# A STUDY OF CURRENT-DEPENDENT RESISTORS IN NONLINEAR CIRCUITS

by

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# ABSTRACT

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Nonlinear electrical circuits can be used to model fluid flow in pipe networks when the resistance of any element in the network is assumed to be dependent on the flow rate through that element. This relationship is often assumed in classical models of pressure drops at orifices and through valves. More recently, it has also been used to model blood flow through vessels, and may potentially have applications in nano-fluid systems. Motivated by these applications, in this thesis we investigate circuits where the resistors have linear and affine (linear plus offset) dependence on current. Rules for their reduction in series and parallel are derived for the general case as well as for special cases of their linear coefficients and offset terms. Other adapted circuit analysis and manipulation techniques are also discussed, including mesh current analysis and delta-wye transformation, and avenues for further investigation of this topic are illuminated. The methods developed in this thesis may have potential applications in simplifying the analysis of complex nonlinear flow networks in cardiovascular systems, especially those at the nano-scale.

## ACKNOWLEDGEMENTS

I would like to thank Dr. Marwan Simaan for providing me with guidance, advice and constructive criticism during my first two graduate years at the University of Pittsburgh, for encouraging me to enrich my studies with courses outside of the Department and for introducing me to the interesting research topic upon which this thesis is based. I would also like to thank the members of my thesis defense committee for generously giving their time and for providing some interesting perspectives on this research.

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## 1.0 A NONLINEAR CIRCUIT MODEL: MOTIVATIONS AND IMPLICATIONS

Traditional education in electrical engineering begins with coursework in linear circuits and systems, and the behavior of the ordinary linear resistor is usually presented to relate the concepts of voltage and current [7], [8]. It is common knowledge that practical resistors display near-ideal behavior, that they obey Ohm's Law and that, for all intents and purposes, the value of their resistance is constant. It is an obvious conclusion to say that, in its simplest form, the resistance of a resistor in a typical non-switching direct current circuit is not directly dependent on voltage, current or time. This investigation, motivated by modeling applications in some biological and nano-scale fluid flow systems, generalizes this concept and considers several cases in which resistance is a function of current. That is, $R=R(I)$. The following equation assumes a polynomial dependence of resistance on current:

$$R = R(I) = c + d_1 I + d_2 I^2 + \ldots + d_n I^n. \tag{1.1}$$

The traditional linear resistor can be viewed as a special case of (1.1) with $R=c$ and $d_1=d_2=\ldots=d_n=0$.

In a liquid flow network, the pressure drop $\Delta P$ across a length of pipe is usually modeled as having a nonlinear algebraic relationship with the flow rate $Q$ [4]. To approximate a pressure drop at an orifice, through a valve or through a pipe with turbulent flow, the following expression is often used [4], [6]:

$$\Delta P = kQ^2 \tag{1.2}$$

where

$\Delta P$ = pressure difference in newtons per square meter [N/m$^2$],

$Q$ = flow rate in cubic meters per second [m$^3$/s],

$k$ = a proportionality constant that depends on the pipe.

The relationship in (1.2) is typically derived from the Bernoulli equation, and the fluid being described is usually assumed to be inviscid and incompressible [5], [13]. Series and parallel pipe networks implementing models such as that presented in (1.2) are often linearized around an operating point using a Taylor series expansion. The pursuit in this research is to develop methods for dealing with nonlinear models to their manageable limit without linearizing. Some interesting nonlinear networks and useful linearization methods have previously been investigated in [2] and [3]. An electric circuit analog to the above model can be created using nonlinear resistors, where $\Delta P$ is similar to a voltage $V$ and $Q$ is similar to current $I$, as in

$$V = dI^2. \tag{1.3}$$

If we use Ohm's law $V=RI$ to model these circuits, then clearly the resulting resistor in this case would be linearly dependent on the current flowing in it. That is

$$R = dI. \tag{1.4}$$

Simple nonlinear circuit elements obeying laws such as (1.3) are often explored in system dynamics textbook problem sets. A model similar to (1.3) for the pressure drop in a percutaneous ventricular assist system has been developed in [12], but it adds a constant offset to the resistive element, resulting in an affine dependency on current:

$$R = c + dI. \tag{1.5}$$

The appropriate relationship between pressure drop and flow rate becomes

$$\Delta P = cQ + dQ^2, \tag{1.6}$$

and the analogous voltage dependence on current becomes

$$V = cI + dI^2. \tag{1.7}$$

To serve as a realistic model for fluid flow, and to avoid the possibility of encountering a negative resistance value, current-dependent resistors must be dependent on the absolute value of current. That is, equations 1.4 and 1.5 should be $R=d|I|$ and $R=c+d|I|$, respectively. For simplicity, the offset ($c$) and linear ($d$) coefficients are assumed to be nonnegative in this thesis. The units for the constant coefficient $c$ would clearly be Ohms ($\Omega$), just like typical linear resistors, and the units for the linear coefficient $d$ would be Ohms per amp ($\Omega$/A). The results of these requirements are graphically summarized in the following two figures.

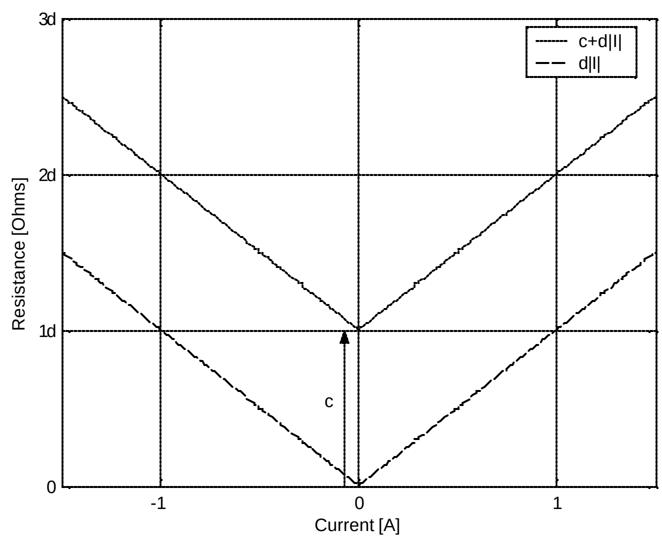


**Figure 1.1** Plot of linear and affine current dependent resistance.

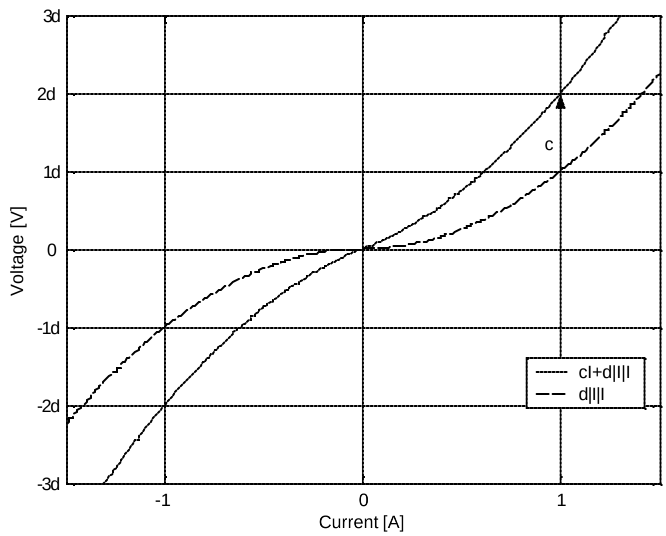


**Figure 1.2** Plot of voltage across resistors with linear and affine current dependence.

In this investigation we will assume all series and parallel resistive circuits obey the passive convention, and only current flow in the positive direction will be analyzed, allowing for the use of the simpler equations 1.4 and 1.5. Therefore, we will not use the absolute value notation throughout the remainder of this thesis, unless the possibility exists of the current reversing its direction of flow.

Another justification for modeling nonlinear circuit elements that obey laws such as (1.7) can be derived by assuming, quite naturally, that the voltage drop across an element is some arbitrary function of the current through it. That is, $V=f(I)$. This generalization is illustrated in Figure 1.3.

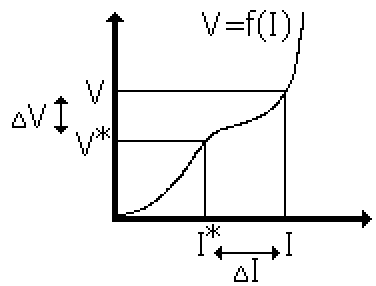


**Figure 1.3** *V* as an arbitrary function of *I*.

Approximating the function *f*(*I*) near $I^*$ with a second-order Taylor series expansion gives

$$f(I) = f(I^*) + \left.\frac{d}{dI}\right|_{I^*}(I - I^*) + \left.\frac{d^2}{dI^2}\right|_{I^*}\frac{(I - I^*)^2}{2!}. \tag{1.8}$$

It is obvious that $f(I)\text{-}f(I^*)=\Delta V$, and since the derivative terms evaluated at $I^*$ are constant coefficients, a relationship similar to (1.7) results:

$$\Delta V = c\Delta I + d(\Delta I)^2$$

where

$$c = \left.\frac{df}{dI}\right|_{I^*} \quad \text{and} \quad d = \frac{1}{2!}\left.\frac{d^2 f}{dI^2}\right|_{I^*}.$$

When one attempts to solve simple nonlinear electric circuits or reduce networks of current-dependent resistors, it quickly becomes apparent that such pursuits can be algebraically impossible. For example, consider the circuit in Figure 1.4, wherein two hypothetical resistors exhibiting affine current dependence are connected in parallel. The goal is to find an equivalent resistor that may be affine dependent on the total network current ($I_{tot}$) in order to reduce the two-element network down to one. The element parameters are summarized as follows:

$$R_0 = c_0 + d_0 I_0,$$
$$R_1 = c_1 + d_1 I_1,$$
$$R_{eq} = c_{eq} + d_{eq} I_{tot}.$$

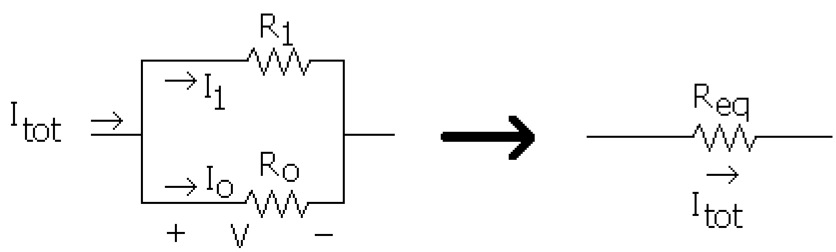


**Figure 1.4** Two current-dependent resistors in parallel.

$I_0$ and $I_1$ are the currents that pass through resistors $R_0$ and $R_1$, respectively and thus are the currents on which each resistance is dependent. The total current $I_{tot}$ into the parallel network would be given by $I_{tot}=I_0+I_1$. By Kirchoff's Voltage Law (KVL), the voltage drop ($V$, given by the passive convention) across each resistor is equal.

$$V = R_1 I_1 = R_0 I_0$$
$$V = (c_1 + d_1 I_1) I_1 = (c_0 + d_0 I_0) I_0$$

The next step towards finding $R_{eq}$ is to obtain an equation relating $V$ and $I_{tot}$. By solving the resultant quadratic equations relating $I_0$, $I_1$ and $V$, the following equation for $I_{tot}$ in terms of $V$ results:

$$I_{tot} = \left(-\frac{c_1}{2d_1}\right) \pm \sqrt{\left(\frac{c_1}{2d_1}\right)^2 + \left(\frac{V}{d_1}\right)} + \left(-\frac{c_0}{2d_0}\right) \pm \sqrt{\left(\frac{c_0}{2d_0}\right)^2 + \left(\frac{V}{d_0}\right)} \qquad (1.9)$$

Note this relationship is only valid when $d_0 \neq 0$ and $d_1 \neq 0$. The leftmost network node in Figure 1.4 must be made to obey Kirchoff's Current Law (KCL), and in doing so the subtractive cases can be discarded from (1.9). In other words, if $I_{\text{tot}}$ is assumed to enter, the exiting currents $I_0$ and $I_1$ must add positively on the right-hand side of the equation. This results in the following equation, the generalized current-voltage relationship between two resistors with affine current dependence in parallel.

$$I_{tot} = \left(-\frac{c_1}{2d_1}\right) + \sqrt{\left(\frac{c_1}{2d_1}\right)^2 + \left(\frac{V}{d_1}\right)} + \left(-\frac{c_0}{2d_0}\right) + \sqrt{\left(\frac{c_0}{2d_0}\right)^2 + \left(\frac{V}{d_0}\right)} \tag{1.10}$$

It is now obvious that the pursuit of a direct expression for $R_{\text{eq}}$ would require the consolidation of the variable $V$, which is present in two radical terms from which it is not necessarily separable. So, depending on the values of $c_1$, $c_0$, $d_1$ and $d_0$, this could be an algebraically impossible task.

In this thesis, we consider a dependence of the resistance on current only in the form $R=c+dI$. In chapter 2, the special case when $c=0$ is considered. This special case is much simpler to treat and provides a stepping-stone for the general case where $c \neq 0$, which is considered in chapter 3. Additional considerations of the general case are discussed in chapter 4, and some concluding remarks are given in chapter 5.

# 2.0 RESISTORS WITH LINEAR CURRENT DEPENDENCE

A purely linear dependence of resistance on current would be exemplified by equation 1.1 in the case that $c=d_2=d_3=\ldots=d_n=0$, and $d_1$ is nonzero. The current through one such resistor can be calculated as shown below, using the passive convention to relate voltage and current. Because the current is assumed to be flowing from the positive node to the negative node of the resistor, the negative root can be thrown out.

$$V = RI$$
$$V = (d_1 I)I$$
$$I = \sqrt{\frac{V}{d_1}}$$

Such elements will be dubbed "linearly dependent resistors" (or LDRs) for the rest of this thesis. In this chapter, we consider adding $n$ such resistors first in series, then in parallel. Along with this, we investigate the resulting voltage and current divider rules. We also provide several examples to illustrate the various steps involved.

## 2.1 Linearly Dependent Resistors in Series

Consider $n$ of these resistors in series, as in Figure 2.1, where each resistor $R_i = d_i I$. The equivalent resistance $R_{eq}$ for such a network is desired.

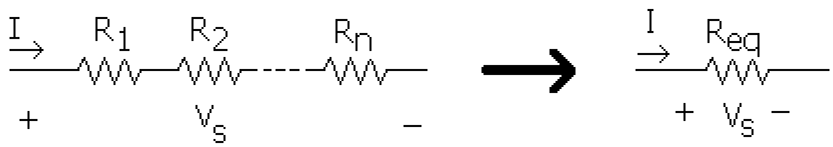


**Figure 2.1** The reduction of $n$ LDRs in series. Each resistor $R_i = d_i I$.

Each voltage $V_i$ drop across resistor $R_i$ can be expressed as $V_i = d_i I^2$, and KVL applied around the circuit results in the following relationship:

$$V_s = d_1 I^2 + d_2 I^2 + \ldots + d_n I^2,$$

which yields

$$R_{eq} = \frac{V_s}{I} = \left(d_1 + d_2 + \ldots + d_n\right) I.$$

Therefore, the **equivalent resistance of *n* linearly LDRs in series** is

$$R_{eq} = \left(\sum_{i=1}^{n} d_i\right) I, \tag{2.1}$$

and an equivalent linear coefficient can be expressed as

$$d_{eq} = \sum_{i=1}^{n} d_i .$$

In this case, $R_{eq}$ has been found to also be linearly dependent on the current through the series network, according to equation 2.1. A similar result for valves obeying equation 1.2 in series has been found before [4].

## 2.2 Voltage Divider Rule

Now we wish to derive a voltage divider rule for LDRs in series as in Figure 2.1. Again, the voltage across resistor $R_i$ is found using $V_i = d_i I^2$, and an expression for $I$ can be found:

$$I^2 = \frac{V_s}{(d_1 + d_2 + ... + d_n)}.$$

The resulting form for the **voltage divider rule for linearly dependent resistors** is described by:

$$V_i = \frac{d_i}{(d_1 + d_2 + ... + d_n)} V_s. \qquad (2.2)$$

Upon examination, this relationship is very similar to the voltage divider rule for current-independent resistors, in that the source voltage divides as the ratio of each linear coefficient to the sum of all coefficients. Thus, the rule can be stated as follows:

**The voltage across *n* LDRs in series divides proportional to the ratio of each coefficient $d_i$ to the sum of all the coefficients of the series combination.**

To illustrate these ideas, consider the simple series circuit in Figure 2.2. The current $I$ passes through the two LDRs in the clockwise direction and is calculated to be $(10/8)^{1/2}$, or 1.118A.

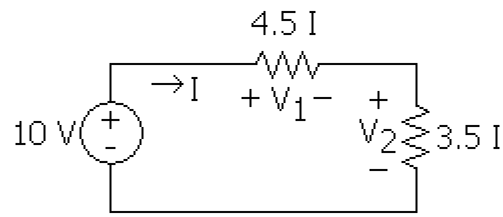


**Figure 2.2** Example circuit containing LDRs to illustrate the voltage divider rule.

The total series resistance in the circuit is (4.5+3.5)$I$, or 8$I$. Using the voltage divider rule for LDRs, the voltages $V_1$ and $V_2$ are calculated to be:

$$V_1 = \frac{4.5}{4.5+3.5}10 = 5.625\text{V},$$
$$V_2 = \frac{3.5}{4.5+3.5}10 = 4.375\text{V}.$$

## 2.3 Linearly Dependent Resistors in Parallel

Now consider $n$ LDRs in parallel, again where $R_i = d_i I_i$ ($d_i$ nonnegative for simplicity). Referring to Figure 2.3, the equivalent resistance $R_{eq}$ is desired, as is a generalized current divider rule for such a network.

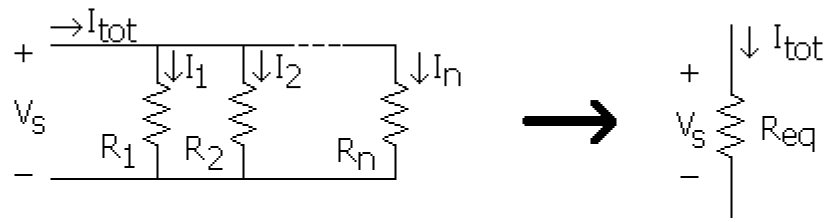


**Figure 2.3** The reduction of $n$ LDRs in parallel. Each resistor $R_i = d_i I_i$

In order to obtain an equivalent resistance for this network, KCL must be applied at the upper node, and an expression directly relating $V_s$ and the total current $I_{tot}$ must be derived.

$$I_i = \sqrt{\frac{V_s}{d_i}}$$

$$I_{tot} = I_1 + I_2 + \ldots + I_n = \sqrt{V_s}\left(\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}} + \ldots + \frac{1}{\sqrt{d_n}}\right)$$

$$V_s = \frac{I_{tot}{}^2}{\left(\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}} + \ldots + \frac{1}{\sqrt{d_n}}\right)^2}$$

The above equations reveal an expression for the **equivalent resistance of *n* linearly dependent resistors in parallel**, which follows as

$$R_{eq} = \frac{1}{\left(\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}} + \ldots + \frac{1}{\sqrt{d_n}}\right)^2} I_{tot}. \tag{2.3}$$

The equivalent linear coefficient is therefore

$$d_{eq} = \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}\right)^2}.$$

Again, the equivalent resistance is found to be linearly dependent upon the total current $I$, but the form of its coefficient differs slightly from the current-independent analog. We mention that in the special case of two LDRs in parallel, expression (2.3) simplifies to:

$$d_{eq} = \frac{d_1 d_2}{\left(\sqrt{d_1} + \sqrt{d_2}\right)^2}.$$

Now consider the example of three LDRs in a parallel network such as that of Figure 2.3. Suppose the three resistors, which are dependent on their respective currents, $I_1$, $I_2$ and $I_3$, are valued $10I_1$, $50I_2$ and $100I_3$. Applying expression (2.3), we find the equivalent LDR to be

$$R_{eq} = \frac{1}{\left(\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{50}} + \ldots + \frac{1}{\sqrt{100}}\right)^2} I_{tot}$$
$$R_{eq} = 3.216 I_{tot}.$$

The following figure illustrates the relationship of $R_{eq}$ to $R_1$, $R_2$ and $R_3$ as a function of current.

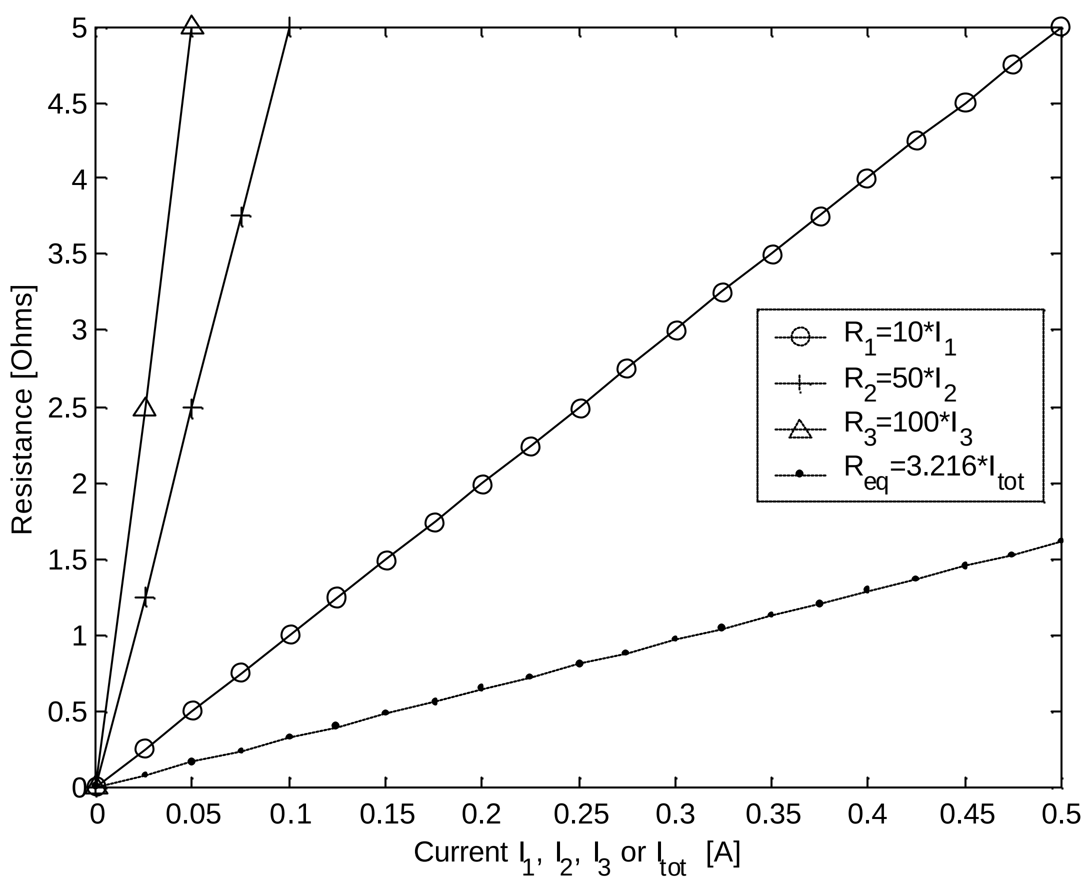


**Figure 2.4** Equivalent resistance calculated for three example LDRs.

## 2.4 Current Divider Rule

The development of a current divider rule for LDRs is now possible. The current $I_i$ through resistor $R_i$, after disregarding the negative case, can be expressed as

$$I_i = \sqrt{\frac{R_{eq} I_{tot}}{d_i}}\,.$$

Equation 2.3 is then substituted into the above equation:

$$I_i = \sqrt{\frac{{I_{tot}}^2}{d_i \left( \sum_{j=1}^{n} \frac{1}{\sqrt{d_j}} \right)^2}}\,.$$

It is now clear that the following expression can be considered the **current divider rule for LDRs**:

$$I_i = \frac{1}{\sqrt{d_i} \left( \sum_{j=1}^{n} \frac{1}{\sqrt{d_j}} \right)} I_{tot}\,. \tag{2.4}$$

The current divider rule can therefore be stated as follows:

> **The current through *n* LDRs in parallel divides proportional to the inverse of the product of each coefficient's square root and the sum of all of the inverted coefficients' square roots.**

We mention that in the special case of two LDRs in parallel (2.4) reveals the following expressions for the two currents:

$$I_1 = \frac{\sqrt{d_2}}{\sqrt{d_1} + \sqrt{d_2}} I_{tot},$$
$$I_2 = \frac{\sqrt{d_1}}{\sqrt{d_1} + \sqrt{d_2}} I_{tot}.$$

These expressions are similar to those found with current-independent resistors, except that they involve the square roots of the linear coefficients. Other relationships can be deduced for LDRs in a similar fashion. For example, the **power consumed by a linearly dependent resistor** turns out to be cubic in $I$:

$$P_i = V_i \cdot I_i = d_i \left| I_i \right| I_i^2. \qquad (2.5)$$

Maintaining a positive resistor value despite current direction avoids the unwanted situation wherein the power consumed is negative.

The calculations required to derive the rules and relations above were algebraically simple, because there was essentially only one quantity left underneath the radicals in equation 1.3. The complications that arise when this is no longer the case are the motivation for chapter 4. First, a numerical example of the concepts developed above is presented.

## 2.5 Example Problem

To illustrate all the above concepts, let us consider the circuit shown in Figure 2.5. It is desired to determine all the currents and voltages in this circuit.

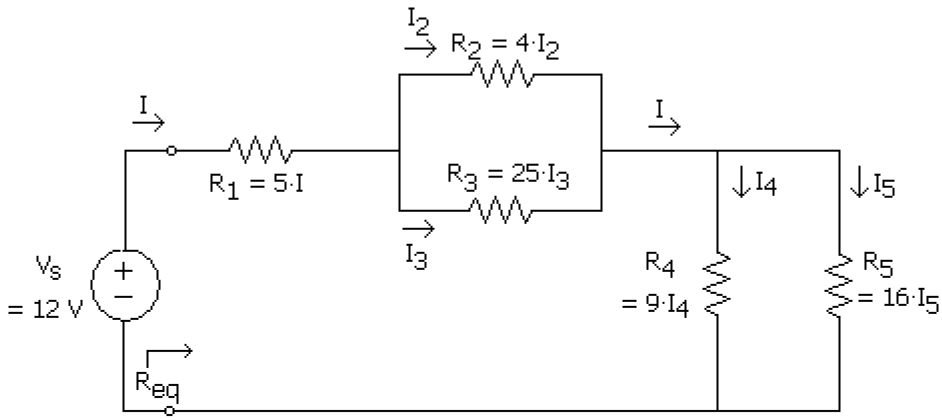


**Figure 2.5** Example circuit containing LDRs.

To find the equivalent resistance $R_{eq}$ for the entire resistive network, one must begin by finding $R_{2,3}$ and $R_{4,5}$, the equivalent resistances for the two pairs of parallel resistors. Using equation 2.3, we have:

$$R_{2,3} = \left( \frac{1}{\sqrt{d_2}} + \frac{1}{\sqrt{d_3}} \right)^{-2} I$$
$$R_{2,3} = \left( \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{25}} \right)^{-2} I = \left( \frac{100}{49} \right) I.$$

Similarly,

$$R_{4,5} = \left( \frac{144}{49} \right) I.$$

Then, equation 2.1 must be used to reduce the resulting series network and obtain $R_{eq}$.

$$R_{eq} = R_1 + R_{2,3} + R_{4,5}$$
$$R_{eq} = \left[ 5 + \left( \frac{100}{49} \right) + \left( \frac{144}{49} \right) \right] I = \left( \frac{489}{49} \right) I$$
$$R_{eq} \approx 10I$$

The currents $I_2$, $I_3$, $I_4$ and $I_5$ can be found using the current divider rule of LDRs. First, the total source current I must be calculated.

$$I = \frac{V_s}{R_{eq}}$$

$$I = \sqrt{\frac{12 \cdot 49}{489}} = 1.1\text{A}$$

Then equation 2.4 is implemented to find each current. For example,

$$I_3 = \frac{\sqrt{4}}{\sqrt{4} + \sqrt{25}} I$$

$$I_3 = \frac{2}{7} I$$

$$I_3 = 0.31\text{A}.$$

In the same way, $I_2$, $I_4$ and $I_5$ are found to be 0.78A, 0.57A and 0.43A, respectively. These results are intuitive, as we expect smaller portions of the total current to flow through the resistors with larger linear dependence.

The voltage divider rule (equation 2.2) is now utilized across the outside loop which consists of $R_1$, $R_{2,3}$ and $R_{4,5}$ to find each of the series voltages in the circuit. For example,

$$V_{R4,5} = \left[ \frac{\left(\frac{144}{49}\right)}{5 + \left(\frac{144}{49}\right) + \left(\frac{100}{49}\right)} \right] 12$$

$$V_{R4,5} = 3.53\text{V}.$$

Similarly, $V_{R2,3}$ and $V_{R1}$ are found to be 2.45V and 6.01V, respectively.

# 3.0 RESISTORS WITH AFFINE CURRENT DEPENDENCE

An affine dependence of resistance on current would be described by (1.5) when the values $c$ and $d$ are nonzero. In a network of $n$ such resistors, each resistance will be of the form $R_i = c_i + d_i I_i$. If the constant $c_i$ is zero the resistor is of the type discussed in chapter 2 (linearly dependent on current), and if the constant $d_i$ is zero the resistor is of the normal type found in classical linear circuit theory (no dependence on current). It is anticipated that any rules or reduction techniques found for these resistors would be even more general forms of those describing the behavior of both linearly dependent and current-independent resistors. Similarly, if a network was composed of a combination of both linearly dependent and current-independent resistors, it would be possible to solve using any techniques discovered for these resistors with affine (linear plus offset) dependence on current. The resistors discussed in this chapter will intuitively be dubbed “affine dependent resistors” (or ADRs) for the rest of this thesis.

In this chapter, we consider combining $n$ ADRs first in series, then in parallel. It is found that parallel networks of such resistors cannot be generalized analytically, so specific cases of the individual resistor coefficients are investigated for reducibility. A general voltage divider rule is found for these resistors in series, and the rules for parallel current division for each reducible case are also found. We also provide several examples to illustrate the various steps involved.

### 3.1 Affine Dependent Resistors in Series

Consider $n$ ADRs in series, where each resistor $R_i = c_i + d_i I_i$, as in the following figure. Note that all of the individual currents ($I_i$) through the resistors, upon which they are each dependent, are equal to the same single branch current ($I$).

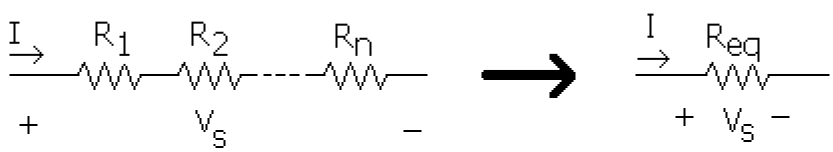


**Figure 3.1** The reduction of $n$ ADRs in series. Each resistor $R_i = c_i + d_i I$.

Each voltage drop $V_i$ across resistor $R_i$ can be expressed as $V_i = R_i I = c_i I + d_i I^2$, assuming the current is flowing in the reference direction indicated in Figure 3.1. KVL applied around the circuit results in the following relationships:

$$V_s = V_1 + V_2 + \ldots + V_n,$$
$$V_s = (c_1 + c_2 + \ldots c_n)I + (d_1 + d_2 + \ldots d_n)I^2.$$

This yields an expression for the equivalent series resistance:

$$R_{eq} = \frac{V_s}{I} = (c_1 + c_2 + \ldots c_n) + (d_1 + d_2 + \ldots d_n)I.$$

Therefore, the **equivalent resistance of *n* affine dependent resistors in series** is, in closed form,

$$R_{eq} = \sum_{i=1}^{n} c_i + \left(\sum_{i=1}^{n} d_i\right) I, \tag{3.1}$$

and the equivalent linear and offset coefficients for $R_{eq}=c_{eq}+d_{eq}I$ can be expressed as

$$c_{eq} = \sum_{i=1}^{n} c_i \text{ and } d_{eq} = \sum_{i=1}^{n} d_i \ .$$

In this case, $R_{eq}$ has been found to also be affine dependent on the current through the series network, according to equation 3.1. This expression is similar to the results obtain for both current-independent resistors and LDRs in series, and can be used to compute the equivalent resistance of a series network which includes both of those types of resistors.

### 3.2 Voltage Divider Rule

As was previously done for linearly dependent resistors, we now consider deriving a voltage divider rule for the case of affine dependence. Any voltage $V_i$ across a particular affine dependent resistor $R_i$ should be possible to find by simply using the resistor coefficients and the source voltage $V_s$. It will be seen that the result is not quite as easy to obtain as that of the previously investigated case (equation 2.2). The first intuitive step is to determine an expression for the loop current $I$.

$$I = \frac{V_s}{\sum_{i=1}^{n} c_j + \left( \sum_{j=1}^{n} d_j \right) I}$$

$$\left( \sum_{j=1}^{n} d_j \right) I^2 + \left( \sum_{i=1}^{n} c_j \right) I - V_s = 0$$

This is a second-order equation in $I$, and its two roots are found to be

$$I = \frac{-\sum_{j=1}^{n} c_j \pm \sqrt{\left(\sum_{j=1}^{n} c_j\right)^2 + 4\left(\sum_{j=1}^{n} d_j\right)V_s}}{2 \cdot \left(\sum_{j=1}^{n} d_j\right)}.$$

By assuming that the current $I$ flows into the positive node of the voltage $V_s$ in the series network of ADRs (as is done with current-independent resistors) the subtractive case above can be discarded. Each resistor voltage $V_i$ can then be calculated by simply multiplying the above result for the loop current by the value of the resistor in question, $R_i = c_i + d_i I$. It is observed that the indirect calculation of the current in such a series circuit is unavoidable, as its substitution into Ohm's Law does not simplify. This produces the following expression for the **voltage divider rule for affine dependent resistors**:

$$V_i = (c_i + d_i I)I \tag{3.2}$$

where

$$I = \frac{-\sum_{i=1}^{n} c_j + \sqrt{\left(\sum_{i=1}^{n} c_j\right)^2 + 4\left(\sum_{i=1}^{n} d_j\right)V_s}}{2\left(\sum_{j=1}^{n} d_j\right)}.$$

We can state the rule as follows:

**The voltage drop across one resistor of *n* ADRs in series is found by using Ohm's Law with the total series current, which is determined to be a root of a quadratic equation whose constant term is the source voltage**.

As an illustrative example of the above concepts, consider the series connection of a DC voltage source two ADRs in Figure 3.2. We wish to determine the equivalent resistance of the two series resistors, the loop current $I$ and the voltage drop across each resistor $V_1$ and $V_2$.

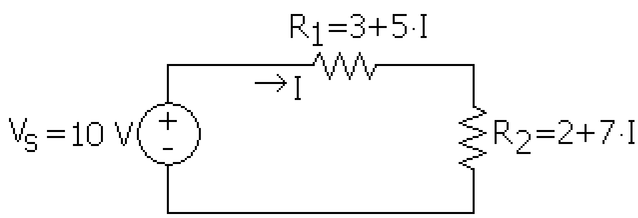


**Figure 3.2** Circuit for example problem.

Using equation 3.1, the equivalent series resistance is found to be (3+2)+(5+7)$I$=5+12$I$. By the expression in (3.2), the total current is found to be

$$I = \frac{-(3+2)+\sqrt{(3+2)^2+4(5+7)10}}{2(5+7)}$$
$$I = 0.728\text{A}.$$

Again, using equation 3.2,

$$V_1 = (3+5\cdot 0.728)0.728 = 4.834\text{V},$$
$$V_2 = (2+7\cdot 0.728)0.728 = 5.166\text{V}.$$

### 3.3 Affine Dependent Resistors in Parallel

In chapter 1 the generalized current-voltage relationship for two ADRs ($R_0=c_0+d_0I_0$ and $R_1=c_1+d_1I_1$) in parallel was found to be in the form of the following equation, where $I_{\text{tot}}$ is the current into their common node and $V$ is the voltage across the parallel network:

$$I_{tot} = \left(-\frac{c_0}{2d_0}\right)+\sqrt{\left(\frac{c_0}{2d_0}\right)^2+\left(\frac{V}{d_0}\right)}+\left(-\frac{c_1}{2d_1}\right)+\sqrt{\left(\frac{c_1}{2d_1}\right)^2+\left(\frac{V}{d_1}\right)}. \tag{3.3}$$

Note that if this expression is valid only if $d_0 \neq 0$ and $d_1 \neq 0$. Still, we will attempt to make progress toward a generalized solution. While attempting to extract an expression for $R_{\text{eq}}$ in this situation, one finds that collecting the $V$ terms can be algebraically complicated, as each is offset by different amounts underneath two different radicals. It is obvious that for $n$ such resistors in parallel, the above relationship would involve the addition of $n$ separate radical terms. Therefore, the values of the linear and offset coefficients contribute directly to the complexity, or the simplicity, of the algebraic reduction. In the following investigations of two (and more) ADRs in parallel, it will become apparent that an analytic expression for the equivalent parallel resistance $R_{\text{eq}}$ will cannot be obtained and that it will not always be affine dependent. First, characterization of the general case will be pursued, and the limited availability of closed-form results will become apparent. Then, after considering as much as we possibly can in the general case, we will examine only three special cases.

- In Case 1 (equal affine dependency), $R_{\text{eq}}$ for two equal ADRs is found and generalized to $n$ equivalent resistors.
- In Case 2 (equal linear coefficients), the $R_{\text{eq}}$ for two ADRs with equal linear coefficients and unequal constant terms is found.

- In Case 3 (special case involving a proportionality constant a), a scenario is examined where the offset and linear coefficients ($c_i$ and $d_i$) of $R_1$ through $R_{n-1}$ are related to those of some reference resistor ($R_0$) in some manner by a constant $a_i$.

Whenever possible, corresponding current divider rules are derived for the various cases under investigation.

### 3.3.1 General Case

A symbolic manipulation software package such as MATLAB© (The MathWorks, Inc.) can be used to find a direct expression for $V$ in equation 3.3. After solving for $V$ and collecting terms in $I_{tot}$, the two following solutions result, $V_1$ being additive and $V_2$ being subtractive (note $d_1 \neq d_0 \neq 0$):

$$V_{1,2} = \left[\frac{d_1^2 d_0 + d_1 d_0^2}{\left(-d_0 + d_1\right)^2}\right] I_{tot}^2$$
$$+\left[\frac{d_0 d_1 c_1 + d_1^2 c_0 + c_1 d_0^2 + d_0 d_1 c_0 \pm d_1 d_0 \times \left(c_1^2 + c_0^2 + 4I_{tot}^2 d_1 d_0 + 4I_{tot} d_0 c_1 + 4I_{tot} d_1 c_0 + 2c_0 c_1\right)^{\frac{1}{2}}}{\left(-d_0 + d_1\right)^2}\right] I_{tot}$$
$$+\frac{d_1 c_0^2 + c_0 d_1 c_1 \pm \left(c_0 d_1 + c_1 d_0\right)\left(c_1^2 + c_0^2 + 4I_{tot}^2 d_1 d_0 + 4I_{tot} d_0 c_1 + 4I_{tot} d_1 c_0 + 2c_0 c_1\right)^{\frac{1}{2}} + c_1^2 d_0 + c_1 d_0 c_0}{2\left(-d_0 + d_1\right)^2}.$$

This expression is not yet entirely simplified as a polynomial in $I_{tot}$. One way to simplify the expression is to reduce the quantity whose square root appears twice in the polynomial; we name this quantity $s(I_{tot})$ and display it in a collected form by the following equation:

$$s(I_{tot}) = \left[(4d_1 d_0) I_{tot}^2 + 4(d_0 c_1 + d_1 c_0) I_{tot} + \left(c_1^2 + 2c_0 c_1 + c_0^2\right)\right]^{\frac{1}{2}} \qquad (3.4)$$

One way we can exploit $s(I_{tot})$ to simplify the expressions for $V$ is by setting the discriminant of the enclosed quadratic equation to zero. This yields two real and equal roots, one of which can be canceled by the outer square root operation. It can be shown that the discriminant $D$ can be factored in the following way:

$$D = (d_0 - d_1)(-d_1 c_0^{\,2} + c_1^{\,2} d_0)\ . \qquad (3.5)$$

Two conditions then arise that can insure the discriminant in (3.5) is zero:

(I) $\quad d_0 = d_1,$

(II) $\quad \dfrac{d_1}{d_0} = \dfrac{c_1^{\,2}}{c_0^{\,2}}.$

These two conditions correspond with two of the cases investigated later in this section. Case 2 considers two parallel resistors with equal linear coefficients ($d_0$ and $d_1$) and therefore corresponds with the first condition, and Case 3 assumes $c_1=ac_0$ and $d_1=a^2d_0$, corresponding with (II). Regardless of the technique used to make the discriminant zero, when this is done $s(I_{tot})$ becomes

$$s(I_{tot}) = I_{tot} + \frac{c_1 d_0 + c_0 d_1}{2 d_1 d_0}.$$

Substituting this simplified form of $s(I_{tot})$ in for its original form (as equation 3.4) in the solutions for $V$ yields two simplified solutions that are true quadratic functions of $I_{tot}$. They are displayed in the following equation:

$$V_{1,2} = \frac{d_1 d_0 \left(\pm 1 + d_1 + d_0\right)}{\left(-d_0 + d_1\right)^2} I_{tot}^{\;2} + \frac{\left(\pm 1 + d_1 + d_0\right)\left(c_0 d_1 + c_1 d_0\right)}{\left(-d_0 + d_1\right)^2} I_{tot} + \frac{\left(c_0 d_1 + c_1 d_0\right)\left[2 d_1 d_0 (c_1 + c_0) \pm \left(c_1 d_0 + c_0 d_1\right)\right]}{4 d_1 d_0 \left(-d_0 + d_1\right)^2}. \qquad (3.6)$$

Before moving further, one must recall the objective of this investigation is to find an equivalent resistance for any two parallel ADRs, which is also affine dependent on the total current $I_{tot}$ being divided by the two resistors. This clearly requires that the constant terms in equation 3.6 equal zero, in order to be able to factor out $I_{tot}$ and form an instance of Ohm's Law. Setting the constant term equal zero results in two possibilities:

(1) $$\frac{d_1}{d_0} = -\frac{c_1}{c_0}$$

(2) $$2d_1 d_0 (c_1 + c_0) \pm (c_1 d_0 + c_0 d_1) = 0.$$

The first option, which applies to both solutions of $V$, is irrelevant because the coefficients are assumed to be positive. The second option, for $V_1$, is impossible unless either the linear or the offset coefficients are both zero, and it is only possible for $V_2$ if either both linear coefficients are zero or $d_0$=$d_1$=1/2. Putting the expression for (2) above in the following equivalent vector product form easily displays these results:

$$\left[(2d_1 d_0 \pm d_0) \quad (2d_1 d_0 \pm d_1)\right] \cdot \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = 0.$$

All of the requirements uncovered while forcing the constant term in (3.6) to zero fall under cases examined elsewhere in this investigation. We have exploited the general case for two ADRs in parallel to its applicable limits.

Next, the equivalent resistance of parallel networks of ADRs with equal linear and constant terms will be determined.

### 3.3.2 Case 1: Equal Affine Dependency

Consider the circuit in Figure 3.3, which is composed of two equivalent ADRs in parallel.

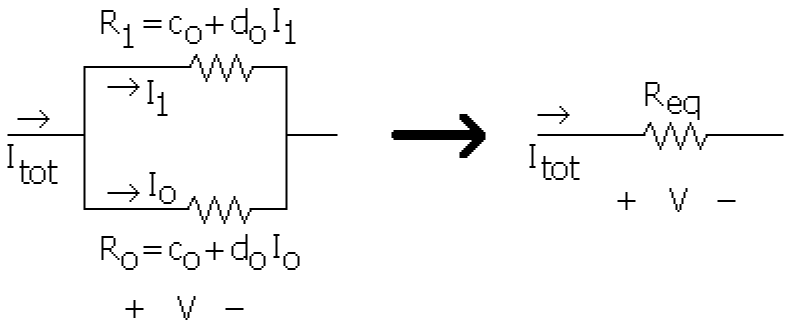


**Figure 3.3** Two equal ADRs in parallel (Case 1).

Currents $I_0$ and $I_1$ pass through the resistors, and the total current $I_{\text{tot}}$ is the sum of those two currents, which is expressed using equation 3.3 as:

$$I_{tot} = -\frac{c_0}{d_0} + 2\sqrt{\left(\frac{c_0}{d_0}\right)^2 + \frac{V}{d_0}}\,. \tag{3.7}$$

It is anticipated that an expression for $R_{\text{eq}}$ will exhibit affine dependence on the total current. To achieve this, both sides of (3.7) are squared, and $V$ is isolated.

$$\left(I_{tot} + \frac{c_0}{d_0}\right)^2 = 4\left[\left(\frac{c_0}{2d_0}\right)^2 + \left(\frac{V}{d_0}\right)\right]$$

$$V = \left(\frac{c_0}{2} + \frac{d_0}{4} I_{tot}\right) I_{tot}$$

The $R_{\text{eq}}$ in this case is therefore calculated using the following equation, the equivalent resistance for two equal affine dependent resistors in parallel.

$$R_{eq} = \frac{c_0}{2} + \frac{d_0}{4} I_{tot} \tag{3.8}$$

It is suspected that the above result can be generalized to $n$ equal resistors. That is, for $i$=1,2,…,$n$,

$$R_i = c + dI_i$$

$$I_i = \left(-\frac{c}{2d}\right) + \sqrt{\left(\frac{c}{2d}\right)^2 + \left(\frac{V}{d}\right)}$$

$$I_{tot} = \sum_{i=1}^{n} I_i = \left(-\frac{nc}{2d}\right) + n\sqrt{\left(\frac{c}{2d}\right)^2 + \left(\frac{V}{d}\right)}.$$

By algebraic manipulation, the parallel voltage $V$ can again be isolated on the left hand side, yielding an instance of Ohm's Law:

$$V = \left(\frac{c}{n} + \frac{d}{n^2} I_{tot}\right) I_{tot}.$$

The reduced expression for $R_{eq}$ has been obtained on the right hand side of the above equation, and it is given by the following expression to be the **equivalent resistance of *n* equal affine dependent resistors in parallel**,

$$R_{eq} = \frac{c}{n} + \frac{d}{n^2} I_{tot}, \tag{3.9}$$

where $I_{tot}$ is the total current flowing into the parallel resistor network. Figure 3.4 demonstrates how the $R_{eq}$ of such networks varies as the number of equivalent ADRs connected in parallel is increased, for the simple case when $c$=$d$=1.

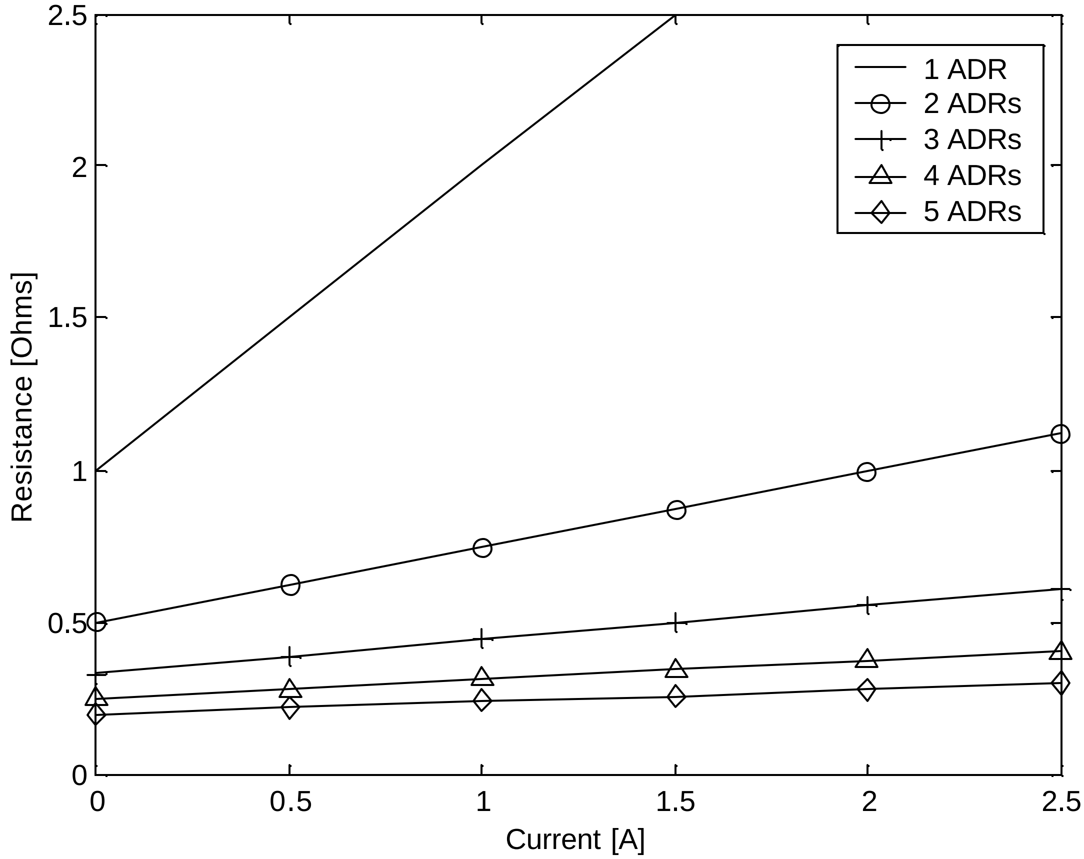


**Figure 3.4** Graph of $R_{eq}$ as size of parallel (Case 1) ADR network increases. $c=d=1$.

It is intuitive to expect each of the $n$ individual branch currents in a network of equally affine dependent resistors to be equal the total source current divided by $n$. This turns out to be true by means of simple algebraic substitution. Each voltage $V_i$ is equal to the voltage across the parallel network, labeled $V$ in Figure 3.3. Solving for each branch current $I_i$ in terms of the network voltage $V$ in the same way equation 3.7 was found,

$$I_i = \frac{-c + \sqrt{c^2 + 4dV}}{2d}.$$

Since $V = R_{eq}I_{tot}$, the following expression for $I_i$ in terms of $I_{tot}$ can be found:

$$I_i = \frac{-c + \sqrt{c^2 + 4d\left(\frac{c}{n} + \frac{d}{n} I_{tot}\right) I_{tot}}}{2d}.$$

Bringing the denominator up into the radical, and observing that a perfect square results, one observes

$$\begin{aligned} I_i &= -\frac{c}{2d} + \sqrt{\frac{c^2}{4d^2} + \frac{c}{dn} I_{tot} + \frac{1}{n^2} I_{tot}^2} \\ &= -\frac{c}{2d} + \sqrt{\left(\frac{1}{n} I_{tot} + \frac{c}{2d}\right)^2}. \end{aligned}$$

This gives the anticipated result, which can be considered a current divider rule for *n* equal ADRs in parallel:

$$I_i = \frac{1}{n} I_{tot}. \tag{3.10}$$

### 3.3.3 Case 2: Equal Linear Coefficients

The I-V relationships of six ADRs with the same linear coefficient ($d$=1/2) and different constant terms ($c_0, c_1, \ldots, c_5$) are plotted in Figure 3.5. The objective in this case study is to determine an expression for the equivalent resistance, and its corresponding I-V behavior, of two or more parallel ADRs with this type of behavior. It will be seen that the result is not necessarily affine.

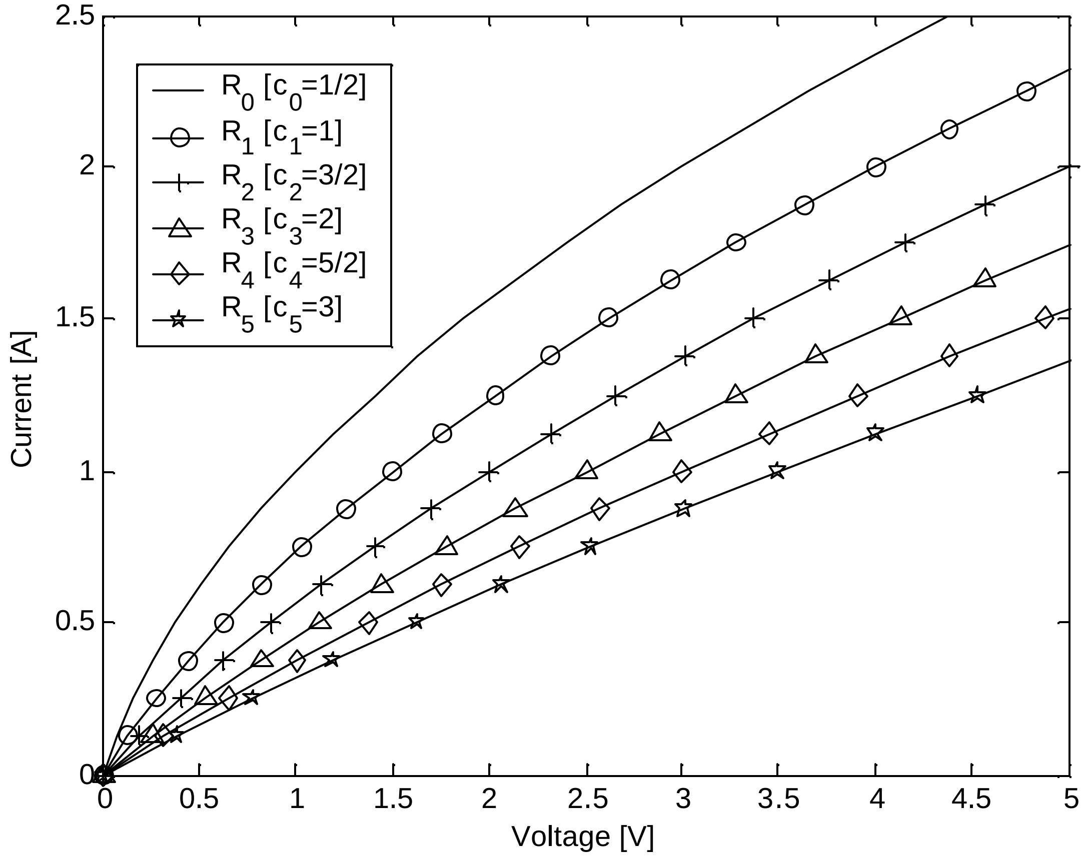


**Figure 3.5** Calculated I-V curves for six example (Case 2) ADRs.

To begin the analysis of this special case, consider the circuit in Figure 3.6, which is composed of two parallel ADRs with equal linear coefficients (both are $d_0$) and different offset values ($c_0$ and $c_1$). Again, the objective here is to obtain a current-dependent expression for $R_{eq}=R(I_{tot})$ and to determine the precise type of dependency that results. The investigation of this case was motivated by a result found while solving the general problem of two ADRs in parallel.

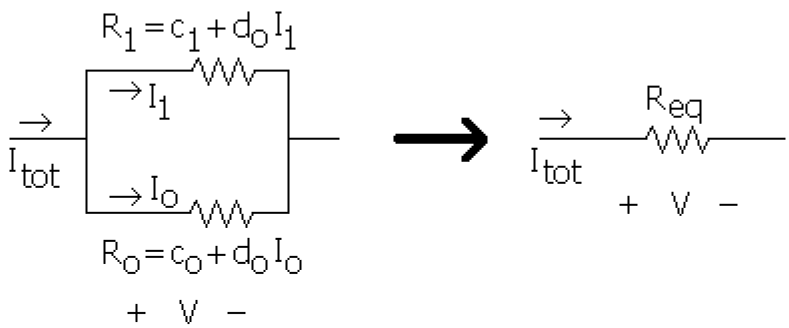


**Figure 3.6** Two parallel ADRs with equal linear coefficients (Case 2).

First, expressions relating the branch currents and the network voltage are found to be

$$I_0 = \frac{-c_0 + \sqrt{c_0^2 + 4d_0 V}}{2d_0} \quad \text{and} \quad I_1 = \frac{-c_1 + \sqrt{c_1^2 + 4d_0 V}}{2d_0}.$$

Using Kirchoff's Current Law, the total current into the network is the sum of the two branch currents, or

$$I = I_0 + I_1 = \frac{-c_0 + \sqrt{c_0^2 + 4d_0 V}}{2d_0} + \frac{-c_1 + \sqrt{c_1^2 + 4d_0 V}}{2d_0}.$$

After some algebraic manipulations, we obtain the following equation:

$$\frac{1}{4}\left[\left(2d_0 I + c_0 + c_1\right)^2 - c_0^2 - c_1^2 - 8d_0 V\right]^2 = \left(c_0^2 + 4d_0 V\right)\left(c_1^2 + 4d_0 V\right),$$

which can be further processed in order to attempt to isolate polynomials purely dependent on $V$ and $I$ on either side. By expanding the square and product in the above equation and collecting terms, we then obtain the following first-order polynomial in $V$:

$$\begin{aligned}0 = &(-4d_0 c_0^2 - 4d_0 c_1^2 - 16d_0^2 c_1 I_{tot} - 16d_0^2 c_0 I_{tot} - 16d_0^3 I_{tot}^2 - 8d_0 c_1 c_0)V \\ &+12d_0^2 c_0 c_1 I_{tot}^2 + 4d_0 c_1 c_0^2 I_{tot} + 4d_0^4 I_{tot}^4 + 4d_0 c_1^2 c_0 I_{tot} + 4d_0^2 c_0^2 I_{tot}^2 \\ &+8d_0^3 c_0 I_{tot}^3 + 8d_0^3 c_1 I_{tot}^3 + 4d_0^2 c_1^2 I_{tot}^2.\end{aligned}$$

It can be shown that by isolating $V$ on the left-hand side, one can obtain the reduced expression by factorization:

$$V = I_{tot} \frac{(c_1 + d_0 I_{tot})(c_0 + d_0 I_{tot})(c_0 + c_1 + d_0 I_{tot})}{(c_0 + c_1 + 2d_0 I_{tot})^2}. \tag{3.11}$$

Treating equation 3.11 as an instance of Ohm's Law, it is obvious that the fraction is **the equivalent resistance for two parallel ADRs with equal linear coefficients**. This is displayed in the following equation.

$$R_{eq} = \frac{(c_1 + d_0 I_{tot})(c_0 + d_0 I_{tot})(c_0 + c_1 + d_0 I_{tot})}{(c_0 + c_1 + 2d_0 I_{tot})^2} \tag{3.12}$$

Its dependence on $I_{tot}$ is not necessarily a purely affine one. Inspection of expression 3.12 reveals a polynomial of order $n$+1 in the numerator, and a polynomial of order $n$ in the denominator. This suggests that for special cases of the linear coefficients and offsets in the ADRs, a resulting parallel equivalent resistance $R_{eq}$ can be recognized as having affine dependence on the total current $I_{tot}$ by polynomial division. It is a simple exercise to show that if all the constant terms $c_i$ are equal, (3.12) gives the same result as expression 3.8.

Equation 3.12 can be used to compute the I-V curves of equivalent resistances that would arise in parallel networks of the example ADRs used to create Figure 3.5. When (3.12) is used to compute the $R_{eq}$ for $R_0=c_0+d_0I_0$ and $R_1=c_1+d_0I_0$ in parallel, its resulting I-V characteristic is graphed in Figure 3.7 against those of $R_0$ and $R_1$.

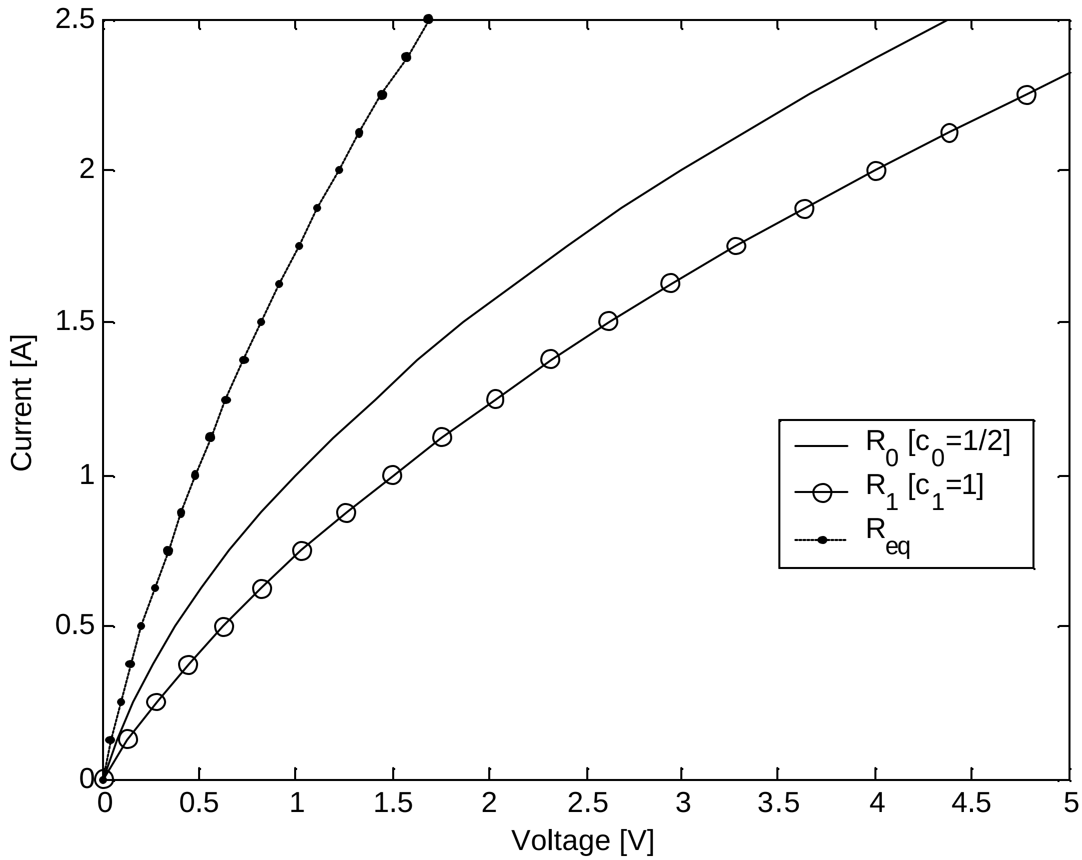


**Figure 3.7** I-V characteristic for $R_{eq}$ of $R_0$ and $R_1$ from Figure 3.5.

### 3.3.4 Case 3: Special Case Involving a Proportionality Constant *a*

As mentioned during the investigation of the general case, another situation which may facilitate the reduction of two or more ADRs in parallel is generalized in the following way:

$$
\begin{array}{ll}
c_1 = a_1 c_0 & d_1 = a_1^{\,2} d_0 \\
c_2 = a_2 c_0 & d_2 = a_2^{\,2} d_0 \\
\quad\downarrow & \\
c_n = a_n c_0 & d_n = a_n^{\,2} d_0 .
\end{array}
\tag{3.13}
$$

To implement this case in parallel ADR circuit design or reduction, one ADR must be chosen whose coefficients will serve as the reference $c_0$ and $d_0$. Then the corresponding $a_i$ ($i$=1,…,$n$-1) for the remaining $n$-1 resistors must be assigned or computed. The hypothetical I-V curves of six such ADRs, where $R_0=c_0+d_0I_0$ serves as the reference resistor, are shown in Figure 3.8.

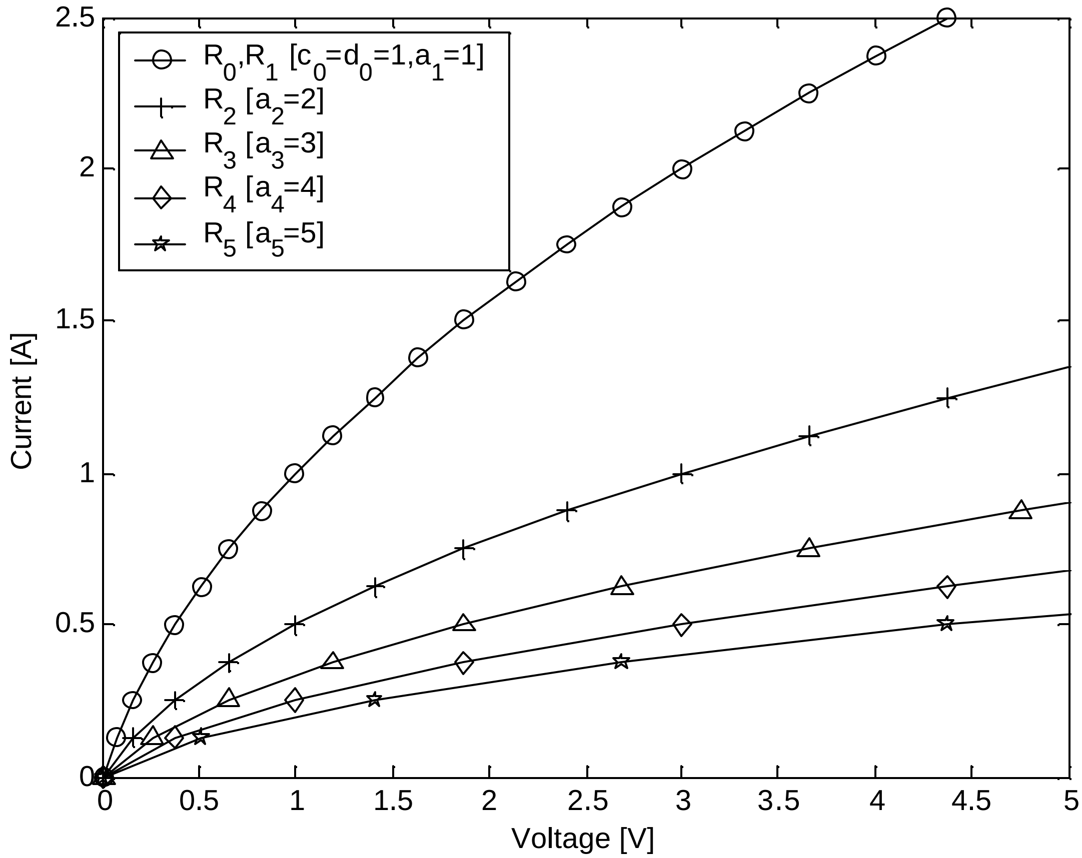


**Figure 3.8** Calculated I-V curves for six example (Case 3) ADRs. Note a= $\alpha$ .

To begin the analysis of this case, consider the simple circuit of Figure 3.9. Two Case 3 ADRs are in parallel, and the objective is again to find an equivalent resistance that is affine dependent on the total network current $I_{tot}$. The useful results obtained for this circuit will then be generalized to even larger parallel networks of Case 3 ADRs, and an appropriate current divider rule will be discovered.

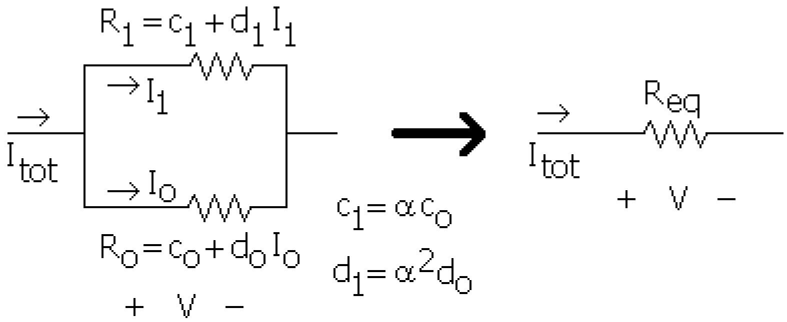


**Figure 3.9** Two parallel ADRs with coefficients related by $a$ (Case 3).

By applying KCL, an equation in a modified form of (3.3) arises to describe this simple circuit:

$$I_{tot} = \left(-\frac{c_0}{2d_0}\right) + \frac{\sqrt{c_0^{\ 2} + 4d_0V}}{2d_0} + \left(-\frac{ac_0}{2a^2d_0}\right) + \frac{\sqrt{(ac_0)^2 + 4a^2d_0V}}{2a^2d_0}.$$

After isolating the simplest form of the radical terms on the right hand side (and before squaring both sides), the following is obtained:

$$\frac{2ad_0I_{tot} + (a+1)c_0}{(a+1)} = \sqrt{c_0^{\ 2} + 4d_0V} .$$

By squaring both sides and isolating only $V$ on one side, an instance of Ohm's Law is eventually obtained in the following fashion:

$$V = \frac{a^2d_0I_{tot}^{\ 2} + a(a+1)c_0I_{tot}}{(a+1)^2},$$

$$V = I_{tot}\left(\frac{a^2d_0}{(a+1)^2}I_{tot} + \frac{ac_0}{a+1}\right).$$

This leads to the following expressions for the **equivalent resistance of two Case 3 ADRs in parallel**:

$$R_{eq} = c_{eq} + d_{eq} I_{tot}$$

where

$$c_{eq} = \frac{a}{a+1} c_0 \quad \text{or} \quad \frac{c_0 c_1}{c_0 + c_1} \tag{3.14}$$

$$d_{eq} = \left( \frac{a}{a+1} \right)^2 d_0 \quad \text{or} \quad \frac{d_0 d_1}{(\sqrt{d_0} + \sqrt{d_1})^2} .$$

It is interesting to observe that the constant term of the equivalent ADR is the product of the two parallel constant terms divided by their sum. This shows that if both linear coefficient terms are zero, equations 3.14 give the same result as that often used in traditional linear circuit analysis for two parallel resistors. It is also obvious that similar results are obtained when the $d$ terms are zero in (3.1), (3.8), (3.9) and (3.12).

The same technique which led to (3.14) will now be attempted for larger networks of parallel ADRs, in order to pursue a closed-form result for any number of resistors. Suppose three ADRs are in parallel and described by the following equations:

$$R_0 = c_0 + d_0 I_0$$
$$R_1 = c_1 + d_1 I_1$$
$$R_2 = c_2 + d_2 I_2$$

where

$$c_1 = a_1 c_0 \quad d_1 = a_1^{\,2} d_0$$
$$c_2 = a_2 c_0 \quad d_2 = a_2^{\,2} d_0 .$$

The KCL equation becomes

$$\mathrm{I}_{\mathrm{tot}} = \frac{-\mathrm{c}_0 + \sqrt{c_0^{\ 2} + 4d_0V}}{2d_0} + \frac{-\mathrm{c}_1 + \sqrt{c_1^{\ 2} + 4d_1V}}{2d_1} + \frac{-\mathrm{c}_2 + \sqrt{c_2^{\ 2} + 4d_2V}}{2d_2},$$

and after substitution it can be rewritten as:

$$I_{tot} = \frac{-c_0 + \sqrt{c_0^{\ 2} + 4d_0V}}{2d_0}\left(1 + \frac{1}{a_1} + \frac{1}{a_2}\right). \tag{3.15}$$

The form of expression 3.15 is intuitive – the total current turns out to be a scalar multiple of the current through the reference ADR, $R_0$. It is not difficult to conclude that for $n$ of these resistors, the result will be:

$$I_{tot} = \frac{-c_0 + \sqrt{c_0^{\ 2} + 4d_0V}}{2d_0}\left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}}\right). \tag{3.16}$$

After algebraically manipulating (3.15) to retrieve the $R_{\mathrm{eq}}$ in terms of the constants $a_1$ and $a_2$, the results are summarized in following expressions:

$$R_{eq} = c_{eq} + d_{eq}I_{tot}$$

where

$$c_{eq} = \frac{a_1a_2}{a_1a_2 + a_1 + a_2}c_0 \tag{3.17}$$

$$d_{eq} = \left(\frac{a_1a_2}{a_1a_2 + a_1 + a_2}\right)^2 d_0.$$

When these results are compared with equations 3.14, a pattern begins to emerge. Using (3.16) to likewise investigate four of these ADRs in parallel gives the following equations,

$$c_{eq} = \frac{a_1a_2a_3}{a_1a_2a_2 + a_1a_2 + a_2a_3 + a_1a_3}c_0 \quad \text{and}$$

$$d_{eq} = \left(\frac{a_1a_2a_3}{a_1a_2a_2 + a_1a_2 + a_2a_3 + a_1a_3}\right)^2 d_0. \tag{3.18}$$

The closed form of the equivalent resistance for any size parallel network of this brand of ADR is now clear. For a network of *n* parallel resistors arranged as in the following figure,

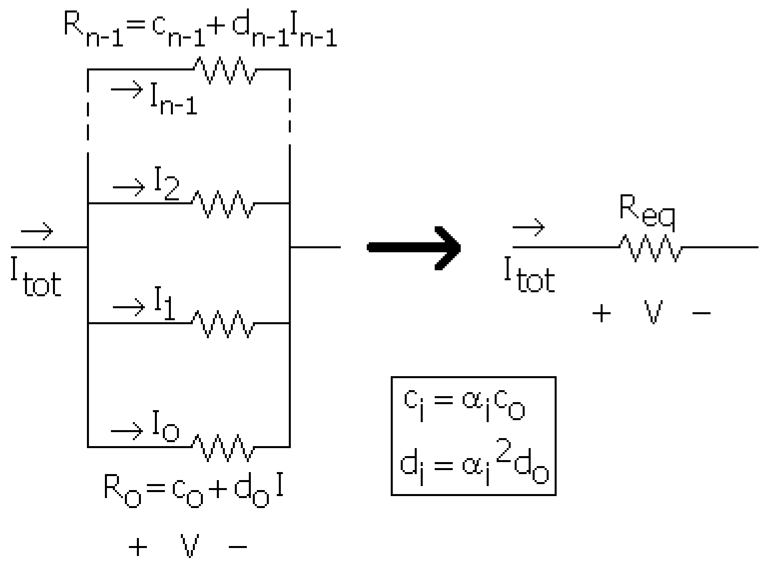


**Figure 3.10** Finding the $R_{\text{eq}}$ for *n* (Case 3) ADRs in parallel.

the equivalent resistance is always affine dependent on the total current $I_{\text{tot}}$, and its offset and linear coefficients are displayed found as:

$$c_{eq} = \left(1 + \sum_{k=1}^{n-1} \frac{1}{a_k}\right)^{-1} c_0 \quad \text{and} \quad d_{eq} = \left(1 + \sum_{k=1}^{n-1} \frac{1}{a_k}\right)^{-2} d_0 \,. \tag{3.19}$$

The expressions in (3.19) can be written in terms of only the $c_{\text{i}}$ and $d_{\text{i}}$ coefficients, and not the proportionality constants $a_{\text{i}}$, by substitution of $a_{\text{i}} = c_{\text{i}}/c_0 = (d_{\text{i}}/d_0)^{1/2}$. The results are displayed in the following manner, which is the **equivalent resistance for *n* (Case 3) ADRs in parallel**.

$$R_{eq} = c_{eq} + d_{eq} I_{tot} ,$$

where

$$c_{eq} = \left( \sum_{k=0}^{n-1} \frac{1}{c_k} \right)^{-1} \quad \text{and} \quad d_{eq} = \left( \sum_{k=0}^{n-1} \frac{1}{\sqrt{d_k}} \right)^{-2} . \tag{3.20}$$

It is fairly straightforward to obtain a corresponding current divider rule for this particular case of ADRs. Recall that any individual branch current ($I_i$) desired can be written as:

$$I_i = \frac{-c_i + \left(c_i^{\,2} + 4 d_i V\right)^{\frac{1}{2}}}{2 d_i} ,$$

assuming it flows into the positive reference node of the voltage across the network. Within this expression, the network voltage $V$ can be written as a product of the equivalent resistance $R_{eq}$ and the total current $I_{tot}$:

$$V = I_{tot} \left( c_{eq} + d_{eq} I_{tot} \right) .$$

Therefore, the **current divider rule for (case 3) ADRs** would take the following form, where $c_{eq}$ and $d_{eq}$ are calculated using (3.20):

$$I_i = \frac{-c_i + \left[c_i^{\,2} + 4 d_i I_{tot} \left( c_{eq} + d_{eq} I_{tot} \right)\right]^{\frac{1}{2}}}{2 d_i} . \tag{3.21}$$

Note that the total current $I_{tot}$, and therefore the individual branch currents $I_i$, must follow the passive convention with respect to the voltage drop across the ADR network to avoid complex results. An application of the results derived thus far in the chapter can be seen in the following example problem.

## 3.4  Example Problem

We wish to solve for all the voltages and currents present in the circuit of Figure 3.11. The network is composed of a DC voltage source and a total of six ADRs.

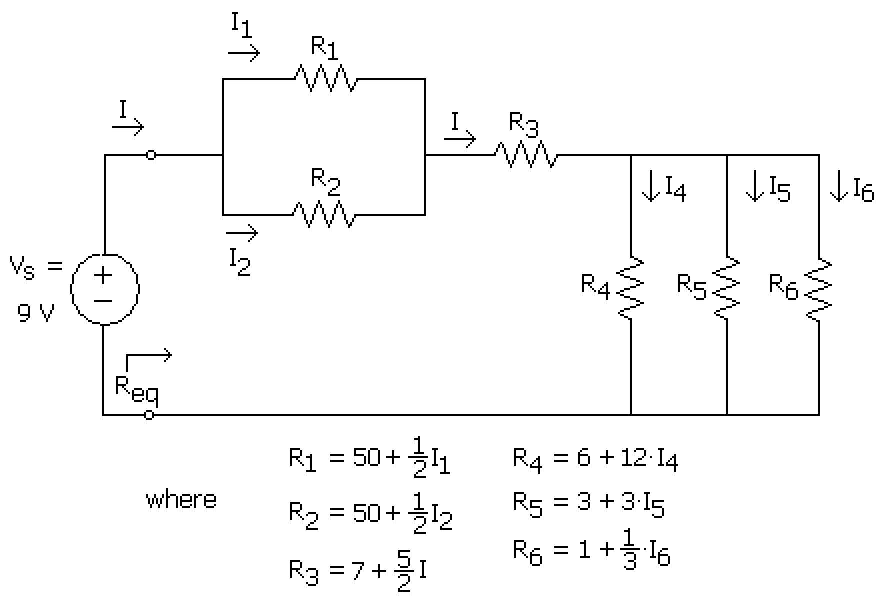


**Figure 3.11**  Example circuit containing ADRs.

To calculate $R_{eq}$, one must reduce the two parallel networks of ADRs in Figure 3.11 and add the total series resistance.  The parallel combination of $R_1$ and $R_2$ will be called $R_{1,2}$, and it will be dependent on the total source current $I$.  Using (3.8),

$$R_{1,2} = 25 + \frac{1}{8} I.$$

The network of $R_4$, $R_5$ and $R_6$ falls into the third special case of ADRs discussed in this chapter, where $c_6$ (=1) and $d_6$ (=1/3) are the reference coefficients, and the corresponding proportionality constants for $R_4$ and $R_5$ are $a_4$ (=6) and $a_5$ (=3), respectively. Equations 3.19 and/or 3.20 can be used to calculate the linear and constant coefficients of $R_{4,5,6}$.

$$R_{4,5,6} = c_{eq} + d_{eq} I$$

where

$$c_{eq} = \left(1 + \frac{1}{3} + \frac{1}{6}\right)^{-1} = \frac{2}{3}$$

$$d_{eq} = \left(1 + \frac{1}{3} + \frac{1}{6}\right)^{-2} \frac{1}{3} = \frac{4}{27}$$

Finally, using (3.1),

$$R_{eq} = \left(25 + 7 + \frac{2}{3}\right) + \left(\frac{1}{8} + \frac{5}{2} + \frac{4}{27}\right) I = 32.67 + 2.77I.$$

The voltage drops across $R_{4,5,6}$, $R_{1,2}$ and $R_3$ are calculated using (3.2), the voltage divider rule for ADRs. The total current $I$ is found to be 0.27A, and therefore

$$V_{R4,5,6} = \left(\frac{2}{3} + \frac{4}{27} 0.27\right) 0.27 = 0.19\text{V}.$$

Similarly,

$$V_{R3} = 2.07\text{V},$$
$$V_{R1,2} = 6.76\text{V}.$$

The currents $I_1$ and $I_2$ are each half of the total current $I$, that is $I_1$=$I_2$=0.135A. To directly calculate the remaining currents, the current divider rule for case 3 ADRs is needed (3.21). To calculate $I_5$, we must realize that $c_i$ and $d_i$ correspond to the characteristics of $R_5$, and $c_{eq}$ and $d_{eq}$ correspond to those of $R_{4,5,6}$.

$$I_5 = \frac{-3 + \left[3^2 + 4 \cdot 3 \cdot 0.27(0.67 + 0.15 \cdot 0.27)\right]^{\frac{1}{2}}}{2 \cdot 3} = 0.06\text{A}.$$

Similarly,

$$I_4 = 0.03\text{A},$$
$$I_6 = 0.18\text{A}.$$

These results obey KVL around each loop and KCL at each node.

# 4.0 TOPICS FOR FUTURE RESEARCH AND CONCLUDING REMARKS

The rules and techniques developed in this study for circuit reduction and signal computation in networks of current-dependent resistors can possibly be used to model fluid flow in certain pipe networks. Certain flow resistance characteristics allow some series and parallel networks to be greatly simplified; however, just as robust linear electric circuit analysis also implements techniques to analyze resistive circuits without the need for reduction (i.e. mesh current and node voltage analysis), a few analogous tools are now pursued in the realm of the particular nonlinear circuit discussed in this thesis. In this chapter, we mention possible future research problems that involve using tools developed in this thesis to derive solution methods such as mesh and node analysis and to perform operations such as delta-wye transformations. We conclude this chapter with remarks that highlight the various contributions made in this thesis.

## 4.1 Possible Topics for Future Research

### 4.1.1 Mesh and Node Analysis

Mesh and node analysis methods are very important for solving circuits in general [7], [8]. To illustrate some of the issues that arise when one attempts to apply these methods for circuits with LDRs and ADRs, let us consider the following example circuit of LDRs in Figure 4.1. The circuit-reduction techniques established in chapter 2 will be tested against an adapted mesh analysis solution. The resistors are labeled according to their degree of linear dependence upon the current, but the specific currents upon which they depend have not yet been selected.

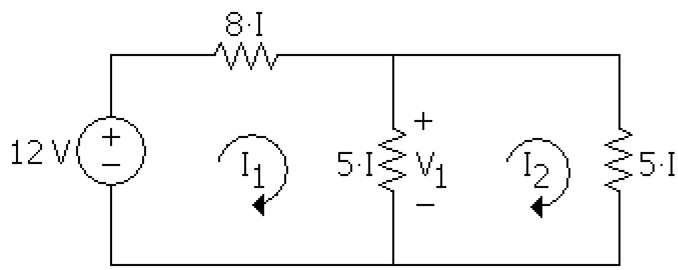


**Figure 4.1** Two-mesh circuit containing LDRs.

The mesh currents will first be calculated using rules from chapter 2. Using (2.3), the equivalent resistance of the two equal LDRs in parallel is found to be

$$R_{1,2} = \left( \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \right)^{-2} I_1 = \frac{5}{4} I_1 .$$

By (2.1), the total series resistance is found to be $(37/4)I_1$, and the total source current $I_1$ is then computed (using the source voltage and the total resistance) to be $I_1$=1.14A. Because of the equivalence of the parallel LDRs, $I_2$ is clearly $I_2=I_1/2$ = 0.57A.

In order to construct two mesh equations for this circuit, one must first choose a reference direction for the voltage drop across the shared resistor and keep that consistent when writing both equations, just as is done in mesh analysis of linear circuits. The reference direction for the net current through the leftmost $5I$ resistor will be downward, and therefore the voltage drop $V_1$ will be

$$V_1 = 5(|I_1 - I_2|)(I_1 - I_2).$$

The resulting nonlinear system of mesh currents for the circuit is found using traditional techniques:

$$8|I_1|I_1 + 5(|I_1 - I_2|)(I_1 - I_2) = 12 \qquad \text{for mesh 1, and}$$

$$-5(|I_1 - I_2|)(I_1 - I_2) + 5|I_2|I_2 = 0 \qquad \text{for mesh 2.}$$

Because of the absolute values in these equations, the solution procedure must take into consideration all possible sign variations of the current variables. In this problem, we know that the reference directions for $I_1$ and $I_2$ will yield positive values for these currents. We also know that $I_1 > I_2$. So the absolute values in the equations can be removed to yield:

$$8I_1^2 + 5(I_1 - I_2)^2 = 12 \tag{4.1a}$$

$$-5(I_1 - I_2)^2 + 5I_2^2 = 0. \tag{4.1b}$$

In general, however, we may not know the sign of the current variables in all resistors. For example, in this problem, suppose we did not know *a priori* that $I_1 > I_2$. Then in addition to the above two equations, the case where $I_1 < I_2$ must also be considered, which itself yields the following two different equations:

$$8I_1^2 - 5(I_1 - I_2)^2 = 12 \tag{4.2a}$$

$$5(I_1 - I_2)^2 + 5I_2^2 = 0. \tag{4.2b}$$

Both sets of equations 4.1 and 4.2 must be solved and only the consistent solutions ($I_1 < I_2$ for 4.1 and $I_1 > I_2$ for 4.2) will be valid. In this case we note that equations 4.2 do not admit any solutions since (4.2) requires that $I_1 = I_2 = 0$ and (4.2a) cannot be satisfied with these values. Therefore (4.1) are the only valid mesh equations for the circuit, and we should expect the solution to satisfy $I_1 > I_2$. Because of the nonlinear nature of these relationships, numerical methods must be used to solve such equations [1].

We note, however, that equations 4.1a and 4.1b are coupled quadratic equations of the Riccati type, and extensive studies on the numerical solution methods for such equations can be found in [9], [10] and [11]. Regarding this example, we realize that equation 4.1b yields

$$I_2 = I_1/2,$$

and when this is substituted into (4.1a) we get

$$37I_1^2 = 48$$

or $I_1$=1.14 and $I_2$=0.57A. Note, as a final check, that this solution is consistent with $I_1 > I_2$.

The above example suggests that mesh currents within larger networks of current-dependent resistors will need to be solved using numerical techniques, but it also exposes the following complication: mesh current calculation depends on the initial reference direction assumption for currents flowing through resistors that are shared among several meshes. Similar results are expected for node-voltage analysis of such networks as well. The pursuit of mathematical techniques for solving circuits despite this complication could open up avenues for future research.

### 4.1.2 Delta-Wye Transformation of Linearly Dependent Resistors

Another nonlinear circuit modification technique worthy of consideration is the corresponding delta-wye transformation for current-dependent resistors. The generalized circuit is found in Figure 4.2. The same transformation requirements that govern current-independent resistors apply:

$$R_A + R_B = R_1 \parallel (R_2 + R_3)$$

$$R_B + R_C = R_2 \parallel (R_1 + R_3) \tag{4.3}$$

$$R_A + R_C = R_3 \parallel (R_1 + R_2).$$

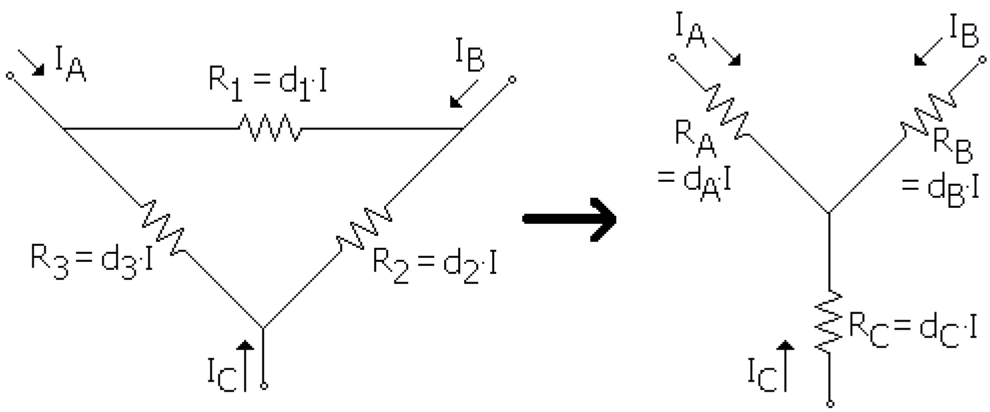


**Figure 4.2** The delta-wye transformation for LDRs.

Upon inspection it is clear that the requirements given by equations 4.3 complicate the possibility of deriving a delta-wye transformation for the ADRs discussed in chapter 3, so the transformation is calculated here only for networks of linearly dependent resistors. The requirements of (4.3) become

$$R_A + R_B = d_1I_1 \parallel (d_2I_2 + d_3I_3)$$

$$R_B + R_C = d_2I_2 \parallel (d_1I_1 + d_3I_3) \tag{4.4}$$

$$R_A + R_C = d_2I_2 \parallel (d_1I_1 + d_2I_2)$$

where $I_1$, $I_2$ and $I_3$ are the clockwise currents passing through resistors $R_1$, $R_2$ and $R_3$, respectively. Based on these requirements, it is intuitive to suspect that resistors $R_A$, $R_B$ and $R_C$ will be dependent on one or more of the three major network currents $I_A$, $I_B$ and $I_C$. After calculating the necessary equivalent series and parallel resistances and performing some algebraic manipulation, intuition proves correct. The following relationships result:

$$R_A = \frac{1}{2}\left[\left(\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2 + d_3}}\right)^{-2} I_A - \left(\frac{1}{\sqrt{d_2}} + \frac{1}{\sqrt{d_1 + d_3}}\right)^{-2} I_B + \left(\frac{1}{\sqrt{d_3}} + \frac{1}{\sqrt{d_1 + d_2}}\right)^{-2} I_C\right],$$

$$R_B = \frac{1}{2}\left[\left(\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2 + d_3}}\right)^{-2} I_A + \left(\frac{1}{\sqrt{d_2}} + \frac{1}{\sqrt{d_1 + d_3}}\right)^{-2} I_B - \left(\frac{1}{\sqrt{d_3}} + \frac{1}{\sqrt{d_1 + d_2}}\right)^{-2} I_C\right],$$

$$R_C = \frac{1}{2}\left[-\left(\frac{1}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2 + d_3}}\right)^{-2} I_A + \left(\frac{1}{\sqrt{d_2}} + \frac{1}{\sqrt{d_1 + d_3}}\right)^{-2} I_B + \left(\frac{1}{\sqrt{d_3}} + \frac{1}{\sqrt{d_1 + d_2}}\right)^{-2} I_C\right].$$

We observe that each wye resistor is found to be linearly dependent on not only its own current but on the currents through the other two wye resistors as well. Though they represent an interesting result, $R_A$, $R_B$ and $R_C$ cannot be considered LDRs by the definition given in this thesis. One avenue for future investigation in this type of circuit transformation can be to pursue analogous results for ADRs.

## 4.2 Concluding Remarks

Techniques similar to those that govern resistor network reduction and solution in linear circuit analysis have been investigated and developed for networks of resistors whose values depend on current in a linear or affine manner. We derived expressions for combining such resistors in series and in parallel and obtained the resulting voltage and current divider rules. It is interesting result to note that in the case of affine dependent resistors, some of the results can be considered generalizations of those of traditional linear circuit analysis. The offset term in these resistors represents the same sort of fixed, current-independent behavior that typical resistors exhibit, and the inclusion of the linear term takes one step in the direction of generalizing circuit reduction for elements whose voltage can be written as a polynomial function of current. It is clear that if the linear coefficients were zero in (3.1), (3.9), (3.12) and (3.20), the same exact results for the combination of current-independent resistors in series and parallel are given. We demonstrated that in the most general case of circuits with all ADRs, there are no closed-form expressions for combining such resistors in parallel. Consequently, we derived the expressions for various special cases of interest. Throughout the thesis, we illustrated the concepts with simple examples to make it easy for the reader to appreciate the difficulties involved in dealing with such nonlinear circuits. We also pointed out several possible research problems that can still be explored as a follow-up to this thesis.

We believe that as the field of nanotechnology evolves, circuits with current-dependent resistors may become very important components in modeling many nano-scale systems that involve fluid flow. The tools and techniques developed in this thesis may provide a theoretical basis for the analysis of such systems.

# APPENDIX

# Appendix A

## MATLAB M-files Investigating Affine Dependent Resistors

```
% Two affine-dependent resistors in parallel, general case

clear all;close all;clc;

syms d0 d1 c0 c1 It V;

solve('It=-c1/(2*d1)+((c1/(2*d1))^2+(V/d1))^(1/2)-
c0/(2*d0)+((c0/(2*d0))^2+(V/d0))^(1/2)','V');

% 2 roots are output for the above task.
%    first try to simplify & collect terms of [1,1]

[V1,how]=simple(-1/2*(d1*c0^2-1/(-d0+d1)*(4*It*d1*d0+2*c1*d0+2*c0*d1+...
     2*(c1^2*d0^2+c0^2*d0^2+4*It^2*d1*d0^3+4*It*d0^3*c1+4*It*d1*d0^2*c0+...
     2*c1*d0^2*c0)^(1/2))*d0*It*d1-1/2/(-
d0+d1)*(4*It*d1*d0+2*c1*d0+2*c0*d1+...
     2*(c1^2*d0^2+c0^2*d0^2+4*It^2*d1*d0^3+4*It*d0^3*c1+4*It*d1*d0^2*c0+...
     2*c1*d0^2*c0)^(1/2))*c0*d1-1/2/(-d0+d1)*(4*It*d1*d0+2*c1*d0+2*c0*d1+...
     2*(c1^2*d0^2+c0^2*d0^2+4*It^2*d1*d0^3+4*It*d0^3*c1+4*It*d1*d0^2*c0+...
     2*c1*d0^2*c0)^(1/2))*d0*c1+2*It^2*d1*d0^2+2*It*d0^2*c1+2*It*d1*d0*c0+...
     c1*d0*c0)/d0/(-d0+d1));

[V2,how]=simple([ -1/2*(d1*c0^2-1/(-d0+d1)*(4*It*d1*d0+2*c1*d0+2*c0*d1-...

2*(c1^2*d0^2+c0^2*d0^2+4*It^2*d1*d0^3+4*It*d0^3*c1+4*It*d1*d0^2*c0+...
          2*c1*d0^2*c0)^(1/2))*d0*It*d1-1/2/(-d0+d1)*(4*It*d1*d0+2*c1*d0+...
          2*c0*d1-2*(c1^2*d0^2+c0^2*d0^2+4*It^2*d1*d0^3+4*It*d0^3*c1+...
          4*It*d1*d0^2*c0+2*c1*d0^2*c0)^(1/2))*c0*d1-1/2/(-
d0+d1)*(4*It*d1*d0+...
          2*c1*d0+2*c0*d1-
2*(c1^2*d0^2+c0^2*d0^2+4*It^2*d1*d0^3+4*It*d0^3*c1+...
          4*It*d1*d0^2*c0+2*c1*d0^2*c0)^(1/2))*d0*c1+2*It^2*d1*d0^2+...
          2*It*d0^2*c1+2*It*d1*d0*c0+c1*d0*c0)/d0/(-d0+d1)]);

V1 = collect(1/2*(d1*c0^2+2*It^2*d1^2*d0+2*It*d1*d0*c1+2*It*d1^2*c0+...
     2*It*d1*d0*(c1^2+c0^2+4*It^2*d1*d0+4*It*d0*c1+4*It*d1*c0+...
     2*c0*c1)^(1/2)+c0*d1*c1+c0*d1*(c1^2+c0^2+4*It^2*d1*d0+4*It*d0*c1+...
     4*It*d1*c0+2*c0*c1)^(1/2)+c1^2*d0+c1*d0*(c1^2+c0^2+4*It^2*d1*d0+...
     4*It*d0*c1+4*It*d1*c0+2*c0*c1)^(1/2)+2*It^2*d1*d0^2+2*It*d0^2*c1+...
     2*It*d1*d0*c0+c1*d0*c0)/(-d0+d1)^2,It);

V2 = collect(1/2*(d1*c0^2+2*It^2*d1^2*d0+2*It*d1*d0*c1+2*It*d1^2*c0-...
     2*It*d1*d0*(c1^2+c0^2+4*It^2*d1*d0+4*It*d0*c1+4*It*d1*c0+...
     2*c1*c0)^(1/2)+c0*d1*c1-c0*d1*(c1^2+c0^2+4*It^2*d1*d0+4*It*d0*c1+...
     4*It*d1*c0+2*c1*c0)^(1/2)+c1^2*d0-c1*d0*(c1^2+c0^2+4*It^2*d1*d0+...
     4*It*d0*c1+4*It*d1*c0+2*c1*c0)^(1/2)+2*It^2*d1*d0^2+2*It*d0^2*c1+...
     2*It*d1*d0*c0+c1*d0*c0)/(-d0+d1)^2,It);

% In the resulting solutions for V, there appears a polynomial in
%    It, which I will call s(It).  The polynomial is ((4*d1*d0)*It^2+
%    4*(d0*c1+d1*c0)*It+(c1^2+2*c0*c1+c0^2))^(1/2).
%    Now I will simplify this...

roots_s =
solve('(4*d1*d0)*It^2+4*(d0*c1+d1*c0)*It+(c1^2+2*c0*c1+c0^2)=0','It');

[root_s,how]=simple(1/8/d1/d0*(-4*c1*d0-4*c0*d1+4*(c1^2*d0^2+d1^2*c0^2-...
```

```
     d1*d0*c1^2-d0*d1*c0^2)^(1/2)));

% Taking the condition for two real & equal roots by setting the
discriminant=0,
%    s(It) has become (It+(c1*d0+c0*d1)/(2*d1*d0)).
%    This only happens for two conditions:  d0=d1 and (c1^2/c0^2)=(d1/d0).
%    The results of these two conditions are examined in other m-files.
%    V# can be simplified further, first by re-collecting It terms...

V1 = collect((d1^2*d0+d1*d0^2)/(-d0+d1)^2*It^2+(d1*d0*c1+d1^2*c0+...
     c1*d0^2+c0*d1*d0+d1*d0*(It+(c1*d0+c0*d1)/(2*d1*d0)))/(-d0+...
     d1)^2*It+(1/2*d1*c0^2+1/2*c0*d1*c1+1/2*c0*d1*(It+...
     (c1*d0+c0*d1)/(2*d1*d0))+1/2*c1^2*d0+1/2*c1*d0*(It+...
     (c1*d0+c0*d1)/(2*d1*d0))+1/2*c1*d0*c0)/(-d0+d1)^2,It);

V2 = collect((d1^2*d0+d1*d0^2)/(-d0+d1)^2*It^2+(d1*d0*c1+d1^2*c0+...
     c1*d0^2+c0*d1*d0-d1*d0*(It+(c1*d0+c0*d1)/(2*d1*d0)))/(-d0+d1)^2*It+...
     (1/2*d1*c0^2+1/2*c0*d1*c1-1/2*c0*d1*(It+(c1*d0+c0*d1)/(2*d1*d0))+...
     1/2*c1^2*d0-1/2*c1*d0*(It+(c1*d0+c0*d1)/(2*d1*d0))+...
     1/2*c1*d0*c0)/(-d0+d1)^2,It);

% Simplifying the resulting coefficients of the 2nd order polynomial in It

V1_sq = factor((d1*d0/(-d0+d1)^2+(d1^2*d0+d1*d0^2)/(-d0+d1)^2))
V1_lin = factor(((1/2*c0*d1+1/2*c1*d0)/(-d0+d1)^2+(d1*d0*c1+...
     d1^2*c0+c1*d0^2+c0*d1*d0+1/2*c0*d1+1/2*c1*d0)/(-d0+d1)^2))
V1_con = factor((1/2*d1*c0^2+1/2*c0*d1*c1+1/4*c0*(c0*d1+...
     c1*d0)/d0+1/2*c1*d0*c0+1/2*c1^2*d0+1/4*c1*(c0*d1+c1*d0)/d1)/(-d0+d1)^2)

V2_sq = factor((-d1*d0/(-d0+d1)^2+(d1^2*d0+d1*d0^2)/(-d0+d1)^2))
V2_lin = factor(((-1/2*c1*d0-1/2*c0*d1)/(-d0+d1)^2+(d1*d0*c1+d1^2*c0+...
    c1*d0^2+c0*d1*d0-1/2*c1*d0-1/2*c0*d1)/(-d0+d1)^2))
V2_con = factor((1/2*d1*c0^2+1/2*c0*d1*c1-1/4*c0*(c1*d0+c0*d1)/d0+...
    1/2*c1*d0*c0+1/2*c1^2*d0-1/4*c1*(c1*d0+c0*d1)/d1)/(-d0+d1)^2)
```

```
% Two affine-dependent resistors in parallel
% Investigate consequence of linear coefficients being equal
%  d0 = d1 = d;

clear all;close all;clc;

syms c0 c1 d0 d1 d It;

solve('It=-c1/(2*d)+((c1/(2*d))^2+(V/d))^(1/2)-
c0/(2*d)+((c0/(2*d))^2+(V/d))^(1/2)','V');

collect(It*(c0^2*It*d+c0^2*c1+It*d*c1^2+2*It^2*d^2*c1+2*It^2*d^2*c0+...
     c1^2*c0+3*It*d*c1*c0+It^3*d^3)/(2*It*d+c1+c0)^2,It);

% The result of the above collection is a product of It and the resulting
Req(It),
%    which is an (n+1) order polynomial divided by a nth order polynomial,
resulting
%    (in some cases) in an affine Req as a function of It.

% This is the numerator of the Req function, factored:
factor(It^3*d^3+(2*d^2*c0+2*d^2*c1)*It^2+...
     (c0^2*d+3*d*c1*c0+d*c1^2)*It+c0^2*c1+c1^2*c0);

% The result, for two resistors is

%        (It*d+c0)*(It*d+c1)*(It*d+c1+c0)
%  Req = --------------------------------
%               (2*It*d+c1+c0)^2
```

```
% Two affine-dependent resistors in parallel
% Investigate case # 3
% c1=(a*c0), d1=(a^2*d0);

clear all;close all;clc;

syms c0 c1 d0 d1 d It a;

solve('It=-(a*c0)/(2*(a^2*d0))+(((a*c0)/(2*(a^2*d0)))^2+(V/(a^2*d0)))^(1/2)-
c0/(2*d0)+((c0/(2*d0))^2+(V/d0))^(1/2)','V');

% Two roots result for V; their It terms will be collected below:

collect(It*a*(It*a*d0+c0+c0*a)/(a+1)^2,It)
collect(a*(c0^2+It^2*a*d0^2+It*d0*c0+It*a*d0*c0)/(a-1)^2/d0,It)

% The result of the first collection is a product of It and the resulting
Req(It),
%    which is an affine Req as a function of It.
% The result of the second collection above requires c0=0 for an affine-
dependent
%    equivalent resistance, so it will be ignored.

% The result, for two resistors is
%
%         (a*c0)    (a^2*d0)
%  Req = ------ +  --------It
%         (a+1)     (a+1)^2
```

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