Middle Grades Geometry and Measurement: Examining Change in Knowledge Needed for Teaching through a Practice-Based Teacher Education Experience

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ABSTRACT

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Geometry and measurement represent topics of great significance in mathematics; however, efforts to teach this content in the middle grades have been formulaic, with students memorizing formulas and definitions without conceptual understanding. Moreover, students and teachers demonstrate gaps and misconceptions in their knowledge of geometry and measurement, particularly with respect to relationships between measurable quantities of geometric figures and proof. This study investigated changes in knowledge needed for teaching geometry and measurement through engagement in a practice-based course for preservice and practicing teachers.

Pre- and post-course measures showed significant teacher growth along all three aspects of knowledge needed for teaching. Teachers grew in their ability to attack non-routine problems relating dimension, perimeter, and area and dimension, surface area, and volume; and in their use of multiple solution methods, multiple representations, and production of mathematically sophisticated solutions. Teachers also grew in content knowledge for teaching, becoming more representationally fluent and increasingly able to modify tasks to target key geometry ideas and about the affordances of different formulas for area and volume, and in knowledge of proof, including identification of the key aspects of the definition of proof, the role of proof in the classroom, and creation of proofs and proof-like arguments.

Teachers grew in knowledge of mathematics for student learning as conceptualized by the five practices for productive use of student thinking: anticipating student solutions to a mathematical task, the use of high-level questions to assess and advance student thinking, selecting and sequencing student work to share, and connecting that work in ways that targeted the big mathematical ideas. Teachers also grew in their identification of routines, an example of practices that support teaching. Qualitative analysis of the course tied these results to opportunities to learn in the course.

The results suggest that teachers can grow in their knowledge of content and pedagogy through practice-based teacher education experiences. The results suggest a value for focusing methods courses on particular slices of mathematical content. The design principles articulated in the analysis predicted teacher learning, and generalize to the design of teacher education experiences that enhance knowledge needed for teaching mathematics.

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DEDICATION

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You taught me so much about mathematics and teaching, despite my own arrogance. Every elementary mathematics teacher with whom I will work in my career owes you a debt of gratitude for broadening my mind.

But more importantly, you taught me about joy, friendship, honor, and loyalty, and shaped my character in ways that I am only now beginning to understand. As one phase of my worldly journey ends, your eternal journey begins.

1. INTRODUCTION

> "The mere memorization of a demonstration in geometry has about the same education value as the memorization of a page from the city directory. And yet it must be admitted that a very large number of our pupils do study mathematics in just this way. There can be no doubt that the fault lies with the teaching."
>
> Young (1925), pp. 4-5

John Wesley Young's assessment of the state of school geometry is nearly a century old, but is no less applicable today. Geometry and measurement represent topics of great significance in the field of mathematics. Geometric study was the foundation of Greek mathematics, and much of the work of Greek mathematicians such as Euclid and Pythagoras still holds a significant place in school mathematics. The development of geometry and measurement skills help students develop spatial relations, allows them to make sense of objects in the world around them, and provides a connection between the numeric/algebraic and the spatial/visual domains (National Council of Teachers of Mathematics [NCTM], 2000; Sarama, Clements, Swaminathan, McMillen, & Gómez, 2003). But perhaps most importantly, geometry and measurement provides a window into the fundamental structures upon which knowledge in the domain of mathematics is built: deductive reasoning and proof.

Geometry and measurement are typically treated as two distinct content areas in mathematics education. However, there is significant overlap between the two, and the content at the intersection of geometry and measurement is particularly salient in the middle grades. Measurement experiences at the elementary grades typically focus on empirical measurements and calculation of perimeter, area, and volume. In order to develop generalized understandings of these quantities, formulas and concepts need to be developed that connect measurement to the general features of geometric figures. For example, understanding that all prisms contain two bases and a height allows students to conceptualize the volume of any prism as the area of its base times the height. Moreover, the misconceptions students often exhibit about geometry and measurement lie at this intersection. Students often have difficulty reasoning about and from general characteristics of shapes, instead relying on empirical results even when they have made a solid deductive argument (Knuth, 2002a; Mayberry, 1983). As such, students studying geometry and measurement topics in the middle grades need to progress from observation to informal deduction to arguments that set the stage for formal deduction; from examples of shapes to considering their properties and relationships between them; from numeric calculations of perimeter, area, and volume to the development of patterns, generalizations, and links to algebra (NCTM, 2000).

To date, student achievement in geometry and measurement at the middle school level has been poor. Results from the 1999 Third International Mathematics and Science Study Repeat (TIMSS-R) show that United States eighth-grade students scored lower in the content areas of geometry and measurement than any other mathematics content area (National Center for Educational Statistics [NCES], 2000). Moreover, the United States ranked 27th of 38 nations surveyed in geometry and 23rd of 38 nations in measurement (NCES, 2000). Domestically, the picture is not significantly better – results from the 2000 National Assessment of Educational Progress show minimal gains in geometry and measurement, with average scale score gains of only 13 points since 1990, representing the lowest scores of any content area and scores that are well below the proficiency threshold (Sowder, Wearne, Martin, & Strutchens, 2004). This low performance in geometry and measurement represents a historical trend over the last 15 years (Kenney & Kouba, 1997; Lindquist & Kouba, 1989a, 1989b; Strutchens & Blume, 1997).

Specifically, eighth grade students had difficulty with items related to giving non-examples of specific geometric figures; the construction of nets for a rectangular solid; measurement items related to perimeter, area, surface area and volume; and items requiring deductions based on a set of geometric properties (Martin & Strutchens, 2000; Sowder, Wearne, Martin, & Strutchens, 2004).

One significant reason for the disappointing student performance in geometry and measurement at the middle grades is the limited opportunities made available to students to engage in meaningful learning of these topics. Only 65 percent of students tested on the 1999 TIMSS-R had teachers who reported that their students were taught geometry in eighth grade. A survey of 12 popular middle grades textbooks, both traditional and reform-oriented, shows that while most curricula provide adequate coverage of geometry skills, only 1 was given a rating of "most content" with respect to geometry concepts (AAAS, 2000). Results from the 2000 NAEP showed 25% of the students assessed had teachers who reported a placing heavy emphasis on geometry and measurement, up 1% from 1996; however, this increase in emphasis was not commensurate with the nominal student achievement gains in these content areas (Grouws, Smith, & Sztajn, 2004; Sowder, Wearne, Martin, & Strutchens, 2004). Moreover, it is not clear from these data how increased emphasis translates in terms of classroom instruction.

While what students have the opportunity to learn is determined by what teachers actually teach and how they teach it, decisions regarding what is taught and how are guided in part by teachers' knowledge and experience with specific content. Middle grades teachers typically have very little experience in geometry and measurement beyond their K-12 schooling, and often exhibit significant gaps in their knowledge of geometry and measurement (e.g. Fuys, Geddes, & Tischler, 1988; Hershkowitz, Bruckheimer, & Vinner, 1987; Usiskin, 1987). Surveys accompanying the 2000 NAEP indicated that only 65% of grade 8 students had mathematics teachers who reported taking one or more college courses in geometry, and 30% had teachers who reported in-service training in geometry; 6% had teachers who reported that they had little or no exposure to the topic (Grouws, Smith, & Sztajn, 2004). The picture is even more grim for measurement, with 43% of students having teachers who reported taking one college course or more, 33% of students having teachers who reported in-service training in measurement, and 10% of students having teachers who indicated little or no exposure to measurement topics (Grouws, Smith, & Sztajn, 2004). Teachers surveyed under the auspices of the 1999 TIMSS-R were asked to indicate their comfort teaching specific mathematical topics; of the 12 topics surveyed, 4 of the lowest-rated 5 topics related to geometry and measurement: measurement–units, instruments, and accuracy; geometric figures–definitions and properties; geometric figures–symmetry; and coordinate geometry (NCES, 2000).

Another possible reason for poor student performance in geometry and measurement relates to the type of instruction typically associated with geometry and measurement. Teachers tend to teach geometry and measurement topics in ways that emphasize rote memorization of properties and rules rather than ways that develop conceptual understanding and gradually move students toward developing formal deductive reasoning (Fawcett, 1938/2004; Fuys, Geddes, & Tischler, 1988; NCTM, 2000). This often results in students developing fragile, procedurally-based understandings that can serve as impediments to meaningful understanding of geometry and measurement topics. For example, a series of studies related to the volume of three-dimensional cube buildings found that students who learned the volume formula $l \times w \times h$ without an attachment to meaning had more misconceptions and greater difficulties in developing a robust understanding of volume (Battista & Clements, 1996, 1998; Battista, 2002). Teaching methods that involve such rote learning do not capitalize on students' existing understandings and build on student thinking, an aspect of pedagogy recommended by recent reform documents (e.g., NCTM, 2000) and shown to be effective in increasing student learning, particularly with master teachers (e.g., Lampert, 2001).

Given the importance of geometry and measurement in the mathematical domain, the historically poor performance by middle grades students on national and international tests in these content areas, and the weak background of many middle grades teachers in geometry and measurement, it seems clear that both preservice and practicing teachers could benefit from additional opportunities to consider both the content and the pedagogy of geometry and measurement. The following sections provide an argument regarding the geometry and measurement content that should comprise such a course and the way in which a course could be designed so as to maximize teachers' opportunities to learn this content.

1.1. Knowledge Needed for Teaching Geometry and Measurement

What knowledge do teachers need to teach geometry and measurement in the middle grades in ways that facilitate deep and significant student learning? To explore this question, I first turn to a more general one: what is the nature of the knowledge needed for teaching mathematics? Begle's (1979) analysis of mathematics teachers' college coursework showed that more exposure to mathematics content does not necessarily translate into improved student learning. Since Begle's analysis, several researchers have studied and attempted to characterize specific facets of knowledge needed in the teaching of mathematics. Conceptualizations such as Shulman's (1986, 1987) *pedagogical content knowledge*, Stein, Grover, and Henningsen's (1996) *cognitive demands of mathematical tasks*, Ma's (1999) *profound understanding of fundamental mathematics*, Leinhardt and colleagues' *instructional explanations* and *instructional* *dialogues* (Leinhardt, 2001; Leinhardt & Steele, 2005), and Sherin's (2002) *content knowledge complexes* are useful ways to think about the unique character of the knowledge teachers need to use and access in the act of teaching mathematics.

One way of conceptualizing and unifying this unique blend of mathematical and pedagogical knowledge is as *knowledge needed for teaching* (Ball, Bass & Hill, 2004; Ball, Lubienski, & Mewborn, 2001). Ball, Bass, & Hill (2004) describe knowledge needed for teaching as the "mathematical knowledge entailed by the work of teaching mathematics" (p. 6). To engage in the meaningful study of the knowledge needed for teaching, researchers need to be able to describe this knowledge and situate it in the actual work of teaching, specifying where and how this knowledge is brought to bear on mathematics teaching in the classroom. Knowledge needed for teaching encompasses knowledge of mathematical content, pedagogical content knowledge that resides at the intersection of mathematics and teaching, and more general pedagogical knowledge that supports the structures and activities inherent in the teaching of mathematics.

In seeking to measure knowledge needed for teaching, Ball and colleagues articulated two types of knowledge that I term *knowledge of mathematics and mathematical activities*, and *knowledge of mathematics for student learning*. Knowledge of mathematics and mathematical activities includes *content knowledge in the domain*, the knowledge of mathematics and the domain of mathematics that learners and doers of mathematics possess, and *content knowledge for teaching*, which includes examples, explanations, representations, and multiple solutions for the key problems and concepts within the mathematical domain (Ball, Bass, & Hill, 2004). Content knowledge for teaching is a type of knowledge that is unique to the work of teaching; most mathematicians and other doers of mathematics have little need for such knowledge.

Knowledge of mathematics for student learning relates to the way a specific population of students might think about and do mathematics. The knowledge needed to analyze student work and the questions one might ask to move a student's work forward are examples of knowledge of mathematics for student learning. Each of these facets of knowledge needed for teaching can be measured with respect to particular mathematical content; for example, the knowledge of mathematics needed for teaching patterns and functions in algebra differs from the knowledge of mathematics needed for teaching geometry or measurement.

In addition to the two facets of knowledge needed for teaching identified by Ball and colleagues, there exist a set of practices that are seen in expert teachers that help them do the work of teaching; I term this facet of the knowledge base *knowledge of practices that support teaching*. This is the knowledge that teachers use to manage daily classroom activity. These practices that support teaching can take many forms. One particular practice that has been investigated by researchers across content areas and grade levels is the use of routines. Routines are socially shared and scripted pieces of behavior that serve a variety of functions in the classroom (Bromme, 1982; Bromme & Brophy, 1986; Leinhardt & Steele, 2005; Leinhardt, Weidman, & Hammond, 1987; Yinger, 1979, 1980, 1987). Specifically, routines can serve to support the activity and learning of students in the classroom, facilitate and organize the exchange of information in the form of classroom discourse, and provide tools for the management of students and materials in the classroom. Routines are particularly interesting to examine, as previous research has found that expert teachers have a range of well-developed routines that that have specific goals and facilitate effective teaching, while novices appear to have a more limited range of routines that are less often linked to goals and are more likely to break down over time (Leinhardt, Weidman, & Hammond, 1987). Moreover, routines seem to transcend teaching style; teachers in more traditional didactic mathematics classrooms exhibit the same general categories of routines as do teachers in more reform-oriented, discourse-based classrooms, although the distribution and specific content of the routines may vary (Leinhardt & Steele, 2005; Leinhardt, Weidman, & Hammond, 1987). Routines are more general with respect to mathematical content than the other two facets of knowledge needed for teaching, as they tend to be similar across topics and even across content areas. However, routines do manifest during the teaching of content and operate in the service of particular content goals, thus serving to move the mathematical activity in the classroom forward.

As this study seeks to explore knowledge needed for teaching in the context of middle grades geometry and measurement topics, it is necessary to examine each of the three facets of knowledge needed for teaching in that particular content. The sections below provide additional detail about each of these facets of knowledge in the context of middle grades geometry and measurement, linking to previous research relevant to each facet.

1.1.1. Knowledge of Mathematics and Mathematical Activities

A logical starting point for any examination of knowledge needed for teaching a piece of mathematical content is to determine what students and teachers know and need to know with respect to the content itself. Research indicates several topics are particularly problematic at the middle grades for both students and teachers: relationships between measurable quantities of geometric figures (e.g., dimension, perimeter, and area; dimension, surface area, and volume), and deductive reasoning and proof. These mathematical topics are discussed with respect to both content knowledge of the domain and content knowledge for teaching.

<u>1.1.1.1. Content knowledge in the domain: Relationships between measurable attributes of geometric figures</u>. Measurable attributes of geometric figures – dimension, perimeter, and

area; and dimension, surface area, and volume – are keystones of the measurement strand of school mathematics in the elementary grades. Elementary students learn how to calculate area and perimeter of specific examples of two-dimensional shapes, often through the use of a formula. Students entering the middle grades are also likely to have had empirical experience determining volume of shapes and comparing the volume of specific shapes by filling solids with sand or rice (NCTM, 2000). In the middle grades, empirical measurement knowledge can be connected to geometric properties of shapes to develop more generalized and robust understandings of the relationships between these measurable attributes. In so doing, students use their knowledge of empirical measurement attributes and the relationships between them to generalize about relationships between geometric figures. For example, what happens to the surface area if one doubles the height of a rectangular prism? What happens to its volume? Are the results the same if we consider a cylinder? A triangular prism? A square pyramid? And most importantly, what are the characteristics of these shapes that make these relationships generalizable? To date, there is substantial evidence that middle grades students and teachers have a limited understanding of perimeter, area surface area, and volume, both as individual topics (as previously noted) and with respect to the connections between dimension, perimeter, and area, and dimension, surface area, and volume.

Several research studies have sought to understand what students and teachers know about perimeter and area. One significant line of research has focused on the van Hiele levels of geometric understanding, a framework developed by a pair of Dutch mathematics educators in the 1950's. (Additional detail on the van Hiele levels can be found in Chapter Two.) A comprehensive study of the van Hiele levels by Fuys, Geddes, & Tischler (1988) showed that above-average sixth and ninth grade students exhibited significant misconceptions related to perimeter, area, and volume and had difficulty distinguishing proper units for each measurement. Similar results have been reported by other researchers (e.g. Bright & Hoeffner, 1993; Clements & Battista, 1989; Chappell & Thompson, 1999; Hoffer, 1983; Martin & Strutchens, 2000; Sarama et al., 2003). Understanding of area and perimeter in these studies was largely constrained to the rote application of formulas and procedures; often, students confused formulas for perimeter and area and did not have a conceptual understanding that allowed them to untangle the confusion. This difficulty in calculating and distinguishing between area and perimeter suggests that the concepts are not well-connected to properties of the shapes in question.

With respect to surface area and volume, several studies investigating spatial sense have revealed that students have significant difficulty coordinating the numeric models and operations with visual models; this difficulty impedes students' development of robust understandings of surface area and volume (e.g. Battista & Clements, 1998; Clements, Battista, Sarama, Swaminathan, & McMillen, 1997). Specifically, students have difficulty coordinating a net-based, two-dimensional representation of the surface area of a rectangular cube building (rectangular prism) with an isometric representation of the same cube building to find the number of cubes (volume) needed to construct the building. In calculating volume, students often initially count the squares in the net representation, which results in an over-count of cubes with outside faces, and an under-count (or no count at all) of the interior squares (Battista, 1999; Battista & Clements, 1996, 1998). The application of the typical formula for volume of a rectangular prism, $V=l\times w\times h$, is of limited utility, as the formula by itself does nothing to help students coordinate the views of the rectangular cube building and the notions of surface area and volume; in fact, rote application of the formula may hinder the development of spatial structuring (Battista, 1999, 2002). Only after students are able to make sense of the rectangular cube building as a layering structure and coordinate the net and isometric views are they consistently successful in calculating volume, which represents the first step towards being able to understand how changes in the attributes of the rectangular prism impact both surface area and volume.

Research into teachers' understandings of content knowledge related to dimension, perimeter, and area and dimension, surface area, and volume has shown similar results. Fuys, Geddes, & Tischler (1988) also examined teachers' understandings of perimeter and area; results indicated that teachers held misconceptions about area and perimeter that mirrored their students' own difficulties. Specifically, teachers had difficulty with area concepts, often thinking that the knowledge that the sum of the three angles in a triangle is 180° would help them determine the area of a triangle (Fuys, Geddes, & Tischler, 1988). Teachers also had significant difficulty determining the proper units for perimeter, area, and volume, mirroring a classic student difficulty in geometry and measurement. This result echoed the findings of Hershkowitz & Vinner (1984), who concluded that teachers held conceptions of geometry that were similar to typical middle school students, and Mayberry (1983), whose study of teacher knowledge with respect to the van Hiele levels revealed that teachers' own understandings of the content resided largely at the first two of the five levels. In a more detailed study conducted by Simon & Blume (1994), teachers exhibited confusion between perimeter and area and had difficulty finding and understanding each quantity when asked to cover a desk with regular rectangular index cards.

<u>1.1.1.2. Content knowledge for teaching: Relationships between measurable attributes of geometric figures</u>. In addition to being able to understand how changing one measurable attribute of a geometric figure impacts the other attributes, teachers also need to understand the

misconceptions, strategies, and developmental trajectories that students are likely to exhibit in their exploration of these ideas. Very few studies have systematically examined teacher knowledge of geometry and measurement topics; still fewer have investigated the sort of content knowledge that teachers need to teach about the relationships between measurable attributes of geometric figures. One study which did seek to investigate such knowledge with respect to geometry was conducted by Swafford, Jones, and Thornton (1997), utilizing the van Hiele levels as a component of a framework for an intervention designed to increase teacher knowledge in geometry. This intervention, aimed at practicing middle grades teachers, involved a content course in geometry and a research seminar that aimed to acquaint teachers with the van Hiele levels. Teachers showed significant growth in geometry knowledge on van Hiele assessments, which included perimeter, area, surface area, and volume, and the relationships between such quantities and the attributes of geometric figures. (Specific results by topic were not reported.) Teachers also became more aware of the van Hiele levels and their potential impact on classroom instruction, and were more willing to make modifications to geometry tasks in a way that raised the level of thinking required of students. Classroom observations also showed teachers exhibiting a greater confidence and a willingness to try new ideas and instructional methods in geometry. The increased use and quality of classroom discourse was cited as a particularly important factor in the changes seen in teachers' geometry instruction. These results, particularly with respect to teachers' selection and modification of textbook tasks, connect to more general notions of the importance of using mathematical tasks that require a high level of cognitive demand (e.g. Stein, Grover & Henningsen, 1996). Based on the descriptions of the van Hiele levels, tasks that require an increased level of student thinking as specified by the van Hiele levels are also tasks that would be categorized as requiring a high level of cognitive demand. Identifying these tasks, or modifying existing tasks to embody high cognitive demand, is a critical aspect of content knowledge for teaching.

Another notable result is derived from the research on spatial sense conducted by Battista, Clements, and colleagues. Specifically, Battista & Clements (1998) found that teaching the formula for volume of a rectangular prism could be problematic if it occurred before students were able to coordinate representations of rectangular prisms. Their work revealed that when 5th grade students were asked to find the surface area and volume of rectangular prisms constructed of cubes, no student who used a formula was able to connect the formula to a spatial structuring of the cube building. While their study did not focus on teachers, their findings identify the sort of content knowledge that teachers need to effectively teach concepts of surface area and volume. By understanding how spatial reasoning develops, and the misconceptions that may develop through the premature use of the volume formula, teachers will be better able to select and sequence instructional tasks and respond to student thinking in ways that foster deep conceptual understanding of volume and surface area.

1.1.1.3. Content knowledge in the domain: Reasoning and proof. While proof is often conceptualized as a particular exervise exemplified by the two-column deductive form often used in high school geometry, reasoning and proof represents a mathematical practice that transcends mathematical content areas (Hanna, 1989, 1991, 1995; NCTM, 2000; Schoenfeld, 1994). Making informal and formal deductive arguments about properties of geometric shapes is cited as a critical skill at the middle school level, particularly in setting the stage for success in later study of geometry (NCTM, 2000). With respect to content knowledge in the domain, research has shown that both students and teachers have difficulty with deductive reasoning and proof practices.

The van Hiele level study conducted by Fuys, Geddes, and Tischler (1988) also investigated issues of deductive reasoning and proof. Students were asked both to make deductive arguments involving properties of geometric shapes and to listen to and evaluate oral deductive arguments made by the interviewer. Even when correct deductive arguments were made by students or by interviewers, students generally remained unconvinced of the generality of statements. Instead, they felt the need to continue to use examples to test the rule, or conjectured that the rule would most likely hold, but not always. Senk (1985, 1989), in a study of proof with respect to van Hiele levels, found very few 9th and 10th grade students able to construct deductive arguments of any sort, even after a high school course in geometry. Chazan (1993) found that 9th and 10th grade students had a difficult time understanding the role of proof, particularly in distinguishing between proof and evidence. The notion of a deductive proof ruling out counterexamples and holding for a general class of cases was particularly problematic.

Similar results were noted for teachers in a study by Knuth (2002a). Teachers were shown various examples of proofs, both complete deductive proofs and various other proof-like arguments, such as the empirical testing of examples. Many teachers favored the empirical arguments over the deductive ones, finding them more convincing or easier to follow. Perhaps more important was the finding that nearly half of teachers surveyed indicated that even after proving a statement deductively, one might still be able to find a counterexample to nullify the proof.

<u>1.1.1.4. Content knowledge for teaching: Reasoning and proof</u>. In addition to his study of teacher knowledge of proof, Knuth (2002b) conducted a companion study regarding teachers' conceptions of the role of proof in school mathematics. The findings indicate that teachers hold a limited view of the utility of proof in secondary school. Semi-structured interviews of

secondary mathematics teachers showed that teachers view proof largely as a specific topic of study rather than as a tool for doing mathematics or as a stance towards mathematics in general (Knuth, 2002b). This contrasts with the practice of mathematicians, where deductive reasoning and proof are used to justify new mathematical results as well as to verify known or proposed results (Hanna, 1989). This limited view of proof held by teachers likely exacerbates the view of proof as a form and activity, and thus the classic procedural replication of two-column geometry proofs that is pervasive in American schools and has been for close to a century (Fawcett, 1938/2004).

In addition to the purposes that reasoning and proof serve in mathematics (justification and verification), reasoning and proof serve an additional purpose in mathematics education: explanation (Hanna, 1989). This role is exemplified in the classroom through the establishment of a classroom climate that both demands and values justification for responses, evaluation and critique of the responses of others, and a press for mathematical meaning (e.g., Boaler & Humphreys, 2005; Lampert, 2001). To support a discourse-based classroom in which proof and reasoning are part of the norms, a teacher needs more than knowledge of mathematics; a teacher also needs knowledge of mathematics for student learning, and a set of practices that support teaching.

1.1.2. Knowledge of Mathematics for Student Learning

In order to create opportunities for a group of students to engage in meaningful mathematics, teachers need knowledge about their students and their understandings in relation to the mathematical content. Knowledge of mathematics for student learning cannot be taught directly, as it relates to a particular group of students; however, the collection and use of such knowledge can be modeled, and strategies can be presented to teachers that facilitate the collection and use of knowledge of mathematics for student learning. One promising model for considering such knowledge is the five practices for productive use of student thinking in discussions (Stein, Engle, Hughes, & Smith, submitted). These five practices include *anticipating student solutions*, *monitoring student work* (including questioning to assess and advance student thinking), *selecting, sequencing,* and making *connections* between student responses for public display and class discussion. While these practices have not been studied in the context of geometry and measurement, some are implicit in other interventions related to geometry and measurement content (e.g. Swafford, Jones & Thornton, 1997), and have shown promise in the context of practice-based courses related to proportional reasoning and algebra as the study of patterns and functions in the middle grades (Hughes & Smith, 2004; Stein, Engle, Hughes, & Smith, submitted). These practices are described in further detail in Chapter Two.

1.1.3. Knowledge of Practices that Support Teaching

In addition to possessing knowledge of mathematics and mathematical activities and knowledge of mathematics for student learning described in the previous sections, teachers also need ways of structuring and organizing the everyday activity of the classroom that advance the mathematical activity of the class. Effectively accessing and applying knowledge needed for teaching in the classroom requires structures that condense and routinize the recurring components of teaching, making them automatic and implicit rather than a recurring focal point of the lesson for teachers and students (Leinhardt & Greeno, 1986; Leinhardt & Steele, 2005). This facet of the knowledge base for teaching, which operates in the service of the facets previously discussed, can be conceptualized as *knowledge of practices that support teaching*.

In their investigation of the instructional dialogues of Magdalene Lampert, a highly skilled teacher whose classroom embodies meaningful learning and uses student thinking in a powerful way, Leinhardt and Steele (2005) noted several practices that Lampert used that supported her teaching. One type of practice that was particularly salient was her use of routines. Routines are shared socially scripted behaviors that serve to organize the activity of the classroom in particular ways, and have been shown to be characteristic of expert teaching in that such teachers organize repetitive tasks through scripts that embody both the procedure of the task and the goal (Leinhardt, Weidman, & Hammond, 1987). The routines identified in Lampert's teaching, and the teaching of other expert teachers, could be placed in three general categories: *support*, *exchange*, and *management* routines. (These categories are described in detail in Chapter Two.)

These moves operate in the context of the classroom and in the service of content goals, and as such, are best observed and understood while embedded in a classroom teaching episode that features specific mathematical goals. While the gestalt of routines tends to be consistent across content areas, the ways in which they serve to advance mathematical goals may differ depending on the content. These practices have been observed in numerous expert teachers across years, academic contexts, and content areas (e.g. Leinhardt, 1993, 2001; Leinhardt, Weidman, & Hammond, 1987). Teachers seem to develop these practices largely through experience, as novice teachers do not exhibit these practices when they exit the academy and enter the teaching profession. These routines and the ways in which they relate to the teacher's goals for the mathematical activity of the classroom are often tacit, yet deep links between routines and goals tend to exist in expert teachers. Routines often go unexamined in the examination of teaching, even when discord between a teacher's goals and their routines exists. Such conflicts frequently occur when experienced teachers change their practices and views of learning, while retaining routines that adhere to previous models.

Leinhardt and colleagues suggest that making novice teachers aware of routines, and specifically asking them to identify routines in the teaching of more experienced teachers, may be a fruitful avenue for fostering such routines in the novice's own teaching (Leinhardt, Weidman, & Hammond, 1997). By modeling these instructional moves embedded in the mathematical work of the classroom, and raising them as objects of inquiry after the moves have been made, it is hypothesized that novice teachers will see the utility of and develop competency with these practices at an earlier stage in their career. In the next section, I describe how these practices, as well as the other two facets of knowledge for teaching mathematics, can be woven into a teacher learning experience.

1.2. Impacting Knowledge for Teaching Mathematics: Teacher Learning

While preservice teacher education programs and on-going professional development experiences are the primary vehicles for teacher learning, both have limitations. In addition to the limited contact that college and university educators have with preservice teachers, there is evidence that typical coursework at the academy is not the most significant influence on teacher knowledge and ultimately teachers' instructional practice (e.g., Brown & Borko, 1992). One of the most common explanations for the academy's limited impact is the notion that the types of experiences provided by courses are not closely related and applicable to the work of teachers in the classroom (e.g., Cochran-Smith & Lytle, 1999). Recent surveys of professional development experiences offered to practicing teachers find that these experiences are generic, limited in scope and utility, and account for a very small number of contact hours per year (Grouws & Smith, 2000; Smith, 2001a). These professional development experiences tend to be additive, seeking to give teachers new mathematical and pedagogical ideas to supplement their existing classroom practice. In contrast, Thompson and Zeuli (1999) argue that to create meaningful teacher change, teachers need to be engaged in transformative experiences that allow them to examine their own knowledge, beliefs, and habits of practice as an object of inquiry rather than adding to an existing repertoire of skills.

One promising avenue for engaging teachers in these transformative experiences is through the use of practice-based materials. Practice-based materials are situated in the artifacts and practice of classroom teaching, and thus hold great promise as powerful resources for teacher learning (Ball & Cohen, 1999; Shulman, 1986). Practice-based materials are materials created for or during the practice of teaching, and include lesson plans, student work, and other artifacts of the practice of teaching (Ball & Cohen, 1999; Smith, 2001a). Some recent efforts to improve teacher education in the United States have placed practice-based materials at the center of the professional education of teachers. A significant line of this work has focused on the use of case-based materials, in the form of narrative or video accounts of teachers engaging in lessons with their students. The use of cases with teachers offers opportunities to focus in on the particulars of an episode of teaching, to reflect critically on the teaching event with respect to teachers' own practice, and to draw general conclusions about the act of teaching (Sykes & Bird, 1992). The study of teacher learning from cases is still in its infancy, but preliminary research suggests that cases represent particularly powerful sites for teacher learning at the intersection of content and pedagogy (e.g. Barnett, 1991, 1998; Lundeberg, Levin, & Harrington, 1999; Smith, 2001b). Because cases and other practice-based materials bring the work of teaching to the fore as an object of inquiry, these materials hold great potential for teachers to develop the unique facets of knowledge needed for teaching.

While the issue of design and design principles has received extensive attention with respect to classroom teaching (e.g., Brown, 1992; Cobb, 2001), the design principles behind professional development experiences, and specifically the class of practice-based professional development experiences, have largely remained implicit. Despite the fact that principles may not be explicitly stated, the views that a teacher educator holds about learning, both in general and specific to particular content, impact the design and enactment of the professional development experiences. Design principles can serve a number of important functions in educational research; they can serve as a means for testing and refining theory (e.g., Cobb, 2001) for articulating new theories (e.g., Edelson, 2002, Kelly & Lesh, 2000), or to articulate a set of principles that serve as a framework that describes the characteristics of a particular learning experience designed to produce a particular set of learning outcomes (van den Akker, 1999). It is in van den Akker's sense that design principles can serve an important role in teacher education, by unifying the activity selection and instructional decisions in the professional development experience and linking these decisions to general learning theories and to particular intended learning outcomes for teachers. The articulation of a set of design principles can also allow other teacher educators to create similar professional development experiences in ways that go beyond the appropriation of particular activities.

1.3. Purpose of the Study

This study aims to measure teachers' knowledge needed for teaching geometry and measurement before and after participation in an intervention designed to engage preservice and practicing teachers in the exploration of geometry and measurement content, the consideration of ways to further student understanding of geometry and measurement, and reflection on their own teaching practice. The study focuses specifically on knowledge of mathematics content identified in the literature as problematic both for teachers and students. These topics include relationships between measurable attributes of geometric figures (e.g., dimension, perimeter, and area; and dimension, surface area and volume), proof and deductive reasoning. The study also seeks to understand what teachers learn with respect to knowledge of mathematics for student learning primarily through examining narrative and video cases of teaching that feature geometry and measurement content. Finally, the study seeks to raise awareness of the practices that support teaching and are characteristic of expert teachers, and how these practices serve to advance the mathematical activity of the classroom.

These learning goals were realized through the design, enactment, and analysis of a course on geometry and measurement in the middle grades taught to preservice and practicing teachers at the graduate level. The design and implementation of the course utilized the *knowledge needed for teaching* framework and takes into consideration the findings summarized previously relating to student and teacher knowledge of geometry and measurement. The course was designed and implemented based on an implicit set of design principles for practice-based teacher education. Materials for the course included a variety of cognitively challenging mathematical tasks (Stein, Grover, & Henningsen, 1996; Stein, Smith, Henningsen, & Silver, 2000), narrative and video cases of teaching that feature middle grades geometry and measurement content (Smith, Silver, Stein, Boston, & Henningsen, 2005), other artifacts of teaching including student work, and research articles related to teaching and the content of the course.

1.4. Research Questions

The study aimed to answer the following questions:

1a. What knowledge of mathematics and mathematical activities relating to geometry and measurement identified in (i), (ii), and (iii) do teachers have before and after participation in a course focused on these ideas?

i. relationships between dimension, perimeter, and area?

ii. relationships between dimension, surface area and volume?

iii. concepts of proof and deductive reasoning?

1b. To what extent do teachers' experiences in a course on geometry and measurement appear to influence changes in key aspects of knowledge of mathematics and mathematical activities?

2a. What knowledge of mathematics for student learning relating to geometry and measurement do teachers have before and after participation in a course focused on these ideas?

2b. To what extent do teachers' experiences in a course on geometry and measurement appear to influence changes in key aspects of knowledge of mathematics for student learning?

3a. What knowledge of practices that support teaching in the context of geometry and measurement do teachers have before and after participation in a course focused on these ideas?

3b. To what extent do teachers' experiences in a course on geometry and measurement appear to influence changes in key aspects of teachers' knowledge of practices that support teaching?

4. In what ways are teachers who participated in a course on geometry and measurement different from teachers who did not participate in the course with respect to knowledge needed for teaching in the domain of geometry and measurement?
5. What are the design principles that undergird the planning and enactment of a practice-based teacher education experience?

Research questions 1, 2, and 3 were designed to investigate changes in the three facets of knowledge needed for teaching for teachers enrolled in a practice-based course focused on middle grades geometry and measurement content. Each of these three questions contains two parts, one which sought to characterize teachers' knowledge pre-course and post-course and second which sought to trace these changes to experiences in the course. Question 1 focused on knowledge of mathematics and mathematical activities, specifically related to relationships between measurable attributes of geometric figures and proof and reasoning. Question 2 focused on knowledge of mathematics for student learning, specifically the five practices for productive use of student work. Question 3 focused on knowledge of practices that support teaching, and specifically on identifying routines and reflecting on the purpose of routines in the classroom. Question 4 compared the post-course knowledge of teachers enrolled in the course to a contrast group comprised of teachers with a similar background to further contextualize teacher learning as a result of the course experience. Question 5 sought to articulate a set of design principles for practice-based teacher education experiences, to make salient the aspects of teaching and learning theory that were operationalized by the course, and to aid in the development of subsequent practice-based experiences by other educators and researchers.

1.5. Contribution to the Field

This study contributes to the field of mathematics education in a number of ways. First, it adds to the knowledge base about teachers' content knowledge of geometry and measurement, an area that has traditionally been under-researched. Second, it adds to the understandings of teachers' knowledge of students and mathematics, specifically as it relates to the development of robust student understandings of geometry and that may support students' development of deductive reasoning. Third, it built significantly on existing research programs while bringing additional theoretical frameworks to bear on them. In particular, the course built on a model of instructional design developed as part of the ASTEROID (A Study in Teacher Education: Research on Instructional Design) Project, a National Science Foundation-funded project (ESI-0101799) directed by Margaret S. Smith and housed in the School of Education at the University of Pittsburgh. Additionally, the study used materials developed as part of the NSF-funded COMET (Cases of Mathematics to Enhance Teaching) Project. Both these projects built on the work of the QUASAR (Qualitative Understanding: Amplifying Student Achievement and Reasoning) Project. From this foundation, this study also adapts and integrates the body of work on the van Hiele levels of geometric thought, spatial reasoning, theories on the development of deductive reasoning and proof, and teaching frameworks related to the development of a discourse-based student-centered pedagogy into a coherent model of the knowledge needed for teaching geometry and measurement. By integrating these perspectives, this study adds to the literature on teacher knowledge, teacher learning, and teacher change.

1.6. Limitations

This study has several limitations. First, it was conducted on a sample of convenience, consisting of Masters level students at a competitive university in the Northeast United States. This sample may not be representative of preservice and practicing teachers in the United States in general. Second, the mathematical content in the course represented a slice of the content of geometry and measurement. The results of the study may not generalize to the remainder of the domain or other content areas; however, previous similar studies conducted as part of the ASTEROID Project suggests a measure of generalizability. Third, the instructor of the course was a relative novice teacher educator. A more experienced teacher educator may produce different results. Finally, this study did not follow teachers into their classrooms to assess the impact of the course on their actual classroom practice. This is an avenue that merits investigation but was beyond the scope of the current study.

This document is organized in five chapters. Chapter One provides a general introduction to the study. Chapter Two reviews the body of literature related to student learning in geometry and measurement, teacher knowledge in geometry and measurement, and teacher learning more generally. Chapter Three outlines the methods used in the study and the data collected. Chapter Four provides the results of the study. Chapter Five discusses implications of the study and suggests avenues for future research.

2. REVIEW OF THE LITERATURE

In this chapter, a review of the literature relevant to the current study is presented. First, an argument is made for designing a course for teachers focused on the content of middle grades geometry and measurement. This includes a discussion of the state of curriculum, instruction, and assessment with respect to middle grades geometry and measurement as discrete topics and considering what lies at the intersection of the two. Then the *knowledge needed for teaching* framework used in this study is justified, and each facet of the knowledge needed for teaching framework will be elaborated with respect to relevant research. This includes knowledge of mathematics and mathematical activities, knowledge of mathematics for student learning, and knowledge of practices that support teaching. Finally, a review of relevant research on teacher learning and the design of teacher education experiences is presented.

2.1. Why Focus on Middle Grades Geometry and Measurement?

Geometry and measurement share many characteristics with respect to their positioning in K-12 education. Both geometry and measurement involve ways of making sense of the physical world: geometry through reasoning about characteristics and relationships of shapes and spaces around us, and measurement through the assignment of numerical values to attributes of objects. Both geometry and measurement are often taught as discrete topics, in isolation from other mathematics, with concepts and procedures being reviewed year after year in near-identical manners (Fuys, Geddes, & Tischler, 1988). Both geometry and measurement are scarce in the high school grades, with geometry often restricted to a single course in 9^{th} or 10^{th} grade and measurement opportunities located largely in the science disciplines (NCTM, 2000; Senk & Thompson, 2003). Both geometry and measurement have been subject to superficial, procedural treatment in typical textbooks and curricula until the mid-1990s (Fuys, Geddes, & Tischler, 1988; Senk & Thompson, 2003; Thompson & Senk, 2003). And perhaps most importantly, student performance on national and international assessments have consistently shown poor performance on items related to geometry and measurement (NCES, 2003; Sowder et al., 2004), limited opportunities for students to learn geometry and measurement (Grouws, Smith, & Sztajn, 2004; NCES, 2003), and relatively low levels of teacher knowledge as measured by coursework experiences, professional development, and teaching comfort level in geometry and measurement (Grouws, Smith, & Sztajn, 2004).

In the sections below, additional detail is provided on the standards and curriculum in the middle grades for geometry and measurement separately. This is followed by an argument for integrating aspects of both topics in a single learning experience.

2.1.1. Geometry in the middle grades: Standards, curriculum and instruction

According to the most recent NCTM Standards (2000), geometry in the middle grades should focus on "investigat[ing] relationships by drawing, measuring, visualizing, comparing, transforming, and classifying geometric objects," and provide a venue for the development of reasoning in mathematics, particularly "inductive and deductive reasoning, making and validating conjectures, and classifying and defining geometric objects" (p. 233). The Standards go on to state that "[m]any topics treated in the Measurement Standard for the middle grades are closely connected to students' study of geometry" (NCTM, 2000, p. 233). Moreover, the Standards recommend that the middle grades provide substantial experiences for students to make connections between geometry and algebra. With extensive experience in elementary grades classifying two- and three-dimensional shapes and finding perimeter, area, and volume, geometric work in the middle grades is positioned to leverage students' measurement skills in the context of comparing and generalizing measurements and relating them to general properties of shapes, elaborating relationships between measurements (e.g. area and perimeter), and creating mathematical arguments that move from specific examples to classes of shapes, from the particular to the general. These activities would serve to position students well for more formal geometric work that is typically a core aspect of the high school curriculum.

These calls for reform represented a considerable shift from the state of middle grades geometry and measurement curriculum at the time of publication of the original NCTM Standards (NCTM, 1989). A text analysis conducted by Fuys, Geddes, & Tischler (1988) of three popular and widely-used K-8 geometry texts in 1980-81 revealed significant deficiencies in the materials available to teachers and students. The analysis showed that few exercises required a level of thinking beyond visual identification and differentiation of shapes. Most topics were repeated year to year in a way that did not significantly develop the content; the authors characterized this treatment as circular, in contrast to a spiraling curriculum that builds concepts from year to year (Fuys, Geddes, & Tischler, 1988). Very few questions required students to write even a sentence to justify their answer.

The text analysis also included a focus on two geometry topics of interest: relationships between properties of shapes, and area. With particular respect to the notion of relationships between properties of shapes, the study found the treatment to be erratic and rote. Often, exercises could be found in different parts of the texts that presented conflicting views of relationships between classes of shapes. In many cases, the teacher was asked to present the relationships for student memorization, rather than having students make sense of the relationships in any way: "[S]tudents may memorize a sentence which they do not really understand, and they are not expected to interpret or apply it in subsequent exercises" (Fuys, Geddes, & Tischler, 1988). With respect to the development of the specific concept of area, very little evidence was found that the texts attended to the development of a conceptual meaning of area formulas or the relationships between area formulas for different shapes. Instead, tackling area was more a matter of finding an area initially by counting squares on a grid, followed by the presentation of a formula for each shape under consideration and repeated application of the formula. Students could complete these activities "by memorizing some fact they may not understand" (Fuys, Geddes, & Tischler, 1988).

While recent reforms in the content of textbook series improved the amount and quality of geometry content for the middle grades, current text series still fail to provide opportunities to engage in the rich geometry learning experiences described in the NCTM Standards. A detailed evaluation of middle grades texts conducted by the American Association for the Advancement of Science (AAAS) in 1999 compared both reform and traditional curricula, including 4 of the 5 NSF-funded Standards-based reform curricula, on a number of content and process dimensions. The two relevant dimensions with respect to geometry are geometry skills, composed of computing circumferences/perimeters, areas, and volumes of common geometric shapes and solids; and geometry concepts, which relate to identifying and working with general properties of shapes and linking these general properties to the particulars of computation. While all curricula but one scored in the highest category (*most content*) with respect to geometry skills, only one of the 12 scored in the highest category with respect to geometry concepts, with two of the curricula scored in the lowest (*minimal content*) of the three categories (AAAS, 2000). This suggests a current picture not unlike the findings of Fuys, Geddes, and Tischler (1988): curricula are available that allow for the practicing of basic computational geometry skills; however, attention to richer conceptual ideas is still limited.

The textbook serves as the primary resource for most teachers of mathematics, and thus, the information contained in the textbook has a profound influence on what mathematics is taught. It is also important to consider how geometry lessons are taught in the middle grades. NAEP self-report surveys indicate that 88% of students assessed have teachers who report giving geometry moderate or heavy emphasis in their Grade 8 classrooms (Grouws, Smith, & Sztajn, 2004). However, data from the Third International Math and Science Study contradicts this self-report data. The 1999 TIMSS video study collected videotaped data from classrooms in 7 different countries, comparing the instruction across a number of dimensions. Of the lessons studied from the U.S., only 9% involved geometry content, excluding measurement (NCES, 2003). The other 6 countries ranged from 19% to 73% (NCES, 2003).

Given the emphasis on skills over concepts in the AAAS results, one might expect what geometry instruction there is to be focused primarily on computational skill and not conceptual understanding. Again, TIMSS supports this conclusion. The TIMSS study also categorized lessons as high, medium, and low processing complexity. Of the two-dimensional geometry lessons observed, only 13% of U.S. lessons were rated as high complexity, 5th among the 7 nations. Moreover, two-dimensional geometry problems seemed to jump from topic to topic in the U.S. sample without making connections to previous knowledge. Excluding the first problem, 43% of problems in U.S. classrooms were unrelated to the previous problem, and an additional 31% were categorized as repetition. This represents the highest percentage of unrelated problems across the 7 nations, and the lowest combined percentage of the other two possible categories, problems that were mathematically and thematically related (NCES, 2003).

These data paint a picture of actual classroom instruction that mirrors the textbook analyses: low level problems requiring little in the way of cognitive demand, featuring repetition and a repertoire of unconnected, unrelated exercises.

2.1.2. Measurement in the middle grades: Standards, curriculum, and instruction

The NCTM Standards (2000) define measurement as “the assignment of a numerical value to an attribute of an object” (p. 44). The study of measurement in the early grades of the K-12 sequence involve understanding what counts as an attribute, and selecting and using a variety of tools in the service of measuring the attribute. Much of this work occurs in the context of measuring everyday objects, then transitioning to measuring particular geometric forms in two and three dimensions. At the middle grades, students need to continue to develop and use formulas for measuring these attributes of shapes, such as perimeter, area, surface area, and volume, in a way that fosters “understanding [of] how these formulas relate to the attribute being measured” (NCTM, 2000, p. 46). The Standards cite experiences transforming and decomposing two-dimensional shapes and creating nets for three-dimensional solids and building solids from nets as key experiences for developing measurement sense. Additionally, the Standards suggest that middle grades students develop understanding of scale and proportionality, which implies direct connections between measurement and geometry. Taken together, this set of experiences serve to connect measurement with geometry primarily through the exploration of characteristics of and relationships between attributes and general properties of geometric figures.

Despite the natural integration opportunities for measurement with geometry and other content areas, measurement is often treated as a separate topic at a particular point in the middle grades curriculum, not to be revisited in the context of other content (NCTM, 2000). Little specific data exist on the quantity and quality of measurement concepts in current middle grades mathematics curricula. A survey of unit descriptions of the 5 NSF-funded Standards-based reform curricula, provided by the ShowMe Center (2005), show that 22 units out of the 146 cataloged across three grades in four of the curricula include measurement as a content objective in some way. Of the 22 identified units, only three include a goal that relates to the relationship between measurement quantities, with all three relating area and perimeter, and only eight explore measurement in the context of scale or similarity, with four of these appearing in a single curricula (Middle School Mathematics through Applications). The remaining units specify goals that are more closely aligned to the traditional treatment of measurement: identification of measurement tools, practice with various units of measurement and conversion, and the measurement and computation of perimeter, area, surface area, and volume in isolation. Additionally, the work on measurement in middle grades curricula decreases from grades 6 to 8. The ShowMe Center survey (2005) shows that of the 22 units identifying measurement as a content strand, 12 are 6th grade units, 6 are 7th grade units, and only 4 are 8th grade units. These data suggest that even in reform-oriented curriculum, which might be expected to be the closest curricular embodiment of the NCTM Standards, the treatment of measurement still leaves much to be desired. From a curricular perspective, measurement is treated as computational, with little effort to draw conceptual connections between measurable quantities and a decrease in attention at the upper middle grades.

Given the resources available to teachers with respect to measurement, it is interesting to consider whether actual classroom instruction attempts to address the conceptual aspects of measurement. The fact that 82% of teachers report moderate to heavy emphasis on measurement at the 8th grade level, combined with the narrow emphasis on measurement in the reform-oriented curricula outlines above, leaves one to wonder what measurement looks like in middle grades classrooms. The 1999 TIMSS Video study identified that 13% of U.S. lessons related to measurement, and more specifically perimeter and area (NCES, 2003). This represented the highest percentage of the 7 countries participating in the survey. However, given than only 6% of all U.S. lessons were rated as high complexity, it is unlikely that the work done in measurement dealt with complex ideas such as the relationship between measurements, or the connection between measurement and abstract ideas such as geometric properties or the development of generalizeable formulas with conceptual understanding.

2.1.3. Summary: Working at the intersection of geometry and measurement

The content of geometry and measurement provides for ample opportunities for students who have advanced past routine measurement of attributes and calculation of perimeter, area, and volume. Indeed, the Standards recommend that a significant part of students' experiences in both geometry and measurement should include activities that lead students to consider the relationships between measurable attributes of geometric figures, and the generalization of these attributes as formulas and rules relating change between quantities such as area and perimeter in a class of shapes. The current state of curriculum and teaching in the middle grades indicates that students may not have opportunities to engage in such tasks. Given also that teachers may have had limited opportunities to engage in tasks that require more than a basic understanding of geometry and measurement skills, focusing on this particular slice of content at the intersection of geometry and measurement has the potential to provide meaningful teacher learning experiences for teachers. Moreover, as is detailed in the sections following, it is at this intersection that many student (and potentially teacher) misconceptions lie. Students often have difficulty reasoning about and from general characteristics of shapes, instead relying on empirical results even when they have made a solid deductive argument (Knuth, 2002a;

Mayberry, 1983). Based on these data, it clear that students use concepts of either geometry or measurement rather than using them in combination to understand the underlying mathematics.

2.2. Knowledge Needed for Teaching Geometry and Measurement

In this section, the knowledge needed for teaching framework is described and elaborated in the context of geometry and measurement at the middle grades. First, the facets of the framework are described, with specific attention to the adaptations that this study brings to the framework and the research-based reasons behind the adaptations. Each facet of the framework is then explored with respect to the specific facets of knowledge needed for teaching geometry and measurement. This includes discussion of knowledge of mathematics and mathematical activities, knowledge of mathematics for student learning, and knowledge of practices that support teaching.

2.2.1. The Knowledge Needed for Teaching Framework

The knowledge that a teacher brings to his or her classroom has a direct impact on what the students in the classroom learn (Armour-Thomas, Clay, Domanico, Bruno, & Allen, 1989; National Commission on Teaching and America's Future, 1996; Hill, Rowan, & Ball, 2004). Researchers have long known that the knowledge needed for teaching consists of more than mathematical knowledge and general content-independent pedagogical strategies – the two poles between which U.S. teacher education has historically vacillated over the past 150 years (Shulman, 1986). Indeed, there is a body of knowledge that lives at the intersections of mathematics and pedagogy, mathematics and students, and pedagogy and students, that is necessary for successful teaching (Shulman, 1986, 1987; Lampert, 2001). This is not simply a third bin of knowledge to be filled; rather, it represents the connective tissue between mathematical and pedagogical knowledge. All aspects of teaching are grounded in the context of a classroom and a particular group of students, from the plans that a teacher creates prior to a class session to the enactment of a lesson to the teacher's reflections on the events of the lesson and thoughts about how to proceed. Moreover, all these aspects are also tied to a particular piece of content that the teacher is aiming for students to learn. As such, any model of knowledge needed for teaching should not just take into account the three facets of teacher knowledge described above, but pay careful attention to the context of the classroom and the mathematical content to be taught as strands through the facets of knowledge needed for teaching.

The *knowledge needed for teaching* framework, proposed by Ball, Bass, and Hill (2004), holds great potential for characterizing the knowledge needed for teaching. This framework is based on several years of work with a rich database of videotaped records of teaching. It incorporates previous research into specific aspects of teacher knowledge, including content knowledge (e.g. Begle, 1979; Ma, 1999; Sherin, 2002) and pedagogical content knowledge (e.g. Wilson, Shulman, & Richert, 1987; Wilson & Berne, 1999), and is resonant with practice-based theories of teaching. The framework is compatible with the situative theory of learning (Greeno & MMAP, 1997), which encompasses aspects of behaviorist and cognitive theory, as it focuses on the interactions between teacher and mathematical task, students and mathematical task, and teacher and students.

The knowledge needed for teaching framework proposed by Ball, Bass, & Hill (2004) contained two categories: knowledge of mathematics and knowledge of students and mathematics. Knowledge of mathematics encompasses content knowledge, both the more common content knowledge needed by most doers of mathematics, and more specialized content knowledge specific to teaching. Knowledge of students and mathematics is a category that encompasses knowledge that depends on students' ways of thinking about and making sense of the mathematics with respect to a specific group of students. This may involve knowing a set of appropriate examples, typical errors, and meaningful definitions are for a particular slice of mathematical content, with respect to the specific group of students being taught (Ball, Bass, & Hill, 2004). Knowledge of students and mathematics also includes ways of making sense of students' current understandings of the mathematics and finding ways to move those understandings forward in the service of the mathematical goal of the lesson. This category of knowledge resides in large part in the enactment of a lesson, in which the teacher must respond dynamically to the understandings of students with respect to the intended mathematical trajectory for the class.

The framework as presented holds great promise for studying teaching. It divides knowledge needed for teaching into two related facets that can be described clearly, observed in classroom settings, and assessed in teachers. However, the framework as presented has some limitations. First, knowledge of mathematics is labeled narrowly. The label *knowledge of mathematics* connotes knowledge of the processes and concepts related to solving problems in the mathematical domain. Indeed, this knowledge as defined by Ball, Bass, and Hill (2004) is broader, encompassing the specialized content knowledge needed for teaching, including knowledge that allows teachers to evaluate and select mathematical tasks for use in their classroom. Knowledge such as how to assess the cognitive demands of a mathematical task in a curriculum is certainly an important component of teachers' specialized mathematical knowledge (Stein, Grover, & Henningsen, 1996). A more explicit label for this category of knowledge is proposed: *Knowledge of mathematics and mathematical activities*.

Second, the label of "knowledge of students and mathematics" is perhaps overly general. Much of the knowledge that Ball, Bass, and Hill (2004) describe as residing within this category involve the moves a teacher makes in the enactment of a mathematics lesson, in the service of moving student understandings forward and promoting student learning. As the ultimate goal of this facet of knowledge needed for teaching is to assess and advance students' learning of the mathematics at hand, the name *Knowledge of mathematics for student learning* is a more descriptive label.

Finally, the framework may not capture some of aspects of knowledge needed for teaching that are more general yet related to the mathematical activity of the classroom; specifically, knowledge of how to organize and manage the daily activity of the classroom. One such construct noted previously is that of routines. Routines have a generalized character, in that they operate similarly across different mathematical content. However, routines always manifest within the context of the mathematics classroom and in the service of a particular lesson's enactment. The effective use of routines has been shown to move the mathematical activity of the classroom forward, in the service of student learning. Routines also have been shown to be characteristic of expert teachers, whose students have proven track records of success, and absent or less developed in novice teachers (Leinhardt & Steele, 2005; Leinhardt, Weidman, & Hammond, 1987).[1] Hence, general pedagogical moves such as routines can be considered as a part of the knowledge needed for teaching framework.

The enhanced knowledge needed for teaching framework incorporates these elements, and is summarized in Table 1. In the sections that follow, additional details are provided about

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[1] Also included in this facet of knowledge needed for teaching are practices such as setting the intellectual climate of the classroom (e.g. Lampert, 2001; Leinhardt & Steele, 2005) and metatalk (Leinhardt & Ohlsson, 1990; Leinhardt & Steele, 2005). These constructs, while important, are beyond the scope of this study.

each of the three facets of the framework in the context of middle grades geometry and measurement.

Table 1. Knowledge needed for teaching framework.

Knowledge Needed for Teaching		
Knowledge of Mathematics and Mathematical Activities	**Knowledge of Mathematics for Student Learning**	**Knowledge of Practices that Support Teaching**
Content knowledge of the domain: knowledge that users of mathematics outside teaching would need to know and do, such as finding area, perimeter, volume, etc. ***Content knowledge for teaching***: knowledge that is specific to the act of teaching, such as the selection of tasks; the set of examples, representations, and solution strategies for a given task; and knowledge of the nature of the domain	knowledge that relates specifically to the way a population of students might think about and do mathematical problems and content includes anticipating student solutions to a task, monitoring and questioning students as they work, selecting and sequencing student solutions for public sharing, and connecting solutions to highlight key understandings (Stein, Engle, Hughes, & Smith, submitted)	knowledge of aspects of teaching practice that automatize and structure the work of teaching includes: - routines - metatalk - intellectual climate of the classroom

(adapted from Ball, Bass, & Hill, 2004)

2.2.2. Knowledge of Mathematics and Mathematical Activities

Whether we consider content knowledge in the domain or content knowledge for teaching, there is substantial evidence that teachers have not had the opportunity to learn the mathematical knowledge needed for teaching geometry and measurement. Specifically, data from the 2000 National Assessment of Education Progress (NAEP) teacher surveys indicates that exposure to geometry and measurement content through college coursework and in-service workshops is fairly low.

Of the grade 8 students assessed[2] through the 2000 NAEP, only 43% had teachers who indicated having taken a full college course or more in measurement, with 10% having teachers who report little to no exposure to the topic. In geometry, 35% reported no college coursework, with 94% indicating little or no exposure at all to geometry content (Grouws, Smith, & Sztajn, 2004). Teachers of eighth grade students surveyed on the 2000 NAEP rated their feelings of preparedness to teach geometry and measurement as fifth and seventh respectively of seven content areas (Grouws, Smith, & Sztajn, 2004). These data suggest that beyond their own K-12 educational experiences, teachers may not have had opportunities to develop significantly different content knowledge understandings from those of their own students. Research confirms that teachers hold many of the same misconceptions and limitations to their content knowledge with respect to geometry and measurement (e.g. Fuys, Geddes, & Tischler, 1988; Knuth, 2002a). Because of this fact, and the lack of studies aimed specifically at teacher knowledge of geometry and measurement, results are included from both student and teacher populations in the survey of research that follows to describe the knowledge needed for teaching geometry and measurement.

As noted in the framework in Table 1, knowledge of mathematics and mathematical activities can be divided into two distinct but related aspects: the content knowledge in the domain, and the content knowledge for teaching. In the sections that follow, content knowledge in the domain and content knowledge for teaching are described with respect to two key mathematical topics within geometry and measurement: relationships between measurable quantities of geometric figures and reasoning and proof. In each section, a brief sketch of the knowledge is provided, followed by a summary of the research in each of these areas related to student and teacher learning.

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[2] NAEP assesses a representative cross-section of students nationwide. While the teachers of these students may not itself be a representative sample of teachers nationwide, it is the closest such data that is available on a national scale.

2.2.2.1. Content knowledge in the domain: Relationships between measurements. Historically, student work in geometry has focused on the computation of area, perimeter, and volume through the application of formulas memorized and recalled by rote. The standards currently promoted for students' understandings of geometry and measurement in the middle grades extend beyond computation and application of formulas; students in the middle grades should be able to understand how measurable quantities such as dimension, perimeter, and area or dimension, surface area, and volume are impacted by the particular properties and characteristics of shapes. It is by making these comparisons and generalizations that students can come to develop meaning for the quantities, an understanding of the similarities and differences between particular properties of shapes, and make connections between the symbolic formulas and the general characteristics of shapes (NCTM, 2000). In addition, these skills are practically useful in the work of doers of mathematics from a variety of careers. For example, understanding the relationships between dimension, surface area, and volume, and how changing the attributes of the shape of a room impact each of those quantities, has implications for many of the workers involved in the design and construction of physical spaces such as homes and offices, or cities and towns. The relationship between attributes of three-dimensional solids and their surface area and volume are relevant in a variety of jobs, from the manufacturing of containers for food and drink that balance an optimal volume with minimal surface area, to high-tech endeavors such as the design of spacecraft that have enough volume to carry necessary personnel and equipment while minimizing surface area that needs to be covered with tiles that can withstand the high heat of atmospheric re-entry.

One particularly salient example with which many grocery shoppers will be familiar is the packaging of 12-packs of 12-ounce aluminum soda cans. The height of a typical 12-ounce soda can is almost exactly twice its diameter. Traditionally, these cans have been packaged in a 4 x 3 x 1 can height configuration, as shown in Figure 1a below. This results in a package with dimensions of 12 in. across, 9 in. deep, and 6 in. high.

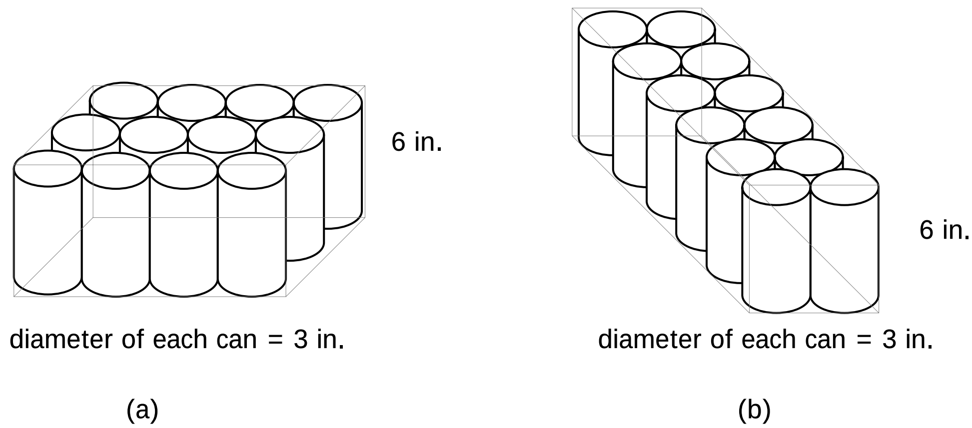


Figure 1. Two configurations of 12-packs of soda cans.

Recently, the packaging of soda cans has changed to a configuration that is easier to storage in refrigerators, shown in Figure 1b. The prism in Figure 1b has a half the length of the prism in Figure 1a and twice the width. How does this change, ostensibly done for consumer convenience, impact the volume? How does it impact the surface area? As an individual in the decision-making chain at a soft-drink company, these issues are of critical importance in being able to make the decision.

An understanding of surface area, volume, and the attributes of a rectangular prism help to answer these questions. One could calculate the volumes of the two rectangular prisms and find them to be the same. However, a simpler (and more mathematically elegant) argument can be made. Since both prisms hold the same number of cans, stacked in a similar manner with adjacent diameters along two dimensions, one can also argue that both prisms hold the same volume. Is the same true with respect to surface area? One can use a number of solution strategies to determine that the surface area in Figure 1b is more than that of Figure 1a. An executive making the case for the "fridge pack" configuration in Figure 1b would have to argue that the cost to manufacture the new boxes, increased with respect to materials needed, will be offset by increased sales of the soft drink. It's not difficult to understand why less expensive brands at the local grocery store still appear with packages like those in Figure 1a.

In sum, understanding the relationship between the properties of geometric figures and measurable quantities of those figures, such as dimension, perimeter, and area or dimension, surface area, and volume, is at the core of the sort of mathematical knowledge advocated for middle grades students by the NCTM Standards (2000). In order to understand these complex relationships between the quantities, teachers (and students) need a conceptual understanding of these quantities, and to be able to find them for a variety of figures, whether it be via a formula committed to memory or through methods that rely on a more conceptual understanding. Moreover, teachers need to be able to articulate how the dimensions of a figure, either two- or three-dimensional, contribute to the quantities, and how changes in these dimensions can impact each quantity, and the quantities in combination. For example, doubling the length of a rectangle doubles the area and increases but does not necessarily double the perimeter. Understanding these relationships is the gateway to moving students from considering specific measurements of specific figures to deducing more general relationships and developing or making meaning for formulas for these quantities.

This knowledge is even more critical for teachers, as they are charged with creating instructional situations in which students can explore these relationships. The relationships should be approached from a variety of directions; that is, one should also be able to explain the impact on dimensions and surface area if one creates a new rectangular prism with double the volume of the original. These understandings should manifest across a variety of representations: teachers should be able to explain how dimensions impact perimeter and area, or surface area and volume, based on the formula or a symbolic representation, through the use of a diagram, through changes in numeric examples, and through narrative explanation. Knowledge of these relationships between measurable quantities is practically useful for a variety of professional doers of mathematics, from mathematicians and engineers to carpenters, manufacturers, and marketing executives. As this is critical content knowledge for students, it can also be considered critical content knowledge for teachers.

Despite the importance of relationships between measurable attributes of geometric figures, students historically have shown poor performance with respect to these mathematical ideas. When faced with problems that require more general conceptual understandings of measurements of geometric figures, and when asked to compare how measurements such as area and perimeter are related with respect to a class of geometric figures, student responses are fraught with misconceptions. Assessments and research have shown that understanding the concepts of area, perimeter, surface area, and volume and making connections between them with respect to geometric figures are challenging tasks.

The National Assessment of Education Progress has shown a clear pattern of difficulty in these areas during the 1990s. While performance in the geometry and measurement content areas improved significantly in 2000 as compared to the previous three administrations of the test, the scale scores are the lowest of the 5 content areas assessed (272 and 273 respectively), well below the threshold for the proficient level of 299 (Sowder et al., 2004). Performance on individual items showed little change, and was extremely poor on items that dealt with conceptual understandings and relationships between measurements. For example, only 14% of 8^{th} grade students were able to determine the number of square tiles needed to cover the area of a region, with only 9% able to determine the number of boxes of tiles needed (Sowder et al., 2004). When asked to show different ways that a region can be partitioned in order to find the area, only 11% could answer successfully (Sowder et al., 2004). Finding the surface area of a rectangular solid was only possible for 25% of 8th grade students tested (Sowder et al., 2004). When asked about properties of shapes and similarity, students showed similarly poor results. Only 36% could identify a type of triangle based on properties, and a mere 7% of students were able to draw a figure similar to a given figure on the basis of a ratio of the area of the figures (Sowder et al., 2004). These items are exactly the types of problem that the NCTM (2000) Standards advocates that middle grades students should be able to solve.

Several research studies have sought to examine particular aspects of students' knowledge of geometry and measurement, particularly with respect to relationships between measurements of geometric figures. Many of these studies, including a seminal piece of work by Fuys, Geddes, & Tischler (1988), utilize the van Hiele levels of geometric thinking. A brief description of the van Hiele framework follows.

The van Hiele framework is a level-based characterization of geometric understanding, developed in the late 1950s by a pair of Dutch mathematics teachers. This framework became the basis for a major overhaul of the teaching of geometry in the former Soviet Union in the 1960s and 1970s (Fuys, Geddes, & Tischler, 1988). The van Hiele model describes five levels of geometric understanding. (The original numbering is used here; subsequent studies particularly in the United States have converted the levels to a 1-5 numbering.) At Level 0, students name, classify, compare, and operate on geometric figures according to appearance. Students analyze figures by components and relationships between components, as well as discover rules and properties of classes of shapes at Level 1. Level 2 features the first use of informal arguments to logically relate properties and rules for geometric figures. Deductive reasoning first appears at Level 3, in which students prove theorems deductively and establish relationships between theorems. Level 4 represents the most sophisticated form of geometric reasoning, where students analyze the affordances of different axiomatic systems. Middle grades geometry traditionally aims at concepts in the first three levels of the framework; levels 3 and 4 are reserved for work in high school and beyond. The stages are taken to be hierarchical, with progression through the levels being a function of biological development, ability, and previous geometry experience (Fuys, Geddes, & Tischler, 1988).

Early English-language research related to the van Hiele levels was produced in the 1980s, and focused primarily on empirical verification of the levels and their hierarchical nature. One of the earliest studies by Mayberry (1983) examined undergraduate preservice elementary teachers and attempted to place them at a van Hiele level through a multiple choice test across topics. Results across topics were diverse, and while results with respect to relationships between measurable attributes were not reported, teachers were generally rated at Levels 0 and 1, exhibiting significant difficult with concepts at Levels 1 and 2. Burger and Shaughnessy (1986) investigated the nature of the levels through a comprehensive assessment of K-12 students and college students, including preservice teachers. While results were not reported by specific topic, the bulk of students resided at Levels 1 and 2, with only 1 student being classified as Level 3 and none as Level 4. Of particular note was the finding the levels were not discrete; the researchers found that several students exhibited behaviors that crossed levels depending on the task. The researchers also posit that some students who have been away from geometry for a number of years regress in level.

The more in-depth study of the van Hiele levels conducted by Fuys, Geddes, & Tischler (1988) included a number of components: individual interviews with sixth and ninth grade students in which they worked on instructional activities in three modules created by the research team; an analysis of three of the most common K-8 mathematics texts for geometry content related to the van Hiele levels; and interviews with teachers in which they worked on the same three instructional models as students. The results for both students and teachers are discussed in this section, with particular emphasis on the third module that dealt with ideas of area. (Modules 1 and 2 dealt with shape identification and angle measurement respectively.)

In the 6^{th} grade phase of the study, 16 average and above-average students were interviewed. Student performance on the instructional modules, which were designed to be traversed in a linear manner with instruction and support from the clinical interviewer, divided the cohort into three groups by performance. The first group (n = 3) showed little understanding of geometry at all, and completed Module 1 and part of Module 2. Student understanding of properties of shapes and relationships between them was weak, with all three students being located at Level 0 on the van Hiele scale, only progressing to some Level 1 understanding with explicit guidance from the interviewer. Two of the three students began Module 3, and made no significant progress on area activities. The second group (n = 6) completed Module 1 and parts of Modules 2 and 3, with most students only progressing to Level 1 with respect to properties of shapes in Module 1 with guidance from the interviewer. Only 2 of the 6 students engaged in part of the Module 3 activities and made any substantive progress. One student confused area and perimeter, and the other could only find the area of a rectangle through counting squares, both representing Level 0 understandings. The student who was able to find the area of a rectangle through multiplication learned the skill by rote, and was unable to explain why the procedure worked. When faced with a triangle and asked to find the area, the student simply multiplied the measures of all three sides. The third group (*n* = 7) completed the first two modules and most made progress in Module 3. Of these 7 students, all had a conception of area that related to covering space and knew the area rule for a rectangle; however, 2 could not explain why the rule worked and 4 others only explained it through specific examples. None knew a rule for triangles, but 2 students did manage to derive it, relating it to the area of a rectangle. While 5 students were able to make deductive arguments related to generating area rules for various shapes, all of these students did so only with explicit guidance from the interviewer. Most of this work still can be classified as van Hiele level 1. In sum, most 6^{th} graders were unable to meaningfully discuss and understand relationships between properties of shapes, particularly in the service of developing formulas for area of various shapes. Those who did make progress only did so through a series of carefully designed instructional activities that required significant guidance from a knowledgeable interviewer.

Students in the 9^{th} grade cohort also were divided into 3 groups by performance: Group 4 (*n* = 2) showed almost no geometric understanding and made limited progress on Modules 1 and 2; Group 5 (*n* = 7) showed some geometric understanding, with most completing Modules 1 and 2 and making some progress on Module 3; Group 6 (*n* = 7) showed the greatest understanding and completed all three modules. Work on Module 1 showed very little understanding of general properties of shapes, with students having difficulty identifying general properties from which to classify shapes. Of the 2 students in Group 4, 1 made no significant progress on Module 3, with the other initially confusing area and perimeter and making random guesses. Eventually, this student was able to find area of irregular shapes and surface area of an open box with guidance from the interviewer. Most of the work done by both students was classified as Level 0 performance. Students in Group 5 exhibited slightly better performance, but had to be prompted to identify all the relevant properties of shapes during Module 1 activities. Additionally, many were not able to generalize that a square was a special case of a rectangle, not understanding inclusion relationships across properties. Only 5 of the 7 students completed all or part of Module 3; of those students, 2 confused area and perimeter, and another talked about the "diameter" of a square when asked about area. No student could explain why the formulas for perimeter or area worked. Two students had difficulty calculating surface area, confusing it with formulas for area and perimeter or just multiplying dimensions together without understanding why. Similarly, students had great difficulty establishing that the area of a right triangle is half the area of the corresponding rectangle, with one student simply thinking it was one dimension or the other of the rectangle. Overall, Group 5 showed Level 1 reasoning, with 2 students making some progress towards Level 2. Group 6 students covered the greatest amount of material and showed the most sophisticated understandings with respect to relationships between measurable attributes of figures. Students were able to reason about properties of shapes and create hierarchical class categories, including some students who held memorized incorrect definitions of the individual shapes. However, these students still exhibited serious misconceptions in Module 3 area activities, including confusing the sum of the measures of interior angles and area, and length of the base of a shape and its area. Students had difficulty seeing that two tangram blocks, when combined, conserved area, and understanding surface area, instead combining computations of area and perimeter in a haphazard manner. Overall, students in Group 6 made progress towards Level 2 thinking, but still exhibited some misconceptions with respect to relationships between measurable attributes of figures. These misconceptions were also present in the two lower-achieving groups.

Fuys, Geddes, and Tischler (1988) also investigated teacher knowledge in their study, engaging 8 preservice and 5 practicing teachers in clinical interviews in which they worked through the same instructional modules. There interviews found that teachers generally entered the tasks at van Hiele Levels 0 and 1, making progress to at least Level 1, with several attaining Level 2 reasoning. No teachers showed evidence of progressing past Level 2. The misconceptions that teachers exhibited were consistent with many of the student misconceptions: teachers grappled with identifying a set of properties that characterized a set of shapes, and with determining hierarchical relationships between classes; they had difficulty with concepts of area, often confusing linear and square units; only one teacher was able to give a convincing deductive argument that supported the area formula for a triangle. Additionally, many preservice teachers showed misconceptions when asked to find the area of more complex shapes, including multiplying adjacent side measures in a parallelogram to find area and using angle sums to determine the area of a triangle. Only after intervention by the interviewer did teachers come to a deeper understanding of area formulas and the relationships between side measures, the concept of area, and area formulas.

Additional work related to the van Hiele levels included the examination of the levels with respect to young children's concept of shape. Clements and colleagues (Clements, Swaminathan, Hannibal, & Sarama, 1999) investigated children's conceptions of shape, attempting to determine if a level existed prior to Level 0. The research found evidence of a pre-visual level prior to level 0, and also posited that Level 1 needs to be redefined, as evidence was present that children recognize components and properties of shapes earlier than Level 2.

A study conducted by Swafford, Jones, and Thornton (1997) focused on increasing teachers' content knowledge of the van Hiele levels. Although results were not specifically reported with respect to content, the study's results with respect to teacher content knowledge indicate that teacher content knowledge did improve with respect to their van Hiele levels. Moreover, the results confirmed that the van Hiele levels are not biologically developmental and that instruction can improve them; however, the results also raised the concern that the levels may be more sensitive to recall knowledge and unreliable for people who have been away from geometry instruction for an extended period of time (Swafford, Jones, & Thornton, 1997), echoing previous results from more than a decade earlier (Burger & Shaughnessy, 1986).

One final line of research related to the van Hiele levels comes from Gutierrez and his colleagues (Gutierrez & Jaime, 1999; Gutierrez, Jaime, & Fortuny, 1991). Gutierrez, Jaime, and Fortuny (1991) sought to develop an alternative paradigm for the acquisition of van Hiele levels by proposing a scale of acquisition for each level. The study was restricted to the assessment of 3-D geometry concepts and found support for an acquisition scale; however, the study contains significant flaws. First, the acquisition scale seems entirely arbitrary and not grounded in a theoretical basis, but rather on empirical patterns from other van Hiele studies. Second, the limitation of the content being assessed to 3-D geometry, a topic often overlooked in the K-12 curriculum, casts doubt on the generalizability of the findings. The second study, by Gutierrez & Jaime (1999), explored preservice elementary teacehrs' conceptions of altitude with respect to the van Hiele levels. The study utilizes Hershkowitz & Vinner's (1984) work regarding the concept images and concept definitions that students develop from their elementary teachers. The study's findings showed that many preservice elementary teachers held poor concept images of the altitude concept. Given that the study was conducted in classrooms in Spain, and that altitude is a focus of the Spanish curriculum, the study implies that elementary teachers may hold weak concept images in other geometry content.

The studies utilizing the van Hiele framework allows us to draw some general conclusions. In general, student and teacher knowledge related to measurable attributes of geometric figures and relationships between them indicate that many middle school students and teachers operate strictly on visual and categorical levels. Students and teachers alike have difficulty generalizing properties across classes of shapes, understanding the concept of area and distinguishing it from perimeter, and generalizing about area formulas across a variety of shapes. These skills are similar to those needed to understand relationships between measurable attributes of geometric figures. For example, to understand how a change in the area of a rectangle impacts its perimeter, one must understand the properties of a rectangle, how those properties impact the individual measurement of sides, how those side measurements impact perimeter and area, and how a change in the dimensions of the rectangle impacts the measurable quantities. Clearly, a student or teacher who has difficulty untangling perimeter and area will not be able to engage in a deep consideration of how one quantity impacts the other.

Coordinating relationships between measurable attributes of geometric figures in two dimensions focuses largely on ideas such as measurement of sides, perimeter, and area. When students begin to consider figures in three dimensions, the coordination of relationships becomes more complex. Students must coordinate between three linear dimensions, physical manifestations of solids, orthographic and isometric representations of three-dimensional solids, and measurable attributes of surface area and volume. This territory is fraught with potential student misconceptions, and requires students to make sense of the physical attributes of three dimensional objects. A group of researchers led by the work of Battista and Clements have investigated aspects of the development of spatial sense.

Specifically, Battista, Clements, and colleagues have conducted a series of studies examining how students build mental representations of rectangular cube buildings (prisms), and how these understandings relate to the ability to compute surface area and volume. Early work in this domain by Ben-Chaim, Lappan, and Houang (1985) identified four common errors made by middle school students with respect to finding the number of cubes (volume) of a rectangular cube structure represented isometrically: counting the visible cube faces, counting the visible cube faces and doubling, counting the number of cubes shown in the diagram, and counting the number of cubes shown in the diagram and doubling. The authors suggest that these errors result from students simply seeing 2-D pictures and being unaware of the third dimension, not visualizing the hidden parts of the diagram, or having trouble understanding what solids and isometric diagram represents. Battista and Clements (1996) investigated this conjecture by engaging 5th graders in tasks related to the volume of rectangular cube buildings in a clinical interview setting, presenting both isometric representations of the cube buildings, physical cubes, and nets representing the surface area of the cube buildings. Their results identified four main strategies for making sense of the buildings: cubes as a rectangular array in layers, cubes as space-filling but without a layering scheme, cubes as faces of the solid, and the use of the volume formula $V=l \times w \times h$. Only students using the first strategy, conceptualizing cubes as layers, were consistently successful in finding the correct number of cubes for each building. Students exhibited particular difficulty coordinating the net view of the cube building with the isometric or physical views; this often resulted in students thinking that each square on the net counted a separate cube rather than understanding that several squares cover the same cube depending on position (and that some cubes do not correspond to squares on the net). The researchers posit that the development of *spatial structuring*, which entails the coordination of views and an eventual layering structure, is fragile and gradual. Developing this understanding requires coordinating multiple physical views, as well as coordinating the enumeration of cubes with the physical views. Discrepancies between predictions based on diagrams and actual counts from constructing physical cube buildings may aid in the development of such spatial sense. Finally, the researchers indicate that teaching the volume formula first may in fact be detrimental to the development of spatial sense, as no students who used the formula exhibited elements of the spatial sense required to correctly predict the volume of a cube building based on its diagrams.

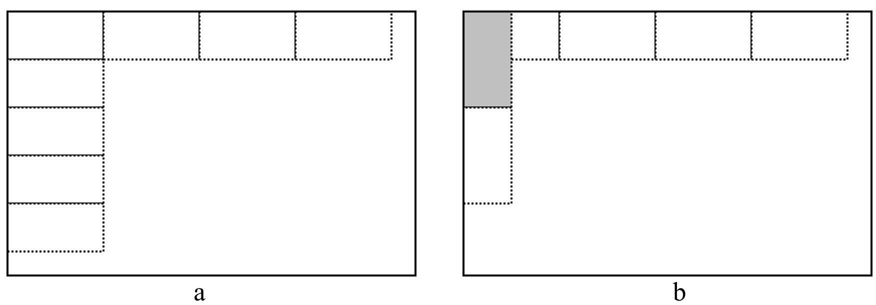
In a continuation of the work on volume, surface area, and spatial sense, the researchers described a similar set of activities designed to enhance understanding of volume concepts. These activities were done in classroom settings with students in grades 3-5 (Battista & Clements, 1998). The results were similar to those cited in the previous study: students had difficulty coordinating different orthogonal views of the cube buildings, which led to difficulty predicting how to fill a net with cubes without counting faces and arriving at incorrect answers. Once again, the key role of prediction prior to constructing actual buildings was cited; by engaging in prediction before building the cube structures, students are allowed opportunities for reflection and cognitive conflict with respect to their enumeration strategies and spatial sense. Reflection and cognitive conflict appear to aid in the development of spatial sense. Additionally, the inadvisability of teaching the volume formula first was cited; students who used the formula for memory had increased difficulty developing spatial sense.

Continued work related to the notion of reflection and cognitive conflict leading to the development of spatial sense examined dyads of 5th grade students engaged in a similar set of activities to those described above (Battista, 1999). Findings indicate that pairs of students who dealt explicitly with numeration strategies, discussing them and making them accessible to one another, were more successful in coordinating views and accurately finding the volume of rectangular cube buildings. The role of prediction prior to enumeration was again cited as a powerful influence on learning. Battista (2002) produced a Web-based applet designed to provide support for developing a layer conception of volume. The dynamic nature of the applet, in which students could add singleton, rows, columns, and layers of cubes to an empty rectangular prism, fold up and unfold the sides of the rectangular prism, and examine the relationships between length, width, height, layers, and volume, fostered the numeration strategies and coordination of views described in previous studies. Additionally, Battista points out that the layering conception is potentially more powerful than a formula, as it can be generalized to any prism and sets the stage for concepts of integral calculus.

The work by Battista, Clements, and colleagues related to spatial sense was also extended back into work with two-dimensional figures. Clements, Battista, Sarama, Swaminathan, & McMillen (1997) investigated how students developed concepts of length during a Logo-based unit from the Investigations in Number, Data, and Space curriculum (Russell, Tierney, Mokros, & Economopoulos, 1998). Students engaged in a variety of written, physical, and computer-based activities designed to foster conceptions of path length. Students initially had difficulty linking spatial measurement and numeric worlds, particularly when asked to use numbers to calculate missing measurements. The integration of spatial and numeric worlds happened largely through dynamic movement activities where students had to take an active, first-person stance towards the path either on the computer or through physical movements. This sense of dynamic movement activities that coordinate the spatial and numerical domains resonates with the 3-D cube building work in which students had difficulty coordinating the enumeration and spatial domains.

Another study by Battista and colleagues (Battista, Clements, Arnoff, Battista, & Borrow, 1998) examined the spatial structuring exhibited by primary grades students with respect to two-dimensional arrays of squares. Students were asked to complete a series of tasks in a clinical interview setting, with interviewers probing student thinking where appropriate. Results indicate that students followed a similar pattern to the one exhibited in the work with three-dimensional cube buildings: initially, students had difficulty reconciling different views of the two-dimensional arrays, making progress towards structuring the arrays as a column-row composite. Progress was slow and nascent understandings were tentative; however, it was noted that "sweeping motions" (Battista et al., 1998, p. 530) and perturbations in the form of differences between predictions or drawings and actual counts of squares served as organizing actions that moved understandings forward. Sarama and colleagues (Sarama, Clements, Swaminathan, McMillen, & Gomez, 2003) continued the work in a study that investigated 4th grade students' performance on the Sunken Ships and Grid Patterns unit from Investigations in Number, Data, and Space (Clements, Battista, Akers, Rubin, & Woolley, 1995). The research focused specifically on how students develop the mathematical ideas of two-dimensional rectilinear space. Developing a conception of rectilinear space requires coordinating the numeric and spatial systems once again, this time in the service of locating points in space on the rectilinear grid. Hurdles that the researchers observed in the development of this understanding included seeing the lines on the grid as physical entities rather than guide marks, and developing both extrinsic, birds-eye views of the grid and intrinsic views, where locating points in space is approached from a first-person standpoint. These two studies together suggest the importance of developing two-dimensional spatial sense through exposure to a variety of approaches, both third- and first-person, and coordinating numerical and physical space.

Two-dimensional spatial sense was the focus of a study by Simon and Blume (1994) that investigated the development of multiplicative reasoning in preservice elementary teachers. One of the key problems that teachers explored involved measuring the area of a rectangular table using small rectangular cards. Teachers grappled with the issue of whether to turn the cards when measuring different dimensions (see Figure 2), and ultimately what the number of rectangles it takes to cover the table means, if anything.



(adapted from Simon & Blume, 1994)

Figure 2. Measuring the table with a cardboard rectangle.

Initially, many teachers claimed that the rectangles should be aligned as shown in Figure 2a, and that the sum total of the rectangles needed to cover the table had no particular meaning. Simon and Blume hypothesize that the challenges teachers experienced in creating meaning for area stems from teachers seeing the rectangles as "one-dimensional" and not coordinating length and width of the table as they relate to the rectangular cards. While teachers understood that one multiplies length and width of a rectangle to find area and could express this area in square units, the researchers claim that "for some of these [teachers], 'square units' do not conjure up an image of a square" (Simon & Blume, 1994, p. 485). To develop an understanding of the relationship between area, length, and width requires attending to the two linear units as descriptors of the size and shape of the rectangular region. Being able to coordinate the two dimensions into a coherent conception of area and connect this conception with the numeric formula for finding area is critical in the development of a robust understanding of area. This mirrors the results reported by Battista, Clements and colleagues with respect to surface area and volume.

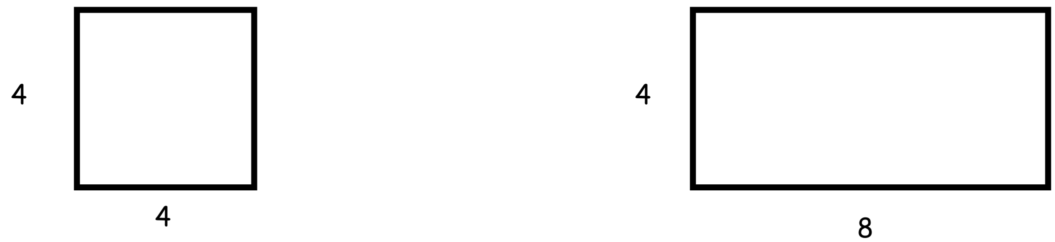
At the start of this section, an argument was made that understanding the relationship between measurable quantities of geometric figures is key content knowledge for doers of mathematics. Students and teachers alike, as everyday users of mathematics, should understand the concepts of dimension, perimeter, area, surface area and volume, have a means of finding values for each of these quantities, understand how the quantities relate to one another, use a variety of representations to articulate these relationships, and understand how changes to one quantity impacts the others. Students need to understand these relationships to be able to use this knowledge in a variety of context, both in and out of school. The body of research that investigates student and teacher understandings of these relationships shows that students and teachers have difficulty with tasks that require the coordination of the spatial or visual domain with other domains, including abstract properties, measurement of physical length, or enumeration of surface area and volume of rectangular cube buildings. The foundational concepts of area and perimeter are also challenging for students and teachers, who often confuse the means of computing and appropriate units for each. Rote memorization of formulas appears to be common, and in some cases represents an impediment to meaningful learning and coordination of multiple perspectives. Also of note is the role of physical motions, such as the sweeping motions described by Battista et al. (1998), and physical constructions in three-dimensional work. This research suggests that coordination of physical, spatial, and numeric representations contributes to increased understanding of relationships between measurable quantities.

In order to aid students in developing these important mathematical understandings, teachers need to understand these ideas from two perspectives. They need to understand the relationships between measurable quantities of geometric figures as doers and users of mathematics, such as their students, do. Beyond that understanding, teachers also need to understand these ideas from the perspective of a teacher in order to create opportunities for students to develop their knowledge of these relationships.

2.2.2.2. Content knowledge for teaching: Relationships between measurements. Liping Ma (1999), in an examination of the knowledge needed for teaching elementary mathematics in the US and China, states, "To empower students with mathematical thinking, teachers should be empowered first" (p. 105). With respect to geometry and measurement, Ma (1999) investigated how teachers make sense of students' understandings of the relationship between measurable quantities of geometric figures. Ma asked teachers in the US and China to respond to the scenario shown in Figure 3.

Scenario

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:



perimeter = 16 cm
area = 16 square cm

perimeter = 24 cm
area = 32 square cm

Ma, 1999, p. 84

Figure 3. Ma's Area and Perimeter Scenario.

Ma’s interviews with US teachers around this scenario revealed that two of the 23 teachers simply accepted the statement without question. Among teachers who did not accept the statement, three prevailing strategies were exhibited for dealing with this scenario. Five teachers indicated that they would consult a book to determine the correctness of the statement; four of these teachers indicated that they needed to consult a book because they did not recall how to compute perimeter and area. The fifth teacher knew how to compute the area and perimeter by formula, but admitted to not knowing what the formulas meant, making the interpretation of the student’s answer impossible. The second strategy that 13 teachers suggested was to ask the student to provide additional examples. Ma claims that this response is based on “everyday experience, rather than mathematical insight” (Ma, 1999, p. 86). The third strategy, utilized by the remaining three teachers, was to investigate the problem mathematically. Only one of the three teachers interviewed was able to successfully do so. These approaches stood in contrast to the responses of the 72 Chinese teachers, of which 72% pursued an investigation of the problem at some level, either for the purpose of explaining the underlying ideas to or exploring them with the student (Ma, 1999).

The results of Ma's study underscore the importance of teacher content knowledge. Teachers who did not have a firm grasp of the mathematical content underlying the scenario were not able to productively foster students' engagement with the mathematical ideas (Engle & Conant, 2002; Ma, 1999). Moreover, these results reveal the interconnected nature of content knowledge in the domain and content knowledge for teaching. To be able to productively respond to students and foster continued inquiry into important mathematical ideas, teachers certainly need sound content knowledge. However, the content knowledge is not enough; teachers also need to understand strategies, both specific to a problem and more generally suited to the mathematical domain, for helping students move their understanding forward and to foster mathematical inquiry. It is this type of knowledge that is the content knowledge for teaching.

To date, very few researchers have engaged in the systematic study of the specialized content that teachers might need to effectively teach concepts related to the relationship between measurable attributes of geometric figures in the middle grades. Much of the effort to study and enhance content knowledge for teaching geometry has centered on the van Hiele levels. A number of teacher education efforts have attempted to teach teachers to structure instruction around the van Hiele levels, from the overhaul of the Soviet mathematics curriculum in the 1960s (Fuys, Geddes, & Tischler, 1988) to more current popular methods publications for middle grades teachers (e.g., van de Walle, 2005). A study by Swafford, Jones, and Thornton (1997) represents the seminal piece of research on specialized content knowledge for teaching of geometry and measurement. The authors recognize that in the area of middle grades geometry in general, “research on teachers’ knowledge of student cognition is virtually nonexistent” (Swafford, Jones, & Thornton, 1997, p. 470).

The study made use of the van Hiele levels in a two-pronged intervention aimed at practicing middle grades teachers. Teachers engaged in two parallel courses – one on geometry content organized using the van Hiele levels, and a second that considered how knowledge of the van Hiele levels might impact teacher practices such as lesson planning, modification of instructional tasks, and risk-taking with respect to ideas and instructional methods in the classroom. The content knowledge results were discussed briefly in the previous section.

The study’s results with respect to teacher practices imply that knowledge of the van Hiele levels may have an impact on the lesson planning and instructional practice of teachers. The researchers report that teachers’ lesson plans improved with respect to the van Hiele levels of the activities planned after the course experiences. Additionally, teachers spent more time on geometry in class, modified textbook tasks more often and in ways that increased the potential van Hiele level of the task, and showed more risk-taking behaviors and confidence in the classroom with respect to geometry instruction. Student discourse also improved in classrooms after the course.

This research attributes the change in teacher practice to knowledge of the van Hiele levels and their potential use in selecting and enacting activities with students. However, some particular aspects of the study's design raise questions regarding how much credit can be given to the van Hiele levels. First, the relationship between the questions used to evaluate changes in teacher knowledge in the study and the van Hiele level descriptors as conceptualized by Fuys, Geddes, & Tischler (1988) is unclear. The descriptions of indicators of the van Hiele levels presented in the Fuys, Geddes, & Tischler study are very general, and represent interpretations and refinements made by the researchers based on study results. Mapping the items used to measure teacher knowledge onto the original descriptors of the van Hiele levels requires a high level of inference. Second, given the fact that teachers entered the study with such low levels of geometry knowledge, it is likely that any additional opportunity to consider geometry concepts, and not specifically the van Hiele framework, would produce changes in teacher knowledge. Significant attention to the topic in the course experience and the presence of researchers in the teachers' classrooms may have produced a Hawthorne effect that resulted in increased instructional time spent on geometry and more extensive planning practices surrounding the geometry lessons. The effect of researchers in the classroom may have also produced the improvements in student discourse in classrooms cited by the researchers.

Thus, while the van Hiele levels represent a significant milestone in understanding students' reasoning in geometry and measurement, they do have significant limitations as content knowledge for teaching. The theory is built on models of student knowledge that are outdated; for example, "ability" is cited as one of the factors determining a student's placement in the framework (Fuys, Geddes, & Tischler, 1988, p. 12). Specifically, Wirzup (1976) characterizes maturation of student thinking with respect to the van Hiele levels as a process of apprenticeship, where students learn primarily by observing a skilled other and taking on simple aspects of more complex tasks. This conceptualization of geometric learning stands in sharp contrast to the work of Fawcett's (1938/2004) students almost 40 years earlier, and of limited utility with respect to the current social constructivist theories of learning. Additionally, the model has been tested largely on students of average or better mathematical achievement; in the Fuys, Geddes, & Tischler (1988) study, low achievers were either excluded from the sample or showed little to no progress through 8 hours of clinical interviews. The instructional models developed from the van Hiele framework have the limitations of featuring a large amount of telling and demonstration on the part of the interviewer; when students exhibit misconceptions, the modules often specify particular ways to directly alleviate the misconception rather than exploring the roots of the misconception with the student. In this manner, the instructional models are limited in their ability to build on student knowledge. Finally, the instructional models developed using the van Hiele framework have been tested in narrow educational settings in the United States. Very few of the instructional models have been tested in classroom practice (Burger & Shaughnessy, 1986; Denis, 1987; Fuys, Geddes, & Tischler, 1988; Usiskin, 1982).

In addition to the limitations as a template for instruction noted previously, a handful of studies have produced results that cast the integrity of the van Hiele levels themselves in doubt. These studies have shown growth in student thinking that did not correspond to the van Hiele levels. Kay (1987) instructed first graders starting with the general case of quadrilaterals and narrowing to particular subclasses based on properties. The results showed that students could divine characteristics at the end of instruction and about half could form hierarchical relationships, which was not a part of instruction. de Villiers (1987) followed a similar instructional trajectory with 8th and 9th graders and found that students' geometric thinking was more dependent on teaching strategy than on van Hiele level. de Villiers claims that hierarchical class organization and deductive thinking actually develop separately, contrary to the claims of the van Hiele framework.

There are some promising aspects of the van Hiele framework that could be integrated into more contemporary theories of geometry and measurement knowledge. The question of whether the van Hiele levels are discrete has been largely answered in the negative (Burger & Shaughnessy, 1986; Gutierrez, Jaime, & Fortuny, 1991); thus, it may be useful for teachers to consider the ways of thinking described by the levels as particular aspects of student thinking to look for and encourage in students rather than complete frameworks for instruction.

Another line of research that illuminates potential content knowledge for teaching is the extensive research on spatial sense conducted by Battista, Clements, and colleagues. By understanding the conceptual trajectory that most students take through making sense of two- and three-dimensional arrays, teachers can organize instruction and support students in ways that foster their understanding. Specifically, teachers should attend to the use of particular physical objects and motions that help to coordinate the numerical, measurement, and spatial worlds. The notion of particular aspects of motion that support students' reasoning is consistent with Simon's (1996) notion of transformative reasoning. Simon, investigating the work of preservice elementary teachers in experiences related to multiplicative reasoning and geometry, describes a dynamic process that teachers used to visualize a situation, which then allowed teachers to transition from specific inductive cases to generalized deductive cases. One example of such reasoning was a teacher who, when asked to demonstrate that the base angles of an isosceles triangle are congruent, began by envisioning three segments whose endpoints are joined at right angles, forming a U shape. By picturing the ends of the U moving together at an equal rate, the teacher could visualize the equal angles formed at the base of the triangle when the two segments met. This enabled her to create a deductive argument that supported the conjecture. Specifically, Simon characterizes transformational reasoning as:

> "the mental or physical enactment of an operation or set of operations on an object or set of objects that allows for one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformative reasoning is the ability to consider not a static state, but a dynamic process by which a new state or a continuum of states are generated." (Simon, 1996, p. 201).

Indeed, early work by Greeno (1979) that examined geometry problem solving through constructions identifies the potential of considering a sequence of spatial transformations that correspond to a set of algebraic manipulations as a means to lend greater understanding to challenging concepts, such as understanding the area formula for a trapezoid.

Thus, the question of what the content knowledge for teaching in middle grades in geometry and measurement should be is a relatively open question. Battista summarizes the general character of this knowledge as follows: "Only by thoroughly understanding the pedagogical approach and the usual paths students take in learning particular mathematical ideas – including stumbling blocks and learning plateaus – can teachers know when to intervene" (Battista, 1999). Certainly the content knowledge for teaching that relates to the relationships between measurable attributes of figures includes trajectories for making sense of two- and three-dimensional arrays. Additionally, teachers need to be aware of the notion of transformational reasoning and activities that might support the development of transformational reasoning. Transformational reasoning may also aid in the development of deductive reasoning strategies, discussed in the next section.

Given that there is little research regarding the content knowledge for teaching in middle grades geometry and measurement, the characteristics of content knowledge for teaching in general provide some suggestions about what this knowledge might look like. One important feature of content knowledge for teaching is having access to a range of examples for a given piece of content; thus, it is reasonable to expect that teachers should be familiar with a class of examples that have the potential for students to consider the relationships between measurable attributes of geometric figures. Additionally, it is reasonable to expect that teachers consider a variety of representations and tools, such as Geometer's Sketchpad (Jackiw, 1991), that may be helpful in exploring and elucidating these relationships. One might also wish teachers to have knowledge of typical misconceptions related to these relationships between measurements; in this area, the literature summarized earlier in this chapter identified several of the common misconceptions relating to area, perimeter, surface area, and volume. These misconceptions have the potential to impede student progress in considering the relationship between measurable attributes of geometric figures and should be addressed in instruction. Additional detail on a set of instructional experiences that may enhance a teacher's knowledge of content for teaching geometry and measurement can be found in Chapter 3.

2.2.2.3. Content knowledge of the domain: Reasoning and Proof. Deductive reasoning and proof are related ideas that are fundamental to the domain of mathematics, from the work of mathematicians to the work of students in school mathematics. The most recent revision of the NCTM (2000) Standards elevated reasoning and proof to a process standard that spans all grade levels, not just 9-12 as in the previous version of the Standards (NCTM, 1989). However, proof has typically been treated less as a practice central to mathematics and more as a particular form and type of activity to be modeled and practiced in high school geometry courses (e.g. Chazan, 1993; Hanna, 1989, 1991, 1995; Schoenfeld, 1994). These researchers and others provide us with insights into what students need to know and be able to do with respect to reasoning and proof.

Reasoning and proof is fundamental to mathematics because it provides conclusive arguments for mathematical ideas in which the assumptions and mathematical principles used to prove the conjecture are clear and the proof is immune to challenge (Hanna, 1991; Lakatos, 1976). To appreciate the utility of proof and its role in the domain of mathematics, students need a number of skills related to proof. First, students need to be able to identify deductive proofs and arguments, as compared to other arguments (e.g. use of examples, overgeneralization) that are not conclusive. Additionally, students need to understand how to make and challenge mathematical arguments, moving from conjecture to identifying evidence for claims, and generalizing them into deductive arguments (e.g., Fawcett, 1938/2004; Miyazaki, 2000; Senk, 1985).

Substantial research in the last 20 years has identified students' difficulties with the process of proof as well as their limited understandings of the utility of proof (Chazan, 1993; Miyazaki, 2000; Senk, 1985, 1989). In contrast to this limited view, Fawcett (1938/2004) describes a teaching experiment conducted almost 70 years ago that used proof as the framework for all classroom activity in a secondary school geometry course. This teaching experiment is described below, followed by a summary of the recent research regarding students' understanding of issues of reasoning and proof.

In the early 1900s, the place of a dedicated deductive geometry course in the high school curriculum was hotly debated. While it was claimed that the study of geometry developed an "understanding and appreciation of a deductive proof and the ability to use this method of reasoning where it is applicable" (National Committee on Mathematical Requirements, 1923, p. 48), the greater educational value and generalizability of deductive reasoning and proof was called into question. Poor results on standardized testing at the time supported the notion that students were not seeing a general use for proof and deductive reasoning, seemingly abandoning reasoning when approaching geometry problems (Fawcett, 1938/2004). The backlash against proof described by Fawcett and colleagues has parallels to similar questions about the nature and importance of proof raised during the later part of the century. The "new math" reforms of the 1950s and 1960s aimed to build students' mathematical understandings through the development of the mathematical system from a shared set of axioms to more complex deductions. This approach's failure, which had less to do with the instructional approach than it did teachers' abilities to implement it, allowed critics to once again call into question the utility of proof, pushing it to the background in the curriculum (Hanna, 1989). Hanna (1995) also documents alternative conceptions of proof that arose during this time, including proof by exhaustion through computers, a method that was impractical until the advent of modern computing.

In response to this challenge to deductive reasoning and proof, a number of prominent mathematicians and math educators of the time emphasized the need for experiences in deductive reasoning and proof to focus on students proving their own conjectures and discussing the nature of proof rather than just the activity of proof (Fawcett, 1938/2004). It is with these tenets in mind that Fawcett designed an innovative geometry course experience for students. Specifically, Fawcett considered the following four factors to be evidence that students understand the nature of deductive proof:

1. The place and significance of undefined concepts in proving any conclusion.
2. The necessity for clearly defined terms and their effect on the conclusion.
3. The necessity for assumptions or unproved propositions.
4. That no demonstration proves anything that is not implied by the assumptions.

(Fawcett, 1938/2004, p. 10)

Fawcett used these milestones in the design of his course, which aimed to identify the following behaviors in students to take as evidence that they understand the aspects of proof:

1. He will select the significant words and phrases in any statement that is important to him and ask that they be carefully defined.
2. He will require evidence in support of any conclusion he is pressed to accept.
3. He will analyze that evidence and distinguish fact from assumption.
4. He will recognize stated and unstated assumptions essential to the conclusion.
5. He will evaluate these assumptions, accepting some and rejecting others.
6. He will evaluate the argument, accepting or rejecting the conclusion.
7. He will constantly re-examine the assumptions which are behind his beliefs which guide his actions.

(Fawcett, 1938/2004, p. 11)

Fawcett's course, enacted in a combined class of students grades 9-11, began with a careful consideration of definition in the context of a contemporary school controversy on awards for “outstanding achievement.” While the need for definition in the abstract was initially opaque to students, the consideration of an issue that was relevant to them made salient the need for a clear definition about what one meant by “outstanding achievement,” and allowed them to see the importance of clearly defining a construct. This segued into the creation of a theory of space through successive definition, assumption, and proving of various conjectures. Students decided on their own definitions and assumptions and developed their own conjectures to prove towards building a theory of space. Students were expected to press one another for evidence, to decide on definitions and proofs as a group, and to record their results but not memorize them. Teacher intervention was minimal, with the instructor occasionally suggesting a direction to proceed and supporting the public conversations, but without setting the direction of the course. Fawcett contends that a teacher should “consider himself nothing more than a guide who directs *towards* the discovery and develops within the pupil increasing power to discover for himself” (Fawcett, 1938/2004, p. 62).

Through their experiences in the course, students came to understand a number of things about the nature of reasoning and proof. First, through debate about what elements of their theory of space should be assumed, students came to understand that if one changes the assumptions in a system, the range of conjectures that can be proven is changed. For example, students initially took that the sum of angles in a triangle is 180° to be an assumption, but soon came to understand that in order to make progress developing the system, this assumption needed to be proved. Students also came to understand the utility of reasoning and proof as a means of generalizing about relationships and establishing shared truths within a community. Similarly, students came to value the social aspect of proof and the notion that a mathematical system is dynamic, with assumptions, definitions, and proofs changing in response to changes in the understanding of the mathematical community. Data from the course showed that students generated the bulk of the geometric concepts and theorems on their own that would comprise a typical high school geometry course. Student learning measures indicate that students in the experimental class outperformed control group students with respect to both measures of geometric knowledge and non-mathematical reasoning assessments. Surveys of parents and students indicate more positive attitudes towards geometry and an increased awareness of the utility of deductive reasoning beyond the geometry classroom.

Are experiences like the course described by Fawcett common in today's K-12 school environment? Data from more recent studies of students' understanding of proof indicate that such experiences are more the exception than the rule. Early work in examining students' deductive reasoning aimed to correlate students' proof-writing abilities with the van Hiele levels of geometric thought. Mayberry's (1983) study of the van Hiele framework asserted that students would not benefit from high school geometry unless they entered at Level 2. Senk (1985, 1989) created an assessment that included subtests for each of the five van Hiele levels, assessing students' abilities to complete partial proofs and prove simple theorems. Of high school students who had completed a formal geometry course, only about 30% were able to achieve 75% mastery in proof writing (Senk, 1985). About 25% of students demonstrated virtually no competence in writing proofs, scoring 0 correct proofs (Senk, 1985). Subsequent work by Usiskin (1987) produced similar numbers. Senk called for increased instructional attention to starting a chain of deductive reasoning, on the meaning of proof, and of who, when and why one transforms a diagram in the service of a proof (Senk, 1985). With respect to the van Hiele levels, Level 2 is the demarcation point for the development of formal deductive reasoning. The results of Senk's (1989) study showed differential performance for students below Level 2 and students at Level 2 or above; however, despite dedicated subtests for each level, results did not distinguish Level 2 students from those at Level 3 or 4. Additionally, the subtests had very low reliability coefficients; Senk attributes this to low numbers of items. Another possible reason for the lack of differentiation at Levels 3 and 4 is a lack of specificity of the van Hiele model at those levels. When geometry achievement was used as a covariate for results, the student's identified van Hiele level accounted for very little of the variance. Senk's studies demonstrate that students have great difficulty completing and constructing proofs even after an educational experience aimed at proof writing. Her results also indicate that the van Hiele model has limited predictive power for assessing a student's deductive reasoning ability.

Research into the cognitive structures of problem solving in geometry by Greeno (1980) suggests that a production system for solving problems in geometry, including proofs, consists of three types of productions. The first type are propositions that are used to make inferences; for example, a statement such as "Vertical angles are congruent." Second, perceptual concepts are used to identify patterns within the antecedents of the propositions. In the previous example, this would entail labeling the concept of vertical angles as an identifiable and important feature of a geometric diagram. Finally, strategic principles are used to set goals and plan. One possible explanation for poor proof-writing performance is that while students are able to recall the propositions and mark identifiable concepts within them (the first two types of production), they lack the strategic principles to set goals and form and execute a plan to put a chain of deduction together. The literature is rife with examples of students being able to successfully memorize or otherwise access geometric properties; it is the coordination of these properties in the service of a goal that may be an obstacle in successfully forming a deductive argument.

With respect to students' understanding of the nature of proof, early research showed that students in general have serious difficulty understanding the generality of proof and the distinction between deduction and empirical examples. Work by Lovell (1971), Mayberry, (1983), Martin & Harel (1989), and Goetting (1995) showed that most high school students and close to 80% of preservice elementary school teachers considered a series of examples confirming a conjecture to suffice as a proof. From the opposite perspective, studies by Galbraith (1981), Fischbein & Kedem (1982), Vinner (1983), Porteous (1986), and Martin & Harel (1989) showed that many high school and college students don't appreciate the generality of proof; specifically, they failed to understand that a deductive proof guards from counterexamples (and that counterexamples serve to disprove a general statement), that proof is sufficiently general that one does not need to examine a specific case, and that the generality of a proof extends it beyond any particular geometric figure used in the proof. Some students alternatively believe that the proof is sufficiently general if they chose their example at random (Harel & Sowder, 1998).

Chazan (1993) examined some of these issues in detail with a sample of high school geometry students. Specifically, Chazan sought to untangle the confusion between the measurement of particular examples and deductive proof through interviews with high school students from two different schools. Two possible explanations were posited for this confusion, consistent with the literature findings noted previously: students think that evidence is proof, or students think that proof is simply a kind of evidence. The students interviewed by Chazan exhibited several of the aforementioned misconceptions. Some students had no understanding of the generalizability of deductive proof; some felt that since proof rests on assumptions that it was not general; several felt that deductive proof did not eliminate the possibility of counterexamples, and had difficulty understanding what the "given" is in a proof and what its role is with respect to the proof. While only a few students preferred deductive proof to examples in terms of explanatory power, the students who valued deductive proof came from a classroom where deduction and proof were held in high regard by the instructor. All students recognized some of the positive aspects of deductive proof in having explanatory power for individual students to understand, yet only 1 student referenced the potential explanatory power of proof with respect to others. (This notion of proofs that explain will be revisited in the section on content knowledge for teaching.) In sum, Chazan's study reinforces the finding that students have difficulty understanding the role of proof in mathematics.

Knuth (2002a) engaged in a study with similar goals to those of Chazan (1993) with practicing secondary school mathematics teachers. This study sought to investigate teachers' conceptions of proof from the stance of users and doers of mathematics. Data was collected through interviews in which teachers examined and commented on a variety of proofs and pseudo-proofs (empirical arguments) that varied with respect to form, formality, and generality. Knuth examined teachers' understandings using the following framework for the roles of proof in mathematics:

- to verify that a statement is true;
- to explain why a statement is true;
- to communicate mathematical knowledge
- to discover or create new mathematics, and
- to systematize statements into an axiomatic system

(Knuth, 2002a, p. 63)

Knuth found that all 16 teachers in the sample discussed four of the elements of the framework to varying degrees. With respect to establishing truth, several teachers spoke of this only in general terms, and only 4 indicated that proof establishes truth no matter what, with 6 teachers indicating that it may still be possible to find a counterexample for a deductive proof. There was no evidence that teachers understood the explanatory power of proof; no teachers discussed this aspect of proof. Twelve teachers discussed proof as a way of communicating mathematics, acknowledging that proof is a social product. Eight teachers discussed proof as a way of creating new mathematics and, to a lesser extent, as a way to systematize statements into an axiomatic system.

When teachers examined proofs and pseudo-proofs, approximately 1/3 of teachers rated non-proofs as proofs. In general, teachers were concerned about proofs offering sufficient detail and that they were mathematically sound in evaluating proofs; valid methods and required knowledge (the less math required to understand the better) were also factors in teachers' evaluations of proofs. Teachers often found the non-proofs more convincing; Knuth identified correctness, familiarity, generality, and showing why as factors teachers used to decide which proofs were more convincing. Proofs with a visual representation were consistently found to be more convincing than those without. These data show that teachers held a view that proof was fallible, with their confidence in a proof increasing only after they tested specific examples on their own. The elements that teachers found convincing in a proof were more about the form than the substance, being more convinced by empirical arguments and visual representations.

In sum, what is the content knowledge in the domain that students and teachers need with respect to reasoning and proof? Students and teachers alike need to be able to identify deductive arguments and distinguish them from empirical ones, as discussed by Knuth (2002a), Senk (1985), and others. They need to be able to develop chains of deductive reasoning that use established knowledge and build towards new understandings, echoing the work of Greeno (1979). These chains of reasoning can take a variety of forms, and need not be limited to the formal two-column proof. Moreover, students and teachers need to understand the roles of proof in the domain, as identified by Knuth (2002a): to verify a statement, to explain why a statement is true, to communicate mathematical knowledge, to create new mathematics, and to systematize the mathematical domain.

The findings regarding students' and teachers' content knowledge in the domain with respect to reasoning and proof show a clear pattern over the last 30 years. Students, even after explicit instruction in deductive reasoning and proof, are in general unable to create a chain of deductive reasoning in the form of a proof. Moreover, both students and teachers lack clear understandings of the nature and utility of proof. To return to the anecdote that began this section, Fawcett's description of a course designed and taught nearly 70 years ago stands in sharp relief to the research results from the last 30 years of research; through a carefully designed set of experiences, Fawcett's students were able not only to see a utility for proof and understand its nature, but also to discuss, create, and defend proofs of conjectures in the domain of geometry. In the next section, I consider the content knowledge needed for teaching reasoning and proof, with an eye towards creating classroom experiences that mirror those of Fawcett.

2.2.2.4. Content knowledge for teaching: Reasoning and Proof. Based on the findings related to content knowledge of reasoning and proof, and particularly Fawcett's (1938/2004) work, there is clear evidence that fostering a classroom environment where reasoning and proof are everyday activities, not just a particular form of exercise, and where the nature and utility of proof are made public will benefit students' abilities to engage in deductive reasoning and mathematical proof. Teachers must counter the beliefs that students often develop that proof is simply a finished product, and one only engages in proving theorems that have already been proven or that are intuitively obvious (Alibert & Thomas, 1991; Schoenfeld, 1994).

In Hanna's (1989, 1995) defense of the importance of proof in the mathematics curricula, she notes that mathematicians create and use proofs for two particular purposes: justifying new results and verifying the results of others. There is no single criterion for a good proof in the mathematical field; convincing proofs arise as a result of exploration and revision and are accepted by the community primarily when they are found to contain compelling arguments and results that are useful in the field. Hanna (1995) argues that in mathematics education, proofs serve an additional purpose; they should serve to explain important mathematics. Not all proofs contain this explanatory power; in fact, one of the functions of proof in the mathematical community is to compress arguments for efficient communication. This conception of using reasoning and proof as an exploratory and explanatory tool in the classroom resonates with the experiences of Fawcett's students and holds great implications for teaching: when engaging in an activity that requires reasoning or proof, teachers should consider what the explanatory benefits will be for students. This supports Greeno's (1980) conception of the role of the teacher in geometry problem solving: the teacher should facilitate students' own exploration, construction, and development of strategies for making sense of mathematics (Clements & Battista, 1992).

Knuth (2002b), in a companion study to his investigation of teachers' conceptions of proof as doers of mathematics, investigated teachers' conceptions of the role of proof in the secondary school mathematics curriculum. Once again using the framework related to the role of proof, Knuth sought to investigate teachers' conceptions of the pedagogy of proof through a series of interviews. Responses fell into three general categories: what constitutes proof in school mathematics, the nature of proof in school mathematics, and the role of proof in school mathematics. With respect to what constitutes proof, 9 of the 16 teachers made a firm distinction between formal proof and other proofs, with 4 teachers identifying the two-column proof form as the formal proof standard. Ten teachers discussed a notion of less formal proof that was mathematically sound but not as rigorous, yet still established truth for the general case. All teachers discussed informal proof in the form of explanations, empirical arguments, and proof by example. With respect to the nature of proof in school mathematics, 14 of 16 teachers did not consider proof as a central idea in school mathematics, and felt that proof was not appropriate for students of all abilities. However, all teachers felt that informal proof experiences were central and appropriate for all students. Of the teachers surveyed, 5 used proof exclusively in the context of geometry, with 9 teachers using proof only in upper level mathematics (geometry and above). They did not consider algebraic arguments used in other courses as proof, and felt the need "not to bother" students in lower classes with proofs of the general case (Knuth, 2002b, p. 76). With respect to the role of proof in school mathematics, 13 teachers saw the development of logical thinking as the primary role, with 10 teacher identifying communicating mathematics as an important role. Four teachers saw proof as a way of displaying one's thinking, and 7 identified proof as a way of explaining why an answer is true. Notably missing, however, was a conception of proof as an explanatory tool for mathematical relationships; teachers' conceptions of the explanatory power of proof was limited to serving as a justification for individual students' answers.

Knuth's study calls for additional research related to how we might better prepare and support teachers in changing their conception of proof as it relates to classroom instruction. Teachers need to understand the role of proof in all of mathematics, beyond the two-column form popularized in formal high school geometry classes. Specifically, reasoning and proof allows students to think logically, communicate mathematics, justify their thinking, and explain a mathematical statement or result. Teachers need to understand that engaging students in reasoning and proof can serve as a means of explaining important mathematics concepts. In this sense, reasoning and proof has a role at all levels of mathematics and for all students. Harel & Sowder (1998) cite discourse-based classrooms in which the norms of the classroom include the construction of new mathematics as a classroom community, consistent presses for students to provide evidence to back up their claims, and the requirement of students to explain their thinking and critique the mathematical arguments of others as promising venues for the development of mathematical reasoning and proof. Teachers need to understand student discourse as a promising tool to support reasoning and proof.

Classroom cultures similar to those described by Ball (1993), Cobb et al. (1991), and Lampert (1990, 2001) have the potential to integrate reasoning and proof in a meaningful way across mathematical topics. A set of practices that have been shown to be effective in aiding teachers in creating these sorts of classroom environments are discussed in the next two sections, knowledge of mathematics for student learning and knowledge of practices that support teaching.

2.2.3. Knowledge of Mathematics for Student Learning

The publication of the NCTM Professional Teaching Standards (NCTM, 1991) represented a national call for changes in the ways that teachers teach. The Teaching standards encouraged teachers to follow models exemplified by Ball (1993), Cobb et al. (1991), and Lampert (1990), in which the teacher serves as a facilitator who organizes, guides, and bounds student exploration of mathematical topics rather than lecturing and having students practice modeled procedures. Along with these suggestions, many teacher leaders and educational researchers advocated a shift from the Madeline Hunter (1982) lesson structure model of anticipatory set, teaching strategies, guided practice, individual practice, and closure to a format more consistent with the suggestions of the Standards: launch of the task, exploration of the task, and discussion and summarization or share-and-discuss phases (Baxter & Williams, in press; Lappan, Fey, Fitzgerald, Friel & Phillips, 1998a, b; Lampert, 2001; Sherin, 2002; Stigler & Hiebert, 1999). The common theme for these recommended reforms was that teachers should find ways to honor and make productive use of student thinking in the classroom.

Early changes to practice in response to this call for reform met with mixed results. One of the most common issues experienced by teachers transitioning to a student-centered pedagogy was the misconception that teachers should honor all student thinking, potentially leaving misconceptions and erroneous statements unchallenged (Leinhardt & Steele, 2005). Teachers also suffered from a loss of efficacy, not understanding what their new role in the classroom should be if they were not supposed to "tell" students mathematical concepts and procedures (Chazan & Ball, 1999; J.P. Smith, 1996). Teachers were told what not to do, but were not presented with a set of teaching practices that supported the ideas established by the reform documents.

In response to these issues, several lines of research attempted to develop a canonical set of practices that would aid teachers in understanding student thinking and moving it forward during students' exploration of mathematical ideas, and using them productively in whole-class discussions in the service of a particular mathematical goal (Ball, 1993, 2000; Chazan & Ball, 1999; Chapin, O'Connor, & Anderson, 2003; Engle & Conant, 2002; Hodge & Cobb, 2003; Lampert, 2001; Nelson, 2001; Silver & Smith, 1996; Wood & Turner-Vorbeck, 2001). One particularly promising framework for developing these abilities in teachers is the five practices that support productive use of student thinking described by Stein, Engle, Hughes, and Smith (submitted). Descriptions and examples of each of these practices are given below.

2.2.3.1. Anticipating Students' Mathematical Responses. The practice of anticipating students' mathematical responses to a task involves a teacher working and thinking through the task in a variety of ways. Actively envisioning how a student might approach the problem has been shown to be an effective first step in facilitating meaningful student learning (e.g., Fernandez & Yoshida, 2004; Lampert, 2001; Schoenfeld, 1998; J. P. Smith, 1996; Stigler & Hiebert, 1999). This has several potential benefits related to student thinking. First and foremost, it allows the teacher to work through the task with the lens of identifying a range of strategies that students might use, as well as the potential misconceptions students might have. With this information available, teachers gain the ability to make strategic decisions with respect to the ways in which the task is launched. For example, returning to the task shown in Figure 1 from Simon & Blume (1994), one of the features of the task that allows students to consider the nature of area is the ambiguity with respect to the alignment of rectangles. If a teacher wishes for students to produce these multiple approaches and use those ideas in a discussion of the task, the teacher must make sure that nothing in the setup of the task leads students to approach the problem in one of the two particular ways illustrated in Figure 2. Moreover, when a teacher anticipates solutions to a task in a variety of ways, the teacher can make initial decisions about how to organize both small-group and whole-group work and discussion such that the mathematical aspects of the task that correspond to the goals are highlighted. This may include making a range of solution strategies and/or representations available and organizing them in a particular way such that students build from their current mathematical understandings to new insights about the mathematics at hand.

In order to anticipate student solutions to a task, a teacher must take a variety of stances toward the problem. This requires having a sense of the current understandings of a group of students. For example, a group of students who have access to the volume formula for a rectangular prism may approach a volume task in a very different way than a group of students who have not been exposed to the formula. The ability to anticipate student solutions lies at the intersection of content knowledge in the domain, content knowledge for teaching in the form of understanding mathematical features and affordances of tasks and in the form of particular ways that students might be likely to approach the content, and knowledge of a particular group of students.

2.2.3.2. Monitoring Student Work. The practice of monitoring is not new; teachers have been monitoring student progress for decades. However, monitoring takes on a novel character when it is done with the goal of assessing and advancing student thinking (e.g., Brendefur & Frykholm, 2000; Hodge & Cobb, 2003; Lampert, 2001; Nelson, 2001; Schoenfeld, 1998; Shifter, 2001). Rather than assess whether students are progressing along a single prescribed path towards an answer, monitoring while students work on rich problems with multiple solution paths involves assessing where the student's current understanding is, deciding on an appropriate direction to move student thinking in the service of the mathematical goal, and choosing a way of communicating with the student or group of students that moves them towards the goal while honoring their existing thinking and without prescribing a particular pathway to follow (Stein, Engle, Hughes, & Smith, submitted).

In addition to assessing and advancing student thinking, an important aspect of monitoring relates to the data that teachers gain from examining student thinking during the exploration of a task. By closely attending to the student work, and particularly the solution paths, representations, and mathematical ideas with which students are grappling, a teacher can gain insight into how they might engineer the discussion phase of the lesson. This form of monitoring requires that teachers pay close attention to student work, including written products as students work, finished products and work with manipulatives; elucidate student thinking through requests for evidence and justification for mathematical ideas; and make physical or mental notes of the range of student strategies and understandings in the service of engineering a whole-class discussion (Stein, Engle, Hughes, & Smith, submitted). Classroom norms that make explicit the requirement that students justify their thinking and provide explanations for their mathematical ideas have the potential to enhance the monitoring process (Lampert, 2001).

2.2.3.3. Purposefully Selecting Student Responses for Public Discussion. When teachers select and use problems that offer a range of solution strategies, it is quite possible that groups of students will develop different understandings based on their individual work. Thus, the public display and use of solutions during a whole-class discussion and summarization phase of the lesson is critical in moving all students towards the mathematical goal of the lesson (Lampert, 2001). There are a variety of methods for the selection of student responses for public discussion: the teacher can use information from the monitoring phase to select particular responses in a particular order to share, either alerting students beforehand or not; alternatively, the teacher can ask for volunteers to share their solutions, often having a particular student or students in mind whose understandings they know to be sound. Whichever method teachers use, the selection and sharing of student responses is subject to the tension of honoring the variety of student thinking and pursing a particular mathematical trajectory that the teacher hopes to accomplish (Lampert, 2001; Leinhardt & Steele, 2005; Stein, Engle, Hughes, & Smith, submitted).

2.2.3.4. Purposefully Sequencing Student Responses for Public Discussion. In addition to selecting the responses that a teacher wants shared and discussed, the teacher has to determine what order, if any, the responses will be shared in. Sequencing responses in a particular way can allow for a progression to form that highlights and builds specific mathematical ideas in a particular way (Schoenfeld, 1998; Stein, Engle, Hughes, & Smith, submitted). Solutions may be sequenced to bring to light a particular mathematical idea, or alternatively to build in complexity from solutions that are more common to those that are more unique. The teacher may also select contrasting solutions in order to allow students to compare them mathematically (Stein, Engle, Hughes, & Smith, submitted). The sequencing of student responses on the part of the teacher can once again take a variety of forms, from a teacher asking for volunteers to the teacher selecting responses during monitoring and asking for them to be presented in a specific sequence.

2.2.3.5. Making Connections Between Student Responses. Having students present a series of responses holds great potential for students to understand the relationship between responses and making connections between them, giving them a more robust understanding of the underlying mathematics. However, just seeing a range of responses and discussing them does not guarantee that such connections will be made. Connections between responses can be made in a variety of ways, from explicit teacher comments to targeted questions designed to relate particular aspects of solutions to draw out a mathematical idea. Solutions can also be compared in terms of the affordances and mathematical elegance of the solution in terms of representation, mathematical knowledge required to execute the solution, or the explanatory power of the solution with respect to the mathematical ideas (Hodge & Cobb, 2003; Stein, Engle, Hughes, & Smith, submitted). Allowing students to make comparisons between solutions affords students the opportunity to reflect on other solutions with respect to their own. This allows the opportunity to make revisions (material or otherwise) to their own work and connects particular aspects of their work to other solutions in order to enhance their mathematical understandings (Brendefur & Frykholm, 2000; Engle & Conant, 2002; Stein, Engle, Hughes, & Smith, submitted).

2.2.3.6. Summary. Together, the five described practices allow teachers to capitalize on students' independent or group work on a task in the service of creating a whole-class discussion that has the potential to illuminate the important mathematical understandings inherent in the task. These practices help to operationalize the content knowledge that teachers might have, bringing it to bear on the moment-to-moment conditions of the classroom and leveraging it in support of student learning. Giving teachers access to and experience with these practices holds the potential for teachers to take steps towards realizing the call of the mathematics reforms to place student thinking and reasoning at the center of classroom activity. These practices occur at the intersection of a mathematical task and a particular set of student work on the task. Thus, teaching these practices to teachers requires a professional education experience which immerses teachers in the examination of the practice of teaching and builds on, or works in concert with the development of, knowledge of mathematics and mathematical activities. Such an experience might entail teachers engaging in the solution of a particular mathematical task, examining a set of student work and discussing the understandings inherent in the work and questions they might ask, and planning for a whole-class discussion around particular pieces of the student work.

2.2.4. Knowledge of Practices that Support Teaching

The five practices for productive use of student thinking described in the previous section are designed to aid teachers in making the transition from individual student work to whole-class understandings of important mathematical concepts. The work of the five practices occurs at the intersection of student work and a mathematical task, and allows the teacher to organize the classroom activity in the service of a particular mathematical understanding. However, these practices alone are not enough to organize the entirety of classroom activity. Teachers need practices at their disposal to structure the activity of the classroom, particularly the discussion in which students ideas are exchanged.

In systematic studies of expert teachers, teachers whose students showed learning above and beyond other students in their schools, Leinhardt and colleagues identified a number of practices that teachers utilized to structure the activity of the classroom (e.g. Leinhardt & Ohlsson, 1990; Leinhardt, Weidmann, & Hammond, 1987; Leinhardt & Steele, 2005). These practices are stable across teachers, grade levels, and content, and are not present or not consistently present in novice teachers. One particularly promising and stable practice observed in teachers is the use of routines.

Routines are small, socially shared, scripted pieces of behavior (Bromme, 1982; Bromme & Brophy, 1986; Leinhardt, Weidman, & Hammond, 1987; Yinger, 1979, 1980, 1987) that seem to be used in all classrooms and serve many functions in organizing classroom activity. Routines evolve over time and are jointly built by teachers and students. Previous research into the practice of expert teachers has identified three classes of routines. The first, and in many cases most common, set of routines are those designed to serve a management function. These routines help to move students around in predictable ways–for physical movement within the school, for small-group formation and reformation, and to control inappropriate behavior. They also serve to maintain discipline and take care of housekeeping tasks, such as attendance-taking and dismissal. These routines are present in some sense for all teachers, novice and expert; however, novice management routines tend to be underdeveloped and focus largely on behavior control, a great concern to novice teachers.

The other two classes of routines provide support for lesson presentations and classroom work. Support routines define and specify for students the types of actions that are necessary for teacher-student learning exchanges to occur (Leinhardt, Weidman, & Hammond, 1987). These include the distribution of resources, defining the starts and ends of particular activity structures, directing students to the location for the next exchange, and helping students find particular locations in the instructional materials. These routines differ from management routines in that they are closely tied to the academic work of the classroom. The third class of routines is exchange routines, which help to foster, structure, and clarify classroom discourse. Exchange routines set the parameters for classroom exchanges by specifying the types of communications that are permitted and encouraged between teacher and students, and the types of communications that are not (Leinhardt, Weidman, & Hammond, 1987). Often these routines are specific to different types of activity in the classroom. Exchange routines are present in all classrooms, traditional and "reform-oriented"; traditional exchange routines tend to follow the initiation-response-evaluation (IRE) format, whereas more discourse-oriented classrooms often feature exchange routines that govern who is called on and in what manner, that press students for justification of mathematical ideas, and that set parameters for challenging and revising claims (Leinhardt & Steele, 2005).

One particularly interesting routine, seen in the teaching of Magdalene Lampert as studied by Leinhardt and Steele (2005), is the *revise* routine. In Lampert's classroom, students routinely offer up conjectures for group consideration. At any time, even out of turn, the revise routine affords the student who offered the conjecture the opportunity to interject and revise the claim based on new understandings. This revision, as one might imagine, is also required to be backed up with mathematical evidence. The procedure of the routine is clear – the students understand the rules for yielding the floor and their privileged place as the maker of an original claim. This routine is enacted by Lampert with a particular goal in mind: the fostering of the notion that mathematics is dynamic and that there is nothing shameful or erroneous about modifying one's conjecture in the face of additional evidence. Routines always consist of a procedure to be followed, but may or may not also represent an instructional goal (Leinhardt, Weidman, & Hammond, 1987). While the goal need not be (and often is not) made visible to the student, routines that instantiate particular instructional goals aid the teacher in creating a consistent classroom environment, and serve to efficiently compress the conscious work of the teacher on these instructional goals, much in the way that definitions and deductive proofs serve to compress mathematical arguments.

The types of routines available for use in a traditional classroom, such as IRE, are well-understood and have a history of being taught to preservice teachers. In contrast, routines that organize complex discourse in the classroom, such as those utilized by Lampert, are not as well-understood, and as a result are not generally taught to preservice teachers.[3] Moreover, expert teachers tend to have deep-seeded reasons for the routines they use that extend beyond the pragmatic, and range from beliefs about teaching and learning to the values and culture of particular student populations. Preservice teachers will develop routines in their classrooms regardless of explicit instruction; *a priori* instruction regarding routines and helping preservice teachers come to understand the reasons upon which the routines of many expert teachers are based has the potential to provide these novice teachers with a skill useful in the work of teaching. The examination of routines also holds potential value for practicing teachers. Practicing teachers are likely to possess well-developed routines already, which serve to organize

[3] One notable exception are the routines embedded in the Principles of Learning (Institute for Learning, 2003), particularly the notion of Accountable Talk[SM].

and advance the mathematical activity of the classroom. However, teachers engaged in changing their practice often maintain the same set of routines, even when the beliefs about learning upon which the routines rest are antithetical to the philosophies guiding the changes in practice. For example, if a teacher is trying implement a more discourse-based pedagogy and retains an IRE routine for soliciting student contributions, the routine is likely to work against the level of student discourse that the teacher is trying to promote.

Given the evidence from previous research that expert teachers make use of these routines, and the current recommendations for discourse-based, student-centered classrooms, preservice and practicing teachers may benefit from an awareness and examination of exchange routines in particular, and all three types of routines more generally. Specifically, enabling teachers to identify routines in the instruction of other teachers and in their own instruction, and to understand how routines can be used in the service of instructional goals, has the potential to be a powerful experience for teacher learning.

In the next section, the three facets of knowledge needed for teaching are brought together in the description of an intervention for preservice and practicing teachers in which the practice of teaching is the central organizing component.

2.3. Practice-Based Teacher Education and Professional Development

In order to have an impact on teacher practice, professional education learning experiences must transform teachers' understandings about teaching and learning in ways that are closely connected to classroom practice (Ball & Chazan, 1999; Smith, 2001a; Thompson & Zeuli, 1999). This notion stands in contrast to typical teacher education experiences, which have taken a theory-into-practice approach (Leinhardt, Young, & Merriman, 1995, Cochran-Smith & Lytle, 1999). In this view, the education of teachers is aimed at introducing them to theories of teaching and learning, leaving the matter of determining how to apply these theories to the practice of teaching largely unaddressed.

Similarly, professional development experiences aimed at practicing teachers have typically consisted of isolated episodes that are often elective and focused on teaching tips, tricks, new curriculum materials, or specific procedures that the instructor is advocating (Smith, 2001a; Remillard & Geist, 2002). These types of experiences are designed to be additive in nature, grafting new knowledge or practices onto teachers' existing classroom practice (Thompson & Zeuli, 1999). However, in order to meaningfully change teachers' beliefs and practices, transformative experiences are required that can give teachers cause to re-examine their beliefs and practices with respect to teaching and learning (Thompson & Zeuli, 1999).

One promising means of providing transformative experiences for teachers is practice-based teacher education (Ball & Cohen, 1999; Shulman, 1986; Smith, 2001a). In this view, teacher learning is situated in the practice of teaching; that is, it is defined by activities central to teaching practice, such as the selection of instructional tasks, interpretations of student thinking, and assessment of students learning, and makes use of materials that depict the authentic work of teaching. These practice-based materials are created for or during the practice of teaching, including lesson plans, student work, and other classroom artifacts (Ball & Cohen, 1999; Smith, 2001a). Through the use of these practice-based materials in a teacher education setting, the everyday work of teaching becomes an object for ongoing investigation and thoughtful inquiry. In contrast to the theory-into-practice approach, the practice-based approach allows teachers to develop understandings of subject matter, of pedagogy, and of student work at the intersection of subject matter and pedagogy through the exploration of materials and situations that are authentic to the work of teaching. General principles and theories about teaching are then seen as emerging from the examination of the particulars of teaching practice.

Some recent efforts to improve teacher education in the United States have placed practice-based materials at the center of the professional training of teachers with positive results. The ASTEROID (A Study in Teacher Education: Research on Instructional Design) project in particular has developed two courses for teachers that use mathematical tasks that represent high cognitive demand from reform-based mathematics curricula, narrative and video cases of teaching, student work artifacts drawn from national assessments, and teacher lesson plans as the foundational materials for the study of both mathematical content and the teaching of mathematics. Preliminary results from the analyses of ASTEROID courses show that both preservice and practicing teachers experienced growth in their knowledge of mathematics, knowledge of content for teaching, and knowledge of mathematics for student learning (Engle & Smith, in preparation; Hughes & Smith, 2004; Smith, Silver, Leinhardt, & Hillen, 2003, in preparation; Steele, 2005; Steele, Hillen, Engle, Smith, Leinhardt, & Greeno, in preparation; Stein, Engle, Hughes, & Smith, submitted).

This study builds on the instructional design of the ASTEROID courses, using a set of narrative cases related to geometry and measurement in the middle grades as the centerpiece for the practice-based teacher education experience for preservice and practicing teachers. The present study makes several significant additions to the ASTEROID instructional design model. First, the present study uses an adaptation of the knowledge needed for teaching framework (Ball, Bass, & Hill, 2004) as an organizing framework for the design and implementation of the course. Additionally, the course adds the practices described in the knowledge of practices that support teaching section into the course design. This is done with the intention of raising the instructor's own pedagogical moves to an increased level of salience for teachers and having them reflect on the purposes of such moves, how the moves might impact student learning, and how such moves might be useful in the teachers' own practices. Finally, the course explores a new content area that has traditionally been underrepresented in teacher education and professional development: geometry and measurement in the middle grades.

As with any professional education experience, the design of a practice-based teacher education experience is guided by principles held by the designer. These design principles operationalize theoretical perspectives on teaching, learning, and assessment by providing general parameters for the design and implementation of activities. The principles also guide the choices that an instructor makes in the enactment of the planned activities. A number of educational researchers (e.g., Brown, 1992; Cobb, 2001) have endeavored to design instructional interventions grounded in a particular set of design principles. While many proponents of practice-based professional education have articulated the theoretical underpinnings of the approach (e.g., Ball & Cohen, 1999; Shulman, 1996; Smith, 2001a; Sykes & Bird, 1992), few[4] have explicitly articulated a set of design principles that might serve to guide other teacher educators in designing similar interventions. However, such principles are implicit in the work of these educators and researchers. The explicit articulation of a set of principles that bridge the theoretical bases for teaching and learning and the implementation of a practice-based teacher education experience would serve to guide other educational researchers in designing similar interventions, making studies that use such an approach both more replicable and more transparent with respect to the decisions made by the instructor.

Regardless of the design used, professional education experiences provide teachers the *opportunity to learn* the content that is embodied in the instructor's goals for the experience.

[4] One notable exception is the work of Smith, Stein, Silver, Hillen, and Heffernan (2001).

Opportunity to learn is a construct that is used in a variety of ways to measure learning in classrooms, from K-12 to college and beyond. Early conceptions of opportunity to learn in the 1960s centered on content to be assessed on a written examination (Wang, 1998). Hallinan (1987) identifies two key components of opportunity to learn: the amount and quality of exposure to new knowledge. Connecting these components to the situative perspective, which considers learning as both an individual and a participatory practice, one might consider also that the opportunity to learn must include a participatory component and an individual one. Thus, to measure teacher learning in a practice-based teacher education experience grounded in the situative view, teachers would have to have significant exposure to new knowledge, both in terms of amount of time and in the quality of the mathematical or pedagogical ideas, and this exposure should occur both in public discussion and through individual work allowing teachers to reflect on and internalize their understandings.

In addition to being grounded in theoretical perspectives on learning, design principles frame the learning opportunities in the course and predict particular types of learning outcomes on the part of the teachers engaged in the professional development experience. To that end, particular types of learning gains for teachers in the course can be predicted by the design principles, assessed through pre- and post-course measures, and attributed to the course experience through the analysis of opportunity to learn. This provides a cohesive trace that links learning outcomes to the design and implementation of the course and provides an opportunity to test conjectures made in the articulation of the design principles.

The next chapter details the design of the practice-based course and methodology used to measure student learning, which has been developed in such a manner consistent with the literature summarized in this chapter and has been designed to examine changes in the knowledge needed for teaching geometry and measurement.

3. METHODS

In this chapter, the design and methods used in the study are described. The first section describes the purpose of the study. The next section details the design of the study, the population, and the instruments designed to assess knowledge needed for teaching geometry and measurement, matching the data sources to the research questions. Data analysis methods are described in the third section, and the final section describes the design of the intervention.

3.1. Purpose of the Study

The purpose of this study was to examine changes in the knowledge needed for teaching of teachers participating in a practice-based course on geometry and measurement in the middle grades. Specifically, the study sought to study teacher growth in the three facets of the knowledge needed for teaching framework: knowledge of mathematics and mathematical activities, knowledge of mathematics for student learning, and knowledge of practices that support teaching. Both knowledge of mathematics and mathematical activities and knowledge of mathematics for student learning were assessed through teachers' written work on a pre/post assessment, through the work created and publicly shared in the course sessions, through the written work of teachers on course assignments, and through individual interviews with teachers. Knowledge of practices that support teaching were examined primarily through a pre- and post-course assessment using a video case of teaching, and secondarily through selected course discussions. The study also aimed to make explicit a set of design principles for practice-based teacher education experiences, and to investigate how those design principles might predict teacher learning in the course. The design of instruments to assess each of these three facets of teacher knowledge and articulate the set of design principles, the relationship between the instruments and the research questions, and the methods for selection of data sources and data analysis are discussed below, following a description of the population for the study.

3.2. Design of the Study

This study used a quasi-experimental design with a convenience sample and a nonequivalent contrast group (Christensen, 2001). The treatment sample consisted of teachers who registered for the geometry and measurement in the middle grades course either as an elective or a course requirement. Teachers in the treatment group participated in the instructional intervention that consisted of a 6-week course related to knowledge needed for teaching geometry and measurement in the middle grades. The sections that follow describe the population, data sources collected, and analyses performed.

3.2.1. Population

The population for the study consisted of 25 teachers enrolled in the geometry and measurement in the middle grades course, offered summer session 2005, who consented to participate. All teachers agreed to provide written work and participate in the videotaping; 5 teachers declined to participate in the out-of-class interviews due to issues of personal scheduling. Table 2 shows the population by subgroup.

Table 2. Population of the geometry and measurement course.

Subgroup	Number of Teachers Enrolled	Number of Teachers Interviewed
Secondary MAT (preservice)	9	8
Elementary MAT (preservice)	3	3
Secondary M.Ed. (practicing)	10	7
Ed.D. (teacher leaders)	3	2

The first subgroup in the course was a group of secondary mathematics Masters of Arts in Teaching (MAT) students, for whom the course was a required capstone experience. The

MAT teachers were in the final month of a full-year internship in secondary mathematics classrooms. Teachers in this subgroup held undergraduate degrees in mathematics or the equivalent. They had also completed five courses related to student-centered mathematics teaching, including a practice-based course focused on proportional reasoning in the middle grades. Several of the pedagogical activities and ideas in the geometry and measurement course were not new to this subgroup; however, these activities and ideas were explored in the context of geometry and measurement, a content area that had not been the focus of prior coursework.

The second subgroup was practicing secondary mathematics teachers pursuing a Masters of Education (M.Ed.). This subgroup consists of teachers who were in general early in their careers (5-10 years experience). The geometry and measurement course was an elective for this subgroup. This subgroup also held undergraduate degrees in mathematics or mathematics education.

The third subgroup was elementary MAT teachers. These teachers were also in the final month of a full-year internship; however, their experience had been in an elementary classroom rather than a secondary classroom teaching subjects that included, but were not limited to, mathematics. The course was an elective for this subgroup. Elementary MAT teachers who elected to take mathematics courses had a particular interest, but not an undergraduate degree, in mathematics.

The final subgroup consisted of teacher leaders pursuing an Ed.D. in mathematics education, who held coaching or professional development positions in the region. This group of teachers had extensive classroom experience (14-25 years) in addition to their work as professional developers. The course was an elective for this subgroup.

A non-equivalent contrast group was recruited. This group consisted of teachers who had prior practice-based learning experiences similar to the geometry and measurement course, but with different content, as well as teachers who had no practice-based learning experiences. Preservice and practicing teachers with varying levels of experiences were recruited. The diversity of the contrast group was intended to approximate the anticipated diversity of the treatment group. Two preservice teachers were recruited out of another teacher certification program at the same university. Eleven practicing teachers with between 1 and 10 years' experience were recruited both from past preservice teacher cohorts at the university and through a professional development experience in which the principal investigator was involved. Participation in the contrast group was voluntary.

3.2.2. Data Sources

The geometry and measurement course in which the treatment group participated was a 6-week experience which met twice a week for 3 hours each meeting. Six data sources were collected from the treatment group in the service of answering the research questions. The two primary data sources were a pre- and post-course written assessment, administered to all teachers in the course, and two semi-structured interviews, one between the first and third meetings of the course and the other following the final course meeting (referred to as *first* and *second* interviews), in which 20 teachers participated. Additional data sources included videotaped records of all course meetings, copies of teachers' notebooks, all co-constructed artifacts from the course enactment, and a planning diary kept by the instructor. All teachers were assigned pseudonyms, interviews were audio recorded and transcribed by the principal investigator, and all artifacts containing teacher names were blinded with pseudonyms substituted.

The contrast group was administered the pre-course written assessment and the first interview. These data sources were collected to establish to what degree performance of the treatment group was representative of group of teachers from similar backgrounds who were not enrolled in the course. If the two groups were equivalent based on analysis of the written and interview data, any learning gains shown by the treatment group could be attributed to the geometry and measurement course.

Each of the six data sources were collected to answer the research questions that frame the study. Tables 3, 4, and 5 match the instruments and specific items within the instruments with the first three research questions investigating knowledge needed for teaching being assessed. Research Question 4 was answered through the analysis of the contrast group data and comparison to the treatment group data, and Research Question 5 was answered through the use of the instructor's planning diary and selected data from the videotaped course meetings and interview transcripts. Following the tables, the data sources are described in additional detail, including specific information about how each data source relates to the knowledge needed for teaching framework and aids in answering the research questions.

Table 3. Data Sources for Knowledge of Mathematics and Mathematical Activities.

	Pre-Course Written Assessment	*First Interview*	*Course Discussions*	*Course Assignments*	*Second Interview*	*Post-Course Written Assessment*
Content Knowledge in the Domain						
Relating Area and Perimeter						
Understand that area and perimeter have a non-constant relationship (no evidence of misconception; measure quality of explanation)	Part C, Tasks 1, 2a	Task 3	X	TTAL	Task 4	Part C, Tasks 1, 2a
Explain how changes to dimensions of a figure impact perimeter and/or area (including transformations on a plane figure)	Part C, Task 1, 2b, 2c	Task 3	X		Task 4	Part C, Task 1, 2b, 2c
Explain the relationships between linear and square units and utilize these relationships to make sense of area and perimeter			X		Task 1	
Demonstrate understanding of the meaning of area and perimeter using a variety of tools and representations	Part C, Task 1, 2a		X			Part C, Task 1, 2b, 2c
Relating Edge Length, Surface Area, and Volume						
Understand the relationship between edge length, surface area and volume, including that surface area and volume have a non-constant relationship (no evidence of misconception; quality of explan.)	Part C, Task 3a, 3b, 3c		X	LL4		Part C, Task 3a, 3b, 3c
Explain how changes to the dimensions of a 3-D figure (specifically a rectangular prism) impact surface area and volume	Part C, Task 3d, 3e		X			Part C, Task 3d, 3e
Link the concepts of surface area and volume to spatial structuring and the composition of a 3-D figure	Part C, Task 3; Part D, Task 6	Task 1	X	LL3	Tasks 1, 2	Part C, Task 3; Part D, Task 6
Demonstrate understanding of the meaning of surface area and volume using a variety of tools and representations	Part C, Task 3; Part D, Task 6		X			Part C, Task 3; Part D, Task 6(?)
Reasoning and Proof						
Define proof		Task 2a	X		Task 3a	
Identifying proofs and non-proofs		Task 2b	X		Task 3b	
Constructing mathematical arguments	Part C, Task 4		X			Part C, Task 4
Understand the roles of proof in mathematics: Verify a stmt is true, explain why a stmt is true, communicate math knowl., create new math, systematize the domain		Task 2a	X		Task 3a	

Table 3, con't.

Content Knowledge for Teaching						
Identifying the big ideas in middle grades geometry and measurement and tasks that provide student with opportunities to explore these ideas	Part B	Task 4	X	LL2, 4	Task 1; Task 4	Part B
Relating Measurable Attributes of Geometric Figures						
Identify misconceptions about area and perimeter	Part C, Task 1; Part D, Task 5	Task 3	X		Task 4	Part C, Task 1; Part D, Task 5
Use a range of representations to explain the relationship between dimension, area, and perimeter	Part C, Task 1, 2		X			Part C, Task 1, 2
Identify and/or create mathematical tasks that provide students with opportunities to explore the big ideas in geometry and measurement	Part C, Task 3a		X	LL4		Part C, Task 3a
Use a range of representations to explain the relationship between edge length, surface area, and volume	Part C, Task 3	Task 1	X		Task 2	Part C, Task 3
Identifying strategies for spatial structuring and tasks and pedagogical approaches that support the development of students' spatial structuring (includes use of volume formulas)	Part D, Task 6		X	LL3	Task 1	Part D, Task 6
Reasoning and Proof						
Explanatory power of proof		Task 2	X		Tasks 1, 3	
nderstand and articulate the role of R&P in school mathematics, including: verifying truth, explaining why, communicating knowledge, creating new math, systematizing the domain, generalization		Task 2	X		Tasks 1, 3	
Identifying discourse as a promising tool to support reasoning and proving		Task 2a	X	LL1	Task 3a	

Table 4. Data Sources for Knowledge of Mathematics for Student Learning.

Course Goals	*Pre-Course Written Assessment*	*First Interview*	*Course Discussions*	*Course Assignments*	*Second Interview*	*Post-Course Written Assessment*
Anticipating student solutions		Task 3	X	TTAL	Tasks 1, 4	
Monitoring student work	Part D, Task 5	Task 3	X	TTAL	Tasks 1, 4	Part D, Task 5
Selecting responses for whole-group discussion	Part D, Task 7	Task 3	X	TTAL	Tasks 1, 4	Part D, Task 7
Sequencing responses for whole-group discussion	Part D, Task 7	Task 3	X	TTAL	Tasks 1, 4	Part D, Task 7
Connecting responses shared in whole-group discussion		Task 3	X	TTAL	Tasks 1, 4	

Table 5. Data Sources for Knowledge of Practices that Support Teaching.

Course Goals	*Pre-Course Written Assessment*	*First Interview*	*Course Discussions*	*Course Assignments*	*Second Interview*	*Post-Course Written Assessment*
Understanding role of routines in teaching	Part A		X	Identifying routines	Task 1	Part A
Identifying routines in teaching	Part A		X	Identifying routines	Task 1	Part A

3.2.2.1. Pre- and Post-Course Written Assessment. The pre- and post-course written assessment (see Appendix A) consists of four sections: Part A related to routines, Part B related to the big ideas in geometry and measurement, Part C related to mathematics, and Part D related to student learning. The items on the assessment are designed to assess all three facets of the knowledge needed for teaching framework. Items are grounded in the content of middle grades geometry and measurement; specifically, the relationship between measurable quantities of geometric figures (dimension, perimeter, and area; dimension, surface area, and volume) and reasoning and proof.

Part A assesses with teacher knowledge of practices that support teaching; specifically, routines. Teachers watched a videotaped excerpt of a teacher facilitating an exploration of the surface area of a cylinder (Boaler & Humphreys, 2005). On the pre-course assessment, teachers were asked to identify moves that the teacher made that organized or supported classroom activity, identifying the move, the lines in the video transcript in which the move was evidenced, and the impact of the move in supporting classroom activity. On the post-course assessment, teachers were asked to identify routines the teacher used, again providing line numbers and describing the impact of the move. This change in terminology between pre- and post-assessment is due to the fact that teachers may not have had the label of "routine" available to them prior to the course, and the label was introduced explicitly during the course. This task assessed which teacher moves teachers attended to before and after the course, and their descriptions of how the moves support the structuring of classroom activity. Teachers were also asked to label the routines as support, exchange, or management on the post-course assessment.

Part B asks teachers to describe what the important ideas of geometry and measurement are in the middle grades. This represents a coarse-grained assessment of teachers' knowledge of the big conceptual ideas in geometry and measurement. Understanding what teachers consider important knowledge related to middle grades geometry and measurement assessed an aspect of content knowledge for teaching.

Part C contains 5 tasks related primarily to teachers' knowledge of mathematics and mathematical activities. Tasks 1 and 2 assess teachers' knowledge of mathematics related to area and perimeter. Task 1 is designed to evaluate teachers' understanding of how a figure with a fixed perimeter can have a range of areas. The task, titled "Fence in the Yard," asks teachers to determine the maximum area for a rectangular enclosure with a 36 foot perimeter. This task is taken from the National Assessment of Educational Progress and was administered on the 1996 NAEP (Kenney & Lindquist, 2000). The task assesses the ability to determine the rectangle that maximizes area holding perimeter constant. This task can be solved using a variety of strategies and representations, from sketching and listing of possibilities to the use of the derivative. Additionally, the task has the potential to expose misconceptions related to area and perimeter; specifically that a fixed perimeter implies a fixed area, and the error of confusing area and perimeter.

The three parts of Task 2 assess teachers' understandings of how a figure with a fixed area can have a range of perimeters. Task 2a asks teachers to determine if a parallelogram with a fixed base measurement and a fixed area can have different perimeters. This task assesses teachers' understandings of the relationship between area and perimeter, as well as the representations and examples they use to justify their response. It also has the potential to uncover the error of confusing area and perimeter. Tasks 2b and 2c also assess teachers' understanding that a constant area can yield a changing perimeter. Both tasks involve the set of geometric tiles known as tangrams. A diagram of the tangram tiles and two pictures of rearrangements of the seven tiles are included. Task 2b asks teachers to determine whether the two figures have the same area. This task is designed to assess the misconception that rearranging a figure impacts the figure's area. Task 2c asks teachers to determine whether the two figures have the same area. This task is designed to assess the misconception that two figures with the same area have the same perimeter. Both tasks also offer opportunities to assess the methods used by teachers to determine their answers, and specifically the generality of such methods.

Task 3 contains a series of subtasks designed to assess teachers' content knowledge related to surface area and volume. The first two subtasks, a and b, involve a situation where two people are painting rooms in a house known to have the same floor space (area of the base). The first task asked teachers to determine whether the two people will need the same amount of paint; that is, do the rooms contain the same surface area. This task was designed to assess teachers' spatial structuring and their understanding of the relationship between area of the base of a prism and its surface area. Additionally, the task offered an opportunity to assess the notion that a rectangle (the floor) with a fixed area can have more than one perimeter. Subtask b asks teachers to determine whether the two rooms have the same volume. This task assessed a variety of understandings related to surface area and perimeter. First, it had the potential to reveal how teachers calculate the area of a rectangular prism, including representational use. For this task, the most straightforward way to find volume is to recognize that volume can be computed as the product of the area of the base and the height. If teachers did not have the understanding of volume as area of the base times height, teachers had to specify at least one set of dimensions for the floor and use the traditional length × width × height formula for volume. A lack of understanding of spatial structuring as the area of the base times height was evident if teachers need to try a number of combinations for length and width of the area of floor to compute volume. The area of the base times height spatial structuring corresponds to the layering structure reported by Battista, Clements, and colleagues (e.g. Battista & Clements, 1998). Finally, the task assessed teachers' conceptions of the relationship between surface area and volume; specifically, the task will assess teachers' understanding that three-dimensional shapes with different surface areas can have the same volume. Subtask c provides a second assessment of this understanding directly, by asking if surface area can be found if the volume of a rectangular prism is known. This controls for teacher performance on subtasks a and b; if teachers are not successful calculating the volume of the rectangular prism in subtask b, possible misconceptions with respect to the relationship between surface area and perimeter may not be evidence.

Subtasks d and e ask teachers how they might adjust the dimensions of a known rectangular prism to double its volume and its surface area. This task assessed teachers' understandings of the relationships between the dimensions (edge length) of the rectangular prism, surface area, and volume. These subtasks also evaluated the representations and types of reasoning that teachers use to respond to the tasks. For example, a teacher might reason about a rectangular prism with double the original’s volume using a numeric example, a symbolic argument relying on a formula, a written explanation, or a combination of one or more of these representations. Teachers who had a more conceptual and well-connected understanding of the relationships between measurable quantities were likely to create responses that connect more than one representation and are more general rather than relying on a single example.

Task 4 assessed teachers’ content knowledge of the domain with respect to proof. The task asked teachers to construct proofs of the formula for the area of a parallelogram and the area of a rectangle. First and foremost, this task assessed teachers' ability to construct a deductive argument in the domain of geometry and measurement. At a more detailed level, teachers’ explanations could be evaluated with respect to the generality and completeness of their argument, and particularly the explanatory power of the argument used. Additional data related to teachers’ understanding of proof was provided through the pre- and post-course interviews.

Part D (Tasks 5, 6, and 7) measured knowledge of mathematics for student learning and the content knowledge for teaching aspect of knowledge of mathematics and mathematical activities. Task 5 assessed content knowledge for teaching related to area and perimeter, as well as knowledge of mathematics for student learning. The task presented a student response containing an overgeneralization regarding the relationship between area and perimeter. The recognition of this misconception assessed content knowledge for teaching. The way in which teachers chose to respond to the student assessed knowledge of mathematics for student learning; for example, whether teachers chose to directly correct the student's misconception as compared to addressing the student's misconception by probing student thinking. This task also assessed use of monitoring; specifically, the types of questions that teachers might use assess and advance student thinking that centers on a misconception.

Task 6 assessed content knowledge for teaching with respect to the use of formulas for area and volume. Subtask a asked teachers to consider the formulas for area of a rectangle, *Area* = *length* × *width* and *Area* = *base* × *height*. The former is specific to rectangles and the more traditional format for the area of a rectangle, whereas the latter generalizes to parallelograms as well and may aid in alleviating misconceptions related to the area of a parallelogram (Fuys, Geddes, & Tischler, 1988). This assessed teachers' understanding of the meaning of each formula and their affordances in the teaching of school mathematics. Subtask b asked teachers to consider two formulas for the volume of a rectangular prism, *Volume* = *length* × *width* × *height* and *Volume* = *Area of base* × *height*. The former corresponds to a convenient way of finding the volume based on measurements, whereas the latter corresponds more closely to a helpful spatial structuring of the rectangular prism, and also generalizes to any prism regardless of the shape of its base (Battista & Clements, 1998). Results of this task indicated how teachers were making sense of rectangular prisms (numerically or spatially), as well as their understanding of the affordances of the formulas with respect to student learning.

Task 7 presented teachers with student work from the Fence in the Yard task solved earlier in the written assessment. Six samples of student work were presented, representing both correct and incorrect answers, a range of arguments, and a range of solution strategies. The task asked teachers to imagine that their students produced these responses, and to select a subset of responses to be shared in a whole-class discussion. The task also asked teachers to specify what order they would want the solutions shared. This task directly assessed knowledge of mathematics for student learning, and specifically addresses two of the five practices for productive use of student thinking in whole-class discussions: selecting student responses to be shared in a whole-group discussion and sequencing the shared responses (Stein, Engle, Hughes, & Smith, submitted). Teachers' choices of solutions to share, the order in which they were shared, and the rationales provided for selection and sequencing were examined.

Taken together, the tasks included on the written assessment assessed teachers' content knowledge, both with respect to the domain and with respect to teaching, in the key areas of relationships between measures (dimension, perimeter, and area; dimension, surface area, and volume) and reasoning and proof. The specific aspects of content knowledge assessed are also summarized in the first column of Table 3. Additionally, the written instrument provided opportunities for teachers to demonstrate knowledge of mathematics for student learning in the form of using three of the five practices for productive use of student thinking in whole-class discussions (Stein, Engle, Hughes, & Smith, submitted). The assessment also provided information on the types of routines to which teachers attended in a video record of teaching, as well as their conceptions of how routines advance the mathematical activity of the classroom.

3.2.2.2. First Interview. The first interview (see Appendix B) was conducted during in the first week of the course, and was designed to serve as another baseline measure of knowledge needed for teaching geometry and measurement. The semi-structured interview provided teachers with opportunities to discuss how they make sense of key geometry and measurement topics and the teaching of those topics without prompting or leading them in the directions of interest with respect to the research questions. The interview consisted of three tasks which were derived from previous research.

Task 1 explored teachers' mathematics content knowledge related the volume of rectangular prisms, and the relationship between linear measurement of the dimensions of a prism and its volume. The task is adapted from Battista (1998). Teachers were presented with a cardboard box with the inside panels marked with the width and height of two unit cubes (see Figure 4). Teachers were also be provided with 10-12 of the two unit cube clusters, called "packages." The task asked teachers to determine the number of packages that will fit in the box.

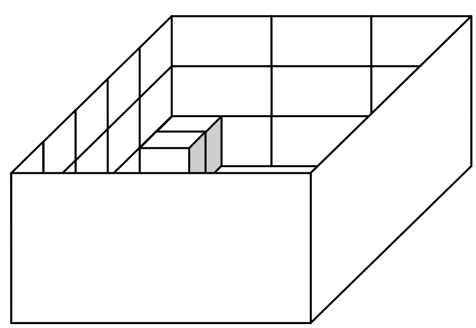


Figure 4. The unit cube box.

This task was designed to assess a number of aspects of teachers' understanding of volume. First, teachers' methods of finding volume were assessed (where volume in this case is operationalized in terms of number of packages). A common method of finding the volume of such a box, as reported by Battista (1998), is to multiply 3 (length) by 5 (width) by 5 (height), ignoring the fact that the units of measure are not consistent. This is similar to the measuring the desk activity described by Simon and Blume (1994) and shown in Figure 2, but in three dimensions rather than two. Of specific interest were methods that represented a full or partial layering strategy (Battista, 1998), which suggested attention to the conceptual meaning of volume, strong spatial sense, and an ability to coordinate the individual dimensions of the box with a notion of volume.

The second part of this task asked teachers to find the surface area of the box (including a top). This task is provided additional detail on teachers' strategies for finding surface area that were not apparent based on their work on the pre- and post-course assessment. Specifically, the type of units teachers used to determine the surface area and the methods they selected were examined. Teachers had rulers and transparent inch grids with which to cover the box to use as tools for finding the surface area.

Task 2 assessed teachers' content knowledge, both of the domain and of teaching, related to proof. This task was adapted from a more detailed examination of teachers' conception of proof by Knuth (2002a, b). The first portion of the task assessed teachers' conceptions of what proof is, what purpose proof serves in the mathematical domain, what makes an argument a proof, and the fallibility of proof (or lack thereof). These issues are identified by Knuth (2002a) and others as areas in which teachers typically hold a limited conception of proof. The second part of the task presented 8 explanations that differ in terms of whether they are deductive proofs, explanations, or other arguments, in terms of explanatory power, and in terms of form. Teachers were asked to identify which of these are proofs and explain their reasoning. Teachers were then asked to rank the proofs on a scale of 1 to 4, borrowing from the work of Senk (1985, 1989). This categorization allowed a finer level of detail regarding what types of arguments teachers find convincing as proofs and which they do not. This task provided the opportunity to examine the ways in which teachers characterized proofs, whether they preferred particular representations as noted in previous research (Chazan, 1993; Knuth, 2002a), and to examine the features teachers valued with respect to the explanatory power of a proof.

Task 3 assessed teachers' planning practices in planning a geometry and measurement lesson, specifically with respect to knowledge of mathematics for student learning. Teachers were given a mathematical task related to geometry and measurement (the Minimizing Perimeter task; see Appendix B) and were asked to sketch a lesson plan based on the task. Teachers were given a general target mathematical goal for the lesson and were able to modify the task if they so choose. By planning the lesson and describing their decisions in the planning process, teachers had the opportunity to show evidence of use of the five practices described by Stein, Engle, Hughes, and Smith (submitted).

<u>3.2.2.3. Course activities and discussions</u>. All 12 class sessions were videotaped, all public written artifacts were collected and preserved, and all individual teacher assignments were duplicated. These artifacts were examined in light of results from the analyses of the assessments and interviews to demonstrate opportunity to learn and to provide additional illustrative detail. In addition, the Thinking Through a Lesson assignment was used as a primary data source to assess use of the five practices for productive use of student thinking.

The Thinking Through a Lesson (TTAL) assignment (see Appendix F) was designed as tool for teachers to apply knowledge of mathematics for student learning in the context of lesson planning, and specifically the five practices for productive use of student thinking, in the context of lesson planning. The TTAL assignment in the current study provided teachers with a task related to two-dimensional geometry and measurement that builds on the ideas pursued to that point in the course, and ask teachers to plan a detailed lesson around the task. The sources for these tasks included the National Council of Supervisors of Mathematics Great Tasks bank and Navigations series, tasks released from the Balanced Assessment project, tasks from articles in Mathematics Teaching in the Middle School, and tasks from the middle grades Connected Mathematics Project. All tasks addressed some aspect of the relationship between measurable quantities of geometric figures. The TTAL Protocol is designed to instantiate the five practices described by Stein, Engle, Hughes, and Smith (submitted), and provided evidence of teachers' abilities to plan a geometry and measurement lesson using those practices when asked.

3.2.2.4. Second Interview. The second interview, included as Appendix C and conducted after the course, was designed to serve as a measure of change in teachers' knowledge needed for teaching geometry and measurement. The semi-structured interview mirrors the structure of the pre-course interview, including all three tasks from the first interview and a fourth task designed to aid teachers in reflecting on their learning as a result of the course. The second interview consisted of four tasks which are derived in part from previous research. Tasks 2, 3, and 4 are identical to Tasks 1, 2, and 3 on the first interview. The new task, Task 1, is described below.

Task 1 directly solicited teachers' own thinking with respect to changes in their knowledge of mathematics, mathematics for student learning, and practices that support teaching. Teachers were presented with a course map that contains representations of the major activities in the course, and were asked to identify ideas that they know or understand that they did not know or understand, or understood differently, prior to the start of the course and map those ideas onto the course activities. The task asked teachers to identify ideas that they have come to know and understand with respect to mathematics, students' mathematical learning, and routines or other pedagogical ideas. This task provided self-report data about teacher learning that was used to contextualize changes in learning observed through other analyses.

3.2.2.5. Other Data Sources from Teachers. In addition to the data sources listed, all notebooks from consenting teachers were duplicated. Additionally, artifacts from classroom discussions were collected. These data sources were used as secondary data sources to support the analysis of the primary data sources listed in the preceding section.

3.2.2.6. Planning and Reflection Journal. During the course, the instructor kept a planning and reflection journal. This journal was intended to document the instructor's decisions related to the choice of tasks, strategies for enactment, and key understandings to facilitate with teachers. Additionally, the journal provided a forum for reflecting on class sessions, including the documentation of the instructor's view of the enactment, unplanned events, understandings that were and were not developed, and how the enactment of one class impacted the planning of the next. Entries were made pre- and post-course, as well as before and after each class meeting. These data served as the primary source for the articulation of design principles; principles were then refined through further reflection and discussion with colleagues familiar with practice-based teacher education and issues of instructional design.

3.2.2.7. Summary. The written pre- and post-course assessments and transcribed interviews served as the primary data sources for answering research questions 1, 2, and 3, with the course videotapes, public artifacts, and other teacher-level data contextualizing those results and demonstrating opportunity to learn. Written assessment and interview data were collected for the contrast group and used to answer research question 4. Research question 5 was addressed through the instructor's planning diary, which documented instructional decisions and reflections on class sessions. Tables 3, 4, and 5 presented earlier correlate the data sources described in this section with knowledge of mathematics and mathematical activities, knowledge of mathematics for student learning, and knowledge of practices that support teaching.

3.3. Data Analysis

Analysis of the data incorporated both quantitative and qualitative methods, and followed a grounded theoretical approach (Corbin & Strauss, 1990). Rubrics were developed for the pre- and post-course written assessments and interview transcripts that sought to characterize performance with respect to the goals identified in Tables 3, 4, and 5. Specifically, these rubrics were designed to assess correctness of responses to mathematical tasks and particular features of responses (generality or mathematical sophistication of response, single or multiple solutions, number and types of representations, rationale for response) for mathematical tasks; level of attention to pedagogical issues (e.g., the five practices) for tasks related to teaching; and emergent categories for open response tasks such as identifying the big mathematical ideas in the domain (See Appendix A, Part B). Rubrics were created to categorize teacher responses to questions about reasoning and proof in the first and second interview, using categories adapted from previous research (e.g., Chazan, 1993; Knuth 2002a, b; Senk, 1989). Statistical tests were then used to compare pre- and post-course performance. Categorical data such as representational use or categorical rubric scores were compared using chi-square analyses. For situations in which the expected value was less than 5, Yates' correction was employed. Numerical data, such as the mean number of representations used, was compared using paired *t*-tests; all *t*-tests were one-tailed unless otherwise noted. On rubrics which measured the presence or absence of a particular type of response or response attribute, McNemar's test was used to compare the number of teachers pre- and post-course who exhibited the response or attribute. On rubrics that yielded ordinal data, the Wilcoxon sign-rank test was used to determine change in performance. Interview transcripts were coded, with lines of text coded as evidence of particular constructs counted. Differences in lines of interview text were tested in three ways: for differences in the number of teachers mentioning or not mentioning an attribute using

McNemar's test; for differences in the mean percentage of lines of text for a certain question devoted to a phenomenon of interest using paired *t*-tests; and for differences in the number of lines of text from first to second interview using a chi-square analysis. The complete set of rubrics used, including relevant examples, is included in Appendix E.

Qualitative data was collected from videotapes and artifacts from course meetings, transcripts of the first and second interviews, individual written assignments, and a planning and reflection journal kept by the instructor. These data were used to trace the opportunity to learn for goals that showed significant growth through the quantitative analysis, and to make explicit the design principles for the course. Building off the criteria articulated by Hallinan (1987) and taking into account a situative view of learning, opportunity to learn was defined as follows: An opportunity to learn with respect to the geometry and measurement course consists of the identification of a mathematical or pedagogical idea for study and engagement with that idea through a single activity or series of activities which provide an opportunity for individual and small-group work, for which entry is available for teachers with a range of prior experiences, in which the mathematical or pedagogical ideas are publicly discussed, and for which there is an opportunity to reflect on and/or expand on the ideas discussed through a written assignment or individual oral interview.[5] For goals which showed significant change, opportunity to learn was traced using the videotaped course meetings and individual and co-constructed course artifacts based on the criteria stated.

All audiotaped interviews and videotaped discussions deemed relevant to change in knowledge needed for teaching were transcribed and stored electronically. All teacher

[5] For the purposes of this study, only activities with whole-group discussions of 20 minutes or more in length were considered for analysis. Given the size of the population in the course (n=25), it is reasonable to expect that discussions of 20 minutes or more gave all teachers significant opportunities to contribute and to wrestle with the ideas at hand.

assignments were acquired electronically where possible; when electronic copies are not available, photocopied print copies were used. With respect to all coding described in the subsequent sections, electronic or print artifacts were coded by the principal investigator, with a second rater coding 20% of all data. Rubrics which were unclear or for which reliability was low were discussed and revised, with the data recoded. Final inter-rater reliability for the rubrics ranged from 88% to 100%.

The sections that follow provide additional detail regarding the development of rubrics to assess knowledge needed for teaching. These descriptions utilize the goals articulated in Tables 3, 4, and 5, and detail how rubrics were developed to assess change along these dimensions for knowledge of mathematics and mathematical activities, knowledge of mathematics for student learning, and knowledge of practices that support teaching.

3.3.1. Knowledge of Mathematics and Mathematical Activities

Data related to knowledge of mathematics was coded in such a way as to make salient learning of the mathematical concepts listed in the first column of Table 3. For items on written assessments, rubrics were developed to characterize teachers' solution methods and the correctness of and level of explanation in their solutions. The rubrics also identified misconceptions related to geometry and measurement concepts. Brief descriptions of the phenomena of interest are presented below, grouped by the categories used in Table 3; brief descriptions of rubrics and coded examples relevant to the analyses can be found in the appropriate sections of Chapter Four, with complete rubrics included in Appendix E.

3.3.1.1. Content Knowledge in the Domain: Relating Area and Perimeter. Items on the written pre- and post-test, as well as relevant evidence from the planning tasks in the pre- and post-course interviews, was coded for four main mathematical ideas:

- Understand that area and perimeter have a non-constant relationship
- Explain how changes to dimensions of a figure impact perimeter and/or area
- Explain the relationships between linear and square units and utilize these relationships to make sense of area and perimeter
- Demonstrate understanding of the meaning of area and perimeter using a variety of tools and representations

The rubrics used sought to characterize the correctness of teachers' responses to the mathematical tasks in the pre- and post-course written assessments with respect to these ideas, and also to identify any misconceptions teachers held with respect to the relationships between dimension, perimeter, and area. For example, in assessing teachers' responses to the *Fence in the Yard* Task (see Appendix A, Part C, Task 1), a rubric was used to assess the correctness of teachers' responses, the level at which teachers connected the notion of changing dimensions to perimeter and area, and any misconceptions teachers may have held regarding the notion that a constant perimeter does not imply a constant area.

In addition, the rubrics created sought to categorize the strategies teachers use to make their arguments and the level of generality with which the arguments were made. For example, teachers who were able to use the general characteristics of the figure to make general conclusions about the relationship between area and perimeter exhibited a more sophisticated understanding than teachers who needed to generate several examples to understand the relationship. Finally, the types of representations teachers used to solve mathematical tasks were coded.

Data from the interviews was coded for evidence of the understandings and/or misconceptions related to relating area and perimeter. Class discussions were examined to trace opportunity to learn these ideas to course discussions in order to understand how and why teacher change occurred in these areas, addressing research question 1b.

3.3.1.2. Content Knowledge in the Domain: Relating Edge Length, Surface Area, and Volume. In a similar procedure to the one described in the previous section, rubrics were created for relevant problems on the written pre- and post-test that assess the relationship between edge length, surface area, and volume. Data from the relevant interview tasks and from course discussions was considered. These data were coded for evidence of four mathematical ideas:

- Understand the relationship between dimension, surface area, and volume, including that surface area and volume have a non-constant relationship
- Explain how changes to to the dimensions of a 3-D figure impact surface area and volume
- Link the concepts of surface area and volume to spatial structuring and the composition of a 3-D figure
- Demonstrate understanding of the meaning of surface area and volume using a variety of tools and representations

Similar to the rubrics for dimension, perimeter, and area, the rubrics for written work first sought to determine the correctness of teachers' responses to mathematical tasks, to categorize the level of generality in teachers' responses, and to identify any misconceptions. For example, the rubric created to evaluate the *Painting the Living Room* task evaluated the correctness of teacher solutions, the degree to which they made connections between dimension, surface area, and volume, and any evidence of the misconception that two rectangular prisms with the same area of the base and height would have the same surface area, or would not have the same volume.

The types of strategies used, specifically for the interview task related to surface area and volume, were categorized using emergent categories based on teachers' responses. In addition, responses to both the written assessment and interview tasks were also examined for evidence of layering structures, as described by Battista and colleagues (e.g. Battista, 1998) to determine the volume of rectangular prisms. Finally, representational use was coded for all written tasks related to dimension, surface area, and volume. Class discussions were examined to trace opportunity to learn these ideas to course discussions in order to understand how and why teacher change occurred in these areas, addressing research question 1b.

3.3.1.3. Content knowledge in the domain: Reasoning and proof. The primary data sources for evaluating growth with respect to content knowledge of reasoning and proof were the proof discussion and evaluation task in the pre- and post-course interview and the constructing proofs task (Appendix A, Part C, Tasks 4a, 4b) in the pre- and post-course written assessment. These two data sources were coded for evidence of four mathematical ideas:

- Define proof
- Identify proofs and non-proofs
- Construct deductive mathematical arguments
- Understand the role of proof in mathematics

Data from the written assessment item asking teachers to construct deductive arguments were coded using a rubric that quantified both the completeness and the generality of the mathematical argument. The rubric also evaluated the level of explanatory detail in the proof. A proof does not have to be explanatory to be correct; however, explanatory proofs are useful tools in the classroom.

Responses to the identifying proofs interview task were coded for accuracy in identifying the proofs and non-proofs. Growth was examined with respect to teachers' ability to identify the proof arguments. Transcripts were coded for evidence of Knuth's (2002a) framework for the roles of proof in school mathematics. Teachers' ratings of the proofs with respect to most and least proof-like were also be analyzed for changes between the pre-course and post-course interviews. Based on similar research by Senk (1989), it was anticipated that teachers would initially score proofs that are symbolic and familiar higher than proofs that are in less familiar forms, but have a greater explanatory power. Rubrics were also created to code teachers' reasons for identifying proofs and non-proofs and in rating the proofs, with coding categories derived from the roles of proof in the mathematical domain as identified by Knuth (2002a). These roles are *verifying* that a statement is true, *explaining* why a statement is true, *communicating* mathematical knowledge, *discovering* or creating new mathematics, and *systematizing* statements into an axiomatic system.

Course discussions were examined for evidence of public discussion of these roles for proof. These data provided evidence of the opportunity to learn with respect to reasoning and proof, since there were few dedicated 'proof' activities in the course design. These data specifically aided in answering research question 1b.

<u>3.3.1.4. Content Knowledge for Teaching: Relating Measurable Attributes of Geometric Figures</u>. Five concepts relevant to concept knowledge for teaching with respect to relating measurable aspects of geometric figures were assessed:

- Identify the bid ideas in middle grades geometry and measurement
- Identify and/or create mathematical tasks that provide students with opportunities to explore the big mathematical ideas

- Use a range of representations to explain the relationships between measurable attributes of geometric figures
- Understand the affordances and constraints of different formulas for area, perimeter, surface area, and volume
- Identify strategies for spatial structuring and tasks and pedagogical approaches that support the development of students' spatial structuring

Teachers' understandings of the big ideas were assessed through the creation of emergent codes based on responses to the big ideas task (Appendix A, Part B). These categories were guided in part by the identified goals for teacher learning in the course. Representational use was measured through the coding of both the number and type of representations, and coding for the use of single as compared to multiple representations in teachers' responses to mathematical tasks. Teachers' ability to identify and modify mathematical tasks in ways that targeted the big mathematical ideas in geometry and measurement was measured through the lesson planning task on the first and second interview. Rubrics were created to categorize the types of modifications teachers made to the mathematical task for which they were asked to plan a lesson; coding categories were based on factors that support or inhibit the cognitive demands of mathematical tasks (Stein, Smith, Henningsen, & Silver, 2000).

The affordances of and reasons for using different formulas were assessed primarily through task 6 on the pre- and post-test. Teachers' explanations were examined for this item and coded with respect to the reasons teachers gave for preferring one formula over the other. Emergent categories were designed for this item based on teacher responses. These categories included familiarity to students, generalizability to related shapes, and personal teacher preference, and are listed in their entirety in Appendix E. Finally, teachers' responses to tasks related to three-dimensional figures in both the written assessments and interviews was coded for evidence of attention to layering strategies. Layering strategies are important with respect to content knowledge for teaching as they have been shown to be fruitful in aiding the development of students' spatial sense and coordination of two- and three-dimensional quantities.

Class discussions were examined to trace opportunity to learn these ideas to course discussions in order to understand how and why teacher change occurred in these areas, addressing research question 1b.

3.3.1.5. Content Knowledge for Teaching: Reasoning and Proof. With respect to reasoning and proof, teachers were assessed with respect to four concepts:

- Identify the big mathematical ideas in middle grades geometry and measurement related to proof
- Understand the explanatory power of proof and its relationship to mathematics teaching
- Understand the role of reasoning and proof in school mathematics
- Understand the relationship between reasoning and proof and classroom discourse

Data regarding teachers' conceptions of the big ideas related to proof in geometry and measurement came from responses to the big ideas task in the written assessment (Appendix A, Part B), and were coded using emergent categories based on teachers' responses and guided by the goals of the course with respect to reasoning and proof. A complete list of the categories used can be found in Appendix E.

The primary data source for the remaining three mathematical concepts was the proof task in the pre- and post-course interview. Interview transcripts were coded for evidence of teachers discussing the explanatory power of proof, the role of reasoning and proof in school mathematics again using categories from Knuth (2002a, 2002b), and the relationship between reasoning and proof and classroom discourse.

Course discussions were examined for evidence of public discussion of these roles for proof related to teaching. These data provided evidence of the opportunity to learn, specifically aiding in answering research question 1b.

3.3.2. Knowledge of Mathematics for Student Learning

With respect to knowledge of mathematics for student learning, the five practices for productive use of student work were used as a framework for coding. Teachers' pre- and post-course written assessments, interview transcripts, and course assignments were coded for evidence of the five practices: *anticipating student solutions*, *monitoring student work* (including questioning to assess and advance student thinking), *selecting, sequencing,* and making *connections* between student responses for public display and class discussion.

Rubrics were created to identify the degree to which each of the five practices were evident in teachers' work around a lesson planning task; these rubrics were applied to the Thinking Through a Lesson assignment and the lesson planning task on the first and second interview. For the purposes of these tasks, monitoring was operationalized through the questions teachers intended to ask during the monitoring phase of a lesson. A rubric was created to categorize the types of questions asked based on factors that support or inhibit students' engagement with cognitively demanding tasks (Stein, Smith, Henningsen, & Silver, 2000).

Finer-grained rubrics were used to assess the items on the written assessment that specifically targeted particular practices (Appendix A, Part D, Tasks 5 and 7). With respect to Task 5, the types of questions asked by teachers in response to the student misconception were coded as evidence of the monitoring practice. A rubric was developed based on previous research by Ma (1999) related to a similar task. With respect to Task 7, the selection and sequencing of student responses were tallied and compared; the reasons for that selection and sequencing were coded using emergent categories based on general features of teachers' responses.

Course discussions were examined for evidence of discussion of the five practices to establish opportunity to learn related to the five practices. These data served to answer research question 2b.

3.3.3. Knowledge of Practices that Support Teaching

This study sought to investigate the role of a particular practice that supports teaching, routines. As routines can only be examined directly in the context of an enacted lesson, a video case of an expert teacher was used as the primary means of assessing teachers' attention to routines. At the start of the course, teachers viewed a video case of Cathy Humphreys' teaching related to surface area (Boaler & Humphreys, 2005) and completed the routines pre-assessment (Appendix A, Part A). This assessed teachers' abilities to identify routines prior to course experiences in which the instructor's own routines are explored as an object of inquiry, and to identify how the teachers' routines served to advance the mathematical activity of the classroom. During the course, routines were discussed and the three types of routines – support, exchange, and management – were identified. Teachers viewed another video case of Cathy Humphreys' teaching related to volume and completed the Identifying Routines activity (See Appendix C). At the close of the course, the first video case was re-viewed and teachers completed the routines post-assessment (Appendix A, Part A). The principal investigator and second coder viewed the video clip multiple times and identified the routines used and their function. Teachers' responses to the written assessment were coded for identification of the routines; post-course assessments were also coded for the categorizations (support, exchange, management) identified for each routine. Change in the types of routines identified was compared pre- and post-course, and the categorization of routines post-course was compared to the researchers' categorizations.

Course discussions where routines are made visible were examined for evidence of talk, both by the instructor and by participating teachers, related to the issue of routines in general, the specific types of routines, examples of how routines manifest in classroom teaching, and consideration of the role of routines in organizing classroom activity and supporting student learning. These data were used to identify opportunities to learn with respect to routines, answering research question 3b.

3.3.4. Contrast Group Data Analysis

The contrast group was recruited to determine the extent to which treatment group pre-course performance was comparable to a group of teachers with similar backgrounds and experiences but not enrolled in the course. If the treatment group and contrast group performed similarly on pre-course measures, then an argument could be made that the treatment group was representative of the class of teachers with similar backgrounds. This allowed improvement in the treatment group to be attributed to the course experience. This comparison is particularly important for members of the contrast group who have had similar practice-based coursework and professional development experiences, as it may suggest a value-added in engaging teachers in practice-based experience focused on particular mathematical content. A comparison between similar groups discounts the notion that gains on post-course assessments by the treatment group were due to extraordinary deficits in knowledge in the treatment group at the start of the course.

Analysis of the contrast group data proceeded in two phases. In the first phase, contrast group data were coded using the same rubrics and measures detailed in the previous sections, with performance compared to the treatment group's pre-course data using standard *t*-tests and chi-square analyses. All differences in performance in which the treatment group outperformed the contrast group were noted. Differences in which the contrast group outperformed the treatment group merited further analysis, as these differences may have suggested a particular deficit in the treatment group.

The second phase only involved differences in which the contrast group outperformed the treatment group. In these cases, treatment group post-course performance was compared to the contrast group's performance using appropriate statistical tests. If no significant differences were found, this indicated that the contrast group performed similarly to the post-course outcomes of the treatment group. If differences were found favoring the contrast group, this indicated a fundamental disparity between the two groups. These results are noted, reported, and discussed in Chapters Four and Five.

3.3.5. Design Principles Analysis

The articulation of the design principles in the course drew on elements of the constant comparative method (Glaser & Strauss, 1967). In the instructor's planning diary, a series of principles were articulated in the pre-course entry. These principles were then revisited and compared with experiences in the enactment of the course, with revisions and additions noted in the planning diary. In addition, emergent principles that were not a part of the original set were articulated in the planning diary as they arose in reflecting on the course experiences. Discussions with other experienced teacher educators led to continued refinement of the principles. Following the course, a set of six principles were identified and related to theoretical constructs in the literature on teacher learning and practice-based teacher education. These principles are reported in Chapter Four.

3.4. Design of the Instructional Intervention

The design of the geometry and measurement course was based strongly in the tradition of practice-based learning experiences for teachers (e.g. Ball & Cohen, 1999; Smith, 2001a). In a practice-based learning experience, the authentic work of teachers is examined as a tool for learning and reflection about the practice of teaching in general and teachers' own practice in particular. In the case of the geometry and measurement course, middle grades mathematics tasks, narrative and video cases of teaching, student work on mathematical tasks, and lesson plans and were used as the primary materials. The structure and design of the course represented an evolution of the instructional design created by the ASTEROID Project. These design features built on current theoretical and pragmatic conceptions of teacher education experiences, both in the realm of practice-based teacher education in general, and the specific manifestation of a content-focused methods course as conceptualized by ASTEROID.

Activities in the course were organized in *constellations*, clustered around the cases of teaching, which included the four narrative cases contained in Smith, Silver, Stein, Boston, and Henningsen (2005) and a video case from Boaler and Humphreys' (2005), all portraying urban middle grades classrooms. (Constellations are defined as activities linked by a common mathematical thread or problem; for example, the consideration of a mathematical task, examination of a case of teaching in which the task or a similar task is portrayed, and consideration of student work produced on an identical or similar task is such a cluster of activities.) Prior to examining each case, teachers engaged in solving the mathematical task featured in the case (or one similar to it) in small groups. A whole-group discussion of teachers' solutions followed. These discussions were designed to enhance teachers' knowledge of the mathematics embedded in the task, as well as enhancing their content knowledge for teaching – that is, the set of solution paths, possible misconceptions, representations, and connections between representations and solutions that their own students might use in the classroom. Following the exploration of the mathematical task, teachers examined the narrative or video case of teaching which corresponded to the task. The goal of studying each case was to gain insights into the understandings of the students described in the case and the teacher decisions and moves that either supported or inhibited the development of students' understandings of the mathematics. As such, the examination and discussion of each case of teaching had the potential to add to teachers' knowledge of mathematics for student learning, particularly as the cases represented the specific teacher moves that support or inhibit student learning. Following the case discussion, teachers had the opportunity to engage in an activity or series of activities related to the practice of teaching, such as analyzing student work for the purpose of advancing student thinking or engineering a whole-class discussion, analyzing similar mathematical tasks, or planning a lesson based on a related task.

The mathematical content of these constellations built in sophistication with respect to the relationship between measures of geometric figures. The first two cases (both narrative) dealt with increasingly complex relationships between dimension, perimeter, and area of two-dimensional figures. The next two cases (one narrative, one video) dealt with the relationships between dimension, surface area, and volume for rectangular cube buildings (rectangular prisms) and extending the notion of volume to other three-dimensional shapes. The final case dealt with methods of estimating the volume of the room (rectangular prism) using different measurement units and different methods.

The mathematical content of reasoning and proof was built across cases and throughout course activities. Early in the course, teachers discussed how they define proof, what reasoning and proof is, and what its role is in the teaching of middle grades mathematics. Throughout the discussion of the mathematical tasks and cases, teachers were pressed to justify and make transparent their reasoning about both mathematics and pedagogy. Periodically, teachers' conception of proof as described early in the course was revisited. Several of the cases contained particularly robust illustrations of teachers engineering whole-class discussions where students are asked consistently for justification and deductive arguments are built. Through these activities, it was expected that teachers would develop both a broader notion of proof from a mathematical standpoint, a clearer understanding of the role of reasoning and proof in the domain, and an understanding of the potential role of reasoning and proof in the mathematics classroom.

During the course, the instructor modeled the use of routines to structure the class activity, and specifically the whole-class discussions. The instructor regularly raised teaching moves as an object of inquiry and reflection. These discussions were designed to raise teachers' awareness of the nature of routines and their function in the classroom. Teachers were also given the opportunity to examine a video case of teaching in order to identify routines, and consider how they use routines in their own teaching (see Appendix C).

The instructor's role in the course had the potential to influence the nature of the data, particularly the whole-class discussions, and also reveals the rationale behind design and implementation decisions. These decisions were guided by a set of design principles that the instructor held with respect to practice-based teacher education. In order to capture these nuances and explicitly articulate the design principles, the instructor documented his planning process. The instructor wrote pre- and post-course reflections, as well as anticipations and reflections of the decision-making process and of teacher learning before and after each course meeting. These data served as the primary source for articulating design principles, and a secondary source to contextualize and add further explanatory detail to the analysis of teacher change during the course. It also served to compare the instructor's expectations and interpretations of teacher learning to teachers' own conceptions of what they learned.

The next chapter contains the results of the study. Data relating to each of the facets of the knowledge needed for teaching framework are presented in turn, including examples of teacher work and discourse from course meetings that support the analysis.

4. RESULTS

The description of the results of the study begins with an articulation of the design principles, which framed the learning opportunities in the course, and a definition of the criteria used to determine "opportunity to learn" with respect to the course. Changes in teachers' knowledge as measured by the written assessments and interview protocols are then presented for each of the aspects of the knowledge needed for teaching framework in turn, along with a description of how the course provided an opportunity to learn and how the design principles framed teacher learning. Finally, learning in the treatment group is compared to measures from the contrast group.

4.1. Learning, course design, and opportunities to learn

The design of the geometry and measurement course and the goals for teacher learning reflects a situative perspective on learning. This perspective maintains that learning extends beyond constructing and organizing information in mental structures within a particular content domain. Learning also encompasses participation in the practices of a community in which a group actively engages in thought within a particular content domain (Greeno & MMAP, 1997). As such, a robust characterization of teacher learning should include changes in their individual knowledge, a trace of the practices in which they engaged during the course experience, and evidence to link individual learning and engagement in learning practices of the community.

The knowledge targeted by the geometry and measurement course crossed two overlapping domains – mathematics and pedagogy. These domains are tightly linked in the work of teaching, and learning experiences that target one domain without making contact with the other are less useful to teachers. The geometry and measurement course was designed to provide opportunities for teachers to learn through engagement in learning practices and through the development of individual knowledge structures, and to tightly link the mathematical and pedagogical ideas such that they would have the greatest potential to be useful in the work of teaching.

The sections that follow articulate the 6 principles that undergird the design of the course, which provided the context for learning. The principles served both to inform the design of the course, and also to predict what teachers might learn through engagement in a course based upon them. These design principles represent refinements or explicit articulations of previous work on practice-based teacher education experiences, and were made explicit for the purpose of contextualizing learning in the course. They are built on both general theories of learning, practice-based teacher education and specific work on the previous courses designed as a part of the ASTEROID Project (Smith et al., 2001). Each principle is accompanied by a short description of the potential of the design feature to support teacher learning.

4.1.1. Design Principle 1: Engaging Teachers in Public Discourse Practices

> Having **public discussions about key ideas** in the course (both related to mathematics and teaching) **gives all teachers access to the ideas** of other teachers, as well as allowing them to question, challenge, and debate these ideas. Conducting these discussions in the public arena of the course affords both active participants and passive listeners the opportunity to learn and to participate in the practice of discussing the key issues at a level of their choice.

The first, and perhaps most important, design principle in the geometry and measurement course was that teachers would engage in public discourse around the tasks and activities in the course. This principle is important for a number of reasons. First, learning in the situative view is a collaborative practice, and engagement in public thinking around issues of mathematics and pedagogy with a group of practitioners constitutes learning. Considering learning as participation in these discourse practices also allows teachers to enter the learning space in a variety of different ways. Teachers who have more experience or skill to bring to bear on a particular task have the opportunity to contribute more centrally, while teachers with less experience or skill have access to the ideas of others (Lave & Wenger, 1991). Previous research has shown that in a community of learners, even learners who participate minimally, or not at all, in the public discourse can demonstrate learning at an individual level with respect to mathematical content (Hatano & Inagaki, 1991; Hillen, 2005; Inagaki, Hatano, & Morita, 1998). Finally, the establishment of social norms for discussions of mathematics and mathematics teaching may hold implications for teachers' own work in their classrooms. Teachers have the potential to learn not only about the content of the public discussions, but also about how to structure such discussion in their own classrooms, thus making the notion of learning through engagement in practices generative.

4.1.2. Design Principle 2: Engaging Teachers in Mathematical Tasks

> Engaging teachers in work on authentic mathematical tasks of high cognitive demand that could be used in the K-12 classroom can have the potential to enhance both their **content knowledge in the domain** related to being able to solve the mathematical tasks and their **content knowledge for teaching** (e.g., specific strategies, misconceptions, and other mathematical nuances that are likely to arise in work on similar mathematical ideas in the classroom with students).

As noted previously, the course was intended to help teachers develop both mathematical and pedagogical knowledge, as conceptualized by the Knowledge Needed for Teaching framework. This design principle relates to the types of mathematical knowledge that teachers need to develop, and is the fundamental articulation of the value of engaging teachers in mathematical tasks in the context of a teacher education course. The principle can be viewed as predicting two particular learning outcomes from designing and enacting a course in this way: that teachers will be able to learn from engaging in mathematical tasks from the K-12 classroom, and this learning will be useful not only in enhancing their mathematical understandings, but also in building the specialized content knowledge that is unique to the work of teaching. Previous professional development work in mathematical education has provided empirical evidence to support the conjecture put forth by this principle (e.g., Schifter & Fosnot, 1993; Simon & Schifter, 1991). Building this specialized content knowledge that is unique to teaching is the first step in connecting learning in the mathematical and pedagogical domains. In the context of the middle grades geometry and measurement course, this design principle predicts that teachers will grow in both their content knowledge in the domain and content knowledge for teaching with respect to middle grades geometry and measurement through participation in the course.

4.1.3. Design Principle 3: Constellations of Practice-Based Activities

> Designing an experience in which **teachers engage in a mathematical task**, **examine a narrative (or video) case of teaching** with the same or similar task, and **engage in additional work** on the same or similar task closely **related to the work of teaching** (e.g. analyzing student work, planning a lesson of their own, considering what the teacher might do the next day) **allows teachers to examine facets of mathematics teaching that might otherwise be hidden from view**. Specifically, it allows teachers to experience the mathematical ideas inherent in the task as students of mathematics, as observers of someone else's teaching and other students' learning, and as practicing professionals responsible for fostering student learning.

It is imperative for teachers to have knowledge bases for mathematics and for pedagogy that are well-connected and are operable and interactive in the work of teaching. As such, teacher education experiences should be designed such that teachers have the potential to make these connections. Design Principle 3 specifies how mathematical and pedagogical activities are sequenced in such a way as to promote teachers' learning of mathematics, of students as learners of mathematics, and of issues around the teaching of mathematics. We have come to call these sequences *constellations* of activities. Using a mathematical task as a starting point and as a common thread throughout the constellation, teachers progress from explorations of the mathematics through solving and discussing the task to explorations of pedagogical issues through artifacts of teaching in which the same mathematical ideas are embedded. For example, teachers may engage in an authentic middle grades mathematical task that explores the relationships between dimension, area, and perimeter, read and discuss a narrative case featuring a teacher working with students around the same task, and then engage in a discussion regarding how one might plan a lesson in advance that made use of the open-ended nature of the task. These constellations of activities are expected to be rich sites for teacher learning in comparison to isolated mathematical or pedagogical activities.

4.1.4. Design Principle 4: Building on Prior Knowledge

> Instructional activities and practices that **allow participants to articulate prior knowledge and build on that knowledge**, with explicit comparison and links back to the previously articulated prior knowledge, have the potential to provide **meaningful opportunities to learn** and for knowledge to be **integrated** with and **well-connected** to existing understandings.

The notion of new learning experiences building on prior knowledge and experiences is widely accepted and prominent across a variety of conceptualizations of teaching and learning (e.g., Cobb, 1994; Greeno, 1991; Piaget, 1952, 1973a,b, 1977, 1978; Vygotsky, 1962, 1978). In order for teachers to develop meaningful knowledge needed for teaching, this knowledge must build on their prior knowledge both of mathematics and of teaching. Given that the course population was diverse in experience with respect to both mathematics and teaching, building on prior knowledge posed a number of challenges. These challenges were addressed through the use of mathematical tasks that allowed for multiple points of entry, the use of narrative cases of teaching as grounding for discussions about pedagogy (Steele, 2005), and continuous assessment and reflection on teacher knowledge on the part of the course instructor. Selecting tasks that provided multiple points of entry and narrative cases that afforded a common pedagogical experiences allowed access to teachers with different knowledge and experience in the course.

4.1.5. Design Principle 5: Revisiting

> **Revisiting complex, overarching mathematical ideas** (e.g. the definition of function, use of representations, the notion of proof) with respect to multiple mathematical and pedagogical experiences **allows teachers to build, grow, and change their understanding of the idea in a way that a single discussion may not afford**.

The principle of revisiting has the potential to promote meaningful learning in a number of different ways. First, the principle can be seen as a specific instantiation of design principle 4 which considered the value of building on prior knowledge. In revisiting mathematical ideas across multiple conversations in a course experience, teachers are positioned to have the opportunity to build on the understandings articulated in the previous conversation(s) to enhance their understanding, both individually and collectively, of the mathematical ideas at play. The notion of revisiting also supports teachers' metacognition and cognitive structuring (e.g., A.L. Brown, 1975; Flavell, 1973), as revisiting opportunities explicitly request that teachers monitor changes in their thinking about a particular idea based on new experiences and in comparison to specifically-marked previous discussions. Finally, revisiting provides opportunities to engage in thinking practices around a mathematical idea, which both provides for opportunities for their learning and also is likely to provide teachers with a model for engaging their students in similar thinking practices. The notion of revisiting could be considered both a means of continuing to explore a mathematical concept with the goal of individual knowledge growth, and a normative sociomathematical practice in which learners engage around complex mathematical content (e.g., Yackel & Cobb, 1996). This design principle predicts that frequent revisiting of a mathematical topic – in the case of the geometry and measurement course, proof – will continue to produce new ideas in discussion, and will result in richer teacher learning.

4.1.6. Design Principle 6: Modeling Good Pedagogy

> The instructor of a course for teachers should **model good instruction and make visible his own pedagogical decisions** in order to support teachers' development of good pedagogical instructional techniques in the service of developing students' mathematical understandings.

Proponents of practice-based professional development argue that the examination of teaching, through the analysis and discussion of authentic artifacts of teaching, has the potential to lead to learning about pedagogy and potentially changes in teacher practices (e.g., Ball & Cohen, 1999; Smith, 2001a). A logical extension to this argument is that teachers have the potential to learn about teaching through the examination of the pedagogy of professional development experiences in which they engage. As such, a teacher education experience such as the geometry and measurement course should both model the sort of student-centered, inquiry-oriented pedagogy that we wish for teachers to develop and provide teachers with opportunities to critically analyze the pedagogical moves of the instructor. From a cognitive perspective, general pedagogical principles can surface through repeated reflection upon and discussion of the instructor's moves. It is then hoped that these moves generalize sufficiently that they are useful in teachers' classrooms. From a situative perspective, the practice of analyzing and reflecting on pedagogy is a practice that facilitates learning about teaching, and one that is potentially powerful for teachers to engage in with respect to their own teaching. It is anticipated that discussions in which the instructor makes his pedagogy visible will be meaningful sites for teacher learning.

4.1.7. Opportunities to learn

These six principles together frame the design and enactment of activities in the geometry and measurement course. Individually and in combination with one another, they predict particular types of teacher learning. Through their engagement in the activities designed and enacted using these principles as a guide, teachers had opportunities to learn knowledge needed for teaching geometry and measurement.

The analysis of the written assessments, interviews, and videotapes of the course sessions discussed in the sections that follow provide evidence of teacher learning. In order to link learning results more closely to activities in the course, evidence must be provided that teachers had the opportunity to learn the knowledge needed for teaching for which they demonstrated growth. *Opportunity to learn* is defined in the following way for the purposes of this study: An opportunity to learn with respect to the geometry and measurement course consists of the identification of a mathematical or pedagogical idea for study and engagement with that idea through a single activity or series of activities which provide an opportunity for individual and small-group work, for which entry is available for teachers with a range of prior experiences, in which the mathematical or pedagogical ideas are publicly discussed, and for which there is an opportunity to reflect on and/or expand on the ideas discussed through a written assignment or individual oral interview.

The sections that follow answer research questions 1, 2, and 3 by describing teacher learning with respect to the three facets in the knowledge needed for teaching framework: *knowledge of mathematics and mathematical activities*, including content knowledge in the domain and content knowledge for teaching; *knowledge of mathematics for student learning*, including the five practices for productive use of student thinking; and *practices that support* *teaching*, specifically with respect to routines. Table 6 below summarizes the characteristics of each of the three facets of the framework, and describes the content targeted in the current study.

Table 6. The Knowledge Needed for Teaching Framework and Assessed Content.

Knowledge Needed for Teaching		
Knowledge of Mathematics and Mathematical Activities	**Knowledge of Mathematics for Student Learning**	**Knowledge of Practices that Support Teaching**
Content knowledge in the domain: knowledge that everyday users of mathematics would need to know and do ***Content knowledge for teaching***: knowledge that is specific to the act of teaching, such as the selection of tasks; the set of examples, representations, and solution strategies for a given task; and knowledge of the nature of the domain *Content Assessed:* - relationship btwn. dimension, area, & perimeter - relationship btwn. dimension, surface area, and volume - reasoning and proof	knowledge that relates specifically to the way a population of students might think about and do mathematical problems and content *Content Assessed:* Five practices for productive use of student work - anticipating solutions - monitoring student work - selecting and sequencing student solutions - connecting solutions (Stein, Engle, Hughes, & Smith, submitted)	knowledge of aspects of teaching practice that automatize and structure the work of teaching includes: - routines - metatalk - intellectual climate of the classroom *Content Assessed:* routines

Each section begins a presentation of results from individual assessment measures (written and interview). Following this presentation, the course activities in which each content topic was addressed are described, along with relevant excerpts from course discussions that demonstrate how work on the course activities provided opportunities for teacher learning. At the close of each section is a brief discussion of how the results compare to learning predicted by the design principles. Figure 5 shows the course map, which contains the complete set of activities in the course, with shapes indicating different types of activities.

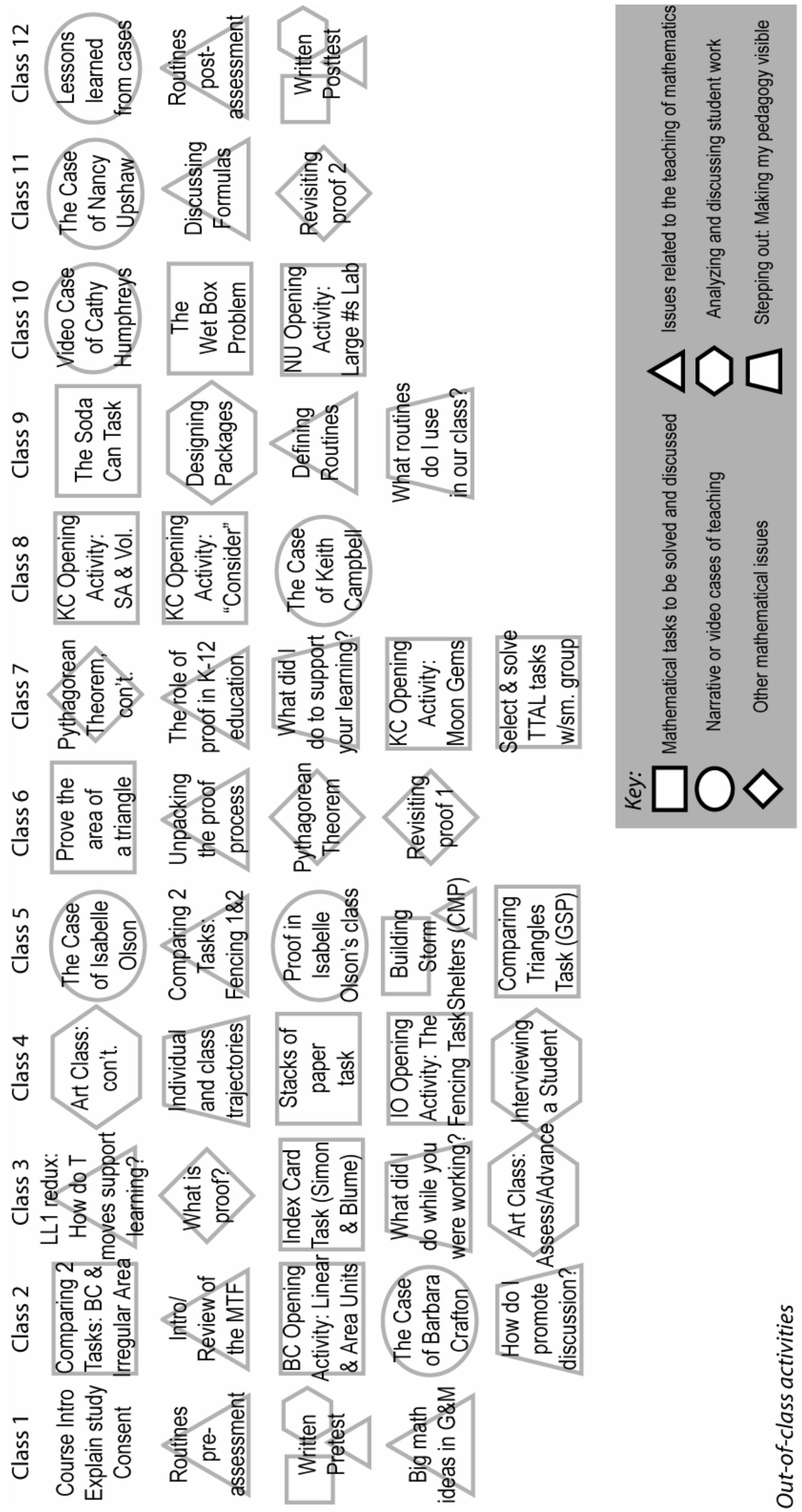


Figure 5. Geometry and Measurement Course Map

4.2. Knowledge of Mathematics and Mathematical Activities

The geometry and measurement course was designed to address three primary areas of mathematical content: relationships between dimension, perimeter, and area in two-dimensional figures; relationships between dimension, surface area, and volume in three-dimensional figures; and reasoning and proof. Within these mathematical content areas, teachers were assessed in two sub-domains, content knowledge in the domain and content knowledge for teaching. In this section, results are presented for each of the three content areas with respect to content knowledge in the domain and content knowledge for teaching.

4.2.1. Dimension, perimeter, and area: Growth in content knowledge

The activities that provided opportunities for teachers to learn about dimension, perimeter, and area were concentrated in Classes 2-6 of the geometry and measurement course, and included engaging in mathematical tasks, reading narrative cases of teaching, examining student work, reading related research articles, and writing to reflect on and extend the in-class conversations. The mathematical ideas discussed built in complexity from issues of the meaning of linear and square units, to conceptual understandings of perimeter and area, and finally to relationships between dimension, perimeter, and area. Table 7 lists the specific aspects of the relationship between dimension, perimeter, and area targeted in the course.

Table 7. Knowledge of mathematics and mathematical activities related to dimension, perimeter, and area addressed in the course.

Content knowledge in the domain	Content knowledge for teaching
Understand that area and perimeter have a non-constant relationship	Identifying the big ideas in middle grades geometry and measurement related to dimension, perimeter, and area
Explain how changes to dimensions of a figure impact perimeter and/or area (including transformations on a plane figure)	Identify and/or create mathematical tasks that provide students with opportunities to explore the big ideas related to dimension, perimeter, and area
Explain the relationships between linear and square units and utilize these relationships to make sense of area and perimeter	Use a range of representations to explain the relationship between dimension, area, and perimeter
Demonstrate understanding of the meaning of area and perimeter using a variety of tools and representations	Understand the affordances and constraints of different formulas for geometry and measurement concepts

Six items on the pre- and post-course written assessment and one item on the pre- and post-course interview assessed aspects of *content knowledge in the domain* and *content knowledge for teaching* related to the relationships between measurable attributes of two-dimensional geometric figures. These items measured teachers' ability to articulate the non-constant relationship between area and perimeter and describe how changes in dimensions impact area and perimeter of two-dimensional figures, and are described below[6].

<u>4.2.1.1. Performance on the *Fence in the Yard* Task</u>. The *Fence in the Yard* task (see Figure 6) presented teachers with a situation in which a rectangular dog pen was to be built with a fixed amount of fence. The task asked teachers to determine what the best configuration for the pen was such that the most space was created inside the pen. In responding to the task, teachers had to recognize that a rectangle with a fixed perimeter could have multiple areas, understand how changes to the dimensions of the rectangle impact both perimeter and area, and represent their solution in a way that was clear and understandable to others.

Fence in the Yard
Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

What is the largest area that Julie can enclose with 36 feet of fencing?

Support your answer by showing the work that would convince Julie that your area is the largest.

(From 1996 NAEP, as cited in Kenney & Lindquist, 2000)

Figure 6. Fence in the Yard task.

The *Fence in the Yard* task afforded opportunities to measure both teachers' content knowledge in the domain and content knowledge for teaching. With respect to content

[6] The complete written assessment can be found in Appendix A.

knowledge in the domain, the task afforded an opportunity for teachers to demonstrate that area and perimeter have a non-constant relationship, to explain how changes to dimensions impact perimeter and area, and to use mathematical representations to describe the relationships between dimension, perimeter, and area.

Understanding that area and perimeter have a non-constant relationship is the first step in understanding the relationship between dimension, perimeter, and area, in that teachers must recognize that figures can have the same area and different perimeter, and vice versa. A correct response to the *Fence in the Yard* task, with an explanation of why the answer maximized area, would provide evidence of an understanding that area and perimeter have a non-constant relationship. Responses were coded for correctness using five categories, shown in Table 8.

Table 8. Area-Perimeter Relationship Coding for *Fence in the Yard*.

Correct-1	Correct pen configuration chosen (9 x 9 pen) Shows at least 2 examples of pens that have the same perimeter and different areas
Correct-2	Correct pen configuration chosen (9 x 9 pen) Does not show 2 examples of pens that have the same perimeter and different areas
Incorrect-1	Incorrect pen configuration chosen Shows clear evidence of the misconception that a pen of perimeter of 36 ft. can only have a single area
Incorrect-2	Incorrect pen configuration chosen Does not show evidence of the misconception that a pen of perimeter of 36 ft. can only have a single area
Vague/Inconclusive	Cannot be classified or response is incomplete

Responses to the *Fence in the Yard* task on both the pre- and post-course assessment indicated that teachers understood the non-constant relationship between area and perimeter in the context of the task. No responses were coded as **Incorrect-1** at either time point, and only 2 teachers on the pre-course assessment and 1 teacher on the post-course assessment provided responses coded as **Incorrect-2**. In fact, most teachers were able to demonstrate the non-constant relationship; 21 out of 25 teachers on both the pre-course and post-course assessment provided responses coded as **Correct-1**, which indicates a clear demonstration of the non-constant relationship.

Fence in the Yard also provided opportunities for teachers to describe how changes to the dimensions of a figure impact the figure's perimeter and area, a key step in understanding the dynamic relationships between dimension, perimeter, and area. Responses to the *Fence in the Yard* task were examined to determine whether teachers described how changes to the length and width of the pen in the task impacted perimeter (remains constant) and area (increases or decreases). This could be accomplished through three different representations: a written explanation such as the one shown in Figure 8, a table that shows dimension, perimeter, and area (as compared to leaving perimeter off the table)[7], or a graph that showed the relationship between dimension and area or perimeter and area.

> As the numbers/length of fencing gets further from one another the area decreases. This is the greatest area one can have w/36 ft of fencing. All the sides are the same, thus making it a square. As we discussed in class a square maximizes the area. ***(posttest response)***

Figure 7. Example of written explanation to the *Fence in the Yard* task showing the impact of changing the dimensions on the perimeter and area.

Written explanations, tables, or graphs that made salient this relationship were categorized as **A** responses. Responses that were correct but did not make salient the relationship were categorized as **B** responses, and incorrect responses were categorized as **C** responses. When responses were categorized in this manner, teachers showed a significant increase from pre to post in **A** responses, $\chi^2(1, 46)$, = 4.21, p = 0.04. When results were disaggregated by representation, there was also a significant increase in teachers using a table to create an **A** response, $\chi^2(1, 50)$, = 6.65, p = 0.01.

[7] An table containing columns for dimension or length, perimeter, and area is particularly interesting, as the perimeter column in such a table is static, showing 36 ft for each row of the table. Tables of this nature suggest that teachers saw a value added for including the static perimeter column in there table.

The last facet of the *Fence in the* Yard task related to content knowledge in the domain is representational use.[8] The categories of representation used were **Table**, **Graph**, **Symbolic/Formula**, **Written Explanation**, and **Diagram**. These five categories were used consistently across all tasks coded for representational use. Only representations that were directly related to the teacher's solution were counted; if it could not be determined how the representation contributed to the response to the task, the representation was not coded. On the *Fence in the Yard* task, no differences in the categories were observed between pre- and post-course assessment. Tables and written explanations were by far the most favored representation, with 19 and 18 of the 25 teachers using each, respectively. The heavy use of written explanations is not surprising, considering the question explicitly asked for an explanation of the teacher's answer.

The results related to content knowledge in the domain for *Fence in the Yard* suggest that in general, teachers had little difficulty solving the task. To determine possible changes in content knowledge for teaching related to this task, the representations that teachers used on pre- and post-course assessments were examined for changes in the use of single or multiple representations and in the mean number of representations used. These measures suggest the level of representational fluency possessed by teachers with respect to this task.

In addition to their successful performance on the task, most teachers on both pre- and post-course assessments used multiple representations; 19 teachers used multiple representations in solving the task on the pre- However, there was a significant increase in the mean number of representations used by teachers, $t(24) = -2.11$, $p = 0.02$, on the post-course assessment. Multiple representations were not required to correctly solve the *Fence in the Yard* task; thus, the

[8] The use of specific representations to respond to a task is considered *content knowledge in the domain*. Representational fluency – the use of multiple representations and connections between them – is considered *content knowledge for teaching*.

increase in the number of representations used by all teachers indicates that teachers appeared to acquire increased representational fluency with respect to the relationship between dimension, area, and perimeter on this task.

4.2.1.2. Performance on the *Area of a Parallelogram* Task. The *Area of a Parallelogram* task (See Figure 8) presented teachers with a scenario in which the base and area of a parallelogram were given. The task asked teachers to determine whether or not these two quantities determined a parallelogram of unique perimeter. In order to successfully respond to this task, teachers needed to understand the aspects of the dimensions of a parallelogram that contribute to perimeter and area, and how those dimensions are or are not constrained in the problem situation. Additionally, they were asked to justify their responses using an example. With respect to content knowledge in the domain, the task afforded an opportunity for teachers to demonstrate that area and perimeter have a non-constant relationship and to use mathematical representations to describe the relationships between dimension, perimeter, and area.

True or false: A parallelogram with a base of 6 cm and an area of 24 cm^2 will always have the same perimeter. Provide at least one example to support your answer.

Figure 8. Area of a Parallelogram task.

The *Area of a Parallelogram* task assessed the non-constant relationship between area and perimeter in a slightly more complex manner, adding the constraints of fixing two attributes of the parallelogram and the area. Coding for the task utilized similar categories to *Fence in the Yard*; these coding classifications are shown in Table 9.

Table 9. Area-Perimeter Coding for *Area of a Parallelogram*.

Correct-1	Correct response (statement is false) Shows at least 2 examples that demonstrate why, or a generalization that explains why
Correct-2	Correct response (statement is false) Does not provide examples or a generalization that demonstrates why
Incorrect-1	Incorrect response (statement is true) Evidence that the teacher thinks there is only one possible parallelogram with the specified base
Incorrect-2	Incorrect response (statement is true) No evidence that the teacher thinks there is only one possible parallelogram with the specified base OR Correct response, erroneous reason
Vague/Inconclusive	Cannot be classified or response is incomplete

Responses coded as **Correct-1** are taken as evidence that teachers understand the non-constant relationship between area and perimeter in the context of the task. Responses coded as **Incorrect-1** are taken as evidence of a misconception related to the non-constant relationship between area and perimeter in this case. Figure 9 shows an example of a response coded **Incorrect-1:**

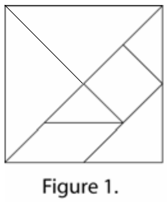
True
$24\ cm^2 = bh$
$24\ cm^2 = 6h$
$4 = h$
only one h possible

Figure 9. Example of Incorrect-1 response to *Area of a Parallelogram* task.

Many teachers responded correctly to this item, producing a ceiling effect and little change within categories. There was an increase in responses coded as **Correct-1** from pre- to post-course assessment, but this increase was not significant ($\chi^2(1, 50) = 3.00$, $p = 0.08$). Changes in **Correct-2**, **Incorrect-1**, and **Incorrect-2** responses were minor and not significant. There was a significant decrease in the number of teachers whose responses were coded as **Vague/inconclusive** ($\chi^2(1, 50) = 4.74$, $p = 0.03$ with Yates' correction). With respect to representational use, *Area of a Parallelogram* responses were coded for the same set of representations used for *Fence in the Yard*. There was a significant increase in the number of written explanations used among all responses, $\chi^2(1, 50)$, = 3.95, $p = 0.05$.

With respect to content knowledge for teaching, the task was examined for the mean number of representations used by teachers in responding to the task. Across all responses, most teachers used multiple representations for this task on the pre-course assessment, causing a ceiling effect. However, there was a significant difference in the mean number of representations used per teacher, $t(24) = -2.19$, $p = 0.02$, on the post-course assessment from a mean of 1.75 to 2.21. Taken together with the fact that teachers performed well both pre-course and post-course on this item, this suggests that teachers came to see a value in using a variety of representations in articulating the relationships between dimension, perimeter, and area. This result resonates with teachers' performance on the *Fence in the Yard* task as well.

4.2.1.3. Performance on the *Tangrams* Tasks. The *Tangram* tasks (see Figure 10) presented a situation based on tangram tiles, a set of 7 polygonal tiles which can be arranged in a square. Two different arrangements of the tangram tiles were presented, with teachers asked to decide which of the arrangements had the greatest area and which had the greatest perimeter. To respond correctly to this task, teachers needed to realize that the rearrangement of the tiles does not change the area, but has the potential to change the perimeter. Additionally, teachers were asked to justify their responses, giving them the opportunity to use generalized mathematical principles to make the comparisons between the two arrangements. As such, this task measured content knowledge in the domain; specifically, explaining how changes in dimension impact perimeter and area.



Tangrams are a special set of 7 geometric tiles shown below in Figure 10.
The shapes in Figures 2 and 3 were formed using all the tangram tiles.
Which figure, 2 or 3, has the greater area? Justify your answer.
Which figure, 2 or 3, has the greater perimeter? Justify your answer.

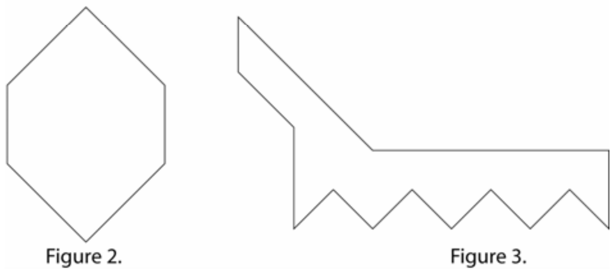


Figure 10. Tangrams task.

The impact of changing dimension on area and perimeter was measured through teachers' construction of the argument as to how the rearrangement of the tiles impacted perimeter and area. Teachers could correctly respond to the task in a variety of ways. Teachers could use a visual estimation or "guess" to make a judgment; teachers could use a mathematical argument to support a correct response; or teachers, could create an *ad hoc* measurement device to calculate perimeter.[9] Teachers had plastic tangram tiles at their disposal for this task and were able to obtain rulers upon request. Responses to the task were rated on rubrics designed to make distinctions between the methods used to justify their conclusions, as shown in Table 10.

[9] Although it is conceivable that teachers could also have created an *ad hoc* measurement device for area, such as a transparent grid, there was no evidence in teachers' written responses that they would consider doing so, nor did any teachers during the course ask for the tools necessary to construct such a device.

Table 10. Rubrics for *Tangrams* task.

Area Question	Perimeter Question
Score Point 2: • Response is correct (both figures have the same area) • Justification is correct and uses the concept of area (e.g., the figure are made from the same set of tiles, and unless they overlap, area cannot change)	**Score Point 3:** • Response is correct (Figure 3 has the greater perimeter) • Justification is correct and uses the concept of perimeter and the arrangement of the tiles (e.g., there are more exposed edges in Figure 3)
Score Point 1: • Response is correct (both figures have the same area) • Justification is based on qualitative observation, or no justification is provided	**Score Point 2:** • Response is correct (Figure 3 has the greater perimeter) • Justification is correct and uses a form of empirical measurement
Score Point 0: • Response is incorrect, or; • No response is given, or; • Response cannot be determined based on work provided	**Score Point 1:** • Response is correct (Figure 3 has the greater perimeter) • Justification is based on qualitative observation, or no justification is provided
	Score Point 0: • Response is incorrect, or; • No response is given, or; • Response cannot be determined based on work provided

Rubric scores were compared using the Wilcoxon Sign-Rank test, which assesses aggregate differences in the ordinal scores; changes in individual rubric categories were assessed using a chi-square test. Performance on the area question was strong pre-course; teachers had little difficulty determining that both arrangements had an equal area. Twenty-three of 25 teachers responded correctly on the pre-course assessment, and 24 of 25 responded correctly on the post-course assessment. Most teachers produced a '2' response both pre- and post-course; there was no significant difference in rubric scores on the Wilcoxon Sign-Rank test.

The perimeter question paints a different picture. Teachers generally responded correctly – 23 of 25 on the pre-course assessment and 24 of 25 on the post-course assessment – but there were differences in the rubric scores. There was a significant improvement in scores from pre- to post-course assessment, and a significant increase in the number of '3' responses and a significant decrease in ‘1’ responses (Wilcoxon sign-rank test, W=-54, $n_{s/r}$=11, Z=-2.38, $p = 0.009$; $\chi^2(1, 50), = 6.35$, $p = 0.01$; $\chi^2(1, 50), = 4.5$, $p = 0.03$). These results suggest growth in teachers’ use of general explanations in describing how changes to a complex figure’s dimensions impacts perimeter. Most teachers initially justified their response to the perimeter question in a way that relied on qualitative judgment; on the post-course assessment however, teachers used more general and mathematical principles to justify their response.

4.2.1.4. Performance on the *Area and Perimeter: Responding to Student Claims* Task. The *Area and Perimeter: Responding to Student Claims* task (see Figure 11) was primarily designed to assess knowledge of mathematics for student learning: specifically, how teachers would respond to an overgeneralized conjecture related to the relationship between area and perimeter.[10] The task also had the potential to assess teachers’ content knowledge in the domain: if their response indicated agreement with the overgeneralized claim, this would be taken as evidence of a misconception related to the relationship between dimension, perimeter, and area. The task features a student who claims that as the perimeter of a rectangle increases, the area also increases. The claim is based on changing one dimension and holding the other constant, and thus is an overgeneralization of the impact of changing dimensions on perimeter and area.

[10] Results for this task with respect to knowledge of mathematics for student learning can be found in section 4.3.

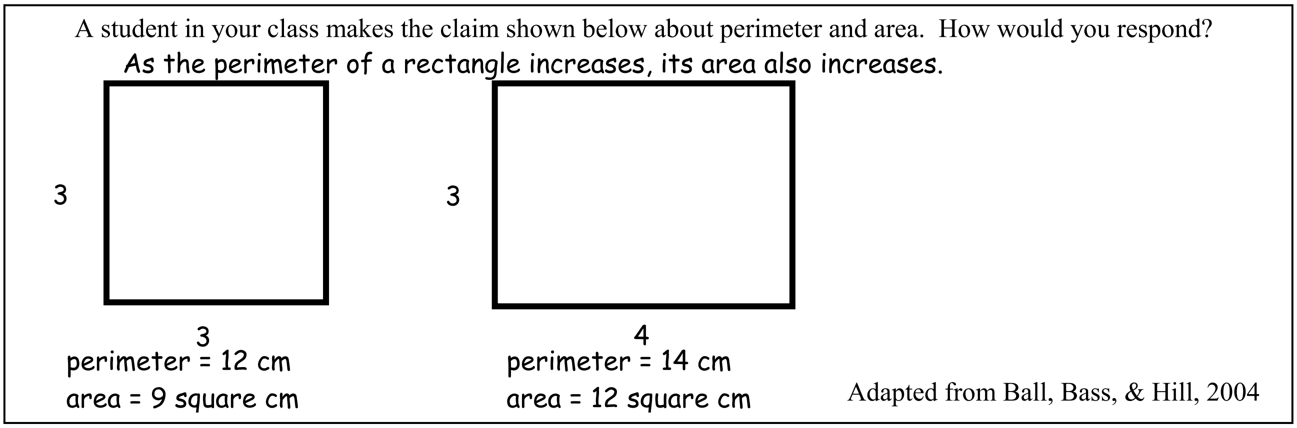


Figure 11. Area and Perimeter: Responding to Student Claims task.

In order to assess content knowledge in the domain, responses to the task were coded for evidence of teachers' support of the student misconception that as perimeter of a rectangle increases, area increases. Support of the student's conjecture included responses to the claim that suggested that the student's claim was correct. On the pre-course assessment, 3 teachers exhibited this misconception; on the post-course assessment, no teacher exhibited this misconception. This trend was marginal but not statistically significant ($\chi^2(1, 49) = 3.06$, $p = 0.08$).

4.2.1.5. Performance on the *Considering Formula Use* Task. The *Considering Formula Use* task asked teachers to consider two formulas that can be used to find the area of a rectangle, and two formulas that can be used to find the volume of a rectangular prism. Teachers were asked to select which formula they would use with a middle grades classroom, and to explain the reasons for their preference. Figure 12 shows the text of the task.

a.	There are two common forms that textbooks use for the volume of a rectangular prism: *Volume* = *length* × *width* × *height* and *Volume* = *Area of base* × *height* Is there a difference between the two formulas? If so, describe the difference. Which would you choose to use with students, and why?	
b.	There are two common forms that textbooks use for the area of a rectangle: *Area* = *length* × *width* and *Area* = *base* × *height* Is there a difference between the two formulas? If so, describe the difference. Which would you choose to use with students, and why?	

Figure 12. Considering Formula Use task.

This task assesses teachers' understandings of the affordances, constraints, and conditions of use for various formulas, an important facet of content knowledge for teaching. This notion is particularly salient in geometry and measurement, as almost every measurable attribute of geometric figures has one or more formulas that are useful in quickly calculating the attribute. Part b of the task assesses teachers' understandings of formulas related to the area of a rectangle.

Results for this item showed a significant increase in the number of teachers preferring the $A=bh$ formula on the post-course assessment (McNemar's Test, $p = 0.012$). There was also significant change in the reasons cited for teachers' formula preference. There was a significant increase in the number of teachers citing the more general nature of the $A=bh$ formula as the reason for their selection, $\chi^2(1, 49), = 7.41$, $p = 0.006$. There was a marginal decrease, from 5 teachers to 0, in the number of teachers citing ease of use as the reason for selecting the $A=lw$ formula, $\chi^2(1, 49), = 5.35$, $p = 0.056$ with Yates' correction.

The results of this item suggest that teachers' formula preferences and reasons for holding those preferences changed following the course. More teachers stated a preference for the $A=bh$ formula. Fewer teachers cited surface-level features, such as the ease of finding length and width, in justifying their selection, and more teachers cited the notion that the formula was more general. In attending to the issue of generality across a variety of two-dimensional shapes, teachers identified a reason for their formula preference that is grounded in the mathematical relationships both within and between attributes of two-dimensional figures.

4.2.1.6. Performance on the *Big Ideas* Task. On the pre- and post-course written assessment, teachers were asked to identify the key ideas that middle grades students should learn related to two-dimensional shapes, area, and perimeter (see Appendix A, Part B, Task 1), measuring a key component of content needed for teaching. Teachers' responses were examined for commonalities, and a series of general categories emerged from the examination of teacher responses. The categories are shown and described in Table 11.

Table 11. Perimeter and Area coding categories for *Identifying the Big Ideas*.

Relationship btwn A&P – General	The relationship between area and perimeter with no additional explanation about the nature of this relationship or examples of this relationship.
Relationship btwn A&P – Specific	The relationship between area and perimeter with a specific (and correct) aspect of this relationship specified.
Calculate/find A&P	Finding or calculating area and/or perimeter using a formula, counting, estimating, or measuring.
Use/apply A&P	Using or applying area and perimeter in problems, real-world situations, or high-level tasks, including applying the formulas or calculating area or perimeter in the service of a context. Does not necessarily imply understanding of the meaning of area and perimeter.
Understand A&P conceptually	Understanding the meaning of area and perimeter from a conceptual standpoint, including responses such as "knowing what area and perimeter mean," "understanding area and perimeter," "concepts of area and perimeter."
Diff. btwn linear & square units	Understanding the difference between linear and square units, including that perimeter is measured in linear units, area is measured in square units, or how units relate to perimeter and area.
Names, characteristics of 2-D shapes	Knowing the names, characteristics, or properties of 2-D shapes, including knowing terms or names for shapes, classifying shapes, knowing core properties of classes of shapes, or distinguishing different 2-D shapes.
Manipulate/decompose shapes	Change, manipulate, decompose, or recompose 2-D shapes, including finding area through partitioning, understanding how to transform one shape into the next (e.g., lopping off one end of a parallelogram and moving it over to make a rectangle), or other transformations.
Memorize/use formulas	The use and/or memorization of formulas for area or perimeter, including understanding what elements of the formula stand for (knowing that the *h* in the area of a triangle formula stands for height, and where to find height on the triangle).
Generate, develop, or explain formulas	Creating or explaining formulas (formal symbolic or informal rules or methods) for area and perimeter based on understandings about what area and perimeter are conceptually. Does imply conceptual understanding of the basis for the formula.
Perimeter as distance around	Conceptual understanding of perimeter specifically as the distance around a shape or as surrounding a shape.
Area as covering	Conceptual understanding of area specifically as covering or the space "inside" or "contained by" a shape.
Visualization/ spatial sense/ sketching	Development of visualization skills or spatial sense with students, or cites sketching of shapes as a way to develop visualization or spatial sense, including creating models for developing spatial sense or visualization skills.
Unit conversion	Converting one set of measurement units to another. Distinct from understanding the relationship between linear and square units; limited to the conversion between units of the same dimension (e.g., cm to in., ft^2 to m^2).
Find missing sides w/A&P	Given a shape with one (or more) dimensions provided, one dimension missing, and the area or perimeter, finding the missing dimension. For example, find length given the width and area of a rectangle.
Difference btwn A&P	Know the difference between area and perimeter. Reserved for statements that do not expand on what that difference is, or only identifies a "relationship" between the two.

When teacher responses were coded using these categories, three significant changes were noted. On both the pre- and post-course assessment, teachers mentioned the relationship between area and perimeter as a big idea that student should learn; however, there was a shift in <u>how</u> teachers described the relationship. There was a significant increase in the number of teachers who talked about this relationship in a way that specifically articulated the nature of the relationship (e.g., one can have shapes with the same area and different perimeters), $\chi^2(1, 50)$, = 17.0, $p < 0.001$) on the post-course assessment. There was also a significant increase in the number of teachers identifying the difference between linear and square units as a key idea for middle grades students, $\chi^2(1, 50)$, = 10.3, $p = 0.001$. Finally, teachers were significantly less likely to cite knowing the names and characteristics of two-dimensional shapes on the post-course assessment, $\chi^2(1, 50)$, = 7.71, $p = 0.005$.

4.2.1.7. Performance on the *Minimizing Perimeter Lesson Plan* Task. The *Minimizing Perimeter Lesson Plan* task (see Figure 13 for a short version; see Appendix B, Task 3 and Appendix C, Task 4 for full protocol) was a part of the interview in which 20 teachers engaged. The first interview was conducted concurrent with the first week of the course experience, and the second interview was conducted following the close of the course. This task was designed primarily to assess teachers' knowledge of mathematics for student learning in the context of planning a lesson around a middle grades geometry and measurement task.[11] However, teachers' work on and talk around the task that was the focus of the lesson plan afforded some important insights into teachers' content knowledge in the domain and content knowledge for teaching.

> Your final task is to plan a lesson around this problem. I'm going to give you 5 to 8 minutes to write down your ideas about how you might implement a lesson with this problem. Your target mathematical goal will be to get students to understand the relationships between area and perimeter. You are free to modify the problem in any way. I'm going to turn off the recorder while you plan. Do you have any questions?

Figure 13. Excerpt from *Minimizing Perimeter Lesson Planning* protocol.

The *Minimizing Perimeter* task asked teachers to create a graph that related dimension and perimeter, and many teachers exhibited misconceptions about this relationship. In the *Fence in the Yard* scenario, where the perimeter was fixed and the area varied, a graph of length vs.

[11] Results for this task with respect to knowledge of mathematics for student learning can be found in section 4.3.

area produces a parabola. However, in a fixed area/changing perimeter situation, the graph of length vs. perimeter is not parabolic. Transcripts of the 20 pre- and post-course interviews were examined for statements that expressed uncertainty as to the nature of the relationship, and statements that indicated the existence of a misconception regarding the nature of the relationship between dimension and perimeter in this task.[12] The number of teachers exhibiting uncertainty and misconceptions was tested using McNemar's test; the number of lines of interview text related to uncertainty or misconception was also compared using a chi-square test.

There was a significant decrease from first to second interview in the number of teachers exhibiting either uncertainty or a misconception related to the graph (McNemar's test, $p = 0.02$). There were also significant decreases from pre-course to post-course interview in the number of lines coded as uncertainty, $\chi^2(1, 3156), = 6.83$, $p = 0.009$, and the number of lines coded as misconception, $\chi^2(1, 3156), = 32.9$, $p < 0.001$. This result suggests that prior to the course, a significant number of teachers held misconceptions related to the relationship between dimension, perimeter, and area in this task, or did not know what the relationship would look like. They also spoke significantly more in the interview setting about such misconceptions or uncertainties about the graph on the first interview. Following the course, teachers gained some insight into the relationship between length, perimeter, and area and how would be represented on a graph when area is held constant.

Two areas of teachers' performance on the lesson planning task in the pre- and post-course interview were salient with respect to content knowledge for teaching. One aspect is the notion of task modification – the ability to modify the published version of a mathematical task in order to afford students more opportunities for entry to the task through the removal of explicit

[12] It is important to note that teachers were encouraged to explore the task prior to the interview sessions, and were permitted to continue to explore the task mathematically in the interview setting if they chose to do so.

pathways, to focus the task more clearly on a particular mathematical idea, and to avoid proceduralizing the task for students. These characteristics are factors that support student thinking with respect to cognitively demanding tasks (Stein et al., 2000). The second aspect of content knowledge for teaching that this item measured was teachers' articulation of the lesson goals.

Interviews were first coded for evidence of three types of task modification: removing explicit pathways, targeting the big mathematical idea, and proceduralizing the task. Of these three, the first two have the potential to support student learning; the third has the potential to inhibit student learning. Lines of interview text were coded for each of these three types, and analyzed in two different ways: comparing the number of lines of text coded as each type across both interviews, and comparing the mean of the percentage of each teacher's interview that was devoted to each of the three modification types. When results were compared in terms of number of teachers using McNemar's test, significantly more teachers targeted the big ideas on the second interview ($p < 0.01$), and significantly fewer teachers proceduralized the task on the second interview ($p < 0.05$). Significant differences were found in the number of lines of text devoted to removing explicit pathways, $\chi^2(1, 3156) = 89.3$, $p < 0.001$, targeting the big idea, $\chi^2(1, 3156) = 71.7$, $p < 0.001$, and proceduralizing the task, $\chi^2(1, 3156) = 27.0$, $p < 0.001$, with removing explicit pathways and targeting the big mathematical idea increasing, and proceduralizing the task showing a decrease. Significant differences in the same directions were also found for the mean percentage of lines devoted to removing explicit pathways, $t(19) = -2.08$, $p = 0.03$; targeting the big mathematical idea, $t(19) = -2.75$, $p = 0.007$; and proceduralizing the task, $t(19) = 2.08$, $p = 0.03$. These results show that in the context of a task that targeted the relationship between dimension, perimeter, and area, teachers showed an increased ability to modify the cognitively demanding task in ways that supported student thinking and meaningful student learning of the relationship between dimension, perimeter, and area.

In planning the *Minimizing Perimeter* lesson, teachers were given a generic goal for students: to understand the relationship between area and perimeter (see also Appendix B, Task 3 and Appendix C, Task 4). After teachers described the lesson they had planned, they were asked what they hoped students would learn as a result of engaging in the lesson. Responses to this prompt were grouped into general categories and compared between the two interviews. Several notable results emerged from this analysis. First, there was a significant increase in the average number of goals for student learning that teachers articulated between the two interviews, $t(19) = -2.76$, $p = 0.006$. Teachers in the first interview stated 2.15 goals on average; on the post-interview, this average increased to 3.1 goals. There were two significant changes with respect to the types of goals identified by teachers: a significant increase in the number of teachers citing the relationship between length, width, and perimeter, $\chi^2(1, 40) = 4.33$, $p = 0.04$; and a significant increase in the number of teachers specifically describing the relationship between area and perimeter as a goal, $\chi^2(1, 40) = 6.14$, $p = 0.01$. These two goals are notable, as they are more specific versions of the general goal given to teachers at the start of the task.[13] These three significant results together suggest that following the course, teachers were able to articulate more goals for students and showed a specific increase in two goals that are closely related to the mathematics that is at the heart of the *Minimizing Perimeter* task.

4.2.1.8. Summary. Table 12 summarizes the results discussed in this section, aligning the findings with the aspects of knowledge of mathematics and mathematical activities intended to be assessed in the geometry and measurement course.

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[13] To qualify for coding as the relationship between area and perimeter as a goal, teachers had to articulate the nature of this relationship in order to differentiate from the general statement given in the task prompt.

Most teachers entered the course with the ability to solve problems related to dimension, perimeter, and area. Changes tended to be in the quality of explanations and their use of representations. The changes in the quality of explanations made more salient the relationships between the three quantities and tying changes in the specific tasks to general properties of the figures and quantities. Performance on the *Area of a Parallelogram* and *Fence in the Yard* tasks are particularly representative of these changes in teacher knowledge. Few teachers exhibited misconceptions related to these relationships between dimension, perimeter, and area on the pre-course assessment, and those misconceptions were almost entirely eliminated on the post-course assessment. A notable exception is the graphical representation of a length vs. perimeter relationship when area is constant; many teachers who engaged in the interviews had confusions or misconceptions about this relationship and representation at the start of the course. These misconceptions significantly decreased in the second interview.

More changes were noted in tasks that were closer to the work of teaching. Representational fluency, measured by the ability to move across representations and use multiple representations in the service of problem solving, increased for teachers from pre- to post-course, as evidenced by the changes to the mean numbers of representations used. Teachers became more attuned to what the big ideas are in geometry and measurement in the middle grades, and were better able to plan a lesson around a cognitively demanding geometry and measurement task that brought these ideas to light and had the potential to support students' engagement in the task. Specifically, performance on the *Minimizing Perimeter* task showed that teachers were better able to plan a lesson that removed explicit pathways for solving a task and to identify goals that clearly described the relationships between dimension, perimeter, and area. Removing explicit pathways is a key aspect of content knowledge for teaching, as it offers more students the ability to make progress the task and approach the task in multiple ways. The decrease in teachers' plans that proceduralized the task also resonates with this result. Teachers also showed growth in the number of goals for student learning articulated, and in the articulation of two goals in particular: the relationship between length, width, and perimeter, and the specific nature of the relationship between area and perimeter. These two goals represent a refinement of the generic goal that teachers were given to focus their work on the task.

One last notable result was the change in teachers' understandings of the affordances, constraints, and conditions of use for various formulas. This notion is particularly salient in geometry and measurement, as almost every measurable attribute of geometric figures has one or more formulas that are useful in quickly calculating the attribute. The results of the *Considering Formula Use* item suggest that teachers' formula preferences and reasons for holding those preferences changed following the course, moving away from favoring a formula for ease of use or surface-level features and instead favoring a formula for mathematical reasons.

Table 12 notes that one goal related to content knowledge in the domain was not assessed, explaining the relationships between linear and square units. This goal was not intended to be a significant focus of the course and thus was not assessed on the written assessment or interview protocol. However, the course discussions and teachers' self-reports of learning in the course suggest that significant learning of this idea did occur. Learning about the relationships between linear and square units is discussed in the context of opportunity to learn in Section 4.2.2.

Table 12. Knowledge of mathematics and mathematical activities related to dimension, perimeter, and area: Summary of results.

Content knowledge in the domain	**Findings**	**Tasks**	**Opportunity to Learn**
Understand that area and perimeter have a non-constant relationship	Strong understanding of this idea pre-course No evidence of misconceptions related to the non-constant relationship on post-course items Decrease in vague responses	*Fence in the Yard* *Area and Perimeter: Responding to Student Claims* *Area of a Parallelogram*	Stacks of Paper Task Fencing Task/The Case of Isabelle Olson activities Building Storm Shelters Comparing Triangles Task
Explain how changes to dimensions of a figure impact perimeter and/or area (including transformations on a plane figure)	Increase in ability to describe the relationships between dimension, perimeter, and area Increase in general explanations describing how changes to a complex figure impact perimeter Decrease in misconceptions and uncertainty related to a constant area, changing perimeter relationship	*Fence in the Yard* *Tangrams* *Minimizing Perimeter Lesson Planning*	Stacks of Paper Fencing Task/The Case of Isabelle Olson activities Building Storm Shelters Comparing Triangles Task Interviewing a Student
Explain the relationships between linear and square units and utilize these relationships to make sense of area and perimeter	(Not directly measured on written assessment or interview)		Linear and Area Units/The Case of Barbara Crafton activities Index Card Task
Demonstrate understanding of the meaning of area and perimeter using a variety of tools and representations	Increase in written explanations	*Area of a Parallelogram*	All tasks related to area and perimeter
Content knowledge for teaching			
Identifying the big ideas in middle grades geometry and measurement related to dimension, perimeter, and area	Increase in big ideas: - specific description of relationship between dimension, perimeter, and area - difference between linear and square units Decrease in big ides: - names and characteristics of two-dimensional shapes	*Big Ideas*	All tasks related to area and perimeter
Identify and/or create mathematical tasks that provide students with opportunities to explore the big ideas related to dimension, perimeter, and area	Increase in modifications that: - removed explicit pathways for solving the task - targeted the big mathematical ideas - described the relationship between length, width, and perimeter or area and perimeter as a goal Decrease in modifications that proceduralized the task	*Minimizing Perimeter Lesson Planning*	Big ideas in geometry and measurement Comparing Fencing Tasks What did I do while you were working? Art Class Assess/Advance
Use a range of representations to explain the relationship between dimension, area, and perimeter	Increase in the mean number of representations used to respond to tasks	*Fence in the Yard* *Area of a Parallelogram*	All tasks related to area and perimeter
Understand the affordances of constraints of different formulas for geometry and measurement concepts	Increase in preference for $A=bh$ formula due to its generality; decrease in selecting $A=lw$ for reasons of ease of use	*Considering Formula Use*	Linear and Area Units Stacks of Paper task

Performance on these 7 tasks indicates that teachers grew in meaningful ways in their knowledge of mathematics and mathematical activities related to dimension, area, and perimeter. It is clear that teachers demonstrated a greater breadth of knowledge on items that measured the relationships between dimension, area, and perimeter, and were able to apply their knowledge to tasks related to the work of teaching, such as articulating goals and modifying tasks. In sum, these data make a compelling argument that teachers acquired knowledge of mathematics and mathematical activities related to dimension, area, and perimeter as a result of the course. In the section that follows, data are presented that illustrate how the experiences in the course may have led to this learning.

4.2.2. Dimension, perimeter, and area: Opportunities to Learn

In this section, the results discussed previously are linked to the design principles and opportunity to learn through selected excerpts from course discussions, interview data in which teachers described their learning, data from other written sources including written assignments and the instructor's planning diary. Table 12 aligns the results of the analysis of written artifacts with particular activities that constituted opportunities for teachers to learn the knowledge described. This section describes the course activities that provided an opportunity to learn about dimension, perimeter, and area, and provides artifacts from discussions and written work that provide evidence of opportunities to learn the knowledge described in Table 12. Figure 14 highlights all course activities that related to dimension, perimeter, and area.

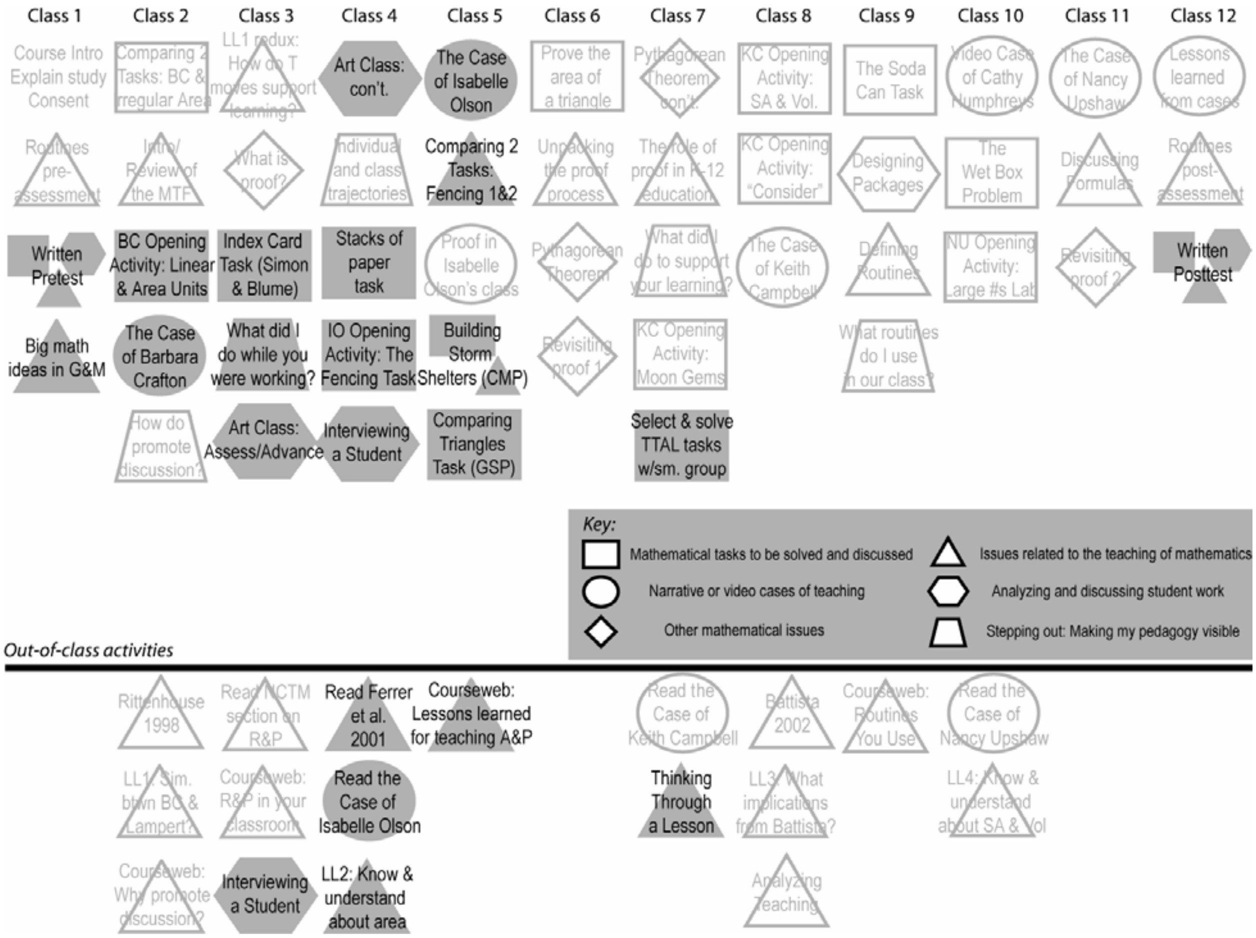


Figure 14. Course activities focused on the relationships between dimension, perimeter, and area.

Based on the criteria identified in section 4.1.7, two constellations of activities provided teachers with opportunities to learn related to dimension, perimeter, and area. The first constellation, the set of activities around The Case of Barbara Crafton, focused on issues of linear and square units and basic understandings of the concepts of perimeter and area. A short series of tasks (from the Index Card Task through Stacks of Paper) provided the connective tissue between the first and second constellation and were designed to explore the conceptual underpinnings of perimeter and area in depth.[14] The second constellation, the set of activities around The Case of Isabelle Olson, expanded on teachers' conceptual understandings of area and

[14] Because they are closest in content to the activities in the second constellation, the opportunities to learn based on these activities are described at the start of the Constellation 2 section below.

perimeter, addressed issues of relationships between measurable quantities of geometric figures, and provided opportunities to consider multiple representations and the connections between these representations. For both of these constellations, two additional activities framed the opportunity to learn. The discussion at the beginning of the course around the big ideas in geometry and measurement introduced the ideas into the public space, and the Thinking Through a Lesson discussion and assignment in Class 7 provided teachers a final opportunity to reflect on their understandings related to both constellations in a written form that connected to the work of teaching.

4.2.2.1. Course opening activities. Two activities at the start of the course were important in setting the stage for teachers' engagement with activities that followed in the two constellations related to dimension, perimeter, and area. These two activities made public a set of issues for study related to the mathematical content, and represent the start of the sequence of activities that provided opportunities to learn. The first activity was teachers' work on the pre- and post-course assessments (see Appendix A for the complete text). By engaging with the assessment during the first course meeting, teachers were likely to have become attuned to the mathematical ideas that were to be the focal points for mathematical learning in the course. Following engagement in the pre- and post-course assessment, teachers were engaged in a discussion of what they thought the big ideas were in geometry and measurement in the middle grades. (Note that this discussion mirrors Part B of the pre-course assessment.) Teachers identified a number of ideas related to dimension, perimeter, and area, as shown in Table 13.

Table 13. Big ideas related to dimension, perimeter, and area identified in Class 1.

Big Ideas Identified	Selected teacher talk
Connections between area and perimeter	Nancy: Um, basically just the relationship between them, um like if one increases how the other increased. Instr.: Daulton? Daulton: I would say they need to know what area and perimeter actually are.
What area and perimeter are and how to use them (more than just formulas)	Instr.: Does anyone want to expand on what Daulton just said? What does it mean to know what area and perimeter are? Chuck? Chuck: I know I just did it in pre-algebra, some kids have no idea like, what square units are. They know to put it at the end of area but they really don't, especially like when we got into volume too, cubic units, they have no idea. They know it has to be cubed because we're talking about volume, and square because we're talking about area. And some of them don't know that it has to be an inch by an inch or, whatever your units are. Instr.: Noelle? Noelle: It's also to know how they use them, I had a student ask if they had to memorize all the formulas for the section on area and perimeter, and I told him you don't need a formula for perimeter, and he couldn't understand, that he didn't need to know a formula. Instr.: So the interesting thing that I heard that Noelle just said is that this understanding has to go beyond just formulas. It can't be that you just know the formula, so you know perimeter.
Dimensions	Kelly: Going back to the first bullet there, I think students need to understand dimensions. So perimeter is dealing in one dimension, area is dealing in two dimensions, and how to visualize it.
Connections between figures (properties)	Florence: I think a lot of those things, and we kind of said it but we haven't written it, is making connections between figures, so all different figures, you should be able to connect them. I taught all my kids to connect everything back to a rectangle in some way. No matter how simple a rectangle seemed, we spent the whole day understanding why a rectangle has the formulas it has. And then break it, cut it, change it fold it, make new figures… Instr.: So one thing I hear that's common to what Florence and Kelly and Kelsey said about a rectangle… these are all things that are general properties of the shapes. And by talking about formulas like area and concepts like area that are based on the general properties, we're buying something there, that we may not get by taking rectangle one, here's the length, here's the width, multiply them, there's my area, I'm done. Rectangle two.
Relationships and characteristics of shapes	Kelsey: Just the idea that when a shape is defined as a parallelogram, you have these specific characteristics and knowing all those different relationships and characteristics, it just gives you so much information about, that knowing those characteristics helps you know those formulas and construct proofs and whatnot.

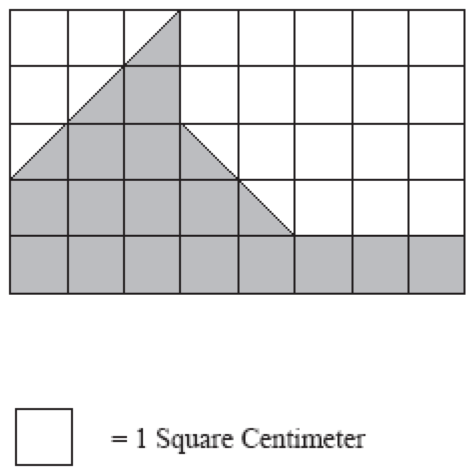
The big mathematical ideas related to dimension, perimeter, and area that teachers identified were generally reflective of the key mathematical ideas that it was hoped teachers would learn, as reflected in Table 12. In fact, the instructor was struck by the extent to which teachers identified these ideas with respect to dimension, area, and perimeter.

> I was surprised a bit that many of the big ideas in geometry and measurement that related to comparing measurable quantities of geometric figures came out in our chart. Specifically, I recall that the first idea offered was connecting area and perimeter. Much was said about these relationships both between area and perimeter and surface area and volume (but less so for the latter). Teachers also talked a lot about understanding the meaning of dimension, and the differences between linear and square units of measure. This sets us up nicely for the first set of activities, which involve this very idea, around The Case of Barbara Crafton. I was a bit concerned that starting with this particular task, which isn't terribly complex, might keep the expectations low at the start of the class, but I think this will be an issue that teachers will grab on to and we will have good conversations about it.
>
> Course Planning Diary, Class 1 Reflection, Lines 182-191

The content of the pre-course assessment, coupled with big ideas discussion, identified the key mathematical ideas for which teachers would be provided an opportunity to learn. In particular, the discussion about the big ideas in geometry and measurement afforded teachers the opportunity to consider mathematical ideas related to dimension, perimeter, and area at a general level, grounding the activities in the next two constellation of activities in prior knowledge and connecting them to issues of teaching.

<u>4.2.2.2. Constellation 1: Activities around The Case of Barbara Crafton</u>. The constellation of activities around The Case of Barbara Crafton provided teachers with opportunities to grapple with issues of linear and square units, basic issues of the conceptual and computational meanings of area, and a variety of representations for units and area. Three activities in this constellation contributed to teachers' opportunities to learn. Teachers first engaged in solving the Area of Irregular Figures II task (the opening activity for The Case of Barbara Crafton), read and discussed the case, read the Ferrer et al. (2001) article which discusses the implementation of related tasks with students, and in Class 5 returned to these issues in their writing assignment in which they reflected on issues related to the teaching of concepts of dimension, perimeter, and area. Both the task and the case were considered individually, then discussed in a whole-group setting, and the writing assignment provided opportunity for continued reflection on the issues.

Teachers solved the Area of Irregular Figures II task (see Figure 15) after having considered the affordances of this task in comparison to a more routine area of irregular figures task from a traditional middle grades textbook. The purpose of comparing the tasks was to give teachers a language and set of criteria for discussing the cognitive demands of tasks (Stein, Smith, Henningsen, & Silver, 2000).



1. Find the area of the shaded region of the irregular shape shown above in square centimeters and square millimeters.

2. How many square inches in a square foot? Draw a diagram and write a few sentences to explain how you determined your answer.

Illustration adapted from Mathematics Learning Center (1991)

Figure 15. Area of Irregular Figures II Task.

The engagement in and discussion of this task provided teachers the opportunity to learn about the relationships between linear and square units, to demonstrate an understanding of the meaning of area using a variety of representations, and to identify the big ideas related to dimension, perimeter, and area that this task had the potential to develop. These notions were evident in the instructor's pre-class entry in the planning diary.

> I think teachers will have little difficulty with the mathematics of the task and are likely to produce the variety of solutions… which are a range of both visual and numeric solutions. I intend to organize the discussion of the task in such a way that the following important mathematical ideas come out: that area is measured by square units; that when deciding on units for area and relationships between them, that the size of the linear unit [upon which the square unit is based] is important; that area measurements depend on and vary by the size of the units chosen. I hope that these are the mathematical ideas that teachers identify when we discuss the mathematics they worked on during the task. Additionally, I intend to use questions such as, "So what's the relationship between your two answers for part 1?" to get teachers to consider these mathematical issues.
>
> Course Planning Diary, Class 2 Pre-Class Entry, Lines 320-329

During the sharing of the task, these notions were discussed, as is evident in the following excerpts in which teachers reflect on the mathematics embedded in their work on the task:

Instr.:	So what math ideas did we work on?
Lana:	We worked on converting from a unit of 1 to smaller units. So the first one it was one square centimeter, you had to convert down to millimeters, the second was one square foot, you had to convert down to square inches.
Instr.:	Thanks. Other math ideas?
Melinda:	How geometry relates to multiplication.
Instr.:	Can you say more about that?
Melinda:	Well the way I solved it like the foot one, I made an array in the box, 12 by 12, and how the area relates to, multiplication.
…	
Kelly:	Visualizing the difference between linear measures and square measures. I think that helped a lot because this was drawn to scale…
Instr.:	Can you say again how the task helped you get at that idea?
Kelly:	Depending on how you solved it, you might have just crossed here where this is a box, and that would be counting linear measures-
Instr.:	Similar to Barrett's approach-
Kelly:	-or back here when you guys were doing the how tall the triangle was, those were linear measures. Or if you were looking here at the square centimeter, counting how many were shaded, that's a square measure, I think that really, makes you make a distinction between the two.

Excerpts from Class 2

Following exploration of the task, teachers were asked to read and discuss The Case of Barbara Crafton, a narrative case of a middle school teacher implementing the task with a group of students. The discussion of the case was primarily aimed at pedagogical issues; however, in exploring how the teacher (Barbara Crafton) supported student learning, important issues of mathematical content were raised. Table 13 shows excerpts from the written record of the discussion in which teachers identified particular pedagogical moves by Ms. Crafton and linked them to important mathematical understandings:

Table 14. Pedagogical moves by Barbara Crafton and their relation to mathematical learning.

Pedagogical Move	How the move supported student learning
using gum and cards to measure (comparing)	Extended understanding of mathematics and measurement and connected to the real world
Selection of the task	Different way other than formulas; cleared up misconceptions
Natalie shows what she means on the diagram	Link abstract ideas to a mathematical model – better understanding

The final two activities related to content knowledge in the constellation were the reading of the Ferrer et al. (2001) article and Learning Log 2, in which teachers were asked to reflect on what they thought students should know and understand about area and perimeter. The article was intended to add a research-based voice to the conversation and to show additional classroom implementations and issues related to dimension, perimeter, and area – specifically, the relationships between units, conceptual understandings of area, and the use of a variety of representations. The Learning Log assignment was viewed as the culminating activity for both this constellation and the next, giving teachers the opportunity to reflect on their experiences in relation to classroom practice in the context of the content knowledge that students should develop. The excerpts below from the Learning Logs of several teachers provide evidence that teachers were attending to issues of dimension, perimeter, and area that arose in the exploration of the Area of Irregular Figures II task and the discussion of The Case of Barbara Crafton. These two excerpts, the first from an experienced practicing teacher and the second from a preservice teacher, are representative of the level of reflection found in the assignments.

> In order to have a thorough understanding of area, students need to understand measurement and units, the conceptual meaning of area and perimeter, formulas and concepts of composition and decomposition, relationships between area and perimeter, variable measures, and methods to prove their conjectures.

> First, I'll explain what I mean by measurement and units. Students should be able to use graph paper to develop the idea of area and to get exact or estimate areas of figures in square units. (B. Crafton) When given figures not on graph paper, they should be able to calculate the measures of unknown dimensions by using other dimensions in the figure. For example, opposite sides of a rectangle are congruent. When measuring height, students should understand that is must be perpendicular to the base. They need to be able to use a ruler to get linear measurements. (Beyond this, they need to take the leap from counting discrete blocks to dealing with these linear measures to calculate area.) (NCTM conference) Students also need a clear understanding of units; they need to be able to visualize the difference between linear and square units, and know when those units are required. They need to understand how changing a linear dimension affects area. (B. Crafton) Students also need to understand equivalence of units, select appropriate units, and know that when you use smaller units, you need more of them.
>
> Excerpt from Learning Log 2, Kelly

> They need to understand that area is represented by square units, because of the multiplicative relationship between dimensions, whereas linear units are used to represent perimeter because of the additive relationship. This concept can be developed by allowing the students the opportunity to engage in a task similar to Barbara Crafton's Reasoning about Units for Linear and Area Measure. Students will be able to use grid blocks to count area and perimeter measurements of irregular shapes, and then focus on the concept of linear and square units, and the situations for which each is appropriate to use. They will also be allowed the chance to develop a concrete understanding of square units, as well as why they are used for area measurements.
>
> Students need to be introduced to the concept of area by exploring situations in which they are able to cover different surfaces, concretely and abstractly, and then describe how they arrived at their solutions. By engaging in these types of activities, students will develop a deep level of understanding of the meaning of area. Students should also be allowed the opportunity to discover the area formulas for different shapes by generalizing them themselves rather than being told only what they are and not why they work. When students generalize a formula themselves, they are able to understand how it came to be and why it works. As a result, they will not need to memorize it because they can easily gain access to it through reasoning.
>
> Excerpt from Learning Log 2, Debra

As represented by their Learning Log entries, the activities in the constellation around The Case of Barbara Crafton were influential in developing teachers' understandings of content knowledge in the domain and content knowledge for teaching related to dimension, perimeter, and area. Given the nature of the work and discourse in the activities and the connections expressed in their Learning Log assignments, it is reasonable to conclude that this set of activities represented an opportunity to learn.

Teachers were also asked to reflect on their learning during the second interview, first being asked to identify what they had learned through their participation in the course, and second to link those understandings to particular course activities. Of the 20 teachers interviewed, 9 cited the relationship between dimension, area, and perimeter as something that they learned when asked to describe their learning. All four of the activities described above were cited by at least some teachers as significant sources for learning. The most significant was The Case of Barbara Crafton; of the 20 teachers interviewed, 11 identified reading and discussing the case as a source of learning. Seven of 20 teachers identified the Solving Area of Irregular Figures II as a source of learning, and 4 teachers each identified the reading of the Ferrer et al. (2001) article and the writing of Learning Log 2 as significant sources of their learning. This supports the conjecture that these activities provided teachers in the course with significant opportunities for learning.

4.2.2.2. Constellation 2: Activities around The Case of Isabelle Olson. The constellation of activities around The Case of Isabelle Olson built on the understandings developed in the previous constellation. This set of activities provided teachers with opportunities to consider a variety of aspects of the relationships between dimension, perimeter, and area; to consider the affordances of particular representations, particularly formulas; and to link conceptual understandings of dimension, area, and perimeter to a variety of representations. Teachers also had the opportunity to consider a variety of mathematical tasks purported to address similar mathematical content, and how those tasks were similar and different in their ability to address the key mathematical ideas at play. A set of five activities (from Index Card to Stacks of Paper in Figure 14) served as connective tissue between Constellations 1 and 2; while they are not officially a part of the second constellation, the opportunities to learn from these activities are described in this section.

A total of 15 activities (5 connective activities and 10 within the constellation) in Classes 3 through 5 together met the criteria as opportunities to learn about the relationships between dimension, perimeter, and area in this constellation (see Figure 14). Seven of these activities focused most closely on mathematical issues, and will be highlighted in this section: the Index Card Task, the Stacks of Paper Task, the Fencing Task (opening activity for The Case of Isabelle Olson), reading and discussing The Case of Isabelle Olson, Comparing the 2 Rabbit Pens Tasks, the Building Storm Shelters Task, and the Comparing Triangles Task. These tasks, as well as the activities surrounding them in the constellation, provided teachers with numerous opportunities to consider the relationships between dimension, area, and perimeter.

The Index Card Task (see Figure 16) was designed to bridge the experiences with dimension, perimeter, and area in the first constellation to the activities to come in the second constellation. The instructor describes how the task relates to previous work in the pre-course planning diary entry for Class 3:

> Mathematically, the work in the last session related primarily to coordinating linear and square units of measure, and the relationships between the two particularly with respect to conversion. The mathematics in the rabbit pens task deals with a constant perimeter and changing area. In order to bridge those two ideas, I am hoping to use the index card task from Simon & Blume (1994) to coordinate the idea of linear and square units of measure with perimeter and area. This is not exactly the tact that Simon and Blume took – they used this task in the service of understanding multiplicative relationships. As such, I've added a second piece to the task, as described below, and am hoping to "tune" the conversation to deal with area and perimeter.
>
> Course Planning Diary, Class 3 Pre-Class Entry, Lines 548-556

The second part of the task was a question asking teachers to use the index card to calculate the perimeter of the table. It was hoped that this addition would make salient the multiplicative nature of area and the additive nature of perimeter through the use of a common referent. Teachers would have to consider the concepts of area and perimeter, their relationships to the dimensions of the table, and the relationships between dimension, perimeter, area, and the tool used to measure all three. The text of the task as presented to teachers is shown below.

Use an index card to determine how many rectangles of the same size and shape as the index card can fit on your table. Rectangles may not overlap each other, they may not overlap the table edge, and they may not be cut. Explain how you determined your answer and what your answer means.

Figure 16. The Index Card task.

During their small-group explorations of the task, teachers were pressed to describe how they calculated area and perimeter, why their methods were valid for finding area and perimeter, and what the numbers they came up with meant in the context of the task. These ideas were made salient in the whole-group discussion, as evidenced by the excerpts below:

Florence: Alright we did – we put our cards, [draws a representation] that way along our table.
Instr.: So what's the that way?
Florence: So we had four of our cards going lengthwise and four going widthwise, and that gave us the least amount of leftover room. So we figured that we could to that construction of lengthwise all the way across, and that was 19 rows across of all those cards. So 19 times those 4 is 76 notecards.
Instr.: So I'm going to stop you for a minute here. Why 19 times 4, why multiply?
Florence: Because these 4 cards occur 19 times.
Instr.: So didn't you just use times to me to justify why you multiply? Can you explain it to me a different way?
Florence: There are 19 groups of these 4 cards.
Instr.: So why does that mean multiply?
Florence: Because I could add them up 19, over and over again. I could add that group of 4, plus 4 plus 4 plus 4… 19, times.
Instr.: Florence, do you remember what you said the first time I asked you this question a few minutes ago?
Florence: I said rows times colums.
Instr.; Nina, do you remember what you said when I asked you this question?
Nina: I said area length times width.
Instr.: Can somebody connect those two?
Florence: Because that fills. When I have 19 groups of my 4s, that fills the entire space, hence it covers the whole area.
Instr.: Can anyone else say that a different way? [pause] Are you convinced?
Betsy: Because I think an area model connects to an array, so if you're thinking in rows and columns, especially when you're, well I guess whether you use non-standard units or not, because you're thinking along the width you can think of so many groups of inches being your length, and so many groups of inches being your width which you're covering with inch units instead. In this case, you're thinking of cards, making so many rows of cards and so many columns of cards, which is the same as covering your area by covering the length and the width.

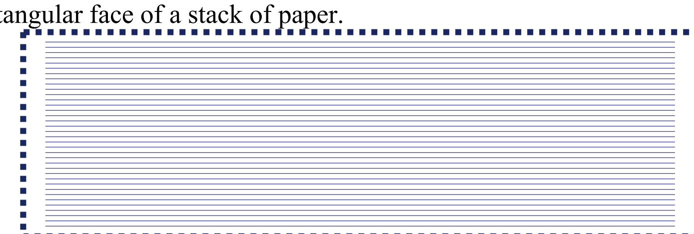
(Ed explains his method for finding perimeter)
Instr.: I wondered something about the solution compared to Florence's how many cover the table solution. It seemed like in the mathematics that Florence did there was a lot more multiplication and I really pressed her to tell me why the multiplication was there. It seems like in Ed's answer there was some multiplication because of the grouping that was used, but there's a lot more addition there. And the other thing I noticed was that, with Florence's we were very concerned about groupings that were the same and that were repeatable. And this we've got, we've definitely got a pattern that repeats on one side, but we've got a pattern that doesn't repeat on the other side, and it looks like we might even have an overlap issue on the corner. So I was wondering what people thought about that.
Cameron: Area you're measuring the space inside, but perimeter you can almost do it, outside of the table, so you just want to know how long, each edge is.
Ed: With the area, you're using the card like you said to measure the inside space, whereas when you're measuring perimeter you're just using this kind of like a ruler you could

> technically- you're not overlapping anything, you could take them on the outside, and use them as a 5 inch like that. And that's a way to see you're not overlapping anything.
>
> Excerpts from Class 3

These excerpts illustrate three issues that were at the heart of the discussion, which built on understandings from the constellations of activities around The Case of Barbara Crafton: that perimeter is a linear measurement based on additive relationships between the dimensions; that area is a square measurement that is based on multiplicative relationships between the dimensions, and that the same tool can be used in differing ways to aid in finding both quantities. These experiences laid the groundwork for understanding the complex relationships between dimension, perimeter, and area.

The Stacks of Paper task was intended to build on these understandings through the examination of a set of transformations on a two-dimensional figure. The task was also intended for teachers to consider which quantities remained invariant and which were variant as they transformed the initial rectangle, setting the stage for making connections between the visual figure, their descriptions of the quantities, and symbolic representations of formulas for finding area and perimeter. The task is shown in Figure 17.

Consider the rectangular face of a stack of paper.



Part 1
Change the shape of the side of the stack in any way you choose.
Do all the shapes you can create have the same area? Justify your answer.

Part 2
Divide the rectangular face into two equal parts in at least two different ways.
Draw a sketch of how you divided the rectangle below.

Figure 17. The Stacks of Paper task.

In sharing and discussing the Stacks of Paper task, teachers came upon a variety of ways to describe the transformation of the stack of paper, the notion that the area remained constant, and the quantities (side lengths) that stayed the same and those that changed. Teachers then made links across the classes of shapes that could be formed through transforming the rectangle (including both common and unusual shapes) and considered links to symbolic formulas for area of a variety of different shapes. These relationships between dimension, perimeter, and area are exemplified by the excerpts from the whole-class discussion below.

> Maura: Can you say that the area is preserved as the shape changes?
> Instr.: If you say what you mean by preserved.
> Maura: Yeah, I don't know. [class laughts]
> Instr.: Sierra?
> Sierra: We kind of did the scientific principle of conservation of matter, except it's conversation of surface area? Like even if we're shifting here- like if you have your original rectangle and you're shifting it, whatever doesn't exist over here now exists over here. So it still exists, just in a different space.
> Instr.: Nick?
> Nick: Like the one thing that I was- they were talking about something that I was just examining, is why the area was staying the same even when we were making a parallelogram, because if we have a regular rectangle, that- a shape that was hinged at all four corners and you tilted it, that would change the area, it's really not a conservation of area because the height was changing… but with this one [the stacks of paper] these [the corners] aren't connected, there's 500 separate little pieces, so when you slide it, it's not pulling the shape down at the same time that it's moving, so I measured, I think it's close to 2 inches high, so when you slide, the height is still 2 inches.
>
> Excerpt from Class 4

The Fencing Task, the opening activity in The Case of Isabelle Olson, marked the official start of Constellation 2, and provided teachers with an opportunity to continue to explore the relationships between dimension, perimter, and area. This task was intended to build on teachers' previous experiences with the relationships between dimension, perimeter, and area and to provide opportunities to experience and make connections between a variety of representations:

> In this task, teachers will be pressed for [a] generalization, and the sequence from index cards to stacks of paper to the rabbit pens task represents a progression from concrete measurement to abstract generalization. I hope to bring that out during the solving of the task. Additionally, this task brings us closer to examining the relationship between dimension, area, and perimeter. The

> index card task (and stacks of paper) will have touched on this, but fairly indirectly. This task will require teachers to examine the impact of changing one on the other... I'll be looking for tabular, diagrammatic, and graphical/symbolic solutions to this task, potentially in that order, to be presented and connected.
>
> Course Planning Diary, Class 4 Pre-Class Entry, Lines 711-721

The *Solve* portion of the fencing task was designed to engage teachers in exploring the relationships between dimension, area, and perimeter at a general level, as no starting dimensions or constraints (aside from using one side of the house) are given. The task also afforded the use of multiple approaches and multiple representations. The *Consider* questions pressed teachers to generalize conclusions about dimension, area, and perimeter based on their explorations. The complete task is shown in Figure 18.

<u>Solve</u>
You are going to build a rectangular pen for your rabbit, Euclid. You have decided to build the pen using some portion of the back of your house as one side of the pen and enclosing the other three sides with the fencing that was left over from another project. If you want Euclid to have as much room as possible (after all he spends most of the day in his pen), what would the length and width of the pen be?

<u>Consider</u>
1. Can you have 2 figures that have the same perimeter but different areas?
2. Can you have 2 figures with the same area but different perimeters?
3. Can you determine the perimeter if the area is known?
4. Can you determine the area if the perimeter is known?

Figure 18. The Fencing Task.

In solving the task and sharing their solutions, teachers considered the relationships between dimension, perimeter, and area using a variety of representations. The work samples in Figure 19 represent some of the solutions that were shared during the whole-class discussion and the order in which they were shared. Questions from the instructor pressed teachers to make connections between these representations and to the *Consider* questions.

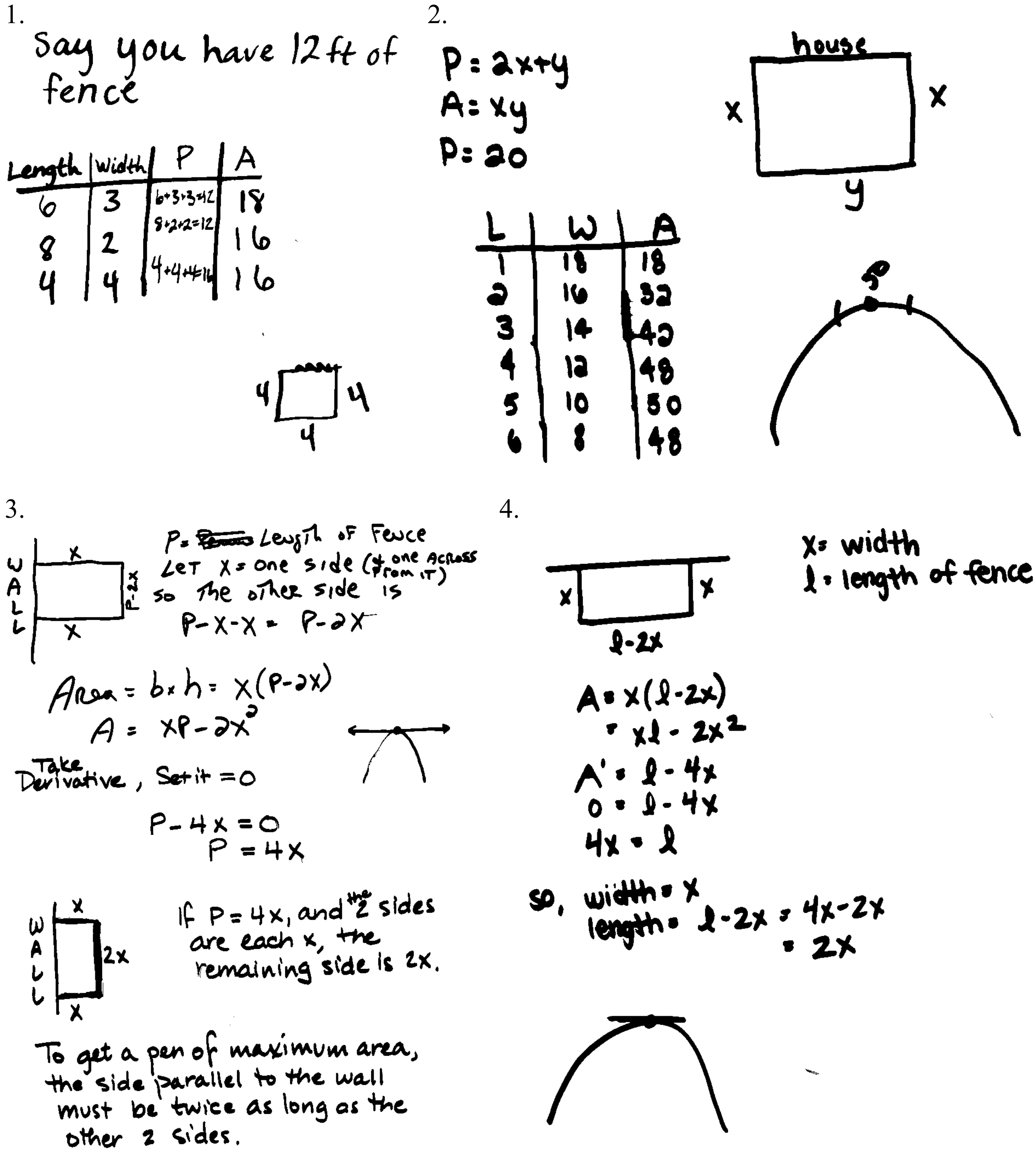


Figure 19. Shared responses to the Fencing Task.

Noelle explains Solution 2 and the connection to Solution 1

Noelle: …and I found that 5 and 10 was my maximum area, and at 6 and 8 it starts to decreasing. And so it's 2 by- the width is twice the length.

Instr.: So how does that relate to what Bridget was doing?

Noelle: It's pretty much the same as what she said, she just had hers in a different order. She found that 6 and 3 were the maximum area, it's the same thing as if I did, my perimeter is all the same, so if I did 1 plus 1 plus 18-

Instr.: So you've gone – you've increased your lengths by 1 each time. And you've told me that 5 by 10 gives you the greatest area, and 6 by 8 goes down. Could there be something that happens between 5 and 6 that gives you an area greater than 50?

Noelle: No.

Instr.: Say why.

Noelle: Well we did it over there, we did it algebraically, and it shows that it equals that. So the way I did it before which led me to the table, I found that it has to be that. And I could pick 5.5 and see.

Instr.: Lana?

Lana: I think because you see that it's going up by 2, and also it's going down by 2, you know that there's not going to be another thing in between, higher than 50 between 5 and 6.

Lana explains Solution 4 and takes questions

Instr.: I'll ask if anyone has any questions for you. Bridget?

Bridget: I'm sorry, I don't understand why *l* is minus 2x.

Lana: *l* minus 2x here? *l* represents the length of the fence I have. So if this is my fence, I have x here and x here, so the rest of the fence here is *l* minus 2x, the entire fence, minus the two sides.

[overlapping talk]

Lana: *l* is perimeter, yeah.

Kelsey: Just the fence or is it the whole perimeter:

Several: Just the fence.

Instr.: So l is the 12 that Bridget had before.

Excerpts from Class 4

The solving and discussion of the Fencing task led directly into the reading and discussion of The Case of Isabelle Olson, which featured a middle school teacher using a similar task with a group of students over the course of two days. The case featured a number of key issues related to content knowledge, including the role of understandings of the relationships between dimension, perimeter, and area in approaching the Fencing task. Teachers were asked to read the case and identify the pedagogical moves that Isabelle Olson made and how those moves supported or inhibited student learning. Small groups identified a number of issues relevant to content knowledge, particularly with respect to how the teacher's changes to the content of the task impacted the implementation of and supported stuednts' engagement in the cognitively demanding task. Selected responses to the prompt that were shared during the whole-class discussion are shown in Table 15.

Table 15. Pedagogical moves identified in the discussion of The Case of Isabelle Olson.

Pedagogical Move Made	Paragraph	How the move supported/inhibited student learning
Changing the task	9	Support: explore more, use own ideas Inhibit: students didn't know what to do; appropriate for students?
Asked Tommy's group to share their insights	32-35	Support: benefited students who were struggling, send back into groups; "didn't give away the farm", kept authority with students and let them struggle Inhibit: Would students have to come to conclusions on their own?
Ms. Olson changed questioning strategies	16, 22	Support: led Tommy's group in a productive direction; "Prove it to me" – students have to consider a way to justify their claim
Almost went back to original problem, but didn't	19	Support: stuck to her goal, didn't take the easy way out

Following their exploration of the Fencing Task and the discussion of The Case of Isabelle Olson, teachers were asked to consider a particular aspect of the case related to knowledge of mathematics and mathematical activities; namely, how the framing of a mathematical task affords opportunities to target particular mathematical ideas. Teachers considered the two versions of the Rabbit Pens task described in the case, shown in Figure 20.

The Original Fencing Problem

Each of the 7th-grade classes in Franklin Middle School will raise rabbits for their spring science fair. Each class will use the school building as one of the sides of its rectangular rabbit pen, and each class wants its rabbits to have as much room as possible.

Ms. Brown's class has 24 feet of fencing to enclose the other three sides of the pen. If Ms. Brown's class wants the rabbits to have as much room as possible, what would the area of the pen be? How long would each of the three sides of the pen be? Try to organize your work so that someone else who reads it will understand it.

Isabelle Olson's Fencing Problem

Ms. Olson's 7th-grade classes at Roosevelt Middle School will raise rabbits for their spring science fair. The class will use some portion of the school building as one of the sides of its rectangular rabbit pen, and will use the fencing that was left over from the school play to enclose the other three sides of the pen.

If Ms. Olson's class wants the rabbits to have as much room as possible, what would the dimensions of the pen be? Try to organize your work so that someone else who reads it will understand it.

Figure 20. Comparing two versions of the Rabbit Pens task.

This discussion was not initially planned to take up a significant amount of time during class; in fact, the instructor made few notes regarding the activity in the planning diary. However, the discussion of the two tasks was quite robust, with several issues related to the selection and implementation of cognitively demanding mathematical tasks, as represented by the excerpts below.

Instr.: Were both of these tasks high-level? Were either or both of these tasks high level tasks?
Noelle: Um, I would say yes, but they're different types. The first one I think is procedures with connections, the second is doing math.
Instr.: Could you say more about– first of all can you say a little bit about what procedures with connections is for people who may not know, and then say why you think the first one is procedures with connections.
Noelle: Procedures with connections is doing something you know how to do, a procedure but you're connecting it with a situation. So what they have to do here is use the formula for area, to make the connection that the optimal size is 24, and hopefully at the end of the task they'll think beyond the 24, but I think the connection is that the length is twice the width. But with the Isabelle Olson task, it doesn't lead them through the steps.
(slight gap from tape change)
Instr.: When Isabelle Olson revised this task, she actually strips the word area out, and I thought that was an interesting choice. I think the point that there really might be a procedure implied by the first one is an important point to take.
Kelsey: I was kind of thinking about the article, the unit or square unit thing, the words area and perimeter how they exist in math class, but then outside of math class, we talk about covering, we talk about going around, you know what I mean?
Barrett: And another thing with that, with what Noelle said with procedures with connections is, I noticed that right away and then here it doesn't say area and it was the first think I looked for, as a student area you think length times width, but then here you say dimensions, it makes the student struggle with, is it the perimeter or the area, what are we looking to find and differentiate them.

Excerpts from Class 5

The final two tasks in the constellation that targeted knowledge of mathematics and mathematical activities were the Building Storm Shelters (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998c) and Comparing Triangles tasks. These tasks aimed to extend teachers' consideration of the relationships between dimension, perimeter, and area in two specific ways: to consider a constant area/changing perimeter situation (Storm Shelters), and to consider these relationships using a different geometric figure in a dynamic computer environment (Comparing Triangles). The primary aim in considering the Building Storm Shelters task was for teachers to identify the mathematical residue that could be left with students following their engagement with the task. Teachers were asked to identify the big mathematical ideas of the task both before solving it and after, with the intent that additional ideas would become more salient following engagement in the task. The task and some sample responses that were shared in the whole-class discussion are shown in Figure 21.

Problem 2.1

The rangers in Great Smoky Mountains National Park want to build several storm shelters. The shelters must have rectangular shaped floors with 24 square meters of floor space.

A. Experiment with different *rectangular* shapes. Sketch each possible floor plan on grid paper. Record your data in a table. Look for patterns in the data.

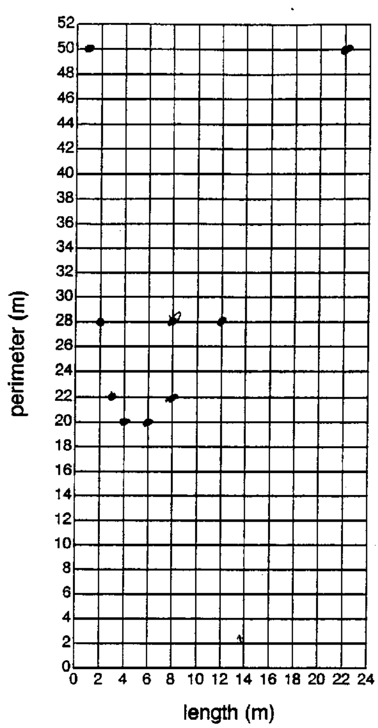
Rectangle	Length	Width	Perimeter	Area

B. Suppose the walls are made of flat rectangular sections that are 1 meter wide.

1. What determines how many wall sections are needed, area or perimeter? Explain.
2. Which design would require the most wall sections? Explain why.
3. Which design would require the least wall sections? Explain why.

C. 1. Make a graph of the length and the perimeter for various rectangles with area 24 square meters.

2. Describe the shape of the graph. How do the patterns that you observed in your table show up in the graph?



D. 1. Suppose you want to consider a rectangular floor space of 36 square meters. Which design would have the smallest perimeter? The largest perimeter? Explain your reasoning.

2. In general, describe the rectangular shape that has the largest perimeter for a fixed area. Which rectangular shape has the smallest perimeter for a fixed area?

Math Ideas (before)

finding area and perimeter
factor pairs
make & analyze a graph (part c)
compare length & perimeter
maximize & minimize
fixed area → perimeter?
different shapes with same area

Math Ideas (after)

multiple representations (part c)
dealing with square root and decimal values

Figure 21. The Building Storm Shelters task and teachers' responses to the task.

The Comparing Triangles task was a dynamic exploration using Geometer's Sketchpad (Jackiw, 1991) in which teachers were confronted with several situations designed to bring to light how changing particular dimensions of triangles impacted perimeter and area. The activity consisted of three explorations. The first consisted of three triangles of equal base and height and allowed teachers to drag one vertex and observe changes to dimension, perimeter, and area as the base and height remained constant. The second exploration allowed teachers to see how the height of a triangle shifted when the vertices were dragged. The third consisted of a single triangle showing all three heights; teachers could drag the vertices and rotate the triangle and observe how the three heights and the corresponding area calculations changed.

Following an open-ended exploration of the three scenarios, teachers were asked in what ways, if any, the exploration added to their understanding of the relationships between dimension, perimeter, and area. Excerpts from the closing discussion are shown below:

Instr.: So other than playing with the cool toys, what did this experience add to our understanding of area?

Kelsey: Well the first one about how the base and the height, the base and the height stays constant and the area stays constant and the perimeter change is similar to what we were doing with rectangles.

Betsy: I think the comment that somebody made today about shapes can look so different, and yet have the same area. You could see that very easily here.

Instr.: With rectangles, whatever you do to a rectangle, they're always going to pretty much look like rectangles. With triangles, with the applet, you can make these triangles as wild as you want…

Excerpt from Class 5

The work and discussion across these 7 tasks show that several of the ideas related to knowledge of mathematics and mathematical activities on which teachers showed growth were at the heart of teachers' work in the second constellation of activities. Teachers were able to explore the relationships between dimension, perimeter, and area in a number of different settings and using a variety of representations; were able to consider how the use and linking of a variety of solutions and representations might have potential to enhance learning of these ideas;

and were able to examine how the big mathematical ideas related to dimension, perimeter, and area can be made available for students through the design of mathematical tasks. The additional activities that served as the connective tissue for the constellation provided teachers opportunities to reflect and make connections to their own practice, enhancing the opportunity to learn.

Teachers viewed several of these 7 tasks as significant sources of their learning. Table 16 shows the number of teachers out of the 20 interviewed who identified each activity[15] as a significant source of learning. The table also disaggregates the data to show how many teachers saw the activity as a source of learning about mathematics, students as learners of mathematics, and the teaching of mathematics.

Table 16. Teacher learning data for connective and Constellation 2 activities.

Activity	Total teachers identifying	Mathematics	Students as learners	Teaching of mathematics
Index Card Task	17	11	12	5
Stacks of Paper Task	10	5	7	2
Fencing Task	12	7	9	7
Read The Case of Isabelle Olson	12	3	5	11
Discuss The Case of Isabelle Olson	9	3	5	6
Comparing Rabbit Pens Tasks	9	6	5	3
Building Storm Shelters	3	1	2	1
Comparing Triangles	4	2	2	0

These data show that the majority of teachers identified the key mathematical activities in and around this constellation as a source for learning. The extent to which teachers identified the activities and the categories under which they classified their learning varied; however, the fact that these activities constituted an opportunity to learn is clear, both from the content of the activities and teachers' own reflections on their learning. In fact, the Index Card Task was identified by more teachers than any other activity in the course as a source of their learning.

[15] Eight activities are listed in the table, as reading and discussing The Case of Isabelle Olson were distinguished as separate activities.

4.2.2.4. Thinking Through a Lesson. The Thinking Through a Lesson assignment asked teachers to engage with one of six mathematical tasks in small groups and plan a lesson for a middle grades classroom around the task. The assignment was designed primarily to assess teachers' knowledge of mathematics for student learning, and will be discussed further in section 4.3. However, all 6 tasks selected focused on some aspect of the relationships between dimension, perimeter, and area; thus provided an opportunity for teachers to write about and reflect upon these relationships. This meets the final criteria for a set of activities that provide an opportunity to learn.

Teachers did engage in solving the Thinking Through a Lesson tasks in small groups, and all groups produced multiple solutions to the task representing a variety of approaches, representations, and some reflecting possible misconceptions that students might have regarding the relationships between dimension, perimeter, and area. In addition, teachers were asked to write their mathematical goals for the lesson and describe how their enactment of the lesson would support those goals. Teachers in general were quite successful in writing goals that clearly articulated the mathematical relationships, were aligned with the task, and were supported by their description of the task enactment. Figure 22 shows The Atrium Task which was assigned to one group of teachers, and one teacher's articulation of the mathematical goals for the task.

The Atrium Task Adapted from Jones, Thornton, McGehe, & Colba, 1995 A local architect is working on an atrium design for a new hotel in town. Each room in the hotel is to open onto a walkway overlooking a central atrium area, which would be rectangular in shape. This design included a protective and decorative brass railing around the edges of the overlook. The architect is on a set budget, and the cost of brass is fairly high. She is restricted to using 650 feet of railing around each floor. Determine the dimensions of the railing so that the guests on each floor could enjoy the maximum area of scenic view of the atrium below.	**Goals articulated by Kelly:** My mathematical goals of the lesson are: 1) to understand that this real-life situation involves maximizing area when given a <u>fixed perimeter,</u> 2) to understand that a fixed perimeter can yield rectangles of many different dimensions and areas, 3) to understand that a square is the rectangle that yields the maximum area for any given perimeter, 4) to use a process that "proves" that all cases are considered when making a conclusion, as well as to make more secure the mathematical ideas listed above that are prerequisite skills with which students may still be struggling.

Figure 22. The Atrium Task and one teacher's goals on the Thinking Through a Lesson assignment.

Although the Thinking Through a Lesson tasks (work in small groups in class and the writing of the assignment) occurred outside the constellation structure, these tasks represented a culmination of the opportunity to learn knowledge of mathematics and mathematical activities with respect to dimension, perimeter, and area. In revisiting tasks that featured these relationships and engaging with them at a mathematical and a pedagogical level, teachers were offered the opportunity both to enhance and to reflect on the content knowledge in the domain and content knowledge for teaching that may have been developed through their course experiences.

<u>4.2.2.5. Summary</u>. The two constellations of activities around the two narrative cases, coupled with the opening course activities and the opportunity for reflection on the Thinking Through a Lesson assignment, represented significant opportunity to learn content knowledge related to dimension, perimeter, and area. In revisiting Table 12, the experiences described in the previous section link strongly to the results with respect to gains in learning about content knowledge in the domain and content knowledge for teaching. Teachers had opportunities to

engage with each of the 8 ideas in the table through individual work, small-group and whole-group discourse, and written reflection in the form of Learning Logs or larger assignments, and in doing so showed growth in their content knowledge in the domain and content knowledge for teaching.

4.2.3. Dimension, surface area, and volume: Growth in content knowledge

The activities related to knowledge of mathematics and mathematical activities in the second half of the course focused on the relationships between dimension, surface area, and volume, and included engaging in mathematical tasks, reading narrative cases of teaching, examining student work, reading related research articles, and writing to reflect on and extend the in-class conversations. The mathematical ideas discussed built in complexity from issues of the calculation of surface area and volume for simple cube buildings, to conceptual understandings of surface area and volume, and finally to relationships between dimension, surface area, and volume. Table 17 lists the specific aspects of the relationship between dimension, surface area, and volume measured in this study that were targeted in the highlighted course activities.

Table 17. Knowledge of mathematics and mathematical activities related to dimension, surface area, and volume addressed in the course.

Content knowledge in the domain	Content knowledge for teaching
Understand the relationship between dimension, surface area and volume, including that surface area and volume have a non-constant relationship	Identifying the big ideas in middle grades geometry and measurement related to dimension, perimeter, and area
Explain how changes to the dimensions of a 3-D figure (specifically a rectangular prism) impact surface area and volume	Identify and/or create mathematical tasks that provide students with opportunities to explore the big ideas in geometry and measurement
Link the concepts of surface area and volume to spatial structuring and the composition of a 3-D figure	Use a range of representations to explain the relationship between dimension, surface area, and volume
Demonstrate understanding of the meaning of surface area and volume using a variety of tools and representations	Identifying strategies for spatial structuring and tasks and pedagogical approaches that support the development of students' spatial structuring (includes use of volume formulas)

Four items on the pre- and post-course written assessment and one item on the pre- and post-course interview assessed aspects of *content knowledge in the domain* and *content knowledge for teaching* related to the relationships between measurable attributes of three-dimensional geometric figures. These items measured teachers' ability to articulate the non-constant relationship between surface area and volume and describe how changes in dimensions impact surface area and volume of three-dimensional figures, and are described below.

4.2.3.1. Performance on the *Painting the Living Room* task. The *Painting the Living Room* task (see Figure 23) presented teachers with a scenario in which two living rooms with the same floor space and ceiling height were to be painted. The task asked teachers assuming similar paint coverage, would the two rooms require the same amount of paint? In responding to the task, teachers had to recognize that even though the two rooms had the same floor area and height of walls, this did not guarantee the same area of the lateral walls. At the heart of this understanding is the notion that knowing the area of the base and the height of a rectangular prism does not imply knowing the surface area. The second question asked teachers if the volumes of the two rooms were the same; responding to this task correctly required an understanding that the area of the base and height of a rectangular prism does imply a specific volume. Teachers also had to justify their responses to both questions, affording the use of a variety of representations to do so.

> Jim and John are both painting their living rooms in their homes (walls only, not the floor or ceiling). They helped each other put new wood floors in the living rooms last summer, and they know that each floor has an area of 400 ft^2. The ceilings in both rooms are 8 ft high.
>
> a. Will Jim and John need to buy the same amount of paint? (Assume an equal number of coats and equal coverage per gallon.) Explain your answer.
>
> b. Do the living rooms have the same volume? Explain how you know.

Figure 23. The Painting the Living Room task.

The *Painting the Living Room* task measured both teachers' content knowledge in the domain and content knowledge for teaching. With respect to content knowledge in the domain, the task measured teachers' understandings of the non-constant relationship between dimension, surface area, and volume, and demonstrated an understanding of the meaning of surface area and volume using a variety of representations. With respect to content knowledge for teaching, the task measured teachers' use of multiple representations in explaining the relationship between dimension, surface area, and volume.

Questions 1 and 2 were coded using rubrics to differentiate correct and incorrect answers of various types. These categorizations were designed to evaluate whether teachers understood the non-constant relationship between surface area and volume and the quality and generality of the explanation. Responses were compared for change in each individual category and collapsed by Correct/Incorrect. The rubrics used are shown in Table 18.

Table 18. Surface Area-Volume Relationship Coding for *Painting the Living Room*.

Part 3a: Surface Area	
Correct-1	Correct response chosen, includes a correct explanation of why the amount of paint needed is not the same Clear evidence that two prisms with the same area of the base does not imply the same surface area Contains a clear statement that links the dimensions to the surface area
Correct-2/3	Correct response chosen, includes a correct explanation of why the amount of paint needed is not the same Clear evidence that two prisms with the same area of the base does not imply the same surface area May contain vague statement relating dimensions and surface area or a counterexample with no explanation
Correct-4	Correct response chosen, no explanation of why the amount of paint needed is not the same No evidence that two prisms with the same area of the base does not imply the same surface area
Incorrect-1	Incorrect response chosen Clear evidence of the misconception that two prisms with the same area of the base will have the same surface area
Incorrect-2	Incorrect response chosen No evidence of the misconception that two prisms with the same area of the base will have the same surface area
Vague/Inconclusive	Cannot be classified or response is incomplete
Part 3b: Volume	
Correct-1	Correct response chosen, includes a correct explanation of why the volume of both rooms is the same Clear evidence that two prisms with the same area of the base implies the same volume Contains a clear statement that links the dimensions to the volume
Correct-2/3	Correct response chosen, includes a correct explanation of why the volume of both rooms is the same Clear evidence that two prisms with the same area of the base implies the same volume May contain vague statement relating dimensions and volume or a calculated examples for each scenario with no explanation
Correct-4	Correct response chosen, with no explanation of why the volume is the same No evidence that two prisms with the same area of the base implies the same volume
Incorrect-1	Incorrect response chosen Clear evidence of the misconception that two prisms with the same area of the base may have different volumes
Incorrect-2	Incorrect response chosen No evidence of the misconception that two prisms with the same area of the base may have different volumes
Vague/Inconclusive	Cannot be classified or response is incomplete

In general, teachers were able to solve the task correctly, with no significant differences in the categories of responses. When collapsed across correct/incorrect responses, there were increases in correct answers (18 to 23 on task 3a; 21 to 24 on task 3b), but these trends were not significant. This indicates that most teachers understood that two rectangular prisms with identical base areas does not guarantee the same surface areas, but does guarantee the same volumes. There was no firm evidence that teachers held a misconception related to this relationship, mirroring the results from the *Fence in the Yard* task for two-dimensional figures.

Teachers' responses to the tasks were also examined for representational use – tables, written explanations, symbolic/formula, diagram, and graph – and for the use of multiple representations. There were no significant differences in the types of representations used from pre- to post-course assessment. Most teachers used multiple representations in responding to the task; this is not surprising, as diagrams are particularly useful in making sense of the relationships. The most popular representations on both pre- and post-course assessments were diagrams and written explanations. For task 3b, the volume question, teachers' formula use was tracked. Formula use is particularly salient for this question, as teachers' use of the $V=Bh$ form of the volume formula for a rectangular prism makes this task very straightforward. There was a significant increase in teachers' use of the $V=Bh$ formula from pre- to post-course assessment, $\chi^2(1, 50)$, = 4.37, $p = 0.037$. This difference suggests that teachers changed in the way in which they approached the volume portion of the task, increasing in their use of an alternative form of a formula which had particular relevance to the task. This may indicate an increased flexibility in teachers' conceptions of volume, particularly with respect to the symbolic representation of volume of a rectangular prism.

4.2.3.2. Performance on *Surface Area and Volume Additional Questions*. Following the painting task, three additional questions were asked that targeted the relationships between dimension, surface area, and volume of a rectangular prism (see Figure 24). These questions were designed to probe content knowledge in the domain; specifically, teachers' understandings of the quantities that contribute to surface area and volume, and how changes to the dimensions of a rectangular prism would impact surface area and volume.

c.	If you know the volume of a box (rectangular prism), can you find its surface area?
d.	If you have a box of known dimensions and volume, how would you create a new box with exactly double the volume?
e.	If you have a box of known surface area, how would you create a new box with exactly four times the surface area?

Figure 24. Surface Area and Volume Additional Questions

In general, teachers struggled with these tasks, with several unable to complete d and e on the pre- and post-course assessments. The only result of note relates to part c, designed to assess teachers' abilities to articulate the non-constant relationship between surface area and volume by describing what knowing each quantity does or does not tell you about the dimensions of the rectangular prism. The question also held the potential to reveal any misconceptions held by teachers; specifically, the notion that a rectangular prism with a particular volume has a fixed surface area.

Responses to the part c were coded in four primary categories, shown in Table 19. Incorrect responses were coded as either showing evidence of the misconception that knowing volume implies being able to find surface area, or lack of evidence. Correct responses were coded as either containing a clear and accurate explanation of why knowing volume did not imply knowing surface area (**Correct-1**), or as being correct without a clear explanation (**Correct-2**). There was a significant increase from pre- to post-course assessment in **Correct-1** responses, $\chi^2(1, 50) = 4.37$, $p = 0.037$. This difference suggests that teachers became more adept at describing the relationship between dimension, surface area, and volume at the end of the course, particularly with respect to the implications of knowing one quantity (volume) on knowing the others (dimension and surface area).

Table 19. Coding for Part c of *Surface Area and Volume Additional Questions*.

Correct-1	Indicates that the surface area is not known. Contains a clear explanation of why knowing volume does not imply knowing surface area that links the two quantities via the dimensions of the rectangular prism.
Correct-2	Indicates that the surface area is not known. Does not contain a clear explanation of why knowing volume does not imply knowing surface area that links the two quantities via dimensions.
Incorrect-1	Indicates the surface area is known. Shows clear evidence of the misconception that knowing the volume implies knowing the surface area.
Incorrect-2	Indicates the surface area is known. Does not show clear evidence of the misconception that knowing the volume implies knowing the surface area.
Vague/Inconclusive	Cannot be classified or response is incomplete

4.2.3.3. Performance on the *Considering Formula Use* Task. The *Considering Formula Use* task asked teachers to consider two formulas that can be used to find the area of a rectangle, and two formulas that can be used to find the volume of a rectangular prism, $V=lwh$ and $V=Bh$. Teachers were asked to select which formula they would use with a middle grades classroom, and to explain the reasons for their preference. Figure 25 shows the text of the task. The first question, related to the volume of a rectangular prism formulas, is analyzed here. One notable difference between the two formulas is that the $V=Bh$ version supports a layering conception of volume (e.g., Battista & Clements, 1998).

a. There are two common forms that textbooks use for the volume of a rectangular prism: *Volume* = *length* × *width* × *height* and *Volume* = *Area of base* × *height*
Is there a difference between the two formulas? If so, describe the difference.
Which would you choose to use with students, and why?

b. There are two common forms that textbooks use for the area of a rectangle:
Area = *length* × *width* and *Area* = *base* × *height*
Is there a difference between the two formulas? If so, describe the difference.
Which would you choose to use with students, and why?

Figure 25. *Considering Formula Use* task.

The item was designed to evaluate teachers' formula preferences and their reasons for holding those preferences. Teachers' formula preferences were tallied, and the reasons for their choice were coded across a variety of categories which emerged from examination of the data. Results showed a significant decrease in teacher preference for using the $V=lwh$ form of the volume formula exclusively, from 5 teachers on the pre-course assessment to 0 teachers on the post-course assessment, $\chi^2(1, 49) = 5.35$, $p = 0.05$ with Yates' correction. These 5 teachers either shifted to a preference for the $V=Bh$ form of the formula exclusively (1 teacher), or using both and helping students see the relationship between them (4 teachers). There were also significant changes in teachers' reasons for their stated formula preferences. There were significant increases in teachers stating that they preferred the $V=Bh$ formula because it was more general, meaning that it could be applied to any solid in the prism family ($\chi^2(1, 49) = 7.78$, $p = 0.005$), in teachers stating that the $V=Bh$ formula helps students visualize a rectangular prism ($\chi^2(1, 49) = 9.98$, $p = 0.002$), and a marginal increase in teachers stating that the $V=Bh$ formula specifically helped students develop a layering perspective with respect to volume ($\chi^2(1, 49) = 3.20$, $p = 0.07$). There was also a significant decrease in teachers selecting the $V=lwh$ formula because it was easier for students to calculate, $\chi^2(1, 49) = 5.35$, $p = 0.05$ with Yates' correction.

These results represent a significant shift in teachers' preferences towards the volume formula they would use with middle grades students. The shift away from the $V=lwh$ form, combined with the changes in reasons for selecting a formula, suggest that teachers came to understand and appreciate the affordances of the $V=Bh$ formula. Similar to the results from the area formula task, teachers cited more reasons for their formula choice that related to the underlying mathematical relationship after the course than they had prior to the course. The notions of the $V=Bh$ formula having a generality beyond rectangular prisms, along with the idea that the *V*=*Bh* formula could facilitate visualization and the use of layering strategies, represent key mathematical issues in fostering students' conceptions of volume. The decrease in the number of teachers citing ease of use as a reason for selecting the *V*=*lwh* formula represents a shift away from a more surface-level feature, that the formula was easier for students to identify the measurements that they needed to use, and only required a single step, as opposed to finding the area of the base, then multiplying by height. One teacher's response on the pre-course assessment emphasizes the procedural nature of this reason for selecting the *V*=*lwh* formula:

> The area of the base is length × width & therefore the 2nd formula skips a step in the process. This formula assumes that students already understand the process for finding the area of a 2-D shape & that the base of the 3-D shape is just a 2-D shape. The problem is if the student doesn't have this prior knowledge the latter formula might become more confusing for students. Even if the students do have prior knowledge the latter formula doesn't explain the process as much.
>
> Melinda, pre-course assessment

It is also interesting to note that following the course, 80% of the teachers who had favored the *V*=*lwh* formula on the pre-course assessment indicated that they would use both formulae, but show the relationship between them.

4.2.3.4. Performance on the *Big Ideas* Task. On the pre- and post-course written assessment, teachers were asked to identify the key ideas that middle grades students should learn related to three-dimensional shapes, area, and perimeter (see Appendix A, Part B, Task 2), measuring a key component of content needed for teaching. Teachers' responses were examined for commonalities, with a series of general categories emerging from the examination of teacher responses. The categories are shown and described in Table 20.

Table 20. Surface Area and Volume coding categories for *Identifying the Big Ideas*.

Relationship btwn SA&V – General	The relationship between surface area and volume but no additional explanation about the nature of this relationship or examples of this relationship.
Relationship btwn SA&V – Specific	The relationship between surface area and volume with a specific (and correct) aspect of this relationship is specified.
Calculate/find SA&V	Finding the surface area or volume using a formula, counting, estimating, or measuring. Does not necessarily imply an understanding of the meaning of surface area and volume
Use/apply SA&V	Using or applying surface area and volume in problems, real-world situations, or high-level tasks. May include applying formulas or calculating surface area or volume in the service of a context. Does not imply an understanding of the meaning of surface area and volume.
Understand SA&V conceptually	Understanding the meaning of surface area and volume from a conceptual standpoint. This may include responses such as "knowing what surface area and volume mean," "understanding surface area and volume," "concepts of surface area and volume."
Connection btwn 2-D & 3-D	Connection between 2-D and 3-D objects. Includes statement that there is a connection, more detailed elaboration of the connection, or surface area as two-dimensional area.
Names, characteristics of 3-D shapes	Knowing the names, characteristics, or properties of 3-D shapes, including knowing terms or names for shapes, classifying shapes, knowing core properties of classes of shapes, or distinguishing different 3-D shapes.
Understand square &/or cubic units	Understanding the difference between square and cubic units, including that surface area is measured in square units, volume is measured in cubic units, or how units relate to surface area and volume.
Memorize/use formulas	The use and/or memorization of formulas for surface area or volume, including understanding what elements of the formula stand for.
Generate, develop, or explain formulas	Creating or explaining formulas (formal symbolic or informal rules or methods) for surface area and volume based on understandings about what surface area and volume are conceptually. Does imply conceptual understanding of the basis for the formula
Surface area as wrapping/covering	The conceptual understanding of surface area as the wrapping or covering of a 3-D object.
Volume as filling	The conceptual understanding of volume as the filling of a 3-D object.
Volume as layering	The conceptual understanding of volume as layering or stacking, where the area of the base of a prism is visualized as being stacked or layered through the height of the prism.
Visualization/ spatial sense/ sketching	The development of visualization skills or spatial sense with students, or cites sketching of shapes as a way to develop visualization or spatial sense. This may also include creating models for the purpose of developing spatial sense or visualization skills.
Difference btwn SA&V	The difference between surface area and volume; reserved for statements that do not expand on what that difference is, or identifies a "relationship" between the two.
Find missing dimensions w/SA&V	Given a shape with one (or more) dimensions provided, one dimension missing, and the surface area or volume, finding the missing dimension.
Relationship btwn volume formulas	Understanding the difference and/or relationship between the two most common formulas for volume of a rectangular prism: $V = lwh$ and $V = Bh$.
Represent/ decompose SA using nets	Representing surface area using nets, decomposing 3-D objects into nets, or otherwise creating or thinking about surface area of 3-D objects using nets that build or cover the 3-D object
Use manipulatives/ build 3-D objects	Development of visualization skills or spatial sense with students, or cites sketching of shapes as a way to develop visualization or spatial sense, including creating models for developing spatial sense or visualization skills.
Use/understand diff. arrangements of SA, V	Use, create, and/or understand different arrangements of surface area and volume. Includes the idea that one can configure a 3-D object that contains smaller 3-D objects (e.g., a large box with smaller boxes inside, or a 12-pack of soda cans) in different ways that impact volume and surface area.
Other (specify)	

When teacher responses were coded using these categories, three significant changes were noted in teachers' responses to the task. On both the pre- and post-course assessment, teachers mentioned the relationship between surface area and volume as a big idea that student should learn; however, there was a shift in how teachers described the relationship. There was a significant increase in the number of teachers who talked about this relationship in a way that specifically articulated the nature of the relationship (e.g., one can have shapes with the same surface area and different volumes), $\chi^2(1, 50) = 7.01$, $p < 0.01$ on the post-course assessment. There was also a significant increase in the number of teachers identifying understanding surface area and volume conceptually as a key idea for middle grades students, $\chi^2(1, 50) = 4.37$, $p = 0.04$. Finally, fewer teachers cited knowing the names and characteristics of three-dimensional shapes on the post-course assessment, $\chi^2(1, 50) = 3.95$, $p = 0.047$.

4.2.3.5. Performance on *The Box Task*: Relating Surface Area and Volume. Teachers who engaged in the pre- and post-course interviews were asked to engage in a task in which they were to determine the number of rectangular 2 in^3 "packages" would fit in a box, and to determine the surface area of the box (see Figure 26; Appendix B, Task 1; Appendix C, Task 2). A successful performance on the volume portion of the task requires teachers to carefully consider how the rectangular packages might fill the box and the relationships between the dimensions of the package and the dimensions of the box. The rote application of the $V=lwh$ formula without consideration of the particular context would result in an erroneous answer. Additionally, this task was particularly conducive to a layering approach to making sense of the three-dimensional box. Teachers were pressed for as many different approaches to the task as they could provide; these performances were categorized both by correctness and type of approach.

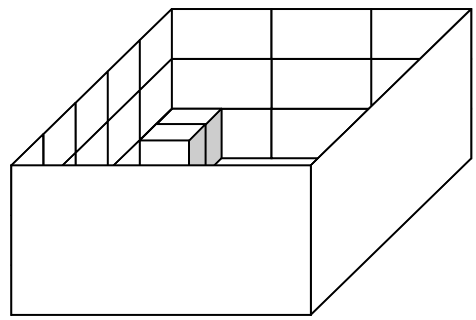


Figure 26. The unit cube box.

Transcripts of teachers' responses to this task were coded for a number of aspects of knowledge of mathematics and mathematical activities. With respect to content knowledge in the domain, approaches to the volume and surface area portions of the task were coded as correct or incorrect. Additionally, lines of interview text were coded for evidence of misconceptions related to the relationships between dimension, surface area, and volume. With respect to content knowledge for teaching, the number and types of strategies used were tallied and coded.

There were several significant changes in teachers' performances on this task between the first and second interview, beginning with correct/incorrect strategies. On the volume portion of the task, there was a significant increase in correct strategies on the second interview as compared to the first, $\chi^2(1, 139) = 6.09$, $p = 0.014$. Lines of interview text were coded for evidence of misconceptions on the task; there was a significant decrease in the number of lines of interview coded as showing a misconception on the second interview, $\chi^2(1, 1897) = 71.1$, $p < 0.001$. The number of lines coded as misconceptions dropped from 94 to 5, with only 1 teacher on the second interview showing evidence of the misconception as compared to 6 on the first interview. In the second interview, several teachers alluded to the possible misconception, with some realizing that they had approached the task incorrectly on the first interview.

On the surface area portion of the task, teachers could either calculate the surface area in square inches or use the faces of the packages as the unit of measure. Teachers were generally successful in correctly determining surface area using one of these two types of units. Teachers favored using square inches by approximately a 3:1 margin, with no significant change in the units used from first to second interview. There was a decrease in the number of incorrect responses across both types of units, but this difference was not significant.

When types of strategies were examined, there was a significant increase in the number of teachers using strategies that involved the blocks or the gridlines drawn on the inside of the box, $\chi^2(1, 40) = 10.2$, $p = 0.001$. This increase was across both the square inch and the package face calculations, but also held within the subgroup of teachers who found the surface area in square inches, $\chi^2(1, 40) = 4.91$, $p = 0.03$.

Following their experiences in the course, teachers appeared to exhibit a stronger conceptual understanding of volume as it relates to the box task. This is evidenced by the increase in the number of correct strategies used and the decrease in misconceptions related to the volume portion of the task. Teachers were in general able to successfully use a method to find the surface area. The changes in teacher performance on the surface area question notable for several reasons. The increase in strategies using the blocks or gridlines suggests that teachers came to see a utility for the manipulatives and/or the gridline representation in finding surface area. This suggests that teachers were better able to coordinate the three-dimensional nature of the blocks and the box with the two-dimensional nature of surface area on the second interview. Additionally, the increase in use of block or gridline strategies for teachers calculating in square inches was a bit surprising, neither the blocks nor the gridlines were in square inches; the face of the blocks and the gridlines were both 2 in^2 rectangles. The increase in teachers' abilities to use the blocks and gridlines to successfully arrive at an answer in square inches suggests that teachers not only coordinated the two- and three-dimensional aspects of the tasks, but were more willing to move flexibly between units and representations of surface area.

With respect to content knowledge for teaching, changes were noted in the number and type of strategies used to approach both parts of the box task. Teachers were pressed to solve the task in as many ways as they could, and had a variety of tools (blocks, ruler, transparent grid) at their disposal. The number of different strategies that teachers used were tallied and categorized by type of strategy. Additionally, interview transcripts were coded with respect to the number of lines of text in which teachers talked about particular strategies. There were a number of significant differences in the number and type of strategies that teachers used on the box task between the first and second interviews.

There was a significant increase in the average number of strategies used by teachers on the second interview, for both the volume ($t(19)=-1.78$, $p=0.04$) and the surface area ($t(19) = -2.27$, $p = 0.02$) portions of the task. The average number of solutions increased from 3.15 to 3.8 on the volume portion of the task, and from 2.25 to 2.95 on the surface area portion of the task. This indicates that following the course, teachers on average had more distinct strategies at their disposal to solve this non-routine task relating dimension, surface area, and volume.

The type of strategies used by teachers also changed from the first to the second interview. For the volume portion of the task, there was a significant increase in the number of layering strategies used by teachers on the task, $\chi^2(1, 139) = 7.77$, $p = 0.005$. Layering strategies which featured teachers using blocks also showed a significant increase on the second interview, $\chi^2(1, 139) = 4.96$, $p = 0.03$. When the interview transcripts were coded for the number of lines related to layering strategies, there was a significant increase in the percentage of lines related to layering, $t(19)=-6.00$, $p<0.001$, and in the number of lines of text, $\chi^2(1, 1897) = 117.2$, $p < 0.001$. This difference is striking; across 20 teachers, 277 of 1021 lines, or 27%, were coded as related to layering on the first interview, and 450 of 876 lines, or 51%, were coded as related to layering on the second interview.

These results suggest a number of interesting conclusions with respect to teachers' content knowledge for teaching related to dimension, surface area, and volume. The increase in the number of strategies for both the volume and surface area portions of the task suggests that teachers had a wider array of strategies available to them for use on a non-routine task that related dimension, surface area, and volume following the geometry and measurement course. Specifically, teachers showed growth in their ability to apply layering strategies to the volume portion of the task, a strategy particularly useful for this task in helping structure their work and avoid misconceptions.

<u>4.2.3.6. Summary</u>. Table 21 summarizes the results discussed in the previous sections, aligning the findings with the aspects of knowledge of mathematics and mathematical activities intended to be assessed in the geometry and measurement course.

Teachers in general entered the course with an understanding of the non-constant relationship between surface area and volume, as evidenced on the *Painting the Living Room* task. However, results from the *Surface Area & Volume Additional Questions* and the *Box Task* suggest that these understandings were limited and not easily applied to non-routine situations. Teachers did show an improvement in their ability to articulate the relationship between dimension, surface area, and volume for a rectangular prism (*Additional Questions)* and to articulate this relationship in a non-routine problem that required conceptual understandings of dimension, volume, and surface area (*Box Task*). These results suggest that in a non-routine situation, teachers' understandings were much more frail than initially suspected on the basis of their work on the written assessment.

Similar to the results for dimension, perimeter, and area, tasks that were more closely related to teaching showed greater changes. Changes in the big ideas teachers identified related to surface area, volume, and 3-D geometry mirror the growth in content knowledge in the domain. The fact that teachers created more specific descriptions of the relationships between dimension, surface area, and volume and cited conceptual understanding of surface area and volume as key ideas in middle grades geometry and measurement suggests that they may have developed a stronger understanding of and appreciation for these ideas and their role in understanding geometry and measurement. While there was no three-dimensional analog to the *Minimizing Perimeter Lesson Plan* task, teachers also were exposed to a range of tasks that made these mathematical ideas salient, allowing them to see how particular tasks brought to light the big ideas related to relationships between measurable quantities in three-dimensional figures.

Across several tasks, the notion of layering and the formula that best encapsulates that approach, $V=Bh$, were pivotal in teachers' learning. Teachers developed an increased preference for the $V=Bh$ formula as compared to the $V=lwh$ form of the formula following the course experience. Teachers also showed improvement in their ability to apply the $V=Bh$ formula to a situation that was particularly well-suited to the use of that formula, namely the volume portion of the painting task. The most stark change in teachers' practices related to layering was their performance on the non-routine box task. A layering approach is highly useful for the box task, as it helps teachers address the complex relationship between the dimensions of the non-square package and the dimensions of the box. Moreover, a layering approach sidesteps the misconception related to the rote application of the *V=lwh* formula to the volume portion of the box task, a misconception exhibited by a number of teachers on the first interview. The fact that teachers increased their use of layering strategies suggests that they saw a utility for the layering strategy in approaching the box problem. Some teachers even went so far as to flag the fact that the rote use of the *V=lwh* formula could cause a misconception during the second interview, and explained ways to use the formula in ways that compensated for the relationship between the dimensions of the package and the dimensions of the box:

> Nina: Ok. I don't know how to use that I don't know how I would use that [appears to be referring to grid] but. The first thing I did last time I blew it, is I just counted 3 across and 5 down,
> Int.: Ok.
> Nina: And I said oh! 15 along the bottom and 3 layers of 15. And now I realize that, it is—it would be 3 packages along side, because the dimensions of the package are different, it would actually be 10 along this side. So it would be, 30 that filled- er now. 3, 10 yeah. So it would be thhhhhhhhh- 30 that filled it, and 3 high so it would be 90 packages.
> Nina, Interview 2, Lines 176-183

In general, teachers held adequate understandings of surface area and volume to solve problems related to the two quantities, as indicated in the *Painting the Living Room* results. However, their conceptions may have been limited initially, as revealed by their early performance on the box task and their adherence to the *V=lwh* formula. The data from the *Additional Questions* and *Box Task* make a compelling argument that teachers acquired knowledge of mathematics and mathematical activities related to dimension, surface area, and volume as a result of the course.

Table 21. Knowledge of mathematics and mathematical activities related to dimension, surface area, and volume: Summary of results.

Content knowledge in the domain	**Findings**	**Tasks**	**Opportunity to Learn**
Understand the relationship between dimension, surface area and volume, including that surface area and volume have a non-constant relationship	Improved ability in relating dimension, surface area, and volume Decrease in misconceptions related to volume on non-routine task	*Painting the Living Room* *Surface Area & Volume Additional Questions* *Box Task*	Moon Gems/The Case of Keith Campbell activities Soda Can Task Wet Box Task
Explain how changes to the dimensions of a 3-D figure impact surface area and volume	No changes on pre/post course instruments	*Surface Area & Volume Additional Questions*	Soda Can Task Wet Box Task Large Numbers Lab
Link the concepts of surface area and volume to spatial structuring and the composition of a 3-D figure	Increase in correct strategies for finding volume and decrease in misconceptions related to volume on a non-routine task Increase in use of layering strategies	*Box Task*	Battista (2002)/Learning Log 3 The Wet Box Task Large Numbers Lab/The Case of Nancy Upshaw
Demonstrate understanding of the meaning of surface area and volume using a variety of tools and representations	Increased use in strategies related to blocks/gridlines to find surface area on a non-routine task	*Box Task*	All mathematical tasks Discussing Formulas
Content knowledge for teaching			
Identifying the big ideas in middle grades geometry and measurement related to dimension, perimeter, and area	Increase in big ideas: - specific description of relationship between dimension, surface area, and volume - conceptual understanding of surface area & volume Decrease in big ides: - names & characteristics of three-dimensional shapes	*Big Ideas*	All mathematical tasks Learning Log 4
Identify and/or create mathematical tasks that provide students with opportunities to explore the big ideas in geometry and measurement	No directly measured outcomes		All mathematical tasks Designing Packages Learning Log 4
Use a range of representations to explain the relationship between dimension, surface area, and volume	Increase in mean number of strategies used for volume and surface area on non-routine task	*Box Task*	Moon Gems/The Case of Keith Campbell activities Soda Can Task Wet Box Task
Identifying strategies for spatial structuring and tasks and pedagogical approaches that support the development of students' spatial structuring (includes use of volume formulas)	Increase in preference for $V=Bh$ formula Shift towards using both formulas and linking Increase in use of layering strategies on non-routine volume task	*Considering Formula Use* *Box Task*	Designing Packages Battista (2002)/Learning Log 3 Discussing Formulas

4.2.4. Dimension, surface area, and volume: Opportunities to Learn

In this section, the results discussed previously are linked to the opportunity to learn through selected excerpts from course discussions, interview data in which teachers described their learning, data from other written sources including written assignments and the instructor's planning diary. Table 21, presented previously, aligns the results of the analysis of written artifacts with particular activities that constituted opportunities for teachers to learn the knowledge described. This section describes the course activities that provided an opportunity to learn about dimension, surface area, and volume, and provides artifacts from discussions and written work that provide evidence of opportunities to learn the knowledge described in Table 21. Figure 27 highlights all course activities that related to dimension, surface area, and volume.

Based on the criteria identified in section 4.1.7, two constellations of activities provided teachers with opportunities to learn related to dimension, perimeter, and area. The first constellation, the set of activities around The Case of Keith Campbell, focused on issues of finding surface area and volume and developing a conceptual understanding of the two quantities and the relationships between them. The second constellation, the set of activities around The Case of Nancy Upshaw, addressed issues of measurement using cubic units and further developed the relationships between dimension, surface area, and volume. For both of these constellations, the discussion at the beginning of the course around the big ideas in geometry and measurement introduced the ideas into the public space, framing the opportunity to learn.

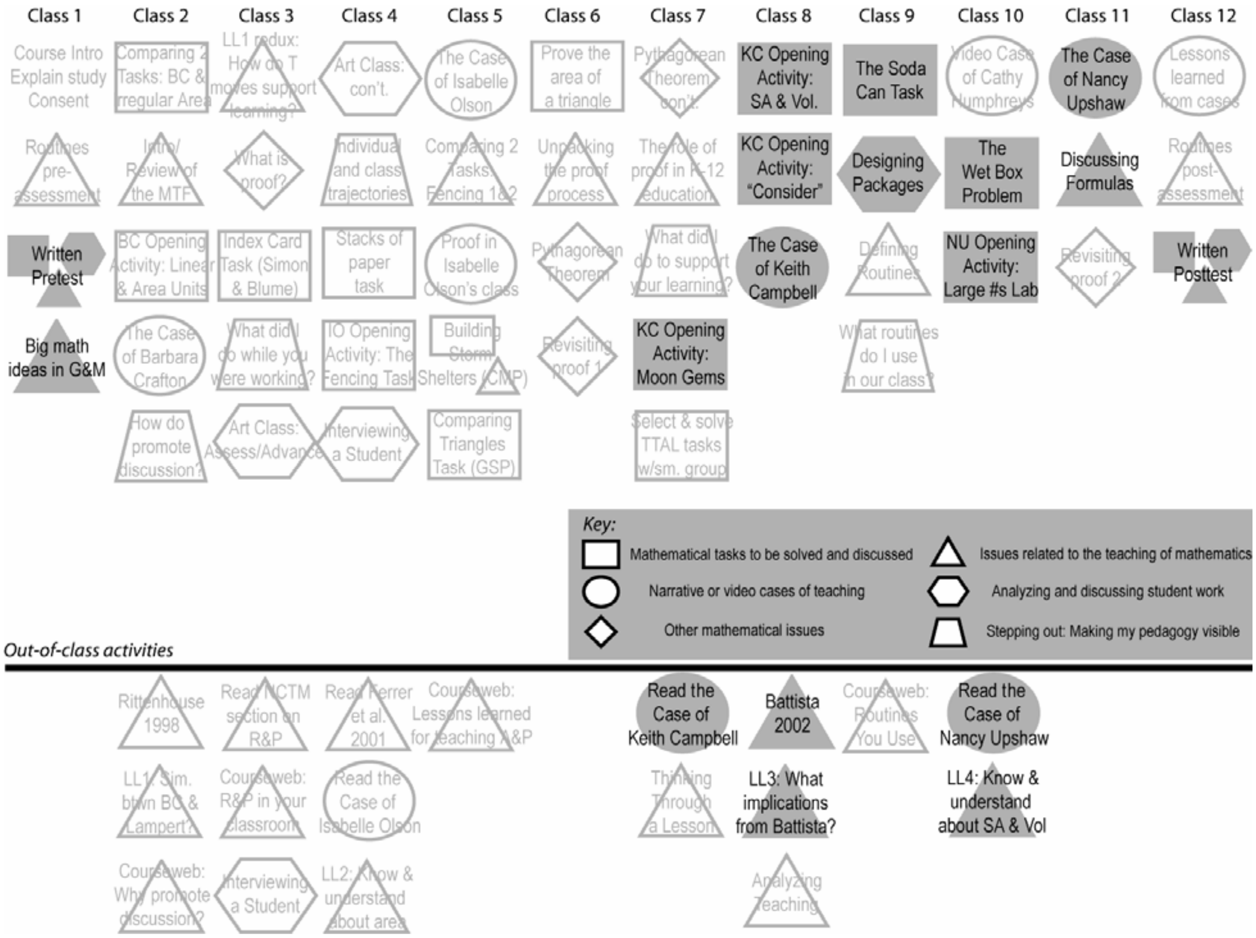


Figure 27. Course activities focused on the relationship between dimension, surface area, and volume.

<u>4.2.4.1. Course opening activities</u>. Two activities at the start of the course were important in setting the stage for teachers' engagement with activities that followed in the two constellations related to dimension, surface area, and volume. These two activities made public a set of issues for study related to the mathematical content, and represent the start of the sequence of activities that provided opportunities to learn. The first activity was teachers' work on the pre- and post-course assessments (see Appendix A for the complete text). By engaging with the assessment during the first course meeting, teachers were likely to have become attuned to the mathematical ideas that were to be the focal points for mathematical learning in the course. Following engagement in the pre- and post-course assessment, teachers were engaged in a discussion of what they thought the big ideas were in geometry and measurement in the middle

grades. (Note that this discussion mirrors Part B of the pre-course assessment.) Teachers identified a number of ideas related to dimension, surface area, and volume, as shown in Table 22 below.

Table 22. Big ideas identified related to dimension, surface area, and volume in Class 1.

Big Ideas Identified	Selected teacher talk	
3-D objects on 2-D planes (nets)	Noelle:	I think working with 3-dimensional objects, but working with them on a 2-dimensional plane, and manipulating them.
	Instr.:	Can you give me an example of that?
	Noelle:	Just like drawing a cube, or a rectangular prism, being able to see the drawing on paper, and understanding which part is the back and the sides and the front and the bottom. Where if you hold it in the air, they would actually be able to see that, and transfer that to the drawing on the paper.
	Instr.:	Why might that be important? Florence?
	Florence:	Because you want them to be able to, um like open it up make a net figure, to be able to understand what it is they're working with, to take the two dimensions to three dimensions, and if they can actually see those three dimensions when they're looking at it when it's drawn, then that gives them an idea of how to break it up into a net figure. And then they can make the relationships between what they know from a rectangle to how to work with a, rectangular prism.
2-D to 3-D connections	Uri:	I think it's kind of important that they figure out as far as three-dimensional objects how they can get that formula by using the knowledge that they have from two-dimensional objects.
	Instr.:	Can you say a little bit more about that? Can someone pick up on Uri's idea?
	Ed:	I was just going to say less emphasis on memorizing formulas and greater emphasis on actually understanding where they come from. I think if they understand it, you don't have to memorize it, and that's how I have my students think of it. There's some formulas you need to know, that you have to know, in your memory. But like she was doing on that video with the surface area, if you know what it's made up of, you don't have to memorize the formula.
Relationships and characteristics of shapes	Kelsey:	Just the idea that when a shape is defined as a parallelogram, you have these specific characteristics and knowing all those different relationships and characteristics, it just gives you so much information about, that knowing those characteristics helps you know those formulas and construct proofs and whatnot.

The content of the pre-course assessment, coupled with big ideas discussion, identified the key mathematical ideas for which teachers would be provided an opportunity to learn. The big mathematical ideas related to dimension, surface area, and volume that teachers identified were generally reflective of the key mathematical ideas that it was hoped teachers would learn, as reflected in Table 21, although not as well-developed as the entries related to dimension, perimeter, and area as shown in Table 13. The discussion about the big ideas in geometry and measurement afforded teachers the opportunity to consider mathematical ideas related to dimension, surface area, and volume at a general level, grounding the activities in the next two constellation of activities in prior knowledge and connecting them to issues of teaching. The content of the pre-course assessment, coupled with the big ideas teachers identified, suggest that teachers' engagement in these two opening tasks set the stage for the exploration of these mathematical ideas in the two constellations of activities that followed.

4.2.4.2. Constellation 3: Activities around The Case of Keith Campbell. The constellation of activities around The Case of Keith Campbell marked the transition point into dealing with issues of three-dimensional geometry. This constellation of activities pressed teachers to first consider the conceptual meaning of surface area and volume, and then explore a number of geometric situations grounded in real-world settings that made salient the relationships between dimension, surface area, and volume across a variety of contexts and representations. Seven activities in this constellation contributed to teachers' opportunities to learn. Teachers began by solving the *Moon Gems* task (the opening activity for The Case of Keith Campbell), read and discussed The Case of Keith Campbell, read the Battista (2002) article which discusses issues how students make sense of volume through spatial visualization, examined student work from the Designing Packages task solved the Soda Can Task, solved the Wet Box task, and reflected on this set of experiences in writing Learning Logs 3 and 4. Together with the making public of the key ideas in the opening course conversation, these activities in the constellation constitute an opportunity to learn ideas related to dimension, surface area, and volume.

The first activity in which teachers engaged that supported their learning about the relationships between dimension, surface area, and volume was solving the *Arranging Cubes* task. The task, shown in Figure 28, asks teachers to find as many different rectangular prisms as possible using 8, 9, 10, 11, and 12 cubes. Teachers then had to find the surface area and volume of each cube structure that they described, and considered how they knew they had found all arrangements, and to explain two formulas for surface area and volume, $SA=2lw + 2wh + 2lh$ and $V=lwh$.

<u>Solve.</u>
Find all of the ways that following fixed numbers of cubes can be arranged into rectangular prisms: 8, 9, 10, 11, and 12. For each fixed number of cubes, sketch the rectangular prisms you create, and record their dimensions, volume, and surface area. You may want to organize your information into a table.
<u>Consider</u>
1. For each fixed number of cubes, how do you know that you have found all the rectangular prisms that can be constructed?
2. Explain why the formulas $SA = 2lw + 2lh + 2wh$ and $V = l \times w \times h$ can be used to determine the surface area and volume (respectively) of a rectangular prism.
3. For each of the fixed number of cubes, compare the prism with the greatest surface area to the one with the least surface area. Make observations about the characteristics of these prisms that appear to affect their surface area. Would the observations you made continue to be true for any set of rectangular prisms that share a constant volume?

Adapted from Shroyer & Fitzgerald (1986)

Figure 28. The Arranging Cubes task.

The engagement in and discussion of this task was designed to bring the conceptual ideas of surface area and volume to the fore through a concrete model for developing understandings about surface area, volume, and their relationships to dimension, as reflected in the instructor's entry preceding Class 8:

> In returning to the Arranging Cubes task, I want to unpack how teachers arrived at their answers for surface area and volume, and on how the dimensions of the rectangular prism relate to the surface area and volume. I also want to find out if teachers have had additional insight into how to find the number of possible rectangular prisms when given a number of cubes. Finally, I hope to touch on the idea that a prism of a fixed volume can have a number of dimensions and thus, a number of different surface areas. Following this, I hope to make connections to more general ideas by discussing the "consider" questions.
>
> Course Planning Diary, Class 8 Pre-Class Entry, Lines 1418-1424

These relationships dominated the discussion during the first half of Class 8, with teachers being pressed to make connections between their methods of finding surface area and volume, the features of the prism, and general conceptual understandings of surface area and volume. These connections, and the press to make them, are particularly evident in this short excerpt from Class 8, in which Noelle describes how she found the surface area for a 4 by 2 by 1 rectangular prism.

Instr.: So Noelle was saying…
Noelle: You take- because she found the area of each side. For the 1 times 2, 1 times 2, 4 times 2 four times. So I just did 1- I just pulled the 2 out.
Instr.: So tell me the mathematical expression that you were-
Noelle: 2(1+ 1+ 4+ 4)
Instr.: [writing] Is that it? That's it? Alright. So where-
Noelle; That's not he whole surface area.
Instr.: Ok, so what is it.
Noelle: It's the area around. It's the sides – it's not the tops.
Instr.: Come show me so I've got it correct.
Noelle: [Points to four lateral sides]
Instr.: So when we think about…
Kelly: It's like she unfolded the sides, and found the area of one large rectangle.
Instr.: So if we unfold this, this is the front – let me put a box around this so we know it's the front. So there's the front, there are the two sides kind of like wings, let's put the back out here… so, now. Noelle, now that I've drawn this, can you tell me where the two 1s and two 4s came from?
Noelle: The 2 is the height of each face, and the 1 is the length of the two, sides.
Instr.: And the 4s?
Noelle: The 4s go to the front.
Instr.: So this is pretty complicated. How did you come about doing it this way? I know we already talked about the top and the bottom, we'll do that in a minute.
Noelle: Because I noticed that when we found the area of each side, that the height was the same, for each one and I could multiply- I did 2 times 1 plus 2 times 1 plus 2 times 4 plus 2 times 4.
Instr.: So the 2 stayed constant. Ok. So now let's deal with the top and bottom. What did you do next?
Noelle: So the top was 4 by 1, and the bottom.
Instr.: So you just did plus 4 times 1 plus 4 times 1, and this came out to be what?
Noelle: 28.
Instr.: 28. So that matches… does everybody understand how Noelle went about this?

Excerpt from Class 8

The excerpt above suggests that Noelle has a strong conceptual understanding of how to find the surface area of this prism built from cubes. Later, the instructor asked teachers if they had used the formula, and discussion centered on the issue of *using* the formula as compared to *understanding* it. Noelle chimed in with a startling revelation based on her previous work.

Noelle: Um, this is- like the opposite of it, I kind of have to use the formula. I have a really hard time doing that [visualizing} I can't see a three dimensions figure, cut it, and open it up. So if I know the formula, for every single prism, I can- I know I'm going to get it right. Or like I didn't know- I couldn't see that it made a rectangle. This is easier but if I couldn't see it, I might mess it up. It's kind of opposite of what you want to hear [laugh] But memorizing a formula kind of helps me.
(instructor asks Noelle to explain her formula again)

Noelle: So you're finding… the area of each side and adding it together. So when you find the area of each face-
Instr.: So why does that work for any prism?
Noelle: Because any prism has… I don't know if I can explain it. [Instructor draws a triangular prism to use as reference] The area of all the sides, the three sides, and add them together and add the area of the two bases.
Instr.: So say again the formula?
Noelle: Height, times the perimeter of the base, plus twice the area of the base.
Instructor writes $h(P_{base}) + 2(A_{base})$

Excerpt from Class 8

Noelle reveals in this excerpt that she relies on her memorization of the formula to find the surface area, and cannot visualize unfolding the 3-D object to understand how the formula finds the surface area. Following this excerpt, the class continued to walk through Noelle's formula again, this time with respect to the generic triangular prism the instructor had drawn on the screen. This made available for everyone, and Noelle in particular, another opportunity to connect the formula introduced by Noelle with a conceptual understanding of the surface area of a prism.

Following exploration of the *Arranging Cubes* task, teachers were asked to read and discuss The Case of Keith Campbell, a narrative case of a middle school teacher implementing the task with a group of students. A particularly salient feature in this case is the notion that Mr. Campbell steered his students towards a particular conceptualization of surface area and volume in order to arrive at a particular formula for volume of a rectangular prism, $V=l\times w\times h$. The implications of this move provided teachers an opportunity to begin to consider the mathematical affordances of different formulas, as well as the implications of moving students towards a specific understanding. In analyzing the case, teachers were asked to identify the mathematical ideas that students were working on, and moves that Mr. Campbell made that supported or inhibited their learning. Table 23 shows excerpts from the written record of the discussion that represent the mathematical ideas teachers identified, along with the pedagogical moves discussed that supported students' learning of the mathematical content and specific paragraph numbers that provide evidence.

Table 23. Math ideas and teacher moves identified in the discussion of The Case of Keith Campbell.

Mathematical Idea	**Pedagogical move that supported/inhibited student learning**	**Paragraph**
Justification/proof	How do you know you found all the packages? Pressed for a reason.	20
Finding all the combos for volume	Generalized from specific cases – didn't press to know why	30
Connecting student responses/ideas	List of observations to reference and build on Write in journal, think about relationship between surface area and volume (gives Keith Campbell more insight into student thinking, lets students reflect on what they did)	37
Defining and finding formula for volume	Couldn't rephrase A's explanation – turned to class… then said too much? "this means that…" Builds formula for observation Not clear what students understand about volume - $l \times w \times h$ might not be helpful? (limited to a particular context – how vs. why) How are you getting the volume? arranging/filling up/building up	29 31 28
Surface area	Task selection: visualization, put terms in context 12 blocks – had groups do other numbers Allowed students to explore in multiple ways – cubes, paper, drawing	4 24 25

The next activity related to content knowledge in the constellation was the reading of the Battista (2002) article, an online article that featured research findings related to students' conceptions of volume, along with interactive applets allowing teachers to engage with the tasks and the issues in the article. Learning Log 3 asked teachers to identify any implications for teaching from the Battista article. This provided teachers with an opportunity to reflect on the potential value added for a teacher to engage his or her students in examining multiple methods and representations for a particular mathematical concept; in this case, that of volume. The excerpts below from the Learning Logs of several teachers provide evidence that teachers were attending to issues of multiple methods and representations and ways to support students' spatial structuring.

> When I was reading the case of Keith Campbell, I paused to consider the implications of only using the formula to teach the concepts of surface area and volume. If this method is used to teach these concepts, a student most likely will only come out of this with the knowledge of what the

formula is and how to use it. This leaves the student with an incomplete understanding of the ideas of surface area and volume.
Learning this way does not require the students to use the four critical mental processes that are mentioned in the Battista article. Completing an activity like the one described in the article provides the students with the opportunity of seeing what volume really is. In this activity, students had to visualize how many blocks fit into a space. As a result of completing this task, students could have a real idea of what volume is. This activity allows the students to not only visualize volume, but it shows the students where the unit of measure comes from. If this idea is taught using the "formula only" method, then the students may not understand what cubic units represent. By assigning the students to figure out how many blocks can fit in a certain space, this makes the students think about the concept mentally before they can continue to solve the problem.

Excerpt from Learning Log 3, Daulton

Michael Battista's article shed a different light onto teaching surface area and volume than Keith Campbell's case study did. The idea that resonated most with me is when Battista writes, "…personal construction of meaning for mathematical ideas is something that happens internally in students' minds. As we teach, we cannot create constructions for students—we cannot even control the constructions. Students themselves construct these ideas as they actively manipulate objects and ideas…" That is an idea that Campbell and a lot of teachers have trouble totally believing. That concept makes teachers seem obsolete… Planned activities hopefully allow students to make conceptual connections with lesson in any subject, but Battista is arguing we as teachers have no control over our students' conceptualizations.

Excerpt from Learning Log 3, Bridget

Michael Battista believes that as teachers we cannot construct three dimensional figures for students, rather they must work to construct the figures themselves. Simply giving students an equation and asking them to find volume and surface area is not sufficient. Students need to be able to conceptually understand the figures they are working with. I believe that students using Battista's method will gain a better understanding of volume and surface area by asking them to consider many different cases like the activity provided in the article. Battista's approach also lends itself well to students creating formulas on their own that will make the investigation more meaningful as well as have them construct the shapes graphically to improve their spatial reasoning.

Excerpt from Learning Log 3, Barrett

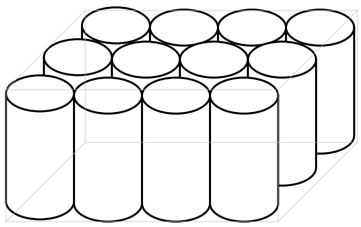
Having grappled with methods of understanding volume and the implications for student learning, teachers then were presented with a set of student work from the *Designing Packages* task in the Connected Mathematics Project (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998b). The *Designing Packages* task is similar in form and concept to the *Arranging Cubes* task. While this activity was designed primarily to build knowledge of mathematics for student learning related to selecting and sequencing student responses, the activity also provided teachers with an opportunity to consider how the ideas inherent in the *Arranging Cubes* task, in The Case of Keith Campbell, and in the Battista (2002) article did or did not play out in students' own work. This allowed teachers to make connections to their own practice in considering how they might support their own students' learning of ideas related to dimension, surface area, and volume, as evidenced by the connections made to classroom practices in the excerpts form Learning Log 3.

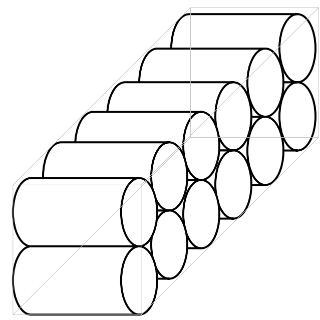
The final two tasks were designed to extend teachers' thinking about the relationships between dimension, surface area, and perimeter, the representations one might use to make sense of these relationships, and issues of spatial reasoning. These tasks were related to the constellation but not officially a part of it; they instead served as connective experiences between Constellations 3 and 4. Both tasks featured a real-world situation related to creating and optimizing packages with non-cubic contents – soda cans and rectangular boxes containing iPod[16] digital music players.

The Soda Can task is shown in Figure 29. In responding to the questions, teachers were asked to coordinate a number of understandings: how to represent surface area and volume in the context of the cans and cardboard box; calculating surface area for different configurations of cans; deciding on what assumptions were reasonable to make about the cylindrical cans; determining alternative configurations and the constraints and affordances of each; and making generalized arguments about the relationships between edge length, surface area, and volume. Teachers had a number of tools at their disposal to work on the task, including an example of each of the two types of soda can boxes brought by the instructor. These were intended to serve as illustrations to ensure that all teachers understood the context; however, the most compelling solution to the first part of the task made use of these boxes in an unanticipated way.

[16] iPod™ is a trademark of Apple Computer, Inc.

Historically, 12-oz soda cans have been sold in 12-packs packaged like the diagram shown below:





In 2001, the Coca-Cola Company unveiled a new design for a 12-pack, designed to fit into refrigerators more easily. A typical 12-oz. soda can is 6 in. tall, with a diameter of 3 in.

1. What is the difference in volume between the two types of 12-packs? Explain how you know.
2. What is the difference in surface area between the two types of 12-packs? (Ignore the flaps and overlaps that are needed to construct the box.) Explain how you know.
3. Considering your answers to 1 and 2, what do you think the pros and cons were for Coca-Cola in moving to the new refrigerator packs?
4. Are there any other ways to construct a 12-pack? Describe any ways you can find, and how the surface area and volume relate to the two other 12-packs.
5. What is the relationship between the edge lengths of the box, the volume of the box, and the surface area of the box?

Figure 29. The Soda Can task.

One small group, which featured an elementary, a middle school, and two high school teachers, asked the instructor if they would be able to have the “traditional” box and a pair of scissors. The instructor obliged. The group then deconstructed the box and reassembled it in the form of the “refrigerator pack” box. Given that the refrigerator pack required more surface area, there were two prominent gaps in their reconstruction of the box. This demonstration served to bolster their numeric calculations of surface area, and display in no uncertain terms to the rest of the class why the refrigerator pack requires 36 more square inches of surface area, also providing an illustration of where the difference comes from and how big in terms of the box the difference is. The excerpt below is from the group’s demonstration to the class.

Instr.: So, in terms of explaining where this difference of 36 square inches comes from, I mean Debra talked about the dimensions changing, we talked a little bit about that, correct me if I'm wrong, but my construction crew in the back has come up with a way of talking about that 36 a little bit, would you guys share that?

Bridget: Do you want us to share the weird yellow overhead or do you want her just to talk about it?

Instr.: Whichever way you think is gonna be best.

Ivy: Ok so, Bridget kept talking about how we could just cut the box and make it fit the other one so if we have our original box, and we think of it with the cans inside, and we cut it in half and move it like this, then we have our new box. And then you see that this part right here, where these two pieces are overlapping, which Bridget also- using these cans and how they were placed in here found out that these would be 6 by 6s if we had the dimensions on this paper. So this block here is a 6 by 6 and this block here is a 6 by 6. So if we cut those pieces and put them on this part that's empty, we found out that these would be 6 by 9 squares, using the dimensions of our can. So we would have these two 6 by 6s, and we could cut, and place where our 6 by 9s are but then we're still short, like for this side a 6 by 3 and this side a 6 by 3, and that's where we found our 36.

[Debbie holds up the cut-up box pieces and illustrates where the 6 by 6 pieces fill part of the 6 by 9 gaps]

Instr.: So Debbie's helped out, it's truly a team effort-

Ivy: So that's a 6 by 3 that's missing and that's the other 6 by 3 that's missing.

Excerpt, Class 9

This excerpt shows Bridget and Ivy making the links between the dimensions of the soda cans, the edge lengths of the two boxes, surface area, and volume across multiple representations. The sharing of alternative soda can box configurations in response to question 4 served to further explore these relationships, which were generalized in response to question 5, as illustrated in the following excerpt.

Instr.: So the last question on here, which I'd like to address briefly, is what is the relationship between the edge length of the box, the surface area of the box, and the volume of the box? What do you think? [long pause]

Uma: The tighter together the lengths are, the smaller the surface area. 6, 9, and 12 versus 6, 6, and 18, 6- 3, 6, and 72. the closer the numbers are together the smaller the surface area.

Instr.: Ok, other thoughts about that question? Debra?

Debra: The only thing I'd add to what Uma said was the volume always stays the same.

Instr.: The volume always stays the same.

Excerpt, Class 9

The Wet Box task attacked the relationships between dimension, surface area, and volume from a related direction, also integrating some of the misconceptions highlighted by Battista (2002) and in the Box Task on the interview protocol. The Wet Box task presented teachers with a scenario in which small rectangular boxes were to be placed in a larger box, and to minimize risk of shipping damage, the number of small boxes touching the outside of the larger shipping box. The text of the task is shown in Figure 30.

Part 1

A shipper is shipping large boxes of iPods from Shanghai. Each large box measures 36” × 18” × 16”. It is filled completely with iPods, which are in boxes that are 3” × 4” × 6”.

One box falls off the boat and submerges completely, but is pulled out of the water almost immediately.

Upon talking with Apple, the shipper concludes that all iPods that were touching the outside of the wet box will have to be returned to Apple to check if they still work. How many iPods will have to go back? Be sure to explain how you arrived at your answer, and use words, symbols, and/or diagrams to support your explanation.

Part 2

Upon hearing the bad news, Apple decides that they need to ship their iPods in a larger box. They want to design a box that has twice the volume of the original and that minimizes the number of iPods that would be damaged in a similar accident.

What are the dimensions of a box that serves this purpose?

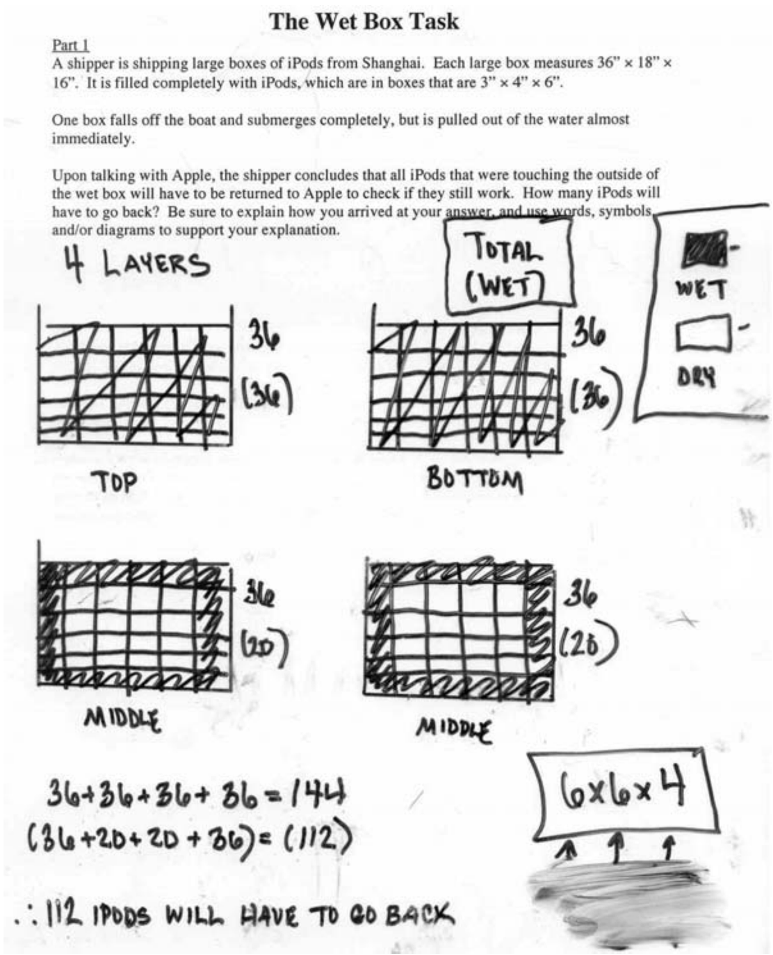
If Apple’s box supplier charges by the square inch of surface area, how much more will the new box cost?

Consider:

1. How does this task relate to the ideas of dimension, surface area, and volume?
2. What other mathematical ideas does this task address?

Figure 30. The Wet Box task.

Based on the parameters of the task, several arrangements of iPods were possible, with two possible correct solutions to the task. In arriving at their answers, teachers had to consider the linear dimensions of both the iPod box and the shipping container, and coordinate some form of two-dimensional representation of the three-dimensional rectangular prism – a move specifically identified in the literature (e.g., Battista, 1998) as challenging for students. Most groups approached this task through counting strategies – counting the number of iPod box faces that touched each face then compensating for double- and triple-counted iPods. However, one teacher identified a strategy that appropriated the layering ideas from Battista (2002) not for finding volume, but for coordinating the two- and three-dimensional representations. Cameron’s illustration and an excerpt from his explanation are shown below.



Cameron: Mine, I've done a 6 by 6 by 4, so I had 4 layers, and I was worried about the double-counting part-
Instr: Can you say what you mean by layers?
Cameron: Well, I figured out how many fit on the bottom, and then there was going to be- it was going to be a 6 by 6 so there was 36 iPods, that would fit on each of these. And then there's gonna be 4 layers that would fit on top of each other. So then I was talking to Noelle and we were getting confused about which ones we were taking out and which ones we were actually leaving in, so I figured if I drew the four out, the four layers out it would be easier to see so, the top and bottom are obviously gonna get wet, and then this ring around here are going to get wet, so, the ones in the middle are going to be the ones that are dry.
Instr.: So how does this relate to Barrett's approach? Uma?
Uma:: Barrett looked at the sides themselves, and it seems like Cameron's looking, almost top down.
Instr.: So you've got 36 for the top and bottom, and then how did you figure out the two middle rows?
Cameron: Well the outside part of it, because if you stack them on top of each other, this outside ring is still going to get wet, because this outside part is still going to touch the outside of the box, so this middle part here, this rectangle here is the only part that's covered.
Instr.: So Barrett mentioned that it's real easy to double-count things. So how are you sure that you didn't double-count in your solution?
Cameron: Because there's 144 total iPods there, and I have all 144 drawn out there. So each rectangle accounted for 1 iPod, so I just went in and shaded the ones that are going to be wet.
Instr.: So there's no chance that one of the rectangles you have on one of those diagrams is a side of another iPod that you've counted. Is everyone convinced?
[several unsuccessful attempts to justify]
Ivy: So there's 4 layers of iPods stacked, right? So he's taking one layer out, it's like he's laying out those four layers. So if one's drawn in one picture, then it can't be drawn in another picture, because they're different layers of iPods… whereas Barrett was looking at faces.

Figure 31. Cameron's solution.

Learning Log 4 offered teachers a culminating opportunity to reflect on their experiences with respect to dimension, surface area, and volume and consider how those ideas might play out in the classroom. Similar to the excerpts from Learning Log 3, teachers' writings in Learning

Log 4 showed significant attention to the notion of relationships between dimension, surface area, and volume, and issues of building spatial sense with students. The excerpts below in particular show that teachers' experiences in the course were influential in their thinking about the mathematical content with respect to their classrooms:

> In class, it took a lot of effort for some of us to think about the unseen faces of figures… Many of these teachers probably did not have experiences with nets prior to this class. Many of them said they never built models of geometric figures prior to the class.
> Students also need to examine, build, compose and decompose complex two- and three-dimensional shapes. They need to build three-dimensional shapes beginning with cubes and rectangular prisms and have opportunities to talk about the layers of the shape. They need opportunities to look at the base and build up from the base and realize that there are other ways to decompose the cube or rectangular prism and recompose it. According to Battista, this will be meaningful to students only if they have developed "properly structured mental models for the array" that is the base.
>
> Excerpt from Learning Log 4, Betsy

> I believe that students in the middle grade *must* have the proper tools to explore and experience the relationships between length, area, and volume. I think manipulatives are extremely important and contribute enormously to student understanding *and* retention. Battista indicates that Units-locating is important to student understanding. Units-locating "refers to locating individual cubes in a 3-D array by coordinating the spatial information from the three perpendicular directions that describe the array." Battista then goes on to discuss a large cube made up of many smaller cubes. I found it difficult in class to determine the surface area of the cargo container with the Ipods. [*sic*] I needed to use all of the blocks that were provided to us to set up the scenario and count all of the Ipods that were getting wet. I was able to do this by drawing the box and figuring out the surface area of each face, but I was not very confident in my answer until I actually constructed the cube and inspected each side of the container. The use of these tools allowed me to inspect the cube and understand that there would be some cubes that were counted more than once (the Ipods [*sic*] at the corners and along the edges).
>
> Excerpt from Learning Log 4, Nick

The work and discussion across these 7 tasks show that several of the ideas related to knowledge of mathematics and mathematical activities on which teachers showed growth were at the heart of teachers' work this constellation of activities. Teachers were able to explore the relationships between dimension, surface area, and volume in a number of different settings and using a variety of representations; were able to consider how the use and linking of a variety of solutions and representations might have potential to enhance learning of these ideas; and were able to consider specific ways of fostering spatial sense with students. The additional activities that served as the connective tissue for the constellation provided teachers opportunities to reflect and make connections to their own practice, enhancing the opportunity to learn.

Teachers viewed several of these 7 tasks as significant sources of their learning. Table 24 shows the number of teachers out of the 20 interviewed who identified each activity in the constellation as a significant source of learning. (Note that the Arranging Cubes task and discussion is represented in three rows due to the fact that the task was discussed across two classes in three distinct discussion episodes.) The table also disaggregates the data to show how many teachers saw the activity as a source of learning about mathematics, students as learners of mathematics, and the teaching of mathematics.

Table 24. Teacher learning data for Constellation 3 and connecting activities.

Activity	Total teachers identifying	Mathematics	Students as learners	Teaching of mathematics
Arranging Cubes: Solving	10	6	6	3
Arranging Cubes: Discuss SA & Vol.	12	7	7	1
Arranging Cubes: Consider ?s	7	4	5	0
Read The Case of Keith Campbell	7	1	4	5
Discuss The Case of Keith Campbell	11	4	8	4
Read Battista (2002)	9	3	5	3
Learning Log 3	6	3	3	3
Designing Packages	6	2	6	1
Learning Log 4	5	4	3	0
The Soda Can Task	15	10	9	3
The Wet Box Task	13	11	7	4

The activities in and around the constellation constituted an opportunity to learn, and the data in Table 24 support that notion in showing that the many teachers identified the activities in this constellation as a source for learning. The extent to which teachers identified the activities and the categories under which they classified their learning varied; however, the fact that these activities constituted an opportunity to learn is clear, both from the content of the activities and teachers' own reflections on their learning. In fact, the Soda Can Task the second-most identified activity in the course as a source of their learning.

4.2.4.3. Constellation 4: Activities around The Case of Nancy Upshaw. The constellation of activities around The Case of Nancy Upshaw afforded teachers the opportunity to continue to consider relationships between dimension, surface area, and volume in ways that integrated the understandings developed in the previous constellation. Due to its position at the end of the course and other course activities taking more time than anticipated, the number of activities in the constellation was small: engaging in the Large Numbers Lab task and discussing the task, reading and discussing The Case of Nancy Upshaw, discussing issues related to formulas for volume, and writing Learning Log 4 (also discussed in the previous constellation).

The Large Numbers Lab is a task that on the surface appears to deal primarily with issues of measurement and estimation, but provides deeper opportunities to consider issues of spatial structuring, of the affordances of different volume formulas, and the impact of using a three-dimensional object as a volume measurement tool. The instructor's course planning diary for Class 10 summarizes the issues he hoped to raise:

> I anticipate many groups measuring both the room and the blocks/tennis balls to make their determination. I'm hoping that some groups can describe a layering process, which will connect to our discussion of formulas in Class 11. I'm also hoping that some groups proceed simply by dividing the smaller area into the larger such that we can discuss what the implications for this might be in terms of partial blocks or tennis balls. Finally, I'm hoping that by comparing the two methods for estimating the volume, we come to some interesting discussion regarding the shape of the items relative to the shape of the room.
>
> Course Planning Diary, Class 10 Pre-Class Entry, Lines 1826-1832

The *Solve* portion of the Large Numbers Lab was intended to press teachers first to estimate how many cubes and balls might fit in the classroom, and then to use these tools (2-cm cubes and tennis balls were provided) to calculate the number of each object that would fit into the room. Teachers were pressed through the *Consider* questions to consider alternate methods for calculating the number of blocks and balls that would fit in the room, and to compare their methods. The final question asked teachers to consider the impact of a differently-sized ball – one with twice the diameter. This question was designed to press teachers to develop the relationship between the diameter of the ball and the length, width, and height of the room. The complete task is shown in Figure 32.

Solve.

1000 Cubes

Picture a big block that has been formed by making a 10×10×10 cube from 1000 small 2-cm cubes. Circle the number that you think would be closest to the number of big blocks it would take to fill up this room:

100 1000 10,000 100,000 1,000,000

Determine as accurately as you can how many big blocks are needed to fill up this room. Explain your method.

1000 Balls

Answer the following questions using the balls provided at your table. Circle the number that you think would be closest to the number of balls it would take to fill up this room:

100 1000 10,000 100,000 1,000,000

Determine as accurately as you can how many balls are needed to fill up this room. Explain your method.

Consider

Think about the methods you used to determine the number of blocks and the number of balls needed to fill the room. Are your two methods mathematically related? If so, how are they related? If not, how are they different?
Think about another way to determine the number of blocks or balls needed to fill the room. What are the mathematical similarities and differences between this method and the method(s) you had originally used?
Suppose we wanted to fill the room with balls that were twice the diameter of the ball that was originally used. How would this change the number of balls that would fit in the room? What if we used a ball that was half the diameter of the original ball?

Adapted from Hatfield (1994)

Figure 32. The Large Numbers Lab.

In the discussion of the task, a diverse range of methods surfaced for solving the Large Numbers Lab. Figure 33 shows the public written record of the methods shared an described by teachers for both the cubes question and the tennis ball question:

Cubes:

1. Found volume of cube, volume of room, divided.
2. Figure out how many cubes fit along each measurement.
3. Find bottom layer & find how many layers (maximum).
4. Find measurements, divide by 20 cm, truncate (drop the decimal) – remainder is empty space.

Balls:

Any of the above strategies

5. Make a "ball cube" and measure it

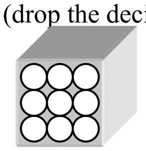
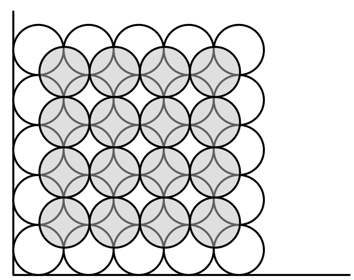


Figure 33. Solution Strategies for Large Numbers Lab.

This range of solutions represents a number of different understandings of the relationships between dimension and volume. Discussion of the different solutions flagged the first as potentially problematic, as it does not take into account partial cubes or balls, and assumes that cubes and balls can be packed exactly into the space. This assumption is valid for cubes and a rectangular prism room, but not for balls. Strategies 2 and 3 are tied closely to the size of the measurement unit, with strategy 3 bringing in the layering idea first encountered in the Battista article. Strategy 4 is an interesting way of blending the purely numerical strategy (1) and the tool-based strategies (2 and 3) in a way that helped to make sense of the issue of partial cubes or balls and the issue of empty space. While teachers agreed that all four strategies cited for the cubes would also work for the tennis balls, there were issues related to the spherical nature of the ball. One response to the issue was creating a 5th strategy, which essentially created a ball structure equivalent to the 20 × 20 × 20 cm cube. There was a great deal of discussion about layering being problematic with the balls. One teacher, Emily, suggested an adaptation to the layering approach to compensate for the spheres:

> Emily: I was thinking a little differently than that. I was finding how many tennis balls you could fit on the whole floor. And I found it to be approximately 144 going this way and 72 that way. And then if you think about laying the next layer of tennis balls in the crevices, there would be one less going this way, and one less going this way. So you could think about shifting them over one to fall into the crevices.
>
> Instr: [sketches] Ok so if we fill the bottom of the room and assume this keeps going, if these start sitting here, right? Now let me use a different color - you're gonna have, one less this way, and one less this way than you'd have, here, right? Is that accurate?
>
> 
>
> Emily: That's what I'm saying. And then I found how many would be in both of those layers, and then if I could take a ruler and just like set four down and put the fifth one in there and measure how tall that was, then that would be the new how many layers would fit in the room.
>
> Instr.: So instead of, when we did the cubes we talked about layers of cubes, right. So instead of just a layer of tennis balls, what Emily's talking about is a layer- kind of a full carpet and a nested layer, and then another full layer and nested layer, and accounting for those as a unit, and then moving up.
>
> Class 11, Whole-class discussion

These approaches to the task show teachers engaged in considering the relationships between the dimensions of the three-dimensional measuring tools (balls and blocks) and the dimensions of the room, making connections between the measurements, the manipulatives, and the methods of calculating volume.

Following their work on the Large Numbers Lab, teachers read and considered The Case of Nancy Upshaw, a narrative description of a middle school teacher implementing the Large Numbers Lab task. The focus for the discussion of the case was what students did to make sense of the task, and how Ms. Upshaw supported or inhibited students' efforts and ultimately their learning. A number of mathematical ideas were raised that extended the mathematical issues teachers grappled with during their exploration of the task. Table 25 lists teachers' responses that focused on key mathematical ideas related to dimension and volume.

Table 25. Mathematical ideas shared in the discussion of The Case of Nancy Upshaw.

Math explored by students	**What Nancy Upshaw did to support students' explorations**	**How the move supported student learning**
Jose, Lucia argued over whether cubic meant the block had to be a cube	Let the students discuss & verified Lucia was correct	Allowed the group to move w/o being stuck on one issue
Sts. used manipulatives (balls, cubes, rulers) to build models & estimate the # of blocks/balls that would fit	Had half the class work on balls lab while other half worked on blocks lab, then switched	More students had access to the manipulatives – didn't have to share w/as many others
Use estimated value of 25 cm.	Glad to see acceptable error	Concentrate on conceptual ideas
Students were reasoning through what their block of 1000 cubes would be	Let them struggle & asked them to sketch their thoughts. Redirected ?s back to group.	Got them to realize their mistakes & finally move forward with task

The table shows teachers identifying mathematical aspects of the task that students grappled with, and that were particularly problematic. These ideas were closely related to their own experiences in making mathematical progress on the task, allowing them to arrive at the range of solutions shown in Figure 33.

Following the case discussion, teachers engaged in a brief discussion of the two most common forms for the volume formula for a rectangular prism, $V=lwh$ and $V=Bh$. This discussion occurred after teachers had applied these formulas both formally and informally in the service of the *Arranging Cubes*, *Soda Can*, *Wet Box*, and *Large Numbers Lab* tasks. The issue of the formula was also relevant to their reading of *The Case of Nancy Upshaw*. Teachers were asked to compare the formulas with a particular eye towards how they might use the formulas in a middle grades classroom. Teachers identified a number of similarities and differences between the formulas, as shown in the public record of the discussion reproduced in Table 26.

Table 26. Public record of the Comparing Volume Formulas (*V*=*lwh* and *V*=*Bh*) discussion.

Similarities	**Differences**
You'll get the same answer	*Bh*: more conceptual/easier to visualize
Both require knowing *h*	*Bh:* extends to other figures
Both have cubic units	Multiplying 1D×1D×1D vs. 2D×1D
Both are related to "grouping"	*Bh* helps to explain visually (make sense of) *lwh*
	need to understand what's meant by *B*- stuck if you don't
	lwh reinforces cubic units
	Bh: layering/chunking (Battista)

Table 25 shows teachers connecting the issue of which formula to use to the conceptual aspects of volume and the measurable attributes of three-dimensional figures. In particular, the list of differences in the formulas identifies the key affordances of one formula over the other in relation to understandings of dimension and volume. The notions of visualization, listed first on the chart, and layering, listed last on the chart, provide a common mathematical strand through all the experiences in this constellation, connecting these issues with solving a task, teaching a lesson, and comparing mathematical formulae.

As mentioned earlier, Learning Log 4 offered teachers opportunities to reflect on their understandings from both the Keith Campbell and the Nancy Upshaw constellations. This final written assignment, allowing for individual writing and reflection, completes the set of activities that qualify this constellation as an opportunity to learn.

Although this constellation of activities which primarily targeted the relationships between dimension and volume was short, the activities were seen by a number of teachers as a source of learning, as shown in Table 27. The table also disaggregates the data to show how many teachers saw the activity as a source of learning about mathematics, students as learners of mathematics, and the teaching of mathematics.

Table 27. Teacher learning data for Constellation 4 activities.

Activity	Total teachers identifying	Mathematics	Students as learners	Teaching of mathematics
Solve the Large Numbers Lab	11	6	6	5
Discuss the Large Numbers Lab	8	3	6	4
Discuss The Case of Nancy Upshaw	8	1	4	4
Comparing Volume Formulas	8	3	6	1
Learning Log 4	5	4	3	0

4.2.4.3. Summary. The two constellations of activities around the two narrative cases, coupled with the opening course activities, represented significant opportunity to learn content knowledge related to dimension, perimeter, and area. In revisiting Table 20, the experiences described in the previous section link strongly to the results with respect to gains in learning about content knowledge in the domain and content knowledge for teaching. Teachers had opportunities to engage with each of the 8 ideas in the table through individual work, small-group and whole-group discourse, and written reflection in the form of Learning Log 4, and in doing so showed growth in their content knowledge in the domain and content knowledge for teaching.

4.2.5. Reasoning and Proof: Growth in Content Knowledge

Unlike the tasks related to measurable quantities of geometric figures, tasks related to proof were threaded throughout the course, with a concentration of work on proof falling between the two-dimensional and three-dimensional sections of the course. This set of activities did not feature a narrative case, but instead featured repeated conversations about the nature of proof, its role in the classroom, a task in which teachers created a proof, and one in which teachers evaluated proofs of the Pythagorean Theorem. In addition, teachers were asked to return to notions of proof following other mathematical activities that were designed to highlight different aspects of proof and its role in school mathematics. Table 28 lists the specific aspects of reasoning and proof measured in this study that were targeted in the highlighted course activities.

Table 28. Knowledge of mathematics and mathematical activities related to reasoning and proof addressed in the course.

Content knowledge in the domain	Content knowledge for teaching
Define proof	Explanatory power of proof
Identify proofs and non-proofs	Understand and articulate the role of R&P in school mathematics, including: logical thinking, communicating math, showing thinking, explaining an answer
Construct mathematical arguments	Identify discourse as a promising tool to support reasoning and proving
Understand the roles of proof in mathematics: Verify a stmt is true, explain why a stmt is true, communicate math knowl., create new math, systematize the domain	

One item on the pre- and post-course written assessment and one item (with several parts) on the pre- and post-course interview assessed aspects of content knowledge in the domain and content knowledge for teaching related to proof. These items measured teachers' ability to define proof, identify and construct proofs or proof-like mathematical arguments, and to consider the role of proof in the mathematical domain and in the K-12 classroom.

4.2.5.1. Performance on the *Defining Proof* interview task. The first of the series of activities related to proof in the pre- and post-course interview asked teachers several questions related to defining proof. (See Appendix B, Task 2a and Appendix C, Task 3a for the interview protocol relevant to defining proof.) Teachers' responses to the questions on proof were coded with respect to four categories, which represent four important aspects of proof as discussed in Chapter 4: generality, mathematical argument, establishes truth, and based on accepted mathematical facts. The number of lines of text in which teachers spoke to each of these four aspects of the definition was compared between first and second interview, as was the average percentage of teacher talk that related to the four aspects of the definition of proof.

On both first and second interview, most teachers discussed the definition of proof in ways that touched on several, if not all, of the four aspects. Thus, no significant differences were seen in the number of teachers addressing any of the four aspects of proof from first to second interview. However, there were notable changes in the depth and extent to which they discussed the four aspects of the definition of proof. In comparing the proportion of teacher talk devoted to each of the four aspects of the definition of proof, significant differences were found from first to second interview in the mean proportion of lines in which teachers discussed generality ($t(19) = -4.04$, $p < 0.001$) and based on accepted mathematical facts ($t(19) = -3.18$, $p = 0.002$). When the number of lines of interview text devoted to each of the four aspects were examined, there were significant increases from first to second interview in the number of lines in which teachers discussed generality ($\chi^2(1, 2444) = 49.6$, $p < 0.001$), mathematical argument ($\chi^2(1, 2444) = 16.7$, $p < 0.001$), establishing truth ($\chi^2(1, 2444) = 7.78$, $p = 0.05$), and based on accepted mathematical facts ($\chi^2(1, 2444) = 39.2$, $p < 0.001$).

These results suggest that the ways in which teachers discussed the four key aspects of the definition of proof changed between the first and second interview. The notion of proof as being general and as based on accepted mathematical facts appeared to become extremely salient to teachers on the second interview as compared to the first. The aspects of proof as being a mathematical argument and proof as establishing truth also increased in terms of the number of lines of text, but not in the mean proportion of talk. These disparate results may be due in part to the fact that some teachers did not talk about these two aspects at all in the second interview

(3 for mathematical argument, 5 for establishing truth), which lowered the overall mean for each category. This was not the case for the other two categories; all teachers discussed generality in the second interview, and all but one teacher discussed accepted mathematical facts.

Below are several examples of the same teachers talking about mathematical argument and establishing truth on the first and second interviews, illustrating the additional detail and certainty teachers exhibited in the second interview.

Mathematical Argument

Int.: So what does it mean to prove something?

LC: When you prove something you show, basically why it is that it works. So what makes this statement true, what makes your answer, in fact true. **Um, [pause] yeah it's sort of uh- it's a mathematical explanation for a given- for a- an universally accepted statement or, something- like a theorem or uhhhh** [pause] [laughs]

Lana, Interview 1, Lines 249-254, emphasis added

Int.: So what does proof mean to you?

LC: Ok proof to me means, [pause] you- you are forming, a- a mathematical argument to back up why something works, so you're using reasoning and, you're justifying, with words with symbols with pictures even, why something works, why something is true.

Lana, Interview 2, Lines 434-438

Establishing Truth

Int.: So the first question I have is, what does proof mean to you?

UT: **Um, [pause] I guess it's just um, it's basically just a way to, take something and, show that, it's true all the time.** Y'know, depending on what the situation is you can say, for this type of, triangle, y'know Pythagorean Theorem will work all the time. **And then, I guess it's just way of proving that it will- I mean, showing that it will work all the time. Y'know under certain circumstances I guess. [pause] Yeah.**

Uri, Interview 1, Lines 111-117, emphasis added

Int.: So what does proof mean to you?

UT: [pause] Hm. Um, proof means to me that, I guess it's just [pause] it's a way of taking an, an idea or ah, a statement, and showing that it'll work for, [pause] any circumstance that that idea discusses or entails. So, it, it's only limited to what the statement is actually, [pause] y'know involved in. But ah, basically yeah it's just, showing that that statement is true for any situation um, and will work for any situation. And is true for any situation um, as long as it, it's within the guidelines of that statement.

Int.: Ok. What does it mean to prove something.

UT: [pause] Um, basically same idea, to prove something means to show that that, [pause] um, that statement is true for, [pause] um, certain guidelines um say y'know if they say- if there's a statement, so I can prove that that'll work for anything, that statement is true for any situation, that um is in the guidelines discussed in the statement. [pause]

Uri, Interview 2, Lines 441-452

These excerpts demonstrate that in addition to the significant results regarding the amount of talk relating to the definition of proof, the quality of talk across the key aspects of the definition of proof improved from first to second interview.

4.2.5.2. Performance on the *Role of Proof* interview task. In addition to the questions regarding the definition of proof, the 20 teachers interviewed were asked about the role of proof. (See Appendix B, Task 2a and Appendix C, Task 3a for the interview protocol relevant to the role of proof in the mathematical domain, and Appendix B, Task 2e and Appendix C, Task 3e for the interview protocol relevant to the role of proof in the classroom.) Teachers were first asked about the role of proof in the mathematical domain (content knowledge in the domain), and about the role of proof in K-12 education (content knowledge for teaching).

Responses were coded using five categories from previous research by Knuth (2002a): verify truth, explain why, communicate knowledge, create new mathematics, and systematize the domain. McNemar's test was used to determine significant changes in the number of teachers mentioning or not mentioning each of the five roles of proof. In order to tease out more subtle changes, lines of interview transcript text were coded for each of the five roles of proof. Pared *t*-tests were used to compare the percentage of talk coded as each of the roles and chi-square analyses to compare number of lines of text.

In general, teachers tended to mention *verify truth* and *explain why* at a much greater rate than any other role for proof. All five of the roles of proof showed significant change between the first and second interview. Table 29 shows the results; the direction of the significant difference is shown in the second row, with subsequent rows indicating for which test or tests the difference was significant.

Table 29. Changes in teachers' conceptions of the role of proof in mathematics.

Role	Verify truth	Explain why	Communicate knowledge	Create new mathematics	Systematize the domain
Significant difference:	Increase	Decrease	Increase	Increase	Decrease
# of teachers (McNemar's)			‡		
Proportion t_{19}	-1.11	1.91†	-3.42‡	-3.20‡	1.42
# of lines $\chi^2_{(1,2444)}$	22.3*	33.8*	75.9*	20.8*	13.5*

$^{\dagger}p < 0.05$ $^{\ddagger}p < 0.01$ $^{*}p < 0.001$

These results show that there was a significant increase in teachers' attention to three aspects of the role of proof, verifying truth, communicating knowledge, and creating new mathematics. In the latter two cases, mentions of these two roles on the first interview were very low, with only 5 and 33 lines respectively coded for communicating knowledge and creating new mathematics. Thus, the increases in these two roles represent attention to them in the second interview where there was little evidence of such attention in the first interview. It is also interesting to note that the notions of communicating knowledge and creating new mathematics are particularly consonant with an inquiry-oriented, discourse-based class environment, a philosophy that was emphasized throughout the course.

The decreases in the roles of explaining why and systematizing the domain are challenging to interpret. In the case of systematizing the domain, the numbers of lines coded on both first and second interviews were extremely low: 20 and 3, respectively. This coding represented a minority of teachers; 3 teachers spoke to this role in the first interview, with only 1 teacher speaking to this role in the second interview. Similar to Knuth's (2002b) findings, the notion of systematizing the domain was in general not relevant for teachers. One possible explanation for the decrease in the *explain why* role relates to the class discussions on the nature and role of proof. There was great debate during class discussions as to whether explaining why was a necessary aspect of a proof, or merely a helpful feature that may or may not be included. (Additional detail regarding these conversations can be found in section 4.2.9.) The decrease in teachers mentioning explaining why as a role of proof may be due to a number of teachers deciding that the level of explanation was not a necessary feature of proof.

Teachers were also given an opportunity in the first and second interview to respond to 2 questions regarding the role of proof in the classroom. The first question focused on the role of proof in the classroom was asked at the close of the set of proof activities, when teachers were asked to determine whether they felt that proof should be a central idea in middle and high school mathematics. The second opportunity to discuss the role of proof in the classroom occurred when teachers were asked more generally about the role of proof in mathematics; often this question elicited responses about the role of proof in the classroom. Thus, responses to this question were also considered. Teacher talk related to the role of proof in the classroom was coded for seven aspects of the role of proof in the classroom. The first five – verify truth, explain why, communicate knowledge, create new mathematics, and systematize the domain – mirror the aspects of the role of proof in the mathematical domain adapted from Knuth (2002a). Two additional aspects – facilitating generalization with students and promoting discourse in the classroom – were also assessed. These two aspects were selected because of their salience within course discussions and their applicability in classroom settings. As with the previous analysis regarding the role of proof in the mathematical domain, three analyses were performed: McNemar's test to compare changes in the number of teachers mentioning or not mentioning a particular aspect, pairwise t-tests to compare the proportion of talk, and chi-square comparisons of total of number of lines coded as evidence of each of the seven conceptions. Due to the small numbers of lines devoted to the role of proof in the classroom, percentages for the t-tests were extremely small, with means and variances being unreliable. Results for chi-square and McNemar's tests are shown in Table 30.

Table 30. Changes in teachers' conceptions of the role of proof in the K-12 classroom.

Role	**Verify truth**	**Explain why**	**Communicate knowledge**	**Create new mathematics**	**Systematize the domain**	**General-ize**	**Discourse**
Significant difference:	No change	No change	Increase	Increase	Increase	Increase	Increase
# of teachers (McNemar's)				†			
# of lines $\chi^2_{(1,7890)}$	1.39	2.69	9.37‡**	1.42	25.9*	36.0*	9.37‡**

† $p < 0.05$ ‡ $p < 0.01$ * $p < 0.001$ ** *Yates' correction employed*

These results should be interpreted with caution, as teachers tended not to talk at length about the role of proof in the classroom. Across 20 teachers, only one category (*explain why*) resulted in more than 50 lines of coded text. However, the results do suggest a number of important conclusions. Teachers paid more attention in their talk to five aspects of the role of proof in the classroom: communicating knowledge, creating new mathematics, systematizing the domain, generalizing, and discourse. The results for communicating knowledge and creating new mathematics mirror the increases in teacher talk around the role of proof in the mathematical domain. This suggests that teachers viewed those two aspects of the role of proof as not only being relevant in the mathematical domain, but also in the classroom. Teachers also spoke more about the role of proof in supporting students' generalization skills and promoting discourse in the classroom. These two aspects stand in contrast to the typical positioning of proof in the classroom as a pencil-and-paper exercise that serves to verify already-known results. Moreover, the notion of generalization may suggest logical connections between geometry and measurement and other content strands, such as algebra, for which generalization is central.

The results with respect to systematizing the domain contrast with the results regarding the role of proof in the mathematical domain, which saw a decrease in teacher talk around this idea. One possible reason for this difference is that the notion of systematizing the domain was more relevant for the classroom. Several teachers in class discussions on proof focused on the idea of proof resting on previously proven mathematical facts accepted by the particular mathematical community for which the proof would serve. It is possible that teachers' increase in talk around systematizing the domain was related to the notion of flagging and structuring this set of facts for a mathematics classroom. Additional qualitative detail regarding these course conversations can be found in section 4.2.6.

4.2.5.3. Performance on the *Identifying Proofs and Non-proofs* interview task. The 20 interviewed teachers were asked to examine 8 explanations of mathematical conjectures, which varied in the degree to which they fulfilled the conditions for a proof. Teachers were asked to identify which explanations they thought were proofs and which were not proofs, to give reasons for their classifications, and to rate the proofs on a scale of 1 to 4, 1 being the least proof-like and 4 being the most proof-like (see Appendix B, Tasks 2b and 2c; Appendix C, Tasks 3b and 3c). Teachers' classifications, ratings, and reasons for both classification and rating were compiled, coded, and examined.

The number of teachers classifying each explanation correctly is shown below in Table 31. With respect to the classification of the explanations as proofs or non-proofs, no significant differences were found across the 20 teachers on any of the 8 explanations.

Table 31. Teachers' classifications of the 8 explanations.

Explanation	**1**	**2**	**3a**	**3b**	**3c**	**4**	**5**	**6**
Interview 1	19	10	8	17	8	12	8	15
Interview 2	18	14	8	19	6	15	6	17

Although there was no significant differences in teachers' classifications of proofs and non-proofs, the reasons teachers cited for their classifications shows insight into teachers' conceptions of the requirements for a proof. When these reasons were examined, a number of significant results emerged. Reasons for classification were coded using categories derived from Knuth (2002a): convincing argument, concrete features of the proof, familiarity with the proof, sufficient level of detail, shows why, uses a valid method, is sufficiently general, and teacher did a supporting example (see rubrics in Appendix E for extended descriptions). The eight codes were designed to identify the level of attention to the deep mathematical features of the proof, as well as identify possible misconceptions about proof in general. *Convincing argument* and *concrete features* are behaviors that rely on surface-level features rather than an in understanding of the mathematics and the underlying argument. The *did supporting example* code was designed to identify an underlying misconception about proof: that a proof was not a sufficiently general argument, and that more supporting examples served to bolster the proof's validity. The *sufficient level of detail* and *valid method* codes were designed to identify the depth of description and methods respectively that teachers did or did not think were necessary for an explanation to be classified as a proof. *Shows why* identifies a dimension of proof upon which explanations vary greatly and often depends on the mathematical knowledge that the reader brings; that is to say, a proof that shows why for a university mathematician may not show why for a university student. Finally, *sufficiently general* relates to the notion that a proof has a particular level of generality and holds for a class of relationships rather than just a single example or set of isolated cases.

Lines of interview text were coded for evidence of these categories, and the data were analyzed three ways: McNemar's test to determine any changes in the number of teachers either mentioning or not mentioning a particular category; paired *t*-tests for differences in the mean proportion of talk for each category; and chi-square analysis comparing raw numbers of coded lines between first and second interview. Significant results are shown in Table 32.

Table 32. Changes in teachers' rationale for classifying proofs and non-proofs.

Reason for Categorization	**Convincing Argument**	**Concrete Features**	**Familiarity with Proof**	**Sufficient Level of Detail**	**Shows Why**	**Valid Method**	**General**	**Did Example**
Significant difference:	Decrease	Decrease	Increase	No change	Increase	Increase	Increase	Decrease
# of teachers (McNemar's)	†							
Proportion t_{19}	1.73‡	4.63*	-1.82†	-0.23	-3.00†	-2.58†	-5.04†	0.90
# of lines $\chi^2_{(1,3008)}$	12.3*	37.5*	3.43	1.76	63.4*	39.6*	90.1*	4.02†

†$p < 0.05$ ‡$p = 0.05$ *$p < 0.001$

A number of interesting conclusions emerge from these data. First, the three categories in which there was an increase in either number of teachers and/or amount of teacher talk represented surface-level features of the proofs – convincing argument, a code for comments that did not back up their opinion with mathematical evidence; concrete features, such as the fact that the proof was laid out in two columns (indicated a proof) or the fact that the argument did not use symbols or variables (indicated a non-proof); and the use of a supporting example, which may provide a case that supports the proof but does not make the argument a proof (note that the special case of a counterexample was excluded in this coding). Three of the four categories that showed a significant increase in talk represent key conceptual facets of proof: the idea that the proof should show why a mathematical statement is true, should use a valid mathematical method, and should be general. Note that these echo three of the four aspects of the definition of proof mentioned in the previous section. The final increase, familiarity with the proof, is likely due to the fact that teachers examined the same set of explanations in both interviews, and that Explanation 6 was featured and discussed during the course. In general, these data suggest that in justifying their classification of the explanations as proofs or non-proofs, teachers became less reliant on surface-level featured, and instead increased in their consideration of aspects related to the mathematical definition of proof to classify the explanations.

In rating the explanations, similar trends were observed. Teachers' 1 (least proof-like) to 4 (most proof-like) ratings of the explanations were tallied and the average ratings compared using paired *t*-tests. The only explanation that showed a significant difference was Explanation 3a, which exhibited a significant increase in rating, from 2.325 to 2.8, $t(19)$ -2.20, $p = 0.02$. This is particularly notable, as Explanation 3a is a collection of examples and not a proof (see Appendix B), yet teachers came to see this explanation as more proof-like.

When examining the rationale for rating the proofs, the same eight categories were used. Results were remarkably similar to those for identifying proofs and non-proofs, and are shown in Table 33.

Table 33. Changes in teachers' rationale for rating the explanations.

Reason for Categorization	**Convincing Argument**	**Concrete Features**	**Familiarity with Proof**	**Sufficient Level of Detail**	**Shows Why**	**Valid Method**	**General**	**Did Example**
Significant difference:	Decrease	Decrease	No change	Increase	Increase	No change	Increase	Decrease
# of teachers (McNemar's)	†							
Proportion t_{19}	2.49†	2.98†	-0.91	-1.63	-1.94†	-0.64	-2.55‡	-1.51
# of lines $\chi^2_{(1,1374)}$	18.5*	32.6*	1.50	14.0*	18.9*	1.99	11.9*	7.00†

†$p < 0.05$ ‡$p < 0.01$ *$p < 0.001$

As with the results for classifying proofs, the categories in which a significant decrease was exhibited represent surface-level features of the explanations, whereas the three categories in which a significant increase was shown represent aspects of the explanation that are mathematical and well-connected to the key characteristics of proof. In this case, there was no change in valid method, but there was a change in sufficient level of detail.

4.2.5.4. Performance on the *Reasoning and Proof* task. On the pre- and post-course written assessments, teachers were asked to create proofs of the formula for area of a triangle and the area of a parallelogram. These proofs were chosen because the shapes and formulae would be accessible to most teachers, even those whose mathematical backgrounds were not particularly strong with respect to geometry. Moreover, teachers could make progress towards proving the validity of the formulas in a variety of ways, including through the use of diagrams. Teachers' mathematical arguments were rated on a 7-point scale, from 0 to 6, with 6 being highest. The rubric (see Appendix E) assessed the completeness, explanatory power, and generality of the explanation. Scores on the pre- and post-course assessment were examined for change using the Wilcoxon sign-rank test and paired *t*-tests for differences in mean scores, and the number of teachers scoring in particular categories were examined for change using chi-square analysis.

On the area of a parallelogram task, there were no significant differences in teachers' scores from pre-course to post-course assessment. Mean scores showed a modest but insignificant change, from 2.96 to 3.08. This mean score, combined with the fact that only 1 teacher scored at the top-level rating on the pre-course assessment, indicate that there was indeed room for improvement. The Wilcoxon sign-rank test showed no significant change in teachers' scores ($W = -9$, $n_{s/r} = -15$, $Z = -0.24$, $p = 0.4$). Chi-square analyses showed a significant change in the number of teachers scoring in the 3 category ($\chi^2(1, 50) = 8.01$, $p < 0.01$), which was the lowest score that indicated a complete mathematical argument. These data suggest that teachers towards the lower end of performance on the pre-course assessment made some progress towards producing complete explanations, but teachers who scored on the higher end of the rubric in the pre-course assessment did not make significant progress towards producing more complete proofs on the post-course assessment.

The area of a triangle task produced very different results. Mean scores showed a significant change[17], $t(23) = -3.86$, $p < 0.001$, from 2.84 to 4. The Wilcoxon sign-rank test also showed significant increases in teachers' scores ($W = -125$, $n_{s/r} = 17$, $Z = -2.95$, $p = 0.002$). Individual were too widely distributed across the 7 score points to draw conclusions using the chi-square; however, two trends are notable. First, the number of explanations rated as 6 – complete and general proofs with high explanatory power – increased from 1 to 5 between pre- and post-course assessments. On the other end of the spectrum, the number of explanations rated as 0 decreased from 3 to 0, and the number of explanations rated 1 decreased from 3 to 1. Taken together with the increase in basic complete mathematical arguments with respect to the area of a parallelogram task, these data suggest that teachers grew in their ability to construct a proof or proof-like mathematical argument between the pre- and post-course assessments.

4.2.5.5. Summary. Teachers showed significant growth across the aspects of content knowledge in the domain related to reasoning and proof: defining proof, articulating the role of proof, identifying and rating proofs and non-proofs, and creating proofs and mathematical arguments. With respect to content knowledge for teaching, teachers grew in understanding of the explanatory power of proof, as evidenced the criteria used to evaluate proofs; in their abilities to create a proof or more proof-like deductive argument; and in their conceptions of the role of proof both in the mathematical domain and in the classroom. Table 34 summarizes findings related to reasoning and proof, matching the goals for teacher learning with the results and

[17] One teacher's pre- and post-course responses were dropped, as the teacher was not feeling well and opted not to complete the area of a triangle proof.

opportunities to learn. In the section that follows, the opportunities to learn about reasoning and proof that were provided in the course are described.

Table 34. Knowledge of mathematics and mathematical activities related to reasoning and proof: Summary of results.

Content knowledge in the domain	**Findings**	**Tasks**	**Opportunity to Learn**
Define proof	Increases in talk about all 4 key aspects of proof	*Interview: Defining Proof*	Defining Proof & Revisiting
Identify proofs and non-proofs	Decreases in use of surface-level features to identify proofs and non-proofs Increase in mathematically-related features to identify proofs and non-proofs	*Interview: Classifying and Rating Explanations*	Considering proofs of the Pythagorean Theorem
Construct mathematical arguments	Increase in proofs and proof-like arguments	*Reasoning and Proof Task*	Prove area of a triangle Unpack the proof process
Understand the roles of proof in mathematics	Increase in verify truth, communicate knowledge, and create new math; decrease in explain why and systematize the domain	*Interview: Proof Questions*	Read NCTM section on reasoning and proof
Content knowledge for teaching			
Explanatory power of proof	Increase in talk about proof as an explanatory tool; increase	*Interview: Proof Questions*	Defining Proof and Revisiting Role of proof in K-12 education
Understand and articulate the role of R&P in school mathematics	Increases in talk about communicating knowledge, creating new mathematics, systematizing the domain (logical thinking), generalization	*Interview: Proof Questions*	Proof in Isabelle Olson's class The Case of Keith Campbell Read NCTM section on reasoning and proof Courseweb posting
Identify discourse as a promising tool to support reasoning and proving	Increase in talk about the role of discourse	*Interview: Proof Questions*	Role of proof in K-12 education What did I do to support your learning?

4.2.6. Reasoning and Proof: Opportunities to Learn

In this section, the activities that were most likely to have contributed to teachers' learning about reasoning and proof are described. As noted earlier, opportunities to learn reasoning and proof were not grouped in constellations, but rather threaded through the course and revisited after key activities that may have caused teachers to change their thinking about proof. All course activities related to proof are shown in Figure 34. The content of the activities are described in clusters below, with artifacts from the course meetings to ground the descriptions and justify their roles as opportunities for teacher learning.

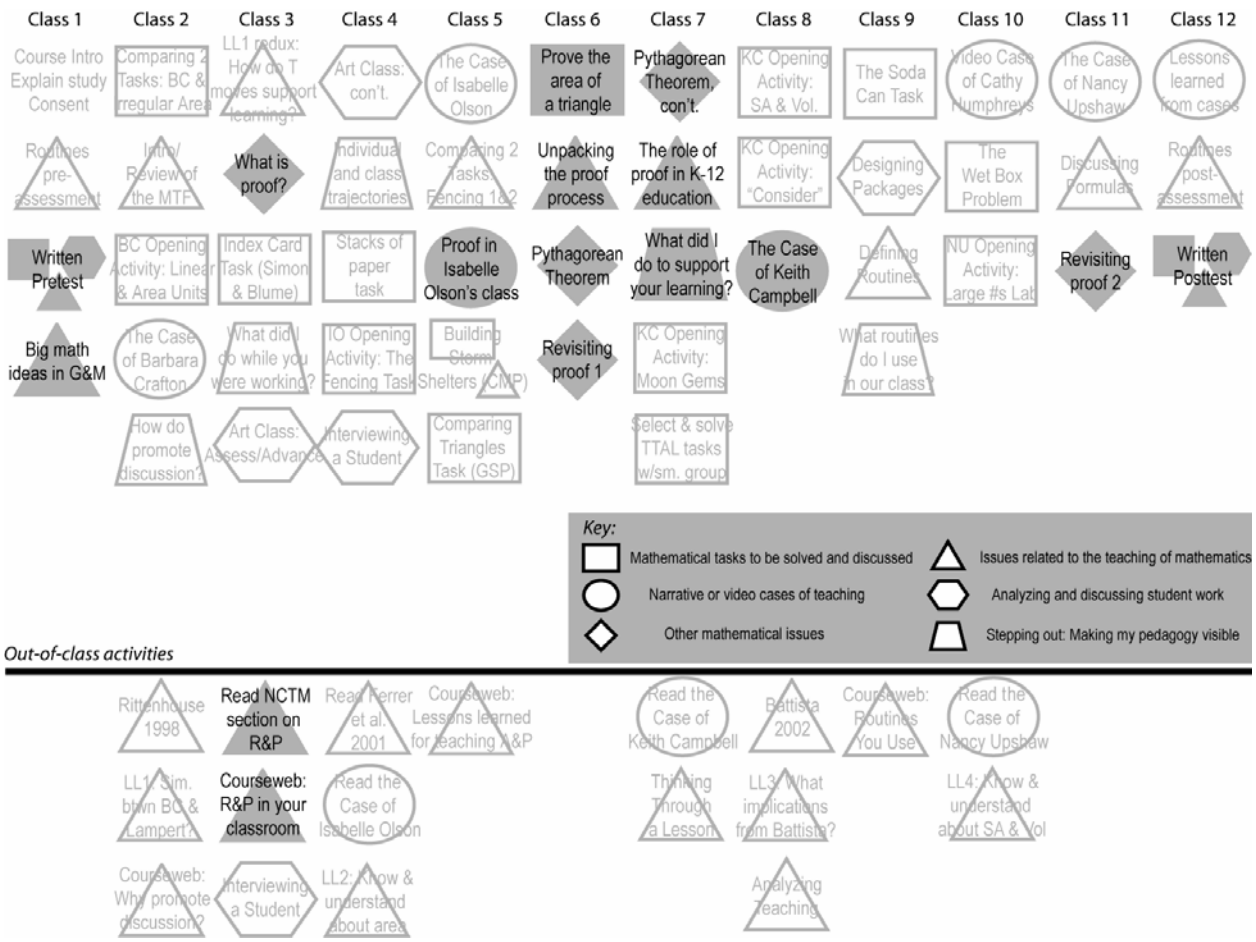


Figure 34. Course activities related to reasoning and proof.

Based on the criteria for opportunity to learn described in section 4.1.7, a subset of the activities in Figure 34, taken together, represented an opportunity to learn with respect to the learning goals articulated in Table 34. This set of activities included the opening course activities and a discussion about the big ideas in middle grades geometry and measurement, discussions regarding the definition of proof, discussions regarding the role of proof in mathematics and in the classroom (including selected case discussions), and activities related to creating and evaluating proofs. Opportunities for teachers to reflect on their experiences in writing included the Courseweb posting in Class 3 and the Unpacking the Proof Process activity in Class 6.

4.2.6.1. Course opening activities. Similar to the cases of learning with respect to dimension, perimeter, and area and dimension, surface area, and volume, the activities at the start of the course marked the start of the opportunities to learn about reasoning and proof by making the key ideas related to reasoning and proof public. The first activity was teachers' work on the pre- and post-course assessments (see Appendix A for the complete text). By engaging with the assessment (specifically Part B, Item 3 and Part C, Item 4) during the first course meeting, teachers were likely to have become attuned to the mathematical ideas related to reasoning and proof that were to be the focal points for mathematical learning in the course. Following engagement in the pre- and post-course assessment, teachers participated in a discussion of what they thought the big ideas were in geometry and measurement in the middle grades. At the close of the discussion, teachers identified a number of ideas related to reasoning and proof, as shown in Table 35.

At the close of the discussion, the instructor flagged the idea of proof as one that would be returned to throughout the course, and noted that the idea of proof had been discussed, what was meant by proof had not been clearly defined. The instructor wrote, "What is proof?" in large letters on the bottom of the chart paper of the big ideas in geometry and measurement and posted the chart prominently for the duration of the course, setting teachers up to anticipate substantive work on the idea of proof.

Table 35. Big ideas related to reasoning and proof identified in Class 1.

Big Ideas Identified	Selected teacher talk
Ideas leading to proof Informal/less formal proof Justifying a claim Using reasoning	Kelsey: Just the idea that when a shape is defined as a parallelogram, you have these specific characteristics and knowing all those different relationships and characteristics, it just gives you so much information about, that knowing those characteristics helps you know those formulas and construct proofs and whatnot. Instr.: Well Kelsey just used a word, that I hadn't heard anyone talk about. She said the properties that you could then use in proof. What about proof, does proof belong on this list? I heard some people saying yes, and I peeked at some tests and I know some people might disagree with that. Who'd like to offer an opinion? Ivy, go ahead.

Ivy: One of the things that goes back to a few things that were already said, understanding what things mean, like if you want to talk about volume, understanding what volume means, like how many things fit inside this. Then using that to build a formula, I think it leads to reasoning through proofs because you reason through what the formula means, you're not just given a formula. So I think the same things that are used to find a formula are the same things used to prove in geometry…

Instr.: Noelle?

Noelle: I think along those lines, justifying your reasoning as you're developing your proof also leads to proof. So you're saying you know this and this is why, so it leads into the forms of proof. So you know which order you have to have, how to order your thinking also leads to proofs.

Instr.: Ed?

Ed: I think on a lower school level, not in a set geometry class in the high school, they need to do proof but more like on an informal basis, kind of get the idea behind it. I don't think a lot of – I'm not going to blame my middle school teachers, but I know in my classes, we get into a little bit of proof and more formalized proof and students have a lot of trouble trying to prove things. I think if they got a little more exposure to it on a less formal basis on a lower school level, just knowing ok I have to show a reason why this is true, it's not just true, I think that would help them succeed in a more formalized geometry class.

…

Instr.: Sierra?

Sierra: Do you mean like a formal proof?

Instr.: That's a good question.

Sierra: Like do you mean, if they say, this is my claim, this is how I find it, this is an example, is that ok, is that a proof, or do they have to do a step-by-step proof at the middle school level?

Instr.: That's a good question. Florence?

Florence: I think cognitively, at the middle school level, that is a proof. This is my claim this is why it works. I don't think it's cognitively appropriate to expect them to write the kind of proof that a high school student or a college student is writing, like a two-column proof. I mean, I have 8th graders I have to do it but, when they're first being introduced to it, I think when they're being introduced to it, that's an appropriate way to do it. And just that it's not broken down into two columns with a claim and justification and anything, doesn't mean it's not a proof.

Instr.: Ed?

Ed: I was just going to add to that, I think they should know the difference between what proves it in all cases, and what proves it for just one example, like the difference between deductive and inductive reasoning. I think that's important that they know, what is a proof and what is just I looked at a bunch of examples. Because you can look at a thousand examples, but if you miss that one that's a counterexample, it's not a proof.

Noelle: I think you have to start with the general proof before you get to the specific proof, because in my experience, I just went into writing proofs, and I can't write proof and my kids can't either, because they just memorize the proof of how to prove, like two lines are parallel. So if they forget it, they can't reason through it, so if you just give them two-column proofs, they're not going to get that reasoning experience.

4.2.6.2. Defining and revisiting proof. Consistent with Design Principle 5, the instructor had intended to provide teachers with an opportunity to define proof early on in the course, and then return to the definition periodically so that teachers could consider revising it based on subsequent mathematical experiences. During Class 3, the issue of proof was raised by the instructor in the context of The Case of Barbara Crafton and the Rittenhouse (1998) article describing the teaching of Magdalene Lampert. Both readings portrayed classrooms in which students engaged in argumentation and classroom discourse around key mathematical ideas; the intent in raising the question was to link to the idea of discourse as a means of allowing students to construct mathematical arguments, both individually and collectively. Shortly into the discussion, one teacher raised an objection to the question:

Instr.: So I ask again, can we call any of these activities proof? [pause] Emily?
Emily: I would say, formally no, but informally yes. To me, proof is just being able to back up your reasons, as why you said something. So if you continuously ask students why, why why, then informally, yes I would think that's a proof.
Instr: Ok, Cameron?
Cam.: I think before we establish that we have to establish what your definition, of proof is going to be, because it's kind of like walking into a classroom and saying, is that a good lesson or is that not a good lesson? Well, if you don't know what their goals were- what the teacher's goals were, you don't know whether they met those goals. So you have to have some kind of agreement on, what the thing is.
Instr: So Cameron's arguing, it sounds like, that we should define proof, before we can really have a conversation about whether what happened was proof or not. So let's do that.

Whole-class discussion, Class 3

The instructor used this comment as an opportunity to launch into creating a definition of proof. Teachers were first asked to consider what proof was individually, then to discuss it in their small groups. Following a brief period of small-group discussion, groups were called upon to share their ideas, which were recorded on chart paper. Figure 35 is a transcription the public written record of the discussion after Class 3.

What is proof?
grounded in previous knowledge (accurate): axioms, basic facts, things we "accept" depends on class/group
using prev. knowledge to show something "new"
general [not a specific example] "for all possible examples"
more than explaining
justification using valid (true) ideas/arguments
must clearly communicate math ideas (audience)

Figure 35. Defining proof – public record following Class 3.

It is interesting to note that many of the key aspects of the definition of proof, such as based on true ideas and generality, are included in this initial chart. Additionally, several important aspects of the role of proof, such as explaining and communicating mathematical ideas were also recorded as a result of the initial discussion. All ideas were accepted and recorded, regardless of agreement by the class or the instructor. In particular, the notion of "audience" was one that was open to interpretation, and was frequently challenged in subsequent conversations.

This chart was displayed in public view for the duration of the class, and referenced and revised in subsequent discussions of proof. The next such opportunity to revisit followed the activities in Classes 6 and 7 in which teachers created proofs, examined a set of proofs related to the Pythagorean Theorem, and discussed the proof process. The final opportunity to reconsider the definition of proof came in the final two classes, Classes 11 and 12. Through these subsequent discussions, teachers came to revise the class' posted definition of proof. The final version, completed in Class 12, is shown in Figure 36.

What is proof? An argument that is/does:
grounded in previous knowledge (accurate): axioms, basic facts, things we "accept" depends on class/group
using prev. knowledge to show something "new"
general [not a specific example] → For all examples asked for in the original statement (class) "for all possible examples"
more than explaining
justification using valid (true) ideas/arguments "math facts" – could be many arguments
must clearly communicate math ideas (audience)
specifies the limitations of the conjecture
not just words

Figure 36. Defining proof – public record following Class 12.

The revised record at the close of the course adds several key notions to the definition and role of proof. First, the line added at the top identifies proof as a mathematical argument, helping to contextualize the criteria listed below. This stemmed from an objection that one teacher had that our list was characteristics of a proof rather than a definition of proof. Second, the notion of the conjecture having limitations that are specified could be considered a refinement of the notion of generality, as can the clause added after general on the poster. Finally, the notion that accepted mathematical facts depends on the class or group who are the audience for the proof clarifies the notion of audience that was initially controversial. Aspects of all four of the key elements of the definition – generality, mathematical argument, based on established mathematical facts, and establishes truth – are all visible in the final version of the definition of proof. The artifact, as a representation of the three public discussions related to the definition of proof, represents an opportunity to learn the key aspects of the definition of proof.

4.2.6.3. The role of proof in the mathematical domain and the classroom. A short sequence of the activities shown in Figure 33 aimed to provide teachers with opportunities to consider the role of proof in the mathematical domain and in the classroom. Goals for teacher learning with respect to the role of proof in the mathematical domain included that proof validates, explains, builds new knowledge, communicates new math ideas, and promotes logical thought and connects ideas by systematizing the mathematical domain. With respect to the classroom, goals for teacher learning were to establish proof as having explanatory power under certain conditions, and to identify discourse as a vehicle for reasoning and proof. The discussion of *What is proof?* in Class 3 began to drift into the terrain of why one might engage students in proof activities in the classroom. The instructor charted teachers' initial thoughts on the subject in Class 3, then asked them to read and consider the NCTM Grades 6-8 standard for reasoning and proof, and to reflect on how they engaged their own students in reasoning and proof through an online message board posting.

Teachers had the opportunity to revisit how proof might play out in the classroom when they were asked if any of the activities in which the teacher and students engaged in The Case of Isabelle Olson constituted proof. This brief discussion was charted; the ideas identified by teachers are shown in Figure 37.

Any proof in Isabelle's class?
expected [students] to show evidence (multiple examples) – "Prove it to me!"
disproving fixed [perimeter] = fixed area
general – no [specific] perimeter
proved for specific cases (work by indiv. groups)
→ lead to pattern/generalization "mini-proofs"
proof that may happen the next day

Figure 37. Proof in The Case of Isabelle Olson.

This artifact shows evidence of several of the understandings that were goals of the discussions of proof in the classroom – the notion of discourse, the idea of showing evidence, and the value of engaging in proof-like experiences in the service of solving a mathematical task.

Three final activities in Classes 7 and 8 represented the last of the opportunities to learn about the role of proof in the classroom. Following the engagement in creating and examining proofs in Class 6 and 7, teachers were asked to reflect on how the instructor supported their learning, and asked again to consider the role of proof in the K-12 classroom. Following these discussions in Class 8, The Case of Keith Campbell was also examined for any evidence of proof-related activities. As a result of these three conversations, two additional public artifacts were produced and/or revised. The first was a chart of responses to the question, *What purpose does proof serve?* This chart was begun in Class 3 and subsequently revised following the discussions in Class 7 and 8. An additional chart to track responses to, *Why teach proof?* was also created. These two charts are reproduced in Figure 38.

What purpose does proof serve?	**Why teach proof?**
communicating math ideas	learn to think logically
connect ideas/build new knowledge	organize thinking
validity	verbalize thinking – "metacognition"
understanding	ask why/give reasons
extends beyond math	understand when to use math ideas
supported by exploration	understand why
generalize an intuition	see new relationships
organization	argue in the "math system" & what rules are
logical thought	work systematically (backwards)
EXPLAIN	EXPLAIN

Figure 38. Responses to *What purpose does proof serve?* and *Why teach proof?*

The final version of the artifacts from the whole-class discussions represent several of the key ideas with respect to the role of proof in the classroom, as well as key aspects of the purpose of proof in the mathematical domain. With respect to the mathematical domain, teachers identified that proof validates, explains, builds new knowledge, communicates new math ideas, and promotes logical thought and connects ideas. These ideas represent the key understandings related to the role of proof in mathematics as identified in previous sections. In the classroom, teachers identified proof as a way to get students to verbalize their thinking, to organize their mathematical work and understand the mathematical system, and to ask, explain, and understand why. These ideas represent the key understandings with respect to proof in the classroom that were goals for teacher learning related to proof. These artifacts, as representations of the whole-class discussions, constitute significant opportunities for teachers to learn about the role of proof in the mathematical domain and in the classroom.

4.2.6.4. Constructing and evaluating proofs. The cluster of activities in Classes 6 and 7 around proof represented a concentrated opportunity to learn with respect to constructing and evaluating proofs. Teachers began this cluster of activities by proving that the area of a triangle equals $\frac{1}{2}bh$. Teachers were asked to work individually for an extended period of time on the proof, and were then allowed to consult with their small-group members to complete their proofs (or in some cases, proof-like arguments) and to ensure that all group members understood their proofs. Several arguments were then shared and discussed by the whole class. These explanations varied along a number of dimensions – representations used (diagrams, manipulatives cut from index cards, verbal explanations, symbolic proofs), amount of prior knowledge needed, and explanatory power. Following the discussion teachers were asked to consider four questions to reflect on their proof-writing experiences. This activity, entitled *Unpacking the Proof Process*, is shown in Figure 39.

<u>Unpacking the Proof Process</u>

1. How did you start your proof?

something that is known	*looking at specific examples (→ general)*
thought about the "why"	*picked a specific "entry point"*
intuition	

2. What steps did you take on your way to constructing the proof?

different "cases"	*think of ways to disprove an idea*
using what was known to link new ideas	*("might end up with a few blind alleys")*
diagrams and labels to explain text	

3. How did you ensure that everyone would understand?

words to explain mathematics	*made manipulatives to help explain*
no steps skilled: reread, shared w/community	*used math language*
precision audience	*"language of the discipline"*
"mathematical community"	*"compression"*

4. How did you know when you were done?

convinced yourself: valid reasons for steps	*no more questions about "thought path"*
no cases against	*clear to others*
links got you from start to finish	

Figure 39. Unpacking the Proof Process Questions and Shared Responses.

The responses to these questions that were shared revealed several key features that distinguish a proof from a non-proof argument. The notion of generality and being immune to counterexamples arises in response to question 4, suggesting the notion of generality. The idea of a proof communicating mathematics clearly was also salient in questions 3 and 4, as was the notion of building on existing mathematical knowledge and using the "language of the discipline." The idea and building blocks of a mathematical argument can be seen across a number of the questions.

The final task in this sequence involved teachers considering 8 explanations of the Pythagorean Theorem, which varied with respect to their completeness as a proof, the representations used, and the level of detail in the explanation. This activity was designed to tease out the features of proof that were more or less helpful in understanding the underlying mathematics, and more generally the notion that proofs can vary with respect to how well they explain the mathematics at play.

Teachers were asked to consider the 8 explanations individually, in small-group discussion, and finally through a whole-class vote whether the explanation was or was not a proof. Following the vote, teachers had to identify what features of the proof they thought were promising, and what needed improvement or clarification about the proof. Among the proofs was Euclid's landmark proof of the Pythagorean Theorem, widely regarded as an extremely concise and elegant mathematical proof. Surprisingly, all 25 teachers in the course voted that this explanation was not a proof. Their misidentification of the explanation as a non-proof provided a rich entry into a conversation about the explanatory power of proof. Excerpts from that conversation are shown below.

Instr.: So this is one of the original proofs of the Pythagorean Theorem. So why did we vote this down? Debbie?
Debbie: I couldn't understand it to be honest with you so, I couldn't make any sense of it, I couldn't understand it I couldn't follow it so, I couldn't accept it as a proof.
Instr.: Ivy?
Ivy: I think that goes directly with what we said before, that a proof has to communicate the math ideas to the audience, so maybe whoever did it gets it, if we're the audience and we don't get it, we can't consider it a proof.
Instr.: Emily?
Emily: It uses the word shearing.
Class: Yeah. [much agreement and overlapping speech]
Emily: How much are we shearing?
Voice: What is shearing?
Instr.: Kelsey asked the-
Kelsey: Is that proven already, so like this idea of shearing, is that proof that shearing the rectangles makes them- y'know what I'm saying, did somebody prove that that's legit, because if that's proven than maybe this one is ok.
Instr.: So if there's a particular, technical definition behind shearing-
Class: [murmur]
Instr.: that's a good analogy, so if shearing had a meaning like squaring, which had a particular mathematical understanding behind it, then this might be legitimate but it's hard to tell.
Class: [murmur]
[instructor uses electronic applet to manipulate rectangle to demonstrate shearing]
Kelsey: Stop right there! Why is that a shear?
Instr.: Ok, so why does this work?
[overlapping speech and discussion of the move]
Instr.: Emily?
Emily: Maura brought this up, the area of a parallelogram is base times height, the base isn't changing, the height isn't changing, so you're just moving the area
…
Flor.: ..like we said one before would be good for my 7th grade class, and one would be good for my 8th grade class, this wouldn't be good for any of my classes!
…
Instr.: [walks to proof posters] There's something that I'm hearing consistently, and I think the fact that we voted pretty convincingly that this was not a proof… allows me to add this on to our list on behalf of the community in the classroom, what purpose does proof serve? I think a purpose that it serves is to explain. And I think the reason, that there's this great objection to D although we may recognize the diagram as something we've seen before, even though I told you where it came from, and that didn't seem to convince anyone, which I think is great, because it didn't explain to you, what was happening, why it was happening, and it wasn't convincing. So I think an important purpose that proof serves, and this is dependent on the community like we talked about here, is that proof explains. And when we talk about teaching, isn't one thing we want to do for our students and that we want our students to do for us, is to explain why something is true.

Whole-class discussion, Class 6

4.2.6.5. Summary. The series of activities described in the previous sections demonstrates that the course provided meaningful opportunities for teachers to learn content knowledge in the domain and content knowledge for teaching related to reasoning and proof. The conversations about the nature of proof and its role in the classroom made salient the key characteristics of proof, the role of proof in the mathematical domain, and the work that proof can do when students are engaged in proof-related experiences. The examination of the Pythagorean Theorem proofs and the creation of the area of a triangle proof gave teachers an opportunity to experience proof creation, convince other teachers of their proof's validity, reflect on the proof process, and identify particular features of proofs that made them more or less mathematically convincing. These set of experiences taken together in light of the results illustrate an opportunity to learn.

Of particular note is the result in which teachers demonstrated growth in their ability to construct deductive proofs. In terms of opportunity to learn, teachers engaged in only one activity in which they were asked to create a mathematical argument that would qualify as a proof. Moreover, not every member of the class managed to produce a complete deductive proof. Despite this limited exposure to proof creation, teachers still showed significant growth in their ability to create a proof or a more proof-like argument as compared to their work at the start of the course. This result will be discussed further in Chapter Five.

In general, teachers saw the set of activities related to proof as contributing to their own learning. In the second interview, all 20 named proof as something that they learned about during the course, with 15 identifying it as a source of mathematical learning, and 4 each identifying it as an opportunity to learn about student thinking and the teaching of mathematics. The extent to which individual activities in the constellation were identified by teachers as contributing to their learning of proof is shown in Table 36.[18]

[18] The discussions about proof in The Case of Isabelle Olson and The Case of Keith Campbell were not identified as a separate activities on the course map given to teachers, and thus do not appear in this table.

Table 36. Teacher learning data for proof activities.

Activity	Total teachers identifying	Mathematics	Students as learners	Teaching of mathematics
What is proof (Class 3)	12	11	2	3
Read NCTM proof chapter	5	4	3	1
Courseweb prompt	6	5	0	1
Prove the Area of a Triangle	12	9	3	4
Unpacking the proof process	8	7	1	1
Pythagorean Theorem Proofs	14	12	4	2
Revisiting what is proof (Class 6)	12	10	2	2
Pythagorean Theorem, con't.	7	6	1	1
Role of proof in K-12 education	5	4	2	1
Revisiting what is proof (Class 11/12)	9	9	1	2

Taken together as a complete set, the course activities around proof included individual and small group work, public class discussions, and the opportunity to reflect in writing, and thus constitutes an opportunity to learn.

4.2.7. Connecting to Design Principles

The previous nine sections have provided an accounting of the results of teacher learning with respect to knowledge of mathematics and mathematical activities and have described how the course activities provided teachers with opportunities to learn. To close the discussion of knowledge of mathematics and mathematical activities, this section returns to the design principles articulated at the start of the chapter to describe how the design principles did or did not predict the learning that occurred with respect to mathematics and mathematical activities.

Design Principles 1, 2, 3, 4, and 5 bear particular relevance to the results presented in the previous sections with respect to teachers' knowledge of mathematics and mathematical activities. Each of these design principles will be discussed in the context of how the principle may have served to predict the learning that took place, and whether the learning that did take place matched the prediction.

4.2.7.1. Public Discussions. Design Principle 1 suggests that having public discussions about key ideas in the course gives all teachers access to the ideas, and allows them to question, challenge, and debate those ideas. The evidence presented in the sections detailing teachers' opportunity to learn suggests that the ideas upon which teachers showed growth were indeed available in the public space. It is also interesting to note that in several cases, particular solutions or ideas presented by a single teacher became touchstones for other teachers that changed their thinking about the mathematical ideas at play. The following excerpt from an experienced secondary teacher from Learning Log 2 serves as an example of such a touchstone.

> In the activity we did in class on Thursday, my group started off assuming that the largest area would be a square. But as we sat and played with the problem through different approaches, we came to the conclusion that this was not the correct solution. I thought that is was quite funny that a kindergarten teacher came up with the most universal solution verses secondary mathematics teachers who used calculus to solve the problem. I was very impressed that a simple chart could solve the same task as a complex procedure.
>
> Uma, Learning Log 2

In the second interview, teachers were asked if there was anything about the teaching of the course itself that helped them come to know or understand something about mathematics teaching. It was in the context of discussing that question in which Bridget, a preservice elementary teacher, clearly identified the public discussions as a means of supporting the learning of teachers in the course:

Bridget: I think it was interesting too how- am I talking about your pedagogy or the course or, what.
Int.: Yeah, both of- either of those (xxxxx).
Bridget: Ok um, I don't want to get (xxxxxx) about your pedagogy. [laughs]
Int.: You're more than welcome to.
Bridget: Um, I really enjoyed and I think it was frustrating though for some people because you were almost hesitant to say something 'cuz I- you KNEW that you were going to ask well say more about that or [both laugh] or y'know, could you revoice that or, whatever somebody else said when you agree with somebody you're thinking why do I agree with them because I know he's gonna ask me y'know. [both laugh] (xxx xxx xx said). But um, I think the whole building, of our own definition of proof and reasoning, was y'know frustrating but, a good frustrating, 'cuz it really was like, well, this is what I think. This is why. And now why is somebody else disagreeing with it and y'know, having us have a, forum for debate but it wasn't y'know like a malicious forum for any means so, it was, interesting [pause] to build, like I would have an idea and then someone else could build on that idea and [pause] make that idea so much better than, what I could've come up with on my own y'know so, [pause] just building that whole definition [pause] was interesting y'know…

Bridget, Interview 2, Lines 543-560

4.2.7.2. Engaging in mathematical tasks. Design Principle 2 suggests that engaging in authentic mathematical tasks from the classroom has the potential to enhance both content knowledge in the domain and content knowledge for teaching. Based on this principle, one might have expected significant results for both aspects of knowledge of mathematics and mathematical activities with respect to the areas of content assessed. Indeed, the results reported earlier do show growth in teacher knowledge along both these aspects for all three content foci – relationships between dimension, perimeter, and area; relationships between dimension, surface area, and volume; and reasoning and proof.

While this claim was not tested empirically, some data to support this claim can be derived from the second interview, in which teachers were asked provide self-report data regarding their learning in the course. Teachers were asked to describe their learning in three categories: knowledge of mathematics, knowledge of students as learners of mathematics, and knowledge of mathematics teaching. The first category relates to content knowledge in the domain, while the last two relate to content knowledge for teaching. Table 37 shows teachers' responses for the main mathematical tasks in the course. As can be seen on the table, all the mathematical tasks in the course were identified by some teachers as sources of learning related to content knowledge in the domain as well as content knowledge for teaching.

Table 37. Teacher learning data for mathematical tasks.

Activity	Total teachers identifying	Mathematics	Students as learners	Teaching of mathematics
Solve Area of Irregular Figures II	7	2	6	3
Solve the Index Card Task	17	11	12	5
Solve the Stacks of Paper Task	10	5	7	2
Solve the Building Rabbit Pens Task	12	7	9	7
Solve Building Storm Shelters Task	3	1	2	1
Solve Comparing Triangles Task	4	2	2	0
Prove the Area of a Triangle Formula	12	9	3	4
Solve the Arranging Cubes Task	10	6	6	3
Solve the Soda Can Task	15	10	9	3
Solve the Wet Box Task	13	11	7	4
Solve the Large Numbers Lab	11	6	6	5

4.2.7.3. Constellations as rich sites for learning. Design Principle 3 offers the notion of constellations as being rich sites for learning. Two of the content areas (relationships between measurable quantities) contained activities organized in constellations. Activities involving reasoning and proof, as well as the activities related to the other facets of knowledge needed for teaching, were not organized in constellations. Thus, the design principle suggests that teacher learning would be greater in the constellation activities.

As with Design Principle 2, no empirical data were collected to test differences in learning with respect to constellations. However, one source of teacher self-report data suggests that constellations may have been rich sites for learning. In the second interview, teachers were asked if there was anything about the pedagogy of the course that contributed to their learning. This open-ended question was designed to give teachers an opportunity to articulate notable design features of the course in their mind. Two teachers talked briefly about constellations.

> And um, [long pause] y'know the idea of doing the- doing the- doing the mathematics think about the mathematics first and then read a case or, watch a video clip where they're enacting it up, is one that helps you understand something on different levels. Um, so all those- those are kind of the pedagogical aspects I see.
>
> Maura, Interview 2, Lines 309-313
>
> And the, maybe the way the class was organized in the, giving us time to think about a problem and then actually solve it before we talk about the cases that we do.
>
> Ivy, Interview 2, Lines 172-173

One possible reason that more teachers did not identify this notion of constellations may have been the fact that the 9 preservice secondary MAT teachers in the course had just completed a similarly-structured course prior to the geometry and measurement coursework experience. In responding to the question, a number of teachers explicitly flagged that they were only identifying features unique to the geometry and measurement course. Moreover, the value added in creating a constellation of activities that meets the criteria in Design Principle 3 suggests that teachers may not learn more as a result of their engagement in the constellation, but may learn something different. Specifically, constellations may provide broader coverage of different facets of the knowledge needed for teaching framework. Investigating differences in constellation as compared to non-constellation activities is a ripe area for further study.

4.2.7.4. Building on prior knowledge. Design Principle 4 echoes a widely held tenet across a variety of conceptions of learning – that building on prior knowledge allows well-integrated, meaningful connections to be made between existing knowledge and newly created knowledge. Course activities related to knowledge of mathematics and mathematical activities were designed to address issues of prior knowledge in two ways. First, tasks were selected that afforded entry to learners who possessed a wide range of prior knowledge. For example, the Index Card Task was approachable by all teachers in that the only prior knowledge needed was a basic understanding of what was meant by area and perimeter. Teachers with only this basic understanding could make significant progress on the task. But as demonstrated in section 4.2.2, teachers with relatively deep understandings of perimeter and area were able to build on those understandings to investigate the meaning of the measurements and the implications of their procedures, affording them an opportunity to learn.

The second way in which the issue of prior knowledge was addressed was in the sequencing of tasks that focused on similar content. As noted in the descriptions of opportunity to learn, tasks were selected and sequenced in ways that built in mathematical complexity and built on one another, allowing teachers to build not only on their prior knowledge acquired prior to the course, but also on their evolving understandings as a result of their engagement and participation in the course. This notion did not escape the attention of teachers; the excerpt below is from the Learning Log 2 of Melinda, an elementary teacher who was often concerned about differences in her mathematical background as compared to the rest of the class.

> All of these tasks involve area and how it is determined or how it relates to the shape when the shape is changing. I believe that the order that they were given to us was not accidental. The first task establishes what area should look like. Most of the students in the Crafton article understood there were one hundred small squares in a square centimeter; each of these small squares being a square millimeter. The other tasks dealt with area/perimeter relationships.
> What we have discussed thus far in class involves students examining what area really is. Students should know that what the square units in area are and why they are called square units. They should also understand how perimeter and area relate to one another, because without perimeter there would be no area to configure.
>
> Melinda, Learning Log 2

4.2.7.5. Revisiting a complex mathematical idea. Design Principle 5 suggests that by threading a mathematical idea through a course and revisiting the idea following new experiences, teachers are able to build a richer and more nuanced mathematical understanding than would be afforded by a single conversation. The notion of proof was positioned in this way in the geometry and measurement course.

Section 4.2.6.2 related to defining proof shows the growth in teachers' conceptions of proof between the initial conversation and following the final conversation about proof. It is clear from the written record that several key aspects of proof were added as a result of the revisiting conversations in Classes 6 and 11/12. Additionally, teachers continued to see the revisited discussions as opportunities to learn; 12 teachers identified the initial conversation regarding proof as an opportunity to learn, and 12 and 9 teachers respectively identified the two revisiting conversations also as opportunities to learn. The additions to teachers' conceptions of proof also suggest links between the intervening activities and the refinements made to the charted response to *What is proof?*. Specifically, the issue of a mathematical argument reflects the conversations around proof in Isabelle Olson's class, as well as the activity in which teachers were asked to create a proof of the area of a triangle. The issue of audience – a controversial one when introduced in Class 3 – was refined by the close of the course, with the refinement in part echoing the issues discussed during the Unpacking the Proof Process activity (see Figure 39). These refinements to teachers' ideas regarding proof and the evidence from the activities in which teachers engaged that may have influenced their thinking suggests that in positioning proof as an object of inquiry to be revisited, teachers' opportunities to learn related to proof were enhanced.

4.3. Knowledge of Mathematics for Student Learning

In addition to enhancing teachers' knowledge of mathematics and mathematical activities, the geometry and measurement course aimed to enhance teachers' knowledge of mathematics that support students' learning related to middle grades geometry and measurement. This knowledge was targeted in the form of the five practices for productive use of student thinking (Stein et al., submitted): anticipating student solutions, monitoring student work, selecting student responses for whole-group discussion, sequencing student responses for whole-group discussion, and connecting student responses in whole-group discussions. In planning for and engaging in these practices, teachers are more likely to teach in ways that support the maintenance of high cognitive demands in the classroom (Stein, Grover, & Henningsen, 1996;

Stein et al., submitted). The practices, either explicitly or implicitly, were likely known to the majority of teachers in the course. The course provided an opportunity to examine these practices in detail, and to consider how these practices operationalized with respect to middle grades geometry and measurement content.

4.3.1. Five Practices: Growth in knowledge

Table 38 lists the five practices that were targeted in the course and measured by pre- and post-course instruments.

Table 38. Knowledge of Mathematics for Student Learning: The Five Practices for Productive Use of Student Thinking.

Anticipating student solutions
Monitoring student work, including questions that assess & advance student thinking
Selecting responses for whole-group discussion
Sequencing responses for whole-group discussion
Connecting responses shared in whole-group discussion

Four data sources were used to assess growth with respect to the five practices for productive use of student thinking. The first and second interview asked 20 teachers to plan a lesson around the *Minimizing Perimeter* task. Their description of the lesson was examined for evidence of each of the five practices. On the pre- and post-course written assessment, the *Responding to Student Claims* task assessed aspects of monitoring through the questions teachers chose to ask in response to the student claim. The *Considering Student Work* task asked teachers to examine a set of student work from the *Fence in the Yard* task and select solutions to share and determine a sequence for sharing the solutions. Finally, the *Thinking Through a Lesson* assignment asked teachers to plan a lesson around a geometry and measurement task. The protocol used for the assignment explicitly included all five practices. Twenty-one of the 25 teachers made their *TTAL* assignments available; this task was assigned in Class 7 and collected in Class 10. Since the assignment was a one-time event, examination of the *TTAL* provides a single snapshot of teacher performance on the five practices when explicitly asked for rather than a pre- and post-course evaluation of their ability to engage in the practices.

Two methods were used to measure teacher growth across the four tasks. First, a rubric was developed for assessing the *Thinking Through a Lesson* assignment for the five practices; this rubric was adapted for use in analyzing the interview protocols for the *Minimizing Perimeter Lesson Plan* task. Second, responses to the *Responding to Student Claims* and *Considering Student Work* tasks were examined for emergent categories related to monitoring (questioning) and reasons for selecting and sequencing solutions. These rubrics provided a finer-grained examination of the teachers' engagement in these three specific practices. Given the small number of tasks related to the five practices, the results with respect to knowledge of mathematics for student learning are grouped by the practices rather than the tasks.

<u>4.3.1.1. Anticipating Student Solutions</u>. Two sources of data were examined to determine change in evidence of teachers anticipating the range of solutions that students might produce on a geometry and measurement task, both of which involved lesson planning. Transcripts and written artifacts were examined from the two interviews, and the Thinking Through a Lesson assignments were examined for evidence of anticipating solutions. For both assignments, artifacts were considered as anticipating student solutions if the teacher articulated a specific solution or category of solution for which they would watch during the enactment of the lesson.

Transcripts of the interview in which asked teachers to plan a lesson around the *Minimizing Perimeter* task (see Appendix B, Task 3; Appendix C, Task 4) were coded for talk that showed evidence of teachers' consideration of particular solution strategies that students might produce. Most teachers showed evidence of some sort of talk relating to anticipating student solutions, so there was no significant change in the number of teachers attending to this practice. There was a significant increase in both the average proportion of talk devoted to anticipating solutions ($t(19) = -2.29$, $p = 0.02$) and in the number of lines of transcript in which teachers discussed anticipating solutions ($\chi^2(1, 3156) = 25.2$, $p < 0.01$). These data indicate that teachers participating in the interviews paid significantly more attention to the issue of anticipating solutions.[19]

Teachers were also asked to anticipate solutions as part of the Thinking Through a Lesson assignment (see Appendix F). In contrast to the interview task, anticipating solutions was a required portion of the assignment, which was a significant component of students' grades in the course. Teachers were assigned in groups to work on particular tasks with a geometry and measurement focus and to collaborate on a variety of solutions to be turned in with the assignment. Assignments were scored on a comprehensive rubric (see Appendix F) which included a 3-point scale to evaluate anticipated student solutions. Solutions were evaluated with respect to variety of approaches and attention to possible student misconceptions or difficulties. To earn a 3, teachers had to consider a broad mathematical range of solutions as appropriate to the task. Of the 21 TTAL assignments made available, teachers scored an average of 2.62 on the 3-point scale. This included only three scores of 1, two of which resulted from a group that had particular challenges with their task. These data indicate that when asked, teachers could produce a broad range of solutions to a middle grades mathematical task that represented a variety of plausible approaches that students may take to the task, including flawed approaches and misconceptions. Together with the results from the interview, these data suggest that

[19] Teachers were not asked specifically to anticipate student solutions in their written work, as this was determined to be too leading. Additionally, teacher talk in the interview often referred to general classes of solutions or specific features of a solution. Counting the number of individual solutions anticipated was impossible in this context. Thus, no data were available regarding the number or type of solutions anticipated by teachers.

teachers grew in their ability to anticipate solutions on geometry and measurement tasks following the course.

4.3.1.2. Monitoring Student Work. Two particular aspects of monitoring were targeted in the course: creating questions that assess and advance student thinking; and evaluating student work. Three sources of data were examined to determine change in the types of questions teachers asked around a particular piece of student work or student solution strategy. Transcripts and written artifacts were examined from the lesson planning task in the interviews for evidence of questioning. The Thinking Through a Lesson assignments were examined to determine the types of questions teachers asked. Finally, the *Responding to Student Claims* item on the pre- and post-course assessment (see Appendix A, Part D, Task 5) was examined to determine how teachers chose to respond to a student's erroneous claim.

A four-point rubric was developed to code teacher questions in the *Minimizing Perimeter* planning task from the interview. This rubric mirrored the rubric used to evaluate questioning on the Thinking Through a Lesson protocol, which can be found in Appendix F. Questions were identified and coded on a scale of 1 to 4, with 4 representing questions that were tied to the goal of the lesson and either probed specific strategies for solving the task or pushed students towards a generalization. The rubric used is shown in Table 39.

Table 39. Rubric for evaluating questions on *Minimizing Perimeter Lesson Plan* task.

Score Point 4	Question is tied to a particular strategy or approach which is clearly articulated in the interview transcript (e.g., *if students do _______, I will ask ________*) ***or*** Question is designed to connect specific strategies or approaches, or press towards a generalization ***and*** Question is clearly related to the target mathematical goal for the lesson
Score Point 3	Question is loosely tied to a particular strategy or approach ***or*** Question is designed to probe or advance student thinking, but is overly general, and/or the conditions for use are not clear ***and*** Question is related to the target mathematical goal for the lesson
Score Point 2	Question is not tied to specific strategy ***or*** Question is general and/or it is not clear how the question serves to focus, assess, or advance student thinking, ***and*** Questions are related to the target mathematical goal for the lesson
Score Point 1	Question is not tied to a specific strategy or approach ***and*** Question is not related to the target mathematical goal for the lesson ***or*** Question is procedural in nature/serves to reduce the demands of the task

Overall, there was a significant increase in the number of lines coded as questions of any type between the first and second interview, $\chi^2(1, 3156) = 41.2$, $p < 0.01$. There were also significant differences in the types of questions asked by teachers, as shown in Table 40.

Table 40. Changes in the types of questions asked by teachers.

Question Score	**Four**	**Three**	**Two**	**One**
Significant difference:	Increase	No change	No change	Decrease
# of teachers (McNemar's Test)				†
Proportion t_{19}	-2.11†	-0.07	-1.19	2.47†
# of lines $\chi^2_{(1,381)}$	17.8*	2.26	0.10	32.2*

† $p < 0.05$ * $p < 0.001$

The results from the lesson plan interview task indicate that teachers articulated more questions in the process of describing their plan following the course, and the quality of these questions increased overall. There was an increase in the number of lines coded as evidence of questions that were most closely tied to the mathematical goal and either linked to specific strategies or a generalization about the mathematical relationship, and a decrease in questions that were not tied to a specific strategy and either not related to the mathematical goal or were procedural in nature.

In the Thinking Through a Lesson (TTAL) assignment, teachers were asked to write questions to focus, assess, and advance student thinking, constructs that were discussed explicitly in the course. Questions were expected either to make contact with specific strategies or misconceptions that students were likely to produce, or serve to advance student thinking towards generalization. Teachers' questions produced on the TTAL assignment were assessed holistically on the same rubric used to code the lesson planning interview transcripts. Of the 21 TTAL assignments made available, teachers scored an average of 3.33 on the 4-point scale. This included no scores of 1, which would indicate procedural questions only. These data indicate that when asked, teachers could produce a set of questions around a geometry and measurement task that targeted the mathematical goal of the task and were at least loosely tied to particular student strategies or pushed towards generalization.

The final data source with respect to monitoring student work was Task 5 on the pre- and post-course assessment, which asked teachers to respond to an erroneous student claim regarding the relationship between area and perimeter. Responses were coded into categories that built on those used by Ma (1999); these categories distinguished responses that pressed students to investigate the mathematical relationship in some way from responses that featured teachers telling the student what to do next or providing them with a particular example or non-example. Table 41 shows a brief list of categories used, grouped by complexity (see Appendix E for a comprehensive list of categories). Depending on the extent of the response, teachers' answers could be coded as more than one category.

Table 41. Question categories used for the *Responding to Student Claims* task.

High level questions	Press students to investigate the mathematical relationships further Press students to justify the claim (general) Make connection to proof (*e.g.*, do two examples prove your claim?)
Promising questions	Ask students to produce a specific type of example (*e.g.*, can you change both dimensions?) Ask students to provide a counterexample Ask the class what they think about the claim Probe the student's thinking (general)
Directive responses	Provide a specific example for the student Provide a counterexample for the student
Unproductive responses	Tell the student they are correct Tell the student they are incorrect

Results showed a significant increase in the number of teachers who pressed students to investigate the mathematical relationship in some way on the post-course assessment, both with the unit of analysis as being number of teachers ($\chi^2(1, 49) = 15.6$, $p < 0.001$) and number of responses ($\chi^2(1, 112) = 13.9$, $p < 0.001$). There was also a significant decrease in the number of teachers who provided a directive or unproductive response, by providing or telling the student something (statement, example, etc.) to prompt them to correct the misconception, $\chi^2(1, 49) = 5.95$, $p = 0.01$.

These results suggest that teachers became more attuned to ways in which they could advance student thinking about a particular mathematical relationship, and in turn felt less compelled to do some of the thinking for the student by providing them with an example or an explanation that dispelled the misconception. In sum, teachers showed significant change in two aspects of monitoring student work. Teachers produced more and higher quality questions in the service of describing a lesson plan around a geometry and measurement task following the course. There were significantly more questions in the highest-rated category, and significantly fewer questions that were procedural in nature. Teachers also showed change in their response to student work exhibiting a misconception; they were more inclined to ask a question or suggest a pathway for investigating the mathematical relationship after the course, and they were less likely to provide an example or to provide the student with an explanation of why the conjecture was incorrect. These changes represent growth in monitoring practices that could serve to support students in maintaining a high level of cognitive demand for students during their work on a mathematical task.

4.3.2.3. Selecting and sequencing responses to share. The practices of selecting and sequencing are closely related, and are thus results with respect to selecting and sequencing are presented in tandem. Three sources of data were examined to determine change in evidence of teachers selecting and sequencing specific student responses to share following student work on a mathematical task. Transcripts and written artifacts from the lesson planning task from the two interviews and the Thinking Through a Lesson assignments were examined for evidence of selecting and sequencing responses. Finally, the pre- and post-course assessment item that asked teachers to select and sequence responses to the *Fence in the Yard* task were examined for evidence of the reasons teachers cited for selecting particular solutions.

Transcripts of the interview question which asked teachers to plan a lesson around the *Minimizing Perimeter* task (see Appendix B, Task 3; Appendix C, Task 4) were coded for talk that showed evidence of teachers' selection and sequencing of particular solution strategies that students for sharing in a whole-class discussion. Most teachers on both the first and second interview discussed solutions that they would have shared with the class; thus, no differences were found in the number of teachers discussing the selection of solutions as compared to not discussing selection. There was a significant increase in the number of lines of transcript in which teachers discussed selecting particular solutions to share ($\chi^2(1, 3156) = 27.8$, $p < 0.001$), although the average proportion of talk devoted to selecting solutions was not significant ($t(19) = -1.33$, $p = 0.10$). The marginal nature of the change for the pairwise t-test may be due in part to the relatively low percentages of lines devoted to the selection of particular student solutions. There was a significant increase in the number of lines of transcript in which teachers discussed sequencing solutions ($\chi^2(1, 3156) = 95.6$, $p < 0.001$) and in the average proportion of talk devoted to sequencing solutions ($t(19) = -3.66$, $p < 0.001$). McNemar's test also shows a significant increase in the number of teachers who spoke about sequencing in the second interview, $p = 0.01$. These data suggest that more teachers paid attention to the sequencing of solutions at the close of the course, and those who were attentive to selecting and sequencing in the first interview spoke more about selecting and sequencing in the second interview.

The Thinking Through a Lesson assignment explicitly asked teachers to articulate which solutions they would share in a whole-class discussion of their task and the order in which they would want to share them. Responses to this aspect of the assignment were rated on a 3-point scale that also evaluated the selection and sequencing of the responses and their connection to the mathematical goal of the task. Of the 21 TTAL assignments made available at the close of the course, teachers scored an average of 2.52 on the 3-point scale for selecting and sequencing student responses. These data suggest that when asked, teachers were able to select specific student responses for the share-and-discuss phase of a lesson, and were able to connect those selections to the mathematical goal for the task.

The *Considering Student Work* task on the pre- and post-course assessment (see Appendix A, Part D, Task 7) provided teachers with a set of student work from the *Fence in the Yard* task that teachers had solved earlier in the assessment and asked teachers to select and sequence student responses for discussion, and to indicate reasons for their selection. These reasons were examined and compiled into emergent categories and analyzed for change in the the mean number of reasons per teacher (paired t-test) and number of teachers citing each reason

(McNemar's test). The categories were *representation*, *organizational feature* (e.g., a solution was laid out in a particular way), *mathematical feature, likely to be a common solution, starting point for further discussion, connects to another solution, ask follow-up question, discuss/draw out misconception*, and *exclude wrong/limited solutions.* There was a significant increase in the mean number of reasons cited per teacher for their selection and sequencing of particular responses ($t(23) = -5.39$, $p < 0.001$). Two categories showed significant change: *selecting a response because of a particular mathematical feature* ($p = 0.01$), and *connecting to another solution* ($p = 0.008$). The *exclude wrong/limited solutions*, used for teachers who explicitly stated that they would not share solutions that were either limited in understanding or incorrect, decreased from 4 on the pre-course assessment to 0 on the post-course assessment. Taken together, these data suggest that teachers increased in articulation of a particular rationale for the selection and sequencing of solutions, and were more apt to select solutions for whole-class discussion in ways that provided for relating solutions to one another (*connecting to other solutions*) and targeting the key mathematical understandings in the problem (*mathematical feature*).

Teachers' sequencing of solutions for the *Considering Student Work* task was also examined.[20] The task asked teachers to select responses to share from 6 pieces of student work that varied both in terms of strategy, correctness, and level of explanation. Teachers' sequencing of solutions were examined for patterns. A number of patterns were seen; however, because teachers' choices of work to share and their sequencing was so disparate, no trends were statistically significant, but one was particularly interesting. The trend that was closest to significant was placing solution A and solution M adjacent to one another in the ordering, χ^2(1,

[20] It should also be noted that teachers' rationales for selecting responses also tended to include information about sequencing. As it was impossible to separate out the rationales for selection as compared to sequencing, the results in the previous section with respect to rationale also provide insight into the issue of sequencing.

49) = 3.71, $p = 0.054$. This is particularly interesting, as A is an extremely well-formed solution and M shows a particular misconception about the relationship between dimension, perimeter, and area.

These results together suggest that following the course, teachers exhibited an increased attention to issues around the selection and sequencing of student responses in the context of a mathematical lesson. Data from the interview task show an increase in attention to these two practices in general. The *Considering Student Work* task results suggest that teachers showed growth in their reasons for selection in ways that focused on the mathematical features of the task and that made connections between solutions, and became more receptive to the notion of selecting a solution that features a misconception and positioning it near to another strategy that helps to draw out the misconception.

4.3.1.4. Making connections among shared student responses. The fact that student solutions are shared and discussed, with the teacher selecting particular solutions and sequencing them in a particular way, does not guarantee that students will connect the mathematical features of particular solutions in ways that support learning of the key mathematical ideas in the task. Thus, the practice of connecting student solutions is critical in garnering the greatest value from a sharing session. The practice of connecting student solutions was examined in two data sources, the lesson planning interview task and the Thinking Through a Lesson assignment.

No differences were found in the number of teachers attending to connections, as the majority (12 of 20) talked in some way about connections in the first interview. However, significant increases were found in the mean proportion of teacher talk devoted to connecting student solutions ($t(19) = -3.07$, $p = 0.03$) and in the number of lines of text coded as evidence of teachers connecting shared solutions ($\chi^2(1, 3156) = 49.5$, $p < 0.001$). Teachers increased significantly in their consideration of what they might do or ask to connect student solutions in the context of the lesson planning task.

In the Thinking Through a Lesson assignment, teachers were asked explicitly to describe how they might connect shared solutions. Connections were scored on a 2-point rubric (see Appendix F) that evaluated the appropriateness of the connections and their potential to support learning of the mathematical goal. Of the 21 TTAL assignments made available at the close of the course, the average score on connecting student solutions was 1.86. Three teachers of the 21 scored 1 point, and all remaining teachers scored 2 points. These data demonstrate that when asked, teachers were able to articulate how they might connect student solutions in ways that supported learning of the target mathematical goal.

4.3.2. Five Practices: Opportunities to Learn

In sum, the data in the previous section suggests that teachers grew in their ability to engage in each of the five practices related to the productive use of student work: anticipating student solutions, monitoring student work, selecting and sequencing responses for whole-class discussion, and connecting student responses in support of mathematical learning. This is particularly notable since many of the teachers in the course had been exposed to these practices previously in the context of other university coursework or professional development sessions. It is clear that many teachers took key aspects of knowledge of mathematics for student learning away from the course; when asked what they learned as a result of the course, 10 teachers of the 20 interviewed shared ideas that related to questioning, discourse, and discussion.

Course activities rarely featured the discussion of the five practices in the abstract; instead, the practices were integrated into activities in which teachers analyzed student work or planned lessons. Additionally, these practices were modeled by the instructor when teachers solved mathematical tasks; teachers were often given the opportunity to openly reflect on and discuss the instructor's practices following their own on these mathematical tasks. One assignment – Thinking Through a Lesson – occurred in the latter half of the course and integrated all five practices in the context of the planning of a lesson around a middle grades geometry and measurement task. Figure 40 shows the course activities in which teachers had the opportunity to consider one or more of the five practices.

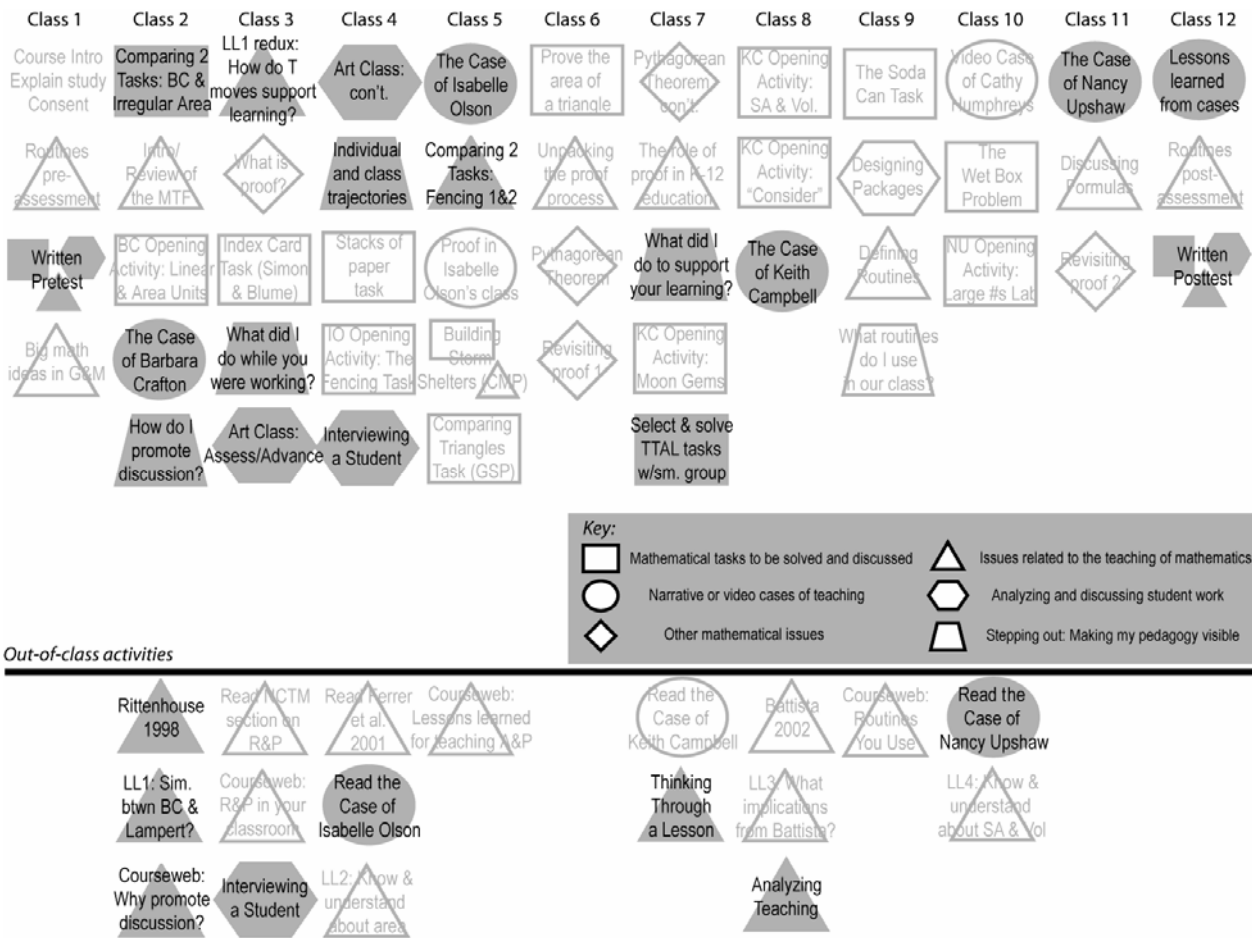


Figure 40. Course activities focused on the five practices for productive use of student thinking.

The activities shown in Figure 40 varied greatly in the degree to which they addressed the five practices. Activities like the case discussions included implicit discussion of issues such as anticipating student solutions and monitoring, although these practices were not the foci of the discussions. In contrast, activities in which teachers examined student work contained more central connections to the five practices; teachers were asked to write questions that assessed or advanced the understanding of a student based on their work (Art Class), or considered which student solutions to share and in what order (Designing Packages). In this section, course activities that most directly targeted the five practices are described as evidence of the opportunity to learn knowledge of mathematics for student learning. These descriptions are supported by data from the second interview in which teachers tied their learning to particular course activities, and selected excerpts of course discussions. The ideas related to the five practices were first made public through the framing of activities focused on analyzing student work and the cases; these issues were then discussed in the public space through the activities, case discussions, and analyses of the instructor's pedagogy. The *Thinking Through a Lesson* written assignment provided the opportunity for teachers to write and reflect on their understandings of the five practices.

4.3.2.1. *Art Class Work: Assessing and Advancing Questions*. The first of the two activities designed to target one of the five practices, in this case monitoring student work, occurred during classes 3 and 4. Following exploration of the *Index Card* task, teachers examined student work from the *Art Class* task, a task that is very similar mathematically to the *Index Card* task. The *Art Class* task was a short task that asked students to determine how many 3 inch by 5 inch pictures would fit on a 12 inch by 15 inch piece of cardboard. Teachers were asked to examine three pieces of work which varied in the understandings evident in the work, and to create questions that would first assess the understanding of the student who produced the work, and then would advance the student's thinking forwards towards the target mathematical goal.

In the post-course interview task in which teachers were asked which activities contributed their learning, only 4 teachers out of the 20 interviewed identified the *Art Class* task as a significant site of learning. However, it is clear that engagement in this activity provided teachers a meaningful opportunity to create and discuss how questions can serve to assess and advance student thinking in the service of a mathematical goal. Table 42 shows the shared list of questions produced during the whole-class discussion of the task. Note that these questions tend to be conceptual in nature, probe student thinking, and were well-connected to the mathematics inherent in the task

Table 42. Assessing and advancing questions for *Art Class* student work.

Work	**Assessing questions**	**Advancing questions**
A	What units would you use to label?	How would you solve for 20 + 16? 22 + 17?
	Why 15 ÷ 5 and 12 ÷ 3? What does it tell you?	What would happen at 15 × 15?
	How do the card dimensions relate to 12 + 15?	Is there another rectangular piece that would hold 12 cards?
	What do you mean by "self-explanatory"?	
	What's the T and the S?	
D	How come you have extra space in the picture?	3 into 12, 5 into 15: what did that tell you?
	Could we put other cards in the spaces?	Do the pictures all have to go in the same direction?
	Explain your process of multiplying 3 into 12.	
	What methods were you considering? Which one did you use?	Could you help someone understand without the picture?
	Why 3 into 12, 5 into 15?	Could you get 12 without the picture?
	How did you know to orient the cards this way?	
	Doesn't the 3 also go into the 15?	
J	What do the 4 and the 3 represent?	Could you draw a picture? Would it support your answer?
	Where are the 4 and the 3 on the cardboard?	
	What does it mean to cover?	How many rows and columns is it?
	Why did you divide? Add?	

4.3.2.2. *Student Work: Designing Packages*. The second activity designed to target knowledge of mathematics for student learning was the *Designing Packages* task. This task occurred after teachers had explored the *Arranging Cubes* task. The *Designing Packages* task comes from the Connected Mathematics Project (Lappan et al., 1998b), and is a revision of the *Arranging Cubes* task featured in *The Case of Keith Campbell*. Teachers were asked to consider several pieces of student work, decide which ones they would like to have shared in a whole-class discussion, and what order they would share them. Hence, the task mirrored the *Considering Student Work* item on the pre- and post-course assessment.

Of the 20 teachers interviewed, 6 teachers cited this activity as a significant source of learning. All 6 marked it as a source of learning about students as learners of mathematics, which resonates with the notion that this activity held the potential for teachers to develop knowledge of mathematics for student learning. After each group had finished selecting and sequencing responses to be shared, the instructor produced a chart of the choices each group had produced, and asked teachers to consider similarities and differences across the choices. Figure 41 shows the public record of the discussion in which teachers shared their selections and sequences and the questions they intended to ask about the most popular choices:

	1	2	3	4	5	6
Group 1	I	D	E	H		
Group 2	I	D	B	G	H	
Group 3	I	A	D	B	H	
Group 4	A	H	I	D	G	
Group 5	A	D	B			
Group 6	C	E	I	H		

D: Why does surface area grow less?	H: Observations about the table?
Explain 'compact'	Does 4 x 2 x 3 take up less space?
Why multiply by 4? by 2?	Which measurement relates to how much space?
What's the difference?	Why is the volume constant throughout?
Where did the 24 come from?	How do you know?
What makes it an RP?	How much space do the others take up?
What makes it easier to stack?	Can you describe your steps?
What will the units be?	Can you draw what you did?
	How do you know you found them all?
	What do you mean by compact?

Figure 41. Selecting and sequencing *Designing Packages* responses.

The record shown in Figure 41 shows teachers selecting particular responses, sequencing them in particular ways (the arrows, boxes, and underlines note similarities across groups), and identifying reasons for selecting particular responses in the form of questions asked. Teachers also addressed the similarities and differences in their sequencing; excerpts from the discussions of sequencing from the whole-class discussion are shown below.

> Kelsey: We liked that the calculation methods for D are different than those from I even though they were the same measurements they calculated with, we thought that would be a nice discussion. Then we liked how B kind of generalized something related to a formula thought it would be good to kind of compare their generalization back to D, kind of go back and forth between the general formula and the specific calculations. And with D we liked the sentence at the end that says as the shapes get more compact the surface area grows less. So that would be a good one to bring up there and you could kind of carry that through the discussion.
>
> …
>
> Nick: I because they used all those possibilities and then we went to A, because when we talked about I, we talked about with I, would there be an easier way to come up with these boxes, are there 26 or are there fewer. A does this, then kind of gets into this we counted each face, we counted each face, but is there another way… and then D, where they started to get into a formula, but they're not stating it… so D would come off of A, which would be a faster way to get to the surface area.
>
> Excerpts from Class 9

4.3.2.3. Reading and Discussing *The Case of Isabelle Olson*. *The Case of Isabelle Olson* was a significant source of learning related to knowledge of mathematics and mathematical activities, but the case also featured a number of salient pedagogical issues that relate to knowledge of mathematics for student learning. The case features a teacher who modifies a mathematical task to be more open-ended, and during the task's enactment grapples with how best to support students as they struggle to make progress on the task. This practice relates both to anticipating student solutions (Ms. Olson anticipates her students would have some key aspects of prior knowledge that they did not exhibit) and monitoring student work (asking questions that help to assess and advance student thinking without removing the challenge from the task, and being watchful of what students were doing in order to inform instructional decision-making). Of the 12 teachers who identified reading the case as a significant source for student learning, 11 of these teachers stated that the reading of the case helped them to learn something about the teaching of mathematics.

After reading the case, teachers were asked to identify the pedagogical moves that Isabelle Olson made using evidence from the case, and to identify how these moves served to support or inhibit student learning. Table 43 shows the list of ideas generated during the class discussion.

Table 43. Pedagogical moves identified in the discussion of The Case of Isabelle Olson.

Pedagogical Move Made	Paragraph	How the move supported/inhibited student learning
Changing the task	9	Support: explore more, use own ideas Inhibit: students didn't know what to do; appropriate for students?
Asked Tommy's group to share their insights	32-35	Support: benefited students who were struggling, send back into groups; "didn't give away the farm", kept authority with students and let them struggle Inhibit: Would students have to come to conclusions on their own?
Ms. Olson changed questioning strategies	16, 22	Support: led Tommy's group in a productive direction; "Prove it to me" – students have to consider a way to justify their claim
Almost went back to original problem, but didn't	19	Support: stuck to her goal, didn't take the easy way out

The responses shown in Table 43 suggest that teachers were paying particular attention to issues of anticipating student solutions and monitoring. They identified changes to the initial task as both supportive of student engagement and as inhibiting learning, as the teacher may not have adequately anticipated what students would be able to do on the task. Teachers also key in on Ms. Olson's questioning strategies and her decision to share an in-progress solution as supporting students' learning.

4.3.2.4. Reading and Discussing *The Case of Keith Campbell*. While *The Case of Isabelle Olson* portrayed a teacher who grappled with decision-making and ultimately took steps to maintain the cognitive demand of a task for her students, *The Case of Keith Campbell* portrayed a teacher who ultimately narrowed his students' understanding of the mathematical ideas at play. The case features a teacher implementing the *Arranging Cubes* task, and a combination of factors lead him to press for and then endorse a particular solution strategy with his students. While there are a number of pedagogical ideas that can be taken away from engaging with the case, one important issue that arises is the importance of allowing multiple solution strategies or ways of thinking about a particular task to arise, and using those ideas productively in moving students towards the mathematical goal.

In discussing the case, the instructor's intent was to tease out the importance of discussing multiple ways of thinking and multiple solutions. Teachers were asked to consider the mathematical goals of the activity, and then to identify moves that supported or inhibited students in making progress towards those mathematical goals. Table 44 shows selections from list of math ideas and pedagogical moves identified by teachers during the whole-class discussion.

Table 44. Math ideas and teacher moves identified in the discussion of The Case of Keith Campbell.

Mathematical Idea	Pedagogical move that supported/inhibited student learning	Paragraph
Justification/proof	How do you know you found all the packages? Pressed for a reason.	20
Finding all the combos for volume	Generalized from specific cases – didn't press to know why	30
Connecting student responses/ideas	List of observations to reference and build on Write in journal, think about relationship between surface area and volume (gives Keith Campbell more insight into student thinking, lets students reflect on what they did	37
Defining and finding formula for volume	Couldn't rephrase A's explanation – turned to class… then said too much? "this means that…" Builds formula for observation Not clear what students understand about volume - $l \times w \times h$ might not be helpful? (limited to a particular context – how vs. why) How are you getting the volume? arranging/filling up/building up	29 31 28
Surface area	Task selection: visualization, put terms in context 12 blocks – had groups do other numbers Allowed students to explore in multiple ways – cubes, paper, drawing	4 24 25

This chart shows that teachers did indeed attend to the issue of honoring and sharing multiple ways of thinking as an important feature of the case, and identified a number of critical moves during Mr. Campbell's monitoring that supported or inhibited student progress. Of the 11 teachers who identified the discussion of *The Case of Keith Campbell* as an important site for learning, 8 of those teachers cited the discussion as a source of learning about students as learners of mathematics.

4.3.2.5. Making the instructor's pedagogy visible. At several key points in the course, the instructor asked teachers to critically examine the moves he made that supported their learning. As the instructor was intent on modeling good practices, and specifically the five practices for productive use of student thinking, these conversations were intended for teachers to identify moves related to the five practices. Examples of such conversations occurred in Class 3 following the Index Card task, and in Class 7 following the proof explorations. Excerpts from each of those two conversations are shown below.

Instr.:	So I told you I like to step out of my role once in a while, so I'd like you to think about this. What was I doing while you were working on the index card task?
Daulton:	Just kind of walking around listening to peoples' ideas, and every once in a while asking a question about how the group in general was solving the problem.
Cameron:	Stepping in, stepping out.
Instr.:	Can you say more about what that means?
Cameron:	That's what the Rittenhouse article was talking about. You were kind of stepping in when- sometimes you were asking questions to kind of, either expand our thinking or clarify something we were saying, you'd say you didn't understand what we were saying because you didn't think we understood it, or you were, stepping out you were just sitting there silently observing because we were engaged in a good discussion and there was no need for you to step in and get in the middle of that and stop whatever creative process was going on so we could explain to you, what we were thinking because you had missed the first part of the conversation.
Kelsey:	You were keeping track of our different strategies and planning what order you wanted them presented in.
Bridget:	You made students explicitly explain what they were doing up front, like Florence said times and you made her clarify, but you didn't necessarily butt in and say, "Well I think you mean," I mean you just let her explain it like, "Can you explain it in a different way."

Excerpts from Class 3

Instr.:	So I'd like you to think back on our experiences over the last 2 classes about proof, and I'd like you to think, what did I do to support your learning related to proof?
Nancy:	Um, I just said that you made me think about what proof is and it might be… so everyday I don't think to myself, "What is proof?" I've been thinking what is proof! You've just helped me open my mind up and think about it more deeply.
Ed:	You gave us all these proofs to look at, kind of put us in a student's position where we're looking at something on the board which we may or may not make sense of, and it kind of gave us the impression that sometimes we may put what we call a proof up on the board that we may understand and we say this this this this this, there's the proof we're done, and the kids in the class are going like this. What in the heck is this guy talking about? So maybe a proof to us isn't exactly a proof, in our students mind. It may prove to them that they may not understand a lick of what we're doing.
Melinda:	Um you asked the question every day, or every few classes, but you gave us new information to go along with that question. So every time you'd add something new to it, what's proof now that we've talked about this.
Instr.:	So in asking it over and over it wasn't just repeating-
Melinda:	No you would add- give us new information to add to our understanding.
Kelly:	Um, I think just the selection of tasks really tried to pinpoint what the criteria was for us to think about at the time.
Uma:	It seemed like when we were working in our groups and you would walk around, you always had a good question. You'd listen and then you would ask something to make us, either clarify what our thoughts were or rethink what we were doing at the time. So you kept us moving in that same direction.
Daulton:	Also you just refused to tell us what you thought a proof was.
Instr.:	And still do.

Excerpts from Class 7

4.3.2.6. The *Thinking Through a Lesson* assignment. The Thinking Through a Lesson assignment was designed to provide teachers with an opportunity to plan a mathematical lesson around a high-level geometry and measurement task which focused on the use of the five practices for productive use of student work. Appendix F contains a copy of the assignment as distributed to teachers; the parameters of the assignment asked teachers to engage in each of the five practices as a part of their planning process. The Thinking Through a Lesson activity represented an opportunity for teachers to operationalize the five practices in a way that was closely tied to the work of teaching, through the preparation of a lesson plan. The data in the previous sections shows evidence that when asked, teachers were able to demonstrate use of the five practices in the service of planning a lesson around a geometry and measurement task.

It should be noted that for at least 15 of the teachers in the class, the Thinking Through a Lesson protocol was familiar to them from previous university coursework. Despite this fact, 6 of the 20 teachers interviewed identified the Thinking Through a Lesson assignment as an opportunity to learn, with 5 of those teachers identifying it specifically as an opportunity to learn about mathematics teaching.

4.3.2.7. Summary. The activities described above show evidence that teachers had opportunities during the course to consider and practice using the five practices that constitute knowledge of mathematics for student learning. Through the examination of student work, reading and discussion of the narrative cases, and planning of a lesson around a high-level task, teachers were able to see the practices in use, discuss the implications of the practices on student learning, and rehearse planning for engagement in these practices with students. Additionally, the modeling of these practices by the instructor made salient to teachers how the practices had the potential to impact their own learning. Given the progress that teachers made on each of the five practices, it is likely that these five activities contributed to teachers' growth in their abilities to engage in these practices.

4.3.3. Connecting to Design Principles

The previous sections have provided an accounting of the results of teacher learning with respect to knowledge of mathematics for student learning and have described how the course activities provided teachers with opportunities to learn. This section returns again to the design principles articulated at the start of the chapter to describe how the design principles did or did not predict the learning that occurred with respect to the five practices for productive use of student thinking.

Design Principles 1, 3, and 6 bear particular relevance to the results presented in the previous sections with respect to teachers' knowledge of mathematics for student learning. Each of these design principles will be discussed in the context of how the principle may have served to predict the learning that took place, and whether the learning that did take place matched the prediction.

4.3.3.1. Public Discussions. Design Principle 1 suggests that having public discussions about key ideas in the course gives all teachers access to the ideas, and allows them to question, challenge, and debate those ideas. The evidence presented in the sections detailing teachers' opportunity to learn suggests that the ideas upon which teachers showed growth were indeed available in the public space. Perhaps the most salient examples of these understandings being available in the public space occurred during the discussions of the instructor's pedagogy. Kelsey, an experienced secondary teacher who had engaged in a great deal of work at the university with like-minded faculty, particularly around ideas related to the five practices, made the following statement in her second interview.

> Int.: Ok. Um, so my last question, for you on this, you've mentioned a few things already about this um, but I was wondering if there was anything else about the pedagogy of the course itself that helped you come to know or understand something about teaching mathematics.
>
> Kelsey: [pause] I mean I think just the way you stopped and had us think about it. Every time. [pause] I mean just, making things explicit, that way, instead of just kinda, [pause] y'know like modeling and hoping that we get, what you wanted us to. Um, I- I mean I think just doing that, [pause] is gonna buid a habit of, being reflective, on your own, d'y'know what I mean?
>
> Int.: Mhm.
>
> Kelsey: So, [pause] like I'm more like to, consider that question about myself. After having done that.
>
> Kelsey, Interview 2, Lines 407-418

Kelsey's comments emphasize the value of the public discussions around the five practices, particularly as they pertained to the instructor's moves in making his pedagogy visible and an object of discussion.

4.3.3.2. Constellations as rich sites for learning. Design Principle 3 offers the notion of constellations as being rich sites for learning. The section discussing knowledge of mathematics and mathematical activities suggest that these constellations might be rich sites for learning about mathematics, students as learners of mathematics, and mathematics teaching. In addition, the artifacts from the discussions of The Case of Isabelle Olson (Table 43) and The Case of Keith Campbell (Table 44) show teachers identifying the pedagogical moves a teacher makes to the mathematics learning of students in the classroom. One might wonder if the same pedagogical moves would have been as clearly identifiable had teachers not grappled with the mathematical task featured in the case prior to engaging with the case. As such, this design principle and the results in Tables 43 and 44 suggest that teacher learning about pedagogy, and specifically instantiations of the five practices, was enhanced by their engagement in a mathematical task as the start of a constellation of activities.

4.3.4.3. Modeling good pedagogy. Design Principle 6 suggests that modeling good pedagogy and making that pedagogy visible held the potential for teachers to learn about pedagogy. The instructor's intention in the course was to model good pedagogy, as conceptualized by the five practices. Further, by stepping to the side and making this pedagogy visible at points where the use of one or more of the practices was likely to support teachers' learning, the instructor intended to make his use of the practices an object of inquiry for teacher discussion and learning.

At the close of the course map activity in the second interview, in which teachers were asked to trace their learning in the course, the instructor asked teachers if there was anything about the pedagogy of the course itself that helped them to learn something about mathematics. Several teachers mentioned the instructor's pedagogy as a model in a variety of different contexts. The response of Sierra, a preservice elementary teacher, suggests that these discussions about the instructor's pedagogy were valuable sources of learning with the potential to translate into the teachers' classroom practice.

Instr.: Ok. So, my last question related to the course is, I was wondering if there was anything um, about the pedagogy of the course itself that helped you come to know or understand something about teaching mathematics.
Sierra: [pause] What do you mean.
Instr.: Um, anything about, how the course, was taught or enacted that helped you, understand something about teaching math.
Sierra: Like, in how you taught it or-
Instr.: Mhm.
Sierra: -how it was, laid out or [pause]
Instr.: Any of those would be fair game.
Sierra: [very long pause] Well I think it's important um, in the way that you worked through the tasks and whatnot with us I mean, I took a cl- a class with Dr. Smith and, she was similar in that she didn't just tell us, that we should present our tasks in this way. She presented the tasks in the way we should present them to our students and you did the same, type of thing um, in the course. You gave us a task and let us work with it and, laid out carefully y'know, the specific questions you were gonna ask us, to further our thinking, and how you would have us present our ideas in order to build off of each other's ideas in the class, you didn't just tell us that's what we're supposed to do, you know.
Instr.: Mhm.
Sierra: You actually demonstrated how we should be teaching our students. You know of course [pause] collegiate y'know math teachers are gonna, come up with different ways of solving than our kids will come up with, but they're still different from each other, so. You can still present and teach the task in the same way that we should be doing them in our room. So I think that's useful, in helping us, like once again it's just seeing how, y'know, professionals who have more experience than us, lay out, their lessons, and that can be useful to us in how we should, work to be able to- well, what we should be working towards, in our own classroom practices.

Sierra, Interview 2, Lines 479-505

4.4. Practices that Support Teaching

As noted in earlier chapters, the design of the course was also intended to provide teachers with an opportunity to examine practices that support teaching. In contrast to the practices categorized as knowledge of mathematics for student learning, these practices are not tied to particular student work or mathematical tasks. Rather, the practices are more general structures that support the everyday work of teaching. The example of one such practice that was selected to be addressed in the geometry and measurement course was routines.

Throughout the course, the instructor made use of particular routines to facilitate classroom discourse, to support students' engagement in the work of the classroom, and to manage the assets (both material and human resources) of the classroom. These routines varied in scope and frequency; some, such as the use of a file folder system for distribution and collection of papers, were used every day in predictable ways at predictable moments; while others, like presses for teachers to say more about their thinking, were more context-sensitive. All of these routines, however, operated in the service of the instructor's particular goals for the class at that moment.

4.4.1. Routines: Growth in teacher knowledge

During the course pre-assessment, teachers watched a video clip of a lesson from Cathy Humphreys (Boaler & Humphreys, 2005), an experienced middle school teacher, and were asked to identify moves the teachers made that supported classroom activity (see Appendix A, Part A). The term *routines* was not used at this point, as it was unlikely that teachers had an understanding of the specific definition of the term with respect to educational practice. During the post-course assessment, the same clip was shown, and teachers were asked to identify routines and classify them as *exchange*, *support*, or *management*.

The instructor and a second coder identified the routines in the video clip and classified teachers' responses with respect to the routines they had identified. It should be noted that after examining teachers' work and reviewing the clip, two additional routines were added to the master list, as it was deemed that they were indeed valid routines. The routine categories and brief descriptions are shown in Table 45. Teachers' responses on the pre- and post-course assessment were compared to the identified routines.

Table 45. Routines identified in the Cathy Humphreys surface area video.

Routine	Description	Type
Comment	Prompting students to comment on the ideas of others	Exchange
Agree/Disagree	Students take stance with respect to others' ideas and justify	Exchange
Small Group	Teacher directs small groups to debate a topic in a particular way	Exchange
Explain	Teacher presses for justification or explanation	Exchange
Hands-Check	Teacher asks for visual cues to check for understanding	Exchange
Prompt & discuss	Teacher asks for an argument about an idea	Exchange
Revoice	Teacher restates an idea, adding nuance or emphasis	Exchange
Call-on	Selects students to participate by table, and in cooperation	Exchange
Tools	Uses manipulatives & diagrams to facilitate explanations/demonstrations	Support

Prior Knowledge	Teacher connects to prior knowledge	Support
Closure	Teacher primes students to recall key understandings from the lesson	Support
Crediting	Teacher gives credit for ideas using student names	Support
Shift	Teacher flags an unplanned direction for the lesson (change in difficulty)	Support
Terms	Prompts for mathematical language to be used	Support
Good Question	Flags a question as significant	Support
Understand	Flags constructing one's own meaning as important in math	Support
Listen	Get the attention of the class focused on the presenters	Management
Hands-Cue	Raising of hands is used as the cue for wanting to speak	Management

Teachers showed change in 3 categories of routines from pre-course to post-course assessment. The number of teachers identifying *tools* that supported mathematical activity, such as manipulatives, diagrams, and the use of the overhead projector, increased significantly from pre-course to post-course assessment, $\chi^2(1, 50) = 11.5$, $p < 0.001$. The number of teachers identifying the *hands-cue*, or the particular method the teacher used to manage student interactions, increased significantly, $\chi^2(1, 50) = 11.0$, $p < 0.03$ with Yates' correction. The number of teachers who identified the flagging of *understanding* a mathematical idea decreased significantly, $\chi^2(1, 50) = 18.1$, $p < 0.001$. When examining the categories (*exchange, support*, and *management*) of routines identified, there was a significant increase in the number of management routines identified by teachers on the post-course assessment, $\chi^2(1, 50) = 10.7$, $p = 0.001$. On the post-course assessment, teachers' classifications of routines were compared to the instructor's classifications; 78% of the routines teachers identified were classified correctly as *exchange*, *support*, or *management*.

These results suggest a number of interesting conclusions. First, the significant increase in identifying *tools* and *hands-cue* are particularly notable, as these two types of routines may have been less apparent than other routines in the video, such as presses for explanations from students, the use of small-group interactions, and the repeated checks for understanding through agree/disagree votes. The use of the manipulative tools, the diagrams, and the systematic method for soliciting student contributions operated in the background of the classroom activity. The fact that teachers became more attuned to these routines suggests that they may have developed a deeper understanding and appreciation for the role of routines during the course. The decrease in the *understanding* routine is curious, since the teacher spends a great deal of time at the end of the clip flagging the key mathematics ideas as something to be understood rather than memorized.

Finally, the increase in management routines identified by teachers is surprising. Management routines tend to be the most ubiquitous, as they are useful across a wider range of pedagogical styles as compared to exchange or support routines. Thus, one may have expected teachers to be more able to identify these types of routines initially, as they were the most likely to manifest in their own teaching.

4.4.2. Routines: Opportunities to learn

Figure 42 shows the course activities related to routines in the course. The bulk of the activity around routines occurred during Classes 9 and 10, when routines were discussed and examined through the analysis and discussion of a video case of teaching. The instructor began the conversation by defining routines, and asked teachers to identify routines that he used during class. As a follow-up, teachers were asked to identify the routines they used in their own classrooms in an online discussion. Finally, teachers watched a video clip of Cathy Humphreys (Boaler & Humphreys, 2005) teaching a geometry lesson and were asked to identify routines, say how the routine supported student learning, and classify the routine as either *exchange*, *support*, or *management*.

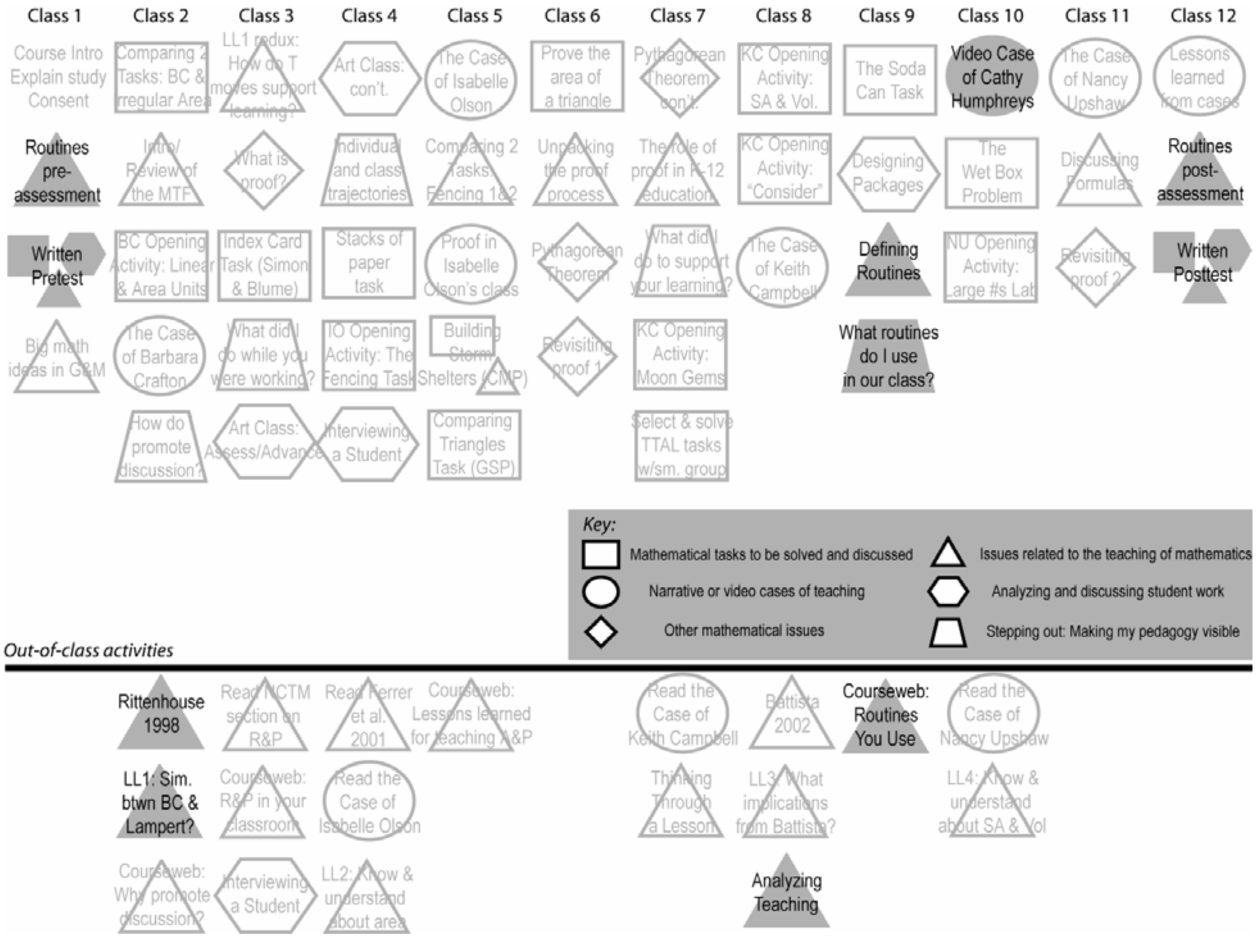


Figure 42. Course activities focused on routines.

These activities taken together constituted an opportunity to learn. Teachers were first made aware of routines (without the label attached) during the pre-course assessment through the analysis of the video case of Cathy Humphreys' teaching. The reading of the Rittenhouse (1998) article which examined the teaching of Magdalene Lampert made reference to routines, although this aspect of the article was not discussed. Classes 9 and 10 included a focused set of activities that defined routines, asked teachers to discuss the routines that the instructor used as well as those identified in a second clip of Cathy Humphreys' teaching, and then asked teachers to reflect on routines in writing through the identification and discussion of routines they used in their classroom through an online message board posting. Thus, teachers were able to discuss routines individually and in small groups, take part in whole-group discussions about routines, and reflect on routines in a way that connected to their own practice.

Despite the modest gains on the assessment, routines appeared to be a significant source of learning in the course. Of the 20 teachers interviewed, 10 teachers identified routines as something that they learned about during the course. In tracing their learning to particular activities, 7 identified the initial discussion of routines, 10 identified the discussion of the instructor's routines, 7 identified the online discussion of their own routines, and 9 identified the analysis of the video clip as contributing to their learning in the course. Given these data, it is clear that these activities provided teachers with the opportunity to learn about routines and how those routines can help to further the mathematical activity of the class. The excerpt below identifies some of the routines that teachers identified in the instructor's own teaching, which provides additional evidence of teachers' ability to identify routines.

Instr.: What routines do I use with you guys? Debbie?
Debbie: The one we like, keeping our papers in that thing [file box] so when we walk in we know to pick up our papers and sit where our [name cards] are.
Instr.: What purpose does it serve?
Debbie: More efficient, time saved.
…
Uma: At the beginning of class, you tend to go over, what we should have and that gives us an idea of what order we'll be using them in.
Ed: You always have us come up and present, when we're working on a problem or something like that. It's kind of supporting our different ideas and supporting us to think about our ideas.
Nancy: You usually have us work individually, then work as a group. It lets us think about it by ourselves, to see what ideas we have, and when we work with others, it lets us see different solution paths and just like how different people think, and that makes us knowledgeable about how our students might think when doing a similar problem.
…
Emily: Whenever we do something or are asking a question, you never tell us if we're wrong.
Noah: Along a very similar line you often don't tell us if we're right, and that promotes self-confidence in what we're doing.
…
Debbie: Wrapping everything up at the end of class and talking about pedagogy, we do that pretty much every class.
Instr.: So kind of reflecting on how I taught. [pause] There's something I say every once in a while that makes certain members of the class snicker, that might be a routine.
Kelsey: Say more about that, [entire class laughs] is that it?
Instr.: Is that a routine?
Kelsey: Yes.
Instr.: So say more about that. [laughter] So what kind of routines are these. What purpose do they serve?

Excerpts from Class 9

4.4.3. Connecting to Design Principles

The portion of the course focused on routines was more limited than those which sought to address knowledge of mathematics and mathematical activities and knowledge of mathematics for student learning. Two design principles – Design Principles 1 and 6 – are of particular relevance to the results related to routines. Design Principle 1 discusses the notion of public discussions as contributing to teachers' learning. The idea of public discussion forces the instructor to consider a particular set of exchange routines – in particular, exchange routines that support the ability of all teachers to enter the conversation, the establishment of conversational norms that support a positive intellectual climate, and that move the mathematical activity of the classroom forward. By using Design Principle 1 as a guiding influence in the design and enactment of the course, the instructor provided teachers with an opportunity to be exposed to a set of routines that supported the discourse-based class environment.

Design Principle 6 suggests that modeling good pedagogy and making that pedagogy visible held the potential for teachers to learn about pedagogy. As with the five practices, the instructor deliberately modeled a series of routines – exchange, support, and management – which he hoped through discussion would be made visible to teachers. The design principle might suggest that this modeling would have provided a source for teacher learning. As the excerpts from Class 9 above show, teachers were able to identify a variety of routines in the instructor's own pedagogy. Of the 20 teachers interviewed, 10 identified the activity in which teachers identified the routines the instructor used as a source of learning. This excerpt from Bridget's post-course interview emphasizes that this modeling was instrumental to teacher learning in a way that might not have otherwise been visible.

> Instr.: So you've- you've mentioned a few things already about the pedagogy of the course itself, that helped you to come to understand some things about teaching mathematics. You talked about the, doing the task and then doing the cases and- and the routines that I used. So I was wondering if- was there anything else um, about the pedagogy of the course itself that helped you come to know or understand something about teaching math.
>
> Bridget: [very long pause] Well the routines that, you used, are- were- I mean they were a little bit blatant some of them but, others, that were like more kind of like, I think support-
>
> Instr.: Mhm.
>
> Bridget: -were not so obvious and when we talked about them, [pause] I thought, Mike does that, he does, when he does this and that. And um, I think I do that when I do this and that, and this, in my classroom so. [long pause] I lost my train of thought. So discussing what the routines are and- and y'know, [pause] the purpose. [pause] They do serve a purpose.
>
> Bridget, Interview 2, Lines 530-541

4.5. Comparing Course Teachers with a Contrast Group

As noted in Chapter Three, a contrast group was recruited to determine what differences, if any, there were between the mathematical knowledge for teaching of a group that participated in the course and a group of teachers that did not. In order to determine differences, the contrast group's performance was compared with the treatment group's pre-course performance. On dimensions that showed no significant difference, a stronger argument can be made attributing gains in the treatment group to the course experience. Differences in which the treatment group outperformed the contrast group pre-course were noted. These differences suggest that on these dimensions, the treatment group's performance may have been better than the average teacher with equivalent background. Differences in which the contrast group outperformed the treatment group's pre-course performance were pursued further. Subsequent tests were performed to compare the contrast group to the treatment group's post-course performance on these measures.

Teachers in the contrast group matched the two major demographic groups in the treatment cohort: preservice and practicing secondary teachers. In total, 11 teachers completed some or all of the assessment instruments. Nine teachers – 2 preservice teachers and 7 practicing teachers (2-10 years' experience) – completed the pre-course assessment. Four teachers – 2 preservice and 2 practicing teachers (2-4 years' experience) completed the interview. Two of these teachers completed both instruments, one preservice and one practicing.

All written assessments and interview transcripts from the contrast group were coded in an identical manner to the treatment group. Results were compared using chi-square analyses for categorical data and *t*-tests for numeric data. Because of the relatively small size of the contrast group, particularly with respect to the interviews, Yates' correction for chi-square was employed to compensate. In all, 375 statistical comparisons were performed. There were 11 measures on which the treatment group outperformed the contrast group and 6 measures on which the contrast group outperformed the treatment group. Differences on those measures are presented below, organized using the knowledge needed for teaching framework.

4.5.1. Knowledge of Mathematics and Mathematical Activities

In general, teachers in the contrast group exhibited similar performances on mathematical tasks as their treatment group counterparts. There were no significant differences in rubrics which assessed the correctness of responses on the pre- and post-course assessments or the level of detail in teachers' responses.

Three tasks showed minor differences in representational use. On the *Area of a Parallelogram* task, the mean number of representations used by teachers was significantly greater for the contrast group, both for all responses ($t(17) = -2.96$, $p = 0.087$) and for correct responses only ($t(6) = -2.63$, $p = 0.039$). When individual representations were examined, the contrast group showed significantly higher use of the symbolic/formula representation ($\chi^2(1,34) = 9.67$, $p < 0.05$ with Yates' correction). The *Fence in the Yard* task also saw the contrast group with significantly higher use of the symbolic/formula representation ($\chi^2(1,34) = 7.64$, $p < 0.01$ with Yates' correction). Performance on the volume portion of the *Painting the Living Room* task showed that significantly more teachers in the contrast group used multiple representations ($\chi^2(1,34) = 4.74$, $p < 0.05$ with Yates' correction). Together, these differences suggest that the contrast group may have had some additional representational fluency as compared to the treatment group, as evidenced by the increase in the mean number of representations. The significant differences in symbolic/formula use suggest that an increased use of this representation in the contrast group may have resulted in the overall increase in mean number of representations.

Given that the contrast group outperformed the treatment group pre-course, results from the treatment group post-course were compared to the contrast group. On measures related to the symbolic formula use, significant differences were found for the *Area of a Parallelogram* task

($\chi^2(1,34) = 4.74$, $p < 0.05$ with Yates' correction) and the *Fence in the Yard* task (($\chi^2(1,34) = 4.74$, $p < 0.05$ with Yates' correction). This is due largely to the fact that all 9 teachers in the contrast group used a symbolic representation. However, on measures related to the use of multiple representations, the treatment group's post-course performance did not differ significantly from the contrast group's performance, both for all responses ($t(24) = -0.91$, $p = 0.38$) and for correct responses only ($t(6) = -1.43$, $p = 0.20$).

The coding of the interview items showed several significant differences between treatment and contrast group, with most showing the contrast group outperforming the treatment group. These results should be interpreted with caution, however, as only 4 teachers were interviewed in the contrast group. Because of these disparities, results comparing the numbers of lines of interview text coded were deemed inappropriate. Only significant results with respect to the proportion of talk are reported here.

In examining responses to the *Box Task*, the percentage of talk ($t(6) = 3.04$ $p = 0.023$) coded as evidence of a layering strategy was significantly higher for the treatment group. This suggests that the treatment group used more layering strategies at the start of the course as compared to the control group.

On the questions involving reasoning and proof, there were several differences in the talk of the treatment group as compared to the contrast group. Nearly all differences showed the treatment group prior to the course talking more about the key aspects of proof as compared to the contrast group. With respect to the definition of proof, treatment group teachers talked significantly more about proof as a mathematical argument ($t(17) = 2.78$, $p = 0.013$). When teachers' reasons for identifying and rating proofs were coded, teachers in the treatment group talked significantly more about valid method as a means of identifying ($t(13) = 2.77$, $p = 0.016$)

and rating ($t(19) = 2.25$, $p = 0.036$) proofs, and significantly more about concrete features of proofs when rating them ($t(21) = 2.30$, $p = 0.031$). Together, these results suggest that the treatment group was more attentive to the idea of a mathematical argument as a key feature of proof, and of particular methods being more or less valid for proving. They also attended more to the concrete features of a proof in rating it. Note that this result does not imply that the contrast group attended more strongly to mathematical features of the proof, as no significant differences in categories relating to mathematical features of the proof were found.

Finally, when considering the role of proof, significant changes were found favoring the treatment group on three types of talk: proof as verifying truth ($t(22) = 4.07$, $p < 0.001$), proof as explaining why ($t(21) = 2.57$, $p = 0.017$), and proof as creating new mathematics ($t(19) = 2.11$, $p = 0.048$). This is due in large part to the small number of lines coded as related to the role of proof in the contrast group interviews. Only 20 lines out of 247 in the contrast group interviews were coded as evidence of any of the 5 categories related to the role of proof. This suggests that the treatment group was more attuned to the roles of proof in the mathematical domain at the start of the course.

In general, with respect to knowledge of mathematics and mathematical activities, comparison of the treatment and contrast group showed that most differences favored the treatment group. In particular with respect to the battery of questions on proof, the treatment group at the point of the first interview was more attentive to many of the key aspects of the definition and role of proof, as well as criteria for evaluating proofs. The contrast group did show a slightly increased use of representations in tasks related to geometry and measurement, with most of the change being due to an increased attention to the symbolic representation.

These differences in mean numbers of representations were not significant when compared to post-course treatment group performance.

4.5.2. Knowledge of Mathematics for Student Learning

In general, teachers in the contrast group exhibited similar performances on tasks related to the five practices for productive use of student thinking as their treatment group counterparts. Some significant differences were found on the types of questions asked (monitoring), attention to the selection of student responses for presentation, and connecting responses. All differences favored the treatment group.

When teachers' questions were assessed on the *Minimizing Perimeter Lesson Plan* task in the interview, significant changes were found in the number of high-level (score point 4) questions asked favoring the treatment group ($t(19) = 3.16$, $p = 0.005$). This suggests that teachers in the treatment group were more likely to ask higher-level questions as part of a lesson planning task at the start of the course.

With respect to selecting student responses to share, the treatment group talked significantly more about sharing student responses as compared to the contrast group ($t(21) = 3.06$, $p = 0.005$). This is due in large part to the fact that only 5 lines out of 331 in the contrast group were coded as evidence of selecting student responses. Similar results held for connecting responses; no contrast group interview showed evidence of an intent to connect shared student responses, resulting in a significant difference favoring the treatment group ($t(19) = 3.94$, $p < 0.001$).

In examining teachers' responses to the *Considering Student Work* task, in which teachers were asked to select and sequence student work and share their reasons for their selection, no differences were found in the responses selected or the sequences in which they were shared. There was, however, one difference in the reasons cited for sharing responses. Contrast group teachers were significantly more likely to share a response with the intent of sparking further discussion ($\chi^2(1,34) = 4.87$, $p = 0.03$ with Yates' correction). When compared to the treatment group's post-course assessment, there were no significant differences between the treatment and contrast group ($\chi^2(1,33) = 2.12$, $p = 0.15$ with Yates' correction). This suggests that the treatment group did improve, but in a way that matched the level of the contrast group. This result may be due to the fact that of the 9 teachers taking the assessment, 7 took the assessment following a professional development meeting for which the sharing of student responses was a topic of conversation, which may have biased the contrast group on this dimension. Five of the 7 teachers who selected responses with the intent of sparking further discussion were the ones who had participated in the professional development.

In all, the slight differences in teachers' attention to the five practices for productive use of student work suggest that the treatment group was more attentive to the practices at the start of the course. This is not surprising, as many of the teachers in the treatment group had been exposed to coursework in which the practices had been implicitly or explicitly addressed. The same cannot be said for the contrast group.

4.5.3. Practices that Support Teaching

When teachers' responses were examined for the Routines activity, no significant changes were found on any dimension.

4.5.4. Summary

Across the variety of instruments used to assess mathematical knowledge for teaching, few differences were found between the treatment group and the contrast group. Performances of the two groups were only significantly different on less than 5% of the measures tested. When differences were found, the majority favored the treatment group as being more attentive to the knowledge that was the target of the geometry and measurement course. These data show that the treatment group was comparable to the contrast group, suggesting that learning gains seen in the treatment group were not a result of particular deficits as compared to teachers with a similar background; rather, these learning gains were a result of their experiences in the course.

5. DISCUSSION

5.1. Introduction

This study examined changes in teachers' knowledge needed for teaching through their engagement in a practice-based teacher education experience focused on the content of middle grades geometry and measurement. The course featured activities that targeted knowledge of mathematics and mathematical activities, including content knowledge in the domain and content knowledge for teaching; knowledge of mathematics for student learning, specifically the five practices for productive use of student thinking; and knowledge of practices that support teaching, with routines being the specific practice examined. Results showed growth in teachers' knowledge in all three areas.

5.1.1. Knowledge of Mathematics and Mathematical Activities

First, with respect to knowledge of mathematics and mathematical activities, teachers grew in their knowledge of relationships between measurable attributes of geometric figures. The course targeted relationships between two-dimensional attributes – dimension, perimeter, and area – and relationships between three-dimensional attributes – dimension, surface area, and volume. These relationships are typically problematic for both teachers and students (e.g. Battista & Clements, 1996, 1998; Battista, 2002; Bright & Hoeffner, 1993; Clements & Battista, 1989; Chappell & Thompson, 1999; Hoffer, 1983; Martin & Strutchens, 2000; Sarama et al., 2003). Teachers in general were able to understand the idea that perimeter and area have a non-constant relationship prior to the course. However, when confronted with mathematical tasks that probed the relationship more deeply, teachers' understandings were relatively frail. Following the course, teachers improved in their ability to explain how changes to the dimensions of a figure impact perimeter and area for two-dimensional figures and surface area and volume for three-dimensional figures; to explain the relationships between linear and square units and between square and cubic units; and to demonstrate their understandings of perimeter, area, surface area, and volume using a variety of tools and representations. Teachers also grew in their ability to link dimension, surface area, and volume through spatial structuring strategies.

In addition to these basic understandings of content knowledge in the domain, which any professional user of mathematics is likely to need to know, teachers also grew in content knowledge for teaching with respect to relationships between measurable attributes of geometric figures. Teachers were better able to identify the big ideas in geometry and measurement in the middle grades related to dimension, perimeter, area, surface area, and volume following the course, and to modify cognitively demanding mathematical tasks in ways that supported students' engagement and provided students with opportunities to explore those big ideas. They became more representationally fluent in solving problems related to measurable attributes of geometric figures, using more representations and strategies in work on those mathematical tasks. Teachers also were better able to link notions of spatial structuring to particular pedagogical approaches and instructional decisions in ways that had greater potential to support student learning.

With respect to reasoning and proof, teachers conceptions of and abilities to identify, evaluate, and create proofs changed following the course. Teachers grew in their content knowledge in the domain; teachers increased in the extent to which they talked about the 4 key aspects of the definition of proof when asked to define proof and in their ability to articulate the roles of proof in the mathematical domain, relied less on surface-level features in identifying proofs and non-proofs and increased in their use of mathematically-related features, and grew in their abilities to write proofs and proof-like mathematical arguments. Teachers also came to understand aspects of content knowledge for teaching related to reasoning and proof, increasing in their consideration of proof as an explanatory tool; proof as a means of communicating new knowledge, creating knowledge, and systematizing the mathematical domain; and student discourse as supporting work on proof.

Teachers did not show any growth in their ability to identify explanations as proofs or non-proofs. For one explanation that was not a proof, Explanation 3a, the ratings teachers assigned to the explanation actually increased significantly between the first and second interview. One possible reason for the lack of differences in the identification of proofs was that although the essential characteristics of proof were discussed and debated throughout the course, it was only during the final discussion in Class 12 when some of the critical disagreements were discussed and settled. Thus, teachers did not have an opportunity during the course to apply the final agreed-upon criteria to evaluating proofs. Additionally, teachers had limited opportunities to actually construct proofs, with only one activity in the course focused on writing mathematical arguments. This trade-off between providing opportunities to grapple with and revisit the key aspects of proof and having a set of criteria with which to evaluate proofs may have led to this result.

Explanation 3a, which was a set of 3 examples (acute, right, and obtuse) explaining why the sum of the measures of the angles in a triangle is 180 degrees, is an interesting case. One explanation for the increase in rating may have been a result of teachers broadening their view of activities that can lead to proof. Initially, many teachers had a fairly limited view of proof, and did not see a collection of examples as a tool that could ultimately lead to generalization and proof. At the close of the course, teachers seemed to have a much broader view of proof, considering the use of examples as a starting point for making generalizations. This view is supported by classroom conversations, as well as teachers' responses to the *Responding to Student Claims* task, which showed more teachers pressing the student portrayed in the task towards a generalization based on the specific examples.

A second possible explanation for the increase in rating was the notion of generality, which was a highly salient topic of conversation with respect to proof. Several of the discussions regarding the nature of proof revolved around a proof having to hold for a particular class of relationships, or having a level of generality. Explanation 3a contained examples of each of the three types of triangles – acute, right, and obtuse. It is possible that teachers rated this explanation higher because they saw the examples as a step towards generality, in that they represented each of the three classes of a triangle.

5.1.2. Knowledge of Mathematics for Student Learning

The geometry and measurement course also dealt with knowledge of mathematics for student learning, specifically conceptualized as the five practices for productive use of student work (Stein et al., submitted). These practices were either implicitly or explicitly known to many of the teachers in the course through their work in previous courses or professional development experiences. When asked to plan and describe a lesson around a geometry and measurement task at the end of the course, teachers showed growth in their ability to anticipate student solutions, to monitor student work through questions that support high-level engagement with the task, to select and sequence student responses to be shared in a whole-class discussion, and to connect student solutions shared in a whole-class discussion. The course addressed these five practices in a number of ways – through activities that targeted the practices individually or in small clusters through examining and responding to student work, and through activities in which teachers were asked to use the practices to create a complete lesson plan.

5.1.3. Knowledge of Practices that Support Teaching

The course also addressed practices that support teaching. These are practices that are less content-specific than the five practices, but serve to advance the mathematical activity of the classroom. Routines were selected as the focus for the geometry and measurement course and measured through the examination of video records of an exemplary teacher's practice in teaching geometry and measurement lessons in a middle grades classroom. Teachers demonstrated the ability to identify routines in a video record and to link those routines to ways of advancing the mathematical activity of the class. But perhaps most importantly, teachers approached routines as an object of inquiry. Previous work related to routines has demonstrated that expert teachers have deep-rooted reasons for even the smallest of routines in the classroom, and these reasons are frequently linked to beliefs about student learning and/or sociocultural issues. Novice teachers often approach routines simply as a means to solve an organizational or management problem, without necessarily considering the reasons for or implications of the routine. Teachers' increased attention to management routines, and the linking of these routines to the mathematical activity of the classroom, is particularly notable. Of the three categories of routines (support, exchange, and management), one might expect management routines to be the most pragmatic in nature and the least-often linked to issues of student learning. Considering these management routines and their links to issues of learning in the course may give teachers reason to revisit and examine their own routines and consider how they relate, or do not relate, to issues of student learning and sociocultural issues.

5.1.4. Opportunity to Learn

An analysis of course activities demonstrated that the course provided teachers with the opportunity to learn aspects of knowledge needed for teaching on which they showed significant growth. Teachers had opportunities to consider the key mathematical and pedagogical ideas through engagement in course tasks, public discussion in the course, and written activities that offered opportunities for teachers to reflect and connect to their own mathematical and pedagogical experiences. Teachers' classroom discourse, writing assignments, and descriptions of their own learning in post-course interviews suggest that the course activities were influential in providing them with opportunities to learn knowledge needed for teaching related to geometry and measurement.

5.1.5. Contrast Group

Results from the contrast group showed that on over 95% of the measures tested, the pre-course performance of the treatment group matched that of the contrast group. These data show that the treatment group was equivalent to the contrast group, allowing changes in the treatment group's performance to be attributed to the course. With respect to the differences, the treatment group outperformed the contrast group on 11 measures (3%). Many of these differences in performance related to the five practices. The five practices were an area of heavy focus for a number of the teachers in the treatment group, most notably the 9 secondary MAT teachers who had just completed a course for which these ideas were highly valued. This suggests that the treatment group may have differed slightly from a group of teachers of similar background on this particular dimension. However, the treatment group still showed significant gains on these measures post-course.

The contrast group outperformed the treatment group pre-course in three areas: use of multiple representations, use of the symbolic/formula representation, and on the selection of student responses to share for the purpose of further discussion. These differences may have been due to the population from which much of the contrast group was recruited. Of the 11 teachers in the contrast group, 8 of these teachers were participants in a professional development project that targeted ideas including connections among representations and the five practices for productive use of student thinking. The administration of the written assessment to 7 of the teachers occurred following a session that focused on the sharing and discussion of student solutions, and may have biased the contrast group's thinking with respect to these ideas. The increased use of representations may have also been due to teachers' engagement in the professional development project.

5.2. Contextualizing Teacher Learning: Implications and Recommendations

The results of the study demonstrate that teachers grew in their mathematical knowledge for teaching middle grades geometry and measurement topics. Teachers' growth in content knowledge in the domain and content knowledge for teaching hold several implications for the existing research base and for future research regarding teachers' knowledge of geometry and measurement.

5.2.1. Knowledge Needed for Teaching Framework

This study made use of the *knowledge needed for teaching framework*, originally proposed by Ball, Bass, and Hill (2004), with several modifications. Different labels for existing categories were used that captured additional nuances of the facets of the framework. A third category – practices that support teaching – was added in an effort to integrate more content-general teaching ideas with the more mathematical aspects of knowledge needed for teaching. In addition, the five practices for productive use of student thinking (Stein et al., submitted) were used as instantiations of knowledge of mathematics for student learning.

Teacher learning outcomes for the course spanned all three facets of the framework. This suggests a number of important conclusions. First, the framework has the potential to serve as a unifying theme for the design of an instructional intervention for teachers. Second, the inclusion of practices that support teaching, specifically related to routines in the case of this study, has the potential to broaden the scope of the framework to include pedagogical ideas related to mathematics teaching that may not have fit into the framework as previously conceived. Finally, other theoretical constructs, such as the *mathematical tasks framework* (Stein et al., 2000) and the five practices (Stein et al., submitted) can be situated within the knowledge needed for teaching framework. In this way, knowledge needed for teaching serves as a unifying theme that has the potential to bring together a variety of related research-based theoretical constructs in the service of a practice-based teacher education experience.

5.2.2. Knowledge of Relationships between Measurable Quantities

As mentioned in Chapter One, very little research to date has explored teachers' content knowledge related to geometry and measurement. Specifically, very few studies exist that explore how teachers make sense of the relationships between dimension, perimeter, and area and dimension, surface area, and volume. This study suggests that teachers' abilities to solve routine problems that reflect middle grades content is relatively sound. However, when confronted with problems that forced teachers to consider variant and invariant properties of geometric figures, such as the *Area of a Parallelogram* task, or involved non-routine situations, such as the *Box Task*, teacher performance was relatively poor. This suggests that similar to research findings related to teachers' conception of function (e.g., Even, 1993; Leinhardt, Zaslavsky, & Stein, 1990; M.R. Wilson, 1994), teachers' abilities to articulate the relationships between measurable quantities of geometric figures are limited to a narrow class of routine relationships, and do not represent a broad conceptual understanding of these relationships. These results suggest that teacher education experiences should provide increased attention to teachers' content knowledge in the domain related to geometry and measurement, specifically relationships between measurable quantities of geometric figures. Further, these results may imply a connection between students' poor performance on national and international assessments in the domain of geometry and measurement (e.g., Martin & Strutchens, 2000; NCES, 2000; Sowder, Wearne, Martin, & Strutchens, 2004) and the limited state of teacher content knowledge. Future research that explores teacher knowledge and the achievement of those teachers' students would add to the field's knowledge of this relationship between teacher knowledge and student performance.

Moreover, teachers' content knowledge for teaching was shown to be relatively narrow for this group of teachers at the start of the geometry and measurement course. While teachers were able to solve routine problems involving relationships between measurable attributes of geometric figures, they were generally unable to approach the tasks in a variety of ways using a variety of representations. On written work, the ways in which teachers connected (or failed to connect) multiple representations was unclear. Teachers' content knowledge for teaching is likely to impact the opportunities that students have to learn, particularly for classrooms in which teachers engage students in mathematical tasks that could be solved and represented in multiple ways. Once again, further research exploring teachers' content knowledge for teaching and assessing their students' learning would serve to link differences in this knowledge to differences in student outcomes. Work by Hill, Rowan, and Ball (2005) has linked increases in knowledge needed for teaching to gains in student learning; however, additional investigation into how increases in knowledge needed for teaching impacts teachers' practice would add to the field's understanding of the links between teacher learning and student learning.

While this study adds to the field's understanding of teachers' knowledge of mathematics and mathematical activities, further research into teachers' content knowledge in the domain and content knowledge for teaching involving relationships between measurable quantities of geometric figures is warranted. The course addressed the relationships between dimension, perimeter, and area and dimension, surface area, and volume in a number of ways; however, the scope of these activities was limited by the nature of the 6-week course. A more comprehensive set of experiences related to measurable attributes of geometric figures may reveal additional information about teachers' understandings and how they change as a result of their engagement in a learning experience.

5.2.3. Knowledge of Proof

The results of the study with respect to proof are particularly interesting. Proof was a key topic of conversation throughout the course; in particular, the bulk of the course activities related to proof involved identifying what proof is, the purpose proof serves in the mathematical domain, and how proof might be useful in the classroom. As such, the changes in teachers' conceptions of proof and the role of proof both in the domain and in the classroom are not surprising. Few activities asked teachers to create proofs on their own or evaluate the proofs of others, providing teachers with a relatively limited opportunity to learn as compared to other topics in the course. Therefore, the growth exhibited by teachers in creating proofs and proof-like arguments on the pre- and post-course assessment is notable.

First, the fact that teachers showed some improvement in their ability to construct proofs or proof-like arguments is notable given that only once during the course did teachers engage in actually constructing a proof. This activity, featured in Class 6, asked teachers to prove the formula for the area of a triangle; this is a likely explanation for the more significant growth in the post-course assessment on the triangle proof as compared to the parallelogram proof. Another important caveat with respect to the parallelogram proof is that many of the proofs teachers provided were ruled to be not general because they did not consider the case of the "top-heavy" parallelogram (see Appendix E for more information). This special case is rarely seen in illustration, and rarely dealt with in proofs of the area of a parallelogram that rely heavily on diagrams and area preservation arguments; thus, it is plausible that teachers simply did not realize that this case needed to be addressed.

In examining the rubric used to evaluate the explanations, it is worth noting that score point 3 is the threshold for a complete mathematical argument; this argument may not provide a great deal of explanatory power and may not be fully general, but it is general for at least a class of cases and is a sound mathematical argument. In the case of both proofs on the pre- and post-course assessment, the mean score changed from less than 3 to greater than 3. Thus, it is reasonable to state that the average teacher in the course progressed from not being able to construct a fully-formed mathematical argument to being able to construct a mathematical argument that was complete and valid.

What accounts for teacher growth in writing mathematical arguments? One possible explanation for this change is the work done on articulating the key aspects of mathematical proof and the purpose proof serves in the mathematical domain. By debating the nature of proof, teachers addressed key issues such as the generality of proof, the fact that proof is based on established mathematical facts, and the idea of proof as a means of explaining why a mathematical conjecture is true. The debate of these key ideas related to proof may have caused teachers to consider these aspects when creating their own proofs. The fact that the rubric evaluated the level of generality for which teachers created a proof and that generality was a particularly salient discussion point in the course supports this conjecture.

Given this conjecture, one might wonder why teachers did not seem to grow in their identification of mathematical arguments as proofs or non-proofs on the first and second interviews (see Appendix B, Task 2b and Appendix C, Task 3b). There are a number of reasons that may explain why teachers failed to show growth in their ability to classify the explanations. First, classifying explanations as proofs or non-proofs was not a major focus of the course; in fact, there was only one activity (Considering Pythagorean Theorem Proofs; see Figure 34) in which teachers were asked to do so, and the correct classification was not the primary aim of the activity. Second, this task occurred in the interview setting, and teachers had limited time and resources with which to consider the explanations. Several teachers expressed uncertainty about particular mathematical assertions in the proofs, and had no way of verifying or arguing against their validity other than their memory of geometry.

Finally, change in teachers' thinking about proof was likely to have been limited simply due to the short amount of time (5-6 weeks) between interviews. Teachers may have been considering new aspects of the definition of proof, but had not had the opportunity to integrate all these aspects and apply them across a variety of test cases. For example, many teachers in the second interview asserted that Explanation 3a was general because it tested the cases of acute, right, and obtuse triangles, but the explanation only does so for single examples. These teachers were honoring the notion that proof is general, which may not have been a consideration in the first interview, but their conception of general was not developed enough to recognize that testing a single example for each of the three types of triangles does not guarantee generality. This notion of generality and cases may also have accounted for the significant increase in the rating of Explanation 3a, the only significant change in ratings across the 8 explanations.

In all, teachers' growth with respect to reasoning and proof was notable, particularly with respect to previous findings in the literature. One of the canonical findings in the study of students' conceptions of proof is that students saw limited utility for proof (Chazan, 1993; Miyazaki, 2000; Senk, 1985, 1989). Teachers' growth in their conceptions of proof and the role of proof, both in the mathematical domain and in the classroom, has the potential to impact their students' perceptions of the role of proof. Moreover, the criteria teachers used to identify and evaluate proofs broadened to include a variety of forms, focusing more on mathematical aspects and less on surface-level features. This suggests that in teachers' classrooms, they may be more open to accepting and exploring proofs that do not necessarily adhere to a traditional two-column format.

The fact that teachers grew in their consideration of generality and validity as a key characteristics of proof responds to a series of research studies (e.g., Fischbein & Kedem, 1982; Galbraith, 1981; Goetting, 1995; Lovell, 1971; Martin & Harel, 1989; Mayberry, 1983; Porteous, 1986; and Vinner, 1983) which found that many teachers and students did not appreciate the generality of proof and did not see proof as immune to counterexamples. Additionally, the results with respect to proof show growth on several of the dimensions explored by Knuth (2002a). The features identified by Knuth (2002a) as those which teachers most commonly used to evaluate proofs were used to categorize teachers' responses to the identifying and rating proof tasks in the geometry and measurement study. Teachers showed growth on several of the categories related to mathematical features of the proofs, and decline in subjective and surface-level categories. These findings suggest that the teachers who experienced the geometry and measurement course grew in their conceptions of proof in ways that responded to typical deficiencies or misconceptions identified in the literature.

Finally, the notion of proof not being concentrated in a constellation, but being spread across the course in a variety of contexts and revisited may have contributed to the learning that did occur. By having the opportunity to posit ideas, to reflect on those ideas with respect to their own work and new experiences, and revisit those understandings, teachers' conceptions of proof may have grown in ways that would not have been afforded by a similar set of activities in a more concentrated period of time. The use of revisiting, which draws out the consideration of an idea over a longer period of time and in ways that invite metacognition and reflection, may be potentially powerful for mathematical ideas such as proof, which represent both mathematical content and more general mathematical processes. Data from a similar course focused on algebra as the study of patterns and functions positioned the definition of function as the subject of revisiting; analyses of data from that course experience suggests that this experience was also a powerful one for teacher learning (Steele et al., in preparation).

There were, however, some questions raised by the data from this study. The lack of significant change in teachers' ability to identify and rate explanations suggests that work on proof in the course failed to sufficiently address this issue. The fact that contact with teachers was limited to a 6-week coursework experience and that work on proof was not the sole focus of the course suggests that a more comprehensive and sustained teacher education curriculum related to reasoning and proof is needed, as suggested by Stylianides and Silver (2004). Additionally, teachers increased in the amount of consideration given to proof as an explanatory tool in the classroom and the role of discourse in promoting reasoning and proof. Future research should examine whether these changes in teachers' content knowledge for teaching impact their practice; specifically, do these changes in teachers' conceptions of the role of proof help teachers to position proof as a means of facilitating students' exploration, construction, and development of strategies for making sense of mathematics (Clements & Battista, 1992; Greeno, 1980).

The ultimate goal of engaging teachers in teacher education experiences such as the geometry and measurement course is to improve student learning. This improvement can be conceptualized as a three-stage process: improving teacher knowledge of content, broadly defined; showing changes in teacher practice with respect to that content, and tracking improvements in student learning. The results of the geometry and measurement study show teachers taking the first step in the process: showing growth in their content knowledge, as conceptualized by the knowledge needed for teaching framework.

5.2.4. The Five Practices for Productive Use of Student Thinking

As noted previously, teachers grew in their consideration and use of the five practices for productive use of student thinking in the service of planning lessons and considering student work artifacts. These results hold a number of important implications for teacher knowledge.

The geometry and measurement study was the first study that investigated growth in teachers' use of the practices as an *a priori* construct. Prior research regarding the practices identified them as emergent constructs in a content-focused mathematics methods course (Stein et al., submitted). Thus, the results demonstrate that the five practices can be successfully integrated into a teacher education experience and that growth in the practices can be measured and demonstrated.

Growth on the five practices is particularly notable for the population in the geometry and measurement course. Of the 25 teachers in the course, 22 teachers had engaged in either a coursework or professional development experience that targeted the practices in some way, either with a different content focus (proportional reasoning or algebra) or in a content-general way. Despite having previously engaged in work related to the practices, teachers showed significant growth in the practices in the context of geometry and measurements tasks. This finding suggests that there may be a value added in revisiting the five practices with different content foci. In particular, the growth in teachers' considerations of multiple student solutions, their increased ability to write questions that targeted the key mathematical ideas in a geometry and measurement task, and to organize a whole-class discussion focused on particular mathematical goals suggests that integrating the five practices into a sustained consideration of issues related to content knowledge held a particular added value for teachers in considering how they might operationalize that content knowledge in work with students. Increased attention to these ways of modifying and implementing cognitively challenging tasks have the potential to support students' engagements with those cognitive demands at a high level, leading to increased student learning (Stein & Lane, 1996; Stein et al., 2000). Moreover, for teachers who had already had exposure to the five practices, this repeat encounter with these ideas in the context of geometry and measurement appears to have resulted in additional growth in their use of the practices in a geometry and measurement context.

Given that there are few empirical studies related to the five practices (Stein et al., submitted), further research into the practices is warranted. Specifically, integrating the five practices with other mathematical content is an area ripe for investigation. Additionally, investigating the transfer and generalizeability of the practices across content and across grade levels should be examined. The five practices were also embodied in the lesson planning framework (Thinking Through a Lesson Protocol) used in the course. Teachers' use of the TTAL protocol and the impact on knowledge of the five practices on actual classroom teaching practice should be investigated; the former is at present the subject of a research study by Hughes (in preparation).

5.2.5. Routines

The inclusion of routines into the geometry and measurement course represented an attempt to integrate the content-neutral construct into a teacher education in a way that brought to light how routines serve to move the mathematical activity of a classroom forward. Based solely on the results from the pre- and post-course assessments, teachers' views of routines changed. The fact that teachers spoke at length in post-course interviews about routines and the fact the discussion of routines was influential in their learning suggests that teachers did come away from the course with new knowledge related to routines. Specifically, I argue that teachers came to consider the underlying reasons behind particular routines, were able to use particular categories tied to these reasons to classify routines by function, and came to see routines as an object of inquiry with respect to the course and with respect to their own instruction.

Conclusions across the three specific routines for which teachers showed change (increase in *tools* and *hands-cue*; decrease in *understand*) are unclear. However, there are some promising implications to changes in the way routines were framed for teachers. Ordinarily, routines such as a method for soliciting student responses (*hands-cue*) and the availability of particular classroom materials (*tools*) are seen as management and resource issues. One way to interpret teachers' increased attention to these two routines is that they began to see them having a purpose beyond serving as a solution to an organization problem; teachers saw them as a means for advancing students' mathematical understandings and as supporting the mathematical work of the class. This link between teachers' intended practices with respect to student learning and their routines is an issue that Leinhardt and Steele (2005) identify as crucial to reforming teachers' practice: providing teachers with a means to critically consider their classroom routines in the light of changes they wish to make to their practice in the service of fostering students' mathematical understandings. Often when teachers seek to improve their practice with the intent of fostering meaningful learning of mathematics, the routines they have used remain unexamined in light of their new pedagogical goals. In considering routines such as *hands-cue* and *tools*, teachers in the geometry and measurement course may have become more attuned to the range of conditions in the classroom that can support or inhibit mathematical learning.

Another limitation of the findings related to routines was the relatively narrow scope of the discussion of routines in the course. Routines were discussed late in the course, and only for three activities across two classes. Despite this fact, teachers were able to identify a variety of routines in the instructor's own teaching of the course, as well as a range of routines in a video clip of a middle school classroom. A fruitful avenue for further research would be to integrate the discussion of routines in a more pervasive manner into a course or series of course experiences. Additionally, providing teachers with opportunities to document their own routines, to observe the classroom practice of other teachers and identify their routines, and to observe a teacher (either live or through video) longitudinally across a number of lessons may afford more robust results with respect to teacher learning related to routines.

Routines were selected as a particular example of practices that support teaching: aspects of pedagogy that are applicable and recognizable across a wide range of content, yet that serve to advance the mathematical activity of the classroom. Future research should continue to identify these practices, either based on prior content-general teacher research or through the identification of new constructs, and examine how these practices might be raised as objects of inquiry in a teacher education experience. Two examples of other practices that support teaching are metatalk and classroom/intellectual culture (Leinhardt & Steele, 2005). While both practices have been studied and, in the case of classroom culture, integrated into teacher education, a productive direction for future research would be to investigate ways to integrate these practices into other content-focused method courses and to study the impact of learning related to these practices on teachers' classroom practice.

5.3. The Design of Teacher Education Experiences: Implications and Recommendations

In addition to holding implications for teacher learning, the study of the geometry and measurement course holds a number of implications for the design of teacher education experiences. The course adds to the field's understandings of learning from practice-based teacher education experiences, of the potential value of content-focused methods courses, and of bringing teachers from a variety of backgrounds and grade levels together. Additionally, the study of the course built on previous design work related to practice-based teacher education, resulting in a refined set of design principles, which hold the potential for other mathematics educators to leverage in the design of teacher education experiences which meet the needs of varying populations.

5.3.1. Practice-Based Teacher Education and Content-Focused Methods Courses

The design and implementation of the geometry and measurement course was grounded in a research base that suggests that positioning the work and artifacts of teaching as objects of study holds great potential for teacher learning. This practice-based approach is intended to bridge the gaps between theory and practice in teacher education by allowing teachers to examine the content and/or pedagogy to be learned in the course in the context of teaching, and to operationalize that content and/or pedagogy for themselves in the creation of teaching artifacts of their own.

The geometry and measurement course was one such practice-based teacher education experience, and results from this study suggest that teachers did indeed gain knowledge of content and pedagogy from the course. These results resonate with the study of other practice-based teacher education experiences (e.g., Engle, 2004; Hillen, 2005; Smith, Leinhardt, & Silver, 2004; Smith, Silver, Leinhardt, & Hillen, 2003; Steele, 2005; Steele et al., in preparation; Stein et al., submitted). Teachers' engagement with the course and reflections on their own learning suggest that these teachers saw the knowledge gained from the course as useful in their work as teachers. The results of the geometry and measurement study also confirm that teachers made significant gains in learning about both content and pedagogy. These findings confirm that knowledge typically attained in two discrete teacher training settings – methods courses and content courses – can be developed in tandem through practice-based teacher education experiences. Moreover, the practice of integrating content and pedagogy tightly together in experiences such as the geometry and measurement course mirrors current conceptualizations of how teachers access and use knowledge of content and knowledge of teaching (e.g., Ball, Bass, & Hill, 2004; Shulman, 1986; Sherin, 2002).

At a finer grain size, the results of the study of the geometry and measurement course also suggest that this integration of content and pedagogy holds value beyond simply learning about both topics. Specifically, the findings indicate that teachers grew in their content knowledge for teaching, a specialized aspect of mathematics content knowledge that is only useful in the work of teaching. This is content that is not likely to emerge in traditionally-structured mathematics content courses. From a pedagogical standpoint, the growth shown by teachers in the five practices for productive use of student thinking, despite many teachers having demonstrated the ability to integrate the practices into their lesson planning when asked in previous coursework experiences, suggests that examining these practices with respect to particular mathematical content enhances teachers' ability to incorporate the practices into their lesson planning when asked to do so. Integration of the five practices into a course-based lesson planning activity does not guarantee that teachers will engage in these practices with their own students in their own classroom, but their demonstrated ability to engage in these practices is likely prerequisite to their inclusion in classroom teaching. The development and enactment of other *content-focused methods courses,* which address both content and pedagogy in a highly integrated way, is a potentially rich area for future research. In particular, courses that focus on the content of statistics and probability and trigonometry/pre-calculus/calculus would add to the understandings of the field regarding mathematical knowledge for teaching, as these content areas have typically been under-researched with respect to teacher knowledge.

5.3.2. Building K-12 Teacher Education Communities

An interesting feature of the teacher population studied in the geometry and measurement course was the background and experience of the teachers who engaged in the course. The course included preservice middle/secondary (n=9) and elementary (n=3) teachers; practicing middle/secondary (n=10) teachers with between 2 and 11 years' experience, and instructional coaches and teacher leaders (n=3) with between 14 and 25 years' experience in classroom teaching. The grade levels taught by teachers across the course ranged from Kindergarten to college calculus. These populations were brought together to address middle grades issues in part because in the state in which the course was taught, both elementary and secondary certified teachers teach at the middle school level. With the design of the course focused on middle grades geometry and measurement content and with a range of teaching experience present in the course population, one might wonder what the implications for learning might be.

When the results of the study were examined and disaggregated by subgroups, no significant differences between subgroups were found. This is in part due to the relatively small population of the study. However, there was clear evidence that each individual teacher experienced learning in the course. All 25 teachers were contributors to public discussions at one time or another during the course, and every teacher grew individually along one dimension or another as measured by the individual pre- and post-course assessments and the interview transcripts. Moreover, of the 20 teachers interviewed, all identified a number of mathematical and pedagogical ideas that were learned as a result of their engagement in the course experience.

Following the lead of similar courses at the university targeting middle grades content, the geometry and measurement course brought together teachers from a variety of backgrounds, and the discourse was enhanced by the diversity. The middle grades provided a stable center from which teachers could expand out and discuss how content issues played out at both lower and higher grade levels. This was particularly salient in discussions of proof, a topic which has the potential to build for students across K-12 through a variety of mathematical experiences (Stylianides & Silver, 2004). In addition, teachers at different grade levels tended to exhibit different ways of thinking about mathematical content. At several points in the course (see the discussion of the Soda Can Task and the Fencing Task), elementary teachers played key roles in exposing secondary teachers to novel ways of solving mathematical tasks. In return, secondary teachers who tended to rely on more general and symbolic approaches to mathematical tasks were pressed to explain the basis of their thinking in ways that all teachers in the course could understand.

The geometry and measurement course experience, coupled with work from similar courses in other content areas originating from the ASTEROID Project, suggests that bringing together teachers from across the K-12 spectrum in the context of a course focused on middle grades mathematics provides a model for teacher education courses serving diverse groups of learners. Through the use of cognitively challenging mathematical tasks from middle grades curricula and the facilitation of these tasks in ways that encouraged multiple solution paths, made connections between key mathematical ideas across solutions, and fostered deep mathematical understandings, teachers from a variety of backgrounds and with a variety of previous mathematical experiences were able to take away new mathematical understandings. This model has the potential to travel to a variety of teacher education settings: formal coursework, both at small universities looking to provide richer course experiences and large universities looking to improve mathematical outcomes for teachers, and in professional development settings, both university-sponsored and district-based.

With respect to pedagogical issues, the diversity in classroom experience did not appear to engender differential levels of participation or imply differential levels of expertise during discussions about teaching. This was due in large part to the use of narrative cases in the course. When discussing teaching, the instructor set clear norms for discussion such that claims about teaching were grounded in evidence from the case. In this manner, the narrative cases provided teachers with a shared experience and common reference point to ground the discussions of pedagogy. Teachers' personal classroom experiences, often the sole source of evidence in professional conversations about teaching, were used to enhance the discourse rather than to ground it. The use of cases in essence leveled the playing field for preservice and practicing teachers across grade levels, giving them all entry into conversations about the practice of teaching. Continued research into the uses of cases in teacher education (e.g., Smith, 2001b) would add to the field's understandings of how these cases serve to enhance mathematical knowledge for teaching and how they might serve to bring together diverse groups of teachers in ways that support meaningful mathematical and pedagogical learning.

5.3.3. A Structure for Teacher Education: Replication of Instructional Design

One final implication from the study of the geometry and measurement course is the refinement and articulation of a set of design principles that guided the structure and implementation of the course experience. These design principles built on prior work related to practice-based teacher education, and the notion of a content-focused methods course specifically, linking elements of design that had either been explicitly stated (e.g., Smith et al., 2001) or implicit in previous work. As noted in Chapter Four, the results with respect to changes in teacher knowledge resonated with the types of learning suggested by the design principles, both individually and collectively. This suggests that to the extent that the design principles are generalizeable, a course designed and enacted using the same or a similar set of principles is likely to result in similar teacher learning.

These principles did not emerge solely from the design and study of the geometry and measurement course. Indeed, the understandings codified by the principles represent the extension of significant prior work on the design of teacher education experiences (e.g., Smith, 2001; Smith, Stein, Silver, Hillen, & Heffernan, 2001; Smith, Silver, Leinhardt, & Hillen, 2003). The revision and rearticulation of the design principles and their successful prediction of teacher learning marks an important contribution to the field, in that these principles can be used in the design of other teacher education experiences, allowing the work summarized by this study to travel. Additionally, these design principles contribute to the broader body of research regarding design experiments in education (e.g., Brown, 1992; Cobb, 1991). The link between the design principles and tenets of learning theory further serves to bridge the often problematic gap between theory and practice. It is hoped that the articulation of the design principles, taken together with the other results related to the study of the geometry and measurement course, provides teachers educators with the insight and tools to influence the knowledge of mathematics teachers in the future.

APPENDIX A

Written Pre- and Post-Course Assessment

Please note: This version of the assessment contains headers for research purposes; these headers were removed for the instrument's administration.

Measurement & Geometry Pre-Course Assessment

Part A: Analyzing Teaching

While watching the video of Cathy Humphreys' 7th grade lesson on surface area, identify the things that the teacher does (either explicitly or implicitly) to organize or support the classroom activity and student engagement in the classroom activity. For each teacher move you identify, write a brief description of the move, identify its location in the transcript, and describe how the move supports classroom activity.

Description of Move	Line #s in transcript	How does the move support classroom activity?

Part B: Your Work as a Teacher

Please answer each of these questions in as much detail as possible, allowing no more than 5 minutes per question.

1. What do you think middle grades students need to understand and be able to do with respect to 2-D shapes, area and perimeter?

2. What do you think middle grades students need to understand and be able to do with respect to 3-D shapes, surface area, and volume?

3. What kinds of experiences do middle grades students need with respect to reasoning and proof to prepare them for high school geometry?

Part C: Mathematics

Please answer each problem in this section, explaining and justifying your thinking. If you make corrections, please do not erase previous work; instead, draw a single line through the work to be ignored.

1. **Fence in the Yard**

   Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

   She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

   What is the largest area that Julie can enclose with 36 feet of fencing?

   Support your answer by showing the work that would convince Julie that your area is the largest.

   (From 1996 NAEP, as cited in Kenney & Lindquist, 2000)

2. Relating Area and Perimeter

a. True or false: A parallelogram with a base of 6 cm and an area of 24 cm^2 will always have the same perimeter. Provide at least one example to support your answer.

Tangrams are a special set of 7 geometric tiles shown below in Figure 1. The shapes in Figures 2 and 3 were formed using all the tangram tiles.

b. Which figure, 2 or 3, has the greater area? Justify your answer.

c. Which figure, 2 or 3, has the greater perimeter? Justify your answer.

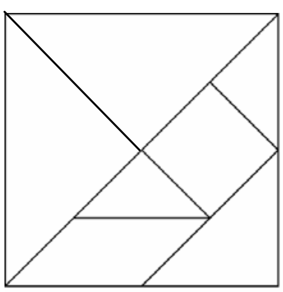


Figure 1.

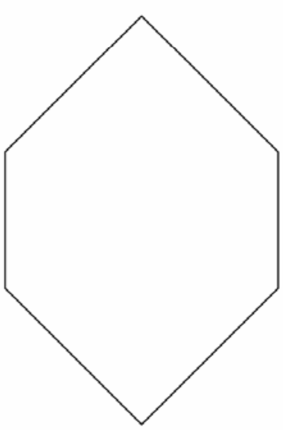


Figure 2.

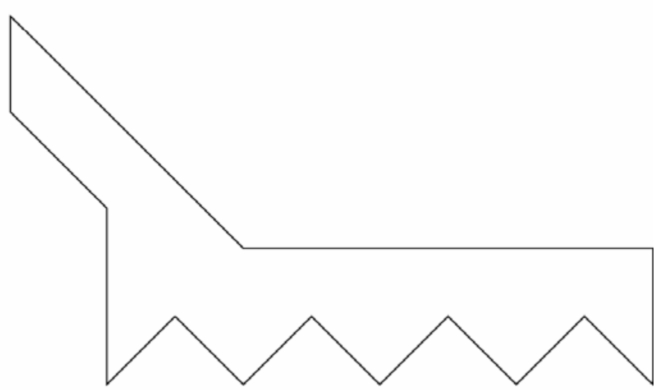


Figure 3.

3. Surface Area and Volume

Jim and John are both painting their dining rooms in their homes (walls only, not the floor or ceiling). They helped each other put new wood floors in the living rooms last summer, and they know that each floor has an area of 400 ft^2. The ceilings in both rooms are 8 ft high.

a. Will Jim and John need to buy the same amount of paint? (Assume an equal number of coats and equal coverage per gallon.) Explain your answer.

b. Do the dining rooms have the same volume? Explain how you know.

c. If you know the volume of a box (rectangular prism), can you find its surface area?

d. If you have a box of known dimensions and volume, how would you create a new box with exactly double the volume?

e. If you have a box of known surface area, how would you create a new box with exactly four times the surface area?

4. Reasoning and Proof

a. Prove that the area of a parallelogram is equal to the base times the height.

b. Prove that the area of a triangle is equal to one half of the base times the height.

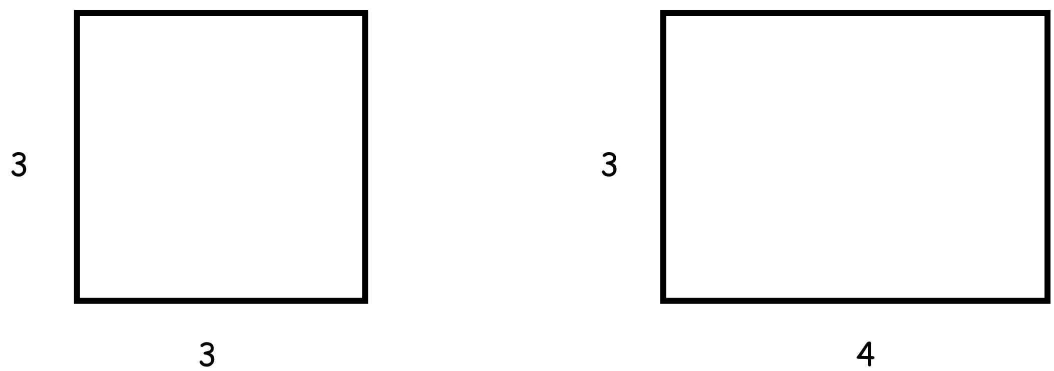
Part D: Student Thinking

In this section, you will be presented with examples of student work and asked to interpret the work and suggest instructional moves. Please respond as you would in your own classroom.

5. Area and Perimeter Responding to Student Claims

A student in your class makes the claim shown below about perimeter and area. How would you respond?

As the perimeter of a rectangle increases, its area also increases.



perimeter = 12 cm
area = 9 square cm

perimeter = 14 cm
area = 12 square cm

Adapted from Ball, Bass, & Hill, 2004

6. Considering the Use of Formulas

a. There are two common forms that textbooks use for the volume of a rectangular prism:
Volume = *length* × *width* × *height* and *Volume* = *Area of base* × *height*
Is there a difference between the two formulas? If so, describe the difference.
Which would you choose to use with students, and why?

b. There are two common forms that textbooks use for the area of a rectangle:
Area = *length* × *width* and *Area* = *base* × *height*
Is there a difference between the two formulas? If so, describe the difference.
Which would you choose to use with students, and why?

7. Considering Student Work on Fence in the Yard

Imagine that you asked your students to work in small groups to complete the Fence in the Yard task. (This is the same task you completed earlier.) Your mathematical goal is for students to understand how changing the dimensions of a rectangle while preserving its perimeter impacts the area of the rectangle.

You want to orchestrate a whole-class discussion of the task, drawing on a subset of the responses produced by your students (shown in A, C, F, H, J, and K).

Determine which responses you wish to have shared in the whole-class discussion, explain why you chose each response, and indicate the order in which you would want them shared.

A

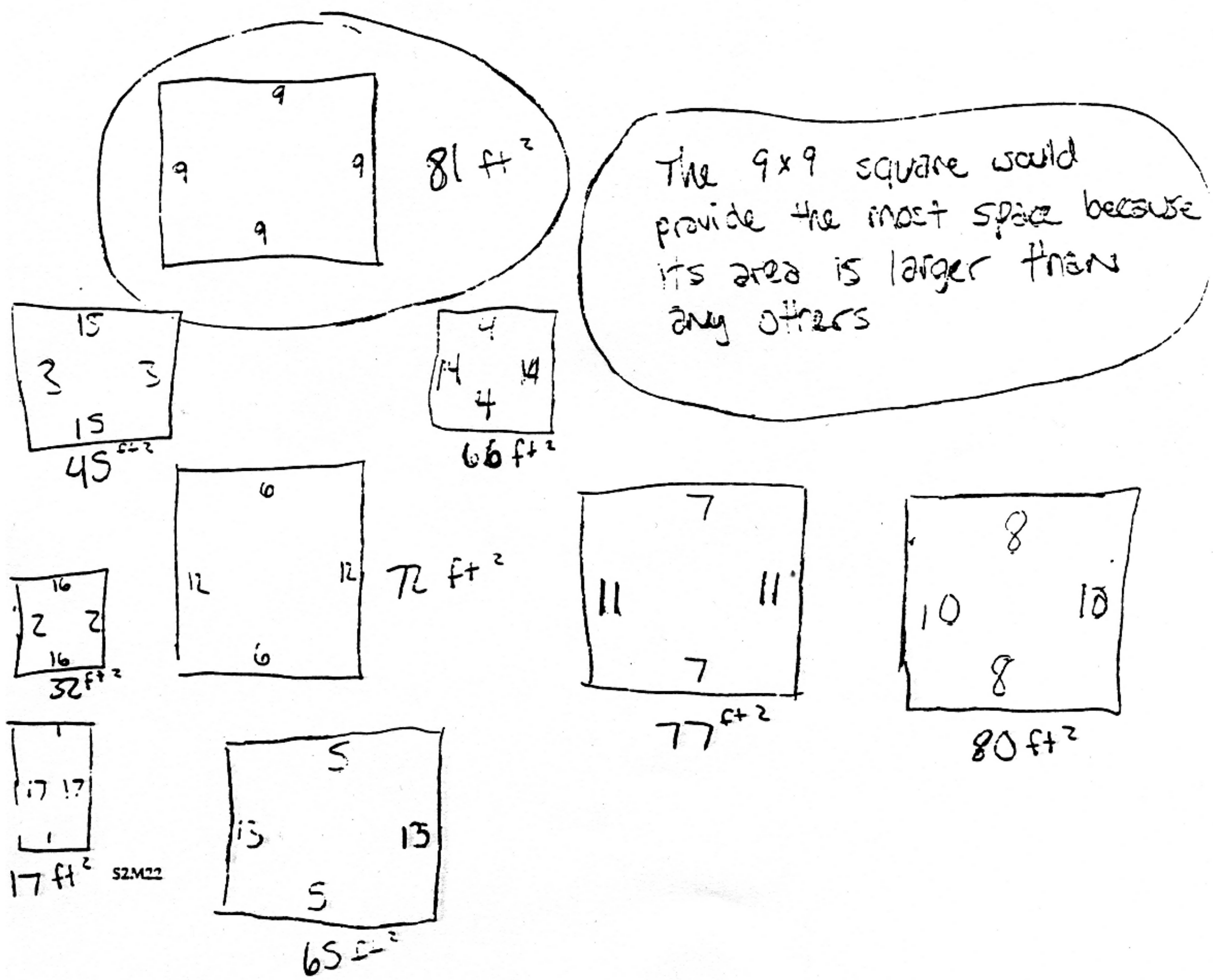
This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show <u>all</u> of your work.

10. Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

    She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

    What is the largest area that Julie can enclose with 36 feet of fencing?

    Support your answer by showing work that would convince Julie that your area is the largest.



C

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show all of your work.

10. Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

What is the largest area that Julie can enclose with 36 feet of fencing?

Support your answer by showing work that would convince Julie that your area is the largest.

$36 \div 4 = 9$ so she could only have a 9x4 area fenced in.

F

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show <u>all</u> of your work.

10. Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

What is the largest area that Julie can enclose with 36 feet of fencing?

Support your answer by showing work that would convince Julie that your area is the largest.

$p = 2l + 2w$

$p = 2(6) + 2(6)$

$p = 18 + 18$

$p = 36$

$\{36\ ft\}$

$a = lw$

$a = 9(4)$

$a = 36$

$\{36\ ft^2\}$

J

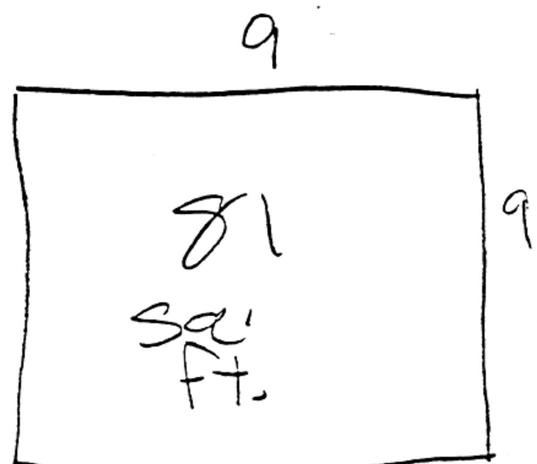
This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show <u>all</u> of your work.

10. Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

What is the largest area that Julie can enclose with 36 feet of fencing?

Support your answer by showing work that would convince Julie that your area is the largest.



$8 \times 10 = 80$
$7 \times 11 = 77$
$6 \times 12 = 72$
$5 \times 13 = 65$
$4 \times 14 = 56$
$3 \times 15 = 45$

$2 \times 16 = 32$
$1 \times 17 = 17$
$0 \times 36 = 0$

K

If you need more room for your work, use the space below.

The largest area would be a square with sides of 9ft. which would allow for 81 $ft.^2$ of area.

This is because of the following:

Width	Length	Area	Perimeter
9	9	81	36
10	8	80	36
11	7	77	36
12	6	72	36
13	5	65	36
14	4	56	36
15	3	45	36
16	2	32	36
19	1	17	36

M

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show <u>all</u> of your work.

10. Julie wants to fence in an area in her yard for her dog. After paying for the materials to build her doghouse, she can afford to buy only 36 feet of fencing.

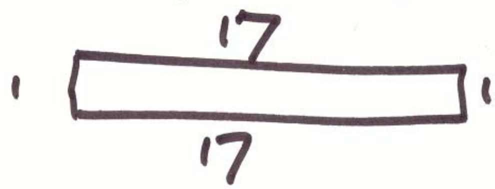
She is considering various different shapes for the enclosed area. However, she wants all of her shapes to have 4 sides that are whole number lengths and contain 4 right angles. All 4 sides are to have fencing.

What is the largest area that Julie can enclose with 36 feet of fencing?

Support your answer by showing work that would convince Julie that your area is the largest.

Area is made up of lenght + width.
The biggest area will have a big lenght + small width.

The smallest width is 1, and 1 and 1 leaves 34 feet. $34 \div 2 = 17$



This is the greatest area: $17+17+1+1=36$.
Anything else will make the lenght smaller and be a smaller area.

APPENDIX B

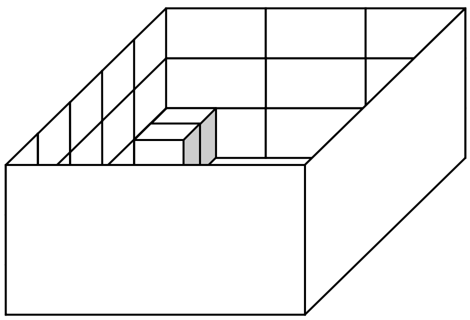
Pre-Course Interview Protocol

This is (<u>YOUR NAME</u>) interviewing (<u>TEACHER'S NAME</u>) on (<u>DATE</u>) and this is Geometry and Measurement Interview #1.

Thank you for participating in this interview. The purpose of this interview is for us to understand your current thinking on ideas related to the course. I have three tasks for you today.

<u>Task 1: Surface Area and Volume</u>

Present the unit cube box.



Say, **The first task involves this special box. I'd like to fill this box with as many packages as I can. These packages are two interlocked cubic inch cubes, like this.**

Hold up the 2 interlocked unit cubes.

Say, **You might notice that the inside of the box is marked with guidelines that are the same size as the packages. I'd like you to look at it and determine how many packages I need to fill the box completely. As you're determining your answer, please verbalize your thinking. I have a few packages and a ruler that you can use if you wish.**

Make 10-12 packages and a ruler available to the teacher as they work.

> *Allow the teacher to take as much time as they need to determine. If the teacher does not verbalize their actions, ask them to describe what they are doing. If this doesn't elicit a response, describe the action yourself by saying, "It looks like you are…"*
>
> *Probe teachers' thinking as they articulate a strategy. If teachers talk about using a formula, make sure the formula is unpacked.*

Ask, **Could you think of any other ways to find the number of packages that fill the box?**

> *Move on only after teachers have offered as many solution methods as they can.*

Say, **Now I'd like you to estimate the surface area of the box. Assume the box has a top as well. You can use the packages if you would like. I also have large pieces of chart paper and grid paper available if you would like to use that.**

Ask, **Could you think of any other ways to find the surface area of the box?**

> *Move on only after teachers have offered as many solution methods as they can.*

Say, **Thank you. Let's move on to the next task.**

<u>Task 2: Teachers' Conceptions of Proof</u>

Say, **Next, I'd like to ask you a few questions about proof. I'd like you to answer these both from your perspective as a math learner and as a math teacher.**

a. What does proof mean to you?

Allow the teacher to say as much as possible without interruption. If the teacher uses particular language or refers to their own students, probe on what the term means or what the student population is. Specifically, if teachers use terms like ***formal proof*** *or* ***informal proof****, probe these terms.*

Use the next question as a follow-up if it hasn't been addressed already.

b. What does it mean to prove something?

Allow the teacher to say as much as possible. Probe any mathematical language.

c. What purpose does proof serve in mathematics?

Allow the teacher to say as much as possible. Probe any mathematical language. If teachers don't understand the question, make it clear that mathematics refers to the mathematics domain as a whole, not just the practice of mathematicians.

d. What makes an argument a proof?

This question is intended to get at the social aspects of proof; if teachers do not offer it, do not lead them in that direction. However, if they use language that suggests a social or community aspect to proof), be sure to probe the language. Examples: "Arguments are proofs only after they're recognized by the math community." "Mathematicians decide what is a proof."

e. Do proofs ever become invalid?

Be sure to probe for how teachers see proofs as becoming invalid if they do not offer an explanation.

When teachers have exhausted the question, hand out the <u>light blue</u> sheets and say,
These sheets contain 8 explanations of mathematical conjectures. I'd like you to look at them and tell me which ones are proofs and which ones are not. I'll give you a few minutes to look them over, then I'll ask you to talk about each one and explain why you think it is or is not a proof. Feel free to think aloud as you examine each proof.

Allow teachers as much time as they need to examine each proof. If they write or gesture, ask them about what they are writing or gesturing about and why. This is particularly important if teachers feel the need to verify the proofs with another example.

After the teacher has had a chance to examine the explanations, ask them to tell whether each one is proof or not and why. Probe any relevant mathematical or pedagogical language. This includes any mathematical terms teachers use, or any description of a particular group of students (e.g. my students, students in high school, gifted students). If teachers name particular student populations, probe what/who they mean by that population.

When finished, hand the teacher the *<u>purple</u>* *sheet say,* **Now I'd like you to rate the 8 explanations on a scale of 1 to 4, with 1 being least proof-like, and 4 being most proof-like. You can rank them on the purple sheet.**

Allow teachers to rate the explanations and ask them to explain the ratings. When they are finished, say: **I'd like you to consider the three explanations of the sum of the measures of the angles in a triangle is equal to 180 degrees. If students in an 8th grade class produced these three explanations, and you only had time to share one with the class in order to promote their understanding of the theorem, which one would you choose, and why?**

When they have finished, ask, **As a teacher, do you think proof should play a central role in secondary school mathematics?**

> *If teachers need clarity on secondary school mathematics, state grades 6-12. Probe as appropriate. If teachers mention the Standards, probe on what they think the Standards say with respect to proof.*

Say, **Thank you. Let's move on to the final task.**

<u>Task 3: Lesson Planning</u>

> *Hand the teacher a copy of the Minimizing Perimeter (**green paper**) if they do not have it with them.*

I'd like you to take out the Minimizing Perimeter problem that [I/Mike] **handed out in the final class that you were asked to consider prior to the interview.**

Your final task is to plan a lesson around this problem. I'm going to give you 5 to 8 minutes to write down your ideas about how you might implement a lesson with this problem. Your target mathematical goal will be to get students to understand the relationships between area and perimeter. You are free to modify the problem in any way. I'm going to turn off the recorder while you plan. Do you have any questions?

> *If teachers have questions, make it clear to them that we only expect an outline of a lesson plan, not a full-scale plan. If they ask if specific ideas should be included, tell them to include whatever they think they would need to enact the lesson.*

> *When the teacher indicates they are finished planning the lesson ask,*

Could you walk me through the lesson you have planned around this task?

> *Probe any language the teacher uses, but try not to introduce any language not used by the teacher. For example, if the teacher indicates that they will ask questions, probe on what questions they intend to ask. If they indicate they will use routines, probe what they mean by routines and what routines they would use. Allow the teacher to speak as much as possible.*

> *If the teacher does not indicate what they expect students to learn, ask:*

What do you hope students will learn through engaging in this lesson?

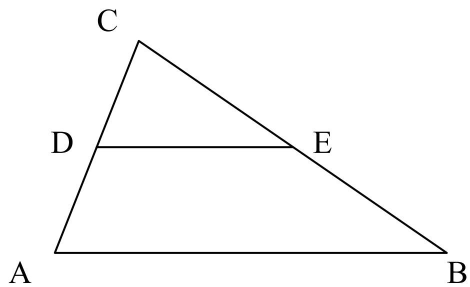
When the teacher finishes talking, ask:

Is there anything else you'd like to say about the lesson?

Probe as above if appropriate.

Say, **Thank you. That's the last task I have for you.**

Explanation 1



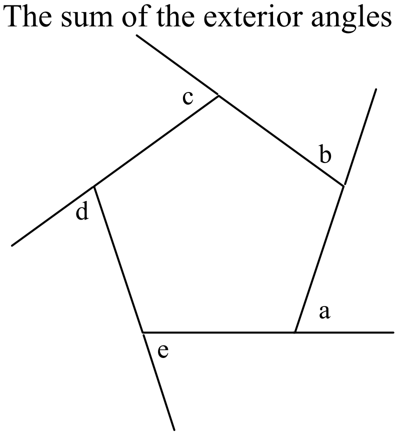
Given: ΔABC and points D and E, which are the midpoints of AC and BC, respectively.

Prove: AB is parallel to DE.

D is the midpoint of AC and E is the midpoint of BC.	Given.
$DC = (1/2)AC$ and $EC = (1/2)BC$	Definition of a midpoint.
$\angle C \cong \angle C$	Reflexive property.
$\Delta ABC \sim \Delta DEC$	If two sides of a triangle are proportional to the two corresponding sides of another triangle, and the included angles are congruent, then the triangles are similar.
$\angle CDE \cong \angle CAB$	Definition of similar triangles.
AB is parallel to DE	If two lines cut by a transversal (AC) form congruent angles with the transversal, then the lines are parallel.

Explanation 2

The sum of the exterior angles of a polygon is 360°.



A regular pentagon is shown above as an example. We know that the sum of the interior angles in a polygon is $(n - 2) \cdot 180°$, so the sum of the interior angles of the pentagon is $(5 - 2) \cdot 180° = 540°$.Since there are 5 interior angles, the measure of each interior angle is $540° \div 5 = 108°$.

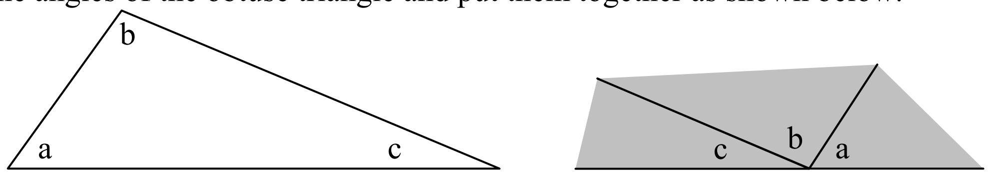
The interior and exterior angles form supplementary pairs that add up to 180°. So each exterior angle measures $180° - 108° = 72°$. Since there are five exterior angles, the sum of the exterior angles is $72° \cdot 5 = 360°$.

Since the formula for the sum of the interior angles is the same for any polygon, this argument holds for any polygon.

Three attempts to prove that the sum of the measures of the interior angles of any triangle is equal to 180°.

Explanation 3a

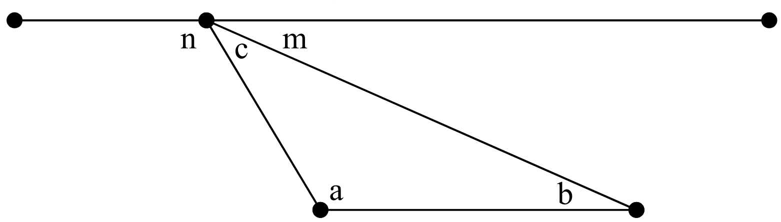
I tore up the angles of the obtuse triangle and put them together as shown below.



The angles came together as a straight line, which is 180°. I also tried it for an acute triangle as well as a right triangle and the same thing happened. Therefore, the sum of the measures of the interior angles of a triangle is equal to 180°.

Explanation 3b

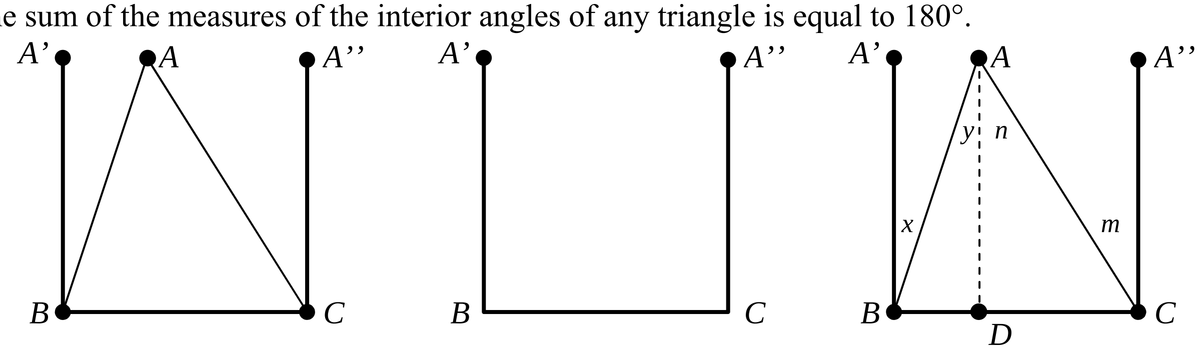
I drew a line parallel to the base of the triangle.



I know $n = a$ because alternate interior angles between two parallel lines are congruent. For the same reason, I also know that $m = b$. Since the angle measure of a straight line is 180°. I know that $n + c + m = 180°$. Substituting a for n and b for m gives $a + b + c = 180°$. Thus, the sum of the measures of the interior angles of a triangle is equal to 180°.

Explanation 3c

Using the diagram below, imagine moving *BA* and *CA* to the perpendicular positions *BA’* and *CA’’*, thus forming the second figure. In reversing this procedure (e.g., moving *BA’* back to *BA*), the amount of the right angle, *A’BC*, that is lost is *x*. This lost amount, however, is gained with angle *y* (*DA* is perpendicular to *BC*). A similar argument can be made for the other case. Thus, the sum of the measures of the interior angles of any triangle is equal to 180°.



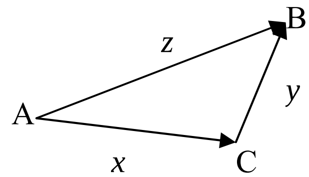
Explanation 3c from “Students Proof Schemes: Results from Exploratory Studies” by G. Harel and L. Sowder in *Research in Collegiate Mathematics Education III* (p. 259), edited by A. Schoenfeld, J. Kaput, and E. Dubinsky, 1998, Washington, DC: Mathematics Association of America (MAA). Copyright 1998 by MAA.

Explanation 4

An attempt to prove the Triangle Inequality:

x, *y* and *z* are sides of a triangle. If *z* is the longest side of the triangle, the length of *z* is shorter than the sum of the lengths of *x* and *y*. Symbolically, $|z| < |x| + |y|$

Consider x, y, and z as distances. They can be represented by a triangle similar to the one below:

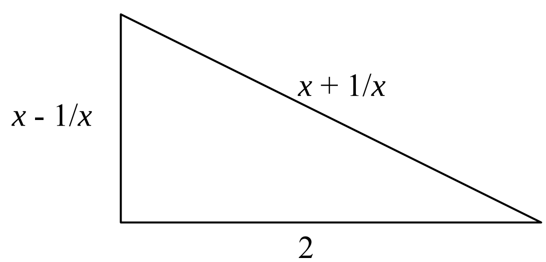


z is the distance between A and B, which is a straight line. $x + y$ also represents a distance between A and B, but it is not a straight line. Since the shortest distance between two points is a straight line, the length of *x* plus the length of *y* will always be larger than the length of *z*.

Explanation 5

An attempt to prove the following: If $x > 0$, then $x + \frac{1}{x} \geq 2$.

We can construct a right triangle with the given sides so that it satisfies the Pythagorean Theorem.



Note: If $0 < x < 1$, then the vertical side has length $\frac{1}{x} - x$.

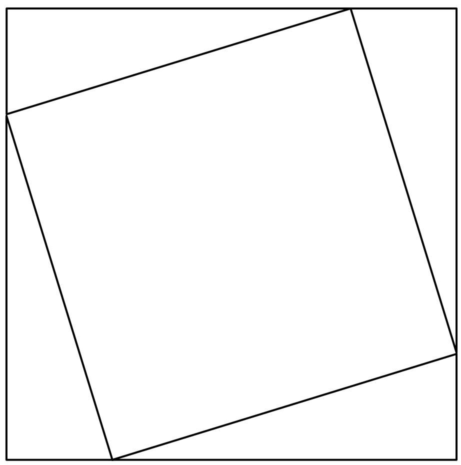
That is, the following is a true statement: $\left(x - \frac{1}{x}\right)^2 + 2^2 = \left(x + \frac{1}{x}\right)^2$

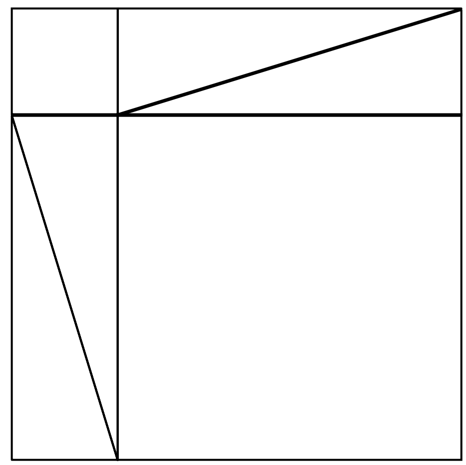
From right triangle geometry, we know that the hypotenuse is longer than either leg.

Thus, $x + \frac{1}{x} \geq 2$

Explanation 6
An argument for the Pythagorean Theorem.

I drew the two squares below, which are congruent. The square on the left contains four right triangles and a square built on the hypotenuse of the right triangle. The square on the right contains the same four right triangles, plus two squares, one built on each leg of the right triangle.





Since the squares are congruent, they have equal area. Canceling out the triangles in each figure, we're left with the square of the hypotenuse (c) equal to the sum of the squares of the two legs (a and b). Thus, $a^2 + b^2 = c^2$.

Minimizing Perimeter

Adapted from Navigating through Geometry in Grades 6-8

The 7th grade class wants to start a small organic school garden to grow vegetables for the cafeteria. The principal has told the class that they can have a 36 ft^2 rectangular area behind the school. The rectangle can be any shape they choose, so long as it is 36 square feet in area.

1. Find the least amount of fencing for a rectangular garden plot that is 36 square feet in area. Organize the information using a table like the one below.

Length (feet)	Width (Feet)	Perimeter (Feet)	Area (Square Feet)

2. Use the data in your table to create a graph of perimeter vs. length.

3. The 6^{th} grade decides they also want to start a small garden. The principal gives them 24 ft^2 to create their garden in any rectangular shape they choose. Find the least amount of fencing for a rectangular garden plot that is 24 square feet in area. Make a table and graph similar to the ones you created above.

4. When they hear of the success of the middle school gardens, the local high school wants to create a garden of their own. Their principal allows the high school to have 100 ft^2. Make a conjecture about the minimum fencing needed for an area of 100 square feet and write a paragraph defending your conjecture.

APPENDIX C

Post-Course Interview Protocol

Note: The proof activity explanations and Minimizing Perimeter task are identical to those in the pre-course interview protocol and are not reproduced in this appendix. Additionally, the course map referred to in Task 1 is not included, as it will not be finalized until the close of the course.

This is (<u>YOUR NAME</u>) interviewing (<u>TEACHER'S NAME</u>) on (<u>DATE</u>) and this is Geometry and Measurement Interview #2.

Thank you for participating in this interview. Now that we are at the end of the course, we'd like to understand how the activities in the course may have impacted your thinking and learning. This will help us provide opportunities for teachers in the future. I have four tasks for you today.

<u>Task 1: Course Map</u>

*Hand the teacher a copy of the course map (**white paper**) if they do not have it with them.*

During the last class, you received a copy of the course map. This map contains the major activities in which you engaged during the course. The point of this task is for us to understand how the activities in the course impacted your learning.

I'm going to ask you about three particular aspects of knowledge. For each one, I'm going to ask you what you know or understand now that you did not know or understand, or understood differently, prior to the course. So let's begin.

<u>1a: Knowledge of Mathematics</u>

My first question is, what do you know or understand now about <u>mathematics</u> that you did not know or understand, or understood differently, prior to the course?

Record each idea that teachers articulate. If appropriate, ask the prompt below after each idea. (You may also wait until all ideas are on the table and ask the prompt with respect to each one.)

Restate each idea in as close to the teacher's own words as possible.

Now I'd like you to mark any activities in the course that helped you to know or understand [restate the idea] **with a BLUE dot.**

As the teacher marks each activity, ask:

How did [name the activity] **help you better understand this idea** [restate idea as appropriate]**?**

When teachers have exhausted all ideas related to knowledge of mathematics, move on to the next topic.

<u>1b: Knowledge of Mathematics for Student Learning</u>

My first question is, what do you know or understand now about <u>students as learners of mathematics</u> that you did not know or understand, or understood differently, prior to the course?

Record each idea that teachers articulate. If appropriate, ask the prompt below after each idea. (You may also wait until all ideas are on the table and ask the prompt with respect to each one.)

Restate each idea in as close to the teacher's own words as possible.
Now I'd like you to mark any activities in the course that helped you to know or understand [restate the idea] **with a RED dot.**

As the teacher marks each activity, ask:

How did [name the activity] **help you better understand this idea** [restate idea as appropriate]**?**

When teachers have exhausted all ideas related to knowledge of mathematics, move on to the next topic.

1c: Knowledge of Practices that Support Teaching
My first question is, what do you know or understand now about mathematics teaching or pedagogy that you did not know or understand, or understood differently, prior to the course?

Record each idea that teachers articulate. If appropriate, ask the prompt below after each idea. (You may also wait until all ideas are on the table and ask the prompt with respect to each one.)

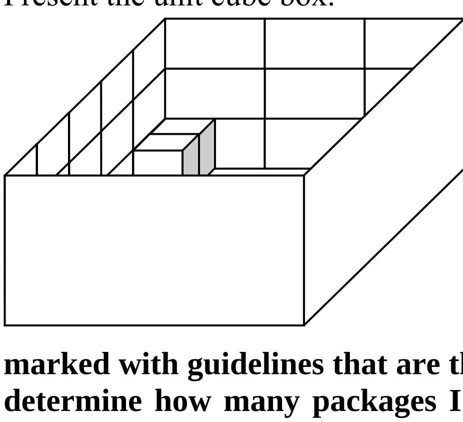
Restate each idea in as close to the teacher's own words as possible.
Now I'd like you to mark any activities in the course that helped you to know or understand [restate the idea] **with a GREEN dot.**

As the teacher marks each activity, ask:

How did [name the activity] **help you better understand this idea** [restate idea as appropriate]**?**

When teachers have exhausted all ideas related to knowledge of mathematics, say, **Thank you. Let's move on to the next task.**

Task 2: Surface Area and Volume
Present the unit cube box.



Say, **The first task involves this special box. I'd like to fill this box with as many packages as I can. These packages are two interlocked cubic inch cubes, like this.**

Hold up the 2 interlocked unit cubes.

Say, **You might notice that the inside of the box is marked with guidelines that are the same size as the packages. I'd like you to look at it and determine how many packages I need to fill the box completely. As you're determining your answer, please verbalize your thinking. I have a few packages and a ruler that you can use if you wish.**

Make 10-12 packages and a ruler available to the teacher as they work.

Allow the teacher to take as much time as they need to determine. If the teacher does not verbalize their actions, ask them to describe what they are doing. If this doesn't elicit a response, describe the action yourself by saying, "It looks like you are…"

Probe teachers' thinking as they articulate a strategy. If teachers talk about using a formula, make sure the formula is unpacked.

Ask, **Could you think of any other ways to find the number of packages that fill the box?**

Move on only after teachers have offered as many solution methods as they can.

Say, **Now I'd like you to estimate the surface area of the box. Assume the box has a top as well. You can use the packages if you would like. I also have large pieces of chart paper and grid paper available if you would like to use that.**

Ask, **Could you think of any other ways to find the surface area of the box?**

Move on only after teachers have offered as many solution methods as they can.

Say, **Thank you. Let's move on to the next task.**

Task 3: Teachers' Conceptions of Proof

Say, **Next, I'd like to ask you a few questions about proof. I'd like you to answer these both from your perspective as a math learner and as a math teacher.**

a. What does proof mean to you?

*Allow the teacher to say as much as possible without interruption. If the teacher uses particular language or refers to their own students, probe on what the term means or what the student population is. Specifically, if teachers use terms like **formal proof** or **informal proof**, probe these terms.*

Use the next question as a follow-up if it hasn't been addressed already.

b. What does it mean to prove something?

Allow the teacher to say as much as possible. Probe any mathematical language.

c. What purpose does proof serve in mathematics?

Allow the teacher to say as much as possible. Probe any mathematical language. If teachers don't understand the question, make it clear that mathematics refers to the mathematics domain as a whole, not just the practice of mathematicians.

d. What makes an argument a proof?

This question is intended to get at the social aspects of proof; if teachers do not offer it, do not lead them in that direction. However, if they use language that suggests a social *or community aspect to proof), be sure to probe the language. Examples: "Arguments are proofs only after they're recognized by the math community." "Mathematicians decide what is a proof."*

e. **Do proofs ever become invalid?**
Be sure to probe for how teachers see proofs as becoming invalid if they do not offer an explanation.

When teachers have exhausted the question, hand out the light blue sheets and say,
These sheets contain 8 explanations of mathematical conjectures. I'd like you to look at them and tell me which ones are proofs and which ones are not. I'll give you a few minutes to look them over, then I'll ask you to talk about each one and explain why you think it is or is not a proof. Feel free to think aloud as you examine each proof.

Allow teachers as much time as they need to examine each proof. If they write or gesture, ask them about what they are writing or gesturing about and why. This is particularly important if teachers feel the need to verify the proofs with another example.

After the teacher has had a chance to examine the explanations, ask them to tell whether each one is proof or not and why. Probe any relevant mathematical or pedagogical language. This includes any mathematical terms teachers use, or any description of a particular group of students (e.g. my students, students in high school, gifted students). If teachers name particular student populations, probe what/who they mean by that population.

When finished, hand the teacher the purple sheet say, **Now I'd like you to rate the 8 explanations on a scale of 1 to 4, with 1 being least proof-like, and 4 being most proof-like. You can rank them on the purple sheet.**

Allow teachers to rate the explanations and ask them to explain the ratings. When they are finished, say: **I'd like you to consider the three explanations of the sum of the measures of the angles in a triangle is equal to 180 degrees. If students in an 8th grade class produced these three explanations, and you only had time to share one with the class in order to promote their understanding of the theorem, which one would you choose, and why?**

When they have finished, ask, **As a teacher, do you think proof should play a central role in secondary school mathematics?**

If teachers need clarity on secondary school mathematics, state grades 6-12. Probe as appropriate. If teachers mention the Standards, probe on what they think the Standards say with respect to proof.

Say, **Thank you. Let's move on to the final task.**

<u>Task 4: Lesson Planning</u>

*Hand the teacher a copy of the Minimizing Perimeter (**green paper**) if they do not have it with them.*

I'd like you to take out the Minimizing Perimeter problem that [I/Mike] **handed out in the final class that you were asked to consider prior to the interview.**

Your final task is to plan a lesson around this problem. I'm going to give you 5 to 8 minutes to write down your ideas about how you might implement a lesson with this problem. Your target mathematical goal will be to get students to understand the relationships between area and perimeter. You are free to modify the problem in any way. I'm going to turn off the recorder while you plan. Do you have any questions?

If teachers have questions, make it clear to them that we only expect an outline of a lesson plan, not a full-scale plan. If they ask if specific ideas should be included, tell them to include whatever they think they would need to enact the lesson.

When the teacher indicates they are finished planning the lesson ask,

Could you walk me through the lesson you have planned around this task?

Probe any language the teacher uses, but try not to introduce any language not used by the teacher. For example, if the teacher indicates that they will ask questions, probe on what questions they intend to ask. If they indicate they will use routines, probe what they mean by routines and what routines they would use. Allow the teacher to speak as much as possible.

If the teacher does not indicate what they expect students to learn, ask:

What do you hope students will learn through engaging in this lesson?

When the teacher finishes talking, ask:

Is there anything else you'd like to say about the lesson?

Probe as above if appropriate.

Say, **Thank you. That's the last task I have for you.**

APPENDIX D

Routines Activity

Practices that Support Teaching: Routines

(To be assigned completed after The Case of Keith Campbell, sometime in the 4th week)

Watch the video of Cathy Humphreys' volume lesson. In watching the video and examining the transcript, identify at least three routines that the teacher uses. Using the chart below, classify each routine you identify as a support, exchange, or management routine, and describe how the routine supported the classroom activity. When describing a routine, be sure to provide evidence of the routine from the transcript.

Finally, identify an exchange or support routine that you use in your own teaching. Describe how you use it now and how the routine supports classroom activity.

Description of Routine	Line number(s) in transcript	Support, exchange, or management?	How does the routine support classroom activity?

Support: define and specify the types of actions that facilitate teacher-student learning exchanges (examples: distributing resources, directing students to the resources for the next activity, defining the starts and ends of activities)

Exchange: foster, structure, and clarify classroom discourse (examples: revoicing, asking students to "say more", asking for justification or evidence)

Management: help students move and act in predictable ways in the classroom (examples: group formation routines, signals for getting student attention and/or quieting down)

APPENDIX E

Selected Coding Rubrics

Coding Rubrics:
Content Knowledge in the Domain: Relating Area and Perimeter

Goal RAP1:
Understand that area and perimeter have a non-constant relationship (includes no evidence of misconceptions related to the goal)

Data sources:
Pre/Post Assessment, Part C Tasks 1, 2a; Part D, Task 5
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
Selected TTAL assignments
Selected Course Discussions (Rabbit Pens, Storm Shelters, Comparing Triangles)

Rubric RAP1.1
Data source: Pre/Post Assessment Part C Task 1
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Fence in the Yard task, as well as to indicate whether or not responses show clear evidence of, or clear lack of evidence of, the misconception that a fixed perimeter indicates a fixed area. It also is designed to identify the misconception of a long and thin rectangle having maximum area. The distinction between Correct-1 and Correct-2 is designed to identify whether the teacher shows clear evidence that they are considering multiple rectangles with the same perimeter and varying areas.

Code responses to Task 1 as **Correct-1**, **Correct-2**, **Incorrect-1**, **Incorrect-2**, **Vague/Inconclusive**, or **No Response**.

Correct-1: Response indicates that the maximum area for the pen is a 9 by 9 square. Response shows 2 or more examples (in any representation) that show that 2 pens can have the same perimeter and different areas.
Example 1:
Perimeter = 36 feet $2l + 2w = 36$ $2(l + w) = 36$ $l + w = 18$

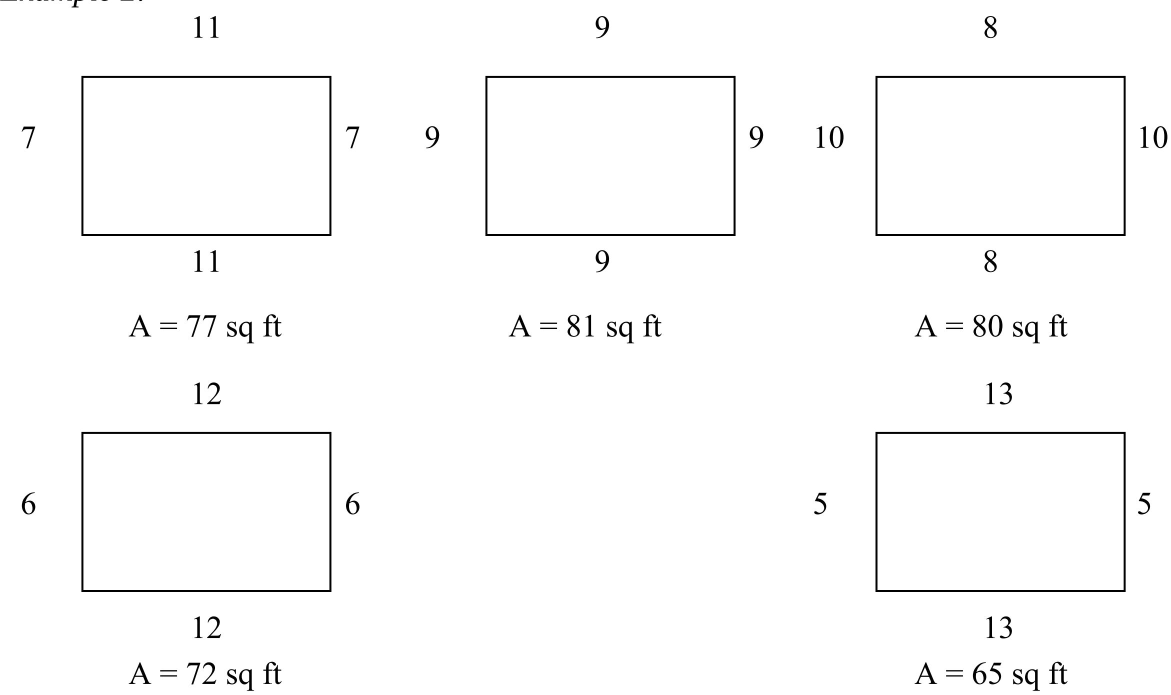
L	W	Perimeter (ft)	Area (ft^2)
1	17	36	17
2	16	36	32
3	15	36	45
4	14	36	56
5	13	36	65
6	12	36	72
7	11	36	77
8	10	36	80
9	9	36	81
10	8	36	80
11	7	36	77

* The largest area that Julie can enclose with 36 ft. of fencing is 81 sq feet

area starts to go down again, so all combinations have been formed

(LC, post)

Example 2:



The area of a square would be the highest. As difference in the length of the width and length increase, the area gets smaller.
(DE, pre)

Correct-2: Response indicates that the maximum area for the pen is a 9 by 9 square. Response does not show 2 or more examples (in any representation) that show that 2 pens can have the same perimeter and different areas.

Example 1:

l = length, w = width, A = area, P = perimeter

P=36 ft
36 = 2l + 2w
A=lw
l=18w
A=w(18 – w)
A = 18w – w^2
A' = 18 – 2w
0 = 18 – 2w
2w = 18
w = 9 ft
26 = 2l + 2(9)
18 = 2l
l = 9 ft
A = 9 * 9 = 81 ft
(NB, post)

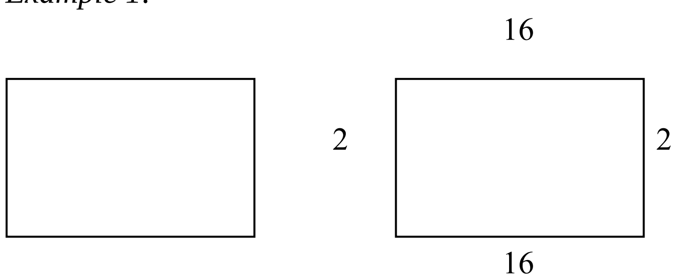
Incorrect-1: Response indicates an answer for the maximum area that is **not** a 9 by 9 square. Teacher shows clear evidence of the misconception that a pen of perimeter 36 can only have a single area.
Example 1:
For a rectangle to have a perimeter of 36, it must be a 9 x 9 x 9 x 9 square.

Incorrect-2: Response indicates an answer for the maximum area that is **not** a 9 by 9 square. Response shows no evidence of the misconception that a pen of perimeter 36 can only have a single area.
Example 1:
Referring to the table you can see that to get the most area her fence should be 10 x 10 x 8 x 8. The two rectangles/squares closed to that dimension are 9 x 9 x 9 x 9 and 11 x 11 x 7 x 7. Those two have a smaller area so the dimensions for 36 ft of fencing that give you the largest perimeter are 10 x 10 x 8 x 8.

dimensions	Perimeter	Area
9 x 9 x 9 x 9	36	36
5 x 5 x 13 x 13	36	65
10 x 10 x 8 x 8	36	80
11 x 11 x 7 x 7	36	77
12 x 12 x 6 x 6	36	72
14 x 14 x 4 x 4	36	56

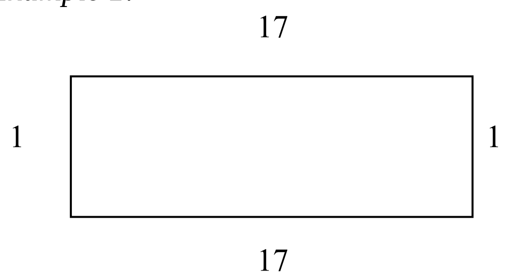
several sketches of rectangles and arithmetic work
(BD, post)

Incorrect-3: Response indicates an answer for the maximum area that is **not** a 9 by 9 square. Response shows evidence that the teacher holds the misconception that a long, thin rectangle will have the maximum area.
Example 1:

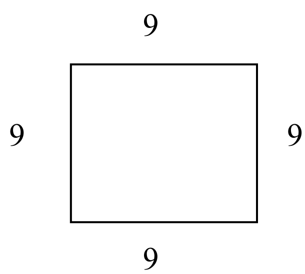


The longer the long sides of the rectangle become the larger the area becomes inside the triangle. I am aware of this relationship in geometry but I am unaware of how to prove this.
(MH, pre)

Vague/Inconclusive: Response does not clearly indicate an answer, but contains work relating to the task.
Example 1:



A = 17 ft^2



A = 81 ft^2

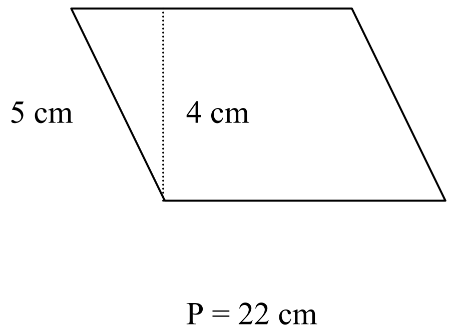
No Response: No response is indicated and little or no written work is evident that pertains to the task.

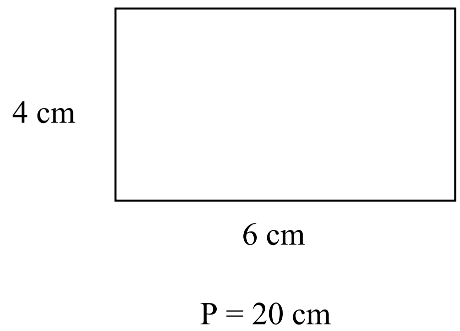
<u>Rubric RAP1.2</u>
Data source: Pre/Post Assessment Part C Task 2a
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Relating Area and Perimeter Parallelogram task, as well as to indicate whether or not responses show clear evidence of, or clear lack of evidence of, the misconception that a fixed area indicates a fixed perimeter.

Code responses to Task 2a as **Correct-1**, **Correct-2**, **Incorrect-1**, **Incorrect-2**, **Vague/Inconclusive**, or **No Response**.

Correct-1: Response indicates that the statement is false, and provides two or more examples that correctly demonstrate why. Alternatively, the examples can be replaced by a generalized argument that correctly demonstrates why the statement is false.
Example 1:
False





(KE, post)

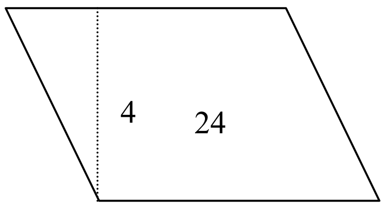
Example 2:
The area of a parallelogram is given by A =- bh ∴ If the base = 6 and the A = 24 then the height = 4. Any parallelogram with the same height & base have the same area but that does not mean the sides are the same.
(EL, post)

Correct-2: Response indicates that the statement is false, but does not provide examples or other reasoning to demonstrate why the statement is false.

Incorrect-1: Response indicates that the statement is true, and provides an example or other indication that there exists only one parallelogram with the specified area and base.
Example 1:
True
24 cm^2 = bh
24 cm^2 = 6h
4 = h
only one h possible

Incorrect-2: Response indicates the statement is true, but does not provide reasoning that clearly indicates that there exists only one parallelogram with the specified area and base. This could include erroneous reasoning, irrelevant reasoning, or no reasoning provided for the response. –OR– Response indicates the statement is false, but provides an erroneous reason.

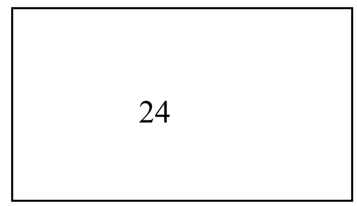
Example 1:
True



6

P = 20

A = bh
24 = 6 x h
4 = h always



4

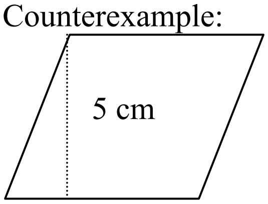
6

P = 20

P = 2b + 2h
P = 12 + 8
P = 20

(DH, pre)

Example 2:
False: the area of a parallelogram is its base x height so the only parallelogram w/base 6 cm that will have an area of 24 cm^2 will have a height of 4 cm.
Counterexample:

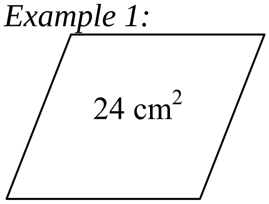


6 cm

A = bh
= 6 x 5
= 30 cm^2

(FY, pre)

Vague/Inconclusive: Response does not clearly indicate the truth value of the statement, or the explanation does not match the question asked.
Example 1:



6

(EH, pre)

No Response: No response is indicated.

Rubric RAP1.3
Data source: Pre/Post Assessment Part D Task 5
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to identify any teachers who, in response to Task 5, did not clearly identify the student's misconception as false.

Code responses to Task 5 as **Misconception**, **No Misconception**, **Unclear**, or **No response**.

Misconception: Response indicates that the teacher believes that the student's conjecture is true.

No Misconception: Response *clearly* indicates that the teacher believes that the student's conjecture is false.

Unclear: Response does not clearly indicate the teacher's belief regarding the misconception.

No response: There is no response to the prompt.

Goal RAP2:
Explain how changes to dimensions of a figure impact perimeter and/or area (including transformations on a plane figure)

Data sources:
Pre/Post Assessment, Part C Tasks 1, 2b, 2c
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
Selected Course Discussions (Comparing Triangles, Stacks of Paper)

Rubric RAP2.1
Data source: Pre/Post Assessment Part C Task 1
Rubric Score Data Type: Categorical, multiple codes possible per response

This rubric is designed to determine whether explanations related to the solution of the Fence in the Yard task contain language that correctly explains how the changes in the dimensions of the fence impact the perimeter and area.

Code responses to Task 1 with all applicable codes from the following list: **Written Explanation-1**, **Written Explanation-2**, **Table-1**, **Table-2**, **Graph**, **No Evidence**. Responses can only be coded as one version of Explanation or Table. Responses coded as No Evidence cannot have multiple codes.

Written Explanation-1: A written explanation is present in the response that describes how changes in the rectangular fence dimensions impact *area* from one rectangle to the next. Responses must connect at least two empirical examples or be generalized statements about how the change to the dimensions impacts area. The impact on perimeter (that perimeter stays constant) may or may not explicitly mentioned.

Example 1:
(table, diagram, and symbols also present in response)
A square will provide the largest area because the sides are the same. Other possibilities require one of the sides to be less than 9, which makes the area decrease.
(NG, pre)
Example 2:
(one diagram and some calculations given)
As the numbers/length of fencing gets further from one another the area decreases. This is the greatest area one can have w/36 ft of fencing. All the sides are the same, thus making it a square. As we discussed in class a square maximizes the area.
(MH, post)

Written Explanation-2: A written explanation is present in the response that describes how the maximum area was found, but does not explicitly address a relationship between dimensions, perimeter, and area, or only addresses the relationship between dimension and perimeter.
Example 1:
A figure that has 4 sides w/4 right angles must be a rectangle. The perimeter must be 36 ft. If all 4 sides total 36 ft, then one length + one width = 18 ft.
(Table with l, w, and area)
This table shows all whole # lengths + widths whose sum is 18. The corresponding areas are also shown, with the max area being 81 ft^2 for a 9 ft x 9 ft rectangle (square).
(KE, pre)

Table-1: A table is present in the response that contains the dimensions of several (at least 3) rectangles, a column for area, and a column for perimeter.

Table-2: A table is present in the response that contains the dimensions of several (at least 3) rectangles, and a column for perimeter. No column for area is present.

Graph: Response contains a graph that relates dimension and perimeter.

No Evidence: Response contains no evidence of an explanation regarding how changes to the dimensions of the rectangle impact perimeter and/or area.

Rubric RAP2.2
Data source: Pre/Post Assessment Part C Task 2b
Rubric Score Data Type: Ordinal (quality of explanation)

This rubric is designed to rate both correctness and quality of explanation related to the tangram rearrangement task. As a reminder, the task asks if two figures, both created from complete sets of non-overlapping tangram tiles, have the same area.

Code responses to Task 2b with Score Point **2**, **1**, or **0**.

Score Point 2:

- Response is correct (both figures have the same area)
- Justification is correct and uses the concept of area (e.g., the figure are made from the same set of tiles, and unless they overlap, area cannot change)

Example 1:
Area is equal because each of the polygons maintains its area in figure 2 and 3 and because you use all the tangrams you add together all the areas to get the irregular figure… in figures 2 and 3 you are adding all the same smaller areas (of each tangram) to get the area of the figures.
(SD, post)

Example 2:
They have the same area, because area is conserved when shapes are rearranged so long as there's no overlap.
(KE, post)

Example 3:
They have the same area because they are the same 7 tiles taking up the same amount of space,
(BD, pre)

Score Point 1:

- Response is correct (both figures have the same area)
- Justification is based on qualitative observation, or no justification is provided

Example 1:
Both of the shapes look like they should have the same area.

Score Point 0:

- Response is incorrect, or;
- No response is given, or;
- Response cannot be determined based on work provided

<u>Rubric RAP2.3</u>
Data source: Pre/Post Assessment Part C Task 2c
Rubric Score Data Type: Ordinal (quality of explanation)

This rubric is designed to rate both correctness and quality of explanation related to the tangram rearrangement task. As a reminder, the task asks if two figures, both created from complete sets of non-overlapping tangram tiles, have the same perimeter.

Code responses to Task 2b with Score Point **3**, **2**, **1**, or **0**.

Score Point 3:

- Response is correct (Figure 3 has the greater perimeter)
- Justification is correct and uses the concept of perimeter and the arrangement of the tiles (e.g., there are more exposed edges in Figure 3)

Example 1:
Figure 3 will have a greater perimeter because more of the edges of the "tiny" tangram tiles are exposed and need to be accounted for
(CD, post)

Example 2:
Fig. 3 has the greater perimeter, because perimeter is lost as pieces are compacted together in Fig. 2. I divided Fig. 3 into its pieces to show how ost pieces lost between 1+2 sides when joined, some only 1. Whereas on Fig. 2 some pieces would have to be completely within the Fig so not part of the perimeter.
(KE, post)

Score Point 2:

- Response is correct (Figure 3 has the greater perimeter))
- Justification is correct and uses a form of empirical measurement

Example 1:
I measured. Fig. 3. has the greater perimeter. (Measurements indicated on figures.)
(NG, pre)

Score Point 1:

- Response is correct (Figure 3 has the greater perimeter))
- Justification is based on qualitative observation, or no justification is provided

Example 1:
Figure 3 looks like it has the greater perimeter.

Score Point 0:

- Response is incorrect, or;
- No response is given, or;
- Response cannot be determined based on work provided

<u>Goal RAP4:</u>
Demonstrate understanding of the meaning of area and perimeter using a variety of tools and representations
Data sources:
Pre/Post Assessment, Part C Tasks 1, 2a
Selected Course Discussions (Index Card, Stacks of Paper)

<u>Rubric RAP4.1</u>
Data source: Pre/Post Assessment Part C Task 1
Rubric Score Data Type: Categorical, multiple codes possible per response

This rubric is designed to determine what representations are used in responding to the Fence in the Yard task.

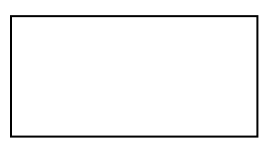
Code responses to Task 1 with all applicable codes from the following list: **Table**, **Symbolic/Formula-1**, **Symbolic/Formula-2**, **Symbolic/Formula-3**, **Written Explanation**,

Diagram, **Graph**, or **No Response/Answer Only**. Responses can only be coded as one version of Symbolic/Formula. Responses coded as No Response/Answer Only cannot have multiple codes. Representations do not have to be correct or accurate to be coded.

Table: A table is present in the response.

Symbolic/Formula-1: A symbolic expression or formula is present and used in the response. The use of the symbolic expression or formula does not qualify under the conditions of Symbolic/Formula-2 or -3, or is unclear.

Symbolic/Formula-2: A symbolic expression or formula is present and used in the response. The use of the symbolic expression or formula is consistent with a method of finding the maximum point using the –b/2a expression ***or*** factoring and finding the roots of the equation (as is typically done in Algebra II or the study of quadratic equations).
Example 1:
largest area with 36’ of fencing and it must be a rectangle, so
P = 36’
P = 2l + 2w
36 = 2l + 2w
18 = l + w
w = 18 – l
Area of rectangle:
A = lw
A = l(18 – l)
A = 18l – l^2
2 zero’s @ 0 & 18
quadratic
Maximum area when l=9 & w=9 from vertex
(MN, pre)

Symbolic/Formula-3: A symbolic expression or formula is present and used in the response. The use of the symbolic expression or formula is consistent with a method of finding the maximum point using calculus (taking the derivative and setting equal to zero).
Example 1:
l = length, w = width, A = area, P = perimeter

P=36 ft
36 = 2l + 2w
A=lw
l=18w
A=w(18 – w)
A = 18w – w^2
A' = 18 – 2w
0 = 18 – 2w
2w = 18
w = 9 ft
26 = 2l + 2(9)

18 = 2l
l = 9 ft
A = 9 * 9 = 81 ft
(NB, post)

Written Explanation: A written explanation is present in the response. This written explanation adds some justification to the response and is not simply a statement of the answer.

Diagram: Response contains diagrams of at least 1 rectangular pen.

Graph: A graph is present in the response.

No Response/Answer Only: Response contains only an answer or there is no response.

Rubric RAP4.2
Data source: Pre/Post Assessment Part C Task 2a
Rubric Score Data Type: Categorical, multiple codes possible per response

This rubric is designed to determine what representations are used in responding to the Relating Area and Perimeter Parallelogram task.

Code responses to Task 2a with all applicable codes from the following list: **Table**, **Symbolic/Formula**, **Written Explanation**, **Diagram**, **Graph**, or **No Response/Answer Only**. Responses coded as No Response/Answer Only cannot have multiple codes. Representations do not have to be correct or accurate to be coded.

Table: A table is present in the response.

Symbolic/Formula: A symbolic expression or formula is present and used in the response.

Written Explanation: A written explanation is present in the response. This written explanation adds some justification to the response and is not simply a statement of the answer.

Diagram: Response contains diagrams that support the conclusion.

Graph: A graph is present in the response that supports the conclusion.

No Response/Answer Only: Response contains only an answer or there is no response.

Relating Edge Length, Surface Area, and Volume

<u>**Goal RSV1:**</u>
Understand the relationship between edge length, surface area and volume, including the fact that surface area and volume have a non-constant relationship (includes no evidence of misconceptions related to the goal)

Data sources:
Pre/Post Assessment, Part C Tasks 3a, 3b, 3c
Learning Log 4
Selected Course Discussions (Arranging Cubes, Soda Can Task, Wet Box Task)

<u>Rubric RSV1.1</u>
Data source: Pre/Post Assessment Part C Task 3a
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Surface Area and Volume task (part a), as well as to indicate whether or not responses show clear evidence of, or clear lack of evidence of, the misconception that a fixed area of the base of a rectangular prism implies a fixed surface area.

Code responses to Task 3a as **Correct-1**, **Correct-2**, **Correct-3**, **Correct-4, Incorrect-1**, **Incorrect-2**, **Vague/Inconclusive**, or **No Response**.

Correct-1: Response indicates that the amount of paint needed is not necessarily the same for both rooms. Response includes a correct explanation of why the amount of paint needed is not the same, showing clear evidence that two prisms with the same area of the base does not guarantee the same surface area and contains a clear statement that links the dimensions to the surface area. For example: Without knowing the dimensions of the floor, we do not know the surface area of the walls.
Example 1:
<u>Case 1: </u>l = 50 ft w = 8 ft 50 x 8 = 400 ft^2
<u>Walls:</u> 50 x 8 = 400 ft^2 (two of these) 2 x 400 = 800 ft^2
8 x 8 = 64 ft^2 (two of these) 2 x 64 = 128 ft^2
Total = 800 + 128 = 928 ft^2
<u>Case 1:</u> lw = 400 l = 20 ft w = 20 ft
<u>Walls:</u> 20 x 8 = 160 ft^2 (four of these) 4 x 160 = 640 ft^2
Jim & John will only need the same amount of paint if their floors have the same dimensions. As shown, they could have different dimensions w/A=400 ft^2 & therefore the surface area of the walls depends on these dimensions.
(FY, pre)

Correct-2: Response indicates that the amount of paint needed is not necessarily the same for both rooms. Response includes a correct **explanation of why** the amount of paint needed is not the same, showing clear evidence that two prisms with the same area of the base does not

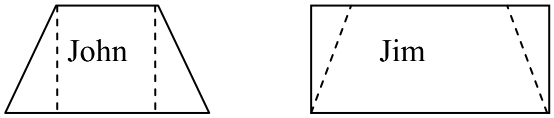
guarantee the same surface area. Response may contain a vague statement relating dimensions and surface area.
Example 1:
(two prisms drawn: 1 x 400, 20 x 20; SA calculated for each: 6416 and 640 ft^2, respectively)
2 possibilities w/different areas of walls. We don't know if we aren't given the dimensions of the floor.
(IT, pre)

Example 2:
A = lw = 400 ft^2
Surface area of 4 walls = 2(8l) + 2(8w) = 16l + 16w = 16 (l + w)
Many combinations of l * w = 400
l = 20, w = 20 => SA = 16(20 + 20) = 640 ft^2
l = 40, w = 10 => SA = 16(40 + 10) = 800 ft^2 & so forth
To answer the question, not necessarily
(NoT, pre)

Correct-3: Response indicates that the amount of paint needed is not necessarily the same for both rooms. Response only includes a counterexample with no additional information provided, or the explanation provided does not explain why or how the examples answer the question; that is, they do not clearly link the notion of different dimensions of the floor to different surface areas and/or the context of the problem (paint needed for the walls).
Example 1:
(two rectangles drawn, 40 x 10 and 20 x 20; SA calculated for each: 800 and 600 respectively)
NO
Counterexample given
(NiT, pre)

Correct-4: Response indicates that the amount of paint needed is not necessarily the same for both rooms. Response does not include an explanation of why the amount of paint needed is not the same, a set of examples, or response does not show evidence that two prisms with the same area of the base does not guarantee the same surface area.

Incorrect-1: Response indicates that the amount of paint needed is the same for both rooms (or is a correct response with incorrect reasoning). Response shows clear evidence of the misconception that the same area of the base implies the same surface area.
Example 1:
The shapes of the rooms may differ & there may be more walls in one room, however if the floor size <u>is</u> equal & the ceilings are the same height. The same amount would be needed. If the walls differ <u>in</u> width the floor space is still the same. (MH, pre)
.
Incorrect-2: Response indicates that the amount of paint needed is the same for both rooms (or is a correct response with incorrect reasoning). Response shows no clear evidence of the misconception that the same area of the base implies the same surface area.
Example 1:
Yes, b/c each room is the same regardless of the floor plan. Example:



Vague/Inconclusive: Response does not clearly indicate an answer, but contains work relating to the task. –OR– Response indicates a correct answer, but the explanation does not support the answer.

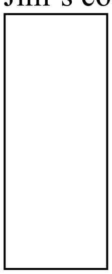
Example 1:

Depends on the shape of the room. If one lives in a pyramid room (square base) with a height of 8, then the slant height would be $\sqrt{(10^2 + 8^2)} = \sqrt{164} = 2\sqrt{41}$. Thus, the area of the walls would be $((2\text{x}\sqrt{41})\text{x}20)/2 = 20 \text{ x } \sqrt{41}$ sq ft. If the other lived in a square room, then the walls are 8 x 20 = 160 sq ft. Both are times 4 walls, so $4 \text{ x } 20\sqrt{41} = 4 \text{ x } 20 \text{ x } 6.5 = 520$ sq ft + 4 x 160 = 640 sq ft.
(NL, pre)

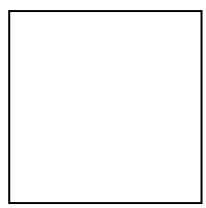
Example 2:

Not definite.

Jim's could be while John's could be



400 ft^2



400 ft^2

or vice versa where neither have the same surface area.
(BI, pre)

Example 3:

A = 20 x 20 160 x 5 = 640

A = 10 x 40 (100 x 8)2 (400 x 8) 2 160 + 640 = 800

if one is a square and the other is a rectangle, they will have different lateral areas
(NB, pre)

No Response: There is no response to the task.

<u>Rubric RSV1.2</u>

Data source: Pre/Post Assessment Part C Task 3b

Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Surface Area and Volume task (part b), as well as to indicate whether or not responses show clear evidence of, or clear lack of evidence of, the misconception that a fixed area of the base of a rectangular prism and a fixed height can produce differing volumes.

Code responses to Task 3b as **Correct-1**, **Correct-2**, **Incorrect-1**, **Incorrect-2**, **Vague/Inconclusive**, or **No Response**.

Correct-1: Response indicates that the volume of both rooms is the same. Response includes a correct explanation of why the volume is the same, showing clear evidence of understanding that two prisms with the same area of the base and height guarantees the same volume. Response specifies the relationship between area of the base, height, and volume; namely, that both prisms have the same volume because they both have the same area of the base (or l*w) and volume.
Example 1:
Yes – volume is found by multiplying l*w*h. We know that both room s have an area of 400 ft^2, which means l*w = 400. Also, we know that both rooms have the same height (8 ft). Therefore, V = lwh for both dining rooms
= 400 * 8
= 3200 ft^2
(LC, pre)

Example 2:
Yes, because if they have the same area (l*w) and the height is 8 ft for both of them then the volume would be
V = l*w*h
V = (l*w)*8
V = 400 * 8 = 3200 ft^3
(UT, pre)

Correct-2: Response indicates that the volume of both rooms is the same. Response includes a correct explanation of why the volume is the same, showing evidence of understanding that two prisms with the same area of the base and height guarantees the same volume, but some details about the nature of this relationship are unclear or omitted. (This evidence may be implicit in the form of an incomplete statement or an annotated formula.) Response may contain a vague or unclear statement regarding the relationship between area of the base, height, and volume.
Example 1:
Yes V = Bh
V = 400 * 8
because the area of the floor is the same
(NB, pre)

Example 2:
Yes, the area of the floor is the same and the height is the same, so volumes are the same.
(EH, pre)

Correct-3: Response indicates that the volume of both rooms is the same. Response only includes calculated examples with no additional information provided, or the explanation provided does not explain why or how the examples answer the question; that is, they do not clearly link the calculations to either the problem situation or to the volume of the rooms.
Example 1:
V = Bh
V = 400 * 8
V = 3200 ft^3

Correct-4: Response indicates that the volume of both rooms is the same. Response does not include a correct explanation of why the volume is the same, does not include examples, or response does not show clear evidence of understanding that two prisms with the same area of the base and height guarantees the same volume.
Example 1:
Yes, they are the same volume.

Incorrect-1: Response indicates that the volume of the rooms are different (or is a correct response with incorrect reasoning). Response shows clear evidence of the misconception that the same area of the base and height does not guarantee the same volume.

Incorrect-2: Response indicates that the volume of the rooms are different (or is a correct response with incorrect reasoning). Response shows no clear evidence of the misconception that the same area of the base and height does not guarantee the same volume.

Vague/Inconclusive: Response does not clearly indicate an answer, but contains work relating to the task. –OR– Response indicates a correct answer, but the explanation does not clearly support or contradict the answer.

No Response: There is no response to the task.

<u>Rubric RSV1.3</u>
Data source: Pre/Post Assessment Part C Task 3c
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Surface Area and Volume task (part c), as well as to indicate whether or not responses show clear evidence of, or clear lack of evidence of, the misconception that a knowing the volume of a rectangular prism implies knowing the surface area of the rectangular prism.

Code responses to Task 1 as **Correct-1**, **Correct-2**, **Incorrect-1**, **Incorrect-2**, **Vague/Inconclusive**, or **No Response**.

Correct-1: Response indicates that the surface area is not known. Response contains a clear explanation of why knowing volume does not imply knowing surface area that links the two quantities via the dimensions of the rectangular prism.
Example 1:
V = lwh / SA = 2lw + 2wh + 2lh
No, there are too many unknown variables, i.e., lwh
(CD, pre)
Example 2:
No: 2 faces will be w*l, 2 faces will be w*h, 2 faces will be h*l
because you can have different factors which you could give different surface areas when you have lengths (factors) that are close you maximize the SA
(two examples included in table form)

(NG, pre)

Correct-2: Response indicates that the surface area is not known. Response does not contain a clear explanation of why knowing volume does not imply knowing surface area that links the two quantities via dimensions.
Example 1:
No. Because you may not know the exact dimensions that are needed to find the surface area. (DE, pre)

Example 2:
No, you need to know the individual l, w, & height.
(KE, pre)

Example 3:
$SA = h*P_b * 2A_b$ $V = h*A_b$
know, you need to know the perimeter of the base
(DH, pre)

Incorrect-1: Response indicates the surface area is known. Response shows clear evidence of the misconception that knowing the volume implies knowing the surface area.
Example 1:
Yes, you could break down the volume until you have the pieces that make up volume. I would assume that the formula works just as well when breaking it up.
(MH, pre)

Incorrect-2: Response indicates the surface area is known. Response does not show clear evidence of the misconception that knowing the volume implies knowing the surface area.
Example 1:
Yes… if you have the 3 measures you can find area of all rectangles that create it
(SD, pre)

Vague/Inconclusive: Response does not clearly indicate an answer, but contains work relating to the task. –OR– Response indicates a correct answer, but the explanation does not clearly support the answer.
Example 1:
$V = ht^2 + w^2$
the volume will allow you to find the ht. & w
(BD, pre)

Example 2:
L x w x h = V
If it is a cube, then L x w x h = L x L x L = V
So L3 = V and $L = \sqrt[3]{V}$. So if all sides are equal, then the SA is $(\sqrt[3]{V} * \sqrt[3]{V})x6$.
(NL, pre)

No Response: There is no response to the task.

<u>Goal RSV2:</u>
Explain how changes to the dimensions of a 3-D figure (specifically a rectangular prism) impact surface area and volume

Data sources:
Pre/Post Assessment, Part C Tasks 3d, 3e
Selected Course Discussions (Soda Can Task, Wet Box Task, Large Numbers Lab)

<u>Rubric RSV2.1</u>
Data source: Pre/Post Assessment Part C Task 3d
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Surface Area and Volume task (part b). Specifically, the rubric is designed to assess the ability to describe how to change the dimensions of a rectangular prism to produce a new prism of twice the volume.

Code responses to Task 1 as **Correct-1 (Double one)**, **Correct-2 (Add same size box)**, **Correct-3 (Factors)**, **Correct-4 (Double specific)**, **Incorrect-1 (Double all dimensions)**, **Incorrect-2 (Other)**, **Vague/Inconclusive**, or **No Response**.

Correct-1 (Double one): Response correctly describes how to create a box of double the original volume by doubling any one dimension.

Correct-2 (Add same size box): Response correctly describes how to create a box of double the original volume by adding a box of the same size on to the original box.

Correct-3 (Factors): Response correctly describes how to create a box of double the original volume through a factor approach in which the volume of the original is doubled, and that result is re-factored into three new dimensions that produce the same product.

Correct-4 (Double specific): Response correctly describes how to create a box of double the original volume by doubling a specific dimension. The specific dimension (length, width, or height) must be clearly mentioned or otherwise indicated.

Incorrect-1 (Double all dimensions): Response incorrectly describes how to create a box of double the original volume by doubling all dimensions.

Incorrect-2 (Other): Response incorrectly describes how to create a box of double the original volume in any other way.

Vague/Inconclusive: Response does not clearly indicate an answer, but contains work relating to the task.

No Response: There is no response to the task.

Rubric RSV2.2
Data source: Pre/Post Assessment Part C Task 3e
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to distinguish correct and incorrect responses on the Surface Area and Volume task (part b). Specifically, the rubric is designed to assess the ability to describe how to change the dimensions of a rectangular prism to produce a new prism of four times the surface area.

Code responses to Task 1 as **Correct-1 (2B + Ph)**, **Correct-Cube (Double each)**, **Incorrect-1 (Double two)**, **Incorrect-2 (Other)**, **Vague/Inconclusive**, or **No Response**.

Correct-1 (Ph + 2B): Response correctly describes how to create a box of four times the original surface area by doubling either both the area of the base and the perimeter of the base, or both the area of the base and the height. (This is based on the SA = 2B + Ph formula.)

Correct-2 (Double each): Response describes how to create a box of four times the original surface area by doubling each dimension.

Incorrect-1 (Double two): Response incorrectly describes how to create a box of four times the original surface area by doubling two dimensions.

Incorrect-2 (Other): Response incorrectly describes how to create a box of four times the original surface area by any other means.

Vague/Inconclusive: Response does not clearly indicate an answer, but contains work relating to the task.

No Response: There is no response to the task.

Goal RSV3:
Link the concepts of surface area and volume to spatial structuring and the composition of a 3-D figure

Data sources:
Pre/Post Assessment, Part C Task 3; Part D, Task 6
Pre-Course Interview, Task 1
Post-Course Interview, Tasks 1, 2
Learning Log 3
Selected Course Discussions
(Soda Can Task, Wet Box Task, Large Numbers Lab, Comparing Formulas, Big Ideas Class 12)

Rubric RSV3.1
Data source: Pre/Post Assessment Part C Task 3, all parts
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to identify language related to connecting spatial structuring to concepts of surface area and volume. The rubric can be applied to all five parts of Task 3.

Code responses to any part of task 3 as **Layering**.

Layering: Response contains language that relates to a layering visualization of a rectangular prism, where the prism is seen as consisting of the area of the base, layered vertically through the distance of the height. Explicit discussion of layering or a layer of the rectangular prism is sufficient. Stacking which describes the building up of base layers is also sufficient. Use of the formula V=Bh without additional explanation is not sufficient to code.

Example 1:
(3b) Yes, because the area of your "base" is always 400 ft^2 and the height is 8 ft. You can think of it like stacking – even if the shape of the "bases" are different, you are still stacking 400 ft^2, 8 ft high.
(CD, pre)

Example 2:
(3b) Volume is defined as l x w x h, and we know l x w = 400 sq ft (regardless of the value of l or w) and we know that the h = 8 ft. 400 x 8 = 3200 ft^3. 400 cubic feet would fill the bottom layer of each room, covering the 400 sq. ft. of floor, then each room has 8 layers of cubes, so they contain 3200 cubes.
(KE, pre)

Rubric RSV3.2
Data source: Pre/Post Assessment Part C Task 3b
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to identify whether teachers used the V=Bh approach with Task 3b, which is much more easily solved using this approach. I take the use of this approach in this problem to show flexibility with respect to the concept of volume.
Code responses to Task 3b as **V=Bh**, **V=lwh**, **Both**, or **No Formula**.
V=Bh: Response uses the formula V=Bh or an explanation that explicitly refers to area of the base, height, and volume to solve the task.

V=lwh: Response uses the formula V=lwh or an explanation that explicitly refers to length, width, height, and volume to solve the task.

Both: Response shows evidence of both volume formulas (or equivalent explanations) used to solve the task.

No Formula: Response does not use a formula or other qualifying explanation.

Rubric RSV3.3
Data source: Pre-course Interview, Task 1a; Post-course Interview, Task 2a
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to layering strategies evident in teachers' work on the box task in the pre- and post-course interview.

Code lines of interview in any part of Task 1a (pre) or Task 2a (post) as **Layering**.

Layering: Strategy used to determine volume relates to a layering visualization of a rectangular prism, where the prism is seen as consisting of the area of the base, layered vertically through the distance of the height. Explicit discussion of layering or a layer of the rectangular prism is sufficient. Stacking which builds up layers of blocks is also sufficient. Use of the formula V=Bh without additional explanation is not sufficient to code.

Example 1:
Ok. [pause] I always, [pause] didn't like how it didn't have the grids on the bottom that, was really hard for me to, n- um not have it on the bottom. [pause] Let's see, so if I want to figure how [pause] how many will fill it, I need to find out how many blocks [pause] will fit in the bottom. So if I turn the blocks so that they're- fit against the smaller side. Um, I can find that 3, blocks can fit across the smaller side. And then the- so it's 3, and how many will fit down the edge, but since each space is 2 blocks wide, but I've got the 1 face which is just 1 inch, so it's gonna be 2, blocks for each square- rectangle. It's 2 4 6 8, 10. So it's 3 by 10, along the bottom so it should be 30, blocks that'll fit, in one layer. So I need to find out how many layers high. So if I stack them, I find that it's 3 layers high, so 3 layers of 30, which is 90. (NB, post, ll. 262-273)

Example 2:
Um, [pause] ok so I'm thinkin' 1 2 3 4 5 it's 5 packages across, and I know that, there'd be, 2 packages in each of these guidelines here that you have- this way, so I know this would be 2 4 6, so there'd be 6 rows of 5 on the bottom layer. And then, [pause] 3 layers high. So that's 6 by 5 is 30, packages, and then, 3 rows of those would be 90 packages altogether. (KE, post, ll. 436-442)

Rubric RSV3.4
Data source: Pre-course Interview, Task 1a; Post-course Interview, Task 2a
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to identify the specific misconception that multiplying length times width times height yields volume when dealing with rectangular units.

Code lines of interview in any part of Task 1a (pre) or Task 2a (post) as **Misconception**.

Misconception: Teacher multiplies the number of packages along the length, width, and height, and arrives at an answer of 45 for the number of packages that fit in the box.

Example 1:
Well, I guess what I would do is just figure out how many, uh, would just fit on this one side. So I'd count up here 3, and then 5, so there's 15 packages, a side. And then we figure out how many go across. And I would just count across, so that'd be 3- 3 sets of 15, so it's 45. (UT, pre, ll. 24-29)

Example 2:
Oh, ok. Ok. So I see that 1 package represents the 1, rectangular mark so I sort of lined them up to see how many packages would follow the length of the box. Which was 1, 2, 3, 4, 5. So I figure you can fit 5 packages this way, and you can stack them by 3 [pause] that looks 3 high [pause] And then this dimension you can fit 1, 2, 3 packages, 3 high, so the base would then by, 15 packages? 3, by 5? [pause] So 15 packages to fill the base of the box, [pause] and they're 3 high, [pause] so 45 packages will fill the whole thing. (NiT, pre, ll. 27-34)

Goal RSV4:
Demonstrate understanding of the meaning of surface area and volume using a variety of tools and representations

Data sources:
Pre/Post Assessment, Part C Tasks 3a, 3b
Selected Course Discussions
(Arranging Cubes, Soda Can Task, Wet Box Task, Large Numbers Lab, Comparing Formulas, Big Ideas Class 12)

Rubric RSV4.1
Data source: Pre/Post Assessment Part C Tasks 3a, 3b
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to determine the representations used to respond to the Surface Area and Volume Task.
Code responses to each part of Task 3 with all applicable codes from the following list: **Written Explanation**, **Symbolic/Formula**, **Example**, **Diagram**, **Graph**, or **No Response/Answer Only**. Responses coded as No Response/Answer Only cannot have multiple codes. Representations do not have to be correct or accurate to be coded.
Symbolic/Formula: A symbolic expression or formula is present and used in the response.

Written Explanation: A written explanation is present in the response. This written explanation adds some justification to the response and is not simply a statement of the answer.

Example: An example is used as a part of the response.

Diagram: Response contains diagrams that support the conclusion.

Graph: A graph is present in the response that supports the conclusion.

No Response/Answer Only: Response contains only an answer or there is no response.

Reasoning and Proof

<u>Goal RPR1:</u>
Define proof

Data sources:
Pre-Course Interview, Task 2a
Post-Course Interview, Task 3a
Selected Course Discussions (Defining and Revisiting Proof)

<u>Rubric RPR1.1</u>
Data source: Pre-Course Interview, Task 2a, and Post-Course Interview, Task 3a
Rubric Score Data Type: Categorical, multiple codes per response

Definition of proof that captures key understandings based on the literature:
A proof is a mathematical argument that is general for a class of mathematical ideas, and establishes the truth of a mathematical statement based on mathematical facts or truths that are accepted or have been previously proven.

Code teachers' responses to the proof questions in Task 2a or 3a using the following codes: **General**, **Mathematical Argument**, **Establishes Truth**, **Based on Mathematical Facts**. Any of the 5 questions asked during Task 2a or 3a are eligible for coding based on the criteria below.

General: Response indicates that a proof has to be general; that is, it has to hold for a class of objects or be more than just an example or series of examples. This also includes the fact that proofs are immune to counterexamples. (If the teacher states that a proof may be nullified by a counterexample, code as an instance of **General**.)

Example 1:
Um and proof is going to, [pause] show you that, it's going to be true for every single case, it's not a case-by-case like, here's a bunch of examples so it's a proof. (EL, pre, 113-115)

Example 2:
Um, then you wanna make sure that, it's true for, all possible cases that you can think of of. I just- you'd want to look at situations where, it might NOT be true um, something you're doing with numbers, [pause] check that it works for positives, negatives and zeroes. (UL, pre, ll. 120-123)

Mathematical Argument: Response indicates that proof has to be a mathematical argument, one that follows rules for argumentation in the mathematical domain. Examples might include that it has a logical structure, steps must be justified, or there must be clear mathematical reasoning behind the proof. (This notion of argument must not necessarily include the word mathematics or mathematical, but must speak to the structure or nature of the argument in some way.)

Example 1:
So you have to like justify everything you're saying, and why- why it makes sense in what you're trying to prove, why it's helping you get closer to, [pause] whatever you're trying to prove? (DH, pre, ll. 201-204)

Example 2:
What makes an argument a proof. [pause] Um, if you, give a reason or justify every- every statement you make, and then it all follows each other and leads up to, the ending truth. (DH, pre, ll. 251-253)

Example 3:
Proof, to me means that, you logically think through a problem and give reasons why something is true. (EL, pre, ll. 112-113) *Double-code as **establishes truth**.*

Establishes Truth: Response indicates that proof must establish a given statement or conjecture as true.

Example 1:
Um, I guess it's the- it's supposed to you understand why, things, are, true or why things are facts or, what makes them work. [pause] (DH, pre, ll. 219-220)

Example 2:
Proof, to me means that, you logically think through a problem and give reasons why something is true. (EL, pre, ll. 112-113) *Double-code as **mathematical argument**.*

Based on Mathematical Facts: Response indicates that proof is based on mathematical facts, previously proven results, or unproved assumptions that are agreed-upon by the person creating the proof and/or the mathematical community at large.

Example 1:
Um, it's gonna be using facts um, and things that you already know to show, something new is true. (EL, pre, 115-116)

Example 2:
Um, [pause] i- in mathematics, um, it would be, [pause] um, um uhhhh, [pause] starting with some basic assumptions and axioms and, use, constructing them, sequencing them in a logical way, [pause] to um, [pause] verify a conclusion. (MN, pre, ll. 154-156) *Double-code as **mathematical argument** and **establishes truth**.*

<u>Goal RPR2:</u>
Identifying proofs and non-proofs

Data sources:
Pre-Course Interview, Task 2b, 2c
Post-Course Interview, Task 3b, 3c
Selected Course Discussions (Pythagorean Theorem Proofs)

Rubric RPR2.1
Data source: Pre-Course Interview, Task 2b, and Post-Course Interview, Task 3b
Rubric Score Data Type: Numerical, one code per response

This rubric is designed to identify the number of proofs and non-proofs that teachers correctly identified on the interview task.

On the recording sheet provided, use the interview transcript to identify whether teachers identified each explanation as **Proof**, **Non-Proof**, or **Undetermined**. If **Undetermined**, prove explanation.

Rubric RPR2.2
Data source: Pre-Course Interview, Task 2b, and Post-Course Interview, Task 3b
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify the features of the proofs that teachers found critical in determining their proof status. The coding used corresponds to the reasons cited for classifying proofs in Knuth (2002b). Using the explanations teachers provided for whether or not each explanation is a proof, code lines of interview as **Convincing Argument**, **Concrete Features**, **Familiarity with Proof**, **Sufficient Level of Detail**, **Shows Why**, **Valid Method**, **General**, and **Do Some Examples**. For each code, start coding at the first line necessary to maintain a context for the comment, and end coding at the last line necessary to maintain context, such that the entire coded segment can stand alone as an intelligible statement. For each code below, code both positive instances (supporting classification as a proof) and negative instances (using lack of the feature to support classification as a non-proof).

Convincing Argument: Response indicates that the argument provided in the explanation was convincing to the teacher. The convincing nature of the argument need not be specified, and could come from a number of sources; for example, the argument could be mathematically convincing, semantically convincing, or aesthetically convincing. If a reason is provided for why the argument is convincing, the statement may need to be coded as **Sufficient Level of Detail**, **Shows Why**, or **Valid Method**.

Example 1 (negative): I mean I thought I was looking at it doesn't prove anything to me. (DH, pre, l. 354)
Example 2: but I think this, without a doubt I- I'd buy this argument. I couldn't argue against it and I think that makes it a valid proof. (IT, pre, ll. 322-323)

Concrete Features: Response indicates that the concrete structural features of the proof, such as the format, caused the teacher to classify the explanation as proof. The most common example of this code will classifying Explanation 1 as a proof because of the two-column format.

Example 1: I also like it because it's the traditional two-column thing which I'm used to. (NiT, pre, l. 217)

Example 2 (negative): So again it wasn't a traditional proof which is what I really go for. (NiT, pre, l. 259)

Familiarity with Proof: Response indicates that teachers' familiarity with the proof, from either using it in the classroom or learning/being told that the explanation was a proof, caused the teacher to classify the explanation as a proof. The most common manifestation of this code will be for Explanation 6, which was experienced by MATs in the Teaching Lab course, is found in Connected Mathematics, and was discussed in class.

Example 1: I like this one. I actually did Explanation 6, and again, on the 8th grade level, proving the Pythagorean Theorem using this very visual representation um, (IT, pre, ll. 347-348)

Example 2: Alright and then this [Explanation 6] is a proof, for sure. A geometric proof. It better work because I do it in my class every year, it better work. (KE, pre, ll. 311-315)

Sufficient Level of Detail: Response indicates that the teacher felt that the explanation provided a sufficient level of detail to convince the reader that the argument is a proof. A negative instance of this code would be a response that indicates that the explanation is not sufficiently detailed to qualify as a proof. This includes notions of "skipping" steps, the proof being confusing, or illustrating how one step or statement follows from the next. If the teacher identifies language that relates to ***explaining*** steps mathematically or providing justification for why a step is "legal," code as **Shows Why**.

Example 1 (negative): Um, [pause] 3a, [pause] it sounds good but, I don't really understand what they did. Like I tore up the angles and put them together as shown below. And it forms a straight like, [pause] I don't know, it's not a straight line. There's like all kind of lines there. Um, so [pause] I don't know. (DH, pre, ll. 308-311)

Example 2 (negative): Everything in the pictures makes sense. Since the 2 squares are congruent their areas are equal. But then they say canceling out the triangles in each fric- in each picture, we're left with the square of the hypotenuse, and that's not really, clear. [pause] Which is equal to the sum of [pause] (xx xx xx xxx) I don't know. I think it's a proof, but just I think it could have been clearly- more clearly explained. (DH, pre, ll. 358-364)

Shows Why: Response indicates that the explanation is a proof because it *shows why* the original statement is true or sufficiently justifies the statement. This may refer to the explanation in its entirety or specific steps. A negative instance of this code would be a response which indicates that the explanation in its entirety, or particular steps, did not sufficiently illustrate why the statement or step is true. Use this code as opposed to **Sufficient Level of Detail** when the teacher identifies that the proof itself does/does not show why the statement is true, or when the teacher identifies that a certain step contains or does not contain an explanation that shows why the step is true or valid.

Example 1 (negative): Um, I don't understand um, why they had to put it in a right triangle. Um, to me that was example again, I'm not sure- just because you CAN make it happen doesn't mean that it has to always happen. So the fact that they used a right triangle confused me 'cuz I wasn't sure why. (NiT, pre, ll. 270-274)

Example 2 (negative): I just don't think there's a lot in here, so it's- I don't see it as being something step by step by step that, gets me to, [pause] that or even why there's a triangle there that would satisfy the Pythagorean Theorem. (IT, post, ll. 521-524)

Valid Method: Response indicates that the explanation, or a specific portion of the explanation, uses a valid method for proving a mathematical statement as true. This can include the method of proving (e.g., backing up statements with known facts, a logical sequence, using algebra to generalize), or the nature of the evidence used (that the rationales are known mathematical facts). This differs from the **Concrete Features** code in that the response targets a specific rationale or method within the proof, not just representational or surface-level features of the proof. A negative instance of this code would be a response that indicates that a step or portion of the explanation is not a valid method of proving. This code is likely to be particularly salient with Explanation 4 (the statement about the shortest distance between two points is a straight line) and Explanation 3c (the dynamic method of demonstrating the angle sum conjecture).

Example 1 (negative): Um, [pause] oh the only thing that bothered me about num- um Explanation 4, [pause] is, it says the shortest distance between 2 points is a straight line, and I'm not sure if that's a theory or a postulate or, I'm not sure that's one of the things that you can use as evidence. (NiT, pre, ll. 264-267)

Example 2 (negative): So this is like if I wanted to show, or I wanted somebody to show me that those angles were 180 degrees, I think that this is a very valid way to do it but I would expect to see something more elegant, maybe in later grades. (IT, post, ll. 481-483)

Example 3: So um, Explanation 1 is a proof because it's all based on, I mean it's traditional it's the (same)- like all these properties and definitions and whatever are true, and have been proven themselves to be true, so, y'know it's all based on true facts and it works so it must be a proof. (KE, pre, ll. 211-214)

General: Response indicates that the explanation proves the statement in a way that is sufficiently general for the statement specified. A negative instance of this code would be a response that indicates that the explanation was not sufficient to generalize to all elements of the class specified by the original statement. (For example, Explanation 2 is an example and the method does not generalize to all polygons.)

Example 1 (negative): Number 2 I (xxx)- I don't think so. Because it says prove, that the sum of the exterior angles of A polygon, are 360 and they used a specific example, of a REGULAR pentagon. And they- and they said well, [pause] um, the formula would, for interior angles is the same for any polygon but I don't think that's true I think it's only the same for regular, where you can divide them evenly. I don't think that's a proof. I think it's an example. (NiT, pre, ll. 224-229)

Example 2 (negative): I think that 2 is not a proof yet. [pause] I think that um, [pause] since they're saying they want to look at a polygon, that this is only looking at one, um, a pentagram- no, a pentagon. (IT, post, ll. 469-472)

Do Some Examples: Response indicates that the explanation would be more convincing if additional examples were included (but not explicitly for the purpose of generality or exhausting cases), <u>or</u> teacher did additional examples to convince themselves of the veracity of the proof.

Example 1: I mean I gue- I guess it works because, even when this is, a negative number it's still squared so. [pause] This is always equal to something. [long pause] (xxxx xxx know). [pause] So give me any number greater than zero. So let's say, ok let's take 3. [pause] [writing] So this is 1, no that's not right. [pause] Oh yeah it is. It's… [long pause] (xxxx xxx4 over 9). [pause] That's 36 over 9. (KE, pre, ll. 262-267)

Example 2: [long pause] Yeah. If I just drew a picture, that would make sense. Because if I had one triangle here, and I had another larger one here. [pause] Um, it wouldn't have to- I'll draw 3 cases (xx xxx can). I'm doing a fairly lousy job at this. Um, if I know these- if I have 2 triangles with better c- I know 2 sides are congruent- There 2 are congruent to these 2, these 2 are congruent these 2 are congruent, I don't know that third side's congruent unless I know that angle there. [pause] So if I was enlarging it here, and I knew these 2 are alike, if I knew that angle that would prove it. So that's a proof. (KT, pre, ll. 317-325)

<u>Rubric RPR2.4</u>
Data source: Pre-Course Interview, Task 2c, and Post-Course Interview, Task 3c
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify the features of the proofs that teachers found critical in rating the proofs. The rubric categories are identical to those contained in Rubric RPR2.2.

<u>Goal RPR3:</u>
Constructing mathematical arguments

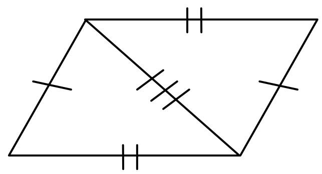
Data sources:
Pre/Post Assessment, Part C Task 4
Selected Course Discussions (Prove the area of a triangle)

<u>Rubric RPR3.1</u>
Data source: Pre/Post Assessment, Part C Task 4a
Rubric Score Data Type: Ordinal, one code per response

This rubric is designed to rate teachers' proofs of area of a triangle using an ordinal rubric. The rubric captures aspects of proof that include correctness, level of detail, representations, and generality.

Code responses to Task 1 as **6**, **5**, **4**, **3**, **2**, **1**, or **0** as described below.

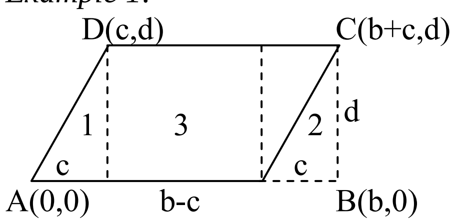
6 *Meets all of the following criteria:*
Response is a fully correct mathematical argument that is general and holds for all parallelograms.
Response provides descriptive detail for each step taken and where appropriate, reasons for taking each step.
Response uses multiple representations in the argument.
Example 1:

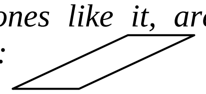


A parallelogram can be cut into 2 congruent triangles (opposite sides of a parallelogram are ≅, and by SSS the triangles are ≅). We know that the area of a triangle is ½bh so the area of the parallelogram has to twice that, or A = 2(½bh) = bh (DN, pre)

5 *Meets the all of the following criteria:*
Response is not sufficiently general. That is, an assumption in the argument limits the proof's generality, such as beginning with a rectangle.
Response is otherwise correct.
Response provides descriptive detail for each step taken and where appropriate, reasons for taking each step.
Response uses multiple representations in the argument.

Example 1:

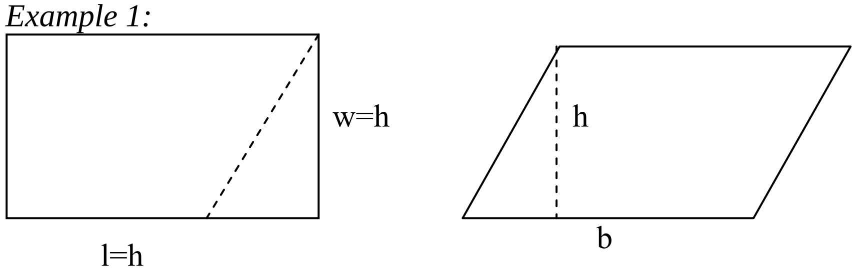


(Diagram also indicates entire base as b)
$A_1 = \frac{1}{2}cd$ $A_2 = \frac{1}{2}cd$ $A_3 = (b-c)d$
$A_T = \frac{1}{2}cd + \frac{1}{2}cd + bd - cd$
$A_T = bd$
$A_{parallelogram} = \text{base} * \text{height} = bd$
(DE, post)
Note: this argument, and ones like it, are not general for parallelograms that can't be partitioned in this way, like so: 

4 *Meets the first criterion and either of the next two:*
Response is a fully correct mathematical argument that is general and holds for all parallelograms.
Response provides some descriptive detail, but it is unclear how one or more steps are justified or how they relate to the argument as a whole.
Note: This includes using a theorem name as justification when the nature of the theorem is not obvious for a casual user of mathematics.
Response uses a single representation in the argument.
No examples found for this code at present time

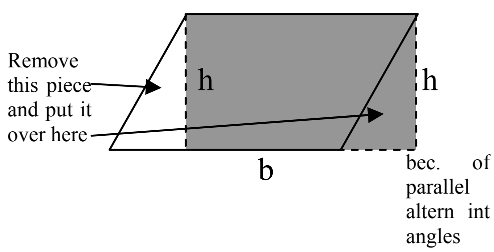
3 *Meets the first criterion and either of the next two:*
Response is not sufficiently general, but is otherwise correct. That is, an assumption in the argument limits the proof's generality, such as beginning with a rectangle.
Response provides some descriptive detail, but it is unclear how one or more steps are justified or how they relate to the argument as a whole.
Note: This includes using a theorem name as justification when the nature of the theorem is not obvious for a casual user of mathematics.
Response uses a single representation in the argument.

Example 1:



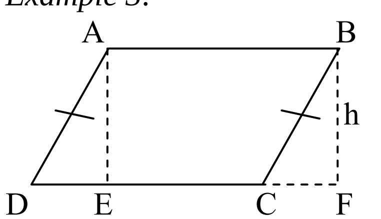
The area is moved the same amt of shape is being covered.
(IT, post)

Example 2:



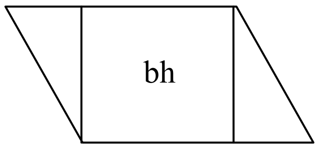
Look at shaded part – it's a rectangle which the area is B*H
MDS notes that the procedure is sound for this type of parallelogram, but the

Example 3:



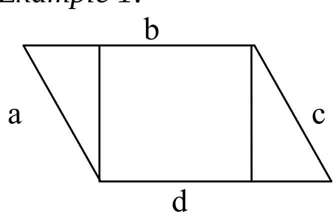
Drop altitu. AE parallel to BF
Rectangle ABFE is formed
area of ABFE is l*w or (EF x h)
triangle AED ≅ triangle BFD by hypo-leg thm
∴ DE ≅ CF by composites
∴ DC ≅ EF by addition of equal lengths
so DC * h ≅ EF * h so area of parallelogram is bh
(NG, post)

Example 2:



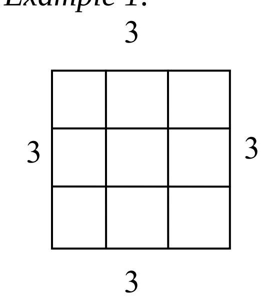
$\frac{1}{2}b_2h + \frac{1}{2}b_2h + b_1h$
$b_2h + b_1h = A$
$b_2 + b_1 = \text{Base}$
$A = (b_1 + b_2)h$
$A = Bh$
(DH, post)

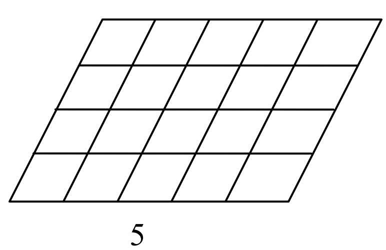
2 *Meets either of the following criteria:*
Response attempts to make a general argument, but suffers from at least one flaw in reasoning (different from limiting assumptions as in score points 5 and 3).
Response is not complete.
Example 1:



has to have 2 parallel sides and equal angles so side a and c will always be equal and side b and d will always be equal the two right triangles shaded can be cut off and turned so the rectangles face out and the shape will make a rectangle and a rectangles are is b^2xh^2=area.
(BD, pre)

1 *Meets the all of the following criteria:*
Response uses a single or multiple empirical examples as the sole means to justify the claim.
Example 1:





A = 3 + 3 + 3 = 9 units^2
or
3 * 3 = 9 units^2
You can count 9 square units

A = 5 + 5 + 5 + 5 = 20 units^2
or
5 * 4 = 20 units^2
or you can count the squares

(BD, post)

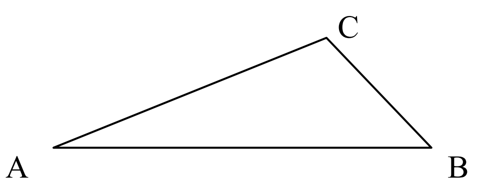
0 *Meets any of the following criteria:*
Response is not a mathematical argument nor an empirical example.
Response shows no evidence of logic or sequence.
No response is given.

Rubric RPR3.2
Data source: Pre/Post Assessment, Part C Task 4b
Rubric Score Data Type: Ordinal, one code per response, plus one categorical code

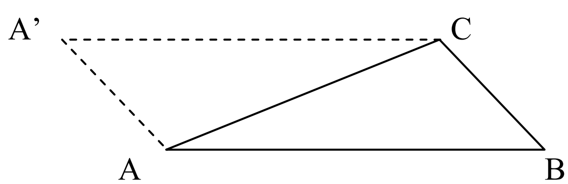
This rubric is designed to rate teachers' proofs of area of a parallelogram using an ordinal rubric. The rubric captures aspects of proof that include correctness, level of detail, representations, and generality.

Code responses to Task 1 as **6**, **5**, **4**, **3**, **2**, **1**, or **0** as described below, and with a **C** as appropriate.

6 *Meets all of the following criteria:*
Response is a fully correct mathematical argument that is general and holds for all triangles.
Response provides descriptive detail for each step taken and where appropriate, reasons for taking each step.
Response uses multiple representations in the argument.
Example 1:



Given triangle ABC, construct BA' such that it is parallel and congruent to AC. Draw AA'.



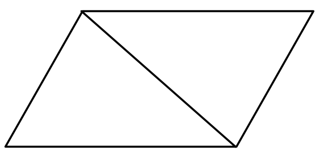
We know AB = AB (reflexive property). We know angle ABA' ≅ angle BAC (alt. int. angles). We know A'B ≅ AC by our construction. So area of triangle ABC is equal to area of triangle ABA'. Therefore Area of triangle ABC = ½ Area of parallelogram A'BAC. Since area of parallelogram = b*h, area of triangle ABC = ½ bh.
(KE, post)

5 *Meets the all of the following criteria:*
Response is not sufficiently general. That is, an assumption in the argument limits the proof's generality, such as beginning with a rectangle.
Response is otherwise correct.

Response provides descriptive detail for each step taken and where appropriate, reasons for taking each step.
Response uses multiple representations in the argument.
No examples found for this code at present time

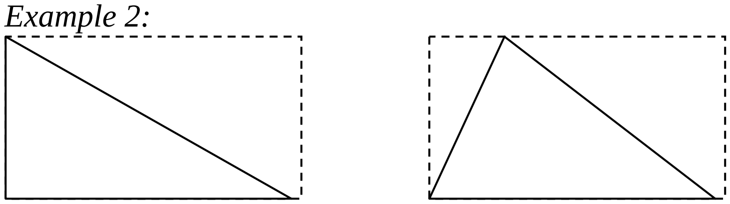
4 *Meets the first criterion and either of the next two:*
Response is a fully correct mathematical argument that is general and holds for all triangles.
Response provides some descriptive detail, but it is unclear how one or more steps are justified or how they relate to the argument as a whole.
Note: This includes using a theorem name as justification when the nature of the theorem is not obvious for a casual user of mathematics.
Response uses a single representation in the argument.

Example 1:



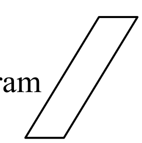
triangle is half (1/2) of a parallelogram
$A_{parallel} = bh$
$A_{triangle} = \frac{1}{2} A_{parallel}$
$= bh$ (MN, pre)

Example 2:



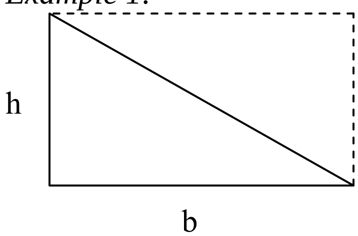
Any triangle can be drawn as half a rectangle. (2 triangles that are the same can be made into a rectangle.) Area of rectangle is length * width. The height of a triangle is the width of a rectangle & base is length. Since it's half a rectangle ½bh. (IT, pre)

3 *Meets the first criterion and either of the next two:*
Response is not sufficiently general. That is, an assumption in the argument limits the proof's generality, such as beginning with a right triangle or neglecting the case in which a parallelogram cannot be build from two triangles (see right).



Response provides some descriptive detail, but it is unclear how one or more steps are justified or how they relate to the argument as a whole.
Note: This includes using a theorem name as justification when the nature of the theorem is not obvious for a casual user of mathematics.
Response uses a single representation in the argument.

Example 1:



A of rectangle is bh
Rectangle is equal to two triangles ~ Area of one triangle is ½bh
(DH, post)

2 *Meets either of the following criteria:*
Response attempts to make a general argument, but suffers from at least one flaw in reasoning (different from limiting assumptions as in score points 5 and 3).
Response is not complete.

1 *Meets the all of the following criteria:*
Response uses a single or multiple empirical examples as the sole means to justify the claim.
Example 1:
(4x4 square ABCD drawn on coordinate plane with diagonal AC)
Triangle ADC = 8 blocks = 8 units^2
ABCD = 16 units^2
Triangle ADC is ½ ABCD so area triangle = ½bh
(EH, post)

0 *Meets any of the following criteria:*
Response is not a mathematical argument nor an empirical example.
Response shows no evidence of logic or sequence.
No response is given.

C Cyclical based on 4a
Add the **C** code to responses if the assumptions in the proof of 4b depend on the result of 4a, ***and*** the assumptions in the proof of 4a depend on the result of 4b.

Goal RPR4:
Understand the roles of proof in mathematics:
Verify a stmt is true, explain why a stmt is true, communicate math knowl., create new math, systematize the domain

Data sources:
Pre-Course Interview, Task 2a
Post-Course Interview, Task 3a
Selected Course Discussions (All proof discussions)
Analyzing Teaching (possibly)

Rubric RPR4.1
Data source: Pre-Course Interview, Task 2a, and Post-Course Interview, Task 3a
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to assess teachers' attention to the five aspects of the role of proof in the domain of mathematics, as identified by Knuth (2002a).

Code lines of transcript from Tasks 2a (Pre) and 3a (Post) with evidence for each of the following: **Verify truth**, **Explain why**, **Communicate knowledge**, **Create new mathematics**, or **Systematize the domain**. Any statement in the responses of the five questions in 2a or 3a is eligible; however, the most likely location for finding evidence will be in the first, second, and third questions (What does proof mean to you, What does it mean to prove something, and What purpose does proof serve in mathematics).

Code as many instances of each role of proof as are merited in the response. While coding, note any additional ideas mentioned or nuances of existing codes with respect to the role of proof for potential use as emergent codes. For each code, start coding at the first line necessary to maintain a context for the comment, and end coding at the last line necessary to maintain context, such that the entire coded segment can stand alone as an intelligible statement.

Verify truth
Teacher indicates that proof serves to verify the truth of mathematical statements. This could include stating that proof serves to verify, check the truth of, or confirm a mathematical statement, theorem, conjecture, fact, or idea. This may also include stating that proof establishes a mathematical statement as unequivocally true (e.g., immune to counterexample), or that proving a statement allows for it to be used "legally" in subsequent mathematical work. This code differs from the **Explain why** code in that the response implies that the proof is simply ***showing that*** something is true, rather than ***showing why*** a mathematical idea might be true. Note that the questions regarding how to disprove and whether proofs ever become invalid might provide additional evidence.

Example 1:
Proof to me, um, pretty much, like I said I think on our first interview is, it's using something you know, to show something else, to be true um, [pause] I think if you show it to be untrue it's not really- it's proving something but it's not proving it to be true I think a proof is proving something to be true not, disproving something or proving it not to be true so that's the first thing I would say. (EL, post, ll. 762-766)

Example 2:
I want to say it's to- to validate your ideas and your findings, um, [pause] that- that's the big thing you can't just, [long pause] I- I guess you just can't say something is just because you've found, [pause] y'know one way that works. You have to, [pause] consider all of the options or all of the counterexamples um, [pause] or else you're gonna have a faulty, [pause] proof or idea um, that you may try to use later down the line but find out that, it's not gonna hold um. [pause] So I think having that proof again is just to validate your ideas, to make sure that they're set and that they're grounded um [pause] so that later on that you can use those ideas um, when you're

trying to prove something else or y'know, when you're trying to- to prove something new. (CD, post, ll. 805-813)

Explain why

Teacher indicates that one purpose of proof is to explain why a mathematical statement is true. This could include stating that proof serves to explain why an argument, conjecture, statement, or mathematical idea is true; to explain one's thinking or reasoning; to justify, back up, or provide reasons for a conjecture, or simply to explain why. This code differs from the **Verify truth** code in that the response implies that the statement is new, either to mathematics in general or to the student or group of students who are putting forth the statement. (That is to say, the statement represents knowledge not previously known by the author.)

Example 1:

So it's an argument that um, [pause] doesn't just explain what's happening, but, it uses, true math facts, to make sense, t- true math facts to um, [pause] to justify, why it happens. (KT, post, ll. 668-670)

Example 2:

Um, to cite examples, to use previous knowledge um, [pause] for a new idea, not necessarily new to, the world, new to you. So, [pause] like if I ask my- my students to prove something, it wouldn't- I would already know, y'know and that- not necessarily some theorem or anything like that but just, [pause] here's this problem, prove that you know the answer kind of thing so, [pause] backing it up using um, previous knowledge and understanding. (MH, post, ll. 345-350)

Communicate knowledge

Teacher indicates that proof serves to communicate mathematical knowledge. This could include stating that proof communicates mathematical knowledge (idea, concept, theorem, statement, conjecture, etc.) to others, that proof helps people/students <u>understand</u> a mathematical idea, concept, theorem, statement, or conjecture, the proof helps to disseminate knowledge that mathematicians or other doers of mathematics create. There does not necessarily have to be a sense of agency or authority in the response; the response could simply state that it helps naïve learners to understand knowledge that others create, with no indication of the level to which the learners will learn and understand that knowledge.

Example 1:

I think to prove something you need to- I think you need to be able to communicate it to your audience um, whether that audience is a group of mathematicians it's different than if it's a group of, your classmates I think that, it's ok that proof is, someone in our class- (CO, post, ll. 526-529)

Create new mathematics

Teacher indicates that proof serves to create new mathematical knowledge. This could include stating that proof develops new mathematical ideas, concepts, facts, or truths; serves to confirm conjectures or nascent ideas; or allows mathematicians or other doers of mathematics to build mathematical knowledge. There does not need to be any sense of a communication of these ideas, simply that proof is the tool that serves to inaugurate a new mathematical idea as part of the knowledge base. In this way, it differs from the **Verify**, **Explain**, and **Communicate** codes.

Example 1:
But so what purpose does it serve, is so that you know, we can condense, some- some mathematical knowledge. I mean otherwise, we just have all these little snippets of ideas, that, maybe fit together but a proof, sort of brings lots of ideas into one more powerful and bigger idea that then can be built on. [pause] That's the purpose that I see it serving. (MN, post, ll. 431-435)

Example 2:
Um, well actually I think in math it serves a big purpose just because, everything in math, kind of builds on itself um, and everything's related. At some point in time you're gonna use geometry and algebra and this and this all together to come up with some of these ideas. And I think proof is a good way to, [pause] I want to say maybe bridge that gap make a couple of those connections 'cuz when you prove something, y'know you're not just using, [pause] one geometry skill. You're using this one and this one and you're kind of pulling them- all these thoughts together in a largical- logical argument to show that that next step is true. Then once you have it, you can jump to the next step. So proof kind acts as those y'know little like, maybe, if you want to say steps leading up to those- the whole set of ideas with um geometry. So it's just one little part of it. (EL, post, ll. 836-845)

Systematize the domain
Teacher indicates that proof serves to impose a logical structure (e.g., differentiating, utilizing, and classifying axioms, theorems, conjectures, etc.) on the domain of mathematics. That is, proof serves to organize results and to catalog them with respect to the underlying axioms and ideas upon which the proofs are built. This is a very specific code, and based on the work of Knuth, is unlikely to be used more than once or twice. (Knuth ended up lumping this code in with Create new mathematics.)

Example 1:
I think, sssssssss- [pause] you know that we want to- we want mathematics to be, a sound system and um, we want it to be able to- to um, serve our intellectual needs. (MN, post, ll. 424-426)

Content Knowledge for Teaching:
Middle Grades Geometry and Measurement Big Ideas

<u>Goal Big1:</u>
Identify the big ideas in middle grades geometry and measurement

Data sources:
Pre/Post Assessment, Part B Tasks 1-3
Learning Log 2, possibly Learning Log 4
Selected Course Discussions (Big Ideas, Classes 1 and 12)

<u>Rubric Big1.1</u>
Data source: Pre/Post Assessment Part B Task 1
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify the big ideas that teachers identified in the sub-domain of 2-D shapes, area, and perimeter.

Code responses to Task 1 as:

Relationship btwn A&P – General	**Relationship btwn A&P – Specific**
Calculate/find A&P	**Use/apply A&P**
Understand A&P conceptually	**Diff. btwn linear & square units**
Names, characteristics of 2-D shapes	**Manipulate/decompose shapes**
Memorize/use formulas	**Generate, develop, or explain formulas**
Perimeter as distance around	**Area as covering**
Visualization/spatial sense/sketching	**Unit conversion**
Find missing sides w/A&P	**Difference btwn A&P**
Other (specify)	

Relationship between area and perimeter – general:
The relationship between area and perimeter is identified as a big idea, but there is no additional explanation about the nature of this relationship or examples of this relationship.

Relationship between area and perimeter – specific:
The relationship between area and perimeter is identified as a big idea, and a specific (and correct) aspect of this relationship is specified. This may include the notion that there is a non-constant relationship between area and perimeter, that if area is held constant perimeter can change, that if perimeter is held constant area can change, that area and perimeter depend on the dimensions of the shape, or other aspects of the relationship.

Calculate or find area and perimeter:
Response identifies an idea related to finding or calculating area and/or perimeter. This includes finding the area or perimeter using a formula, counting, estimating, or measuring. This response does not necessarily imply an understanding of the meaning of area and perimeter; if this is the case, use the **Understand area and perimeter conceptually** code either in replacement or in addition, depending on the response.

Use or apply area and perimeter:
Response identifies an idea related to using or applying area and perimeter in problems, real-world situations, or high-level tasks. This may include applying the formulas or calculating area or perimeter in the service of a context. This response does not necessarily imply an understanding of the meaning of area and perimeter.

Understand area and perimeter conceptually:
Response identifies an idea related to understanding the meaning of area and perimeter from a conceptual standpoint. This may include responses such as "knowing what area and perimeter mean," "understanding area and perimeter," "concepts of area and perimeter." Specific metaphors, such as perimeter as distance around/surrounding or area as covering, should be coded under more specific codes. Understanding related to formulas should also be coded under the code for **Generate/develop formulas**.

Difference between linear and square units:
Response identifies an idea related to understanding the difference between linear and square units. This also includes understanding that perimeter is measured in linear units and that area is measured in square units, or how those units relate to the calculation of perimeter and area. The additive vs. multiplicative nature of perimeter and area does not fall under this code, and should be coded **Other**.

Names, characteristics, or properties of 2-D shapes:
Response identifies an idea related to knowing the names, characteristics, or properties of 2-D shapes. This includes knowing terms or names for shapes, being able to classify shapes, knowing the core properties of classes of shapes, or otherwise being able to distinguish one 2-D shape from another.

Manipulate or decompose shapes:
Response identifies an idea related to being able to change, manipulate, decompose, or recompose 2-D shapes. This may include finding area through partitioning, understanding how to transform one shape into the next (e.g., lopping off one end of a parallelogram and moving it over to make a rectangle), or other such transformations. These transformations are not necessarily limited to those that preserve area.

Memorize or use formulas:
Response identifies an idea related to the use and/or memorization of formulas for area or perimeter. This may include understanding what elements of the formula "stand for" – for example, knowing that the *h* in the area of a triangle formula stands for height, and where to find height on the triangle. Note that this code does not necessarily imply conceptual understanding of the basis for the formula.

Generate, develop, or explain formulas:
Response identifies an idea related to creating or explaining formulas (formal symbolic or informal rules or methods) for area and perimeter based on understandings about what area and perimeter are conceptually. This includes developing generalizations based on patterns or based

on general characteristics of shapes. This code does imply conceptual understanding of the basis for the formula.

Perimeter as distance around:
Response identifies an idea related to the conceptual understanding of perimeter specifically as the distance around a shape or as surrounding a shape. This language or similarly specific language must appear in the response to qualify for coding.

Area as covering:
Response identifies an idea related to the conceptual understanding of area specifically as covering or the space "inside" or "contained by" a shape. This language or similarly specific language must appear in the response to qualify for coding.

Visualization, spatial sense, sketching of 2-D shapes:
Response identifies an idea related to the development of visualization skills or spatial sense with students, or cites sketching of shapes as a way to develop visualization or spatial sense. This may also include creating models for the purpose of developing spatial sense or visualization skills.

Unit conversion:
Response identifies the idea of converting one set of measurement units to another. This is distinct from understanding the relationship between linear and square units, and is limited to the conversion between units of the same dimension (e.g., cm to in., ft^2 to m^2).

Find missing sides of shapes using area and/or perimeter:
Response identifies the idea given a shape with one (or more) dimensions provided, one dimension missing, and the area or perimeter, finding the missing dimension. For example, find length given the width and area of a rectangle.

Difference between area and perimeter:
Response identifies the idea that students should know the difference between area and perimeter. This code is reserved for statements that do not expand on what that difference is, or identifies a "relationship" between the two. In these cases, the response should be coded **Relationship between area and perimeter – general** or one of the more specific alternatives listed above.

Other: Any other response that does not fit the criteria for any of the codes listed above. On the coding sheet (see page 4 of this document), please specify the nature of the response. The **Other** code can be used multiple times for a single response.

<u>Rubric Big1.2</u>
Data source: Pre/Post Assessment Part B Task 2
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify the big ideas that teachers identified in the sub-domain of 3-D shapes, surface area, and volume.

Code responses to Task 2 as:

Relationship btwn SA&V – General
Calculate/find SA&V
Understand SA&V conceptually
Names, characteristics of 3-D shapes
Memorize/use formulas
Surface area as wrapping/covering
Visualization/spatial sense/sketching
Find missing dimensions w/SA&V
Represent/decompose SA using nets
Use/understand diff. arrangements of SA, V
Other (specify)
Relationship btwn SA&V – Specific
Use/apply SA&V
Connection/relationship btwn 2-D & 3-D
Understand square &/or cubic units
Generate, develop, or explain formulas
Volume as filling
Volume as layering
Difference btwn SA&V
Relationship btwn volume formulas
Use manipulatives/build 3-D objects

Relationship between surface area and volume – general:
The relationship between surface area and volume is identified as a big idea, but there is no additional explanation about the nature of this relationship or examples of this relationship.

Relationship between surface area and volume – specific:
The relationship between surface area and volume is identified as a big idea, and a specific (and correct) aspect of this relationship is specified. This may include the notion that there is a non-constant relationship between surface area and volume, that if surface area is held constant volume can change, that if volume is held constant surface area can change, that shapes with the same volume can have differing surface areas, that surface area and volume depend on the dimensions of the object, or other aspects of the relationship.

Calculate or find surface area and volume:
Response identifies an idea related to finding or calculating surface area and/or volume. This includes finding the surface area or volume using a formula, counting, estimating, or measuring. This response does not necessarily imply an understanding of the meaning of surface area and volume; if this is the case, use the **Understand area and perimeter conceptually** code either in replacement or in addition, depending on the response.

Use or apply surface area and volume:
Response identifies an idea related to using or applying surface area and volume in problems, real-world situations, or high-level tasks. This may include applying the formulas or calculating surface area or volume in the service of a context. This response does not necessarily imply an understanding of the meaning of surface area and volume.

Understand surface area and volume conceptually:
Response identifies an idea related to understanding the meaning of surface area and volume from a conceptual standpoint. This may include responses such as “knowing what surface area and volume mean,” “understanding surface area and volume,” “concepts of surface area and volume.” Specific metaphors, such as surface area as warpping or covering, or volume as filling or layering, should be coded under more specific codes. Understanding related to formulas should also be coded under the code for **Generate/develop formulas**.

Understand square and/or cubic units:
Response identifies an idea related to understanding of, and/or the difference between, square and/or cubic units. This also includes understanding that surface area is measured in square units and that volume is measured in cubic units, or how those units relate to the calculation of surface area and volume. If only one unit is mentioned (e.g., only cubic units), it is still appropriate to use this code. (Many teachers mention square units in Task 1 and do not restate it in their response to Task 2.)

Connection or relationship between 2-D and 3-D objects:
Response identifies an idea related the connection between 2-D and 3-D objects. This includes a statement that there is a connection, or more detailed elaboration regarding that connection. This also includes identifying surface area as two-dimensional area. If appropriate, the **Represent or decompose surface are using nets** code may be used in conjunction with this code.

Names, characteristics, or properties of 3-D shapes:
Response identifies an idea related to knowing the names, characteristics, or properties of 3-D shapes. This includes knowing terms or names for shapes, being able to classify shapes, knowing the core properties of classes of shapes, or otherwise being able to distinguish one 3-D shape from another.

Memorize or use formulas:
Response identifies an idea related to the use and/or memorization of formulas for surface area or volume. This may include understanding what elements of the formula "stand for" – for example, knowing that the *h* in the volume formula stands for height of the object, and where to find height on the pnkect. Note that this code does not necessarily imply conceptual understanding of the basis for the formula.

Generate, develop, or explain formulas:
Response identifies an idea related to creating or explaining formulas (formal symbolic or informal rules or methods) for surface area or volume based on understandings about what area and perimeter are conceptually. This includes developing generalizations based on patterns or based on general characteristics of shapes. This code ***does*** imply conceptual understanding of the basis for the formula.

Surface area as wrapping or covering:
Response identifies an idea related to the conceptual understanding of surface area specifically as the wrapping or covering of a 3-D object. This language or similarly specific language must appear in the response to qualify for coding. If the response is not specific, use the **Connection or relationship between 2-D and 3-D objects** code.

Volume as filling:
Response identifies an idea related to the conceptual understanding of volume specifically as the "filling" of a 3-D object or the amount of stuff or material inside a 3-D object. This language or similarly specific language must appear in the response to qualify for coding.

Visualization, spatial sense, sketching of 3-D shapes:
Response identifies an idea related to the development of visualization skills or spatial sense with students, or cites sketching of shapes as a way to develop visualization or spatial sense. This may also include creating models for the purpose of developing spatial sense or visualization skills. If specific aspects of spatial sense are identified, such as volume as layering or the notion of decomposing figures into nets, use the appropriate code in addition to this code.

Volume as layering:
Response identifies an idea related to the conceptual understanding of volume specifically as layering or stacking, where the area of the base of a prism is visualized as being stacked, layered, or extruded through the height of the prism. This language or similarly specific language must appear in the response to qualify for coding. While this particular spatial structuring applies only for prisms and cylinders, that restriction need not appear in the response to qualify for coding.

Find missing sides of shapes using surface area and/or volume:
Response identifies the idea given a shape with some dimensions provided, one dimension missing, and the surface area or volume, finding the missing dimension. For example, find height given the length, width, and volume of a rectangular prism.

Difference between surface area and volume:
Response identifies the idea that students should know the difference between surface area and volume. This code is reserved for statements that do not expand on what that difference is, or identifies a "relationship" between the two. In these cases, the response should be coded **Relationship between surface area and volume – general** or one of the more specific alternatives listed above. If the response specifically deals with units, the response should be coded **Understand square and/or cubic units**.

Represent or decompose surface area using nets:
Response identifies the notion of representing surface area using nets, decomposing 3-D objects into nets, or otherwise creating or thinking about surface area of 3-D objects using nets that build or cover the 3-D object. This code may often be used in conjunction with **Connection or relationship between 2-D and 3-D objects**.

Relationship between volume formulas:
Response identifies the idea of understanding the difference and/or relationship between the two most common formulas for volume of a rectangular prism: $V = lwh$ and $V = Bh$. This code may often be used in conjunction with **Volume as layering**.

Use and/or understand different arrangements of surface area and volume:
Response identifies the idea that students should use, create, and/or understand different arrangements of surface area and volume. This may relate to the idea that one can configure a 3-D object that contains smaller 3-D objects (e.g., a large box with smaller boxes inside, or a 12-pack of soda cans) in different ways that impact volume and surface area. This code may be used in conjunction with **Relationship between surface area and volume – specific** if enough detail is provided to qualify for that code.

Use manipulatives and/or build 3-D objects:
Response identifies that students should use manipulatives to understand 3-D objects and/or build 3-D objects in some way. Response must make specific mention of constructing, building, or the use of manipulatives. Specific details on how manipulatives should be used are not necessary for this code.

Other: Any other response that does not fit the criteria for any of the codes listed above. On the coding sheet (see page 4 of this document), please specify the nature of the response. The **Other** code can be used multiple times for a single response.

<u>Rubric Big1.3</u>
Data source: Pre/Post Assessment Part B Task 3
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify the big ideas that teachers identified in the sub-domain of reasoning and proof.

Code responses to Task 3 as:

Justify/defend thinking, provide evidence	**Explaining why**
Generalizing	**Informal/beginning/simple proofs**
Understand the proof process	**Communicate/discuss thinking with others**
Make mathematical arguments	**Understand requirements/limitations**
Find patterns from cases, observations	**Make conjectures**
Generate formulas	**Prove why formulas work**
Sequence steps in an argument or proof	**Logical thinking**
Understand counterexamples	**Mathematical reasoning**
Construct new knowledge	**Should not be emphasized**
Other – content-specific	**Other – general**

Justify and/or defend thinking, using or providing evidence for claims:
Justifying or defending thinking or using or providing evidence for claims is cited as a big idea. The words justify, defend thinking, or evidence (or appropriate grammatical derivates) should appear somewhere in the response to qualify. This can also include justifying steps taken in a proof or mathematical argument. The difference between justifying, defending, and providing evidence and other similar codes, such as explaining why, is that the response implies that students are to make a conjecture and defend it, as opposed to simply explaining how they arrived at an answer to a problem (that may or may not be high-level). "Conjecture" need not appear in the response to qualify for this code.

Explaining why:
The idea of explaining why is cited as a big idea. The phrase "explain(ing) why" is sufficient to qualify for this code without additional detail. This code can be used in conjunction with the **justify** code above if there is evidence for both explaining and justifying.

Generalizing:
Response indicates that students should generalize, make generalizations, or make general arguments.

Informal, beginning, or simple proofs:
Response indicates that students should engage in informal proofs, beginning proofs, simple proofs, or create parts of proofs. The response must use the term proof; if the response uses language related to mathematical arguments, it should be coded using the **Make mathematical arguments** code. The meaning of phrases such as "informal proof" need not be specified.

Understand the proof process:
Response indicates that students should understand the process for creating a proof, which may include making conjectures, exploring, or creating a logical set of steps that form an argument. If the response discusses the specific sequencing of steps in a proof, it should be coded using the **Sequence steps** code below rather than this code.

Communicate or discuss thinking with others:
Response indicates that students should discuss, communicate, or justify their thinking to others. The response must specifically address communication with other students or peers, not just to a single mathematical authority such as the teacher.

Make mathematical arguments:
Response indicates that students should construct mathematical arguments. This may include proofs in the response, but the statement must be more general than simply proofs. The response must also go beyond making conjectures; if it is limited to making conjectures, it should be coded solely using that code.

Generate formulas:
Response indicates that students should generate formulas as a part of reasoning and proof. Response must indicate that students should generate the formulas rather than simply justifying why the formulas work.

Prove why formulas work:
Response indicates that students should take known formulas and justify why those formulas work. Response must imply that the formulas are known; if the response implies that students should generate and justify the formulas, it should be coded using the **Generate formulas** code.

Sequence steps in an argument or proof:
Response indicates that students should understand how steps in a mathematical argument or proof fit together, or the proper order for steps in an argument or proof. This could include understanding the sequencing through examining a proof done by someone else, or understanding the sequencing of steps through the construction of one's own proofs. This code must address steps specifically; the more general notion of logical or structured thinking should use the **Logical thinking** code described below.

Logical thinking:
Response indicates that students should learn logical or structured thinking. The response need not provide additional detail beyond such a statement to qualify for the code.

Understand counterexample:
Response indicates that students should understand the notion of a counterexample, and what purpose a counterexample serves in proof. The term "counterexample" must be included.

Mathematical reasoning:
Response indicates that students should use mathematical reasoning. The response need not provide any additional detail beyond the words "mathematical reasoning" to qualify. If specific aspects of mathematical reasoning are identified, one of the more specific codes should be used.

Construct new knowledge:
Response indicates that students should construct new knowledge through work on reasoning and proof. The response may state that this knowledge is based on previously learned ideas, but this notion is not a requirement for this code. Response may indicate that this knowledge should develop through the use of mathematical argumentation, conjecturing, proof, justifying thinking, or other such descriptions. In these cases, add the appropriate code in addition to this code.

Should not be emphasized:
Response indicates that reasoning and/or proof should not be emphasized in the middle grades.

Other – content-specific:
Response indicates another skill or concept that should be addressed in conjunction with reasoning and proof that is specific to a mathematical content topic, such as properties of parallel lines or line and angle relationships.

Other – general:
Response indicates another skill or concept that should be addressed in conjunction with reasoning and proof that is not specific to a mathematical content topic or is process-oriented. Examples include if-then relationships, problem-solving, visualization, and algebra skills.

Modifying Tasks

Goal MOD:
Teachers will be able to modify tasks in order to enhance their cognitive demand

Data sources:
Pre-Course Interview, Task 3
Post-Course Interview, Task 4

Rubric MOD1.1
Data sources: Pre-Course Interview, Task 3; Post-Course Interview, Task 4
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify modifications that teachers made to the Minimizing Perimeter Task that serve to enhance or reduce the cognitive demands of the task.

Moves that enhance the cognitive demand include **Removing Explicit Pathways** and **Targeting the Big Mathematical Idea**. Moves that reduce the cognitive demand include **Proceduralizing the Task**. Code lines of interview as **Removing Explicit Pathways**, **Targeting the Big Mathematical Idea**, or **Proceduralizing the Task** as detailed below. Only code a line of text if the teacher indicates that they would <u>modify</u> or <u>set up</u> the task in the specified manner; if they mention the issue while discussing the explore phase of the task, do not code.

For each code, start coding at the first line necessary to maintain a context for the comment, and end coding at the last line necessary to maintain context, such that the entire coded segment can stand alone as an intelligible statement.

Removing Explicit Pathways: Response indicates that the teacher would remove elements of the task that suggest a pathway for approaching the task. In the case of the Minimizing Perimeter task, this entails the removal of the table and/or graph, leaving the prompt to investigate the minimum perimeter for an area of 36 square feet.

Example 1:
And have the students, with- working with the table um, [pause] being able to figure out- actually I wouldn't even give them the table. I would have them trying to figure out um, the amount of fencing, that they're, allowed with the- the- (xx xxx) which you need to find with just, knowing that it has to be 36 square feet. (BI, pre, ll. 430-433)

Example 2:
I would definitely get rid of, [pause] in the question where it says organize information in a table and create the, graph- for perimeter versus the length. (EL, pre, ll. 676-679)

Targeting the Big Mathematical Idea: Response indicates that the teacher wants students to develop an understanding of the non-constant relationship between area and perimeter for a rectangle, and/or the impact that changing the dimensions has on area and perimeter. Code any statement that expresses one or both of these ideas as the important residue for students. *Note: Just stating "the relationship between area and perimeter" does not qualify, as that was given as the lesson goal.*

Example 1:
Um, [pause] well I would hope that they would, learn that there are relationships between some of the variables, I think that's a key thing it's one of the things I pointed out that um, mathematical wise it's an important thing with geo- whole idea of geometry I think I would- they would look at the relationship between area and perimeter. But also um, what's the relationship between the length and the width. How are those changes from one another in a cube. Er, not a cube in a um, rectangle. Or y'know what's the relationship between, the length and the perimeter, um. Things of that nature and I think they can come up with some of the different ideas there. (EL, post, ll. 1194-1201)

Example 2:
So I want them to understand the relationship between area and perimeter. I would first want them, to understand that, rectangles could have the same area but different perimeters. (DH, post, ll. 495-497)

Proceduralizing the Task: Response indicates that the teacher would modify the task in a way that would proceduralize the task by suggesting a path or routinizing the challenging aspects of the task. For the Minimizing Perimeter task, this includes giving the area or perimeter formulas to students, giving them particular numbers to start with, suggesting a method to begin the task as a class, changing the graph to length vs. area, or eliminating the challenge by either deleting the generalization in question 4 or by changing all the areas to perfect squares.

Example 1:
So the first thing is, I like the chart that they have set up. I think it organizes the information pretty well. I also liked how- at first I thought it was stupid, that they kept putting that area. [pause] But I think it's important to find the relationship that they see that perimeter changes and the area doesn't. ***And I thought that might be confusing to kids, so I might ask them to draw, uh 9 by 4 6 by 6, [pause] and, show them, and have- even have dirt if, y'know we're doing it on a small scale and have them see that the same amount of dirt, will cover the areas***. (NiT, pre, ll. 419-425; emphasis added)

Example 2:
The next thing that I would do, is, [pause] establish some sort of- of um, guidelines. The fact that it's 36 square feet what's the largest the length could be? And what's the uh, shortest that the width would be that would still produce, 36 uh, square feet. (NoT, pre, ll. 631-634) *Note: As best can be determined from the context, this still occurs during the task setup.*

Measurable Attributes of Geometric Figures

<u>**Goal MAF1:**</u>
Identify misconceptions about area and perimeter

Data sources:
Pre/Post Assessment, Part C Task 1, Part D Task 5
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
Selected Course Discussions (Case of Barbara Crafton, Case of Isabelle Olson, Art Class work)

<u>Rubric MAF 1.1</u>
Data sources: Pre/Post Assessment, Part C Task 1; Part D Task 5, Pre-Course Interview Task 3, Post-Course Interview Task 4
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to identify evidence of misconceptions related to area and perimeter; specifically, that a rectangle with a fixed perimeter can only have a fixed area, that as area increases, so does perimeter, and that a rectangle with a fixed area can only have one perimeter.

For each of the tasks, code as **Misconception**, **No evidence**, or **No response**.

Part C, Task 1: fixed perimeter, changing area
Use the coding from Rubric RAP1.1. If the response is coded as **Incorrect-1**, code here as **Misconception**. If the response is coded as **Correct-1**, **Correct-2**, **Incorrect-2**, or **Vague/Inconclusive**, code as **No evidence**. If the response is coded as **No Response**, code as **No response**.

Part D, Task 5: as area increases, so does perimeter
Use the coding from Rubric RAP1.3. If the response is coded as **Misconception**, code here as **Misconception**. If the response is coded as **No Misconception** or **Unclear**, code as **No evidence**. If the response is coded as **No response**, code as **No response**.

Pre-Course Interview Task 3/Post-Course Interview Task 4: fixed area, changing perimeter
Examine written work and transcripts from Pre-Course Interview Task 3 and Post-Course Interview Task 4.
Misconception: Written work on task or talk about task indicates that the teacher believes that there is only one rectangle with a given perimeter.
No Evidence: Written work on task or talk about task shows no evidence that the teacher believes that there is only one rectangle with a given perimeter.
No response: No response to the prompt.

Goal MAF2:
Use a range of representations to explain the relationship between dimension, area, and perimeter

Data sources:
Pre/Post Assessment, Part C Tasks 1, 2a
Selected Course Discussions (Index Card, Comparing Triangles, Storm Shelters, Case of IO)

Rubric MAF 2.1
Data sources: Pre/Post Assessment, Part C Tasks 1, 2a
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to determine what representations are used in responding to the Fence in the Yard task and the Relating Area and Perimeter task.

To code these two tasks, aggregate codes from rubrics RAP4.1 and 4.2, as well as indicating whether each response used a single representation or multiple representations. Code Tasks 1 and 2a as **Single Representation**, **Multiple Representations**, or **No Response/Answer Only**. Code all responses except **No Response/Answer Only** as any applicable codes from the following list: **Table**, **Symbolic/Formula**, **Written Explanation**, **Diagram**, and **Graph**. Representations do not have to be correct or accurate to be coded.

Single Representation: Response contains only a single representation used in the service of creating a solution to the task. A written sentence or sentences that simply states the answer, but does not add any mathematical explanation, does not qualify as a second representation.

Multiple Representations: Response contains more than one representation used in the service of creating a solution to the task.

No Response/Answer Only: Response contains only an answer or there is no response.

Table: Response was coded **Table** using RAP4.1 or 4.2.

Symbolic/Formula: Response was coded **Symbolic/Formula, Symbolic/Formula-1, -2,** or **-3** using RAP4.1 or 4.2.

Written Explanation: Response was coded **Written Explanation** using RAP4.1 or 4.2.

Diagram: Response was coded **Diagram** using RAP4.1 or 4.2.

Graph: Response was coded **Graph** using RAP4.1 or 4.2.

No Response/Answer Only: Response contains only an answer or there is no response.

Goal MAF3:
Identify misconceptions about surface area and volume

Data sources:
Pre/Post Assessment, Part C Tasks 3a, 3b
Learning Log 4(?)
Selected Course Discussions (Arranging Cubes, Soda Can, Wet Box)

Rubric MAF 3.1
Data sources: Pre/Post Assessment, Part C Tasks 3a, 3b
Rubric Score Data Type: Categorical, one code per response

This rubric is designed to identify evidence of misconceptions related to area surface area and volume; specifically, that a rectangular prism with a fixed area of the base and height can have differing surface areas, but must have the same volume.

For each of the tasks, code as **Misconception**, **No evidence**, or **No response**.

Part C, Task 3a: fixed base, height, different surface areas
Use the coding from Rubric RSV1.1. If the response is coded as **Incorrect-1**, code here as **Misconception**. If the response is coded as **Correct-1**, **Correct-2**, **Incorrect-2**, or **Vague/Inconclusive**, code as **No evidence**. If the response is coded as **No Response**, code as **No response**.

Part C, Task 3b: fixed base, height, same volume
Use the coding from Rubric RSV1.2. If the response is coded as **Incorrect-1**, code here as **Misconception**. If the response is coded as **Correct-1**, **Correct-2**, **Incorrect-2**, or **Vague/Inconclusive**, code as **No evidence**. If the response is coded as **No Response**, code as **No response**.

<u>Goal MAF4:</u>
Use a range of representations to explain the relationship between edge length, surface area, and volume

Data sources:
Pre/Post Assessment, Part C Tasks 3a, 3b
Selected Course Discussions (Arranging Cubes, Wet Box, Soda Can, Large Numbers Lab)

<u>Rubric MAF 4.1</u>
Data sources: Pre/Post Assessment, Part C Tasks 3a, 3b
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to determine what representations are used in responding to the Relating Surface Area and Volume task.

To code these two tasks, aggregate codes from rubric RSV4.1, as well as indicating whether each response used a single representation or multiple representations. Code Tasks 3a and 3b as **Single Representation**, **Multiple Representations**, or **No Response/Answer Only**. Code all responses except **No Response/Answer Only** as any applicable codes from the following list: **Table**, **Symbolic/Formula**, **Written Explanation**, **Diagram**, and **Graph**. Representations do not have to be correct or accurate to be coded.

Single Representation: Response contains only a single representation used in the service of creating a solution to the task. A written sentence or sentences that simply states the answer, but does not add any mathematical explanation, does not qualify as a second representation.

Multiple Representations: Response contains more than one representation used in the service of creating a solution to the task.

No Response/Answer Only: Response contains only an answer or there is no response.

<u>Goal MAF5:</u>
Identifying strategies for spatial structuring and tasks and pedagogical approaches that support the development of students' spatial structuring (includes use of volume formulas)

Data sources:
Pre/Post Assessment, Part D Task 6
Selected Course Discussions (Arranging Cubes, Wet Box, Soda Can, Large Numbers Lab, Comparing Formulas)
Learning Log 3

Post-Course Interview, Task 1 (ideas identified)

<u>Rubric MAF 5.1</u>
Data sources: Pre/Post Assessment, Part D Task 6a
Rubric Score Data Type: Categorical, multiple codes possible per response

This rubric is designed to analyze teachers' formula preferences given two choices, and their reasons for selecting a preferred formula.

Code each response with the formula preference: **lwh**, **Bh**, **Both**, **Both & Show Relationship**, **No Preference**, **No Response**, **Other.**
(Note: This task asks if there is a difference between the two formulas; on first pass, responses to this part of the question were not meaningful enough to merit coding.)

Code the rationale for each response using one or more of the following as applicable: **More general**, **Easier to visualize (Bh)**, **Layering (Bh)**, **Conceptual understanding (Bh)**, **Relates to dimensions (lwh)**, **Easier to understand/breaks down concept (lwh)**, **Shows relationship between area and volume**, **Other** (specify).

Formula preference codes:
lwh: Response indicates that the teacher would use the *V=lwh* formula.

Bh: Response indicates that the teacher would use the *V=Bh* formula.

Both: Response indicates that the teacher would use both formulas. If the teacher indicates using the two formulas in a specific order, use the **Both** code or the **Both & Show Relationship** code.

Both & Show Relationship: Response indicates that the teacher would use both formulas, and contains explicit mention of showing, demonstrating, or discussing how the two formulas are related to one another. This may also include presenting one formula first and having students use the first to derive the second in some way.

No Preference: Teacher indicates that they have no preference between the two formulas.

No Response: Teacher does not explicitly or implicitly indicate which formula they would use.

Other: Response indicates that the teacher would either use a different formula, or have a conditional means (e.g., it depends on the group of students) of determining their formula choice.

Rationale codes:
More general: Formula chosen is selected because it is more general. This includes applying to directly or relating to other shapes (e.g., *V=Bh* also works for a cylinder) or relating to other formulae (e.g., using *V=Bh* helps make sense of the volume of a pyramid or cone, *V=1/3Bh*).

Example 1:
lwh is for just rectangular prisms, and *Bh* is for all prisms.
I would show how *lwh* is a specific case just as A of a cylinder is $\pi r^2 h$ (NB, post)

Easier to visualize (Bh): *V=Bh* formula is chosen because it helps students visualize either volume, the nature of a rectangular prism, or how volume is found. This code may be used in conjunction with **Layering (Bh)** or **Conceptual understanding (Bh)** as appropriate.

Example 1:
The second allows for more visualization. The students can see the area of the base and then the layers which make up the height. (DN, post)

Example 2:
I think *area of b x h* helps you visualize volume more clearly. (BN, post)

Layering (Bh): *V=Bh* formula is chosen because it helps students understand volume using a layering or stacking metaphor. This might include talking about the base of the prism as being "stacked" up through the height, visualizing a base layer of cubes that are then built up to a particular height, or understanding the rectangular prism as a series of small layers that stack on one another.

Example 1:
I prefer *area of base x height = Volume*. It makes more conceptual sense. You are laying a base, then layering rows on top. (KT, post)

Example 2:
The second allows for more visualization. The students can see the area of the base and then the layers which make up the height. (DN, post)

Conceptual understanding (Bh): *V=Bh* formula is chosen because it helps students to understand the concept or meaning of volume. The words "concept," "conceptual(ly)," "conceptual understanding," or "meaning" (or any other derivative word or phrase) should appear in the response. The response need not expand on what is meant by conceptual understanding.

Example 1:
I prefer *area of base x height = Volume*. It makes more conceptual sense. You are laying a base, then layering rows on top. (KT, post)

Example 2:
I prefer *Volume = Bh*. It transfers to many shapes & offers a conceptual description of volume. (NiT, post)

Relates to dimensions (lwh): *V=lwh* formula is chosen because it helps students understand that volume is three-dimensional. This includes stating that it shows the notion of three dimensions because three quantities are multiplied together, or that the idea of cubic units is emphasized because three quantities are involved. If the response discusses the idea that the formula extends the *A=lw* formula for area of a rectangle, code using the **Shows relationship between area and volume** code instead or in addition, as applicable.

Example 1:
The *l x w x h* may also allow students to see that volume is 3-D a little easier. (UL, post)

Example 2:
The first allows the students to see the units cubed, length x width x height. (DN, post)

Easier to understand or compute, or breaks down the concept (lwh): *V=lwh* formula is chosen; reason cited indicates that the formula is easier to understand, use, or compute, or breaks down the concept. Evidence for this code might include stating that the individual measurements (length, width, height) are easier to find on the rectangular prism, that the formula is more familiar to students, that students identify length, width, and height more easily, that the formula multiplies 3 numbers rather than having to find area of the base first then multiply again, or that the formula more clearly shows the component parts of volume. Additionally, use this code if the response indicates that the *V=Bh* formula is more confusing, more difficult to compute, requires an extra step, or is confusing because of the labels (base and height) used.

Example 1:
The area of the base is length x width & therefore the 2nd formula skips a step in the process. This formula assumes that students already understand the process for finding the area of a 2-D shape & that the base of a 3-D shape is just a 2-D shape. The problem is if the student doesn't have this prior knowledge the latter formula might become more confusing for students. Even if the students do have prior knowledge the latter formula doesn't explain the process as much. (MH, pre)

Example 2:
I would start off using the first formula because I think that it would be easier for the students to understand due to the fact that they could physically see the length, width, & height. (DE, pre)

Shows relationship between area and volume: Response indicates that their preferred formula is selected because it shows the relationship between area and volume. This code may be applied regardless of the preferred formula selected. Evidence for this code includes the notion that the *V=lwh* formula builds on the *A=lw* formula; that the *V=Bh* shows the 2-D to 3-D relationship by multiplying a 2-D quantity by a third dimension; or that the *V=Bh* formula shows that the 2-D base is being stacked or layered through a third dimension (code as **Layering** also).

Example 1:
However, the first formula is the one that I use with my students because I believe that it best lends itself to the idea of volume as the area of the base times the height… which treats area as *lxw*, and volume as *lxwxh* (surf area x *h*). (LC, pre)

Example 2:
I would present $V=A_b*h$ b/c it builds on something they already know (A rect.) and it's easier to see how the V formula works (the A of base h times). (DH, pre)

Other (specify): Response gives a reason for their preferred formula not included in the other codes. When using **Other**, briefly specify the nature of the reason.

<u>Rubric MAF 5.2</u>
Data sources: Pre/Post Assessment, Part D Task 6b
Rubric Score Data Type: Categorical, multiple codes possible per response

This rubric is designed to analyze teachers' formula preferences given two choices, and their reasons for selecting a preferred formula.

Code each response with the formula preference: **lw**, **bh**, **Both**, **Both & Show Relationship**, **No Preference**, **No Response**, **Other.**
(Note: This task asks if there is a difference between the two formulas; on first pass, responses to this part of the question were not meaningful enough to merit coding.)

Code the rationale for each response using one or more of the following as applicable: **More general**, **Stresses the meaning of height (bh)**, **2-D Layering (bh)**, **Confusion w/*V=Bh* (lw)**, **Easier to understand/easier to locate measurements/more common (lw)**, **Keeps terms consistent**, **Shows relationship between area and volume**, **Other** (specify).

Formula preference codes:
lw: Response indicates that the teacher would use the *A=lw* formula.

bh: Response indicates that the teacher would use the *A=bh* formula.

Both: Response indicates that the teacher would use both formulas. If the teacher indicates using the two formulas in a specific order, use the **Both** code or the **Both & Show Relationship** code.

Both & Show Relationship: Response indicates that the teacher would use both formulas, and contains explicit mention of showing, demonstrating, or discussing how the two formulas are related to one another. This may also include presenting one formula first and having students use the first to derive the second in some way.

No Preference: Teacher indicates that they have no preference between the two formulas.

No Response: Teacher does not explicitly or implicitly indicate which formula they would use.

Other: Response indicates that the teacher would either use a different formula, or have a conditional means (e.g., it depends on the group of students) of determining their formula choice.

Rationale codes:
More general: Formula chosen is selected because it is more general. This includes applying to directly or relating to other shapes (e.g., A=*bh* also works for a parallelogram) or relating to other formulae (e.g., using *A=bh* helps make sense of the area of a triangle, *V=1/2bh*).

Example 1:
I would use *A=bh* because it ties together with the formulas for triangles & trapezoids much better. (UL, post)

Example 2:
I would use base x height because, again that formula transfers to other polygons rather than just a rectangle. (EH, pre)

Stresses the meaning of height (bh): *A=bh* formula is chosen because it allows for the opportunity to explore the meaning of height. This might include noting the fact that height is the perpendicular distance between two bases, that height is not always vertical, or that height is not always a side of the figure in question.

Example 1:
Yes because the height is how high an object is from base to base and the width is the legth [sic] of the segment that connects the 2 bases. I would use *bh* because by using *lw* the students may use the length of the edge of a parallelogram instead of the height when finding area (NB, post)

Example 2:
…Later in the course, we'd focus on the term "base" meaning it must be perpendicular to the height. (KT, pre)

2-D Layering (bh):*A=bh* formula is chosen because it helps students understand area using a layering or stacking metaphor. This might include talking about the base of the rectangle as being “stacked” up through the height, visualizing a linear base that is then built up to a particular height, or understanding the rectangle as a series of infinitely thin lines that stack on one another. Might include a reference to the “Stacks of Paper” task in which this idea was explored.

Example 1:
A difference I see is that in *bh* you could have students see how grouping could apply. You

could have them find the # of units in the base and see how many layers tall, or the height of the rectangle. (CD, post)
MDS believes that this is the only instance of this code.

Confusion with *V=Bh* (lw): *A=lw* formula is chosen because it avoids confusion between the *A=bh* and the *V=Bh* formulae. This must include specific reference to the volume formula or a 3-D shape in general (if this is not present, the **Easier to understand** code may be applicable), and indicate that students may confuse the notion of the linear base in *A=bh* with the notion of the area of the base in *V=Bh*.

Example 1:
The second seems confusing after reading the prior question & I believe students might encounter this confusion as well. A two-dimensional shape doesn't have a base the way a 3-D shape does. (MH, pre)

Example 2:
When moving into 3-D figures "height" can be tricky because we now have height of a base (2-D) & height of a figure (3-D). (FY, pre)

Easier to understand, easier to locate measurements, or more common (lw): *A=lw* formula is chosen because it is in general easier for students. Evidence for this code might include stating that the formula is more common; that for formula "relates to the real world"; that length and width measurements are easier or more clear for students to locate on a rectangle than base and height; that the *A=bh* is confusing for students because base and height don't always indicate sides, or that height is not always vertical; or that the *A=bh* formula is confusing because either base or height is vague.

Example 1:
I think *A=l x w* is better to use, as students always seem to identify height with the vertical dimension. (NoT, post)

Example 2:
I think there is a visual difference. If we introduce it the second way students may get caught visualizing in 3-D before they are ready. The first has an easy and established sound to it. (BI, pre)

Keeps terms consistent: Response indicates that the preferred formula is selected because the terms used are consistent with other formulas or figures. This code may be applied regardless of the preferred formula selected. Evidence for this code includes the notion that the *A=bh* formula is preferred because the terms are consistent with other terms used in 2-D or 3-D figures; that *A=lw* is preferred because the terms length and width are consistent with other formulas, concepts, or measurements; or that *A=lw* is preferred because length and width are established terms.

Example 1:
…if you begin w/base x height you can stick w/it instead of renaming them and causing confusion. (EH, post)

Example 2:
I do feel introducing them to base helps them apply to other 2-D shapes as well as whenthey begin to learn about 3-D shapes. (MH, post)

Shows relationship between area and volume: Response indicates that their preferred formula is selected because it shows the relationship between area and volume. This code may be applied regardless of the preferred formula selected. Evidence for this code includes the notion that the $A=lw$ formula can build to the $V=lwh$ formula; that the $A=bh$ formula can build to an understanding of the $V=Bh$ formula; or that the $A=lw$ formula can connect to the $V=Bh$ formula in that the area of the base is found by multiplying length and width.

Example 1:
I would use $A=lxw$ and then move into using the other because lxw is common and helps see the relationship to surface and volume in order to understand the two dimensions. (BI, post)

Other (specify): Response gives a reason for their preferred formula not included in the other codes. When using **Other**, briefly specify the nature of the reason.

Reasoning and Proof

Goal RPR5:
Teachers will understand and articulate the various roles of proof in the K-12 mathematics classroom, including: verifying a statement is true, explaining why/showing thinking, communicating mathematical knowledge, systematizing the domain/promoting logical thinking, and encouraging generalization.

Data sources:
Pre-Course Interview, Task 2 all parts
Post-Course Interview, Task 3 all parts
Selected Course Discussions (Defining and Revisiting Proof)

Rubric RPR5.1
Data source: Pre-Course Interview, Task 2, and Post-Course Interview, Task 3
Rubric Score Data Type: Categorical, multiple codes per response

The purpose of proof in the classroom mirrors the purpose of proof in the domain. The categories listed in the goal mirror the categories Knuth (2002b) identifies as the roles of proof in the classroom, while adding a sixth: encouraging generalization. The first 5 of these 6 categories also mirror the purpose of proof in the domain as coded in Set 3, Rubric RPR4.1.

For any part of the proof questions (Pre-interview Task 2 or Post-interview Task 3), code lines of interview as **Verify Truth**, **Explain Why/Show Thinking**, **Communicate Math**, **Create New Math**, **Systematize/Logical Thinking**, or **Facilitate Generalization**. The descriptors for the first 4 categories are identical to those of Rubric RPR4.1. Recode the interviews using these criteria, but coding **only** lines of interview that discuss these ideas as purposes for proof in the K-12 classroom. Additionally, code for **Facilitate Generalization** as detailed below. The descriptors for the first 5 codes are included with examples from the previous rubric (examples are not specific to K-12 classroom).

Verify truth
Teacher indicates that proof serves to verify the truth of mathematical statements. This could include stating that proof serves to verify, check the truth of, or confirm a mathematical statement, theorem, conjecture, fact, or idea. This code differs from the **Explain why** code in that the response implies that the statement is known and proof serves to check the truth, rather

than explaining why a new mathematical idea might be true. Note that the questions regarding how to disprove and whether proofs ever become invalid might provide additional evidence.

Example 1:
Proof to me, um, pretty much, like I said I think on our first interview is, it's using something you know, to show something else, to be true um, [pause] I think if you show it to be untrue it's not really- it's proving something but it's not proving it to be true I think a proof is proving something to be true not, disproving something or proving it not to be true so that's the first thing I would say. (EL, post, ll. 762-766)

Example 2:
I want to say it's to- to validate your ideas and your findings, um, [pause] that- that's the big thing you can't just, [long pause] I- I guess you just can't say something is just because you've found, [pause] y'know one way that works. You have to, [pause] consider all of the options or all of the counterexamples um, [pause] or else you're gonna have a faulty, [pause] proof or idea um, that you may try to use later down the line but find out that, it's not gonna hold um. [pause] So I think having that proof again is just to validate your ideas, to make sure that they're set and that they're grounded um [pause] so that later on that you can use those ideas um, when you're trying to prove something else or y'know, when you're trying to- to prove something new. (CD, post, ll. 805-813)

Explain Why/Show Thinking
Teacher indicates that one purpose of proof is to explain why a mathematical statement is true. This could include stating that proof serves to explain why an argument, conjecture, statement, or mathematical idea is true; to explain one's thinking or reasoning; to justify, back up, or provide reasons for a conjecture, or simply to explain why. This code differs from the **Verify truth** code in that the response implies that the statement is new, either to mathematics in general or to the student or group of students who are putting forth the statement. (That is to say, the statement represents knowledge not previously known by the author.)

Example 1:
So it's an argument that um, [pause] doesn't just explain what's happening, but, it uses, true math facts, to make sense, t- true math facts to um, [pause] to justify, why it happens. (KT, post, ll. 668-670)

Example 2:
Um, to cite examples, to use previous knowledge um, [pause] for a new idea, not necessarily new to, the world, new to you. So, [pause] like if I ask my- my students to prove something, it wouldn't- I would already know, y'know and that- not necessarily some theorem or anything like that but just, [pause] here's this problem, prove that you know the answer kind of thing so, [pause] backing it up using um, previous knowledge and understanding. (MH, post, ll. 345-350)

Communicate Math
Teacher indicates that proof serves to communicate mathematical knowledge. This could include stating that proof communicates mathematical knowledge (idea, concept, theorem, statement, conjecture, etc.) to others, that proof helps people/students <u>understand</u> a mathematical

idea, concept, theorem, statement, or conjecture, the proof helps to disseminate knowledge that mathematicians or other doers of mathematics create. There does not necessarily have to be a sense of agency or authority in the response; the response could simply state that it helps naïve learners to understand knowledge that others create, with no indication of the level to which the learners will learn and understand that knowledge.

Example 1:
I think to prove something you need to- I think you need to be able to communicate it to your audience um, whether that audience is a group of mathematicians it's different than if it's a group of, your classmates I think that, it's ok that proof is, someone in our class- (CO, post, ll. 526-529)

Create New Math
Teacher indicates that proof serves to create new mathematical knowledge. This could include stating that proof develops new mathematical ideas, concepts, facts, or truths; serves to confirm conjectures or nascent ideas; or allows mathematicians or other doers of mathematics to build mathematical knowledge. There does not need to be any sense of a communication of these ideas, simply that proof is the tool that serves to inaugurate a <u>new</u> mathematical idea as part of the knowledge base. In this way, it differs from the **Verify**, **Explain**, and **Communicate** codes.

Example 1:
But so what purpose does it serve, is so that you know, we can condense, some- some mathematical knowledge. I mean otherwise, we just have all these little snippets of ideas, that, maybe fit together but a proof, sort of brings lots of ideas into one more powerful and bigger idea that then can be built on. [pause] That's the purpose that I see it serving. (MN, post, ll. 431-435)

Example 2:
Um, well actually I think in math it serves a big purpose just because, everything in math, kind of builds on itself um, and everything's related. At some point in time you're gonna use geometry and algebra and this and this all together to come up with some of these ideas. And I think proof is a good way to, [pause] I want to say maybe bridge that gap make a couple of those connections 'cuz when you prove something, y'know you're not just using, [pause] one geometry skill. You're using this one and this one and you're kind of pulling them- all these thoughts together in a largical- logical argument to show that that next step is true. Then once you have it, you can jump to the next step. So proof kind acts as those y'know little like, maybe, if you want to say steps leading up to those- the whole set of ideas with um geometry. So it's just one little part of it. (EL, post, ll. 836-845)

Systematize/Logical Thinking
Teacher indicates that proof serves to impose a logical structure (e.g., differentiating, utilizing, and classifying axioms, theorems, conjectures, etc.) on the domain of mathematics. That is, proof serves to organize results and to catalog them with respect to the underlying axioms and ideas upon which the proofs are built. This is a very specific code, and based on the work of Knuth, is unlikely to be used more than once or twice. (Knuth ended up lumping this code in with Create new mathematics.)

Example 1:
I think, sssssssss- [pause] you know that we want to- we want mathematics to be, a sound system and um, we want it to be able to- to um, serve our intellectual needs. (MN, post, ll. 424-426)

Facilitate Generalization
Response indicates that proof can serve to facilitate students' work towards generalization in the K-12 classroom.

Example 1:
I think it should be woven throughout the curriculum at the middle and the high school level because I think kids need to have, exposure to it and just start- I mean, even just, so much as learning how to reason through, strategies and reason through, situations, and then, make generalizations and prove those generalizations true I think is a very very important skill for them so, [pause] I think it needs to be, an integral part of the curriculum at all levels, and not, JUST in high school it needs to start early. Even as far as elementary school just basic, why does it work, what- how do you know this, kind of things like that, building up to a more formal and general proof. (LC, post, ll. 705-713)

Goal RPR6:
Teachers will identify classroom discourse as a promising tool in supporting students' work with proof.

Data sources:
Pre-Course Interview, Task 2 all parts
Post-Course Interview, Task 3 all parts
Selected Course Discussions (Defining and Revisiting Proof)

Rubric RPR6.1
Data source: Pre-Course Interview, Task 2, and Post-Course Interview, Task 3
Rubric Score Data Type: Categorical, multiple codes per response

Promoting discourse in the classroom can serve as a powerful tool for supporting aspects of the proof process with students. Specifically, discourse between students can press students to justify, explain, and communicate mathematical knowledge. The purpose of this rubric is to measure teachers' attention to the notion that discourse can serve as a tool to support student's work with proof.

Code any applicable part of the proof questions as **Discourse** if the teacher mentions that discourse with or between students as supporting work related to proof.

Example 1:
I think specifically making students accountable for what they're doing or making them explain what they're thinking or doing will help them in the long run when they get into- if they get into more advanced mathematics because, you have to be able to sit down and think through a problem and, [pause] understand what you might get or what you're doing and why you're doing

it in order to be able to do, more advanced mathematics more difficult mathematics. (SD, post, ll. 846-851)

Knowledge of Mathematics for Student Learning: Five Practices for Productive Use of Student Work

Goal ANT:
Anticipating student solutions

Data sources:
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
TTAL assignment

Rubric ANT 1
Data sources: Pre-Course Interview Task 3, Post-Course Interview Task 4
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify evidence of teachers' anticipation of student solutions in the planning of a mathematics lesson surrounding a high level task. Code lines of interview and artifacts from the interview task (the Minimizing Perimeter task sheets) as **Anticipating-Specific** or **Anticipating-General**. Record the number of lines of text for each code.

For each line of the interview for Task 3 (pre) or Task 4 (post), code as **Anticipating-Specific** if in the line of text, the teacher discusses a specific strategy, solution path, or misconception that students might produce in their work on the task.
Note: If the student work discussed is scaffolded to the extent that students are responding to prompts and are not engaging in high-level thinking, do not use this code.

Example 1:
Uh, the- the big thing there is, I would want students to look for the repeating pattern, to see that the dimensions are gonna, come back again. That if you have 9, for the length 4 for the width, that, you can flip that around and make the length 4, and the width 9. (CD, pre, ll. 591-594)

Example 2:
Um, [pause] but while the kids are working, I think that they need to, recognize like I'm thinking that a lot of kids'll automatically go to the 6 by 6 um [pause] garden and think that that's gonna have [pause] the lowest perimeter. (DN, post, ll. 492-494)

Example 3:
So maybe once they sketch it then if they have time go back and make the graph and see if their, thinking about how perimeter and length are actually, um related if they're, sketch did look like their graph. (IT, post, ll. 641-643)

Non-Example 1:

So, there are- I felt um, if- since this is a 7th grade class, er one of the best- one of, [pause] my first approaches would probably be to, have them just throw out a set of numbers. Such that the area- know that area is uh, length times width- Uh, that would come out to 36. And, the 2 numbers I chose for example were 9 and 4. (NoT, pre, ll. 622-626)
MDS notes that while this response suggests what students might do with the task, the idea that the teacher would have them "throw out a set of numbers" suggests that this might not a path that students would produce spontaneously, and this path is heavily scaffolded.

In examining the teacher artifacts from the Minimizing Perimeter task, code an instance of **Anticipating-Specific** if the teacher has included multiple solutions in their own exploration of the task. Code each unique solution as an instance of the code.

For each line of the interview for Task 3 (pre) or Task 4 (post), code as **Anticipating-General** if in the line of text, the teacher discusses at a general level that they would or have anticipated student strategies, solution paths, or misconceptions that students might produce in their work on the task.

<u>Goal MON:</u>
Monitoring student work:
- *questioning strategies (focus, assess, and advance)*
- *evaluating student work*

Data sources:
Pre/Post Assessment, Part D, Task 5
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
TTAL assignment

<u>Rubric MON 1</u>
Data sources: Pre/Post Assessment, Part D, Task 5
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to categorize teachers' responses to the erroneous student conjecture featured in Task 5. This rubric borrows from Ma's (1999) analysis of US and Chinese elementary teachers' responses to the same task.

Code teachers' responses as **Press to justify or mathematically investigate**, **Ask for more examples**, **Provide more examples**, **Ask for counterexample**, **Provide counterexample**, **Ask class to evaluate**, **Demonstrate/tell it is false**, **Correct/Right track**, **Probe student thinking**, **Connect to proof**, **Other question** (specify).

Press to mathematically investigate: Response indicates that the teacher would either ask the student to justify why they think their conjecture is true, or ask the student questions designed to prompt them to investigate the mathematical relationship. These questions might include asking for the student to provide a specific class of example (e.g. one with a greater perimeter, smaller area; same area, different perimeter), press the student for a generalization, or ask them to

explain or consider how they might show the conjecture to be true in a way that is grounded in the mathematical relationship. If the press for justification is general, code as **Press to justify: general**. If the example specified is a counterexample, code as **Ask for counterexample** instead.
Example 1:
What if you changed the dimensions but kept the same perimeter would the area change. (NB, pre)
Note: Weak question, but still gets at the mathematical relationship.

Example 2:
…Then, I would guide them to reword their original simpler assumption and help them generalize our new finding about fixed perimeters yielding various areas. (KT, pre)

Example 3:
I would have the student construct rectangles w/different perimeter but the same areas. (CD, pre)

Press to justify: general: Response indicates that the teacher would ask the student to justify their response, but without specific connections to the mathematical relationship at hand.
Example 1:
I would ask the student how he/she knows that this claim is true, forcing them to explain their thinking. (LC, pre)
Note: This prompts the student to justify as opposed to probing their current understanding.

Example 2:
I would ask the student how they came to this conclusion & what evidence they have to prove this claim. (MH, pre)

Ask for more examples: Response indicates that the teacher would ask the student to provide additional examples. The nature of these examples is either unspecified or unrelated to the mathematical relationship in the task.
Example 1:
Can you draw another shape w/an area of 9? (NL, pre)
Note: By saying "shape", it's not clear that NL is going to get a square or rectangle.

Example 2:
How many ways can you make a sq/rect of area 9? (Hoping they would come up with a 1 x 9). Is the perimeter the same? Is the area the same? (NL, pre)
Note: Since the conjecture is about increase in perimeter implies increase in area, it's unclear how this question would lead the student to consider the mathematical relationship at hand.

Provide more examples: Response indicates that the teacher would provide additional examples for the student to consider.
Example 1:
I would ask what happens (after the presented the 3x4) if I increased the perimeter to 26 cm by making a 1 x 12 rectangle. The area would then stay the same. (CD, pre)

Note: Even though CD is asking the student what happens, he is providing an example of a particular type.

Example 2:
Change the dimensions to different numbers, trying to lead them to fractional side lengths. (EH, pre) *Note: Not sure what EH is getting at here, but she is clearly providing more examples.*

Ask for counterexample: Response indicates that the teacher would ask the student to provide a counterexample to the conjecture.
Example 1:
I would ask students to first investigate this claim, "Can anyone find an example in which this doesn't work." (SD, pre)
Note: In this case, SD is asking the class to evaluate in a specific way, finding a counterexample. This should be double-coded with ***Ask class to evaluate****.*

Example 2:
If students said they thought it was false, I would then ask them why and to come up with a counterexample that shows this. (DE, pre)

Provide counterexample: Response indicates that the teacher would provide a counterexample.
Example 1:
Give an example in which this is not true. (SD, pre)

Example 2:
If after a time, no one came up with an answer, I would do a few examples. Eventually I would get to one that shows that this is not true. (DE, pre)

Ask class to evaluate: Response indicates that the teacher would ask the class to evaluate the conjecture and/or say whether they think it is true or not.
Example 1:
I would ask students to first investigate this claim, "Can anyone find an example in which this doesn't work." (SD, pre)
Note: In this case, SD is asking the class to evaluate in a specific way, finding a counterexample. This should be double-coded with ***Ask for counterexample****.*

Example 2:
I would first ask the class if they thought that was true or false. (DE, pre)

Demonstrate/tell it is false: Response indicates that the teacher would either demonstrate for the student why the conjecture is false, or simply tell the student that their response is false.
Example 1:
Not always, if you think about a different size rectangle, maybe long & thin then the area does not necessarily have to increase if the perimeter does. (UL, pre)

Correct/Right track: Response indicates that the teacher believes the student's conjecture is true or that they are on the right track or correct path.

Example 1:
Explain in more detail. This reasoning is on the right track. (BD, pre)

Example 2:
$P/2 = l + w \quad l = P/2 - w \Rightarrow A = w(P/2 - w) = Pw/2 - w^2$
If perimeter increases then this factor will increase so A increases (MN, pre)

Probe student thinking: Response indicates that the teacher would ask the student to explain what they mean by their conjecture. This could be a generic statement or specific questions designed to probe; in the case of specific questions, code each instance.
Example 1:
I would ask what they meant by the perimeter increasing, trying to get at the point that the perimeter not only has to increase in one dimension. (CD, pre)

Example 2:
I would ask them why they think this happens. (BN, pre)

Connect to proof: Response indicates that teachers would make a connection to the notion of proof. Specifically, teachers might ask if their two examples prove the case, or what they might have to do to prove the conjecture or make sure it holds for all cases.
Example 1:
If there's no counter example, can we prove the area always increases as the perimeter does using only repeated examples? (KE, pre)

Example 2:
Can you prove this? How do you know you have thought of enough scenarios to prove your claim? (BN, pre)
Note: This would count as two instances of the code.

Other: Response specifies a question that does not fit any of the above categories.

<u>Rubric MON 2</u>
Data sources: Pre-Course Interview Task 3, Post-Course Interview Task 4
Rubric Score Data Type: Numerical, multiple codes per response, single code per question

This rubric is designed to identify evidence of teachers' plans for questioning during the planning of a lesson around a high level tasks . Code questions in the interview on the rubric scale below, ranging from 1 to 4. This rubric mirrors the questioning rubric used in the TTAL scoring rubric. Questions can also be in the form of first-person statements (see Example 1 in the 3 pts code).

The target mathematical goal for the lesson was ***for students to understand the relationships between area and perimeter***.

4 pts Question is tied to a particular strategy or approach which is clearly articulated in the interview transcript (e.g., *if students do* _______, *I will ask* ________) ***or***

Question is designed to connect specific strategies or approaches, or press towards a generalization ***and***
Question is clearly related to the target mathematical goal for the lesson

Example 1:
And say ok, if they wanted to make theirs 72 square feet, using what you know NOT making tables or anything, [pause] What do you think would happen? (EL, pre, ll. 730-734)

So y'know, with this question y'know if it's gonna be 72 feet what do you think the length would be you could also say, [pause] well what would happen if I made, [pause] the side length of my garden y'know, [pause] um, well this is gonna be 6 so maybe, what would happen if I made it, 24. Or 18. Or some multiple of what that side length was gonna be from your original 'cuz it's gonna be a 6 by 6. So, if you y'know, make it 3 times bigger, or you could just say y'know from the original problem what would happen if I made the side length 3 times bigger? (EL, pre, ll. 762-768)

Example 2:
Um, I'd (want them) to draw, their fence because they can also- I would expect for most students to say ok 36, so they'd need the factors of 36, um, (xxx xxx) just say 6 by 6. So the length is 6, the width is 6. So ask them, what other factors multiply to give you 36? (NB, pre, ll. 483-485)

Example 3:
And with the 100 feet as well. [pause] And um, [pause] I would say just um, ask them to make observations. Or what did you notice what patterns did you see in all of your tables if you look at them together so hopefully they'd get at well the one with the least, is the most square-like. (IT, post, ll. 633-636)

3 pts Question is loosely tied to a particular strategy or approach ***or***
Question is designed to probe or advance student thinking, but is overly general, and/or the conditions for use are not clear ***and***
Question is related to the target mathematical goal for the lesson

Example 1:
Y'know, compare the two graphs and then from that they should be able to, basically, answer both of these. I would encourage them to, prove to me why they have, um, why these work. (UT, pre, ll. 431-433)

Example 2:
And, on that problem then with the 24 square feet, I would say ok what if your answer, could be decimals or fractions. (NiT, pre, ll. 488-489) *MDS notes that while the question is a good one to advance student thinking, it's not clear under what conditions the question might be used.*

Example 3:
I would um, then once I did that I would, y'know let them walk around the- walk- as I was walking around um, I would, talk to them about how many- how many possibilities can you come up with. (CO, post, ll. 745-747) *MDS notes that the question might be good in general to press students to articulate as many combinations as possible, it's not clear when this question might be used.*

2 pts Question is not tied to specific strategy ***or***
Question is general and/or it is not clear how the question serves to focus, assess,

or advance student thinking, ***and***
Questions are related to the target mathematical goal for the lesson

Example 1:
And also to ask- make sure they understand it's AREA, that area has to be length by width. And for them to draw this, if you make, length 6, what does that mean? One side is 6. (NB, pre, ll. 486-488) *MDS notes that it's not clear how the question is related to understanding area.*

Example 2:
] I guess I would, um, just have them do that first, the lengths and the widths. After that, [pause] is when I would ask them what is perimeter. What- what does perimeter really mean. Hoping for them to say it's the, y'know distance around something. The distance around a figure. (NoT, pre, ll. 653-656)

Example 3:
I would try to prod them in the right direction or, ask questions so that they would, think about what they're doing. (UL, post, ll. 736-738)

1 pt Question is not tied to a specific strategy or approach ***and***
Question is not related to the target mathematical goal for the lesson ***or***
Question is procedural in nature/serves to reduce the demands of the task

Example 1:
Um, [pause] I would make a number 3 though that would be AREA versus length, because-- (xxxx) if they really want to understand the relationship between, area and perimeter they should see the difference between those two graphs, and see um, y'know, what it- what happens as the length increases. (UT, pre, ll. 418-423)
MDS notes that in this case, an area vs. length graph is actually a horizontal line. Thus, the question of what happens when length increases does not clearly serve to advance student thinking about the relationship between area and perimeter.

Example 2:
So what would be a way- if you had, a fence, what would be a way to make the perimeter smaller? You'd take away, a side. So is there any way, you can take away a side, to make it- the perimeter smaller. So the- lead them, to maybe see, using a garden, next to a house. (NB, pre, ll. 490-493) *MDS notes that this line of questioning departs from the conditions of the task, and it's not clear how it might serve the mathematical goal of the lesson.*

Example 3:
The next thing that I would do, is, [pause] establish some sort of- of um, guidelines. The fact that it's 36 square feet what's the largest the length could be? (NoT, pre, ll. 631-633) *MDS notes that this question asked during the setup, may serve to reduce the demands in terms of giving the students guidelines for selecting values.*

<u>Goal SEL:</u>
Selecting responses for whole-group discussion

Data sources:
Pre/Post Course Assessment, Part D, Task 7
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
TTAL assignment

<u>Rubric SEL 1</u>
Data sources: Pre-Course Interview Task 3, Post-Course Interview Task 4
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify evidence of teachers' selection of student responses for sharing in the lesson planning process. Code lines of interview as **Selecting+Monitoring** or **Selecting**.

For each line of the interview for Task 3 (pre) or Task 4 (post), code as **Selecting** if in the line of text, the teacher discusses specific strategies that he or she would select for presentation and/or discussion with the class. Record the number of lines of text for each code.

Example 1:
So then when we started our share and discuss I would just begin it like, what did you do when you started this problem and then, list all of the different um, possible rectangles for 36 and just start with a discussion about that but talk about how the dimensions are factors, of 36 and talk about what areas and how do you know if this rectangle really works, that they multiply together if would be 36. And talk a little bit about area. (IT, post, ll. 643-648)
Example 2:
Now let's see what else I jotted down here, tables, uh I guess then when we got to the share and discuss, I may choose to go with the pictures first. I think like I said a lot of students might start with that, so I'd get that up there so that they can see what the majority had done. (CD, pre, ll. 600-604)

For each line of the interview for Task 3 (pre) or Task 4 (post), code as **Selecting+Monitoring** if in the line of text, the teacher discusses specific strategies that he or she would select for presentation and/or discussion with the class, and that he or she would be watching for these strategies during the lesson phase in which they are monitoring student work..
Example 1:
So then, so I would ask questions like that as I was walking around and I'd be looking specifically for- I would want my order in my um, share and discuss to be, a picture, [pause] and by picture I mean on graph paper where they drew- All the different- drew all the different dimensions [pause] –ensions and then I would want a ta- I'd want a ta- I'd want kids to come up with, this type of table, and then I would want, [pause] I would want the graph- oh, I think I would get- I don't know if I would leave, I think I would take this question off too the graph question- and I would ask kids that are done if they could graph their work. (CO, post, ll. 754-763)

<u>Goal SEQ:</u>
Sequencing responses for whole-group discussion

Data sources:
Pre/Post Course Assessment, Part D, Task 7
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
TTAL assignment

<u>Rubric SEQ 1</u>
Data sources: Pre-Course Interview Task 3, Post-Course Interview Task 4
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify evidence of teachers' sequencing of student responses for sharing in the lesson planning process. Code lines of interview as **Sequencing**.

For each line of the interview for Task 3 (pre) or Task 4 (post), code as **Sequencing** if in the line of text, the teacher discusses the order in which they would want responses shared during the share and discuss phase of the lesson has a particular ordering for discussing other artifacts of the students' work. Record the number of lines of text for each code.

Example 1:
So then when we started our share and discuss I would just begin it like, what did you do when you started this problem and then, list all of the different um, possible rectangles for 36 and just start with a discussion about that but talk about how the dimensions are factors, of 36 and talk about what areas and how do you know if this rectangle really works, that they multiply together if would be 36. And talk a little bit about area. (IT, post, ll. 643-648)
Note: This is a case in which IT talks about how she would organize the discussion around questions, drawing in particular aspects of student work at key points.

Example 2:
Now let's see what else I jotted down here, tables, uh I guess then when we got to the share and discuss, I may choose to go with the pictures first. I think like I said a lot of students might start with that, so I'd get that up there so that they can see what the majority had done. (CD, pre, ll. 600-604)

<u>Goal CON:</u>
Connecting responses shared in whole-group discussion

Data sources:
Pre-Course Interview, Task 3
Post-Course Interview, Task 4
TTAL assignment

Rubric CON 1

Data sources: Pre-Course Interview Task 3, Post-Course Interview Task 4

Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify evidence of teachers' consideration of connecting shared responses in the lesson planning process. Code lines of interview as **Connecting**.

For each line of the interview for Task 3 (pre) or Task 4 (post), code as **Connecting** if in the line of text, the teacher discusses specific plans for making connections between shared responses. Begin the coding segment where the first response is described, and end the coding segment where the teacher states the response to connect and/or the nature of the connection. Record the number of lines of text for each code.

Example 1:

So then, so I would ask questions like that as I was walking around and I'd be looking specifically for- I would want my order in my um, share and discuss to be, a picture, [pause] and by picture I mean on graph paper where they drew- All the different- drew all the different dimensions [pause] –ensions and then I would want a ta- I'd want a ta- I'd want kids to come up with, this type of table, and then I would want, [pause] I would want the graph- oh, I think I would get- I don't know if I would leave, I think I would take this question off too the graph question- and I would ask kids that are done if they could graph their work. So that would be one of the questions I would ask. And then my point with the pic- so then when I had the share and discuss my point would be, I would have kids with the, with the tab- with the picture first. So try and get every kid in the class to be able to- to see the different- to see the problem and to see how, [pause] if all they had y'know, at least with them all drawn out you could actually count ok here's what- here's what the perimeter is, here's what [pause] here- here's what, here's what the area is there's different- there's different ways we could look at it. Then, I would- then I would bring the kid up with the- then I would ask about well this is kind of confusing I'm having a hard time following this, is there any way that we could- that we could look at this and then I would finish with- [pause] with, with the graph. (CO, post, ll. 754-774)

Example 2:

So then when we started our share and discuss I would just begin it like, what did you do when you started this problem and then, list all of the different um, possible rectangles for 36 and just start with a discussion about that but talk about how the dimensions are factors, of 36 and talk about what areas and how do you know if this rectangle really works, that they multiply together if would be 36. And talk a little bit about area. Then after looking at all of them, go back and um, what did you notice about like the rectangles that required the least fencing. So that would get at the generalization. (IT, post, ll. 643-650)

IT starts by asking students to list the possibilities and discuss them, and then makes the connection to a generalization through observing patterns.

Knowledge of Practices that Support Teaching: Routines

<u>Goal ROU:</u>
Teachers will be able to identify routines and understand the role of routines in teaching and advancing student learning

Data sources:
Pre/Post Course Assessment, Part A
Identifying Routines Assignment

<u>Rubric ROU1.1</u>
Data sources: Pre/Post Assessment, Part A
Rubric Score Data Type: Categorical, multiple codes per response

This rubric is designed to identify the routines teachers identified in the Surface Area video clip and compare those routines with the ones identified by the researcher. Code using the routines identified and described below. Code if teachers identify the routine in either the "Description of move/routine" column or the "How does the move/routine support classroom activity" column. On the post-assessment only, indicate whether or not teachers agree with the classification of the code (support, exchange, management) indicated for each routine. In cases where the teacher identified multiple classifications, count as a match if at least one of the responses correlates with the indicated classification.
Teachers were asked to identify line numbers; the exact location in the transcript is less important than the description of the move, so there is no need to correlate time codes and/or line numbers in responses.

For each code below, examples from the video are included.

Comment: Prompting students to comment on the ideas of others ***Exchange***
[01:02] What do you think about that, Artie?

Agree/Disagree: Students take stance with respect to others' ideas and justify ***Exchange***
[01:07] I agree with Brittany, because…

Small Group: Teacher directs small groups to debate a topic in a particular way ***Exchange***
[02:00] Talk about it in your group, write it down and say why.
[03:10] Talk (with your groups).

Explain: Teacher presses for justification or explanation ***Exchange***
[02:18] And why does that make sense.
[ll. 65] Asks Leslie to explain the equation

Hands-Check: Teacher asks for visual cues to check for understanding ***Exchange***
[02:28] How many agree.
[02:55] Why don't you all nod, yes or no.
[05:40] Could you raise your hand if you think you know why this formula makes sense.

[06:40] I want to make sure everyone understands this formula, so how many can understand it?

Prompt and discussion: Teacher asks for an argument about an idea ***Exchange***
[03:10] Can anybody make an argument about why we would need another variable?
[05:05] Can anyone come up and make an argument for the surface area.

Revoice: Teacher restates an idea, adding nuance or emphasis ***Exchange***
[03:00] Some people are saying yes, some people are saying no. *Restates question at issue.*

Call-on: Selects students to participate by table, and in cooperation ***Exchange***
[05:10] This table, go. Actually, both of you go.

Tools: Uses manipulatives & diagrams to facilitate explanations/demonstrations ***Support***
[00:43] Hands student cylinder to use with explanation
[01:10] Artie asks for cylinder to use with his explanation

Prior Knowledge: Teacher connects to prior knowledge ***Support***
[01:50] We'll use a variable **like we have been doing**…

Closure: Teacher primes students to recall key understandings from the lesson ***Support***
[08:00] All I need to know is two things, what two things are they.
[04:35] *Reproduces diagram on overhead for students to use.*

Crediting: Teacher gives credit for ideas using student names ***Support***
[02:40] *Credits Victor and Jesse when discussing previous idea*
[03:52] Pirmin, say what you said again.
[06:53] Jordan and Leslie, thank you so much.

Shift: Teacher flags an unplanned direction for the lesson (change in difficulty) ***Support***
[01:42] I hadn't planned to do this – I don't even know if you **can** do this…
[04:25] I'm jumping ahead because this is our last day on this…

Terms: Prompts for mathematical language to be used ***Support***
[06:12] I hate to interrupt you, Leslie, but let's use correct language. So what would that be?

Good Question: Flags a question as significant ***Support***
[06:45] Do you have to memorize it? Oh, that's a very good question.

Understand: Flags constructing one's own meaning as important in math ***Support***
(closing discussion on memorizing vs. understanding)

Listen: Get the attention of the class focused on the presenters ***Management***
[05:35] So Jordan and Leslie are waiting for your attention…

Hands-Cue: Raising of hands is used as the cue for wanting to speak ***Management***
(throughout)

APPENDIX F

Thinking Through a Lesson Assignment & Scoring Rubric

THINKING THROUGH A LESSON ASSIGNMENT
TTAL

The main purpose of the *Thinking Through a Lesson Protocol* is to prompt you in thinking deeply about a specific lesson that you will be teaching that is based on a cognitively challenging mathematical task.

Part 1: Selecting and Setting up a Mathematical Task

- What are your mathematical goals for the lesson (i.e., what is it that you want students to know and understand about mathematics as a result of this lesson)?

- In what ways does the task build on students' previous knowledge? What definitions, concepts, or ideas do students need to know in order to begin to work on the task?

- What are all the ways the task can be solved?
  - Which of these methods do you think your students will use?
  - What misconceptions might students have?
  - What errors might students make?

- What are your expectations for students as they work on and complete this task?
  - What resources or tools will students have to use in their work?
  - How will the students work -- independently, in small groups, or in pairs -- to explore this task? How long will they work individually or in small groups/pairs? Will students be partnered in a specific way? If so in what way?
  - How will students record and report their work?

- How will you introduce students to the activity so as not to reduce the demands of the task? What will you hear that lets you know students understand the task?

Part 2: Supporting Students' Exploration of the Task

- As students are working independently or in small groups:
  - What questions will you ask to focus their thinking?
  - What will you see or hear that lets you know how students are thinking about the mathematical ideas?
  - What questions will you ask to assess students' understanding of key mathematical ideas, problem solving strategies, or the representations?
  - What questions will you ask to advance students' understanding of the mathematical ideas?

- What questions will you ask to encourage students to share their thinking with others or to assess their understanding of their peer's ideas?

- How will you ensure that students remain engaged in the task?
  - What will you do if a student does not know how to begin to solve the task?
  - What will you do if a student finishes the task almost immediately and becomes bored or disruptive?
  - What will you do if students focus on non-mathematical aspects of the activity (e.g., spend most of their time making a beautiful poster of their work)?

Part 3: Sharing and Discussing the Task

- How will you orchestrate the class discussion so that you accomplish your mathematical goals? Specifically:
  - Which solution paths do you want to have shared during the class discussion? In what order will the solutions be presented? Why?
  - In what ways will the order in which solutions are presented help develop students' understanding of the mathematical ideas that are the focus of your lesson?
  - What specific questions will you ask so that students will:
    - make sense of the mathematical ideas that you want them to learn?
    - expand on, debate, and question the solutions being shared?
    - make connections between the different strategies that are presented?
    - look for patterns?
    - begin to form generalizations?

- What will you see or hear that lets you know that students in the class understand the mathematical ideas that you intended for them to learn?

- What will you do tomorrow that will build on this lesson?

The Thinking Through a Lesson Protocol was developed through the collaborative efforts (lead by Margaret Smith, Victoria Bill and Elizabeth Hughes) of the mathematics team at the Institute for Learning and faculty and students in the School of Education at the University of Pittsburgh.

Smith, M.S. & Bill, V. (2004, January). Thinking Through A Lesson: Collaborative Lesson Planning as a Means for Improving the Quality of Teaching. Presentation at the annual meeting of the Association of Mathematics Teacher Educators, San Diego, CA.

Hughes, E.K., & Smith, M.S. (2004, April). Thinking Through a Lesson: Lesson Planning as Evidence of and a Vehicle for Teacher Learning. Poster presented as part of a symposium, "Developing a Knowledge Base for Teaching: Learning Content and Pedagogy in a Course on Patterns and Functions " at the annual meeting of the American Educational Research Association, San Diego, CA.

TTAL Scoring Rubric

Solving the Task **3 Points**

3 pts Included solutions represent a range of approaches to the task, varying by representation or strategy where appropriate.
Solutions are fully developed and clear
Solutions include incorrect pathways/note possible misconceptions

2 pts Included solutions represent a range of approaches to the task, with some variation by representation or strategy where appropriate.
Solutions are fully developed and clear
Solutions include incorrect pathways/note possible misconceptions

1 pt Included solutions represent a narrow range of approaches to the task with little variation OR incorrect pathways/misconceptions are not included OR solutions are described in a general way rather than representing fully developed solutions

0 pts Solutions are not included

Mathematical Goal **1 Point**

1 pt An appropriate math goal is included

0 pts Math goal is inappropriate or not included

Building on Prior Knowledge **2 Points**

2 pts Prior knowledge that students will have is identified and connected to the mathematical task and the mathematical goal

1 pt Prior knowledge that students will have is identified, but connections to the mathematical task and the mathematical goal are weak or unspecified

0 pts No information about how the task builds on prior knowledge

Expectations for Students **2 Points**

2 pts Resources for students to use are identified
Grouping strategies/formats are specified and the means for reporting work is included

1 pt Either resources or grouping strategies and reporting are not included, or both are included and unclear

0 pts No information about expectations for students

Task Setup **2 Points**

2 pts Information about how the teacher will set up the task is included
This information is explicitly connected to maintaining a high level of cognitive demand for the task

1 pt Information about how the teacher will set up the task is included but is not well-connected to maintaining a high level of cognitive demand for the task

0 pts No information about the task setup

Questions: Focus, Assess, Advance **3 Points**

4 pts A variety of questions are listed that have the potential to focus, assess, and advance student thinking
Questions are tied to particular strategies or approaches
Questions are clearly related to the target mathematical goal for the lesson

3 pts A variety of questions are listed that have the potential to focus, assess, and advance student thinking, but one category may be narrowly represented
Questions are loosely tied to particular strategies or approaches
Questions are related to the target mathematical goal for the lesson

2 pts A variety of questions are listed that have the potential to focus, assess, and advance student thinking, but one category is absent or multiple categories are narrowly represented
Questions are generally not tied to specific strategies
Questions are related to the target mathematical goal for the lesson

1 pt Questions are listed, but it is not clear how the questions have the potential to focus, assess, or advance student thinking
Questions are not tied to specific strategies
Questions are not clearly related to the target mathematical goal for the lesson

0 pts No questions are listed

Ensuring Student Engagement **2 Points**

2 pts Strategies are discussed that address what the teacher will do if students cannot begin the task, if they finish almost immediately, and if they focus on non-mathematical aspects of the task
Strategies presented are sufficiently open in that they do not reduce the demands of the task

1 pt One of the categories in score point 2 is not addressed, OR the strategies presented reduce the cognitive demands of the task

0 pts Ensuring student engagement is not addressed

Selecting and Sequencing Student Responses **3 Points**

3 pts Specific student responses are identified for sharing during the Share & Discuss
A specific ordering for the sharing of responses is specified
Rationale for the selection and ordering is clearly stated and related to the development of students' mathematical understandings
Questions or issues relating to each response are included

2 pts Specific student responses are identified for sharing during the Share & Discuss
A specific ordering for the sharing of responses is specified
Rationale for the selection and ordering is stated and loosely related to the development of students' mathematical understandings
Questions or issues relating to some responses are included

1 pt Specific student responses are identified for sharing during the Share & Discuss
A specific ordering for the sharing of responses is specified

Rationale for the selection and ordering unclear
Questions or issues relating to some responses are included

0 pts Specific responses are not identified for the Share & Discuss phase

Connecting Ideas & Making Sense of the Mathematics **2 Points**

2 pts Specific questions or other comments are presented that connect the mathematical ideas in the shared responses
Connecting questions or comments align with the mathematical goal

1 pt Specific questions or other comments are presented that connect the mathematical ideas in the shared responses
Connecting questions or comments loosely align with the mathematical goal

0 pts No connecting ideas are presented

Students' Understanding of the Math Ideas **2 Points**

2 pts Specific words or work (things the teacher might see or hear) are identified that will help the teacher know if students are understanding the mathematical ideas

1 pt Vague descriptions of talk and work are presented that will help the teacher know if students are understanding the mathematical ideas

0 pts No information is given related to how the teacher will assess students' understandings of the mathematical ideas

Extending to the Next Day **2 Points**

2 pts A task or discussion is described for the next day's work that either promotes deeper engagement with the target mathematical goal for the lesson, or connects the understandings from the lesson to a new but related mathematical goal

1 pt A task or discussion is described for the next day's work, but it is unclear how this task promoted deeper engagement with the mathematical ideas or connects to a new mathematical goal

0 pts No information about the next day's work

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