wealth of expertise on the theory of the Navier-Stokes equations, development and implementation of numerical methods for fluids problems, and scientific programming within a short walk from my office. It is an honor and pleasure to learn from you. Thanks is also warranted to Professor Mike Sussman who has spent many hours discussing and assisting me on many different aspects of my work.

It has been an incredible joy (a joy sometimes understood to be a joy only retroactively after many painstaking hours of puzzlement, deliberation, and false-eurekas, but still a *joy*) investigating the theory and approximation of the beautifully complicated Navier-Stokes equations. Of course, my enjoyment is enabled by the sacrifices of time and generosity of many other people. Thank you, everybody.

1.0 INTRODUCTION

"*No human investigation can claim to be scientific if it doesn't pass the test of mathematical proof*," Leonardo da Vinci (1452-1519) [72].

Typical industrial and biological flows often occur in complicated domains that are either infeasible or impossible to resolve. Alternatives to solving the Navier-Stokes (NS) equations (NSE) for the fluid velocity in the pores of these problems *must* be considered. Constructing efficient and reliable numerical simulations based on rigorous mathematics has tremendous potential to inspire technological advances while reducing material costs. The search for the *right* alternative (fast to solve, reliably accurate, and easily integrated into existing computing platforms) is necessary for practical computing. The focus of our research is to develop a mathematically precise foundation for *fast*, *robust*, and accurate models common in practical flow-transport applications. We list several motivating examples here:

- **Pebble bed nuclear reactors** (PBR) (see e.g. [59, 66] and Figure 1.1(a)) consist of $100,000+$ loosely packed graphite-covered uranium fuel spheres (approximately the size of a tennis ball). PBR's are cooled by a high speed helium gas and operate at high temperatures to promote high thermal efficiencies. One major drawback of PBR's, however, is that heat transfer in the core is poorly understood since no reliable experiments are possible at these high core temperatures. An essential problem is to develop stable and accurate methods to efficiently and reliably predict peak fuel temperatures in PBR's. These predictions should provide guidelines for operating PBR's at maximal thermal power output while avoiding meltdown due to overheating fuel-pebbles. However, due to the internal geometric complexity, it is impossible to approximate the flow through a

PBR with NSE in the pores. Other methods must be considered.

- **Wind power** (see e.g. [26, 60] and references therein and Figure 1.1(b)) is another popular alternative energy source, particularly because it does not emit any adverse byproduct. A pertinent flow problem concerns the mutual interference (due to the generation of large turbulent wakes) between windmills, ultimately leading to a collective reduction in turbine efficiency. Like PBR's, however, it is computationally infeasible to solve for the evolutionary flow field across a wind farm composed of 100's of wind mills.
- **Open-angle glaucoma** (see e.g. [69, 89] and Figure 1.1(c)), representing 85% of all glaucoma cases, typically results as a consequence of increased resistance to fluid flow drainage from the anterior chamber of the eye through a very porous region called the trabecular meshwork (TM). In such a case, intraocular pressure increases which can result in irreversible damage to the optical nerve and ultimately blindness. This problem reduces to as a coupled free flow (anterior chamber)/porous medium (TM) system (albeit, with smaller and even more complicated pores than in PBR's and wind farms). However, the pores in the TM are larger than those in conventional porous media problems (e.g. groundwater flows) in which Darcy's law is applied to compute a filtration velocity. However, the pore-system is far too complicated to solve the NSE for the flow field. Consequently, accurate models must be investigated to monitor flow-rate degradation associated with pressure variations in human eye to assist in the understanding of open-angle glaucoma.

In the remainder of Chapter 1, we provide background on the theory and approximation of the NSE as well as filtration models like Darcy's Law and the Brinkman equation for flows in porous media (Section 1.1). In Section 1.2 we provide a background and overview for our analysis of a linearly extrapolated, Crank-Nicolson (CN) time-stepping approximation for the NSE. In Section 1.3, we provide background and overview for our work on the Brinkman equations for flow in a (non-Darcy) porous media.

In Chapter 2 we provide the mathematical setting for the document. We include here notation for continuous and discrete function spaces along with fundamental approximation properties of the finite element (FE) spaces (Sections 2.1, 2.2), extension theory for inho-

Figure 1.1: (left) Model of gas-cooled nuclear reactor with pebble-bed core, (center) wind farm in Middelgrunden, Denmark, (right) model of the eye

mogeneous boundary data under divergence constraint in Section 2.3, and frequently used estimates Section 2.4. In Section 2.5, we provide an overview for the theory of the NSE. We discuss existence and regularity of NSE solutions for the setting of inhomogeneous problem data, essential to our analysis. For completeness, we also include a proof for stability of stationary (Section 2.5.1) and evolutionary (Section 2.5.2) NSE solutions.

In Chapters 3, 4 we investigate the stability and accuracy of a linearly extrapolated CN time-stepping method for a FE spatial discretization of the NSE (CNLE). The linear extrapolation of the convecting velocity in CNLE eliminates the necessity of multiple, *time-intensive* nonlinear iterations at each time-step required in fully nonlinear CN methods. We prove that the CNLE velocity converges to the NSE velocity as the mesh width h and Δt tend to 0 *without any restriction* on the time-step size Δt. Moreover, under a nonstandard linearization, we prove that the CNLE velocity converges to the NSE velocity in higher order norms without any time-step restriction. Convergence in these higher order norms (in particular, $l^{\infty}(H^1)$ and the discrete time-derivative in $l^2(L^2)$) is the key to proving similar estimates for drag/lift forces a fluid exerts on obstacles obstructing the fluid's flow path and pressure.

In Chapters 5, 6, 7, we investigate the validity and accuracy of the Brinkman model for approximating flows in complicated domains. In Chapter 5 we establish the well-posedness, under specific constraints, for the stationary, nonlinear Brinkman equations for inhomogeneous boundary data and non-zero divergence constraint (both continuous and FE models). In Chapters 6, 7 we investigate the accuracy of the FE approximation of the Brinkman volume penalization (BrVP) equations (both stationary and evolutionary) in approximating viscous, incompressible fluid flows through complicated domains. The motivation is to avoid body-fitted meshes conforming to the internal geometry (e.g. avoid meshing the boundaries of the 100,000's of spheres in the PBR's, Figure 1.1(a)) in order to make use of efficient solvers designed for *structured* (Cartesian) meshes instead. As a first step, when the mesh does conform to the obstacle boundaries, we prove optimal convergence rate (made precise herein) as h, Δt, and penalty parameter $\varepsilon \to 0$. Moreover, BrVP provides a convenient volume integral for computing the forces exerted by the fluid on the embedded obstacles. We prove convergence (in particular norms) of the BrVP forces relative to the actual fluid forces as well.

In Chapter 8, we summarize our main contributions and discuss our future direction for research.

1.1 THEORY AND APPROXIMATION OF FLUID FLOW

Things should be made as simple as possible, but not any simpler, Albert Einstein (1934), paraphrased from [22].

Our understanding of fluid flow has developed slowly. Influenced by the early works of Archimedes (212 BC), the qualitative observations of Leonardo da Vinci (1452-1519), and Isaac Newton's Laws of Motion (1643-1727), physicist Claude-Louis Navier (1785-1836) and mathematician George Stokes (1819-1903) independently formulated the NSE. Today, the NSE is attributed as the definitive model for fluid flow. The theory and approximation of the NSE is the central focus of our current work. We provide the setting here: Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). Fix time $T > 0$, Reynolds number $Re > 0$, and body force $\mathbf{f}$.

- (NSE) For incompressible, Newtonian fluids (e.g. water) (either laminar or turbulent flows), find fluid velocity $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, and pressure $p : \Omega \times (0,T] \to \mathbb{R}$ satisfying

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f} + Re^{-1} \Delta \mathbf{u} - \nabla p, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \text{in } \Omega \times (0,T] \tag{1.1}$$

  subject to boundary and initial conditions

$$\mathbf{u} = \phi \text{ on } \partial\Omega \times (0,T], \qquad \mathbf{u}(\cdot,0) = \mathbf{u}^0 \text{ in } \Omega \tag{1.2}$$

  where $\mathbf{u}^0|_{\partial\Omega} = \phi$ and $\nabla \cdot \mathbf{u}^0 = 0$.

Note that the kinematical viscosity satisfies $\nu \propto Re^{-1}$.

Although physically straightforward, the NSE is mathematically and numerically complex. Proving the well-posedness of the NSE (in 3-dimensions) is an open Clay Prize problem in mathematics [1]. The problem is that, although initially smooth solutions exist and are unique for finite time, it is unknown whether they remain smooth and/or unique as time evolves. Leray introduced the concept of weak NS-solutions that are shown to exist for all time [67, 68] (with improved and simplified proofs provided by Hopf [47]), but the uniqueness of these solutions has not been proved. Moreover, Kolmogorov's K41 theory of turbulence suggests that $\mathcal{O}(Re^{9/4})$ mesh points (per time-step) are required to resolve $3d$-flows where $Re = LU/\nu$ for some characteristic length L and velocity U (see e.g. [63]): i.e.

$$Re = 10^4 \text{ (moderate Re)} \;\Rightarrow\; 10^9 \text{ mesh points!}$$

For flow through porous media (e.g. filtration, groundwater flow, oil extraction), the NSE is impossible or impractical to apply in practice. In these situations, Darcy's equations are often solved for a filtration velocity $\mathbf{u}_D$ and pressure p_D and are given by

$$\mathbf{u}_D = -\nu^{-1} \mathbf{K} \nabla p_D, \qquad \nabla \cdot \mathbf{u}_D = g, \qquad \text{in } \Omega_{ext} \tag{1.3}$$

where $\Omega_{ext} \supset \Omega$ is a simple domain *extended* to contain both the fluid pores and solid matrix, $\mathbf{K}$ is permeability tensor (accounting for pore geometry) and g represents sources/sinks of fluid. Although theoretically and computationally simpler than the NSE, Darcy cannot predict the formation of vortices (important for high velocity flows in very porous media) since $\mathbf{u}_D \propto -\nabla p$ so that $\nabla \times \mathbf{u}_D = 0$.

Brinkman noted that, in general, the viscous effects must also be taken into account to model flow accurately through porous media [14, 15]:

- (Brinkman, 1947) Find filtration velocity $\mathbf{u}_B : \Omega_{ext} \to \mathbb{R}^d$ and pressure $p_B : \Omega_{ext} \to \mathbb{R}$ satisfying

$$-\nabla \cdot (\tilde{\nu} \nabla \mathbf{u}_B) + \nabla p_B + \nu \mathbf{K}^{-1} \mathbf{u}_B = \mathbf{f}, \qquad \nabla \cdot \mathbf{u}_B = g \qquad \text{in } \Omega_{ext} \tag{1.4}$$

  where and $\tilde{\nu}$ is the *effective* fluid viscosity.

In the limiting case $\tilde{\nu} \to 0$ so that $\nabla \cdot (\tilde{\nu} \nabla \mathbf{u}_B) \approx 0$ in (1.4), Darcy's equation (1.3) is recovered. Alternatively, restricted to Ω (the fluid pore region) the Brinkman equation (1.4) reduces exactly to Stokes equation when $\tilde{\nu} = \nu$. However, *passing the limit* $\tilde{\nu} \to 0$ must be done with care especially when changing the order of a differential equation. Theoretical justifications exist for the Brinkman model as an asymptotic approximation to the NSE, e.g. see [2, 49] and references therein. Straughan presents several of the most popular non-Darcy models for flow in porous media in [85] (a well-cited compilation of his and others' contributions to this theory). Along with these theoretical justifications, heuristic generalizations of Darcy's law have been considered to model non-Darcy flows in porous media (e.g. [39, 56, 77, 10]).

The coupling condition between Stokes flow domains and Darcy flow domains (used for flow in porous media) is physically unresolved even though the Beavers-Joseph-Saffman (BJS) interface condition is widely accepted and generally used in practice [9, 81, 64]. Furthermore, the coupled Stokes-Darcy system with the BJS interface condition is well-posed [64], but such a conclusion has not been verified for the nonlinear NS-Darcy coupling for large data (for further unresolved compatibility issues between these flows, see e.g. [56, 77, 10]). It is exactly these shortcomings for Stokes/NS, and Darcy's flow models that are the strengths of Brinkman. The Brinkman approach eliminates the mathematical problem with the interface couplings that corresponds physically the to the BJS interface condition, see [73]. Moreover, it is simple in implementation and easily adapted to existing computing platforms.

The Brinkman model and generalizations have been applied to approximate non-Darcian flows in a variety of contexts; e.g. it is used to model oil filtration flows [52], groundwater flows [19], forced convective flows in metal foam-filled pipes (used in the cooling of electronic equipment) [70], gas diffusion through fuel cell membranes [38], Casson fluid flow in porous media (e.g. blood flow in vessels obstructed by fatty plaques and clots) [20], and interstitial fluid flow through muscle cells [87] with good accuracy. The Brinkman equation is also used to model turbulence in porous media in the macroscopic scales [58] (for a discussion concerning turbulence modeling at the macroscopic versus the microscopic pore level see [75]).

Numerical analysis of a discretization of the Stokes and Brinkman flow model is limited. In [3, 4], Angot et.al provide a beautifully detailed error analysis for the continuous Stokes-Brinkman fluid velocity in fluid-porous and fluid-solid domains compared to Darcy-Stokes velocities. In [88], Xie et.al. provide an innovative numerical analysis of the Stokes-Brinkman equations with a condition that ensures stable FE spaces for the discrete Stokes-Brinkman equation in the limiting condition for high Reynold's number. A DG method for Brinkman is proposed in [23].

1.2 MOTIVATION OF FULLY IMPLICIT, LINEARIZATIONS OF THE NAVIER-STOKES EQUATIONS

"*[Truth] is much too complicated to allow anything but approximations*," John von Neumann (1947) [74].

A central question in practical computational fluid dynamics is: *what is the smallest amount of work permitted to produce a stable and accurate approximation of the flow field.* The method for approximating NS fluid flows is largely influenced by the following:

- *stiffness* of problem in diffusion-dominated flow regions
- lack of and/or *unknown regularity* of true NSE-solution
- high Reynolds number (Re) $\Rightarrow$ many mesh points $\Rightarrow$ *extremely large system of ODE's.*

Implicit time-stepping approximations of the NSE are preferred in practice in order to avoid unnecessary numerical/modeling restrictions on the time-step size. In Chapters 3, 4, we investigate in the stability and accuracy of a linearly extrapolated version of the CN time-stepping scheme for the NSE which eliminates the necessity of multiple, *time-intensive*, nonlinear iterations at each time-step.

The usual CN (in time) FE (in space) discretization of the NSE denoted by CNFE is well-known to be unconditionally and nonlinearly (energetically) stable. The error analysis of the CNFE method is based on a discrete Gronwall inequality which introduces a time-step restriction (for convergence, not for stability) of the form

$$\Delta t \leq \mathcal{O}(Re^{-5/3}h^{2/3}), \qquad \text{or} \qquad \Delta t \leq \mathcal{O}(Re^{-3}) \tag{1.5}$$

(implicitly reported for $W^{1,\infty}$-solutions in [45]). Here $h > 0$ is the mesh width, $\Delta t > 0$ is the time-step size, and $Re > 0$ is the Reynolds number. Condition (1.5)(a) implies *conditional convergence* whereas (1.5)(b) is a *robustness condition* and both are prohibitively restrictive in practice; for example, (1.5)(b) suggests

$$Re = 100 \text{ (low-to-moderate value)} \quad \Rightarrow \quad \Delta t \leq \mathcal{O}(10^{-6}).$$

Consequently, an important open question regards whether condition (1.5) is

- an artifact of imperfect mathematical technique, or
- a special feature of the CN time discretization.

In Chapter 4, we consider the necessity of a time-step restriction in a linear, fully implicit variant of CNFE obtained by extrapolation of the convecting velocity $\mathbf{u}$: for example,

$$\mathbf{u}\cdot\nabla\mathbf{u} \approx (\frac{3}{2}\mathbf{u}^{n-1} - \frac{1}{2}\mathbf{u}^{n-2})\cdot\nabla\frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}, \qquad \mathbf{u}^i := \mathbf{u}(x, t_i). \tag{1.6}$$

This method is often called CNLE and was first studied by Baker [7]. CNLE is linearly implicit, unconditionally (energetically) stable (at least for $\mathbf{u}|_{\partial\Omega} = 0$), and second-order accurate. We show that *no time-step restriction* is required for the convergence of CNLE. Additionally, the error satisfies

$$||error(CNLE)||_{l^\infty(L^2)\cap l^2(H^1)} \leq C(h^k + \Delta t^2), \qquad k = \text{ degree of FE-space.}$$

We also prove convergence estimates in other norms. Under a modest time-step restriction

$$\Delta t \leq h^{1/4}, \quad \text{no } Re\text{-dependence}, \tag{1.7}$$

the CNLE velocity approximation converges optimally in the $l^\infty(H^1)$-norm and the corresponding discrete time derivative of the velocity approximation converges optimally in the $l^2(L^2)$-norm. The restriction (1.7) is not a typical artifact of the discrete Gronwall inequality since it does not depend on Re or other problem data. Correspondingly, (1.7) is much less restrictive than (1.5). The error estimate is obtained through a bootstrap argument that utilizes the error in the energy norm.

In fact, we propose a new extrapolation to replace (1.6) given by

$$\mathbf{u} \cdot \nabla \mathbf{u} \approx (2\frac{\mathbf{u}^{n-1} + \mathbf{u}^{n-2}}{2} - \frac{\mathbf{u}^{n-2} - \mathbf{u}^{n-3}}{2}) \cdot \nabla \frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}. \tag{1.8}$$

so that the time-step restriction (1.7) is avoided completely. Linearization by (1.8) preserves $\mathcal{O}(\Delta t^2)$ of CN like (1.6), but additionally stabilizes the CNLE approximations (see Chapter 3). Our analysis depends on the *extrapolated* convecting velocity in (1.6), careful majorization of associated bi- and trilinear forms, and application of a particular discrete Gronwall inequality.

In Chapter 3, we show that the alternate linearization (1.8) is a sufficient condition to prove stability of CNLE approximations in the case of inhomogeneous boundary data $\mathbf{u}|_{\partial\Omega} \neq 0$. Such a result is (perhaps surprisingly!) unknown for CNLE under the conventional extrapolation (1.6) even for arbitrarily small boundary data $\mathbf{u}|_{\partial\Omega} \neq 0$. Briefly, the problem arises when we lift the boundary data so that $\mathbf{u} = \mathbf{u}_0 + E(\mathbf{u}|_{\partial\Omega})$ for some extension operator $E(\mathbf{u}|_{\partial\Omega})$. Cross-terms from the nonlinearity pollute the RHS of the resulting estimate upon the substitution $\mathbf{u}^n = \mathbf{u}_0^n + E(\mathbf{u}^n|_{\partial\Omega})$. The energy estimate for $\mathbf{u}_0^n$ is obtained by testing CNLE with $\mathbf{v} = \mathbf{u}_0^{n+1/2}$ to get

$$||\mathbf{u}_0^{n+1}||^2 + 2\Delta t \nu \sum_n |\mathbf{u}_0^{n+1/2}|_1^2 + \ldots = -\Delta t \sum_n \int \xi^n(\mathbf{u}_0) \cdot \nabla E(\mathbf{u}^{n+1/2}|_{\partial\Omega}) \cdot \mathbf{u}_0^{n+1/2} + \ldots. \tag{1.9}$$

Without the discrete Gronwall lemma (which introduces an exponential time-dependence on the basic energy estimate for the fluid velocity), standard techniques do not provide a means to absorb $\mathbf{u}_0$ from $\Delta t \sum_n \int \xi^n(\mathbf{u}_0) \cdot \nabla E(\mathbf{u}^{n+1/2}|_{\partial\Omega}) \cdot \mathbf{u}_0^{n+1/2}$ into the RHS of (1.9).

There are many analyses of CN time-stepping methods for the NSE. Heywood and Rannacher [45] provide analysis of CNFE. The 2nd and 3rd order CNLE methods are introduced and analyzed in [7, 8]. Multilevel methods based on CNLE (building on the work in [62] and [32]) are analyzed in [42], [50]. CNLE approximation of a stochastic NSE is analyzed in [21]. The authors in [61] analyze a stabilized CNLE method. Each of these analyses requires, explicitly stated or implicitly, a time-step restriction of the form (1.5) to guarantee convergence. Error analysis for the semi-discrete BE scheme is analyzed in [33]. A 1st order CNLE is used in [53] in conjunction with a coupled multigrid and pressure Schur complement schemes for the NSE. Numerical comparison of various NS time-stepping schemes (excluding CNLE) are provided in [55].

A CN/Adams-Bashforth (CN-AB) time-stepping, scheme is another linear variant of CNFE. Unlike CNLE, CN-AB is explicit in the nonlinearity and only *conditionally* stable [41] (i.e. a time-step restriction of form (1.5)(a) is required for *stability*). CN-AB is a popular method for approximating NS flows because it is fast and easy to implement. Each time-step requires only one discrete Stokes system and linear solve. For example, it is used to model turbulent flows induced by wind turbine motion [84], turbulent flows transporting particles in [71], and reacting flows in complex geometries (e.g. gas turbine combustors) [5].

The CN method is also applied, for example, to a general class of non-stationary partial differential equations encompassing reaction-diffusion type equations including the nonlinear Sobolev equations [76] and the Ginzburg-Landau model [51]. Time-step restrictions of type (1.5)(b) (where Re has a different meaning) are implicitly required in the convergence analyses of these discrete models.

1.3 MOTIVATION OF BRINKMAN FLOW MODELS

"*[All] models are wrong, but some models are useful*," George P. E. Box (1987) [11].

In Chapters 5, 6, 7, we investigate the accuracy of Brinkman volume penalization (BrVP) for modeling stationary and evolutionary incompressible, viscous fluid flows. The motivation behind BrVP is to avoid body-fitted meshes in order to use efficient solvers designed for *structured* (Cartesian) meshes instead. For BrVP, the usual no-slip boundary condition on the solid obstacles Ω_s is replaced by a penalized drag term in the volume Ω_s.

- (BrVP) Find $\mathbf{u}_\varepsilon : \Omega_{ext} \times [0,T] \to \mathbb{R}^d$, and $p_\varepsilon : \Omega_{ext} \times (0,T] \to \mathbb{R}$ satisfying

$$
\begin{aligned}
&\partial_t \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon = \mathbf{f} + \nabla \cdot (\tilde{\nu} \nabla \mathbf{u}_\varepsilon) - \nabla p_\varepsilon - \nu \mathbf{K}^{-1} \mathbf{u}_\varepsilon, && \text{in } \Omega_{ext} \times (0,T] \\
&\nabla \cdot \mathbf{u}_\varepsilon = 0, && \text{in } \Omega_{ext} \times (0,T] \\
&\mathbf{u}_\varepsilon = \phi, && \text{on } \partial\Omega_{ext} \times [0,T] \\
&\mathbf{u}_\varepsilon(\cdot, 0) = \mathbf{u}_\varepsilon^0, && \text{in } \Omega_{ext}
\end{aligned}
\tag{1.10}
$$

where $\chi_s(x) = 0$ for $x \in \Omega$ and $\chi_s(x) = 1$ for $x \in \Omega_s$ and γ_1, $\gamma_2 > 0$.

We consider stationary solutions in Chapters 5, 6 in which $\partial_t \mathbf{u} = 0$ in (1.10) and the evolutionary case in Chapter 7.

In particular, when approximating flows in Ω, we want $\mathbf{u}_\varepsilon$ be as small as possible inside all solid obstacles $\Omega_{solid} \subset \Omega_{ext}$ and recover the no-slip condition associated with NS solutions on each solid interface $\partial\Omega_{solid}$. This is attained to an arbitrary degree by imposing a large Brinkman viscosity $\tilde{\nu}|_{\Omega_s}$ and small permeability $K|_{\Omega_s}$. In addition, in the purely fluid region $\Omega \subset \Omega_{ext}$, there are no obstacles impeding the flow; thus, $K|_{\Omega}^{-1} = 0$. Fix parameter $0 < \varepsilon << 1$ small and set

$$
\tilde{\nu}|_{\Omega_{solid}} = \nu \varepsilon^{-1}, \qquad K^{-1}|_{\Omega_{solid}} = \varepsilon^{-1}, \qquad K^{-1}|_{\Omega_{fluid}} = 0.
$$

We are interested in the asymptotic behavior of solutions $\mathbf{u}_\varepsilon$ to (1.10) as $\varepsilon \to 0$ and the (triple asymptotic) limit of approximate solutions $\mathbf{u}_{\varepsilon,h}$ as ε, h, $\Delta t \to 0$.

We investigate, as a first step, the ideal case of a body-fitted mesh conforming to $\partial\Omega_s$. Let $(\mathbf{u}_{nse,h}, p_{nse,h})$ be discrete approximation of $(\mathbf{u}_{nse}, p_{nse})$ corresponding to the same discretization scheme used to obtain $(\mathbf{u}_{\varepsilon,h}, p_{\varepsilon,h})$. Controlling stress, either discrete or continuous, on $\partial\Omega_s$ is the key to proving the optimal $\mathcal{O}(\varepsilon)$ convergence rate. Indeed, we derive the error equation (presented here for the linear case)

$$
\frac{d}{dt}||\gamma_1 \mathbf{e}||^2_{L^2(\Omega_{ext})} + \nu||\gamma_2 \nabla \mathbf{e}||^2_{L^2(\Omega_{ext})} + \frac{\nu}{\varepsilon}||\mathbf{e}||_{L^2(\Omega_s)} = \int_{\partial\Omega_s} (\sigma \cdot \hat{\mathbf{n}}) \cdot \mathbf{e} \tag{1.11}
$$

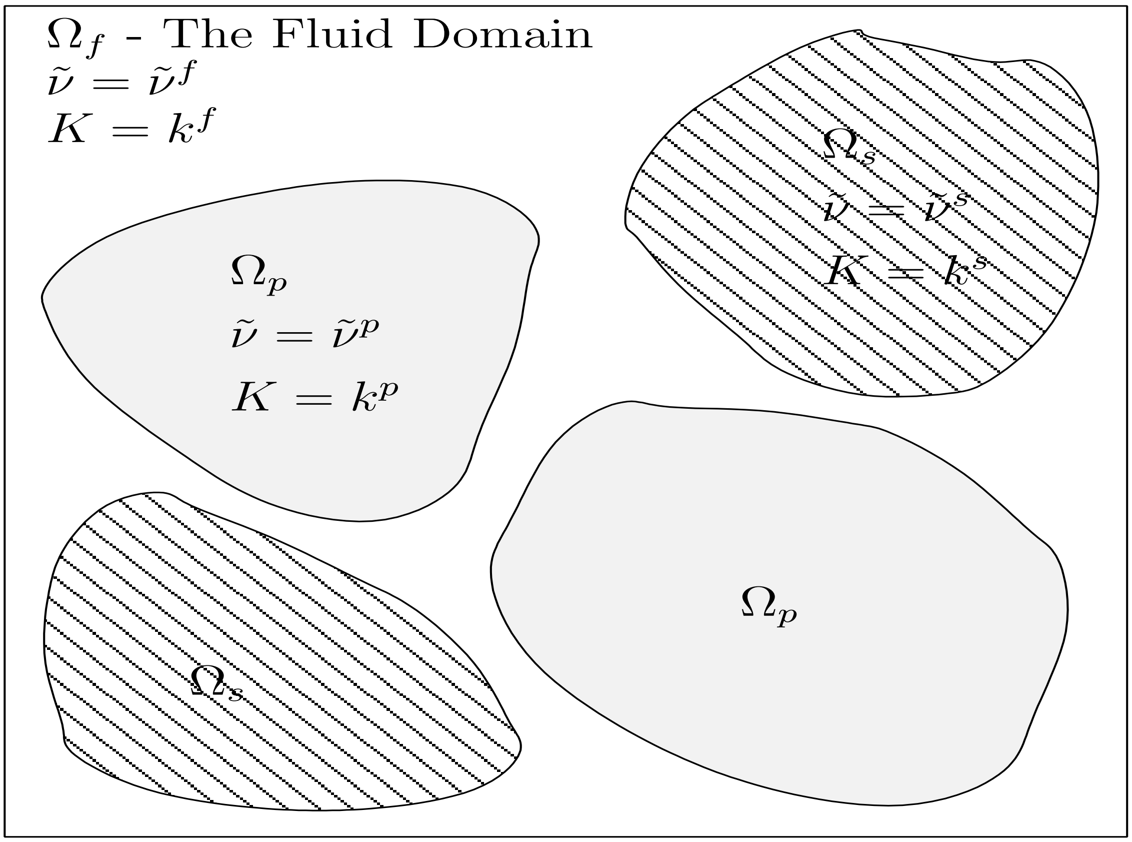


Figure 1.2: Sample heterogeneous Brinkman (extended) domain $\Omega_{ext} = \Omega \cup \overline{\Omega}_s$, fluid region $\Omega = \Omega_f \cup \overline{\Omega}_p$, completely solid region Ω_s, and porous region Ω_p with variable Brinkman viscosity $\tilde{\nu}$ and permeability $\mathbf{K}$

where γ_1, γ_2 are some piecewise constants, and $\sigma \cdot \hat{\mathbf{n}}$ is either the continuous traction vector if $\mathbf{e} = \mathbf{u}_\varepsilon - \mathbf{u}_{nse}$ or a discrete traction vector if $\mathbf{e} = \mathbf{u}_{\varepsilon,h} - \mathbf{u}_{nse,h}$. From (1.11), we see that the error between BrVP and NSE is propagated by the stress σ. The continuous traction vector is given by

$$\sigma_{nse} \cdot \hat{\mathbf{n}} = -\nu(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}_{nse} + p_{nse}\hat{\mathbf{n}}, \quad \text{on } \partial\Omega_s. \tag{1.12}$$

However, the FE space *prohibits* writing an equation analogous to (1.12) for the discrete traction vector since, in general, $-\nu\Delta\mathbf{u}_h + \nabla p_h \notin L^2(\Omega)$ (e.g. C^0-velocity elements and discontinuous pressure elements). Consequently, in the discrete case, we introduce σ_h to approximate (1.12). We prove that σ_h exists in the $\partial\Omega_s$-trace of the FE space and is bounded for both the stationary and evolutionary case. This enables our convergence analysis of velocity, pressure, and fluid forces on Ω_s presented herein.

Analysis of the continuous BrVP method and various perturbations are provided in e.g. [3, 4, 18, 17]. The volume penalization scheme was first introduced in [16]. The authors in [6] first suggested that the Brinkman term $\nu\varepsilon^{-1}\int_{\Omega_s}\mathbf{u}_\varepsilon$ gives the drag and lift coefficients. Convergence analysis of the compressible BrVP is given in [25]. Although applied in practice (see e.g. [58, 82, 57, 31, 36, 80]), rigorous analysis of discrete BrVP schemes is limited.

2.0 MATHEMATICAL PRELIMINARIES

We review common notation used throughout this document here and introduce problem-specific notation when necessary. Define the vector $\mathbf{a} := (a_0, a_1, \ldots, a_{n_0}) \in \mathbb{R}^{n_0+1}$ for some $n_0 \in \{0\} \cup \mathbb{N}$ equipped with the standard l^q norm

$$|\mathbf{a}|_q := \begin{cases} (\sum_{i=0}^{n_0} |a_i|^q)^{1/q}, & \forall q \in [1, \infty) \\ \max_{0 \le i \le n_0} |a_i|, & q = \infty \end{cases} .$$

Let $S \subset \mathbb{R}^d$ be an open, bounded, locally Lipschitzian domain for $d = 2$, 3. Fix $p \geq 1$. Let $L^p(S)$ denote the linear space of all real Lebesgue-measurable functions $\mathbf{u}$ so that

$$||\mathbf{u}||_{L^p(S)} := \begin{cases} (\int_S |\mathbf{u}|^p)^{1/p}, & p < \infty \\ \operatorname{ess\,sup}_S |\mathbf{u}|, & p = \infty \end{cases}$$

are bounded. Denote by $(\cdot, \cdot)_S$ and $|| \cdot ||_S$ the standard $L^2(S)$-inner product and norm. Fix $k \in \mathbb{R}$ and multi-index α. Define the Sobolev space

$$W^{m,p}(S) := \{\mathbf{u} \in L^p(S) \ : \ D^\alpha \mathbf{u} \in L^p(S),\ 0 \le |\alpha| \le m\}$$

equipped with the norm

$$||\mathbf{u}||_{W^{m,p}(S)} := \begin{cases} (\sum_{|\alpha| \le m} ||D^\alpha \mathbf{u}||_{L^p(S)}^p)^{1/p}, & p < \infty \\ \max_{|\alpha| \le m} ||D^\alpha \mathbf{u}||_{L^\infty(S)}, & p = \infty \end{cases} .$$

Identify $|| \cdot ||_{k,p,S} := || \cdot ||_{W^{k,p}(S)}$, $H^k(S) := W^{k,2}(S)$, $|| \cdot ||_{k,S} := || \cdot ||_{W^{k,2}(S)}$ with $| \cdot |_{k,S}$ the corresponding semi-norm. Let the context determine whether $W^{k,p}(S)$ denotes a scalar, vector, or tensor function space. For example let $\mathbf{v} : S \to \mathbb{R}^d$. Then, $\mathbf{v} \in H^1(S)$ implies that $\mathbf{v} \in H^1(S)^d$ and $\nabla \mathbf{v} \in H^1(S)$ implies that $\nabla \mathbf{v} \in H^1(S)^{d \times d}$.

Fix $g \in L^2(S)$ and $\phi \in H^{1/2}(\partial S)$ (an element of the trace of $H^1(S)$ functions) satisfying

$$\int_S g = \int_{\partial S} \phi \cdot \hat{\mathbf{n}}_S \tag{2.1}$$

where $\hat{\mathbf{n}}_S$ is the outward (relative to S) unit normal defined a.e. on ∂S. Indeed, $\hat{\mathbf{n}}_S$ exists a.e. on ∂S for bounded, locally Lipschitzian S. Define

$$\begin{aligned}
H^1_\phi(S) &:= \{\mathbf{v} \in H^1(S) \ : \ \mathbf{v}|_{\partial S} = \phi\} \\
V_\phi(g)(S) &:= \left\{\mathbf{v} \in H^1_\phi(S) \ : \ \nabla \cdot \mathbf{v} = g\right\} \\
(V(S))^0 &:= \left\{\mathbf{f} \in W^{-1,2}(S) \ : < \mathbf{f}, \mathbf{v} >_{W^{-1,2}(S)\times H^1_0(S)} = 0, \ \forall \mathbf{v} \in V_0(0)(S)\right\} \\
(V(S))^\perp &:= \left\{\mathbf{v}^\perp \in H^1(S) : \textstyle\int_S \mathbf{v}^\perp \cdot \mathbf{v} = 0, \ \forall \mathbf{v} \in V_0(0)(S)\right\}.
\end{aligned}$$

Write $V(S) = V_0(0)(S)$, $V_\phi(S) = V_\phi(0)(S)$, and $V(g)(S) = V_0(g)(S)$. Moreover, the dual space of $H^1_0(S)$ is denoted by $W^{-1,2}(S) := (H^1_0(S))'$ and equipped with the norm

$$||\mathbf{f}||_{-1,S} := \sup_{\mathbf{v} \in H^1_0(S), \mathbf{v} \neq 0} \frac{< \mathbf{f}, \mathbf{v} >_{W^{-1,2}(S)\times H^1_0(S)}}{|\mathbf{v}|_1}.$$

Define

$$L^2_0(S) := \left\{q \in L^2(S) : \int_S q = 0\right\}.$$

Note that $(L^2_0(S))' = L^2_0(S)$. If the domain S is omitted in the definitions above, assume that $S = \Omega$ - flow domain. For example, $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $H^1 = H^1(\Omega)$, and $V = V_0(\Omega)(0)$.

Fix time $T > 0$. Let $W^{m,q}(0,T; W^{k,p}(S))$ denote the linear space of all Lebesgue measurable functions from $(0,T)$ onto $W^{k,p}(S)$ equipped with norm

$$||\mathbf{u}||_{W^{m,q}(0,T;W^{k,p}(S))} := \left(\int_0^T \sum_{i=0}^m ||\partial_t^{(i)} \mathbf{u}(\cdot,t)||^q_{W^{k,p}(S)} dt\right)^{1/q}.$$

Write $W^{m,q}(W^{k,p}(S)) = W^{m,q}(0,T; W^{k,p}(S))$ and $C^m(W^{k,p}(S)) = C^m([0,T]; W^{k,p}(S))$.

2.1 FINITE ELEMENT FORMULATION

Fix $h > 0$. Let $\mathcal{T}_h$ be a family of subdivisions (e.g. triangulation) of $\overline{\Omega}_{ext} \subset \mathbb{R}^d$ so that

$$\overline{\Omega}_{ext} = \bigcup_{E \in \mathcal{T}_h} E, \quad h_E \leq h$$

where $h_E := diameter(E)$ and any two closed elements E_1, $E_2 \in \mathcal{T}_h$ are either disjoint or share exactly one face, side, or vertex. Suppose further that $\mathcal{T}_h$ is (uniformly, or possibly quasi-uniformly) regular as $h \to 0$. See [13] (Definition 4.4.13) for a precise definition and treatment of the inherited properties of such a space (see also [34] (Appendix A, Chapter II) for more on this subject in the context of Stokes problem). For example, $\mathcal{T}_h$ consists of triangles for $d = 2$ or tetrahedra for $d = 3$ that are nondegenerate as $h \to 0$.

For our treatment $\Omega_{ext} = \Omega \cup \overline{\Omega}_s \cup \overline{\Omega}_p$ so that Ω, Ω_s, and Ω_p are disjoint polygonal domains. We assume that $\mathcal{T}_h$ conforms to the solid obstacle boundary $\partial\Omega_s$ and porous medium boundary $\partial\Omega_p$: precisely, if $S = \Omega_{ext}$, Ω, Ω_s, or Ω_p then

$$E \in \mathcal{T}_h \quad \Rightarrow \quad E \subset S \text{ or } E \cap S = \emptyset.$$

Let $X_{h,\cdot}(S) \subset H^1(S)^d$ and $Q_{h,\cdot}(S) \subset L^2(S)$ be a mixed finite element (FE) space. For example, let $X_{h,\cdot}(\Omega_{ext})$ and $Q_{h,\cdot}(\Omega_{ext})$ be continuous, piecewise (on each $E \in \mathcal{T}_h$) polynomial spaces. Define $X_{h,\phi_h}(S) \subset X_{h,\cdot}(S)$ and $Q_h(S) \subset Q_{h,\cdot}(S)$ so that

$$X_{h,\phi_h}(S) := \{\mathbf{v} \in X_{h,\cdot}(S) \ : \ \mathbf{v}|_{\partial S} = \phi_h\}, \qquad Q_h(S) := \left\{q \in Q_{h,\cdot}(S) \ : \ \int_S q_h = 0\right\}$$

so that $X_{h,\phi_h}(S) = X_{h,\cdot}(S) \cap H^1_{\phi_h}(S)$ and $Q_h(S) = Q_{h,\cdot}(S) \cap L^2_0(S)$. The discretely divergence-free space is given by

$$V_{h,\phi_h}(g)(S) = \{\mathbf{v}_h \in X_{h,\phi_h}(S) : (q_h, \nabla \cdot \mathbf{v}_h) = (g, q_h) \quad \forall q_h \in Q_{h,\cdot}(S)\}.$$

As usual, write $V_{h,\phi_h}(S) = V_{h,\phi_h}(0)(S)$, $V_h(g)(S) = V_{h,0}(g)(S)$, and $V_h(S) = V_{h,0}(0)(S)$. Note that in general $V_h(S) \not\subset V(S)$ (e.g. Taylor-Hood elements). Preserving an abstract framework for the FE-spaces, we assume that $X_{h,\cdot} \times Q_{h,\cdot}$ inherit several fundamental approximation properties.

Assumption 2.1.1. *The FE-spaces $X_h(S) \times Q_h(S)$ satisfy:*

Uniform inf-sup (LBB) condition

$$\inf_{q_h \in Q_h(S)} \sup_{\mathbf{v}_h \in X_h(S)} \frac{(q_h, \nabla \cdot \mathbf{v}_h)_S}{|\mathbf{v}_h|_{1,S} \ ||q||_S} \geq C > 0 \tag{2.2}$$

FE-approximation

$$\begin{aligned} \inf_{\mathbf{v}_h \in X_h(S)} ||\mathbf{u} - \mathbf{v}_h||_{m+1,S} &\leq Ch^{k-m}||\mathbf{u}||_{k+1,S} \\ \inf_{q_h \in Q_h(S)} ||p - q_h||_{m,S} &\leq Ch^{s+1-m}||p||_{s+1,S} \end{aligned} \tag{2.3}$$

for $m \geq 0$, $k \geq 0$, and $s \geq -1$ when $\mathbf{u} \in H^{k+1}(S) \cap H^1_0(S)$, $p \in H^{s+1}(S) \cap L^2_0(S)$

Inverse-estimate

$$|\mathbf{v}_h|_{1,S} \leq Ch^{-1}||\mathbf{v}_h||_S, \qquad \forall \mathbf{v}_h \in X_h(S). \tag{2.4}$$

The generic constant $0 \leq C < \infty$ in (2.2), (2.3), and (2.4) is independent of $h \to 0$

The well-known Taylor-Hood mixed FE is one such example satisfying Assumption 2.1.1. Although verification of the inf-sup condition (2.2) for $X_h \times Q_h$ is nontrivial, it is essential to ensure uniqueness of a pressure solution to Stokes-type problems. The inverse estimate (2.4) holds when $X_{h,\cdot}$ is a continuous, piecewise polynomial space and $\mathcal{T}_h$ a quasi-uniform triangulation of $\overline{\Omega}_{ext}$.

2.2 TIME STEPPING FORMULATION

Let $0 = t^0 < t^1 < \ldots < t^n = T < \infty$ be a partition of the time interval $[0, T]$ for a constant time-step $\Delta t = t^n - t^{n-1}$. Write $z^n := z(t^n)$ and $z^{n+1/2} := \frac{1}{2}(z(t^{n+1}) + z(t^n))$. Define

$$||\mathbf{u}||_{l^q([m_1,m_2];W^{k,p}(S))} := \begin{cases} (\Delta t \sum_{n=m_1}^{m_2} ||\mathbf{u}^n||^q_{k,p,S})^{1/q}, & q \in [1, \infty) \\ \max_{m_1 \leq n \leq m_2} ||\mathbf{u}^n||_{k,p,S}, & q = \infty \end{cases}$$

for any $0 \leq n = m_1, m_1 + 1, \ldots, m_2 \leq N$. Write $||\mathbf{u}||_{l^q(W^{k,p}(S))} = ||\mathbf{u}||_{l^q([0,N];W^{k,p}(S))}$. Define the discrete time-derivative

$$\partial^{n+1}_{\Delta t}\mathbf{v} := \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t}, \quad (\partial^{(m)}_{\Delta t})^{n+1}\mathbf{v} := \frac{(\partial^{(m-1)}_{\Delta t})^{n+1}\mathbf{v} - (\partial^{(m-1)}_{\Delta t})^n\mathbf{v}}{\Delta t}$$

where $(\partial_{\Delta t}^{(0)})^{n+1}\mathbf{v} := \partial_{\Delta t}^{n+1}\mathbf{v}$. Estimates in (2.5), (2.6), (2.7), (2.10) stated below are used in proving error estimates for time-dependent problems: for any $n = 0, 1, \ldots, N-1$, $k \geq -1$, there exists $0 \leq C < \infty$ so that

$$||\partial_{\Delta t}^{n+1} z||_{k,S}^2 \leq \Delta t^{-1} \int_{t^n}^{t^{n+1}} ||\partial_t z(\cdot, t)||_{k,S}^2 dt \tag{2.5}$$

$$||z^{n+1/2} - z(\cdot, t^{n+1/2})||_{k,S}^2 \leq C\Delta t^3 \int_{t^n}^{t^{n+1}} ||\partial_t^{(2)} z(\cdot, t)||_{k,S}^2 dt \tag{2.6}$$

$$||\partial_{\Delta t}^{n+1} z - \partial_t z(\cdot, t^{n+1/2})||_{k,S}^2 \leq C\Delta t^3 \int_{t^n}^{t^{n+1}} ||\partial_t^{(3)} z(\cdot, t)||_{k,S}^2 dt \tag{2.7}$$

$$||\partial_{\Delta t}^{n+1} z - \partial_t z(\cdot, t^{n+1})||_{k,S}^2 \leq C\Delta t \int_{t^n}^{t^{n+1}} ||\partial_t^{(2)} z(\cdot, t)||_{k,S}^2 dt \tag{2.8}$$

where $\partial_t z \in L^2(H^k(S))$, $\partial_t^{(2)} z \in L^2(H^k(S))$, and $\partial_t^{(3)} z \in L^2(H^k(S))$, $\partial_t^{(2)} z \in L^2(H^k(S))$ is required respectively. Derivation of these estimates follows from application of an appropriate Taylor expansion with integral remainder.

Explicit skew-symmetrization of the convective term in Navier-Stokes (NS)-type equations ensures stability of the corresponding numerical approximation:

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}\left((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})\right). \tag{2.9}$$

Fix $a_i \in \mathbb{R}$ for $i = -1, 0, 1, \ldots, n_0 \geq -1$ and $n \in \{0\} \cup \mathbb{N}$. Define the linearization operator $\xi^n(\mathbf{u})$ so that

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \approx c_h(\xi^n(\mathbf{u}), \mathbf{v}, \mathbf{w}), \qquad \xi^n(\mathbf{u}) := a_{-1}\mathbf{u}^{n+1} + a_0\mathbf{u}^n + \ldots + a_{n_0}\mathbf{u}^{n-n_0}.$$

To summarize,

$$\begin{aligned}
&\text{No linearization} && \Rightarrow \quad a_{-1} = 1, \quad a_i = 0 \text{ for all } i \geq 0 \\
&\text{Linear extrapolation} && \Rightarrow \quad a_{-1} = 0, \quad a_i \neq 0 \text{ for some } i \geq 0
\end{aligned}$$

For example,

$$\begin{aligned}
\xi^n(\mathbf{u}) &= \mathbf{u}^n && \Rightarrow \quad \xi^n(\mathbf{u}) = \mathbf{u}^{n+1} + \mathcal{O}(\Delta t) \\
\xi^n(\mathbf{u}) &= \tfrac{1}{2}(3\mathbf{u}^n - \mathbf{u}^{n-1}) && \Rightarrow \quad \xi^n(\mathbf{u}) = \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t^2) \\
\xi^n(\mathbf{u}) &= 2\mathbf{u}^{n-1/2} - \mathbf{u}^{n-3/2} && \Rightarrow \quad \xi^n(\mathbf{u}) = \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t^2).
\end{aligned}$$

It is convenient to make the following definition:

$$\overline{n}_0 : \max = \{n_0, 0\}$$

The linear extrapolation $\xi^n(\mathbf{u})$ must have minimal approximation property to preserve the Δt-convergence rate of the parent (nonlinear) method. This is made precise by *assuming* (2.10) holds.

Assumption 2.2.1 (Linear Extrapolation)**.** *For some $i \in \mathbb{N}$ and $k \in \mathbb{R}$, $t^{n+1/i} \in [t^n, t^{n+1}]$, and $\partial_t^{(i)}\mathbf{u} \in L^2(H^k)$,*

$$||\xi^n(\mathbf{u}(\cdot,t)) - \mathbf{u}(\cdot, t^{n+1/i})||_k^2 \leq C\Delta t^{2i-1} \int_{t^n}^{t^{n+1}} ||\partial_t^{(i)}\mathbf{u}(\cdot,t)||_k^2 dt \quad \forall n = 0,\ 1, \dots,\ N-1. \tag{2.10}$$

In Assumption 2.2.1, set $i = 1$ and $t^{n+1/i} = t^{n+1}$ for backward-Euler (BE) time-stepping and $i = 2$ and $t^{n+1/i} = t^{n+1/2}$ for Crank-Nicolson (CN) time-stepping.

2.3 FUNCTIONAL SETTING FOR INHOMOGENEOUS DATA

It is convenient in the analysis of problems with inhomogeneous data to introduce the following function spaces:

$$V.(g)(S) := \left\{\mathbf{v} \in H^1(S) \ : \ \nabla \cdot \mathbf{v} = g\right\} \tag{2.11}$$

$$H_g^{1/2}(\partial S) := \left\{\mu \in H^{1/2}(\partial S) \ : \ \int_{\partial S} \mu \cdot \hat{\mathbf{n}}_S = \int_S g\right\}. \tag{2.12}$$

Identify $V.(g) = V.(g)(\Omega)$, $V.(S) = V.(0)(S)$, and $V. = V.(0)(\Omega)$. There exists an extension operator $E : H_g^{1/2}(\partial S) \to V.(g)(S)$ defined so that $E(\lambda)|_{\partial S} = \lambda$ and $\nabla \cdot E(\lambda) = g$ for all $\lambda \in H_g^{1/2}(\partial S)$. Note that the compatibility condition (2.1) is satisfied since

$$\int_S g = \int_S \nabla \cdot E(\lambda) = \int_{\partial S} E(\lambda)|_{\partial S} \cdot \hat{\mathbf{n}}_S = \int_{\partial S} \lambda \cdot \hat{\mathbf{n}}_S$$

as a consequence of the divergence theorem. Indeed, the following result is a consequence of the work provided in [28], pp. 131-132 (proved explicitly when $g = 0$ and left as an exercise in the general case).

Lemma 2.3.1 (Trace Theorem). *Fix $2 \le q < \infty$ and $g \in L^q(S)$. There exists an extension $E : (H_g^{1/2}(\partial S) \cap W^{1-1/q,q}(\partial S)) \to (V_\cdot(g)(S) \cap W^{1,q}(S))$ satisfying*

$$\begin{cases} \mu \in H_g^{1/2}(\partial S) \cap W^{1-1/q,q}(\partial S) \\ E(\mu) \in V_\mu(g)(S) \cap W^{1,q}(S) \\ ||E(\mu)||_{1,q,S} \le C(||\mu||_{1-1/q,q,\partial S} + ||g||_{0,q,S}) \end{cases} \tag{2.13}$$

for some $0 \le C < \infty$.

Remark 2.3.2. *In fact, the above result holds (slightly altered) for $1 < q < \infty$.*

As an immediate consequence, there exists an extension $E : W^{1-1/q,q}(\partial S) \to W^{1,q}(S)$

$$\begin{cases} \mu \in W^{1-1/q,q}(S) \\ E(\mu) \in H_\mu^1(S) \cap W^{1,q}(S) \\ ||E(\mu)||_{1,q,S} \le C||\mu||_{1-1/q,\partial S}. \end{cases}$$

Note that all such extensions satisfy $E(0) \in V(S)$. We also require an analogous results to (2.13) in the case of time-dependent boundary data and divergence constraint:

Lemma 2.3.3. *Fix $g(\cdot,t) : [0,T] \to L^2(S)$. There exists an extension operator $E : H_{g(\cdot,t)}^{1/2}(\partial S) \to V_\cdot(g(\cdot,t))(S)$ satisfying (2.13). In addition, for $m \ge 0$, $k \ge 1$, $p \ge 1$, $q \ge 1$, there exists $0 \le C < \infty$ so that*

$$||\partial_t^{(m)} E(\mu)||_{L^p(W^{k,q}(S))} \le C(||\partial_t^{(m)}\mu||_{L^p(W^{k-1/2,q}(\partial S))} + ||\partial_t^{(m)} g||_{L^p(W^{k-1,q}(S))}) < \infty \tag{2.14}$$

when $g \in W^{m,p}(W^{k-1,q}(S))$ and $\mu \in W^{m,p}(W^{k-1/2,q}(\partial S) \cap H_g^{1/2}(\partial S))$.

We require the discrete analogue to the trace theorems above. Define the discrete trace space of $X_h(S)$ by

$$\begin{aligned} \Lambda_h(\partial S) := \big\{ \lambda_h : H^{1/2}(\partial S) \ : \ \exists \mathbf{v}_h \in X_{h,\cdot}(S) \text{ such that} \\ \lambda_h|_{\partial E \cap \partial S} = \mathbf{v}_h|_{\partial E \cap \partial S} \ \forall E \in \mathcal{T}_h \text{ and } \partial E \cap \partial S \ne \emptyset \big\}. \end{aligned}$$

Next define discrete analogues to $V_\cdot(g)(S)$ and $H_g^{1/2}(\partial S)$ respectively by

$$\begin{aligned} V_{h,\cdot}(g)(S) &:= \left\{ \mathbf{v}_h \in X_{h,\cdot}(S) \ : \ \int_S q_h \nabla \cdot \mathbf{v}_h = \int_S g q_h \ \forall q_h \in Q_{h,\cdot} \right\} \\ \Lambda_{h,g}(\partial S) &:= \left\{ \mu_h \in \Lambda_h(\partial S) \ : \ \int_{\partial S} \mu_h \cdot \hat{\mathbf{n}}_S = \int_S g \right\}. \end{aligned}$$

Identify $V_{h,\cdot}(g) = V_{h,\cdot}(g)(\Omega)$, $V_{h,\cdot}(\Omega) = V_{h,\cdot}(0)(\Omega)$, and $V_{h,\cdot} = V_{h,\cdot}(0)(0)$. These function spaces are particularly convenient for defining a divergence-constrained extension operator of inhomogeneous problem data. It is also useful in ensuring discrete incompressibility for an auxiliary problem defined on a subdomain of the flow problem. We require the following discrete Trace Theorem for the subsequent analysis.

Assumption 2.3.4. *Fix $g(\cdot,t) : [0,T] \to L^2(S)$. There exists an extension operator $E_h : \Lambda_{h,g}(\partial S) \to V_{h,\cdot}(g)(S)$ satisfying*

$$\begin{cases} \mu_h \in \Lambda_{h,g}(\partial\Omega_s) \\ E_h(\mu_h) \in V_{h,\mu_h}(g)(S) \\ ||E_h(\mu_h)||_{1,S} \leq C(||\mu_h||_{1/2,\partial S} + ||g||_{0,S}) \end{cases} \tag{2.15}$$

for some $0 \leq C < \infty$ (independent of h, $\Delta t \to 0$). Moreover, for $m \geq 0$ and $q \geq 1$,

$$||\partial_{\Delta t}^{(m)} E_h(\mu_h)||_{l^q(H^1(S))} \leq C(||\partial_{\Delta t}^{(m)}\mu_h||_{l^q(H^{1/2}(\partial S))} + ||\partial_{\Delta t}^{(m)} g||_{l^q(L^2(S))}) < \infty$$

when $\partial_{\Delta t}^{(m)} g \in l^q(L^2(S))$ and $\partial_{\Delta t}^{(m)}\mu_h \in l^q(H_g^{1/2}(\partial S))$.

As an immediate consequence of Assumption 2.3.4, there exists an extension $E_h : \Lambda_h(\partial\Omega_s) \to X_{h,\cdot}(S)$ so that

$$\begin{cases} \mu_h \in \Lambda_h(\partial\Omega_s) \\ E_h(\mu_h) \in X_{h,\mu_h}(S) \\ ||E_h(\mu_h)||_{1,S} \leq C||\mu_h||_{1/2,\partial S}. \end{cases}$$

The existence of such an extension $E_h : \Lambda_{h,0}(\partial\Omega) \to V_{h,\cdot}$ holds for a large class of FE spaces, see e.g. [37, 83, 12]. Note that all such extensions satisfy $E_h(0) \in V_h(S)$.

2.4 FUNDAMENTALS OF ESTIMATION

The estimates in the following subsections are fundamental to our analysis. Let $C > 0$ be a generic data-independent constant throughout (depending, possibly on Ω). Let $C_* > 0$ be a generic data-dependent constant (depending, possibly, on $\mathbf{f}$, g, ϕ, $\mathbf{u}^0$, ν^{-1}). In the discrete case, C, C_* are independent of h, $\Delta t \to 0$.

2.4.1 The inf-sup condition

The inf-sup condition presented in Lemma 2.4.1 is used to establish existence and uniqueness of pressure given a divergence-free velocity $\mathbf{u} \in V(S)$ for the NS and NS-type (e.g. Stokes, Brinkman, etc.) problems. See [34] (implied in the proof of Theorem I.5.1, referring to Corollary I.2.4 and Lemma I.4.1) for a thorough development.

Lemma 2.4.1. *(LBB-condition) The function space $H_0^1(S) \times L_0^2(S)$ satisfies:*

1. *There exists $\beta > 0$ so that*

$$\inf_{q \in L_0^2(S)} \sup_{\mathbf{v} \in H_0^1(S)} \frac{(q, \nabla \cdot \mathbf{v})_S}{|\mathbf{v}|_{1,S} ||q||_S} \geq \beta. \tag{2.16}$$

2. *The gradient operator $\nabla : L_0^2(S) \to (V(S))^0$ is an isomorphism so that*

$$||\nabla q||_{-1,S} \geq \beta ||q||_S, \quad \forall q \in L_0^2(S).$$

3. *The divergence operator $\nabla \cdot : (V(S))^\perp \to L_0^2(S)$ is an isomorphism so that*

$$||\nabla \cdot \mathbf{v}^\perp||_S \geq \beta |\mathbf{v}^\perp|_{1,S}, \quad \forall \mathbf{v}^\perp \in (V(S))^\perp. \tag{2.17}$$

Remark 2.4.2. *In fact, the conditions (1)-(3) in Lemma 2.4.1 are actually equivalent statements for the abstract variational problem presented in [34] (Chapter I.4.1).*

The following is a consequence of the isomorphism guaranteed in Lemma 2.4.1(3):

Lemma 2.4.3. *For any $q \in L_0^2(S)$ there exists $\mathbf{v}^\perp \in H_0^1(S)$ (unique in $(V(S))^\perp$) satisfying*

$$\nabla \cdot \mathbf{v}^\perp = q, \quad |\mathbf{v}^\perp|_{1,S} \leq \beta^{-1} ||q||_S. \tag{2.18}$$

There is an analogous result for the FE case (see e.g. [34], p.125).

Lemma 2.4.4. *The inf-sup condition (2.2) in Assumption 2.1.1 holds if and only if for any for any $g \in L_0^2(S)$ there exists $\mathbf{v}_h^\perp \in X_h(S)$ (unique in $(V_h(S))^\perp$) so that*

$$(\nabla \cdot \mathbf{v}_h^\perp, q_h) = (g, q_h) \quad \forall q_h \in Q_h. \tag{2.19}$$

2.4.2 The discrete Gronwall Lemma

We investigate fully discrete approximations of the NS equation (NSE) and Brinkman equation: BE in time, FE (BEFE) in space and CN in time, FE (CNFE) in space. Standard error analysis relies on the discrete Gronwall Lemma 2.4.5 which leads to a time-step restriction of the form $\Delta t \kappa^n < 1$ for convergence (not energetic stability). On the other hand, we show that this time-step restriction is avoidable for linearly extrapolated variants of BEFE and CNFE because the second Gronwall Lemma 2.4.6 can be applied instead. See Chapters 3, 4.

Lemma 2.4.5 (Gronwall, Δt-restriction)**.** *Let $D \geq 0$ and κ^n, A^n, B^n, $C^n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A^N + \Delta t \sum_{n=0}^{N} B^n \leq \Delta t \sum_{n=0}^{N} \kappa^n A^n + \Delta t \sum_{n=0}^{N} C^n + D, \quad \forall N \geq 0.$$

Suppose that for all n

$$\Delta t \kappa^n < 1$$

and set $\lambda^n = (1 - \Delta t \kappa^n)^{-1}$. Then,

$$A^N + \Delta t \sum_{n=0}^{N} B^n \leq \exp\left(\Delta t \sum_{n=0}^{N} \lambda^n \kappa^n\right) \left[\Delta t \sum_{n=0}^{N} C^n + D\right], \quad \forall N \geq 0.$$

Lemma 2.4.6 (Gronwall, no Δt-restriction)**.** *Let $D \geq 0$ and κ^n, A^n, B^n, $C^n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A^N + \Delta t \sum_{n=0}^{N} B^n \leq \Delta t \sum_{n=0}^{N-1} \kappa^n A^n + \Delta t \sum_{n=0}^{N} C^n + D, \quad \forall N \geq 0.$$

Then

$$A^N + \Delta t \sum_{n=0}^{N} B^n \leq \exp\left(\Delta t \sum_{n=0}^{N-1} \kappa^n\right) \left[\Delta t \sum_{n=0}^{N} C^n + D\right], \quad \forall N \geq 0.$$

Proof. (Lemmas 2.4.5, 2.4.6) See pp. 369-370 in [45]. □

The following change of indices formula is required to resolve double sums in stability and convergence analysis of linearly extrapolated BEFE and CNFE.

Lemma 2.4.7. *Let κ^n, $\lambda^n \in \mathbb{R}$ for all $n \in \mathbb{N}$, $\alpha^i \in \mathbb{R}$ for all $i = 0, 1, \ldots, n_0$. Then,*

$$\sum_{n=n_0}^{N-1} \kappa^n \left(\sum_{i=0}^{n_0} \alpha^i \lambda^{n-i} \right) = \sum_{n=0}^{N-1} \left(\sum_{i=i_0(n)}^{i_1(n)} \alpha^i \kappa^{n+i} \right) \lambda^n \tag{2.20}$$

where

$$i_0(n) := \begin{cases} 0, & n \geq n_0 \\ n_0 - n, & \text{otherwise} \end{cases}, \quad i_1(n) := \begin{cases} n_0, & n < N-1-n_0 \\ N-n, & \text{otherwise} \end{cases}.$$

Proof. Identity (2.20) follows from a change of indices. □

2.4.3 Estimates in L^q

We use the fact that $||\nabla \cdot \mathbf{v}||_{\Omega_{ext}} \leq \sqrt{d}|\mathbf{v}|_{1,\Omega_{ext}}$ throughout without further reference. Fix $1/q + (1/q') = 1$. We will also make use of the following estimates (see. e.g. [28], Chapter II):

- (Young) For any $a > 0$, $b > 0$, and $\delta > 0$

$$ab \leq \frac{1}{q\delta^{q/q'}} a^q + \frac{\delta}{q'} b^{q'} \tag{2.21}$$

- (Hölder) For any $\mathbf{v} \in L^q(S)$, $\mathbf{w} \in L^{q'}(S)$

$$|\int_S \mathbf{v} \cdot \mathbf{w}| \leq ||\mathbf{v}||_{0,q,S} ||\mathbf{w}||_{0,q',S} \tag{2.22}$$

- (Poincaré) For any $\mathbf{v} \in H^1(S)$ so that $\mathbf{v}|_\Gamma = 0$ for some positive portion $\Gamma \subset \partial S$, there exists $0 < c_p < \infty$ so that

$$||\mathbf{v}||_S \leq c_p |\mathbf{v}|_{1,S} \tag{2.23}$$

- (Ladyzhenskaya) For any $\mathbf{v} \in H^1(S)$,

$$\begin{cases} ||\mathbf{v}||_{0,3,S} & \leq & c_3 ||\mathbf{v}||_S^{1/2} ||\mathbf{v}||_{1,S}^{1/2} \\ ||\mathbf{v}||_{0,4,S} & \leq & c_4 ||\mathbf{v}||_S^{d/4} ||\mathbf{v}||_{1,S}^{(4-d)/4} \\ ||\mathbf{v}||_{0,6,S} & \leq & c_6 ||\mathbf{v}||_{1,S} \end{cases} \tag{2.24}$$

  for constant $0 < c_i < \infty$, $i = 3$, 4, 6.

- (Sobolev) $H^2(S) \hookrightarrow L^\infty(S)$. Moreover, for any $\mathbf{v} \in H^2(S)$, there exists $C > 0$ so that

$$||\mathbf{v}||_{0,\infty,S} \le C||\mathbf{v}||_{2,S}. \tag{2.25}$$

2.4.4 Estimating $\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}$

Formulation of a stable FE discretization of NS and NS-type problems is subtle. We introduced the explicitly skew-symmetric convective term in (2.9) so that, in the case $g = 0$,

$$c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{w}) \approx \int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}, \quad c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0.$$

In fact, as derived in (2.40), we show that $c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}$ to prove the consistency between the explicitly skew-symmetric term $c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{w})$ and its continuous counterpart $\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}$ when $\nabla \cdot \mathbf{u} = 0$. However, when $\nabla \cdot \mathbf{u} = g$ (with appropriate terms vanishing on boundary), we have

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} = c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{w}) - \frac{1}{2} \int_S g \mathbf{v} \cdot \mathbf{w} \tag{2.26}$$

Therefore, inclusion of $\frac{1}{2} \int_S g \mathbf{v} \cdot \mathbf{w}$ along with $c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{w})$ defines a consistent numerical pseudo-skew symmetrization of $\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}$ so that (2.9) must be replaced with

$$c_h(S)(\mathbf{u}, \mathbf{v}, \mathbf{w}) - \frac{1}{2} \int_S g \mathbf{v} \cdot \mathbf{w} \approx \int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}.$$

It is consequently worthwhile working out the identities and estimates associated with both the continuous and (proposed) discrete convective terms.

The following estimates of the convective term are direct consequences of various applications of Hölder's (2.22), Ladyzhenskaya's (2.24), and the Sobolev embedding (2.25) inequalities. See [65] for a comprehensive compilation of associated estimates.

Application of Hölder's inequality (2.22) with $1/p + 1/q + 1/r = 1$ and p, q, $r \ge 1$ gives

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} \le ||\mathbf{u}||_{0,p,S} ||\nabla \mathbf{v}||_{0,q,S} ||\mathbf{w}||_{0,r,S}, \quad \forall \mathbf{u} \in L^p(S),\ \mathbf{v} \in W^{1,q}(S),\ \mathbf{w} \in L^r(S).$$

Using Einstein vector notation, we derive the following identity:

$$\begin{aligned}
\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} &= \int_S u_i v_{j,i} w_j \\
&= \int_S u_i \left((v_j w_j)_{,i} - v_j w_{j,i} \right) \\
&= -\int_S \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v} + \int_S \left((u_i v_j w_j)_{,i} - u_{i,i} v_j w_j \right) \\
&= -\int_S \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v} - \int_S (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{w} + \int_S \nabla \cdot (\mathbf{u} \, (\mathbf{v} \cdot \mathbf{w})).
\end{aligned}$$

Then, application of the divergence theorem gives

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} = -\int_S \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v} - \int_S (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{w} + \int_{\partial S} (\mathbf{u} \cdot \hat{\mathbf{n}}_S) \mathbf{v} \cdot \mathbf{w}. \tag{2.27}$$

We conclude the following from (2.27).

Lemma 2.4.8. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1(S)$ *so that* $(\mathbf{u} \cdot \hat{\mathbf{n}}_S) \mathbf{v} \cdot \mathbf{w}|_{\partial S} = 0$. *Then*

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} = -\int_S \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v} - \int_S (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{w}. \tag{2.28}$$

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{v} = -\frac{1}{2} \int_S (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{v}. \tag{2.29}$$

Additionally, for any $\mathbf{u} \in V_{\cdot}(S)$,

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} = -\int_S \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v}, \tag{2.30}$$

$$\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{v} = 0. \tag{2.31}$$

From the previous lemma, we derive several important majorizations of the trilinear form $\int_S \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}$ required in the analysis of the NSE (and NS-type models).

Lemma 2.4.9. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1(S)$. *Then*

$$|\int_S \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}| \le C \begin{cases} (||\mathbf{u}||_S||\mathbf{u}||_{1,S})^{1/2}|\mathbf{v}|_{1,S}||\mathbf{w}||_{1,S} & \\ ||\mathbf{u}||_{1,S}|\mathbf{v}|_{1,S}(||\mathbf{w}||_S||\mathbf{w}||_{1,S})^{1/2} & \\ ||\mathbf{u}||_S||\mathbf{v}||_{2,S}(||\mathbf{w}||_S||\mathbf{w}||_{1,S})^{1/2} & \forall \mathbf{v}\in H^2(S) \\ (||\mathbf{u}||_S||\mathbf{u}||_{1,S})^{1/2}||\mathbf{v}||_{2,S}||\mathbf{w}||_S & \forall \mathbf{v}\in H^2(S) \\ ||\mathbf{u}||_{0,\infty,S}|\mathbf{v}|_{1,S}||\mathbf{w}||_S & \forall \mathbf{u}\in L^\infty(S) \\ ||\mathbf{u}||_S|\mathbf{v}|_{1,S}||\mathbf{w}||_{0,\infty,S} & \forall \mathbf{w}\in L^\infty(S) \end{cases} \tag{2.32}$$

Moreover, when $(\mathbf{u}\cdot\hat{\mathbf{n}}_S)\mathbf{v}\cdot\mathbf{w}|_{\partial S} = 0$, *we have*

$$|\int_S \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}| \le C \begin{cases} ||\mathbf{u}||_{1,S}(||\mathbf{v}||_S||\mathbf{v}||_{1,S})^{1/2}||\mathbf{w}||_{1,S} & \\ ||\mathbf{u}||_{2,S}||\mathbf{v}||_S||\mathbf{w}||_{2,S} & \forall \mathbf{u},\mathbf{w}\in H^2(S) \\ ||\mathbf{u}||_S(||\mathbf{v}||_S||\mathbf{v}||_{1,S})^{1/2}||\mathbf{w}||_{2,S} & \forall \mathbf{u}\in V_\cdot(S),\ \ \mathbf{w}\in H^2(S) \\ (||\mathbf{u}||_S||\mathbf{u}||_{1,S})^{1/2}||\mathbf{v}||_S||\mathbf{w}||_{2,S} & \forall \mathbf{u}\in V_\cdot(S),\ \ \mathbf{w}\in H^2(S) \\ ||\mathbf{u}||_{0,\infty,S}||\mathbf{v}||_S|\mathbf{w}|_{1,S} & \forall \mathbf{u}\in V_\cdot(S)\cap L^\infty(S) \\ ||\mathbf{u}||_S||\mathbf{v}||_{0,\infty,S}|\mathbf{w}|_{1,S} & \forall \mathbf{u}\in V_\cdot(S),\ \ \mathbf{v}\in L^\infty(S) \end{cases} \tag{2.33}$$

Proof. Estimate (2.32) is a consequence of Hölder's (2.22), Ladyzhenskaya's (2.24), and the Sobolev embedding (2.25) inequalities. Estimate (2.33) follows from these same estimates applied to Identities (2.28), (2.30). □

We also need estimates for the discrete, explicitly skew-symmetric trilinear form presented in (2.9):

$$c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\int_S(\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w} - \mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{v})$$

so that

$$c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w}) = -c_h(S)(\mathbf{u},\mathbf{w},\mathbf{v}) \tag{2.34}$$

$$c_h(S)(\mathbf{u},\mathbf{v},\mathbf{v}) = 0. \tag{2.35}$$

Next, substitute Identity (2.27) into (2.9) to obtain 3 equivalent formulations of $c_h(\cdot,\cdot,\cdot)$:

$$c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_S \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}\int_S(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w} - \frac{1}{2}\int_{\partial S}(\mathbf{u}\cdot\hat{\mathbf{n}}_S)\mathbf{v}\cdot\mathbf{w} \tag{2.36}$$

$$= -\int_S \mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{v} - \frac{1}{2}\int_S(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}\int_{\partial S}(\mathbf{u}\cdot\hat{\mathbf{n}}_S)\mathbf{v}\cdot\mathbf{w}. \tag{2.37}$$

The following is an immediate consequence of (2.36), (2.37)/

Lemma 2.4.10. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1(S)$ *so that* $(\mathbf{u}\cdot\hat{\mathbf{n}}_S)\mathbf{v}\cdot\mathbf{w}|_{\partial S} = 0$. *Then,*

$$c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_S \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}\int_S (\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w} \tag{2.38}$$

$$= -\int_S \mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{v} - \frac{1}{2}\int_S (\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}. \tag{2.39}$$

Additionally, for any $\mathbf{u} \in V_\cdot(S)$,

$$c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_S \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}. \tag{2.40}$$

Similar to the continuous case, we derive several important majorizations of the discrete trilinear form $c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w})$ required in the analysis of the FE approximations of the NSE (and NS-type models).

Lemma 2.4.11. *Fix* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1(S)$. *Then*

$$|c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w})| \le C \begin{cases} (||\mathbf{u}||_S||\mathbf{u}||_{1,S})^{1/2}||\mathbf{v}||_{1,S}||\mathbf{w}||_{1,S} & \\ ||\mathbf{u}||_S||\mathbf{v}||_{2,S}||\mathbf{w}||_{1,S} & \forall \mathbf{v}\in H^2(S) \\ ||\mathbf{u}||_S||\mathbf{v}||_{1,S}||\mathbf{w}||_{2,S} & \forall \mathbf{w}\in H^2(S) \end{cases} \tag{2.41}$$

Additionally, when $(\mathbf{u}\cdot\hat{\mathbf{n}}_S)\mathbf{v}\cdot\mathbf{w}|_{\partial S} = 0$,

$$|c_h(S)(\mathbf{u},\mathbf{v},\mathbf{w})| \le C \begin{cases} ||\mathbf{u}||_{1,S}(||\mathbf{v}||_S||\mathbf{v}||_{1,S})^{1/2}||\mathbf{w}||_{1,S} & \\ ||\mathbf{u}||_{1,S}||\mathbf{v}||_{1,S}(||\mathbf{w}||_S||\mathbf{w}||_{1,S})^{1/2} & \\ ||\mathbf{u}||_{2,S}||\mathbf{v}||_S||\mathbf{w}||_{1,S} & \forall \mathbf{u}\in H^2(S) \\ ||\mathbf{u}||_{2,S}||\mathbf{v}||_{1,S}||\mathbf{w}||_S & \forall \mathbf{u}\in H^2(S) \\ ||\mathbf{u}||_{1,S}||\mathbf{v}||_S||\mathbf{w}||_{2,S} & \forall \mathbf{w}\in H^2(S) \\ ||\mathbf{u}||_{1,S}||\mathbf{v}||_{2,S}||\mathbf{w}||_S & \forall \mathbf{v}\in H^2(S) \end{cases} \tag{2.42}$$

Proof. Estimate (2.41) is an immediate consequence of Hölder's (2.22), Ladyzhenskaya's (2.24) inequalities along with and the Sobolev embedding estimate $||\mathbf{u}||_{0,\infty,S} \le C||\mathbf{u}||_{2,S}$ to the definition (2.9). Estimate (2.42) follows from these same estimates applied to Identities (2.38), (2.39). □

2.5 THE NAVIER-STOKES EQUATIONS

The NSE are well-known to be a physically accurate model for incompressible, viscous fluid flow including turbulence. Although the NSE provide an accurate description of fluid flow, there are many subtle and unresolved mathematical mysteries associated with the existence and smoothness of NS-solutions. The development and implementation of stable, accurate, and robust methods for computing NS-solutions is equally challenging. We first present the equations and then present an overview of the notion of weak and strong NS-solutions. In the subsections that follow, we investigate the stability of steady and evolutionary solutions to the NSE. We are particularly interested in the nonlinear feedback of inhomogeneous boundary data as $T \to \infty$ in stability analysis. See [27, 78, 79, 24] for a thorough treatment (with references) of the existence of NSE solutions for inhomogeneous data.

Suppose that Ω is locally Lipschitz, $\mathbf{f} \in L^2(W^{-1,\infty})$, $\mathbf{u}^0 \in V$, $g \in L^4(L^2)$, and $\phi \in L^4(W^{-1/6,6}(\partial\Omega) \cap H_g^{1/2}(\partial\Omega))$. Then weak NS-solutions $\mathbf{u} \in L^2(V_\phi(g)) \cap L^\infty(L^2)$, $\partial_t \mathbf{u} \in L^{4/d}(H^{-1})$ and $p \in W^{-1,\infty}(L_0^2)$ of (1.1), (1.2) satisfying (2.43), (2.44), (2.45) exist (see e.g. Remarks 2.6, 2.7 in [30], see also [24] for sufficient regularity of data):

- (Weak NSE) Find $\mathbf{u} \in L^2(H_\phi^1) \cap L^\infty(L^2)$ and $p \in W^{-1,\infty}(L_0^2)$ satisfying

$$\frac{d}{dt}(\mathbf{u},\mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v}) + Re^{-1}(\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in H_0^1 \tag{2.43}$$

$$\nabla\cdot\mathbf{u}(\mathbf{x},t) = g(\mathbf{x},t) \quad \text{a.e. } (\mathbf{x},t) \in \Omega\times[0,T] \tag{2.44}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}), \quad \text{a.e. } \mathbf{x}\in\Omega. \tag{2.45}$$

Recall that ϕ must satisfy the compatibility condition

$$\int_{\partial\Omega} \phi\cdot\hat{\mathbf{n}} = \int g$$

where $\hat{\mathbf{n}}$ is the outward unit normal defined on $\partial\Omega$. See, e.g., [30] (and references therein) for a comprehensive discussion and derivation of the existence of weak solutions and regularity results in the case $\phi = 0$, $g = 0$.

When restricted to $\mathbb{R}^2$, weak solutions are known to be unique. However, in $\mathbb{R}^3$, although *weak solutions* of the NSE are known to exist, they might not be unique. Consequently, different methods used to conclude existence of solutions might actually lead to different solutions.

Although a potential theoretical void, non-uniqueness leads to one possible explanation of the physical phenomenon of turbulence. *Strong solutions* of the NSE, a notion for $\mathbb{R}^3$, solve (2.43), (2.44), (2.45) with additional regularity to ensure *uniqueness*. For example weak NS-solutions satisfying $\mathbf{u} \in L^4(H^1)$ are necessarily unique, but are not known to exist for arbitrary time intervals.

For the numerical analysis that follows, we require, as is assuredly a fact for many pertinent flows, strong solutions that satisfy $\mathbf{u} \in L^\infty(H^1)$ on a *given* time interval $[0, T)$. In fact, $\mathbf{u}^0 \in V_\phi \cap H^2$ ensures that $\mathbf{u} \in C^0([0, t'); H^1)$ for restricted $t' \leq T$ possibly small. Along with basic regularity of $\partial\Omega$, these assumptions guarantee $\mathbf{u} \in C^0([0, T); H^2)$, $\partial_t \mathbf{u} \in C^0([0, T); L^2)$, and $p \in C^0([0, T); H^1)$. We require the problem data to (minimally) satisfy

$$\mathbf{u}^0 \in V_\phi \cap H^2, \quad \mathbf{f} \in W^{1,\infty}(L^2), \quad g \in L^4(L^2), \quad \phi \in L^4(H_g^{1/2}(\partial\Omega)). \tag{2.46}$$

(indeed, see [24] for the proper ϕ-regularity; alternative treatments are found in [27, 78, 79]). Alternatively, it has been shown that $||\partial_t \mathbf{u}(\cdot, t)||_{-1}$ and $||p(\cdot, t)||$ diverge to ∞ as $t \to 0^+$ for certain $\mathbf{u}^0 \notin H^2$ (see [46]).

Assumption 2.5.1. *Fix $q \geq 1$ and $k \geq 1$. Then $\mathbf{u} \in C^0(H^k \cap V_\phi(g))$, $\partial_t \mathbf{u} \in C^0(H^{k-2})$, $p \in C^0(H^{k-1} \cap L_0^2)$ so that*

$$||\mathbf{u}||_{L^q(H^k)} + ||\partial_t \mathbf{u}||_{L^q(W^{k-2,2})} + ||p||_{L^q(H^{k-1})} \leq M_* < \infty$$

when the problem data is sufficiently regular so that M_* depends on $\mathbf{f}$, $\mathbf{u}^0$, ϕ, ν^{-1}, g.

In order to avoid the accompanying factor $\min\{t^{-1}, 1\}$ with M_* in Assumption 2.5.1 when $k \geq 3$, the following compatibility condition is necessarily required (see e.g. [43] for the case $g = 0$):

$$\begin{cases} \Delta p(\cdot, 0) = \nabla \cdot (\tilde{\mathbf{f}}(\cdot, 0) - \mathbf{u}(\cdot, 0) \cdot \nabla \mathbf{u}(\cdot, 0)), & \text{in } \Omega \\ \nabla p(\cdot, 0) = (\Delta \mathbf{u}(\cdot, 0) + \tilde{\mathbf{f}}(\cdot, 0) - \mathbf{u}(\cdot, 0) \cdot \nabla \mathbf{u}(\cdot, 0)), & \text{on } \partial\Omega \end{cases} \tag{2.47}$$

where $\tilde{\mathbf{f}}$ depends on both $\mathbf{f}$ and g and $\tilde{\mathbf{f}} = \mathbf{f}$ when $g = 0$. This is an overdetermined Neumann-type problem where $p(\cdot, 0)$ is the solution of (well-posed) Neumann problem

$$\begin{cases} \Delta p(\cdot, 0) = \nabla \cdot (\tilde{\mathbf{f}}(\cdot, 0) - \mathbf{u}(\cdot, 0) \cdot \nabla \mathbf{u}(\cdot, 0)), \text{ in } \Omega, \\ \nabla p(\cdot, 0) \cdot \hat{\mathbf{n}}|_{\partial\Omega} = (\Delta \mathbf{u}(\cdot, 0) + \tilde{\mathbf{f}}(\cdot, 0) - \mathbf{u}(\cdot, 0) \cdot \nabla \mathbf{u}(\cdot, 0)) \cdot \hat{\mathbf{n}}|_{\partial\Omega}. \end{cases}$$

Condition (2.47) is a nonlocal condition relating $\mathbf{u}(\cdot,0)$ and $\tilde{\mathbf{f}}(\cdot,0)$. If not satisfied, then the following estimate is sharp as $t \to 0^+$:

$$||\mathbf{u}(\cdot,t)||_k + ||p(\cdot,t)||_{k-1} \le Ct^{-(k-2)/2}.$$

Note that condition (2.47) is (trivially) satisfied when $\mathbf{u}(\cdot,0) = 0$, $\mathbf{f}(\cdot,0) = 0$ (i.e. starting from rest with zero-force).

2.5.1 Stability of stationary flows

The steady-state analogue of (2.43), (2.44) is relevant to our analysis.

- (Steady NSE) If problem data is time-independent, so that $\phi \in H_0^{1/2}(\partial\Omega)$, find equilibrium solutions $\mathbf{u} \in H^1_\phi$ and $p \in L^2_0$ satisfying

$$(\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H^1_0 \tag{2.48}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } L^2(\Omega). \tag{2.49}$$

Care must be taken to ensure the existence of stationary (formally, $\partial_t\mathbf{u} = 0$) and evolutionary solutions of the NSE in the case of inhomogeneous problem data $\mathbf{u}|_{\partial\Omega} = \phi \neq 0$. To illustrate the problem, define the flux on each *hole* of Ω by

$$\Phi_i = \int_{\partial\Omega_i} \phi\cdot\hat{\mathbf{n}}$$

where $\partial\Omega_i$ for $i = 1, \dots, m \geq 1$ are the connected components of boundary of $\partial\Omega$ (representing the boundaries of the internal *holes* in Ω). It is known that equilibrium solutions exist for arbitrary $\phi \in H_0^{1/2}(\partial\Omega)$ as long as $\Phi_i = 0$ for $i = 1, \dots, m$. On the other hand, it is shown in [29] (Theorem 4.1 in Chapter VIII.4) that $\sum_i |\Phi_i| \leq c_*\nu$ (for a computable constant c_*) is a sufficient condition for the existence of steady NS-solutions. It is the breakdown of Stokes theorem when Ω is not simply connected that prevents the existence of the *solenoidal* extension with the necessary properties. Precisely, in proving the necessary *a priori* estimate for (steady) NSE, we must control the size of the trilinear form

$$|\int \mathbf{w}\cdot\nabla E(\phi)\cdot\mathbf{w}| \leq \delta|\mathbf{w}|_1^2 \tag{2.50}$$

where $E(\phi)$ is a V_ϕ-extension of ϕ so that $0 < \delta < C\nu$ for some constant C. Note that there are actually counter-examples that disprove the validity of (2.50) for $\delta \leq C\nu$ under the weaker condition (2.1) (see e.g. see pp. 22-23 in [29]). In particular, it is an **open question** whether or not equilibrium solutions exist (in either $\mathbb{R}^2$ or $\mathbb{R}^3$) when Ω is *not* simply connected and the compatibility condition $\int_{\partial\Omega} \phi \cdot \hat{\mathbf{n}} = 0$ is satisfied but $\Phi_i \neq 0$.

We consider an alternate manifestation of sources/sinks of fluid within a domain with similar mathematical consequences than described previously. Specifically, let Ω be simply connected, but relax of the conservation of mass equation via (2.51) by introducing a non-zero divergence constraint $g \in L^2(\Omega_p)$.

Lemma 2.5.2 (Stationary-NSE Solutions are Bounded)**.** *Fix $g \in L^2(\Omega_p)$. Suppose that (2.49) is replaced by*

$$\nabla \cdot \mathbf{u} = g. \tag{2.51}$$

Suppose further that $\phi \in H_g^{1/2}(\partial\Omega)$, $\mathbf{f} \in V'$, and the small data condition

$$4\nu^{-1}\left|\int (\mathbf{w}\cdot\nabla\mathbf{w})\cdot E(\phi) - \frac{1}{2}g|\mathbf{w}|^2\right| \leq |\mathbf{w}|_1^2, \quad \forall \mathbf{w} \in V \tag{2.52}$$

is satisfied where $E : H_g^{1/2}(\partial\Omega) \to V.(g)$ is an extension operator. Then

$$||\mathbf{u}||_1 + ||p||^{1/2} \leq \nu^{-1}M_0 \tag{2.53}$$

for some $0 < M_0 = M_0(\mathbf{f}, \phi, g) < \infty$ independent of ν^{-1}.

Remark 2.5.3. *If $g = 0$, for all $\phi \in H_0^{1/2}(\partial\Omega)$ and for any $\delta > 0$ there exists an extension $E_\delta : H_0^{1/2}(\partial\Omega) \to V.$ that satisfies (2.52) via*

$$|\int \mathbf{v}\cdot\nabla E_\delta(\phi)\cdot\mathbf{v}| \leq \delta|\mathbf{v}|_1^2, \quad \forall \mathbf{v} \in H_0^1.$$

If $g \in L_0^2(\Omega_p)$, then sufficiently small g and or ν^{-1} ensures condition (2.52) via

$$\left|\int (\mathbf{w}\cdot\nabla\mathbf{w})\cdot E(\phi) - \frac{1}{2}g|\mathbf{w}|^2\right| \leq C||g||_{\Omega_p}|\mathbf{w}|_1^2$$

is satisfied with no restriction on $\phi \in H_g^{1/2}(\partial\Omega)$. However, avoiding a smallness constraint on ϕ leads to an exponential growth of $||E(\phi)||_{k,q} \leq C\exp(1/\delta)$ for any $k \geq 0$, $q \geq 1$. In either case case, the constant $M_0 = M_0(\delta) \to \infty$ (exponentially) as $\delta \to 0$. On the other hand, existence is guaranteed without this exponential contamination of M_0 for sufficiently small ϕ, g, ν^{-1}.

Proof. Let $E(\phi) \in H^1$ be an extension problem data so that $E(\phi)|_{\partial\Omega} = \phi$ and $\nabla \cdot E(\phi) = g$. Write $\mathbf{u} = \mathbf{w} + E(\phi)$ so that $\mathbf{w}|_{\partial\Omega} = 0$ and $\nabla \cdot \mathbf{w} = 0$. Substitute $\mathbf{w}$ into (2.48) and test with $\mathbf{v} = \mathbf{w}$. Note that $\mathbf{w}|_{\partial\Omega} = 0$ and $E(\phi) \in V_\phi(g)$ together with Identities (2.31) and (2.29) gives $(\mathbf{w} \cdot \nabla \mathbf{w}, \mathbf{w}) = 0$ and $(E(\phi) \cdot \nabla \mathbf{w}, \mathbf{w}) = -\frac{1}{2}(g\mathbf{w}, \mathbf{w})_{\Omega_p}$ so that

$$\begin{aligned}
\nu|\mathbf{w}|_1^2 &= (\mathbf{f}, \mathbf{w}) - \nu(\nabla E(\phi), \nabla \mathbf{w}) \\
&\quad - (E(\phi) \cdot \nabla \cdot E(\phi), \mathbf{w}) - \int (\mathbf{w} \cdot \nabla \cdot E(\phi) \cdot \mathbf{w} - \frac{1}{2} g|\mathbf{w}|^2).
\end{aligned}$$

Apply Identity (2.29) and Hölder's inequality (2.22) to get

$$\begin{aligned}
|(E(\phi) \cdot \nabla E(\phi), \mathbf{w})| &= (E(\phi) \cdot \nabla \mathbf{w}, E(\phi)) - \frac{1}{2}(g\mathbf{w}, E(\phi))_{\Omega_p} \\
&\leq C(||E(\phi)||_{0,4} + \frac{c_4}{2}||g||_{\Omega_p})||E(\phi)||_{0,4}|\mathbf{w}|_1.
\end{aligned}$$

Along with application of Young's inequality (2.21), we get

$$\begin{aligned}
\nu|\mathbf{w}|_1^2 &\leq \left(||\mathbf{f}||_{V'} + \nu|E(\phi)|_1 + (||E(\phi)||_{0,4} + \frac{c_4}{2}||g||_{\Omega_p})||E(\phi)||_{0,4}\right)|\mathbf{w}|_1 \\
&\quad + \left|\int (\mathbf{w} \cdot \nabla \cdot E(\phi) \cdot \mathbf{w} - \frac{1}{2} g|\mathbf{w}|^2)\right| \\
&\leq \nu^{-1}(||\mathbf{f}||_{V'} + \nu|E(\phi)|_1 + (||E(\phi)||_{0,4} + \frac{c_4}{2}||g||_{\Omega_p})||E(\phi)||_{0,4})^2 \\
&\quad + \frac{\nu}{4}|\mathbf{w}|_1^2 + \left|\int \mathbf{w} \cdot \nabla \cdot E(\phi) \cdot \mathbf{w} - \frac{1}{2} g|\mathbf{w}|^2\right|.
\end{aligned}$$

Assume that condition (2.52) is satisfied. Absorb like-terms from right into the left-hand side to get

$$|\mathbf{w}|_1 \leq 2\nu^{-1}(||\mathbf{f}||_{V'} + \nu|E(\phi)|_1 + (||E(\phi)||_{0,4} + \frac{c_4}{2}||g||_{\Omega_p})||E(\phi)||_{0,4})$$

Since $\mathbf{w} = \mathbf{u} - E(\phi)$, application of the triangle inequality gives

$$|\mathbf{u}|_1 \leq \sqrt{2}\nu^{-1}(||\mathbf{f}||_{V'} + \frac{\sqrt{2}+1}{\sqrt{2}}\nu|E(\phi)|_1 + (||E(\phi)||_{0,4} + \frac{c_4}{2}||g||_{\Omega_p})||E(\phi)||_{0,4}).$$

The estimate for $\mathbf{u}$ in (2.53) follows.

To bound the pressure, first solve (2.48) for p. Apply Identities (2.31) and (2.29), the duality estimate on $W^{-1,2} \times H_0^1$, and Hölder's (2.22) and Ladyzhenskaya's (2.24) inequalities to get

$$
\begin{aligned}
(p, \nabla \cdot \mathbf{v}) &= (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\mathbf{f}, \mathbf{v}) \\
&\leq (c_4^2(||\mathbf{u}||_1 + \frac{1}{2}||g||_{\Omega_p})||\mathbf{u}||_1 + \nu|\mathbf{u}|_1 + ||\mathbf{f}||_{-1})|\mathbf{v}|_1.
\end{aligned}
$$

Divide by $|\mathbf{v}|_1$ and apply (2.16) to get

$$
||p|| \leq \beta^{-1}(c_4^2(||\mathbf{u}||_1 + \frac{1}{2}||g||_{\Omega_p})||\mathbf{u}||_1 + \nu|\mathbf{u}|_1 + ||\mathbf{f}||_{-1}).
$$

The estimate for p in (2.53) follows by applying the previously derived estimate for $\mathbf{u}$ in (2.53). $\square$

2.5.2 Stability of evolutionary flows

Stability and hence existence of NSE solutions with inhomogeneous data including non-zero divergence constraint is investigated in [27, 78, 79, 24]. We provide an *a priori* estimate here. The regularity we assume on g and ϕ in Lemma 2.5.4 can be generally relaxed. Indeed, it is shown in [24] that data satisfying (2.46) is enough to ensure existence of solutions. However, this minimal regularity of problem data is not enough to ensure even local in-time uniqueness of solutions. In their proof, the authors construct a *very weak* extension $E(\phi)$ that produces a *weak* NSE solution, but at the cost of a Gronwall (exponential) factor in the *a priori* estimate. Note that our method of proof is similar to that of [24], but differs from the method followed in [27, 78, 79]. Our stability proof is certainly crude, but highlights (without introducing an excess of technical machinery) the key choices and costs associated with constructing data extensions and to either exploit Gronwall's inequality or not!

Lemma 2.5.4 (NSE Solutions are Bounded)**.** *Fix $g \in C^0(L^2)$ and suppose that (2.44). Suppose that $\phi \in C^0(H_g^{1/2}(\partial\Omega))$, $\mathbf{f} \in L^2(V')$, and that*

$$
4\nu^{-1}\left|\int (\mathbf{w}(\cdot,t) \cdot \nabla \mathbf{w}(\cdot,t)) \cdot E(\phi(\cdot,t)) - \frac{1}{2}g(\cdot,t)|\mathbf{w}(\cdot,t)|^2\right| \leq |\mathbf{w}(\cdot,t)|_1^2, \quad \forall \mathbf{w}(\cdot,t) \in V \quad (2.54)
$$

is satisfied where $E : H_g^{1/2}(\partial\Omega) \to V.(g)$ *is an extension operator. Then*

$$||\mathbf{u}||_{L^\infty(L^2)} + \nu^{1/2}||\mathbf{u}||_{L^2(H^1)} \leq \nu^{-1/2}M_0 \tag{2.55}$$

for some $0 < M_0 = M_0(\mathbf{f}, \phi, g) < \infty$ *independent of* ν^{-1}.

Remark 2.5.5. *Similar comments hold from Remark 2.5.3. If* $g = 0$, *for all* $\phi \in W^{1,\infty}(H_0^{1/2}(\partial\Omega))$ *and for any* $\delta > 0$ *there exists an extension* $E_\delta : H_0^{1/2}(\partial\Omega) \to V.$ *that satisfies (2.54). If* $g \in L_0^2(\Omega_p)$, *then sufficiently small* g *and or* ν^{-1} *ensures condition (2.54) without restricting* $\phi \in W^{1,\infty}(H_g^{1/2}(\partial\Omega))$. *Avoiding the smallness constraint on* ϕ *leads to an exponential growth of* $||E(\phi(\cdot,t))||_{k,p} \leq C\exp(1/\delta)$ *for* $k \geq 0$, $p \geq 1$.

Alternatively, we can avoid the smallness assumption on the extension $E(\phi) \in V_\phi(g)$ *by exploiting the Gronwall Lemma (unavailable for the stationary problem in Lemma 2.5.2). Indeed,* $\mathbf{w} \in V$ *gives*

$$(\mathbf{w} \cdot \nabla E(\phi), \mathbf{w}) \leq C\nu^{-3}||E(\phi)||_1^4||\mathbf{w}||_1^2 + \frac{\nu}{2}|\mathbf{w}|_1^2.$$

With slight alterations to the following proof, we derive

$$\begin{aligned}\frac{d}{dt}||\mathbf{w}||^2 + \nu|\nabla\mathbf{w}|_1^2 \leq C(\nu^{-3}||E(\phi)||_1^4 + ||g||_{0,\infty})||\mathbf{w}||^2\\ + 3\nu^{-1}(||\mathbf{f}||_{V'}^2 + ||\partial_t E(\phi)||_{V'}^2 + \nu^2|E(\phi)|_1^2).\end{aligned} \tag{2.56}$$

Introduce the integrating factor $G(t) := \exp(-C(\nu^{-3}||E(\phi)||^4_{L^4(0,t;H^1)} + ||g||_{L^1(0,t;L^\infty)}))$. *Group terms, integrate on* $(0,T)$, *and simplify to get*

$$\begin{aligned}||\mathbf{u}(\cdot,T)||^2 + \nu||\nabla\mathbf{u}||^2_{L^2(L^2)} \leq 3\nu^{-1}G(t)^{-1}(||\mathbf{f}||^2_{L^2(V')} + \nu||\mathbf{u}^0||^2 + \nu||E(\phi(\cdot,0))||^2 + \dots\\ \dots + ||\partial_t E(\phi)||^2_{L^2(V')} + (\nu + \frac{\nu^2 G(t)}{3})||\nabla E(\phi)||^2_{L^2(L^2)}) + ||E(\phi)||^2_{L^\infty(L^2)}.\end{aligned} \tag{2.57}$$

Without (2.54), the a priori estimate for $\mathbf{u}$ *is contaminated by the exponential factor* $\exp(C(T)\nu^{-3})$. *Note that the* $\exp(C(T)\nu^{-3})$*-dependency can be reduced to* $\exp(C(T))$ *with additional regularity of* ϕ. *Even in this case, however, exponential dependence on* ν^{-1} *that grows as* $T \to \infty$ *render such estimates meaningless over long time intervals.*

Proof. Let $E(\phi) \in H^1$ be an extension problem data so that $E(\phi)|_{\partial\Omega} = \phi$ and $\nabla \cdot E(\phi) = g$. Write $\mathbf{u} = \mathbf{w} + E(\phi)$ so that $\mathbf{w}|_{\partial\Omega} = 0$ and $\nabla \cdot \mathbf{w} = 0$. Substitute $\mathbf{u} = \mathbf{w} + E(\phi)$ into (2.43) and test with $\mathbf{v} = \mathbf{w}$. Note that $\mathbf{w}|_{\partial\Omega} = 0$ and $E(\phi) \in V_\phi(g)$ together with Identities (2.31) and (2.29) give $(\mathbf{w} \cdot \nabla\mathbf{w}, \mathbf{w}) = 0$ and $(E(\phi) \cdot \nabla\mathbf{w}, \mathbf{w}) = -\frac{1}{2}(g\mathbf{w}, \mathbf{w})_{\Omega_p}$ so that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + \nu|\mathbf{w}|_1^2 = (\mathbf{f}, \mathbf{w}) - (\partial_t E(\phi), \mathbf{w}) \\
&\quad - \nu(\nabla E(\phi), \nabla\mathbf{w}) - (E(\phi) \cdot \nabla E(\phi), \mathbf{w}) - ((\mathbf{w} \cdot \nabla E(\phi), \mathbf{w}) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{w}|^2).
\end{aligned}$$

Apply Identity (2.29) and Hölder's inequality (2.22) to get

$$\begin{aligned}
|(E(\phi) \cdot \nabla E(\phi), \mathbf{w})| &= (E(\phi) \cdot \nabla\mathbf{w}, E(\phi)) - \frac{1}{2}(g\mathbf{w}, E(\phi))_{\Omega_p} \\
&\le C\left(||E(\phi)||_{0,4} + ||g||_{\Omega_p}\right)||E(\phi)||_{0,4}|\mathbf{w}|_1.
\end{aligned}$$

Apply the duality estimate on $V' \times V$, Hölder's inequality (2.22), and Young's inequality (2.21) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + \nu|\mathbf{w}|_1^2 \\
&\le \left(||\mathbf{f}||_{V'} + ||\partial_t E(\phi)||_{V'} + \nu|E(\phi)|_1 + (||E(\phi)||_{0,4} + ||g||_{\Omega_p})||E(\phi)||_{0,4}\right)|\mathbf{w}|_1 \\
&\quad + \left|\int (\mathbf{w} \cdot \nabla\mathbf{w}) \cdot E(\phi) - \frac{1}{2}g|\mathbf{w}|^2\right| \\
&\le \nu^{-1}(||\mathbf{f}||_{V'} + ||\partial_t E(\phi)||_{V'} + \nu|E(\phi)|_1 + (||E(\phi)||_{0,4} + \frac{c_4}{2}||g||_{\Omega_p})||E(\phi)||_{0,4})^2 \\
&\quad + \frac{\nu}{4}|\mathbf{w}|_1^2 + \left|\int (\mathbf{w} \cdot \nabla\mathbf{w}) \cdot E(\phi) - \frac{1}{2}g|\mathbf{w}|^2\right|.
\end{aligned}$$

Assume that condition (2.54) is satisfied. Absorb like-terms from right into left-hand side of the above estimate to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + \frac{\nu}{2}|\mathbf{w}|_1^2 &\le 5\nu^{-1}||\mathbf{f}||_{V'}^2 + 5\nu|E(\phi)|_1^2 \\
&\quad + 5\nu^{-1}||\partial_t E(\phi)||_{V'}^2 + 5\nu^{-1}(||E(\phi)||_{0,4}^2 + \frac{c_4^2}{4}||g||_{\Omega_p}^2)||E(\phi)||_{0,4}^2.
\end{aligned}$$

Integrating on $(0, T)$ to get

$$\begin{aligned}
||\mathbf{w}(\cdot, T)||^2 + \nu||\nabla\mathbf{w}||_{L^2(L^2)}^2 &\le ||\mathbf{w}(\cdot, 0)||^2 + 5||\mathbf{f}||_{L^2(V')}^2 + 5\nu||\nabla E(\phi)||_{L^2(L^2)}^2 \\
&\quad + 5\nu^{-1}||\partial_t E(\phi)||_{L^2(V')}^2 + \frac{5c_4^2}{4\nu}||g||_{L^\infty(L^2(\Omega_p))}^2||E(\phi)||_{L^2(L^4)}^2 + 5\nu^{-1}||E(\phi)||_{L^4(L^4)}^4.
\end{aligned}$$

Application of the triangle inequality along with $\mathbf{w} = \mathbf{u} - E(\phi)$ gives

$$
\begin{aligned}
&||\mathbf{u}(\cdot,T)||^2 + \nu||\nabla\mathbf{u}||^2_{L^2(L^2)} \le ||\mathbf{u}^0||^2 + 5\nu^{-1}||\mathbf{f}||^2_{L^2(V')} \\
&\quad + 2||E(\phi)||^2_{L^\infty(L^2)} + 6\nu||\nabla E(\phi)||^2_{L^2(L^2)} + 5\nu^{-1}||\partial_t E(\phi)||^2_{L^2(V')} \\
&\quad + \frac{5c_4^2}{4\nu}||g||^2_{L^\infty(L^2(\Omega_p))}||E(\phi)||^2_{L^2(L^4)} + 5\nu^{-1}||E(\phi)||^4_{L^4(L^4)}.
\end{aligned}
$$

Bound $E(\phi)$ via (2.14) to prove (2.55). $\square$

3.0 STABILITY OF CNLE APPROXIMATIONS IS SUBTLE

The Navier-Stokes (NS) equations (NSE) provide an accurate description of fluid flow. However, there are many subtle and unresolved questions regarding existence and smoothness of the NS velocity field $\mathbf{u}$. There are related open problems regarding development and implementation of stable, accurate, robust, and feasible methods for approximating $\mathbf{u}$. Let $\Omega \subset \mathbb{R}^d$ be the problem domain. We consider herein the nonlinear feedback of inhomogeneous boundary data and divergence constraint:

$$\mathbf{u}|_{\partial\Omega} = \phi \neq 0, \quad \nabla \cdot \mathbf{u} = g \neq 0. \tag{3.1}$$

Energetic stability for Crank-Nicolson (CN) time-stepping with linear extrapolation (CNLE) is well established for homogeneous problem data. We show, however, that within current techniques, the standard $\mathcal{O}(\Delta t^2)$ linear extrapolation

$$\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \cdot \nabla \mathbf{u} \approx (\frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}, \quad \mathbf{u}^i := \mathbf{u}(x, t_i) \tag{3.2}$$

presented and analyzed in [7] *does not* lead to a (provable) energetically stable numerical discretization in the case of inhomogeneous problem data for *long-time* solutions. Specifically, stabilisty has not been proven and known methods of proof fail. We propose a *new* alternative $\mathcal{O}(\Delta t^2)$ extrapolation for general data:

$$\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \cdot \nabla \mathbf{u} \approx \xi^n(\mathbf{u}) \cdot \nabla \mathbf{u}, \quad \xi^n(\mathbf{u}) := 2\mathbf{u}^{n-1/2} - \mathbf{u}^{n-3/2}. \tag{3.3}$$

Write $\mathbf{u}^{i+1/2} := \frac{1}{2}(\mathbf{u}^{i+1} + \mathbf{u}^i)$. We show herein that CNLE approximations $\{\mathbf{u}^n\}_n$ obtained with (3.3) are *provably* stable for general data (3.1) so that

$$\max_n ||\mathbf{u}^{n+1}||^2 + \nu\Delta t \sum_n ||\nabla \mathbf{u}^{n+1/2}||^2 \leq C(data) < \infty.$$

Energetic stability of NS solutions (and corresponding discretization) is the key to establishing well-posedness. It is illuminating to introduce the backward-Euler (BE) scheme (stable for general data) to highlight the difficulties of inhomogeneous CNLE. First, the stability analysis for homogeneous data relies on the skew-symmetry of the convective nonlinearity in the NSE:

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \nabla\cdot\mathbf{u} = 0 \quad \Rightarrow \quad \int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u} = 0.$$

Let $i = 1$ for BE with linear extrapolation (BELE) and $i = 2$ for CNLE. The *energy difference* due to the numerical extrapolation

$$\int \mathbf{u}^{n+1/i}\cdot\nabla\mathbf{u}^{n+1/i}\cdot\mathbf{v} \approx \int \xi^n(\mathbf{u})\cdot\nabla\mathbf{u}^{n+1/i}\cdot\mathbf{v}, \quad \xi^n(\mathbf{u}) := a_0\mathbf{u}^n + \ldots + a_{n_0}\mathbf{u}^{n-n_0}$$

must be absorbed into the model viscous term $\nu\sum_n |\mathbf{u}^{n+1/i}|_1^2$ to establish energetic stability for $T\to\infty$. Indeed, we lift the data so that $\mathbf{u} = \mathbf{u}_0 + E(\phi)$ for some extension operator $E(\phi)$ satisfying (3.1) in the case of inhomogeneous data (3.1). Cross-terms from the nonlinearity pollute the RHS of the resulting estimate upon the substitution of $\mathbf{u}^n = \mathbf{u}_0^n + E(\phi^n)$. The energy estimate for $\mathbf{u}_0^n$ is obtained by testing either BELE or CNLE with $\mathbf{v} = \mathbf{u}_0^{n+1/i}$ to get

$$||\mathbf{u}_0^{n+1}||^2 + 2\Delta t\nu\sum_n |\mathbf{u}_0^{n+1/i}|_1^2 + \ldots = -\Delta t\sum_n \int \xi^n(\mathbf{u}_0)\cdot\nabla E(\phi^{n+1/i})\cdot\mathbf{u}_0^{n+1/i} + \ldots. \quad (3.4)$$

Suppose that the extension $E(\phi)$ satisfies

$$|\int \xi^n(\mathbf{u}_0)\cdot\nabla E(\phi^{n+1/i})\cdot\mathbf{u}_0^{n+1/i}| \le \delta|\xi^n(\mathbf{u}_0)|_1|\mathbf{u}_0^{n+1/i}|_1 \quad (3.5)$$

for some $\delta > 0$. For any $\delta > 0$, there exists $E(\phi) = E_\delta(\phi)$ satisfying (3.5) (when $g\equiv 0$) in the continuous case. Suppose that $\xi^n(\mathbf{u}) = \mathbf{u}^n$ for BELE and $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$ for CNLE. We use (3.5) to resolve an *a priori* estimate for $\mathbf{u}_0$ starting with (3.4). One option is to bound (3.5) so that

$$|\int \xi^n(\mathbf{u}_0)\cdot\nabla E(\phi^{n+1/i})\cdot\mathbf{u}_0^{n+1/i}| \le \frac{\delta}{2}\begin{cases} (|\mathbf{u}_0^n|_1^2 + |\mathbf{u}_0^{n+1}|_1^2), & \text{BELE} \\ (|\frac{3}{2}\mathbf{u}_0^n - \frac{1}{2}\mathbf{u}_0^{n-1}|_1^2 + |\mathbf{u}_0^{n+1/2}|_1^2), & \text{CNLE} \end{cases}. \quad (3.6)$$

We can absorb $\frac{\delta}{2}\sum_n(|\mathbf{u}_0^n|_1^2 + |\mathbf{u}_0^{n+1}|_1^2)$ into $\nu\sum_n|\mathbf{u}_0^{n+1}|_1^2$ in (3.4) for BELE after summing from $n = 0$ to $N-1$. However, regardless how small δ is taken, there is no general way to absorb $\frac{\delta}{2}\sum_n |\frac{3}{2}\mathbf{u}_0^n - \frac{1}{2}\mathbf{u}_0^{n-1}|_1^2$ into $\nu\sum_n |\mathbf{u}_0^{n+1/2}|_1^2$ (3.4) for CNLE even after summing from $n = 0$ to $N-1$. Indeed, in the extreme case that $\mathbf{v}^n = -\mathbf{v}^{n+1} \neq 0$, then $|\mathbf{v}^{n+1/2}|_1^2 = 0$ while $|\mathbf{v}^n|_1^2 > 0$ so that no small data restriction on ν or $\phi \neq 0$ will help absorb the latter into the former. Instead, we consider restrictions of the linearizations (3.3) satisfying (3.3). Then the extrapolation (3.3) leads to a resolvable estimate so that (3.6) is now replaced by $\sum_{i=1}^2 |\mathbf{u}_0^{n-i+1/2}|_1^2 + |\mathbf{u}_0^{n+1/2}|_1^2$. For small enough $\delta > 0$, we can absorb $\frac{\delta}{2}\sum_n(\sum_{i=1}^2 |\mathbf{u}_0^{n-i+1/2}|_1^2 + |\mathbf{u}_0^{n+1/2}|_1^2)$ into $\nu\sum_n |\mathbf{u}_0^{n+1/2}|_1^2$ in (3.4) for CNLE after summing from $n = 0$ to $N-1$.

A discrete Gronwall lemma can be applied instead of (3.5), but introduces the factor

$$C(data) \propto \exp(\nu^{-q}\sum_n ||E(\phi^n)||_{W^{q,\infty}}^{2-q}), \quad q = 0, \text{ or } 1 \tag{3.7}$$

so that the *a priori* estimate of CNLE solutions in the energy norm grows exponentially with problem data and T. Ultimately the Gronwall constant gives very poor long-time estimates and, to preserve the applicability of a numerical method, should be avoided for *a priori* energy estimates.

We formulate the discrete setting for analysis in Section 3.1. We consider finite element (FE) spatial discretization in conjunction with BE (BEFE) and CN (CNFE) time-stepping. In Section 3.2 we present and prove stability of BELE and CNLE (with extrapolations of the form (3.3)) for inhomogeneous data. In Section 3.3, we conclude with a numerical investigation in which we compare CNFE (with Newton nonlinear iterations), CNLE with extraploation (3.2), and CNLE with extrapolation (3.3) denoted CNLE(stab). For flow past a 2d cylinder, for a given time-step, the energy dissipation rate for CNLE(stab) approximations more closely models CNFE (with Newton) than CNLE. In fact, for a given time-step, CNLE fails to predict the vortex shedding in the wake of the the cylinder (overly diffusive) whereas CNLE(stab) captures the physics properly.

3.1 SPACE AND TIME DISCRETIZATION

For a locally Lipschitz domain $\Omega \subset \mathbb{R}^d$ for $d = 2$, 3, body forces $\mathbf{f}$, and fluid source/sink g, consider strong solutions satisfying (2.43), (2.44), (2.45): Find $\mathbf{u} \in L^2(H_\phi^1) \cap L^\infty(L^2)$ and

$p \in W^{-1,\infty}(L_0^2)$ satisfying

$$(\partial_t \mathbf{u}, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + Re^{-1}(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0^1 \tag{3.8}$$

$$\nabla \cdot \mathbf{u}(\cdot, t) = g(\cdot, t) \quad \text{in } L^2 \tag{3.9}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \quad \text{in } \Omega. \tag{3.10}$$

The method for approximating NS fluid flows is largely influenced by the following:

- *stiffness* of problem in diffusion-dominated flow regions
- lack of and/or *unknown regularity* of true NSE-solution
- high $Re \Rightarrow$ many mesh points $\Rightarrow$ *extremely large system of ODE's.*

Consequently, low-order and implicit time-stepping approximations of (3.8), (3.9), (3.10) are preferred in practice in order to avoid unnecessary numerical/modeling restrictions on the time-step size $\Delta t > 0$.

We consider a FE discretization of (3.8), (3.9), (3.10). In order to avoid stability issues arising when FE solutions are not exactly divergence free (i.e. when $V_h \not\subset V$), we introduce the explicitly skew-symmetric convective term given in (2.9) and presented again here:

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})). \tag{3.11}$$

Require that the FE space $X_{h,\cdot} \times Q_{h,\cdot}$ satisfies the basic approximation and stability properties summarized in Assumption 2.1.1. Write

$$\xi^n(\mathbf{u}) := a_{-1}\mathbf{u}^{n+1} + a_0\mathbf{u}^n + a_1\mathbf{u}^{n-1} + \ldots + a_{n_0}\mathbf{u}^{n-n_0}, \quad \overline{n}_0 := \max\{n_0, 0\}. \tag{3.12}$$

BE is the simplest implicit time-stepping scheme with Δt-accuracy and excellent stability properties.

Problem 3.1.1 (BEFE/LE)**.** *Let* $\mathbf{u}_h^i \in V_{h,\phi_h^i}(g^i)$ *be a good approximation of* $\mathbf{u}^i$ *for each* $i = 0, 1, \ldots, \overline{n}_0$. *For each* $n = \overline{n}_0, \overline{n}_0 + 1, \ldots, N-1$, *find* $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_{h,\phi_h^{n+1}} \times Q_h$ *satisfying*

$$\begin{aligned}(\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h) - \frac{1}{2}\int g^{n+1}\mathbf{u}_h^{n+1}\cdot\mathbf{v}_h \\ + \nu(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h) - (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) =< \mathbf{f}^{n+1}, \mathbf{v}_h >, \quad \forall \mathbf{v}_h \in X_h \qquad (3.13)\end{aligned}$$

$$(q_h, \nabla\cdot\mathbf{u}^{n+1}) = (q_h, g^{n+1}), \quad \forall q_h \in Q_h. \qquad (3.14)$$

Remark 3.1.2. *Note that* $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^{n+1}$ *defines BEFE and* $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^n$ *defines BELE (see e.g. [33, 44, 35, 86]).*

CN methods are Δt^2-accurate (more accurate than BE), but require consistent initial conditions including pressure. CNLE is a particularly attractive method because it is Δt^2-accurate, implicit in the nonlinearity (a source of stiffness), and linearized which avoids issues of nonlinear solvers converging and greatly reduces the computational cost.

Problem 3.1.3 (CNFE/LE)**.** *Let* $\mathbf{u}_h^i \in V_{h,\phi_h^i}(g^i)$ *be a good approximation of* $\mathbf{u}^i$ *for each* $i = 0, 1, \ldots, \overline{n}_0$ *and* $p_h^{\overline{n}_0} \in Q_h$ *be a good approximation of* $p^{\overline{n}_0}$. *For each* $n = \overline{n}_0, \overline{n}_0 + 1, \ldots, N-1$, *find* $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_{h,\phi_h^{n+1}} \times Q_h$ *satisfying*

$$\begin{aligned}(\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g^{n+1/2}\mathbf{u}_h^{n+1/2}\cdot\mathbf{v}_h \\ + \nu(\nabla\mathbf{u}_h^{n+1/2}, \nabla\mathbf{v}_h) - (p_h^{n+1/2}, \nabla\cdot\mathbf{v}_h) =< \mathbf{f}^{n+1/2}, \mathbf{v}_h >, \quad \forall \mathbf{v}_h \in X_h \qquad (3.15)\end{aligned}$$

$$(q_h, \nabla\cdot\mathbf{u}^{n+1}) = (q_h, g^{n+1}), \quad \forall q_h \in Q_h. \qquad (3.16)$$

Remark 3.1.4. *Note that* $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^{n+1/2}$ *defines the CNFE method analyzed in e.g. [45] and* $\xi^n(\mathbf{u}_h) = \frac{1}{2}(3\mathbf{u}_h^n - \mathbf{u}_h^{n-1})$ *defines the CNLE method of e.g. [7, 40, 61].*

Higher order backward difference methods like BDF2 are considered the *best* choice in general for time-stepping (more stable than CN and Δt^2-accurate). The main disadvantage of BDF2 is the introduction of artificial dissipation (avoided in CN). See [35] (e.g. Chapter 3.16) for an overview of the analysis and treatment of many time-stepping schemes available for approximating NSE-flows with a well-documented discussion of the advantages and disadvantages of each method.

3.2 STABLE LINEARIZATIONS WHEN $\mathbf{U}_H|_{\partial\Omega} \neq 0$

We now proceed to establish energetic stability of BEFE (and BELE) and CNFE (and CNLE) approximations. We require minimal stability properties of the initial iterates. First define

$$F_{ic} := ||\mathbf{u}_h^{\overline{n}_0}||^2 + \begin{cases} 0, & \text{if } a_i = 0 \text{ for } i \geq 0 \\ \nu\Delta t \sum_{i=0}^{n_0} |\mathbf{u}_h^i|_1^2, & \text{if } n_0 \geq 0 \text{ and BELE} \\ \nu\Delta t \sum_{i=0}^{n_0-1} |\mathbf{u}_h^{i+1/2}|_1^2, & \text{if } n_0 \geq 1 \text{ and CNLE} \end{cases} . \tag{3.17}$$

The constants $K_0 > 0$ in Lemma 3.2.1 and Theorem 3.2.3 do not depend on a Gronwall constant $\exp(C(T))$. For example, for BEFE/LE

$$\begin{aligned} K_0 : = C(&\nu^{1/2}F_{ic} + ||\mathbf{f}||_{l^2(\overline{n}_0+1,N;W^{-1,2})} + \nu^{1/2}||\partial_{\Delta t}E_h(\phi_h)||_{l^2(\overline{n}_0+1,N;W^{-1,2})} + \ldots \\ &\ldots + \nu^{1/2}||\nabla E_h(\phi_h)||^4_{l^4(\overline{n}_0+1,N;L^2)} + \nu||E_h(\phi_h)||_{l^\infty(\overline{n}_0+1,N;L^2)} + \ldots \\ &\ldots + \nu^{3/2}||\nabla E_h(\phi_h)||_{l^2(\overline{n}_0+1,N;L^2)} + (\Delta t \sum_{n=\overline{n}_0}^{N-1} ||g^{n+1}||^2_{\Omega_p}||E_h(\phi_h^{n+1})||^2_{0,3})^{1/2}) \end{aligned}$$

for some extension operator $E_h : \Lambda_{h,g}(\partial\Omega) \to V_{h,\cdot}(g)$.

Lemma 3.2.1 (BEFE Solutions are Bounded)**.** *Fix* $g \in C^0(L^2)$*,* $\phi_h \in l^4(\Lambda_{h,g}(\partial\Omega))$*,* $\partial_{\Delta t}\phi_h \in l^2(\Lambda_{h,g}(\partial\Omega))$*, and* $\mathbf{f} \in l^2(W^{-1,2})$*. Suppose that* $\mathbf{u}_h^i \in V_{h,\phi_h^i}(g^i)$ *for* $i = 0, 1, \ldots, \overline{n}_0$ *so that*

$$F_{ic} < \infty, \quad \textit{as } h,\ \Delta t \to 0$$

where F_{ic} *is given in (3.17) and*

$$\begin{cases} |c_h(\xi^n(\mathbf{v}_h), E_h(\phi_h^{n+1}), \mathbf{v}_h^{n+1}) - \dfrac{1}{2}\displaystyle\int_{\Omega_p} g^{n+1}|\mathbf{v}_h^{n+1}|^2| \\ \qquad \le \dfrac{\nu}{4}\left(\dfrac{|\xi^n(\mathbf{v}_h)|_1}{(1+|\mathbf{a}|_2^2)(\overline{n}_0+1)^{1/2}} + |\mathbf{v}_h^{n+1}|_1\right)|\mathbf{v}_h^{n+1}|_1, \\ \forall \{\mathbf{v}_h^n\}_{n=0}^N \subset V_h, \quad \forall n = \overline{n}_0, \overline{n}_0+1, \dots, N-1 \end{cases} \tag{3.18}$$

for some extension operator $E_h : \Lambda_{h,g}(\partial\Omega) \to V_{h,\cdot}(g)$ *satisfying Assumption 2.3.4. Then*

$$||\mathbf{u}_h||_{l^\infty(\overline{n}_0+1,N;L^2)} + \nu^{1/2}||\nabla \mathbf{u}_h||_{l^2(\overline{n}_0+1,N;L^2)} \le \nu^{-1/2}K_0 < \infty \tag{3.19}$$

for some $K_0 > 0$.

Remark 3.2.2. *Note that* $K_0 < \infty$ *uniformly as* h, $\Delta t \to 0$ *is ensured, for example, for* $g = 0$ *and smooth enough* $t \mapsto \phi_h(\cdot, t)$ *under a small data constraint: i.e. either* ϕ_h, ν^{-1}, *or* h *(at least refined near* $\partial\Omega$*) is small. Similar conditions can be established for* $g \neq 0$.

Theorem 3.2.3 (CNFE Solutions are Bounded)**.** *Fix* $g \in C^0(L^2)$, $\phi_h \in l^4(\Lambda_{h,g}(\partial\Omega))$, $\partial_{\Delta t}\phi_h \in l^2(\Lambda_{h,g}(\partial\Omega))$, *and* $\mathbf{f} \in l^2(W^{-1,2})$. *Suppose that* $\mathbf{u}_h^i \in V_{h,\phi_h^i}(g^i)$ *for* $i = 0, 1, \dots, \overline{n}_0$ *so that*

$$F_{ic} < \infty, \quad \textit{as } h,\ \Delta t \to 0$$

where F_{ic} *is given in (3.17) and*

$$\begin{cases} |c_h(\xi^n(\mathbf{v}_h), E_h(\phi_h^{n+1/2}), \mathbf{v}_h^{n+1/2}) - \dfrac{1}{2}\displaystyle\int_{\Omega_p} g^{n+1/2}|\mathbf{v}_h^{n+1/2}|^2| \\ \qquad \le \dfrac{\nu}{4}\left(\dfrac{|\xi^n(\mathbf{v}_h)|_1}{(1+|\mathbf{a}|_2^2)(\overline{n}_0+1)^{1/2}} + |\mathbf{v}_h^{n+1/2}|_1\right)|\mathbf{v}_h^{n+1/2}|_1, \\ \forall \{\mathbf{v}_h^n\}_{n=0}^N \subset V_h, \quad \forall n = \overline{n}_0, \overline{n}_0+1, \dots, N-1 \end{cases}$$

for some extension operator $E_h : \Lambda_{h,g}(\partial\Omega) \to V_{h,\cdot}(g)$ *satisfying Assumption 2.3.4. Then CNFE solutions with* $\xi^n(\mathbf{u}) = \mathbf{u}^{n+1}$ *satisfy*

$$||\mathbf{u}_h||_{l^\infty(\overline{n}_0+1,N;L^2)} + \nu^{1/2}(\Delta t \sum_{n=\overline{n}_0}^{N-1} |\mathbf{u}_h^{n+1/2}|_1^2)^{1/2} \le \nu^{-1/2}K_0 < \infty \tag{3.20}$$

where $0 < K_0 < \infty$ *is a constant depending on* $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$, $\mathbf{f}$, ϕ_h, g, *but independent of* ν. *CNLE solutions satisfy (3.20) when* $\phi_h = 0$. *CNLE solutions with general* $\phi_h \neq 0$ *satisfy (3.20) when*

$$\xi^n(\mathbf{u}) = a_0\mathbf{u}^{n-1/2} + a_1\mathbf{u}^{n-3/2} + \dots + a_{n_0}\mathbf{u}^{n-n_0-1/2}.$$

Remark 3.2.4. *As mentioned previously, the result for CNLE for inhomogeneous data with $\xi^n(\mathbf{v}) = a_0\mathbf{v}^n + \ldots + \mathbf{v}^{n-n_0}$ remains an open question. Of course, $n_0 = 1$ with the alternate extrapolation now refers to a 3-step extrapolation rather than a 2-step to preserve $\mathcal{O}(\Delta t^2)$ accuracy of CN time-stepping.*

Proof of Lemma 3.2.1. Let $E_h(\phi_h^n) \in V_{h,\phi_h}(g)$ be an extension problem data so that $E_h(\phi_h^n)|_{\partial\Omega} = \phi_h^n$ and $(\nabla \cdot E_h(\phi_h^n), q_h) = (g^n, q_h)$ for all $q_h \in Q_h$. Then writing $\mathbf{u}_h^n = \mathbf{w}_h^n + E_h(\phi_h^n)$ we see that $\mathbf{w}_h^n|_{\partial\Omega} = 0$ and $(\nabla \cdot \mathbf{w}_h^n, q_h) = 0$ for all $q_h \in Q_h$. Substitute $\mathbf{u}_h^n = \mathbf{w}_h^n + E_h(\phi_h^n)$ into (3.13) and test with $\mathbf{v} = \mathbf{w}_h^{n+1}$. Apply identity (2.35) so that $c_h(\cdot, \mathbf{v}, \mathbf{v}) = 0$. Then

$$\begin{aligned}
&(\partial_{\Delta t}^{n+1}\mathbf{w}_h, \mathbf{w}_h^{n+1}) + \nu|\mathbf{w}_h^{n+1}|_1^2 = (\mathbf{f}^{n+1}, \mathbf{w}_h^{n+1}) - (\partial_{\Delta t}^{n+1}E_h(\phi_h), \mathbf{w}_h^{n+1}) \\
&\quad - \nu(\nabla E_h(\phi_h^{n+1}), \nabla \mathbf{w}_h^{n+1}) + \frac{1}{2}\int_{\Omega_p} g^{n+1}E_h(\phi_h^{n+1}) \cdot \mathbf{w}_h^{n+1} \\
&\quad - c_h(\xi^n(E_h(\phi_h)), E_h(\phi_h^{n+1}), \mathbf{w}_h^{n+1}) - c_h(\xi^n(\mathbf{w}_h), E_h(\phi_h^{n+1}), \mathbf{w}_h^{n+1}) + \frac{1}{2}\int_{\Omega_p} g^{n+1}|\mathbf{w}_h^{n+1}|^2.
\end{aligned} \tag{3.21}$$

Application of the identity $(\mathbf{a} - \mathbf{b}, \mathbf{a}) = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2)$ gives

$$(\partial_{\Delta t}^{n+1}\mathbf{w}_h, \mathbf{w}_h^{n+1}) = \frac{1}{2\Delta t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{w}_h^n||^2) + \frac{1}{2\Delta t}||\mathbf{w}_h^{n+1} - \mathbf{w}_h^n||^2. \tag{3.22}$$

Apply the duality estimate in $W^{-1,2} \times H_0^1$ to get

$$(\mathbf{f}^{n+1}, \mathbf{w}_h^{n+1}) - (\partial_{\Delta t}^{n+1}E_h(\phi_h), \mathbf{w}_h^{n+1}) \leq (||\mathbf{f}^{n+1}||_{-1} + ||\partial_{\Delta t}^{n+1}E_h(\phi_h)||_{-1})|\mathbf{w}_h^{n+1}|_1. \tag{3.23}$$

Apply Hölder's (2.22), Ladyzhenskaya (2.24), and Poincaré's (2.23) (valid with $\mathbf{w} \in H_0^1(\Omega)$) inequalities to get

$$\begin{aligned}
&(\nabla E_h(\phi_h^{n+1}), \nabla \mathbf{w}_h^{n+1}) + \frac{1}{2}\int_{\Omega_p} g^{n+1}E_h(\phi_h^{n+1}) \cdot \mathbf{w}_h^{n+1} \\
&\qquad \leq (|E_h(\phi_h^{n+1})|_1 + \frac{c_6}{2}||g^{n+1}||_{\Omega_p}||E_h(\phi_h^{n+1})||_{0,3})|\mathbf{w}_h^{n+1}|_1.
\end{aligned} \tag{3.24}$$

An intermediate consequence of Estimate (2.42)(a) gives

$$c_h(\xi^n(E_h(\phi_h)), E_h(\phi_h^{n+1}), \mathbf{w}_h^{n+1}) \leq c_6||\xi^n(E_h(\phi_h))||_1||E_h(\phi_h^{n+1})||_{0,3}|\mathbf{w}_h^{n+1}|_1. \tag{3.25}$$

Application of the above estimates (3.22), (3.23), (3.24), and (3.25) along with Young's inequality (2.21) to (3.21) gives

$$
\begin{aligned}
&\frac{1}{2\Delta t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{w}_h^n||^2) + \frac{1}{2\Delta t}||\mathbf{w}_h^{n+1} - \mathbf{w}_h^n||^2 + \nu|\mathbf{w}_h^{n+1}|_1^2 \\
&\le 5\nu^{-1}||\mathbf{f}^{n+1}||_{-1}^2 + 5\nu^{-1}||\partial_{\Delta t}^{n+1}E_h(\phi_h)||_{-1}^2 + 5\nu|E_h(\phi_h^{n+1})|_1^2 \\
&\quad + 5c_6^2\nu^{-1}||\xi^n(E_h(\phi_h))||_1^2||E_h(\phi_h^{n+1})||_{0,3}^2 + \frac{5c_6^2}{2}\nu^{-1}||g^{n+1}||_{\Omega_p}^2||E_h(\phi_h^{n+1})||_{0,3}^2 \\
&\quad + \frac{\nu}{4}|\mathbf{w}_h^{n+1}|_1^2 - (c_h(\xi^n(\mathbf{w}_h), E_h(\phi_h^{n+1}), \mathbf{w}_h^{n+1}) - \frac{1}{2}\int_{\Omega_p} g^{n+1}|\mathbf{w}_h^{n+1}|^2). \qquad (3.26)
\end{aligned}
$$

Case 1 (BEFE): Suppose that $\xi^n(\mathbf{w}_h) = \mathbf{w}_h^{n+1}$. Apply condition (3.18) to (3.26) with $|\mathbf{a}|_2 = 1$, $\overline{n}_0 = 0$. Absorb like-terms from right into left-hand sides to get

$$
\begin{aligned}
&||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{w}_h^n||^2 + ||\mathbf{w}_h^{n+1} - \mathbf{w}_h^n||^2 + \nu\Delta t|\mathbf{w}_h^{n+1}|_1^2 \\
&\le 10\nu^{-1}\Delta t||\mathbf{f}^{n+1}||_{-1}^2 + 10\nu^{-1}\Delta t||\partial_{\Delta t}^{n+1}E_h(\phi_h)||_{-1}^2 + 10\nu\Delta t|E_h(\phi_h^{n+1})|_1^2 \\
&+ 10c_6^2\nu^{-1}\Delta t||\xi^n(E_h(\phi_h))||_1^2||E_h(\phi_h^{n+1})||_{0,4}^2 + 5c_6^2\nu^{-1}\Delta t||g^{n+1}||_{\Omega_p}^2||E_h(\phi_h^{n+1})||_{0,4}^2. \qquad (3.27)
\end{aligned}
$$

Case 2 (BELE): Suppose that $\xi^n(\mathbf{w}_h) = \sum_{i=0}^{n_0} a_i\mathbf{w}_h^{n-i}$. Young's inequality (2.21) gives $(1 + n_0)^{-1/2}|\xi^n(\mathbf{w}_h)|_1|\mathbf{w}_h^{n+1}|_1 \le \frac{1}{2}((1+n_0)^{-1}|\xi^n(\mathbf{w}_h)|_1^2 + |\mathbf{w}_h^{n+1}|_1^2)$. Apply condition (3.18) to (3.26) with $a_{-1} = 0$ and $\overline{n}_0 = n_0 \ge 0$. Absorb like terms from right into left-hand sides to get

$$
\begin{aligned}
&\Delta t^{-1}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{w}_h^n||^2) + \Delta t^{-1}||\mathbf{w}_h^{n+1} - \mathbf{w}_h^n||^2 \\
&+ \frac{\nu}{2}\left((\frac{3}{2} - \frac{1}{2(1+|\mathbf{a}|_2^2)})|\mathbf{w}_h^{n+1}|_1^2 - (\frac{1}{2(1+|\mathbf{a}|_2^2)(1+n_0)})|\xi^n(\mathbf{w}_h)|_1^2\right) \\
&\le 5\nu^{-1}||\mathbf{f}^{n+1}||_{-1}^2 + 5\nu^{-1}||\partial_{\Delta t}^{n+1}E_h(\phi_h)||_{-1}^2 + 5\nu|E_h(\phi_h^{n+1})|_1^2 \\
&+ 5c_6^2\nu^{-1}||\xi^n(E_h(\phi_h))||_1^2||E_h(\phi_h^{n+1})||_{0,3}^2 + \frac{5c_6^2}{2}\nu^{-1}||g^{n+1}||_{\Omega_p}^2||E_h(\phi_h^{n+1})||_{0,3}^2. \qquad (3.28)
\end{aligned}
$$

From the change of indices identity (2.20), we obtain

$$
\begin{aligned}
\sum_{n=n_0}^{N-1}|\xi^n(\mathbf{w}_h)|_1^2 &\le \sum_{n=n_0}^{N-1}\sum_{i=0}^{n_0}(1+n_0)|a_i|^2|\mathbf{w}_h^{n-i}|_1^2 \\
&= (1+n_0)\sum_{n=0}^{N-1}|\mathbf{w}_h^n|_1^2\sum_{i=i_0(n)}^{i_1(n)}|a_i|^2 \le (1+n_0)|\mathbf{a}|_2^2\sum_{n=0}^{N-1}|\mathbf{w}_h^n|_1^2
\end{aligned}
$$

so that

$$
\begin{aligned}
&(\frac{3}{2} - \frac{1}{2(1+|\mathbf{a}|_2^2)}) \sum_{n=n_0}^{N-1} |\mathbf{w}_h^{n+1}|_1^2 - (\frac{1}{2(1+|\mathbf{a}|_2^2)(1+n_0)}) \sum_{n=n_0}^{N-1} |\xi^n(\mathbf{w}_h)|_1^2 \\
&\geq (\frac{3}{2} - \frac{1}{2(1+|\mathbf{a}|_2^2)}) \sum_{n=n_0}^{N-1} |\mathbf{w}_h^{n+1}|_1^2 - \frac{|\mathbf{a}|_2^2}{2(1+|\mathbf{a}|_2^2)} \sum_{n=0}^{N-1} |\mathbf{w}_h^n|_1^2 \\
&\geq \sum_{n=n_0+1}^{N} |\mathbf{w}_h^n|_1^2 - \frac{|\mathbf{a}|_2^2}{2(1+|\mathbf{a}|_2^2)} \sum_{i=0}^{n_0} |\mathbf{w}_h^i|_1^2. \qquad (3.29)
\end{aligned}
$$

For either Case 1 or 2, sum from $n = \overline{n}_0$ to $n = N-1$ either estimate (3.27) (Case 1) or (3.28) along with (3.29) (Case 2). Simplify to get

$$
\begin{aligned}
&||\mathbf{w}_h^N||^2 + \sum_{n=\overline{n}_0}^{N-1} ||\mathbf{w}_h^{n+1} - \mathbf{w}_h^n||^2 + \nu\Delta t \sum_{n=\overline{n}_0}^{N-1} |\mathbf{w}_h^{n+1}|_1^2 \\
&\quad \leq ||\mathbf{w}_h^{\overline{n}_0}||^2 + \chi_*\nu\Delta t \sum_{n=0}^{\overline{n}_0} |\mathbf{w}_h^n|_1^2 + \chi_* C\nu^{-1}\Delta t \sum_{n=0}^{\overline{n}_0} |E_h(\phi_h^n)|_1^4 \\
&\quad + C\nu^{-1}\Delta t \sum_{n=\overline{n}_0}^{N-1} (||\mathbf{f}^{n+1}||_{-1}^2 + ||\partial_{\Delta t}^{n+1} E_h(\phi_h^{n+1})||_{-1}^2 + \ldots \\
&\quad \ldots + |E_h(\phi_h^{n+1})|_1^4 + \nu^2 |E_h(\phi_h^{n+1})|_1^2 + ||g^{n+1}||_{\Omega_p}^2 ||E_h(\phi_h^{n+1})||_{0,3}^2) \qquad (3.30)
\end{aligned}
$$

where $\chi_* = 0$ for BEFE and $\chi_* = 1$ for BELE. Apply the triangle inequality with $\mathbf{u}_h^n = \mathbf{w}_h^n - E_h(\phi_h^n)$ and (3.30) to get

$$
\begin{aligned}
&\nu||\nabla \mathbf{u}_h||_{l^2(\overline{n}_0+1,N;L^2)}^2 \leq ||\mathbf{u}_h^{\overline{n}_0}||^2 + \chi_*\nu||\nabla \mathbf{u}_h||_{l^2(0,\overline{n}_0;L^2)}^2 \\
&\quad + \chi_*\nu||\nabla E_h(\phi_h)||_{l^2(0,\overline{n}_0;L^2)}^2 + \chi_* C\nu^{-1}||\nabla E_h(\phi_h)||_{l^4(0,\overline{n}_0;L^2)}^4 \\
&\quad + C\nu^{-1}(||\mathbf{f}||_{l^2(\overline{n}_0+1,N;W^{-1,2})}^2 + ||\partial_{\Delta t}^{n+1} E_h(\phi_h)||_{l^2(\overline{n}_0+1,N;W^{-1,2})}^2 + \ldots \\
&\quad \ldots + ||\nabla E_h(\phi_h)||_{l^4(\overline{n}_0+1,N;L^2)}^4 + \nu^2||\nabla E_h(\phi_h)||_{l^2(\overline{n}_0+1,N;L^2)}^2 + \ldots \\
&\quad \ldots + \Delta t \sum_{n=\overline{n}_0}^{N-1} ||g^{n+1}||_{\Omega_p}^2 ||E_h(\phi_h^{n+1})||_{0,3}^2) \qquad (3.31)
\end{aligned}
$$

and

$$
\begin{aligned}
||\mathbf{u}_h^n||^2_{l^\infty(\overline{n}_0+1,N;L^2)} &\leq ||\mathbf{u}_h^{\overline{n}_0}||^2 + \chi_*\nu||\nabla\mathbf{u}_h||^2_{l^2(0,\overline{n}_0;L^2)} \\
&+ \chi_*\nu||\nabla E_h(\phi_h)||^2_{l^2(0,\overline{n}_0;L^2)} + \chi_* C\nu^{-1}||\nabla E_h(\phi_h)||^4_{l^4(0,\overline{n}_0;L^2)} \\
&+ C\nu^{-1}(||\mathbf{f}||^2_{l^2(\overline{n}_0+1,N;W^{-1,2})} + ||\partial^{n+1}_{\Delta t}E_h(\phi_h)||^2_{l^2(\overline{n}_0+1,N;W^{-1,2})} + \ldots \\
&\ldots + ||\nabla E_h(\phi_h)||^4_{l^4(\overline{n}_0+1,N;L^2)} + \nu||E_h(\phi_h)||^2_{l^\infty(\overline{n}_0+1,N;L^2)} + \ldots \\
&\ldots + \nu^2||\nabla E_h(\phi_h)||^2_{l^2(\overline{n}_0+1,N;L^2)} + \Delta t\sum_{n=\overline{n}_0}^{N-1}||g^{n+1}||^2_{\Omega_p}||E_h(\phi_h^{n+1})||^2_{0,3}).
\end{aligned} \tag{3.32}
$$

The estimate (3.19) follows from (3.31), (3.32) under the assumed regularity. □

Proof of Theorem 3.2.3. The proof of Theorem 3.2.3 follows the proof in Lemma 3.2.1 closely. For CNFE, test with $\mathbf{v}_h = \mathbf{w}^{n+1/2}$ to get

$$
\begin{aligned}
\frac{1}{2\Delta t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{w}_h^n||^2) + \nu|\mathbf{w}_h^{n+1}|_1^2 &= (\mathbf{f}^{n+1}, \mathbf{w}_h^{n+1}) - (\partial^{n+1}_{\Delta t}E_h(\phi_h), \mathbf{w}_h^{n+1/2}) \\
&- \nu(\nabla E_h(\phi_h^{n+1/2}), \nabla\mathbf{w}_h^{n+1/2}) + \frac{1}{2}\int_{\Omega_p} g^{n+1/2}E_h(\phi_h^{n+1/2})\cdot\mathbf{w}_h^{n+1/2} \\
&- c_h(\xi^n(E_h(\phi_h)), E_h(\phi_h^{n+1/2}), \mathbf{w}_h^{n+1/2}) - c_h(\xi^n(\mathbf{w}_h), E_h(\phi_h^{n+1/2}), \mathbf{w}_h^{n+1/2}) \\
&+ \frac{1}{2}\int_{\Omega_p} g^{n+1/2}|\mathbf{w}_h^{n+1/2}|^2
\end{aligned} \tag{3.33}
$$

instead of (3.21). The remaining estimates are obtained similar to those in the proof of Lemma 3.2.1. The main difference, aside from exchanging indices $n+1$ with $n+1/2$, concerns the legitimacy of estimate (3.29) in the case of CNLE. When $\phi_h = 0$, there is no problem because there is no contribution from the nonlinearity. However, for general $\phi_h \neq 0$, we require the prescribed form of the linearization $\xi^n(\mathbf{u}) = a_0\mathbf{u}^{n-1/2} + a_1\mathbf{u}^{n-1-1/2} + \ldots + a_{n_0}\mathbf{u}^{n-n_0-1/2}$ which allows the nonlinearity to be absorbed in a similar way as shown in (3.29) for BELE. Proceeding as before, we prove (3.20). □

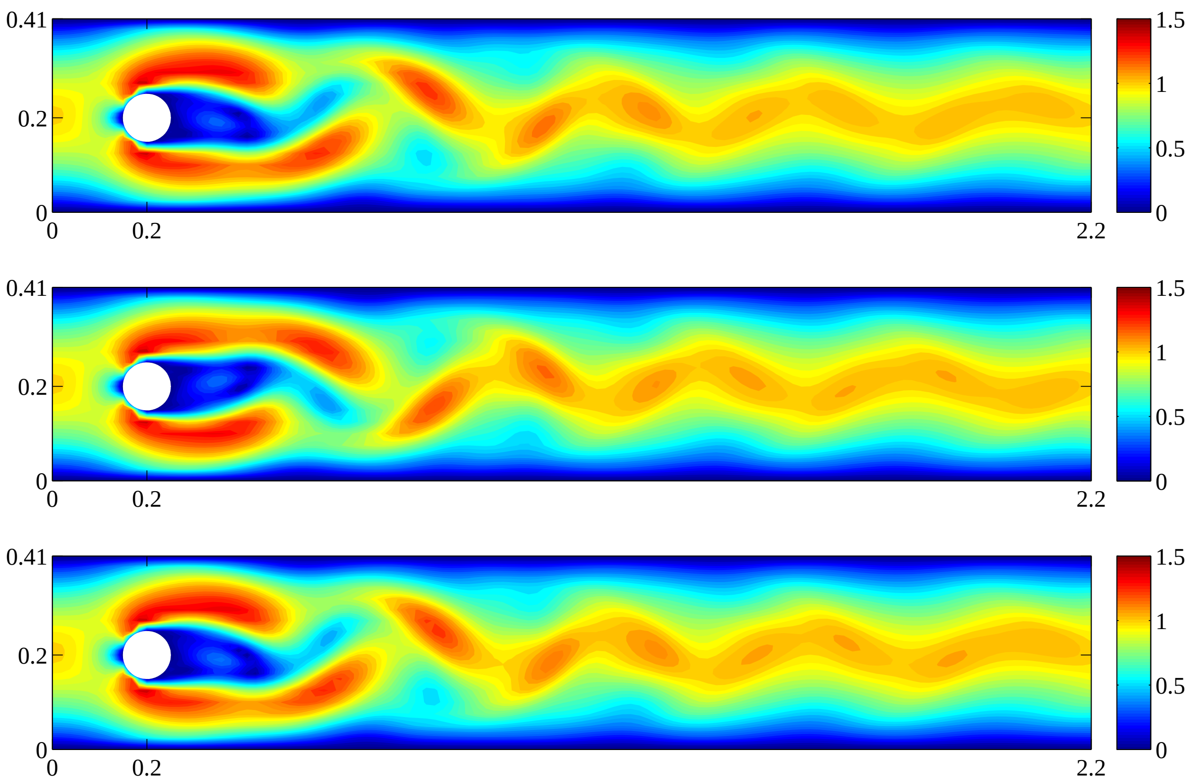


Figure 3.1: Flow past 1 cylinder: magnitude of velocity field computed with CNFE (newton) at (a) $T = 5$, (b) $T = 10$, (c) $T = 15$ with $\Delta t = 0.005$. Notice the distinct and periodic vortex shedding associated with the von Kàrmàn vortex street.

3.3 NUMERICAL INVESTIGATION

In this section we investigate how CNLE(stab) approximations with the the alternate extrapolation

$$\xi^n(\mathbf{u}) = 2\mathbf{u}^{n-1/2} - \mathbf{u}^{n-3/2}$$

improves flow statistics and preserves flow integrity from CNLE obtained with the conventional extrapolation $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$. The energy dissipation rate is given by

$$\varepsilon(t) := \nu|\mathbf{u}(\cdot,t)|_1^2.$$

In the previous discussion, our work suggests that CNLE solutions might have *worse* control on the *size* of $\varepsilon(t)$ than CNLE(stab). To be precise, we compare herein the size of the numerical dissipation rate ε_{cnle}^n for CNLE and CNLE(stab) applied to flow past a 2d cylinder where

$$\varepsilon_{cnle}^{n+1/2} := \nu|\mathbf{u}_h^{n+1/2}|_1^2.$$

For the problem setup, consider the channel $([0,2.2]\times[0,0.41]) - \Omega_s$ where Ω_s is circular obstacle with diameter $= 0.1$ centered at $(0.2, 0.2)$. The flow has boundary conditions:

$$\mathbf{u}(x,y=0) = \mathbf{u}(x,y=0.41) = \mathbf{u}|_{\partial\Omega_s} = 0, \qquad \mathbf{u}(x=0,y) = \mathbf{u}(x=2.2,y)\frac{4}{0.41^2}y(0.41-y).$$

Let the initial data $(\mathbf{u}^0, p^0)$ satisfy the (steady) Stokes problem. For high enough Reynolds number (albeit below turbulence levels) vortices will begin shedding from the wake of Ω_s at a regular frequency (von Kármán vortex street). This is a similar experiment performed in [54], but there with time-dependent boundary conditions and starting from rest.

We compare 3 approximate NSE flows obtained with CNFE, CNLE, and CNLE(stab). We solve each problem on the time interval $[0, 15]$ with Taylor-Hood finite elements on the same mesh. The mesh is generated by Delaunay-Voronoi triangulation in FreeFem++ and contains 143100 velocity degrees of freedom (161168 total degrees of freedom) with 128 vertices on Ω_s. For CNFE, we resolve the nonlinearity with Newton iterations so that the H^1 residual error less than 10^{-12} at each time step. For CNLE and CNLE(stab), the iterates $\mathbf{u}_h^i$

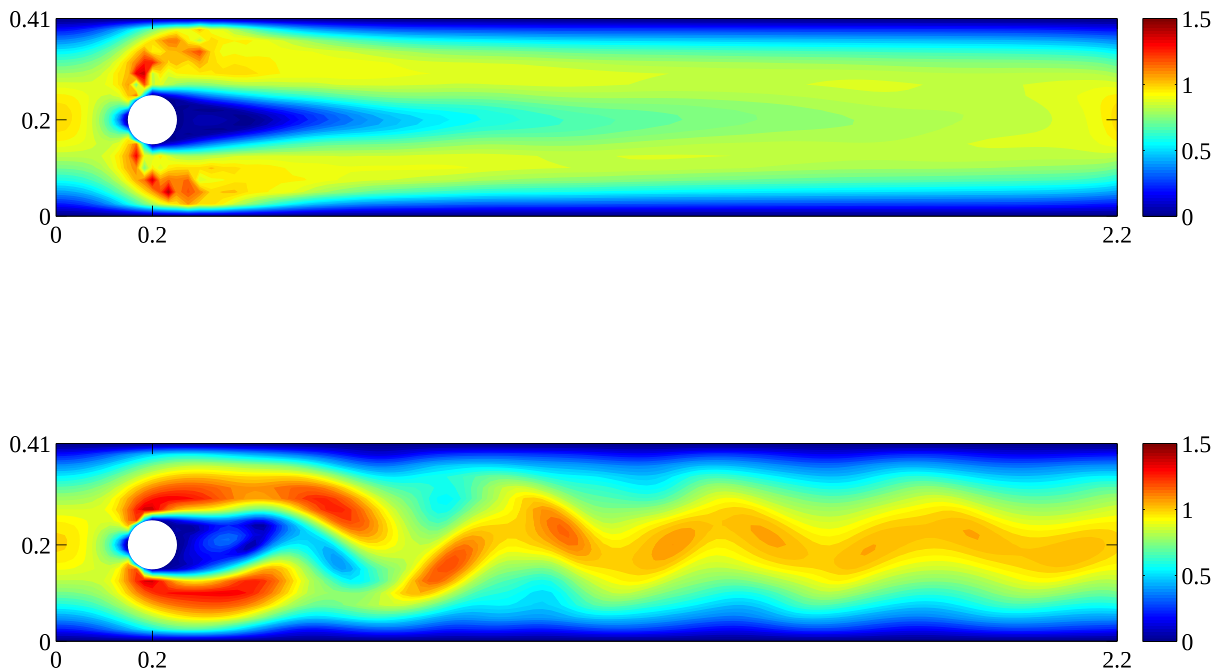
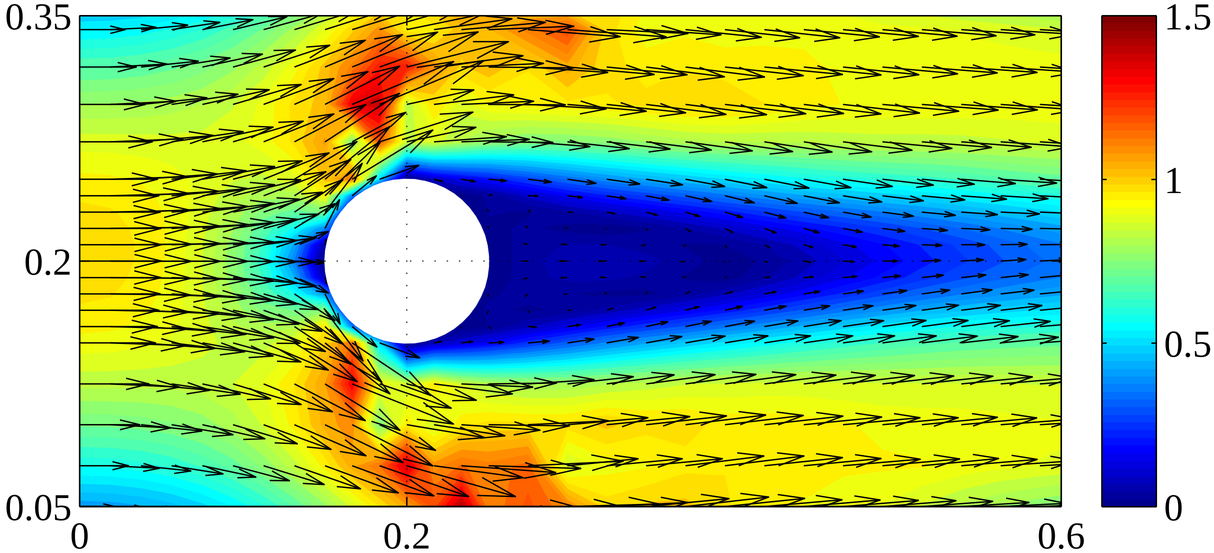


Figure 3.2: Flow past 1 cylinder: magnitude of velocity field at $T = 10$ for (a) CNLE and (b) CNLE (stab) with $\Delta t = 0.002$. Notice that CNLE (a) fails to reproduce the characteristic von Kàrmàn vortex street.



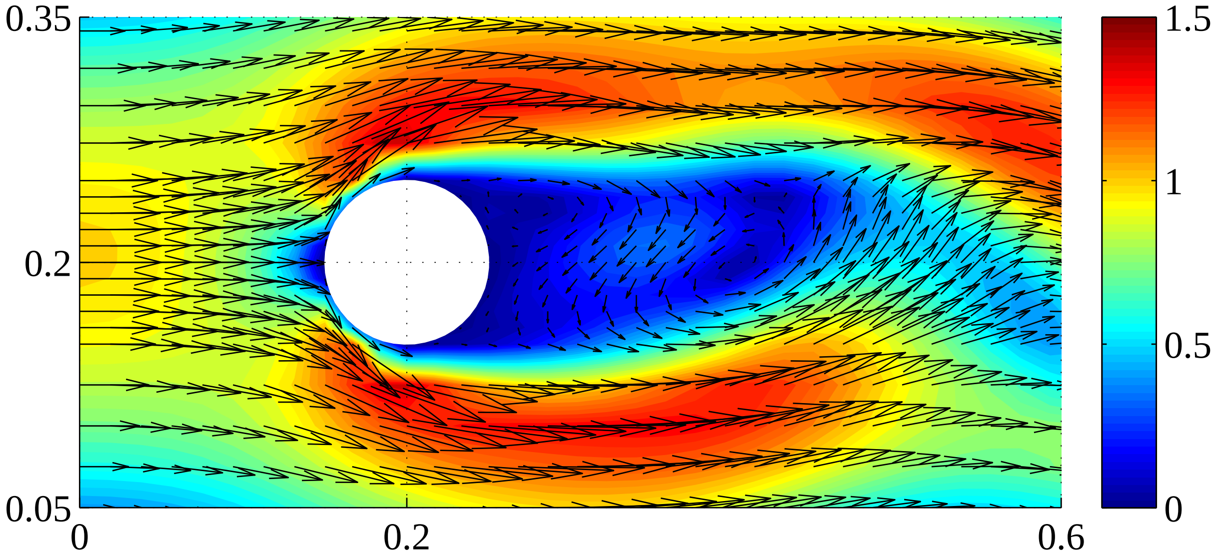


Figure 3.3: Flow past 1 cylinder: magnitude of velocity field at $T = 10$ for (a) CNLE and (b) CNLE (stab) with $\Delta t = 0.002$. Notice that CNLE (a) suppresses all vortex shedding predicted by CNLE (stab) (b).

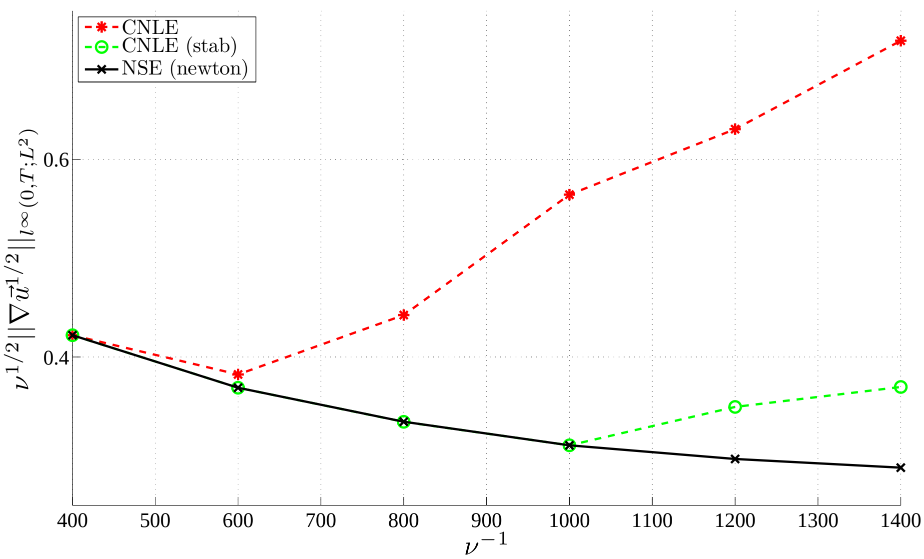
for $i = 1, \ldots, n_0$ are obtained with a fixed point nonlinear iteration so that the H^1 residual error less than 10^{-12}.

We present the magnitude of the velocity field of the CNFE flow for $\nu^{-1} = 1000$ computed with $\Delta t = 0.005$ at $T = 5,\ 10,\ 15$ in Figure 3.1. The characteristic shedding of vortices is captured here as seen by the vortices shedding off the back of the cylinder and then carried down the length of the channel. We present the magnitude of the velocity field of the CNLE and CNLE(stab) flow for the same conditions at $T = 10$ computed with $\Delta t = 0.005$ in Figure 3.2. In this case, the CNLE(stab) method closely models the flow generated by CNFE, but the CNLE method is much over-diffused and fails to capture expected the vortex shedding. The vector field in the near-wake of the cylinder is shown for CNLE and CNLE(stab) in Figure 3.3 to further illustrate this difference.

The degradation of CNLE flow approximation is clearly seen in the plots displayed in Figures 3.4, 3.5. In each plot, we plot a statistic measuring the numerical energy dissipation rate $\varepsilon_{cnle}^{n+1/2}$ over the time interval $[0, 15]$ for $\nu^{-1} = 400,\ 600,\ 800,\ 1000,\ 1200,\ 1400$. In Figure 3.4 we measure the maximum $\varepsilon_{cnle}^{n+1/2}$ on the time interval and in Figure 3.5 we measure the $l^2(0,T)$-norm of $\varepsilon_{cnle}^{n+1/2}$. The curve on each plot for CNFE is the bottom-most curve and decreases as ν^{-1} as expected. The curve for CNLE(stab) matches CNFE when $\Delta t = 0.001$, but deviates slightly starting at $\nu^{-1} = 1200$ when $\Delta t = 0.002$. Conversely, the curve for CNLE deviates from CNFE starting at $\nu^{-1} = 1400$ when $\Delta t = 0.001$, and deviates more significantly starting at $\nu^{-1} = 600$ when $\Delta t = 0.002$.

In Figures 3.6, 3.7 we present the behavior of an alternate measure of the numerical dissipation based on ε_{cnle}^{n} rather than the average $\mathbf{u}^{n+1/2}$ natural for the CN method. Interestingly, the curves for CNFE and CNLE(stab) are comparable for $\varepsilon_{cnle}^{n+1/2}$ and ε_{cnle}^{n} but the curve for CNLE deviates from the expectation even more dramatically for ε_{cnle}^{n}.

In Figures 3.8, 3.9, 3.10 we plot ε_{cnle}^{n} for CNFE ($\Delta t = 0.005$), CNLE ($\Delta t = 0.002$), and CNLE(stab) ($\Delta t = 0.002$) respectively for $\nu^{-1} = 600,\ 800,\ 1000$ with respect to the numerical time levels over $[0, 15]$. The curves for CNFE and CNLE(stab) match closely with a relative decrease between each curve with increasing ν^{-1}. Conversely, the curves for CNLE increases with ν^{-1}.



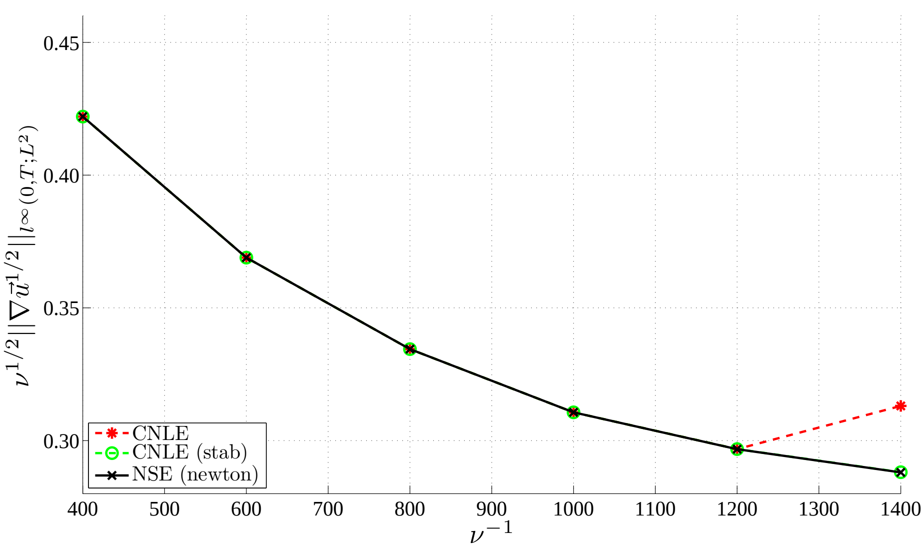


Figure 3.4: Flow past 1 cylinder: maximal energy dissipation rate at $t^{n+1/2}$ versus ν^{-1} for CNFE (newton) solutions computed with $\Delta t = 0.005$ and CNLE, CNLE(stab) solutions with (a) $\Delta t = 0.002$, and (b) $\Delta t = 0.001$.

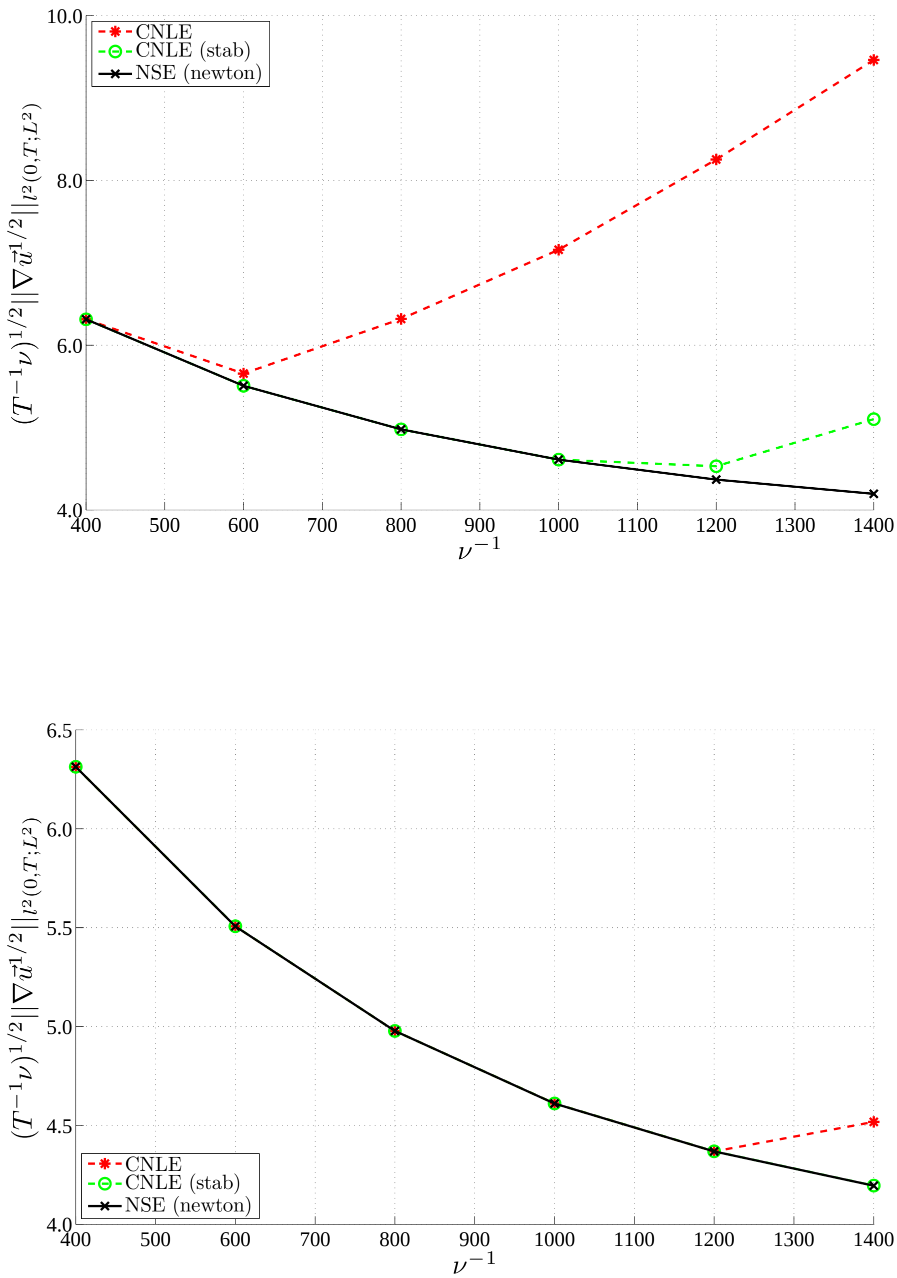
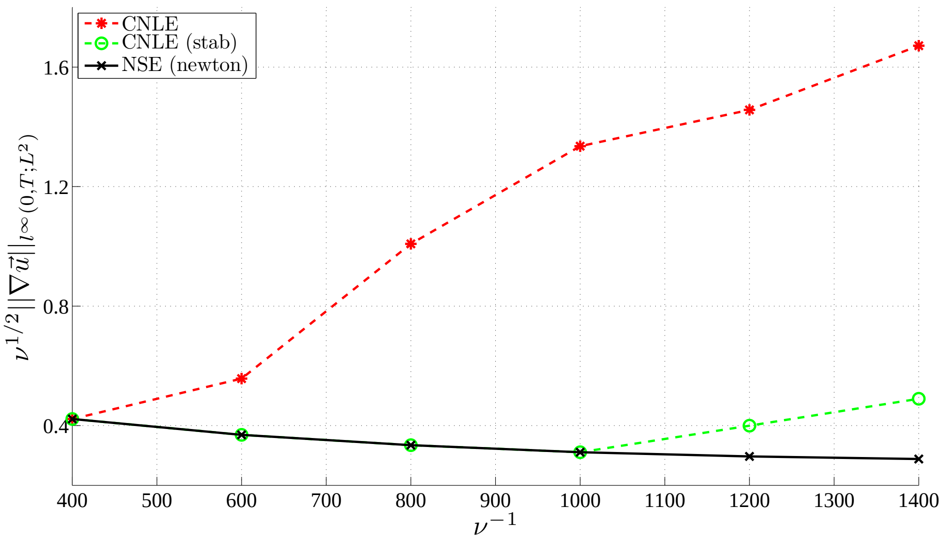


Figure 3.5: Flow past 1 cylinder: time-averaged energy dissipation rate at $t^{n+1/2}$ versus ν^{-1} for CNFE (newton) solutions computed with $\Delta t = 0.005$ and CNLE, CNLE(stab) solutions with (a) $\Delta t = 0.002$, and (b) $\Delta t = 0.001$.



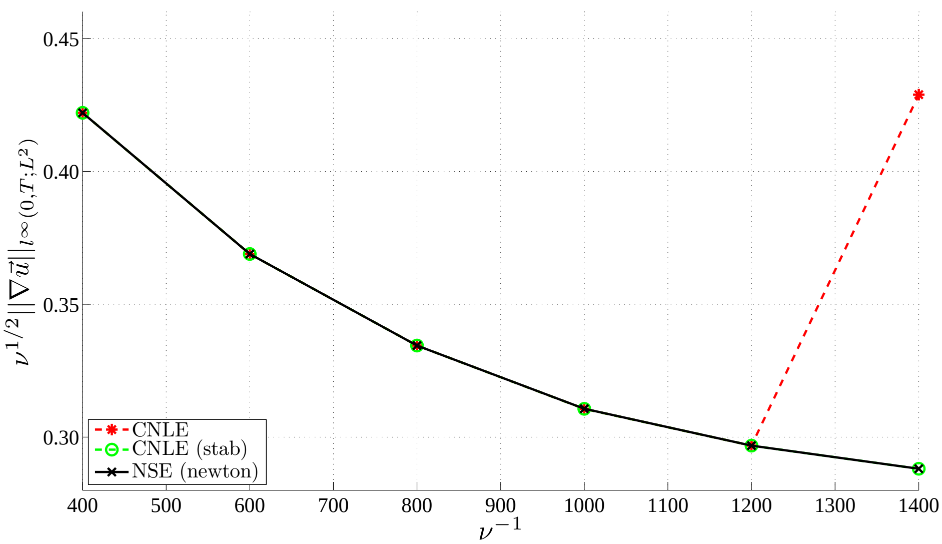
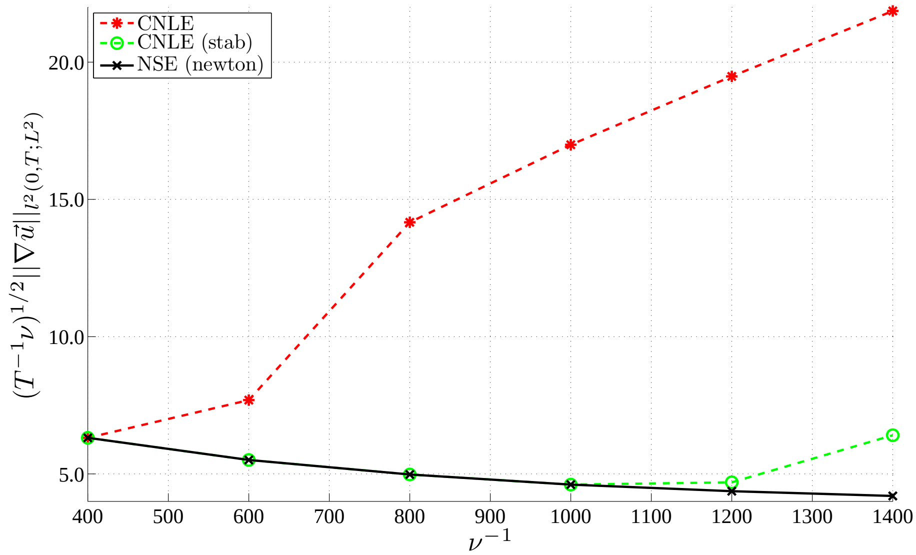


Figure 3.6: Flow past 1 cylinder: maximal energy dissipation rate at t^n versus ν^{-1} for CNFE (newton) solutions computed with $\Delta t = 0.005$ and CNLE, CNLE(stab) solutions with (a) $\Delta t = 0.002$, and (b) $\Delta t = 0.001$.



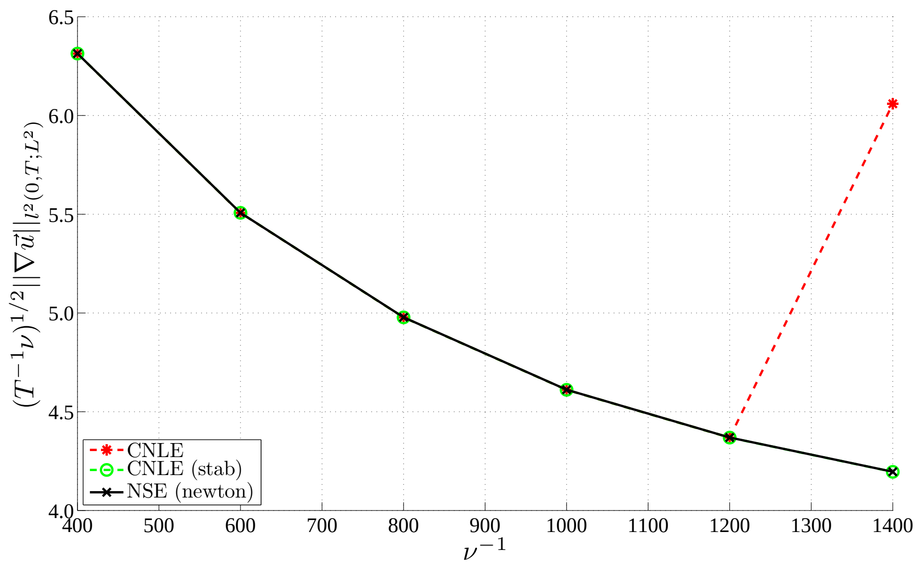


Figure 3.7: Flow past 1 cylinder: time-averaged energy dissipation rate at t^n versus ν^{-1}; for CNFE (newton) solutions computed with $\Delta t = 0.005$ and CNLE, CNLE(stab) solutions with (a) $\Delta t = 0.002$, and (b) $\Delta t = 0.001$.

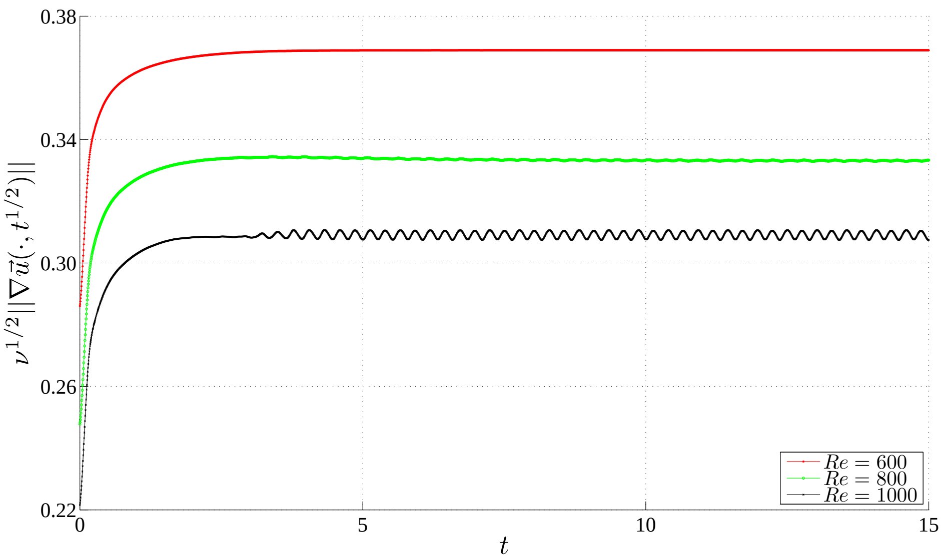


Figure 3.8: Flow past 1 cylinder: energy dissipation rate at t^n versus time for CNFE (newton) solutions computed with $\Delta t = 0.005$.

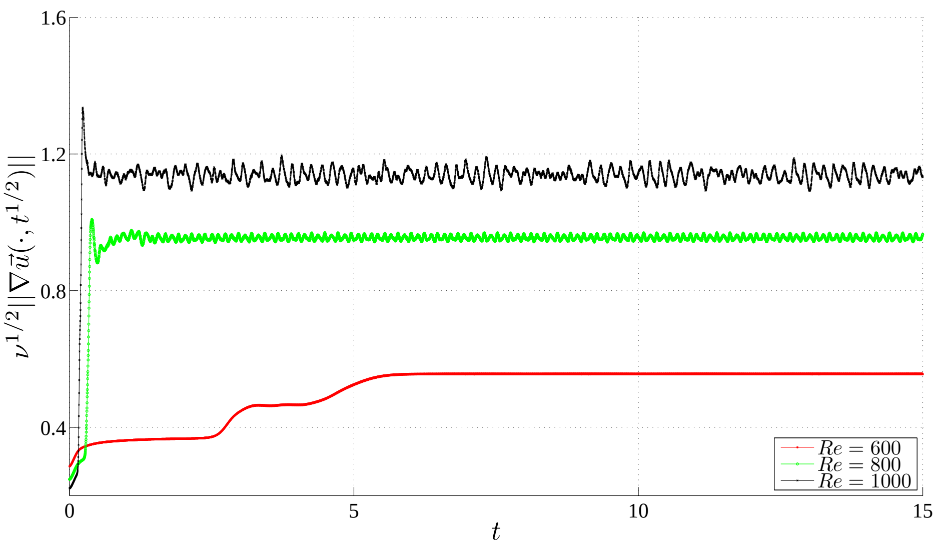


Figure 3.9: Flow past 1 cylinder: energy dissipation rate at t^n versus time for CNLE solutions computed with $\Delta t = 0.002$.

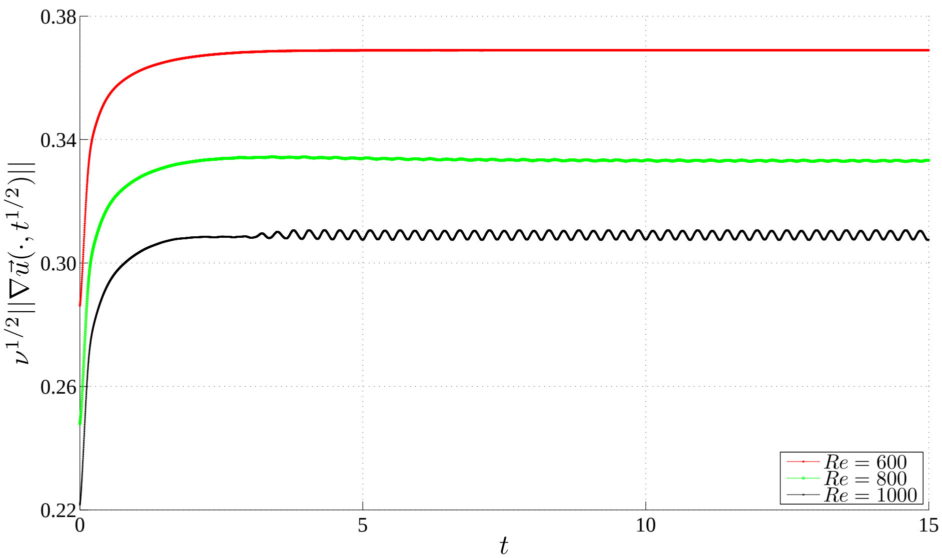


Figure 3.10: Flow past 1 cylinder: energy dissipation rate at t^n versus time for CNLE (stab) solutions computed with $\Delta t = 0.002$.

4.0 CNLE CONVERGES UNCONDITIONALLY

The usual Crank-Nicolson (CN) in time finite element (FE) in space discretization of the Navier-Stokes (NS) equations (NSE) denoted by CNFE is well-known to be unconditionally (energetically) stable. The error analysis of the CNFE method is based on a discrete Gronwall inequality which introduces a time-step restriction (for convergence, not for stability) of the form

$$\Delta t \leq \mathcal{O}(Re^{-5/3}h^{2/3}), \qquad \text{or} \qquad \Delta t \leq \mathcal{O}(Re^{-3}) \tag{4.1}$$

(implicitly reported for $W^{1,\infty}$-solutions in [45]). Here $h > 0$ is the mesh width, $\Delta t > 0$ is the time-step size, and $Re > 0$ is the Reynolds number. Condition (4.1)(a) implies *conditional convergence* whereas (4.1)(b) is a *robustness condition* and both are prohibitively restrictive in practice; for example, (4.1)(b) suggests

$$Re = 100 \text{ (low-to-moderate value)} \quad \Rightarrow \quad \Delta t \leq \mathcal{O}(10^{-6}).$$

Consequently, an important open question regards whether condition (4.1) is

- an artifact of imperfect mathematical technique, or
- a special feature of the CN time discretization.

We consider the necessity of a time-step restriction in a linear, fully implicit variant of CNFE obtained by extrapolation of the convecting velocity $\mathbf{u}$: for example,

$$\mathbf{u} \cdot \nabla \mathbf{u} \approx (\frac{3}{2}\mathbf{u}^{n-1} - \frac{1}{2}\mathbf{u}^{n-2}) \cdot \nabla \frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}, \qquad \mathbf{u}^i := \mathbf{u}(x, t_i). \tag{4.2}$$

This method is often called CNLE and was first studied by Baker [7]. CNLE is linearly implicit, unconditionally (energetically) stable (at least for $\phi = 0$), and second-order accurate.

In this report, we show that *no time-step restriction* is required for the convergence of CNLE (Theorem 4.3.1). Additionally, the error satisfies

$$||error(CNLE)||_{l^\infty(L^2)\cap l^2(H^1)} \leq C(h^k + \Delta t^2), \qquad k = \text{ degree of FE-space}$$

(Theorem 4.3.3). Our analysis depends on the *extrapolated* convecting velocity in (4.2), careful majorization of associated bi- and trilinear forms, and application of a particular discrete Gronwall inequality. The key difference between our convergence proof for CNLE and that of CNFE is the resulting intermediate estimate: for approximations $\mathbf{U}_h^n$ and constants $\kappa^n > 0$,

$$\text{CNFE} \Rightarrow ||\mathbf{U}_h^N||^2 + \ldots \leq \sum_{n=0}^{N} \kappa^n ||\mathbf{U}_h^n||^2 + \ldots \tag{4.3}$$

$$\text{CNLE} \Rightarrow ||\mathbf{U}_h^N||^2 + \ldots \leq \sum_{n=0}^{N-1} \kappa^n ||\mathbf{U}_h^n||^2 + \ldots. \tag{4.4}$$

Notice that the term $||\mathbf{U}_h^N||^2$ is included in the right-hand-side of (4.3), but not of (4.4). Estimates of the form (4.3) require a discrete Gronwall inequality (Lemma 2.4.5) to proceed, which is the source of a time-step restriction. Conversely, estimates of the form (4.4) allow application of an alternate discrete Gronwall inequality (Lemma 2.4.6), which does not require a time-step restriction.

We also prove convergence estimates in other norms. Under a modest time-step restriction

$$\Delta t \leq h^{1/4}, \quad \text{no } Re\text{-dependence}, \tag{4.5}$$

the CNLE velocity approximation converges optimally in the $l^\infty(H^1)$-norm and the corresponding discrete derivative of the velocity approximation converges optimally in the $l^2(L^2)$-norm (Theorems 4.3.4, 4.3.6). The restriction (4.5) is not a typical artifact of the discrete Gronwall inequality since it does not depend on Re or other problem data. Correspondingly, (4.5) is much less restrictive than (4.1). The error estimate is obtained through a bootstrap argument that utilizes the error in the energy norm (Theorems 4.3.1, 4.3.3). In fact, the time-step restriction (4.5) can be removed if we replace the linearization (4.2) with

$$\mathbf{u} \cdot \nabla \mathbf{u} \approx (2\frac{\mathbf{u}^{n-1} + \mathbf{u}^{n-2}}{2} - \frac{\mathbf{u}^{n-2} - \mathbf{u}^{n-3}}{2}) \cdot \nabla \frac{\mathbf{u}^n + \mathbf{u}^{n-1}}{2}. \tag{4.6}$$

Linearization by (4.6) preserves $\mathcal{O}(\Delta t^2)$ of CN like (4.2), but additionally stabilizes the CNLE approximations (see Chapter 3). See Theorem 4.3.7.

We introduced the CNLE approximation in Chapter 3, Problem 3.1.3 with $\xi^n(\mathbf{v}) = a_0\mathbf{v}^n + \ldots + a_{n_0}\mathbf{v}^{n-n_0}$. The governing equations (3.15), (3.16) here for the case $g = 0$: find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_{h,\phi_n} \times Q_h$ satisfying

$$
\begin{aligned}
&(\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) \\
&\qquad + \nu(\nabla \mathbf{u}_h^{n+1/2}, \nabla \mathbf{v}_h) - (p_h^{n+1/2}, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^{n+1/2}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h \qquad (4.7)\\
&(q_h, \nabla \cdot \mathbf{u}^{n+1}) = 0, \quad \forall q_h \in Q_h. \qquad (4.8)
\end{aligned}
$$

Chapter 2 contains a collection of fundamental estimates required for the convergence analysis. We discuss our contributions relative to previous work in Section 4.1 including comments on the necessity of the Stokes projection to ensure optimal error estimate in $l^\infty(H^1)$ and on the our regularity assumptions. In Section 4.2, we provide the mathematical setting for analyzing the NSE with inhomogeneous boundary data. In particular, we derive error estimates for the elliptic and Stokes projection, with particular care given to estimating errors in L^2 and $W^{-1,2}$. A nontrivial extension of the standard Aubin-Nitsche lift argument is applied for functions with non-zero trace. Proofs are contained in the subsections that follow. Our main result is presented in Section 4.3 with proofs in the following subsection. We include an estimate for the discrete pressure in $l^2(L^2)$ in Corollary 4.3.8. We finish with a comment on similar error estimates for backward-Euler time-stepping with extrapolation.

4.1 REMARK ON IMPROVED ESTIMATE

We utilize the elliptic and Stokes projections in the convergence analysis of Theorems 4.3.3 and 4.3.6 respectively. The Stokes projection requires additional regularity of the pressure p, but is necessary in establishing the optimal convergence rate predicted Theorem 4.3.6, error in $l^\infty(H^1)$. The crucial estimate involves the error in the pressure. Let $\tilde{q}_h \approx p^{n+1}$, $\mathbf{v} \in H_0^1$. Then

$$(p^{n+1} - \tilde{q}_h, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \leq \begin{cases} ||p^{n+1} - \tilde{q}_h|| \; ||\nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||, & \text{or} \\ |p^{n+1} - \tilde{q}_h|_1 \; ||\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}|| \end{cases}. \tag{4.9}$$

The first option in (4.9) must be avoided, because we have no control of $||\nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}||$. The second option (4.9) is applicable, but ultimately leads to a suboptimal error estimate. Indeed, approximation theory for FE functions suggests

$$||p^{n+1} - \tilde{q}_h||_m \leq C_* h^{s+1-m} \tag{4.10}$$

so that a factor of h is *lost* in the case $m = 1$. Alternatively, let $(\tilde{\mathbf{v}}_h, \tilde{q}_h) \approx (\mathbf{u}^{n+1}, p^{n+1})$ be the Stokes projection. Then

$$\nu(\nabla(\mathbf{u}^{n+1} - \tilde{\mathbf{v}}_h), \nabla \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - (p^{n+1} - \tilde{q}_h, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) = 0. \tag{4.11}$$

The identity (4.11) simplifies analysis, eliminates the need to bound (4.10) when $m = 1$. The error is shifted to the time derivative instead for the Stokes projection so that we require

$$||\frac{(p^{n+1} - \tilde{q}_h^{n+1}) - (p^n - \tilde{q}_h^n)}{\Delta t}|| \leq C_* h^{s+1}. \tag{4.12}$$

The CNLE method is analyzed in [7] and [61] and the convergence analysis (corresponding to Theorem 4.3.3) assumes that $\mathbf{u} \in L^\infty(W^{1,\infty})$ and a time-step restriction. The conclusions of Theorems 4.3.1, 4.3.3, in addition to those of Theorems 4.3.4, 4.3.6, 4.3.7 are preserved with the regularity condition $\mathbf{u}(\cdot, t) \in H^2$ replaced by $\mathbf{u}(\cdot, t) \in W^{1,\infty}$. Regardless, the analysis of [61] suggests an associated sub-optimal convergence estimate, in the energy norm, $||\mathbf{u} - \mathbf{u}_h||_{l^\infty(L^2) \cap l^2(H^1)} \leq \mathcal{O}(h^k + h^{s+1} + h^{-3/2}\Delta t^4 + \Delta t^{3/2})$. Such an estimate requires, for instance, $\Delta t \leq h^{(3+2k)/8}$ for optimal convergence rate as $h \to 0$, but still predicts suboptimal convergence rate with respect to $\Delta t \to 0$.

The assumptions in Theorems 4.3.1, 4.3.4 hold *a priori* if we assume sufficient smoothness and sufficiently small problem data (see e.g. [30]). Moreover, if $\mathbf{u} \in L^\infty(H^1) \cap L^2(H^2)$, then any NS solution $\mathbf{u}$ is smooth up to the regularity of the problem data $\mathbf{f}$, $\mathbf{u}^0$, $\partial\Omega$ (independent of a small data restriction). Consequently, the regularity suggested of $(\mathbf{u}, p)$ in Theorems 4.3.1, 4.3.4 implies that the solution is actually smooth. Note, however, that we assume Ω is polygonal and hence only C^0. For $k = 2, 3, \ldots$, we implicitly assume *sufficient* regularity of $\mathbf{u}^0(x, \cdot)$ and compatibility between $\mathbf{u}^0$ and $\mathbf{f}$ to achieve the estimates (4.37). The compatibility condition required (implied by Equation (1.5) in [43]) is infeasible to verify in practice, but is satisfied in many practical applications (e.g. startup from rest). Consequently, estimate (4.37) can be formally altered to include a time-dependency factor $t^{(1-k)/2}$ for small time $t \leq 1$ (consequence of Equation (1.6) in [43]).

4.2 APPROXIMATING FUNCTIONS WHEN $\mathbf{U}|_{\partial S} \neq 0$

We define the elliptic and Stokes projections for approximating H^1-functions in $X_{h,\cdot}$. See Section 4.1 for the motivation of using two different projections in the derivation of Theorems 4.3.1 and 4.3.4. Assumption 2.1.1 must be modified in the case of inhomogeneous problem data $\mathbf{u}|_{\partial S} \neq 0$ and $\nabla \cdot \mathbf{u} \neq 0$. We show that errors measured in $V_{h,\cdot}(g)$ can be bounded in the larger space $X_{h,\cdot}$ via (4.14) if the discrete boundary data $\phi_h \approx \phi$ satisfies (4.13). Estimate (4.14) is necessary since the discrete pressure is eliminated from the error analysis for velocity by testing with functions in the discretely divergence free space V_h.

Lemma 4.2.1. *Fix $g \in L^2(S)$. Suppose that the FE-space satisfies Assumption 2.1.1 and that ϕ, $\phi_h \in H^{1/2}_g(\partial S)$ each satisfy the compatibility condition*

$$\int_{\partial S} \phi \cdot \hat{\mathbf{n}}_S = \int_S g = \int_{\partial S} \phi_h \cdot \hat{\mathbf{n}}_S. \tag{4.13}$$

Then, for any $\mathbf{u} \in V_\phi(g)$, there exists a constant $0 < C < \infty$ depending on (2.2) so that

$$\inf_{\mathbf{v}_h \in V_{h,\phi_h}(g)} |\mathbf{u} - \mathbf{v}_h|_1 \leq C \inf_{\mathbf{w}_h \in X_{h,\phi_h}} |\mathbf{u} - \mathbf{w}_h|_1. \tag{4.14}$$

Proof. See [34], proof of the intermediate estimate (1.16) in Theorem 1.1 of Chapter II. Condition (4.13) ensures that $\nabla \cdot (\mathbf{u} - \mathbf{v}_h) \in L^2_0$ for all $\mathbf{v}_h \in X_{h,\phi_h}$. Therefore, (2.19) ensures that there exists a unique $\mathbf{z}_h \in V_h^\perp$ so that $(\nabla \cdot \mathbf{z}_h, q_h) = (\nabla \cdot (\mathbf{u} - \mathbf{v}_h), q_h)$ for all $\mathbf{v}_h \in V_{h,\phi_h}(g)$, $q_h \in Q_h$. The remainder of the proof follows the cited work. $\square$

Error estimates for the elliptic projection (4.18) and Stokes projection (4.21) in L^2 and $W^{-1,2}$ require regularity of solutions to the following auxiliary problem. Note that the continuous auxiliary function $\mathbf{w}_\theta$ solving (4.15) is defined to have the discrete boundary data $\mathbf{w}_\theta|_{\partial\Omega} = \phi_h$ (which is used in the proof of each of these results).

Assumption 4.2.2. *Given $\theta \in W^{-1,2}$, find $(\mathbf{w}_\theta, r_\theta) \in H_0^1 \times L_0^2$ satisfying*

$$(\nabla \mathbf{w}_\theta, \nabla \mathbf{v}) - (r_\theta, \nabla \cdot \mathbf{v}) - (q, \nabla \cdot \mathbf{w}) = (\theta, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in H_0^1 \times L^2. \tag{4.15}$$

This problem is known to be well posed. Suppose further that $(\mathbf{w}_\theta, r_\theta) \in (H^{m+2} \cap V) \times (H^{m+1} \cap L_0^2)$ satisfy

$$||\mathbf{w}_\theta||_{m+2} + ||r_\theta||_{m+1} \leq C||\theta||_m \tag{4.16}$$

when $m = 0$, 1 and $\theta \in H_0^m$ (with $H_0^0 = L^2$).

Indeed, (4.16) is true if Ω is smooth enough.

Define the elliptic projection P_e: fix $\mathbf{u} \in V_\phi(g)$ so that

$$P_e : V_\phi(g) \to V_{h,\phi_h}(g), \qquad (\nabla(\mathbf{u} - P_e(\mathbf{u})), \nabla \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in V_h. \tag{4.17}$$

We present an error estimate for P_e in H_0^1 below as well as L^2, $W^{-1,2}$ for a sufficiently regular domain Ω. Note that with inhomogeneous data $\phi \neq 0$ the estimate (4.19) in L^2, $W^{-1,2}$ depends on $|\mathbf{v} - P_e(\mathbf{v})|_{-m}$ where $\mathbf{v} = E(\phi)$ is an extension of the boundary data. Setting $\mathbf{v} = \mathbf{u}$ in (4.19) gives in the case $m = 1$. In general $|\mathbf{v} - P_e(\mathbf{v})|_{-m}$ represents the error in $\phi_h \approx \phi$ via $X_{h,\phi_h} \approx H_\phi^1$ and $|(\mathbf{u} - \mathbf{v}) - \mathbf{v}_{h0}|_1$ represents the remaining error in Ω via $X_h \approx H_0^1$. Let $|\cdot|_0 = ||\cdot||$ and $|\cdot|_{-1} = ||\cdot||_{-1}$ throughout.

Lemma 4.2.3. *Fix $g \in L^2$. Suppose that FE-space satisfies 2.1.1 and ϕ, $\phi_h \in H_g^{1/2}(\partial\Omega)$. Then P_e given by (4.17) is well-defined and satisfies*

$$|\mathbf{u} - P_e(\mathbf{u})|_1 \leq C \inf_{\mathbf{v}_h \in X_{h,\phi_h}} |\mathbf{u} - \mathbf{v}_h|_1 \tag{4.18}$$

for some $0 \leq C < \infty$ is a constant independent of $h \to 0$ when $\mathbf{u} \in H_\phi^1$. Suppose further that Assumption 4.2.2 is satisfied. Fix $m = 0$ or 1. Then

$$||\mathbf{u} - P_e(\mathbf{u})||_{-m} \leq \inf_{\mathbf{v} \in H_\phi^1} (Ch^{m+1} \inf_{\mathbf{v}_{h0} \in X_h} |(\mathbf{u} - \mathbf{v}) - \mathbf{v}_{h0}|_1 + |\mathbf{v} - P_e(\mathbf{v})|_{-m}). \tag{4.19}$$

Proof. See Section 4.2.1 $\square$

Define the Stokes projection: let $P_s : (V_\phi(g), L^2_0) \to (V_{h,\phi_h}(g), Q_h)$ so that $(\tilde{\mathbf{v}}_h, \tilde{q}_h) := P_s(\mathbf{u}, p)$ satisfies

$$\begin{aligned} &\forall \mathbf{v} \in X_h, \quad \nu(\nabla(\mathbf{u} - \tilde{\mathbf{v}}_h), \nabla \mathbf{v}) - (p - \tilde{q}_h, \nabla \cdot \mathbf{v}) = 0 \\ &\forall q \in Q_h, \quad (q, \nabla \cdot \tilde{\mathbf{v}}_h) = (q, g). \end{aligned} \tag{4.20}$$

We present an error estimate for P_s in H^1_0 below as well as L^2, $W^{-1,2}$ for a sufficiently regular domain Ω. Note that with inhomogeneous data $\phi \neq 0$ the estimate (4.22) in L^2, $W^{-1,2}$ depends on $|\mathbf{v} - P_e(\mathbf{v})|_{-m}$ where $\mathbf{v} = E(\phi)$ is an extension of the boundary data. Write $\tilde{\mathbf{v}}_h := P_{s,1}(\mathbf{u}, p)$. Similar to analysis of P_e in (4.19), $|\mathbf{v} - P_{s,1}(\mathbf{v}, p)|_{-m}$ represents the error in $\phi_h \approx \phi$ via $X_{h,\phi_h} \approx H^1_\phi$ and $|(\mathbf{u} - \mathbf{v}) - \mathbf{v}_{h0}|_1$ represents the remaining error in Ω via $X_h \approx H^1_0$ in (4.22).

Lemma 4.2.4. *Fix $g \in L^2$. Suppose that FE-space satisfies 2.1.1 and ϕ, $\phi_h \in H^{1/2}_g(\partial\Omega)$. Then P_s given by (4.20) is well-defined so that*

$$|\mathbf{u} - P_{s,1}(\mathbf{u}, p)|_1 \leq C\Big(\inf_{\mathbf{v}_h \in X_{h,\phi_h}} |\mathbf{u} - \mathbf{v}_h|_1 + \nu^{-1} \inf_{q_h \in Q_h} ||p - q_h||\Big) \tag{4.21}$$

for some $0 \leq C < \infty$ is a constant independent of $h \to 0$ when $\mathbf{u} \in H^1_\phi$ and $p \in L^2_0$. Suppose further that Assumption 4.2.2 is satisfied. Fix $m = -1$, 0 or 1. Then

$$\begin{aligned} ||\mathbf{u} - P_{s,1}(\mathbf{u})||_{-m} &\leq C\nu^{-1} h^{m+1} \inf_{q_h \in Q_h} ||p - q_h|| \\ &\quad + \inf_{\mathbf{v} \in H^1_\phi} \Big(C h^{m+1} \inf_{\mathbf{v}_{h0} \in X_h} |(\mathbf{u} - \mathbf{v}) - \mathbf{v}_{h0}|_1 + |\mathbf{v} - P_{s,1}(\mathbf{v}, p)|_{-m}\Big). \end{aligned} \tag{4.22}$$

Proof. See Section 4.2.2 $\square$

Assumption 4.2.5 is the proper modification to Assumption 2.1.1 in the case $\mathbf{u}|_{\partial\Omega} \neq 0$ applied to P_e and P_s defined above.

Assumption 4.2.5 (Discrete Boundary Interpolant). *Fix $g \in L^2$. Suppose that the FE-space satisfies Assumption 2.1.1. Suppose further that ϕ, $\phi_h \in H_g^{1/2}(\partial\Omega)$ so that (2.3)(a) holds in a slightly varied form:*

$$||\mathbf{u} - P_e(\mathbf{u})||_{-m} \leq Ch^{k+m+1}||\mathbf{u}||_{k+1} \tag{4.23}$$

when $\mathbf{u} \in H^{k+1} \cap H^1_\phi$ and

$$||\mathbf{u} - P_{s,1}(\mathbf{u}, p)||_{-m} \leq C(h^{k+m+1}||\mathbf{u}||_{k+1} + h^{s+m+2}||p||_{s+1}) \tag{4.24}$$

when additionally $p \in H^{s+1} \cap L^2_0$.

4.2.1 Proof of elliptic projection error

Proof. Lemma 4.2.3

First suppose $\phi = \phi_h = 0$ so that $P_e : V \to V_h$. For $m = -1$, apply Céa's Lemma to get $|\mathbf{u} - \tilde{\mathbf{v}}_h|_1 \leq 2 \inf_{\mathbf{v}_h \in V_h} |\mathbf{u} - \mathbf{v}_h|_1$. To recover infimum over all $\mathbf{v}_h \in X_h$, apply estimate (4.14). To recover estimate for $m = 0$ and 1, follow the procedure in [34] (e.g. Chapter II, Theorem 1.9).

Now suppose ϕ, $\phi_h \neq 0$ so that $P_e : V_\phi(g) \to V_{h,\phi_h}(g)$. Fix $\mathbf{v}_h \in V_{h,\phi_h}(g)$ and set $\mathbf{v} = \tilde{\mathbf{v}}_h - \mathbf{v}_h \in V_h$ in (4.17) to get

$$|\tilde{\mathbf{v}}_h - \mathbf{v}_h|_1^2 = (\nabla(\mathbf{u} - \mathbf{v}_h), \nabla(\tilde{\mathbf{v}}_h - \mathbf{v}_h)). \tag{4.25}$$

Apply the triangle inequality and (4.14) to prove (4.18).

For the L^2 and $W^{-1,2}$-estimates, consider the auxiliary problem: given $\theta \in H_0^m$ for $m = 0$ or 1 let $(\mathbf{w}_\theta, r_\theta) \in (H^{m+2} \cap V) \times (H^{m+1} \cap L_0^2)$ solve (4.15) with estimate (4.16). Fix extension $E(\phi) \in V_\phi(g)$. Recall that $P_e(E(\phi)) \in V_{h,\phi_h}(g)$. Define the expansion

$$\left\{ \begin{array}{rcl} \mathbf{u} - \tilde{\mathbf{v}}_h & = & (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}) + (E(\phi) - P_e(E(\phi))) \\ \mathbf{u}_0 & = & \mathbf{u} - E(\phi) \in V \\ \tilde{\mathbf{v}}_{h0} & = & \tilde{\mathbf{v}}_h - P_e(E(\phi)) \in V_h \end{array} \right. . \tag{4.26}$$

Since $\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0} \in H_0^1 \cap V_{h,\cdot}$ (discretely divergence free), we get

$$\begin{aligned}(\mathbf{u} - \tilde{\mathbf{v}}_h, \theta) &= (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}, \theta) + (E(\phi) - P_e(E(\phi)), \theta) \\ &= (\nabla(\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}), \nabla \mathbf{w}_\theta) - (r_\theta, \nabla \cdot (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0})) + (E(\phi) - P_e(E(\phi)), \theta). \end{aligned} \tag{4.27}$$

Let $P_e|_{\phi_h=0}$ be the elliptic projection with homogeneous problem data so that $P_e|_{\phi_h=0}(\mathbf{v}) \in V_h$ for all $\mathbf{v} \in H^1$. Then

$$\tilde{\mathbf{v}}_{h0} = P_e|_{\phi_h=0}(\mathbf{u}_0), \qquad (\nabla(\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}), \nabla \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in V_h$$

since $\tilde{\mathbf{v}}_h = P_e(\mathbf{u})$. Moreover, $(q_h, \nabla \cdot (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0})) = 0$ for all $q_h \in Q_h$ since $\tilde{\mathbf{v}}_{h0} \in V_h$. For fixed $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$,

$$\begin{aligned}(\mathbf{u} - \tilde{\mathbf{v}}_h, \theta) = (\nabla(\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}), \nabla(\mathbf{w}_\theta - \mathbf{v}_h)) - (r_\theta - q_h, \nabla \cdot (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0})) \\ + (E(\phi) - P_e(E(\phi)), \theta) \\ = (\nabla(\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}), \nabla(\mathbf{w}_\theta - \mathbf{v}_h)) - (r_\theta - q_h, \nabla \cdot (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0})) \\ + (E(\phi) - P_e(E(\phi)), \theta). \end{aligned} \tag{4.28}$$

Apply duality estimate for $W^{-1,2} \times H_0^1$ when $m = 1$ and Cauchy Schwarz (2.22) to (4.27). Take supremum over all $\mathbf{v}_h \in V_h$, $q_h \in Q_h$. Apply estimates (4.14) and (2.3)(a) (via Assumption 2.1.1) for $\phi = \phi_h = 0$. Apply (4.16) and divide by $||\theta||_m$ to get

$$\frac{(\mathbf{u} - \tilde{\mathbf{v}}_h, \theta)}{|\theta|_m} \le Ch^{m+1} |(\mathbf{u} - E(\mathbf{u})) - P_e|_{\phi_h=0}(\tilde{\mathbf{v}}_h - E(\phi))|_1 + |E(\phi) - P_e(E(\phi))|_{-m}.$$

Apply (4.18), (4.14) in the case $\phi = \phi_h = 0$. Take supremum over all $\theta \in H_0^m$ to get

$$|\mathbf{u} - \tilde{\mathbf{v}}_h|_{-m} \le Ch^{m+1} \inf_{\mathbf{v}_h \in X_h} |(\mathbf{u} - E(\mathbf{u})) - \mathbf{v}_h|_1 + |E(\phi) - P_e(E(\phi))|_{-m}.$$

Take the infimum over all extensions $E(\phi) \in V_\phi(g)$ to prove (4.19). $\square$

4.2.2 Proof of Stokes projection error

Proof. Lemma 4.2.4 First suppose that $\phi = \phi_h = 0$ so that $P_{s,1}|_{\phi_h=0} : (V, L_0^2) \to V_h$. For $m = -1$, a similar (but simpler) proof for the error estimate of the nonlinear NSE in Section 4.3.2 proves $|\mathbf{u} - \tilde{\mathbf{v}}_h|_1 \leq 2 \inf_{\mathbf{v}_h \in V_h} |\mathbf{u} - \mathbf{v}_h|_1 + \nu^{-1} \inf_{q_h \in Q_h} ||p - q_h||$. Take the infimum over all $\mathbf{v}_h \in V_{h,\phi_h}(g)$ and all $\tilde{q}_h \in Q_h$ and apply (4.14) to prove (4.21) for $m = -1$. To recover estimate for $m = 0$, 1, follow the procedure in [34] (e.g. Chapter II, Theorem 1.9).

Now suppose ϕ, $\phi_h \neq 0$ so that $P_{s,1} : (V_\phi(g), L_0^2) \to V_{h,\phi_h}(g)$. Fix $\mathbf{v}_h \in V_h(g)$ and $\tilde{q}_h \in Q_h$. Set $\mathbf{v} = \tilde{\mathbf{v}}_h - \mathbf{v}_h$ in (4.20) to get

$$\nu|\tilde{\mathbf{v}}_h - \mathbf{v}_h|_1^2 = \nu(\nabla(\mathbf{u} - \mathbf{v}_h), \nabla(\tilde{\mathbf{v}}_h - \mathbf{v}_h)) - (p - \tilde{q}_h, \nabla \cdot (\tilde{\mathbf{v}}_h - \mathbf{v}_h)). \tag{4.29}$$

The triangle inequality and (4.14) gives

$$|\mathbf{u} - \tilde{\mathbf{v}}_h|_1 \leq C(\inf_{\mathbf{v}_h \in X_{h,\phi_h}} |\mathbf{u} - \mathbf{v}_h|_1 + \nu^{-1} \inf_{q_h \in Q_h} ||p - q_h||) \tag{4.30}$$

to prove (4.21) for $m = -1$.

For the L^2 and $W^{-1,2}$-estimates, consider the auxiliary problem: given $\theta \in H_0^m$ for $m = 0$ or $m = 1$ let $(\mathbf{w}_\theta, r_\theta) \in (H^{m+2} \cap V) \times (H^{m+1} \cap L_0^2)$ solve (4.15) with estimate (4.16). Fix extension $E(\phi) \in V_\phi(g)$. Since $\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0} \in H_0^1 \cap V_{h,\cdot}$ (discretely divergence free), we apply the expansion (4.19) and definition (4.15) to get

$$\begin{aligned}(\mathbf{u} - \tilde{\mathbf{v}}_h, \theta) &= (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}, \theta) + (E(\phi) - P_{s,1}(E(\phi), p), \theta) \\ &= (\nabla(\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0}), \nabla \mathbf{w}_\theta) - (r_\theta, \nabla \cdot (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0})) + (E(\phi) - P_{s,1}(E(\phi), p), \theta). \quad (4.31)\end{aligned}$$

Recall that $P_e|_{\phi_h=0}$ is the elliptic projection so that $P_e|_{\phi_h=0}(\mathbf{v}) \in V_h$ for all $\mathbf{v} \in H^1$. Note that $\tilde{\mathbf{v}}_h = P_{s,1}(\mathbf{u}, p)$ implies $\nu(\nabla(\mathbf{u} - \tilde{\mathbf{v}}_h), \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p) = 0$ for all $\mathbf{v}_h \in V_h$. Moreover, $\nu(\nabla(E(\phi) - P_{s,1}(E(\phi), p)), \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p) = 0$ for all $\mathbf{v}_h \in V_h$. Therefore, $(\nabla(\tilde{\mathbf{v}}_{h0} - \mathbf{u}_0), \nabla \mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in V_h$ so that $\tilde{\mathbf{v}}_{h0} = P_e|_{\phi_h=0}(\mathbf{u}_0)$. Also note that $\tilde{\mathbf{v}}_{h0} \in V_h$ implies $(q_h, \nabla \cdot (\mathbf{u}_0 - \tilde{\mathbf{v}}_{h0})) = 0$ for all $q_h \in Q_h$. Thus, for any $\mathbf{v}_h \in V_h$ and any $q_h \in Q_h$, we derive the same identity (4.28) as in the proof of the elliptic projection except that $\tilde{\mathbf{v}}_h$ is the Stokes projection here rather than the elliptic projection and $P_e(E(\phi))$ is replaced by $P_{s,1}(E(\phi), p)$. Proceed in the same manner, except applying (4.21) instead of (4.18), to prove (4.22). $\square$

4.3 UNCONDITIONAL CONVERGENCE OF CNLE

We first construct the error equation and then state the main results in this section. A list of the constants found in these theorems (referenced throughout this section) are compiled in Section 4.3.1 along with a derivation of the error equation required for their proofs. Recall that NS-solutions satisfy (2.43), (2.44), (2.45). Strong solutions $\mathbf{u} \in C^0(H^1_\phi)$ and $p \in C^0(L^2_0)$ satisfy

$$
\begin{aligned}
&(\partial_t \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v}) + (\mathbf{u}(\cdot, t^{n+1/2}) \cdot \nabla \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v}) + \nu(\nabla \mathbf{u}(\cdot, t^{n+1/2}), \nabla \mathbf{v}) \\
&\quad - (p(\cdot, t^{n+1/2}), \nabla \cdot \mathbf{v}) = (\mathbf{f}(\cdot, t^{n+1/2}), \mathbf{v}), \quad \forall \mathbf{v} \in H^1_0 \qquad (4.32)\\
&\nabla \cdot \mathbf{u}(\cdot, t^{n+1/2}) = 0 \qquad (4.33)\\
&\mathbf{u}(\cdot, t = 0) = \mathbf{u}^0. \qquad (4.34)
\end{aligned}
$$

Theorem 4.3.3 ensures $(\mathbf{u}_h - \mathbf{u}) \in l^\infty(L^2) \cap l^2(H^1)$. Define

$$
n_0{*} := \begin{cases} n_0 + 1, & \text{if } \xi^n(\mathbf{v}) = a_0 \mathbf{v}^{n-1/2} + a_1 \mathbf{v}^{n-3/2} + \ldots + a_{n_0} \mathbf{v}^{n-n_0-1/2} \\ n_0, & \text{otherwise} \end{cases} \qquad (4.35)
$$

and $\mathbf{e}_u = \mathbf{u}_h - \mathbf{u}$.

Theorem 4.3.1. *Suppose that the FE-space and $\phi_h \approx \phi$ satisfy Assumption 4.2.5. Suppose further that $\mathbf{f} \in C^0([t^{n_0*}, T]; W^{-1,2})$ and*

$$
\mathbf{u} \in C^0(H^1), \quad \partial_t \mathbf{u} \in C^0([t^{n_0*}, T]; W^{-1,2}), \quad p \in C^0([t^{n_0*}, T]; L^2_0).
$$

Then

$$
||\mathbf{u} - \mathbf{u}_h||_{l^\infty(n_0*, N; L^2)} + \nu^{1/2}(\nu \Delta t \sum_{n=n_0*}^{N-1} |\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2}|_1^2)^{1/2} \leq K_* \qquad (4.36)
$$

where $K_* = G^N \nu^{-1/2}(C_{*,fem} + C_{*,cn} + C_{*,IC})$, $G^N := C \exp(\Delta t \sum_{n=n_0*}^{N-1} \kappa^n)$, and $\kappa^n := C\nu^{-1}||\mathbf{u}^{n+1/2}||_2^2$ if $\mathbf{u} \in l^2(H^2)$ or $\kappa^n := C\nu^{-3}||\mathbf{u}^{n+1/2}||_1^4$ if $\xi^n(\mathbf{v}) = a_0 \mathbf{v}^{n-1/2} + a_1 \mathbf{v}^{n-3/2} + \ldots + a_{n_0} \mathbf{v}^{n-n_0-1/2}$ and n_0* governed by (4.35). Additionally require $\mathbf{u} \in l^2(H^2)$ if $\kappa^n = \nu^{-1}||\mathbf{u}^{n+1/2}||_2^2$. The constants $C_{*,fem}$, $C_{*,cn}$, $C_{*,IC}$ (depending on $\mathbf{u}$, p, ν^{-1}) are given in (4.51), (4.52), and (4.53) respectively and remain bounded as h, $\Delta t \to 0$.

Proof. See Section 4.3.2. ∎

The convergence result in Theorem 4.3.3 extends Theorem 4.3.1. The initial iterates $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$ and discrete boundary data ϕ_h also must be *good enough* to ensure the optimal convergence rate. We make this precise via the following assumption.

Assumption 4.3.2 (Accuracy of Initial Iterates)**.** *Fix $k \geq 0$, $s \geq -1$. Suppose $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$ satisfy*

$$C_{*,IC} \leq C_{fem,u}h^k + C_{fem,p}h^{s+1} + C_{cn}\Delta t^2$$

where $C_{*,IC}$ is given in (4.53) and $C_{fem,u}$, $C_{fem,p}$, C_{cn} are given in (4.54), (4.55), (4.56).

Note that Assumption 4.3.2 reduces to, when $s = k - 1$,

$$||\mathbf{e}_u||_{l^\infty([0,n_0];L^2)} \leq C(h^k + \Delta t^2)$$

for some constant $C > 0$. For the discrete boundary data, Assumption 4.2.5 is precisely what is needed so that $h^k C_{fem,u}$ in (4.54) can absorb the error due to $\phi_h \approx \phi$. Therefore, under usual regularity conditions ($s = k - 1$)

$$||\mathbf{e}_u||_{l^\infty(L^2)\cap l^2(H^1)} \leq C_*(h^k + \Delta t^2), \quad \textit{without } \Delta t\text{-restriction.}$$

Theorem 4.3.3 (Unconditional convergence)**.** *Fix $k > 0$, $s > -1$. Under the assumptions of Theorem 4.3.1, suppose further that $\partial_t \mathbf{f} \in C^0([t^{n_0}, T]; W^{-1,2})$, $\mathbf{u} \in C^0([t^{n_0}, T]; H^{k-1})$, $\mathbf{u} \in l^\infty(H^k)$, $\mathbf{u} \in l^2(H^{k+1})$, $\partial_t \mathbf{u} \in C^0(H^1)$, $\partial_t^{(2)}\mathbf{u} \in C^0([t^{n_0}, T]; W^{-1,2})$, $p \in l^2(n_0, N; H^{s+1})$, and $\partial_t p \in C^0([t^{n_0}, T]; L^2)$ are satisfied. If the initial data satisfies Assumption 4.3.2, then*

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}_h||_{l^\infty(n_0,N;L^2)} &+ (\nu\Delta t \sum_{n=n_0}^{N-1} |\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2}|_1^2)^{1/2} \\ &\leq G^N \nu^{-1/2}(C_{fem,u}h^k + C_{fem,p}h^{s+1} + C_{cn}\Delta t^2) \end{aligned} \tag{4.37}$$

where $G^N := C\exp(\Delta t \sum_{n=n_0}^{N-1} \kappa^n)$, and $\kappa^n := C\nu^{-1}||\mathbf{u}^{n+1/2}||_2^2$. The constants $C_{fem,u}$, $C_{fem,p}$, C_{cn} (depending on $\mathbf{u}$, p, ν^{-1}) are given in (4.54), (4.55), (4.56) respectively and remain bounded as h, $\Delta t \to 0$.

Proof. See Section 4.3.2. □

An estimate for $\Delta t\sum_n ||(\mathbf{e}_u^{n+1}-\mathbf{e}_u^n)/\Delta t||^2$ is needed in the error analysis for pressure and the drag/lift forces by the fluid on embedded obstacles. Theorem 4.3.4 provides sufficient regularity of $(\mathbf{u}, p)$ solving (4.32), (4.33), (4.34) to ensure $\mathbf{u}_h \in l^\infty(H^1)$ and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$.

Theorem 4.3.4. *Under the assumptions of Theorem 4.3.1, suppose that* $\mathbf{f} \in C^0([t^{n_0}, T]; L^2)$ *and*

$$\mathbf{u} \in C^0(H^2), \quad \partial_t \mathbf{u} \in C^0([t^{n_0}, T]; L^2), \quad p \in C^0([t^{n_0}, T]; H^1)$$

and that

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0. \tag{4.38}$$

Then

$$||\partial_{\Delta t}(\mathbf{u}-\mathbf{u}_h)||_{l^2(n_0,N;L^2)} + \nu||\nabla(\mathbf{u}-\mathbf{u}_h)||_{l^\infty(n_0,N;L^2)} \leq K_* \tag{4.39}$$

where $K_* = G^N(\nu^{-1}F_{*,fem}+F_{*,cn}+F_{*,IC})$, $G^N := C\exp(\Delta t\sum_{n=n_0}^{N-1}\kappa^n)$, *and* $\kappa^n := C\nu^{-1}(||\mathbf{u}^{n+1/2}||_2^2+h^{-1}||\mathbf{e}_u^{n+1/2}||_1^2)$. *The constants* $F_{*,fem}$, $F_{*,cn}$, $F_{*,IC}$ *(depending on* $\mathbf{u}$, p, ν^{-1}*) are given in (4.57), (4.58), (4.59) respectively and remain finite as* h, $\Delta t \to 0$.

Proof. See Section 4.3.3. □

The convergence result in Theorem 4.3.6 extends Theorem 4.3.4. The initial iterates $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$ and discrete boundary data ϕ_h must be chosen to preserve an optimal convergence rate. For the discrete boundary data, Assumption 4.2.5 is precisely what is needed.

Assumption 4.3.5 (Accuracy of initial iterates)**.** *Fix* $k \geq 0$, $s \geq -1$. *Suppose* $\{\mathbf{u}_h^i\}_i^{n_0}$ *satisfy*

$$F_{*,IC} \leq F_{fem,u}h^k + \nu^{-1}F_{fem,p}h^{s+1} + F_{cn}\Delta t^2$$

where $F_{*,IC}$ *is given in (4.59) and* $F_{fem,u}$, $F_{fem,p}$, F_{cn} *are given in (4.60), (4.61), (4.63).*

Note that Assumption 4.3.5 reduces to, when $s = k-1$,

$$||\mathbf{e}_u||_{l^\infty(0,n_0;H^1)} \leq C(h^k + \Delta t^2)$$

for some constant $C > 0$. Therefore, under usual regularity conditions we show in Theorems 4.3.6, 4.3.7 that

$$||\partial_{\Delta t}\mathbf{e}_u||_{l^2(n_0*,N;L^2)} + ||\mathbf{e}_u||_{l^\infty(n_0*,N;H^1)} \leq C_*(h^k + \Delta t^2)$$

as long as either $\Delta t \leq h^{1/4}$ (no ν-dependence) or $\xi^n(\mathbf{v}) = a_0\mathbf{v}^{n-1/2} + \ldots + a_{n_0}\mathbf{v}^{n-n_0-1/2}$ without any Δt restriction!

Theorem 4.3.6. *Fix $k > 0$, $s > -1$. Under the regularity and initial data assumptions of Theorem 4.3.4, suppose further that $\partial_t \mathbf{f} \in C^0([t^{n_0}, T]; L^2)$, $\mathbf{u} \in l^\infty(H^{k+1})$, $\mathbf{u} \in C^0([t^{n_0}, T]; H^k)$, $\partial_t \mathbf{u} \in C^0(H^1)$, $\partial_t^{(2)}\mathbf{u} \in C^0([t^{n_0}, T]; L^2)$, $p \in C^0(H^s)$, $p \in l^\infty(H^{s+1})$, $\partial_t p \in C^0([t^{n_0}, T]; H^1)$ and that*

$$\Delta t \leq h^{1/4} \tag{4.40}$$

is satisfied. Then

$$\begin{aligned} &||\partial_{\Delta t}(\mathbf{u} - \mathbf{u}_h)||_{l^2(n_0,N;L^2)} + \nu^{1/2}|\nabla(\mathbf{u} - \mathbf{u}_h)|_{l^\infty(n_0,N;L^2)} \\ &\qquad \leq G^N(F_{fem,u}h^k + \nu^{-1}F_{fem,p}h^{s+1} + F_{fem,0}(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2)^{1/2} + F_{cn}\Delta t^2) \end{aligned} \tag{4.41}$$

where $G^N := C\exp(\Delta t \sum_{n=n_0}^{N-1} \kappa^n)$, and $\kappa^n := C\nu^{-1}(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}||\mathbf{e}_u^{n+1/2}||_1^2)$. The constants $F_{fem,u}$, $F_{fem,p}$, $F_{fem,0}$, and F_{cn} (depending on $\mathbf{u}$, p, ν^{-1}) are given in (4.60), (4.61), (4.62), and (4.63) respectively and remain finite as h, $\Delta t \to 0$.

Proof. See Section 4.3.3. □

Theorem 4.3.7. *Consider extrapolations of the form*

$$\xi^n(\mathbf{v}) = a_0\mathbf{v}^{n-1/2} + a_1\mathbf{v}^{n-3/2} + \ldots + a_{n_0}\mathbf{v}^{n-n_0-1/2}$$

so that $n_0 = n_0 + 1$ by (4.35). Under the assumptions of Theorem 4.3.4 (with n_0 replaced by $n_0 + 1$), suppose that (4.38) is replaced by the condition*

$$\min\left\{h^{-1}, \Delta t^{-2}\right\} \Delta t \sum_{n=n_0+1}^{N-1} |\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0. \tag{4.42}$$

Then

$$||\partial_{\Delta t}(\mathbf{u} - \mathbf{u}_h)||_{l^2(n_0+1,N;L^2)} + \nu||\nabla(\mathbf{u} - \mathbf{u}_h)||_{l^\infty(n_0+1,N;L^2)} \leq K_* \tag{4.43}$$

where $G^N := C\exp(\Delta t \sum_{n=n_0+1}^{N-1} \kappa^n)$, and $\kappa^n := C\nu^{-1}(||\mathbf{u}^{n+1/2}||_2^2 + \min\{h^{-1}, \Delta t^{-1}\}\, ||\mathbf{e}_u^{n+1/2}||_1^2)$. The constants $F_{*,fem}$, $F_{*,cn}$, $F_{*,IC}$ (depending on $\mathbf{u}$, p, ν^{-1}) are given in (4.57), (4.58), (4.59) respectively and remain finite as h, $\Delta t \to 0$.

Moreover, suppose that the conditions of Theorem 4.3.6 (with n_0 replaced by n_0+1) are satisfied without the restriction (4.40) on Δt. Then (4.42) is satisfied (without Δt, h restriction) and

$$||\partial_{\Delta t}(\mathbf{u}-\mathbf{u}_h)||_{l^2(n_0+1,N;L^2)} + \nu^{1/2}|\mathbf{u}-\mathbf{u}_h|_{l^\infty(n_0+1,N;H^1)} \\ \leq G^N(F_{fem,u}h^k + \nu^{-1}F_{fem,p}h^{s+1} + F_{fem,0}(\Delta t \sum_{n=n_0+1}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2)^{1/2} + F_{cn}\Delta t^2). \quad (4.44)$$

The constants $F_{fem,u}$, $F_{fem,p}$, $F_{fem,0}$, and F_{cn} (depending on $\mathbf{u}$, p, ν^{-1}) are given in (4.60), (4.61), (4.62), and (4.63) respectively and remain finite as h, $\Delta t \to 0$.

Proof. See Section 4.3.4. □

The error estimates of Theorems 4.3.1, 4.3.3, 4.3.4, Theorem 4.3.6 give conditions in which $p_h \in l^2(n_0, N; L^2)$, $\mathbf{u}_h \in l^\infty(n_0, N; H^1)$, and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$. In particular, as a direct consequence of estimate (4.39) and the conditions of Theorem 4.3.4, we have

$$||\partial_{\Delta t}\mathbf{u}_h||_{l^2(n_0,N;L^2)} + \nu||\nabla \mathbf{u}_h||_{l^\infty(n_0,N;L^2)} \leq K_1 < \infty.$$

Estimates for pressure follow as well and are summarized in the next Corollary. Under the conditions of Theorem 4.3.4, estimate (4.45) simplifies to

$$\Delta t \sum_{n=n_0+1}^{N-1} ||p_h^{n+1/2}|| \leq K_1$$

for some data-dependent $K_1 > 0$ independent of h, $\Delta t \to 0$. Under the conditions of Theorem 4.3.6, estimate (4.46) simplifies to

$$\Delta t \sum_{n=n_0+1}^{N-1} ||p^{n+1/2} - p_h^{n+1/2}|| \leq C_*(h^k + h^{s+1} + \Delta t^2)$$

for some data-dependent $C_* > 0$ independent of h, $\Delta t \to 0$.

Corollary 4.3.8. *Under the conditions and conclusions of Theorem 4.3.4,*

$$\Delta t \sum_{n=n_0+1}^{N-1} ||p^{n+1/2} - p_h^{n+1/2}||^2 \leq K_1 < \infty \tag{4.45}$$

so that K_1 is independent of h, $\Delta t \to 0$. Suppose further that the conditions and conclusions of Theorem 4.3.6 are satisfied, then

$$\begin{aligned}\Delta t \sum_{n=n_0+1}^{N-1} ||p^{n+1/2} - p_h^{n+1/2}|| &\leq C(||p||_{l^2(n_0,N;H^{s+1})} h^{s+1} + C_{cn}\Delta t^2 + \dots \\ &\dots + ||\partial_{\Delta t}\mathbf{e}_u||_{l^2(n_0,N;L^2)} + ||\mathbf{u}||_{l^\infty(H^1)}(\Delta t \sum_{n=n_0+1}^{N-1} ||\mathbf{e}_u^{n+1/2}||_1^2)^{1/2} + \dots \\ &\dots + (\Delta t \sum_{n=n_0+1}^{N-1} ||\mathbf{u}_h^{n+1/2}||_1^2)^{1/2} ||\mathbf{e}_u||_{l^\infty(L^2)}^{1/2} ||\mathbf{e}_u||_{l^\infty(H^1)}^{1/2})\end{aligned} \tag{4.46}$$

for some $0 < C < \infty$ and where C_{cn} is given in (4.56).

Remark 4.3.9. *Notice that in general, we must make use of $l^\infty(H^1)$ estimates for the linearly extrapolated error $\xi^n(\mathbf{e}_u)$ since we are only guaranteed estimates on the average errors $\mathbf{e}_u^{n+1/2}$ via (4.37) and not $\mathbf{e}_u^n$.*

Proof. See Section 4.3.5. □

4.3.1 Technical preliminaries

We finish with some technical remarks concerning the error equation referenced throughout in the following proofs and the constants arising in the above Theorems. The consistency error for the time-discretization is given by, for any $\mathbf{v} \in H_0^1$,

$$\begin{aligned}R^{n+1}(\mathbf{v}) := &(\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v}) - (p^{n+1/2} - p(\cdot, t^{n+1/2}), \nabla \cdot \mathbf{v}) \\ &+ c_h(\xi^n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{v}) - (\mathbf{u}(\cdot, t^{n+1/2}) \cdot \nabla \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v}) \\ &+ \nu(\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \nabla \mathbf{v}) + (\mathbf{f}(\cdot, t^{n+1/2}) - \mathbf{f}^{n+1/2}, \mathbf{v}).\end{aligned} \tag{4.47}$$

Recall identity (2.40). Applied (4.32) along with (4.47) to get

$$\begin{aligned}&(\partial_{\Delta t}^{n+1}\mathbf{u}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{v}) - (p^{n+1/2}, \nabla \cdot \mathbf{v}) \\ &\quad + \nu(\nabla \mathbf{u}^{n+1/2}, \nabla \mathbf{v}) = (\mathbf{f}^{n+1/2}, \mathbf{v}) + R^{n+1}(\mathbf{v}), \quad \forall \mathbf{v} \in H_0^1.\end{aligned} \tag{4.48}$$

Decompose the velocity error, for some $\tilde{\mathbf{v}}_h^n \in V_{h,\phi_h}$,

$$\begin{cases} \mathbf{e}_u^n &= \mathbf{u}_h^n - \mathbf{u}^n = \mathbf{U}_h^n - \eta^n \\ \mathbf{U}_h^n &= \mathbf{u}_h^n - \tilde{\mathbf{v}}_h^n \in V_h \\ \eta^n &= \mathbf{u}^n - \tilde{\mathbf{v}}_h^n. \end{cases} \tag{4.49}$$

Fix $\tilde{q}_h^n \in Q_h$. Note that $(p_h, \nabla \cdot \mathbf{v}) = 0$ for any $\mathbf{v} \in V_h$. Subtract (4.48) from (4.7) to get the error equation

$$\begin{aligned} &(\partial_{\Delta t}^{n+1}\mathbf{U}_h, \mathbf{v}) + c_h(\xi^n(\mathbf{u}_h), \mathbf{U}_h^{n+1/2}, \mathbf{v}) + \nu(\nabla \mathbf{U}_h^{n+1/2}, \nabla \mathbf{v}) \\ &= -R^{n+1}(\mathbf{v}) + (\partial_{\Delta t}^{n+1}\eta, \mathbf{v}) - (p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla \cdot \mathbf{v}) + \nu(\nabla \eta^{n+1/2}, \nabla \mathbf{v}) \\ &- c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}_h), \eta^{n+1/2}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h. \end{aligned} \tag{4.50}$$

Specifying different $\mathbf{v}$ in (4.50) results in error estimates in different norms. For instance

$$\begin{aligned} \mathbf{v} = \mathbf{U}_h^{n+1/2} \quad &\in V_h \quad \Rightarrow \quad \text{Theorems 4.3.1, 4.3.3} \\ \mathbf{v} = \partial_{\Delta t}^{n+1}\mathbf{U}_h \quad &\in V_h \quad \Rightarrow \quad \text{Theorems 4.3.4, 4.3.6, 4.3.7} \end{aligned}.$$

For Theorems 4.3.1, 4.3.3, we set the spatial error $C_{*,fem}$, time-error $C_{*,cn}$, and initial condition modeling error $C_{*,IC}$. For Theorem 4.3.1, define

$$\begin{aligned} C_{*,fem} :&= C(||\mathbf{u}||_{l^\infty(H^1)}||\mathbf{u}||_{l^2(H^1)} + \nu^{1/2}||\mathbf{u}||_{l^\infty(n_0,N;L^2)} + \nu^{1/2}C_{*,6}^{1/2} + \ldots \\ &\ldots + h^2||\partial_t \mathbf{u}(\cdot,t)||_{L^2(t^{n_0*},T;H^1)} + ||p||_{l^2(n_0*,N;H^{s+1})}) \end{aligned} \tag{4.51}$$

$$\begin{aligned} C_{*,cn} &:= C(||\partial_t \mathbf{f}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\mathbf{u}||_{l^\infty(H^1)}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^1)} + \ldots \\ &\ldots + ||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\partial_t p||_{L^2(t^{n_0},T;L^2)}) \end{aligned} \tag{4.52}$$

$$C_{*,IC} := ||\mathbf{e}_u^{n_0}|| + C \begin{cases} \nu^{1/2}(\Delta t \sum_{i=0}^{n_0} |\mathbf{e}_u^{i+1/2}|_1^2)^{1/2} + \ldots & \\ \quad \ldots + (\Delta t \sum_{i=0}^{2n_0} \kappa^i)^{1/2}||\mathbf{e}_u||_{l^\infty(0,n_0;L^2)} & \text{if } n_0* = n_0 + 1, \\ (\Delta t \sum_{i=0}^{2n_0-1} \kappa^i)^{1/2}||\mathbf{e}_u||_{l^\infty(0,n_0-1;L^2)} & \text{if } n_0* = n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.53}$$

$$C_{*,6} := C \begin{cases} (\Delta t \sum_{i=0}^{n_0*+n_0-1} \kappa^i)^{1/2}||\mathbf{u}||_{l^\infty(0,n_0-1;H^k)} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases}.$$

and for Theorem 4.3.3 with $k > 0$, $s > -1$,

$$C_{fem,u} := C(||\mathbf{u}||_{l^\infty(H^1)}||\mathbf{u}||_{l^2(H^{k+1})} + \nu^{1/2}||\mathbf{u}||_{l^\infty(n_0,N;H^k)} + \nu^{1/2}C_6 + \dots$$
$$\dots + ||\partial_t \mathbf{u}(\cdot,t)||_{L^2(t^{n_0*},T;H^k)}) \tag{4.54}$$
$$C_{fem,p} := C||p||_{l^2(n_0*,N;H^{s+1})} \tag{4.55}$$
$$C_{cn} := C(||\partial_t^{(2)}\mathbf{f}||_{L^2(t^{n_0*},T;W^{-1,2})} + ||\mathbf{u}||_{l^\infty(H^1)}||\partial_t^{(2)}\mathbf{u}||_{L^2(H^1)} + \dots$$
$$\dots + ||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0*},T;W^{-1,2})} + ||\partial_t^{(2)}p||_{L^2(t^{n_0*},T;L^2)}) \tag{4.56}$$
$$C_6 := C\begin{cases} (\Delta t \displaystyle\sum_{i=0}^{n_0*+n_0-1} \kappa^i ||\mathbf{u}||^2_{l^\infty(0,n_0-1;H^k)})^{1/2} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases}.$$

Analogous to $C_{*,fem}$, $C_{*,cn}$, $C_{*,IC}$ above (for Theorem 4.3.1), define the spatial error $F_{*,fem}$, time-error $F_{*,cn}$, and initial condition modeling error $F_{*,IC}$ for Theorems 4.3.4, 4.3.7 by

$$F_{*,fem} := C(||\mathbf{u}||_{l^\infty(H^2)}(\nu||\mathbf{u}||_{l^2(H^1)} + ||p||_{l^2(L^2)}) + \dots$$
$$\dots + h\nu||\partial_t \mathbf{u}(\cdot,t)||_{L^2(t^{n_0*},T;H^1)} + h||\partial_t p(\cdot,t)||_{L^2(t^{n_0*},T;L^2)}) + \dots$$
$$\dots + C(\nu||\mathbf{u}||_{l^\infty(H^2)} + h^{1/2}||p||_{l^\infty(H^1)})\Delta t \sum_{n=n_0*}^{N-1} ||\mathbf{e}_u^{n+1/2}||_1^2)^{1/2} + \dots$$
$$\dots + (\Delta t \sum_{i=0}^{n_0*+n_0-1} \kappa^i)^{1/2}(\nu||\mathbf{u}||_{l^\infty(0,n_0*-1;H^1)} + ||p||_{l^\infty(0,n_0*-1;L^2)}) \tag{4.57}$$
$$F_{*,cn} := C(||\partial_t \mathbf{f}||_{L^2(t^{n_0*},T;L^2)} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t \mathbf{u}||_{L^2(t^{n_0*},T;H^1)} + \dots$$
$$\dots + ||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0*},T;L^2)} + ||\partial_t p||_{L^2(t^{n_0*},T;H^1)}) \tag{4.58}$$
$$F_{*,IC} := |\mathbf{e}_u^{n_0*}|_1 + C\begin{cases} (\Delta t \displaystyle\sum_{i=0}^{n_0*+n_0-1} \kappa^i)^{1/2}||\nabla \mathbf{e}_u||_{l^\infty(0,n_0*-1;L^2)} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.59}$$

and for Theorems 4.3.6, 4.3.7 with $k > 0$, $s > -1$,

$$F_{fem,u} := C(||\mathbf{u}||_{l^\infty(H^2)}||\mathbf{u}||_{l^2(H^{k+1})} + \dots$$
$$\dots + ||\partial_t \mathbf{u}(\cdot,t)||_{L^2(t^{n_0*},T;H^k)} + (\Delta t \sum_{i=0}^{n_0*+n_0-1} \kappa^i)^{1/2}||\mathbf{u}||_{l^\infty(0,n_0*-1;H^{k+1})}) \tag{4.60}$$

$$F_{fem,p} := C(||\mathbf{u}||_{l^\infty(H^2)}||p||_{l^2(H^{s+1})} + \dots$$
$$\dots + ||\partial_t p(\cdot,t)||_{L^2(t^{n_0*},T;H^s)} + (\Delta t \sum_{i=0}^{n_0*+n_0-1} \kappa^i)^{1/2}||p||_{l^\infty(0,n_0*;H^{s+1})}) \tag{4.61}$$

$$F_{fem,0} := Ch||\mathbf{u}||_{l^\infty(H^2)}(+(\nu||\mathbf{u}||_{l^\infty(H^2)} + ||p||_{l^\infty(H^1)})) \tag{4.62}$$

$$F_{cn} := C(||\partial_t^{(2)}\mathbf{f}||_{L^2(t^{n_0*},T;L^2)} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(H^1)} + \dots$$
$$\dots + ||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0*},T;L^2)} + ||\partial_t^{(2)}p||_{L^2(t^{n_0*},T;H^1)}). \tag{4.63}$$

4.3.2 Proof of $\mathbf{u}_h \to \mathbf{u}$ in $l^2(H^1) \cap l^\infty(L^2)$

Proof. (Theorems 4.3.1, 4.3.3)

Fix $n = n_0, n_0+1, \dots, N-1$. Set $\tilde{\mathbf{v}}_h^n = P_e(\mathbf{u}^n)$ defined by (4.17) in (4.49). Fix $\tilde{q}_h \in Q_h$. Set $\mathbf{v} = \mathbf{U}_h^{n+1/2} \in V_h$ in (4.50). Recall identity (2.35). Then

$$\frac{1}{2\Delta t}\left(||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2\right) + \nu|\mathbf{U}_h^{n+1/2}|_1^2 = (\partial_{\Delta t}^{n+1}\eta, \mathbf{U}_h^{n+1/2})$$
$$- (p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla\cdot\mathbf{U}_h^{n+1/2}) - c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2})$$
$$+ c_h(\xi^n(\mathbf{u}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) + c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) - R^{n+1}(\mathbf{U}_h^{n+1/2}). \tag{4.64}$$

Apply the duality estimate on $W^{-1,2} \times H_0^1$ and Cauchy-Schwarz (2.22) to get

$$(\partial_{\Delta t}^{n+1}\eta, \mathbf{U}_h^{n+1/2}) \le ||\partial_{\Delta t}^{n+1}\eta||_{-1}|\mathbf{U}_h^{n+1/2}|_1 \tag{4.65}$$
$$(p^{n+1/2} - \tilde{q}_h^{n+1/2}, \nabla\cdot\mathbf{U}_h^{n+1/2}) \le \sqrt{d}||p^{n+1/2} - \tilde{q}_h^{n+1/2}||\;|\mathbf{U}_h^{n+1/2}|_1. \tag{4.66}$$

We bound the convective terms and $R^{n+1}(\cdot)$ from (4.47) in the next 2 lemmas.

Lemma 4.3.10. *Suppose that the FE space satisfies Assumption 2.1.1 and* $\mathbf{u} \in l^4(H^1)$. *Then,*

$$
\begin{aligned}
&c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\
&\qquad - c_h(\xi^n(\mathbf{u}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) - c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\
&\leq C(\sum_{i=0}^{n_0}(||\mathbf{u}^{n-i}||_1 + ||\eta^{n-i}||_1))||\eta^{n+1/2}||_1 \ |\mathbf{U}^{n+1/2}|_1 \\
&\quad + C||\mathbf{u}^{n+1/2}||_1(\sum_{i=0}^{n_0}||\eta^{n-i}||_1)|\mathbf{U}^{n+1/2}|_1 + C_1(\xi^n(\mathbf{U}_h))
\end{aligned}
\tag{4.67}
$$

where

$$
C_1(\xi^n(\mathbf{U}_h)) := C \begin{cases} (||\mathbf{u}^{n+1/2}||_2 + \frac{1}{\sqrt{h}}||\eta^{n+1/2}||_1)\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}|||\mathbf{U}_h^{n+1/2}|_1, \ \textit{or} \\ (||\mathbf{u}^{n+1/2}||_1 + ||\eta^{n+1/2}||_1)\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||^{1/2}|\xi^n(\mathbf{U}_h)|_1^{1/2}|\mathbf{U}_h^{n+1/2}|_1 \end{cases}. \tag{4.68}
$$

Proof. Recall that $\xi^n(\mathbf{v}) = a_0\mathbf{v}^n + \ldots a_{n_0}\mathbf{v}^{n-n_0}$ and estimate (2.41)(a)(b). Then

$$
c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C \begin{cases} ||\mathbf{u}^{n+1/2}||_2||\sum_{i=0}^{n_0} a_i\mathbf{U}_h^{n-i}|||\mathbf{U}_h^{n+1/2}|_1, \\ ||\mathbf{u}^{n+1/2}||_1||\sum_{i=0}^{n_0} a_i\mathbf{U}_h^{n-i}||^{1/2}|\xi^n(\mathbf{U}_h)|_1^{1/2}|\mathbf{U}_h^{n+1/2}|_1 \end{cases}. \tag{4.69}
$$

Recall estimate (2.41)(a). Since $\mathbf{u} \in l^\infty(H^1)$ and $\mathbf{U}_h^{n+1/2} \in H_0^1$, we get

$$
c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C||\mathbf{u}^{n+1/2}||_1||\sum_{i=0}^{n_0} a_i\eta^{n-i}||_1|\mathbf{U}_h^{n+1/2}|_1. \tag{4.70}
$$

Recall identity (2.40). Then since $\xi^n(\mathbf{u}) \in V$. and $\mathbf{U}_h^{n+1/2} \in H_0^1$, we can rewrite the remaining nonlinear term:

$$
\begin{aligned}
c_h(\xi^n(\mathbf{u}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) &= (\xi^n(\mathbf{u}) \cdot \nabla\eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\
&\quad - c_h(\xi^n(\eta), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) + c_h(\xi^n(\mathbf{U}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}).
\end{aligned}
$$

Estimate (2.32)(a) gives

$$(\xi^n(\mathbf{u}) \cdot \nabla \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C || \sum_{i=0}^{n_0} a_i \mathbf{u}^{n-i} ||_1 || \eta^{n+1/2} ||_1 \ |\mathbf{U}_h^{n+1/2}|_1 \tag{4.71}$$

and similarly

$$c_h(\xi^n(\eta), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C || \sum_{i=0}^{n_0} a_i \eta^{n-i} ||_1 || \eta^{n+1/2} ||_1 \ |\mathbf{U}_h^{n+1/2}|_1. \tag{4.72}$$

Last, (2.41)(a)(b) together with the inverse estimate (2.4) imply

$$c_h(\xi^n(\mathbf{U}_h), \eta^{n+1/2}, \mathbf{U}_h^{n+1/2}) \leq C \left\{ \begin{array}{l} \dfrac{1}{\sqrt{h}} || \eta^{n+1/2} ||_1 || \displaystyle\sum_{i=0}^{n_0} a_i \mathbf{U}_h^{n-i} || |\mathbf{U}_h^{n+1/2}|_1, \\ || \eta^{n+1/2} ||_1 || \displaystyle\sum_{i=0}^{n_0} a_i \mathbf{U}_h^{n-i} ||^{1/2} |\xi^n(\mathbf{U}_h)|_1^{1/2} |\mathbf{U}_h^{n+1/2}|_1 \end{array} \right. . \tag{4.73}$$

Note that $|| \sum_{i=0}^{n_0} a_i \mathbf{v}_i || \leq \sum_{i=0}^{n_0} |a_i| \ ||\mathbf{v}_i||$. Absorb constants into C above. Then estimates (4.69), (4.70), (4.71), (4.72), and (4.73) imply (4.67), (4.68). □

Lemma 4.3.11. *Suppose that* $\mathbf{f} \in C^0([t^{n_0}, T]; W^{-1,2})$, $\mathbf{u} \in C^0(H^1)$, $\partial_t \mathbf{u} \in C^0([t^{n_0}, T]; W^{-1,2})$, *and* $p \in C^0([t^{n_0}, T]; L^2)$. *Then,*

$$R^{n+1}(\mathbf{U}_h^{n+1/2}) \leq \sqrt{C_{*,\Delta t}^{n+1}} |\mathbf{U}_h^{n+1/2}|_1 \tag{4.74}$$

where

$$\begin{aligned} C_{*,\Delta t}^{n+1} &:= C \Big(|| \mathbf{f}^{n+1/2} - \mathbf{f}(\cdot, t^{n+1/2}) ||_{-1}^2 + \dots \\ &\dots + || \mathbf{u} ||_{l^\infty(H^1)}^2 (|| \mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2}) ||_{0,3}^2 + || \xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2}) ||_{0,3}^2) + \dots \\ &\dots + || \partial_{\Delta t}^{n+1} \mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2}) ||_{-1}^2 + || p^{n+1/2} - p(\cdot, t^{n+1/2}) ||^2 \Big) . \end{aligned} \tag{4.75}$$

We assume that $|\mathbf{u}|_{l^\infty(H^1)} \geq C\nu$ *to simplify the expression.*

Proof. Duality of $W^{-1,2} \times H_0^1$ gives

$$(\partial_{\Delta t}^{n+1}\mathbf{u}^{n+1} - \partial_t \mathbf{u}(\cdot, t^{n+1/2}), \mathbf{U}_h^{n+1/2}) \le ||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2})||_{-1}|\mathbf{U}_h^{n+1/2}|_1 \tag{4.76}$$

$$(\mathbf{f}(\cdot, t^{n+1/2}) - \mathbf{f}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \le ||\mathbf{f}(\cdot, t^{n+1/2}) - \mathbf{f}^{n+1/2}||_{-1}|\mathbf{U}_h^{n+1/2}|_1. \tag{4.77}$$

Cauchy-Schwarz inequality (2.22) gives

$$\nu(\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \nabla\mathbf{U}_h^{n+1/2}) \le \nu|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})|_1 \ |\mathbf{U}_h^{n+1/2}|_1 \tag{4.78}$$

$$(p^{n+1/2} - p(\cdot, t^{n+1/2}), \nabla \cdot \mathbf{U}_h^{n+1/2}) \le \sqrt{d}||p^{n+1/2} - p(\cdot, t^{n+1/2})|| \ |\mathbf{U}_h^{n+1/2}|_1. \tag{4.79}$$

Rewrite the remaining nonlinear terms

$$\begin{aligned}
&(\xi^n(\mathbf{u}) \cdot \nabla\mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) - (\mathbf{u}(\cdot, t^{n+1/2}) \cdot \nabla\mathbf{u}(\cdot, t^{n+1/2}), \mathbf{U}_h^{n+1/2}) \\
&= ((\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})) \cdot \nabla\mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \\
&\qquad + (\mathbf{u}(\cdot, t^{n+1/2}) \cdot \nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \mathbf{U}_h^{n+1/2}).
\end{aligned}$$

Apply Hölder's (2.22) and Ladyzhenskaya's (2.24) inequalities give

$$((\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})) \cdot \nabla\mathbf{u}^{n+1/2}, \mathbf{U}_h^{n+1/2}) \le C||\mathbf{u}||_{l^\infty(H^1)}||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_{0,3}|\mathbf{U}_h^{n+1/2}|_1 \tag{4.80}$$

and similarly

$$\begin{aligned}
&(\mathbf{u}(\cdot, t^{n+1/2}) \cdot \nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \mathbf{U}_h^{n+1/2}) \\
&\le C||\mathbf{u}||_{l^\infty(H^1)}||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_{0,3}|\mathbf{U}_h^{n+1/2}|_1.
\end{aligned} \tag{4.81}$$

Estimates (4.76), (4.77), (4.78), (4.79), (4.80), and (4.81) imply (4.74) with (4.75). □

Bound each term on the RHS of (4.64) with (4.65), (4.66), (4.67), (4.74). Apply Young's inequality (2.21) to get

$$(||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2) + \nu\Delta t|\mathbf{U}_h^{n+1/2}|_1^2 \le C\nu^{-1}\Delta t(C_{*,h}^{n+1} + C_{*,\Delta t}^{n+1}) + C_2(\xi^n(\mathbf{U}_h)) \tag{4.82}$$

where

$$\begin{aligned} C_{*,h}^{n+1} &:= C(||\mathbf{u}||^2_{l^\infty(H^1)}\sum_{i=0}^{n_0}||\eta^{n-i}||_1^2 + \sum_{i=0}^{n_0}(||\mathbf{u}^{n-i}||_1^2 + ||\eta^{n-i}||_1^2))||\eta^{n+1/2}||_1^2 + \dots \\ &\dots + ||\partial_{\Delta t}^{n+1}\eta||_{-1}^2 + ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2) \end{aligned} \tag{4.83}$$

and

$$C_2(\xi^n(\mathbf{U}_h)) : = C\begin{cases} \nu^{-1}(||\mathbf{u}^{n+1/2}||_2^2 + \dfrac{1}{h}||\eta^{n+1/2}||_1^2)\displaystyle\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||^2|, \text{ or} \\ \nu^{-3}(||\mathbf{u}^{n+1/2}||_1^4 + ||\eta^{n+1/2}||_1^4)\displaystyle\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||^2 + C_{2,*}(\xi^n(\mathbf{U}_h)) \end{cases} \tag{4.84}$$

$$C_{2,*}(\xi^n(\mathbf{U}_h)) : = \frac{\nu\Delta t}{2(1+|\mathbf{a}|_2^2)(n_0+1)}|\xi^n(\mathbf{U}_h)|_1^2. \tag{4.85}$$

Absorb Young's constants into C above. Recall definition of n_0* in (4.35).

Lemma 4.3.12. *Suppose that the FE-space and $\phi_h \approx \phi$ satisfy Assumption 4.2.5. Fix $k \ge 0$, $k* \ge 0$, $s \ge -1$. Suppose that $\mathbf{f} \in C^0([t^{n_0*}, T]; W^{-1,2})$, $\mathbf{u} \in l^4(n_0*, N; H^1) \cap \in l^\infty(H^k)$, $\mathbf{u} \in l^2(H^{k+1})$, $\mathbf{u} \in C^0([t^{n_0*}, T]; H^{k*+1})$, $\partial_t\mathbf{u} \in C^0([t^{n_0*}, T]; W^{-1,2})$, $p \in l^2(n_0*, N; H^{s+1})$, and $p \in C^0([t^{n_0*}, T]; L^2)$. Then*

$$\begin{aligned} &||\mathbf{U}_h^N||^2 + \nu\Delta t\sum_{n=n_0*}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2 \\ &\quad \le C_{*,IC}(\mathbf{e}_u^{n_0}) + C_{*,fem}(k,s,k*) + \Delta t\sum_{n=n_0*}^{N-1}(\nu^{-1}C_{*,\Delta t}^{n+1} + ||\mathbf{U}_h^n||^2\sum_{i=0}^{i_1(n)}\kappa^{n+i}) \end{aligned} \tag{4.86}$$

where

$$\kappa^n := C\begin{cases} \nu^{-1}||\mathbf{u}^{n+1/2}||_2^2, & \textit{if } \mathbf{u} \in l^2(H^2) \\ \nu^{-3}||\mathbf{u}^{n+1/2}||_1^4, & \textit{else if } n_* = n_0 + 1 \end{cases} \tag{4.87}$$

and

$$C_{*,fem} := C(h^{2k}(\nu^{-1}||\mathbf{u}||^2_{l^\infty(H^1)}||\mathbf{u}||^2_{l^2(H^{k+1})} + ||\mathbf{u}||^2_{l^\infty(n_0,N;H^k)} + C_6(\mathbf{u}^{n_0})) + \dots$$
$$\dots + h^{2k*+4}\nu^{-1}||\partial_t \mathbf{u}(\cdot,t)||^2_{L^2(t^{n_0*},T;H^{k*+1})} + h^{2s+2}\nu^{-1}||p||^2_{l^2(n_0*,N;H^{s+1})}) \tag{4.88}$$

$$C_{*,IC} := ||\mathbf{e}_u^{n_0}||^2 + C\begin{cases} \nu\Delta t\sum_{i=0}^{n_0}|\mathbf{e}_u^{i+1/2}|_1^2 + \dots & \\ \quad\dots + \Delta t\sum_{i=0}^{n_0*+n_0-1}\kappa^i||\mathbf{e}_u||^2_{l^\infty(0,n_0*;L^2)} & \textit{if } n_0* = n_0+1, \\ \Delta t\sum_{i=0}^{n_0*+n_0-1}\kappa^i||\mathbf{e}_u||^2_{l^\infty(0,n_0-1;L^2)} & \textit{if } n_0* = n_0 \geq 1, \\ 0 & \textit{otherwise} \end{cases}. \tag{4.89}$$

$$C_6 := C\begin{cases} \Delta t\sum_{i=0}^{n_0*+n_0-1}\kappa^i||\mathbf{u}||^2_{l^\infty(0,n_0;H^k)} & \textit{if } n_0 \geq 1, \\ 0 & \textit{otherwise} \end{cases}. \tag{4.90}$$

Proof. The estimate for the elliptic projection error η (4.23) via Assumption 4.2.5 gives

$$||\eta^n||_1 \leq Ch^k||\mathbf{u}^n||_{k+1}. \tag{4.91}$$

Fix $k* \geq 0$. Then (4.23) along with (2.5) gives

$$||\partial^n_{\Delta t}\eta||^2_{-1} \leq Ch^{2k*+4}\Delta t^{-1}\int_{t^{n-1}}^{t^n}||\partial_t\mathbf{u}(\cdot,t)||^2_{k*+1}dt. \tag{4.92}$$

Estimate (2.3)(b) gives

$$\inf_{\tilde{q}_h\in Q_h}||p^{n+1/2} - \tilde{q}_h^{n+1/2}|| \leq Ch^{s+1}||p^{n+1/2}||_{s+1}. \tag{4.93}$$

Apply (4.91), (4.92), (4.93) to (4.82), (4.83), (4.84), (4.85). This proves $C^{n+1}_{*,h} \leq C^{n+1}_{**,h}$ where

$$(||\mathbf{U}_h^{n+1}||^2 - ||\mathbf{U}_h^n||^2) + \nu\Delta t|\mathbf{U}_h^{n+1/2}|_1^2 \leq \Delta t(C^{n+1}_{**,h} + \nu^{-1}C^{n+1}_{*,\Delta t}) + C_3(\xi^n(\mathbf{U}_h)) \tag{4.94}$$

and

$$C_{h,**}^{n+1} := C\nu^{-1}(h^{2k}||\mathbf{u}||_{l^\infty(H^1)}\sum_{i=-1}^{n_0}||\mathbf{u}^{n-i}||_{k+1}^2 + \dots$$

$$\dots + h^{2k*+4}\Delta t^{-1}||\partial_t \mathbf{u}(\cdot,t)||^2_{L^2(t^n,t^{n+1};H^{k*+1})} + h^{2s+2}||p^{n+1/2}||_{s+1}^2) \tag{4.95}$$

$$C_3(\xi^n(\mathbf{U}_h)) := C\begin{cases} \nu^{-1}||\mathbf{u}^{n+1/2}||_2^2\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||^2, \\ \nu^{-3}||\mathbf{u}^{n+1/2}||_1^4\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||^2 + C_{2,*}(\xi^n(\mathbf{U}_h)) \end{cases} \tag{4.96}$$

$$C_{3,*}(\xi^n(\mathbf{U}_h)) := \frac{\nu\Delta t}{2(1+|\mathbf{a}|^2)(n_0+1)}|\xi^n(\mathbf{U}_h)|_1^2. \tag{4.97}$$

To deal with (4.97), consider the case $\xi^n(\mathbf{v}) = a_0\mathbf{v}^{n-1/2} + a_1\mathbf{v}^{n-3/2} + \dots + \mathbf{v}^{n-n_0-1/2}$. From the change of indices identity (2.20), we obtain

$$\begin{aligned}\sum_{n=n_0+1}^{N-1}|\xi^n(\mathbf{U}_h)|_1^2 &\le \sum_{n=n_0+1}^{N-1}\sum_{i=0}^{n_0}(1+n_0)|a_i|^2|\mathbf{U}_h^{n-i-1/2}|_1^2 \\ &= (1+n_0)\sum_{n=1}^{N-1}|\mathbf{U}_h^{n-1/2}|_1^2\sum_{i=i_0(n)}^{i_1(n)}|a_i|^2 \le (1+|\mathbf{a}|_2^2)(1+n_0)\sum_{n=0}^{N-2}|\mathbf{U}_h^{n+1/2}|_1^2\end{aligned}$$

so that

$$\begin{aligned}&\nu\Delta t\sum_{n=n_0+1}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2 - \frac{\nu\Delta t}{2(1+|\mathbf{a}|_2^2)(1+n_0)}\sum_{n=n_0+1}^{N-1}|\xi^n(\mathbf{U}_h)|_1^2 \\ &\ge \nu\Delta t\sum_{n=n_0+1}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2 - \frac{\nu\Delta t}{2}\sum_{n=0}^{N-2}|\mathbf{U}_h^{n+1/2}|_1^2 \\ &\ge \nu\Delta t\sum_{n=n_0+1}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2 - \frac{\nu\Delta t}{2}\sum_{i=0}^{n_0}|\mathbf{U}_h^{i+1/2}|_1^2.\end{aligned} \tag{4.98}$$

Sum from $n = n_0*$ to $n = N-1$ in (4.85) so that $n_0* = n_0$ when $\xi^n(\mathbf{v}) = a_0\mathbf{v}^n + a_1\mathbf{v}^{n-1} + \dots + a_{n_0}\mathbf{v}^{n-n_0}$ or $n_0* = n_0+1$ when $\xi^n(\mathbf{v}) = a_0\mathbf{v}^{n-1/2} + a_1\mathbf{v}^{n-3/2} + \dots + a_{n_0}\mathbf{v}^{n-n_0-1/2}$. Apply Young's inequality (2.21), the change of and the indices identity (2.20) along with (4.98) to bound (4.96):

$$\begin{aligned}&||\mathbf{U}_u^N||^2 + \nu\Delta t\sum_{n=n_0*}^{N-1}|\mathbf{U}_h^{n+1/2}|_1^2 \\ &\le ||\mathbf{U}_h^{n_0}||^2 + C_4(\mathbf{U}^{n_0}) + \Delta t\sum_{n=n_0*}^{N-1}(C_{**,h}^{n+1} + \nu^{-1}C_{*,\Delta t}^{n+1} + ||\mathbf{U}_h^n||^2\sum_{i=0}^{i_1(n)}\kappa^{n+i})\end{aligned} \tag{4.99}$$

and

$$C_4(\mathbf{U}_h^{n_0}) := C \begin{cases} \nu\Delta t \sum_{i=0}^{n_0} |\mathbf{U}_h^{i+1/2}|_1^2 + \dots \\ \quad \dots + \Delta t \sum_{i=0}^{n_0*+n_0-1} \kappa^i ||\mathbf{U}_h||^2_{l^\infty(0,n_0;L^2)} & \text{if } n_0* = n_0+1, \\ \Delta t \sum_{i=0}^{n_0*+n_0-1} \kappa^i ||\mathbf{U}_h||^2_{l^\infty(0,n_0;L^2)} & \text{if } n_0* = n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.100}$$

where κ^n is given in (4.87). Consider (4.99). Apply triangle inequality $||\mathbf{U}_h|| \leq ||\mathbf{e}_u|| + ||\eta||$ to appropriate terms in $C_4(\mathbf{U}_h^{n_0})$ (4.100). Apply estimate for elliptic projection error η (4.23) via Assumption 4.2.5 and absorb appropriate term into $C_{**,h}^{n+1}$ (4.95) to get $C_{*,fem}(k,s,k*)$ (4.88) and $C_6(\mathbf{u}^{n_0})$ (4.90). We assumed that $\nu \leq ||\mathbf{u}||^2_{l^\infty(H^1)}$. The remaining terms involving $||\mathbf{e}_u||$ combine to give $C_{*,IC}(\mathbf{e}^{n_0})$ (4.89) to prove (4.86). □

Apply discrete Gronwall Lemma 2.4.6 to (4.86) and triangle inequality $||\mathbf{e}_u|| \leq ||\mathbf{U}_h|| + ||\eta||$ (absorb η terms into $C_{*,fem}(k,s,k*)$) and simplify to get

$$\begin{aligned} &||\mathbf{e}_u^N||^2 + \nu\Delta t \sum_{n=n_0*}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2 \\ &\quad \leq G^N(C_{*,IC}(\mathbf{e}_u^{n_0}) + C_{*,fem}(k,s,k*) + \nu^{-1}\Delta t \sum_{n=n_0*}^{N-1} C_{*,\Delta t}^{n+1}) \end{aligned} \tag{4.101}$$

where

$$G^N := C\exp(\Delta t \sum_{n=n_0*}^{N-1} \kappa^n). \tag{4.102}$$

We assumed that $||\eta^N||^2 + \nu\Delta t \sum_{n=n_0*}^{N-1} |\eta^{n+1/2}|_1^2 \leq G^N C_{*,fem}(k,s,k*)$. It remains to bound $\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1}$. Consider two cases: the first with minimal regularity (boundedness) for Theorem 4.3.1 and the second for optimal convergence rate (regularity matching the FE and CN approximation degree) for Theorem 4.3.3.

Case 1 (Theorem 4.3.1): Suppose that the regularity of Lemma 4.3.12 is satisfied for $k = 0$, $k* = 0$, and $s = -1$. If $\kappa^n = \nu^{-1}||\mathbf{u}^{n+1/2}||_2^2$, suppose in addition that $\mathbf{u} \in l^2(H^2)$. Then inspecting (4.88), (4.89), and (4.102), we verify that

$$G^N(||\mathbf{e}_u^{n_0}||^2 + C_5(\mathbf{e}_u^{n_0}) + C_{*,fem}) < \infty, \quad \text{as } h\ \Delta t \to 0. \tag{4.103}$$

Suppose further that $\mathbf{u} \in C^1([t^{n_0}, T]; W^{-1,2})$ so that $\partial_t^{(2)}\mathbf{u} \in L^2(t^{n_0}, T; W^{-1,2})$. Then, Taylor expansion with integral remainder gives

$$\Delta t \sum_{n=n_0+1}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2})||_{-1}^2 \le C||\partial_t^{(2)}\mathbf{u}||^2_{L^2(t^{n_0},T;W^{-1,2})}.$$

If $\mathbf{f} \in C^0([t^{n_0}, T]; W^{-1,2})$, $\mathbf{u} \in C^0(H^1)$, $p \in C^0([t^{n_0}, T]; L_0^2)$, similar estimates hold for the remaining terms in (4.75) to ensure

$$\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1} \le C_{*,cn} < \infty, \quad \text{as } h\ \Delta t \to 0 \tag{4.104}$$

where $C_{*,cn}$ is given in (4.52) by

$$\begin{aligned} C_{*,cn} := C(&||\partial_t \mathbf{f}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\mathbf{u}||_{l^\infty(H^1)}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^1)} + \dots \\ &\dots + ||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\partial_t p||_{L^2(t^{n_0},T;L^2)}). \end{aligned}$$

Estimate (4.36) so that $\mathbf{u}_h \in l^\infty(L^2) \cap l^2(H^1)$ via the boundedness ensured in (4.103), (4.104) under the regularity detailed above.

Case 2 (Theorem 4.3.3): Suppose that the regularity of Lemma 4.3.12 is satisfied for $k > 0$, $k* \ge 0$, and $s > -1$. As a consequence, the estimates are bounded as h, $\Delta t \to 0$ when $\kappa^n = \nu^{-1}||\mathbf{u}^{n+1/2}||_2^2$. Suppose that $\partial_t^{(2)}\mathbf{u} \in L^2(H^1)$, $\partial_t^{(3)}\mathbf{u} \in L^2(t^{n_0}, T; W^{-1,2})$, $\partial_t^{(2)} p \in L^2(t^{n_0}, T; L^2)$, and $\partial_t^{(2)}\mathbf{f} \in L^2(t^{n_0}, T; W^{-1,2})$. Then, (2.6), (2.7), and (2.10) (via Assumption 2.2.1) gives

$$(\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1})^{1/2} \le \Delta t^2 C_{cn}$$

where C_{cn} is given in (4.56) by

$$C_{cn} := C(||\partial_t^{(2)}\mathbf{f}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\mathbf{u}||_{l^\infty(H^1)}||\partial_t^{(2)}\mathbf{u}||_{L^2(H^1)} + \dots$$
$$\dots + ||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0},T;W^{-1,2})} + ||\partial_t^{(2)}p||_{L^2(t^{n_0},T;L^2)}).$$

Estimate (4.39) so that $\mathbf{u}_h \to \mathbf{u}$ in $l^\infty(L^2) \cap l^2(H^1)$ with rate $\mathcal{O}(h^k + h^{s+1} + \Delta t^2)$ under the regularity detailed above. □

4.3.3 Proof of $\mathbf{u}_h \to \mathbf{u}$ in $l^\infty(H^1)$, $\partial_{\Delta t}(\mathbf{u}_h - \mathbf{u}) \to 0$ in $l^2(L^2)$

Proof. Theorems 4.3.4, 4.3.6

Fix $n = n_0,\ n_0+1, \dots, N-1$. Set $(\tilde{\mathbf{v}}_h^n, \tilde{q}_h^n) = P_s(\mathbf{u}^n, p^n)$ defined by (4.20) in (4.49). Set $\mathbf{v} = \Delta t^{-1}(\mathbf{U}_h^{n+1} - \mathbf{U}_h^n) \in V_h$ in (4.50). Notice that

$$\begin{aligned} R_h(\mathbf{v})^{n+1} &:= c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}) - c_h(\xi^n(\mathbf{u}), \mathbf{u}^{n+1/2}, \mathbf{v}) \\ &= c_h(\xi^n(\mathbf{e}_u), \mathbf{u}^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}), \mathbf{e}_u^{n+1/2}, \mathbf{v}) - 2c_h(\xi^n(\mathbf{e}_u), \mathbf{e}_u^{n+1/2}, \mathbf{v}). \end{aligned}$$

Then

$$\begin{aligned} &||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \frac{\nu}{2\Delta t}(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2) \\ &\quad = -R^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) - R_h^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) + (\partial_{\Delta t}^{n+1}\eta, \partial_{\Delta t}^{n+1}\mathbf{U}_h). \end{aligned} \tag{4.105}$$

The Cauchy-Schwarz inequality (2.22) implies

$$(\partial_{\Delta t}^{n+1}\eta, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \le ||\partial_{\Delta t}^{n+1}\eta||\ ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. \tag{4.106}$$

Note that $\partial_{\Delta t}^{n+1}\mathbf{U}_h|_{\partial\Omega} = 0$ since $\mathbf{U}_h^n|_{\partial\Omega} = 0$ for all n. The remaining terms in (4.105) are bounded in the next 2 lemmas.

Lemma 4.3.13. *Suppose that the FE space satisfies Assumption 2.1.1 and* $\mathbf{u} \in l^\infty(H^2)$. *Then*

$$\begin{aligned} |R_h^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h)| \le C(h^{-1/2}\sum_{i=0}^{n_0}(||\eta^{n-i}||_1 + ||\mathbf{U}_h^{n-i}||_1)||\mathbf{e}_u^{n+1/2}||_1||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| + \dots \\ \dots + (||\mathbf{u}||_{l^\infty(H^2)}||\mathbf{e}_u^{n+1/2}||_1 + ||\mathbf{u}^{n+1/2}||_2\sum_{i=0}^{n_0}(||\eta^{n-i}||_1 + ||\mathbf{U}_h^{n-i}||_1)||\partial_{\Delta t}^{n+1}\mathbf{U}_h||)). \end{aligned} \tag{4.107}$$

Proof. Recall that $\xi^n(\mathbf{v}) = a_0\mathbf{v}^n + \dots a_{n_0}\mathbf{v}^{n-n_0}$. Consider the decomposition

$$c_h(\xi^n(\mathbf{e}_u), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) = c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) - c_h(\xi^n(\eta), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h).$$

Note that $\partial_{\Delta t}^{n+1}\mathbf{U}_h|_{\partial\Omega} = 0$ since $\mathbf{U}_h^n|_{\partial\Omega} = 0$ for all n. Apply estimate (2.42)(d) along with $\mathbf{u} \in l^\infty(H^2)$ to get

$$\begin{aligned}
&|c_h(\xi^n(\mathbf{u}), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) + c_h(\xi^n(\mathbf{e}_u), \mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)| \\
&\quad \leq C(||\mathbf{u}||_{l^\infty(H^2)}||\mathbf{e}_u^{n+1/2}||_1 + \dots \\
&\quad \dots + (||\sum_{i=0}^{n_0} a_i\eta^{n-i}||_1 + ||\sum_{i=0}^{n_0} a_i\mathbf{U}_h^{n-i}||_1)||\mathbf{u}^{n+1/2}||_2)||\partial_{\Delta t}^{n+1}\mathbf{U}_h||.
\end{aligned} \tag{4.108}$$

Consider the decomposition

$$c_h(\xi^n(\mathbf{e}_u), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) = c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) - c_h(\xi^n(\eta), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h).$$

Apply estimate (2.42)(d) along with (2.4) via Assumption 2.1.1 so that

$$|c_h(\xi^n(\eta), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq Ch^{-1/2}||\sum_{i=0}^{n_0} a_i\eta^{n-i}||_1||\mathbf{e}_u^{n+1/2}||_1||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \tag{4.109}$$

and

$$|c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)| \leq C||\sum_{i=0}^{n_0} a_i\mathbf{U}_h^{n-i}||_1 h^{-1/2}||\mathbf{e}_u^{n+1/2}||_1||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. \tag{4.110}$$

Note that $||\sum_{i=0}^{n_0} a_i\mathbf{v}_i|| \leq \sum_{i=0}^{n_0} |a_i|\ ||\mathbf{v}_i||$. Absorb constants into C above. Estimates (4.108), (4.109) (4.110) prove (4.107). □

Lemma 4.3.14. *Suppose that* $\mathbf{f} \in C^0([t^{n_0}, T]; L^2)$, $\mathbf{u} \in C^0(H^2)$, $\partial_t\mathbf{u} \in C^0([t^{n_0}, T]; L^2)$, *and* $p \in C^0([t^{n_0}, T]; H^1)$. *Then*

$$R^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) \leq \sqrt{C_{*,\Delta t}^{n+1}}||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \tag{4.111}$$

where

$$\begin{aligned}
C_{*,\Delta t}^{n+1} &:= C(||\mathbf{f}(\cdot, t^{n+1/2}) - \mathbf{f}^{n+1/2}||^2 + \nu^2||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_2^2 + \dots \\
&\dots + ||\mathbf{u}||_{l^\infty(H^2)}^2(||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_1^2 + ||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_1^2) + \dots \\
&\dots + ||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t\mathbf{u}(\cdot, t^{n+1/2})||^2 + ||p^{n+1/2} - p(\cdot, t^{n+1/2})||_1^2).
\end{aligned} \tag{4.112}$$

Proof. First, application of Cauchy-Schwarz inequality (2.22) gives

$$(\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t\mathbf{u}(\cdot, t^{n+1/2}), \partial_{\Delta t}^{n+1}\mathbf{U}_h) \leq ||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t\mathbf{u}(\cdot, t^{n+1/2})||\ ||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \tag{4.113}$$

$$(\mathbf{f}(\cdot, t^{n+1/2}) - \mathbf{f}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \leq ||\mathbf{f}^{n+1/2} - \mathbf{f}(\cdot, t^{n+1/2})||\ ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. \tag{4.114}$$

Moreover, (2.22) and $\mathbf{U}_h|_{\partial\Omega} = 0$ along with $\mathbf{u} \in C^0(n_0, N; H^2)$ gives

$$\nu(\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \nabla(\partial_{\Delta t}^{n+1}\mathbf{U}_h)) \leq \nu||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_2||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| \tag{4.115}$$

and with $p \in C^0(n_0, N; H^1)$ gives

$$(p^{n+1/2} - p(\cdot, t^{n+1/2}), \nabla\cdot(\partial_{\Delta t}^{n+1}\mathbf{U}_h)) \leq ||p^{n+1/2} - p(\cdot, t^{n+1/2})||_1||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. \tag{4.116}$$

Rewrite the convective terms

$$\begin{aligned}
&(\xi^n(\mathbf{u})\cdot\nabla\mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) - (\mathbf{u}(\cdot, t^{n+1/2})\cdot\nabla\mathbf{u}(\cdot, t^{n+1/2}), \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\
&= ((\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2}))\cdot\nabla\mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\
&\quad + (\mathbf{u}(\cdot, t^{n+1/2})\cdot\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \partial_{\Delta t}^{n+1}\mathbf{U}_h).
\end{aligned}$$

Recall the Sobolev embedding (2.25). Then estimates (2.32)(d), (2.32)(e) along with $\mathbf{u} \in L^\infty(H^2)$ give

$$\begin{aligned}
&((\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2}))\cdot\nabla\mathbf{u}^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\
&\quad \leq C||\mathbf{u}||_{L^\infty(H^2)}||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_1||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. && (4.117) \\
&(\mathbf{u}(\cdot, t^{n+1/2})\cdot\nabla(\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})), \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\
&\quad \leq C||\mathbf{u}||_{L^\infty(H^2)}||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_1||\partial_{\Delta t}^{n+1}\mathbf{U}_h||. && (4.118)
\end{aligned}$$

Therefore, estimates (4.113), (4.114), (4.115), (4.116), (4.117), and (4.118) imply (4.111), (4.112). □

We bound each term on the RHS of (4.105) with (4.106), (4.107), (4.111). Apply Young's inequality (2.21) to get

$$\Delta t||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2) \leq C\nu^{-1}\Delta t(C_{*,h}^{n+1} + C_{*,\Delta t}^{n+1}) + C_2(\mathbf{U}_h^{n+1/2}) \tag{4.119}$$

where

$$\begin{aligned} C_{*,h}^{n+1} := C(||\mathbf{u}||_{l^\infty(H^2)}^2||\mathbf{e}_u^{n+1/2}||_1^2 + ||\mathbf{u}^{n+1/2}||_2^2\sum_{i=0}^{n_0}||\eta^{n-i}||_1^2 + \dots \\ \dots + \sum_{i=0}^{n_0}(h^{-1}||\eta^{n-i}||_1^2)||\mathbf{e}_u^{n+1/2}||_1^2 + ||\partial_{\Delta t}^{n+1}\eta||^2) \end{aligned} \tag{4.120}$$

and

$$C_2(\xi^n(\mathbf{U}_h)) := C(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}||\mathbf{e}_u^{n+1/2}||_1^2)\sum_{i=0}^{n_0}||\mathbf{U}_h^{n-i}||_1^2. \tag{4.121}$$

Lemma 4.3.15. *Suppose that the FE-space together with and $\phi_h \approx \phi$ satisfies Assumptions 4.2.5. Fix $k \geq 0$, $k* \geq 0$, $s \geq -1$, $s* \geq -1$. Suppose that $\mathbf{f} \in C^0([t^{n_0}, T]; L^2)$, $\mathbf{u} \in l^2(n_0, N; H^2)$, $\mathbf{u} \in (l^2 \cap l^\infty)(H^{k+1})$, $\mathbf{u} \in C^0([t^{n_0*}, T]; H^{k*+1})$, $\partial_t\mathbf{u} \in C^0([t^{n_0*}, T]; L^2)$, $p \in C^0([t^{n_0*}, T]; H^1)$, $p \in C^0(H^{s*+1})$ $p \in (l^2 \cap l^\infty)(H^{s+1})$. Then*

$$\begin{aligned} &\Delta t\sum_{n=n_0}^{N-1}||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu|\mathbf{U}_h^N|_1^2 \\ &\leq F_{*,IC}(\mathbf{e}_u^{n_0}) + F_{*,fem}(k, s, k*, s*) + \Delta t\sum_{n=n_0}^{N-1}(C_{*,\Delta t}^{n+1} + \nu|\mathbf{U}_h^n|_1^2\sum_{i=0}^{i_1(n)}\kappa^{n+i}) \end{aligned} \tag{4.122}$$

where

$$\kappa^n := C\nu^{-1}(||\mathbf{u}^{n+1/2}||_2^2 + h^{-1}||\mathbf{e}_u^{n+1/2}||_1^2) \tag{4.123}$$

and

$$
\begin{aligned}
F_{*,fem} := & \, C||\mathbf{u}||^2_{l^\infty(H^2)}(h^{2k}||\mathbf{u}||^2_{l^2(H^{k+1})} + \nu^{-2}h^{2s+2}||p||^2_{l^2(H^{s+1})}) \\
& + C(h^{2k*+2}||\partial_t \mathbf{u}(\cdot,t)||^2_{L^2(t^{n_0},T;H^{k*+1})} + \nu^{-2}h^{2s*+4}||\partial_t p(\cdot,t)||^2_{L^2(t^{n_0},T;H^{s*+1})}) \\
& + C(||\mathbf{u}||^2_{l^\infty(H^2)} + h(||\mathbf{u}||^2_{l^\infty(H^2)} + \nu^{-2}||p||^2_{l^\infty(H^1)}))\Delta t \sum_{n=n_0}^{N-1} ||\mathbf{e}_u^{n+1/2}||_1^2 \\
& + \Delta t \sum_{i=0}^{2n_0-1} \kappa^i (h^{2k+2}||\mathbf{u}||^2_{l^\infty(0,n_0-1;H^{k+1})} + \nu^{-2}h^{2s+4}||p||^2_{l^\infty(0,n_0-1;H^{s+1})})
\end{aligned}
\tag{4.124}
$$

$$
F_{*,IC} := |\mathbf{e}_u^{n_0}|_1^2 + C \begin{cases} \Delta t \displaystyle\sum_{i=0}^{2n_0-1} \kappa^i ||\nabla \mathbf{e}_u||^2_{l^\infty(0,n_0-1;L^2)} & \textit{if } n_0 \geq 1, \\ 0 & \textit{otherwise} \end{cases} . \tag{4.125}
$$

Proof. The estimate for the Stokes projection error η (4.24) via Assumption 4.2.5 gives

$$
||\eta^n||_1 \leq C(h^k||\mathbf{u}^n||_{k+1} + \nu^{-1}h^{s+1}||p^n||_{s+1}). \tag{4.126}
$$

Fix $k* \geq 0$, $s* \geq -1$. Then (4.24) along with (2.5) gives

$$
||\partial^n_{\Delta t}\eta||^2_{-1} \leq Ch^2\Delta t^{-1}\int_{t^{n-1}}^{t^n}(h^{2k*}||\partial_t\mathbf{u}(\cdot,t)||^2_{k*+1} + \nu^{-2}h^{2s*+2}||\partial_t p(\cdot,t)||^2_{k*+1})dt. \tag{4.127}
$$

Apply (4.126), (4.127), to (4.119), (4.120), (4.121). This proves $C^{n+1}_{*,h} \leq C^{n+1}_{**,h}$ where

$$
\Delta t||\partial^{n+1}_{\Delta t}\mathbf{U}_h||^2 + \nu(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2) \leq \Delta t(C^{n+1}_{**,h} + C^{n+1}_{*,\Delta t}) + C_2(\xi^n(\mathbf{U})) \tag{4.128}
$$

and

$$
\begin{aligned}
C^{n+1}_{h,**} := & \, C(||\mathbf{u}||^2_{l^\infty(H^2)}||\mathbf{e}_u^{n+1/2}||_1^2 + \ldots \\
& \ldots + ||\mathbf{u}||^2_{l^\infty(H^2)}\sum_{i=0}^{n_0}(h^{2k}||\mathbf{u}^{n-i}||^2_{k+1} + \nu^{-2}h^{2s+2}||p^{n-i}||^2_{s+1}) + \ldots \\
& \ldots + \sum_{i=0}^{n_0}(h^{2k-1}||\mathbf{u}^{n-i}||^2_{k+1} + \nu^{-2}h^{2s+1}||p^{n-i}||^2_{s+1})||\mathbf{e}_u^{n+1/2}||_1^2 + \ldots \\
& \ldots + h^{2k*+2}\Delta t^{-1}||\partial_t\mathbf{u}(\cdot,t)||^2_{L^2(t^n,t^{n+1};H^{k*+1})} + \ldots \\
& \ldots + \nu^{-2}h^{2s*+4}\Delta t^{-1}||\partial_t p(\cdot,t)||^2_{L^2(t^n,t^{n+1};H^{s*+1})}).
\end{aligned}
\tag{4.129}
$$

Sum from $n = n_0$ to $n = N-1$ in (4.128) Apply Young's inequality (2.21), the change of and the indices identity (2.20). Simplify to get

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \nu|\mathbf{U}_h^N|_1^2$$
$$\leq \nu|\mathbf{U}_h^{n_0}|_1^2 + C_3(\mathbf{U}^{n_0}) + \Delta t \sum_{n=n_0}^{N-1}(C_{**,h}^{n+1} + C_{*,\Delta t}^{n+1} + \nu|\mathbf{U}_h^n|_1^2 \sum_{i=0}^{i_1(n)} \kappa^{n+i}) \tag{4.130}$$

and

$$C_3(\mathbf{U}_h^{n_0}) := C \begin{cases} \Delta t \sum_{i=0}^{2n_0-1} \kappa^i ||\nabla \mathbf{U}_h||^2_{l^\infty(0,n_0-1;L^2)} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.131}$$

where κ^n is given in (4.123). Consider (4.130). Apply triangle inequality $||\mathbf{U}_h|| \leq ||\mathbf{e}_u|| + ||\eta||$ to appropriate terms in $C_3(\mathbf{U}_h^{n_0})$. Apply estimate for Stokes projection error η (4.24) via Assumption 4.2.5 and absorb appropriate term into $C_{**,h}^{n+1}$ (4.129) to get the term $F_{*fem}(k, s, k*, s*)$ (4.124). We assume that $\nu \leq ||\mathbf{u}||^2_{l^\infty(H^2)}$. The remaining terms involving $||\mathbf{e}_u||$ combine to give $F_{*,IC}(\mathbf{e}^{n_0})$ in (4.125) and proves (4.122). $\square$

Apply discrete Gronwall Lemma 2.4.6 to (4.122), triangle inequality $||\mathbf{e}_u|| \leq ||\mathbf{U}_h|| + ||\eta||$ (absorb η terms into $C_{*,fem}(k, s, k*, s*)$), and simplify to get

$$\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{e}_u||^2 + \nu|\mathbf{e}_u^N|_1^2$$
$$\leq G^N(F_{*,IC}(\mathbf{e}_u^{n_0}) + F_{*,fem}(k, s, k*, s*) + \Delta t \sum_{n=n_0}^{N-1} C_{*,\Delta t}^{n+1}) \tag{4.132}$$

where

$$G^N := C \exp(\Delta t \sum_{n=n_0}^{N-1} \kappa^n). \tag{4.133}$$

We assumed that $|\eta^N|_1^2 + \nu\Delta t \sum_{n=n_0}^{N-1} ||\partial_{\Delta t}^{n+1}\eta||^2 \leq G^N C_{*,fem}(k, s, k*s, *)$. It remains to bound $\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1}$. We consider two cases: the first with minimal regularity (boundedness) for Theorem 4.3.4 and the second for optimal convergence rate (regularity to match FE and CN approximation degree) for Theorem 4.3.6.

Case 1 (Theorem 4.3.4): Suppose that the regularity of Lemma 4.3.15 is satisfied for $k = 0$, $k* = 0$, and $s = -1$, $s* = -1$. Inspect (4.124), (4.125), and (4.133) to verify

$$G^N(||\mathbf{e}_u^{n_0}||^2 + F_{*,IC}(\mathbf{e}_u^{n_0}) + C_{*,fem}) < \infty, \quad \text{as } h,\ \Delta t \to 0 \tag{4.134}$$

as long as

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0.$$

Suppose further that $\mathbf{u} \in C^1([t^{n_0}, T]; L^2)$ so that $\partial_t^{(2)}\mathbf{u} \in L^2(t^{n_0}, T; L^2)$. Then, Taylor expansion with integral remainder gives

$$\Delta t \sum_{n=n_0+1}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t \mathbf{u}(\cdot, t^{n+1/2})||^2 \leq C||\partial_t^{(2)}\mathbf{u}||^2_{L^2(t^{n_0},T;L^2)}.$$

If $\mathbf{f} \in C^0([t^{n_0}, T]; L^2)$, $\mathbf{u} \in C^0(H^2)$, $p \in C^0([t^{n_0}, T]; H^1)$, similar estimates hold for the remaining terms in (4.112) to ensure

$$\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1} \leq C_{*,cn} < \infty, \quad \text{as } h\ \Delta t \to 0 \tag{4.135}$$

where $F_{*,cn}$ is given in (4.52) by

$$\begin{aligned} F_{*,cn} := C(&||\partial_t \mathbf{f}||_{L^2(t^{n_0},T;L^2)} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t \mathbf{u}||_{L^2(t^{n_0},T;H^1)} + \dots \\ &\dots + ||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0},T;L^2)} + ||\partial_t p||_{L^2(t^{n_0},T;H^1)}). \end{aligned}$$

Estimate (4.39) follows since $\mathbf{u}_h \in l^\infty(H^1)$ and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$ under the regularity constraints that ensure (4.134), (4.135).

Case 2 (Theorem 4.3.6): Suppose that the regularity of Lemma 4.3.15 is satisfied for $k > 0$, $k* \geq 0$, $s > -1$, and $s* \geq -1$. Then as long as $\Delta t \leq h^{1/4}$, Theorem 4.3.3 ensures

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0.$$

Suppose that $\partial_t^{(2)}\mathbf{u} \in L^2(H^1)$, $\partial_t^{(3)}\mathbf{u} \in L^2(t^{n_0}, T; L^2)$, $\partial_t^{(2)} p \in L^2(t^{n_0}, T; H^1)$, and $\partial_t^{(2)}\mathbf{f} \in L^2(t^{n_0}, T; L^2)$. Then, (2.6), (2.7), and (2.10) (via Assumption 2.2.1) give

$$(\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1})^{1/2} \leq \Delta t^2 F_{cn}$$

where F_{cn} is given in (4.56) by

$$F_{cn} := C(||\partial_t^{(2)}\mathbf{f}||_{L^2(t^{n_0},T;L^2)} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(H^1)} + \ldots$$
$$\ldots + ||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0},T;L^2)} + ||\partial_t^{(2)} p||_{L^2(t^{n_0},T;H^1)}).$$

Estimate (4.39) follows so that $\mathbf{u}_h \to \mathbf{u}$ in $l^\infty(H^1)$ and $\partial_{\Delta t}(\mathbf{u}_h - \mathbf{u}) \to 0$ in $l^2(L^2)$ with rate $\mathcal{O}(h^k + h^{s+1} + \Delta t^2)$ under the specified regularity constraints as long as $\Delta t \leq h^{1/4}$. $\square$

4.3.4 Another proof of $\mathbf{u}_h \to \mathbf{u}$ in $l^\infty(H^1)$, $\partial_{\Delta t}(\mathbf{u}_h - \mathbf{u}) \to 0$ in $l^2(L^2)$

Proof. Theorem 4.3.4

Fix $n = n_0,\, n_0 + 1, \ldots, N-1$, $n_0 \geq 1$. Suppose that we consider only restrictions of the linearization $\xi^n(\mathbf{v}) = a_0\mathbf{v}^n + a_1\mathbf{v}^{n-1} + \ldots + a_{n_0*}\mathbf{v}^{n-n_0*}$ so that $n_0* = n_0 + 1$ and

$$\xi^n(\mathbf{v}) = a_0\mathbf{v}^{n-1/2} + a_1\mathbf{v}^{n-3/2} + \ldots + a_{n_0}\mathbf{v}^{n-n_0-1/2}.$$

We follow the proof in Section 4.3.3 closely. Set $(\tilde{\mathbf{v}}_h^n, \tilde{q}_h^n) = P_s(\mathbf{u}^n, p^n)$ defined by (4.20) in (4.49). Pick $\mathbf{v} = \partial_{\Delta t}\mathbf{U}_h \in V_h$ in (4.50). Then

$$\begin{aligned} &||\partial_{\Delta t}^{n+1}\mathbf{U}_h||^2 + \frac{\nu}{2\Delta t}(|\mathbf{U}_h^{n+1}|_1^2 - |\mathbf{U}_h^n|_1^2) \\ &\quad = -R^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) - R_h^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h) + (\partial_{\Delta t}^{n+1}\eta, \partial_{\Delta t}^{n+1}\mathbf{U}_h). \end{aligned} \tag{4.136}$$

The estimates in Section 4.3.3 for the proof of Theorems 4.3.4, 4.3.6 remain largely unchanged except for the term exchanging all indices $n-i$ with $n-i-1/2$ (for the alternate extrapolation considered here) and the handling of the term $c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h)$ bounded previously in (4.110). We consider the decomposition

$$\partial_{\Delta t}^{n+1}\mathbf{U}_h = 2\Delta t^{-1}(\mathbf{U}_h^{n+1/2} - \mathbf{U}_h^n)$$

so that

$$\begin{aligned} &c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\ &\quad = 2\Delta t^{-1} c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \mathbf{U}_h^{n+1/2}) - 2\Delta t^{-1} c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \mathbf{U}_h^n). \end{aligned}$$

Recall that $\mathbf{U}_h = \mathbf{e}_u + \eta$ and identity (2.35) so that

$$\begin{aligned} &c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \\ &\quad = 2\Delta t^{-1} c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \eta^{n+1/2}) - 2\Delta t^{-1} c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \mathbf{U}_h^n). \end{aligned}$$

Then estimate (2.41)(a) along with the previous estimate (4.110) gives

$$c_h(\xi^n(\mathbf{U}_h), \mathbf{e}_u^{n+1/2}, \partial_{\Delta t}^{n+1}\mathbf{U}_h) \leq C \begin{cases} ||\sum_{i=0}^{n_0} a_i \mathbf{U}_h^{n-i-1/2}||_1 h^{-1/2} ||\mathbf{e}_u^{n+1/2}||_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||, \\ \Delta t^{-1}(||\sum_{i=0}^{n_0} a_i \mathbf{U}_h^{n-i-1/2}||_1 ||\eta^{n+1/2}||_1 + \dots \\ \quad \dots + ||\sum_{i=0}^{n_0} a_i \mathbf{U}_h^{n-i-1/2}||_1 ||\mathbf{U}_h^n||_1) ||\mathbf{e}_u^{n+1/2}||_1 \end{cases}. \quad (4.137)$$

Estimate (4.107) is replace by

$$\begin{aligned} |R_h^{n+1}(\partial_{\Delta t}^{n+1}\mathbf{U}_h)| &\leq C(+G_1(\mathbf{U}^{n-1/2}) + ||\mathbf{u}^{n+1/2}||_2 \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i-1/2}||_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| + \dots \\ &\dots + h^{-1/2} \sum_{i=0}^{n_0} ||\eta^{n-i}||_1 ||\mathbf{e}_u^{n+1/2}||_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h|| + \dots \\ &\dots + (||\mathbf{u}||_{l^\infty(H^2)} ||\mathbf{e}_u^{n+1/2}||_1 + ||\mathbf{u}^{n+1/2}||_2 \sum_{i=0}^{n_0} ||\eta^{n-i}||_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||)) \end{aligned} \quad (4.138)$$

where

$$G_1(\mathbf{U}^{n-1/2}) := C \begin{cases} \sum_{i=0}^{n_0} ||\mathbf{U}_h^{n-i-1/2}||_1 h^{-1/2} ||\mathbf{e}_u^{n+1/2}||_1 ||\partial_{\Delta t}^{n+1}\mathbf{U}_h||, \\ \sum_{i=0}^{n_0} (||\mathbf{U}_h^{n-i-1/2}||_1 ||\eta^{n+1/2}||_1 + ||\mathbf{U}_h^{n-i-1/2}||_1 ||\mathbf{U}_h^n||_1) \Delta t^{-1} ||\mathbf{e}_u^{n+1/2}||_1 \end{cases}.$$

Note that $||\mathbf{U}_h^{n-i-1/2}||_1 \leq ||\mathbf{e}_u^{n-i-1/2}||_1 + ||\eta^{n-i-1/2}||_1$ in (4.3.4)(b). We have an estimate from Theorems 4.3.4, 4.3.6 that provides an estimate for $\Delta t \sum_n ||\mathbf{e}_u^{n-i-1/2}||_1^2$. This is exactly where the alternate extrapolation is required. Following the proof of Theorems 4.3.4, 4.3.6, we arrive at the following:

Lemma 4.3.16. *Suppose that the FE-space together with and $\phi_h \approx \phi$ satisfies Assumptions 4.2.5. Fix $k \geq 0$, $k* \geq 0$, $s \geq -1$, $s* \geq -1$. Suppose that $\mathbf{f} \in C^0([t^{n_0}, T]; L^2)$, $\mathbf{u} \in l^2(n_0, N; H^2)$, $\mathbf{u} \in (l^2 \cap l^\infty)(H^{k+1})$, $\mathbf{u} \in C^0([t^{n_0*}, T]; H^{k*+1})$, $\partial_t \mathbf{u} \in C^0([t^{n_0*}, T]; L^2)$, $p \in C^0([t^{n_0*}, T]; H^1)$, $p \in C^0(H^{s*+1})$ $p \in (l^2 \cap l^\infty)(H^{s+1})$. Then*

$$\begin{aligned}
&\Delta t \sum_{n=n_0+1}^{N-1} ||\partial_{\Delta t}^{n+1} \mathbf{U}_h||^2 + \nu |\mathbf{U}_h^N|_1^2 \\
&\quad \leq F_{*,IC}(\mathbf{e}_u^{n_0}) + F_{*,fem}(k, s, k*, s*) + \Delta t \sum_{n=n_0+1}^{N-1} (C_{*,\Delta t}^{n+1} + \nu |\mathbf{U}_h^n|_1^2 \sum_{i=0}^{i_1(n)} \kappa^{n+i})
\end{aligned} \tag{4.139}$$

where

$$\kappa^n := C\nu^{-1}(||\mathbf{u}^{n+1/2}||_2^2 + \min\left\{h^{-1}, \Delta t^{-2}\right\} ||\mathbf{e}_u^{n+1/2}||_1^2) \tag{4.140}$$

and

$$\begin{aligned}
F_{*,fem}(k, s, k*, s*) :&= C\nu^{-2} ||\mathbf{u}||_{l^\infty(H^2)}^2 (\nu^2 h^{2k} ||\mathbf{u}||_{l^2(H^{k+1})}^2 + h^{2s+2} ||p||_{l^2(H^{s+1})}^2) \\
&\quad + C\nu^{-2}(\nu^2 h^{2k*+2} ||\partial_t \mathbf{u}(\cdot, t)||_{L^2(t^{n_0+1}, T; H^{k*+1})}^2 + \ldots \\
&\quad\quad \ldots + h^{2s*+4} ||\partial_t p(\cdot, t)||_{L^2(t^{n_0+1}, T; H^{s*+1})}^2) \\
&\quad + C\nu^{-2}(\nu^2 ||\mathbf{u}||_{l^\infty(H^2)}^2 + h ||p||_{l^\infty(H^1)}^2) \Delta t \sum_{n=n_0+1}^{N-1} ||\mathbf{e}_u^{n+1/2}||_1^2 \\
&\quad + \nu^{-2} \Delta t \sum_{i=0}^{2n_0} \kappa^i (\nu^2 h^{2k+2} ||\mathbf{u}||_{l^\infty(0, n_0; H^{k+1})}^2 + h^{2s+4} ||p||_{l^\infty(0, n_0; H^{s+1})}^2)
\end{aligned} \tag{4.141}$$

$$F_{*,IC}(\mathbf{e}_u^{n_0}) := \nu |\mathbf{e}_u^{n_0+1}|_1^2 + C\Delta t \sum_{i=0}^{2n_0} \kappa^i ||\nabla \mathbf{e}_u||_{l^\infty(0, n_0; L^2)}^2. \tag{4.142}$$

Proof. Estimate (4.139) with (4.140), (4.141), (4.142) follows the proof of Lemma 4.3.15 closely. Note that we sum from $n_0 + 1$ to $N - 1$ here so that n_0 is replaced by $n_0 + 1$ throughout from this point. □

Apply discrete Gronwall Lemma 2.4.6 to (4.139), triangle inequality $||\mathbf{e}_u|| \leq ||\mathbf{U}_h|| + ||\eta||$ (absorb η terms into $C_{*,fem}(k,s,k*,s*)$), and simplify to get

$$\begin{aligned} \Delta t \sum_{n=n_0+1}^{N-1} ||\partial_{\Delta t}^{n+1}\mathbf{e}_u||^2 + \nu|\mathbf{e}_u^N|_1^2 \\ \leq G^N(F_{*,IC}(\mathbf{e}_u^{n_0}) + F_{*,fem}(k,s,k*,s*) + \Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1}) \end{aligned} \tag{4.143}$$

where

$$G^N := C\exp(\Delta t \sum_{n=n_0}^{N-1} \kappa^n). \tag{4.144}$$

It remains to bound $\Delta t \sum_{n=n_0+1}^{N-1} C_{*,\Delta t}^{n+1}$. We consider two cases: the first with minimal regularity (boundedness) and the second for optimal convergence rate (regularity to match FE and CN approximation degree).

Case 1: Suppose that the regularity of Lemma 4.3.16 is satisfied for $k = 0$, $k* = 0$, and $s = -1$, $s* = -1$. Inspect (4.141), (4.142), and (4.144) to verify

$$G^N(||\mathbf{e}_u^{n_0+1}||^2 + F_{*,IC}(\mathbf{e}_u^{n_0}) + F_{*,fem}) < \infty, \quad \text{as } h,\ \Delta t \to 0 \tag{4.145}$$

as long as

$$\min\left\{h^{-1}, \Delta t^{-2}\right\} \Delta t \sum_{n=n_0+1}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0.$$

Follow the proof in Section 4.3.3 to bound $F_{*,cn}$ (4.52), now shifted to start at n_0+1,

$$\begin{aligned} F_{*,cn} := C(||\partial_t \mathbf{f}||_{L^2(t^{n_0+1},T;L^2)} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t \mathbf{u}||_{L^2(t^{n_0+1},T;H^1)} + \dots \\ \dots + ||\partial_t^{(2)}\mathbf{u}||_{L^2(t^{n_0+1},T;L^2)} + ||\partial_t p||_{L^2(t^{n_0+1},T;H^1)}). \end{aligned}$$

Estimate (4.43) follows since $\mathbf{u}_h \in l^\infty(H^1)$ and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$ under the suggested regularity constraints.

Case 2: Suppose that the regularity of Lemma 4.3.15 is satisfied for $k > 0$, $k* \geq 0$, $s > -1$, and $s* \geq -1$. Apply error estimate from Theorem 4.3.3 to get

$$\min\left\{h^{-1}, \Delta t^{-2}\right\} \Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2 \leq \min\left\{h^{-1}, \Delta t^{-2}\right\} C_*(h^{2k} + h^{2s+2} + \Delta t^4)$$
$$\leq C_*(h^{2k-1} + h^{2s+1} + \Delta t^2)$$

so that

$$\min\left\{h^{-1}, \Delta t^{-2}\right\} \Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}_u^{n+1/2}|_1^2 < \infty, \quad \text{as } h,\ \Delta t \to 0.$$

Follow the proof in Section 4.3.3 to bound F_{cn} (4.56), now shifted to start at $n_0 + 1$,

$$F_{cn} := C(||\partial_t^{(2)}\mathbf{f}||_{L^2(t^{n_0+1},T;L^2)} + ||\mathbf{u}||_{l^\infty(H^2)}||\partial_t^{(2)}\mathbf{u}||_{L^2(H^1)} + \dots$$
$$\dots + ||\partial_t^{(3)}\mathbf{u}||_{L^2(t^{n_0+1},T;L^2)} + ||\partial_t^{(2)}p||_{L^2(t^{n_0+1},T;H^1)}).$$

Estimate (4.44) follows so that $\mathbf{u}_h \to \mathbf{u}$ in $l^\infty(H^1)$ and $\partial_{\Delta t}(\mathbf{u}_h - \mathbf{u}) \to 0$ in $l^2(L^2)$ with rate $\mathcal{O}(h^k + h^{s+1} + \Delta t^2)$ under the specified regularity constraints *without* any Δt, h restrictions.

□

4.3.5 Proof of $p_h \to p$ in $l^2(L^2)$

Proof. Fix $n = n_0, n_0 + 1, \dots, N-1$. Note that

$$c_h(\xi^n(\mathbf{e}_u), \mathbf{u}^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}_h), \mathbf{e}_u^{n+1/2}, \mathbf{v}) = -c_h(\xi^n(\mathbf{e}_u), \mathbf{u}_h^{n+1/2}, \mathbf{v}) - c_h(\xi^n(\mathbf{u}), \mathbf{e}_u^{n+1/2}, \mathbf{v}).$$

Let $\tilde{q}_h \in Q_h$ be the L^2-projection of p. It is well-known that (e.g. see Lemma I.A.5 in [34]) that if $p \in H^s$ for some $s \geq -1$, then

$$||p - \tilde{q}_h|| \leq Ch^{s+1}||p||_{s+1}. \tag{4.146}$$

Solve for pressure in (4.50) to get, for any $\mathbf{v} \in X_h$,

$$(\tilde{q}_h^{n+1/2} - p_h^{n+1/2}, \nabla \cdot \mathbf{v}) = (\tilde{q}_h^{n+1/2} - p^{n+1/2}, \nabla \cdot \mathbf{v}) + (\partial_{\Delta t}^{n+1}\mathbf{e}_u, \mathbf{v})$$
$$+ \nu(\nabla \mathbf{e}_u^{n+1/2}, \nabla \mathbf{v}) + c_h(\xi^n(\mathbf{e}_u), \mathbf{u}_h^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}), \mathbf{e}_u^{n+1/2}, \mathbf{v}) - R^{n+1}(\mathbf{v}). \tag{4.147}$$

Application of Hölder's inequality (2.22) and the duality estimate on $W^{-1,2} \times H_0^1$ gives

$$|(\tilde{q}_h^{n+1/2} - p_h^{n+1/2}, \nabla \cdot \mathbf{v})| \leq (\sqrt{d}||\tilde{q}_h^{n+1/2} - p^{n+1/2}|| + ||\partial_{\Delta t}^{n+1}\mathbf{e}_u||_{-1} + \nu|\mathbf{e}_u^{n+1/2}|_1)|\mathbf{v}|_1$$
$$+ |c_h(\xi^n(\mathbf{e}_u), \mathbf{u}_h^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}), \mathbf{e}_u^{n+1/2}, \mathbf{v}) - R^{n+1}(\mathbf{v})|. \tag{4.148}$$

Supposing that $\mathbf{u} \in l^\infty(H^1)$, we can majorize the convective terms in (4.148) via (2.41)(a) and (2.42)(b) to get

$$|c_h(\xi^n(\mathbf{e}_u), \mathbf{u}_h^{n+1/2}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}), \mathbf{e}_u^{n+1/2}, \mathbf{v})|$$
$$\leq C(||\mathbf{u}_h^{n+1/2}||_1||\xi^n(\mathbf{e}_u)||^{1/2}||\xi^n(\mathbf{e}_u)||_1^{1/2} + ||\mathbf{u}||_{l^\infty(H^1)}||\mathbf{e}_u^{n+1/2}||_1|\mathbf{v}|_1. \tag{4.149}$$

Estimate (4.74), (4.75) in Lemma 4.3.11 gives

$$R^{n+1}(\mathbf{v}) \leq \sqrt{C_{*,\Delta t}^{n+1}}|\mathbf{v}|_1 \tag{4.150}$$

where

$$C_{*,\Delta t}^{n+1} := C\left(||\mathbf{f}^{n+1/2} - \mathbf{f}(\cdot, t^{n+1/2})||_{-1}^2 + \dots\right.$$
$$\dots + ||\mathbf{u}||_{l^\infty(H^1)}^2(||\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})||_1^2 + ||\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})||_1^2) + \dots$$
$$\left.\dots + ||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_t\mathbf{u}(\cdot, t^{n+1/2})||_{-1}^2 + ||p^{n+1/2} - p(\cdot, t^{n+1/2})||^2\right). \tag{4.151}$$

We assume that $|\mathbf{u}|_{l^\infty(H^1)} \geq C\nu$ to simplify the expression. Apply estimates estimates (4.150), (4.151) and (4.149) and (4.146) to (4.148). Divide by $|\mathbf{v}|_1$ and apply the discrete inf-sup condition (2.2) via Assumption 2.1.1 to get

$$||p_h^{n+1/2} - \tilde{q}_h^{n+1/2}|| \leq C(h^{s+1}||p^{n+1/2}||_{s+1} + ||\partial_{\Delta t}^{n+1}\mathbf{e}_u||_{-1} + \nu|\mathbf{e}_u^{n+1/2}|_1 + \dots$$
$$\dots + ||\mathbf{u}_h^{n+1/2}||_1||\xi^n(\mathbf{e}_u)||^{1/2}||\xi^n(\mathbf{e}_u)||_1^{1/2} + ||\mathbf{u}||_{l^\infty(H^1)}||\mathbf{e}_u^{n+1/2}||_1 + (C_{*,\Delta t}^{n+1})^{1/2}). \tag{4.152}$$

Apply the triangle inequality $||p^{n+1/2} - \tilde{q}_h|| \leq ||p^{n+1/2} - \tilde{q}_h^{n+1/2}|| + ||p_h^{n+1/2} - \tilde{q}_h^{n+1/2}||$ and estimate (4.146) once again. Square each side of (4.152) multiply by Δt and sum from $n = n_0$ to $N - 1$ to get

$$\Delta t \sum_{n=n_0+1}^{N-1} ||p^{n+1/2} - \tilde{q}_h^{n+1/2}||^2 \leq C\Delta t \sum_{n=n_0+1}^{N-1} (h^{s+1}||p^{n+1/2}||_{s+1}^2 + ||\partial_{\Delta t}^{n+1}\mathbf{e}_u||_{-1}^2 + \nu|\mathbf{e}_u^{n+1/2}|_1^2 + \dots$$
$$\dots + ||\mathbf{u}_h^{n+1/2}||_1^2||\xi^n(\mathbf{e}_u)||||\xi^n(\mathbf{e}_u)||_1 + ||\mathbf{u}||_{l^\infty(H^1)}^2||\mathbf{e}_u^{n+1/2}||_1^2 + C_{*,\Delta t}^{n+1}). \tag{4.153}$$

Apply estimates from Theorem 4.3.4 to (4.153) with bound on $C^{n+1}_{*,\Delta t}$ via (4.151) and (4.52) to prove (4.45). Apply estimates from Theorem 4.3.6 to (4.153) with bound on $C^{n+1}_{*,\Delta t}$ via (4.151) and (4.56) to prove (4.46). □

4.4 COROLLARY FOR BACKWARD-EULER

The proof for convergence of BELE methods (see e.g. [86]) provided in Problem 3.1.1 is similar to that of CNLE given in Theorems 4.3.1, 4.3.3, Theorems 4.3.4, 4.3.6. First recall the BELE-problem: Let $\mathbf{u}_h^i \in V_{h,\phi_h}(g)$ be a good approximation of $\mathbf{u}^i$ for each $i = 0, 1, \dots, \overline{n}_0 = \max\{0, n_0\}$. For each $n = \overline{n}_0, \overline{n}_0 + 1, \dots, N-1$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_{h,\phi_h} \times Q_h$ satisfying

$$(\partial^{n+1}_{\Delta t}\mathbf{u}_h, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g^{n+1}\mathbf{u}_h^{n+1}\cdot\mathbf{v}_h$$

$$+ \nu(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h) - (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) =< \mathbf{f}^{n+1}, \mathbf{v}_h >, \quad \forall \mathbf{v}_h \in X_h \tag{4.154}$$

$$(q_h, \nabla\cdot\mathbf{u}^{n+1}) = (q_h, g^{n+1}), \quad \forall q_h \in Q_h. \tag{4.155}$$

Remark 4.4.1. *Recall that $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^n$ defines the linearly extrapolated backward-Euler, Finite Element method with supporting literature provided in e.g. [33, 44, 35].*

The main difference in the error equation for the BEFE and CNLE methods resides in time-consistency error $R^{n+1}(\mathbf{v}_h)$ and the fact that the energy norm for CN estimates velocity averages $\frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)$. Briefly, the consistency error for the BE time-discretization is given by, for any $\mathbf{v} \in H_0^1$,

$$R^{n+1}(\mathbf{v}) := (\partial^{n+1}_{\Delta t}\mathbf{u} - \partial_t\mathbf{u}(\cdot, t^{n+1/2}), \mathbf{v}) \tag{4.156}$$

so that we can rewrite (4.48)

$$(\partial^{n+1}_{\Delta t}\mathbf{u}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}), \mathbf{u}^{n+1}, \mathbf{v}) - (p^{n+1}, \nabla\cdot\mathbf{v})$$

$$+ \nu(\nabla\mathbf{u}^{n+1}, \nabla\mathbf{v}) = (\mathbf{f}^{n+1}, \mathbf{v}) + R^{n+1}(\mathbf{v}), \quad \forall \mathbf{v} \in H_0^1. \tag{4.157}$$

Use the notation of (4.49). Fix $\tilde{q}_h^n \in Q_h$. Note that $(p_h, \nabla \cdot \mathbf{v}) = 0$ for any $\mathbf{v} \in V_h$. Subtract (4.157) from (4.154) to get the error equation

$$\begin{aligned}
&(\partial_{\Delta t}^{n+1}\mathbf{U}_h, \mathbf{v}) + c_h(\xi^n(\mathbf{u}_h), \mathbf{U}_h^{n+1}, \mathbf{v}) + \nu(\nabla \mathbf{U}_h^{n+1}, \nabla \mathbf{v}) \\
&= -R^{n+1}(\mathbf{v}) + (\partial_{\Delta t}^{n+1}\eta, \mathbf{v}) - (p^{n+1} - \tilde{q}_h^{n+1}, \nabla \cdot \mathbf{v}) + \nu(\nabla \eta^{n+1}, \nabla \mathbf{v}) \\
&- c_h(\xi^n(\mathbf{U}_h), \mathbf{u}^{n+1}, \mathbf{v}) + c_h(\xi^n(\eta), \mathbf{u}^{n+1}, \mathbf{v}) + c_h(\xi^n(\mathbf{u}_h), \eta^{n+1}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h. \qquad (4.158)
\end{aligned}$$

We conclude without further proof.

Theorem 4.4.2. *Suppose that the FE-space and $\phi_h \approx \phi$ satisfy Assumption 4.2.5. Then for sufficiently smooth NSE solutions $(\mathbf{u}, p)$, the BELE approximation $\mathbf{u}_h$ satisfies*

$$||\partial_{\Delta t}(\mathbf{u} - \mathbf{u}_h)||_{l^2(n_0,N;L^2)} + \nu||\mathbf{u} - \mathbf{u}_h||_{l^\infty(n_0,N;H^1)} \leq K_* < \infty \qquad (4.159)$$

where K_* is a data-dependent constant that remains bounded as h, $\Delta t \to 0$.

For convergence of $BELE$ as h, $\Delta t \to 0$, we require that the initial data satisfy minimal approximation properties.

Assumption 4.4.3 (Accuracy of Initial Iterates)**.** *Suppose $\{\mathbf{u}_h^i\}_i^{n_0}$ for BELE satisfy*

$$||\mathbf{e}_u||_{l^\infty(0,n_0;H^1)} \leq C(h^k + h^{s+1} + \Delta t), \quad \forall i = 0, 1, \ldots, n_0$$

for some $C > 0$.

Corollary 4.4.4 (Unconditional convergence)**.** *Suppose that the FE-space and $\phi_h \approx \phi$ satisfy Assumption 4.2.5. Then for sufficiently smooth NSE solutions $(\mathbf{u}, p)$ and if the initial data satisfies Assumption 4.4.3, then BELE-solutions $\mathbf{u}_h$ satisfy*

$$\begin{aligned}
&||\partial_{\Delta t}(\mathbf{u} - \mathbf{u}_h)||_{l^2(n_0,N;L^2)} + \nu^{1/2}||\mathbf{u} - \mathbf{u}_h||_{l^\infty(n_0,N;H^1)} \\
&\leq C_*(h^k + h^{s+1} + ||\nabla \mathbf{e}_u^{n+1}||_{l^2(n_0,N;l^2)} + \Delta t) \qquad (4.160)
\end{aligned}$$

where C_* is a data-dependent constant that remains bounded as h, $\Delta t \to 0$.

The error estimates of Theorem 4.4.2, Corollary 4.4.4 gives conditions in which the BELE approximations $(\mathbf{u}_h, p_h)$ satisfy $p_h \in l^2(n_0, N; L^2)$, $\mathbf{u}_h \in l^\infty(n_0, N; H^1)$, and $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$. In particular, as a direct consequence of estimate (4.159) and the conditions of Theorem 4.4.2, we have

$$||\partial_{\Delta t}\mathbf{u}_h||_{l^2(n_0,N;L^2)} + \nu||\nabla \mathbf{u}_h||_{l^\infty(n_0,N;L^2)} \leq K_1 < \infty. \tag{4.161}$$

Estimates for pressure follow as well and are summarized in the next Corollary.

Corollary 4.4.5. *Under the conditions and conclusions of Theorem 4.4.2,*

$$||p_h||_{l^2(n_0,N;L^2)} + ||p - p_h||_{l^2(n_0,N;L^2)} \leq K_1 < \infty \tag{4.162}$$

so that K_1 is independent of h, $\Delta t \to 0$. Suppose further that the conditions and conclusions of Corollary 4.4.4 are satisfied, then

$$||p - p_h||_{l^2(n_0,N;L^2)} \leq C_*(h^k + h^{s+1} + \Delta t^2) \tag{4.163}$$

for some data-dependent $0 < C_* < \infty$ independent of h, $\Delta t \to 0$.

5.0 STATIONARY FLOWS IN COMPLICATED DOMAINS

The approximation of high velocity flow through complex geometries with relatively large pores is important in many applications (including flow through pebble bed nuclear reactors (PBR), wind turbines, and the trabecular meshwork of the eye as discussed in Chapter 1). Common to these applications, the fluid velocities are too large to model accurately with a filtration model like Darcy's equation and the pore geometry is too complex to feasibly, or at least efficiently, approximate by the Navier-Stokes (NS) equation (NSE) in the pore region with no-slip boundary conditions on the solid obstacles. Moreover, each of these problems lacks symmetry so that they are inherently 3D-problems. See [66] (and references therein) for an explanation of the mammoth task of approximating flow in PBR's (in particular, they consider a scaled experiment involving flow past pebbles in a wind tunnel). Appropriate for this setting, we propose the Brinkman model, beginning with the equilibrium case.

- (Stationary Brinkman) For an incompressible, viscous fluid, find velocity $\mathbf{u}_B : \Omega_{ext} \to \mathbb{R}^d$ and pressure $p_B : \Omega_{ext} \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla \cdot (\tilde{\nu} \nabla \mathbf{u}_B) + \mathbf{u}_B \cdot \nabla \mathbf{u}_B + \nabla p_B + \nu \mathbf{K}^{-1} \mathbf{u}_B &= \mathbf{f}, \quad \text{in } \Omega_{ext} \\ \nabla \cdot \mathbf{u}_B &= g, \quad \text{in } \Omega_{ext} \\ \mathbf{u}_B &= \phi, \quad \text{in } \partial\Omega_{ext}. \end{aligned} \tag{5.1}$$

Recall that $\Omega_{ext} \subset \mathbb{R}^d$ is an open domain for $d = 2$ or 3 consisting of both the pores Ω and solid obstacles Ω_{solid} so that

$$\Omega_{ext} = \Omega \cup \overline{\Omega}_{solid}.$$

Additionally, $\mathbf{f}$ represents body forces, g represents sources and/or sinks, $\mathbf{K}$ is the permeability tensor, $\nu \propto Re^{-1}$ is the kinematical viscosity with Reynolds number (of the fluid) $Re > 0$, and $\tilde{\nu}$ is the Brinkman viscosity. In general, $\mathbf{K}$, $\tilde{\nu}$ are (highly) discontinuous coefficients.

For model parameters $\tilde{\nu}$ and $\mathbf{K}$ of $\mathcal{O}(1)$, the numerical analysis of the Brinkman model fits within the framework for the abstract error analysis of the NSE (e.g. [34] for a finite element approximation). However, the targeted applications of the Brinkman model are often highly non-generic flows involving

- complex geometries, i.e. dense swarm of porous and solid obstacles
- highly discontinuous parameters $\tilde{\nu}$ and K
- inhomogeneous problem data:

$$\mathbf{u}_B|_{\partial\Omega_{ext}} = \phi \neq 0, \qquad \nabla\cdot\mathbf{u}_B = g \neq 0. \tag{5.2}$$

Inhomogeneous conditions in (5.2) are necessary for many natural and industrial flows in porous media and lead to non-trivial complications in analysis. Moreover, although the Brinkman model for fluid flow was initially formulated as a filtration model similar to Darcy, we will also consider another application known as Brinkman volume penalization (BrVP). For BrVP, we fix $0 < \varepsilon << 1$ and set $\tilde{\nu} = 1/\varepsilon$ and $\mathbf{K} = K = \varepsilon$ in the solid obstacles of Ω_{ext} and $\tilde{\nu} = \nu$ and $\mathbf{K}^{-1} = K^{-1} = 0$ in the purely fluid parts of Ω_{ext}. In this case, we will use the suggestive notation $\mathbf{u}_B = \mathbf{u}_\varepsilon$ and $p_B = p_\varepsilon$. Thus, we consider herein the numerical analysis associated with the asymptotic limits and rates of convergence as the discretization parameter $h > 0$ and penalty parameter $\varepsilon > 0$ tend to 0.

We introduce some simplifying notation in 5.1 and propose the weak and finite element (FE) formulation of the (nonlinear) Brinkman model in Section 5.2. We address several subtle issues that arise with application of Poincaré's inequality and duality estimates on subsets of the problem domain in Section 5.3 required in our analysis. In Section 5.4 we compile several key continuity and coercivity-type estimates required in the main analysis. In Section 5.5.1, we investigate the coupling between $\mathbf{u}_B|_{\partial\Omega_{ext}}$ and $\nabla\cdot\mathbf{u}_B = g \neq 0$ that prevents a general existence result. Our analysis is based on the construction of an extension operator $E(\phi)$ of boundary data ϕ satisfying the constraint $\nabla\cdot E(\phi) = g$. We show that if $g \in L^2(\Omega_{ext})$ and $\phi \in (H^{1/2}(\partial\Omega_{ext}))^d$ satisfy

- $g \equiv 0$, or
- g has compact support in Ω_{ext}, $\int_{\Omega_{ext}} g = 0$, and g small enough, or

- g has compact support in Ω_{ext}, and g, ϕ are small enough

then there exists a solution $(\mathbf{u}_B, p_B) \in (H^1(\Omega_{ext}))^d \times L_0^2(\Omega_{ext})$ to (5.1), see Theorem 5.5.1. We present a similar result for the existence of FE approximations $(\mathbf{u}_{B,h}, p_{B,h})$ of $(\mathbf{u}_B, p_B)$ given in (5.5), see Theorem 5.5.4. We also show that all Brinkman-solutions $\mathbf{u}_B$ or $\mathbf{u}_{B,h}$ satisfy *a priori* estimates given in (5.11), (5.12) and (5.15), (5.16) respectively so that the continuous and discrete solutions $\mathbf{u}_B = \mathbf{u}_\varepsilon$, $\mathbf{u}_{B,h} = \mathbf{u}_{\varepsilon,h}$ are uniformly stable with respect to $\varepsilon \to 0$. In Section 5.6, we conclude with several numerical experiments that confirm the expected convergence rates for $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}_h$ and $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}$. We also generate a model nonsolenoidal Brinkman flow.

5.1 PROBLEM FORMULATION

Refer to Figure 1.2 for an illustration of the following problem setup. Let $\Omega_{ext} \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be an open and connected domain. Decompose the *flow-domain* $\Omega = \Omega_f \cup \overline{\Omega}_p$ where Ω_f is the purely fluid subdomain (no flow obstruction), and Ω_p is a porous medium (some flow obstruction). Then with Ω_s representing the solid obstacles (complete flow obstruction)

$$\Omega_{ext} = \Omega \cup \overline{\Omega}_s, \qquad \Omega = \Omega_f \cup \overline{\Omega}_p$$

where $\partial\Omega_{ext}$, $\partial\Omega$, $\partial\Omega_f$, $\partial\Omega_p$, and $\partial\Omega_s$ represent the corresponding boundaries of the indicated subdomains. Assume that Ω_p and Ω_s consist of open and connected subsets of Ω_{ext}, and Ω_p and Ω_s are disjoint and bounded away from $\partial\Omega_{ext}$:

$$\overline{\Omega}_p \cap \overline{\Omega}_s = \emptyset, \qquad (\overline{\Omega}_p \cup \overline{\Omega}_s) \cap \partial\Omega_{ext} = \emptyset.$$

Require that

$$\partial\Omega = \partial\Omega_{ext} \cup \partial\Omega_s.$$

Write Ω_* for $* = f$, p, s, fp, and ext where

$$\Omega_{fp} := \Omega.$$

Assume that $\nu > 0$ is constant in Ω. Also, $\tilde{\nu} > 0$ is piecewise constant and constant in Ω_{ext} so that

$$\tilde{\nu}_f := \tilde{\nu}|_{\Omega_f} = \nu, \qquad \tilde{\nu}_p := \tilde{\nu}|_{\Omega_p} = \nu, \qquad \tilde{\nu}_s := \tilde{\nu}|_{\Omega_s} = \nu\varepsilon^{-1}.$$

Let $K > 0$ be a constant scalar on each subdomain Ω_f, Ω_p, and Ω_s and write

$$K_f^{-1} := K^{-1}|_{\Omega_f} = 0, \qquad K_p := K|_{\Omega_p} > 0, \qquad K_s := K|_{\Omega_s} = \varepsilon << 1.$$

In porous regions Ω_p, the Brinkman viscosity K_p and $\tilde{\nu}_p$ should have moderate values. We suppose that K_p depends on the domain geometry (e.g. see [10]). The Brinkman viscosity $\tilde{\nu}$ in Ω_p depends on fluid properties (i.e. on ν), but also the geometry of the pores; hence, it is not generally known how to choose $\tilde{\nu}$ beyond data-fitting. We set $\tilde{\nu}|_{\Omega} \equiv \nu$ which is a common choice in both engineering practice and analytical theory. For more on $\tilde{\nu}$, see [10, 56] and references therein.

Let $\nabla \cdot \mathbf{u}_B = g$ have compact support so that

$$g \equiv 0 \text{ in } \Omega_{ext} - \Omega_p.$$

Furthermore, require compatibility between g and $\mathbf{u}|_{\partial\Omega_{ext}} = \phi$ so that

$$\int_{\Omega_p} g = \int_{\partial\Omega_{ext}} \phi \cdot \hat{\mathbf{n}}_{\partial\Omega_{ext}} \tag{5.3}$$

where $\hat{\mathbf{n}}_{\partial\Omega_{ext}}$ is the outward unit normal on $\partial\Omega_{ext}$. Write $\hat{\mathbf{n}} = \hat{\mathbf{n}}_{\partial\Omega_*}$ when no confusion exists concerning the boundary defining the normal vector.

5.2 WEAK AND FE-FORMULATION

Refer to the notation and formulations of Chapter 2. Note that $\nabla \cdot (\nabla \mathbf{u})^t = \nabla(\nabla \cdot \mathbf{u})$. Then $\nabla \cdot \mathbf{u} = g$ implies

$$\nabla \cdot (\nabla \mathbf{u}_B + (\nabla \mathbf{u}_B)^t) = \Delta \mathbf{u}_B + \nabla g.$$

The term ∇g is data. Therefore, if the viscous term $-\nu \Delta \mathbf{u}_B$ is replaced by the deformation-tensor formulation $-\nu \nabla \cdot (\nabla \mathbf{u}_B + (\nabla \mathbf{u}_B)^t)$, then $\mathbf{f}$ should be replaced by ∇g in the RHS of (5.1)(a). Since this does not greatly affect the analysis, we omit this possibility (or suppose that ∇g is absorbed into the forcing function $\mathbf{f}$).

Next, define the necessary bi/tri-linear functionals. Fix $* = f,\ p,\ s,\ fp,\ ext$:

$$\begin{aligned}
a_*(\cdot,\cdot) \quad &: H^1(\Omega_{ext}) \times H^1(\Omega_{ext}) \to \mathbb{R}, \quad a_*(\mathbf{u},\mathbf{v}) &&:= (\tilde{\nu} \nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_*} + (\nu K^{-1} \mathbf{u}, \mathbf{v})_{\Omega_* \cap \Omega_s} \\
b_*(\cdot,\cdot) \quad &: H^1(\Omega_{ext}) \times L^2(\Omega_{ext}) \to \mathbb{R}, \quad b_*(\mathbf{v},q) &&:= -(q, \nabla \cdot \mathbf{v})_{\Omega_*} \\
l_{2,*}(\cdot) \quad &: L^2(\Omega_{ext}) \to \mathbb{R}, \quad l_{2,*}(q) &&:= -(g,q)_{\Omega_*}.
\end{aligned}$$

Further, define the linear form $l_{1,*}(\cdot) : H_0^1(\Omega_{ext}) \to \mathbb{R}$,

$$l_{1,*}(\mathbf{v}) := < \mathbf{f}, \mathbf{v} >_{W^{-1,2}(\Omega_*) \times H_0^1(\Omega_*)}$$

and trilinear form $c_*(\cdot,\cdot,\cdot) : H^1(\Omega_{ext}) \times H^1(\Omega_{ext}) \times H^1(\Omega_{ext}) \to \mathbb{R}$,

$$c_*(\mathbf{u},\mathbf{v},\mathbf{w}) := \int_{\Omega_{ext}} \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}.$$

We can now write the weak formulation of (5.1).

- (Stationary Weak-Brinkman) Given $\mathbf{f} \in W^{-1,2}(\Omega_{ext})$, $g \in L^2(\Omega_{ext})$, and $\phi \in H_g^{1/2}(\partial\Omega_{ext})$, find $\mathbf{u}_B \in H_\phi^1(\Omega_{ext})$, $p_B \in L_0^2(\Omega_{ext})$ so that

$$\begin{aligned}
\forall \mathbf{v} \in H_0^1(\Omega_{ext}), \qquad & a_{ext}(\mathbf{u}_B,\mathbf{v}) + b_{ext}(\mathbf{v},p_B) + c_{ext}(\mathbf{u}_B,\mathbf{u}_B,\mathbf{v}) = l_{1,ext}(\mathbf{v}) \\
\forall q \in L^2(\Omega_{ext}), \qquad & b_{ext}(\mathbf{u}_B,q) = l_{2,ext}(q).
\end{aligned} \tag{5.4}$$

The FE-approximation of $(\mathbf{u}_B, p_B)$ is a projection into the FE-spaces $X_h(\Omega_{ext})$, $Q_h(\Omega_{ext})$:

- (Stationary FE-Brinkman) Given $\mathbf{f} \in W^{-1,2}(\Omega_{ext})$, $g \in L^2(\Omega_{ext})$, and $\phi_h \in \Lambda_{h,g}(\partial\Omega_{ext})$, find $\mathbf{u}_{B,h} \in X_{h,\phi_h}(\Omega_{ext})$, $p_{B,h} \in Q_h(\Omega_{ext})$ satisfying

$$\begin{aligned} \forall \mathbf{v}_h \in X_h(\Omega_{ext}), \qquad & a_{ext}(\mathbf{u}_{B,h}, \mathbf{v}_h) + b_{ext}(\mathbf{v}_h, p_{B,h}) \\ & \qquad + c_{h,ext}(\mathbf{u}_{B,h}, \mathbf{u}_{B,h}, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g(\mathbf{u}_{B,h} \cdot \mathbf{v}_h) = l_{1,ext}(\mathbf{v}_h) \qquad (5.5) \\ \forall q_h \in Q_h(\Omega_{ext}), \qquad & b_{ext}(\mathbf{u}_{B,h}, q_h) = l_{2,ext}(q_h). \end{aligned}$$

5.3 CALCULUS ON SUBDOMAINS Ω_{EXT}

We often decompose Ω_{ext} into its physical components: the purely fluid region Ω_f, the porous region Ω_p, and solid region Ω_s. In doing so, we must be careful when applying Poincaré's Inequality (since we require that functions vanish on a set of positive measure on the domain boundary) and duality estimates (since $W^{-1,2}(\Omega_{ext})$ is dual to $H^1_0(\Omega_{ext})$ and *not* to $H^1(\Omega_{ext})$). We state these estimates in the context of analysis of the Brinkman problem.

Theorem 5.3.1 (Poincaré's Inequality)**.** *For any $\mathbf{v} \in H^1_0(\Omega_{ext})$, there exists $C_p^* > 0$ such that*

$$||\mathbf{v}||_{\Omega_*} \leq C_p^* |\mathbf{v}|_{1,\Omega_*}, \ \textit{for } * = f, fp, fs, fps.$$

We generically write $C = C_p^*$ for all $*$.

Note that this result is *not* applicable in Ω_s or Ω_p since boundary data is generally not provided on $\partial\Omega_p$ or $\partial\Omega_s$. Additionally, the functional $\mathbf{f} \in W^{-1,2}(\Omega_{ext})$ acts on elements from its dual space $\mathbf{v} \in H^1_0(\Omega_{ext})$. Again, since boundary data is generally not provided on $\partial\Omega_p$ or $\partial\Omega_s$ and

$$||\mathbf{f}||_{-1,\Omega_{ext}} = \sup_{\mathbf{v} \in H^1(\Omega_{ext}), \mathbf{v} \neq 0} \frac{< \mathbf{f}, \mathbf{v} >_{W^{-1,2}(\Omega_{ext}) \times H^1(\Omega_{ext})}}{||\mathbf{v}||_{1,\Omega_{ext}}}$$

then $< \mathbf{f}, \mathbf{v} >_{W^{-1,2}(\Omega_*) \times H^1_0(\Omega_*)} \leq ||\mathbf{f}||_{-1,\Omega_*} ||\mathbf{v}||_{1,\Omega_*}$ only applies when $* = fps$.

5.4 FUNDAMENTALS OF EXISTENCE AND ESTIMATION

We now proceed with some important bounds on the functionals previously defined in Section 5.2. These estimates are required in the proceeding stability and error analysis.

Lemma 5.4.1. *The linear functionals $l_{1,ext}(\cdot)$ and $l_{2,ext}(\cdot)$ are continuous on $H_0^1(\Omega_{ext})$ and $L^2(\Omega_{ext})$ respectively. In particular,*

$$\begin{array}{rcl} l_{1,ext}(\mathbf{v}) & \leq & ||\mathbf{f}||_{-1}|\mathbf{v}|_{1,\Omega_{ext}}, \ \forall \mathbf{v} \in H_0^1(\Omega_{ext}) \\ l_{2,ext}(q) & \leq & ||g||_{\Omega_p}||q||_{\Omega_{ext}}, \ \forall q \in L^2(\Omega_{ext}) \end{array}.$$

Proof. Linearity for the functionals is obvious. Continuity follows by a direct follows from duality of $W^{-1,2}(\Omega_{ext}) \times H_0^1(\Omega_{ext})$ for $l_{1,ext}$ and Cauchy-Schwarz (2.22) for $l_{2,ext}$. □

Lemma 5.4.2. *The bilinear functional $b_{ext}(\cdot,\cdot)$ is continuous on $H^1(\Omega_{ext}) \times L^2(\Omega_{ext})$. In particular,*

$$b_{ext}(\mathbf{v}, q) \leq \sqrt{d}|\mathbf{v}|_{1,\Omega_{ext}}||q||_{\Omega_{ext}}, \quad \forall \mathbf{v} \in H^1(\Omega_{ext}), \ q \in L^2(\Omega_{ext}).$$

Proof. Bilinearity is obvious. Continuity follows by applying Cauchy-Schwarz (2.22) and estimate $||\nabla \cdot \mathbf{v}||_{\Omega_{ext}} \leq \sqrt{d}|\mathbf{v}|_{1,\Omega_{ext}}$. □

Lemma 5.4.3. *The bilinear functional $a_{ext}(\cdot,\cdot)$ is continuous on $H^1(\Omega_{ext}) \times H^1(\Omega_{ext})$ and coercive on $H_0^1(\Omega_{ext})$. In particular,*

$$\begin{array}{rcll} a_{ext}(\mathbf{u}, \mathbf{v}) & \leq & \alpha_0 \nu ||\mathbf{u}||_{1,\Omega_{ext}}||\mathbf{v}||_{1,\Omega_{ext}}, & \forall \mathbf{u}, \ \mathbf{v} \in H^1(\Omega_{ext}) \\ a_{ext}(\mathbf{v}, \mathbf{v}) & \geq & \alpha_1 \nu ||\mathbf{v}||_1^2 & \forall \mathbf{v} \in H_0^1(\Omega_{ext}) \end{array} \tag{5.6}$$

where $\alpha_0 := \max_{x\in\Omega_{ext}} \{\tilde{\nu}(x)/\nu, \ K^{-1}(x)\}$ and $\alpha_1 := \frac{1}{2}\min_{x\in\Omega_{ext}} \{\tilde{\nu}(x)/\nu\} \min\{1, 1/c_p\}$ where $c_p > 0$ is the Poincaré constant from (2.23).

Proof. Bilinearity is obvious. Continuity follows from application of Cauchy-Schwarz (2.22) and bounding problem parameters ν, $\tilde{\nu}$, and K. The coercivity estimate follows from application of the Poincaré inequality (2.23) and bounding the problem parameters. □

Lemma 5.4.4. *The trilinear functional* $c_{ext}(\cdot,\cdot,\cdot)$ *is continuous on* $H^1(\Omega_{ext}) \times H^1(\Omega_{ext}) \times H^1(\Omega_{ext})$*. In particular,*

$$c_{ext}(\mathbf{u},\mathbf{v},\mathbf{w}) \le \alpha_2 ||\mathbf{u}||_{1,\Omega_{ext}} |\mathbf{v}|_{1,\Omega_{ext}} ||\mathbf{w}||_{1,\Omega_{ext}}, \quad \forall \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in H^1(\Omega_{ext}) \tag{5.7}$$

where

$$\alpha_2 := \inf_{0 \neq \mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega_{ext})} \frac{c_{ext}(\mathbf{u},\mathbf{v},\mathbf{w})}{||\mathbf{u}||_{1,\Omega_{ext}} |\mathbf{v}|_{1,\Omega_{ext}} ||\mathbf{w}||_{1,\Omega_{ext}}}.$$

Moreover, if $\nabla \cdot \mathbf{u} = g$ *and* $(\mathbf{u} \cdot \hat{\mathbf{n}}(\mathbf{v} \cdot \mathbf{w}))|_{\partial\Omega_{ext}} = 0$*, then*

$$c_{ext}(\mathbf{u},\mathbf{v},\mathbf{w}) = -c_{ext}(\mathbf{u},\mathbf{w},\mathbf{v}) - \int_{\Omega_{ext}} g(\mathbf{v}\cdot\mathbf{w}), \qquad c_{ext}(\mathbf{u},\mathbf{v},\mathbf{v}) = -\frac{1}{2}\int_{\Omega_{ext}} g|\mathbf{v}|^2.$$

Proof. See the derivations in Section 2.4.4. □

We call $c_{ext}(\cdot,\cdot,\cdot)$ is generally *pseudo-skew symmetric* since

$$(\mathbf{u},\mathbf{v},\mathbf{w}) \mapsto c_{ext}(\mathbf{u},\mathbf{v},\mathbf{w}) + \frac{1}{2}\int_{\Omega_{ext}} g(\mathbf{v}\cdot\mathbf{w})$$

is skew-symmetric. Moreover, if $\nabla \cdot \mathbf{u} = 0$ in Ω_{ext}, then $c_{ext}(\cdot,\cdot,\cdot)$ is actually *skew-symmetric*, see (2.31).

Recall that the explicitly skew-symmetric discrete convective term introduced in (2.9) is given by

$$c_{h,ext}(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}(c_{ext}(\mathbf{u},\mathbf{v},\mathbf{w}) - c_{ext}(\mathbf{u},\mathbf{w},\mathbf{v})).$$

Lemma 5.4.5. *The trilinear functional* $c_{h,ext}(\cdot,\cdot,\cdot)$ *is continuous on* $H^1(\Omega_{ext}) \times H^1(\Omega_{ext}) \times H^1(\Omega_{ext})$*. In particular, fix* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w} \in H^1(\Omega_{ext})$*. Then*

$$c_{h,ext}(\mathbf{u},\mathbf{v},\mathbf{w}) \le \alpha_3 ||\mathbf{u}||_{1,\Omega_{ext}} ||\mathbf{v}||_{1,\Omega_{ext}} ||\mathbf{w}||_{1,\Omega_{ext}} \tag{5.8}$$

where

$$\alpha_3 := \inf_{0 \neq \mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega_{ext})} \frac{c_{h,ext}(\mathbf{u},\mathbf{v},\mathbf{w})}{||\mathbf{u}||_{1,\Omega_{ext}} ||\mathbf{v}||_{1,\Omega_{ext}} ||\mathbf{w}||_{1,\Omega_{ext}}}.$$

Moreover,

$$c_{h,ext}(\mathbf{u},\mathbf{v},\mathbf{w}) = -c_{h,ext}(\mathbf{u},\mathbf{w},\mathbf{v}), \qquad c_{h,ext}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0.$$

Proof. See the derivations in Section 2.4.4. □

5.5 WELL-POSEDNESS OF STEADY BRINKMAN

Existence of $(\mathbf{u}_B, p_B) \in H^1_\phi(\Omega_{ext}) \times L^2_0(\Omega_{ext})$ solving (5.4) is subtle. The strategy implemented here relies on a lift of the inhomogeneous problem data $E(\phi) \in V_\phi(g)(\Omega_{ext})$ and application of the Leray-Schauder fixed-point theorem. The major difficulty is associated with the derivation of an *a priori* estimate for $\mathbf{u}_B$ solving (5.4). In particular, similar to the NSE case discussed in Section 2.5.1, the key estimate for Brinkman is analogous to (2.50), now with general divergence constraint:

$$|\int \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w} - \int_{\Omega_p} g|\mathbf{w}|^2| \leq \delta |\mathbf{w}|_1^2 \tag{5.9}$$

for some $\delta < \nu$. We present a derivation of the Leray-Hopf extension $E(\phi) \in V_\phi(\Omega)$ is Section 5.5.1 with generalization to $E(\phi) \in V_\phi(g)(\Omega)$. When $\partial\Omega$ consists of exactly one connected component and $g = 0$, the Leray-Hopf extension is exactly what is needed to ensure we can take $\delta < \nu$ in (5.9) for any $\phi \in H^{1/2}(\partial\Omega_{ext})$. However, when $g \neq 0$, there is a nontrivial coupling between the divergence constraint $\nabla \cdot \mathbf{u}_B = g$ and boundary data $\mathbf{u}_B|_{\partial\Omega_{ext}} = \phi$ even when $\partial\Omega_{ext}$ consists of a single connected component. We show that when $g \in L^2_0(\Omega_{ext})$ that the small data constraint decouples: i.e. existence is ensured for arbitrary $\phi \in H^{1/2}(\partial\Omega_{ext})$ as long as g is small enough.

First, as a consequence of Theorem 5.5.1, we show that, all solutions $(\mathbf{u}_B, p)$ satisfy

$$|\mathbf{u}_B|_{1,\Omega_{ext}} + ||p_B|| \leq C < \infty$$

and for BrVP,

$$|\mathbf{u}_\varepsilon|_{1,\Omega_{ext}} \leq \mathcal{O}(\nu^{-1}), \qquad ||\mathbf{u}_\varepsilon||_{1,\Omega_s} \leq \mathcal{O}(\nu^{-1}\sqrt{\varepsilon}) \quad \text{(sub-optimal result).}$$

For inhomogeneous data (5.2), the key step is to choose extension $E(\phi)$ that vanishes in Ω_s. Moreover, the estimate $\mathbf{u}_\varepsilon \to 0$ as $\varepsilon \to 0$ in $H^1(\Omega_s)$ can be improved to $\mathcal{O}(\varepsilon)$. We prove this result in Proposition 6.2.3 and Theorem 6.2.5 for the discrete problem.

Theorem 5.5.1 (Well-posedness, Steady Brinkman). *Fix $g \in L^2(\Omega_{ext})$, $\phi \in H_g^{1/2}(\partial\Omega_{ext})$. Then there is at least one pair $(\mathbf{u}_B, p_B) \in (H_\phi^1(\Omega_{ext}), L_0^2(\Omega_{ext}))$ satisfying (5.4) as long as (a possible small data restriction on ϕ, g and/or ν^{-1})*

$$|\int_{\Omega_{ext}} \mathbf{v} \cdot \nabla E(\phi) \cdot \mathbf{v} - \frac{1}{2}\int_{\Omega ext} g|\mathbf{v}|^2| \leq \frac{\nu}{4}|\mathbf{v}|_1^2, \qquad \forall \mathbf{v} \in V(\Omega_{ext}) \tag{5.10}$$

is satisfied for some extension $E(\phi) \in V_\phi(g)(\Omega_{ext})$. Moreover, all BrVP-solutions satisfying (5.10) also satisfy

$$||\mathbf{u}_B||_{1,\Omega_{ext}} + ||p_B||^{1/2} \leq \nu^{-1} M_{B,0} \tag{5.11}$$

$$(\varepsilon\gamma_{2,s})^{1/2}|\mathbf{u}_B|_{1,\Omega_s} + ||\mathbf{u}_B||_{\Omega_s} \leq \nu^{-1} M_{B,0}\varepsilon^{1/2} \tag{5.12}$$

where $\gamma_{2,s} = \tilde{\nu}/\nu$, $M_{B,0} = C(||\mathbf{f}||_{-1,\Omega_{ext}} + \nu|E(\phi)|_1 + \nu K_p^{-1/2}||E(\phi)||_{\Omega_p}^2 + ||E(\phi)||_{0,4}^2)$, and $E(\phi)|_{\overline{\Omega}_s} = 0$. There is at most one such solution $(\mathbf{u}_B, p_B)$ when the additional small data condition is satisfied:

$$M_{B,0} \leq \frac{\alpha_1}{2\alpha_2} \tag{5.13}$$

where $\alpha_1 > 0$ is the coercivity constant for $a_{ext}(\cdot,\cdot)$ in (5.6) and $\alpha_2 > 0$ is the continuity constant for $c_{ext}(\cdot,\cdot,\cdot)$ in (5.7).

Remark 5.5.2. *In the case of L^2-penalization in which $\tilde{\nu}_s = \nu$, we have that $||p_B||_{\Omega_s} \leq \nu^{-2}M_{B,0}\varepsilon^{-1/4} \to \infty$ as $\varepsilon \to 0$. In the case of H^1-penalization in which $\tilde{\nu}_s = \varepsilon^{-1}\nu$, we are ensured that*

$$||p_B||_{\Omega_s} \leq \nu^{-2}M_{B,0}^2 < \infty, \qquad \textit{uniformly as } \varepsilon \to 0.$$

See Proposition 6.2.3.

Remark 5.5.3. *In Section 5.5.1, we construct possible extensions $E(\phi) \in V_\phi(g)(\Omega_{ext})$ in Proposition 5.5.6 that can be used in Theorem 5.5.1. In particular, if $g \in L_0^2(\Omega_p)$, then existence of Brinkman-solutions holds for any $\phi \in H_0^{1/2}(\partial\Omega_{ext})$. Otherwise, we need $||g||_{\Omega_p} + ||\phi||_{1/2,\partial\Omega_{ext}}$ bounded by ν.*

Proof. See Section 5.5.2. ∎

Proving well-posedness (and boundedness) of the discrete problem governed by (5.5) closely follows the proof of Theorem 5.5.1. The extension $E(\phi)$ of Theorem 5.5.1 must be replaced by a discrete extension in the FE-space; e.g. $E_h(\phi_h) \in V_{h,\phi_h}(g)(\Omega_{ext})$. The construction of such a discrete extension so that (5.10) is satisfied for all restricted $\mathbf{v}_h \in V_h(\Omega_{ext})$ ultimately depends on the physical domain Ω_f as well as the mesh $\mathcal{T}_h$, FE-space X_{h,ϕ_h}, and element size h. If ϕ_h and g are small enough, then the discrete analogue to (5.10) will be satisfied. In particular, the uniqueness constraint (5.13) for the continuous problem is enough to ensure existence of $(\mathbf{u}_{B,h}, p_{B,h}) \in X_{h,\phi_h}(\Omega_{ext}) \times Q_h(\Omega_{ext})$ to (5.5). We conclude the following without further proof it follows the proof of Theorem 5.5.1 so closely.

Theorem 5.5.4 (Well-posedness, FE-Brinkman)**.** *Fix $g \in L^2(\Omega_{ext})$, $\phi_h \in \Lambda_{h,g}(\partial\Omega_{ext})$. Then there is at least one pair $(\mathbf{u}_{B,h}, p_{B,h}) \in X_{h,\phi_h}(\Omega_{ext}) \times Q_h(\Omega_{ext})$ satisfying (5.5) as long as*

$$|\int_{\Omega_{ext}} \mathbf{v} \cdot \nabla E_h(\phi_h) \cdot \mathbf{v} - \frac{1}{2}\int_{\Omega ext} g|\mathbf{v}|^2| \leq \frac{\nu}{4}|\mathbf{v}|_1^2, \qquad \forall \mathbf{v} \in V_h(\Omega_{ext}) \tag{5.14}$$

is satisfied for some extension $E_h(\phi_h) \in V_{h,\phi_h}(g)(\Omega_{ext})$. Moreover, all BrVP-solutions satisfying (5.14) also satisfy

$$||\mathbf{u}_{B,h}||_{1,\Omega_{ext}} + ||p_{B,h}||^{1/2} \leq \nu^{-1} K_{B,0} \tag{5.15}$$

$$(\varepsilon\gamma_{2,s})^{1/2}|\mathbf{u}_{B,h}|_{1,\Omega_s} + ||\mathbf{u}_{B,h}||_{\Omega_s} \leq \nu^{-1} K_{B,0}\varepsilon^{1/2} \tag{5.16}$$

where $\gamma_{2,s} = \varepsilon^{-1}$ for H^1-penalization and $\gamma_{2,s} = 1$ for L^2-penalization, $K_{B,0} = C(||\mathbf{f}||_{-1,\Omega_{ext}} + \nu|E_h(\phi_h)|_{1,\Omega_{ext}} + \nu K_p^{-1/2}||E_h(\phi_h)||^2_{\Omega_p} + ||E_h(\phi_h)||^2_{0,4})$, and $E_h(\phi_h)|_{\overline{\Omega}_s} = 0$. There is at most one such solution $(\mathbf{u}_{B,h}, p_{B,h})$ when the additional small data condition is satisfied:

$$K_{B,0} \leq \frac{\alpha_1}{2\alpha_3} \tag{5.17}$$

where $\alpha_1 > 0$ is the coercivity constant for $a_{ext}(\cdot,\cdot)$ in (5.6) and $\alpha_3 > 0$ is the continuity constant for $c_{h,ext}(\cdot,\cdot,\cdot)$ in (5.8).

5.5.1 Decoupling $\nabla \cdot \mathbf{u} = g$ and $\mathbf{u}|_{\partial\Omega_{ext}} = \phi$

It is necessary to rewrite (5.1) in terms of a divergence-free velocity vanishing on the boundary $\partial\Omega_{ext}$. We do so by *lifting* the inhomogeneous data. Recall the following function spaces $V_{\cdot}(g)(S)$ and $H_g^{1/2}(\partial S)$ defined in (2.11) and (2.12) respectively. These function spaces are particularly convenient for defining a divergence-constrained extension operator of inhomogeneous problem data. The following result is a provided in [28], pp. 131-132.

Lemma 5.5.5 (Trace Theorem). *Fix $g \in L^2(\Omega_{ext})$ so that $g|_{\Omega_{ext}-\Omega_p} \equiv 0$. There exists an extension $E : H_g^{1/2}(\partial\Omega_{ext}) \to V_{\cdot}(g)(\Omega_{ext})$ satisfying*

$$\begin{cases} \phi \in H_g^{1/2}(\partial\Omega_{ext}) \\ E(\phi) \in V_\phi(g)(\Omega_{ext}) \\ ||E(\phi)||_{1,\Omega_{ext}} \leq C(||\phi||_{1/2,\partial\Omega_{ext}} + ||g||_{0,\Omega_p}) \end{cases} \tag{5.18}$$

for some $0 \leq C < \infty$.

We can construct such an extension for problem data $g \in L^2(\Omega_{ext})$, $\phi \in H_g^{1/2}(\partial\Omega_{ext})$. Indeed, there exists $\mathbf{u}_\phi \in H^1_\phi(\Omega_{ext})$ and $C > 0$ satisfying

$$||\mathbf{u}_\phi||_{1,\Omega_{ext}} \leq C||\phi||_{1/2,\partial\Omega_{ext}} \tag{5.19}$$

(e.g. solution of Laplace problem with ϕ-boundary data). Apply $\phi \in H_g^{1/2}(\partial\Omega_{ext})$ along with application of the divergence theorem to get $\int_{\Omega_{ext}}(g - \nabla \cdot \mathbf{u}_\phi) = 0$. Hence, $g - \nabla \cdot \mathbf{u}_\phi \in L^2_0(\Omega_{ext})$. Lemma 2.4.1 ensures that $\nabla \cdot : V(\Omega_{ext})^\perp \to L^2_0(\Omega_{ext})$ defines an isomorphism; in particular, recalling estimate (2.18), there exists a unique $\mathbf{u}^\perp \in V(\Omega_{ext})^\perp \subset H^1_0(\Omega_{ext})$ satisfying

$$\nabla \cdot \mathbf{u}^\perp = g - \nabla \cdot \mathbf{u}_\phi, \text{ and } |\mathbf{u}^\perp|_{1,\Omega_{ext}} \leq \beta^{-1}(||g||_{\Omega_p} + \sqrt{d}|\mathbf{u}_\phi|_{1,\Omega_{ext}}). \tag{5.20}$$

Hence, we can look for $\mathbf{w} \in V(\Omega_{ext})$ rather than $\mathbf{u}_B \in V_\phi(\Omega_{ext})(g)$ solving (5.4). Moreover,

$$E(\phi) := \mathbf{u}_\phi + \mathbf{u}^\perp \in V_\phi(g)(\Omega_{ext}), \qquad ||E(\phi)||_{1,\Omega_{ext}} \leq C(||\phi||_{1/2,\partial\Omega_{ext}} + ||g||_{\Omega_p}) \tag{5.21}$$

for some constant $C > 0$ depending on the LBB-constant β^{-1}. Note that solving (5.4) for $\mathbf{u}_B \in V_\phi(\Omega_{ext})(g)$ is equivalent to solving the same equations for $\mathbf{w} \in V(\Omega_{ext})$ where

$$\mathbf{w} = E(\phi) - \mathbf{u}_B \in V(\Omega_{ext}). \tag{5.22}$$

Unfortunately, bound (5.21) is unsatisfying. Indeed, as discussed in Section 2.5.1, in the case of inhomogeneous problem data $\mathbf{u}|_{\partial\Omega} = \phi \neq 0$, care must be taken to ensure the existence of stationary and evolutionary solutions of the NSE. Precisely, in proving the necessary *a priori* estimate for (steady) NSE, we must control the size of the convective term, see e.g. (2.50). We must control a similar convective term in the Brinkman case:

$$|\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w}| \leq \delta |\mathbf{w}|^2_{1,\Omega_{ext}}, \qquad \forall \mathbf{w} \in V(\Omega_{ext}) \tag{5.23}$$

where $E(\phi)$ is a V_ϕ-extension of ϕ so that $0 < \delta < \nu$. With application of Holder's inequality (2.22), we can show, as one possibility, that $\delta = C||E(\phi)||_{0,4}$. In general, however, we are not guaranteed control of $||E(\phi)||_{0,4}$. The breakdown of Stokes theorem when Ω_{ext} is not *simply* connected that prevents the existence of the *solenoidal* extension with the necessary properties. This result builds upon the subtle and technical work of Leray rigorously compiled by Hopf in [48] and elegantly presented by Galdi in [29]. In these works, they consider inhomogeneous boundary data for the steady NSE with divergence-free constraint $\nabla \cdot \mathbf{u} = 0$. See Section 2.5.1 for an overview and Chapter VIII in [29] for a complete treatment of well-posedness of the inhomogeneous, stationary NSE.

We apply the following proposition to establish existence of solutions to (5.4) for arbitrary data $\phi \in H^{1/2}_g(\partial\Omega_{ext})$ under the general divergence constraint $\nabla \cdot \mathbf{u}_B = g$. In addition to estimating (5.23) we also require the constructed extension to vanish in Ω_s so that $E(\phi)|_{\overline{\Omega}_s} = 0$ to avoid contamination by the volume penalty term $\varepsilon^{-1} >> 0$ on the RHS of our stability and error estimates.

Proposition 5.5.6. *Let Ω_{ext} be an open, connected domain with Lipschitz boundary $\partial\Omega_{ext}$. Suppose that $g \in L^2(\Omega_{ext})$, $\phi \in H^{1/2}_g(\partial\Omega_{ext})$, and $g|_{\Omega_{ext}-\Omega_p} \equiv 0$.*

1. *There exists an extension $E(\phi) \in V_\phi(g)(\Omega_{ext})$ satisfying*

$$\begin{cases} E(\phi)|_{\partial\Omega} = \phi, \quad \nabla \cdot E(\phi) = g, \quad E(\phi)|_{\overline{\Omega}_s} = 0 \\ |\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w}| \leq C(||g||_{\Omega_p} + ||\phi||_{1/2,\partial\Omega_{ext}})|\mathbf{w}|^2_1, \qquad \forall \mathbf{w} \in V(\Omega_{ext}) \end{cases}$$

2. *Suppose further that $g \in L^2_0(\Omega_{ext})$. Then for any $\delta > 0$, there exists an extension $E(\phi) \in V_\phi(g)(\Omega_{ext})$ satisfying*

$$\begin{cases} E(\phi)|_{\partial\Omega} = \phi, \quad \nabla \cdot E(\phi) = g, \quad E(\phi)|_{\overline{\Omega}_s} = 0 \\ |\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w}| \leq (\delta + C||g||_{\Omega_p})|\mathbf{w}|_1^2, \qquad \forall \mathbf{w} \in V(\Omega_{ext}) \end{cases}$$

Remark 5.5.7. *Note that for $g \equiv 0$ in Ω_{ext}, we can take $\delta < \nu$ so that (5.23) is satisfied and can be absorbed into $\nu|\mathbf{w}|_1^2$ to establish an a priori estimate for $\mathbf{u}_B$ for all $\phi \in H_0^{1/2}(\partial\Omega_{ext})$. However, for general $g \in L^2(\Omega_{ext})$, then g and ϕ are coupled via the necessary compatibility condition (5.3); hence, there is no obvious way to control the size of (5.23). Moreover, using the notation in the following proof, we can show that in Case 1 with general $g \in L^2(\Omega_{ext})$ (not necessarily in $L^2_0(\Omega_{ext})$), that $\delta^{-1/p+1/6}||\mathbf{u}_{\phi,\delta}||_{0,p,\Omega_{ext}} + \delta^{-1}\exp(-1/\delta)|\mathbf{u}_{\phi,\delta}|_{1,\Omega_{ext}} \leq C||\phi||_{1/2,\partial\Omega_{ext}}$ so that*

$$|E(\phi)|_{1,\Omega_{ext}} \leq C\exp(1/\delta) \to \infty, \qquad \text{as } \delta \to 0.$$

In Case 2, $g \in L^2_0(\Omega_{ext})$. Even though (5.23) is properly resolved, one can show that

$$||E(\phi)||_{\Omega_{ext}} \leq C\exp(1/\delta) \to \infty, \qquad \text{as } \delta \to 0.$$

Proof. Our proof follows the work of Galdi [29] (Chapter VIII and references therein) and Raviart and Girault [34] (Chapter IV, Section 2.1 and references therin) for divergence-free velocities and inhomogeneous Dirichlet boundary conditions. Recall that we consider a Lipschitz continuous, bounded domain Ω_{ext}. Fix $\delta > 0$ and define

$$d(x) = dist(x, \partial\Omega_{ext}), \qquad \gamma(\delta) := \exp(-1/\delta).$$

Then Lemma 6.2 of [34] (along with [28], Lemma III.6.2) ensures the existence of $\theta_\delta \in C^2(\overline{\Omega}_{ext})$ so that

$$\begin{cases} \theta_\delta(x) = 1, & \forall x \in \Omega_{ext} \text{ when } d(x) \leq \gamma(\delta)^2/2\kappa_1 \\ \theta_\delta(x) = 0, & \forall x \in \Omega_{ext} \text{ when } d(x) \geq 2\gamma(\delta) \\ |\theta_\delta(x)| \leq 1 & \forall x \in \Omega_{ext} \\ |\nabla\theta_\delta| \leq \kappa_2\delta/d(x), & \forall x \in \Omega_{ext} \end{cases}$$

for some constants κ_1, κ_2 depending only on the problem dimension. In particular, Lemma 2.5 of [34] ensures the existence of some constant $C > 0$ (depending only on Ω_{ext}) so that

$$||\mathbf{v}/d(x)|| \leq C|\mathbf{v}|_1, \qquad \forall \mathbf{v} \in H_0^1(\Omega_{ext}). \tag{5.24}$$

The *cut-off* function θ_δ is obtained by mollifying

$$\phi_\delta(t) := \begin{cases} 1, & 0 \leq t < \gamma(\delta)^2 \\ \delta \ln(\gamma(\delta)/t), & \gamma(\delta)^2 \leq t < \gamma(\delta) \\ 0, & t \geq \gamma(\delta) \end{cases} .$$

Case 1: Let $\mathbf{u}_\phi \in H^1_\phi(\Omega_{ext})$ satisfying $||\mathbf{u}_\phi||_{1,\Omega_{ext}} \leq C||\phi||_{1/2,\partial\Omega_{ext}}$. Define

$$\mathbf{u}_{\phi,\delta}(x) := \theta_\delta(x)\mathbf{u}_\phi(x).$$

Take $\delta > 0$ small enough to ensure that $\mathbf{u}_{\phi,\delta} = 0$ in $\Omega_{ext} - \Omega_f$. Next, notice that, since $\mathbf{u}_{\phi,\delta}|_{\Omega_s} = 0$ and $g|_{\Omega_s} = 0$ that

$$\int \nabla \cdot \mathbf{u}_{\phi,\delta} = \int_{\partial\Omega_{ext}} \phi \cdot \hat{\mathbf{n}} = \int_{\Omega_p} g \quad \Rightarrow \quad g - \nabla \cdot \mathbf{u}_{\phi,\delta} \in L^2_0.$$

Recall that Ω is connected. Then Lemma 2.4.1 ensures that $\nabla\cdot : V^\perp \to L^2_0$ defines an isomorphism; in particular, recalling estimate (2.18), there exists a unique $\mathbf{u}^\perp \in V^\perp \subset H^1_0$ satisfying

$$\nabla \cdot \mathbf{u}^\perp = g - \nabla \cdot \mathbf{u}_{\phi,\delta}, \text{ and } |\mathbf{u}^\perp|_1 \leq C(||g||_{\Omega_p} + |\mathbf{u}_{\phi,\delta}|_1).$$

We can extend $\mathbf{u}^\perp \in V(\Omega_{ext})^\perp$ so that $\mathbf{u}^\perp|_{\Omega_s} = 0$. Set $E(\phi) = \mathbf{u}_{\phi,\delta} + \mathbf{u}^\perp$. Therefore, we can estimate the problem term (5.23) with Hölder's inequality (2.22) and (5.24) to get

$$\begin{aligned} |\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w}| &= |\int_{\Omega_f} \mathbf{w} \cdot \nabla \mathbf{w} \cdot (\theta_\delta \mathbf{u}_\phi) + \int \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u}^\perp| \\ &\leq (\delta + ||\mathbf{u}^\perp||_{0,3})|\mathbf{w}|^2_{1,\Omega_{ext}}, \qquad \forall \mathbf{w} \in V(\Omega_{ext}). \end{aligned}$$

However, there is no control of $||\mathbf{u}^\perp||_{0,3}$ independent of $||\nabla \cdot \mathbf{u}_{\phi,\delta}||$ (this is a sharp result, see Remark VIII.4.1 in [29]). Therefore, we have that

$$|\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w}| \leq C(||g||_{\Omega_p} + ||\phi||_{1/2,\partial\Omega_{ext}})|\mathbf{w}|^2_{1,\Omega_{ext}}, \qquad \forall \mathbf{w} \in V(\Omega_{ext})$$

which completes the proof for general $g \in L^2(\Omega_p)$.

Case 2: Suppose now that $g \in L^2_0(\Omega_p)$. Suppose further that $\partial\Omega_{ext}$ is connected or $\int_{\partial\Omega_i} \phi \cdot \hat{\mathbf{n}} = 0$ on each connected component $\partial\Omega_i \subset \partial\Omega$. There exists a potential $\Psi \in H^2(\Omega_{ext})$ (scalar in 2d) so that $\mathbf{u}_\phi := \nabla \times \Psi \in V_\phi(\Omega_{ext})$ (scalar curl in 2d) (see e.g. Theorem I.3.1, Corollary I.3.3 in [34]). Define

$$\mathbf{u}_{\phi,\delta}(x) := \nabla \times (\theta_\delta(x)\mathbf{u}_\phi(x)).$$

Take $\delta > 0$ small enough to ensure that $\mathbf{u}_{\phi,\delta} = 0$ in $\Omega_{ext} - \Omega_f$. Recall that $g \in L^2_0(\Omega_p)$ and Ω is connected. Lemma 2.4.1 ensures that $\nabla\cdot : V^\perp \to L^2_0$ defines an isomorphism; in particular, recalling estimate (2.18), there exists a unique $\mathbf{u}^\perp \in V^\perp \subset H^1_0$ satisfying

$$\nabla \cdot \mathbf{u}^\perp = g, \text{ and } |\mathbf{u}^\perp|_1 \leq C||g||_{\Omega_p}.$$

We can extend $\mathbf{u}^\perp \in V(\Omega_{ext})^\perp$ so that $\mathbf{u}^\perp|_{\Omega_s} = 0$. Set $E(\phi) = \mathbf{u}_{\phi,\delta} + \mathbf{u}^\perp$. Therefore, we can estimate the problem term (5.23) with Hölder's inequality (2.22) and (5.24) to get

$$\begin{aligned} |\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla E(\phi) \cdot \mathbf{w}| &= |\int_{\Omega_f} \mathbf{w} \cdot \nabla \mathbf{w} \cdot (\theta_\delta \mathbf{u}_\phi) + \int \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u}^\perp| \\ &\leq (\delta + C||g||_{\Omega_p})|\mathbf{w}|^2_{1,\Omega_{ext}}, \qquad \forall \mathbf{w} \in V(\Omega_{ext}) \end{aligned}$$

which completes the proof for $g \in L^2_0(\Omega_p)$. □

5.5.2 Proof of Well-Posedness, Theorem 5.5.1

We prove existence of $(\mathbf{u}_B, p_B) \in H^1_\phi \times L^2_0$ solving (5.4) using the Leray-Schauder fixed point theorem. We first introduce some notation. Fix $g \in L^2(\Omega_{ext})$, $\phi \in H^{1/2}_g(\partial\Omega_{ext})$, and $E(\phi) \in V_\phi(g)(\Omega_{ext})$. Define:

1. $T : W^{-1,2}(\Omega_{ext}) \to V_\phi(g)(\Omega_{ext})$ so that $T(\mathbf{y}) := \mathbf{w}$ where $\mathbf{w} \in V_\phi(g)(\Omega_{ext})$ solves

$$\forall \mathbf{v} \in V(\Omega_{ext}), \qquad a_{ext}(\mathbf{w}, \mathbf{v}) =< \mathbf{y}, \mathbf{v} >_{W^{-1,2}(\Omega_{ext}) \times H^1_0(\Omega_{ext})} \tag{5.25}$$

2. $N : V_\phi(g)(\Omega_{ext}) \to W^{-1,2}(\Omega_{ext})$ so that

$$N(\mathbf{w}) := \mathbf{f} - \mathbf{w} \cdot \nabla \mathbf{w} \tag{5.26}$$

3. $F : V_\phi(g)(\Omega_{ext}) \to V_\phi(g)(\Omega_{ext})$ so that

$$F := T \circ N. \tag{5.27}$$

In order to apply the Leray-Schauder fixed point theorem, we show that F is a continuous, compact operator. We prove this through the following lemmas.

Lemma 5.5.8. *T is a well-defined linear, continuous operator.*

Proof. T is clearly linear. Well-posedness and boundedness of T follows from from the continuity of $\mathbf{v} \in V(\Omega_{ext}) \mapsto < \mathbf{y}, \mathbf{v} >$ and $(\mathbf{u}, \mathbf{v}) \in V(\Omega_{ext}) \times V(\Omega_{ext}) \mapsto a_{ext}(\mathbf{u}, \mathbf{v})$ and coercivity of $\mathbf{v} \in V(\Omega_{ext}) \mapsto a_{ext}(\mathbf{v}, \mathbf{v})$ established in Lemmas 5.4.1, 5.4.3. In particular, let $E(\phi) \in V_\phi(g)(\Omega_{ext})$. Write $\mathbf{w}_0 = \mathbf{w} - E(\phi) \in V(\Omega_{ext})$ to get $a_{ext}(\mathbf{w}_0, \mathbf{v}) = l_{1,ext}(\mathbf{y}) + a_{ext}(E(\phi), \mathbf{v})$ for all $\mathbf{v} \in V(\Omega_{ext})$. In addition to the previous bounds, we have $\mathbf{v} \in V(\Omega_{ext}) \mapsto < \mathbf{y}, \mathbf{v} > + a_{ext}(E(\phi), \mathbf{v})$ continuously so that application of Lax-Milgram ensures the existence and uniqueness of $\mathbf{w}_0 \in V(\Omega_{ext})$. Existence and uniqueness of $\mathbf{w} = \mathbf{w} + E(\phi) \in V_\phi(g)(\Omega_{ext})$ follows. It follows that T is continuous as a linear and bounded operator. □

Lemma 5.5.9. *For any $\mathbf{w} \in V_\phi(g)(\Omega_{ext})$, then $N(\mathbf{w}) \in W^{-d/4,2}(\Omega_{ext})$ and N maps $V_\phi(g)(\Omega_{ext}) \to W^{-d/4,2}$ continuously.*

Proof. The Ladyzhenskaya inequality (2.24) give $||\mathbf{w}||_{0,4,\Omega_{ext}} \le C||\mathbf{w}||_{1,\Omega_{ext}}$ for any $\mathbf{w} \in H^1(\Omega_{ext})$. The Sobolev embedding $W^{d/4,2}(\Omega_{ext}) \hookrightarrow L^4(\Omega_{ext})$ for $d = 2, 3$ gives $||\mathbf{v}||_{0,4,\Omega_{ext}} \le C||\mathbf{v}||_{d/4,\Omega_{ext}}$ for any $\mathbf{v} \in W^{d/4,2}(\Omega_{ext})$. Therefore, together with Hölder's inequality (2.22), we get

$$\int_{\Omega_{ext}} \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{v} \le ||\mathbf{w}||_{0,4,\Omega_{ext}} |\mathbf{w}|_{1,\Omega_{ext}} ||\mathbf{v}||_{0,4,\Omega_{ext}} \le C||\mathbf{w}||_1^2 ||\mathbf{v}||_{d/4,\Omega_{ext}}.$$

Moreover,

$$< \mathbf{f}, \mathbf{v} > \le ||\mathbf{f}||_{\Omega_{ext}} ||\mathbf{v}||_{\Omega_{ext}} \le C||\mathbf{f}||_{\Omega_{ext}} ||\mathbf{v}||_{d/4,\Omega_{ext}} \qquad \forall \mathbf{v} \in V(\Omega_{ext})$$

so that

$$\begin{aligned} ||N(\mathbf{w})||_{-d/4,\Omega_{ext}} :&= \sup_{0 \neq \mathbf{v} \in W^{d/4,2}(\Omega_{ext})} \frac{|\int_{\Omega_{ext}} N(\mathbf{w}) \cdot \mathbf{v}|}{||\mathbf{v}||_{d/4,\Omega_{ext}}} \\ &\le C(||\mathbf{f}||_{\Omega_{ext}} + ||\mathbf{w}||_{1,\Omega_{ext}}^2) < \infty, \qquad \forall \mathbf{w} \in V_\phi(g)(\Omega_{ext}). \end{aligned}$$

Therefore, $N(\mathbf{w}) \in W^{-d/4,2}(\Omega_{ext})$. We now show that N is a continuous operator. Fix $\mathbf{w}_1$, $\mathbf{w}_2 \in V_\phi(g)(\Omega_{ext})$. Then

$$\begin{aligned}
\int_{\Omega_{ext}} (N(\mathbf{w}_1) - N(\mathbf{w}_2)) \cdot \mathbf{v} &= \int_{\Omega_{ext}} (\mathbf{w}_1 \cdot \nabla \mathbf{w}_1 - \mathbf{w}_2 \cdot \nabla \mathbf{w}_2) \cdot \mathbf{v} \\
&= \int_{\Omega_{ext}} (\mathbf{w}_1 - \mathbf{w}_2) \cdot \nabla \mathbf{w}_1 \cdot \mathbf{v} + \int_{\Omega_{ext}} \mathbf{w}_2 \cdot \nabla (\mathbf{w}_1 - \mathbf{w}_2) \cdot \mathbf{v} \\
&\leq C(||\mathbf{w}_1||_{1,\Omega_{ext}} + ||\mathbf{w}_2||_{1,\Omega_{ext}})||\mathbf{w}_1 - \mathbf{w}_2||_{1,\Omega_{ext}}||\mathbf{v}||_{d/4,\Omega_{ext}}.
\end{aligned}$$

Therefore, dividing by $||\mathbf{v}||_{d/4,\Omega_{ext}}$, and taking sup over all $0 \neq \mathbf{v} \in W^{d/4,2}(\Omega_{ext})$ proves that N maps $V_\phi(g)(\Omega_{ext}) \to W^{-d/4,2}(\Omega_{ext})$ continuously. □

Proposition 5.5.10. *$F : V_\phi(g)(\Omega_{ext}) \to V_\phi(g)(\Omega_{ext})$ is a continuous, compact operator.*

Proof. For $d = 2$ or 3, $W^{-d/4,2}(\Omega_{ext})$ is compactly embedded (hence continuously) in $W^{-1,2}(\Omega_{ext})$. Hence, we summarize,

$$H^1 \underset{N,\text{ cont.}}{\longmapsto} W^{-d/4,2} \underset{compact}{\hookrightarrow} W^{-1,2} \underset{T,\text{ cont.}}{\longmapsto} H^1.$$

Then F is continuous and compact as a composition of continuous and continuous/compact operators. □

We are now ready to prove the well-posedness of (5.4)

Theorem 5.5.1. Fix $E(\phi) \in V_\phi(g)$ and define

$$F_0 : V(\Omega_{ext}) \to V(\Omega_{ext}), \quad F_0(\mathbf{v}_0) := F(\mathbf{v}_0 + E(\phi)) - E(\phi).$$

where $F : V_\phi(g)(\Omega_{ext}) \to V_\phi(g)(\Omega_{ext})$ is defined in (5.27) via (5.25), (5.26). Then F_0 is a continuous, compact operator on the linear space V. Note that if $\mathbf{v}_0$ is a fixed point of F_0, then $\mathbf{v}_0 + E(\phi)$ is a fixed point of F solving (5.4). Consider the family of fixed point problems: for any $0 \leq \lambda \leq 1$, find $\mathbf{u}_\lambda \in V_\phi(g)(\Omega_{ext})$ so that $\mathbf{w}(\lambda) = \mathbf{u}_\lambda - E(\phi)$ satisfy $\mathbf{w}(\lambda) = \lambda F_0(\mathbf{w}(\lambda)) \in V(\Omega_{ext})$. Fix $\mathbf{v} \in V(\Omega_{ext})$ Then $\mathbf{w}(\lambda)$ satisfies

$$\begin{aligned}
a_{ext}(\mathbf{w}(\lambda), \mathbf{v}) + \lambda c_{ext}(\mathbf{w}(\lambda), \mathbf{w}(\lambda), \mathbf{v}) &= \lambda < \mathbf{f}, \mathbf{v} > -\lambda a_{ext}(E(\phi), \mathbf{v}) \\
&- \lambda c_{ext}(E(\phi), \mathbf{w}(\lambda), \mathbf{v}) - \lambda c_{ext}(\mathbf{w}(\lambda), E(\phi), \mathbf{v}) - \lambda c_{ext}(E(\phi), E(\phi), \mathbf{v}). \qquad (5.28)
\end{aligned}$$

Suppose that the extension satisfies $E(\phi)|_{\Omega_s} = 0$ so that $a_{ext}(E(\phi), \mathbf{v}) = (\tilde{\nu}\nabla E(\phi), \nabla \mathbf{v})$. Recall that $c_{ext}(\mathbf{w}(\lambda), \mathbf{w}(\lambda), \mathbf{w}(\lambda)) = 0$ via (2.31) since $\mathbf{w}(\lambda) \in V(\Omega_{ext})$. Moreover, (2.29) gives $c_{ext}(E(\phi), \mathbf{w}(\lambda), \mathbf{w}(\lambda)) = -\frac{1}{2}\int_{\Omega_p} g|\mathbf{w}(\lambda)|^2$. Testing with $\mathbf{v} = \mathbf{w}(\lambda)$ and applying these identities to (5.28) gives

$$\begin{aligned}\alpha_1\nu|\mathbf{w}(\lambda)|_1^2 &= \lambda < \mathbf{f}, \mathbf{w}(\lambda) > -(\tilde{\nu}\nabla E(\phi), \nabla \mathbf{w}(\lambda)) \\ &- \int E(\phi)\cdot\nabla E(\phi)\cdot\mathbf{w}(\lambda) - (\int \mathbf{w}(\lambda)\cdot\nabla E(\phi)\cdot\mathbf{w}(\lambda) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{w}(\lambda)|^2). \qquad (5.29)\end{aligned}$$

Recall that $\lambda \leq 1$. Application of the duality between $W^{-1,2}(\Omega_{ext}) \times H_0^1(\Omega_{ext})$, and Hölder's (2.22) and Young's (2.21) inequalities give, after simplification,

$$\begin{aligned}\frac{\nu}{2}|\mathbf{w}(\lambda)|_1^2 &\leq C\nu^{-1}(||\mathbf{f}||_{-1,\Omega_{ext}}^2 + \nu^2|E(\phi)|_{1,\Omega_{ext}}^2 + ||E(\phi)||_{0,4}^4) \\ &+ |\int \mathbf{w}(\lambda)\cdot\nabla E(\phi)\cdot\mathbf{w}(\lambda) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{w}(\lambda)|^2|. \qquad (5.30)\end{aligned}$$

Require that $E(\phi)$ (via a possible small data constraint on ϕ and g or possibly ν^{-1}) satisfies (5.10):

$$\left|\int \mathbf{w}(\lambda)\cdot\nabla E(\phi)\cdot\mathbf{w}(\lambda) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{w}(\lambda)|^2\right| \leq \frac{\nu}{4}|\mathbf{w}(\lambda)|_1^2.$$

Then

$$|\mathbf{w}(\lambda)|_1^2 \leq C\nu^{-2}(||\mathbf{f}||_{-1,\Omega_{ext}}^2 + \nu^2|E(\phi)|_{1,\Omega_{ext}}^2 + ||E(\phi)||_{0,4}^4).$$

Therefore, $\mathbf{w}(\lambda)$ is uniformly bounded in $H^1(\Omega_{ext})$ with respect to $0 \leq \lambda \leq 1$. Since F_0 is continuous, compact (inherited from F via Proposition 5.5.10), we have the necessary bound uniform in λ to conclude existence of $\mathbf{u}_0 \in V(\Omega_{ext})$ satisfying $\mathbf{u}_0 = F(\mathbf{u}_0 + E(\phi)) - E(\phi)$ via Leray-Schauder. Therefore $\mathbf{u}_B := \mathbf{u}_0 + E(\phi) \in V_\phi(g)(\Omega_{ext})$ satisfies $\mathbf{u}_B = F(\mathbf{u}_B)$ and hence (5.4).

The method for deriving a stability bound for any solution $\mathbf{u}_B \in V_\phi(g)(\Omega_{ext})$ is similar and leads to the same result as for $\mathbf{u}_\lambda$, but we also desire an estimate in Ω_s for BrVP-solutions. Suppose that $\mathbf{f} \in L^2(\Omega_{ext})$. Recall $E(\phi)|_{\Omega_s} \equiv 0$. Set $\mathbf{v} = \mathbf{w}$ in (5.28) and suppress λ dependence (set explicit $\lambda = 1$). Note here that

$$a_{ext}(\mathbf{w}, \mathbf{w}) = ||\tilde{\nu}\nabla\mathbf{w}||_{\Omega_{ext}}^2 + \frac{\nu_p}{K_p}||\mathbf{w}||_{\Omega_p}^2 + \frac{\nu}{\varepsilon}||\mathbf{w}||_{\Omega_s}^2.$$

We estimate the RHS-terms similarly as above to conclude that

$$
\begin{aligned}
&\frac{\nu}{2}|\mathbf{w}|_1^2 + \frac{\nu_s}{2}|\mathbf{w}|_{1,\Omega_s}^2 + \frac{\nu_p}{K_p}||\mathbf{w}||_{\Omega_p}^2 + \frac{\nu}{\varepsilon}||\mathbf{w}||_{\Omega_s}^2 \\
&\leq C\nu^{-1}(||\mathbf{f}||_{-1,\Omega_{ext}}^2 + \nu^2|E(\phi)|_{1,\Omega_{ext}}^2 + ||E(\phi)||_{0,4}^4) \\
&\quad + \left| \int \mathbf{w}(\lambda)\cdot\nabla E(\phi)\cdot\mathbf{w}(\lambda) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{w}(\lambda)|^2 \right|.
\end{aligned} \tag{5.31}
$$

Again require that $E(\phi)$ (via a possible small data constraint on ϕ and g or possibly ν^{-1}) satisfies (5.10) to conclude

$$
\begin{aligned}
&|\mathbf{w}|_1^2 + \frac{\nu_s}{\nu}|\mathbf{w}|_{1,\Omega_s}^2 + K_p^{-1}||\mathbf{w}||_{\Omega_p}^2 + \varepsilon^{-1}||\mathbf{w}||_{\Omega_s}^2 \\
&\leq C\nu^{-2}(||\mathbf{f}||_{-1,\Omega_{ext}}^2 + \nu^2|E(\phi)|_{1,\Omega_{ext}}^2 + ||E(\phi)||_{0,4}^4).
\end{aligned} \tag{5.32}
$$

Estimate (5.11) follow from the triangle inequality $|\mathbf{u}_\varepsilon|_1 \leq |\mathbf{w}|_1 + |E(\phi)|_1$ noting that $\mathbf{u}_\varepsilon|_{\Omega_s} = \mathbf{w}|_{\Omega_s}$ since $E(\phi)|_{\Omega_s} \equiv 0$.

To establish uniqueness, suppose $\mathbf{u}_1$, $\mathbf{u}_2 \in V_\phi(\Omega_{ext})(g)$ are two distinct solutions satisfying (5.4). Then for fixed $\mathbf{v} \in V(\Omega_{ext})$, subtracting the corresponding equations and writing $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ provides

$$a_{ext}(\mathbf{w},\mathbf{v}) + c_{ext}(\mathbf{w},\mathbf{u}_1,\mathbf{v}) + c_{ext}(\mathbf{u}_2,\mathbf{w},\mathbf{v}) = 0.$$

Set $\mathbf{v} = \mathbf{w} \in V(\Omega_{ext})$. Recall that $c_{ext}(\mathbf{u},\mathbf{v},\mathbf{w}) \leq \alpha_2||\mathbf{u}||_{1,\Omega_{ext}}||\mathbf{v}||_{1,\Omega_{ext}}||\mathbf{w}||_{1,\Omega_{ext}}$ along with *a priori* estimate (5.11). We conclude that:

$$\alpha_1||\mathbf{w}||_1^2 \leq 2\alpha_2 M_{B,0}||\mathbf{w}||_1^2.$$

Therefore, if small data condition (5.13) is satisfied, we have $||\mathbf{w}||_1 \leq 0$ so that $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2 = 0$ which ensures uniqueness of solutions $\mathbf{u}_B$.

We next investigate the existences and boundedness of p_B. From above, we have that there exists a unique $\mathbf{u}_B$ (under possible restrictions on the problem data) so that

$$a_{ext}(\mathbf{u}_B,\mathbf{v}) + c_{ext}(\mathbf{u}_B,\mathbf{u}_B,\mathbf{v}) - <\mathbf{f},\mathbf{v}> = 0 \qquad \forall \mathbf{v} \in V(\Omega_{ext}).$$

Therefore

$$-\nabla\cdot(\tilde{\nu}\nabla\mathbf{u}_B) + \mathbf{u}_B\cdot\nabla\mathbf{u}_B + \nu K^{-1}\mathbf{u}_B - \mathbf{f} \in (V(\Omega_{ext}))^0.$$

From Lemma 2.4.1, we have that $\nabla : L_0^2(\Omega_{ext}) \to (V(\Omega_{ext}))^0$ defines an isomorphism: in particular, there exists a unique $p_B \in L_0^2(\Omega_{ext})$ so that $(\mathbf{u}_B, p_B) \in H_\phi^1(\Omega_{ext}) \times L_0^2(\Omega_{ext})$ solve (5.4). As for an estimate, solve (5.4)(a) for p_B:

$$\begin{aligned}\frac{|b_{ext}(\mathbf{v}, p_B)|}{|\mathbf{v}|_{1,\Omega_{ext}}} &= \frac{1}{|\mathbf{v}|_{1,\Omega_{ext}}}| < \mathbf{f}, \mathbf{v} > -a_{ext}(\mathbf{u}_B, \mathbf{v}) - c_{ext}(\mathbf{u}_B, \mathbf{u}_B, \mathbf{v})| \\ &\leq C(||\mathbf{f}||_{-1,\Omega_{ext}} + \nu||\mathbf{u}_B||_1 + \nu_s|\mathbf{u}_B|_{1,\Omega_s} + \nu\varepsilon^{-1}||\mathbf{u}_B||_{\Omega_s} + ||\mathbf{u}_B||^2_{1,\Omega_{ext}}).\end{aligned}$$

Recall Estimate (5.11). Application of the inf-sup condition on $H_0^1(\Omega_{ext}) \times L_0^2(\Omega_{ext})$ (see Estimate (2.16 in Lemma 2.4.1)) we conclude after simplification

$$||p_B||_{\Omega_{ext}} \leq CM_{0,B}^2\nu^{-2}\varepsilon^{-1/2}.$$

Unfortunately, the RHS diverges to ∞ as $\varepsilon \to 0$. On the other hand, we can restrict $\mathbf{v}|_{\Omega_s}$ in (5.5.2) so that

$$a(\mathbf{u}_B, \mathbf{v}) + c(\mathbf{u}_B, \mathbf{u}_B, \mathbf{v}) - < \mathbf{f}, \mathbf{v} >= 0 \qquad \forall \mathbf{v} \in V.$$

Therefore

$$-\nabla \cdot (\tilde{\nu}\nabla\mathbf{u}_B) + \mathbf{u}_B \cdot \nabla\mathbf{u}_B + \nu K^{-1}\mathbf{u}_B - \mathbf{f} \in V^0.$$

From Lemma 2.4.1, we have that $\nabla : L_0^2 \to V^0$ defines an isomorphism: in particular, there exists a unique $p_B \in L_0^2$ so that $(\mathbf{u}_B, p_B) \in H_\phi^1 \times L_0^2$ solve (5.4) for restricted $\mathbf{v} \in V$. Now we estimate p_B on Ω similarly as above to get

$$\begin{aligned}\frac{|b(\mathbf{v}, p_B)|}{|\mathbf{v}|_1} &= \frac{1}{|\mathbf{v}|_1}| < \mathbf{f}, \mathbf{v} > -a(\mathbf{u}_B, \mathbf{v}) - c(\mathbf{u}_B, \mathbf{u}_B, \mathbf{v})| \\ &\leq C(||\mathbf{f}||_{-1} + \nu||\mathbf{u}_B||_1 + ||\mathbf{u}_B||^2_{1,\Omega}).\end{aligned}$$

Therefore, Estimate (5.11). along with the inf-sup condition on $H_0^1 \times L_0^2$ gives

$$||p_B|| \leq CM_{0,B}^2\nu^{-2}. \tag{5.33}$$

□

5.6 NUMERICAL RESULTS

We consider three distinct numerical experiments in this section. First, we confirm the convergence rate ($h \to 0$) for BrVP-solutions. Next, we demonstrate the robust capability of our proposed FE-BrVP to handle a source and inhomogeneous boundary conditions ($\nabla \cdot \mathbf{u}_{\varepsilon,h} \neq 0$ and $\int_{\partial\Omega_{ext}} \mathbf{u}_{\varepsilon,h} \cdot \hat{\mathbf{n}} \neq 0$). We then consider flow past a non-uniform array of solid obstacles to test the rate of convergence ($h,\ \varepsilon \to 0$) for Stokes-Brinkman to Stokes with no-slip velocity condition imposed at each obstacle interface.

We utilize Taylor-Hood mixed FE's (piecewise quadratics for velocity and piecewise linear pressure) for the discretization. Note that the optimal convergence rate for steady NS and Stokes velocity approximations is of order $\mathcal{O}(h^2)$ in $H^1(\Omega_{ext})$ and $\mathcal{O}(h^3)$ in $L^2(\Omega_{ext})$. We use a Picard iteration to solve the nonlinear BrVP: i.e. set $\mathbf{u}_{\varepsilon,h}^{(0)} = 0$, solve for $\mathbf{u} = \mathbf{u}_{\varepsilon,h}^{(n+1)}$ lagging the convective term by $\mathbf{u}_{\varepsilon,h}^{(n)} \cdot \nabla \mathbf{u}_{\varepsilon,h}^{(n+1)}$. We use the FreeFem++ software for each of our simulations.

Experiment 1: For the first experiment, we consider $\Omega_{ext} = [0,1]^2$ with $\Omega_s = ([0,0.5] \times [0,0.5]) \cup ([0.5,1] \times [0.5,1])$, $\nu = 10^{-2}$, $\varepsilon = 10^{-2}$, $\tilde{\nu}_s = \nu/\varepsilon$, $\tilde{\nu}_f = \nu$, $K_f = 1/\varepsilon$, $K_s = \varepsilon$ and true velocity and pressure given by

$$\mathbf{u}_\varepsilon = \begin{bmatrix} 0.01\pi sin(\pi x) cos(\pi y) \\ -0.01\pi cos(\pi x) sin(\pi y) \end{bmatrix}, \qquad p_\varepsilon = 0.25(x-0.5)(y-0.5).$$

Note that since the velocity is smooth and K, $\tilde{\nu}$ are discontinuous, it follows that $\mathbf{f}$ must be discontinuous. A uniform triangular mesh is used. The results for this experiment are compiled in Table 5.1. Notice that the H^1-convergence rate is optimal $\mathcal{O}(h^2)$ supporting the basic effectiveness of the proposed FE-discretization of the BrVP and confirming the predictions of the convergence analysis.

Experiment 2: Now we consider $\Omega_{ext} = [0,2] \times [0,1]$, $\nu = 10^{-2}$, $\tilde{\nu}_s = \nu/\varepsilon$, $K_f = 1/\varepsilon$, $K_s = \varepsilon$. Here, we consider a 0.2×0.2 source $g = 1$ centered in the domain Ω_{ext} and the resulting flow around two square obstacles as shown in Figure 5.1 with imposed Dirichlet boundary conditions

$$\mathbf{u}_\varepsilon|_{x=0} = -0.12y(1-y), \qquad \mathbf{u}_\varepsilon|_{x=2} = 0.12y(1-y), \qquad \mathbf{u}_\varepsilon|_{y=0,1} = 0.$$

Table 5.1: Convergence rate data for the first experiment

h	$\|\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}_\varepsilon\|\|$	$\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}_\varepsilon\|_1$	Rate (H^1)
0.09428	3.2033e-6	4.2548e-3	—
0.04714	4.0143e-7	1.0663e-4	5.3
0.02357	5.0215e-8	2.6674e-5	2.0
0.01179	6.2781e-9	6.6696e-6	2.0

A uniform triangular mesh is used. The velocity plot in Figure 5.1 shows the BrVP-approximation to the proposed flow for Experiment 2 corresponding with our intuition. To quantify the accuracy of the approximation, we list the L^2 norm of $\mathbf{u}_{\varepsilon,h}$ in Ω_s and H^1 semi-norm in Ω_s and Ω_{ext} for several combinations of h and ε-values in Table 5.2. Notice that $||\mathbf{u}_{\varepsilon,h}||_{\Omega_s}$ and $|\mathbf{u}_{\varepsilon,h}|_{1,\Omega_s}$ converge at a rate $\mathcal{O}(\varepsilon)$ for each indicated h as expected. Also note that $|\mathbf{u}_{\varepsilon,h}|_1$ remains bounded (relatively constant in fact) with h and ε.

Experiment 3: We investigate the relation between the velocity field predicted by Stokes-Brinkman and that predicted by Stokes with no-slip boundary condition imposed at each solid interface. We consider $\Omega_{ext} = [0,2] \times [0,1]$, $\nu = 10^2$, $\mathbf{f} = 0$, $g = 0$, $\tilde{\nu}_s = \nu/\varepsilon$, $K_f = 1/\varepsilon$, $K_s = \varepsilon$. Here, we consider the non-uniform array of square obstacles as shown in Figure 5.2 with imposed Dirichlet boundary conditions

$$\mathbf{u}_\varepsilon|_{x=0} = y(1-y), \qquad \mathbf{u}_\varepsilon|_{x=2} = y(1-y), \qquad \mathbf{u}_\varepsilon|_{y=0,1} = 0.$$

The Stokes velocity used for comparison is obtained by approximating the Stokes equation with the Taylor-Hood mixed FE's for pressure and velocity with a fine mesh, $h_{max} = 0.018760$. The mesh is constructed by FreeFem++ based on the Delaunay triangulation. We solve Stokes-Brinkman on a coarser, uniform triangular mesh. As illustrated in Table 5.3, there appears to be a degradation in the convergence rate of $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}$ in L^2 as $h \to 0$ for

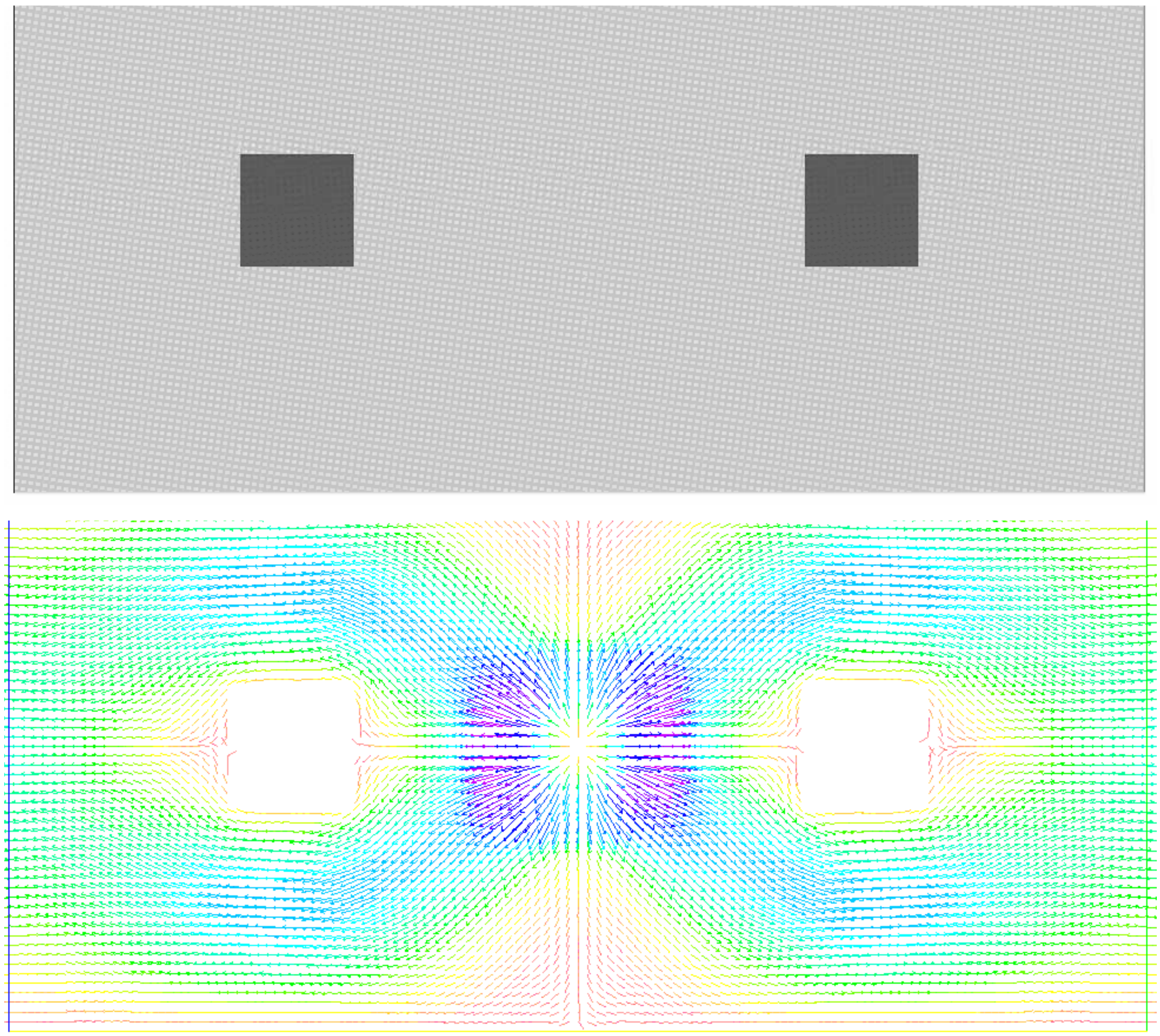


Figure 5.1: Experiment 2: (top) problem domain, dark squares represent solid obstacles , (bottom) NS-Brinkman velocity approximation

Table 5.2: Convergence rate data for the second experiment

	h	$\|\|\mathbf{u}_{\varepsilon,h}\|\|_{\Omega_s}$	$\|\mathbf{u}_{\varepsilon,h}\|_{1,\Omega_s}$	$\|\mathbf{u}_{\varepsilon,h}\|_1$
$\varepsilon = 10^{-5}$	0.1414	4.6299e-5	2.1236e-6	0.2973
	0.07071	4.7124e-5	2.3095e-6	0.2992
$\varepsilon = 10^{-10}$	0.1414	4.6543e-10	2.1334e-11	0.2980
	0.07071	4.4738e-10	2.3206e-11	0.2999
$\varepsilon = 10^{-15}$	0.1414	4.6543e-15	2.1335e-16	0.2980
	0.07071	4.7377e-15	2.3296e-16	0.2999

larger $\varepsilon = 10^{-5}$. For $\varepsilon = 10^{-10}$ and 10^{-15}, the Stokes-Brinkman velocity appears to converge to the Stokes velocity with $h \to 0$ twice as fast in the L^2 norm than H^1 semi-norm, as one would expect. Our results compiled in Table 5.4 also indicates that $\mathbf{u}_{\varepsilon,h} \to 0$ in Ω_s as $\varepsilon \to 0$ at a rate $\mathcal{O}(\varepsilon)$, once again, as expected.

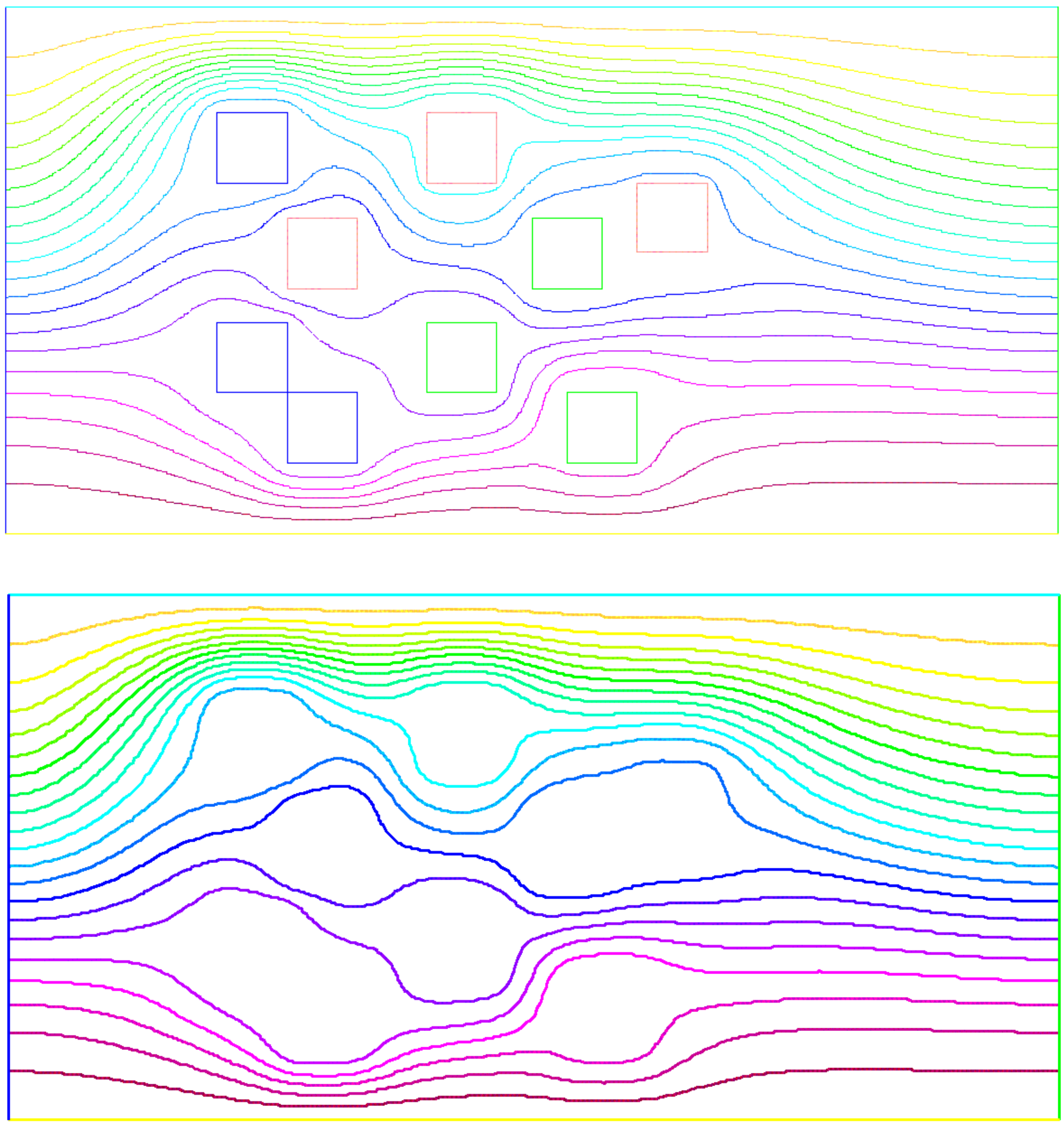


Figure 5.2: Experiment 3: Squares in top plot represent outlines of solid obstacles, (top) Stokes velocity approximation, streamlines , (bottom) Stokes-Brinkman velocity approximation, streamlines

Table 5.3: Convergence rate data for the third experiment

	h	$\|\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}\|\|$	Rate L^2	$\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}\|_1$	Rate H^1
$\varepsilon = 10^{-5}$	0.09428	2.6968e-2	—	1.4324e-0	—
	0.04714	1.1757e-2	1.2	8.6499e-1	0.73
	0.02357	7.0509e-3	0.74	4.9880e-1	0.79
$\varepsilon = 10^{-10}$	0.09428	2.4180e-2	—	1.4413e-0	—
	0.04714	7.9136e-3	1.6	8.8644e-1	0.70
	0.02357	2.0529e-3	1.9	4.8989e-1	0.86
$\varepsilon = 10^{-15}$	0.09428	2.4180e-2	—	1.4413e-0	—
	0.04714	7.9154e-3	1.6	8.6437e-1	0.74
	0.02357	2.0529e-3	1.9	4.8989e-1	0.82

Table 5.4: Convergence rate data for the third experiment; $h = 0.02357$

	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-15}$
$\|\|\mathbf{u}_{\varepsilon,h}\|\|_{\Omega_s} =$	2.0353e-3	2.0983e-08	2.0983e-13
$\|\mathbf{u}_{\varepsilon,h}\|_{1,\Omega_s} =$	7.8208e-5	8.0846e-10	8.0846e-15

6.0 CONSISTENCY ANALYSIS OF STATIONARY BRVP

Herein we derive error estimates for approximations of steady Navier-Stokes (NS) equations (NSE) by through Brinkman volume penalization (BrVP). BrVP approximations correspond to solutions (possibly weak) of the stationary Brinkman equation previously presented in (5.1): Find velocity $\mathbf{u}_\varepsilon : \Omega_{ext} \to \mathbb{R}^d$ and pressure $p_\varepsilon : \Omega_{ext} \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla \cdot (\tilde{\nu} \nabla \mathbf{u}_\varepsilon) + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon + \nabla p_\varepsilon + \nu \mathbf{K}^{-1} \mathbf{u}_\varepsilon &= \mathbf{f}, \quad \text{in } \Omega_{ext} \\ \nabla \cdot \mathbf{u}_\varepsilon &= g, \quad \text{in } \Omega_{ext} \\ \mathbf{u}_\varepsilon &= \phi, \quad \text{in } \partial\Omega_{ext}. \end{aligned} \tag{6.1}$$

The change in notation for BrVP focuses attention on the volume penalization drag term $\varepsilon^{-1}\nu\chi_s\mathbf{u}_\varepsilon$ that replaces explicit enforcement of no-slip boundary conditions on the solid matrix of a porous medium. As $\varepsilon \to 0$, it has been shown for the linear, homogeneous Brinkman case in [3] that $\mathbf{u}_\varepsilon \to \mathbf{u}$ at a rate $\mathcal{O}(\varepsilon)$ in $H^1(\Omega_{ext})$ if $\tilde{\nu}_s = \nu\varepsilon^{-1}$ and $\mathcal{O}(\varepsilon^{1/4})$ in $H^1(\Omega_{ext})$ if $\tilde{\nu}_s = \nu$. The time-dependent/nonlinear problem is analyzed in [4] for homogeneous data, starting from rest with similar estimates (discussed more in Chapter 7). We generalize their results to steady-state, nonlinear solutions in $\mathbb{R}^2$ or $\mathbb{R}^3$ and inhomogeneous data $\mathbf{u}_\varepsilon|_{\partial\Omega_{ext}} = \phi \neq 0$ and $\nabla \cdot \mathbf{u}_\varepsilon = g \neq 0$.

Let the boundary of the solid obstacles be denoted $\partial\Omega_s$. The analysis for $\varepsilon \to 0$ is delicate for $\mathbf{u}_\varepsilon$ in (6.1). In particular, *passing the limit* requires regularity of the approximating solution (e.g. the NSE-solution) that may or may not be available. We are interested in finite element (FE) approximations of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ and denoted $(\mathbf{u}_{\varepsilon,h}, p_\varepsilon)$. In either case, *controlling* stress on the internal solid matrix $\partial\Omega_s$ is paramount in proving control on $\varepsilon^{-1}\nu\chi_s\mathbf{u}_\varepsilon$ as $\varepsilon \to 0$. In the discrete case, however, the notion of stress on the boundary $-\nu(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}_h + p_h\hat{\mathbf{n}}|_{\partial\Omega_s}$ is not well-defined since this requires, in general, that $-\nu\Delta\mathbf{u}_h + \nabla p_h \in L^2$ (which is not true, e.g., for C^0-velocity elements and L^2-pressure elements).

Let $(\mathbf{u}, p)$ be the NSE velocity/pressure and $(\mathbf{u}_h, p_h)$ be the corresponding FE-approximation. We show that $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}_h$ in Ω_{ext} and $\mathbf{u}_{\varepsilon,h} \to 0$ in the solid obstacles $\Omega_s \subset \Omega_{ext}$ as $\varepsilon \to 0$; precisely, for $\mathcal{O}(1)$-model parameters

$$||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_{H^1} \leq \mathcal{O}(\varepsilon), \qquad ||\mathbf{u}_{\varepsilon,h}||_{H^1(\Omega_s)} \leq \mathcal{O}(\varepsilon). \tag{6.2}$$

See estimates (6.20), (6.21). We use this estimate in conjunction with convergence analysis of the FE NSE-approximation to show that, for $\mathcal{O}(1)$-model parameters,

$$||\mathbf{u}_{\varepsilon,h} - \mathbf{u}||_{H^1} \leq \mathcal{O}(\varepsilon + h^k)$$

for $k \in \mathbb{N}$, polynomial degree of velocity space. See estimate (6.25). We conclude our report with numerical validations of our theory in Section 5.6.

In order to obtain $\mathcal{O}(\varepsilon)$-convergence rate in $H^1(\Omega_{ext})$ in (6.2), we consider the expansion $\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h = \varepsilon(\mathbf{w}_{\varepsilon,h} + \mathbf{w}_h)$ where $\mathbf{w}_h$ is the solution of a *carefully* chosen dual problem, see Section 6.2.2. We then show *a priori* that $|\mathbf{w}_{\varepsilon,h}|_1 \leq C_*$ independent of $\varepsilon \to 0$.

The main difference between the continuous and discrete analysis arises when comparing $\mathbf{u}_{\varepsilon,h}$ and $\mathbf{u}_h$ since the variational problem for $\mathbf{u}_h$ is formulated with test-functions that vanish on $\partial\Omega_s$ whereas the variational problem for $\mathbf{u}_{\varepsilon,h}$ is formulated with test-functions that are *not* restricted on $\partial\Omega_s$:

$$\begin{array}{lcl} \text{Test Functions (NSE)} & \Rightarrow & \mathbf{v}|_{\partial\Omega_s} = 0 \\ \text{Test Functions (Brinkman)} & \Rightarrow & \mathbf{v}|_{\partial\Omega_s} \neq 0 \end{array}.$$

In order to subtract the NS and BrVP equations to obtain the error equation we need the test functions to be unrestricted on $\partial\Omega_s$. In the continuous case, the integral equation for $\mathbf{u}$ can be extended to include test functions that do not vanish on $\partial\Omega_s$ by including a boundary integral of the pseudo-stress tensor $\sigma(\mathbf{u}, p)$:

$$\int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v}, \quad \forall \mathbf{v} \in H^{1/2}(\partial\Omega_s), \qquad \sigma(\mathbf{u}, p) := -\nu\nabla\mathbf{u} + p\mathbb{I} \tag{6.3}$$

where $\hat{\mathbf{n}}$ is the outward normal on $\partial\Omega_s$, and p is the NS-pressure, see (2.48). For $\mathbf{u}_h$, we cannot simply replace (6.3) by restricting $\mathbf{v}$ to the velocity FE space for BrVP and replacing $\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}$ with $-\nu(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}_h + p_h\hat{\mathbf{n}}$. Instead, we replace (6.3) with

$$\int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h, \quad \forall \mathbf{v}_h \text{ in BrVP FE trace-space,} \tag{6.4}$$

see (6.14). We show that σ_h exists via Riesz Representation theorem. Furthermore, we define a discrete $H^{-1/2}$-trace norm (6.7) and show that σ_h is bounded in this norm, see (6.8). The key to this estimate is the discrete Trace Theorem ensuring the existence of continuous extensions in the FE-space of $H^{1/2}(\partial\Omega_s)$ functions, Assumption 2.3.4. The bound for σ_h in the discrete $H^{-1/2}$-trace norm is used in the proof of Theorem 6.2.5.

In Section 6.1, we define and derive the necessary properties for the discrete traction vector σ_h. In Section 6.2 we state our main convergence theorems for the convergence of BrVP velocity, pressure, and drag/lift forces. In Section 6.3, we conclude with several numerical experiments that confirm our theory. We also investigate therein BrVP approximations on a uniform mesh by simulating flow around an single 2d cylinder and an array of 40 2d cylinders.

6.1 EXTENSION OF FE-NSE FOR NONVANISHING TEST FUNCTIONS

We limit our analysis to the case $K_p^{-1} = 0$. Incorporating the porous regions $K_p^{-1} \neq 0$ is possible at the cost of added notational complexity. Note that the NSE is formulated in Ω variationally with a test function space vanishing on $\partial\Omega_{ext}$ but not $\partial\Omega_s$. Consequently, the variational BrVP and NSE cannot subtracted directly to formulate an error equation. Herein, we develop the necessary theory to analyze the error between the NSE and BrVP.

First consider 0-extensions of NSE-spaces H^1_ϕ, X_{h,ϕ_h} so that

$$\begin{aligned} \mathbf{v} \in H^1_\phi \subset H^1_\phi(\Omega_{ext}) \;&\Rightarrow\; \mathbf{v}|_{\Omega_s} \equiv 0 \\ \mathbf{v}_h \in X_{h,\phi_h} \subset X_{h,\phi_h}(\Omega_{ext}) \;&\Rightarrow\; \mathbf{v}_h|_{\Omega_s} \equiv 0. \end{aligned}$$

Also define the pseudo-traction vector $\sigma(\mathbf{u}, p) : \Omega_{ext} \to \mathbb{R}^{d\times d}$ by

$$\sigma(\mathbf{u}, p) := \nu \cdot \nabla\mathbf{u} - p\mathbb{I}.$$

The NS-problem is given by: Find $\mathbf{u} : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ satisfying

$$-\nabla \cdot (\nu \nabla \mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = g, \qquad \mathbf{u}|_{\partial\Omega_{ext}} = \phi, \qquad \mathbf{u}|_{\partial\Omega_s} = 0. \tag{6.5}$$

For regular enough data so that $\mathbf{f} \in L^2$ and $\nabla \cdot (\nu \nabla \mathbf{u}) - \nabla p \in L^2$, any $\mathbf{u}$, p satisfying (6.5) also satisfy

$$\begin{aligned} \forall \mathbf{v} \in H_0^1(\Omega_{ext}), \qquad & a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) \\ & \qquad + \int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v} = < \mathbf{f}, \mathbf{v} > \\ \forall q \in L^2, \qquad & b(\mathbf{u}, q) = l_2(q). \end{aligned} \tag{6.6}$$

Note that the weak formulation of the NSE is posed to *find* $\mathbf{u} \in H^1$ satisfying $\mathbf{u}|_{\partial\Omega_{ext}} = \phi$, $\mathbf{u}|_{\partial\Omega_s} = 0$ and $p \in L_0^2$ so that (6.6) is satisfied for test functions $\mathbf{v} \in H_0^1$; hence, $\int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v} = 0$ in (6.6). The form (6.6) with test functions $\mathbf{v}|_{\partial\Omega_s} \neq 0$ is the key to proving the $\mathcal{O}(\varepsilon)$-convergence of $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $H^1(\Omega_{ext})$. Angot proved this for linear, homogeneous Stokes-Brinkman; i.e. for $\mathbf{u} \cdot \nabla \mathbf{u} = 0$, $g \equiv 0$, and $\phi \equiv 0$ in [3]. The extension to inhomogeneous, nonlinear BrVP requires a small data restriction on the problem data including ϕ, g, $\mathbf{f}$. The proof is very similar to the proof of the discrete case in Theorem 6.2.5 and therefore is omitted for the sake of brevity.

In order to prove Proposition 6.2.3 and Theorem 6.2.5, we need to extend the FE-NSE (6.14) in a similar way to the continuous model (6.6). We prove such an extension through the following lemmas. First, it is convenient to define the discrete $H^{-1/2}(\partial\Omega_s)$-norm implicit in defining σ_h above:

Definition 6.1.1. *Let $\mu : \partial\Omega_s \to \mathbb{R}^d$. Define*

$$||\mu||_{h,-1/2,\partial\Omega_s} := \sup_{0 \neq \lambda_h \in \Lambda_h(\partial\Omega_s)} \frac{|\int_{\partial\Omega_s} \mu \cdot \lambda_h|}{||\lambda_h||_{1/2,\partial\Omega_s}}. \tag{6.7}$$

Lemma 6.1.2. *Under the assumptions of Lemma 6.2.2, suppose further that $\mathbf{f} \in L^2$ and Assumption 2.3.4 is satisfied so that $E_h : \Lambda_h(\partial\Omega) \to X_{h,\cdot}$ satisfies the discrete trace inequality (2.15). Then*

$$\sup_{0 \neq \lambda_h \in \Lambda_h(\partial\Omega_s)} \frac{| < A_{ext}(\mathbf{u}_h), E(\lambda_h) > |}{||\lambda_h||_{1/2,\partial\Omega_s}} \leq \nu^{-2} K_0. \tag{6.8}$$

We formally identified $< A_{ext}(\mathbf{u}_h), \mathbf{v}_h >:= a(\mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g\,(\mathbf{u}_h \cdot \mathbf{v}_h) + b(\mathbf{v}_h, p_h) - < \mathbf{f}, \mathbf{v}_h >$

Proof. Fix $\lambda_h \in \Lambda_h(\partial\Omega)$ so that $\lambda_h|_{\partial\Omega_{ext}} = 0$. Apply Cauchy-Schwarz (2.22), Poincaré (2.23), and Ladyzhenskaya (2.24) inequalities to (6.14). Then, simplification and (6.18) gives

$$\begin{aligned}
&| < A_{ext}(\mathbf{u}_h), E(\lambda_h) > | \\
&= - < \mathbf{f}, E(\lambda_h)) > + a(\mathbf{u}_h, E(\lambda_h)) + b(E(\lambda_h), p_h) \\
&\qquad + c_h(\mathbf{u}_h, \mathbf{u}_h, E(\lambda_h)) - \frac{1}{2}\int_{\Omega_p} g\,(\mathbf{u}_h \cdot E(\lambda_h)) \\
&\leq C(||\mathbf{f}|| + ||p_h|| + (\nu + ||g||_{\Omega_p} + |\mathbf{u}_h|_1)|\mathbf{u}_h|_1)|E(\lambda_h)|_1 \\
&\leq C\nu^{-2}K_0|E(\lambda_h)|_1.
\end{aligned}$$

Assumption 2.3.4 ensures that there exists a particular extension $E_h : \Lambda_h(\partial\Omega) \to X_{h,\cdot}$ satisfying

$$|E_h(\lambda_h)|_1 \leq C||\lambda_h||_{1/2,\partial\Omega_s}.$$

Then Estimate (6.8) follows. Note that C is absorbed into K_0. □

Proposition 6.1.3. *Under the assumptions of Lemma 6.2.2, suppose further that* $\mathbf{f} \in L^2$ *and Assumption 2.3.4 is satisfied so that* $E_h : \Lambda_h(\partial\Omega) \to X_{h,\cdot}$ *satisfies the discrete trace inequality (2.15). Then there exists* $\sigma_h \in \Lambda_h(\partial\Omega_s)$ *so that*

$$\begin{aligned}
&\nu\int \nabla\mathbf{u}_h : \nabla\mathbf{v}_h + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\
&\quad - \frac{1}{2}\int_{\Omega_p} g(\mathbf{u}_h \cdot \mathbf{v}_h) + \int p_h\nabla\cdot\mathbf{v}_h + \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h = < \mathbf{f}, \mathbf{v}_h >, \qquad \forall \mathbf{v}_h \in X_h(\Omega_{ext})
\end{aligned} \tag{6.9}$$

so that

$$||\sigma_h||_{h,-1/2,\partial\Omega_s} \leq \nu^{-2}K_0 < \infty. \tag{6.10}$$

Proof. Let $E_h : \Lambda_h(\partial\Omega) \to X_{h,\cdot}$ be an extension satisfying Assumption 2.3.4. From Lemma 6.1.2, we showed that $\lambda_h \mapsto < A_{ext}(\mathbf{v}_h), E_h(\lambda_h) >$ is a bounded linear functional on $\Lambda_h(\partial\Omega_s)$. By the Riesz Representation Theorem, there exists a unique $\sigma_h \in (\Lambda_h(\partial\Omega_s))'$ satisfying

$$\begin{aligned} \forall\lambda_h \in \Lambda_h(\partial\Omega_s), \qquad & \int_{\partial\Omega_s} \sigma_h \cdot E_h(\lambda_h) =< \mathbf{f}, E_h(\lambda_h) > -a(\mathbf{u}_h, E_h(\lambda_h)) \\ & -c_h(\mathbf{u}_h, \mathbf{u}_h, E_h(\lambda_h)) + \frac{1}{2}\int_{\Omega_p} g\,(\mathbf{u}_h \cdot E_h(\lambda_h)) - b(E_h(\lambda_h), p_h) \end{aligned} \tag{6.11}$$

where it is understood that boundary data is 0-extended for input to extension operator E_h so that $\lambda_h|_{\partial\Omega_{ext}} = 0$. This actually holds for $E_h(\lambda_h)$ (for any $\lambda_h \in \Lambda_h(\partial\Omega)$ replaced with *any* $\mathbf{v}_h \in X_{h,\cdot}$ so that $\mathbf{v}_h|_{\partial\Omega_{ext}} = 0$. Indeed, fix $\mathbf{v}_h \in X_{h,\cdot}$ restricted so that $\mathbf{v}_h|_{\partial\Omega_{ext}} = 0$. Set $\mu_h := \mathbf{v}_h|_{\partial\Omega_s} \in \Lambda_h(\partial\Omega_s)$ and extended so that $\mu|_{\partial\Omega_{ext}} = 0$. Since $\mathbf{v}_h - E_h(\mu_h) \in X_h$ (notably $(\mathbf{v}_h - E_h(\mu_h))|_{\partial\Omega} = 0$), it follows that

$$\begin{aligned} & a(\mathbf{u}_h, \mathbf{v}_h - E_h(\lambda_h)) + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h - E_h(\lambda_h)) \\ & \quad - \frac{1}{2}\int_{\Omega_p} g\,(\mathbf{u}_h \cdot (\mathbf{v}_h - E_h(\lambda_h))) + b_{ext}(\mathbf{v}_h - E_h(\lambda_h), p_h) =< \mathbf{f}, \mathbf{v}_h - E_h(\lambda_h) > . \end{aligned} \tag{6.12}$$

Since $\mathbf{v}_h = E_h(\lambda_h) + (\mathbf{v}_h - E_h(\lambda_h))$, it follows that

$$\begin{aligned} \forall\mathbf{v}_h \in X_h(\Omega_{ext}), \qquad & \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h =< \mathbf{f}, \mathbf{v}_h > -a(\mathbf{u}_h, \mathbf{v}_h) \\ & - c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \frac{1}{2}\int_{\Omega_p} g\,(\mathbf{u}_h \cdot \mathbf{v}_h) - b_{ext}(\mathbf{v}_h, p_h). \end{aligned} \tag{6.13}$$

Therefore, existence of $\sigma_h \in \Lambda_h(\partial\Omega_s)$ satisfying (6.9) follows. The estimate (6.10) is a consequence of estimates similar to those used in the proof of Lemma 6.1.2 and Definition 6.1.1. $\square$

6.2 CONVERGENCE ANALYSIS

Our method of proof to show $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}_h$ as $\varepsilon \to 0$ requires 2-steps. The first conclusion is presented in Proposition 6.2.3. We derive a sub-optimal convergence rate in $H^1(\Omega_{ext})$. In Theorem 6.2.5, we attain the optimal $\mathcal{O}(\varepsilon)$-convergence rate in $H^1(\Omega_{ext})$. We prove this in 2-steps because each proof contains useful approaches: the first utilizes the discrete traction vector σ_h derived in Proposition 6.1.3 and the second additionally requires an asymptotic expansion of the penalization error and the definition of an auxiliary problem. Although sub-optimal, the first estimate (6.20) has a weaker ν-dependence $\mathcal{O}(\nu^{-3})$ than the $\mathcal{O}(\nu^{-5})$-dependency derived in the second (6.20). The derived estimate (6.21) leads to the convergence estimate in Theorem 6.2.7.

For the analysis of the discrete problem, we define the FE approximation of the NSE in the fluid region $(\mathbf{u}, p)$ from (6.6): Given $\mathbf{f} \in L^2$, find $\mathbf{u}_h \in X_{h,\phi_h}$, $p_h \in Q_h$ satisfying

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \qquad & a(\mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ & \quad -\frac{1}{2}\int_{\Omega_p} g(\mathbf{u}_h \cdot \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = < \mathbf{f}, \mathbf{v}_h > \\ \forall q_h \in Q_h, \qquad & b(\mathbf{u}_h, q_h) = l_2(q_h). \end{aligned} \tag{6.14}$$

We proved stability (and ultimately existence) of stationary NS-solutions in Lemma 2.5.2. The proof of stability (and existence) of $(\mathbf{u}_h, p_h) \in X_{h,\phi_h} \times Q_h$ solving (6.14) is similar. The small data condition for existence is reasonably satisfied when $(\mathbf{u}, p) \in H^1_\phi \times L^2_0$ is unique (which requires a small data condition on $\mathbf{f}$, ν^{-1}, ϕ, and g via the stability constant M_0). Otherwise, the mesh $\mathcal{T}_h$ (especially near $\partial\Omega_s$) is intricately related to the existence of the Leray-Hopf extension of ϕ. Since the proof for the discrete case is similar, we simply state the lemmas here without further exposition.

Lemma 6.2.1 (Stationary NSE solutions are bounded)**.** *Fix $g \in L^2(\Omega_p)$, $\phi \in H^{1/2}_g(\partial\Omega)$, $\mathbf{f} \in V'$. Suppose that the small data condition*

$$4\nu^{-1}\left|\int (\mathbf{w}\cdot\nabla\mathbf{w})\cdot E(\phi) - \frac{1}{2}g|\mathbf{w}|^2\right| \leq |\mathbf{w}|_1^2, \quad \forall \mathbf{w} \in V \tag{6.15}$$

is satisfied where $E : H_g^{1/2}(\partial\Omega) \to V_\cdot(g)$ *is an extension operator satisfying the trace inequality (2.13). Then any* $(\mathbf{u}, p)$ *satisfying (6.6) also satisfies*

$$||\mathbf{u}||_1 + ||p||^{1/2} \leq \nu^{-1} M_0 \tag{6.16}$$

for some $0 < M_0 = M_0(\mathbf{f}, \phi, g) < \infty$.

Lemma 6.2.2 (Stationary FE-NSE solutions are bounded)**.** *Suppose that the FE-space satisfies Assumptions 2.1.1. Fix* $g \in L^2(\Omega_p)$, $\phi_h \in \Lambda_{h,g}(\partial\Omega)$, $\mathbf{f} \in (V_h)'$. *Suppose that the small data condition*

$$4\nu^{-1} \left| \int (\mathbf{w}_h \cdot \nabla \mathbf{w}_h) \cdot E_h(\phi_h) - \frac{1}{2} g |\mathbf{w}_h|^2 \right| \leq |\mathbf{w}_h|_1^2, \quad \forall \mathbf{w} \in V \tag{6.17}$$

is satisfied where $E_h : \Lambda_{h,g}(\partial\Omega) \to V_{h,\cdot}(g)$ *is an extension operator satisfying the trace inequality (2.15). Then any* $(\mathbf{u}_h, p_h)$ *satisfying (6.14) also satisfies*

$$||\mathbf{u}_h||_1 + ||p_h||^{1/2} \leq \nu^{-1} K_0 \tag{6.18}$$

for some $0 < K_0 = K_0(\mathbf{f}, \phi_h, g) < \infty$.

Proposition 6.2.3 (Consistency, Part I)**.** *Suppose that the FE-space satisfies Assumption 2.1.1. Fix* $\mathbf{f} \in L^2(\Omega_{ext})$, $g \in L^2(\Omega_p)$, *and* $\phi_h \in \Lambda_{h,g}(\partial\Omega)$. *Let* $(\mathbf{u}_{B,h}, p_{B,h}) \in X_{h,\phi_h}(\Omega_{ext}) \times Q_h(\Omega_{ext})$ *satisfy the conditions of Theorem 5.5.4 and* $(\mathbf{u}_h, p_h) \in X_{h,\phi_h} \times Q_h$ *satisfy the conditions of Lemma 6.2.2. Suppose further that*

$$2|c_h(\mathbf{v}_h, \mathbf{u}_h, \mathbf{v}_h) - \frac{1}{2} \int_{\Omega_p} g |\mathbf{v}_h|^2| \leq \nu |\mathbf{v}_h|_1^2, \qquad \forall \mathbf{v}_h \in V_h(\Omega_{ext}) \tag{6.19}$$

is satisfied. Then,

$$\varepsilon^{1/2} |\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h|_1 + ||\mathbf{u}_{\varepsilon,h}||_{1,\Omega_s} \leq \nu^{-3} K_0 \varepsilon. \tag{6.20}$$

Remark 6.2.4. *The small data condition (6.19) on* $\mathbf{f}$, g, ϕ_h *and/or* ν^{-1} *is essentially a uniqueness condition for the stationary FE-NSE solution* $\mathbf{u}_h$.

Proof. See Section 6.2.1. ∎

We are motivated by Angot's analysis of the continuous, homogeneous Stokes-Brinkman problem in [3] (see Section 4.2, p. 1407-1410 for details) to recover $\mathcal{O}(\varepsilon)$-convergence in $H^1(\Omega_{ext})$.

Theorem 6.2.5 (Consistency, Part II)**.** *Under the assumptions of Proposition 6.2.3, suppose further that the FE-space satisfies Assumption 2.3.4. Then*

$$||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_{1,\Omega_{ext}} \leq \nu^{-5} K_{B,0}\varepsilon \tag{6.21}$$

where $K_{B,0}$ is actually the maximum of K_0 and $K_{B,0}$.

Proof. See Section 6.2.2. □

Theorem 6.2.6. *Suppose that the FE-space satisfies Assumption 2.1.1. Fix $\mathbf{f} \in L^2(\Omega_{ext})$, $g \in L^2(\Omega_p)$, and $\phi_h \in \Lambda_{h,g}(\partial\Omega)$. Let $(\mathbf{u}_{B,h}, p_{B,h}) \in X_{h,\phi_h}(\Omega_{ext}) \times Q_h(\Omega_{ext})$ satisfy the conditions of Theorem 5.5.4 and $(\mathbf{u}_h, p_h) \in X_{h,\phi_h} \times Q_h$ satisfy the conditions of Lemma 6.2.2. Then*

$$||p_{\varepsilon,h} - p_h|| \leq \nu^{-1} K_{B,0} ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_1 \tag{6.22}$$

If the FE-space also satisfies Assumption 2.3.4, then

$$\Big|\int_{\partial\Omega_s} \sigma_h \cdot \hat{\mathbf{c}}_* - \int_{\Omega_s} (\frac{\nu}{\varepsilon}\mathbf{u}_{\varepsilon,h} - \mathbf{f}) \cdot \hat{\mathbf{c}}_*\Big| \leq \nu^{-1} K_{B,0} ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_{1,\Omega_{ext}} \tag{6.23}$$

where $K_{B,0}$ is actually the maximum of K_0 and $K_{B,0}$ and $\hat{\mathbf{c}}_*$ is a constant unit vector on $\partial\Omega_s$.

Proof. See Section 6.2.3. □

Under suitable regularity of NS-solutions $(\mathbf{u}, p)$, a *nice enough* FE-space yields the estimate

$$||\mathbf{u} - \mathbf{u}_h||_1 + ||p - p_h|| \leq C_*(h^{s+1} + h^k) \tag{6.24}$$

for some constant $C_* > 0$ independent of $h \to 0$ (although, depending on problem data including ν^{-1}, $||\mathbf{u}||_{k+1}$, and $||p||_s$).

Theorem 6.2.7 (Convergence, FE-BrVP)**.** *Under the assumptions of Theorem 6.2.5, suppose further that for some* $k,\ s \geq 0$, $\mathbf{u} \in H^1_\phi \cap H^{k+1}$, $p \in L^2_0 \cap H^s$ *are solutions of (6.5). Then*

$$|\mathbf{u}_{\varepsilon,h} - \mathbf{u}|_1 + ||p_{\varepsilon,h} - p|| \leq C_*(h^{s+1} + h^k + \varepsilon) \tag{6.25}$$

where $0 < C_* < \infty$ *is independent of* $h \to 0,$.

Proof. The triangle inequality gives $||\mathbf{u}_{\varepsilon,h} - \mathbf{u}||_1 \leq ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_1 + ||\mathbf{u}_h - \mathbf{u}||_1$ and $||p_{\varepsilon,h} - p|| \leq ||p_{\varepsilon,h} - p_h|| + ||p_h - p||$. Then (6.25) follows from an application of (6.21), (6.22), (6.24). $\square$

6.2.1 Proof of Velocity Error, Proposition 6.2.3

Proposition 6.2.3. Recall that $\mathbf{u}_h|_{\Omega_s} \equiv 0$. Let $\mathbf{e}_u = \mathbf{u}_h - \mathbf{u}_{\varepsilon,h} \in V_h(\Omega_{ext})$. Note that $\mathbf{e}_u|_{\Omega_s} = \mathbf{u}_{\varepsilon,h}|_{\Omega_s}$. Fix $\mathbf{v}_h \in V_h(\Omega_{ext})$. Subtracting (5.5) from (6.9) to get

$$\begin{aligned}
&\nu \int_\Omega \nabla \mathbf{e}_u : \nabla \mathbf{v}_h + \nu\varepsilon^{-1} \int_{\Omega_s} (\nabla \mathbf{u}_{\varepsilon,h} : \nabla \mathbf{v}_h + \mathbf{u}_{\varepsilon,h} \cdot \mathbf{v}_h) \\
&= \int_{\Omega_s} \mathbf{f} \cdot \mathbf{v}_h - \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h - c_h(\mathbf{e}_u, \mathbf{u}_h, \mathbf{v}_h) - c_{h,ext}(\mathbf{u}_{\varepsilon,h}, \mathbf{e}_u, \mathbf{v}_h) + \frac{1}{2}\int_{\Omega_p} g\left(\mathbf{e}_u \cdot \mathbf{v}_h\right).
\end{aligned} \tag{6.26}$$

Set $\mathbf{v} = \mathbf{e}_u \in V_h(\Omega_{ext})$ in (6.26) and apply identity (2.35) to (6.26). We get

$$\nu|\mathbf{e}_u|_1^2 + \nu\varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}||^2_{1,\Omega_s} = \int_{\Omega_s} \mathbf{f} \cdot \mathbf{e}_u - \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{e}_u - (c_h(\mathbf{e}_u, \mathbf{u}_h \mathbf{e}_u) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{e}_u|^2). \tag{6.27}$$

Recall the definition of the $H^{1/2}(\partial\Omega_s)$-norm

$$||\mathbf{e}_u||_{1/2,\partial\Omega_s} := \inf_{0 \neq \mathbf{v} \in H^1_{e_h}(\Omega_s)} ||\mathbf{v}||_{1,\Omega_s} \leq ||\mathbf{e}_u||_{1,\Omega_s}. \tag{6.28}$$

Since $\mathbf{f} \in L^2(\Omega_{ext})$, we apply Hölder's (2.22), Young's (2.21), and Ladyzhenskaya's (2.24) inequalities to (6.27) along with estimate (6.7) to get after simplification

$$\begin{aligned}
&\nu|\mathbf{e}_u|_1^2 + \frac{\nu}{2\varepsilon}||\mathbf{u}_{\varepsilon,h}||^2_{1,\Omega_s} \\
&\leq \varepsilon\nu^{-1}||\mathbf{f}||^2_{\Omega_s} + \varepsilon\nu^{-1}||\sigma_h||^2_{h,-1/2,\partial\Omega_s} + (c_h(\mathbf{e}_u, \mathbf{u}_h \mathbf{e}_u) - \frac{1}{2}\int_{\Omega_p} g|\mathbf{e}_u|^2).
\end{aligned} \tag{6.29}$$

Application of the small data condition (6.19) to (6.29) gives after absorbing like-terms from right to left-hand sides and simplification

$$|\mathbf{e}_u|_1^2 + \varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}||_{1,\Omega_s}^2 \leq C\nu^{-2}\varepsilon||\mathbf{f}||_{\Omega_s}^2 + C\nu^{-2}\varepsilon||\sigma_h||_{h,-1/2,\partial\Omega_s}^2. \tag{6.30}$$

Application of estimate (6.10) to (6.30) gives (6.20). Note that we absorb constant factor $C > 0$ into K_0. □

6.2.2 Proof of Velocity Error, Theorem 6.2.5

Theorem 6.2.5. $(\omega_*, \pi_*) \in V_h(\Omega_{ext}) \times Q_h$ be the solution of an auxiliary problem (to be defined later). Formally substitute $\mathbf{e}_u = \varepsilon(\omega + \omega_*)$ into (6.26). Divide by ε, and group terms to get

$$\begin{aligned}
&\nu \int_\Omega \nabla\omega : \nabla\mathbf{v}_h + \nu\varepsilon^{-1} \int_{\Omega_s} (\nabla\omega : \nabla\mathbf{v}_h + \omega \cdot \mathbf{v}_h) \\
&= -c_h(\omega, \mathbf{u}_h, \mathbf{v}_h) - (c_{h,ext}(\mathbf{u}_{\varepsilon,h}, \omega, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g\omega \cdot \mathbf{v}_h) \\
&- \nu \int_\Omega \nabla\omega_* : \nabla\mathbf{v}_h - c_h(\omega_*, \mathbf{u}_h, \mathbf{v}_h) - (c_{h,ext}(\mathbf{u}_{\varepsilon,h}, \omega_*, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g\omega_* \cdot \mathbf{v}_h) \\
&- \varepsilon^{-1}(\nu \int_{\Omega_s} \nabla\omega_* : \nabla\mathbf{v}_h + \int_{\Omega_s} \omega_* \cdot \mathbf{v}_h + \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h - \int_{\Omega_s} \mathbf{f} \cdot \mathbf{v}_h).
\end{aligned} \tag{6.31}$$

The idea is to pick ω_* so that the ε^{-1}-term in (6.31) is annihilated. Moreover, we need ω and ω_* to be bounded in $H^1(\Omega_{ext})$ independent of ε so that $||\mathbf{e}_u||_{1,\Omega_{ext}} = \varepsilon||\omega + \omega_*||_{1,\Omega_{ext}} \leq C_*\varepsilon$.

- (Weak Formulation) Find $\omega_* \in X_h(\Omega_{ext})$ and $\pi_* \in Q_h(\Omega_{ext})$ satisfying

$$\begin{aligned}
&\nu \int_{\Omega_s} \nabla\omega_{*,s} : \nabla\mathbf{v} - \varepsilon \int_{\Omega_s} \pi_{*,s}\nabla \cdot \mathbf{v} \\
&\quad + \nu \int_{\Omega_s} \omega_{*,s} \cdot \mathbf{v} = \int_{\Omega_s} \mathbf{f} \cdot \mathbf{v} + \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in X_{h,\cdot}(\Omega_s)
\end{aligned} \tag{6.32}$$

$$\int_{\Omega_s} q\nabla \cdot \omega_{*,s} = 0, \qquad \forall q \in Q_{h,\cdot}(\Omega_s) \tag{6.33}$$

and

$$\nu \int \nabla\omega_{*,f} : \nabla\mathbf{v} - \int \pi_{*,f}\nabla \cdot \mathbf{v} = 0, \qquad \forall \mathbf{v} \in X_h \tag{6.34}$$

$$\int q\nabla \cdot \omega_{*,f} = 0, \qquad \forall q \in Q_h \tag{6.35}$$

$$\omega_{*,f}|_{\partial\Omega_s} = \omega_{*,s}|_{\partial\Omega_s}, \qquad \omega_{*,f}|_{\partial\Omega_{ext}} = 0 \tag{6.36}$$

where $\omega_*|_{\Omega_s} := \omega_{*,s}$, $\omega_*|_\Omega := \omega_{*,f}$, $\pi_*|_{\Omega_s} := \pi_{*,s}$, and $\pi_*|_\Omega := \pi_{*,f}$. The above problem for $(\omega_*, \pi_*) \in X_h(\Omega_{ext}) \times Q_h(\Omega_{ext})$ is well-posed since the sub-problem in Ω_s given by (6.32), (6.33) and in Ω given by (6.34), (6.35), (6.36) are well-posed. Indeed, $H^1(\Omega_{ext})$ is preserved by the continuity requirement $\omega_{*,s}|_{\partial\Omega_s} = \omega_{*,f}|_{\partial\Omega_s}$. Moreover, solutions ω_* of (6.32), (6.33), (6.34), (6.35), (6.36) are stable in the sense summarized in the following lemma.

Lemma 6.2.8. *Suppose that the FE-space satisfies Assumption 2.3.4. Then any $\omega_* \in X_h(\Omega_{ext})$ satisfying (6.32), (6.33) (6.34), (6.35), (6.36) also satisfies*

$$||\omega_*||_{1,\Omega_{ext}} \leq \nu^{-3} K_0. \tag{6.37}$$

Proof. Test (6.32) with $\mathbf{v} = \omega_* \in V_h(\Omega_{ext})$. Application of the discrete $H^{-1/2}(\partial\Omega_s)$-norm given in (6.7) and definition of the $H^{1/2}(\partial\Omega_s)$-norm along Hölder's inequality (2.22) gives

$$||\omega_*||_{1,\Omega_s} \leq \nu^{-1}(||\mathbf{f}||_{\Omega_s} + ||\sigma_h||_{h,-1/2,\partial\Omega_s}). \tag{6.38}$$

We next estimate ω_* in Ω. Write $\lambda_h := \omega_*|_{\partial\Omega_s}$. Assumption 2.3.4 provides existence of data extension $E_h(\lambda_h) \in X_h(\Omega_{ext})$ satisfying

$$|E_h(\lambda_h)|_1 \leq C||\lambda_h||_{1/2,\partial\Omega_s} := \inf_{\mathbf{v}\in X_{h,\lambda_h}(\Omega_s)} ||\mathbf{v}||_{1,\Omega_s} \leq C||\omega||_{1,\Omega_s}. \tag{6.39}$$

Write $\omega_{0,f} = \omega_{*,f} + E_h(\lambda_h) \in V_h$. Test (6.34) with $\mathbf{v} = \omega_{0,f} \in V_h$ and apply Cauchy-Schwarz (2.22) to get

$$|\omega_{0,f}|_1 \leq |E_h(\lambda_h)|_1. \tag{6.40}$$

Application of the triangle inequality $|\omega_*|_1 \leq |\omega_{0,f}|_1 + |E_h(\lambda_h)|_1$ along with (6.39) to (6.40) gives

$$|\omega|_1 \leq C||\omega||_{1,\Omega_s}. \tag{6.41}$$

Estimates (6.38), (6.41) together with (6.10) proves (6.37). Note that we absorb constants $C > 0$ into K_0. $\square$

Let ω_* solve (6.32), (6.33) (6.34), (6.35), (6.36). Then (6.31) simplifies:

$$\begin{aligned}
&\nu\int_\Omega \nabla\omega : \nabla\mathbf{v}_h + \nu\varepsilon^{-1}\int_{\Omega_s}(\nabla\omega : \nabla\mathbf{v}_h + \omega\cdot\mathbf{v}_h)\\
&= -c_h(\omega, \mathbf{u}_h\mathbf{v}_h) - (c_{h,ext}(\mathbf{u}_{\varepsilon,h}, \omega, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g\omega\cdot\mathbf{v}_h)\\
&- c_h(\omega_*, \mathbf{u}_h\mathbf{v}_h) - (c_{h,ext}(\mathbf{u}_{\varepsilon,h}, \omega_*, \mathbf{v}_h) - \frac{1}{2}\int_{\Omega_p} g\omega_*\cdot\mathbf{v}_h).
\end{aligned} \tag{6.42}$$

Now test (6.42) with $\mathbf{v}_h = \omega \in V_h(\Omega_{ext})$. Identity (2.35) gives

$$\begin{aligned}
&\nu|\omega|_1^2 + \nu\varepsilon^{-1}||\omega||_{1,\Omega_s}^2\\
&= -c_h(\omega_*, \mathbf{u}_h, \omega) - c_{h,ext}(\mathbf{u}_{\varepsilon,h}, \omega_*, \omega) + \frac{1}{2}\int_{\Omega_p} g\omega_*\cdot\omega - (c_h(\omega, \mathbf{u}_h\omega) + \frac{1}{2}\int_{\Omega_p} g|\omega|^2)
\end{aligned} \tag{6.43}$$

so that estimate (2.41)(a) along with Hölder's (2.22), Ladyzhenskaya (2.24), Poincaré's (2.23), and Young's (2.21) inequalities give

$$\frac{3\nu}{4}|\omega|_{1,\Omega_{ext}}^2 \leq C\nu^{-1}(|\mathbf{u}_h|_1^2 + ||\mathbf{u}_{\varepsilon,h}||_{1,\Omega_{ext}}^2 + ||g||_{\Omega_p}^2)|\omega_*|_1^2 + |c_h(\omega, \mathbf{u}_h\omega) - \frac{1}{2}\int_{\Omega_p} g|\omega|^2|. \tag{6.44}$$

Supposing that the small data condition (6.19) is satisfied, we get

$$|\omega|_{1,\Omega_{ext}} \leq C\nu^{-1}(|\mathbf{u}_h|_1 + ||\mathbf{u}_{\varepsilon,h}||_{1,\Omega_{ext}} + ||g||_{\Omega_p})|\omega_*|_1. \tag{6.45}$$

Apply the triangle inequality to get $||\mathbf{e}_u||_{1,\Omega_{ext}} \leq \varepsilon(||\omega||_{1,\Omega_{ext}} + ||\omega_*||)$. Along with (6.37), (6.20), (5.15), (6.45) we prove (6.21). Note that the constant $C > 0$ is absorbed into K_0, $K_{B,0}$. □

6.2.3 Proof of Pressure, Drag, and Lift Error, Theorem 6.2.6

Theorem 6.2.6. Subtract (6.9) and (5.5) and solve for the pressure error:

$$\int (p_{\varepsilon,h} - p_h)\nabla \cdot \mathbf{v}_h = \int_{\Omega_s} \mathbf{f} \cdot \mathbf{v}_h - \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h + a_s(\mathbf{u}_{\varepsilon,h}, \mathbf{v}_h) - \int_{\Omega_s} p_{\varepsilon,h}\nabla \cdot \mathbf{v}_h + c_{h,s}(\mathbf{u}_{\varepsilon,h}\mathbf{u}_{\varepsilon,h}\mathbf{v}_h)$$
$$+ \nu \int \nabla(\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h) : \nabla\mathbf{v}_h - c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_{\varepsilon,h}, \mathbf{v}_h). \tag{6.46}$$

Restrict $\mathbf{v}_h \in X_h$ so that $\mathbf{v}_h|_{\overline{\Omega}_s} \equiv 0$. Then (6.46) simplifies to

$$\int (p_{\varepsilon,h} - p_h)\nabla \cdot \mathbf{v}_h = \int \nabla(\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h) : \nabla\mathbf{v}_h - c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_{\varepsilon,h}, \mathbf{v}_h). \tag{6.47}$$

We can rewrite the nonlinear term

$$c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_{\varepsilon,h}, \mathbf{v}_h) = c_h(\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}, \mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_h - \mathbf{u}_{\varepsilon,h}, \mathbf{v}_h). \tag{6.48}$$

Apply Cauchy-Schwarz (2.22) and (2.41)(a) to (6.47). Divide by $||\nabla\mathbf{v}_h||$, take the supremum over $\mathbf{v}_h$, and apply the discrete inf-sup condition (2.2) via Assumption 2.1.1 to get

$$||p_{\varepsilon,h} - p_h|| \leq C(\nu + ||\mathbf{u}_h||_1 + ||\mathbf{u}_{\varepsilon,h}||_1)|\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h|_1. \tag{6.49}$$

Estimates (6.37), (6.20), (5.15) applied to (6.49) proves (6.22). Note that we absorb constant $C > 0$ into K_0, $K_{B,0}$ and assumed $\nu \leq \nu^{-1}(K_0 + K_{B,0})$.

Next, rearrange the error equation (6.46) with (6.48) to get

$$\int_{\partial\Omega_s} \sigma_h \cdot \mathbf{v}_h - \int_{\Omega_s} (\frac{\nu}{\varepsilon}\mathbf{u}_{\varepsilon,h} - \mathbf{f}) \cdot \mathbf{v}_h = \frac{\nu}{\varepsilon}\int_{\Omega_s} \nabla\mathbf{u}_{\varepsilon,h} : \nabla\mathbf{v}_h + \int_{\Omega_s} p_{\varepsilon,h}\nabla \cdot \mathbf{v}_h$$
$$+ c_{h,s}(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_{\varepsilon,h}, \mathbf{v}_h) + \nu \int \nabla(\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h) : \nabla\mathbf{v}_h + \int (p_h - p_{\varepsilon,h})\nabla \cdot \mathbf{v}_h$$
$$- c_h(\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}, \mathbf{u}_h, \mathbf{v}_h) - c_h(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_h - \mathbf{u}_{\varepsilon,h}, \mathbf{v}_h). \tag{6.50}$$

Pick $\mathbf{v}_h|_{\Omega_s} \equiv \mathbf{c}_h$ to be constant so that $|\mathbf{c}_h| = 1$. By Assumption 2.3.4, there exists an extension operator $E_h : \Lambda_h(\partial\Omega_s) \to X_h$ so that $E_h(\mathbf{c}_h|_{\partial\Omega_s}) \in X_h$ and $E_h(\mathbf{v}_h|_{\partial\Omega_s}) = \mathbf{c}_h$ in $\overline{\Omega}_s$. Write $\mathbf{c}_h|_\Omega := E_h(\mathbf{c}_h|_{\partial\Omega_s})$. Then $\nabla \cdot \mathbf{v}_h|_{\Omega_s} = 0$, $\nabla \cdot \mathbf{v}_h = 0$ so that (6.50) simplifies to

$$\int_{\partial\Omega_s} \sigma_h \cdot \mathbf{c}_h - \int_{\Omega_s} (\frac{\nu}{\varepsilon}\mathbf{u}_{\varepsilon,h} - \mathbf{f}) \cdot \mathbf{c}_h = c_{h,s}(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_{\varepsilon,h}, \mathbf{c}_h) + \nu \int \nabla(\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h) : \nabla\mathbf{c}_h$$
$$+ \int (p_h - p_{\varepsilon,h})\nabla \cdot \mathbf{c}_h - c_h(\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}, \mathbf{u}_h, \mathbf{c}_h) - c_h(\mathbf{u}_{\varepsilon,h}, \mathbf{u}_h - \mathbf{u}_{\varepsilon,h}, \mathbf{c}_h). \tag{6.51}$$

Apply Cauchy-Schwarz (2.22), (2.41)(a), and (6.49) to (6.51) to get

$$|\int_{\partial\Omega_s} \sigma_h \cdot \mathbf{c}_h - \int_{\Omega_s} (\frac{\nu}{\varepsilon}\mathbf{u}_{\varepsilon,h} - \mathbf{f}) \cdot \mathbf{c}_h| \leq C|\mathbf{c}_h|_1(\nu + ||\mathbf{u}_{\varepsilon,h}||_1 + ||\mathbf{u}_h||_1)|\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}|_1. \tag{6.52}$$

Estimates (6.37), (6.20), (5.15) applied to (6.52) proves (6.23). Note that we absorb constant $C > 0$ into K_0, $K_{B,0}$ and assumed $\nu \leq \nu^{-1}(K_0 + K_{B,0})$.

□

6.3 NUMERICAL INVESTIGATION

In this section we show that the $\mathcal{O}(\varepsilon)$ convergence rate suggested by the theory for BrVP is observed in practice. We also conduct a computational analysis of BrVP solved on a uniform mesh. We will show that the mesh has a clear impact of flow phenomena at the boundary of the penalized obstacle which can have a considerable global impact to the flow dynamics unless the mesh is sufficiently resolved.

For the problem setup, consider the channel $([0,4] \times [0,1]) - \Omega_s$ where Ω_s are the flow obstructions. The flow has boundary conditions:

$$\mathbf{u}(x, y = 0) = \mathbf{u}(x, y = 0.41) = \mathbf{u}|_{\partial\Omega_s} = 0, \qquad \mathbf{u}(x = 0, y) = \mathbf{u}(x = 2.2, y)\frac{4}{0.41^2}y(0.41 - y).$$

We compare herein a stationary NSE approximation with stationary BrVP approximations. We solve each problem with Taylor-Hood finite elements. In one case, we solve both the NSE and BrVP on the same mesh is generated by Delaunay-Voronoi triangulation extended into Ω_s for BrVP. We also solve the BrVP on a uniform triangular mesh.

Table 6.1: Mesh refinement levels

h_{level}	NSE mesh (nodes on $\partial\Omega_s$)	Total DOF	uniform mesh (Total DOF)
1	32	11636	12396
2	64	44251	37667
3	128	162335	149059
4	256	621941	593027
5	512	2470015	2365699
6	1024	9829082	—

6.3.1 Flow past 1 2D cylinder

Let Ω_s consist of 1 2d cylinder with diameter $= 0.25$ centered at $(0.5, 0.5)$. We summarize the mesh details in Table 6.1. We analyze the case when $\nu^{-1} = 100$ and 800.

The magnitude of the computed velocity field when $\nu^{-1} = 100$ is presented in Figure 6.1. BrVP solutions are presented here on a coarse mesh $h_{level} = 2$ with good results for BrVP on the boundary conforming mesh when $\varepsilon = 10^{-8}$. The BrVP solution on a uniform mesh also appears to be a good approximation, but there are jagged contour lines near the boundary of the cylinder indicating *dead zones* of flow arising as numerical artifacts of the inconsistent mesh. In Figure 6.2 we show the velocity vector field overlaying the magnitude of velocity near the cylinder. The NSE solution is given in the top-left plot. Refinements of BrVP on a uniform mesh are provided for $h_{level} = 2$, 3, 4. Here we clearly see the influence of the inconsistent mesh on the flow field. In the upper-right plot associated with BrVP when $h_{level} = 2$, the dead-flow zone around the cylinder has a square-ish bottom-left and upper-right corner. The outline of the dead-flow zone follows the underlying FE mesh. As $h \to 0$, we see this effect decrease, but there is still a jagged boundary effect resembling a staircase along the actual $\partial\Omega_s$. In Figure 6.3 we compare NSE, BrVP, and BrVP on a uniform mesh when $\nu = 10^{-8}$. Once again, we see that BrVP with a conforming mesh very closely resembles the NSE flow, whereas BrVP on a uniform mesh appears to approximate

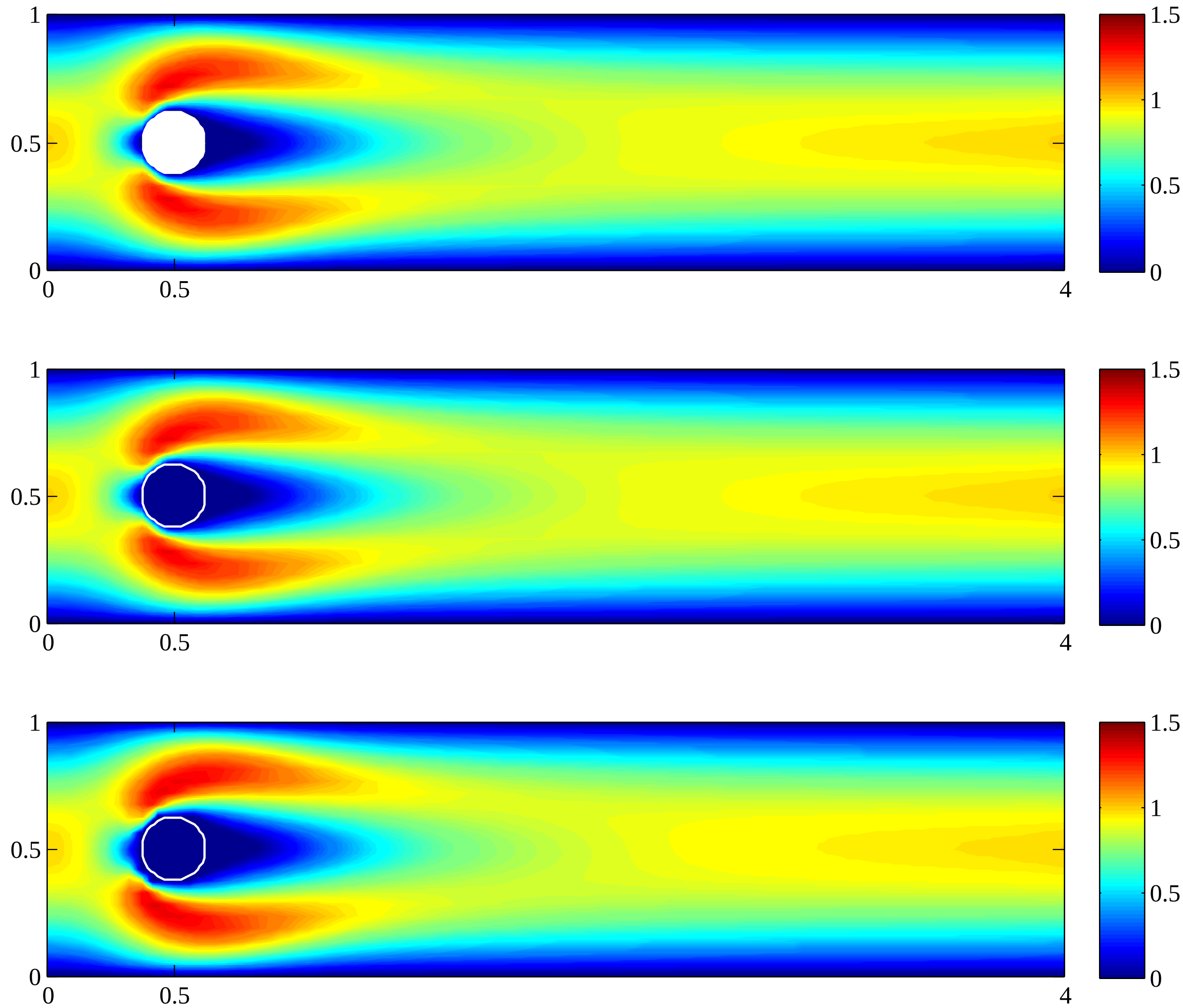


Figure 6.1: Flow past 1 cylinder: magnitude of velocity field when $\nu^{-1} = 100$ for (a) NSE $h_{level} = 6$, (b) BrVP, $\varepsilon = 10^{-8}$, $h_{level} = 2$, (c) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 2$

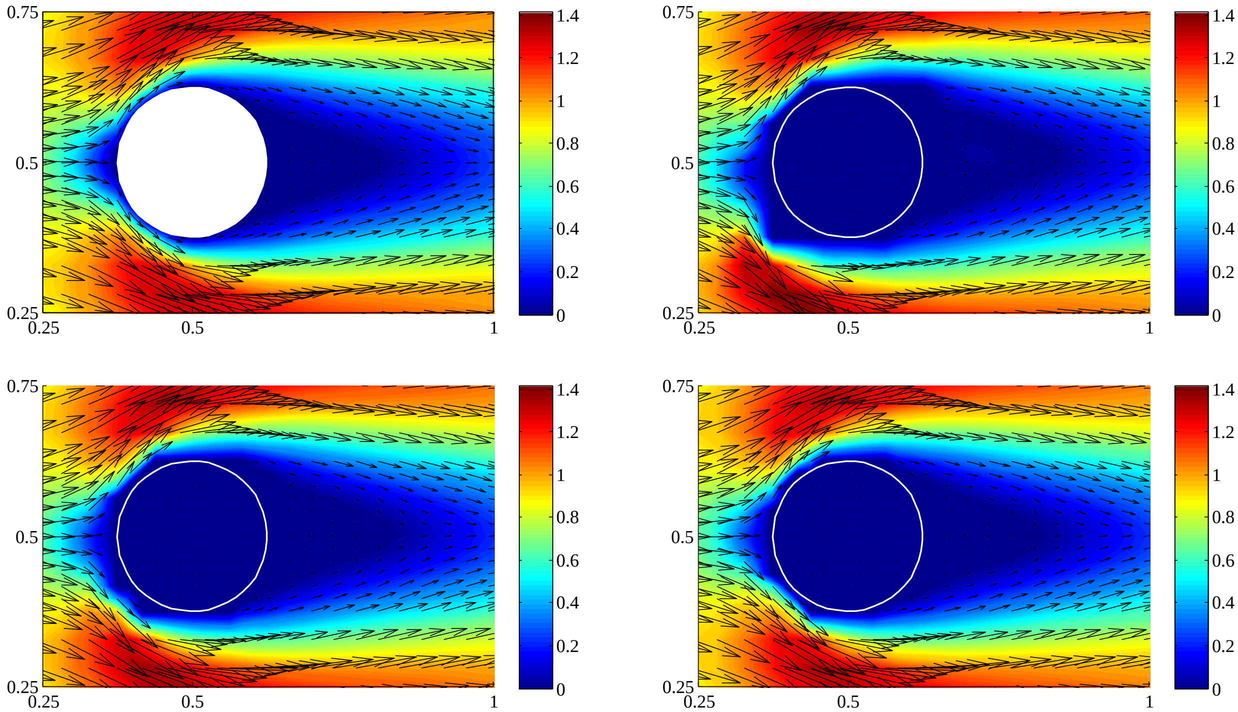


Figure 6.2: Flow past 1 cylinder: velocity field when $\nu^{-1} = 100$ for (top-left) NSE $h_{level} = 6$, (top-right) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 2$, (bottom-left) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 3$, (bottom-right) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 4$

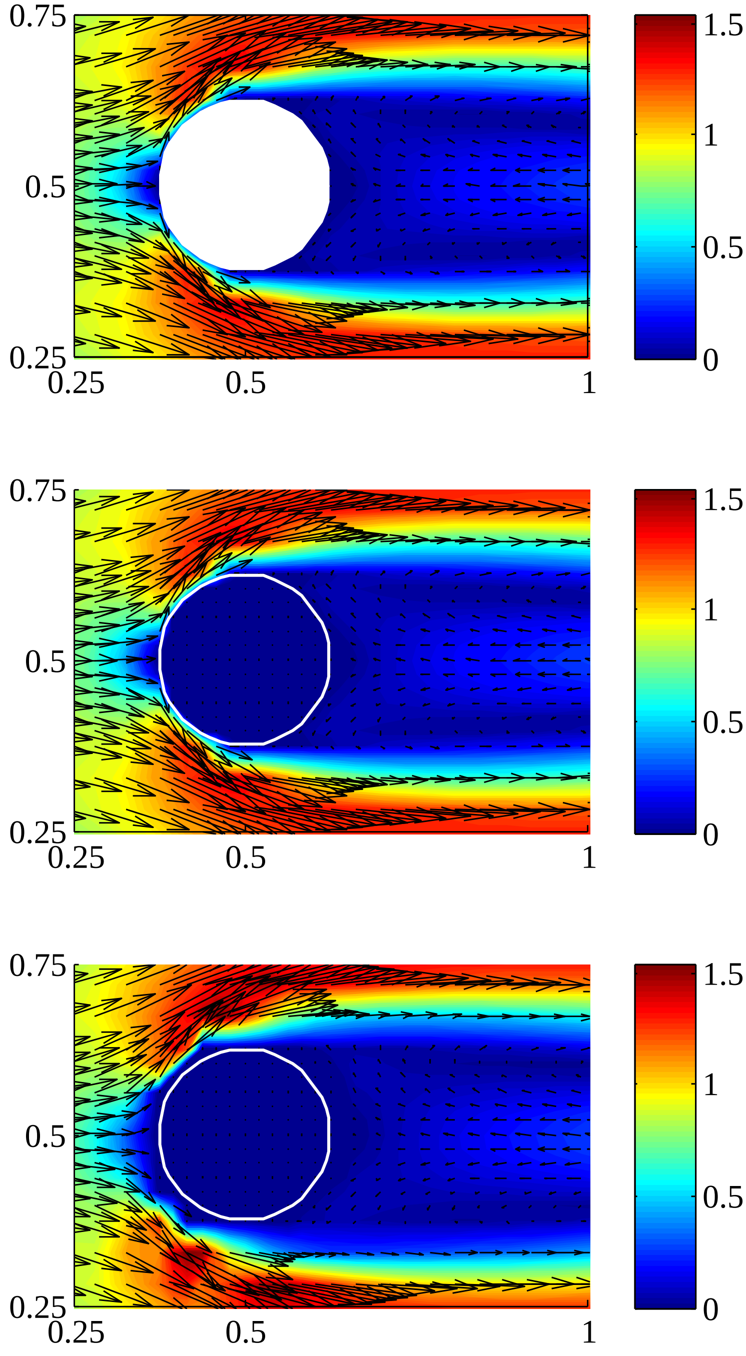


Figure 6.3: Flow past 1 cylinder: velocity field when $\nu^{-1} = 800$ for (a) NSE $h_{level} = 6$, (b) BrVP, $\varepsilon = 10^{-8}$, $h_{level} = 2$, (c) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 2$

Table 6.2: Flow past 1 cylinder: convergence of BrVP velocity in Ω, $\nu^{-1} = 800$, $h_{level} = 5$

ε	$\|\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}\|\|$	Rate	$\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}\|_1$	Rate
1e-4	2.902e-1	—	4.355	—
1e-5	1.617e-1	0.254	2.379	0.263
1e-6	5.532e-3	1.48	7.663e-2	1.49
1e-7	5.368e-4	1.00	7.719e-3	1.00
1e-8	5.372e-5	1.00	7.725e-4	1.00
1e-9	5.373e-6	1.00	7.726e-5	1.00
1e-10	5.373e-7	1.00	7.741e-6	1.00

the bulk flow well but has a clear mismatch at $\partial\Omega_s$.

In the theory developed previously, we emphasize that the discrete analogue of the traction vector $-\nu(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}_h + p_h\hat{\mathbf{n}}$ is not well-defined since $\mathbf{u}_h$ is generally not H^2 and p_h is generally not H^1 which is required for defining the corresponding trace on $\partial\Omega_s$. We prove the well-posedness of the bounded, linear functional

$$\sigma_h(\mathbf{v}_h) := -\nu(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) - c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + (p_h, \nabla \cdot \mathbf{v}_h). \tag{6.53}$$

We write σ_h as a functional here on $X_h(\Omega_{ext})$ because this is the most convenient form for computations. Now we define the drag

$$\begin{aligned} &\text{NSE, Method 1}: && D^{(1)} := \textstyle\sum_e \int_{e\cap\partial\Omega_s} (-\nu(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}_h + p_h\hat{\mathbf{n}}) \cdot [1,0]^t \\ &\text{NSE, Method 2}: && D^{(2)} := \sigma_h([1,0]^t) \\ &\text{BrVP}: && D_\varepsilon := \textstyle\int_{\Omega_s} \frac{\nu}{\varepsilon}\mathbf{u}_h \cdot [1,0]^t \end{aligned}$$

where $e = E \cap \partial\Omega_s$ for any $E \in \mathcal{T}_h$ are the boundary elements on $\partial\Omega_s$ derived from the mesh $\mathcal{T}_h$. Let $\mathbf{v}_h \in X_h(\Omega_{ext})$ be such that $\mathbf{v}_h|_{\partial\Omega_s} = [1,0]^t$. Since σ_h is a well-defined functional on FE velocity functions restricted to $\partial\Omega_s$, we can compute $\sigma_h(\mathbf{v}_h)$ uniquely by (6.53) for any $\mathbf{v}_h = [1,0]^t$ in a small *ring* around $\partial\Omega_s$ inside Ω. Tables 6.2, 6.3, 6.4 providing confirmation

Table 6.3: Flow past 1 cylinder: convergence of BrVP velocity in Ω_s, $\nu^{-1} = 800$, $h_{level} = 5$

ε	$\|\|\mathbf{u}_{\varepsilon,h}\|\|_{\Omega_s}$	Rate	$\|\mathbf{u}_{\varepsilon,h}\|_{1,\Omega_s}$	Rate
1e-4	4.954e-2	—	6.075e-3	—
1e-5	5.432e-3	0.960	6.136e-4	0.996
1e-6	5.510e-4	0.994	6.094e-5	1.00
1e-7	5.520e-5	1.00	6.090e-6	1.00
1e-8	5.521e-6	1.00	6.086e-7	1.00
1e-9	5.521e-7	1.00	6.089e-8	1.00
1e-10	5.521e-8	1.00	6.089e-9	1.00

Table 6.4: Flow past 1 cylinder: convergence of BrVP pressure in Ω, $\nu^{-1} = 800$, $h_{level} = 5$

ε	$\|\|p_{\varepsilon,h} - p\|\|$	Rate	$\|p_{\varepsilon,h} - p\|_1$	Rate
1e-4	1.736e-1	—	4.990e-1	—
1e-5	6.666e-2	0.416	6.088e-1	-0.0864
1e-6	3.055e-3	1.34	9.211e-3	1.82
1e-7	3.077e-4	0.997	9.315e-4	0.995
1e-8	3.080e-5	1.00	9.326e-5	1.00
1e-9	3.080e-6	1.00	9.327e-6	1.00
1e-10	3.081e-7	1.00	9.332e-7	1.00

Table 6.5: Flow past 1 cylinder: convergence of BrVP drag $\partial\Omega_s$, $\nu^{-1} = 800$, $h_{level} = 5$

ε	$\lvert D^{(1)} - D_\varepsilon \rvert$	Rate	$\lvert D^{(2)} - D_\varepsilon \rvert$	Rate
1e-4	1.571e-2	—	1.572e-2	—
1e-5	2.458e-3	0.806	2.4687-3	0.804
1e-6	2.974e-4	0.917	3.076e-4	0.904
1e-7	2.080e-5	1.16	3.103e-5	0.996
1e-8	7.122e-6	0.465	3.107e-6	0.999
1e-9	9.918e-6	-0.144	3.123e-7	0.998
1e-10	1.020e-5	-0.0121	3.274e-8	0.979

Table 6.6: Flow past 1 cylinder: convergence of BrVP velocity in Ω as ε, $h \to 0$, $\nu^{-1} = 100$, compared to NSE solution when $h_{level} = 6$

ε	(h_{level})	h_{max}	$\lvert\lvert \mathbf{u}_{\varepsilon,h} - \mathbf{u} \rvert\rvert$	Rate	$\lvert \mathbf{u}_{\varepsilon,h} - \mathbf{u} \rvert_1$	Rate
1e-4	(1)	0.127	3.320e-2	—	4.576e-1	—
1e-5	(2)	0.0642	3.685e-3	3.23	7.676e-2	2.63
1e-6	(3)	0.0471	4.280e-4	6.95	2.035e-2	4.28
1e-7	(4)	0.0213	5.745e-5	2.53	6.496e-3	1.45
1e-8	(5)	0.0117	8.466e-6	3.20	1.802e-3	2.145

Table 6.7: Mesh refinement levels

h_{level}	NSE mesh (nodes on $\partial\Omega_s$)	Total DOF	uniform mesh (Total DOF)
1	16	28402	9619
2	32	107942	37667
3	64	406096	149059
4	128	1593438	593027
5	256	6312540	2365699

of the $\mathcal{O}(\varepsilon)$ convergence rate predicted by our theory for BrVP to NSE on a fixed mesh in Ω. The rate is seen immediately from $\varepsilon = 10^{-4}$ to $\varepsilon = 10^{-5}$ for convergence of $\mathbf{u}_{\varepsilon,h}$, but not until $\varepsilon = 10^{-5}$ to $\varepsilon = 10^{-6}$ for $p_{\varepsilon,h}$. Table 6.5 also confirms the theory for convergence of BrVP drag. Indeed, although D_ε initially approximates the NSE drag computed by both Methods 1 and 2, as $\varepsilon \to 0$, we see the trend that BrVP actually approximates $D^{(2)}$ computed via the discrete traction functional σ_h as predicted by our theory. Lastly, Table 6.6 confirms $\mathcal{O}(\varepsilon + h^3)$ convergence rate for $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}$ as ε, $h \to 0$ in L^2 and $\mathcal{O}(\varepsilon + h^2)$ in H^1 as suggested by our theory.

6.3.2 Flow past 40 2D cylinders

Let Ω_s consist of 40 2d cylinders each with diameter $= 0.11$ in a uniform array with centers on the lattice defined by the x-values $x = 0.22,\ 0.405,\ 0.59,\ 0.775,\ 0.96,\ 1.145,\ 1.33,\ 1.515$ and y-values $y = 0.13,\ 0.315,\ 0.50,\ 0.685,\ 0.870$. We summarize the mesh details in Table 6.7. We analyze the case when $\nu^{-1} = 200$.

In Figure 6.4, we compare NSE flow with $h_{level} = 5$ to BrVP on a uniform mesh with $h_{level} = 1$ and 3 and $\varepsilon = 10^{-8}$. Failure for the computational mesh to align with the obstacles effectively squeezes-off the flow in the narrow fluid pores. Indeed, the fluid velocities peak high in the middle plot for BrVP on the coarse uniform mesh and slightly less high on the bottom plot for BrVP on the finer uniform mesh. In Figure 6.5, we illustrate how refining

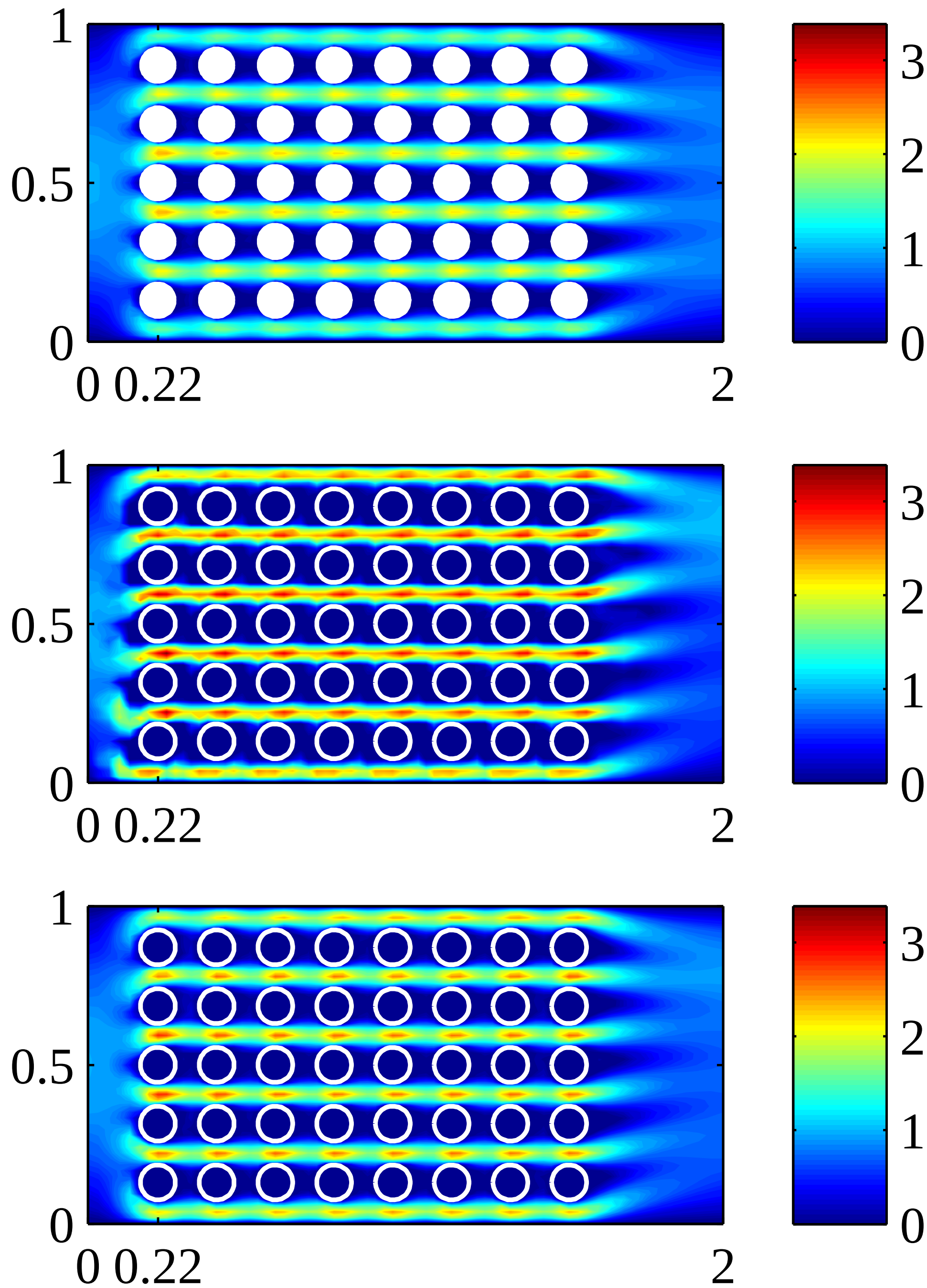


Figure 6.4: Flow past 40 cylinders: magnitude of velocity field when $\nu^{-1} = 200$ for (a) NSE $h_{level} = 5$, (b) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 1$, (c) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 3$

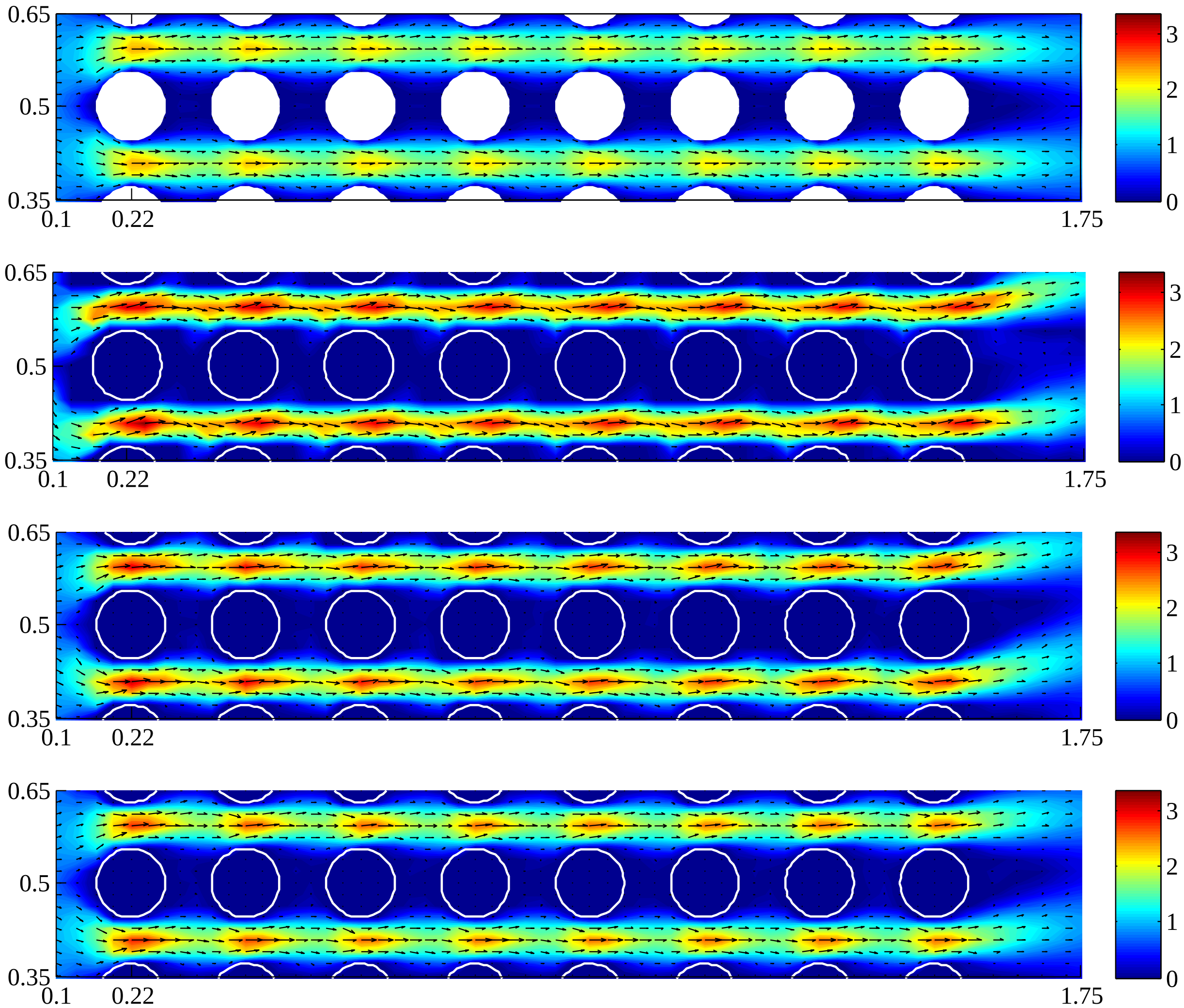


Figure 6.5: Flow past 40 cylinders: velocity field when $\nu^{-1} = 200$ for (a) NSE $h_{level} = 5$, (b) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 1$, (c) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 2$, (d) BrVP on uniform mesh, $\varepsilon = 10^{-8}$, $h_{level} = 3$

Table 6.8: Flow past 40 cylinders: convergence of BrVP velocity in Ω, $\nu^{-1} = 200$, $h_{level} = 1$

ε	$\|\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h\|\|$	Rate	$\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h\|_1$	Rate
1e-4	4.028e-1	—	20.51	—
1e-5	7.195e-2	0.748	3.633	0.752
1e-6	7.821e-3	0.964	3.939e-5	0.965
1e-7	7.890e-4	0.996	3.973e-2	0.996
1e-8	7.897e-5	1.00	3.977e-3	1.00
1e-9	7.898e-6	1.00	3.977e-4	1.00
1e-10	7.898e-7	1.00	3.977e-5	1.00

the uniform mesh for BrVP approximations recovers local NSE behavior by focusing on the 2 channels above and below the centerline. First note that the NS velocity field through these channels is mainly concentrated in the x-direction. Alternatively, at $h_{level} = 1$, the BrVP approximation has an oscillatory behavior in the y-direction. Ultimately, the flow exits from the upper channel skewed upward and failing to preserve the symmetry predicted in by the NSE approximation. As the mesh is refined to $h_{level} = 2$ and $h_{level} = 3$, the BrVP velocity field flattens through the channel.

We finish with convergence statistics between BrVP and NSE on the same, boundary-conforming mesh. First define the pressure drop by

$$\Delta p = p(x = 0.165, y = 0.5) - p(x = 1.57, y = 0.5).$$

Tables 6.8, 6.9 confirm the $\mathcal{O}(\varepsilon)$ convergence rate for velocity in Ω and Ω_s predicted by our theory. Table 6.10 confirm the $\mathcal{O}(\varepsilon)$ convergence rate for pressure in Ω and Ω_s predicted by our theory. We include the estimate for pressure drop because it is a useful statistic for practical applications. Ultimately, by showing $\mathcal{O}(\varepsilon)$ dependence of pressure drop between 2 points suggests control of pressure in $C^0(\Omega)$.

Table 6.9: Flow past 40 cylinders: convergence of BrVP velocity in Ω_s, $\nu^{-1} = 200$, $h_{level} = 1$

ε	$\lvert\lvert\mathbf{u}_{\varepsilon,h}\rvert\rvert_{\Omega_s}$	Rate	$\lvert\mathbf{u}_{\varepsilon,h}\rvert_{1,\Omega_s}$	Rate
1e-4	2.128e-1	—	4.307e-3	—
1e-5	3.746e-2	0.754	8.145e-4	0.723
1e-6	4.056e-3	0.965	8.952e-5	0.959
1e-7	4.090e-4	0.996	9.041e-6	0.996
1e-8	4.093e-5	1.00	9.050e-7	1.00
1e-9	4.094e-6	1.00	9.051e-8	1.00
1e-10	4.094e-7	1.00	9.051e-9	1.00

Table 6.10: Flow past 40 cylinders: convergence of BrVP pressure in Ω, $\nu^{-1} = 200$, $h_{level} = 1$

ε	$\lvert\lvert p_{\varepsilon,h} - p_h\rvert\rvert$	Rate	$\lvert p_{\varepsilon,h} - p_h\rvert_1$	Rate	$\Delta(p_{\varepsilon,h} - p_h)$	Rate
1e-4	10.63	—	10.08	—	7.129	—
1e-5	1.876	0.753	2.016	0.699	2.016	0.699
1e-6	2.031e-1	0.966	2.235e-1	0.955	2.234e-1	0.955
1e-7	2.047e-2	0.996	2.259e-2	0.995	2.259e-2	0.995
1e-8	2.049e-3	1.00	2.262e-3	1.00	2.262e-3	1.00
1e-9	2.049e-4	1.00	2.262e-4	1.00	2.262e-4	1.00
1e-10	2.049e-5	1.00	2.263e-5	1.00	2.263e-5	1.00

7.0 EVOLUTIONARY FLOWS IN COMPLICATED GEOMETRIES

We investigate the accuracy of Brinkman volume penalization (BrVP) for modeling *evolutionary*, incompressible, viscous fluid flows. The motivation behind BrVP is to avoid body-fitted meshes in order to use efficient solvers designed for *structured* (Cartesian) meshes instead. For BrVP, the usual no-slip boundary condition on solid obstacle Ω_s is replaced by a penalized drag term in the volume Ω_s. Let $\Omega_{ext} = \Omega \cup \overline{\Omega}_s$ extend the fluid domain Ω in $\mathbb{R}^d$ for $d = 2$ or 3.

- (BrVP) Find $\mathbf{u}_\varepsilon : \Omega_{ext} \times [0, T] \to \mathbb{R}^d$, and $p_\varepsilon : \Omega_{ext} \times (0, T] \to \mathbb{R}$ satisfying

$$\begin{aligned}
&\gamma_1 \partial_t \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon = \mathbf{f} + \nu \nabla \cdot (\gamma_2 \nabla \mathbf{u}_\varepsilon) - \nabla p_\varepsilon - \frac{\nu \chi_s}{\varepsilon} \mathbf{u}_\varepsilon, && \text{in } \Omega_{ext} \times (0, T] \\
&\nabla \cdot \mathbf{u}_\varepsilon = 0, && \text{in } \Omega_{ext} \times (0, T] \\
&\mathbf{u}_\varepsilon = \phi, && \text{on } \partial\Omega_{ext} \times [0, T] \\
&\mathbf{u}_\varepsilon(\cdot, 0) = \mathbf{u}^0_\varepsilon, && \text{in } \Omega_{ext}
\end{aligned} \tag{7.1}$$

where $\chi_s(x) = 0$ for $x \in \Omega$ and $\chi_s(x) = 1$ for $x \in \Omega_s$ and γ_1, $\gamma_2 > 0$.

Equation (7.1)(a) reduces exactly to the Navier-Stokes (NS) equations (NSE) (1.1) in Ω when $\gamma_1|_\Omega$, $\gamma_2|_\Omega = 1$. Let $(\mathbf{u}_{nse}, p_{nse})$ be velocity and pressure solving the NSE. It is well-established for $\gamma_2|_{\Omega_s} = \varepsilon^{-1}$ that

$$error(\mathbf{u}_\varepsilon - \mathbf{u}_{nse}) \leq C_* \varepsilon \tag{7.2}$$

in the energy norm for some data-dependent $C_* > 0$, see e.g. [4]. Analysis of corresponding discretization of (7.1) in space and/or time is limited. Let $h > 0$ be a characteristic mesh width and $\Delta t > 0$ the time-step size. Herein, we investigate a finite element (FE) in space, backward-Euler (BE) in time discretization of (7.1) with solution $(\mathbf{u}_{\varepsilon,h}, p_{\varepsilon,h})$. We establish analogous estimates to (7.2) for discrete velocity, pressure, drag/lift forces on $\partial\Omega_s$ with optimal $\mathcal{O}(\varepsilon)$ convergence as h, $\Delta t \to 0$.

We investigate, as a first step, the ideal case of a body-fitted mesh conforming to $\partial\Omega_s$. Let $(\mathbf{u}_{nse,h}, p_{nse,h})$ be discrete approximation of $(\mathbf{u}_{nse}, p_{nse})$ corresponding to the same discretization scheme used to obtain $(\mathbf{u}_{\varepsilon,h}, p_{\varepsilon,h})$. Controlling stress, either discrete or continuous, on $\partial\Omega_s$ is paramount in proving the optimal $\mathcal{O}(\varepsilon)$ convergence rate. Indeed, we derive the error equation (presented here for the linear case)

$$\frac{d}{dt}||\gamma_1 \mathbf{e}||^2_{L^2(\Omega_{ext})} + \nu||\gamma_2 \nabla \mathbf{e}||^2_{L^2(\Omega_{ext})} + \frac{\nu}{\varepsilon}||\mathbf{e}||_{L^2(\Omega_s)} = \int_{\partial\Omega_s} (\sigma \cdot \hat{\mathbf{n}}) \cdot \mathbf{e} \tag{7.3}$$

where $\sigma \cdot \hat{\mathbf{n}}$ is either the continuous traction vector if $\mathbf{e} = \mathbf{u}_\varepsilon - \mathbf{u}_{nse}$ or a discrete traction vector if $\mathbf{e} = \mathbf{u}_{\varepsilon,h} - \mathbf{u}_{nse,h}$. From (7.3), we see that the error between BrVP and NSE is controlled the *forcing function* concentrated on $\partial\Omega_s$ and propagated by the stress σ.

The continuous traction vector is given by

$$\sigma_{nse} \cdot \hat{\mathbf{n}} = -\nu(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}_{nse} + p_{nse}\hat{\mathbf{n}}, \quad \text{on } \partial\Omega_s. \tag{7.4}$$

The FE space *prohibits* writing an equation analogous to (7.4) for the discrete traction vector since, in general, $-\nu\Delta\mathbf{u}_h + \nabla p_h \notin L^2(\Omega)$ (e.g. C^0-velocity elements and discontinuous pressure elements). Consequently, in the discrete case, we introduce σ_h to approximate (7.4). We prove that σ_h exists in the $\partial\Omega_s$-trace of the FE space and is bounded (Proposition 7.1.1). Moreover,

$$error(\sigma_h - \sigma_{nse} \cdot \hat{\mathbf{n}}) \leq C_*(h^k + \Delta t) \tag{7.5}$$

(Proposition 7.1.3). As a consequence, we prove

$$error(\mathbf{u}_{\varepsilon,h} - \mathbf{u}_{nse}) \leq C_*(h^k + \Delta t + \varepsilon) \tag{7.6}$$

in the energy norm for the velocity error where $k > 0$ is the degree of approximation for the FE space (Theorem 7.63). Let $D = \int_{\partial\Omega_s}(\sigma_{nse} \cdot \hat{\mathbf{n}}) \cdot \mathbf{c}_d$ and $L = \int_{\partial\Omega_s}(\sigma_{nse} \cdot \hat{\mathbf{n}}) \cdot \mathbf{c}_l$ be the drag and lift forces on Ω_s for constant vectors $\mathbf{c}_d$, $\mathbf{c}_l \in \mathbb{R}^d$. We prove

$$error(D - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h} \cdot \mathbf{c}_d) + error(L - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h} \cdot \mathbf{c}_l) \leq C_*(h^k + \Delta t + \varepsilon) \tag{7.7}$$

and

$$error(p - p_{\varepsilon,h}) \leq C_*(h^k + \Delta t + \varepsilon) \tag{7.8}$$

for time-averaged errors (Theorem 7.2.8).

7.0.3 Importance of volume-penalization methods

Analysis of BrVP and various perturbations are provided in e.g. [3, 4, 18, 17]. The volume penalization scheme was first introduced in [16]. The authors in [6] first suggested that the Brinkman term $\nu\varepsilon^{-1}\int_{\Omega_s}\mathbf{u}_\varepsilon$ gives the drag and lift coefficients. Convergence analysis of the compressible BrVP is give in [25]. Although applied in practice (see e.g. [58, 82, 57, 31, 36, 80]), rigorous analysis of discrete BrVP schemes is limited.

We present the weak formulations with corresponding *a priori* estimates in Sections 7.0.4, 7.0.5 for the continuous and discrete NSE, BrVP. For the discrete problem we analyze both the fully implicit, nonlinear BE method (BEFE) as well as fully implicit, linearly extrapolated BE method (BELE). BELE is obtained from BEFE by extrapolating the convecting velocity $\mathbf{u}$: e.g.

$$\mathbf{u}\cdot\nabla\mathbf{u}\approx\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1}, \qquad \mathbf{u}^i := \mathbf{u}(x,t^i). \tag{7.9}$$

BEFE/LE is analyzed in e.g. [33, 44, 35, 86]. Estimates (7.5), (7.6), (7.7), (7.8) hold *without* any restriction on h, Δt for BELE, see [86]. For BEFE, the usual Δt-restriction arising from the discrete Gronwall inequality is required (at least within the framework of current analytical techniques). In Section 6.1, we define and investigate σ_h. Section 7.2 contains our main results with proofs in the corresponding subsections.

Our method of proof in deriving estimates (7.6), (7.7) is based on the decomposition

$$\mathbf{u}_{\varepsilon,h}-\mathbf{u}_{nse}=(\mathbf{u}_{\varepsilon,h}-\mathbf{u}_{nse,h})+(\mathbf{u}_{nse,h}-\mathbf{u}_{nse}) \tag{7.10}$$

where $\mathbf{u}_{\varepsilon,h}-\mathbf{u}_{nse,h}$ is the error in the discretization scheme from volume penalization and $\mathbf{u}_{nse,h}-\mathbf{u}_{nse}$ is the error in approximating the continuous NSE with BEFE/LE. For regular enough $(\mathbf{u}_{nse},p_{nse})$, $||\mathbf{u}_{nse,h}-\mathbf{u}_{nse}||_{l^\infty(L^2)\cap L^2(H^1)}\le C_*(h^k+\Delta t)$ see [86]. Our analysis follows these steps:

- Prove existence of σ_h (Riesz Representation theorem)
- Prove σ_h is bounded and satisfies (7.5) (define discrete $H^{-1/2}(\partial\Omega_s)$-norm)
- Expand the error $\mathbf{u}_\varepsilon-\mathbf{u}_{nse}=\varepsilon(\omega_*+\omega)$
- Define extension ω_* of σ_h, bounded as h, Δt, $\varepsilon\to 0$
- Prove (for particular ω_*) that ω bounded as h, Δt, $\varepsilon\to 0$

- Conclude

$$||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_{nse,h}||_{l^\infty(L^2)\cap L^2(H^1)} \leq C_*\varepsilon. \tag{7.11}$$

- Apply (7.10), (7.11) to prove (7.6)

We carefully track ν-dependence and regularity assumptions throughout our analysis.

For completeness, we include analysis of the continuous BrVP in Section 7.3. Although the ultimate $\mathcal{O}(\varepsilon)$ estimates are unaltered from previous analysis, our investigation is unique in several ways. First, we do not assume that the initial data of NSE and BrVP coincide. In this case, penalizing the time-derivative in (7.1)(a) with $\gamma_1|_{\Omega_s} = \varepsilon^{-1}$ requires that $\mathbf{u}_\varepsilon^0|_{\Omega_s} = \mathcal{O}(\varepsilon^{3/2})$. We also track the propagation of ν and $\sigma(\mathbf{u}, p)\cdot\hat{\mathbf{n}}$ through the error estimates. For optimal $\mathcal{O}(\varepsilon)$ estimates, we require that $\sigma(\mathbf{u}, p)\cdot\hat{\mathbf{n}} \in H^1(0, T; H^{-1/2}(\partial\Omega_s))$. We note that the auxiliary problem for ω_* must be chosen carefully to extend $\sigma(\mathbf{u}, p)\cdot\hat{\mathbf{n}}|_{\partial\Omega_s}$ in a way that does not require more regularity of NS solutions or a more restrictive ν-dependence than necessary. In this sense, our analysis is an improvement from previous reports. In Section 7.4, we investigate the accuracy of the BrVP force $\int \nu\varepsilon^{-1}\mathbf{u}_{\varepsilon,h}$. Drag and lift approximations are generally quite poor when generated by low order methods like BE. Herein, we apply BrVP with Crank-Nicolson time-stepping with good results.

7.0.4 The continuous error equation

We define the continuous problems for NSE and BrVP here. Fix $\phi(\cdot, t) \in H_0^{1/2}(\partial\Omega)$. Strong solutions $(\mathbf{u}, p) \in H^1_\phi \times L^2_0$ of the NSE satisfy

$$(\partial_t\mathbf{u}, \mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) + Re^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) =< \mathbf{f}, \mathbf{v} >, \quad \forall \mathbf{v} \in H_0^1 \tag{7.12}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega\times[0, T] \tag{7.13}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \quad \text{in } \Omega. \tag{7.14}$$

A similar weak formulation is derived for BrVP (7.1):

- (Weak BrVP) Find $\mathbf{u}_\varepsilon : [0,T] \to H^1_\phi(\Omega_{ext})$ and $p_\varepsilon : (0,T] \to L^2_0(\Omega_{ext})$ for each $t \in (0,T]$ satisfying

$$\int_{\Omega_{ext}} \gamma_1 \partial_t \mathbf{u}_\varepsilon \cdot \mathbf{v} + \int_{\Omega_{ext}} \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon \cdot \mathbf{v} + \nu \int_{\Omega_{ext}} \gamma_2 \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v}$$
$$- \int_{\Omega_{ext}} p_\varepsilon \nabla \cdot \mathbf{v} + \nu \varepsilon^{-1} \int_{\Omega_s} \mathbf{u}_\varepsilon \cdot \mathbf{v} = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H^1_0(\Omega_{ext}) \tag{7.15}$$
$$\nabla \cdot \mathbf{u}_\varepsilon = 0, \quad \text{in } \Omega_{ext} \times [0,T] \tag{7.16}$$
$$\mathbf{u}_\varepsilon(x,0) = \mathbf{u}^0_\varepsilon(x), \quad \text{in } \Omega_{ext}. \tag{7.17}$$

Different values of γ_i lead to different penalization methods:

$$\gamma_1 := 1 + (\gamma_{1,s} - 1)\chi_s(x), \qquad \gamma_2 := 1 + (\gamma_{2,s} - 1)\chi_s(x)$$

and

$$\begin{aligned}
\gamma_{1,s} &= 1, & \gamma_{2,s} &= 1 & &\Rightarrow \quad L^2\text{-penalization} \\
\gamma_{1,s} &= \varepsilon^{-1}, & \gamma_{2,s} &= 1 & &\Rightarrow \quad L^2\text{-}L^\infty\text{-penalization} \\
\gamma_{1,s} &= 1, & \gamma_{2,s} &= \varepsilon^{-1} & &\Rightarrow \quad H^1\text{-penalization} \\
\gamma_{1,s} &= \varepsilon^{-1}, & \gamma_{2,s} &= \varepsilon^{-1} & &\Rightarrow \quad H^1\text{-}L^\infty\text{-penalization.}
\end{aligned}$$

Well-posedness of BrVP follows the NSE theory (with similar unresolved gaps in $\mathbb{R}^3$), see e.g. [30]. For L^2-penalization, the discontinuous drag term $\nu\varepsilon^{-1}\chi_s \mathbf{u}_\varepsilon$ prevents greater than $H^2(\Omega_{ext})$ spatial regularity. For H^1-penalization the discontinuous viscous term $\nabla \cdot (\tilde{\nu} \nabla \mathbf{u}_\varepsilon)$ prevents greater than $H^1(\Omega_{ext})$ spatial regularity. Smoothness in the subdomains Ω, Ω_s corresponds to usual NSE-theory.

We derived an *a priori* estimate for $(\mathbf{u}, p)$ satisfying (7.12), (7.13), (7.14) in Lemma 2.5.4. A similar estimate holds for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfying (7.15), (7.16), (7.17). The main difficulty for BrVP is to bound $||\mathbf{u}_\varepsilon||_{l^\infty(L^2(\Omega_{ext})) \cap l^2(H^1(\Omega_{ext}))} < M_{B,0}$ independent of $\varepsilon \to 0$. The proof requires the existence of an extension operator $E(\phi) \in V_\phi(g)$ so that $E(\phi)|_{\Omega_s} = 0$. We conclude stability of BrVP solutions without further proof.

Lemma 7.0.1 (NSE Solutions are Bounded)**.** *Fix $\phi \in C^0(H^{1/2}_0(\partial\Omega))$, $\mathbf{f} \in L^2(V')$. Then all solutions $\mathbf{u}$ of (7.12), (7.13), (7.14) also satisfy $\mathbf{u} \in L^\infty(L^2) \cap L^2(V_\phi)$ and*

$$||\mathbf{u}||_{L^\infty(L^2)} + \nu^{1/2} ||\mathbf{u}||_{L^2(H^1)} \leq \nu^{-1/2} M_0 \tag{7.18}$$

for some $0 < M_0 = M_0(\mathbf{f}, \phi) < \infty$ independent of ν^{-1}.

Lemma 7.0.2 (BrVP Solutions are Bounded)**.** *Under the conditions of Lemma 7.0.1, all solutions of (7.15), (7.16), (7.17) also satisfy* $\mathbf{u}_\varepsilon \in L^\infty(L^2(\Omega_{ext})) \cap L^2(V_\phi(\Omega_{ext}))$ *and*

$$||\mathbf{u}_\varepsilon||_{L^\infty(L^2(\Omega_{ext}))} + \nu^{1/2}||\mathbf{u}_\varepsilon||_{L^2(H^1(\Omega_{ext}))} \leq \nu^{-1/2} M_{B,0} \tag{7.19}$$

for some $0 < M_{B,0} = M_{B,0}(\mathbf{f}, \phi) < \infty$ *independent of* ν, ε.

Note that NSE test functions in (7.12) vanish on $\partial\Omega_s$. Consequently, the (7.15) and (2.43) cannot be subtracted directly to formulate an error equation. To resolve this difficulty, first consider 0-extensions of H^1_ϕ, X_{h,ϕ_h} so that

$$\begin{array}{lcl} \mathbf{v} \in H^1_\phi \subset H^1_\phi(\Omega_{ext}) & \Rightarrow & \mathbf{v}|_{\overline{\Omega}_s} \equiv 0 \\ \mathbf{v}_h \in X_{h,\phi_h} \subset X_{h,\phi_h}(\Omega_{ext}) & \Rightarrow & \mathbf{v}_h|_{\overline{\Omega}_s} \equiv 0 \end{array}. \tag{7.20}$$

Define the pseudo-traction vector $\sigma(\mathbf{u}, p) : \Omega_{ext} \times (0, T] \to \mathbb{R}^{d\times d}$ by

$$\sigma(\mathbf{u}, p) := \nu\nabla\mathbf{u} - p\mathbb{I}.$$

Under NSE-regularity Assumption 2.5.1, NS solutions satisfy

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{f} + \nabla\cdot\sigma(\mathbf{u}, p), \qquad \nabla\cdot\mathbf{u} = 0, \qquad \text{in } \Omega \times (0, T]. \tag{7.21}$$

Then $\mathbf{u}$ and p satisfy

$$\begin{aligned} \int \partial_t\mathbf{u}\cdot\mathbf{v} + \nu\int\nabla\mathbf{u} : \nabla\mathbf{v} + \int\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{v} + \int p\nabla\cdot\mathbf{v} + \int q\nabla\cdot\mathbf{u} \\ + \int_{\partial\Omega_s}(\sigma(\mathbf{u}, p)\cdot\hat{\mathbf{n}})\cdot\mathbf{v} = \int\mathbf{f}\cdot\mathbf{v}, \qquad \forall\mathbf{v} \in H^1_0(\Omega_{ext}),\ \forall q \in L^2. \end{aligned} \tag{7.22}$$

Recall the weak NSE (7.12), (7.13) is posed to *find*, for each $t \in (0, T]$, $\mathbf{u}(\cdot, t) \in H^1_\phi$ satisfying $\mathbf{u}(\cdot, t)|_{\partial\Omega_s} = 0$ and $p(\cdot, t) \in L^2_0$ so that (7.22) is satisfied for test functions $\mathbf{v} \in H^1_0$; hence, $\int_{\partial\Omega_s}(\sigma(\mathbf{u}(\cdot, t), p(\cdot, t))\cdot\hat{\mathbf{n}})\cdot\mathbf{v} = 0$ for all $t \in (0, T]$. The form (7.22) with test functions $\mathbf{v}|_{\partial\Omega_s} \neq 0$ is the key to proving the $\mathcal{O}(\varepsilon)$-convergence for H^1-penalization and $\mathcal{O}(\varepsilon^{3/4})$-convergence for L^2-penalization of $\mathbf{u}_\varepsilon(\cdot, t) \to \mathbf{u}(\cdot, t)$ in $H^1(\Omega_{ext})$.

We derive the error equation between NSE and BrVP by subtracting (7.22) and (7.15). First define

$$\mathbf{e}_u := \mathbf{u}_\varepsilon - \mathbf{u}, \qquad e_p := p_\varepsilon - p.$$

Then, for any $\mathbf{v} \in H_0^1(\Omega_{ext})$,

$$\int_{\Omega_{ext}} \gamma_1 \partial_t \mathbf{e}_u \cdot \mathbf{v} + \int_{\Omega_{ext}} \mathbf{e}_u \cdot \nabla \mathbf{e}_u \cdot \mathbf{v} + \nu \int_{\Omega_{ext}} \gamma_2 \nabla \mathbf{e}_u : \nabla \mathbf{v} - \int_{\Omega_{ext}} e_p \nabla \cdot \mathbf{v}$$
$$+ \frac{\nu}{\varepsilon} \int_{\Omega_s} \mathbf{u}_\varepsilon \cdot \mathbf{v} = - \int \mathbf{e}_u \cdot \nabla \mathbf{u} \cdot \mathbf{v} - \int \mathbf{u} \cdot \nabla \mathbf{e}_u \cdot \mathbf{v} + \int_{\partial \Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v}. \tag{7.23}$$

7.0.5 Discrete formulation

Let $\{(\mathbf{u}_h^n, p_h^n)\}_{n=0}^N$ approximate NS solutions by a finite element (FE) spatial discretization and backward-Euler (BE) time-stepping (BEFE) given in (7.24), (7.25). First define

$$\xi^n(\mathbf{v}) = a_{-1}\mathbf{v}^{n+1} + a_0 \mathbf{v}^n + \dots a_{n_0} \mathbf{v}^{n-n_0}, \quad \overline{n}_0 = \max\{n_0, 0\}.$$

Problem 7.0.3 (BEFE). *Let $\mathbf{u}_h^i \in V_{h,\phi_h^i}$ be a good approximation of $\mathbf{u}^i$ for each $i = 0, 1, \dots, \overline{n}_0$. For each $n = \overline{n}_0, \overline{n}_0 + 1, \dots, N-1$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_{h,\phi_h^{n+1}} \times Q_h$ satisfying*

$$(\partial_{\Delta t}^{n+1} \mathbf{u}_h, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h)$$
$$+ \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) = < \mathbf{f}^{n+1}, \mathbf{v}_h >, \quad \forall \mathbf{v}_h \in X_h \tag{7.24}$$
$$(q_h, \nabla \cdot \mathbf{u}^{n+1}) = 0, \quad \forall q_h \in Q_h. \tag{7.25}$$

We consider an analogous approximation for BrVP solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfying (7.1).

Problem 7.0.4 (BEFEb). *Let $\mathbf{u}_{\varepsilon,h}^i \in V_{h,\phi_h}(\Omega_{ext})$ be a good approximation of $\mathbf{u}_\varepsilon^i$ for each $i = 0, 1, \dots, \overline{n}_0$. For each $n = \overline{n}_0, \overline{n}_0 + 1, \dots, N-1$, find $(\mathbf{u}_{\varepsilon,h}^{n+1}, p_{\varepsilon,h}^{n+1}) \in X_{h,\phi_h^{n+1}}(\Omega_{ext}) \times Q_h(\Omega_{ext})$ satisfying*

$$\int_{\Omega_{ext}} \gamma_1 (\partial_{\Delta t}^{n+1} \mathbf{u}_{\varepsilon,h}) \cdot \mathbf{v}_h + c_{h,ext}(\xi^n(\mathbf{u}_{\varepsilon,h}), \mathbf{u}_{\varepsilon,h}^{n+1}, \mathbf{v}_h)$$
$$+ \nu \int_{\Omega_{ext}} \gamma_2 \nabla \mathbf{u}_{\varepsilon,h}^{n+1} : \nabla \mathbf{v}_h + \nu \varepsilon^{-1} \int_{\Omega_s} \mathbf{u}_{\varepsilon,h}^{n+1} \cdot \mathbf{v}_h - \int_{\Omega_{ext}} p_{\varepsilon,h}^{n+1} \nabla \cdot \mathbf{v}_h$$
$$= (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h(\Omega_{ext}) \tag{7.26}$$
$$\int_{\Omega_{ext}} q_h \nabla \cdot \mathbf{u}_{\varepsilon,h}^{n+1} = 0, \quad \forall q_h \in Q_h(\Omega_{ext}). \tag{7.27}$$

In the case $a_{-1} = 0$, we consider have a linear extrapolated version of BEFE (BELE) and BEFEb (BELEb).

Well-posedness of BEFEb follow a similar argument as for BEFE. Indeed *a priori* estimate for BEFEb can be established in a similar manner as done for BEFE. As in the continuous case, particular care must be taken with terms involving ε^{-1}. Since this is handled in the more involved error estimates of Section 7.2, we omit the proof of the following result. First define

$$C_{ic}(\mathbf{v}) := ||\mathbf{u}_h^{\overline{n}_0}|| + \begin{cases} 0, & \text{if } a_i = 0 \text{ for } i \geq 0 \\ \nu^{1/2}(\Delta t \sum_{i=0}^{n_0} |\mathbf{u}_h^i|_1^2)^{1/2}, & \text{if } n_0 \geq 0 \end{cases}. \tag{7.28}$$

Lemma 7.0.5 (BEFE Solutions are Bounded)**.** *Fix $\phi_h \in C^0(\Lambda_{h,0}(\partial\Omega))$ and $\mathbf{f} \in l^2(W^{-1,2})$. Suppose $\mathbf{u}_h^i \in V_{h,\phi_h^i}$ for $i = 0, 1, \ldots, \overline{n}_0$ so that*

$$C_{ic} < \infty, \quad \text{as } h,\ \Delta t \to 0$$

where C_{ic} is given in (7.28) and

$$\begin{aligned} |c_h(\xi^n(\mathbf{v}_h), E_h(\phi_h^{n+1}), \mathbf{v}_h^{n+1})| &\leq \frac{\nu}{2(1+|\mathbf{a}|_2^2)(\overline{n}_0+1)^{1/2}} |\xi^n(\mathbf{v}_h)|_1 |\mathbf{v}_h^{n+1}|_1, \\ \forall\, \{\mathbf{v}_h^n\}_{n=0}^N \subset V_h, &\quad \forall n = \overline{n}_0, \overline{n}_0+1, \ldots, N-1 \end{aligned} \tag{7.29}$$

for some $E_h : \Lambda_{h,0}(\partial\Omega) \to V_{h,\cdot}$ satisfying Assumption 2.3.4. Then all solutions of Problem 7.0.3 also satisfy

$$||\mathbf{u}_h||_{l^\infty(\overline{n}_0+1,N;L^2)} + \nu^{1/2}||\nabla \mathbf{u}_h||_{l^2(\overline{n}_0+1,N;L^2)} \leq \nu^{-1/2} K_0 < \infty \tag{7.30}$$

for some $0 < K_0 < \infty$ that depends on $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$, $\mathbf{f}$, ϕ_h, but independent of ν, h, and Δt.

Define

$$C_{ic,\varepsilon} := ||\mathbf{u}_{\varepsilon,h}^{\overline{n}_0}|| + \gamma_{1,s}^{1/2}||\mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_s} + \begin{cases} 0, & \text{if } a_i = 0 \text{ for } i \geq 0 \\ \nu^{1/2}(\Delta t \sum_{i=0}^{n_0} ||\nabla \mathbf{u}_{\varepsilon,h}^i||^2)^{1/2} & \\ \quad +(\nu\gamma_{2,s})^{1/2}(\Delta t||\nabla \mathbf{u}_{\varepsilon,h}^n||_{\Omega_s}^2)^{1/2}, & \text{if } n_0 \geq 0 \end{cases}. \tag{7.31}$$

Lemma 7.0.6 (BEFEb Solutions are Bounded). *Fix* $\phi_h \in C^0(\Lambda_{h,0}(\partial\Omega_{ext}))$ *and* $\mathbf{f} \in l^2(W^{-1,2}(\Omega_{ext}))$. *Suppose* $\mathbf{u}^i_{\varepsilon,h} \in V_{h,\phi^i_h}(\Omega_{ext})$ *for* $i = 0, 1, \ldots, \overline{n}_0$ *so that*

$$C_{ic,\varepsilon} < \infty, \quad \text{as } h,\ \Delta t,\ \varepsilon \to 0$$

where $C_{ic,\varepsilon}$ *is given in (7.31) and*

$$\begin{aligned} |c_{h,ext}(\xi^n(\mathbf{v}_h), E_h(\phi_h^{n+1}), \mathbf{v}_h^{n+1})| &\leq \frac{\nu}{2(1+|\mathbf{a}|_2^2)(\overline{n}_0+1)^{1/2}} |\xi^n(\mathbf{v}_h)|_1 |\mathbf{v}_h^{n+1}|_1, \\ \forall \{\mathbf{v}_h^n\}_{n=0}^N \subset V_h(\Omega_{ext}), &\quad \forall n = \overline{n}_0, \overline{n}_0+1, \ldots, N-1 \end{aligned} \tag{7.32}$$

for some $E_h : \Lambda_{h,0}(\partial\Omega) \to V_{h,\cdot}$ *satisfying Assumption 2.3.4 and* $E_h(\phi_h)|_{\Omega_s} = 0$. *Then all solutions of Problem 7.0.4 also satisfy*

$$||\mathbf{u}_\varepsilon||_{l^\infty(\overline{n}_0+1,N;L^2(\Omega_{ext}))} + \nu^{1/2}||\nabla \mathbf{u}_h||_{l^2(\overline{n}_0+1,N;L^2(\Omega_{ext}))} \leq \nu^{-1/2} K_{B,0} < \infty \tag{7.33}$$

for some $0 < K_{B,0} < \infty$ *that depends on* $\left\{\mathbf{u}^i_{\varepsilon,h}\right\}_{i=0}^{n_0}$, $\mathbf{f}$, ϕ_h, g, *but independent of* ν, h, Δt, ε.

7.1 EXTENSION OF FE-NSE FOR NONVANISHING TEST FUNCTIONS

Herein, we develop the necessary theory to investigate the error between the discrete BrVP and NSE. We extend (3.13) to relax the restriction $\mathbf{v}_h|_{\partial\Omega_s} = 0$. This is done by introducing the discrete pseudo-traction vector $\sigma_h \approx \sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}$. As a consequence of Proposition 7.1.1, the error equation between BEFE (7.36) and BEFEb (7.26) is, for any $\mathbf{v}_h \in X_h(\Omega_{ext})$

$$\begin{aligned} &\int_{\Omega_{ext}} \gamma_1(\partial_{\Delta t}^{n+1}\mathbf{e}_{uh}) \cdot \mathbf{v}_h + \nu \int_{\Omega_{ext}} \gamma_2 \nabla \mathbf{e}_{uh}^{n+1} : \nabla \mathbf{v}_h + \nu\varepsilon^{-1} \int_{\Omega_s} \mathbf{u}_{\varepsilon,h}^{n+1} : \mathbf{v}_h \\ &\quad = \int_{\Omega_{ext}} e_{ph}^{n+1} \nabla \cdot \mathbf{v}_h + \int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \mathbf{v}_h \\ &\quad - 2c_{h,ext}(\xi^n(\mathbf{e}_{uh}), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) - c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}_h^{n+1}, \mathbf{v}_h) - c_h(\xi^n(\mathbf{u}_h), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) \end{aligned} \tag{7.34}$$

where

$$\mathbf{e}_{uh} := \mathbf{u}_{\varepsilon,h} - \mathbf{u_h}, \qquad e_{ph} := p_{\varepsilon,h} - p_h.$$

If $\mathbf{u} \in C^0(H^2)$, $p \in C^0(H^1)$, $\mathbf{u}_t \in C^0(L^2)$, then we are guaranteed (see e.g. [86]) that BEFE solutions satisfy

$$||\mathbf{u}_h||_{l^\infty(H^1)} + ||\partial_{\Delta t}\mathbf{u}||_{L^2(L^2)} + ||p_h||_{l^2(L^2)} \leq K_1 < \infty \tag{7.35}$$

without a Δt restriction for BELE, but only guaranteed for $\Delta t \leq C\nu$ in the case of BEFE.

Proposition 7.1.1. *Under the conditions of Lemma 7.0.5, suppose further that* $\mathbf{f} \in l^2(L^2)$ *and the FE-space satisfies Assumptions 2.1.1, 2.3.4. Then for each* $n = \overline{n}_0, \overline{n}_0 + 1, \ldots N-1$, *there exists a unique* $\sigma_h^{n+1} \in \Lambda_h(\partial\Omega_s)$ *so that*

$$\begin{aligned}(\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h)&\\ + (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) + \int_{\partial\Omega_s}\sigma_h^{n+1}\cdot\mathbf{v}_h = (\mathbf{f}^{n+1}, \mathbf{v}_h), \qquad &\forall\mathbf{v}_h \in X_h(\Omega_{ext}).\end{aligned} \tag{7.36}$$

Moreover, if (7.35) is satisfied,

$$(\Delta t\sum_{n=\overline{n}_0}^{N-1}||\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s})^{1/2} \leq K_1^{1/2}(K_1^{1/2} + \nu^{-5/4}K_0^{3/2}) < \infty. \tag{7.37}$$

Proof. See Section 7.1.1. □

Remark 7.1.2. *We assumed that* $\nu^{1/4} \leq (K_0K_1)^{1/2}$ *to simplify (7.37). Recall that* K_1 *depends on* ν^{-1}.

We actually prove a generalization of estimate (7.36) in Section 7.1.1: for any $q \in [1, 2]$,

$$\begin{aligned}\Delta t\sum_{n=\overline{n}_0}^{N-1}||\sigma_h^{n+1}||^q_{h,-1/2,\partial\Omega_s} &\leq C(||\mathbf{f}||^q_{l^q([\overline{n}_0+1,N];L^2)} + ||\partial_{\Delta t}\mathbf{u}_h||^q_{l^q([\overline{n}_0+1,N];L^2)} + \ldots\\ &\ldots + ||p_h||^q_{l^q([\overline{n}_0+1,N];L^2)} + \nu^q||\mathbf{u}_h||^q_{l^q([\overline{n}_0+1,N];H^1)} + \ldots\\ &\ldots + ||\mathbf{u}_h||^{q/2}_{l^\infty(L^2)}||\mathbf{u}_h||^{r'}_{l^\infty(H^1)}||\mathbf{u}_h||^r_{l^r(H^1)}) < \infty\end{aligned} \tag{7.38}$$

where $r = 3q/2$, $r' = 0$ for $q \in [1, 4/3]$ and $r = 2$, $r' = (3q-4)/2$ for $q \in (4/3, 2]$.

Existence of $\{\sigma_h^n\}_{n=\overline{n}_0+1}^N$ actually follows by a dimensional argument, but we prefer the method in the previous proof because it sheds light on a generalization of $\sigma(\mathbf{u}, p)$ in the continuous case. Indeed, a similar procedure shows that, for any $q \in [1,2]$, there exists **sigma** $\in L^q(H^{-1/2}(\partial\Omega_s))$ that is a generalization of $\sigma(\mathbf{u}(\cdot,t),p(\cdot,t))\cdot\hat{\mathbf{n}}$ satisfying

$$\int_0^T ||\sigma||^q_{-1/2,\partial\Omega_s} \leq C(||\mathbf{f}||^q_{L^q(L^2)} + ||\partial_t\mathbf{u}||^q_{L^q(L^2)} + \dots$$
$$\dots + ||p||^q_{L^q(L^2)} + \nu^q||\mathbf{u}||^q_{L^q(H^1)} + ||\mathbf{u}||^{q/2}_{L^\infty(L^2)}||\mathbf{u}||^{r'}_{L^\infty(H^1)}||\mathbf{u}||^r_{L^r(H^1)}) < \infty \tag{7.39}$$

where $r = 3q/2$, $r' = 0$ for $q \in [1,4/3]$ and $r = 2$, $r' = (3q-4)/2$ for $q \in (4/3,2]$. In particular, for $q = 4/3$, when $\mathbf{u} \in L^\infty(L^2)\cap L^2(H^1)\cap W^{1,4/3}(W^{-1,2})$, and $p \in L^{4/3}(L^2)$ we can show $\sigma \in L^{4/3}(H^{-1/2}(\partial\Omega_s))$. Moreover, for $q = 2$, when $\mathbf{u} \in L^\infty(L^2)\cap L^3(H^1)\cap H^1(W^{-1,2})$ and $p \in L^2(L^2)$ we have $\sigma \in L^2(H^{-1/2}(\partial\Omega_s))$ so that

$$(\int_0^T ||\sigma||^2_{-1/2,\partial\Omega_s})^{1/2} \leq M_1^{1/2}(M_1^{1/2} + \nu^{-5/4}M_0^{3/2}) < \infty. \tag{7.40}$$

In addition to (7.35) for a nice enough FE space and sufficiently regular $(\mathbf{u},p)$,

$$||\mathbf{u}-\mathbf{u}_h||_{l^\infty(H^1)} + ||\partial_{\Delta t}(\mathbf{u}-\mathbf{u}_h)||_{L^2(L^2)} + ||p-p_h||_{l^2(L^2)} \leq C_*(h^k + h^{s+1} + \Delta t) \tag{7.41}$$

where $k \geq 1$, $s \geq 0$ is the degree of the velocity, pressure FE spaces. Estimate (7.41) holds without Δt-restriction for BELE, but is only guaranteed for $\Delta t \leq C\nu$ in the case of BEFE (see e.g. [86]). We apply (7.41) to prove an error estimate between σ_h and $\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}$ in the following proposition.

Proposition 7.1.3. *Under assumptions of Proposition 7.1.1, suppose that the approximations $(\mathbf{u}_h, p_h)$ satisfy (7.41) and $\xi^n(\cdot)$ satisfies Assumption 2.2.1. Then*

$$(\Delta t\sum_{n=\overline{n}_0}^{N-1} ||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1},p^{n+1})\cdot\hat{\mathbf{n}}||^2_{h,-1/2,\partial\Omega_s})^{1/2} \leq C_*(h^k + h^{s+1} + \Delta t). \tag{7.42}$$

Proof. See Section 7.1.2. □

7.1.1 Proof of Extended NSE, Proposition 7.1.1

Proposition 7.1.1. Define, for any $\mathbf{v}_h \in X_{h,\cdot}$

$$< A^{n+1}(\mathbf{u}_h), \mathbf{v}_h >:= (\mathbf{f}, \mathbf{v}_h) - (\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h)$$
$$- c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h) - \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) + (p_h^{n+1}, \nabla \cdot \mathbf{v}_h).$$

Fix $\lambda_h \in \Lambda_h(\partial\Omega)$ so that $\lambda_h|_{\partial\Omega_{ext}} = 0$. There exists an extension $E_h(\lambda_h) \in X_{h,\lambda_h}$. Apply Hölder's (2.22), Poincaré's (2.23), and Ladyzhenskay's (2.24) along with estimate (2.41)(a) to get

$$\frac{| < A^{n+1}(\mathbf{u}_h), E_h(\lambda_h) > |}{|E_h(\lambda_h)|_1} \leq C(||\mathbf{f}^{n+1}|| + ||\partial_{\Delta t}^{n+1}\mathbf{u}_h|| + ||p_h^{n+1}|| + \dots$$
$$\dots + \nu|\mathbf{u}_h^{n+1}|_1 + ||\xi^n(\mathbf{u}_h)||^{1/2}||\xi^n(\mathbf{u}_h)||_1^{1/2}||\mathbf{u}_h^{n+1}||_1).$$

We are ensured that there exists a particular extension operator E_h satisfying the discrete trace inequality $|E_h(\lambda_h)|_1 \leq C||\lambda_h||_{1/2,\partial\Omega_s}$ via Assumption 2.3.4. Young's inequality (2.21) gives $||\xi^n(\mathbf{u}_h)||_1^{1/2}||\mathbf{u}_h^{n+1}||_1 \leq C(||\xi^n(\mathbf{u}_h)||_1^{3/2} + ||\mathbf{u}_h^{n+1}||_1^{3/2})$. Therefore

$$\sup_{0 \neq \lambda_h \in \Lambda_h(\partial\Omega_s)} \frac{| < A^{n+1}(\mathbf{u}_h), E_h(\lambda_h) > |}{||\lambda_h||_{1/2,\partial\Omega_s}} \leq C(||\mathbf{f}^{n+1}|| + ||\partial_{\Delta t}^{n+1}\mathbf{u}_h|| + \dots$$
$$\dots + ||\xi^n(\mathbf{u}_h)||^{1/2}(||\xi^n(\mathbf{u}_h)||_1^{3/2} + ||\mathbf{u}_h^{n+1}||_1^{3/2}) + \nu||\mathbf{u}_h^{n+1}||_1 + ||p_h^{n+1}||) < \infty. \tag{7.43}$$

Indeed, the RHS is bounded since $\mathbf{u}_h^n \in X_{h,\cdot} \subset H^1$, $p^n \in Q_{h,\cdot} \subset L^2$ for each n. By the Riesz Representation Theorem, for each $n = \overline{n}_0, \overline{n}_0 + 1, \dots, N-1$ there exists a unique $\sigma_h^{n+1} \in (\Lambda_h(\partial\Omega_s))' = \Lambda_h(\partial\Omega_s) \subset L^2(\partial\Omega_s)$ satisfying, for all $\lambda_h \in \Lambda_h(\partial\Omega_s)$,

$$\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot E_h(\lambda_h) = (\mathbf{f}^{n+1}, E_h(\lambda_h)) - (\partial_{\Delta t}^{n+1}\mathbf{u}_h, E_h(\lambda_h))$$
$$- (\nabla \mathbf{u}_h^{n+1}, \nabla E_h(\lambda_h)) - c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, E_h(\lambda_h)) + (p_h^{n+1}, \nabla \cdot E_h(\lambda_h)).$$

This actually holds for $E_h(\lambda_h)$ replaced with $\mathbf{v}_h \in X_h(\Omega_{ext})$. Indeed, fix $\mathbf{v}_h \in X_h(\Omega_{ext})$. Set $\mu_h := \mathbf{v}_h|_{\partial\Omega_s} \in \Lambda_h(\partial\Omega_s)$. Then since $(\mathbf{v}_h - E_h(\mu_h))|_\Omega \in X_h$ so that $(\mathbf{v}_h - E_h(\lambda_h))|_{\partial\Omega} = 0$, it follows from (3.13) that

$$(\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h - E_h(\lambda_h)) + (\nabla \mathbf{u}_h^{n+1}, \nabla(\mathbf{v}_h - E_h(\lambda_h)))$$
$$+ c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h - E_h(\lambda_h)) + (\nabla \cdot (\mathbf{v}_h - E_h(\lambda_h)), p_h^{n+1}) = (\mathbf{f}^{n+1}, \mathbf{v}_h - E_h(\lambda_h)).$$

Since $\mathbf{v}_h = E_h(\lambda_h) + (\mathbf{v}_h - E_h(\lambda_h))$, it follows that

$$\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \mathbf{v}_h = (\mathbf{f}^{n+1}, \mathbf{v}_h) - (\partial_\Delta^{n+1}\mathbf{u}_h, \mathbf{v}_h)$$
$$- (\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) - c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{v}_h) + (\nabla \cdot \mathbf{v}_h, p_h^{n+1}), \qquad \forall \mathbf{v}_h \in X_h(\Omega_{ext}).$$

Rearranging terms, we have that $\{(\mathbf{u}_h^n, p_h^n)\}_{n=0}^N$ from (7.24), (7.25) satisfy (7.36). We previously defined the discrete $H^{-1/2}(\partial\Omega_s)$-norm in (6.7). Replacing $< A^{n+1}(\mathbf{u}_h), E_h(\lambda_h) >$ with $\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \mathbf{v}_h$ in the analysis above, we conclude that

$$||\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s} \leq C(||\mathbf{f}^{n+1}|| + ||\partial_{\Delta t}^{n+1}\mathbf{u}_h|| + \dots$$
$$\dots + ||\xi^n(\mathbf{u}_h)||^{1/2}(||\xi^n(\mathbf{u}_h)||_1^{3/2} + ||\mathbf{u}_h^{n+1}||_1^{3/2}) + \nu||\mathbf{u}_h^{n+1}||_1 + ||p_h^{n+1}||).$$

Consequently, for any $q \in [1, 2]$, take the q-th power of each side of the estimate above, multiply by Δt and sum from $n = \overline{n}_0$ to $N-1$ to get (7.43). Apply estimate (7.30) along with those from (7.35) to (7.43) so that

$$\Delta t \sum_{n=\overline{n}_0}^{N-1} ||\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s} \leq C(||\mathbf{f}||^2_{l^2([\overline{n}_0+1,N];L^2)} + ||\partial_{\Delta t}\mathbf{u}_h||^2_{l^2([\overline{n}_0+1,N];L^2)} + \dots$$
$$\dots + ||\mathbf{u}_h||^2_{l^\infty(H^1)}||\mathbf{u}_h||^2_{l^2(H^1)} + \nu^2||\mathbf{u}_h^{n+1}||^2_{l^2([\overline{n}_0+1,N];H^1)} + ||p_h||^2_{l^2([\overline{n}_0+1,N];L^2)}).$$

This gives (7.38). Estimate (7.37) follows after simplification. Note that we absorbed constants C into K_0, K_1. $\square$

7.1.2 Proof of Discrete Boundary Stress Error, Proposition 7.1.3

Proposition 7.1.3. Fix $\lambda_h \in \Lambda_h(\partial\Omega)$ so that $\lambda_h|_{\partial\Omega_{ext}} = 0$. There exists $E_h : \Lambda_h(\partial\Omega) \to X_{h,\cdot}$ so that $E_h(\lambda_h) \in X_{h,\lambda_h}$ and the discrete trace inequality $|E_h(\lambda_h)|_1 \le C||\lambda_h||_{1/2,\partial\Omega_s}$ is satisfied via Assumption 2.3.4. Subtract (7.36) and (7.22) with $\mathbf{v} = \mathbf{v}_h = E_h(\lambda_h) \in X_h(\Omega_{ext})$ and solve for $\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}$ to get after simplification,

$$\begin{aligned}\int_{\partial\Omega_s} (\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}) \cdot E_h(\lambda_h) &= (\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_{\Delta t}^{n+1}\mathbf{u}_h, E_h(\lambda_h)) \\ &- ((\partial_t \mathbf{u})^{n+1} - \partial_{\Delta t}^{n+1}\mathbf{u}, E_h(\lambda_h)) - (p^{n+1} - p_h^{n+1}, \nabla \cdot E_h(\lambda_h)) \\ &+ \nu(\nabla(\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}), \nabla E_h(\lambda_h)) + c_h(\xi^n(\mathbf{u}) - \xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, E_h(\lambda_h)) \\ &+ c_h(\mathbf{u}^{n+1} - \xi^n(\mathbf{u}), \mathbf{u}_h^{n+1}, E_h(\lambda_h)) + c_h(\mathbf{u}^{n+1}, \mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}, E_h(\lambda_h)).\end{aligned} \tag{7.44}$$

First, apply Hölder's inequality (2.22) to get

$$\begin{aligned}&|\int \left((\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_{\Delta t}^{n+1}\mathbf{u}_h) - ((\partial_t \mathbf{u})^{n+1} - \partial_{\Delta t}^{n+1}\mathbf{u})\right) \cdot E_h(\lambda_h)| \\ &\le C(||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_{\Delta t}^{n+1}\mathbf{u}_h|| + ||(\partial_t \mathbf{u})^{n+1} - \partial_{\Delta t}^{n+1}\mathbf{u}||)||\lambda_h||_{1/2,\partial\Omega_s}\end{aligned} \tag{7.45}$$

and

$$\begin{aligned}&|\int (p^{n+1} - p_h^{n+1})\nabla \cdot E_h(\lambda_h) + \nu \int \nabla(\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}) : \nabla E_h(\lambda_h) \\ &\le C(||p^{n+1} - p_h^{n+1}|| + \nu|\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}|_1)||\lambda_h||_{1/2,\partial\Omega_s}.\end{aligned} \tag{7.46}$$

Majorization of the convective terms in (7.44) remains. Estimate (2.41)(a) gives

$$|c_h(\xi^n(\mathbf{u}) - \xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1}, E_h(\lambda_h))| \le C||\mathbf{u}_h||_{l^\infty([\overline{n}_0+1,N];H^1)}||\xi^n(\mathbf{u}) - \xi^n(\mathbf{u}_h)||_1 \; ||\lambda_h||_{1/2,\partial\Omega_s} \tag{7.47}$$

and

$$|c_h(\mathbf{u}^{n+1} - \xi^n(\mathbf{u}), \mathbf{u}_h^{n+1}, E_h(\lambda_h))| \le C||\mathbf{u}_h||_{l^\infty([\overline{n}_0+1,N];H^1)}||\mathbf{u}^{n+1} - \xi^n(\mathbf{u})||_1 \; ||\lambda_h||_{1/2,\partial\Omega_s} \tag{7.48}$$

and

$$|c_h(\mathbf{u}^{n+1}, \mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}, E_h(\lambda_h))| \le C||\mathbf{u}||_{l^\infty([\overline{n}_0+1,N];H^1)}||\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}||_1 \; ||\lambda_h||_{1/2,\partial\Omega_s}. \tag{7.49}$$

Application of (7.45), (7.46), (7.47), (7.48), and (7.49) to (7.44) gives

$$
\begin{aligned}
||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||_{h,-1/2,\partial\Omega_s} &\leq C(||\partial_{\Delta t}^{n+1}\mathbf{u} - \partial_{\Delta t}^{n+1}\mathbf{u}_h|| + \dots \\
&\dots + ||(\partial_t \mathbf{u})^{n+1} - \partial_{\Delta t}^{n+1}\mathbf{u}|| + ||p^{n+1} - p_h^{n+1}|| + ||\mathbf{u}||_{l^\infty([\overline{n}_0+1,N];H^1)} + \dots \\
&\dots + ||\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}||_1 + \nu||\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}||_1 + \dots \\
&\dots + ||\mathbf{u}_h||_{l^\infty([\overline{n}_0+1,N];H^1)}(||\xi^n(\mathbf{u}) - \xi^n(\mathbf{u}_h)||_1 + ||\mathbf{u}^{n+1} - \xi^n(\mathbf{u})||_1)). \qquad (7.50)
\end{aligned}
$$

Square both sides of (7.50), multiply by Δt, and sum from $n = \overline{n}_0$ to $N-1$ to get

$$
\begin{aligned}
&\Delta t \sum_{n=\overline{n}_0}^{N-1} ||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||^2_{h,-1/2,\partial\Omega_s} \\
&\leq C(||\partial_{\Delta t}\mathbf{u} - \partial_{\Delta t}\mathbf{u}_h||^2_{l^2([\overline{n}_0+1,N];L^2)} + ||p - p_h||^2_{l^2([\overline{n}_0+1,N];L^2)} + \dots \\
&\quad \dots + (||\mathbf{u}||^2_{l^\infty([\overline{n}_0+1,N];H^1)} + ||\mathbf{u}_h||^2_{l^\infty([\overline{n}_0+1,N];H^1)})||\mathbf{u} - \mathbf{u}_h||^2_{l^2(H^1)} + \dots \\
&\quad \dots + ||(\partial_t \mathbf{u}) - \partial_{\Delta t}\mathbf{u}||^2_{l^2([\overline{n}_0+1,N];L^2)} + ||\mathbf{u}_h||^2_{l^\infty([\overline{n}_0+1,N];H^1)}\Delta t \sum_{n=\overline{n}_0}^{N-1} ||\mathbf{u}^{n+1} - \xi^n(\mathbf{u})||_1^2). \qquad (7.51)
\end{aligned}
$$

We assume here that $\nu \leq ||\mathbf{u}||_{l^\infty([\overline{n}_0+1,N];H^1)}$. Apply (7.30), (7.35), (7.41), (2.8), and (2.10) via Assumption 2.2.1 to conclude the proof. □

7.2 CONVERGENCE ANALYSIS, BEFEB

In this section, we investigate the convergence of $\mathbf{u}_{\varepsilon,h} \to \mathbf{u}$ as h, Δt, $\varepsilon \to 0$. In Proposition 7.2.3 we show that

$$
\begin{aligned}
H^1 - \text{penalization} \quad \Rightarrow \quad & ||\mathbf{u}_{\varepsilon,h}||_{l^2(H^1(\Omega_s))\cap l^\infty(L^2(\Omega_s))} && \leq C_*\varepsilon, \\
& ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_{l^2(H^1)\cap l^\infty(L^2)} && \leq C_*\varepsilon^{1/2} \\
L^2 - \text{penalization} \quad \Rightarrow \quad & ||\mathbf{u}_{\varepsilon,h}||_{l^2(L^2(\Omega_s))} && \leq C_*\varepsilon^{3/4}, \\
& ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_{l^2(H^1(\Omega_{ext}))\cap l^\infty(L^2(\Omega_{ext}))} && \leq C_*\varepsilon^{1/4}.
\end{aligned} \qquad (7.52)
$$

To guarantee the convergence rates in (7.52), the initial data $\left\{\mathbf{u}_{\varepsilon,h}^{i}\right\}_{i=0}^{\overline{n}_0}$ must be a *good* approximation of $\{\mathbf{u}_h^i\}_{i=0}^{\overline{n}_0}$. We make this precise in Assumption 7.2.1. First define

$$\begin{aligned} F_{ic} :=\ & ||\mathbf{u}_h^{\overline{n}_0} - \mathbf{u}_{\varepsilon,h}^{\overline{n}_0}|| + \gamma_{1,s}^{1/2}||\mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_s} \\ & + \begin{cases} ||\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}||_{l^\infty(0,n_0-1;L^2)}(\Delta t \sum_{n=n_0}^{2n_0-1} \mu^n)^{1/2} & \text{if } n_0 > 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{7.53}$$

Assumption 7.2.1. *Fix $i = 0, 1, \ldots, \overline{n}_0$. The data $\mathbf{u}_{\varepsilon,h}^i \approx \mathbf{u}_h^i$ so that*

$$F_{ic} \leq C(\gamma_{2,s}^{-1}\varepsilon)^{1/4} \tag{7.54}$$

for some $C > 0$ where F_{ic} is given in (7.53).

Suppose that $n_0 \leq 0$. If $\gamma_{1,s} = 1$, then Assumption 7.2.1 implies that $||\mathbf{u}_h^{\overline{n}_0} - \mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_{ext}} \leq \mathcal{O}(\varepsilon^{1/4})$ for L^2-penalization and $||\mathbf{u}_h^{\overline{n}_0} - \mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_{ext}} \leq \mathcal{O}(\varepsilon^{1/2})$ for H^1-penalization. If $\gamma_{1,s} = \varepsilon^{-1}$ then we additionally require $||\mathbf{u}_h^{\overline{n}_0} - \mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_s} \leq \mathcal{O}(\varepsilon^{3/4})$ for L^2-penalization and $||\mathbf{u}_h^{\overline{n}_0} - \mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_s} \leq \mathcal{O}(\varepsilon)$.

In Theorem 7.3.6 we improve the estimate in Proposition 7.2.3 so that

$$H^1 - \text{penalization} \quad \Rightarrow \quad ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_{l^\infty(L^2(\Omega_{ext}))\cap l^2(H^1(\Omega_{ext}))} \leq C_*\varepsilon. \tag{7.55}$$

For this optimal $\mathcal{O}(\varepsilon)$ to hold, the initial data $\left\{\mathbf{u}_{\varepsilon,h}^{i}\right\}_{i=0}^{\overline{n}_0}$ must be a *better* approximation of $\{\mathbf{u}_h^i\}_{i=0}^{\overline{n}_0}$ than required for (7.52). We make this precise in the next assumption.

Assumption 7.2.2. *Fix $i = 0, 1, \ldots, \overline{n}_0$. The data $\mathbf{u}_{\varepsilon,h}^i \approx \mathbf{u}_h^i$ so that*

$$F_{ic} \leq C\varepsilon \tag{7.56}$$

for some $C > 0$ where F_{ic} is given in (7.53).

Suppose that $n_0 \leq 0$. If $\gamma_{1,s} = \varepsilon^{-1}$, then Assumption 7.2.2 requires that $||\mathbf{u}_h^{\overline{n}_0} - \mathbf{u}_{\varepsilon,h}^{\overline{n}_0}||_{\Omega_s} \leq \mathcal{O}(\varepsilon^{3/2})$. This is more restrictive than the predicted $\mathcal{O}(\varepsilon)$ error estimate.

For BEFEb-approximations, we require a time-step restriction (depending on problem data, not h) to ensure the convergence (7.52), (7.55). For BELEb-approximations, we avoid any time-step restriction by exploiting the second discrete Gronwall Lemma 2.4.6. Note that the constant $C_* > 0$ in (7.52) and (7.55) depends on problem data but is independent of h, Δt, $\varepsilon \to 0$ as long as $\mathbf{u}_h \in l^4(H^1)$ (or $\mathbf{u}_h \in l^\infty(H^1)$ which is guaranteed for BELE for smooth enough NSE solutions in [86], see (7.35)). In particular, it is convenient to introduce

$$\begin{cases} \mu^{n+1} := C\nu^{-3}||\mathbf{u}_h^{n+1}||_1^4, \quad \lambda^{n+1} := 1/(1 - a_{-1}\Delta t \mu^{n+1}) \\ \qquad G^N := C\exp(\Delta t \sum_{n=\overline{n}_0}^{N-1} \lambda^{n+1}\mu^{n+1}) \end{cases} \tag{7.57}$$

since $C_* \propto G^N$. Recall that $a_{-1} = 0$ for BELEb so that $\lambda^{n+1} = 1$ in this case.

Proposition 7.2.3 (Consistency, Part I)**.** *Fix $r = 0$, 1 and let $\gamma_{2,s} = \varepsilon^{r-1}$. Suppose that the FE-space satisfies Assumptions 2.1.1. For each $n \geq \overline{n}_0$, suppose that $\mathbf{u}_h^{n+1}$ solves Problem 7.0.3 and the conditions of Proposition 7.1.1 with σ_h given in (7.36). Suppose further $\mathbf{u}_{\varepsilon,h}^{n+1}$ solves Problem 7.0.4 under the conditions of Lemma 7.0.6. In the case of BEFEb, require $\Delta t \mu^{n+1} < 1$ for each $n \geq \overline{n}_0$. Then*

$$\begin{aligned} &(\gamma_{1,s}\varepsilon)^{1/2}||\mathbf{u}_{\varepsilon,h}||_{l^\infty([\overline{n}_0+1,N];L^2(\Omega_s))} + \nu^{1/2}||\mathbf{u}_{\varepsilon,h}||_{l^2([\overline{n}_0+1,N];H^r(\Omega_s))} \\ &+ \varepsilon^{1/2}||\mathbf{e}_{uh}||_{l^\infty([\overline{n}_0+1,N];L^2)} + \nu^{1/2}\varepsilon^{1/2}||\nabla \mathbf{e}_{uh}||_{l^2([\overline{n}_0+1,N];L^2(\Omega_{ext}))} \leq G^N(F_{ic} + F_\sigma \varepsilon^{(3+r)/4}) \end{aligned} \tag{7.58}$$

where

$$F_\sigma := \begin{cases} ||\sigma_h||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} & \text{if } r = 1 \\ ||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} + \dots & \\ \quad \dots + \varepsilon^{1/4}h^{-1}||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}} - \sigma_h||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} & \text{if } r = 0 \end{cases}$$

and G^N, μ^{n+1} are given in (7.57).

Remark 7.2.4. *In the case $r = 0$, we require that $(\mathbf{u}, p)$ be a strong solution of (7.22) so that $\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}} \in l^2([\overline{n}_0+1, N]; h, H^{-1/2}(\partial\Omega_s))$. There is no h-ε restriction for H^1-penalization. For L^2-penalization it is enough to pick $\varepsilon = \varepsilon(h) = \mathcal{O}(h^4)$ to ensure convergence as h, Δt, $\varepsilon(h) \to 0$.*

We improve the estimate in the previous proposition in the case of H^1-penalization. We require the existence of an extension of either the continuous stress $\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}|_{\partial\Omega_s}$ or discrete stress σ_h summarized in the next assumption.

Assumption 7.2.5. *Fix $\gamma=\gamma_{1,s}\varepsilon$ or $\gamma=0$, and $\mathbb{T}^n=\sigma_h^n$ or $\mathbb{T}^n=\sigma(\mathbf{u}^n,p^n)\cdot\hat{\mathbf{n}}$ for $n=0,1,\dots,N$. There exists $\omega_*^n\in V_h(\Omega_{ext})$ so that $\omega_*^n|_{\Omega_s}\in V_{h,\cdot}(\Omega_s)$ and*

$$\int_{\Omega_s}(\gamma\partial_{\Delta t}^n\omega_*\cdot\mathbf{v}_h+\nu\nabla\omega_*^n:\nabla\mathbf{v}_h+\nu\omega_*^n\cdot\mathbf{v}_h)=\int_{\partial\Omega_s}\mathbb{T}^n\cdot\mathbf{v}_h\qquad\forall\mathbf{v}_h\in V_{h,\cdot}(\Omega_s).\tag{7.59}$$

Moreover, for each $n=\overline{n}_0,\dots,N-1$ and any $\mathbf{v}\in V_h(\Omega_{ext})$,

$$\begin{aligned}|(\partial_{\Delta t}^{n+1}\omega_*,\mathbf{v})+\nu(\nabla\omega_*^{n+1},\nabla\mathbf{v})+(\gamma_{1,s}-\frac{\gamma}{\varepsilon})\int_{\Omega_s}\partial_{\Delta t}^{n+1}\omega_*\cdot\mathbf{v}| &\leq L_{\omega*}^{n+1}|\mathbf{v}|_{1,\Omega_{ext}}\\ |c_h(\xi^n(\omega_*),\mathbf{u}_h^{n+1},\mathbf{v})+c_{h,ext}(\xi^n(\mathbf{u}_{\varepsilon,h}),\omega_*^{n+1},\mathbf{v})| &\leq M_{\omega*}^{n+1}|\mathbf{v}|_{1,\Omega_{ext}}\end{aligned}\tag{7.60}$$

for some $L_{\omega*}$, $M_{\omega*}\in l^2(\overline{n}_0,N)$.

We prove existence of ω_* satisfying Assumption 7.2.5 in Proposition 7.2.9. Note that $||\mathbb{T}-\sigma_h||_{l^2([n_0+1,N];h,H^{-1/2}(\partial\Omega_s))}=0$ in $F_{\omega*}$ in Theorem 7.2.6 when $\mathbb{T}=\sigma_h$. However, this requires boundedness of the discrete stress $\partial_{\Delta t}\sigma_h\in l^2(h,H^{-1/2}(\partial\Omega_s))$. Choosing $\mathbb{T}=\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}|_{\partial\Omega_s}$ keeps the regularity restriction on $(\mathbf{u},p)$. The cost, however, is requiring (h^k+h^{s+1}), $\Delta t\leq\varepsilon^{1/2}$ to preserve $\mathcal{O}(\varepsilon)$-convergence. The worst case prediction is $\mathcal{O}(\varepsilon^{1/2})$-convergence so that the results of Proposition 7.2.9 prevail.

Theorem 7.2.6 (Consistency, Part II)**.** *Let ω_*, $L_{\omega*}$, $M_{\omega*}$, and $\mathbb{T}$ satisfy the properties of Assumption 7.2.5. Suppose that the conditions of Proposition 7.2.3 are satisfied with $\gamma_{2,s}=\varepsilon^{-1}$ $(r=1)$. Then,*

$$||\mathbf{u}_h-\mathbf{u}_{\varepsilon,h}||_{l^\infty([\overline{n}_0+1,N];L^2(\Omega_{ext}))}+\nu^{1/2}||\mathbf{u}_h-\mathbf{u}_{\varepsilon,h}||_{l^2([\overline{n}_0+1,N];H^1(\Omega_{ext}))}\leq G^N(F_{ic}+F_{\omega*}\varepsilon)\tag{7.61}$$

where

$$\begin{aligned}F_{\omega*}:=&\,||\omega_*||_{l^\infty(L^2)}+\nu^{1/2}\varepsilon||\omega_*||_{l^2(H^1)}+\gamma_{1,s}^{1/2}\varepsilon||\omega_*^0||_{\Omega_s}\\&+C\nu^{-1/2}(\Delta t\sum_{n=\overline{n}_0}^{N-1}((L_{\omega*}^{n+1})^2+(M_{\omega*}^{n+1})^2))^{1/2}\\&+C\nu^{-1/2}\varepsilon^{-1/2}||\mathbb{T}-\sigma_h||_{l^2([n_0+1,N];h,H^{-1/2}(\partial\Omega_s))})+C_1(\omega_*^{n_0})\end{aligned}$$

and $C_1(\omega_*^{n_0})=||\omega_*||_{l^\infty(0,n_0-1;L^2)}(\Delta t\sum_{n=0}^{2n_0-1}\mu^n)^{1/2}$ if $n_0>1$ and $C_1(\omega_*^{n_0})=0$ otherwise.

Proof. See Section 7.2.2. □

Under suitable regularity of NS-solutions $(\mathbf{u}, p)$, a *nice enough* FE-space yields the estimate

$$||\mathbf{u} - \mathbf{u}_h||_{l^\infty([\overline{n}_0+1;N];L^2)} + ||\mathbf{u} - \mathbf{u}_h||_{l^2([\overline{n}_0+1;N];H^1)} \leq C_*(h^{s+1} + h^k + \Delta t) \tag{7.62}$$

for some constant $C_* > 0$ independent of $h \to 0$ (although, depending on problem data including ν^{-1}, $||\mathbf{u}||_{k+1}$, and $||p||_{s+1}$). Estimate (7.62) holds without Δt-restriction for BELE, but requires $\Delta t \leq C\nu$ for BEFE (see e.g. [86]).

Theorem 7.2.7 (Convergence, FE-Brinkman)**.** *Under the assumptions of Theorem 7.2.6, suppose further that for some* $k \geq 0$, $s \geq -1$, $\mathbf{u}(\cdot, t) \in H^1_\phi \cap H^{k+1}$, $p(\cdot, t) \in L^2_0 \cap H^{s+1}$ *are solutions of (1.1). Then*

$$||\mathbf{u}_{\varepsilon,h} - \mathbf{u}||_{l^\infty([\overline{n}_0;N];L^2)} + ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}||_{l^2([\overline{n}_0;N];H^1)} \leq C_*(h^{s+1} + h^k + \Delta t + \varepsilon) \tag{7.63}$$

where $0 < C_* < \infty$ *is independent of* h, Δt, $\varepsilon \to 0$.

Proof. The triangle inequality gives $||\mathbf{u}_{\varepsilon,h} - \mathbf{u}||_1 \leq ||\mathbf{u}_{\varepsilon,h} - \mathbf{u}_h||_1 + ||\mathbf{u}_h - \mathbf{u}||_1$. Then (7.63) follows from an application of (7.61), (7.62). □

We also investigate the approximability of discrete Darcy drag contribution $\nu\varepsilon^{-1}\int_{\Omega_s} \mathbf{u}_\varepsilon$ to the discrete stress σ_h on $\partial\Omega_s$ in addition to an error estimate for $p_{\varepsilon,h}|_\Omega \to p_h$. These results are derived from the velocity error estimates presented earlier for $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^\infty(L^2) \cap L^2(H^1)$ in Proposition 7.2.3, Theorem 7.2.6. Define first the discrete time-averaging operator by

$$< \theta >_N := \frac{\Delta t}{N} \sum_{n=\overline{n}_0}^{N-1} \theta^{n+1}. \tag{7.64}$$

Estimates (7.65), (7.66) provide long-time $T \to \infty$ estimate for the discrete pressure and drag/lift consistency error in modeling with BEFEb. Under the conditions of Theorem 7.2.6 and for fixed $0 < T < \infty$, $0 < N = N(T) < \infty$, the time-averaged error of pressure satisfies

$$\mathrm{error}(p_h - p_{\varepsilon,h}) \leq C\varepsilon.$$

The discrete drag and lift coefficients are given by $D_h = \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{c}_d$ and $L_h = \int_{\partial\Omega_s} \sigma_h \cdot \mathbf{c}_l$ for some constant vectors $\mathbf{c}_d$, $\mathbf{c}_l$. Under the same conditions as above, the discrete time-averaged error of drag/lift on Ω_s satisfies

$$\text{error}(D_h - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}\cdot\mathbf{c}_d) + \text{error}(L_h - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}\cdot\mathbf{c}_l) \leq C\varepsilon.$$

Theorem 7.2.8. *Suppose that the conditions of Proposition 7.2.3 are satisfied. Then*

$$|| < p_h - p_{\varepsilon,h} >_N || \leq N^{-1}E_1 + \max\{K_0, K_{B,0}\} < |\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}|_1^2 >_N^{1/2} \tag{7.65}$$

where $E_1 := C(||\mathbf{u}_h^N - \mathbf{u}_{\varepsilon,h}^N|| + ||\mathbf{u}_h^0 - \mathbf{u}_{\varepsilon,h}^0||)$*. Additionally, for any constant unit vector* $\hat{\mathbf{c}} : \partial\Omega_s \to \mathbb{R}^d$*, we have,*

$$\begin{aligned} &| < \int_{\partial\Omega_s} (\sigma_h \cdot \hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}\cdot\hat{\mathbf{c}} >_N | \leq N^{-1}E_2 \\ &\quad + \nu < |\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}|_1 >_N^{1/2} + C(< |\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}|_1^2 >_N^{1/2} + < |\mathbf{u}_h|_1^2 >_N^{1/2})||\mathbf{u}_h - \mathbf{u}_{\varepsilon,h}||_{l^\infty(L^2)} \end{aligned} \tag{7.66}$$

where $E_2 := C(E_1 + \gamma_{1,s}(||\mathbf{u}_{\varepsilon,h}^N||_{\Omega_s} + ||\mathbf{u}_{\varepsilon,h}^0||_{\Omega_s}))$*.*

Proof. See Section 7.2.3. □

We finish with a proof of particular ω_* satisfying Assumption 7.2.5.

Proposition 7.2.9 (Auxiliary estimate)**.** *Let* $\mathbb{T}^n = \sigma_h^n$ *or* $\mathbb{T}^n = \sigma(\mathbf{u}^n, p^n)\cdot\hat{\mathbf{n}}$ *for* $n = 0, 1, \ldots, N$*. Under the hypotheses of Theorem 7.2.6, there exists* ω_*^n *satisfying Assumption 7.2.5. In particular, for* $\gamma = 0$*,* $\gamma_{1,s} = 1$ *and* $m = 0,\ 1$*,*

$$||(\partial_{\Delta t}^{(m)})^{n+1}\omega_*||_{\Omega_{ext}} \leq \nu^{-1}||(\partial_{\Delta t}^{(m)})^{n+1}\mathbb{T}||_{h,-1/2,\partial\Omega_s}. \tag{7.67}$$

Moreover, $F_{\omega*}$ *in Theorem 7.2.6 can be replaced by*

$$\begin{aligned} F_{\omega*} &:= \nu^{-1}(\nu^{1/2}||\mathbb{T}||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} + \ldots \\ &\quad \ldots + ||\partial_{\Delta t}\mathbb{T}||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} + \nu^{-1/2}\max\{K_0, K_{B,0}\}\,||\mathbb{T}||_{l^\infty(h,H^{-1/2}(\partial\Omega_s))} + \ldots \\ &\quad \ldots + C\nu^{1/2}\varepsilon^{-1/2}||\mathbb{T} - \sigma_h||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} + C_2(\mathbb{T}^{n_0})) \end{aligned}$$

where $C_2(\mathbb{T}^{n_0}) = \nu^{-1}||\mathbb{T}||_{l^\infty([0,n_0-1];h,H^{-1/2}(\partial\Omega_s))}$ *when* $n_0 > 0$ *and* $C_2(\mathbb{T}^{n_0}) = 0$ *otherwise. .*

Proof. See Section 7.2.4. □

7.2.1 Proof of Velocity Error, Proposition 7.2.3

Propositions 7.2.3. Set $\mathbf{v}_h = \mathbf{e}_{uh}^{n+1}$ in (7.34). Then application of the identity $(\mathbf{a}-\mathbf{b},\mathbf{a}) = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2)$ gives

$$\begin{aligned}
&\frac{1}{2\Delta t}(||\mathbf{e}_{uh}^{n+1}||^2 - ||\mathbf{e}_{uh}^{n}||^2 + ||\mathbf{e}_{uh}^{n+1} - \mathbf{e}_{uh}^{n}||^2) \\
&\quad + \frac{\gamma_{1,s}}{2\Delta t}(||\mathbf{u}_{\varepsilon,h}^{n+1}||^2_{\Omega_s} - ||\mathbf{u}_{\varepsilon,h}^{n}||^2_{\Omega_s} + ||\mathbf{u}_{\varepsilon,h}^{n+1} - \mathbf{u}_{\varepsilon,h}^{n}||^2_{\Omega_s}) \\
&\quad + \nu|\mathbf{e}_{uh}^{n+1}|_1^2 + \nu(\gamma_{2,s}|\mathbf{u}_{\varepsilon,h}^{n+1}|^2_{1,\Omega_s} + \varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}^{n+1}||^2_{\Omega_s}) \\
&\quad = -\int_{\partial\Omega_s} \sigma_h^{n+1}\cdot \mathbf{e}_{uh}^{n+1} - c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}_h^{n+1}, \mathbf{e}_{uh}^{n+1}).
\end{aligned} \tag{7.68}$$

We first need to majorize each term on the RHS of (7.68). Suppose that $\mathbf{u}_h \in l^\infty(H^1)$. We bound the terms in (7.69). Recall that $\xi^n(\mathbf{v}) = a_{-1}\mathbf{v}^{n+1} + a_0\mathbf{v}^n + \ldots + a_{n_0}\mathbf{v}^{n-n_0}$ where $a_{-1} = 1$, $a_i = 0$ for all $i \geq 0$ for BEFEb and $a_{-1} = 0$, $a_i \neq 0$ for some $i \geq 0$ for BELEb. Estimate (2.41)(a) and Young's inequality (2.21) give

$$\begin{aligned}
&|c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}_h^{n+1}, \mathbf{e}_{uh}^{n+1})| \\
&\quad \leq C\nu^{-3}||\mathbf{u}_h^{n+1}||_1^4||\xi^n(\mathbf{e}_{uh})||^2 + \frac{\nu}{2}(\frac{(1-a_{-1})}{\overline{n}_0+1}\sum_{i=0}^{\overline{n}_0}|\mathbf{e}_{uh}^{n-i}|_1^2 + a_{-1}\ |\mathbf{e}_{uh}^{n+1}|_1^2).
\end{aligned} \tag{7.69}$$

Bounding $\int_{\partial\Omega_s}\sigma_h^{n+1}\cdot\mathbf{e}_{uh}^{n+1}$ in (7.68) remains.

Lemma 7.2.10. *Suppose that the FE-space satisfies 2.1.1. Then,*

$$\begin{aligned}
&|\int_{\partial\Omega_s}\sigma_h^{n+1}\cdot\mathbf{u}_{\varepsilon,h}^{n+1}| \leq \frac{\nu}{2}(\gamma_{2,s}|\mathbf{u}_{\varepsilon,h}|^2_{1,\Omega_s} + \varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}||^2_{\Omega_s}) \\
&\quad + \varepsilon^{1/2}(\kappa_1||\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s} + \kappa_2||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1})\cdot\hat{\mathbf{n}}||^2_{h,-1/2,\partial\Omega_s})
\end{aligned} \tag{7.70}$$

where

$$\kappa_1 := C\begin{cases}\varepsilon^{1/2} & \text{if } \gamma_{2,s} = \varepsilon^{-1}\\ 1 & \text{if } \gamma_{2,s} = 1\end{cases} \qquad \kappa_2 := C\begin{cases}0 & \text{if } \gamma_{2,s} = \varepsilon^{-1}\\ \varepsilon h^{-2} & \text{if } \gamma_{2,s} = 1\end{cases}. \tag{7.71}$$

Proof. We consider the case of H^1- and L^2-penalization separately.

Case 1 (H^1-penalization): Suppose that $\gamma_{2,s} = \varepsilon^{-1}$. Recall the discrete $H^{-1/2}(\partial\Omega_s)$-norm in (6.7). Then

$$|\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \mathbf{u}_{\varepsilon,h}^{n+1}| \leq ||\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||_{1,\Omega_s}. \tag{7.72}$$

Then application of Young's inequality (2.21) to (7.72) gives

$$|\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \mathbf{u}_{\varepsilon,h}^{n+1}| \leq C\nu^{-1}\varepsilon ||\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s} + \frac{\nu}{2\varepsilon}||\mathbf{u}_{\varepsilon,h}^{n+1}||^2_{1,\Omega_s}. \tag{7.73}$$

Case 2 (L^2-penalization): Suppose that $\gamma_{2,s} = 1$. Write

$$\sigma_h^{n+1} = \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}} + (\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}).$$

Recall (6.7). Then

$$\begin{aligned} |\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \mathbf{u}_{\varepsilon,h}^{n+1}| &\leq C||\sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||_{\partial\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||^{1/2}_{\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||^{1/2}_{1,\Omega_s} \\ &+ ||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||_{h,-1/2,\partial\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||_{1,\Omega_s}. \end{aligned} \tag{7.74}$$

Application of Young's inequality (2.21) gives

$$\begin{aligned} &||\sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||_{\partial\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||^{1/2}_{\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||^{1/2}_{1,\Omega_s} \\ &\leq C\nu^{-1}\varepsilon^{1/2} ||\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s} + \frac{\nu}{4}||\mathbf{u}_{\varepsilon,h}^{n+1}||^2_{1,\Omega_s} + \frac{\nu}{4\varepsilon}||\mathbf{u}_{\varepsilon,h}^{n+1}||^2_{\Omega_s}. \end{aligned} \tag{7.75}$$

Application of the inverse (2.4) and Young's (2.21) inequalities give

$$\begin{aligned} &||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||_{h,-1/2,\partial\Omega_s} ||\mathbf{u}_{\varepsilon,h}^{n+1}||_{1,\Omega_s} \\ &\leq C\nu^{-1}\varepsilon^{1/2}(\varepsilon^{1/2}h^{-2}||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||^2_{h,-1/2,\partial\Omega_s}) + \frac{\nu}{4\varepsilon}||\mathbf{u}_{\varepsilon,h}^{n+1}||^2_{\Omega_s}. \end{aligned} \tag{7.76}$$

Application of (7.73), (7.74) (7.75) (7.76) proves Lemma 7.2.10. □

Apply (7.69), and (7.70) to (7.68) to get

$$
\begin{aligned}
&||\mathbf{e}_{uh}^{n+1}||^2 - ||\mathbf{e}_{uh}^{n}||^2 + \gamma_{1,s}(||\mathbf{u}_{\varepsilon,h}^{n+1}||_{\Omega_s}^2 - ||\mathbf{u}_{\varepsilon,h}^{n}||_{\Omega_s}^2) \\
&\quad + \nu\Delta t((2-a_{-1})|\mathbf{e}_{uh}^{n+1}|_1^2 - (1-a_{-1})\frac{\nu}{\overline{n}_0+1}\sum_{i=0}^{\overline{n}_0}|\mathbf{e}_{uh}^{n-i}|_1^2) \\
&\quad + \nu\Delta t(\gamma_{2,s}|\mathbf{u}_{\varepsilon,h}^{n+1}|_{1,\Omega_s}^2 + \varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}^{n+1}||_{\Omega_s}^2) \\
&\le \mu^{n+1}\Delta t||\xi^n(\mathbf{e}_{uh})||^2 \\
&\quad + \varepsilon^{1/2}\Delta t(\kappa_1^{n+1}||\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s}^2 + \kappa_2^{n+1}||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1},p^{n+1})\cdot\hat{\mathbf{n}}||_{h,-1/2,\partial\Omega_s}^2)
\end{aligned} \tag{7.77}
$$

where

$$\mu^{n+1} := C\nu^{-3}||\mathbf{u}_h^{n+1}||_1^4.$$

Note that

$$\sum_{n=\overline{n}_0}^{N-1}\frac{\nu}{\overline{n}_0+1}\sum_{i=0}^{\overline{n}_0}|\mathbf{e}_{uh}^{n-i}|_1^2 \le \nu\sum_{n=\overline{n}_0}^{N-1}|\mathbf{e}_{uh}^{n+1}|_1^2 + \nu\sum_{i=0}^{\overline{n}_0}|\mathbf{e}_{uh}^{i}|_1^2.$$

Sum from $n = \overline{n}_0$ to $N-1$ and simplify to get

$$
\begin{aligned}
&||\mathbf{e}_{uh}^{N}||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^{N}||_{\Omega_s}^2 \\
&\quad + \nu\Delta t\sum_{n=\overline{n}_0}^{N-1}|\mathbf{e}_{uh}^{n+1}|_1^2 + \nu(\gamma_{2,s}\Delta t\sum_{n=\overline{n}_0}^{N-1}|\mathbf{u}_{\varepsilon,h}^{n+1}|_{1,\Omega_s}^2 + \varepsilon^{-1}\Delta t\sum_{n=\overline{n}_0}^{N-1}||\mathbf{u}_{\varepsilon,h}^{n+1}||_{\Omega_s}^2) \\
&\quad \le ||\mathbf{e}_{uh}^{0}||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^{0}||_{\Omega_s}^2 + (1-a_{-1})\nu\Delta t\sum_{i=0}^{\overline{n}_0}|\mathbf{e}_{uh}^{i}|_1^2 + \Delta t\sum_{n=\overline{n}_0}^{N-1}\mu^{n+1}||\xi^n(\mathbf{e}_{uh})||^2 \\
&\quad + \varepsilon^{1/2}\Delta t\sum_{n=\overline{n}_0}^{N-1}(\kappa_1||\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s}^2 + \kappa_2||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1},p^{n+1})\cdot\hat{\mathbf{n}}||_{h,-1/2,\partial\Omega_s}^2).
\end{aligned} \tag{7.78}
$$

Case 1 (BEFE): Suppose that $\xi^n(\mathbf{e}_{uh}) = \mathbf{e}_{uh}^{n+1}$ so that $n_0 = -1$, $\overline{n}_0 = 0$. Define

$$\lambda^{n+1} := \frac{1}{1-\Delta t\mu^{n+1}}, \quad G^N := \exp(\Delta t\sum_{n=0}^{N-1}\lambda^{n+1}\kappa^{n+1}).$$

Suppose that $\Delta t \kappa^{n+1} < 1$ for all $n = 0, 1, \ldots, N-1$. Then the discrete Gronwall Lemma 2.4.5 applied to (7.78) gives

$$
\begin{aligned}
&||\mathbf{e}_{uh}^N||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^N||_{\Omega_s}^2 + \nu||\nabla \mathbf{e}_{uh}||_{l^2(1,N;L^2)}^2 \\
&\quad + \nu\gamma_{2,s}||\nabla \mathbf{u}_{\varepsilon,h}||_{l^2(1,N;L^2(\Omega_s))}^2 + \nu\varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}||_{l^2(1,N;L^2(\Omega_s))}^2 \\
&\leq G^N(||\mathbf{e}_{uh}^0||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^0||_{\Omega_s}^2 + \ldots \\
&\quad \ldots + \varepsilon^{1/2}(\kappa_1||\sigma_h||_{l^2([1,N];h,H^{-1/2}(\partial\Omega_s))}^2 + \kappa_2||\sigma_h - \sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||_{l^2([1,N];h,H^{-1/2}(\partial\Omega_s))}^2)).
\end{aligned} \tag{7.79}
$$

Case 2 (BELE): Suppose now that $\xi^n(\mathbf{e}_{uh}) = a_0\mathbf{e}_{uh}^n + a_1\mathbf{e}_{uh}^{n-1} + \ldots + \mathbf{e}_{uh}^{n-n_0}$ so that $\overline{n}_0 = n_0 + 1$. Identity (2.20) gives

$$
\begin{aligned}
\sum_{n=n_0+1}^{N-1} \mu^{n+1}||\xi^n(\mathbf{e}_{uh})||^2 &= \sum_{n=n_0+1}^{N-1} \mu^{n+1}||\sum_{i=0}^{n_0} a_i\mathbf{e}_{uh}^{n-i}||^2 \\
&\leq \sum_{n=n_0+1}^{N-1} \mu^{n+1}\sum_{i=0}^{n_0} |a_i|^2(n_0+1)||\mathbf{e}_{uh}^{n-i}||^2 \\
&\leq (n_0+1)\sum_{n=0}^{N-1} ||\mathbf{e}_{uh}^n||^2 \sum_{i=i_0(n)}^{i_1(n)} |a_i|^2\mu^{n+1+i}.
\end{aligned} \tag{7.80}
$$

Apply (7.80) to (7.78). Then

$$
\begin{aligned}
&||\mathbf{e}_{uh}^N||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^N||_{\Omega_s}^2 \\
&\quad + \nu\Delta t\sum_{n=n_0}^{N-1} |\mathbf{e}_{uh}^{n+1}|_1^2 + \nu(\gamma_{2,s}\Delta t\sum_{n=n_0}^{N-1} |\mathbf{u}_{\varepsilon,h}^{n+1}|_{1,\Omega_s}^2 + \varepsilon^{-1}\Delta t\sum_{n=n_0}^{N-1} ||\mathbf{u}_{\varepsilon,h}^{n+1}||_{\Omega_s}^2) \\
&\quad \leq ||\mathbf{e}_{uh}^{n_0}||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^{n_0}||_{\Omega_s}^2 + \Delta t\sum_{n=0}^{n_0-1} ||\mathbf{e}_{uh}^n||^2 \sum_{i=i_0(n)}^{i_1(n)} \mu^{n+1+i} \\
&\quad + \varepsilon^{1/2}\Delta t\sum_{n=n_0}^{N-1}(\kappa_1||\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s}^2 + \kappa_2||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1},p^{n+1})\cdot\hat{\mathbf{n}}||_{h,-1/2,\partial\Omega_s}^2)
\end{aligned} \tag{7.81}
$$

where $\sum_{n=0}^{n_0-1} c_n = 0$ if $n_0 < 1$. Define

$$
G^N := C\exp(\Delta t\sum_{n=n_0}^{N-1} \mu^{n+1}).
$$

Apply the discrete Gronwall Lemma 2.4.6 to (7.81) so that

$$
\begin{aligned}
&||\mathbf{e}_{uh}^N||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^N||^2_{\Omega_s} \\
&\quad + \nu||\nabla \mathbf{e}_{uh}||^2_{l^2(n_0+1,N;L^2)} + \nu(\gamma_{2,s}||\nabla \mathbf{u}_{\varepsilon,h}||^2_{l^2(n_0+1,N;L^2(\Omega_s))} + \varepsilon^{-1}||\mathbf{u}_{\varepsilon,h}||^2_{l^2(n_0+1,N;L^2(\Omega_s))}) \\
&\quad \leq G^N(||\mathbf{e}_{uh}^{n_0}||^2 + \gamma_{1,s}||\mathbf{u}_{\varepsilon,h}^{n_0}||^2_{\Omega_s} + \Delta t \sum_{n=0}^{n_0-1} ||\mathbf{e}_{uh}^n||^2 \sum_{i=i_0(n)}^{i_1(n)} \mu^{n+1+i} + \dots \\
&\quad \dots + \varepsilon^{1/2}\Delta t \sum_{n=n_0}^{N-1} (\kappa_1||\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s} + \kappa_2||\sigma_h^{n+1} - \sigma(\mathbf{u}^{n+1}, p^{n+1}) \cdot \hat{\mathbf{n}}||^2_{h,-1/2,\partial\Omega_s})). \qquad (7.82)
\end{aligned}
$$

Then (7.79) for BEFEb and (7.82) for BELEb proves Proposition 7.2.3.

□

7.2.2 Proof of Velocity Error, Theorem 7.2.6

Theorem 7.2.6. Consider the ε-order expansion of the BEFEb velocity and pressure:

$$\mathbf{u}_{\varepsilon,h}^n = \mathbf{u}_h^n + \varepsilon(\omega^n + \omega_*^n), \qquad p_{\varepsilon,h}^n = p_h^n + \varepsilon(\pi^n + \pi_*^n)$$

so that $\omega^n \in V_h(\Omega_{ext})$ for each $n = 0, 1, \dots, N$ and $\omega^i = \varepsilon^{-1}(\mathbf{u}_{\varepsilon,h}^i - \mathbf{u}_h^i) - \omega_*^i$ for each $i = 0, 1, \dots, \overline{n}_0$. Note that

$$
\begin{aligned}
&c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_h), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) \\
&\quad + c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}^{n+1}, \mathbf{v}_h) = c_h(\xi^n(\mathbf{u}_{\varepsilon,h}), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) + c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}^{n+1}, \mathbf{v}_h).
\end{aligned}
$$

Substitute into (7.34) and divide by ε to get, for all $\mathbf{v}_h \in X_h(\Omega_{ext})$,

$$
\begin{aligned}
&\int_{\Omega_{ext}} \gamma_1 \partial_{\Delta t}^{n+1}\omega \cdot \mathbf{v}_h + \nu \int_{\Omega_{ext}} \gamma_2 \nabla \omega^{n+1} : \nabla \mathbf{v}_h - \int_{\Omega_{ext}} \pi^{n+1} \nabla \cdot \mathbf{v}_h + \nu\varepsilon^{-1} \int_{\Omega_s} \omega^{n+1} \cdot \mathbf{v}_h \\
&= -c_h(\xi^n(\mathbf{u}_{\varepsilon,h}), \omega^{n+1}, \mathbf{v}_h) - c_h(\xi^n(\omega), \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\
&\quad - c_h(\xi^n(\omega_*), \mathbf{u}_h^{n+1}, \mathbf{v}_h) - c_h(\xi^n(\mathbf{u}_{\varepsilon,h}), \omega_*^{n+1}, \mathbf{v}_h) \\
&\quad - \left[\int \partial_{\Delta t}^{n+1}\omega_* \cdot \mathbf{v}_h + \nu \int \nabla \omega_*^{n+1} : \nabla \mathbf{v}_h - \int \pi_*^{n+1} \nabla \cdot \mathbf{v}_h\right] \\
&\quad - \varepsilon^{-1}\left[\gamma_{1,s}\varepsilon \int_{\Omega_s} \partial_{\Delta t}^{n+1}\omega_*^{n+1} \cdot \mathbf{v}_h + \nu \int_{\Omega_s} \nabla \omega_*^{n+1} : \nabla \mathbf{v}_h - \varepsilon \int_{\Omega_s} \pi_*^{n+1} \nabla \cdot \mathbf{v}_h + \nu \int_{\Omega_s} \omega_*^{n+1} \cdot \mathbf{v}_h\right] \\
&\quad + \varepsilon^{-1} \int_{\partial\Omega_s} \mathbb{T}^{n+1} \cdot \mathbf{v}_h + \varepsilon^{-1} \int_{\partial\Omega_s} (\mathbb{T}^{n+1} - \sigma_h^{n+1}) \cdot \mathbf{v}_h. \qquad (7.83)
\end{aligned}
$$

The objective is to choose $\{(\omega_*^n, \pi_*^n)\}_{n=0}^N$ so that the RHS of the above equation is $\mathcal{O}(1)$ with respect to ε. In particular, we choose (ω_*^n, π_*^n) so that $\varepsilon^{-1}\int_{\partial\Omega_s} \mathbb{T}^{n+1}\cdot \mathbf{v}_h$ is annihilated. Substitute identity (7.59) into (7.83) to get after simplification, for any $\mathbf{v}_h \in V_h(\Omega_{ext})$

$$\begin{aligned}
&\int_{\Omega_{ext}} \gamma_1 \partial_{\Delta t}^{n+1}\omega\cdot \mathbf{v}_h + \nu\int_{\Omega_{ext}} \gamma_2 \nabla\omega^{n+1} : \nabla \mathbf{v}_h + \nu\varepsilon^{-1}\int_{\Omega_s} \omega^{n+1}\cdot \mathbf{v}_h \\
&= -L(\omega_*^{n+1}; \mathbf{v}_h) - M(\omega_*^{n+1}; \mathbf{v}_h) + \varepsilon^{-1}\int_{\partial\Omega_s} (\mathbb{T}^{n+1} - \sigma_h^{n+1})\cdot \mathbf{v}_h \\
&\quad - c_h(\xi^n(\mathbf{u}_h), \omega^{n+1}, \mathbf{v}_h) - c_h(\xi^n(\omega), \mathbf{u}_h^{n+1}, \mathbf{v}_h)
\end{aligned} \tag{7.84}$$

where

$$\begin{aligned}
L(\omega_*^{n+1}; \mathbf{v}_h) &:= \int \partial_{\Delta t}^{n+1}\omega_*\cdot \mathbf{v}_h + \nu\int \nabla\omega_*^{n+1} : \nabla \mathbf{v}_h + \varepsilon^{-1}(\gamma_{1,s}\varepsilon - \gamma)\int_{\Omega_s} \partial_{\Delta t}^{n+1}\omega_*\cdot \mathbf{v}_h \\
M(\omega_*^{n+1}; \mathbf{v}_h) &:= c_h(\xi^n(\omega_*), \mathbf{u}_h^{n+1}, \mathbf{v}_h) + c_h(\xi^n(\mathbf{u}_{\varepsilon,h}), \omega_*^{n+1}, \mathbf{v}_h).
\end{aligned}$$

Test (7.84) with $\mathbf{v}_h = \omega^{n+1} \in V_h(\Omega_{ext})$. Recall Identity (2.35). Then

$$\begin{aligned}
&\frac{1}{2\Delta t}(||\omega^{n+1}||^2 - ||\omega^n||^2 + ||\omega^{n+1} - \omega^n||^2) \\
&\quad + \frac{\gamma_{1,s}}{2\Delta t}(||\omega^{n+1}||^2_{\Omega_s} - ||\omega^n||^2_{\Omega_s} + ||\omega^{n+1} - \omega^n||^2_{\Omega_s}) + \nu|\omega^{n+1}|_1^2 + \nu\varepsilon^{-1}||\omega^{n+1}||^2_{1,\Omega_s} \\
&= -L(\omega_*^{n+1}; \omega^{n+1}) - M(\omega_*^{n+1}; \omega^{n+1}) + \varepsilon^{-1}\int_{\partial\Omega_s} (\mathbb{T}^{n+1} - \sigma_h^{n+1})\cdot \omega^{n+1} \\
&\quad - c_h(\xi^n(\omega), \mathbf{u}_h^{n+1}, \omega^{n+1}).
\end{aligned} \tag{7.85}$$

It remains now to bound each term on the right-hand side of (7.85) and either absorb terms involving ω^{n+1} into the LHS side or with the discrete Gronwall lemma. Application of Young's inequality (2.21) to (7.60) gives

$$|L(\omega_*^{n+1}; \omega^{n+1}) + M(\omega_*^{n+1}; \omega^{n+1})| \le C\nu^{-1}((L_{\omega*}^{n+1})^2 + (M_{\omega*}^{n+1})^2) + \frac{\nu}{6}|\omega^{n+1}|^2_{1,\Omega_{ext}}. \tag{7.86}$$

Estimate (2.41)(a) and Young's inequality (2.21) give

$$\begin{aligned}
&|c_h(\xi^n(\omega), \mathbf{u}_h^{n+1}, \omega^{n+1})| \\
&\qquad \le \nu^{-3}||\mathbf{u}_h^{n+1}||_1^4||\xi^n(\omega)||^2 + \frac{\nu}{6}|\omega^{n+1}|_1^2.
\end{aligned} \tag{7.87}$$

Recall (6.7). Then

$$|\int_{\partial\Omega_s}(\mathbb{T}^{n+1}-\sigma_h^{n+1})\cdot\omega^{n+1}| \leq C\nu^{-1}||\mathbb{T}^{n+1}-\sigma_h^{n+1}||^2_{h,-1/2,\partial\Omega_s}+\frac{\nu}{6}||\omega^{n+1}||^2_{1,\Omega_s}. \tag{7.88}$$

Apply (7.86), (7.87), and (7.88) to (7.85). Then

$$\begin{aligned}&||\omega^{n+1}||^2-||\omega^n||^2+\gamma_{1,s}(||\omega^{n+1}||^2_{\Omega_s}-||\omega^n||^2_{\Omega_s})+\nu\Delta t|\omega^{n+1}|^2_1+\nu\varepsilon^{-1}\Delta t||\omega^{n+1}||^2_{1,\Omega_s}\\&\leq \Delta t\mu^{n+1}||\xi^n(\omega)||^2+C\nu^{-1}\Delta t((L^{n+1}_{\omega*})^2+(M^{n+1}_{\omega*})^2+\varepsilon^{-1}||\mathbb{T}^{n+1}-\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s})\end{aligned} \tag{7.89}$$

where

$$\mu^{n+1}:=C\nu^{-3}||\mathbf{u}_h^{n+1}||_1^4.$$

Recall $\omega=\varepsilon^{-1}\mathbf{e}_{uh}-\omega_*$. Sum from $n=\overline{n}_0$ to $N-1$ in (7.89) to get after simplification

$$\begin{aligned}&||\omega^N||^2_{\Omega_{ext}}+\nu\Delta t\sum_{n=\overline{n}_0}^{N-1}|\omega^{n+1}|^2_{1,\Omega_{ext}}\leq\varepsilon^{-1}||\mathbf{e}^0_{uh}||+\gamma_{1,s}^{1/2}\varepsilon^{-1}||\mathbf{u}^0_{\varepsilon,h}||_{\Omega_s}+||\omega_*^0||+\gamma_{1,s}^{1/2}||\omega^0_*||_{\Omega_s}\\&+\Delta t\sum_{n=\overline{n}_0}^{N-1}\mu^{n+1}||\xi^n(\omega)||^2+C\nu^{-1}\Delta t\sum_{n=\overline{n}_0}^{N-1}((L^{n+1}_{\omega*})^2+(M^{n+1}_{\omega*})^2+\varepsilon^{-1}||\mathbb{T}^{n+1}-\sigma_h^{n+1}||_{h,-1/2,\partial\Omega_s}).\end{aligned} \tag{7.90}$$

Case 1 (BEFE): Suppose that $\xi^n(\mathbf{e}_{uh})=\mathbf{e}^{n+1}_{uh}$ so that $n_0=-1$, $\overline{n}_0=0$. Define

$$\lambda^{n+1}:=\frac{1}{1-\Delta t\kappa^{n+1}},\quad G^N:=C\exp(\Delta t\sum_{n=0}^{N-1}\lambda^{n+1}\kappa^{n+1}).$$

Suppose that $\Delta t\kappa^{n+1}<1$ for all $n=0,1,\ldots,N-1$. Then the discrete Gronwall Lemma 2.4.5 applied to (7.90) gives

$$\begin{aligned}&||\omega^N||_{\Omega_{ext}}+\nu||\omega||_{l^2(1,N;H^1(\Omega_{ext}))}\\&\leq G^N(\varepsilon^{-1}||\mathbf{e}^0_{uh}||+\gamma_{1,s}^{1/2}\varepsilon^{-1}||\mathbf{u}^0_{\varepsilon,h}||_{\Omega_s}+||\omega^0_*||+\gamma_{1,s}^{1/2}||\omega^0_*||_{\Omega_s}+\ldots\\&\quad\ldots+\nu^{-1/2}(\Delta t\sum_{n=0}^{N-1}(L^{n+1}_{\omega*})^2+(M^{n+1}_{\omega*})^2)^{1/2}+\ldots\\&\quad\ldots+\nu^{-1/2}\varepsilon^{-1/2}||\mathbb{T}-\sigma_h||_{l^2([1,N];h,H^{-1/2}(\partial\Omega_s))}).\end{aligned} \tag{7.91}$$

Recall that $\mathbf{e}_{uh} = \varepsilon(\omega - \omega_*)$. Apply the triangle inequality $||\mathbf{e}_{uh}|| = \varepsilon||\omega|| + \varepsilon||\omega_*||$ along with (7.91) to get

$$\begin{aligned}
&||\mathbf{e}^N_{uh}||_{\Omega_{ext}} + \nu||\mathbf{e}_{uh}||_{l^2(1,N;H^1(\Omega_{ext}))} \le G^N(||\mathbf{e}^0_{uh}|| + \gamma_{1,s}^{1/2}||\mathbf{u}^0_{\varepsilon,h}||_{\Omega_s} + \dots \\
&\quad \dots + \varepsilon||\omega_*||_{l^\infty(L^2(\Omega_{ext}))} + \gamma_{1,s}^{1/2}\varepsilon||\omega^0_*||_{\Omega_s} + \nu^{1/2}||\omega_*||_{l^2(1,N;H^1(\Omega_{ext}))} + \dots \\
&\quad \dots + \nu^{-1/2}\varepsilon(\Delta t\sum_{n=0}^{N-1}(L^{n+1}_{\omega*})^2 + (M^{n+1}_{\omega*})^2)^{1/2} + \dots \\
&\quad \dots + \nu^{-1/2}\varepsilon^{1/2}||\mathbb{T} - \sigma_h||_{l^2([1,N];h,H^{-1/2}(\partial\Omega_s))}).
\end{aligned} \tag{7.92}$$

Case 2 (BELE): Suppose now that $\xi^n(\mathbf{e}_{uh}) = a_0\mathbf{e}^n_{uh} + a_1\mathbf{e}^{n-1}_{uh} + \ldots + \mathbf{e}^{n-n_0}_{uh}$ so that $\overline{n}_0 = n_0 \ge 0$, and $a_{-1} = 0$. Recall $\omega = \varepsilon^{-1}\mathbf{e}_{uh} - \omega_*$. Identity (2.20) gives

$$\begin{aligned}
\Delta t\sum_{n=n_0}^{N-1}\mu^{n+1}||\xi^n(\omega)||^2 &\le \Delta t\sum_{n=n_0}^{N-1}\sum_{i=0}^{n_0}\mu^{n+1}(n_0+1)|a_i|^2||\omega^{n-i}||^2 \\
&= (n_0+1)\Delta t\sum_{n=0}^{N-1}||\omega^n||^2\sum_{i=i_0(n)}^{i_1(n)}|a_i|^2\mu^{n+i+1}.
\end{aligned} \tag{7.93}$$

Apply (7.93) to (7.90). Then, after simplification, we get

$$\begin{aligned}
&||\omega^N||^2_{\Omega_{ext}} + \nu\Delta t\sum_{n=\overline{n}_0}^{N-1}|\omega^{n+1}|^2_{1,\Omega_{ext}} \\
&\quad \le \varepsilon^{-1}||\mathbf{e}^0_{uh}|| + \gamma_{1,s}^{1/2}\varepsilon^{-1}||\mathbf{u}^0_{\varepsilon,h}||_{\Omega_s} + \varepsilon^{-2}\Delta t\sum_{n=0}^{n_0-1}||\mathbf{e}^n_{uh}||^2\sum_{i=i_0(n)}^{i_1(n)}\mu^{n+i+1} \\
&\quad + ||\omega^0_*|| + \gamma_{1,s}^{1/2}||\omega^0_*||_{\Omega_s} + \Delta t\sum_{n=0}^{n_0-1}||\omega^n_*||^2\sum_{i=i_0(n)}^{i_1(n)}\mu^{n+i+1} \\
&\quad + C\nu^{-1}\Delta t\sum_{n=\overline{n}_0}^{N-1}((L^{n+1}_{\omega*})^2 + (M^{n+1}_{\omega*})^2 + \varepsilon^{-1}||\mathbb{T}^{n+1} - \sigma^{n+1}_h||_{h,-1/2,\partial\Omega_s}).
\end{aligned} \tag{7.94}$$

Identify $\sum_{n=0}^{n_0-1}c_n = 0$ when $n_0 < 1$. Define

$$G^N := C\exp(\Delta t\sum_{n=n_0}^{N-1}\mu^{n+1}).$$

Apply the discrete Gronwall Lemma 2.4.6 to (7.94). Then

$$
\begin{aligned}
&||\omega^N||_{\Omega_{ext}} + \nu^{1/2}||\omega||^2_{l^2(n_0+1,N;H^1(\Omega_{ext}))} \\
&\quad \leq G^N(\varepsilon^{-1}||\mathbf{e}^0_{uh}|| + \gamma_{1,s}^{1/2}\varepsilon^{-1}||\mathbf{u}^0_{\varepsilon,h}||_{\Omega_s} + \varepsilon^{-1}||\mathbf{e}_{uh}||_{l^\infty(0,n_0-1;L^2)}(\Delta t \sum_{n=0}^{2n_0-1} \mu^n)^{1/2} + \ldots \\
&\quad \ldots + ||\omega^0_*|| + \gamma_{1,s}^{1/2}||\omega^0_*||_{\Omega_s} + ||\omega_*||_{l^\infty(0,n_0-1;L^2)}(\Delta t \sum_{n=0}^{2n_0-1} \mu^n)^{1/2} + \ldots \\
&\quad \ldots + C\nu^{-1}(\Delta t \sum_{n=\overline{n}_0}^{N-1}((L^{n+1}_{\omega*})^2 + (M^{n+1}_{\omega*})^2))^{1/2} + \ldots \\
&\quad \ldots + C\nu^{-1/2}\varepsilon^{-1/2}||\mathbb{T} - \sigma_h||_{l^2([n_0+1,N];h,H^{-1/2}(\partial\Omega_s))}).
\end{aligned}
\tag{7.95}
$$

Recall that $\mathbf{e}_{uh} = \varepsilon(\omega - \omega_*)$. Then the triangle inequality gives

$$
\begin{aligned}
&||\mathbf{e}^N_{uh}||_{\Omega_{ext}} + \nu^{1/2}||\mathbf{e}_{uh}||^2_{l^2(n_0+1,N;H^1(\Omega_{ext}))} \\
&\quad \leq G^N(||\mathbf{e}^0_{uh}|| + \gamma_{1,s}^{1/2}||\mathbf{u}^0_{\varepsilon,h}||_{\Omega_s} + ||\mathbf{e}_{uh}||_{l^\infty(0,n_0-1;L^2)}(\Delta t \sum_{n=0}^{2n_0-1} \mu^n)^{1/2} + \ldots \\
&\quad \ldots + \varepsilon||\omega_*||_{l^\infty(L^2)} + \nu^{1/2}\varepsilon||\omega_*||_{l^2(H^1)} + \ldots \\
&\quad \ldots + \gamma_{1,s}^{1/2}\varepsilon||\omega^0_*||_{\Omega_s} + \varepsilon||\omega_*||_{l^\infty(0,n_0-1;L^2)}(\Delta t \sum_{n=0}^{2n_0-1} \mu^n)^{1/2} + \ldots \\
&\quad \ldots + C\nu^{-1}\varepsilon(\Delta t \sum_{n=\overline{n}_0}^{N-1}((L^{n+1}_{\omega*})^2 + (M^{n+1}_{\omega*})^2))^{1/2} + \ldots \\
&\quad \ldots + C\nu^{-1/2}\varepsilon^{1/2}||\mathbb{T} - \sigma_h||_{l^2([n_0+1,N];h,H^{-1/2}(\partial\Omega_s))}).
\end{aligned}
\tag{7.96}
$$

Estimates (7.92) for BEFEb and (7.96) for BELEb prove Theorem 7.2.6. □

7.2.3 Proof of Pressure, Drag, and Lift Error, Theorem 7.2.8

First, let $\hat{\mathbf{c}} \in \Lambda_{h,0}(\partial\Omega_s)$ be a constant unit vector on $\overline{\Omega}_s$. Indeed, it is easily verified that $\int_{\partial\Omega_s} \hat{\mathbf{c}} \cdot \hat{n} = 0$. Let $E_h(\hat{\mathbf{c}}) \in V_h(\Omega_{ext})$ extend $\hat{\mathbf{c}}$ to a bounded, discretely divergence-free function in Ω. Then letting $\mathbf{v}_h = \hat{\mathbf{c}}$ in (7.34), rearranging, and simplifying gives

$$\begin{aligned}\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}^{n+1} \cdot \hat{\mathbf{c}} &= -\int_{\Omega_{ext}} \gamma_1(\partial_{\Delta t}^{n+1}\mathbf{e}_{uh}) \cdot \mathbf{v}_h \\ &- c_{h,ext}(\xi^n(\mathbf{e}_{uh}), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) - \nu \int \gamma_2 \nabla \mathbf{e}_u : \nabla E(\hat{\mathbf{c}}) \\ &- c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}_h^{n+1}, E(\hat{\mathbf{c}}) - c_h(\xi^n(\mathbf{u}_h), \mathbf{e}_{uh}^{n+1}, E(\hat{\mathbf{c}}).\end{aligned} \tag{7.97}$$

Sum (7.152) from $n = \overline{n}_0$ to $N-1$, multiply by Δt, and simplify to get

$$\begin{aligned}\Delta t \sum_{n=\overline{n}_0}^{N-1} (\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}^{n+1} \cdot \hat{\mathbf{c}}) &= -\int_{\Omega_{ext}} \gamma_1(\mathbf{e}_{uh}^N - \mathbf{e}_{uh}^0) \cdot \mathbf{v}_h \\ - \Delta t \sum_{n=\overline{n}_0}^{N-1} c_{h,ext}(\xi^n(\mathbf{e}_{uh}), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) &- \nu\Delta t \sum_{n=\overline{n}_0}^{N-1} \int \nabla \mathbf{e}_u : \nabla E(\hat{\mathbf{c}}) \\ - \Delta t \sum_{n=\overline{n}_0}^{N-1} c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}_h^{n+1}, E(\hat{\mathbf{c}}) &- \Delta t \sum_{n=\overline{n}_0}^{N-1} c_h(\xi^n(\mathbf{u}_h), \mathbf{e}_{uh}^{n+1}, E(\hat{\mathbf{c}}).\end{aligned} \tag{7.98}$$

Successive applications of Hölder's inequality (2.22) with respect to $\int_\Omega(\cdot)$ and with $E(\hat{\mathbf{c}}) \in L^\infty(\Omega_{ext})$ gives

$$\begin{aligned}&|\Delta t \sum_{n=\overline{n}_0}^{N-1} (\int_{\partial\Omega_s} \sigma_h^{n+1} \cdot \hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}^{n+1} \cdot \hat{\mathbf{c}}| \\ &\leq C(||\mathbf{e}_{uh}^N|| + ||\mathbf{e}_{uh}^0|| + \gamma_{1,s}(||\mathbf{u}_{\varepsilon,h}^N||_{\Omega_s} + ||\mathbf{u}_{\varepsilon,h}^0||_{\Omega_s}) + \dots \\ &\quad \dots + \nu\Delta t \sum_{n=\overline{n}_0}^{N-1} |\mathbf{e}_{uh}^{n+1}|_1 + \Delta t \sum_{n=\overline{n}_0}^{N-1} (|\mathbf{e}_{uh}^{n+1}|_1 + |\mathbf{u}_h^{n+1}|_1)||\mathbf{e}_{uh}||_{l^\infty(L^2)}).\end{aligned} \tag{7.99}$$

Division by N and application of Hölder's inequality (2.22) with respect to $\Delta t \sum_{n=\overline{n}_0}^{N-1}(\cdot)$ gives

$$
\begin{aligned}
&|\frac{\Delta t}{N}\sum_{n=\overline{n}_0}^{N-1}(\int_{\partial\Omega_s}\sigma_h^{n+1}\cdot\hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s}\mathbf{u}_{\varepsilon,h}^{n+1}\cdot\hat{\mathbf{c}}| \\
&\leq C(N^{-1}||\mathbf{e}_{uh}^N|| + ||\mathbf{e}_{uh}^0|| + N^{-1}\gamma_{1,s}(||\mathbf{u}_{\varepsilon,h}^N||_{\Omega_s} + ||\mathbf{u}_{\varepsilon,h}^0||_{\Omega_s}) + \dots \\
&\quad \dots + \nu(\Delta t\sum_{n=\overline{n}_0}^{N-1}||\mathbf{e}_{uh}^{n+1}|_1)^{1/2} + \dots \\
&\quad \dots + \left((\Delta t\sum_{n=\overline{n}_0}^{N-1}|\mathbf{e}_{uh}^{n+1}|_1)^{1/2} + (\frac{\Delta t}{N}\sum_{n=\overline{n}_0}^{N-1}|\mathbf{u}^{n+1}|_1^2)^{1/2}\right)||\mathbf{e}_{uh}||_{l^\infty(L^2)}).
\end{aligned}
\tag{7.100}
$$

Estimate (7.66) follows.

To estimate the pressure error, set $\mathbf{v}_h \in X_h$ in (7.34) extended so that $\mathbf{v}_h|_{\Omega_s} \equiv 0$. Rearrange, sum from $n = \overline{n}_0$ to $N-1$, and simplify to get

$$
\begin{aligned}
(\Delta t\sum_{n=\overline{n}_0}^{N-1}p_h^{n+1} - p_{\varepsilon,h}^{n+1}, \nabla\cdot\mathbf{v}_h) &= -\int(\mathbf{e}_{uh}^N - \mathbf{e}_{uh}^0)\cdot\mathbf{v}_h \\
&- \Delta t\sum_{n=\overline{n}_0}^{N-1}c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h) - \nu\Delta t\sum_{n=\overline{n}_0}^{N-1}\int\nabla\mathbf{e}_u : \nabla\mathbf{v}_h \\
&- \Delta t\sum_{n=\overline{n}_0}^{N-1}c_h(\xi^n(\mathbf{e}_{uh}), \mathbf{u}_h^{n+1}, \mathbf{v}_h - \Delta t\sum_{n=\overline{n}_0}^{N-1}c_h(\xi^n(\mathbf{u}_h), \mathbf{e}_{uh}^{n+1}, \mathbf{v}_h.
\end{aligned}
\tag{7.101}
$$

Similarly as above successive applications of Hölder's inequality (2.22) gives

$$
\begin{aligned}
\frac{|(\Delta t\sum_{n=\overline{n}_0}^{N-1}(p_h^{n+1} - p_{\varepsilon,h}^{n+1}), \nabla\cdot\mathbf{v}_h|}{|\mathbf{v}_h|_1} &\leq ||\mathbf{e}_{uh}^N|| + ||\mathbf{e}_{uh}^0|| \\
&+ \nu\Delta t\sum_{n=\overline{n}_0}^{N-1}|\mathbf{e}_{uh}^{n+1}|_1 + \Delta t\sum_{n=\overline{n}_0}^{N-1}(|\mathbf{u}_h^{n+1}|_1 + |\mathbf{e}_{uh}^{n+1}|_1)|\mathbf{e}_{uh}^{n+1}|_1.
\end{aligned}
\tag{7.102}
$$

Note that $\Delta t\sum_{n=\overline{n}_0}^{N-1}(p_h^{n+1} - p_{\varepsilon,h}^{n+1}) \in Q_h$. Then application of the discrete inf-sup condition (2.2) along with division by N and then Hölder's inequality (2.22) with respect to $\Delta t\sum_{n=\overline{n}_0}^{N-1}(\cdot)$ gives

$$
\begin{aligned}
&||\frac{\Delta t}{N}\sum_{n=\overline{n}_0}^{N-1}(p_h^{n+1} - p_{\varepsilon,h}^{n+1}|| \leq C(N^{-1}||\mathbf{e}_{uh}^N|| + N^{-1}||\mathbf{e}_{uh}^0|| + \nu(\frac{\Delta t}{N}\sum_{n=\overline{n}_0}^{N-1}|\mathbf{e}_u^{n+1}|_1^2 dt)^{1/2} + \dots \\
&\quad \dots + ((\Delta t\sum_{n=\overline{n}_0}^{N-1}|\mathbf{u}_h^{n+1}|_1^2)^{1/2} + (\Delta t\sum_{n=\overline{n}_0}^{N-1}|\mathbf{u}_{\varepsilon,h}^{n+1}|_1^2)^{1/2})(\frac{\Delta t}{N}\sum_{n=\overline{n}_0}^{N-1}|\mathbf{e}_{uh}^{n+1}|_1^2)^{1/2}.
\end{aligned}
\tag{7.103}
$$

Estimate (7.65) follows.

7.2.4 Proof of Auxiliary Estimate, Proposition 7.3.7

Proposition 7.3.7. Let $\mathbb{T}^n = \sigma(\mathbf{u}^n, p^n) \cdot \hat{\mathbf{n}}$ or $\mathbb{T}^n = \sigma_h^n$. For $n = 0, 1, \ldots, N$, let $\omega_{*,s}^n := \omega_*^n|_{\Omega_s} \in X_{h,\cdot}(\Omega_s)$, $\pi_{*,s}^n := \pi_*^n|_{\Omega_s} \in Q_{h,\cdot}(\Omega_s)$ satisfy

$$\nu \int_{\Omega_s} \nabla \omega_{*,s}^n : \nabla \mathbf{v}_h - \varepsilon \int_{\Omega_s} \pi_{*,s}^n \nabla \cdot \mathbf{v}_h$$

$$+ \nu \int_{\Omega_s} \omega_{*,s}^n \cdot \mathbf{v}_h = \int_{\partial \Omega_s} \mathbb{T}^n \cdot \mathbf{v}_h, \qquad \forall \mathbf{v}_h \in X_{h,\cdot}(\Omega_s) \tag{7.104}$$

$$\int_{\Omega_s} q_h \nabla \cdot \omega_{*,s}^n = 0, \qquad \forall q_h \in Q_h(\Omega_s) \tag{7.105}$$

and let $\omega_{*,f}^n := \omega_*^n|_{\Omega} \in X_{h,\cdot}$, $\pi_{*,f}^n := \pi_*^n|_{\Omega} \in Q_h$ satisfy

$$\nu \int \nabla \omega_{*,f}^n : \nabla \mathbf{v}_h - \int \pi_{*,f}^n \nabla \cdot \mathbf{v}_h = 0, \qquad \forall \mathbf{v}_h \in X_h \tag{7.106}$$

$$\int q_h \nabla \cdot \omega_{*,f}^n = 0, \qquad \forall q_h \in Q_h \tag{7.107}$$

$$\omega_{*,f}^n|_{\partial \Omega_s} = \omega_{*,s}^n|_{\partial \Omega_s}, \quad \omega_{*,f}^n|_{\partial \Omega_{ext}} = 0. \tag{7.108}$$

It is clear that ω_*^n satisfies the conditions of Assumption 7.2.5 with $L(\omega_*^n; \mathbf{v}_h) = -(\partial_{\Delta t}^{n+1} \omega_*, \mathbf{v}_h) - \nu(\nabla \omega_*^n, \nabla \mathbf{v}_h) - \gamma_{1,s} \int_{\Omega_s} \partial_{\Delta t}^{n+1} \omega_* \cdot \mathbf{v}_h$ for all $\mathbf{v}_h \in V_h(\Omega_{ext})$. The existence and uniqueness of $\omega_*^n \in X_h(\Omega_{ext})$ for regular enough $\mathbb{T}^n$ follows a standard argument. In the following lemmas, we estimate the size of ω_*^n and ultimately trace the feedback to the ε-error in the BEFEb problem.

Lemma 7.2.11. *Fix $m = 0$ or 1. For regular enough $\{\mathbb{T}^n\}_{n=0}^N$, all solutions $\omega_{*,s}^n$ satisfying (7.104), (7.105) also satisfy*

$$||(\partial_{\Delta t}^{(m)})^{n+1} \omega_{*,s}^n||_{1,\Omega_s}^2 \leq C \nu^{-1} ||(\partial_{\Delta t}^{(m)})^{n+1} \mathbb{T}||_{h,-1/2,\partial \Omega_s}. \tag{7.109}$$

Proof. Test (7.104) with $\mathbf{v}_h = \omega_{*,s}^n \in X_{h,\cdot}(\Omega_s)$ to get

$$||\omega_{*,s}^n||_{1,\Omega_s} \leq C \nu^{-1} ||\mathbb{T}^n||_{h,-1/2,\partial \Omega_s}. \quad \forall n \geq 0.$$

For higher order estimates, discretely differentiate (7.104) with respect to t and test with $\mathbf{v}_h = \partial_{\Delta t}^{n+1} \omega_{*,s} \in V_{h,\cdot}(\Omega_s)$. Then (7.109) for $m = 1$ is proved by following a proof similar to the case when $m = 0$ above. □

Lemma 7.2.12. *Fix $m = 0$ or 1. For regular enough $\{\mathbb{T}^n\}_{n=0}^N$, suppose that the FE-space satisfies Assumption 2.3.4. Then all solutions $\omega_{*,f}^n$ satisfying (7.106), (7.107), (7.108) also satisfy*

$$||(\partial_{\Delta t}^{(m)})^{n+1}\omega_*||_{1,\Omega_{ext}} \leq C\nu^{-1}||(\partial_{\Delta t}^{(m)})^{n+1}\mathbb{T}||_{h,-1/2,\partial\Omega_s}, \quad \forall n \geq 0. \tag{7.110}$$

Proof. Next, let $\lambda_h^n := \omega_{*,s}^n|_{\partial\Omega_s}$. Let $E_h : \Lambda_{h,0}(\partial\Omega) \to V_{h,\cdot}$ be a discrete extension operator. Substitute $\omega_{*,f}^n = \omega_0^n + E_h(\lambda_h^n)$, into (7.106): find $\omega_0^n \in V_h$ satisfying

$$\nu\int \nabla\omega_0^n : \nabla\mathbf{v}_h = -\nu\int \nabla E_h(\lambda_h^n) : \nabla\mathbf{v}_h, \qquad \forall\mathbf{v}_h \in V_h. \tag{7.111}$$

Test (7.111) with $\mathbf{v}_h = \omega_0^n \in V_h$, apply Cauchy-Schwarz (2.22), and simplify to derive $|\omega_0^n|_1 \leq |E_h(\lambda_h^n)|_1$ Then since $\omega_{*,f}^n = \omega_0^n + E_h(\lambda_h^n)$, application of the triangle inequality gives, for $m = 0$,

$$|(\partial_{\Delta t}^{(m)})^{n+1}\omega_{*,f}|_1 \leq 2|(\partial_{\Delta t}^{(m)})^{n+1}E_h(\lambda_h)|_1. \tag{7.112}$$

For higher order estimates, discretely differentiate (7.106), (7.107), and (7.108) with respect to t and test with $\mathbf{v}_h = \partial_{\Delta t}^{n+1}\omega_0$. Then Estimate (7.112) for $m = 1$ is proved by following a proof similar to the case when $m = 0$ above.

There exists a particular extension $E_h : \Lambda_{h,0}(\partial\Omega) \to V_{h,\cdot}$ via Assumption 2.3.4 satisfying:

$$|E_h(\mu_h)|_1 \leq C||\mu_h||_{1/2,\partial\Omega_s}.$$

Recall the definition of the $H^{1/2}$-norm, $||\lambda_h^n||_{1/2,\partial\Omega_s} := \inf_{0\neq\mathbf{v}_h\in X_{h,\lambda_h^n}(\Omega_s)} ||\mathbf{v}_h||_{1,\Omega_s}$ Apply these estimates to (7.112) to prove (7.110) for $m = 0$ or 1. ☐

It remains to estimate $L_{\omega*}$ and $M_{\omega*}$ in Theorem 7.2.6. Set $\gamma = 0$. Apply Cauchy-Schwarz (2.22) to obtain

$$(L_{\omega*}^{n+1})^2 = C(||\partial_{\Delta t}^{n+1}\omega_*||^2 + \gamma_{1,s}||\partial_{\Delta t}^{n+1}\omega_*||_{\Omega_s}^2 + \nu|\omega_*^{n+1}|_1^2). \tag{7.113}$$

Apply (2.41)(a) to get

$$(M_{\omega*}^{n+1})^2 = C||\mathbf{u}_h^{n+1}||_1^2|\xi^n(\omega_*)|_1^2 + C||\xi^n(\mathbf{u}_{\varepsilon,h})||_1^2|\omega_*^{n+1}|_1^2. \tag{7.114}$$

Successive applications of (7.110) allow us to replace $F_{\omega *}$ in Theorem 7.2.6 with

$$\begin{aligned} F_{\omega *} &:= \gamma_{1,s}^{1/2}\nu^{-3/2}\big(\nu^{1/2}||\mathbb{T}^0||_{-1/2,\partial\Omega_s} + ||\partial_{\Delta t}\mathbb{T}||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))}\big) \\ &\quad + \nu^{-3/2}||\partial_{\Delta t}\mathbb{T}||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} + \nu^{-3/2}||\mathbb{T}||_{l^2([\overline{n}_0+1,N];h,H^{-1/2}(\partial\Omega_s))} \\ &\quad + \nu^{-3/2}\big(\nu^{1/2} + ||\mathbf{u}_h||_{l^2([\overline{n}_0+1,N];H^2)} + ||\mathbf{u}_{\varepsilon,h}||_{l^2(H^1(\Omega_{ext}))}\big)||\mathbb{T}||_{l^\infty(h,H^{-1/2}(\partial\Omega_s))}\big) \\ &\quad + C\nu^{-1/2}\varepsilon^{-1/2}||\mathbb{T}-\sigma_h||_{l^2([n_0+1,N];h,H^{-1/2}(\partial\Omega_s))} + C_2(\mathbb{T}^{n_0}). \end{aligned}$$

Apply *a priori* estimates (7.30), (7.33) to simplify $F_{\omega *}$ in Proposition 7.3.7. We assume here that $\nu \leq 1$. $\square$

7.3 CONVERGENCE ANALYSIS, BRVP

In this section, we investigate the convergence of $\mathbf{u}_\varepsilon \to \mathbf{u}$. In Proposition 7.3.3, we show that

$$\begin{array}{llll} H^1-\text{penalization} & \Rightarrow & ||\mathbf{u}_\varepsilon||_{L^2(H^1(\Omega_s))\cap L^\infty(L^2(\Omega_s))} & \leq C_*\varepsilon, \\ & & ||\mathbf{u}_\varepsilon-\mathbf{u}||_{L^2(H^1)\cap L^\infty(L^2)} & \leq C_*\varepsilon^{1/2} \\ L^2-\text{penalization} & \Rightarrow & ||\mathbf{u}_\varepsilon||_{L^2(L^2(\Omega_s))} & \leq C_*\varepsilon^{3/4}, \\ & & ||\mathbf{u}_\varepsilon-\mathbf{u}||_{L^2(H^1(\Omega_{ext}))\cap L^\infty(L^2(\Omega_{ext}))} & \leq C_*\varepsilon^{1/4}. \end{array} \quad (7.115)$$

Estimates (7.115) holds if the initial condition $\mathbf{u}_\varepsilon^0$ is a *good* approximation of $\mathbf{u}_\varepsilon^0$. We make this precise in the Assumption 7.3.1. First define

$$F_{ic} := ||\mathbf{u}_\varepsilon^0 - \mathbf{u}^0|| + \gamma_{1,s}^{1/2}||\mathbf{u}_\varepsilon^0||_{\Omega_s}. \quad (7.116)$$

Assumption 7.3.1. *The data* $\mathbf{u}_\varepsilon^0 \approx \mathbf{u}^0$ *so that*

$$F_{ic} \leq C(\gamma_{2,s}^{-1}\varepsilon)^{1/4}$$

for some constant $C > 0$ *where* F_{ic} *is given in (7.116).*

For example, if $\gamma_{1,s} = 1$, H^1 penalization requires $||\mathbf{u}_\varepsilon^0 - \mathbf{u}^0||_{\Omega_{ext}} \leq \mathcal{O}(\varepsilon^{1/2})$ and L^2-penalization requires $||\mathbf{u}_\varepsilon^0 - \mathbf{u}^0||_{\Omega_{ext}} \leq \mathcal{O}(\varepsilon^{1/4})$. On the other hand, if $\gamma_{1,s} = \varepsilon^{-1}$, H^1-penalization additionally requires $||\mathbf{u}_\varepsilon^0 - \mathbf{u}^0||_{\Omega_s} \leq \mathcal{O}(\varepsilon)$ and L^2-penalization additionally requires $||\mathbf{u}_\varepsilon^0 - \mathbf{u}^0||_{\Omega_s} \leq \mathcal{O}(\varepsilon^{3/4})$.

In Theorem 7.3.6 we establish the *improved* estimate

$$H^1 - \text{penalization} \quad \Rightarrow \quad ||\mathbf{u}_\varepsilon - \mathbf{u}||_{L^\infty(L^2(\Omega_{ext})) \cap L^2(H^1(\Omega_{ext}))} \leq C_* \varepsilon. \tag{7.117}$$

For optimal $\mathcal{O}(\varepsilon)$-estimates in (7.117), $\mathbf{u}_\varepsilon^0$ must be a *better* approximation of $\mathbf{u}_\varepsilon^0$ than required for (7.115). We make this precise in the next assumption.

Assumption 7.3.2. *The data $\mathbf{u}_\varepsilon^0 \approx \mathbf{u}^0$ so that*

$$F_{ic} \leq C\varepsilon$$

for some constant $C > 0$ where F_{ic} is given in (7.116).

If $\gamma_{1,s} = \varepsilon^{-1}$, Assumption 7.3.2 suggests $||\mathbf{u}_\varepsilon^0||_{\Omega_s} \leq \mathcal{O}(\varepsilon^{3/2})$. This condition is more restrictive than the $\mathcal{O}(\varepsilon)$ accuracy guaranteed by the method.

Although suboptimal, estimate (7.115)(a) (Proposition 7.3.3) requires that $\sigma(\mathbf{u}, p) \cdot \hat{n} \in L^2(H^{-1/2}(\partial\Omega_s))$ whereas (7.117) (Theorem 7.3.6) requires $\sigma(\mathbf{u}, p) \cdot \hat{n} \in H^1(H^{-1/2}(\partial\Omega_s))$ (see Proposition 7.3.7). Moreover, Proposition 7.3.3 gives an estimate for both L^2- and H^1-penalization, whereas the result of Theorem 7.3.6 is restricted to H^1-penalization. Although not predicted by current theory, $\mathcal{O}(\varepsilon)$-convergence is reported for $\mathbf{u}_\varepsilon \to \mathbf{u}$ in the energy norm $L^\infty(L^2(\Omega_{ext})) \cap L^2(H^1(\Omega_{ext}))$ for L^2-penalization (see e.g. [4]).

The constant $C_* > 0$ corresponding with estimates (7.115), (7.117) is finite if $(\mathbf{u}, p)$ is regular enough. In particular, it is convenient to introduce

$$\mu(t) := C \begin{cases} |\mathbf{u}(\cdot, t)|_{1,\infty}, & \text{if} \quad \mathbf{u} \in L^1(W^{1,\infty}) \\ \nu^{-1/3}||\mathbf{u}(\cdot, t)||_2^{4/3}, & \text{else if} \quad \mathbf{u} \in L^{4/3}(H^2) \\ \nu^{-3}|\mathbf{u}(\cdot, t)|_1^4, & \text{else if} \quad \mathbf{u} \in L^4(H^1) \end{cases} \tag{7.118}$$

since $C_* \propto \exp(\int_0^T \mu(t)dt)$.

Proposition 7.3.3 (Consistency, Part I)**.** *Let* $\gamma_{2,s} = \varepsilon^{r-1}$ *for* $r = 0,\ 1$. *Suppose that* $\mathbf{u}$ *is a strong solution satisfying (7.22) so that* $\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}|_{\partial\Omega_s} \in L^2(H^{-r/2}(\partial\Omega_s))$. *Suppose further that* $\mathbf{u}_\varepsilon$ *solves (7.15), (7.16), (7.17). Then*

$$\begin{aligned}\varepsilon^{1/2}||\mathbf{u} - \mathbf{u}_\varepsilon||_{L^\infty(L^2(\Omega_{ext}))} + \varepsilon^{1/2}\nu^{1/2}||\nabla(\mathbf{u} - \mathbf{u}_\varepsilon)||_{L^2(L^2)} \\ + (\gamma_{1,s}\varepsilon)^{1/2}||\mathbf{u}_\varepsilon||_{L^\infty(L^2(\Omega_s))} + \nu^{1/2}||\mathbf{u}_\varepsilon||_{L^2(H^r(\Omega_s))} \leq G(T)(F_{ic} + F_\sigma \varepsilon^{(3+r)/4})\end{aligned} \tag{7.119}$$

where

$$F_\sigma := \nu^{-1/2}||\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}||_{L^2(H^{-r/2}(\partial\Omega_s))}$$

and $G(t) := C\exp(C\mu(t))$ *with* $\mu(t)$ *given in (7.118),* F_{ic} *is given in (7.116).*

Remark 7.3.4. *Restricted to the* $\mathbb{R}^2$*-case, then we can replace (7.118) with* $\mu(t) := C\nu^{-1}|\mathbf{u}(\cdot, t)|_1^2$. *Moreover, the uniqueness condition for steady-state solutions (5.13) leads to* $\mu(t) \equiv 0$.

Proof. See Section 7.3.1. □

We can improve the estimate in the previous proposition in the case of H^1-penalization. We require an extension of the stress $\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}|_{\partial\Omega_s}$ with properties summarized in the next assumption.

Assumption 7.3.5. *Fix* $\gamma = \gamma_{1,s}\varepsilon$ *or* $\gamma = 0$. *There exists a function* $\omega_* : \Omega_{ext} \times [0, T] \to \mathbb{R}^d$ *satisfying* $\omega_*(\cdot, t) \in H_0^1(\Omega_{ext})$, $\nabla \cdot \omega_*(\cdot, t) = 0$ *and*

$$\int_{\Omega_s} (\gamma\partial_t\omega_* \cdot \mathbf{v} + \nu\nabla\omega_* : \nabla\mathbf{v} + \nu\omega_* \cdot \mathbf{v}) = \int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V_\cdot(\Omega_s). \tag{7.120}$$

Moreover, for any $\mathbf{v} \in V(\Omega_{ext})$,

$$\begin{aligned}|(\partial_t\omega_*, \mathbf{v}) + \nu(\nabla\omega_*, \nabla\mathbf{v}) + (\gamma_{1,s} - \frac{\gamma}{\varepsilon})\int_{\Omega_s} \partial_t\omega_* \cdot \mathbf{v}| &\leq L_{\omega*}(t)|\mathbf{v}|_{1,\Omega_{ext}} \\ |(\omega_* \cdot \nabla\mathbf{u} + \mathbf{u}_\varepsilon \cdot \nabla\omega_*, \mathbf{v})_{\Omega_{ext}}| &\leq M_{\omega*}(t)|\mathbf{v}|_{1,\Omega_{ext}}\end{aligned} \tag{7.121}$$

for some $L_{\omega*}$, $M_{\omega*} \in L^2(0, T)$.

We prove existence of ω_* satisfying Assumption 7.3.5 in Proposition 7.3.7.

Theorem 7.3.6 (Consistency, Part II)**.** *Let* $\gamma_{2,s} = \varepsilon^{-1}$ *and* ω_*, $L_{\omega*}$, $M_{\omega*}$ *satisfy the properties of Assumption 7.3.5. Suppose that* $(\mathbf{u}, p)$ *is a strong solution satisfying (7.22) and that* $\mathbf{u}_\varepsilon$ *solves (7.15), (7.16), (7.17). Then,*

$$||\mathbf{u} - \mathbf{u}_\varepsilon||_{L^\infty(L^2(\Omega_{ext}))} + \nu^{1/2}||\mathbf{u} - \mathbf{u}_\varepsilon||_{L^2(H^1(\Omega_{ext}))} \leq G(T)(F_{ic} + F_{\omega*}\varepsilon). \tag{7.122}$$

where

$$\begin{aligned} F_{\omega*} :&= \gamma_{1,s}^{1/2}||\omega_*(\cdot, 0)||_{\Omega_s} + ||\omega_*||_{L^\infty(L^2(\Omega_{ext}))} \\ &\quad + \nu^{1/2}||\nabla\omega_*||_{L^2(L^2(\Omega_{ext}))} + \nu^{-1/2}||L_{\omega*}||_{L^2(0,T)} + \nu^{-1/2}||M_{\omega*}||_{L^2(0,T)} \end{aligned}$$

and $G(t) := C\exp(C\mu(t))$ *with* $\mu(t)$ *given in (7.118), and* F_{ic} *is given in (7.116).*

Proof. See Section 7.3.2. □

Proposition 7.3.7 (Auxiliary Estimate)**.** *Under the hypotheses of Theorem 7.3.6, there exists* $\omega_*(\cdot, t) \in H_0^1(\Omega_{ext})$ *satisfying Assumption 7.3.5. In particular, pick* $\gamma = 0$, $\gamma_{1,s} = 1$ *so that for* $m = 0,\ 1$,

$$||\partial_t^{(m)}\omega_*(\cdot, t)||_{\Omega_{ext}} \leq \nu^{-1}||\partial_t^{(m)}\sigma(\mathbf{u}(\cdot, t), p(\cdot, t)) \cdot \hat{\mathbf{n}}||_{-1/2,\partial\Omega_s}. \tag{7.123}$$

Moreover, $F_{\omega*}$ *in Theorem 7.3.6 can be replaced by*

$$\begin{aligned} &F_{\omega*} := \nu^{-1}(\nu^{1/2}||\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}||_{L^2(H^{-1/2}(\partial\Omega_s))} + \ldots \\ &\quad \ldots + ||(\partial_t\sigma(\mathbf{u}, p)) \cdot \hat{\mathbf{n}}||_{L^2(H^{-1/2}(\partial\Omega_s))} + \nu^{-1/2}\max\{M_0, M_{B,0}\}\,||\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}||_{L^\infty(H^{-1/2}(\partial\Omega_s))}). \end{aligned}$$

Proof. See Section 7.3.3. □

Last, we investigate the approximability of Darcy drag contribution $\nu\varepsilon^{-1}\int_{\Omega_s}\mathbf{u}_\varepsilon$ to the actual drag/lift on $\partial\Omega_s$ as well as an error estimate for $p_\varepsilon|_\Omega \to p$. These results are derived from the velocity error estimates presented earlier for $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^\infty(L^2)\cap L^2(H^1)$ in Proposition 7.3.3, Theorem 7.3.6. Define first the time-averaging operator by

$$< \theta >_T := \frac{1}{T}\int_0^T \theta(t)dt. \tag{7.124}$$

Estimates (7.125), (7.126) provide long-time $T\to\infty$ estimate for the pressure and drag/lift consistency error in modeling with BrVP. Under the conditions of Theorem 7.3.6 and for fixed $0<T<\infty$, the time-averaged error of pressure satisfies

$$||\int_0^T (p(\cdot,t)-p_\varepsilon(\cdot,t))dt||. \le C_*\varepsilon$$

The drag and lift coefficients on $\partial\Omega_s$ are given by $D=\int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\mathbf{c}_d$ and $L=\int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\mathbf{c}_l$ for some constant vectors $\mathbf{c}_d$, $\mathbf{c}_l$. Then under the same conditions above, the time-averaged error of drag/lift on Ω_s satisfies

$$|\int_0^T (D(t)-\frac{\nu}{\varepsilon}\int_{\Omega_s}\mathbf{u}_\varepsilon(\cdot,t)\cdot\mathbf{c}_d)dt| + |\int_0^T (L(t)-\frac{\nu}{\varepsilon}\int_{\Omega_s}\mathbf{u}_\varepsilon(\cdot,t)\cdot\mathbf{c}_l)dt| \le C_*\varepsilon.$$

Theorem 7.3.8. *Suppose that $(\mathbf{u},p)$ is a strong solution satisfying (7.21) so that at least one of the regularity conditions associated with (7.118) is satisfied. Suppose that $(\mathbf{u}_\varepsilon,p_\varepsilon)$ solves (7.15), (7.16), (7.17). Then*

$$|| < p-p_\varepsilon >_T || \le T^{-1}E_1 + \max\{M_0, M_{B,0}\} < |\mathbf{u}-\mathbf{u}_\varepsilon|_1^2 >_T^{1/2} \tag{7.125}$$

where $E_1 := C(||\mathbf{u}(\cdot,T)-\mathbf{u}_\varepsilon(\cdot,T)||+||\mathbf{u}(\cdot,0)-\mathbf{u}_\varepsilon(\cdot,0)||)$. Additionally, for any constant unit vector $\hat{\mathbf{c}}:\partial\Omega_s\to\mathbb{R}^d$, we have,

$$\begin{aligned} | < \int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s}\mathbf{u}_\varepsilon\cdot\hat{\mathbf{c}} >_T | &\le T^{-1}E_2 \\ + \nu < |\mathbf{u}-\mathbf{u}_\varepsilon|_1 >_T^{1/2} + C(< |\mathbf{u}-\mathbf{u}_\varepsilon|_1^2 >_T^{1/2} &+ < |\mathbf{u}|_1^2 >_T^{1/2})||\mathbf{u}-\mathbf{u}_\varepsilon||_{L^\infty(L^2)} \end{aligned} \tag{7.126}$$

where $E_2 := C(E_1+\gamma_{1,s}(||\mathbf{u}_\varepsilon(\cdot,T)||_{\Omega_s}+||\mathbf{u}_\varepsilon^0||_{\Omega_s}))$.

Proof. See Section 7.3.4. □

7.3.1 Proof of Velocity Error, Proposition 7.3.3

Proposition 7.3.3. Set $\mathbf{v} = \mathbf{e}_u$ in (7.23) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||\mathbf{e}_u||^2 + \gamma_{1,s}||\mathbf{u}_\varepsilon||^2_{\Omega_s}\right) + \nu\left(||\nabla \mathbf{e}_u||^2 + \gamma_{2,s}||\nabla \mathbf{u}_\varepsilon||^2_{\Omega_s}\right) + \frac{\nu}{\varepsilon}||\mathbf{u}_\varepsilon||^2_{\Omega_s} \\
&= -\int \mathbf{e}_u \cdot \nabla \mathbf{u} \cdot \mathbf{e}_u - \int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{e}_u.
\end{aligned} \tag{7.127}$$

First, we bound the convective terms in (7.127).

Lemma 7.3.9. *Suppose that* $\mathbf{u}$ *is a strong solution. Then*

$$|\int \mathbf{e}_u(\cdot,t) \cdot \nabla \mathbf{u}(\cdot,t) \cdot \mathbf{e}_u(\cdot,t)| \le \mu(t)||\mathbf{e}_u(\cdot,t)||^2 + \frac{\nu}{2}|\mathbf{e}_u(\cdot,t)|_1^2 \tag{7.128}$$

where $\mu(t)$ *is given in (7.118).*

Proof. Apply estimate (2.32)(a) and Young's (2.21) inequality to get

$$|\int \mathbf{e}_u \cdot \nabla \mathbf{u} \cdot \mathbf{e}_u| \le C\nu^{-3}|\mathbf{u}|_1^4||\mathbf{e}_u||^2 + \frac{\nu}{4}|\mathbf{e}_u|_1^2.$$

Alternatively, (2.32)(c) with $\mathbf{u} \in L^{4/3}(H^2)$ gives

$$|\int \mathbf{e}_u \cdot \nabla \mathbf{u} \cdot \mathbf{e}_u| \le C\nu^{-1/3}||\mathbf{u}||_2^{4/3}||\mathbf{e}_u||^2 + \frac{\nu}{2}|\mathbf{e}_u|_1^2.$$

If $\mathbf{u} \in L^1(W^{1,\infty})$, we have yet another alternative

$$|\int \mathbf{e}_u \cdot \nabla \mathbf{u} \cdot \mathbf{e}_u| \le C|\mathbf{u}|_{1,\infty}||\mathbf{e}_u||^2.$$

Estimate (7.128) follows from the above derivations. □

It remains to estimate the boundary integral in (7.127). If $\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}} \in H^{-1/2}(\partial\Omega_s)$, then the duality estimate on $H^{-1/2}(\partial\Omega_s)\times H^{1/2}(\partial\Omega_s)$ along with application of the $H^{1/2}(\partial\Omega_s)$-norm and Young's inequality (2.21) gives

$$|\int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\mathbf{u}_\varepsilon| \leq \frac{1}{2\nu\gamma_{2,s}}||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||^2_{-1/2,\partial\Omega_s} + \frac{\nu\gamma_{2,s}}{2}||\mathbf{u}_\varepsilon||^2_{1,\Omega_s}.$$

Alternatively, suppose now that $\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}} \in L^2(\partial\Omega_s)$ then the Trace Theorem gives $||\mathbf{u}_\varepsilon||_{\partial\Omega_s} \leq C||\mathbf{u}_\varepsilon||^{1/2}_{\Omega_s}||\nabla\mathbf{u}_\varepsilon||^{1/2}_{\Omega_s}$. Apply Cauchy Schwarz (2.22) and Young's inequality (2.21) twice to get

$$\begin{aligned}
|\int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\mathbf{u}_\varepsilon| &\leq C||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||_{\partial\Omega_s}||\mathbf{u}_\varepsilon||^{1/2}_{\Omega_s}||\nabla\mathbf{u}_\varepsilon||^{1/2}_{\Omega_s}\\
&\leq C\nu^{-1/3}\varepsilon^{1/3}||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||^{4/3}_{\partial\Omega_s}||\nabla\mathbf{u}_\varepsilon||^{2/3}_{\Omega_s} + \frac{\nu}{2\varepsilon}||\mathbf{u}_\varepsilon||^2_{\Omega_s}\\
&\leq \frac{C\varepsilon^{1/2}}{\nu\gamma_{2,s}^{1/2}}||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||^2_{\partial\Omega_s} + \frac{\nu\gamma_{2,s}}{2}||\nabla\mathbf{u}_\varepsilon||^2_{\Omega_s} + \frac{\nu}{2\varepsilon}||\mathbf{u}_\varepsilon||^2_{\Omega_s}. \qquad (7.129)
\end{aligned}$$

Fix $r=0$ or 1. Application of (7.128), (7.3.1) or (7.136) to (7.127) gives

$$\begin{aligned}
&\frac{d}{dt}\left(||\mathbf{e}_u||^2 + \gamma_{1,s}||\mathbf{u}_\varepsilon||^2_{\Omega_s}\right) + \nu\left(||\nabla\mathbf{e}_u||^2 + \gamma_{2,s}||\nabla\mathbf{u}_\varepsilon||^2_{\Omega_s}\right) + \frac{\nu}{\varepsilon}||\mathbf{u}_\varepsilon||^2_{\Omega_s}\\
&\leq \mu(t)||\mathbf{e}_u||^2 + \frac{C\varepsilon^{1/2}}{\nu\gamma_{2,s}^{1/2}}||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||^2_{-r/2,\partial\Omega_s}. \qquad (7.130)
\end{aligned}$$

Multiply (7.130) by the integrating factor $\exp(-\int_0^t\mu(t')dt')$, group of terms, integrate on $(0,t)$, and simplify to get

$$\begin{aligned}
&||\mathbf{e}_u(\cdot,t)|| + \gamma_{1,s}^{1/2}||\mathbf{u}_\varepsilon(\cdot,t)||_{\Omega_s}\\
&\quad + \nu^{1/2}\int_0^t \exp(\frac{1}{2}\int_{t'}^t\mu(t')dt')(||\nabla\mathbf{e}_u(\cdot,t')|| + \gamma_{2,s}^{1/2}||\nabla\mathbf{u}_\varepsilon(\cdot,t')||_{\Omega_s} + \varepsilon^{-1/2}||\mathbf{u}_\varepsilon(\cdot,t')||_{\Omega_s})dt'\\
&\leq \exp(\frac{1}{2}\int_0^t\mu(t')dt')(||\mathbf{e}_u^0|| + \gamma_{1,s}^{1/2}||\mathbf{u}_\varepsilon^0||_{\Omega_s})\\
&\quad + \frac{C\varepsilon^{1/4}}{\nu^{1/2}\gamma_{2,s}^{1/4}}\int_0^t\exp(\frac{1}{2}\int_{t'}^t\mu(t')dt')||\sigma(\mathbf{u}(\cdot,t'),p(\cdot,t'))\cdot\hat{\mathbf{n}}||_{-r/2,\partial\Omega_s}dt'. \qquad (7.131)
\end{aligned}$$

We conclude (7.119) from (7.131). □

7.3.2 Proof of Velocity Error, Theorem 7.3.6

Theorem 7.3.6. Consider the ε-order expansion of the BrVP velocity and pressure:

$$\mathbf{u}_\varepsilon = \mathbf{u} + \varepsilon(\omega + \omega_*), \qquad p_\varepsilon = p + \varepsilon(\pi + \pi_*)$$

so that

$$\omega|_{\partial\Omega_{ext}} = 0, \qquad \omega(\cdot, t = 0) = \varepsilon^{-1}(\mathbf{u}^0_\varepsilon - \mathbf{u}^0) - \omega_*(\cdot, 0), \qquad \nabla \cdot \omega = 0$$

where $\omega_*(\cdot, t) \in H^1_0(\Omega_{ext})$ satisfies the conditions of Assumption 7.3.5. Note that

$$\begin{aligned}
&\int_{\Omega_{ext}} \mathbf{e}_u \cdot \nabla\mathbf{e}_u \cdot \mathbf{v} + \int_{\Omega_{ext}} \mathbf{u} \cdot \nabla\mathbf{e}_u \cdot \mathbf{v} + \int_{\Omega_{ext}} \mathbf{e}_u \cdot \nabla\mathbf{u} \cdot \mathbf{v} \\
&= \int_{\Omega_{ext}} \mathbf{u}_\varepsilon \cdot \nabla\mathbf{e}_u \cdot \mathbf{v} + \int_{\Omega_{ext}} \mathbf{e}_u \cdot \nabla\mathbf{u} \cdot \mathbf{v}.
\end{aligned}$$

Substitute into (7.23) and divide by ε to get, for all $\mathbf{v} \in H^1_0(\Omega_{ext})$,

$$\begin{aligned}
&\int_{\Omega_{ext}} \gamma_1 \partial_t \omega \cdot \mathbf{v} + \nu \int_{\Omega_{ext}} \gamma_2 \nabla\omega : \nabla\mathbf{v} - \int_{\Omega_{ext}} \pi \nabla \cdot \mathbf{v} + \nu\varepsilon^{-1} \int_{\Omega_s} \omega \cdot \mathbf{v} \\
&= -\int_{\Omega_{ext}} \mathbf{u}_\varepsilon \cdot \nabla\omega \cdot \mathbf{v} - \int \omega \cdot \nabla\mathbf{u} \cdot \mathbf{v} - \int \omega_* \cdot \nabla\mathbf{u} \cdot \mathbf{v} - \int_{\Omega_{ext}} \mathbf{u}_\varepsilon \cdot \nabla\omega_* \cdot \mathbf{v} \\
&\quad - \Big(\int \partial_t\omega_* \cdot \mathbf{v} + \nu \int \nabla\omega_* : \nabla\mathbf{v} - \int \pi_* \nabla \cdot \mathbf{v}\Big) \\
&\quad - \varepsilon^{-1}\Big(\gamma_{1,s}\varepsilon \int_{\Omega_s} \partial_t\omega_* \cdot \mathbf{v} + \nu \int_{\Omega_s} \nabla\omega_* : \nabla\mathbf{v} - \varepsilon \int_{\Omega_s} \pi_* \nabla \cdot \mathbf{v} + \nu \int_{\Omega_s} \omega_* \cdot \mathbf{v}\Big) \\
&\quad + \varepsilon^{-1} \int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v}.
\end{aligned} \tag{7.132}$$

The objective is to choose (ω_*, π_*) so that the RHS of the above equation is $\mathcal{O}(1)$ with respect to ε. We choose (ω_*, π_*) so that $\varepsilon^{-1} \int_{\partial\Omega_s} (\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}) \cdot \mathbf{v}$ is annihilated and (ω_*, π_*) are bounded in the energy norm independent of $\varepsilon \to 0$.

Substitute identity (7.120) via Assumption 7.3.5 into (7.132) to get after simplification, for any $\mathbf{v} \in V(\Omega_{ext})$

$$\begin{aligned}
&\int_{\Omega_{ext}} \gamma_1 \partial_t \omega \cdot \mathbf{v} + \nu \int_{\Omega_{ext}} \gamma_2 \nabla\omega : \nabla\mathbf{v} + \nu\varepsilon^{-1} \int_{\Omega_s} \omega \cdot \mathbf{v} \\
&= -L(\omega_*; \mathbf{v}) - M(\omega_*; \mathbf{v}) - \int \omega \cdot \nabla\mathbf{u} \cdot \mathbf{v} - \int \mathbf{u}_\varepsilon \cdot \nabla\omega \cdot \mathbf{v}
\end{aligned} \tag{7.133}$$

where

$$L(\omega_*; \mathbf{v}) := \int \partial_t \omega_* \cdot \mathbf{v} + \nu \int \nabla \omega_* : \nabla \mathbf{v} + \varepsilon^{-1}(\gamma_{1,s}\varepsilon - \gamma) \int_{\Omega_s} \partial_t \omega_* \cdot \mathbf{v}$$
$$M(\omega_*; \mathbf{v}) := \int \mathbf{u}_\varepsilon \cdot \nabla \omega_* \cdot \mathbf{v} + \int \omega_* \cdot \nabla \mathbf{u} \cdot \mathbf{v}.$$

Test (7.133) with $\mathbf{v} = \omega \in V(\Omega_{ext})$. Recall identities (2.29), (2.31). Then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\omega||^2 + \frac{\gamma_{1,s}}{2}\frac{d}{dt}||\omega||^2_{\Omega_s} + \nu|\omega|_1^2 + \nu\varepsilon^{-1}||\omega||^2_{1,\Omega_s} \\ &= -L(\omega_*; \omega) - M(\omega_*; \omega) - \int \omega \cdot \nabla \mathbf{u} \cdot \omega. \end{aligned} \tag{7.134}$$

Application of Young's inequality (2.21) and (7.121) give

$$|L(\omega_*; \mathbf{v}) + M(\omega_*; \mathbf{v})| \leq \nu^{-1}(L^2_{\omega*} + M^2_{\omega*}) + \frac{\nu}{4}|\mathbf{v}|^2_{1,\Omega_{ext}}. \tag{7.135}$$

Estimates (2.32)(a)(d) and Hölder's inequality (2.22) along with Young's inequality (2.21) give

$$|\int \omega \cdot \nabla \mathbf{u} \cdot \omega| \leq \frac{\nu}{4}|\omega|_1 + C\mu||\omega||^2 \tag{7.136}$$

where

$$\mu(t) := \begin{cases} \nu^{-3}||\mathbf{u}(\cdot,t)||_1^4 \\ \nu^{-1/3}||\mathbf{u}(\cdot,t)||_2^2 \\ ||\mathbf{u}(\cdot,t)||_{1,\infty} \end{cases}.$$

Apply estimates (7.135), (7.136) to (7.134). Absorb like-terms from the right into left-hand-side to get

$$\frac{d}{dt}||\omega||^2 + \gamma_{1,s}\frac{d}{dt}||\omega||^2_{\Omega_s} + \frac{\nu}{2}|\omega|^2_{1,\Omega_{ext}} \leq \mu(t)||\omega(\cdot,t)||^2 + \nu^{-1}(L^2_{\omega*} + M^2_{\omega*}). \tag{7.137}$$

Multiply (7.137) by the integrating factor $\exp(-\int_0^t \mu(t')dt')$, group terms, integrate on $(0,t)$. Recall that $\omega(\cdot,0) = \varepsilon^{-1}(\mathbf{u}^0_\varepsilon - \mathbf{u}^0) - \omega_*(\cdot,0)$. Then simplifying gives

$$\begin{aligned} &||\omega(\cdot,t)||_{\Omega_{ext}} + \nu^{1/2}||\nabla\omega||_{L^2(0,t;L^2(\Omega_{ext}))} \\ &\leq (\varepsilon^{-1}||\mathbf{u}^0_\varepsilon - \mathbf{u}^0|| + \gamma_{1,s}^{1/2}\varepsilon^{-1}||\mathbf{u}^0_\varepsilon||_{\Omega_s} + ||\omega_*(\cdot,0)|| + \gamma_{1,s}^{1/2}||\omega_*(\cdot,0)||_{\Omega_s} + \ldots \\ &\quad \ldots + \nu^{-1/2}(||L_{\omega*}(\cdot)||_{L^2(0,t)} + ||M_{\omega*}(\cdot)||_{L^2(0,t)}))\exp(\frac{1}{2}\int_0^t \mu(t')dt'). \end{aligned} \tag{7.138}$$

Recall $\mathbf{u}_\varepsilon - \mathbf{u} = \varepsilon(\omega + \omega_*)$. Application of the triangle inequality bound on initial data in Assumption 7.3.2 gives (7.122), (7.122). □

7.3.3 Proof of Auxiliary Estimate, Proposition 7.3.7

Proposition 7.3.7. Let $\omega_* : \Omega_{ext} \times [0,T] \to \mathbb{R}^d$, and $\pi_* : \Omega_{ext} \times (0,T] \to \mathbb{R}$ satisfy

$$-\nu\Delta\omega_{*,s} + \varepsilon\nabla\pi_{*,s} + \nu\omega_{*,s} = 0, \qquad \nabla\cdot\omega_{*,s} = 0 \qquad \text{in } \Omega_s \times [0,T] \tag{7.139}$$

subject to

$$(-\nu\nabla\omega_{*,s}\cdot\hat{\mathbf{n}} + \varepsilon\pi_{*,s}\hat{\mathbf{n}})|_{\partial\Omega_s} = -\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}|_{\partial\Omega_s} \tag{7.140}$$

and

$$-\nu\Delta\omega_{*,f} + \nabla\pi_{*,f} = 0, \qquad \nabla\cdot\omega_{*,f} = 0 \qquad \text{in } \Omega \times [0,T] \tag{7.141}$$

subject to

$$\omega_{*,f}|_{\partial\Omega_s} = \omega_{*,s}|_{\partial\Omega_s}, \qquad \omega_{*,f}|_{\partial\Omega_{ext}} = 0 \tag{7.142}$$

where $\omega_*|_{\Omega_s} := \omega_{*,s}$, $\omega_*|_{\Omega} := \omega_{*,f}$, $\pi_*|_{\Omega_s} := \pi_{*,s}$, and $\pi_*|_{\Omega} := \pi_{*,f}$. We consider the following weak formulation of the above problem.

- (Weak Formulation) Find $\omega_* : [0,T] \to H_0^1(\Omega_{ext})$, and $\pi_* : \Omega_{ext} \times [0,T] \to L^2$ satisfying

$$\begin{aligned}\nu\int_{\Omega_s}\nabla\omega_{*,s} : \nabla\mathbf{v} - \varepsilon\int_{\Omega_s}\pi_{*,s}\nabla\cdot\mathbf{v} - \int_{\Omega_s} q\nabla\cdot\omega_{*,s}\\ + \nu\int_{\Omega_s}\omega_{*,s}\cdot\mathbf{v} = \int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\mathbf{v}, \quad \forall\mathbf{v}\in H^1(\Omega_s),\ \forall q\in L^2(\Omega_s)\end{aligned} \tag{7.143}$$

and

$$\int\partial_t\omega_{*,f}\cdot\mathbf{v} + \nu\int\nabla\omega_{*,f} : \nabla\mathbf{v} - \int\pi_{*,f}\nabla\cdot\mathbf{v} - \int q\nabla\cdot\omega_{*,f} = 0, \quad \forall\mathbf{v}\in H_0^1,\ \forall q\in L^2 \tag{7.144}$$

$$\omega_{*,f}|_{\partial\Omega_s} = \omega_{*,s}|_{\partial\Omega_s}, \qquad \omega_{*,f}|_{\partial\Omega_{ext}} = 0. \tag{7.145}$$

It is clear that ω_* satisfies the requirements of Theorem 7.3.6 with $L(\omega_*;\mathbf{v}) = -(\partial_t\omega_*,\mathbf{v}) - \nu(\nabla\omega_*,\nabla\mathbf{v}) - \gamma_{1,s}\int_{\Omega_s}\partial_t\omega_*\cdot\mathbf{v}$ for all $\mathbf{v}\in V(\Omega_{ext})$. Given $\sigma(\mathbf{u}(\cdot,t),p(\cdot,t))\cdot\hat{\mathbf{n}}\in H^{-1/2}(\partial\Omega_s)$ existence and uniqueness of $\omega_*(\cdot,t)\in H_0^1(\Omega_{ext})$ for the *linear* problem follows a standard argument. In the following lemmas, we estimate the size of ω_* and ultimately trace the feedback to the ε-error in the BEFEb problem.

Lemma 7.3.10. *Fix $m = 0$, 1. Suppose that $\partial_t^{(m)}\sigma(\mathbf{u}(\cdot,t), p(\cdot,t)) \cdot \hat{\mathbf{n}} \in H^{-1/2}(\partial\Omega_s)$. Then all solutions $\omega_{*,s}$ satisfying (7.139), (7.140) also satisfy*

$$||\partial_t^{(m)}\omega_{*,s}(\cdot,t)||_{1,\Omega_s} \leq C\nu^{-1}||\partial_t^{(m)}\sigma(\mathbf{u}(\cdot,t), p(\cdot,t)) \cdot \hat{\mathbf{n}}||_{-1/2,\partial\Omega_s}. \tag{7.146}$$

Proof. Test (7.143) with $\mathbf{v} = \omega_{*,s} \in H^1(\Omega_s)$ to get

$$||\omega_{*,s}(\cdot,t)||_{1,\Omega_s} \leq C\nu^{-1}||\sigma(\mathbf{u}(\cdot,t), p(\cdot,t)) \cdot \hat{\mathbf{n}}||_{-1/2,\partial\Omega_s}.$$

For higher order estimates, differentiate (7.139) with respect to t (permissible for smooth enough $\sigma(\mathbf{u}, p) \cdot \hat{\mathbf{n}}$) and test with $\mathbf{v} = \partial_t\omega_{*,s} \in V.(\Omega_s)$. Estimate (7.146) for $m = 1$ is proved by following a proof similar to the case when $m = 0$ above. □

Lemma 7.3.11. *Fix $m = 0$ or 1. Suppose that $\partial_t^{(m)}\sigma(\mathbf{u}(\cdot,t), p(\cdot,t)) \cdot \hat{\mathbf{n}} \in H^{-1/2}(\partial\Omega_s)$. Then all solutions ω_* satisfying (7.139), (7.140), (7.141), (7.142) also satisfy*

$$||\partial_t^{(m)}\omega_*||_{1,\Omega_{ext}} \leq C\nu^{-1}||\partial_t^{(m)}\sigma(\mathbf{u}(\cdot,t), p(\cdot,t)) \cdot \hat{\mathbf{n}}||_{-1/2,\partial\Omega_s}. \tag{7.147}$$

Proof. Let $\lambda := \omega_{*,s}|_{\partial\Omega_s}$. Let $E : H_0^{1/2}(\partial\Omega) \to V.$ be an extension operator. Substitute $\omega_{*,f} = \omega_0 + E(\lambda)$, into (7.144): find $\omega_0 \in V$ satisfying

$$\nu\int \nabla\omega_0 : \nabla\mathbf{v} = -\nu\int \nabla E(\lambda) : \nabla\mathbf{v}, \qquad \forall\mathbf{v} \in V. \tag{7.148}$$

Test (7.148) with $\mathbf{v} = \omega_0 \in V$, apply Cauchy-Schwarz (2.22), and simplify to derive $|\omega_0|_1 \leq |E(\lambda)|_1$. Then since $\omega_{*,f} = \omega_0 + E(\lambda)$, application of the triangle inequality gives, for $m = 0$,

$$|\partial_t^{(m)}\omega_{*,f}|_1 \leq 2|\partial_t^{(m)}E(\lambda)|_1. \tag{7.149}$$

For higher order estimates, differentiate (7.141), (7.142) with respect to t and test with $\mathbf{v} = \partial_t\omega_0$. Then estimate (7.149) for $m = 1$ is proved by following a proof similar to the case when $m = 0$ above.

There exists a particular extension $E : H_0^{1/2}(\partial\Omega) \to V.$ satisfying

$$|E(\mu)|_1 \leq C||\mu||_{1/2,\partial\Omega_s}, \quad \forall\mu \in H_0^{1/2}(\partial\Omega).$$

Estimate (7.147) for $m = 0$ or 1 follows then by applying the definition of the $H^{1/2}(\partial\Omega)$ norm along with (7.146). □

It remains now to bound $L_{\omega*}$ and $M_{\omega*}$ in Theorem 7.3.6. Set $\gamma = 0$. Apply Cauchy-Schwarz (2.22) to obtain

$$L_{\omega*}(t)^2 = ||\partial_t \omega_*(\cdot,t)||^2 + \gamma_{1,s}||\partial_t \omega_*(\cdot,t)||^2_{\Omega_s} + \nu|\omega_*(\cdot,t)|_1^2. \tag{7.150}$$

Apply (2.32)(a) to obtain

$$M_{\omega*}(\cdot,t)^2 = C(||\mathbf{u}(\cdot,t)||_1^2 + ||\mathbf{u}_\varepsilon(\cdot,t)||_1^2)|\omega_*(\cdot,t)|_1^2. \tag{7.151}$$

Successive applications of (7.147) allow us to replace $F_{\omega*}$ in Theorem 7.3.6 with

$$\begin{aligned} F_{\omega*} := \gamma_{1,s}\nu^{-1}(||\sigma(\mathbf{u}(\cdot,0),p(\cdot,0))\cdot\hat{\mathbf{n}}||_{-1/2,\partial\Omega_s} + ||(\partial_t\sigma(\mathbf{u},p))\cdot\hat{\mathbf{n}}||_{L^2(H^{1/2}(\partial\Omega_s))}) \\ + \nu^{-1/2}||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||_{L^2(H^{1/2}(\partial\Omega_s))} + \nu^{-1}||(\partial_t\sigma(\mathbf{u},p))\cdot\hat{\mathbf{n}}||_{L^2(H^{1/2}(\partial\Omega_s))} \\ + (\nu^{-1} + ||\mathbf{u}||_{L^2(H^1)} + ||\mathbf{u}_\varepsilon||_{L^2(H^1)})||\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}}||_{L^\infty(H^{1/2}(\partial\Omega_s))}. \end{aligned}$$

Apply *a priori* estimates to simplify $F_{\omega*}$ in Proposition 7.3.7. We assume here that $\nu \le 1$. □

7.3.4 Proof of pressure, drag, and lift error

First, let $\hat{\mathbf{c}} \in H_0^{1/2}(\partial\Omega_s)$ be a constant unit vector on $\overline{\Omega}_s$. Indeed, it is easily verified that $\int_{\partial\Omega_s}\hat{\mathbf{c}}\cdot\hat{n} = 0$. Let $E(\hat{\mathbf{c}}) \in V(\Omega_{ext})$ extend $\hat{\mathbf{c}}$ to a bounded, divergence-free function in Ω. Then letting $\mathbf{v} = \hat{\mathbf{c}}$ in (7.23), rearranging, and simplifying gives

$$\begin{aligned} \int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s}\mathbf{u}_\varepsilon\cdot\hat{\mathbf{c}} = -\int_{\Omega_{ext}}\gamma_1\partial_t\mathbf{e}_u\cdot E(\hat{\mathbf{c}}) - \int_{\Omega_{ext}}\mathbf{e}_u\cdot\nabla\mathbf{e}_u\cdot E(\hat{\mathbf{c}}) \\ - \nu\int\gamma_2\nabla\mathbf{e}_u : \nabla E(\hat{\mathbf{c}}) + \int\mathbf{e}_u\cdot\nabla\mathbf{u}\cdot E(\hat{\mathbf{c}}) + \int\mathbf{u}\cdot\nabla\mathbf{e}_u\cdot E(\hat{\mathbf{c}}). \end{aligned} \tag{7.152}$$

Integrate (7.152) from $t = 0$ to T,

$$\begin{aligned} \int_{\partial\Omega_s\times[0,T]}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s\times[0,T]}\mathbf{u}_\varepsilon\cdot\hat{\mathbf{c}} = -\int_{\Omega_{ext}}\gamma_1(\mathbf{e}_u(\cdot,T) - \mathbf{e}_u(\cdot,0))\cdot E(\hat{\mathbf{c}}) \\ - \int_{\Omega_{ext}\times[0,T]}\mathbf{e}_u\cdot\nabla\mathbf{e}_u\cdot E(\hat{\mathbf{c}}) - \nu\int_{\Omega\times[0,T]}\nabla\mathbf{e}_u : \nabla E(\hat{\mathbf{c}}) \\ + \int_{\Omega\times[0,T]}\mathbf{e}_u\cdot\nabla\mathbf{u}\cdot E(\hat{\mathbf{c}}) + \int_{\Omega\times[0,T]}\mathbf{u}\cdot\nabla\mathbf{e}_u\cdot E(\hat{\mathbf{c}}). \end{aligned} \tag{7.153}$$

Successive applications of Hölder's inequality (2.22) with respect to $\int_\Omega(\cdot)$ and with $E(\hat{\mathbf{c}}) \in L^\infty(\Omega_{ext})$ gives

$$
\begin{aligned}
&|\int_{\partial\Omega_s\times[0,T]}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s\times[0,T]}\mathbf{u}_\varepsilon\cdot\hat{\mathbf{c}}| \\
&\leq C(||\mathbf{e}_u(\cdot,T)|| + ||\mathbf{e}_u(\cdot,0)|| + \gamma_{1,s}(||\mathbf{u}_\varepsilon(\cdot,T)||_{\Omega_s} + ||\mathbf{u}_\varepsilon^0||_{\Omega_s}) + \dots \\
&\quad \dots + \nu\int_0^T|\mathbf{e}_u(\cdot,t)|_1 dt + \int_0^T(|\mathbf{e}_u(\cdot,t)|_1 + |\mathbf{u}(\cdot,t)|_1)dt||\mathbf{e}_u||_{L^\infty(L^2)}). \qquad (7.154)
\end{aligned}
$$

Division by T and application of Hölder's inequality (2.22) with respect to $\int_0^T(\cdot)$ gives

$$
\begin{aligned}
&|T^{-1}\int_0^T(\int_{\partial\Omega_s}(\sigma(\mathbf{u},p)\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{c}} - \frac{\nu}{\varepsilon}\int_{\Omega_s}\mathbf{u}_\varepsilon\cdot\hat{\mathbf{c}})| \\
&\leq C(T^{-1}||\mathbf{e}_u(\cdot,T)|| + T^{-1}||\mathbf{e}_u(\cdot,0)|| + T^{-1}\gamma_{1,s}(||\mathbf{u}_\varepsilon(\cdot,T)||_{\Omega_s} + ||\mathbf{u}_\varepsilon^0||_{\Omega_s}) + \dots \\
&\quad \dots + \nu(\frac{1}{T}\int_0^T|\mathbf{e}_u(\cdot,t)|_1^2 dt)^{1/2} + \dots \\
&\quad \dots + \left((\frac{1}{T}\int_0^T|\mathbf{e}_u(\cdot,t)|_1^2 dt)^{1/2} + (\frac{1}{T}\int_0^T|\mathbf{u}(\cdot,t)|_1^2 dt)^{1/2}\right)||\mathbf{e}_u||_{L^\infty(L^2)}). \qquad (7.155)
\end{aligned}
$$

Estimate (7.126) follows.

To estimate the pressure error, set $\mathbf{v} \in H_0^1$ in (7.23) extended so that $\mathbf{v}|_{\Omega_s} \equiv 0$. Rearrange, integrate (7.101) from $t = 0$ to T, and simplify to obtain

$$
\begin{aligned}
\int\int_0^T(p(\cdot,t) - p_\varepsilon(\cdot,t))\nabla\cdot\mathbf{v} &= -\int(\mathbf{e}_u(\cdot,T) - \mathbf{e}_u(\cdot,0))\cdot\mathbf{v} \\
&- \nu\int_{\Omega\times[0,T]}\nabla\mathbf{e}_u : \nabla\mathbf{v} + \int_{\Omega\times[0,T]}\mathbf{e}_u\cdot\nabla\mathbf{u}\cdot\mathbf{v} + \int_{\Omega\times[0,T]}\mathbf{u}_\varepsilon\cdot\nabla\mathbf{e}_u\cdot\mathbf{v}. \qquad (7.156)
\end{aligned}
$$

Similarly as above successive applications of Hölder's inequality (2.22) gives

$$
\begin{aligned}
\frac{|\int\int_0^T(p(\cdot,t) - p_\varepsilon(\cdot,t))dt\nabla\cdot\mathbf{v}|}{|\mathbf{v}|_1} &\leq ||\mathbf{e}_u(\cdot,T)||_{-1} + ||\mathbf{e}_u(\cdot,0)||_{-1} \\
&+ \nu\int_0^T|\mathbf{e}_u(\cdot,t)|_1 dt + \int_0^T(|\mathbf{u}(\cdot,t)|_1 + |\mathbf{u}(\cdot,t)|_1)|\mathbf{e}_u(\cdot,t)|_1 dt. \qquad (7.157)
\end{aligned}
$$

Note that $\int_0^T (p(\cdot,t) - p_\varepsilon(\cdot,t))dt \in L_0^2$. Then application of the inf-sup condition (2.2) along with division by T and then Hölder's inequality (2.22) with respect to $\int_0^T(\cdot)$ gives

$$
\begin{aligned}
||T^{-1}\int_0^T (p(\cdot,t) - p_\varepsilon(\cdot,t))dt|| &\leq C(T^{-1}||\mathbf{e}_u(\cdot,T)||_{-1} + T^{-1}||\mathbf{e}_u(\cdot,0)||_{-1} + \dots \\
&\dots + \nu(T^{-1}\int_0^T |\mathbf{e}_u(\cdot,t)|_1^2 dt)^{1/2} + \dots \\
&\dots + ((\int_0^T |\mathbf{u}(\cdot,t)|_1^2 dt)^{1/2} + (\int_0^T |\mathbf{u}_\varepsilon(\cdot,t)|_1^2 dt)^{1/2})(T^{-1}\int_0^T |\mathbf{e}_u(\cdot,t)|_1^2 dt)^{1/2}). \qquad (7.158)
\end{aligned}
$$

Estimate (7.125) follows.

7.4 NUMERICAL INVESTIGATION

In this section we investigate how well BrVP predicts the velocity field, pressure, and drag and lift forces exerted by a fluid on a solid obstacle. We consider the same problem investigated in [54] and compare our lift and drag coefficients with the benchmark results presented therein. For accurate drag and lift calculations, it is generally preferable to use higher order time-stepping and spatial discretization, see e.g. [54]. For this reason, we consider Crank-Nicolson time-stepping for BrVP.

Problem 7.4.1 (CNFEb)**.** *Let* $\mathbf{u}_{\varepsilon,h}^0 \in V_{h,\phi_h}(\Omega_{ext})$, $p_{\varepsilon,h}^0 \in Q_h(\Omega_{ext})$ *be a good approximation of* $\mathbf{u}_\varepsilon^0$, p_ε^0. *For each* $n = 0, 1, \dots, N-1$, *find* $(\mathbf{u}_{\varepsilon,h}^{n+1}, p_{\varepsilon,h}^{n+1}) \in X_{h,\phi_h^{n+1}}(\Omega_{ext}) \times Q_h(\Omega_{ext})$ *satisfying*

$$
\begin{aligned}
&\int_{\Omega_{ext}} \gamma_1(\partial_{\Delta t}^{n+1}\mathbf{u}_{\varepsilon,h}) \cdot \mathbf{v}_h + c_{h,ext}(\mathbf{u}_{\varepsilon,h}^{n+1/2}, \mathbf{u}_{\varepsilon,h}^{n+1}, \mathbf{v}_h) \\
&\qquad + \nu \int_{\Omega_{ext}} \gamma_2 \nabla \mathbf{u}_{\varepsilon,h}^{n+1} : \nabla \mathbf{v}_h + \nu\varepsilon^{-1}\int_{\Omega_s} \mathbf{u}_{\varepsilon,h}^{n+1} \cdot \mathbf{v}_h - \int_{\Omega_{ext}} p_{\varepsilon,h}^{n+1} \nabla \cdot \mathbf{v}_h \\
&\qquad = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h(\Omega_{ext}) \qquad (7.159) \\
&\int_{\Omega_{ext}} q_h \nabla \cdot \mathbf{u}_{\varepsilon,h}^{n+1} = 0, \quad \forall q_h \in Q_h(\Omega_{ext}). \qquad (7.160)
\end{aligned}
$$

We investigate Crank-Nicolson time stepping for the NSE in Chapters 3, 4.

In the theory developed above, we emphasize that the discrete analogue of the traction vector $-\nu(\hat{\mathbf{n}}\cdot\nabla)\mathbf{u}_h^n + p_h\hat{\mathbf{n}}$ is not well-defined since $\mathbf{u}_h$ is generally not H^2 and p_h is generally not H^1 which is required for defining the corresponding trace on $\partial\Omega_s$. The existence of σ_h for CNFEb is similarly guaranteed and given by the same formula:

$$\sigma_h^{n+1}(\mathbf{v}_h) := -\left(\partial_{\Delta t}^{n+1}\mathbf{u}_h, \mathbf{v}_h\right) - \nu(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h) - c_h(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) \quad (7.161)$$

as long as Δt is sufficiently restricted to guarantee $\mathbf{u}_h \in l^\infty(H^1)$. Otherwise, $\sigma_h^{n+1/2}(\mathbf{v}_h)$ is well-defined. We write σ_h as a functional here on $X_h(\Omega_{ext})$ because this is the most convenient form for computations.

For the problem setup, consider the channel $([0,2.2]\times[0,0.41]) - \Omega_s$ where Ω_s is circular obstacle with diameter $= 0.1$ centered at $(0.2, 0.2)$. Fix the time interval $[0, 8]$. The flow has boundary conditions:

$$\begin{gathered}\mathbf{u}(x, y=0) = \mathbf{u}(x, y=0.41) = \mathbf{u}|_{\partial\Omega_s} = 0 \\ \mathbf{u}(x=0, y, t) = \mathbf{u}(x=2.2, y, t)\frac{6}{0.41^2}y(0.41-y)\sin(\frac{\pi t}{8}).\end{gathered}$$

For high enough Reynolds number (albeit below turbulence levels) vortices will begin shedding from the wake of Ω_s at a regular frequency (von Kármán vortex street). As reported in [54], as the flow rate increases, 2 vortices develop in the wake of the cylinder that separate between $t = 4$ and 5. Set $\nu = 10^{-3}$.

Now we define the drag

$$\begin{aligned}
&\text{NSE, Method 1}: && D^{n+1/2} := \textstyle\sum_e \int_{e\cap\partial\Omega_s}(-\nu(\hat{\mathbf{n}}\cdot\nabla)\mathbf{u}_h^{n+1/2} + p_h^{n+1/2}\hat{\mathbf{n}})\cdot[1,0]^t \\
&\text{NSE, Method 2}: && D^{n+1/2} := \sigma_h^{n+1/2}([1,0]^t) \\
&\text{BrVP}: && D_\varepsilon^{n+1/2} := \textstyle\int_{\Omega_s}\frac{\nu}{\varepsilon}\mathbf{u}_h^{n+1/2}\cdot[1,0]^t
\end{aligned}$$

and lift

$$\begin{aligned}
&\text{NSE, Method 1}: && L^{n+1/2} := \textstyle\sum_e \int_{e\cap\partial\Omega_s}(-\nu(\hat{\mathbf{n}}\cdot\nabla)\mathbf{u}_h^{n+1/2} + p_h^{n+1/2}\hat{\mathbf{n}})\cdot[0,1]^t \\
&\text{NSE, Method 2}: && L^{n+1/2} := \sigma_h^{n+1/2}([0,1]^t) \\
&\text{BrVP}: && L_\varepsilon^{n+1/2} := \textstyle\int_{\Omega_s}\frac{\nu}{\varepsilon}\mathbf{u}_h^{n+1/2}\cdot[0,1]^t
\end{aligned}$$

where $e = E \cap \partial\Omega_s$ for any $E \in \mathcal{T}_h$ are the boundary elements on $\partial\Omega_s$ derived from the mesh $\mathcal{T}_h$. Let $\mathbf{v}_h \in X_h(\Omega_{ext})$ be such that $\mathbf{v}_h|_{\partial\Omega_s} = [1,0]^t$ or $[0,1]^t$. Since σ_h is a well-defined functional on FE velocity functions restricted to $\partial\Omega_s$, we can compute $\sigma_h^{n+1/2}(\mathbf{v}_h)$ uniquely by (7.161) for any $\mathbf{v}_h = [1,0]^t$ or $[0,1]^t$ in a small *ring* around $\partial\Omega_s$ inside Ω.

We compare NSE approximation with BrVP approximation each obtained with Crank-Nicolson time-stepping and $\Delta t = 0.01$. We solve each problem on the time interval $[0,8]$ with Taylor-Hood finite elements on the same mesh extended into Ω_s for BrVP. The mesh is generated by Delaunay-Voronoi triangulation in FreeFem++ and contains 143100 velocity degrees of freedom (161168 total degrees of freedom) in Ω with 128 vertices on Ω_s. We resolve the nonlinearity with Newton iterations so that the H^1 residual error less than 10^{-12} at each time step.

In Figures 7.1, 7.2, 7.3 we present snapshots of the magnitude of the velocity field for both the NSE and BrVP flows at $T = 4$, 6, and 8 respectively. The BrVP fails to produce a match to the NSE flow when $\varepsilon = 10^{-3}$, but matches the NSE profile well for $\varepsilon = 10^{-6}$, 10^{-9}. Indeed, when $\varepsilon = 10^{-3}$ the BrVP flow does not provide much flow-resistance in Ω_s. Figure 7.4 focuses attention on the BrVP flow field developed in the wake of the cylinder at $T = 6$. The center of the vortex in the middle plot for $\varepsilon = 10^{-6}$ is shifted slightly to the right of the vortex predicted by NSE and BrVP with $\varepsilon = 10^{-9}$. Indeed, BrVP with $\varepsilon = 10^{-6}$ does not resist flow enough allowing the wake to extend further than predicted by the NSE approximation.

We provide convergence rate data as $\varepsilon \to 0$ in Tables 7.1, 7.2. For a fixed mesh, we observe the characteristic $\mathcal{O}(\varepsilon)$ convergence rate expected in Ω and Ω_s. The following reference intervals for drag, lift, and pressure drop (across the cylinder) are provided in [54]:

$$\max(D) \in [0.1465,\ 0.1485], \qquad \max(L) \in [0.0235,\ 0.0245]$$
$$\Delta p(t=8) \in [-0.115,\ -0.105].$$

Pressure drop is computed by

$$\Delta p(t) = p(x = 0.15, y = 0.2, t) - p(x = 0.25, y = 0.2, t).$$

(Note that the values reported here for D and L are differ by a factor of 20 from those in [54] because of definition (2) and (3) in [54] for the calculation of the drag and lift coefficients

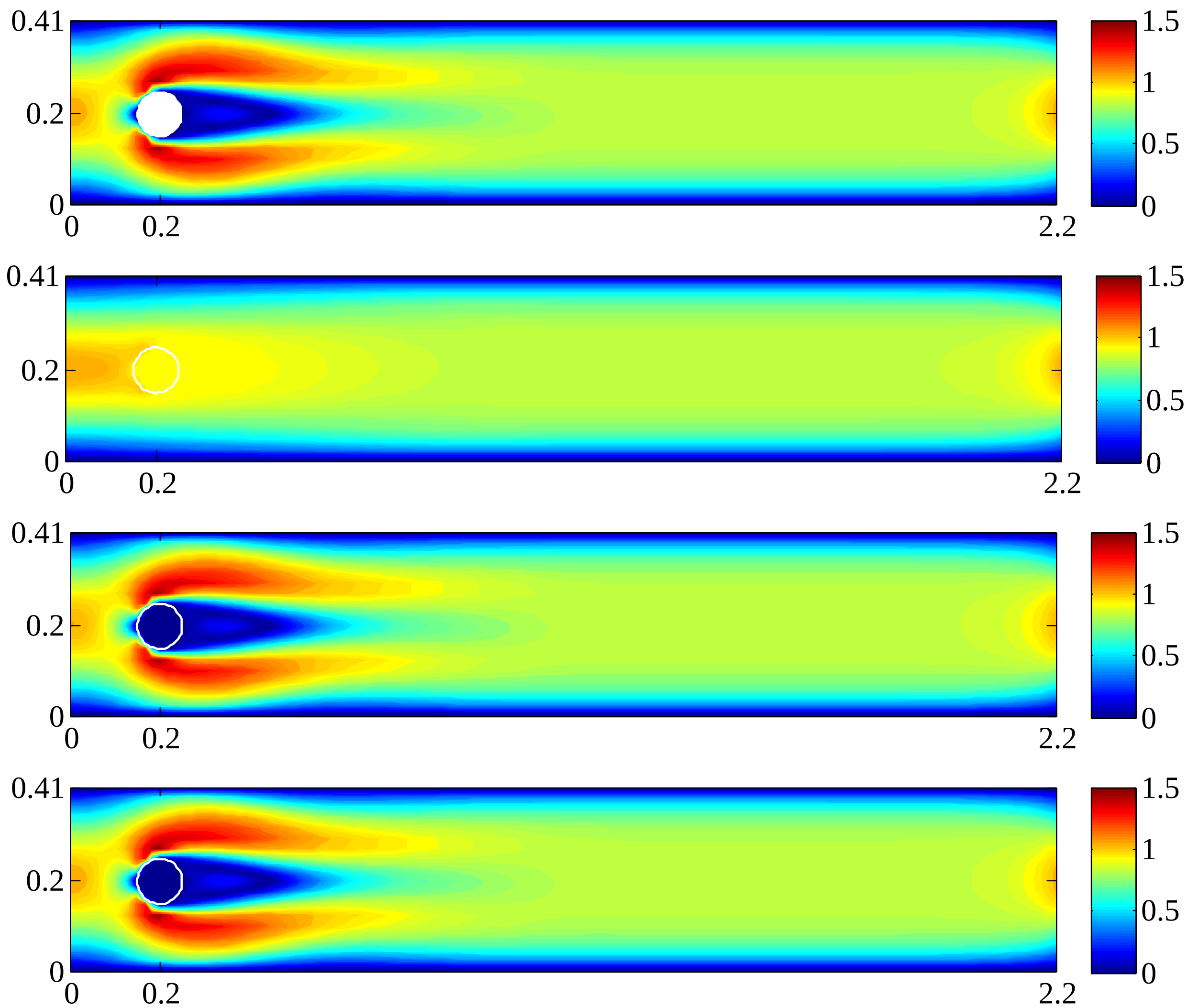


Figure 7.1: Evolutionary flow past 1 cylinder: magnitude of velocity field at $T = 4$ for (a) NSE, (b) BrVP, $\varepsilon = 10^{-3}$, (c) BrVP, $\varepsilon = 10^{-6}$, (d) BrVP, $\varepsilon = 10^{-9}$

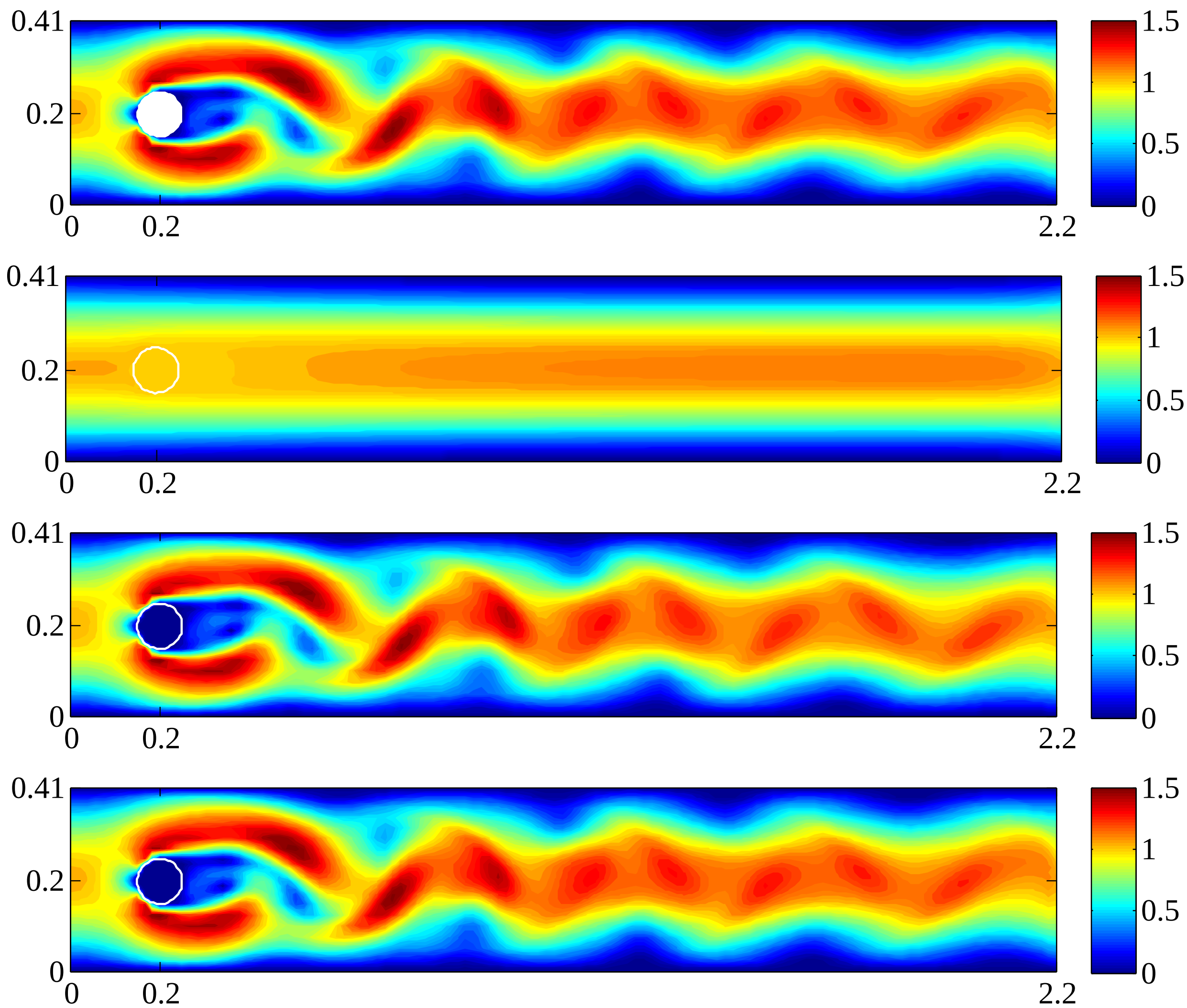


Figure 7.2: Evolutionary flow past 1 cylinder: magnitude of velocity field at $T = 6$ for (a) NSE, (b) BrVP, $\varepsilon = 10^{-3}$, (c) BrVP, $\varepsilon = 10^{-6}$, (d) BrVP, $\varepsilon = 10^{-9}$

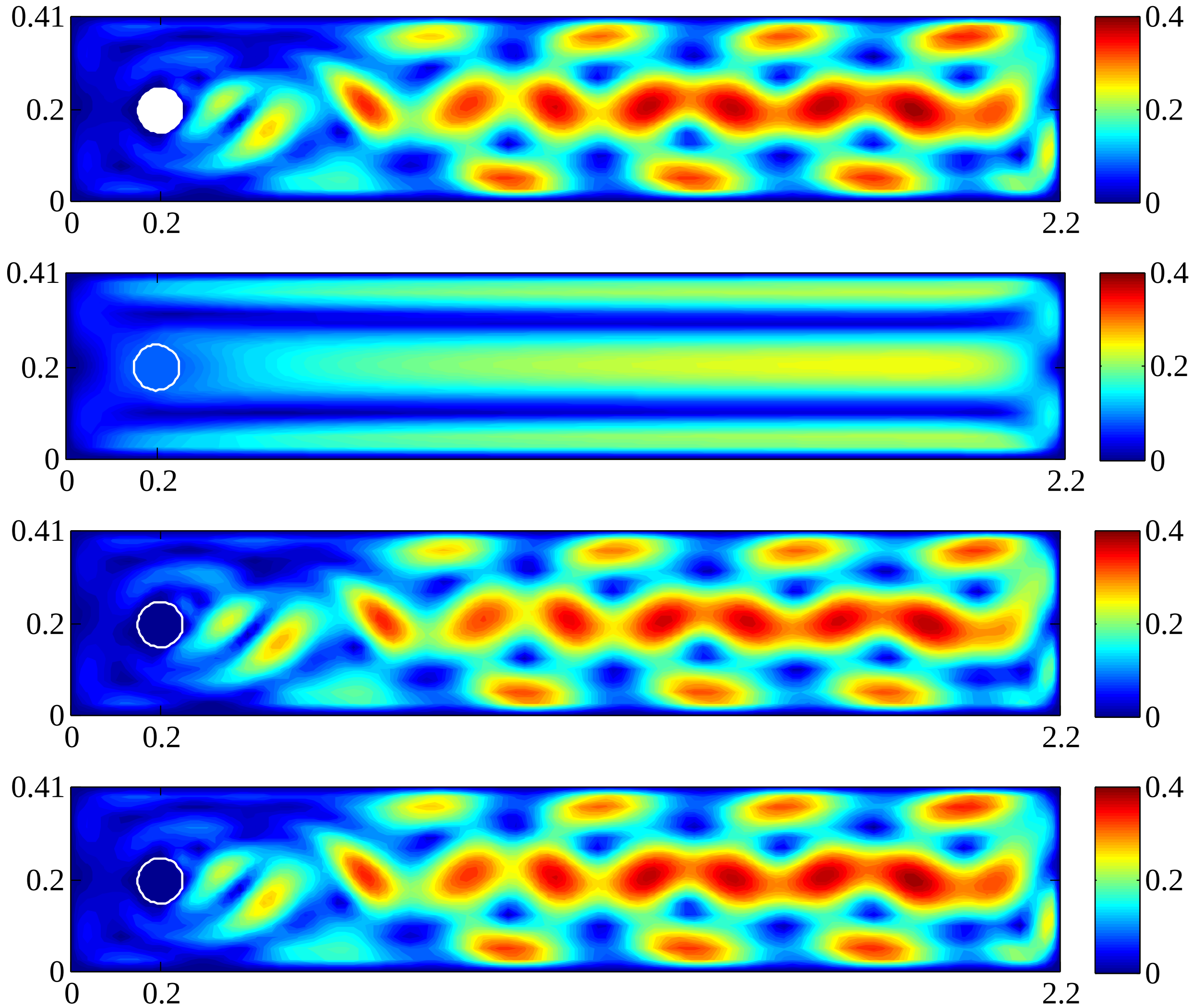


Figure 7.3: Evolutionary flow past 1 cylinder: magnitude of velocity field at $T = 8$ for (a) NSE, (b) BrVP, $\varepsilon = 10^{-3}$, (c) BrVP, $\varepsilon = 10^{-6}$, (d) BrVP, $\varepsilon = 10^{-9}$

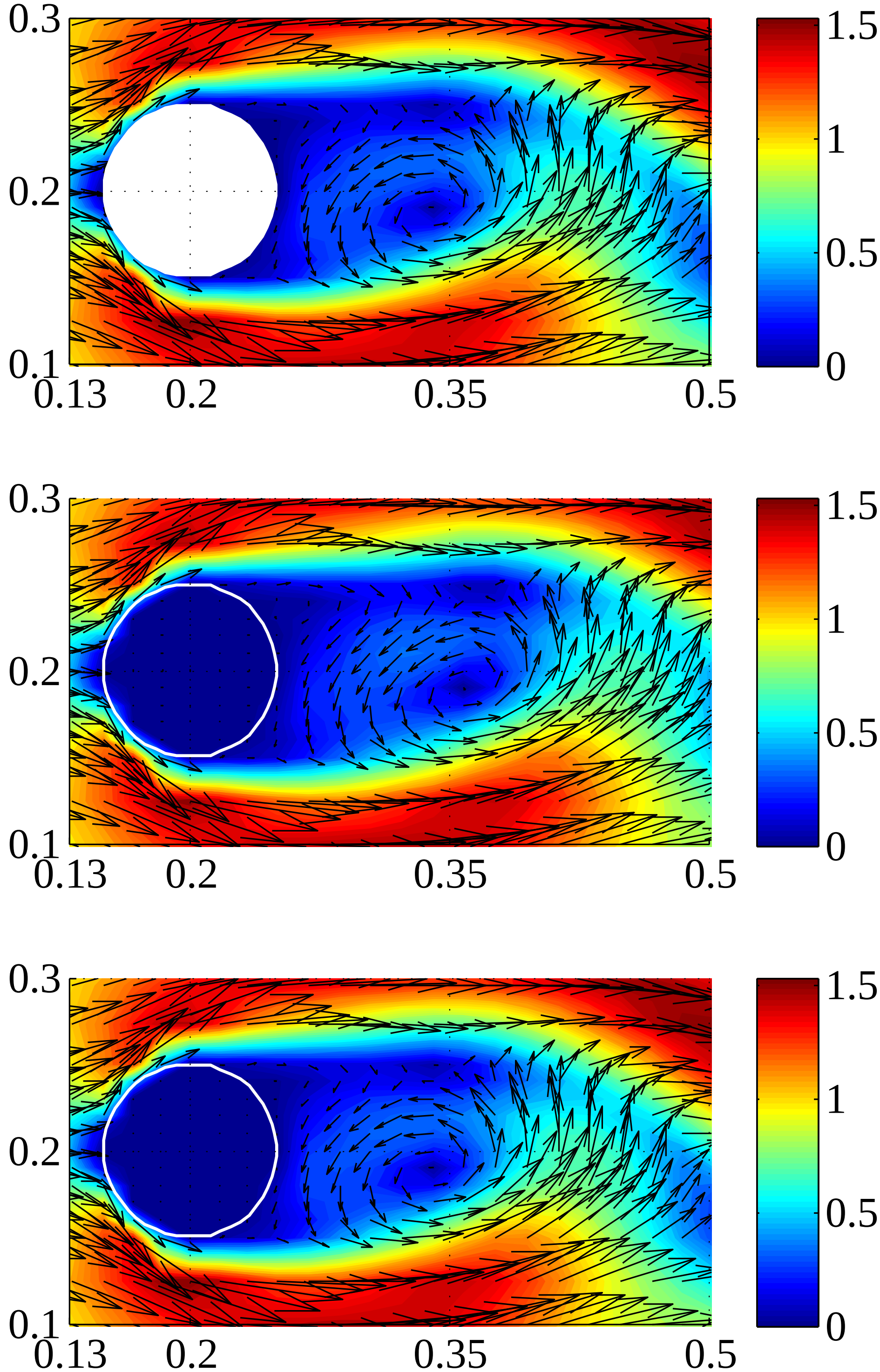


Figure 7.4: Evolutionary flow past 1 cylinder: velocity field at $T = 6$ for (a) NSE, (b) BrVP, $\varepsilon = 10^{-6}$, (c) BrVP, $\varepsilon = 10^{-9}$

Table 7.1: Evolutionary flow past 1 cylinder: convergence of BrVP velocity in Ω

ε	$\|\|\mathbf{u}_{\varepsilon,h} - \mathbf{u}\|\|_{l^\infty(L^2)}$	Rate	$\|\|\nabla(\mathbf{u}_{\varepsilon,h} - \mathbf{u})\|\|_{l^2(L^2)}$	Rate
1e-3	4.125e-1	—	1.3468e1	
1e-6	1.495e-1	0.147	6.549e-1	0.143
1e-9	1.649e-4	0.986	7.425e-3	0.985
1e-12	1.649e-7	1.000	7.433e-7	1.000

Table 7.2: Evolutionary flow past 1 cylinder: convergence of BrVP velocity in Ω_s

ε	$\|\|\mathbf{u}_{\varepsilon,h}\|\|_{l^\infty(L^2(\Omega_s))}$	Rate	$\|\|\nabla\mathbf{u}_{\varepsilon,h}\|\|_{l^2(L^2(\Omega_s))}$	Rate
1e-3	1.253e-1	—	1.323e-3	—
1e-6	1.651e-3	0.627	1.294e-5	0.670
1e-9	1.719e-6	0.994	1.296e-8	1.000
1e-12	1.741e-9	0.998	1.296e-11	1.000

Table 7.3: Evolutionary flow past 1 cylinder: NS computed drag and lift at $t^{n+1/2}$

	$D^{1/2}_{max}$	t_{max}	$L^{1/2}_{max}$	t_{max}	$\Delta p(t = 7.995)$
Method 1	0.147471	3.935	0.0239886	5.925	0.110948
Method 2	0.147505	3.935	0.0239927	5.925	—

Table 7.4: Evolutionary flow past 1 cylinder: BrVP drag and lift at $t^{n+1/2}$

ε	$D^{1/2}_{\varepsilon,max}$	t_{max}	$L^{1/2}_{\varepsilon,max}$	t_{max}	$\Delta p_{\varepsilon}(T=7.995)$
$1e$-3	0.0110989	4.055	1.57959e-5	1.045	0.040746
$1e$-6	0.146248	3.935	0.0186180	5.895	0.108082
$1e$-9	0.147505	3.935	0.0239882	5.925	0.110938
$1e$-12	0.147505	3.935	0.0239939	5.925	0.110948

therein). Throughout, let t_{max} be the time at which either the maximal computed drag or lift occurs. Define

$$D_{max} = \max_{0<n\le N} |D^n|, \quad D^{1/2}_{max} = \max_{0<n\le N} |D^{n+1/2}|$$
$$L_{max} = \max_{0<n\le N} |L^n|, \quad L^{1/2}_{max} = \max_{0<n\le N} |L^{n+1/2}|.$$

The drag, lift, and pressure drop statistics reported for NSE in Tables 7.3 and BrVP in 7.4 correspond well as $\varepsilon \to 0$. As suggested by our theory, the BrVP force calculation is a better approximation of the approximate NSE force computed by Method 2.

Drag, lift, and pressure drop statistics are reported for NSE in Tables 7.5 and BrVP in 7.6 for at time levels t^n rather than averaged $t^{n+1/2}$. The lift reported for NSE and for BrVP as $\varepsilon \to 0$ falls within the reference range, but not the drag and pressure drop values. This is not surprising, however, since CN schemes compute approximations of average velocities $\mathbf{u}^{n+1/2}$ without a guarantee on the accuracy of $\mathbf{u}^n$. Note that once again the computed BrVP approximations are in good agreement with the NSE approximation as $\varepsilon \to 0$.

Table 7.5: Evolutionary flow past 1 cylinder: NS computed drag and lift at t^n

ε	D_{max}	t_{max}	L_{max}	t_{max}	$\Delta p(t = 8)$
Method 1	0.154210	3.93	0.0240147	5.92	0.197969
Method 2	0.154301	3.93	0.0240185	5.92	—-

Table 7.6: Evolutionary flow past 1 cylinder: BrVP drag and lift at t^n

ε	$D_{\varepsilon,max}$	t_{max}	$L_{\varepsilon,max}$	t_{max}	$\Delta p_{\varepsilon}(T = 8)$
1e-3	0.011099	4.06	1.57981e-5	1.04	0.0785283
1e-6	0.146248	3.94	0.0186472	5.90	0.149763
1e-9	0.152351	3.93	0.0240187	5.92	0.175569
1e-12	0.154281	3.93	0.0240206	5.92	0.197938

8.0 CONCLUSIONS

Deriving mathematical theory lending itself to the reliable approximation of practical problems in fluid dynamics is a major undertaking. Flows in complicated domains like pebble bed reactor cores and wind farms are inherently time-dependent problems that require the solving of many (proportional to the physical time interval of interest) large linear systems (proportional to the number of degrees of freedom required to resolve the flow). We focused herein on investigating (1) a linear time stepping method (to avoid extra linear solves at *each* time step) and (2) a simple volume penalization technique easily integrated to existing computing platforms that provides a way to solve NS-type problems on a uniform mesh. Rigorous mathematical formulation of these techniques is necessary for designing fast, stable, and robust numerical methods for simulating fluid flow in complicated domains.

In Chapters 3, 4 we investigated the stability and accuracy of a linearly extrapolated Crank-Nicolson (CN) time-stepping method for a finite element (FE) spatial discretization of the NSE (CNLE). We proved that the CNLE velocity converges without any restriction on the time-step size to the NSE velocity as the mesh width h and time step size Δt tend to 0. Moreover, under a (novel) nonstandard linear extrapolation of the convecting velocity, we also proved that the CNLE velocity converges to the NSE velocity in higher order norms without any time-step restriction. Convergence in these higher order norms (in particular, $l^\infty(H^1)$ and the discrete time-derivative in $l^2(L^2)$) is the key to proving similar estimates for drag, lift, and pressure. The numerical results in Chapter 3 confirm that the alternate extrapolation for CNLE we propose herein is advantageous.

In Chapters 5, 6, 7, we investigated the validity and accuracy of the Brinkman model for approximating flows in complicated domains. In Chapter 5 we established the well-posedness, under specific constraints, for the stationary, nonlinear Brinkman equations for inhomogeneous boundary data and non-zero divergence constraint (both continuous and FE models). In Chapters 6, 7 we investigated the accuracy of the FE approximation of the Brinkman volume penalization (BrVP) equations (both stationary and evolutionary) in approximating viscous, incompressible fluid flows through complicated domains. Moreover, BrVP provides a convenient volume integral for computing the forces exerted by the fluid on the embedded obstacles. We proved convergence (in particular norms) of the BrVP forces relative to the actual fluid forces as well. Our numerical results for BrVP flow confirm that, under suitable conditions, the predicted $\mathcal{O}(\varepsilon)$ convergence rates are observed in practice. We also investigated how BrVP flow approximated on a uniform mesh deviates the corresponding NSE approximation. For a fine enough mesh, the global influence of the boundary non-conforming mesh is diminished. We simulated flow past a tightly packed array of 2d spheres. The BrVP approximation on a uniform mesh predicts higher pore velocities than NSE. Again, this affect is diminished for finer meshes.

Many questions arise from our research that will be investigated as continuations of this work.

Concerning Brinkman as a porous media model:

Modeling PBR flow may be best approximating by a completely homogenized fluid model. Darcy porous media models have been applied in the past without reliability. There is reason to expect Brinkman (linear or nonlinear) to perform better (or at least differently!). As a first consideration, we must setup a simple, testable, model problem. For example, we could simulate flow through the 40 2d cylinder array presented earlier in this document. We can compute the NS-flow as a baseline for comparison. Given the outflow as a constraint (or some other physically reasonable constraint), we can then calculate the permeability tensor for Darcy flow by a constrained optimization procedure (matching flow output). Given this permeability, we then compute the Brinkman viscosity $\tilde{\nu}$ by a similar constrained optimization procedure (same constraint as Darcy). The questions is: if the cylinders are each heat sources,

- how does the heat distribution *evolve* for the NS, Darcy, and Brinkman flow
- how does the heat distribution compare between NS, Darcy, and Brinkman

Last, we also consider the following fundamental problem:

Problem Statement: *Do stationary BrVP exist for all $g \in L^2(\Omega)$? This question is open for the NSE as well when $\tilde{\nu} = \nu$ and $\mathbf{K}^{-1} = 0$. It is worth noting that for general sources/sinks in Ω, examples exist showing that the Leray-Hopf-extension of ϕ mentioned above fails to exists domains with embedded sources/sinks in Ω, see e.g. [29].*

Concerning Brinkman with volume penalization:

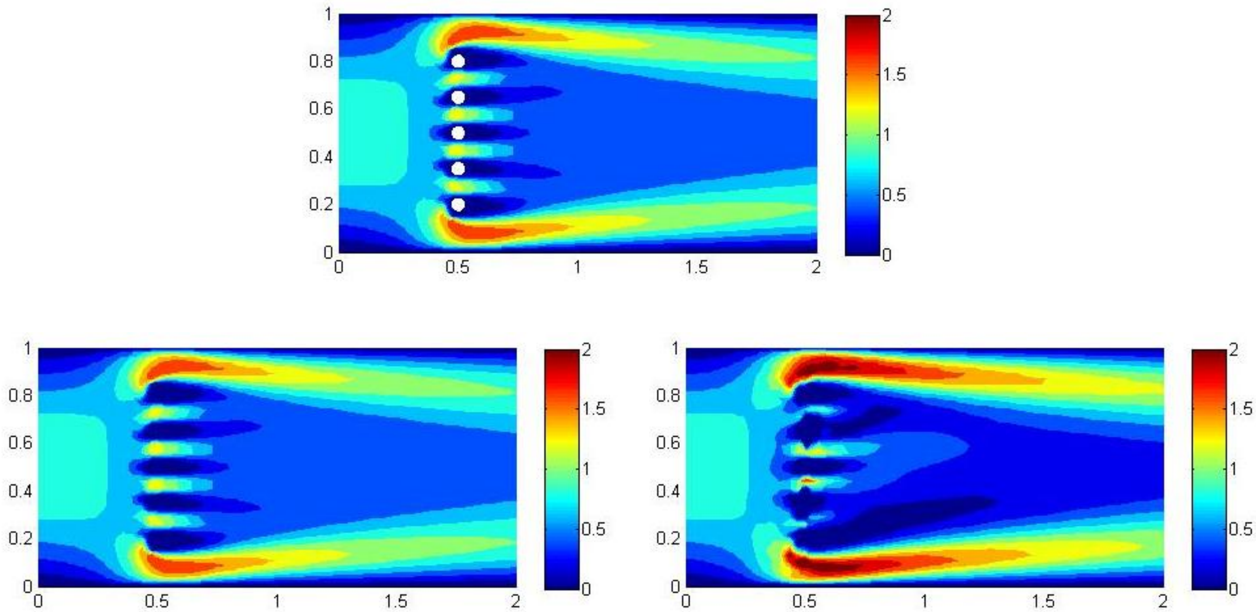


Figure 8.1: Speed profile for (a) (top) NSE-flow, (b) (bottom-left) BrVP-flow with boundary conforming mesh, (c) (bottom-right) BrVP-flow with uniform mesh

The main advantage of approximating flows with BrVP is that solutions computable on structured meshes (without conforming to the complicated pore geometry). However, the accurate implementation of BrVP depends strongly on the impossible problem of meshing the pores. For instance, in Figure 8.1 notice that BrVP with boundary conforming mesh closely predicts appropriate speed profiles and symmetry for flow past an array of cylinders, but BrVP with a uniform mesh incorrectly predicts choked flow through the array and a non-symmetric speed distribution.

Problem Statement: *Given a sphere distribution, modify BrVP to incorporate the domain geometry. For example, I am considering the incorporation of an additional set of velocity test functions (in the FE-framework) that has compact support on each sphere. This accounts for the lost geometry from BrVP whereas BrVP ensures that the velocity inside the spheres is small. This process can be viewed as exact numerical homogenization - a discrete analog to the homogenization technique used to derive filtration models like Darcy and Brinkman porous media equations.*

Problem Statement: *Is $\mathcal{O}(\varepsilon)$ convergence in $H^1(\Omega_{ext})$ preserved for L^2-penalization so that only the zero-order term $\nu K^{-1}\mathbf{u}_\varepsilon$ forces $\mathbf{u}_\varepsilon|_{\Omega_{solid}} \approx 0$. In practice, the $\mathcal{O}(\varepsilon)$-convergence is generally observed, but the estimate has not been shown theoretically for the continuous or discrete problem.*

Although an approximation of the velocity field is required for properly determining the thermodynamics in a PBR, the temperature of the pebbles (to avoid overheating and a possible nuclear accident) and outlet temperature of the reactor vessel (essential in determining the efficiency of the reactor plant) are the primary variables. Therefore, we consider the heat equation (with convection and diffusion)

$$\partial_t \theta_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \theta_\varepsilon - \nabla \cdot (\kappa \nabla \theta_\varepsilon) = g, \quad \text{in } \Omega$$

where θ_ε is temperature (generated with BrVP-velocity $\mathbf{u}_\varepsilon$), κ is the thermal conductivity, and g is the heat source/sink.

Problem Statement: *Analyze consistency of steady and evolutionary temperature approximations achieved with $\mathbf{u}_\varepsilon$ rather than $\mathbf{u}$. Analyze the convergence of steady and evolutionary discretization of BrVP-temperature approximations as ε, h, $\Delta t \to 0$.*

Problem Statement: *Investigate the phenomenon of natural convection be including a buoyancy term $\beta\theta$ in the NSE and BrVP equations. Investigate efficient, robust, and stable* *decoupling strategies for the resulting velocity-pressure-temperature system. Analyze limiting behavior ε, h, $\Delta t \to 0$.*

Problem Statement: *In each case, investigate interplay of spatial mesh and domain geometry.*

Optimal placement of filters subject to a specified constraint is a common problem in engineering. For example, wind turbines placement on a fixed plot to maximize energy conversion while minimizing noise pollution is an important question. This constrained-optimization problem can be formally stated: Let X be the velocity space and W is the set of all possible windmill configurations, and define a functional $J : X \times W \to \mathbb{R}$ representing the total momentum of wind in a domain containing the windmills; e.g. for a particular windmill configuration $w \in W$, let $\Omega_w \subset \mathbb{R}^3$ be the smallest connected set containing all windmills on the wind farm and

$$J\left(u, w\right) = \int_{\Omega_w} \left|u\right| .$$

Let $\Omega \subset \mathbb{R}^3$ be bounded. Consider $W_0 \subset W$ restricted to N-finite number of windmills contained in Ω. Find the control $w^* \in W_0$ satisfying the maximization problem:

$$(u^*, w^*) = \text{ arg-max } _{w \in W_0} J\left(u(w), w\right)$$

where u(w) is a solution of the variational Brinkman equation for windmill configuration $w \in W_0$. There are certainly other possibilities for J that must be considered.

Concerning CNLE methods:

Note that unconditional stability ensures that the energy norm of the velocity remains bounded for all time. Lyapunov stability implies that small perturbations of a given solution asymptotically return to that given solution. Lyapunov stability analysis is important in practice since discretization error introduces small, or possibly large, perturbations from the actual solution at each time-step.

Problem Statement: *We know that NSE, CNFE, and CNLE (at least for homogeneous boundary data) are each unconditionally stable in terms of energy. The question of Lyapunov* *stability for these methods should also be considered. What is the region of Lyapunov stability for NSE, CNFE, and CNLE and how do they relate? What is the* Δt *and* h *dependency for the discrete schemes?*

BIBLIOGRAPHY

[1] *Millennium Prize Problems*. http://www.claymath.org/millennium/, 2010.

[2] G. Allaire, *Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes*, Arch. Ration. Mech. Anal., 113 (1991), pp. 209–259.

[3] P. Angot, *Analysis of singular perturbations on the Brinkman problem for fictious domain models of viscous flows*, Math. Methods Appl. Sci., 22 (1999), pp. 1395–1412.

[4] P. Angot, C.H. Bruneau, and P. Fabrie, *A penalization method to take into account obstacles in incompressible viscous flows*, Numer. Math., 81 (1999), pp. 497–520.

[5] G. Apte, S. Canstantinescu, F. Ham, G. Iaccarino, and P. Mahesh, K. Moin, *Large-eddy simulation of reacting turbulent flows in complex geometries*, J. Appl. Mech., 73 (2006), pp. 364–371.

[6] E. Arquis and J. P. Caltagirone, *Sur les conditions hydrodynamiques au voisinage d'une interface milieu fluide-milieux poreux: application à la convection naturelle*, in C.R. Acad. Sci. Paris, vol. 299 of Série II, Paris, 1984, pp. 1–4.

[7] G.A. Baker, *Galerkin approximations for the Navier-Stokes equations*, tech. report, Havard University, 1976.

[8] G.A. Baker, V. Dougalis, and O. Karakashian, *On a higher order accurate, fully discrete galerkin approximation to the Navier-Stokes equations*, Math. Comp., 39 (1982), pp. 339–375.

[9] G.S. Beavers and D.D. Joseph, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech., 30 (1967), pp. 197–207.

[10] A. Bejan and D.A. Nield, *Convection in Porous Media*, Springer, New York, third ed., 2006.

[11] G.E.P. Box and N.R. Draper, *Empirical Model-Building and Response Surfaces*, John Wiley & Sons, Inc., New York, 1987.

[12] J. H. Bramble, J. E. Pasciak, and P. S. Vassilevski, *Computational scales of Sobolev norms with application to preconditioning*, Math. Comp., 69 (1999), pp. 69–463.

[13] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, Berlin, second ed., 2002.

[14] H.C. Brinkman, *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles*, Appl. Sci. Res., A1 (1947), pp. 27–34.

[15] ——, *On the permeability of media consisting of closely packed porous particles*, Appl. Sci. Res., A1 (1947), pp. 81–86.

[16] J. P. Caltagirone, *Sur l'interaction fluide-milieux poreux: application au calcul des efforts exercés sur un obstacle par un fluide visqueux*, in C.R. Acad. Sci. Paris, vol. 318 of Série II, Paris, 1994, pp. 571–577.

[17] G. Carbou, *Brinkman model and double penalization method for the flow around a porous thin layer*, J. Math. Fluid Mech., 10 (2008), pp. 126–158.

[18] G. Carbou and P. Fabrie, *Boundary layer for a penalization method for incompressible flow*, Adv. Differential Equations, 8 (2003), pp. 1453–1480.

[19] D.B. Das, *Hydrodynamic modelling for groundwater flow through permeable reactive barriers*, Hydrol. Process., 16 (2002), pp. 3393–3418.

[20] R.K. Dash, K.N. Mehta, and G. Jayaraman, *Casson fluid flow in a pipe filled with a homogeneous porous medium*, Int. J. Engng. Sci., 34 (1996), pp. 1145–1156.

[21] Y. Duan, W. Wang, and X. Yang, *The approximation of a Crank-Nicolson scheme for the stochastic Navier-Stokes equations*, J. Comput. Appl. Math., 225 (2009), pp. 31–43.

[22] A. Einstein, *On the method of theoretical physics*, Philos. Sci., 1 (1934), pp. 163–169.

[23] R.E. Ewing, O. Iliev, R.D. Lazarov, I. Rybak, and J. Willems, *A simplified method for upscaling composite materials with high contrast of the conductivity*, SIAM J. Sci. Comput., (2009), pp. 2568–2586.

[24] R. Farwig, H. Kozono, and H. Sohr, *Global weak solutions of the Navier-Stokes equations with nonhomogeneous boundary data and divergence*, Rend. Semin. Ma. Univ. P., (2010).

[25] E. Feireisl, J. Neustupa, and J. Stebel, *Convergence of a Brinkman-type penalization for compressible fluid flows*, J. Differential Equations, 250 (2011), pp. 596–606.

[26] S. Frandsen, R. Barthelmie, S. Pryor, O. Rathmann, S. Larsen, J. Højstrup, and M. Thøgersen, *Analytical modelling of wind speed deficit in large offshore wind farms*, Wind Energy, 9 (2006), pp. 39–53.

[27] A.V. Fursikov, M.D. Gunzburger, and L.S. Hou, *Inhomogeneous boundary value problems for the three-dimensional evolutionary Navier-Stokes equations*, J. Math. Fluid Mech., 4 (2002), pp. 45–75.

[28] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, vol. I, Springer-Verlag, New York, 1994.

[29] ——, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, vol. II, Springer-Verlag, New York, 1994.

[30] G.P Galdi, *An Introduction to the Navier-Stokes Initial-Boundary Value Problem*, Birkhauser, Basel, New York, 2000.

[31] M. Gazzola, O.V. Vasilyev, and P. Koumoutsakos, *Shape optimization for drag reduction in linked bodies using evolution strategies*, Comput. Struct., In Press, Corrected Proof (2010), pp. –.

[32] V. Girault and J.-L. Lions, *Two-grid finite-element schemes for the transient Navier-Stokes problem*, Model. Math. Anal. Numer., 35 (2001), pp. 945–980.

[33] V. Girault and P.A. Raviart, *Finite Element Approximations of the Navier-Stokes Equations*, Lecture Notes in Mathematics, Springer-Verlag, New York, 1979.

[34] ——, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.

[35] P. M. Gresho, R. L. Sani, and M. S. Engelman, *Incompressible Flow and the Finite Element Method*, John Wiley & Sons, LTD, New York, 1998.

[36] M. Griebel and M. Klitz, *Homogenization and numerical simulation of flow in geometries with textile microstructures*, SIAM MMS, 8 (2010), pp. 1439–1460.

[37] M. D. Gunzburger and J. S. Peterson, *On conforming finite element methods for the inhomogeneous stationary Navier-Stokes equations*, Numer. Math., 42 (1983), pp. 173–194.

[38] V. Gurau, Hongtan Liu, and S. Kakac, *Two-dimensional model for proton exchange membrane fuel cells*, AIChE J., 44 (1988), pp. 2410–2422.

[39] M.H Hamdan, *Single-phase flow through porous channels a review of flow models and channel entry conditions*, Appl. Math. Comput., 62 (1994), pp. 203–222.

[40] Y. He, *Two-level method based on finite element and Crank-Nicolson extrapolation for the time-dependent Navier-Stokes equations*, SIAM J. Numer. Anal., 41 (2003), pp. 1263–1285.

[41] Y. He and W. Sun, *Stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the time-dependent Navier-Stokes equations*, Math. Comput., 76 (2007), pp. 115–136.

[42] ——, *Stabilized finite element method based on the Crank-Nicolson extrapolation scheme for the time-dependent Navier-Stokes equations*, Math. Comput., 76 (2007), pp. 115–136.

[43] J.G Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem, I. Regularity of solutions and second-order spatial discretizations*, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[44] ——, *Finite element approximation of the nonstationary Navier-Stokes problem, II. Stability of solutions and error estimates uniform in time*, SIAM J. Numer. Anal., 23 (1986), pp. 750–777.

[45] ——, *Finite element approximation of the nonstationary Navier-Stokes problem, IV. Error analysis for second-order time discretization*, SIAM J. Numer. Anal., 19 (1990), pp. 275–311.

[46] J.G Heywood and O.D. Walsh, *A counter-example concerning the pressure in the Navier-Stokes equations, as $t \to 0^+$*, Pacific J. Math., 164 (1994), pp. 351–359.

[47] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr., (1950-1951), pp. 213–231.

[48] ——, *On non-linear partial differential equations*, Lecture Series of the Symposium on Partial Differential Equations, (1955).

[49] U. Hornung, ed., *Homogenization and porous media*, vol. 6, Springer, Berlin, 1997.

[50] Y. Hou and Q. Liu, *A two-level finite element method for the Navier-Stokes equations based on a new projection*, Applied Mathematical Modelling, 34 (2010), pp. 383–399.

[51] Y. Huang and M. Mu, *An alternating Crank-Nicolson method for decoupling the Ginzburg-Landau equations*, SIAM J. Numer. Anal., 35 (1998), pp. 1740–1761.

[52] O. Iliev and V. Laptev, *On numerical simulation of flow through oil filters*, Comput. Vis. Sci., 6 (2004), pp. 139–146.

[53] V. John, *A comparison of parallel solvers for the incompressible Navier-Stokes equations*, Computing and Visualization in Science, 1 (1999), pp. 193–200.

[54] ——, *Reference values for drag and lift of a two-dimensional time-dependent flow around a cylinder*, Int. J. Numer. Meth. Fl., 44 (2004), pp. 777–788.

[55] V. John, M. Gunar, and J. Rang, *A comparison of time-discretization/linearization approaches for the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 5995–6010.

[56] M. Kaviany, *Principles of Heat Transfer in Porous Media*, Springer-Verlag, New York, 1991.

[57] G.H. Keetels, U. D'Ortona, W. Kramer, H.J.H. Clercx, K. Schneider, and G.J.F. van Heijst, *Fourier spectral and wavelet solvers for the incompressible Navier-Stokes equations with volume-penalization: Convergence of a dipole-wall collision*, J. Comput. Phys., 227 (2007), pp. 919–945.

[58] N.K.-R. Kevlahan and J.M. Ghidaglia, *Computation of turbulent flow past an array of cylinders using a spectral method with Brinkman penalization*, Eur. J. Mech. B Fluids, 20 (2001), pp. 333– 350.

[59] M.-H. Kim, H.-S. Lim, and W.J. Lee, *Computational fluid dynamics assessment of the local hot core temperature in a pebble-bed type very high temperature reactor*, J. Eng. Gas Turb. Power, 131 (2009), p. 012905.

[60] A. Kusiak and Z.Zhe Song Song, *Design of wind farm layout for maximum wind energy capture*, Renewable Energy, 35 (2010), pp. 685–694.

[61] A. Labovsky, W. Layton, C. Manica, M. Neda, and L. Rebholz, *The stabilized, extrapolated trapezoidal finite element method for the Navier-Stokes equations*, Comput. Methods Appl. Mech. Eng., 198 (2009), pp. 958–974.

[62] W. Layton, *A two-level discretization method for the Navier-Stokes equations*, Comput. Math Appl., 26 (1993), pp. 33–38.

[63] W. Layton and M. Neda, *A similarity theory of approximate deconvolution models of turbulence*, J. Math. Anal. Appl., 333 (2007), pp. 416–429.

[64] W.J. Layton, F. Schieweck, and I. Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal., 40 (2003), pp. 2195–2218.

[65] W. Layton and L. Tobiska, *A two-level method with backtracking for the Navier-Stokes equations*, SIAM J. Numer. Anal., 35 (1998), pp. 2035–2054.

[66] J.-Y. Lee and S.-Y. Lee, *Flow visualization in the scaled up pebble bed of high temperature gas-cooled reactor using particle image velocimetry method*, J. Eng. Gas Turbines Power, 131 (2009), p. 064502.

[67] J. Leray, *Essai sur les Mouvements Plan d'un Liquide Visqueux que Limiten des Parois*, J. Math. Pures Appli., 13 (1934), p. 331.

[68] ——, *Sur le Mouvements d'un Liquide Visqueux Emplissant l'Espace*, Acta. Math., 63 (1934), p. 193.

[69] A. Llobet, X. Gasull, and A. Gual, *Understanding Trabecular Meshwork Physiology: A Key to the Control of Intraocular Pressure*, News Phys. Sci, 18 (2003), pp. 205–209.

[70] W. Lu, C.Y. Zhao, and S.A. Tassou, *Thermal analysis on metal-foam filled heat exchangers. Part I: Metal-foam filled pipes*, Int. J. Heat Mass Transf., 49 (2006), pp. 2751–2761.

[71] K. Matsuzaki, M. Munekata, H. Ohba, and Ushijima, *Numerical study on particle motions in swirlling flows in a cylinder separator*, J. Therm. Sci., 15 (2006), pp. 181–186.

[72] W.J. Meyer, *Concepts of Mathematical Modeling*, Dover Publications, New York, 2004.

[73] G. Neale and W. Nader, *Practial significance of Brinkman's extension of Darcy's Law: Coupled parallel flows within a channel and a bounding porous medium*, Can. J. Chem. Eng., 52 (1974), pp. 475–478.

[74] J. von Neumann, *The mathematician*, in The Works of the Mind, The University of Chicago Press, Chicago, Ill., 1947, pp. 180–196. Edited for the Committee on Social Thought by Robert B. Heywood.

[75] D.A. Nield, *Alternative models of turbulence in a porous medium, and related matters*, J. Fluids Eng., 123 (2001), pp. 928–931.

[76] M.R. Ohm, H.Y. Lee, and J.Y. Shin, *L^2-error estimates of the extrapolated Crank-Nicolson discontinuous Galerkin approximations for nonlinear Sobolev equations*, J. Inequal. Appl., 2010 (2010), pp. 1–17.

[77] I. Pop and B. Derek, *Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Media*, Pergamon, New York, first ed., 2001.

[78] J.-P. Raymond, *Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions*, Anal. Non Linéaire, 24 (2007), pp. 921–951.

[79] ——, *Stokes and Navier-Stokes equations with a nonhomogeneous divergence condition*, Discret. Contin. Dyn. S. B, 14 (2010), pp. 1537–1564.

[80] D. Rossinelli, M. Bergdorf, G. Cottet, and P. Koumoutsakos, *GPU accelerated simulations of bluff body flows using vortex particle methods*, J. Comput. Phys., 229 (2010), pp. 3316–3333.

[81] P.G. Saffman, *On the boundary condition at the surface of a porous medium*, Stud. Appl. Math., L (1971), pp. 93–101.

[82] K. Schneider and M. Farge, *Numerical simulation of the transient flow behaviour in tube bundles using a volume penalization method*, J. Fluids Structures, 20 (2005), pp. 555–566.

[83] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[84] J.N. Sorensen and W.Z Shen, *Numerical modeling of wind turbine wakes*, J. Fluids Eng., 124 (2002), pp. 393–399.

[85] B. Straughan, *Stability and Wave Motion in Porous Media: Applied Mathematical Sciences*, Springer, New York, first ed., 2008.

[86] M. Tabata and D. Tagami, *Error estimates for finite element approximations of drag and lift in nonstationary Navier-Stokes flows*, Japan J. Indust. Appli. Math., 17 (2000), pp. 371–389.

[87] Shigeru Tada and J.M. Tarbell, *Interstitial flow through the internal elastic lamina affects shear stress on arterial smooth muscle cells*, Am. J. Physiol Heart Circ. Physiol, 278 (2000), pp. H1589–H1597.

[88] X. Xie, J. Xu, and G. Xue, *Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models*, J. Comp. Math., 26 (2008), pp. 437–455.

[89] I. Zuhaila and Alistair F. D., *Mathematical modelling of flow in Schlemm's canal and its influence on primary open angle glaucoma*, in International Converence on Science & Technology: Applications in Industry & Education (2008), Universiti Teknologi MARA, 2008, pp. 1967–1973.