

**DYNAMICS OF ASSET PRICE CHANGES:
STATISTICAL AND DIFFERENTIAL EQUATIONS
MODELS**

by

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This dissertation is comprised of two related tracts: (i) Quantitative Modeling and (ii) Analysis of Asset Flow Differential Equations. In the former a data set of over 100,000 daily closed-end fund prices is analyzed using mixed-effects regressions with the objective of understanding price dynamics. This analysis provides strong statistical evidence that relative daily price change is positively influenced by valuation, recent price trend, short term volatility, volume trend, and the M2 money supply. There is a strong nonlinearity in the influence of the price trend, so that a significantly large recent uptrend has a negative influence on the subsequent day's relative price change. The nonlinearity is the key to an understanding of the competing role of price trend, since a single large data set exhibits both under- and overreaction in different regimes of the independent variables. The role of long term volatility is not a clear-cut risk/return inverse relation; rather there is an ambiguous and complicated relationship between volatility and return. Standardization of the independent regression variables allows for a more direct comparison of each factor's influence on the return.

In the latter a two-group asset flow model of a financial instrument with one group focused on price trend, the other on value, is considered. The existence of both stable and unstable regions for the system of differential equations is proven. It is shown that a strong motivation based on (recent) price trend is associated with instability. Numerical computations using a set of typical parameters describe regions of stability and instability. A precise limiting connection between the discrete and differential equations is also established.

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PREFACE

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1.0 INTRODUCTION

The dynamics of asset prices and the underlying motivations have been of great interest for both theorists and practitioners. The classical theory has, as its cornerstone, the Efficient Market Hypothesis (EMH), which in one of its forms states that no publicly available information, specifically price history, can be used to obtain a risk adjusted profit beyond a market index. Fischer Black (of the famous Black-Scholes Equation) indicated in his 1986 article "Noise" [3] that markets were efficient 90% of the time. He defines efficient to mean that the price is between one-half and two times the actual fundamental value. Thus, a market is deemed efficient if, for example, a stock valued at \$100 has a price between \$50 and \$200 - a rather large deviation.

Another key tenet of the classical theory is that while investors may be influenced by behavioral biases and/or cognitive errors, any pricing mistakes are immediately exploited by much more experienced practitioners with ample (usually assumed to be infinite amounts of) capital. In addition, since all investors have access to the same information, they share the same notion of valuation. Thus, this theory stipulates that prices should fluctuate randomly about this unique valuation and leads to the classical price equation

$$\frac{dP}{P} = \mu dt + \sigma dX \tag{1.1}$$

where P is the price, μ the risk-free rate of return, and $dX = N(0, dt)$.

More recently, Behavioral Finance has grown in popularity. This theory seeks insights into the psychological motivations behind investors' decisions to buy and sell. For example, the concepts of under- and overreaction and anchoring (which will be discussed later) have been utilized in discussing investor behavior. Publications such as the Wall Street Journal

consistently publish articles referring to momentum (i.e. the idea that a recent trend in price tends to continue) and the behavioral state of investors (e.g., cautious, weary, tentative, etc.).

The role of investor psychology in trading is marginalized in classical finance and economics on the grounds that any bias is exploited by completely rational, well informed traders with ample capital. However, the recent (2008) chain of events that started with the bankruptcy of Lehman Brothers and evolved into a crisis of confidence in the world's financial markets reminds us that the behavioral aspects of markets cannot be neglected. Huge daily drops cascading throughout the world's stock indexes were rarely reported without attributing to investors fear, panic, and other nouns that are nowhere to be found in finance or economics textbooks or research papers. The hedge funds that were supposed to exploit these shortcomings of investors were selling alongside these investors in order to meet margin calls. Thus, the lack of infinite arbitrage capital intertwined with investor fear of plummeting prices and created a situation with no explanation in classical work.

While the classical theory can be regarded as a default hypothesis, its inability to explain stock market bubbles and crashes is a significant limitation. Indeed, stock market bubbles and crashes have an impact that extends far beyond shareholders. These bubbles, in which stock prices soar drastically above the valuation of the stock, pose a challenge to classical finance's EMH, which stipulates that the stock price should randomly fluctuate about a unique valuation on which all traders agree. The tumultuous financial markets of September 2008 are a dramatic reminder of the diverse forces driving markets. These recent upheavals capping a prolonged housing bubble are the most recent of a series of modern bubble/bust cycles. Among these are the internet/high-tech bubble of the late 1990's and the Japanese stock bubble of the late 1980's. Due to their sheer magnitude, these episodes have had an impact far beyond the immediate shareholders who lost trillions. The ensuing years of economic slowdowns and job losses were among the consequences of these boom/bust cycles.

Yet it is surprising that only a relative handful of works focus on these phenomena, and finance textbooks hardly mention them. Papers that discuss motivations beyond valuation are often expository in nature, and rarely have direct contact with market data. As the effects of these episodes can be severe and far-reaching, it is important to study the forces that drive markets. To this end, one needs to understand the motivations, in addition to

the classical concept of price deviation from the perceived value, behind traders' decisions to buy and sell stock. Any meaningful resolution of these problems must be quantitative. The complexity of the problem necessitates powerful mathematical and statistical techniques.

This dissertation is comprised of two related sections. Section I (Chapter 2), Quantitative Modeling, utilizes linear mixed effects regressions with appropriately defined variables to determine the effect various factors have on the return. The inclusion of a valuation variable as an independent variable in the regressions accounts for changes in the valuation thereby unmasking the effects of other variables such as the recent trend in price, volatility, and the amount of money in the system.

Section II (Chapter 3), Analysis of Asset Flow Differential Equations, examines a model developed by Caginalp and collaborators since 1990. This model contains two key features: (i) it incorporates the notion of finiteness of assets (both shares and cash) and (ii) it allows for investment reasons beyond the classical reason of valuation. The latter is achieved via the creation of sentiment functions that model an investor group's decision to buy or sell. This section considers the two investor group model where one group is focused solely on the valuation and the other solely on the recent trend in price. Focus on the trend is justified via Section I results. Theorems demonstrating the existence and stability of equilibria are presented. And, the use of practical values for parameters modeling investor groups' strategies allows the presentation of several graphs depicting the effect these parameters have on equilibria and stability.

The remainder of this thesis is organized as follows. Chapter 2 covers the Quantitative Modeling portion of this dissertation. An introduction including prior related work followed by a discussion of the data set is included. A review of two studies, one without [8] and one with [9] standardized independent regression variables, is presented. Chapter 3 encapsulates a research article [10] that provides an introduction to the Asset Flow Differential Equations along with both theoretical and practical results. This dissertation is concluded in Chapter 4 with a brief summary of the results and a listing of related future projects/problems.

2.0 QUANTITATIVE MODELING

2.1 BACKGROUND AND PRIOR WORK

The process whereby investors and traders react to new announcements, as well as new supplies of investment capital or cash, leads to a complicated dynamical problem beyond merely valuation. Traders are also aware of the reactions of others' assessments through price and volume changes. The question of whether a significant portion of traders are influenced by these factors is an empirical one. If the investors who base their decisions solely on the valuations constitute the vast majority – measured through the percentage of the assets owned by this group – then statistical methods should confirm this to be the case. If, on the other hand, a significant fraction base decisions partly on the reactions of others, then it should be possible, in principle, to extract some results that reveal the strategy in their decisions. For example, concepts such as underreaction and overreaction have been utilized in discussing investor behavior. Without a clear quantitative methodology for distinguishing these two opposing modes, it is very difficult, if not impossible to utilize these concepts in finance. In other words, knowing only that the market dynamics sometimes exhibit overreaction and sometimes underreaction does not provide much insight into trader motivation nor is it useful in trading decisions.

The goal of this Quantitative Modeling methodology is to provide some quantitative answers to these questions. Such a task has always faced a key obstacle, namely noise, which can be defined as the randomness in the stream of information updating the valuation of an asset. From the perspective of an investor or trader this information is stochastic. As Fischer Black [3] noted, "noise makes it very difficult to test either practical or academic theories about the way economic or financial markets work." And, as previously mentioned

in Section 1, he states that markets are efficient at least 90% of the time, defining efficiency as a market price that is "more than half of value and less than twice value." This point is consistent with frequent assertions by practitioners that there are important issues beyond valuation. However, the attempts to uncover these issues are often stymied by the presence of noise, as Black has noted. This leads to the question of whether there are methodologies whereby this noise can be greatly reduced to permit additional analysis of price dynamics.

One way to overcome this obstacle has been to perform a large scale statistical study in the hope of extracting a result that is perhaps too small to be useful to traders, but establishes the effect through statistical significance. Toward this end, Poterba and Summers [41] established that stock returns display small positive serial correlations for short time periods and are negatively autocorrelated over long time periods.

An early study [7] highlighting the importance of trend examined two essentially identical closed-end funds, Future Germany Fund and Germany Fund. Closed-end funds, unlike open-end funds, trade as other company stocks on the exchanges (see, e.g., [1]). Thus the price can be higher or lower than the net asset value (NAV), and can vary independently of this value. In the case of these two clone funds, any change in valuation will be identical in both funds. In terms of valuation, the ratio would be constant. By considering the ratio of these funds as a time series they were able to extract all noise attributable to valuation. They found that tomorrow's price is not predicted well by today's price, as efficient markets would suggest. But rather, tomorrow's price is halfway between today's price and the price obtained by continuing the pure trend from yesterday to today.

The persistence of a trend can be viewed as an underreaction, as it suggests that there is a delay in reaching a particular price. On the other hand, the behavioral finance community has also stipulated that participants overreact to news, as recent information tends to overshadow more established facts (see, e.g., [28]).

To study the concepts of underreaction and overreaction, Madura and Richie [34] performed a statistical study on the daily opening and closing prices of exchange traded funds (ETFs) during the time period August 1998 to August 2002. They define underreaction as positive [alternatively, negative] cumulative abnormal returns following large positive [alternatively, negative] price movements. Similarly, overreaction is defined as reversal of returns

following large price movements. They found evidence of overreaction after extreme price changes of greater than 5% in either direction within normal or after hours trading periods. If stocks are more likely to increase on a particular day if they had increased the previous day, then one can claim that there is evidence of underreaction. In other words, investors are not reacting fully on the first day. For instance, a group of investors may remain unconvinced of the new information and are waiting for further confirmation as manifested in higher prices. Once confirmed, these investors submit their buy order further increasing demand and thereby driving prices higher. On the other hand, if price increases tend to be followed by price declines, then one can classify this as part of overreaction. In aggregate, they found some evidence for overreaction. Overreaction has also been observed on a larger time scale (see, e.g., [19]).

In an effort to quantify overreactions, Sturm [47] showed that a stock with positive returns and/or increasing book value per share tends to rebound after a large drop of 10% percent, supporting the Madura and Richie findings. However, a stock already on a downtrend or suffering from certain negative fundamentals (e.g., declining book value per share) exhibits no such rebound. Further bolstering the empirical evidence for overreaction, Duran and Caginalp [22] studied 134,406 data points representing daily closing prices for a set of closed-end funds. Focusing on the deviation between the market price and the NAV, they examined changes in this deviation, which they divided into several distinct threshold levels (e.g., 5% to 7.5%). With a large statistical significance they found that large deviations from the net asset value led to a significant price movement in the opposite direction. More surprisingly, they found precursors to large deviations. In addition, Caginalp and Ilieva [12], using hybrid difference equations and regressions, found that the recent price trend is a statistically significant factor (with positive coefficient) in predicting tomorrow's price change.

2.2 DATA AND VARIABLES

2.2.1 Data Set: Daily Closing Price for Closed-end Funds

We use data on 125 closed-end funds (28 Generalized, 66 Specialized, and 31 World funds) consisting of 119,260 daily prices during the time period October 26, 1998 through January 30, 2008. As noted in Section 2.1, we consider these funds because their NAV is reported on a daily basis. Thus, estimating the value of an asset is not necessary, as it would be for most corporate stocks, thereby eliminating additional error.

Studying closed-end funds presents an opportunity to subtract out the random changes in valuation appropriately, and thereby eliminate a large part of the "noise" discussed above. Since closed-end funds regularly report their "net asset value" (NAV) based upon the current value of the investments, they offer a substantial advantage in this regard compared with stocks of most corporations. The set of closed-end funds we consider are those that report their NAV on a daily (rather than weekly) basis. Thus the 125 equity closed-end funds we study have assets that are sufficiently liquid, and there is a sound basis for daily evaluation of the underlying assets. However, for our purposes the liquidity of the underlying assets is not as important as the trading volume of the actual stock (i.e., the closed-end fund). After all, the underlying assets of a large company such as GE or IBM are usually not very liquid, but the stock is very active, and traders are interested in knowing the direction of the stock price. While the discount (defined as the percentage that the trading price of the closed-end fund is below its NAV) has been the subject of many papers (see [1] for a survey), almost all of these focus on reasons that are essentially steady state issues, e.g., tax liability, corporate structure or liquidity of underlying assets. In other words, even if the discount is larger due to tax liability or the underlying illiquidity of the assets, this situation does not change from one day to the next.

Studying the daily price changes circumvents many of these steady state issues that do not change on a daily basis. If valuation were the only factor in price movements, then the volatility of the trading price of closed-end funds would be similar to that of the NAV. However, Pontiff [39] finds that even though closed-end funds underreact to changes in NAV,

their prices are on average 64% more volatile than their assets. So the volatility of closed-end funds cannot be completely attributed to changes in the NAV. Thus the questions posed by market dynamics can be addressed effectively through the study of daily price changes of closed-end funds while compensating for changes in valuation.

We note that while the data set involves closed-end funds, there is little reason to believe that the market price dynamics of these stocks differ significantly from the average stock on the exchanges. Key daily trading features such as trading volume, market capitalization, and ratio of institutional to individual ownership are similar to most mid-cap stocks. The average weekly volume of the set of closed-end funds we consider is about 400,000 shares traded. Hence, the group of stocks we consider is comparable in activity to many ordinary stocks, bonds and options. Our methods can be applied to a broad range of stocks upon the adoption of a valuation model that is already well-understood in finance. However, from a scientific perspective, the ability to use an unambiguous quantity such as the net asset value provided by closed-end funds enhances the credibility of the methodology. Analogously, our methods are not restricted to daily changes, and one can implement them on different time scales of interest. Once again, however, the daily changes minimize the "noise" that is present so that the dynamics of asset prices can be determined more precisely. For example, an analysis of yearly changes in the US not only involves fewer data points, but also suffers from diminished credibility since the statistical results may simply be artifacts of particular eras such as the depression of the 1930's or the high-tech bubble of the late 1990's. The analysis of daily data minimizes such possibilities. Furthermore, using data involving 1000 trading days, for example, instead of 20 yearly data points enhances the statistical power of the tests.

We discuss the variables utilized in the regressions.

2.2.2 Definitions

Relative Price Change

The basic quantity of interest is the Relative Price Change that we define as

$$R(t) = \frac{P(t) - P(t-1)}{P(t-1)},$$

which is sometimes called the "return" for day t . The Relative Price Change for day $t + 1$, namely $R(t + 1)$, will be used as the dependent variable in the regressions.

With the exception of the M2 Money Supply, the following variables, which are utilized as independent variables in the regressions, are based upon the above mentioned daily close prices and NAVs.

Valuation

As noted above, although the value (NAV) of a closed-end fund is known, it seldom trades at that price. In fact, these funds may trade at a persistent premium (price exceeds NAV) or discount (price is below NAV). So, if a fund has been consistently trading at a discount of 10% and subsequently trades at a 5% discount, value investors are not likely to regard it as a bargain despite the discount. Thus, we define the relative premium (or discount) as $\frac{NAV(t)-P(t)}{NAV(t)}$, and then subtract the weighted average of the relative premium/discount over the past 10 days from this value as

$$D(t) = \frac{NAV(t) - P(t)}{NAV(t)} - \frac{1}{3.2318} \sum_{k=1}^{10} \frac{NAV(t-k) - P(t-k)}{NAV(t-k)} e^{-0.25k},$$

where $NAV(t)$ is the net asset value of the fund on day t and $P(t)$ is the fund's share price on day t . The $e^{-0.25k}$ factor is a smoothing factor that (i) emphasizes more recent deviations¹ between the NAV and share price and (ii) gradually reduces the impact of large deviations in the past. Large relative deviations between NAV and price (that occur with nontrivial frequency) will therefore not drop out of this Valuation variable abruptly, but rather fade slowly from the equation. The coefficient $(3.2318)^{-1} = (\sum_{k=1}^{10} e^{-0.25k})^{-1}$ is used to normalize the variable.

Price Trend

¹Theoretical studies have used the concept of a declining exponential in gauging investor sentiment and incorporated it into differential equations models (see Chapter 3 and references therein, specifically [6]). There is experimental evidence [28] that individuals tend to emphasize recent events more heavily than earlier events in their decisions.

As noted in Section 2.1, one possible motivation for buying a stock is that the price is in an uptrend (and analogously for selling). Since prices are fluctuating and changing direction frequently, the definition requires a choice of time scale, which we take as ten days, and a scaling factor that determines the weightings of recent days relative to earlier days. We use the exponential factor $e^{-0.25k}$ for the weighting of the relative price change k days ago, as in the Valuation variable above. Thus with the normalization factor above, we define the Price Trend as

$$T(t) = \frac{1}{3.2318} \sum_{k=1}^{10} \frac{P(t-k+1) - P(t-k)}{P(t-k)} e^{-0.25k}.$$

For both the Price Trend and the Valuation variables, tests of robustness have shown similar results with a longer time scale (e.g., 25 days) and different weighting factors [12].

Money Supply

Previous theoretical [6], experimental ([15] and [16]), and empirical [12] studies have shown that an increase in the money supply bolsters asset prices. We obtain the weekly M2 money data² (not seasonally adjusted) for the time period of study from the Federal Reserve website [www.federalreserve.gov/releases/h6/hist/]. We then performed a linear interpolation to obtain daily data. The M2 Money Supply variable is defined as the relative change in this statistic on a daily basis, i.e.,

$$M2(t) = \frac{M(t) - M(t-1)}{M(t-1)}.$$

Volatility

The volatility in the model is computed as the standard deviation of the relative price change, $R(t) = \frac{P(t) - P(t-1)}{P(t-1)}$, for the past X number of days including the current day. If t represents today, then we define today's volatility as the standard deviation of the relative price change values over the past $X + 1$ days, i.e., $Volatility(t) = \{variance(R[t - X, t])\}^{1/2}$ where $R[t - X, t]$ represents a column vector containing the relative price change values over

²M2 includes: Currency, Traveler's checks, demand and other checkable deposits, retail MMMFs (money market mutual funds), savings, and small time deposits. M2 is measured in trillions for this paper.

the past $X + 1$ days. We set $X = 10$ for Short Term Volatility or $X = 251$ for 52 Week Volatility, i.e., we compute the standard deviation of the past 11 (or 252) days (including today). We use the unbiased estimator of the variance: $\frac{1}{X} \sum_{i=t}^{t-X} (R(i) - \text{Mean}(R[t - X, t]))^2$.

This definition determines the deviation of the relative price change about the growth curve of the share price. If the relative price change is constant, then the price is an exponential function of time. Indeed, representing the relative price change in a limiting form such as $\frac{1}{P} \frac{dP}{dt}$ yields the differential equation $\frac{1}{P} \frac{dP}{dt} = C$ which implies $P(t) = K e^{Ct}$ via separation of variables where K and C are constants. For example, if the stock price follows the pattern $e^{0.02t}$ (i.e., $K = 1$ and $C = 0.02$), then the relative price change would be constant ($e^{0.02} - 1 \simeq 0.02$) and the Volatility would be zero. Thus, with the above definition the trend in price would not contribute to the volatility.

52 Week Price Trend

As with the Volatility variable, we are not only interested in the short term price trend, but also the longer term trend. As such, we determine the 52 Week Price Trend variable as follows: (i) fit a straight line to the past 252 Relative Price Change values (including the current day t), i.e., $R([t - 251, t])$; (ii) take the slope of this line and multiply it by 252 for conversion to annual units. The resulting value is then denoted $LT(t)$.

Volume Trend

We incorporate the trading volume for each fund into the model by considering the trend of the volume in the same manner as the Price Trend. The Volume Trend, $VT(t)$, is defined as

$$VT(t) = \frac{1}{3.2318} \sum_{k=1}^{10} \frac{Vol(t - k + 1) - Vol(t - k)}{Vol(t - k)} e^{-0.25k}$$

where $Vol(t)$ represents the trading volume³ for day t .

Time

³Note that the Volume may be zero on certain days. This would cause the Volume Trend to be infinite on those days. As such, any records in the data set with a volume of zero are excluded for Study 1. For Study 2, any fund with a zero Volume record was excluded from the data set.

To ensure the regression results are not artifacts of a specific time period, a historical time variable is included as an independent regression variable. This time variable, $Time(t)$, is defined as the approximate number of months since October 26, 1998 (the earliest date included in the data set). More precisely, this variable is calculated by determining the number of days from date t to October 26, 1998, dividing this number by the average number of working days per month (i.e., $\frac{252}{12} = 21$), and then rounding down to the closest integer (i.e., if the resulting number is 15.8, then the variable is assigned a value of 15). Finally, the variable is normalized by dividing by the total approximate number of months in the data set, 143 (this number corresponds to Regression 5 which includes this variable). Also, $Time^2(t)$ and $Time^3(t)$ are considered to determine the nature of the relationship (e.g., linear, quadratic, or cubic).

Resistance

The possibility that stock prices tend to decline after approaching a recent (in this case quarterly) high is called Resistance which we identify by an indicator variable and use as an independent variable in a regression. The recent quarterly high is defined by $H(t) := \max \{P(s)\}$ for $s \in [t - 63, t - 16]$. The Resistance Indicator, $Q(t)$, is set if the following conditions are satisfied: (1) for s in $[t - 15, t - 10]$, $P(s) \leq 0.85H(t)$ and (2) $0.85H(t) \leq P(t) \leq H(t)$ (note that there is no condition on $P(s)$ for s in $[t - 9, t - 1]$). Thus, we interpret Resistance as having occurred on day t if the share price on day t is between 85 – 100% of the recent quarterly high.

2.2.3 Discussion of Variables

The variables defined above are chosen due to their role in theory, investment practice or experimental settings, rather than as a consequence of data-mining. We discuss below the motivation for the inclusion of several of these variables.

Price Trend

The influence of price trend on future price changes has been of interest from many perspectives. Traders often express their belief in momentum. For example the phrases "the

trend is your friend" or "don't fight the tape" are old sayings on Wall Street. The apparent persistence of a trend provides some support for the hypothesis of underreaction. Yet the hypothesis of overreaction also receives much attention. A quantitative methodology for distinguishing these two competing motivations is necessary to transform these philosophical ideas into finance. We are able to accomplish this in several ways using a single set of large data.

Various theories such as the "affect heuristic" have suggested that prices rise excessively as investors focus on a salient feature of a company or its product [45]. As more investors are attracted to the stock, one expects a positive trend term in the short run, consistent with our findings. However, when viewed on a longer time scale, it is clear that, at some point, the fundamentals will become more evident, resulting in a return to more modest prices. This perspective can be regarded as a basis for overreaction on a longer time horizon, and has some statistical backing from the DeBondt and Thaler [19] study. Our results, which encompass the effects of both long and short term trends, are consistent with this study and with that of Poterba and Summers [41].

Money Supply

When applied to any consumer good, the law of supply and demand clearly stipulates that an increase in demand will raise prices. An investment vehicle differs from a consumer good in that there is typically no consumer at the end of the trading chain, and purchase of the asset is solely for the purpose of re-selling at a higher price (or obtaining a stream of dividends). As such, it is not clear that an increase in the money available for possible investment in that asset (analogous to demand for a consumer good) will lead to higher asset prices, as it would for a consumer good. Furthermore, the methods of classical finance (e.g., the no arbitrage hypothesis) would cast doubt on the concept that a larger money supply should lead to higher prices. However, experimental asset markets have provided considerable evidence [15] that a larger ratio of cash to asset leads to higher prices. Mathematical models [6] have indicated that a greater cash supply leads to higher prices. There is also a belief in some investment circles that "cheap money fuels market prices," so that when there is an increase in money supply the increase in available investment funds tends to push up prices.

Volatility

A basic idea in finance is that the risk and reward are inversely correlated so that investors seeking higher return must tolerate higher risk [5]. In the classical literature, risk is identified with volatility in the asset's price. The measure of volatility, however, depends crucially on the time scale for measurement. We consider both short term and long term volatility. Volatility has been studied by other researchers (see [21] and the references therein).

Volume

As with the price trend, volume provides an indication to traders about the beliefs and resources of other traders. In the process of price discovery, a trader who believes that the asset is undervalued will nevertheless strive to purchase it at the lowest possible price. Since he does not know the strategies and assessments of the other traders, his only recourse is to examine the earliest manifestations of their trading decisions which are exhibited in the changes in price and the volume of trading. Rising prices on low volume provide an indication to traders that the buying interest is not as strong as one might believe by examining price changes alone.

Resistance

A concept that is frequently used by traders involves resistance, or the tendency for prices to pull back when approaching a yearly or recent high. Traders are then aware that all other traders know that a higher price could not be attained during that time period. This could be viewed as a manifestation of "anchoring" whereby observers focus on a particular value and neglect to consider other possibilities. Traders have explained the concept of resistance by stating that investors who held the stock through the recent high may experience regret at not having reaped a profit. Thus, as the shares again approach this recent high, these investors seek to recover their perceived "losses" by selling; thereby lowering the stock price and preventing it from breaking through this price barrier. A yearly high, for example, also provides information to traders by setting a price at which supply and demand for the asset were equal after a period of rising prices. Recently, the concept of resistance has received academic attention in a study that indicates that the yearly return is influenced by the stock's

proximity to the yearly high [27] though another study, [48], demonstrates some limitations of these findings and obtains more mixed results.⁴

2.3 STUDY 1

2.3.1 Objectives

The objective of this study is to provide quantitative answers to the questions of whether and to what extent investors are influenced by the following factors: (a) recent price trend, (b) recent valuation, (c) changes in money supply, (d) volatility, (e) long term price trend, (f) recent changes in volume, (g) resistance (i.e., proximity to yearly highs), and (h) effects of time (during the data period). A primary goal of this study is to identify when over- and underreaction occur and to determine a quantitative methodology for distinguishing between the two using nonlinearity in the price trend. Also, we are able to confirm reversion to the mean, a Poterba and Summers [41] finding, by including the longer term trend. The claim that an influx of money bolsters trading prices is supported by experimental [15], theoretical [6], and empirical [12] studies. Thus, it is reasonable to incorporate a variable which represents the amount of available money. Volatility is identified with risk in classical finance. As such, we consider this factor in order to test the effect of volatility on trading prices. Traders often observe the volume, and believe that an uptrend is more likely to continue if it is accompanied by rising volume, and that gradual price rise with declining volume is a sign that the uptrend is likely to falter. Smith, Suchanek, and Williams [46] noted that bids (and thereby volume) tend to diminish shortly before the peak of an experimental bubble. Resistance is included to test the relationship between the current price and a recent high price. The effect of time is considered to ensure our results are not the artifacts of a particular time period. For a more detailed discussion of the motivation behind the inclusion

⁴The Resistance variable can be explored more fully by augmenting this variable within the context of our methodology. In particular, Sturm [48] finds that investors are influenced differently based on how recently the high price was attained (i.e., last month, two months ago, three months ago, etc.). Thus, one could define a variable(s) representing this time difference and include it as an additional independent variable in our regressions.

of these factors, see Section 2.2.3.

Our basic approach is to determine the fractional change in the price of a stock as a regression on a set of variables, including an appropriate valuation variable that effectively subtracts out much of the stochastic noise due to changes in valuation. The set of variables is discussed in Sections 2.2.2 and 2.2.3. The definitions of the valuation, recent price trend, and volume trend variables include a weighting so that the most recent changes have the greatest effect. In addition to the use of related terms in mathematical modeling [6] there is both experimental [28] and empirical [47] evidence that individuals tend to emphasize recent events more heavily than earlier events in their decisions. Furthermore, we consider both the square and cubic price change terms, and obtain a non-linear function indicating that a daily gain of up to 2.78% tends to yield higher prices, but larger gains lead to lower prices. The analogous crossover point for price drops is 2.1%. There is a negative coefficient of smaller magnitude for the long term trend, consistent with mean reversion on this time scale. Further evidence that traders are influenced by the evidence of others' strategies through price patterns is the fact that the price representing the recent quarterly high is associated with a slightly negative coefficient, providing "resistance" to a stock's upward movement. Positive changes in the money supply are associated with increases in the fractional price, consistent with a liquidity (excess cash) perspective of markets ([15] and [16]).

Our regressions indicate an ambiguous role for volatility, indicating that the classical concept of associating volatility with risk is more complicated. In fact, the sign is positive for the short term, suggesting that volatility has a tendency to boost trading prices. However, in the longer term volatility is a negative factor. The recent change in volume is a slightly positive factor, indicating that positive (negative) changes in volume correspond to positive (negative) changes in price. Finally, we utilize a time variable defined as the number of months since the earliest data point. By using regressions that include up to the cubic power in this variable, we can largely extract and eliminate any effects due to a particular time period. The Price Trend, Valuation, M2 Money Supply, Short Term Volatility, and Volume Trend coefficients are essentially unchanged upon introduction of this variable, suggesting that our results for these factors are not artifacts of the changes during this time period.

This study's primary contributions to the literature are summarized by the following:

(1) Identification of the nonlinearity in the price trend provides a method for distinguishing between over- and underreaction. Without such a method it is difficult to argue that a systematic bias exists or that the study of these concepts would be helpful for trading.

(2) The key concept behind these regressions is the inclusion of the Valuation as an independent variable. By including in the regression a function based on the NAV, a significant portion of the price's volatility is eliminated; thereby, removing (or explaining) a great deal of "noise" from the data. Thus, the effects of other factors are no longer camouflaged by the valuation.

(3) The M2 money supply is also a significant factor in price changes, which supports the findings of theoretical, experimental and empirical studies.

(4) The volatility results are quite intriguing as short term volatility actually boosts prices while long term volatility has a small negative effect on price changes.

2.3.2 Methodology

To determine the influence of variables defined above on investor decisions, we consider their effect on the relative price change. This is accomplished by performing regressions with the next day's relative price change, $R(t+1) = \frac{P(t+1)-P(t)}{P(t)}$, as the dependent variable and various subsets of the above factors (using the data of day t and earlier) as the independent variables. For each variable the regression provides a coefficient and its corresponding statistical significance (t- and p-values). The relative changes in the above variables can be adjusted for the fact that some variables have a much larger range than others in order to facilitate comparison of the magnitudes. For example, the average of the absolute value of the Price Trend variable is 0.003698796, while the absolute value of the Volume Trend average is 0.248292 or approximately 67 times larger. To facilitate comparisons of coefficients, the Volume Trend coefficient should be multiplied by a factor of 67. Note that this scaling is essentially a matter of convenience from our perspective, as it does not alter the statistical significance indicators. Whether scaled or not, one can use the computed coefficient of the regression multiplied with the average magnitude of the variable to determine whether it is large enough to be of value to trading or investing (see Table 2.4).

Our data set consists of closing prices from different funds. An ordinary linear regression is an example of a classical modeling technique that assumes observations, in our case closing prices, are independent and identically distributed. However, as our data set consists of closing prices from different funds, this assumption is not necessarily valid. Thus, we need to utilize a method that accounts for the variation within funds and between different funds. To accomplish this, we utilize a mixed-effects model, i.e., a model with both fixed and random effects. A fixed effect is a parameter "associated with an entire population or with certain repeatable levels of experimental factors", while random effects "are associated with individual experimental units drawn at random from a population" [38]. As we wish to draw inferences regarding all closed-end funds based upon our sample, fund is utilized as a random effect in our model, while the other independent variables (see Section 2.2.2) are fixed effects. While an ordinary linear regression would estimate parameters using expected mean squares, our mixed-effects regression utilizes the restricted maximum likelihood method to obtain unbiased estimates of the variance components of the random effects. Thus this procedure implements a methodology that accounts for the grouped (by fund) data, while still providing coefficients and significance values that are representative of the entire data set.

We execute linear regressions of the form:

$$R(t+1) = \alpha_0 + \alpha_1 \text{Variable}_1(t) + \alpha_2 \text{Variable}_2(t) + \dots + \alpha_n \text{Variable}_n(t)$$

where t represents the current day, then $R(t+1) = \frac{P(t+1)-P(t)}{P(t)}$ represents the following day's relative price change, or return, and $\text{Variable}_i(t)$ represents one of the above independent variables. Each regression includes an intercept term (α_0) which may be interpreted as the "drift" term of classical finance. Retaining this intercept term in the regressions adjusts for the average return so that non-zero coefficients for any of the variables demonstrate "abnormal return," i.e., return beyond the small daily average return of a stock within a particular class.

2.3.3 Results

2.3.3.1 Linear Regressions Regression 1. As a baseline, we perform a linear mixed-effects regression with only the independent variable Price Trend. The regression has the form:

$$R(t + 1) = \alpha_0 + \alpha_1 T(t).$$

Results may be found in Table⁵ 2.1.

Table 2.1: Regression 1 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.00043484	0.000039850	10.91202	<.0001
Price Trend	-0.03787525	0.007761571	-4.87984	<.0001

Observations: 117,760; Groups: 125; Degrees of Freedom: 117,634

Although Price Trend is statistically significant, its value is negative and small. In quantitative terms a 1% per day increase during the recent time period, on average, yields a 0.04% decrease today. A significant, small (and most likely not tradeable) Price Trend term is consistent with the Poterba and Summers [41] findings. However, the sign is negative; whereas, Poterba and Summers found a positive trend term.

Regression 2. This baseline regression above is a standard treatment of such data, and as such, does not circumvent the key issue of "noise" due to changes in valuation (i.e., fundamentals). The inclusion of our Valuation variable in the regression above removes

⁵We list the number of observations, groups, and degrees of freedom for each regression, because these may vary depending upon the included independent variables. The calculation of the Price Trend, Valuation, and Volume Trend variables requires the previous ten days' data. Thus, these variables cannot be computed for the first ten days for each fund. Accordingly, the first ten days of data are excluded from the regressions. Similarly, the first 63 days of data for each fund are excluded due to the Resistance and Breakout variables, while the first 251 records of each fund from the regression data set are ignored due to the formulas for 52 Week Volatility and 52 Week Price Trend.

much of this noise and dramatically identifies the role of Price Trend as shown below for the regression:

$$R(t + 1) = \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t).$$

Table 2.2: Regression 2 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.0004893	0.000039393	12.42070	<.0001
Price Trend	0.2270024	0.009145194	24.82204	<.0001
Valuation	0.1988558	0.003739072	53.18320	<.0001

Observations: 117,760; Groups: 125; Degrees of Freedom: 117,633

The results of this regression (see Table 2.2) show that by "subtracting out" the valuation, i.e., including the Valuation in the regression, the Price Trend is not only statistically significant but has a positive coefficient comparable in magnitude to the Valuation coefficient.

Regression 3. The regression above shows the significance of the Price Trend variable. To explore the nonlinearities in the relationship between Price Trend and Relative Price Change, the quadratic and cubic factors of the Price Trend and Valuation variables are added to Regression 2. In addition, the cross terms, e.g., Price Trend multiplied by Valuation, are also included. This regression has the form:

$$R(t + 1) = \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t) + \alpha_3 T^2(t) + \alpha_4 T^3(t) + \alpha_5 D^2(t) \\ + \alpha_6 D^3(t) + \alpha_7 T(t)D(t) + \alpha_8 T^2(t)D(t) + \alpha_9 T(t)D^2(t)$$

Table 2.3: Regression 3 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.0002	0.00004	5.61263	<.0001
Price Trend	0.3581	0.01085	33.00490	<.0001
Valuation	0.2304	0.00422	54.62312	<.0001
(Price Trend) ²	3.8716	0.88137	4.39271	<.0001
(Price Trend) ³	-610.7741	40.74653	-14.98960	<.0001
(Valuation) ²	0.3529	0.05692	6.19928	<.0001
(Valuation) ³	-1.3587	0.11411	-11.90712	<.0001
(Price Trend)(Valuation)	-1.2234	0.44093	-2.77467	0.0055
(Price Trend) ² (Valuation)	-139.8894	27.18508	-5.14581	<.0001
(Price Trend)(Valuation) ²	9.3560	5.39400	1.73452	0.0828

Observations: 117,760; Groups: 125; Degrees of Freedom: 117,626

The inclusion of the additional terms does not appear to affect the significance of the Price Trend and Valuation terms; however, it does increase the magnitudes of their respective coefficients (see Table 2.3). The relationship between Price Trend and the Relative Price Change can be visualized using the three dimensional graph (see Figure 2.1) of Relative Price Change, Valuation, and Price Trend with the coefficients obtained from the regression above:

$$R(T, D) = 0.0002 + 0.3581T + 0.2304D + 3.8716T^2 - 610.7741T^3 \\ + 0.3529D^2 - 1.3587D^3 - 1.2234TD - 139.8894T^2D + 9.3560TD^2$$

with T representing Price Trend and D representing Valuation.

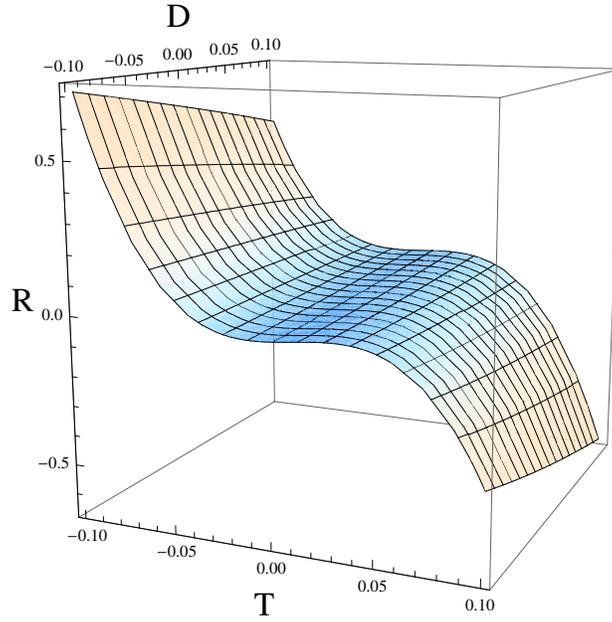


Figure 2.1: The coefficients of the Price Trend variable, T , and the Valuation variable, D , together with all terms up to cubic order (namely, D , T , D^2 , T^2 , DT , D^3 , T^3 , D^2T , DT^2) are used as the independent variables in a regression for the Relative Price Change, R . The coefficients of these terms define a cubic polynomial in D and T that is plotted above. The surface describes the effect on the Relative Price Change on the following day, exhibiting the nonlinear relationship between D , T and R . In particular, a positive trend that is large can influence the following day's Relative Price Change in the opposite direction (analogously for negative trend). The precise point at which the magnitude changes sign depends nonlinearly on the valuation.

From this three dimensional surface, it is clear that the relationship between the Relative Price Change and Price Trend is highly nonlinear. Taking the cross-section of this surface when the Valuation is zero yields the function

$$R(T, 0) = 0.0002 + 0.3581T + 3.8716T^2 - 610.7741T^3$$

plotted in Figure 2.2 below.

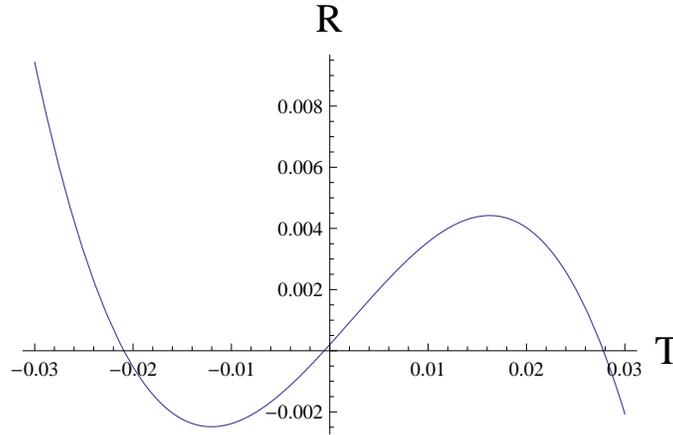


Figure 2.2: The cross-section of the surface in Figure 2.1 obtained by setting the Valuation variable, D , to zero is plotted with the Price Trend variable, T , on the horizontal axis and the Relative Price Change variable, R , on the vertical. The coefficients of the regression terms define a cubic polynomial in D and T that is plotted above. The T intercepts of the above graph are -0.021, -0.000562, and 0.0278 indicating that if the Price Trend is between -2.1 and 2.78, then the Relative Price Change and Price Trend have the same sign. Thus, smaller changes in the Price Trend, tend to push the Relative Price Change in the same direction. However, if the Price Trend is less than -2.1 or greater than 2.78, then tomorrow's price change is more likely to be opposite today's.

This regression involving nonlinear terms shows the complex nature of underreaction and overreaction. Small changes in price tend to continue, indicating that there is underreaction, while large changes tend to be reversed, indicating overreaction. One of the problems in utilizing the concepts of underreaction and overreaction has been the difficulty in distinguishing between the two. In any particular situation, if one cannot determine in advance, using a scientific method, whether there will be overreaction or underreaction, then an efficient market advocate can claim the market is free of bias. A methodology to delineate between these concepts can render them into practical tools rather than philosophical insights.

Note that if the terms with higher (> 1) powers of the Valuation variable are excluded from the regression, then there is still a cubic relationship between Price Trend and Relative

Price Change. However, instead of a change greater than 2.78% producing a negative Relative Price Change, the necessary change is 8.37%, which is not as significant in practical terms. Inclusion of the additional Valuation terms provides a more complete picture of the dynamics between Price Trend and Relative Price Change.

Regression 4. This regression examines the effects of the M2 Money Supply (M2), Short Term Volatility (STVol), 52 Week Volatility (Vol), 52 Week (Long Term) Price Trend (LTT), and Volume Trend (VT) (in addition to the Valuation and Price Trend) on the Relative Price Change⁶:

$$R(t+1) = \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t) + \alpha_3 M2(t) + \alpha_4 STVol(t) \\ + \alpha_5 Vol(t) + \alpha_6 LTT(t) + \alpha_7 VT(t).$$

Table 2.4 shows that the Price Trend and Valuation variables are still statistically significant and of approximately the same magnitude with slightly smaller positive coefficients than in Regression 3, while the Intercept term is only marginally significant. The M2 Money Supply has a positive coefficient (t-value of 9.5). This confirms the experimental findings in [15] that the money supply is a significant factor in the price change and that an infusion of money into the market should cause prices to rise.

The Short Term Volatility coefficient is statistically significant and comparable in magnitude (when scaled with respect to the average magnitude, as discussed above) to the Price Trend, Valuation, and M2 Money Supply variables. The Short Term Volatility typically contributes 0.000842 (the product of the coefficient with the average magnitude of the volatility) compared with 0.0017 for the Valuation.

⁶Note that as previously mentioned, with the inclusion of the 52 Week Volatility, 52 Week Price Trend, and Volume Trend more records are excluded from each fund. The total number of records used is decreased from 117,760 to 88,127 and the number of funds dropped from 125 to 114. The 11 excluded funds had fewer than 253 days' data.

Table 2.4: Regression 4 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.000099	0.000124	0.80097	0.4231
Price Trend	0.234642	0.010836	21.65396	<.0001
Valuation	0.206172	0.004232	48.71275	<.0001
M2 Money Supply	0.385010	0.040603	9.48238	<.0001
Short Term Volatility	0.071211	0.008063	8.83150	<.0001
52 Week Volatility	-0.043829	0.011562	-3.79074	0.0002
52 Week Price Trend	-0.031533	0.019147	-1.64683	0.0996
Volume Trend	0.000267	0.000087	3.08137	0.0021

<i>Term</i>	<i>Value</i>	Mean(<i>Var</i>)	(<i>Value</i>)(Mean(<i>Var</i>))
<i>(Intercept)</i>	0.000099	N/A	N/A
Price Trend	0.234642	0.003698796	0.000868
Valuation	0.206172	0.008276127	0.001706
M2 Money Supply	0.385010	0.0008950968	0.000345
Short Term Volatility	0.071211	0.01183092	0.000842
52 Week Volatility	-0.043829	0.0121939	-0.000534
52 Week Price Trend	-0.031533	0.001934814	-0.000061
Volume Trend	0.000267	0.248292	0.000066

Observations: 88,127; Groups: 114; Degrees of Freedom: 88,006

In addition, it is surprising to note that the coefficient of the Short Term Volatility is positive, indicating that a short uptrend that has high variance leads to higher prices than a steady uptrend. This may indicate that some large spikes in prices tend to attract the attention of buyers, and analogously, sharp drops tend to induce more selling compared with the same magnitude of change that is more evenly distributed.

In contrast to this evidence that Short Term Volatility actually boosts the Relative Price Change, we find that the coefficient of the 52 Week Volatility is negative and statistically

significant (t-value of -3.79), indicating that an increase in the annual volatility of a fund forces the Relative Price Change down. Thus, while short term volatility does not appear to discourage buying, we find evidence that longer term volatility does make funds less attractive to investors.

The 52 Week Price Trend is marginally significant with a t-value of -1.64 yielding limited support for mean reversion over longer time periods as noted by Poterba and Summers [41] using different methods. It is also consistent with the findings of DeBondt and Thaler [19] who found that those portfolios that performed poorly the previous year tended to outperform the market on average the following year, and vice versa.

We find evidence that the Volume Trend is statistically significant. Its t-value of 3.08 implies strong statistical support for the positivity of this coefficient. This is consistent with trader beliefs that rising volume in an uptrend is a positive sign for the direction of prices. However, the impact as measured in the product of the coefficient with the average magnitude of the variable is one order of magnitude smaller than for most of the other variables.

Regression 5. In any regression spanning several years there is the possibility that the results are influenced by events or characteristics of a particular era or time period. For example, momentum trading may have been popular when the market was rising. In order to discount this possibility we include the time variables (up to third order) in the list of variables in Regression 4. These variables represent the number of months, the square of the number of months, and the cube of the number of months since October 26, 1998. In this way if prices are rising then falling and rising again during the time period considered, the cubic polynomial in time generated by the regression will account for this. For example, if the Price Trend coefficient is entirely due to this time issue, then the coefficient of Price Trend in this new regression would be statistically zero. Hence, this new regression is of the form:

$$\begin{aligned}
R(t + 1) = & \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t) + \alpha_3 M2(t) + \alpha_4 STVol(t) \\
& + \alpha_5 Vol(t) + \alpha_6 LTT(t) + \alpha_7 VT(t) + \alpha_8 Time(t) \\
& + \alpha_9 Time^2(t) + \alpha_{10} Time^3(t).
\end{aligned}$$

Table 2.5: Regression 5 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.00015	0.000371	0.41238	0.6801
Price Trend	0.22643	0.010867	20.83641	<.0001
Valuation	0.20262	0.004247	47.70291	<.0001
M2 Money Supply	0.38763	0.040588	9.55020	<.0001
Short Term Volatility	0.08290	0.008206	10.10215	<.0001
52 Week Volatility	-0.02654	0.012917	-2.05492	0.0399
52 Week Price Trend	-0.06216	0.019528	-3.18314	0.0015
Volume Trend	0.00025	0.000087	2.81859	0.0048
Time	-0.00961	0.002518	-3.81704	0.0001
(Time) ²	0.02821	0.005146	5.48264	<.0001
(Time) ³	-0.02011	0.003068	-6.55358	<.0001

Observations: 88,127; Groups: 114; Degrees of Freedom: 88,003

All three Time terms are statistically significant (see Table 2.5). In addition, the absolute value of the relative percentage changes of the Price Trend, Valuation, M2 Money Supply, and Volume Trend coefficients from Regression 4 to Regression 5 are less than 6.5%, while the percentage change of the Short Term Volatility is 16%. This leads to the conclusion that our results for these variables are not significantly influenced by the particular time period included in this study. The magnitude of the relative percentage changes for the 52 Week

Volatility and 52 Week Price Trend are 40% and 97%, respectively. Since these variables involve data for an entire year, they are most strongly influenced by inclusion of the time variables.

To determine whether the hypothesized phenomenon of resistance is statistically significant we perform two regressions below, the first involving only the basic two variables, the second involving all of the factors we have found to be significant above.

Regression 6. This regression includes the Resistance along with the Price Trend and Valuation as independent variables. It has the form:

$$R(t + 1) = \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t) + \alpha_3 Q(t).$$

Table 2.6: Regression 6 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.0005491	0.000040813	13.45485	<.0001
Price Trend	0.2248558	0.009401159	23.91788	<.0001
Valuation	0.2021202	0.003837561	52.66893	<.0001
Resistance	-0.0011975	0.000520305	-2.30156	0.0214

Observations: 111,135; Groups: 125; Degrees of Freedom: 111,007

Although the Resistance variable appears to be small in magnitude (see Table 2.6), its impact is comparable to the other variables since the indicator variable is 1 when the criteria are met compared with relatively small magnitudes for the other variables (see discussion after Regression 4). Since the criteria for resistance are met infrequently compared to all data points (namely 687 of 111,135) one does not obtain the overwhelming p-values as in the other variables. However, one still has 98% confidence that the coefficient is negative.

Regression 7. Augmenting Regression 6 with the remaining independent variables (M2 Money Supply, Short Term Volatility, Long Term Volatility, Long Term Price Trend, and Volume Trend) yields a regression of the form:

$$R(t+1) = \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t) + \alpha_3 M2(t) + \alpha_4 STVol(t) + \alpha_5 Vol(t) \\ + \alpha_6 LTT(t) + \alpha_7 VT(t) + \alpha_8 Q(t).$$

Table 2.7: Regression 7 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.0000844	0.00012428	0.67890	0.4972
Price Trend	0.2357864	0.01084270	21.74610	<.0001
Valuation	0.2060078	0.00423260	48.67172	<.0001
M2 Money Supply	0.3852189	0.04060101	9.48791	<.0001
Short Term Volatility	0.0717047	0.00806476	8.89111	<.0001
52 Week Volatility	-0.0427853	0.01156665	-3.69902	0.0002
52 Week Price Trend	-0.0308970	0.01914791	-1.61360	0.1066
Volume Trend	0.0002703	0.00008667	3.11880	0.0018
Resistance	-0.0038353	0.00132225	-2.90056	0.0037

Observations: 88,127; Groups: 114; Degrees of Freedom: 88,005

Table 2.7 shows that the statistical significance of the Resistance variable increases from Regression 6 to Regression 7 (p-value increases from 0.0214 to 0.0037) and remains negative (t-value of -2.90056). In addition, the magnitude of the coefficient increases three-fold from Regression 6 to Regression 7. The fact that the significance of Resistance in Regression 7 is much stronger than in Regression 6 demonstrates the importance of including variables that are known to have an effect on the dependent variable. The percentage relative changes

in the other coefficients (excluding Intercept) from Regression 4 (which does not include Resistance) to Regression 7 are less than 2.4%.

Note that the inclusion of the Time variables has little effect on the Resistance (coefficient of -0.0039525 with p-value equal to 0.0028) with negligible differences between the results for the other variables and Regression 5. While the p-value of 0.0037 does not yield the same overwhelming degree of confidence as with some of the other variables, it is nevertheless very significant, attaining the 99.5% level. The reason for the difference (e.g. 3 standard deviations for Resistance instead of 21 for Price Trend) is probably attributable to the fact that only 687 points satisfied our criteria for Resistance. The impact of Resistance, when it does occur is large, as one can see by multiplying the coefficient, -0.0038353 by the value (namely, 1) when the criteria are met. Comparing this with the analogous product for valuation, namely, 0.001706, we see that Resistance asserts a negative influence that is twice as large as a typical positive Valuation change. Stated otherwise, if one is within the Resistance criterion, as we have defined it, the Valuation variable needs to be twice the typical magnitude in the positive direction just to neutralize the effect of Resistance. Valuation aside, the Price Trend needs to be 4.4 times the average magnitude of 0.000868 in order to counteract the Resistance.

The inclusion of the same variable(s) in various regressions can be viewed as a test of robustness. As demonstrated by the results above, once the Valuation variable has been added to the regression, the coefficients and significance values of the other independent variables do not vary significantly from one regression to another.

In analyzing daily closing prices there is always the issue of the bid/ask spread at the end of the trading day. In other words, the close may occur at either the asking price or the bidding price. This tends to introduce some noise into the analysis of trading prices. However, the large statistical significance attained in our linear regressions suggests that this randomness is not a dominating factor. Moreover, if there is a non-random bias for the closing price to be at the asking price, for example, under particular conditions, then it has the same effect as rising prices. The bid/ask spread is usually not very significant for active stocks, as it is often about one cent for a \$30 stock so it is an effect that is about $(3,000)^{-1}$ but could be much larger for less active stocks.

2.3.3.2 Forecasting One advantage of a linear regression is that if it models the dynamics of the dependent variable well, it can be utilized to obtain predictions of that variable's behavior. The S-Plus statistics package provides forecasting functionality. As the entire data set was used to obtain the above results, we perform an in-sample forecast utilizing Regression 4. This forecast predicts the Relative Price Change for the 88,127 data points included in the regression. We compare the sign (i.e., positive or negative) of the predicted Relative Price Change with that of the actual observed Relative Price Change and find that the model successfully predicts the sign of the Relative Price Change 57% of the time (50,641 correctly predicted signs out of 88,127 observations). This provides some evidence that the variables used in this regression play a key role in market price dynamics. Another approach to forecasting using trend and valuation has been implemented in Duran and Caginalp [23] where the coefficients of these terms are optimized using market data.

2.3.3.3 Regression Per Fund We perform a linear regression individually for each of the 125 funds with dependent variable Relative Price Change and the independent variables from Regression 4. A regression is of the form:

$$R(t + 1) = \alpha_0 + \alpha_1 T(t) + \alpha_2 D(t) + \alpha_3 M2(t) + \alpha_4 STVol(t) \\ + \alpha_5 Vol(t) + \alpha_6 LTT(t) + \alpha_7 VT(t).$$

We found that only 17 of the 125 funds have a negative Price Trend coefficient, supporting the above evidence of a positive trend effect. Of these 17 funds, five are related to the Energy industry. In addition, these individual regressions confirm the mixed-effects linear regression results and show that the coefficients are not distorted by a small number of funds. The average values of the coefficients (see Table 2.8) are close to the Regression 4 results for all variables except the 52 Week Volatility, 52 Week Price Trend, and Volume Trend. Thus, it appears that these variables are more dependent upon the characteristics of the individual funds than the others.

Table 2.8: Comparison of individual funds to entire population. Ordinary linear regressions were run on each individual fund. The coefficient values for the short and long term Price Trend, Valuation, M2 Money Supply, short and long term Volatility, and Volume Trend were averaged over all of the funds. These average values are compared to the corresponding coefficient values resulting from the mixed-effects linear regression run for all funds.

Term	Price Trend	Valuation	M2 Money Supply
Average Value	0.298753	0.227028	0.26835
Regression 4 Coefficient	0.234642	0.206172	0.38501
Term	Short Term Volatility	52 Week Volatility	
Average Value	0.086857	1.6558247	
Regression 4 Coefficient	0.071211	-0.043829	
Term	52 Week Price Trend	Volume Trend	
Average Value	-1.248058	0.003503	
Regression 4 Coefficient	-0.031533	0.000267	

2.3.4 Conclusion

Using a methodology that compensates for the randomness of changing valuation, we find strong evidence that the Price Trend, M2 Money Supply, Volatility, and Volume Trend influence investor decisions. Unlike some previous studies, the near elimination of the noise associated with changing fundamentals yields results that are significant in magnitude. Of particular interest, we find that the dependence of (today's) Relative Price Change on (yesterday's) Price Trend is nonlinear, thus supporting both of the ideas of underreaction and overreaction, in a way that distinguishes between them quantitatively. Roughly speaking, when the Price Trend is not large, the price tends to continue in the same direction; but when the Price Trend is large it moves in the opposite direction. This suggests that investors and traders view a gradual trend as information that the asset's value is increasing and are thereby willing to pay more for it. However, when there is a large change, they tend to view

the price change as excessive and implement a strategy of capitalizing on the overreaction, consistent with the findings of [22].

Using the weekly changes in the M2 Money Supply (interpolated for daily changes) as one of the independent variables, in addition to the others that we have already established, we have implemented a regression analysis to demonstrate the positive effect of money supply on asset price changes. There is very clear statistical evidence (t-value greater than 9) that an increase in money supply is a positive factor in price changes. For this variable in particular, there is the possibility that an increasing money supply is associated with a particular time period during our ten year data period. The inclusion of a time variable with terms up to $(Time)^3$ essentially eliminates this possibility, as the coefficient and the statistical significance are virtually unchanged.

Short term and long term volatility both have a statistically significant effect on trading price. Volatility that is persistent appears to deter investors, consistent with the concept of risk aversion. Surprisingly, short term volatility has a positive effect on trading price. This may suggest that volatility may be associated with increased attention on the stock that draws more investors, and hence, increases the demand for the stock, boosting prices.

In our study we find definitive statistical support for the hypothesis that rising volume has a positive influence on price changes, thereby confirming the experimental and practitioner ideas.

We formulate a reasonable mathematical criterion for resistance, the phenomenon by which stock prices tend to decline after approaching a recent high, and find strong support for the assertion that prices are less likely to rise when they are just below the recent highs.

Looking beyond the particular variables (e.g., Price Trend, Money Supply, etc.) studied in this paper, we note that our method is quite general and is capable of addressing other hypotheses that can be formulated quantitatively. The use of assets in which one can objectively define a valuation enables one to compensate for the random noise that is inherent in fundamentals, as illustrated by the first two regressions. Without addressing the issue of changes in valuation, statistical methods will often show that the null hypothesis of no effect cannot be eliminated. For some variables, there may be nonlinearity which can also be understood (as with Price Trend and Valuation in Regression 3) using this methodology.

Nonlinearity can provide an explanation for phenomena that influence the dependent variable in competing directions, e.g., underreactions and overreactions. An important nonlinear term can also appear as a zero coefficient in a linear regression as a consequence of having a positive effect for part of the variable domain and negative on the remainder.

While our study has focused on closed-end funds and daily price changes, the methods can be applied to any particular time horizon, and to more general stocks once a method for valuation is chosen. From the perspective of establishing the methodology and effects of key variables, closed-end funds have the advantage that there is no ambiguity in their valuation. More generally, one would need to use a method of valuation (which is available from classical finance) together with our methodology in order to obtain predictions on relative price changes.

A major challenge in the analysis of financial markets has been the development of a methodology that can establish and quantify the effect of various forces that move prices. Our study has taken a step in this direction, and provides considerable statistical evidence for the implementation of the asset flow differential equations utilizing these concepts. Through optimization of parameters relating to trend, for example, one can use these equations to predict price dynamics (see [23] for the differential equations and [12] for the difference equations).

2.4 STUDY 2: STANDARDIZATION OF INDEPENDENT REGRESSION VARIABLES

2.4.1 Objectives

The dynamics of asset prices and the underlying motivations have been of great interest for both theorists and practitioners. A basic rationale for price movement is due to the changes in the value of the asset. In the absence of any insight into the motivations of investors and traders, one might stipulate that prices should fluctuate randomly about this basic valuation. One can regard this concept as a default hypothesis expressing a distinguished

limit in which there is (a) unlimited information, that ensures all participants share the same notion of valuation, and (b) essentially infinite arbitrage capital, whereby informed investors vie with one another to quickly exploit any deviations from this valuation.

There is little doubt among practitioners that additional factors are at work in markets. The fact that some studies find evidence of overreaction while others demonstrate underreaction has led some to assert that this may be evidence that markets are efficient. Our perspective, however, is that perhaps there are nonlinear relationships between price movements. One of the key goals of this study is to demonstrate this nonlinearity and to establish its form on an empirical and statistical basis. In particular, does an uptrend have a negative impact on daily returns after a particular threshold? And how does the valuation enter into that threshold? A second goal is to have an objective measure of the impact of various competing effects, e.g., price trend, valuation, etc.

Similar to Study 1, these goals are achieved by quantifying the effect of various factors on the daily relative price change. Given the unambiguous nature of the definition of a closed-end fund's valuation (namely, its NAV), we consider a data set comprised of 111,356 daily closing prices for 119 closed-end funds (28 Generalized, 62 Specialized, and 29 World funds)⁷. The records correspond to daily closing prices for the time period October 26, 1998 through January 30, 2008⁸. We perform linear regressions with the relative change in daily price as the dependent variable and various decision-making factors as independent variables. Specifically, we consider (i) the recent and long term trend in fund price; (ii) the valuation of the fund; (iii) the M2 money supply; (iv) the recent and long term volatility of the stock price; (v) the recent trend in volume; and (vi) resistance, i.e., the idea that rising stock prices tend to slow their increase when approaching a recent high price, which acts as a barrier. As shown by the regressions of [12], the incorporation of valuation in the appropriate form will greatly reduce the "noise" inherent in the stochastic nature of the valuation and enable the analysis of these other contributing factors.

⁷While the same data set was used for both Study 1 and Study 2, fewer funds were included in Study 2. This is due to the definition of the Volume Trend variable. In Study 1, any record (daily) with a trading volume of zero was eliminated; whereas for Study 2, any fund with a trading volume of zero on any day was excluded.

⁸Due to the definition of certain independent variables, all funds were required to have at least one year's worth of data.

By standardizing the independent variables, i.e., subtracting the mean values and dividing by the standard deviations, we are then able to easily compare the effects of each variable. Note that the methodology we employ to find quantitative measures of effect can be extended for use on ordinary stocks provided a valuation measure for these stocks is employed. In addition, this process is not limited to the above mentioned variables, but can be used to examine any decision-making factor provided it may be expressed as a variable. An important aspect of this study involves a judicious definition of valuation. Without a mechanism for extracting the valuation, the remaining terms would quite likely be masked by the "noise" inherent in changes in valuation (see Regressions 4 and 5 in Appendix A).

2.4.2 Approach and Rationale for Standardization

The objective of this paper is to determine whether various factors such as Price Trend, Valuation, Money Supply, Volume Trend, and Volatility influence an asset's Relative Price Change, $R(t+1) = \frac{P(t+1)-P(t)}{P(t)}$ where $P(t)$ is the asset price on day t . This is accomplished by executing several linear regressions with the relative price change as the dependent variable and subsets of the other factors as independent variables. The linear regression produces a regression coefficient, p-value, and t-value for each independent variable. The regression coefficient provides the "magnitude" of the effect, while the statistical significance (p-value and t-value) of the dependent variable determines whether the effect is truly present. This approach, however, has two shortcomings: (i) our data set consists of multiple stocks, each with its own individual attributes and (ii) the scales of the variables vary over several orders of magnitude (e.g., the mean value for Price Trend is approximately 0.0004 while the mean value for the Volume Trend is 0.21).

Prior studies [12] have shown that the first issue can be circumvented by performing a mixed effects linear regression which accounts for the unique statistical characteristics of each fund.

The second issue can be addressed by standardizing the data, a methodology whereby all independent variables are placed on a common footing to facilitate comparisons of effect [17]. This is accomplished by subtracting the mean and dividing by the standard deviation.

This makes the variables and resulting regression coefficients unitless by putting them on the scale of standard deviations. It also facilitates comparison, for example, of a two standard deviation event for one variable compared with another. However, as noted by Bring [4], a drawback of this approach to standardization is that the standard deviation unit may vary across groups - in our case funds within the data set. To mitigate this issue we standardize the coefficients by fund. That is, we compute the mean and standard deviation for each variable by fund, and then standardize each fund's data with these values. Ultimately, as noted by Cohen *et al.* [17] the standardized regression coefficient "is often the most useful coefficient for answering questions about the influence of one variable on another, with or without other variables in the equation."

All of the independent variables have been standardized with the exception of the Resistance Indicator, $Q(t)$, and the time variable, $Time(t)$. The Resistance Indicator is a highly skewed binary variable in that it is either "set" (corresponding to 1) or "not set" (corresponding to 0) and less than 1% of the records satisfy the criteria. Consequently, standardizing this binary variable would distort the results.⁹

The Time variables are included in regressions to determine if the historical periods have an effect that could be disguised as one of the other variables. For example, Regression 2 below includes the same variables as Regression 1, but also incorporates the Time variables. Thus, any discrepancies in the output for the variables common to both regressions are due to the inclusion of the Time variables. Thus, while we are concerned with the statistical significance of the Time variables, their coefficients are not directly relevant to this study. Also, the Time variable is already scaled between zero and one. As such, we do not standardize this variable.

⁹Gelman [26] notes that we run the risk of overstating the importance of such a binary variable relative to the other standardized variables if our standardization procedure only divides by one standard deviation. He considers the possibility of division by two standard deviations. He indicates that a binary variable with equal probabilities has a mean of 0.5 and a standard deviation of 0.5. Then, the difference between a "0" and a "1" on the original (unstandardized) scale is actually 2 standard deviations. Now, suppose the binary variable is "highly skewed", which our Resistance variable is - only approximately 1% of the records meet the criteria (so, the mean is 0.01 and the standard deviation is approximately $(0.01 * 0.99)^{\frac{1}{2}}$). Then, the difference between a "0" and a "1" on the original (unstandardized) scale is actually approximately 9.8 standard deviations. Thus, Gelman suggests that by not standardizing the Resistance variable, we might actually overstate its importance because standardization in this case will actually yield larger values, which in turn will result in smaller regression coefficients.

When utilizing standardized coefficients, one has the option to either standardize or not standardize the dependent variable. We choose not to standardize the Relative Price Change to facilitate the interpretation of results. For example, Regression 1 below shows the Price Trend variable to have a regression coefficient of 0.0012. In other words, a positive one standard deviation change in the recent trend will yield, on average, a 0.12% positive change in the daily relative price change (i.e., the Relative Price Change variable).

2.4.3 Results

We perform a set of regressions, each one of the form:

$$R(t + 1) = \beta_0 + \sum_{i=1}^n \beta_i x_i$$

where x_i is one of the above independent variables or a product of these variables. Since the variables have been standardized (as discussed above), β_i may be interpreted as the standardized regression coefficient determined by the mixed effects linear regression. The intercept term, β_0 , is present as we do not standardize the dependent variable, and it may be interpreted as the "drift" of classical finance. In other words, this is the average relative daily price change. The dependent variable is evaluated at day $t + 1$ which indicates that it represents the following day's relative price change or return.

Note that for each regression the total number of observations is 80,351 corresponding to 108 funds¹⁰. The number of observations included in the regressions does not equal the total number of records in the data set because the calculations for some (long term) variables required data from the previous year. For example, the computation of the Long Term Price Trend and Long Term Volatility records required the previous 251 data points.

Regression 1. We consider the Relative Price Change regressed against all of the above mentioned variables with the exception of the Time variables. This regression has the form:

¹⁰The number of degrees of freedom for each regression is dependent upon the number of independent variables. As such, this statistic is included in the results for each regression.

$$R(t + 1) = \beta_0 + \beta_1 T(t) + \beta_2 D(t) + \beta_3 M2(t) + \beta_4 STV(t) + \beta_5 LTV(t) \\ + \beta_6 LTT(t) + \beta_7 VT(t) + \beta_8 Q(t).$$

Table 2.9: Regression 1 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.000417343	0.0000478596	8.72015	<.0001
Price Trend	0.001184948	0.0000578973	20.46639	<.0001
Valuation	0.002658618	0.0000547312	48.57592	<.0001
M2 Money Supply	0.000470137	0.0000478630	9.82257	<.0001
Short Term Volatility	0.000520087	0.0000555387	9.36440	<.0001
Long Term Volatility	-0.000138966	0.0000530404	-2.62000	0.0088
Long Term Trend	-0.000050647	0.0000539793	-0.93827	0.3481
Volume Trend	0.000310028	0.0000481547	6.43816	<.0001
Resistance	-0.000708118	0.0006023445	-1.17560	0.2398

Degrees of Freedom: 80,235

The Valuation, Price Trend, Money Supply, Short Term Volatility, and Volume Trend terms are all highly statistically significant with positive coefficients (refer to Table 2.9). The coefficient of Price Trend indicates that a one standard deviation change in the Price Trend will induce a 0.12% change in the Relative Price Change in the same direction, i.e., if the Price Trend increases, then so will the Relative Price Change (provided all other independent variables are unchanged). The standardization allows a comparison of impact of the different independent variables. Thus, the Price Trend coefficient is approximately one half the magnitude of the Valuation coefficient. The M2 money supply variable has a coefficient that is approximately half the magnitude of the Price Trend. This supports

theoretical [6], experimental ([15] and [16]), and empirical [12] studies which suggest that an influx of cash will bolster the trading price. These studies have helped in resolving the paradox of experimental bubbles (see, e.g., [46] and [40]). The volatility variables are quite interesting in that both are statistically significant, but while the Long Term Volatility has a negative coefficient, the Short Term Volatility coefficient is positive. Thus, recent volatility in the fund price tends to raise the price, while longer term volatility has a negative effect on the price. The Long Term Trend variable is only marginally statistically significant. However, its small negative coefficient agrees with the findings of Poterba and Summers [41] of stock price regression to the mean over longer time frames. The Volume Trend is also statistically significant with a positive coefficient that is approximately one quarter the magnitude of the Price Trend coefficient. This confirms a widely held belief among traders that rising volume is associated with rising prices. Finally, we note that the Resistance variable has a negative coefficient which indicates that when the Resistance criteria are satisfied, the price is pushed downward. However, the statistical significance of this variable is marginal, probably due to the small number of records in the data set that met these criteria (518 out of 80,351). The Intercept term is both statistically significant and positive.

Regression 2. The Time variables are included to ascertain whether any of the results from Regression 1 are artifacts of a particular era. This regression has the form:

$$\begin{aligned}
 R(t+1) = & \beta_0 + \beta_1 T(t) + \beta_2 D(t) + \beta_3 M2(t) + \beta_4 STV(t) + \beta_5 LTV(t) \\
 & + \beta_6 LTT(t) + \beta_7 VT(t) + \beta_8 Q(t) + \beta_9 Time(t) \\
 & + \beta_{10} Time^2(t) + \beta_{11} Time^3(t).
 \end{aligned}$$

Results for Regression 2 are included in Table 2.10. Each Time variable is statistically significant. The inclusion of these variables had little effect on the coefficients of the Price Trend, Valuation, M2 Money Supply, and Resistance (see Table 2.11) and slightly more

Table 2.10: Regression 2 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.00137697	0.000378576	3.63724	0.0003
Price Trend	0.00115478	0.000057919	19.93789	<.0001
Valuation	0.00262333	0.000054807	47.86512	<.0001
M2 Money Supply	0.00047234	0.000047831	9.87515	<.0001
Short Term Volatility	0.00061354	0.000056140	10.92873	<.0001
Long Term Volatility	0.00014445	0.000060335	2.39407	0.0167
Long Term Trend	-0.00014979	0.000054925	-2.72721	0.0064
Volume Trend	0.00031614	0.000048187	6.56077	<.0001
Resistance	-0.00066603	0.000602981	-1.10456	0.2694
Time	-0.01588736	0.002819201	-5.63541	<.0001
Time ²	0.04328314	0.005828924	7.42558	<.0001
Time ³	-0.02985772	0.003508966	-8.50898	<.0001

Degrees of Freedom: 80,232

impact on the Short Term Volatility coefficient. However, the Long Term Volatility and Long Term Trend terms were significantly impacted by the inclusion of the Time variables. This is to be expected as the definition of these variables includes data for the entire previous year. Furthermore, the data set only includes 10 years of data, which essentially amounts to 10 data points. Hence, a few years in which the broad market experienced a highly volatile year followed by a less volatile year, but one in which stock prices are increasing, could explain the change in sign (from negative to positive) of the Long Term Volatility coefficient. The Long Term Trend variable also changed from not statistically significant (p-value of 0.3481) to statistically significant (p-value of 0.0064) with a small negative effect.

Table 2.11: Comparison of the regression coefficients of the variables common to both Regressions 1 and 2. The relative change in magnitudes of the Intercept, Long Term Volatility, and Long Term Trend coefficients are significant (greater than 20%). Also, note that the Long Term Volatility coefficient changed from negative to positive.

Variable	Regression 1 Coefficient	Regression 2 Coefficient	% Change
(Intercept)	0.000417343	0.00137697	229.94
Price Trend	0.001184948	0.00115478	-2.55
Valuation	0.002658618	0.00262333	-1.33
M2 Money Supply	0.000470137	0.00047234	0.47
Short Term Volatility	0.000520087	0.00061354	17.97
Long Term Volatility	-0.000138966	0.00014445	-203.95
Long Term Trend	-0.000050647	-0.00014979	195.75
Volume Trend	0.000310028	0.00031614	1.97
Resistance	-0.000708118	-0.00066603	-5.94
Time	NA	-0.01588736	NA
Time ²	NA	0.04328314	NA
Time ³	NA	-0.02985772	NA

Regression 3. In order to explore the nonlinearity in the Price Trend and Valuation variables, the interactions (multiplicative products) up to third order of these variables are included in the regression. Regression 6, which is presented in Appendix A, includes the additional variables that we exclude here for simplicity. There are relatively minor differences between the two regressions. Note that the Price Trend and Valuation variables are standardized (in both regressions), while the interactions are products of the standardized variables (i.e., the interaction variables are not standardized). This regression has the form:

$$R(t + 1) = \beta_0 + \beta_1 T(t) + \beta_2 D(t) + \beta_3 T^2(t) + \beta_4 T^3(t) + \beta_5 D^2(t) \\ + \beta_6 D^3(t) + \beta_7 T(t)D(t) + \beta_8 T^2(t)D(t) + \beta_9 T(t)D^2(t)$$

Table 2.12: Regression 3 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.000265656	0.00005434210	4.88858	<.0001
Price Trend	0.001668129	0.00006798784	24.53570	<.0001
Valuation	0.002911161	0.00006232596	46.70865	<.0001
Price Trend ²	-0.000117148	0.00003880772	-3.01868	0.0025
Price Trend ³	-0.000105260	0.00001123276	-9.37080	<.0001
Valuation ²	0.000064430	0.00001803873	3.57174	0.0004
Valuation ³	-0.000007622	0.00000160542	-4.74795	<.0001
Price Trend * Valuation	-0.000268997	0.00004854760	-5.54089	<.0001
Price Trend ² * Valuation	-0.000087231	0.00001771199	-4.92498	<.0001
Price Trend * Valuation ²	-0.000008461	0.00000977158	-0.86592	0.3865

Degrees of Freedom: 80,234

With the exception of the interaction of Price Trend with the square of the Valuation, all of the independent variables are statistically significant (see Table 2.12). Using these results, we define a function representing the Relative Price Change as a function of the Price Trend and Valuation:

$$R(T, D) = 0.0003 + 0.0017T + 0.0029D - 0.0001T^2 - 0.0001T^3 \\ + 0.0001D^2 - 0.00001D^3 - 0.0003TD - 0.0001T^2D - 0.00001TD^2.$$

Plotting this function yields the three dimensional surface depicted in Figure 2.3.

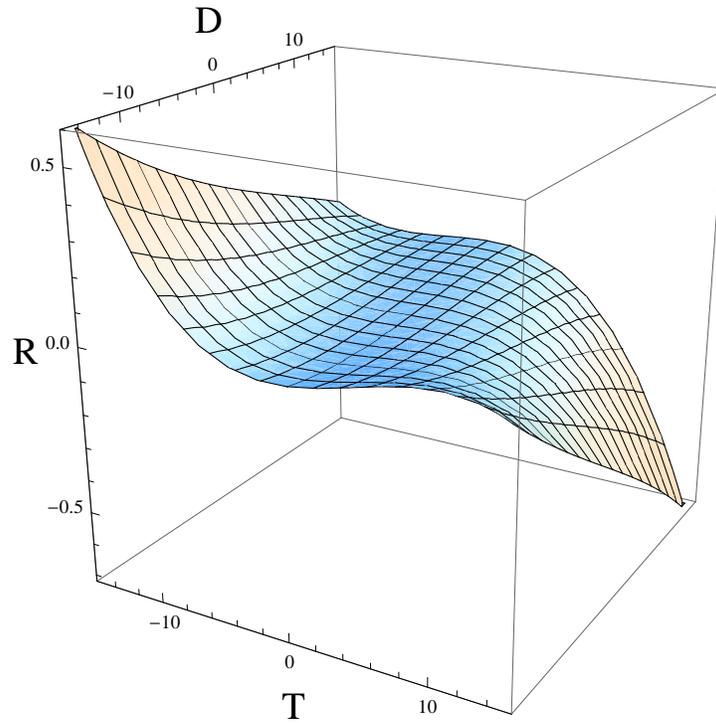


Figure 2.3: The plot was produced using the coefficient values from Regression 3. These coefficients define a cubic polynomial in two variables, Price Trend (T) and Valuation (D). The surface describes the effect on the following day's Relative Price Change (R), and exhibits the nonlinear relationship between D , T , and R . Overreaction is evident as a large positive trend can result in a negative Relative Price Change (analogously for negative trend). The precise point at which the magnitude changes sign depends nonlinearly on the valuation.

To better understand the nonlinear relationship between Price Trend and Valuation we consider the cross-sections of the above graph for $D = -1, 0$, and 1 .

$$R(T, -1) = 0.0003 + 0.0017T + 0.0029(-1) - 0.0001T^2 - 0.0001T^3 + 0.0001(-1)^2 - 0.00001(-1)^3 - 0.0003T(-1) - 0.0001T^2(-1) - 0.00001T(-1)^2$$

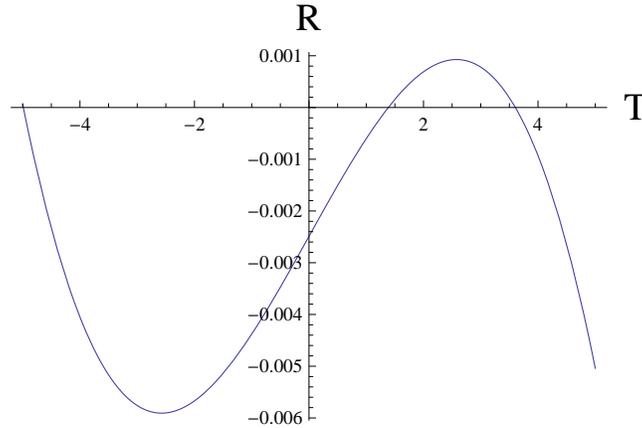


Figure 2.4: With the Valuation held constant at -1, the Relative Price Change may be represented as a cubic function of the Price Trend variable, T . For a large enough positive change in the Price Trend, $T > 3.6044$, the next day's Relative Price Change is negative. Analogously, for a Price Trend < -4.9891 the next day's Relative Price Change is positive.

Figure 2.4 displays the cross-section of the surface in Figure 2.3 with the valuation held constant at $D = -1$. This cubic function intersects the T -axis at $T = -4.9891$, 1.3847 , and 3.6044 . So, for a negative one standard deviation change in the Valuation variable, the Relative Price Change will also be negative for $-4.9891 < T < 1.3847$ and $T > 3.6044$. But, for $1.3847 < T < 3.6044$, the Relative Price Change will be positive implying that the Price Trend will have a greater impact on the price than the Valuation. Approximately 92% of the Price Trend values will actually be less than 1.3847 standard deviations.

$$R(T, 0) = 0.0003 + 0.0017T + 0.0029(0) - 0.0001T^2 - 0.0001T^3$$

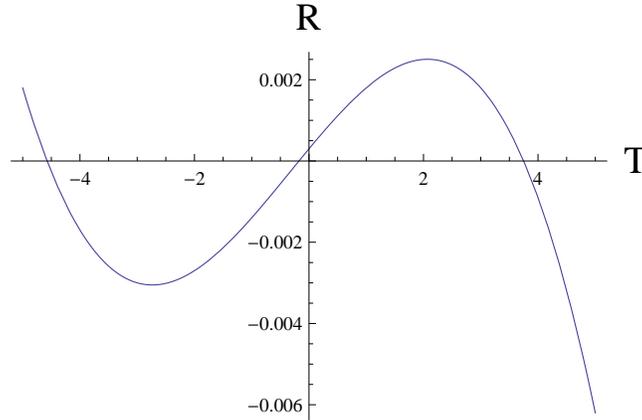


Figure 2.5: A cross-section of Figure 2.3 is represented with the Valuation variable held constant, $D = 0$. As in Figure 2, overreaction is present if the Price Trend variable, T , is large enough in magnitude, i.e., for $T > 3.7486$ the next day's Relative Price Change is negative; while for $T < -4.5736$, the next day's Relative Price Change is positive. Hence an uptrend has a positive influence on price change for T satisfying $-4.5736 < T < 3.7486$.

Figure 2.5 corresponds to the cross-section of Figure 2.3 with $D = 0$. If the Price Trend is positive, then the Relative Price Change is also positive up to 3.7486 standard deviations. However, if the Price Trend is greater than 3.7486 standard deviations, the Relative Price Change is negative. Thus, we see that a large (and unusual) change in the Price Trend produces a negative Relative Price Change supporting the theory of overreaction. Similarly, a negative Price Trend yields a negative Relative Price Change unless the change in the Price Trend is less than -4.5736 standard deviations. Thus, there is evidence for underreaction for $-4.5736 < T < 3.7486$ (approximately 99.9921% of the time) and evidence for overreaction when large (either positive or negative) changes in the Price Trend occur.

Part of the curve lies above the negative T -axis for $-0.17498 < T < 0$ due to the drift (intercept) term from the regression.

$$R(T, 1) = 0.0003 + 0.0017T + 0.0029(1) - 0.0001T^2 - 0.0001T^3 + 0.0001(1)^2 - 0.00001(1)^3 - 0.0003T(1) - 0.0001T^2(1) - 0.00001T(1)^2$$

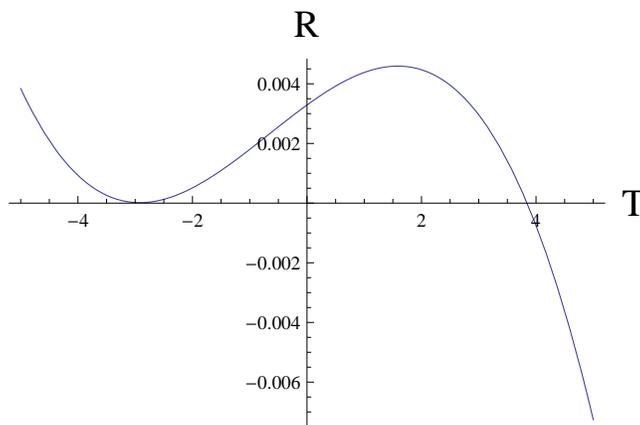


Figure 2.6: A cross-section of Figure 2.3 is represented with the Valuation variable held constant, $D = 1$. Thus, we have the next day's Relative Price Change, R , as a function of the Price Trend, T . If $T > 3.8435$, then the next day's Relative Price Change is negative; thereby, providing evidence to support overreaction in stock prices. However, notice that for $T < 3.8435$, the Relative Price Change is always positive. Thus, a one standard deviation change in the Valuation variable is large enough to counteract the effects of a negative Price Trend.

With the Valuation fixed at 1 in Figure 2.6, we see that the cross-section only crosses the T -axis once at $T = 3.8435$. Thus, a negative Price Trend of any size does not have enough of an impact on the Relative Price Change to counteract the positive one standard deviation change in Valuation. However, a positive change in Price Trend of more than 3.8435 standard deviations results in a negative Relative Price Change and exhibits overreaction.

Figure 2.7 depicts the relationship between Relative Price Change and Valuation (holding Price Trend constant). This relationship is essentially linear suggesting that an increase in value is always positive for stock prices:

$$R(0, D) = 0.0003 + 0.0017(0) + 0.0029D - 0.0001(0)^2 - 0.0001(0)^3 + 0.0001D^2 - 0.00001D^3 - 0.0003(0)D - 0.0001(0)^2D - 0.00001(0)D^2$$

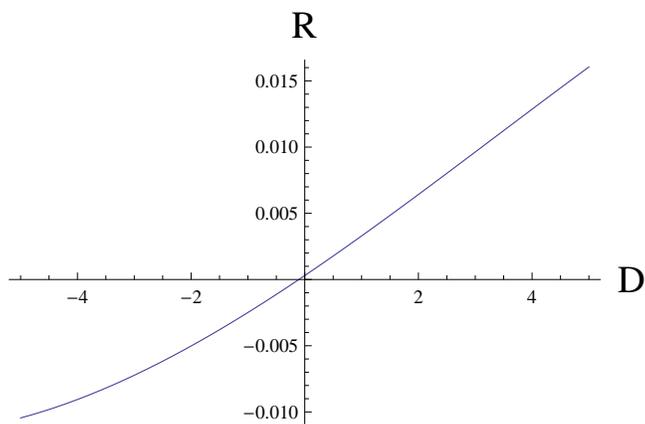


Figure 2.7: A cross-section of Figure 2.3 is represented with the Price Trend variable held constant, $T = 0$. This shows that the Relative Price Change variable, R , is essentially linear in the Valuation variable, D .

Figure 2.8 displays a contour plot of the Relative Price Change function, $R(T, D)$, for $R = -0.01, 0$, and 0.01 , which correspond to relative price changes of $\pm 1\%$ and 0% . These provide an intriguing view of the nonlinear relationship between the Valuation and Price Trend variables.

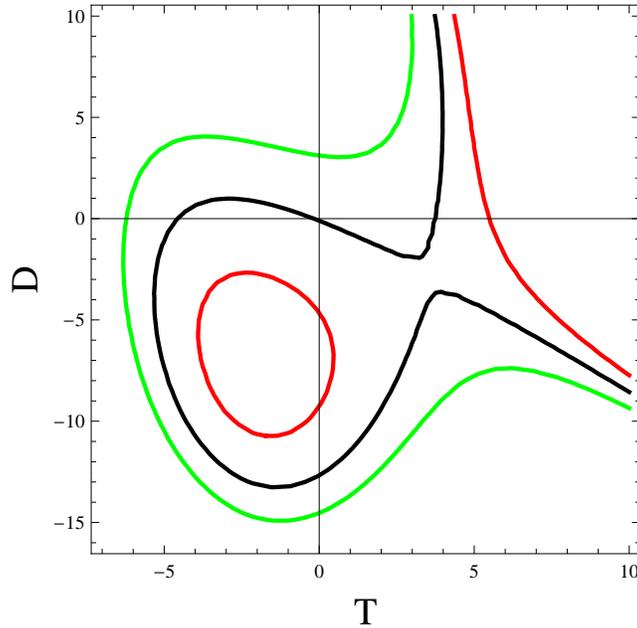


Figure 2.8: Using the coefficient values from Regression 3, one can express the following day's Relative Price Change, R , as a cubic function of two variables, the Price Trend, T , and the Valuation, D . This figure is a contour plot of this function. The curves represent level sets on which R is held fixed at $\pm 1\%$ and 0% . The innermost curve corresponds to $R = -1$, the middle curve to $R = 0$, and the outermost curve to $R = 1$.

2.4.4 Conclusion

We have presented an empirical methodology that is capable of testing almost any hypothesis involving dynamics of asset prices. We have utilized the mixed effects regressions on a set of independent variables: valuation, short and long term trends, short and long term volatility, the M2 money supply, volume trend and resistance. The results exhibit strong statistical support for the assertion that the short term price trend is a factor that tends to increase trading prices. The magnitude of this effect is almost half of that for valuation. Unlike some previous studies in which raw data was analyzed, displaying a tiny effect for trend, our study shows that the effect of the trend is very important. This is largely due to a methodology in which the changes in valuation (amounting to noise) are considered in a multi-regression

in an appropriate form. Similarly, positive statistically significant coefficients were found for short term volatility, volume trend and the money supply. The latter confirms the assertions of the asset flow theory (supported by experiments and empirical studies cited in Sections 2.2.2 and 2.2.3) that additional cash fuels trading price increases.

The positive coefficient obtained for short term volatility is surprising in the context of classical finance since the inverse risk-reward relationship stipulates that high volatility should be interpreted as greater risk that would diminish the price that traders would pay for the stock. The positive coefficient for short term volatility may be explained by the hypothesis that traders are attracted to higher volatility as it offers the opportunity for greater profits. As more capital is attracted the increased level of cash (as shown with the money supply and prior studies) would tend to bolster prices. The role of long term volatility is quite small compared to trend. Also, it is more ambiguous and complicated. While we obtain a negative coefficient (provided the historical time is not taken into account) consistent with the expectations of classical finance, as shown in Regression 1, we also find that the coefficient is positive in Regression 2, where the time variables are included. However, the coefficient has only about one-tenth of the magnitude of the short term trend. A basic classical finance tenet is that investors are rational and seek to avoid risk and increase return (see e.g., [5]). This is followed by the assumption that risk can be identified with volatility. Our study shows that the only interpretation in which this classical concept is upheld is that a time period exhibiting high volatility is followed by a period of slightly more negative price changes.

The methodology uses a technique of standardizing the data prior to executing the linear regressions. This has the salient feature that it allows direct comparison of the distinct hypothesized factors. In particular, it allows us to go beyond the question of whether these effects are actually present to the possibility of quantifying, for the first time, their relative importance. In particular the aggregate effect of short term trend, short term volatility and M2 money supply is comparable to that of the valuation.

Another feature of our methodology is that the data is standardized with respect to individual stocks. This tends to mitigate the distortion introduced by the large variations in magnitudes of the independent variables. As a practical application, one should be able to

obtain more accurate results by performing the regressions on only one stock of interest at each time. Although the statistical significance and scientific impact would be diminished, the practical results for prediction would be enhanced.

While our sample is inadequate to make a strong statistical assertion on resistance (the tendency for prices to move down while nearing a recent high), the fact that the size of the coefficient is comparable to most of the other variables (except short term trend and valuation) suggests that a larger sample could result in establishing this as an important factor.

The growing evidence for factors influencing asset market dynamics may appear, upon cursory analysis, to be contradictory. For example, there are studies that demonstrate the presence of underreaction, exhibited by the continuation of a price trend. There are also studies showing that overreaction is present as prices reverse course. The statistical analysis involving 111,356 data points supports our assertion that the presence of both under- and overreaction is a manifestation of the underlying nonlinearity of trader motivations. In particular, Figure 2.3, which displays the daily return as a function of trend and valuation, shows that while an uptrend is positive for stocks in one region (e.g., the uptrend is not too large and the valuation range is far from zero), it may be negative in another region (e.g., when the trend is very large or the valuation change is large). Thus, nonlinearity is the key to understanding competing motivations.

By incorporating the square and cubic price trend and valuation terms as well as the interactions (up to third order) of these two key variables, we are able to express the relative price change as a nonlinear function of the price trend and valuation. Plotting this function in various ways ($3d$ and $2d$ with valuation constant, as well as level sets) illuminates the nonlinear relationship between these variables and renders a quantitative and empirical explanation for how the competing effects of overreaction and underreaction can coexist within the same data set. For example, it yields the precise information that a positive change in valuation at the level of one standard deviation will not be counteracted, on average, by *any* short term trend. However, a similar negative change in valuation will be balanced by a positive trend at the level of 1.38 standard deviations.

3.0 ANALYSIS OF ASSET FLOW DIFFERENTIAL EQUATIONS

3.1 INTRODUCTION

Recent events in the world's financial markets have demonstrated the need for developing mathematical models that are capable of addressing issues of market dynamics and stability. Standard theories of asset management and options assume near equilibrium conditions. In particular, asset and option valuation theories are based on the equation

$$\frac{dP}{P} = \sigma dX + \mu dt \quad (3.1)$$

where P is the asset price at time t , while dX is a normal random variable (mean 0 and variance dt), σ^2 is the variance, and μ is the drift, so that μdt is the expected return on the investment in time dt .

This formalism is based upon the idea that a sufficiently large proportion of investors are informed of the realistic value of the asset and act solely on that basis. It assumes that there is an infinite amount of arbitrage capital that is ever-present to quickly exploit any deviation from fundamental value. However, it has been noted (see [44]) that there exist practical limitations to arbitrage in real markets. Thus, while an arbitrage opportunity might exist, arbitrageurs might not be able to act on it due to risk of losses and the need to liquidate the portfolio due to pressure from investors. Furthermore, on a larger time scale the assumption of rapid assimilation of information leads to an inverse relationship between volatility, defined as σ and mean expected return, μ . A standard optimization result [5] is that all investors should invest in a mixture of a risk-free asset (e.g., Treasury bills) and a single portfolio of risky assets consisting of stocks and bonds. The fraction of wealth

invested in risky assets will differ among investors who differ in their utility (or preference) functions, and consequently, risk tolerances.

The classical theories are idealizations that are limiting cases. In particular, the capital owned by knowledgeable investors focused on value is assumed to be infinite. Starting from a theory in which a crucial quantity is infinite, it is often difficult to design a unique generalization or modification in which it is finite. The expectation, or hope, in the research community is that more refined theories with smaller effects will augment an existing, established theory. Thus it is difficult, if not impossible, to build on a theory that does not recognize a key finite quantity. One must model such effects from first principles. Yet utilizing a perspective of basic modeling that does not emanate from a familiar theory encounters difficulties as a result of the sociology of scientific research.

Two perspectives in addressing this issue are noted below. (i) One can consider first the modeling of an experimental asset market, pioneered by Vernon Smith (2002 Economics Nobel Laureate) and collaborators, where one or two dozen participants trade an asset defined by the experimenter with the objective of earning real money based on their trades. Within this setting one knows precisely the quantity of cash, asset, payouts etc. Thus, one can develop and test mathematical models of these experiments. (ii) Upon generalizing such a model to several groups with varying motivations and assuming some randomness one can try to recover the classical models as a limit of the value oriented group having a preponderance of the assets. This is an issue that we consider in Appendix B of this paper.

We summarize an approach that has been developed by Caginalp and collaborators since 1990 (see www.ssrn.com). This model is flexible enough to account for a variety of factors that may affect investor sentiment, allow for multiple investor groups, and incorporate concepts such as the gradual diffusion of information across the investor population. If we focus on a simple experimental asset market setting, there are a set number of traders who trade a single asset whose worth is defined by its payouts or dividends. The experimenter has the option of allowing these dividends to be traded during the experiment, or deferred to the end of the experiment. Thus we may define M as the constant representing the total cash in the system (assuming for the time being that dividends are deferred), and N as the

(constant) number of shares of the asset. We let

$$B := \frac{NP}{NP + M} \quad (3.2)$$

be the fraction of total wealth invested in the stock in terms of the trading price, $P(t)$, at time t . Then one also has

$$1 - B = \frac{M}{NP + M} \quad \text{and} \quad \frac{B}{1 - B} = \frac{NP}{M} =: \frac{P}{L}. \quad (3.3)$$

The liquidity, $L := M/N$, is a key variable introduced in [6] that has units of dollars per share. Thus, the quantities L , P , and the fundamental value of the asset, $P_a(t)$, determined by the definition of the asset, all have these units, and one can measure price in natural units of liquidity. The modeling of asset price dynamics starts by using a simple equation that stipulates that price moves in proportion to the imbalance between demand, D , and supply, S :

$$\tau P^{-1} \frac{dP}{dt} = \frac{D - S}{S}, \quad (3.4)$$

where τ is a time scale [50]. Within a closed system one can express the demand for the asset as $D = k(1 - B)$ and the supply as $S = (1 - k)B$, where k is a transition rate. One can interpret k as the probability of a unit of cash being submitted as a buy order per unit time. Classical game theory ([37] and [25]) would suggest that k should depend solely on the valuation, since each trader is aware of the same information. However, the dependence of k on other quantities such as the trend in price (price trend or trend) can be determined experimentally [15] and empirically (in the case of world markets) ([8], [9], and [41]). Given a particular motivation, one can use the discrete version of these differential equations in conjunction with statistical methods to determine whether there is statistical support for it [13]. With these definitions, one can rewrite 3.4 as

$$\tau P^{-1} \frac{dP}{dt} = \frac{k(1 - B)}{(1 - k)B} - 1. \quad (3.5)$$

The key issue is the dependence of k on investor strategy and behavior. The investor sentiment can be described by a function $\zeta(t)$ with range in \mathbb{R} , and can be written in terms

of a finite set of components: $\zeta(t) = \zeta_1(t) + \zeta_2(t) + \dots + \zeta_n$. Since k must have values in $[0, 1]$ a simple way to map values of ζ into k is through a function such as

$$k := \{1 + \tanh(\zeta_1 + \zeta_2 + \dots + \zeta_n)\} / 2. \quad (3.6)$$

Considering valuation, $P_a(t)$, and just one other motivation, namely trend, we have

$$\begin{aligned} \zeta_1(t) &= q_1 c_1 \int_{-\infty}^t e^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau; \\ \zeta_2(t) &= q_2 c_2 \int_{-\infty}^t e^{-c_2(t-\tau)} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} d\tau \end{aligned} \quad (3.7)$$

where q_i and c_i characterize the magnitudes and time scales of the two motivations for investing [11]. The first of these indicates that investors are influenced by the change in price but are more strongly influenced by recent changes. A small value for the time scale $1/c_1$ indicates that investors are focused on short term changes in the trend. The $\zeta_2(t)$ parameter represents investors' focus on the deviation between the asset price and its fundamental value. Again, more recent changes have a greater effect on investor decisions. A large value for $1/c_2$ indicates that investors take action slowly when there is an over- or undervaluation. An alternative explanation for the $1/c_i$ parameters is that they regulate the rate at which information diffuses to the investor population. The magnitudes of q_1 and q_2 represent the significance of these motivations in trading decisions for the aggregate (homogeneous) investor population. Note that other motivations, $\zeta_3(t)$, $\zeta_4(t)$, etc. are possible. For example, statistical studies ([8] and [9]) have shown that the M2 Money Supply and asset price volatility also impact investor decisions to buy and sell.

One can also show (see equations 2.6 and 3.5 in [6]) that the change in the fraction of wealth invested in the asset is given by

$$\frac{dB}{dt} = k(1 - B) + (k - 1)B + B(1 - B) \frac{1}{P} \frac{dP}{dt}. \quad (3.8)$$

The system of equations 3.5 - 3.8 is a complete set of ODEs that have been studied in a number of papers. We will study generalizations of these equations to two or more groups.

One can simplify these equations by taking the limit of short time scale and using the approximation $\tanh(x) \simeq x$, leading to the approximation

$$\frac{k}{1-k} = 1 + 2\frac{q_1\tau}{P} \frac{dP}{dt} + 2q_2 \left(1 - \frac{P}{P_a}\right). \quad (3.9)$$

One of the concepts that has emerged from this analysis is that the equilibrium price depends not only on P_a (which would be the case in classical economics) but also on the liquidity, L . The equilibrium price was found to be

$$\frac{P_{eq}}{L} = \frac{1 + 2q_2}{1 + 2q_2 \left(\frac{L}{P_a}\right)}. \quad (3.10)$$

When q_2 is large (i.e. there is a great deal of importance attached to valuation), the equilibrium price, P_{eq} , is close to P_a . However, if q_2 is small, the equilibrium price is close to L , as the valuation is marginalized.

This approach to modeling incorporates the issues of trend, overreaction, underreaction, disparate information and motivation of distinct groups as well as the effect of finiteness of assets within a single mathematical model. Thus it is evident that these effects arise from the dynamics of trading when the assumptions are more realistic than the classical idealizations. In the years since 1990 several works have implemented various aspects of this approach. In particular, [31] models overreaction, underreaction, and momentum via the interaction between two heterogeneous investor groups: news watchers and momentum traders. In other words, members of the fundamental group are called "news watchers" since they update the fundamental value via news reports. Similarly, an asset price's tendency to follow a recent trend in price has been studied ([32], [33]) along with the competing concepts of overreaction and underreaction.

In addition several models including [18] attempt to explain momentum profits by investors' inherent biases in interpreting information. They consider overconfident investors who overreact to private information. Within our models, this corresponds to one group setting a higher price on the fundamental value. The Barberis, Shleifer, and Vishny [2] model incorporates the psychological concepts of representativeness [49] and conservatism [24]. Grinblatt and Han [29] explained momentum via the concept of market disposition, i.e. investors are more likely to sell to realize a gain than a loss ([43], [42]).

Note that trend effects need not be exclusively due to overreaction or underreaction. For example, due to the finiteness of assets, a large buyer attempting to build a position without drastically pushing up prices could cause an upward trend in price. Another possible explanation could be the observation that others are profiting, which causes even the value-based investor to act [13].

Duran and Caginalp [23] utilized optimization methods to estimate the parameters of the asset flow differential equations using market data. These equations were applied to typical secondary stock offerings by Caginalp and Merdan [14], who obtained quantitative results relating an influx of shares with lower prices within this model. A sensitivity analysis of parameters was performed in [21]. Recently, liquidity issues were further explored in [35] and [36].

In this paper we present a detailed derivation of the discrete and continuum equations for asset dynamics with disparate groups with differing assessments of valuation and motivational characteristics. We determine the region of equilibrium points given the basic parameters governing the investor group motivations and assets (see Theorem 4 in Section 3.4). Theorem 10 (see Section 3.4) establishes regions of stability and instability. In Section 3.5 numerical computations for typical parameters display precise regions of stability and instability.

3.2 THE DISCRETE EQUATIONS

We consider two groups of investors (that can easily be generalized to an arbitrary number of groups) so that group i ($i = 1, 2$) is endowed at time T_j with $N^{(i)}(T_j)$ shares and $M^{(i)}(T_j)$ in cash. The probability that a unit of cash is submitted by investor group i for purchase during the time interval (T_j, T_{j+1}) of the single equity in the system is given by $k_*^{(i)}$ and may depend on many factors including the price history, as discussed below. Similarly, $\tilde{k}_*^{(i)}$ is the probability that a unit of stock will be submitted for sale. The demand (in terms of dollars) at time interval $\delta T := T_{j+1} - T_j$ (assume a uniform spacing so that δT is independent of j) is given by $D(\delta T) = k_*^{(1)}(\delta T) M^{(1)} + k_*^{(2)}(\delta T) M^{(2)}$ while the supply (in terms of dollars) is

$S := \tilde{S}P$ with $\tilde{S}(\delta T) := \tilde{k}_*^{(1)}(\delta T)N^{(1)} + \tilde{k}_*^{(2)}(\delta T)N^{(2)}$. An equilibrium trading price is one for which the supply, $\tilde{S}(T_j)$, and demand, $D(T_j)$, balance, i.e.

$$P(T_j) = F(T_j; k_*^{(i)}(P(T_0), \dots, P(T_j)), M^{(i)}(T_j), N^{(i)}(T_j)) \quad (3.11)$$

where

$$\begin{aligned} & F(T_j; k_*^{(i)}(P(T_0), \dots, P(T_j)), M^{(i)}(T_j), N^{(i)}(T_j)) \\ & := \frac{D(T_j; \dots)}{\tilde{S}(T_j; \dots)} \\ & = \frac{k_*^{(1)}M^{(1)}(T_j) + k_*^{(2)}M^{(2)}(T_j)}{\tilde{k}_*^{(1)}N^{(1)}(T_j) + \tilde{k}_*^{(2)}N^{(2)}(T_j)} \end{aligned} \quad (3.12)$$

and, for brevity, we have replaced $k_*^{(i)}(P(T_0), \dots, P(T_j))$ by $k_*^{(i)}$ on the right hand side. We define τ as a relaxation time that determines the rate at which a non-equilibrium situation returns to equilibrium. For simplicity we can assume $\tilde{k}_*^{(i)} := 1 - k_*^{(i)}$ though the modeling is similar in the more general case. Next, we describe the dynamics of price. Similar to [30] we postulate that the change in price in a unit time is proportional to the extent to which it is away from equilibrium:

$$\frac{1}{P(T_j)} \frac{P(T_{j+1}) - P(T_j)}{T_{j+1} - T_j} = \frac{1}{\tau} \left(\frac{F(T_j)}{P(T_j)} - 1 \right), \quad (3.13)$$

where we have suppressed the dependence of F on the other variables. As in earlier papers (see [14] and references therein) we consider a limiting form of this microeconomic equation, i.e. 3.24 below. Unlike past work, however, we derive this limiting form precisely in Section 3.3. Note that by using the definition of F , we see that the right hand side is the excess demand, $(D - S)/S$. Hence, this equation is compatible with basic price theory [50] as well as observations in experimental asset markets [46].

The time scale. Since δT is simply the time interval on which we choose to perform the computations in the discrete case, a natural choice for δT is given by $\delta T := \tau$ since the imbalance is restored on this time scale. Then 3.13 is simply

$$P(T_{j+1}) = F(T_j). \quad (3.14)$$

This indicates that if the supply and demand do not balance, then a new price, defined as $P(T_{j+1})$, is discovered. This price restores the balance between supply and demand, i.e.

$$\frac{k_*^{(1)} M^{(1)} + k_*^{(2)} M^{(2)}}{\left(\tilde{k}_*^{(1)} N^{(1)} + \tilde{k}_*^{(2)} N^{(2)}\right) P(T_{j+1})} = 1, \quad (3.15)$$

within a time frame that is the natural relaxation time for the problem.

Equation 3.13 states that if the price is not at an equilibrium value then it moves toward this price on a time scale given by τ . We can regard $\delta T := T_{j+1} - T_j$ as the time interval between trades in a discrete system, e.g. a market in which bids and asks are submitted at noon each day or an experimental setup with discrete periods.

The choice $\delta T = \tau$ is perhaps the most natural and closest to an efficient market since it suggests that the basic time scale for readjustment to equilibrium is the one period time scale defined by the market institution.

Prior to focusing on $\delta T = \tau$ we consider the implications of choosing $\delta T := a\tau$ for $a \neq 1$. In this case 3.13 can be written as

$$\frac{P(T_{j+1})}{P(T_j)} - 1 = a \left(\frac{F(T_j)}{P(T_j)} - 1 \right) \quad (3.16)$$

or

$$P(T_{j+1}) = F(T_j) + \delta \{F(T_j) - P(T_j)\} \quad (3.17)$$

where $\delta := a - 1$.

For $-1 < \delta < 0$, i.e. $0 < a < 1$, when $F(T_j) \neq P(T_j)$ the price at time T_{j+1} does not attain $F(T_j)$ as it does for $\delta = 0$. For example, if $\delta = -1/2$, then

$$P(T_{j+1}) = \frac{1}{2}F(T_j) + \frac{1}{2}P(T_j)$$

so that $P(T_{j+1})$ is only the average of the old price, $P(T_j)$, and the "target equilibrium price," $F(T_j)$.

Interpreting $\delta \neq 0$ within the context of experimental markets, one can consider $\tau = 3$ and $\delta T = 1$ so that $a = 1/3$ and $\delta = -2/3$. This means that the time scale of adjustment to equilibrium is 3 units while the trading time scale is 1. Hence by 3.17 the price adjustment at time T_{j+1} attains 1/3 of the deviation, $F(T_j) - P(T_j)$, from equilibrium.

Unless specified otherwise, we assume $\delta T = \tau$ below. For $\delta T \neq \tau$ the equations 3.18 and 3.19 below must be modified by replacing $P(T_{j+1})$ by $F(T_j)$.

The equation 3.13 remains valid if there is an influx or outflow of cash or shares during the time period between T_j and T_{j+1} with the assumption that this new cash or shares cannot be submitted until time T_{j+1} . At this point we distinguish between a totally conserved system in which no shares or cash are added or subtracted versus one in which there is an influx or outflow of shares or cash. In the latter case, we can define $m_*^{(i)}(T_j)\delta T$ and $n_*^{(i)}(T_j)\delta T$ as the net inflows of cash and shares during this time period between T_j and T_{j+1} . Note that negative values for either of these variables denote outflows. Thus we can write the basic conservation law as

$$\begin{aligned} M^{(i)}(T_{j+1}) - M^{(i)}(T_j) &= -k_*^{(i)}(\delta T) M^{(i)}(T_j) \\ &+ \tilde{k}_*^{(i)}(\delta T) N^{(i)}(T_j)P(T_{j+1}) + (\delta T) m_*^{(i)}(T_j). \end{aligned} \quad (3.18)$$

In other words, during the time period from T_j to T_{j+1} the cash decreases due to the fraction of existing cash, $k_*^{(i)}\delta T$, that is submitted for purchase of the equity, and increases due to the fraction of shares, $\tilde{k}_*^{(i)}\delta T$, that is submitted for sale. Setting $m_*^{(i)} = 0$ for all time and each group, i , specifies the special case in which we have no additional cash entering or leaving the system. Similarly, for each group i , the change in the number of shares owned must satisfy:

$$\begin{aligned} P(T_{j+1}) \{N^{(i)}(T_{j+1}) - N^{(i)}(T_j)\} &= k_*^{(i)}(\delta T) M^{(i)}(T_j) \\ &- \tilde{k}_*^{(i)}(\delta T) N^{(i)}(T_j)P(T_{j+1}) \\ &+ (\delta T) n_*^{(i)}(T_j)P(T_{j+1}). \end{aligned} \quad (3.19)$$

For each group i , the change in the value of stock owned must be balanced by the flow plus the influx of cash or shares. In fact, comparing the right hand sides of 3.18 and 3.19 we observe:

$$\begin{aligned} P(T_{j+1}) \{N^{(i)}(T_{j+1}) - N^{(i)}(T_j)\} &= -\{M^{(i)}(T_{j+1}) - M^{(i)}(T_j)\} \\ &+ (\delta T) m_*^{(i)}(T_j) + (\delta T) n_*^{(i)}(T_j)P(T_{j+1}). \end{aligned} \quad (3.20)$$

The system of equations 3.13, 3.18, and 3.19 can be studied as a system of difference equations upon specifying initial conditions together with $k_*^{(i)}$ as a function of $(P(T_0), \dots, P(T_j))$ and possibly other variables. In other words, these equations can be solved algebraically for $(P(T_{j+1}), M^{(i)}(T_{j+1}), N^{(i)}(T_{j+1}))$ in terms of the variables at earlier times. The choice of $k_*^{(i)}$ involves assumptions on the motivations of the traders. The dependence of these functions on various factors characterizes the trader population.

Remark 1. *In the case of no additional cash or asset, if we sum either 3.18 or 3.19 over all groups i , then we obtain from $\sum_i N^{(i)}(T_j) = \text{Constant}$ or $\sum_i M^{(i)}(T_j) = \text{Constant}$ that the left hand sides vanish. As a result, we obtain simply the definition of $F(T_j)$ so it is consistent with conservation. With non-zero $m_*^{(i)}$ or $n_*^{(i)}$ one has from 3.18*

$$\begin{aligned} & \sum_i \left\{ \frac{M^{(i)}(T_{j+1}) - M^{(i)}(T_j)}{\delta T} \right\} \\ &= \sum_i \left\{ -k_*^{(i)} M^{(i)}(T_j) + \tilde{k}_*^{(i)} N^{(i)}(T_j) P(T_{j+1}) + m_*^{(i)}(T_j) \right\}, \end{aligned} \quad (3.21)$$

yielding

$$\frac{M^{(1)}(T_{j+1}) - M^{(1)}(T_j)}{\delta T} + \frac{M^{(2)}(T_{j+1}) - M^{(2)}(T_j)}{\delta T} = m_*^{(1)}(T_j) + m_*^{(2)}(T_j) \quad (3.22)$$

for $n = 2$, i.e. two investor groups, using $P(T_{j+1}) = F(T_j)$ and the definition of F .

Remark 2. *The relationship between the discrete asset flow equations 3.13, 3.18, and 3.19 and the stochastic asset pricing equation 3.1 can be understood formally in the following sense. One considers two groups, one of which has the preponderance of assets and focuses on value, buying slightly below and selling slightly above the valuation, $P_a(t) = P_a(0) e^{\mu t}$, where μ is the risk-free rate. The other group employs a variety of strategies that cancel out one another. The additional funds entering into the system are a random process leading to a price dynamics that is stochastic and similar to 3.1. Further details are provided in Appendix B.*

3.3 THE CONTINUUM LIMIT

In deriving the continuum limit it is essential to examine the roles of δT and τ . The relaxation parameter τ is intrinsic to the system and describes the time scale on which the traders react to the changes in price and other variables. On the other hand, δT is a computational time scale so that we can consider the limit $\delta T \rightarrow 0$. In practice, δT is small compared to the overall time period, e.g., daily prices within a year of trading.

We can regard the discrete equations (particularly with $\delta T = \tau$) as the exact equations, with the continuum as an approximation. Thus, we can approximate the ratio $\{P(T_{j+1}) - P(T_j)\} / \delta T$ by $P'(T_j)$. More precisely, for a smooth function, P , we have

$$P(T_{j+1}) = P(T_j) + (\delta T) P'(T_j) + \frac{(\delta T)^2}{2} P''(\zeta), \text{ for some } \zeta \in (T_j, T_{j+1}) \quad (3.23)$$

which we can write as

$$\{P(T_{j+1}) - P(T_j)\} / \delta T - P'(T_j) = O[\delta T]$$

where $f(x) = O[x]$ signifies

$$\limsup_{x \rightarrow 0} \frac{|f(x)|}{|x|} < \infty.$$

Hence we replace the discrete derivative on the left hand side of 3.13 with a very small error and obtain the continuum limit

$$\tau \frac{1}{P(t)} \frac{dP}{dt}(t) = \frac{F(t)}{P(t)} - 1 \quad (3.24)$$

where we have replaced the discrete variables T_j with the continuum variable t in F (see below) and P . Note that the discarded term is $\tau O[\delta T]$.

Another perspective on this is that in equation 3.13 the discrete derivative is $O[1]$ so that one must have $F/P - 1 = O[\tau]$ in order to maintain equality in this equation. Hence, in the discrete equation with $\delta T := \tau$ as a small parameter, we write 3.13 as

$$\tau \frac{1}{P(T_j)} \{P'(T_j) + O[\delta T]\} = \left\{ \frac{F(T_j)}{P(T_j)} - 1 \right\}.$$

Then each term is of order τ except for the second term $\tau O[\delta T]$ which is order τ^2 and can be neglected, leading to 3.24.

To obtain the continuum approximation for the remaining equations we first define

$$k^{(i)} := k_*^{(i)}\tau, \quad \tilde{k}^{(i)} := \tilde{k}_*^{(i)}\tau, \quad m^{(i)} := m_*^{(i)}\tau, \quad n^{(i)} := n_*^{(i)}\tau.$$

We treat the $m^{(i)}$ and $n^{(i)}$ as known functions that represent the inflow/outflow of cash and shares, respectively, due to macroscopic factors. Following a similar argument as above the equations describing the cash and stock position of each group can be written as

$$\tau \frac{dM^{(i)}(t)}{dt} = -k^{(i)}(t)M^{(i)}(t) + \tilde{k}^{(i)}(t)N^{(i)}(t)F(t) + m^{(i)}(t), \quad (3.25)$$

$$\tau \frac{dN^{(i)}(t)}{dt} = \frac{k^{(i)}(t)M^{(i)}(t)}{F(t)} - \tilde{k}^{(i)}(t)N^{(i)}(t) + n^{(i)}(t). \quad (3.26)$$

We define the continuous version of the function F as

$$F(t) := \frac{k_*^{(1)}M^{(1)}(t) + k_*^{(2)}M^{(2)}(t)}{\tilde{k}_*^{(1)}N^{(1)}(t) + \tilde{k}_*^{(2)}N^{(2)}(t)}.$$

Thus, equations 3.24 - 3.26 can be studied numerically subject to initial conditions upon specifying $k^{(i)}$. By rescaling time we can set τ to unity in each of the three equations.

Equilibrium Conditions. In the totally conserved case (no shares or cash added or withdrawn from the system) setting the time derivatives in 3.24 - 3.26 to zero yields

$$P = F = \text{Constant} \quad (3.27)$$

$$0 = -k^{(i)}M^{(i)} + (1 - k^{(i)})N^{(i)}P,$$

provided we define $\tilde{k}^{(i)} := 1 - k^{(i)}$, so that a necessary and sufficient condition for equilibrium (which we denote now by P_{eq}) is

$$P_{eq} = \frac{k^{(i)}M^{(i)}}{1 - k^{(i)}N^{(i)}} \quad \text{for } i = 1, 2. \quad (3.28)$$

Note that 3.27 and 3.28 imply the identities

$$\frac{k^{(1)}M^{(1)} + k^{(2)}M^{(2)}}{(1 - k^{(1)})N^{(1)} + (1 - k^{(2)})N^{(2)}} = \frac{k^{(1)}M^{(1)}}{1 - k^{(1)}N^{(1)}} = \frac{k^{(2)}M^{(2)}}{1 - k^{(2)}N^{(2)}} \quad (3.29)$$

establishing a compatibility relationship between preferences (determined by $k^{(i)}$) and asset positions of the two groups. Thus, given a compatible set $(k^{(1)}, k^{(2)}, M^{(1)}, M^{(2)}, N^{(1)}, N^{(2)})$ there is a unique equilibrium price P_{eq} . This differs from classical theory where P_{eq} would be uniquely determined by P_a alone. If we regard the $k^{(i)}$ as known, then we have two nonlinear equations, i.e. equations 3.28, for three unknowns, namely $M^{(1)}$, $N^{(1)}$, and P .

Remark 3. *The equations 3.24 - 3.26 reduce to the single group model of [6] when the assets of one group are reduced to zero. To verify this, let $M^{(2)} = N^{(2)} = 0$ and drop the subscript 1 on the first group. Thus, Group 1 controls the total amount of cash, M , and total number of shares, N , in the system; and any trading must be among traders within Group 1. Hence, the definition of F , i.e.*

$$F = \frac{k^{(1)}M^{(1)}}{\tilde{k}^{(1)}N^{(1)}} = \frac{kM}{\tilde{k}N},$$

and the price equation 3.24 with $\tilde{k} = 1 - k$ imply the result

$$\frac{1}{P(t)} \frac{dP(t)}{dt} = \frac{1}{\tau} \left\{ \frac{k}{1-k} \frac{M}{NP(t)} - 1 \right\}.$$

The concept of a liquidity value, $L := M/N$, was introduced in this paper. With this substitution we have the equation

$$\frac{1}{P(t)} \frac{dP(t)}{dt} = \frac{1}{\tau} \left\{ \frac{k}{1-k} \frac{L}{P} - 1 \right\}.$$

In the limit of a single group without influx of additional shares or cash, the variables, M , N , and L are all constants in time. Nevertheless, the fraction of funds in the asset, B , defined by

$$B := \frac{NP}{NP + M},$$

varies in time and satisfies

$$\frac{B}{1-B} = \frac{N}{M}P = \frac{P}{L}$$

yielding the differential equation 3.5

$$\tau P^{-1} \frac{dP}{dt} = \frac{k(1-B)}{(1-k)B} - 1.$$

Hence, for any definition of $k^{(i)}$ the multi-group model defined above reduces to the single group model.

3.4 ANALYSIS OF EQUILIBRIUM AND STABILITY WITH ARBITRARY PARAMETER VALUES

As noted above, the transition rate functions, $k_{*}^{(i)}$, account for the different motivations investors have to buy or sell an asset. Through statistical modeling and regression analysis on market data [8], it has been shown that two such motivations are (i) the asset's recent trend in price and (ii) the asset price's deviation from its fundamental value. These motivations are modeled by equations (3.7). To account for multiple groups, these equations are modified as follows:

$$\zeta_1^{(i)}(t) := q_1^{(i)} c_1^{(i)} \int_{-\infty}^t e^{-c_1^{(i)}(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau \quad (3.30)$$

$$\zeta_2^{(i)}(t) := q_2^{(i)} c_2^{(i)} \int_{-\infty}^t e^{-c_2^{(i)}(t-\tau)} \frac{P_a^{(i)}(\tau) - P(\tau)}{P_a^{(i)}(\tau)} d\tau \quad (3.31)$$

where the superscript $i = 1, 2, \dots, n$ represents the investor group. The parameters, now group-specific, are discussed in Section 3.1. For example, $\zeta_2^{(i)}(t)$ represents group i 's motivation arising from the deviation between the asset's price and group i 's assessment of the asset's fundamental value, $P_a^{(i)}(t)$, when trading [14]. Thus, a quantification of an investor group's motivation to buy/sell is given by $\zeta^{(i)}(t) = \zeta_1^{(i)}(t) + \zeta_2^{(i)}(t)$.

Differentiating equations 3.30 and 3.31 yields the ordinary differential equations

$$\frac{d\zeta_1^{(i)}(t)}{dt} = q_1^{(i)} c_1^{(i)} \frac{1}{P(t)} \frac{dP(t)}{dt} - c_1^{(i)} \zeta_1^{(i)}(t) \quad (3.32)$$

$$\frac{d\zeta_2^{(i)}(t)}{dt} = q_2^{(i)} c_2^{(i)} \frac{P_a^{(i)}(t) - P(t)}{P_a^{(i)}(t)} - c_2^{(i)} \zeta_2^{(i)}(t). \quad (3.33)$$

Adding these to equations 3.24 - 3.26 gives the following system

$$\tau \frac{dN^{(i)}}{dt} = \frac{k^{(i)} M^{(i)}(t)}{F(t)} - \tilde{k}^{(i)} N^{(i)}(t) + n^{(i)}(t) \quad (3.34)$$

$$\tau \frac{dM^{(i)}}{dt} = -k^{(i)} M^{(i)}(t) + \tilde{k}^{(i)} N^{(i)}(t) F(t) + m^{(i)}(t) \quad (3.35)$$

$$\tau \frac{dP}{dt} = F(t) - P(t) \quad (3.36)$$

$$\frac{d\zeta_1^{(i)}(t)}{dt} = q_1^{(i)} c_1^{(i)} \frac{1}{P(t)} \frac{dP(t)}{dt} - c_1^{(i)} \zeta_1^{(i)}(t) \quad (3.37)$$

$$\frac{d\zeta_2^{(i)}(t)}{dt} = q_2^{(i)} c_2^{(i)} \frac{P_a^{(i)}(t) - P(t)}{P_a^{(i)}(t)} - c_2^{(i)} \zeta_2^{(i)}(t) \quad (3.38)$$

where $i = 1, 2, \dots, n$ and

$$F(t) := \frac{k^{(1)}(t)M^{(1)}(t) + k^{(2)}(t)M^{(2)}(t) + \dots + k^{(n)}(t)M^{(n)}(t)}{\tilde{k}^{(1)}(t)N^{(1)}(t) + \tilde{k}^{(2)}(t)N^{(2)}(t) + \dots + \tilde{k}^{(n)}(t)N^{(n)}(t)}.$$

Thus we have a system of $4n + 1$ equations with $5n + 1$ parameters.

For the remainder of Section 3.4 and Section 3.5 we consider the two group system with conserved cash and shares, i.e. $M^{(1)}(t) + M^{(2)}(t) = M_0$ and $N^{(1)}(t) + N^{(2)}(t) = N_0$, where M_0 and N_0 are fixed parameters representing the total amount of cash and the total number of shares in the system. In addition we make the following assumptions:

$$(A1) \quad \tilde{k}^{(i)} = 1 - k^{(i)}$$

(A2) Group 1 is focused solely on the recent trend in price and Group 2 is focused solely on the deviation from its perception of the fundamental value, $P_a^{(2)}(t)$. Classical finance suggests that the price fluctuates randomly about this value [51].

(A3) To simplify calculations we use the Taylor series approximation $\tanh(x) \simeq x$. Reasonable realistic parameter values will typically result in the argument, $\zeta_i^{(i)}(t)$, $i = 1, 2$, lying in the interval $(-1, 1)$, so that differences between the two models, i.e. with or without this approximation, should be negligible. Assumptions (A2) and (A3) imply $k^{(i)} \simeq \frac{1}{2} \left(1 + \zeta_i^{(i)} \right)$, $i = 1, 2$.

$$(A4) \quad P_a^{(2)}(t) = P_a^{(2)} \text{ is constant.}$$

$$(A5) \quad 0 \leq N^{(1)} \leq N_0 \text{ and } 0 \leq M^{(1)} \leq M_0.$$

(A6) As discussed in Sections 3.2 and 3.3, we set $\tau = 1$, i.e. one period. As noted in Section 3.3, through a rescaling of time we may set $\tau = 1$, i.e. one period, which implies $k^{(i)}(t) = k_*^{(i)}(t)$.

$$(A7) \quad M_0, N_0, q_i^{(i)}, c_i^{(i)}, \text{ and } P_a^{(i)}, i = 1, 2 \text{ are strictly positive constants.}$$

With these assumptions the system corresponding to equations 3.34 - 3.38 becomes

$$\frac{dN^{(1)}}{dt} = \frac{1}{2}(1 + \zeta_1^{(1)})M^{(1)} \frac{(-\zeta_1^{(1)} + \zeta_2^{(2)})N^{(1)} + (1 - \zeta_2^{(2)})N_0}{(\zeta_1^{(1)} - \zeta_2^{(2)})M^{(1)} + (1 + \zeta_2^{(2)})M_0} - \frac{1}{2}(1 - \zeta_1^{(1)})N^{(1)} \quad (3.39)$$

$$\frac{dM^{(1)}}{dt} = -\frac{1}{2}(1 + \zeta_1^{(1)})M^{(1)} + \frac{1}{2}(1 - \zeta_1^{(1)})N^{(1)} \frac{(\zeta_1^{(1)} - \zeta_2^{(2)})M^{(1)} + (1 + \zeta_2^{(2)})M_0}{(-\zeta_1^{(1)} + \zeta_2^{(2)})N^{(1)} + (1 - \zeta_2^{(2)})N_0} \quad (3.40)$$

$$\frac{dP}{dt} = \frac{(\zeta_1^{(1)} - \zeta_2^{(2)})M^{(1)} + (1 + \zeta_2^{(2)})M_0}{(-\zeta_1^{(1)} + \zeta_2^{(2)})N^{(1)} + (1 - \zeta_2^{(2)})N_0} - P \quad (3.41)$$

$$\frac{d\zeta_1^{(1)}}{dt} = q_1^{(1)}c_1^{(1)} \frac{1}{P} \frac{dP}{dt} - c_1^{(1)}\zeta_1^{(1)} \quad (3.42)$$

$$\frac{d\zeta_2^{(2)}}{dt} = q_2^{(2)}c_2^{(2)} \frac{P_a^{(2)} - P}{P_a^{(2)}} - c_2^{(2)}\zeta_2^{(2)} \quad (3.43)$$

where we have suppressed the dependence of $M^{(1)}$, $N^{(1)}$, P , $\zeta_1^{(1)}$, and $\zeta_2^{(2)}$ on t . Equations 3.39 and 3.40 give the following relationship

$$\frac{dN^{(1)}}{dt} = -\frac{1}{F(t)} \frac{dM^{(1)}}{dt}. \quad (3.44)$$

We define equilibrium in terms of all time derivatives set to zero. Note that equations 3.42 and 3.43 indicate that existence of equilibria is independent of the parameters $c_1^{(1)}$, $c_2^{(2)}$, and $q_1^{(1)}$. Setting the time derivatives in equations 3.41, 3.39, and 3.40 to zero yields the following relationship between $\hat{N}^{(1)}$ and $\hat{M}^{(1)}$, where the hat notation is used to denote the equilibrium value,

$$\hat{N}^{(1)} = \frac{\hat{M}^{(1)}}{P_{eq}}. \quad (3.45)$$

Thus, at equilibrium the algebraic system of equations admits a one-dimensional curve of equilibrium points that may be parameterized by one of the dynamic variables, i.e. $\hat{N}^{(1)}$, $\hat{M}^{(1)}$, P_{eq} , $\hat{\zeta}_1^{(1)}$, or $\hat{\zeta}_2^{(2)}$. Caginalp and Balenovich [6] proved that the equilibrium price for the one group model is between the liquidity value, L , and the fundamental value, P_a . We prove a similar result (see Theorem 4) for the two group model with conserved cash and shares.

As noted above, the parameter $P_a^{(2)}$ represents Group 2's assessment of the fundamental value of the asset; while the parameter $L = M_0/N_0$ equals the total amount of cash divided by the total number of shares in the system¹.

Theorem 4. *Consider the system 3.39 - 3.43 and assume (A1)-(A7). Let L and P_a be arbitrary in \mathbb{R}^+ where $L = \frac{M_0}{N_0}$ is the liquidity value and P_a is Group 2's assessment of the fundamental value of the asset.*

(i) *Suppose $L \neq P_a$. For any $P_{eq} \in \mathbb{R}^+$, $P_{eq} \in (\min[P_a, L], \max[P_a, L])$ the system has a unique equilibrium point of the form*

$$\begin{aligned} \hat{x} &= (\hat{N}^{(1)}, \hat{M}^{(1)}, P_{eq}, \hat{\zeta}_1, \hat{\zeta}_2) \\ &= \left(\frac{[P_a - q_2(P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2(P_a - P_{eq})] M_0}{2P_{eq}q_2(P_{eq} - P_a)}, P_{eq}\hat{N}^{(1)}, P_{eq}, 0, \right. \\ &\quad \left. q_2 \frac{P_a - P_{eq}}{P_a} \right). \end{aligned}$$

Consider the limiting case, $L = P_{eq} \neq P_a$. For any $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$ the system has a unique equilibrium point of the form

$$\hat{x} = (N_0, M_0, P_a(1 - \frac{\hat{\zeta}_2}{q_2}), 0, \hat{\zeta}_2).$$

(ii) *Suppose $P_{eq} = P_a$. Then $P_{eq} = P_a = L$, and for any $\hat{N}^{(1)} \in \mathbb{R}^+$, $0 \leq \hat{N}^{(1)} \leq N_0$, the system has a unique equilibrium point of the form*

$$\hat{x} = (\hat{N}^{(1)}, \hat{N}^{(1)}P_{eq}, P_{eq}, 0, 0).$$

Moreover, under the above assumptions any equilibrium point of this system must lie in the interval $[\min(P_a, L), \max(P_a, L)]$.

¹For notational convenience we set $P_a = P_a^{(2)}$, $q_2 = q_2^{(2)}$, $\hat{\zeta}_2 = \hat{\zeta}_2^{(2)}$, $\hat{\zeta}_1 = \hat{\zeta}_1^{(1)}$, $c_1 = c_1^{(1)}$, and $c_2 = c_2^{(2)}$ for the remainder of the paper.

Remark 5. *Classical finance suggests that, because (1) all traders have the same information and (2) any advantage is quickly exploited by completely rational, well informed traders with ample capital, there exists a unique equilibrium price for a given time t . For example, the basic Capital Asset Pricing Model (CAPM) assumes all traders are rational mean-variance optimizers and share a common economic view of the world, i.e. the investor population has homogeneous expectations [5]. The CAPM theory then provides a method for calculating the unique equilibrium price, which corresponds to what we have labeled P_a , the fundamental value of the asset. As noted in Appendix B, if Group 2 has the vast majority of the assets, then the formal limit of the solution(s) of the difference equations, 3.13, 3.18, and 3.19, is the solution of the classical price equation 3.1. This suggests the price, P , is a small fluctuation about this unique equilibrium price. However, if neither group has an infinite amount of capital, then our model, as shown in Theorem 4, suggests that there exists a range of equilibrium prices. Thus, our model encompasses the classical theory as a subset, and one may determine from market data and optimization of parameters which assumptions are applicable at any given time.*

Remark 6. *It is appropriate to consider the scenario $L = P_{eq} \neq P_a$ as a distinguished limit of Case (i). Indeed, in Case (i) for given M_0 , N_0 , and P_a the variable $\hat{N}^{(1)}$ can vary, forming a one-dimensional curve of equilibrium points. However, in the limit $L \rightarrow P_{eq}$, i.e. the case $L = P_{eq} \neq P_a$, for given M_0 , N_0 , and P_a , the $\hat{\zeta}_2$ variable is uniquely determined.*

Proof. Let $L, P_a \in \mathbb{R}^+$ be arbitrary.

(i) Suppose the system 3.39 - 3.43 is at equilibrium, i.e. all time derivatives vanish. We solve the resulting algebraic system of equations and denote the solution, i.e. the equilibrium point, by $\hat{x} := (\hat{N}^{(1)}, \hat{M}^{(1)}, P_{eq}, \hat{\zeta}_1, \hat{\zeta}_2)$. Equation 3.42 yields:

$$\hat{\zeta}_1 = 0 \tag{3.46}$$

while equation 3.43 yields:

$$\hat{\zeta}_2 = q_2 \frac{P_a - P_{eq}}{P_a} \tag{3.47}$$

$\hat{\zeta}_2$ is defined as $P_a > 0$ by assumption (A7) and $\hat{\zeta}_2 \neq 0$ as $P_{eq} \in (\min [P_a, L], \max [P_a, L])$.

Next, solve 3.41 for P_{eq} and then substitute this value along with $\hat{\zeta}_1 = 0$ into 3.39. This yields

$$\hat{M}^{(1)} = \hat{N}^{(1)} P_{eq}.$$

Then substituting this result into 3.41 gives

$$\begin{aligned} P_{eq} &= \frac{-\hat{\zeta}_2 \hat{M}^{(1)} + (1 + \hat{\zeta}_2) M_0}{\hat{\zeta}_2 \hat{N}^{(1)} + (1 - \hat{\zeta}_2) N_0} \\ &= \frac{-\hat{\zeta}_2 \hat{N}^{(1)} P_{eq} + (1 + \hat{\zeta}_2) M_0}{\hat{\zeta}_2 \hat{N}^{(1)} + (1 - \hat{\zeta}_2) N_0}. \end{aligned}$$

Solving for $\hat{N}^{(1)}$ yields

$$\hat{N}^{(1)} = \frac{-(1 - \hat{\zeta}_2) N_0 P_{eq} + (1 + \hat{\zeta}_2) M_0}{2 P_{eq} \hat{\zeta}_2}.$$

Using equation 3.47 we obtain

$$\begin{aligned} \hat{N}^{(1)} &= \frac{-(1 - q_2 \frac{P_a - P_{eq}}{P_a}) N_0 P_{eq} + (1 + q_2 \frac{P_a - P_{eq}}{P_a}) M_0}{2 P_{eq} q_2 \frac{P_a - P_{eq}}{P_a}} \quad (3.48) \\ &= \frac{-[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} + [P_a + q_2 (P_a - P_{eq})] M_0}{2 P_{eq} q_2 (P_a - P_{eq})}. \end{aligned}$$

Using equation 3.45 we obtain the form of an arbitrary equilibrium point

$$\begin{aligned} \hat{x} &= \left(\frac{[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2 (P_a - P_{eq})] M_0}{2 P_{eq} q_2 (P_{eq} - P_a)}, P_{eq} \hat{N}^{(1)}, \right. \\ &\quad \left. P_{eq}, 0, q_2 \frac{P_a - P_{eq}}{P_a} \right) \quad (3.49) \end{aligned}$$

for any $P_{eq} \in \mathbb{R}$, $P_{eq} \neq 0$ and $P_{eq} \neq P_a$. Substitution of the above point into system 3.39 - 3.43 shows that the point given by equation 3.49 is indeed an equilibrium point of the system.

Although the equilibrium point given by equation 3.49 holds for all $P_{eq} \in \mathbb{R}$, $P_{eq} \neq 0$ and $P_{eq} \neq P_a$, we must restrict P_{eq} to the interval $[\min(P_a, L), \max(P_a, L)]$ in order to satisfy

assumptions (A1)-(A7). Indeed, suppose the contrary, i.e. $P_{eq} > \max(P_a, L)$ and consider equation 3.48:

$$\begin{aligned}\hat{N}^{(1)} &= \frac{\{P_a - q_2(P_a - P_{eq})\} N_0 P_{eq} - \{P_a + q_2(P_a - P_{eq})\} M_0}{2P_{eq}q_2(P_{eq} - P_a)} \\ &= N_0 \frac{P_a(P_{eq} - L) + q_2(P_{eq} - P_a)(P_{eq} + L)}{2P_{eq}q_2(P_{eq} - P_a)}.\end{aligned}$$

Notice that we have $\hat{N}^{(1)} \geq 0$ as needed. However, we also require $\hat{N}^{(1)} \leq N_0$ by assumption (A5). For this latter condition we require

$$0 \leq \frac{P_a(P_{eq} - L) + q_2(P_{eq} - P_a)(P_{eq} + L)}{2P_{eq}q_2(P_{eq} - P_a)} \leq 1$$

The left inequality is satisfied. Thus we verify the right inequality:

$$\begin{aligned}\frac{P_a(P_{eq} - L) + q_2(P_{eq} - P_a)(P_{eq} + L)}{2P_{eq}q_2(P_{eq} - P_a)} \leq 1 &\Leftrightarrow \\ -1 &\geq q_2\left(1 - \frac{P_{eq}}{P_a}\right) = \hat{\zeta}_2.\end{aligned}$$

This contradicts assumption (A3) as we require $-1 < \hat{\zeta}_2 = q_2\left(1 - \frac{P_{eq}}{P_a}\right) < 1$.

Similarly, one may show $P_{eq} < \min(P_a, L)$ is not permissible.

We have proven that we cannot have $P_{eq} > \max(P_a, L)$ or $P_{eq} < \min(P_a, L)$. Thus, $\min(P_a, L) \leq P_{eq} \leq \max(P_a, L)$. An arbitrary equilibrium point is given by equation 3.49 provided $\min(P_a, L) < P_{eq} < \max(P_a, L)$. The cases where P_{eq} is constant, $P_{eq} = P_a$ and $P_{eq} = L$, are considered in (ii) and (iii) below. These cases also exhibit a one dimensional curve of equilibrium points; however, as P_{eq} is constant, we parameterize this curve by variables $\hat{N}^{(1)}$ and $\hat{\zeta}_2$, respectively.

We next consider the limiting case $P_{eq} = L \neq P_a$ and show the existence of an equilibrium for any $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$.

Suppose $P_{eq} = L$. Equation 3.41 with $\hat{\zeta}_1 = 0$ yields $P_{eq} = \frac{-\hat{\zeta}_2 \hat{M}^{(1)} + (1 + \hat{\zeta}_2) M_0}{\hat{\zeta}_2 \hat{N}^{(1)} + (1 - \hat{\zeta}_2) N_0} = L = \frac{M_0}{N_0}$.

Cross multiplying gives:

$$\hat{\zeta}_2(2M_0N_0 - \hat{M}^{(1)}N_0 - \hat{N}^{(1)}M_0) = 0 \tag{3.50}$$

which implies (1) $\hat{\zeta}_2 = 0$ or (2) $\hat{M}^{(1)} = M_0$ and $\hat{N}^{(1)} = N_0$, i.e. group 1 has all of the system's cash and shares. Indeed, assuming $\hat{\zeta}_2 \neq 0$ and dividing both sides of equation 3.50 by $\hat{\zeta}_2 M_0 N_0$ gives

$$2 - \frac{\hat{M}^{(1)}}{M_0} - \frac{\hat{N}^{(1)}}{N_0} = 0.$$

As $0 \leq \frac{\hat{M}^{(1)}}{M_0} \leq 1$ and $0 \leq \frac{\hat{N}^{(1)}}{N_0} \leq 1$ by assumption (A5), this holds if and only if $\hat{M}^{(1)} = M_0$ and $\hat{N}^{(1)} = N_0$.

If $\hat{\zeta}_2 = 0$, then $P_{eq} = P_a = L$. However, if $\hat{\zeta}_2 \neq 0$, then we must have $\frac{\hat{M}^{(1)}}{M_0} = \frac{\hat{N}^{(1)}}{N_0} = 1$. Thus, equilibrium may exist where $P_{eq} = L \neq P_a^{(2)}$, and an arbitrary equilibrium point has the form

$$\hat{x} = (N_0, M_0, P_a(1 - \frac{\hat{\zeta}_2}{q_2}), 0, \hat{\zeta}_2) \quad (3.51)$$

where $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$ is arbitrary. A simple calculation shows that \hat{x} is an equilibrium point of the system 3.39 - 3.43.

Note this result may also be obtained by taking the limit as $P_{eq} \rightarrow L$ of the arbitrary equilibrium point given by 3.49.

(ii) We show that if the equilibrium price equals group 2's estimation of the fundamental value of the asset, i.e. $P_{eq} = P_a$, then it must also equal the liquidity value, i.e. $P_{eq} = P_a = L = \frac{M_0}{N_0}$.

Indeed, suppose $P_{eq} = P_a$. Then equation 3.47 yields $\hat{\zeta}_2 = 0$ which implies $P_{eq} = \frac{M_0}{N_0} = L$ by considering equation 3.41. Thus, if $P_{eq} = P_a$, then we must have $P_{eq} = P_a = L$. By assumption (A4) P_a and thus P_{eq} is a constant. Therefore, we parameterize the curve of equilibrium points by $\hat{N}^{(1)}$. It is clear from equations 3.42 and 3.43 that $\hat{\zeta}_1 = \hat{\zeta}_2 = 0$. Using equation 3.45 an arbitrary equilibrium point is of the form

$$\hat{x} = (\hat{N}^{(1)}, \hat{N}^{(1)} P_{eq}, P_{eq}, 0, 0) \quad (3.52)$$

for any $\hat{N}^{(1)} \in \mathbb{R}$, $0 \leq \hat{N}^{(1)} \leq N_0$ and $N_0 > 0$.

As in the above case, a simple calculation shows that the point \hat{x} given by equation 3.52 is an equilibrium point of the system 3.39 - 3.43. \square

Combining assumptions (A1)-(A7) with the results from Theorem 4 allows us to rewrite the criteria for equilibrium in Corollary 7. As noted above the parameters c_1 , c_2 , and q_1 do not affect the existence of equilibrium points.

Corollary 7. *Consider the system 3.39 - 3.43 under assumptions (A1)-(A7). Then any equilibrium point(s) must satisfy the following criteria:*

Case (1) $L < P_{eq} < P_a$

$$(a) P_a > P_{eq} > 0, (b) 0 < q_2 < \frac{P_a}{P_a - P_{eq}}, \text{ and}$$

$$(c) \frac{M_0}{P_{eq}} < N_0 \leq \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]}.$$

In the limit $P_{eq} \rightarrow L$ we have the subcase $P_a > P_{eq} = L$.

$$(a) 0 < \hat{\zeta}_2 < 1, (b) q_2 > \hat{\zeta}_2, \text{ and } (c) L = P_a \frac{q_2 - \hat{\zeta}_2}{q_2}.$$

Case (2) $P_a < P_{eq} < L$

$$(a) 0 < P_a < P_{eq}, (b) 0 < q_2 < -\frac{P_a}{P_a - P_{eq}}, \text{ and}$$

$$(c) \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \leq N_0 < \frac{M_0}{P_{eq}}.$$

In the limit $P_{eq} \rightarrow L$ we have the subcase $P_a < P_{eq} = L$.

$$(a) -1 < \hat{\zeta}_2 < 0 \text{ and } (b) L = P_a \frac{q_2 - \hat{\zeta}_2}{q_2}.$$

Case (3) $P_a = P_{eq} = L$

$$(a) 0 \leq \hat{N}^{(1)} \leq N_0 \text{ with } N_0 > 0, (b) P_a = P_{eq} = L, \text{ and } (c) \hat{\zeta}_1 = \hat{\zeta}_2 = 0.$$

Remark 8. For cases (1) and (2) the given criteria may be rewritten to provide bounds for the equilibrium price, P_{eq} . Indeed, define

$$E := \frac{1}{2q_2N_0} \left[N_0P_a(-1 + q_2) - q_2M_0 + \sqrt{4q_2M_0N_0P_a(1 + q_2) + [q_2M_0 + N_0P_a(1 - q_2)]^2} \right].$$

Then for case (1) we have

$$E \geq P_{eq} > \max \left[L, \frac{P_a(q_2 - 1)}{q_2} \right] \quad (3.53)$$

and for case (2) we have

$$E \leq P_{eq} < \min \left[L, \frac{P_a(1 + q_2)}{q_2} \right]. \quad (3.54)$$

Mathematica's Reduce command was utilized to obtain these expressions. Note the quadratic formula was employed and the "negative" solution was disregarded as it conflicted with the assumptions previously made. In addition, the strict inequality is due to assumption (A3) which restricts $-1 < \hat{\zeta}_2 < 1$. Indeed, along the boundary curves $\frac{P_a(q_2-1)}{q_2}$ and $\frac{P_a(1+q_2)}{q_2}$ the variable $\hat{\zeta}_2 = 1$ and -1 , respectively. These curves correspond to the "upper" boundary curves in Figures 3.1 and 3.3.

Proof. Case (1) $L < P_{eq} < P_a$

By assumption (A3) and the condition $L < P_{eq} < P_a$, we have $0 < \hat{\zeta}_2 < 1$. As $\hat{\zeta}_2 = q_2 \frac{P_a - P_{eq}}{P_a}$ by equation 3.49 we have

$$\frac{P_a}{P_a - P_{eq}} > q_2 > 0, \quad (3.55)$$

which is (b). Also, $P_a > P_{eq}$ and $N_0 > \frac{M_0}{P_{eq}}$ imply (a) and the left inequality in (c). Thus, it remains to show:

$$\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \geq N_0.$$

From assumption (A5) we have

$$0 \leq \hat{N}^{(1)} \leq N_0.$$

Using the definition of $\hat{N}^{(1)}$ from equation 3.49 yields

$$0 \leq \frac{[P_a - q_2 (P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2 (P_a - P_{eq})] M_0}{2P_{eq} q_2 (P_{eq} - P_a)} \leq N_0.$$

Multiplying the inequality by $2P_{eq}q_2(P_{eq} - P_a)$ gives

$$0 \geq [P_a - q_2(P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2(P_a - P_{eq})] M_0 \geq 2N_0 P_{eq} q_2 (P_{eq} - P_a),$$

which is equivalent to

$$\begin{aligned} -[P_a - q_2(P_a - P_{eq})] N_0 P_{eq} &\geq -[P_a + q_2(P_a - P_{eq})] M_0 \\ &\geq N_0 P_{eq} [-q_2(P_a - P_{eq}) - P_a]. \end{aligned}$$

Dividing by $-[P_a - q_2(P_a - P_{eq})] P_{eq}$ gives the desired result

$$N_0 \leq \frac{[P_a + q_2(P_a - P_{eq})] M_0}{[P_a - q_2(P_a - P_{eq})] P_{eq}} \leq \frac{N_0 [P_a + q_2(P_a - P_{eq})]}{[P_a - q_2(P_a - P_{eq})]}.$$

Note that from equation 3.55 we have

$$0 < P_a - q_2(P_a - P_{eq}) < P_a.$$

This justifies reversing the inequalities when we divided by $-[P_a - q_2(P_a - P_{eq})] P_{eq}$ above.

We next consider the limiting case $P_a > P_{eq} = L$. The criterion $P_a > P_{eq} = L$ along with assumption (A3) implies (a). Condition (c) holds as equation 3.51 gives $L = P_{eq} = P_a(1 - \frac{\hat{\zeta}_2}{q_2})$. Condition (b) is necessary to ensure $P_{eq} > 0$. Indeed, by equation 3.51

$$P_{eq} = P_a(1 - \frac{\hat{\zeta}_2}{q_2}).$$

The inequality $q_2 \leq \hat{\zeta}_2$ implies $P_{eq} \leq 0$, which is not feasible. Thus, $q_2 > \hat{\zeta}_2$ is necessary.

Case (2) $P_a < P_{eq} < L$

By assumption (A3) and the condition $P_a < P_{eq} < L$ we have $-1 < \hat{\zeta}_2 < 0$. Again, equation 3.49 implies $\hat{\zeta}_2 = q_2 \frac{P_a - P_{eq}}{P_a}$ which leads to

$$-\frac{P_a}{P_a - P_{eq}} > q_2 > 0, \tag{3.56}$$

namely, (b). Also, $P_a < P_{eq}$ and $N_0 < \frac{M_0}{P_{eq}}$ verify (a) and the right inequality in (c). Thus, it remains to show:

$$\frac{M_0 [P_a + q_2(P_a - P_{eq})]}{P_{eq} [P_a - q_2(P_a - P_{eq})]} \leq N_0.$$

By assumption (A5) we have

$$0 \leq \hat{N}^{(1)} \leq N_0.$$

Using the definition of $\hat{N}^{(1)}$ from equation 3.49 yields

$$0 \leq \frac{[P_a - q_2(P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2(P_a - P_{eq})] M_0}{2P_{eq}q_2(P_{eq} - P_a)} \leq N_0.$$

Multiplying the inequality by $2P_{eq}q_2(P_{eq} - P_a)$ gives

$$0 \leq [P_a - q_2(P_a - P_{eq})] N_0 P_{eq} - [P_a + q_2(P_a - P_{eq})] M_0 \leq 2N_0 P_{eq} q_2 (P_{eq} - P_a),$$

which is equivalent to

$$\begin{aligned} -[P_a - q_2(P_a - P_{eq})] N_0 P_{eq} &\leq -[P_a + q_2(P_a - P_{eq})] M_0 \\ &\leq N_0 P_{eq} [-q_2(P_a - P_{eq}) - P_a]. \end{aligned}$$

Dividing by $-[P_a - q_2(P_a - P_{eq})] P_{eq}$ gives the desired result

$$N_0 \geq \frac{[P_a + q_2(P_a - P_{eq})] M_0}{[P_a - q_2(P_a - P_{eq})] P_{eq}} \geq \frac{N_0 [P_a + q_2(P_a - P_{eq})]}{[P_a - q_2(P_a - P_{eq})]}.$$

Note that from equation 3.56 we have

$$2P_a > P_a - q_2(P_a - P_{eq}) > P_a > 0$$

This justifies reversing the inequalities when we divided by $-[P_a - q_2(P_a - P_{eq})] P_{eq}$ above.

We next consider the limiting case $P_a < P_{eq} = L$. It is easily seen that the criterion $P_a < P_{eq} = L$ along with (A3) implies (a). Condition (b) holds because in this case $L = P_{eq} = P_a(1 - \frac{\hat{\zeta}_2}{q_2})$ by equation 3.51.

Case (3) $P_a = P_{eq} = L$

It is clear from the arbitrary form of an equilibrium point, equation 3.52, that criteria (a)-(c) hold. □

Now that criteria for equilibria have been established, we consider the stability of an arbitrary equilibrium point for each of the four cases identified in Corollary 7. For stability, we linearize about an arbitrary equilibrium point and consider the eigenvalues of the corresponding Jacobian matrix evaluated at the equilibrium point. As noted above, this system admits a curve of equilibria. As such, this system will always have a zero eigenvalue with corresponding eigenvector tangent to this curve of equilibria. In addition, we will see that -1 is also a root of the characteristic polynomial in all four cases. Thus, consider the following definition:

Definition 9. *The equilibrium point is considered stable if the remaining three eigenvalues have real parts strictly less than zero and unstable if any one of these eigenvalues has a real part greater than zero.*

Theorem 10 utilizes the Routh-Hurwitz criterion (see Appendix B) to prove the existence of stable and unstable regions of equilibrium points within the appropriate (parameter) spaces for each of the four cases. As 0 and -1 are always roots, we focus on the cubic factor of the characteristic polynomial. Routh-Hurwitz applied to a cubic polynomial ($\alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$) can be stated as: The real parts of all roots are strictly negative if and only if $\alpha_i > 0$, $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ [20].

We define the following constants which will be used in the statement and proof of Theorem 10:

$$\begin{aligned}
A &:= M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] \\
B &:= M_0 [P_{eq}^2 q_2 + P_a^2 (1 + q_2) - P_a P_{eq} (3 + 2q_2)] \\
&\quad + N_0 P_{eq} [-P_a^2 (-1 + q_2) - P_{eq}^2 q_2 + P_a (P_{eq} + 2P_{eq} q_2)] \\
C &:= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq} q_2 + P_a (-2q_1 + q_2)] \\
&\quad - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq} q_2 - P_a (2q_1 + q_2)] \\
D &:= [(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C] [c_2 q_2 B + c_1 \{(P_a - P_{eq}) q_2 A + c_2 C\}]
\end{aligned}$$

Recall that $q_1 > 0$ by assumption (A7). This agrees with empirical studies on both experimental [13] and world market [8] data which show that if the trend is positive (alternatively negative) then the price continues to rise (alternatively fall) due to momentum traders.

Theorem 10. Assume (A1)-(A7). (i) There exists a stable region of equilibrium points, \hat{x} , for the system 3.39 - 3.43. For each of the following cases, one has the following sufficient criteria for the stability of the equilibrium point:

(1) $L < P_{eq} < P_a$:

$$(q_1 < \frac{1}{2}) \text{ and } (c_1 < 1 + c_2);$$

(2) $P_a < P_{eq} < L$:

$$(q_1 < \frac{1}{2}) \text{ and } (c_1 < 1 + c_2);$$

(3) $P_a = P_{eq} = L$:

$$q_1^{(1)} < \min \left[\frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}, \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}} \right].$$

(ii) There exists an unstable region of equilibrium points. In each of the three cases one has the following sufficient criteria for instability:

(1) $L < P_{eq} < P_a$:

$$\begin{aligned} & \frac{-M_0 + N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{P_a}{P_a - P_{eq}} < q_2 < \frac{P_a}{P_a - P_{eq}} \text{ and} \\ q_1 & > \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \text{ and} \\ c_1 & > \frac{-(1 + c_2) (P_a - P_{eq}) q_2 A}{C}; \end{aligned}$$

(2) $P_a < P_{eq} < L$:

$$\begin{aligned} & \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{-P_a}{P_a - P_{eq}} < q_2 < \frac{-P_a}{P_a - P_{eq}} \text{ and} \\ q_1 & > \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \text{ and} \\ c_1 & > \frac{-(1 + c_2) (P_a - P_{eq}) q_2 A}{C}; \end{aligned}$$

(3) $P_a = P_{eq} = L$:

$$q_1^{(1)} > \frac{(1 + c_1 + c_2) N_0}{2c_1 \hat{N}^{(1)}}.$$

In addition, for $L = P_{eq} \neq P_a$, a distinguished limit of cases (1) and (2), equilibrium points are stable provided

$$q_1 < \frac{1 + c_1}{2c_1}$$

and unstable for

$$q_1 > \frac{1 + c_1}{2c_1}.$$

Remark 11. *The conditions for stability are consistent with the intuition of the model, as a small trend coefficient, q_1 , and a large time scale for trend, $1/c_1$, favor stability (see cases (1), (2) of (i)). The stable condition for the distinguished limit case ($L = P_{eq} \neq P_a$) shows that one can have stability for large values of q_1 provided the trend time scale is large. Furthermore, in case (3) stability can also be maintained for larger values of q_1 if the number of shares of the trend-based group is small compared with the total number of shares in the system.*

Remark 12. *In cases (1), (2), and (3) the existence of stable and unstable regions is shown. However, for the case $L = P_{eq} \neq P_a$ we provide necessary and sufficient criteria for the stability and instability of an arbitrary equilibrium point.*

Proof. Let $\hat{x} = (\hat{N}^{(1)}, \hat{M}^{(1)}, P_{eq}, \hat{\zeta}_1, \hat{\zeta}_2)$ be an arbitrary equilibrium point.

Case (1) $L < P_{eq} < P_a$

(i) We first prove existence of a stable region of equilibrium points. Linearizing about

$$\hat{x} = \left(\frac{[P_a^{(2)} - q_2(P_a^{(2)} - P_{eq})]N_0P_{eq} - [P_a^{(2)} + q_2(P_a^{(2)} - P_{eq})]M_0}{2P_{eq}q_2(P_{eq} - P_a^{(2)})}, P_{eq}, \hat{N}^{(1)}, \right. \\ \left. P_{eq}, 0, q_2 \frac{P_a^{(2)} - P_{eq}}{P_a^{(2)}} \right)$$

yields the following characteristic polynomial

$$\lambda(\lambda + 1) (\alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0) \tag{3.57}$$

where

$$\alpha_3 = 1 \\ \alpha_2 = \frac{(1 + c_2)(P_a - P_{eq})q_2A + c_1C}{(P_a - P_{eq})q_2A} \\ \alpha_1 = \frac{c_2q_2B + c_1[(P_a - P_{eq})q_2A + c_2C]}{(P_a - P_{eq})q_2A} \\ \alpha_0 = \frac{c_1c_2B}{(P_a - P_{eq})A}.$$

The zero eigenvalue is due to the curve of equilibria. As such, it does not impact the stability. Thus, \hat{x} is stable if the roots of the cubic factor of the characteristic polynomial have strictly negative real parts. By the Routh-Hurwitz criterion these roots lie in the left half of the complex plane if and only if $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$. Note that

$$\begin{aligned} & \alpha_2\alpha_1 - \alpha_3\alpha_0 \\ &= \frac{-c_1c_2(P_a - P_{eq})q_2^2AB}{(P_a - P_{eq})^2q_2^2A^2} \\ &+ \frac{[(1 + c_2)(P_a - P_{eq})q_2A + c_1C][c_2q_2B + c_1\{(P_a - P_{eq})q_2A + c_2C\}]}{(P_a - P_{eq})^2q_2^2A^2}. \end{aligned}$$

Any equilibrium point must satisfy the criteria given by Corollary 7, which for this case is

$$\begin{aligned} & \text{(a) } P_a > P_{eq} > 0, \text{ (b) } 0 < q_2 < \frac{P_a}{P_a - P_{eq}}, \\ & \text{and (c) } \frac{M_0}{P_{eq}} < N_0 \leq \frac{M_0[P_a + q_2(P_a - P_{eq})]}{P_{eq}[P_a - q_2(P_a - P_{eq})]}. \end{aligned}$$

Using these conditions in conjunction with $L < P_{eq} < P_a$ we show that A , B , and C are positive, which, in turn, implies the existence of a region in which $\alpha_i > 0$ for $i = 1, 2, 3, 4$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ (i.e. \hat{x} is stable). In fact, $A > 0$ by (a) and (b). Next, show B is positive. Indeed,

$$\begin{aligned} B &= M_0[P_{eq}^2q_2 + P_a^2(1 + q_2) - P_aP_{eq}(3 + 2q_2)] \\ &+ N_0P_{eq}[-P_a^2(-1 + q_2) - P_{eq}^2q_2 + P_a(P_{eq} + 2P_{eq}q_2)] \\ &= M_0[q_2(P_a - P_{eq})^2 + P_a(P_a - 3P_{eq})] + N_0P_{eq}[-q_2(P_a - P_{eq})^2 + P_a^2 + P_aP_{eq}] \\ &> M_0[q_2(P_a - P_{eq})^2 + P_a(P_a - 3P_{eq})] \\ &+ N_0P_{eq}\left[-\frac{P_a}{P_a - P_{eq}}(P_a - P_{eq})^2 + P_a^2 + P_aP_{eq}\right] \text{ by (b)} \\ &= M_0[q_2(P_a - P_{eq})^2 + P_a(P_a - 3P_{eq})] + 2N_0P_{eq}P_aP_{eq} \\ &> M_0[P_a(P_a - 3P_{eq})] + 2N_0P_{eq}P_aP_{eq} = M_0[P_a^2 - 3P_aP_{eq}] + 2N_0P_{eq}P_aP_{eq} \\ &> M_0[P_a^2 - 3P_aP_{eq}] + 2M_0P_aP_{eq} \text{ by (c)} \\ &= M_0[P_a^2 - P_aP_{eq}] > 0 \text{ by (a)}. \end{aligned}$$

We claim there exists a region in parameter space where C is also greater than zero. Based upon the current case and assumptions we have $M_0 [P_a + q_2 (P_a - P_{eq})] > 0$ and $N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] > 0$, while $[P_{eq} q_2 - P_a (2q_1 + q_2)]$ is strictly less than zero. Thus, if $[-P_{eq} q_2 + P_a (-2q_1 + q_2)] \geq 0$ (i.e. $q_1 \leq \frac{q_2(P_a - P_{eq})}{2P_a}$), then $C > 0$. Alternatively, suppose $[-P_{eq} q_2 + P_a (-2q_1 + q_2)] \leq 0$ (i.e. $q_1 \geq \frac{q_2(P_a - P_{eq})}{2P_a}$). Then $C > 0$ provided $\frac{M_0}{P_{eq}} < N_0$ and $q_1 < \frac{1}{2}$. Indeed, one has

$$\begin{aligned}
C &= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq} q_2 + P_a (-2q_1 + q_2)] \\
&\quad - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq} q_2 - P_a (2q_1 + q_2)] \\
&> M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq} q_2 + P_a (-2q_1 + q_2)] \\
&\quad - M_0 [P_a - q_2 (P_a - P_{eq})] [P_{eq} q_2 - P_a (2q_1 + q_2)] \\
&= M_0 [P_a + q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) - 2P_a q_1] \\
&\quad + M_0 [P_a - q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) + 2P_a q_1] \\
&= 2M_0 P_a q_2 (P_a - P_{eq}) - 4M_0 q_2 q_1 P_a (P_a - P_{eq}) \\
&= 2M_0 P_a q_2 (P_a - P_{eq}) (1 - 2q_1) \\
&> 0 \text{ provided } 0 < q_1 < \frac{1}{2}.
\end{aligned}$$

So, if $[-P_{eq} q_2 + P_a (-2q_1 + q_2)] \leq 0$, then $\frac{q_2(P_a - P_{eq})}{2P_a} \leq q_1 < \frac{1}{2}$ ensures $C > 0$.

So, $A > 0$ and $B > 0$ for this case and the above assumptions. One also has $C > 0$ provided

$$q_1 < \frac{1}{2}.$$

Furthermore, A and B positive implies $\alpha_0 > 0$; A and C positive implies $\alpha_2 > 0$; and A , B , and C positive give α_1 positive. Finally, consider $\alpha_2 \alpha_1 - \alpha_3 \alpha_0$. With A , B , and C positive, the terms

$$\begin{aligned}
&[(1 + c_2) (P_a - P_{eq}) q_2 A + c_1 C] \\
&\text{and } [c_2 q_2 B + c_1 \{(P_a - P_{eq}) q_2 A + c_2 C\}]
\end{aligned}$$

are positive. Thus, $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ if

$$\begin{aligned} | -c_1c_2(P_a - P_{eq})q_2^2AB | < D &\Leftrightarrow \\ c_1c_2(P_a - P_{eq})q_2^2AB < D & \end{aligned}$$

where

$$D = [(1 + c_2)(P_a - P_{eq})q_2A + c_1C][c_2q_2B + c_1\{(P_a - P_{eq})q_2A + c_2C\}].$$

The product of the first terms in each factor D is

$$(1 + c_2)(P_a - P_{eq})c_2q_2^2AB.$$

Thus, this inequality holds and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ provided $c_1 < 1 + c_2$.

Therefore, if $q_1 < \frac{1}{2}$ and $(c_1 < 1 + c_2)$, then the equilibrium is stable.

(ii) Using the Routh-Hurwitz criterion we show that an unstable region exists by showing $\alpha_2 < 0$. Indeed, one has

$$\alpha_2 = \frac{(1 + c_2)(P_a - P_{eq})q_2A + c_1C}{(P_a - P_{eq})q_2A}$$

where

$$\begin{aligned} A &:= M_0[P_a + q_2(P_a - P_{eq})] + N_0P_{eq}[P_a - q_2(P_a - P_{eq})] \\ C &:= M_0[P_a + q_2(P_a - P_{eq})][-P_{eq}q_2 + P_a(-2q_1 + q_2)] \\ &\quad - N_0P_{eq}[P_a - q_2(P_a - P_{eq})][P_{eq}q_2 - P_a(2q_1 + q_2)]. \end{aligned}$$

A is positive. This implies that both the denominator and the first term in the numerator are positive. Thus C must be negative.

$$\begin{aligned} C < 0 &\Leftrightarrow \\ \frac{(P_a - P_{eq})q_2A}{2P_a\{M_0[P_a + q_2(P_a - P_{eq})] - N_0P_{eq}[P_a - q_2(P_a - P_{eq})]\}} &< q_1 \end{aligned}$$

provided

$$2P_a\{M_0[P_a + q_2(P_a - P_{eq})] - N_0P_{eq}[P_a - q_2(P_a - P_{eq})]\} > 0 \quad (3.58)$$

which holds for

$$\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} > N_0. \quad (3.59)$$

Recall equilibrium conditions (b) and (c) for this case

$$(b) 0 < q_2 < \frac{P_a}{P_a - P_{eq}} \text{ and (c) } \frac{M_0}{P_{eq}} < N_0 \leq \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]}.$$

Note equation 3.59 is the strict version of the right inequality in (c). For equation 3.59 and subsequently 3.58 to hold we require $q_2 \neq \frac{(-M_0 + N_0 P_{eq})}{(M_0 + N_0 P_{eq})} \frac{P_a}{P_a - P_{eq}}$. Thus, using condition (b), for $C < 0$ we restrict q_2 to the interval

$$\frac{-M_0 + N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{P_a}{P_a - P_{eq}} < q_2 < \frac{P_a}{P_a - P_{eq}}$$

and require

$$\begin{aligned} q_1 &> \frac{(P_a - P_{eq}) q_2 A}{2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\}} \\ &> \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}}. \end{aligned}$$

Choosing $c_1 > \frac{-(1+c_2)(P_a - P_{eq})q_2 A}{C}$ gives $\alpha_2 < 0$. Since $\alpha_3 = 1 > 0$ there is at least one sign change in the first column of the Routh array. By the Routh-Hurwitz criterion there is at least one eigenvalue with positive real part. Thus, the equilibrium is unstable.

Case (2) $P_a < P_{eq} < L$

(i) We prove existence of a stable region of equilibrium points. The critical point, \hat{x} , and characteristic polynomial both have the same form as in Case (1). Similar to the proof of Case (1), we utilize the equilibrium conditions given by Corollary 7 to show that $A > 0$, $B < 0$, and $C < 0$. These criteria are:

$$\begin{aligned} (a) 0 < P_a < P_{eq}, (b) 0 < q_2 < -\frac{P_a}{P_a - P_{eq}}, \\ \text{and (c) } \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} &\leq N_0 < \frac{M_0}{P_{eq}}. \end{aligned}$$

Note that $\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} > 0$. Indeed, $[P_a - q_2 (P_a - P_{eq})] > 0$ by (a). In addition, $[P_a + q_2 (P_a - P_{eq})] > 0$ by (b).

Using these equilibrium conditions and assumption (A7) we show that $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ (i.e. \hat{x} is stable). Indeed, $A > 0$ by (a) and (b). Next, we show B is negative. Indeed,

$$\begin{aligned}
B &= M_0 [P_{eq}^2 q_2 + P_a^2 (1 + q_2) - P_a P_{eq} (3 + 2q_2)] \\
&+ N_0 P_{eq} [-P_a^2 (-1 + q_2) - P_{eq}^2 q_2 + P_a (P_{eq} + 2P_e q_2)] \\
&= M_0 [q_2 (P_a - P_{eq})^2 + P_a^2 - 3P_a P_{eq}] + N_0 P_{eq} [-q_2 (P_a - P_{eq})^2 + P_a^2 + P_a P_{eq}] \\
&< M_0 [-\frac{P_a}{P_a - P_{eq}} (P_a - P_{eq})^2 + P_a^2 - 3P_a P_{eq}] + N_0 P_{eq} [0 + P_a^2 + P_a P_{eq}] \text{ by (b)} \\
&= M_0 [-P_a (P_a - P_{eq}) + P_a^2 - 3P_a P_{eq}] + N_0 P_{eq} [P_a^2 + P_a P_{eq}] \\
&< M_0 [-2P_a P_{eq}] + M_0 [P_a^2 + P_a P_{eq}] \text{ by (c)} \\
&= M_0 [-P_a P_{eq} + P_a^2] \\
&< 0 \text{ by (a)}.
\end{aligned}$$

We will show that there exists a region in parameter space where C is also negative. Based upon (a) and (b) and assumption (A7) we have $M_0 [P_a + q_2 (P_a - P_{eq})] > 0$ and $N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] > 0$, while $[-P_{eq} q_2 + P_a (-2q_1 + q_2)]$ is strictly less than zero. Therefore, if $[P_{eq} q_2 - P_a (2q_1 + q_2)] \geq 0$ (i.e. $q_1 \leq \frac{-q_2(P_a - P_{eq})}{2P_a}$), then $C < 0$. Alternatively, suppose $[P_{eq} q_2 - P_a (2q_1 + q_2)] \leq 0$ (i.e. $q_1 \geq \frac{-q_2(P_a - P_{eq})}{2P_a}$). Then $C < 0$ provided $N_0 < \frac{M_0}{P_{eq}}$ and $q_1 < \frac{1}{2}$. Indeed,

$$\begin{aligned}
C &= M_0 [P_a + q_2 (P_a - P_{eq})] [-P_{eq} q_2 + P_a (-2q_1 + q_2)] \\
&- N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [P_{eq} q_2 - P_a (2q_1 + q_2)] \\
&= M_0 [P_a + q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) - 2q_1 P_a] \\
&+ N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) + 2q_1 P_a] \\
&< M_0 [P_a + q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) - 2q_1 P_a] \\
&+ M_0 [P_a - q_2 (P_a - P_{eq})] [q_2 (P_a - P_{eq}) + 2q_1 P_a] \text{ by (c)} \\
&= 2M_0 P_a q_2 (P_a - P_{eq}) - 4M_0 P_a q_2 (P_a - P_{eq}) q_1 \\
&= 2M_0 P_a q_2 (P_a - P_{eq}) [1 - 2q_1] \\
&< 0 \text{ provided } 0 < q_1 < \frac{1}{2}.
\end{aligned}$$

So, if $[P_{eq}q_2 - P_a(2q_1 + q_2)] \leq 0$, then $\frac{-q_2(P_a - P_{eq})}{2P_a} \leq q_1 < \frac{1}{2}$ ensures $C < 0$.

So, $A > 0$ and $B < 0$ for this case and the above assumptions. $C < 0$ provided

$$q_1 < \frac{1}{2}.$$

$A > 0$ and $B < 0$ imply $\alpha_0 > 0$; $A > 0$ and $C < 0$ imply $\alpha_2 > 0$; and $A > 0$, $B < 0$, and $C < 0$ give α_1 positive. Finally, consider $\alpha_2\alpha_1 - \alpha_3\alpha_0$. With $A > 0$, $B < 0$, and $C < 0$ the terms $[(1 + c_2)(P_a - P_{eq})q_2A + c_1C]$ and $[c_2q_2B + c_1\{(P_a - P_{eq})q_2A + c_2C\}]$ are negative. Thus, $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ if

$$\begin{aligned} | -c_1c_2(P_a - P_{eq})q_2^2AB | < D &\Leftrightarrow \\ c_1c_2(P_a - P_{eq})q_2^2AB < D & \\ &= (1 + c_2)(P_a - P_{eq})c_2q_2^2AB \\ &+ (1 + c_2)(P_a - P_{eq})c_1q_2A\{(P_a - P_{eq})q_2A + c_2C\} \\ &+ c_1c_2q_2BC + c_1^2C\{(P_a - P_{eq})q_2A + c_2C\}. \end{aligned}$$

where

$$D = [(1 + c_2)(P_a - P_{eq})q_2A + c_1C][c_2q_2B + c_1\{(P_a - P_{eq})q_2A + c_2C\}].$$

Notice that each term on the right side of this inequality is positive for this case. Further, the first term on the right side of the inequality is greater than the term on the left side of the inequality provided $c_1 < 1 + c_2$. Thus, if $c_1 < 1 + c_2$, then $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$.

Therefore, if $q_1 < \frac{1}{2}$ and $c_1 < 1 + c_2$, then the equilibrium is stable.

(ii) Using the Routh-Hurwitz criterion we show that an unstable region exists by showing $\alpha_2 < 0$. Indeed,

$$\alpha_2 = \frac{(1 + c_2)(P_a - P_{eq})q_2A + c_1C}{(P_a - P_{eq})q_2A}$$

where

$$\begin{aligned} A &:= M_0 [P_a + q_2(P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2(P_a - P_{eq})] \\ C &:= M_0 [P_a + q_2(P_a - P_{eq})] [-P_{eq}q_2 + P_a(-2q_1 + q_2)] \\ &\quad - N_0 P_{eq} [P_a - q_2(P_a - P_{eq})] [P_{eq}q_2 - P_a(2q_1 + q_2)]. \end{aligned}$$

A is positive. This implies that both the denominator and the first term in the numerator are negative. Thus, we need C to be positive. Consider

$$C > 0 \Leftrightarrow q_1 > \frac{(P_a - P_{eq}) q_2 A}{2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\}}$$

provided

$$2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\} > 0 \quad (3.60)$$

which holds for

$$\frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} < N_0. \quad (3.61)$$

Recall equilibrium conditions (b) and (c) for this case

$$(b) \ 0 < q_2 < -\frac{P_a}{P_a - P_{eq}} \quad \text{and} \quad (c) \ \frac{M_0 [P_a + q_2 (P_a - P_{eq})]}{P_{eq} [P_a - q_2 (P_a - P_{eq})]} \leq N_0 < \frac{M_0}{P_{eq}}.$$

Note equation 3.61 is the strict version of the left inequality in (c). For equation 3.61 and subsequently 3.60 to hold we require $q_2 \neq \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{-P_a}{P_a - P_{eq}}$. Thus, using condition (b), for $C > 0$ we restrict q_2 to the interval

$$\frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}} \frac{-P_a}{P_a - P_{eq}} < q_2 < \frac{-P_a}{P_a - P_{eq}}$$

and require

$$\begin{aligned} q_1 &> \frac{(P_a - P_{eq}) q_2 A}{2P_a \{M_0 [P_a + q_2 (P_a - P_{eq})] - N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]\}} \\ &> \frac{A}{-M_0 [P_a + q_2 (P_a - P_{eq})] + N_0 P_{eq} [P_a - q_2 (P_a - P_{eq})]} \frac{1}{2} \frac{M_0 - N_0 P_{eq}}{M_0 + N_0 P_{eq}}. \end{aligned}$$

Choosing $c_1 > \frac{-(1+c_2)(P_a - P_{eq})q_2 A}{C}$ gives $\alpha_2 < 0$. Since $\alpha_3 = 1 > 0$ there is at least one sign change in the first column of the Routh array. By the Routh-Hurwitz criterion there is at least one eigenvalue with positive real part. Thus, the equilibrium is unstable.

Case (3) $L = P_{eq} = P_a$

(i) Linearizing about $\hat{x} = (\hat{N}^{(1)}, \hat{N}^{(1)}P_{eq}, P_{eq}, 0, 0)$ yields the following characteristic polynomial

$$-\lambda(1+\lambda) \left\{ \begin{array}{l} \lambda^3 + \lambda^2[(1+c_1+c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1] \\ + \lambda[c_1 + c_2 + c_1 c_2(1 - 2\frac{\hat{N}^{(1)}}{N_0} q_1) + 2c_2 q_2(1 - \frac{\hat{N}^{(1)}}{N_0})] \\ + c_1 c_2[1 + 2q_2(1 - \frac{\hat{N}^{(1)}}{N_0})] \end{array} \right\} \quad (3.62)$$

where

$$\begin{aligned} \alpha_3 &= 1 \\ \alpha_2 &= (1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 \\ \alpha_1 &= c_1 + c_2 + c_1 c_2(1 - 2\frac{\hat{N}^{(1)}}{N_0} q_1) + 2c_2 q_2(1 - \frac{\hat{N}^{(1)}}{N_0}) \\ \alpha_0 &= c_1 c_2[1 + 2q_2(1 - \frac{\hat{N}^{(1)}}{N_0})] \end{aligned}$$

The zero eigenvalue is due to the curve of equilibria. As such, it does not impact the stability. Thus, \hat{x} is stable if the roots of the cubic factor of the characteristic polynomial have strictly negative real parts. By the Routh-Hurwitz criterion these roots lie in the left half of the complex plane if and only if $\alpha_i > 0$ for $i = 0, 1, 2, 3$ and $\alpha_2 \alpha_1 - \alpha_3 \alpha_0 > 0$. Note that $0 \leq \frac{\hat{N}^{(1)}}{N_0} \leq 1$ by assumption (A5).

Consider $\alpha_2 > 0$, i.e.

$$\begin{aligned} (1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 &> 0 \Leftrightarrow \\ q_1 &< \frac{(1 + c_1 + c_2) N_0}{2c_1 \hat{N}^{(1)}}. \end{aligned}$$

Next, determine a region where $\alpha_1 > 0$. Indeed, if $q_1 < \frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}$, then

$$\begin{aligned} \alpha_1 &= c_1 + c_2 + c_1 c_2(1 - 2\frac{\hat{N}^{(1)}}{N_0} q_1) + 2c_2 q_2(1 - \frac{\hat{N}^{(1)}}{N_0}) \\ &> 0 \text{ (Note this is independent of } q_2\text{.)} \end{aligned}$$

Note that $\alpha_0 > 0$ by assumptions (A5) and (A7). Finally, consider $\alpha_2\alpha_1 - \alpha_3\alpha_0$.

$$\begin{aligned}
& \alpha_2\alpha_1 - \alpha_3\alpha_0 \\
&= \left[(1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 \right] \left[c_1 + c_2 + c_1 c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) + 2c_2 q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&\quad - c_1 c_2 \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&= \left[(1 + c_1 + c_2) - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 \right] \left[c_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} + c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \right] \\
&\quad - c_1 c_2 \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&= (1 + c_1 + c_2) c_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} + (1 + c_1 + c_2) c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \\
&\quad - 2c_1^2 \frac{\hat{N}^{(1)}}{N_0} q_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \\
&\quad - c_1 c_2 \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right]
\end{aligned}$$

If $q_1 < \frac{(1+c_1+c_2) N_0}{2c_1 \hat{N}^{(1)}}$, which ensures $\alpha_2 > 0$, and $q_1 < \frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}$, which ensures $\alpha_1 > 0$, then

$$(1 + c_1 + c_2) c_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} - 2c_1^2 \frac{\hat{N}^{(1)}}{N_0} q_1 \left\{ 1 + c_2 \left(1 - 2 \frac{\hat{N}^{(1)}}{N_0} q_1 \right) \right\} > 0.$$

Thus, we obtain

$$\begin{aligned}
& (1 + c_1 + c_2) c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 \left\{ 1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right\} \\
&\quad - c_1 c_2 \left[1 + 2q_2 - 2 \frac{\hat{N}^{(1)}}{N_0} q_2 \right] \\
&= \left[(1 + c_2) c_2 - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 \right] \left[1 + 2q_2 \left(1 - \frac{\hat{N}^{(1)}}{N_0} \right) \right] \\
&> 0
\end{aligned}$$

provided

$$\begin{aligned}
(1 + c_2) c_2 - 2c_1 \frac{\hat{N}^{(1)}}{N_0} q_1 c_2 > 0 &\Leftrightarrow \\
q_1 < \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}}.
\end{aligned}$$

Thus, the equilibrium point \hat{x} is stable if

$$0 < q_1 < \min\left[\frac{(1 + c_1 + c_2) N_0}{2c_1 \hat{N}^{(1)}}, \frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}, \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}}\right],$$

i.e.

$$0 < q_1 < \min\left[\frac{1}{2} \frac{N_0}{\hat{N}^{(1)}}, \frac{1 + c_2}{2c_1} \frac{N_0}{\hat{N}^{(1)}}\right].$$

as $c_1 > 0$ by assumption (A7).

(ii) With respect to instability, if $q_1 > \frac{(1+c_1+c_2) N_0}{2c_1 \hat{N}^{(1)}}$, then $\alpha_2 < 0$. Since $\alpha_3 = 1 > 0$ there is at least one sign change in the first column of the Routh array. By the Routh-Hurwitz criterion there is at least one eigenvalue with positive real part. Thus, the equilibrium is unstable.

Finally, we consider the limiting case of cases (1) and (2): $L = P_{eq} \neq P_a$. Linearizing about

$$\hat{x} = (N_0, M_0, P_a(1 - \frac{\hat{\zeta}_2}{q_2}), 0, \hat{\zeta}_2)$$

yields the following characteristic polynomial

$$-\lambda(1 + \lambda)(c_2 + \lambda) [\lambda^2 + \{1 + c_1(1 - 2q_1)\} \lambda + c_1]. \quad (3.63)$$

The zero eigenvalue is due to the curve of equilibria. As such, it does not impact the stability. Thus, \hat{x} is stable if the roots of the quadratic factor of the characteristic polynomial have strictly negative real parts. Note that stability only depends on the q_1 and c_1 parameters.

The quadratic formula yields

$$\lambda_{+,-} = \frac{-\{1 + c_1(1 - 2q_1)\} \pm \sqrt{\{1 + c_1(1 - 2q_1)\}^2 - 4c_1}}{2}.$$

Note that

$$\sqrt{\{1 + c_1(1 - 2q_1)\}^2 - 4c_1} < \sqrt{\{1 + c_1(1 - 2q_1)\}^2} = |-\{1 + c_1(1 - 2q_1)\}|.$$

As such, there are two cases to consider.

Case 1. $\{1 + c_1(1 - 2q_1)\}^2 - 4c_1 > 0$ and $-\{1 + c_1(1 - 2q_1)\} < 0$. If these criteria are met then we have $\lambda_{+,-} \in \mathbb{R}$ and $\lambda_{+,-} < 0$. These criteria are met when $q_1 < \frac{-2\sqrt{c_1+1+c_1}}{2c_1}$. Indeed, consider

$$\begin{aligned} \{1 + c_1(1 - 2q_1)\}^2 - 4c_1 > 0 &\Leftrightarrow \\ \{1 + c_1(1 - 2q_1)\}^2 > 4c_1 &\Leftrightarrow \\ q_1 < \frac{-2\sqrt{c_1+1+c_1}}{2c_1}. \end{aligned}$$

Note that $-2\sqrt{c_1+1+c_1} > 0 \Leftrightarrow 1 + c_1 > 2\sqrt{c_1} \Leftrightarrow (1 + c_1)^2 > 4c_1 \Leftrightarrow (1 - c_1)^2 > 0 \Leftrightarrow c_1 \neq 1$. Next, consider

$$\begin{aligned} -\{1 + c_1(1 - 2q_1)\} < 0 &\Leftrightarrow \\ q_1 < \frac{1 + c_1}{2c_1}. \end{aligned}$$

So, for the above criteria to be met we need $q_1 < \min \left[\frac{-2\sqrt{c_1+1+c_1}}{2c_1}, \frac{1+c_1}{2c_1} \right] = \frac{-2\sqrt{c_1+1+c_1}}{2c_1}$ with $c_1 \neq 1$.

Case 2. $\{1 + c_1(1 - 2q_1)\}^2 - 4c_1 \leq 0$ and $-\{1 + c_1(1 - 2q_1)\} < 0$. If these criteria are met then we have $\text{Im}(\lambda_{+,-}) \geq 0$. The second criteria ensures $\text{Re}(\lambda_{+,-}) < 0$. These criteria are met when $\frac{-2\sqrt{c_1+1+c_1}}{2c_1} < q_1 < \frac{1+c_1}{2c_1}$. Indeed, consider

$$\begin{aligned} \{1 + c_1(1 - 2q_1)\}^2 - 4c_1 \leq 0 &\Leftrightarrow \\ q_1 \geq \frac{-2\sqrt{c_1+1+c_1}}{2c_1} \end{aligned}$$

And the second criterion gives

$$\begin{aligned} \Re(\lambda_{+,-}) < 0 &\Leftrightarrow \\ -\{1 + c_1(1 - 2q_1)\} < 0 &\Leftrightarrow \\ q_1 < \frac{1 + c_1}{2c_1}. \end{aligned}$$

Thus, $\text{Re}(\lambda_{+,-}) < 0$ provided $\frac{-2\sqrt{c_1+1+c_1}}{2c_1} \leq q_1 < \frac{1+c_1}{2c_1}$.

Note that since

$$\begin{aligned}\sqrt{\{1 + c_1(1 - 2q_1)\}^2 - 4c_1} &< \sqrt{\{1 + c_1(1 - 2q_1)\}^2} \\ &= |-\{1 + c_1(1 - 2q_1)\}|,\end{aligned}$$

$\text{Re}(\lambda_{+,-}) < 0$ if $0 < q_1 < \frac{1+c_1}{2c_1}$. Further, this also indicates that if $q_1 > \frac{1+c_1}{2c_1}$, then the equilibrium is unstable. \square

3.5 EQUILIBRIUM AND STABILITY FOR SPECIFIC PARAMETER VALUES

The system 3.39 - 3.43 has eight parameters $(\tau, c_1, c_2, q_1, q_2, M_0, N_0, P_a)$ which may assume any real and positive value. Now that the existence of stable and unstable regions has been established, we compute specific regions of stability and instability. To this end we set some of these parameters to specific values in order to facilitate the analysis of equilibrium and stability. We set $c_1 = \frac{1}{10}$ and $c_2 = 1$ indicating that Group 1 focuses on the price trend over the past two weeks, while Group 2 considers the current day's deviation from its assessment of the asset's fundamental value. The values for M_0 and N_0 are set at 30,000 and 3,000 so that $L = 10$. P_a is set to 12, 8, or 10 depending upon the case. These are typical values for L and P_a in that the fundamental value of the asset is set to 120% or 80% of the liquidity value. Note that for L , P_a , and P it is their relative magnitudes rather than absolute values that are essential.

We consider the effect of the q_2 parameter on the existence of equilibrium points, and the effects of the q_1 and q_2 parameters on the stability of these equilibrium points. It is clear from equations 3.42 and 3.43 that the existence of equilibria is independent of the values for the q_1 , c_1 , and c_2 parameters. Given Theorem 4, we determine the conditions for the existence and stability of equilibria for the following four cases: (1) $L < P_{eq} < P_a$, (2) $P_a < P_{eq} < L$, (3) $P_{eq} = P_a = L$, and (4) $L = P_{eq} \neq P_a$, a distinguished limit of cases (1) and (2). Note that assumptions (A1)-(A7) are still applicable to all results in this section.

Refer to Appendix F for a discussion of how stability is affected by the c_1 and c_2 parameters with the q_1 and q_2 parameters fixed.

Case 1. $10 = L < P_{eq} < P_a = 12$

Equilibrium

Let $P_{eq} \in (10, 12)$ be arbitrary. Then equation 3.49 gives the form of an equilibrium point dependent upon the value of q_2 . All P_{eq} between L and P_a yield an equilibrium point of the system 3.39 - 3.43; however, only some (P_{eq}, q_2) pairs yield an equilibrium satisfying assumptions (A1)-(A7). The region of (P_{eq}, q_2) pairs that yield permissible equilibrium points is derived from Corollary 7 and given by

$$L < P_{eq} < P_a \text{ and } \frac{(P_{eq} - L) P_a}{(L + P_{eq})(P_a - P_{eq})} \leq q_2 < \frac{P_a}{P_a - P_{eq}}.$$

For the specific parameter values given above the region is described by

$$10 < P_{eq} < 12 \text{ and } \frac{(P_{eq} - 10) 12}{(10 + P_{eq})(12 - P_{eq})} \leq q_2 < \frac{12}{12 - P_{eq}}$$

and shown in Figure 3.1. Note the "lower" boundary curve is solid indicating it corresponds to permissible equilibrium points of the system. The "upper" boundary curve is dashed indicating it is not part of the equilibrium region.

The shape of this region is analogous to the Caginalp and Balenovich [6] finding for the single group model. Specifically, a large value for q_2 implies P_{eq} is close to P_a , while a small q_2 means that P_{eq} must be close to L . Indeed, suppose Group 2 is strongly influenced by the price's deviation from P_a . If the price is above (below) this value, then Group 2 will sell (buy) the asset; thereby, lowering (raising) the price. Thus, it is reasonable to expect the equilibrium price, P_{eq} , to be close to P_a . Refer to Appendix C for more details regarding the relationship between q_2 , P_{eq} , P_a , and L .

Stability

Linearization about an arbitrary equilibrium point yields a Jacobian matrix, which is included in Appendix E. The characteristic polynomial can be factored into the form:

$$\lambda(\lambda + 1)(\text{cubic polynomial in } \lambda).$$

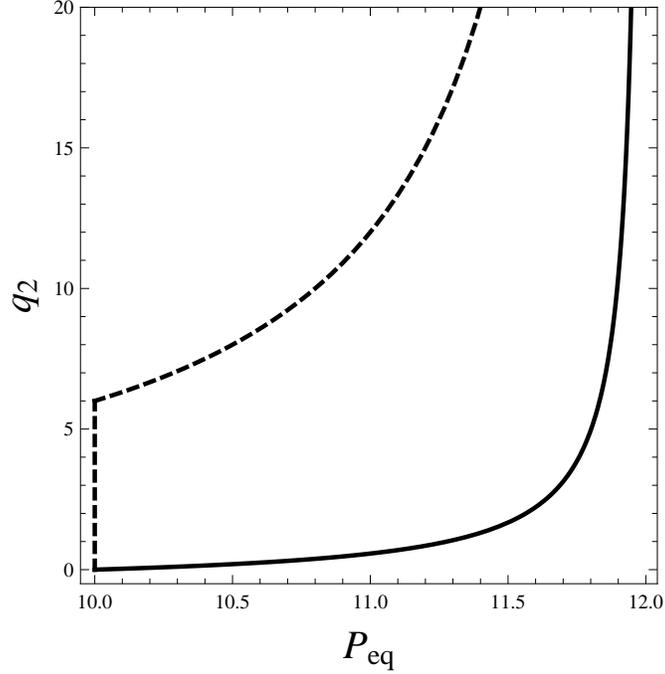


Figure 3.1: Region of (P_{eq}, q_2) -pairs that yield permissible, i.e. satisfy assumptions (A1)-(A7), equilibrium points ($10 = L < P_a = 12$).

Thus, as noted above, two of the jacobian's five eigenvalues are 0 and -1 . We are interested in the signs of the real parts of the remaining three eigenvalues. These eigenvalues correspond to the roots of the cubic factor of the characteristic polynomial which is given by equation 3.57.

We utilize the Routh-Hurwitz criterion to determine the regions of stability in the (P_{eq}, q_2) -plane. To determine the effect q_1 has on stability, we set² $q_1 = 0.447, 10, 20$. These regions, which are nested subsets (the larger subset corresponding to the smaller q_1 value), are shown in Figure 3.2.

The union of the Blue, Red, and Yellow regions, which corresponds to $q_1 = 0.447$, is identical to the equilibrium region plotted in Figure 3.1. Thus, we see that all equilibrium

²Using linear regressions on experimental data, Caginalp and Ilieva [13] found a q_1 of 0.447 and a q_2 of 0.073 for their closed book experiment. Thus, we utilize these values in Cases 1-4 as appropriate.

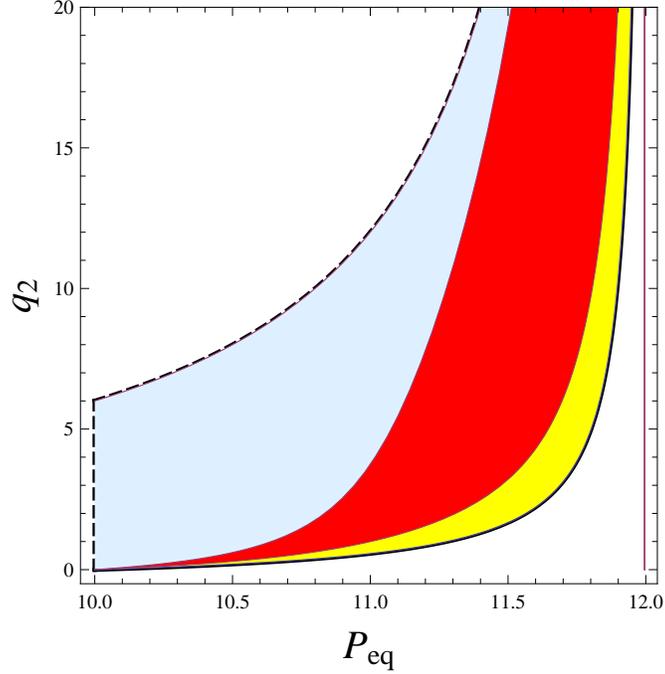


Figure 3.2: Region of (P_{eq}, q_2) pairs for $L = 10$ and $P_a = 12$ corresponding to $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. Areas outside of the designated colored regions but inside the dashed/solid black lines correspond to non-stable equilibria.

points are stable for $q_1 = 0.447$. As previously noted, the region of equilibria does not change as the value of q_1 changes. However, the value of q_1 does affect the regions of stable and non-stable equilibria. For example, for $q_1 = 20$ all equilibrium points inside the Yellow region in Figure 3.2 are stable, while the equilibrium points in the Blue and Red regions are not stable. Note that while the "lower" boundary curve of the stable region(s) is included within the stable region(s), the "upper" boundary curves (i.e. curve separating Blue and Red regions and curve between Red and Yellow regions) are not. Indeed, the Routh-Hurwitz criterion (via Mathematica) indicates the system admits a pair of pure imaginary eigenvalues along these curves. Thus, the equilibrium points along these curves are marginally stable. Refer to Scenario (iii) in Appendix D with $\alpha_i > 0$, $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 = 0$.

The q_2 parameter controls the existence of equilibria, while the q_1 parameter is a key factor in the stability of the equilibria (at least for this parameter region). For example, if q_2 is large, then the equilibrium price, P_{eq} , must be close to P_a for stability. The magnitude of the q_1 parameter determines how close P_{eq} must be to P_a for this equilibrium to be stable. Alternatively, if Group 1 is very strongly focused on the price trend while Group 2 is not strongly affected by the asset price's deviation from the fundamental value (i.e., large q_1 and small q_2), then P_{eq} must be very close to L for stability. Indeed, from Figure 3.2 it is clear in this scenario that the equilibrium is most likely not stable.

Case 2. $8 = P_a < P_{eq} < L = 10$

Equilibrium

This case contains the same curve of equilibrium points (prior to substituting any parameter values) as in Case 1 where $P_{eq} \neq P_a$ may assume any value between P_a and L . Requiring these equilibrium points to satisfy assumptions (A1)-(A7) restricts the set of (P_{eq}, q_2) -pairs that yield equilibrium to the following region

$$P_a < P_{eq} < L \text{ and } \frac{(P_{eq} - L) P_a}{(L + P_{eq})(P_a - P_{eq})} \leq q_2 < \frac{-P_a}{P_a - P_{eq}}$$

which is derived from Corollary 7. For the specific parameter values given above the region is described by

$$8 < P_{eq} < 10 \text{ and } \frac{(P_{eq} - 10) 8}{(10 + P_{eq})(8 - P_{eq})} \leq q_2 < \frac{-8}{8 - P_{eq}}$$

and shown in Figure 3.3. Similar to Case 1, note the "lower" boundary curve is included within the equilibrium region, while the "upper" boundary curve is not part of the equilibrium region.

Large q_2 implies P_{eq} is close to P_a , while small values for q_2 mean P_{eq} must be close to L . Again, this result is analogous to that of [6] for the single group model. Refer to Appendix C for more details regarding the relationship between q_2 , P_{eq} , P_a , and L .

Stability

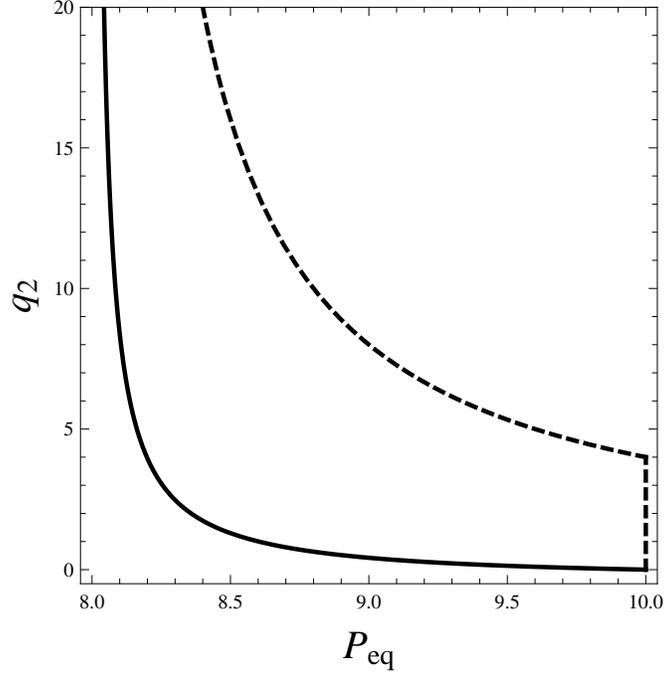


Figure 3.3: Region of (P_{eq}, q_2) -pairs that yield permissible, i.e. satisfy assumptions (A1)-(A7), equilibrium points ($8 = P_a < L = 10$).

The Jacobian matrix and characteristic polynomial for this case are identical (prior to setting parameter values) to those of Case 1. Refer to Appendix E for the Jacobian matrix and equation 3.57 for the characteristic polynomial.

Using the Routh-Hurwitz criterion, we determine regions in the (P_{eq}, q_2) -plane that yield stable equilibrium points. To understand the effect of the q_1 parameter on stability, we produce plots for $q_1 = 0.447, 10, \text{ and } 20$. These plots are displayed in Figure 3.4. As q_1 increases, each corresponding region is nested within the prior region.

The union of the Blue, Red, and Yellow regions corresponds to $q_1 = 0.447$ and is identical to the region of permissible equilibrium points in Figure 3.3. Thus, we see that all equilibrium points are stable for $q_1 = 0.447$. As in Case 1, the q_2 parameter controls the existence of equilibria, while q_1 is a key factor in determining stability. As the value of the q_1 parameter increases, the region of equilibria remains unchanged, while the region of

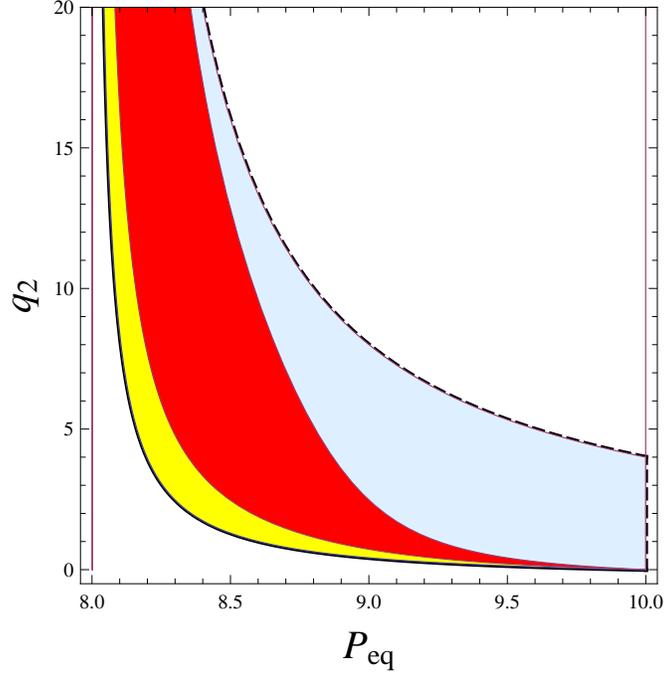


Figure 3.4: Region of (P_{eq}, q_2) pairs for $P_a = 8$ and $L = 10$ corresponding to $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. Areas outside of the designated colored regions but inside the dashed/solid black lines correspond to non-stable equilibria.

(P_{eq}, q_2) -pairs that yield stable equilibria gets smaller. Thus, if $q_1 = 20$, then the Yellow region corresponds to stable equilibria, while the Blue and Red regions corresponds to non-stable equilibria. Note that if Group 2 is strongly focused on the asset price's deviation from the fundamental value and Group 1 is not strongly motivated by the recent price trend (i.e., q_1 is small and q_2 is large), then P_{eq} must remain close to P_a for stability.

Similar to Case 1 note that while the "lower" boundary curve of the stable region(s) is included within the stable region(s), the "upper" boundary curves are not. Indeed, the Routh-Hurwitz criterion (via Mathematica) indicates the system admits a pair of pure imaginary eigenvalues along these curves. Thus, the equilibrium points along these curves are marginally stable. Refer to Scenario (iii) in Appendix D with $\alpha_i > 0$, $i = 0, 1, 2, 3$ and

$$\alpha_2\alpha_1 - \alpha_3\alpha_0 = 0.$$

It has been noted previously through experimental [15], theoretical [6], and empirical ([8] and [9]) studies that an increase in the amount of cash available to purchase an asset boosts the asset's price. We increase the value of L (by increasing M_0 while holding N_0 fixed) to study the effect of an increase of cash on the system. Figure 3.5 contains the plots of the stability regions corresponding to $L = 20$ and 30, respectively.

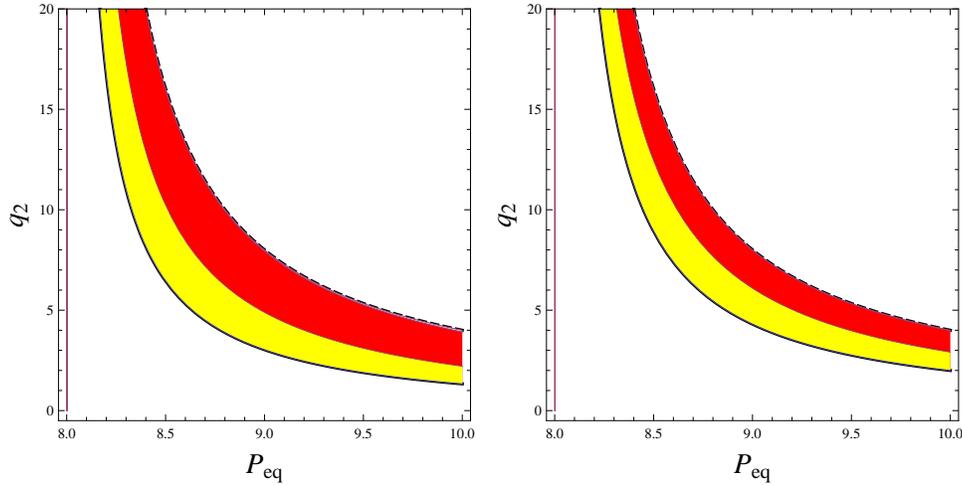


Figure 3.5: Region of (P_{eq}, q_2) pairs for $P_a = 8$ corresponding to $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. The left plot corresponds to $L = 20$ and the right $L = 30$. Compare these with Figure 3.4.

As shown in Figure 3.5, the region of permissible equilibrium points (dashed lines) becomes smaller as L increases, though the overall shape is similar. Note that a similar phenomenon occurs as L is reduced with P_a fixed.

Case 3. $P_a = P_{eq} = L = 10$

Equilibrium

For any $\hat{N}^{(1)}$ such that $0 \leq \hat{N}^{(1)} \leq N_0$ and $N_0 > 0$ equation 3.52 yields an equilibrium point of the system 3.39 - 3.43. In contrast with Cases 1 and 2, any $(\hat{N}^{(1)}, q_2)$ -pair yields an equilibrium point satisfying assumptions (A1)-(A7).

Stability

The Jacobian matrix evaluated at an arbitrary equilibrium point is given in Appendix E while the characteristic polynomial is given by equation 3.62.

Letting q_1 and q_2 assume any real and positive number, the Routh-Hurwitz criterion yields the stable region $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ contained in $(N^{(1)}, q_1, q_2)$ space where

$$\Omega_1 = \left[0 < q_1 < \frac{11}{2} \text{ and } 0 \leq \hat{N}^{(1)} \leq 3,000 \text{ and } q_2 > 0 \right],$$

$$\Omega_2 = \left[q_1 = \frac{11}{2} \text{ and } 0 \leq \hat{N}^{(1)} < 3,000 \text{ and } q_2 > 0 \right],$$

$$\Omega_3 = \frac{11}{2} < q_1 \leq 10$$

$$\text{and } \left\{ \begin{array}{l} \left(0 \leq \hat{N}^{(1)} \leq \frac{16,500}{q_1} \text{ and } q_2 > 0 \right) \\ \text{or } \left(\begin{array}{l} \frac{16,500}{q_1} < \hat{N}^{(1)} < 3,000 \\ \text{and } q_2 > \frac{-544,500,000 + 49,500\hat{N}^{(1)}q_1 - (\hat{N}^{(1)}q_1)^2}{900,000,000 - 300,000\hat{N}^{(1)} - 30,000\hat{N}^{(1)}q_1 + 10(\hat{N}^{(1)})^2q_1} \end{array} \right) \end{array} \right\},$$

and

$$\Omega_4 = q_1 > 10$$

$$\text{and } \left\{ \begin{array}{l} \left(0 \leq \hat{N}^{(1)} \leq \frac{16,500}{q_1} \text{ and } q_2 > 0 \right) \\ \text{or } \left(\begin{array}{l} \frac{16,500}{q_1} < \hat{N}^{(1)} < \frac{30,000}{q_1} \\ \text{and } q_2 > \frac{-544,500,000 + 49,500\hat{N}^{(1)}q_1 - (\hat{N}^{(1)}q_1)^2}{900,000,000 - 300,000\hat{N}^{(1)} - 30,000\hat{N}^{(1)}q_1 + 10(\hat{N}^{(1)})^2q_1} \end{array} \right) \end{array} \right\}.$$

This region is displayed in Figure 3.6 for various q_1 and q_2 values. Note that larger q_1 values yield smaller stable regions while larger q_2 values correspond to larger stable regions.

From the diagram on the left in Figure 3.6 we see that the region of stable equilibrium points increases in size as the value of the q_2 parameter increases. Note that from this diagram we see that for large q_1 if $\hat{N}^{(1)} \approx N_0$, then the equilibrium point is not stable. The diagram on the right in Figure 3.6 shows that for small q_1 all equilibrium points are stable and as q_1 increases the region of stable equilibrium points gets smaller.

For the diagram on the left in Figure 3.6 the "upper" boundary curves for the stable regions are not included within the stable regions. Indeed, along these curves the system

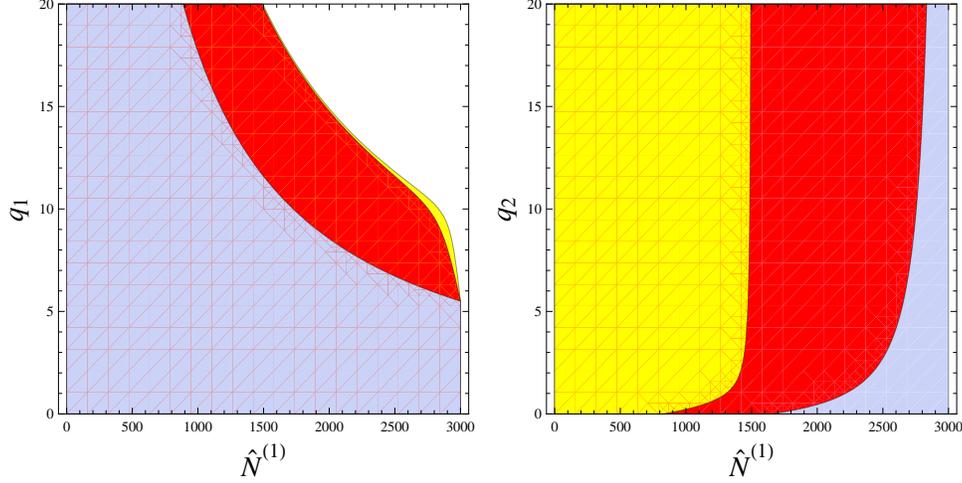


Figure 3.6: Regions in $(\hat{N}^{(1)}, q_1)$ and $(\hat{N}^{(1)}, q_2)$ space for $10 = L = P_{eq} = P_a$ corresponding to stability. Diagram on left shows $(\hat{N}^{(1)}, q_1)$ pairs for $q_2 = 0.073$ (Blue), 10 (Blue and Red), and 20 (Blue, Red, and Yellow) that yield stability. The diagram on the right displays $(\hat{N}^{(1)}, q_1)$ pairs for $q_1 = 0.447$ (Blue, Red, and Yellow), 10 (Red and Yellow), and 20 (Yellow) that yield stability. Note that any point within the plotted region yields an equilibrium point of the system. Areas outside of the designated colored regions correspond to non-stable equilibria.

admits a pair of pure imaginary eigenvalues. For the diagram on the right in Figure 3.6 the equilibrium points along the curves separating the Yellow and Red regions and the Red and Blue regions are not stable. Indeed, along these curves the system admits a pair of pure imaginary eigenvalues. Mathematica's Reduce command was utilized to confirm these results via the Routh-Hurwitz criterion.

Case 4. $P_{eq} = L = 10 \neq P_a$

Equilibrium

For any $\hat{\zeta}_2 \in (-1, 1) \setminus \{0, q_2\}$ an equilibrium point is given by equation 3.51. Note that P_a may be less than or greater than $P_{eq} = L$. As such, we consider two scenarios: $0 < P_a < L$ and $P_a > L$.

If $0 < P_a < L$, then the surface of equilibrium points in (ζ_2, q_2, P_a) - space is given by³

$$-1 < \hat{\zeta}_2 < 0 \text{ and } q_2 > 0 \text{ and } P_a = \frac{10q_2}{q_2 - \hat{\zeta}_2},$$

and if $P_a > L$, then this surface is described by

$$0 < \hat{\zeta}_2 < q_2 \text{ and } P_a = \frac{10q_2}{q_2 - \hat{\zeta}_2}.$$

In both cases we have ensured these equilibrium points satisfy assumptions (A1)-(A7). These regions are depicted graphically in Figure 3.7.

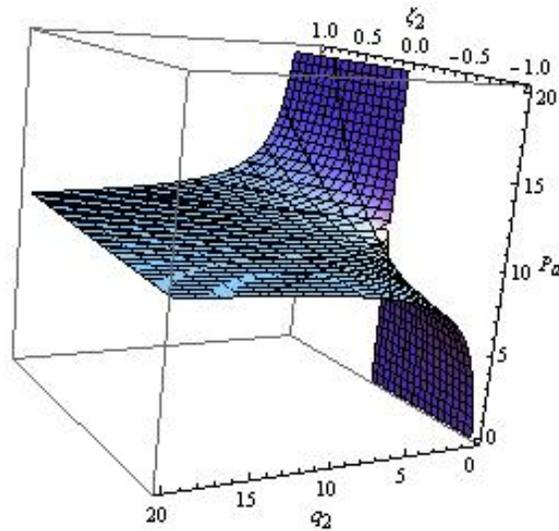


Figure 3.7: Surface of $(\hat{\zeta}_2, q_2, P_a)$ - triples that yield permissible, i.e. satisfy assumptions (A1)-(A7), equilibrium points ($10 = P_{eq} = L \neq P_a$).

Note that Figure 3.7 shows that equilibrium may exist for values of P_a that are drastically different from P_{eq} .

Stability

With $P_a = \frac{10q_2}{q_2 - \hat{\zeta}_2}$ (found by solving the P_{eq} critical point equation for P_a) the Jacobian matrix evaluated at an arbitrary equilibrium point is given by:

³The actual criteria necessary to ensure a valid equilibrium is $\hat{\zeta}_2$ less than zero. However, as noted above we assume $|\hat{\zeta}_2| < 1$ which gives the first inequality.

$$\begin{bmatrix} \frac{1}{2}(-1 + \hat{\zeta}_2) & \frac{1 + \hat{\zeta}_2}{20} & 0 & 0 & 0 \\ 5 - 5\hat{\zeta}_2 & \frac{1}{2}(-1 - \hat{\zeta}_2) & 0 & 0 & 0 \\ -\frac{\hat{\zeta}_2}{300} & -\frac{\hat{\zeta}_2}{3000} & -1 & 20 & 0 \\ -\frac{q_1 \hat{\zeta}_2}{30000} & -\frac{q_1 \hat{\zeta}_2}{300000} & -\frac{q_1}{100} & \frac{1}{10}(-1 + 2q_1) & 0 \\ 0 & 0 & -\frac{q_2}{P_a} & 0 & -1 \end{bmatrix}$$

with characteristic equation:

$$-\frac{1}{10}\lambda(1 + \lambda)^2 \{1 + (11 - 2q_1)\lambda + 10\lambda^2\} = 0.$$

This system has a zero eigenvalue and a negative one eigenvalue with algebraic multiplicity two. Thus, the sign of the real parts of the remaining two eigenvalues will determine the stability of the equilibrium point. Note that the stability is independent of q_2 and only depends upon the value of parameter q_1 .

The remaining eigenvalues are of the form:

$$\lambda_{4,5} = \frac{-11 + 2q_1 \pm \sqrt{81 - 44q_1 + 4q_1^2}}{20}.$$

The real parts of these eigenvalues are negative for $q_1 < \frac{11}{2}$, equal to zero for $q_1 = \frac{11}{2}$, and positive for $q_1 > \frac{11}{2}$. So, the equilibrium point is stable for $q_1 \in (-\infty, \frac{11}{2})$ and unstable for $q_1 \in (\frac{11}{2}, \infty)$. It is interesting to note that the point of transition from stability to instability, $q_1 = \frac{11}{2}$, is also significant in Case 3, $P_{eq} = P_a = L$, in that this is the point where parameter q_1 begins to influence the stability.

3.6 CONCLUSION

The stability of financial markets is of crucial practical importance; however, studying stability is not feasible within classical models that are idealizations based on near equilibrium conditions. A key assumption in classical finance is the existence of an infinite amount of capital controlled by experts, free of bias, and unanimous in their assessment of true value of the asset. Thus any study of price dynamics within this setting leads to prices that evolve

quickly and smoothly to the equilibrium price with some random fluctuations, or noise, along the way. The equations developed by Caginalp and collaborators since 1990 model asset price dynamics by utilizing basic microeconomics of supply and demand but also allowing these functions to depend on motivations beyond valuation. Also inherent in this approach is the finiteness of assets and the possibility that different investor groups (controlling parts of the total asset pool) may also differ in their assessments of the value of an asset. We show in Appendix A that the formal limit of solutions of this model corresponds to solutions of the classical price equation 3.1. Thus, this model encompasses the classical model upon removal of the generalized features.

After a precise derivation of the link between the discrete (difference equations) and the continuum (differential equations) for the multi-group models, we study the stability properties with the objective of determining the parameter regions for equilibrium, stability, and instability. An important difference between the neoclassical dynamics and our asset flow models is that equilibrium is not uniquely determined by the valuation in the latter. Instead it is determined by a set of algebraic equations incorporating the characteristics of each group. In other words, given a particular valuation (held by Group 2 that focuses on value) there are a spectrum of equilibrium prices. One has a unique price given the other parameters such as the cash and share position of each group, etc.

Specifically, for the two group system with conserved cash and shares we prove (Section 3.4) the existence of a 1-dimensional curve of equilibrium points under various parameter regimes. In addition, this equilibrium price, P_{eq} , must lie between the liquidity value,

$$L := (\text{total amount of cash}) / (\text{total number of shares}),$$

and Group 2's assessment of the fundamental value, P_a . In Theorem 10 the existence of local stable and unstable regions for an arbitrary equilibrium point is established. We note that existence of equilibrium is independent of the q_1 parameter, i.e. the magnitude of trend based investing does not impact the existence of equilibrium.

Numerical computations (Section 3.5) with specific parameter values establish the exact stable regions for an equilibrium point. In general, we showed that if Group 2 is strongly focused on valuation, i.e. q_2 is large, then the equilibrium price is close to P_a . The strength

of Group 1's motivation due to the recent price trend determines how close P_{eq} must be to either P_a or L to ensure stability. For instance, if q_1 is large, i.e. Group 1 is very strongly focused on the recent trend in price, and q_2 is small, then P_{eq} must be close to L for stability (see Figure 3.2).

The results show that in Cases 1, 2, and 4 the parameter q_2 controls the existence of equilibria, while in Case 3 existence of equilibria is independent of the q_2 parameter. In all four cases, the q_1 parameter plays a key role in determining the stability. In Cases 1, 2, and 3 as q_1 increases, the region of (P_{eq}, q_2) pairs that yield stable equilibria gets smaller (see Figures 3.2, 3.4, and 3.6). In Case 4 the equilibrium point is stable for $q_1 < 11/2$ and unstable for $q_1 > 11/2$. In Case 3 as the value for q_2 increases, the stable region also grows larger. Figure 3.6 also shows that if Group 1 is strongly focused on the trend, i.e. q_1 large, and owns a vast majority of the shares, then the equilibrium is most likely not stable. Our results regarding the influence of the q_1 parameter are analogous to those of Duran [21] where he finds that the market price is more sensitive to small changes in the q_1 parameter than the q_2 parameter.

Thus the asset flow models are capable of addressing stability issues. The remaining challenge is to establish a stronger connection with world markets by estimating the assets controlled by different groups characterized by their trading strategy. This requires not only estimation of the q_1 and q_2 parameters but also an estimate of the magnitudes of the assets of each group. With these estimates one can develop criteria that lead to instability in markets.

4.0 CONCLUSION AND FUTURE RELATED PROJECTS

The Quantitative Modeling portion of this dissertation provides a quantitative methodology for determining the effect of various factors on the return. Moreover, the effect of any hypothesized factor that can be expressed quantitatively may be determined via this approach. While the studies presented focus on closed-end funds, the methodology may be applied to any set of funds provided a suitable valuation methodology is employed. One of the criticisms of Behavioral Finance is that its insights and ideas are not easily verified with real market data. The approach outlined in this dissertation provides a method to test these philosophical insights thereby transforming them into practical tools.

There is strong statistical support that the return depends significantly on (i) the recent price trend in a nonlinear manner, (ii) recent changes in valuation, (iii) recent changes in money supply (M2), (iv) longer term trend, and (v) recent volume changes. Proximity to a recent high price is a marginally significant factor. The dependence on the volatility is more subtle, as short term volatility has a positive influence, while the longer term is negative. The cubic nonlinearity in the weighted price trend shows that, in the absence of any valuation change, a percentage daily gain of up to 2.78% (Study 1 (Non-standardized variables): Table 2.3) or 3.75 standard deviations (Study 2 (standardized variables): Table 2.12) tends to yield higher prices, but larger gains lead to lower prices. Thus, the nonlinearity of price trend establishes an empirical and quantitative basis for both underreaction and overreaction within one large data set, facilitating an understanding of these competing motivations in markets. Increasing money supply is found to have a significant positive effect on stock price, while proximity to recent high prices has a slight (and marginal) negative effect.

The Analysis of Asset Flow Differential Equations section presents the derivation of the ordinary differential equations model that has been utilized by Caginalp and collaborators

since 1990. Specifically, the two investor group model is considered. This model admits a one-dimensional curve of equilibria, which is parameterized by the equilibrium price, P_{eq} . Three key quantities, all with the same units of dollars per share, are identified in [6]. They are the equilibrium price, the fundamental value of a share of the asset, P_a , and the liquidity value, L , which is defined as the total amount of money in the system divided by the total number of shares. It is proven that any equilibrium of the system must have its equilibrium price between the fundamental value and the liquidity value, i.e. $P_{eq} \in [\min(P_a, L), \max(P_a, L)]$. Moreover, any price between P_a and L corresponds to an equilibrium point. Theorem 10 proves the existence of stable and unstable regions of equilibria within the parameter space.

Future work will include an investigation into the effect of the investor groups' strategies, as determined by the c_i and q_i parameters, on the existence of stable and unstable equilibria. Specifically, the transition from stable to unstable is a function of these parameters. As such, understanding how this transition point behaves will help to illuminate the impact of changes in an investor group's strategy. From a global perspective, these parameters also impact the solution trajectory's "excursion" from unstable to stable equilibrium points. This is particularly interesting from a practical point of view as these excursions can be quite large and, thus, can be directly related to how risky an investment is. For example, consider Figure 4.1 which was produced via Matlab for the following parameter values¹: $M_0 = 100$, $N_0 = 10$, $P_a = 8$, $q_1 = 1/10$, $q_2 = 0.073$, $c_1 = 1/10$, and $c_2 = 1$. Even though the values for M_0 and N_0 differ from those in Section 3.5, their ratio, which is the value of interest, is the same, i.e. $L = M_0/N_0 = 10$.

Consider trajectory 4. While the trajectory's starting price of 9.939 is close to its ending price of 9.834, at its furthest point from the curve of equilibria it has a price of 55.304 which is almost 7 times the fundamental value of the asset. Thus, this model provides an avenue for exploring instability in asset prices and risk. Note the curve of equilibria ranges from $P_{eq} = 9.6953$ to $P_{eq} = 10$. This range is derived according to the expression given in the Remark to Corollary 7.

¹To produce 4.1 assumption (A3) was relaxed, i.e. it is no longer assumed that $\tanh(x) \approx x$.

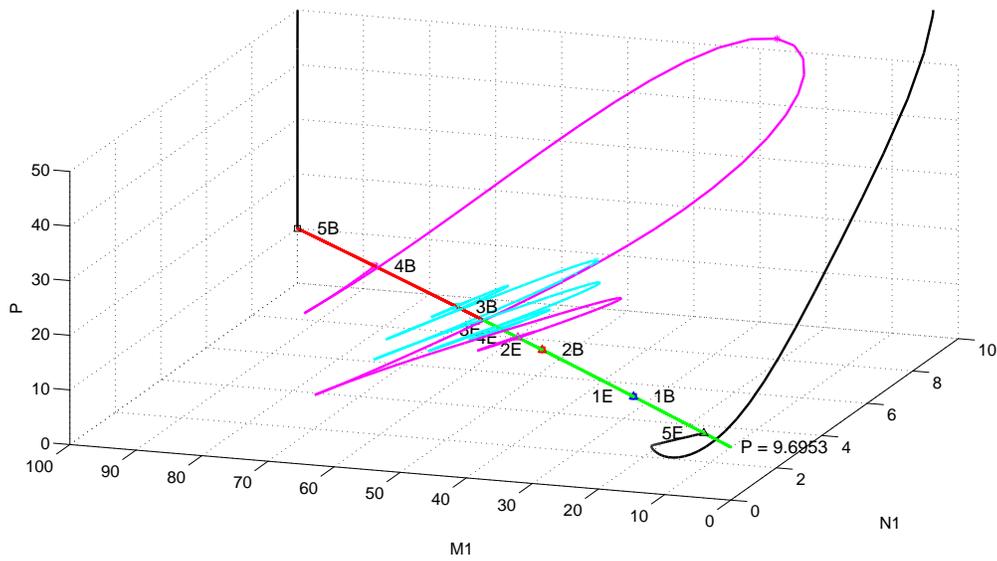


Figure 4.1: Curve of equilibrium points for the case $8 = P_a < P_{eq} < L = 10$. Red points correspond to unstable equilibria, while green correspond to stable. There are 5 sample trajectories (obtained via Matlab’s ode23s solver). Trajectories start at the points 1B, 2B, ..., 5B and end at the points 1E, 2E, ..., 5E, respectively. Note the large excursions trajectories 3, 4, and 5 take before returning to equilibrium.

This investigation should provide a better understanding of the existence and stability of the equilibrium of the two investor group system. Once the nature of the equilibrium is understood, the effect of various parameters and behavioral factors on the dynamics of the system can be explored.

A subsequent major project can be to establish a stronger connection between real world markets and the model discussed in this thesis by using statistical methods to determine appropriate values for the $c_j^{(i)}$ and $q_j^{(i)}$ parameters. These will then be utilized in the discretized version of this model to obtain numerical results. A primary goal of this work will be to verify and expand upon (by considering additional motivations) the results in [13], where the authors used asset market experiments in conjunction with statistical methods and a discretized version of Equation 3.36 to better understand the formation of market bubbles.

Using NYSE data instead of experimental data will provide an interesting challenge not present in [13], as trader-specific data (bids and asks at each price and period) will not be available. Thus, using actual market data, a method for distinguishing the two groups and the sizes of their assets and determining each group's estimate of the value of the asset will need to be established.

There are numerous other avenues of research to consider. Specifically, two such avenues are: (i) the addition of noise to the system of differential equations and (ii) the formation of a valuation methodology for stocks that will allow the approach outlined in the Quantitative Modeling section to be extended to assets other than closed-end funds. With respect to the former, Appendix B contains a formal argument showing the limit of solutions to the discrete two group model corresponds to solutions of the classical price equation, 3.1. Closed-end funds have been the subject of much scrutiny. Refer to [1] for a summary of the literature. Development of a methodology to value funds/stocks (without utilizing the NAV of closed-end funds) will allow the approach outlined in Chapter 2 to be applied to other financial instruments.

APPENDIX A

QUANTITATIVE MODELING STUDY 2: ADDITIONAL REGRESSIONS

As noted in Chapter 2, the key to obtaining meaningful coefficients for the independent variables involves formulating a suitable definition for the valuation. To illustrate this point, suppose that we perform a linear regression in the manner of most financial studies, i.e., without making any attempt to subtract out the valuation. If we consider the most significant of the remaining variables, namely, Price Trend, then we obtain the relation below.

Regression 4. As a baseline we consider the linear regression with the single independent variable, Price Trend:

$$R(t + 1) = \beta_0 + \beta_1 T(t).$$

The regression results are included in Table A1.

Table A1: Regression 4 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.0004126711	0.00004845682	8.516265	<.0001
Price Trend	-0.0001618137	0.00004847659	-3.337976	0.0008

Degrees of Freedom: 80,242

From this we see that although the Price Trend is statistically significant, the coefficient has one-tenth of the magnitude of the previous regressions, and the opposite sign. Without accounting for changes in valuation, a one standard deviation change in the Price Trend variable corresponds to a change of -0.016% in the Relative Price Change.

Regression 5. By incorporating the Valuation variable in the regression, namely,

$$R(t + 1) = \beta_0 + \beta_1 T(t) + \beta_2 D(t),$$

we can readily demonstrate that the true effect of the Price Trend is extracted. Indeed the regression results are summarized in Table A2.

Table A2: Regression 5 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.000412679	0.00004775848	8.64096	<.0001
Price Trend	0.001103913	0.00005451764	20.24873	<.0001
Valuation	0.002630504	0.00005452487	48.24412	<.0001

Degrees of Freedom: 80,241

Hence, the Price Trend is now much more significant (t-value 20.25 versus -3.34), positive, and approximately 10 times larger in magnitude. By accounting for the Valuation, we find that the trend in price is statistically significant and has roughly half the effect of Valuation on the Relative Price Change.

From these two regressions, we can conclude that ignoring the changes in value (as most studies have done in the past) leads to coefficients for Price Trend that are totally inconclusive in terms of practical trading.

Regression 6. To further explore the nonlinear relationship between Price Trend and Valuation we consider a regression that incorporates all of the significant variables from

Regression 2 as well as the Price Trend and Valuation interaction terms. This regression has the form:

$$\begin{aligned}
R(t + 1) = & \beta_0 + \beta_1 T(t) + \beta_2 D(t) + \beta_3 M2(t) + \beta_4 STV(t) + \beta_5 LTV(t) \\
& + \beta_6 LTT(t) + \beta_7 VT(t) + \beta_8 T^2(t) + \beta_9 T^3(t) + \beta_{10} D^2(t) + \\
& + \beta_{11} D^3(t) + \beta_{12} T(t)D(t) + \beta_{13} T^2(t)D(t) + \beta_{14} T(t)D^2(t) \\
& + \beta_{15} Time(t) + \beta_{16} Time^2(t) + \beta_{17} Time^3(t)
\end{aligned}$$

Comparing the results in Table [A3](#) with those from Regression 3, we find that the Intercept, Price Trend², and Price Trend*Valuation² terms are the only variables with significant (i.e., greater than 20%) relative changes in the magnitudes of their coefficients.

Table A3: Regression 6 results

<i>Term</i>	<i>Value</i>	<i>Std. Error</i>	<i>t-value</i>	<i>p-value</i>
<i>(Intercept)</i>	0.00139316	0.000378472	3.681	0.0002
Price Trend	0.00168829	0.00007044	23.96784	<.0001
Valuation	0.00290006	0.0000625270	46.38096	<.0001
M2 Money Supply	0.00047243	0.0000477520	9.89339	<.0001
Short Term Volatility	0.00057453	0.0000591470	9.71363	<.0001
Long Term Volatility	0.00015736	0.0000602920	2.60991	0.0091
Long Term Trend	-0.0001755	0.0000548500	-3.19962	0.0014
Volume Trend	0.00026413	0.0000487820	5.41459	<.0001
Price Trend ²	-0.00022655	0.0000413210	-5.48258	<.0001
Price Trend ³	-0.00012016	0.0000114110	-10.53101	<.0001
Valuation ²	0.00005617	0.0000180640	3.10918	0.0019
Valuation ³	-0.0000079	0.0000016040	-4.92458	<.0001
Price Trend * Valuation	-0.00026923	0.0000488830	-5.50767	<.0001
Price Trend ² * Valuation	-0.00009215	0.0000177630	-5.18788	<.0001
Price Trend * Valuation ²	-0.00000651	0.0000097640	-0.66662	0.505
Time	-0.01582821	0.0028159320	-5.62095	<.0001
Time ²	0.04268599	0.0058229880	7.3306	<.0001
Time ³	-0.02936503	0.0035057630	-8.37622	<.0001

Degrees of Freedom: 80,226

APPENDIX B

THE CLASSICAL ASSET PRICE EQUATION AS A LIMIT OF THE ASSET FLOW DIFFERENTIAL EQUATIONS

We show that solutions of the discrete asset flow equations 3.13, 3.18, and 3.19 have a formal limit as solutions to the classical stochastic asset price equation 3.1. This is accomplished by utilizing the basic assumptions of classical finance within the context of our equations. The key assumption needed to attain the classical limit is that Group 2 is focused solely on the value of the asset, $P_a(t)$, while Group 1 consists of "noise traders," a common assumption in classical economics and finance. In other words, these participants hope to make a profit, but without additional information, they try a variety of strategies that amount to nothing more than noise in the aggregate. While individual traders may have strategies, there are many disparate ideas, and so there is no net dependence on either price or value. This summarizes the viewpoint of classical finance.

Classical finance stipulates the existence of a risk-free interest rate (per year), which we will call μ , and assume that it is constant within the time period we are considering. Since all investors have the same publicly available information, the expectation for the return of any asset is given by

$$P_a(t) = P_a(0) e^{\mu t}. \tag{B.1}$$

We consider a small time frame (compared to one year), and denote by P_a the fundamental value of the stock during that time. With the assumption of non-zero trading costs

we assume that $k^{(2)}$ is given by

$$k^{(2)} := \begin{cases} 1 & \text{if } P < P_a - \delta \\ 1/2 & \text{if } P_a - \delta < P < P_a + \delta \\ 0 & \text{if } P > P_a + \delta \end{cases} \quad (\text{B.2})$$

for some small, positive δ . We can also consider the smoothed version of this function, given by

$$k^{(2)}(P) := \frac{1}{2} \left[1 + \tanh \left(\frac{P_a - P}{\varepsilon} \right) \right] \quad (\text{B.3})$$

for some small, positive ε .

We make the classical assumption that the knowledgeable investors, namely Group 2, represent the vast majority of wealth, expressed by the following.

Assumption A. At the initial time, the endowments to the two groups satisfy the following:

$$M^{(1)}/M^{(2)} \ll 1 \quad \text{and} \quad N^{(1)}/N^{(2)} \ll 1. \quad (\text{B.4})$$

Assumption B. We assume that the available cash and asset are balanced so that the cash supply of Group 2 satisfies

$$L_2(t) := M^{(2)}(t)/N^{(2)}(t) = P_a(t). \quad (\text{B.5})$$

The assumption that Group 1 consists of "noise traders" implies that the net flow of orders by this group does not depend on P_a or P .

Assumption C. The transition rate of Group 1, $k^{(1)}$, is a constant independent of P_a and P and $0 < k^{(1)} < 1$.

Under Assumptions A and B, for $P < P_a - \delta$, one has that $k^{(2)} = 1$ from B.2, so we can write 3.12 as

$$F(T_j) = \frac{k^{(1)}M^{(1)} + M^{(2)}}{(1 - k^{(1)})N^{(1)}} \approx \frac{1}{(1 - k^{(1)})} \frac{M^{(2)}}{N^{(1)}} = \frac{P_a}{(1 - k^{(1)})} \frac{N^{(2)}}{N^{(1)}} = O\left(\frac{N^{(2)}}{N^{(1)}}\right) \quad (\text{B.6})$$

which is large and positive. Hence, by 3.13 the price quickly moves into the range $P > P_a - \delta$. Similarly, when the stock is overvalued, i.e., $P > P_a + \delta$, one has $k^{(2)} = 0$, yielding

$$F(T_j) = \frac{k^{(1)}M^{(1)}}{(1-k^{(1)})N^{(1)} + N^{(2)}} \simeq \frac{k^{(1)}M^{(1)}}{N^{(2)}} = k^{(1)}\frac{M^{(1)}}{M^{(2)}}P_a \ll P_a.$$

Hence,

$$\frac{F(T_j)}{P(T_j)} - 1 \simeq -1, \quad (\text{B.7})$$

so that the price falls back into the range $P < P_a + \delta$ where $k^{(2)}$ no longer vanishes.

Finally, we consider the range $P_a - \delta < P < P_a + \delta$ where the knowledgeable investors are essentially inactive since it is within their trading costs. In this region of price, the "noise traders" dominate due to the absence of Group 2 investors. Using Assumptions A and B for prices $P(T_j)$ near P_a , we have, from 3.12 and B.2,

$$F(T_j) = \frac{M^{(2)}}{N^{(2)}} \frac{1 + 2k^{(1)}M^{(1)}/M^{(2)}}{1 + 2(1-k^{(1)})N^{(1)}/N^{(2)}}$$

yielding, from 3.13, the identity,

$$\begin{aligned} \frac{P(T_{j+1}) - P(T_j)}{P(T_j)} &= \frac{P_a}{P(T_j)} \left\{ \frac{1 + 2k^{(1)}M^{(1)}/M^{(2)}}{1 + 2(1-k^{(1)})N^{(1)}/N^{(2)}} \right\} - 1 \\ &\simeq 2k^{(1)}\frac{M^{(1)}}{M^{(2)}} - 2(1-k^{(1)})\frac{N^{(1)}}{N^{(2)}}. \end{aligned} \quad (\text{B.8})$$

Indeed, as $P_a - \delta < P < P_a + \delta$, we approximate $P_a/P(T_j) \simeq 1$. In addition, we use a geometric series argument and drop the $4k^{(1)}M^{(1)}/M^{(2)}(1-k^{(1)})N^{(1)}/N^{(2)}$ as it is much smaller in magnitude than the other terms due to Assumption A.

The system B.8, 3.18, and 3.19 is not yet close to 3.1, since $M^{(1)}$ and $N^{(1)}$ are not constant.

We now examine $M^{(1)}(t)$ and assume that there is a normal random variable, $X(T)$, governing the influx/outflow of investors and their cash. The parameter M_0 represents the total amount of cash in the system, i.e. $M^{(1)} + M^{(2)} = M_0$.

Assumption E. The cash supply $M^{(1)}(t)$ in 3.18 includes an inflow/outflow term $m^{(1)}(T_j) := \bar{\sigma}X(T_j)M_0$ that dominates the right hand side of B.8.

Thus, we obtain

$$\begin{aligned} \frac{P(T_{j+1}) - P(T_j)}{P(T_j)} &= 2k^{(1)} \frac{m^{(1)}(T_j)}{M^{(2)}} - 2(1 - k^{(1)}) \frac{N^{(1)}}{N^{(2)}} \\ &\simeq \sigma X(T_j) \end{aligned} \tag{A.9}$$

where σ incorporates the additional constants multiplying $\bar{\sigma}$. In other words, the existing "noise traders" effectively cancel out one another as they employ a variety of strategies. More significant, however, is the stochastic influx and exit of new investors that becomes the main stochastic contribution to price change. For example, if there is a net inflow of investor cash (as in the late 1990's) due to macroscopic or demographic changes, then the larger supply of cash leads to higher prices. This holds even assuming that investors, on average, are just as likely to sell as they are to buy. Formally taking the continuum limit of [A.9](#) yields

$$dP = \sigma P dX. \tag{B.9}$$

Since, $P(t)$ and $P_a(t)$ are essentially identical in classical finance, we assume $P(t)$ satisfies the same differential equation, namely

$$dP = \mu P dt. \tag{B.10}$$

Combining [B.9](#) and [B.10](#) yields the stochastic equation [3.1](#), i.e.

$$\frac{dP}{P} = \sigma dX + \mu dt.$$

The main difference is that we are assuming a deterministic motivation once the price has moved beyond δ away from the fundamental value.

APPENDIX C

LIMIT CALCULATIONS FOR EQUILIBRIUM REGIONS IN CASES 1

($L < P_{EQ} < P_A$) AND 2 ($P_A < P_{EQ} < L$)

Intuitively, we expect the equilibrium price, P_{eq} , to move toward the fundamental value, P_a , as $q_2 \rightarrow \infty$ and toward the liquidity value, $L = M_0/N_0$, as $q_2 \rightarrow 0$. This is confirmed graphically in Figures 3.1 and 3.3. Using the expressions in the remark to Corollary 7 we prove these claims.

Lemma 13. *Under the conditions in Corollary 7 we prove for Cases 1 and 2 that (i) as $q_2 \rightarrow \infty$, the equilibrium price, P_{eq} , moves toward the fundamental value, P_a ; and (ii) as $q_2 \rightarrow 0$, the equilibrium price, P_{eq} , moves toward the liquidity value, L .*

Proof. Indeed, consider expressions 3.53 and 3.54 from the remark to Corollary 7. Define

$$E := \frac{1}{2q_2N_0} \left[N_0P_a(-1 + q_2) - q_2M_0 + \sqrt{4q_2M_0N_0P_a(1 + q_2) + [q_2M_0 + N_0P_a(1 - q_2)]^2} \right].$$

For Case 1, $L < P_{eq} < P_a$, we have

$$\max \left[L, \frac{P_a(q_2 - 1)}{q_2} \right] < P_{eq} \leq E \tag{C.1}$$

and for Case 2, $P_a < P_{eq} < L$, we have

$$E \leq P_{eq} < \min \left[L, \frac{P_a(1 + q_2)}{q_2} \right]. \tag{C.2}$$

Consider the limit of the expression E as $q_2 \rightarrow 0$ and as $q_2 \rightarrow \infty$.

First, suppose $q_2 \rightarrow \infty$. Indeed, consider

$$\begin{aligned}
& \lim_{q_2 \rightarrow \infty} \frac{1}{2q_2 N_0} \left[N_0 P_a (-1 + q_2) - q_2 M_0 + \sqrt{4q_2 M_0 N_0 P_a (1 + q_2) + [q_2 M_0 + N_0 P_a (1 - q_2)]^2} \right] \\
&= \frac{N_0 P_a - M_0 + \sqrt{4M_0 N_0 P_a + M_0^2 + N_0^2 P_a^2 - 2M_0 N_0 P_a}}{2N_0} \\
&= \frac{N_0 P_a - M_0 + M_0 + N_0 P_a}{2N_0} \\
&= P_a.
\end{aligned}$$

Next, suppose $q_2 \rightarrow 0$. Indeed, consider

$$\begin{aligned}
& \lim_{q_2 \rightarrow 0} \frac{1}{2q_2 N_0} \left[N_0 P_a (-1 + q_2) - q_2 M_0 + \sqrt{4q_2 M_0 N_0 P_a (1 + q_2) + [q_2 M_0 + N_0 P_a (1 - q_2)]^2} \right] \\
&= \frac{0}{0}.
\end{aligned}$$

Utilization of l'Hôpital's rule leads to

$$\begin{aligned}
& \lim_{q_2 \rightarrow 0} \frac{1}{2q_2 N_0} \left[N_0 P_a (-1 + q_2) - q_2 M_0 + \sqrt{4q_2 M_0 N_0 P_a (1 + q_2) + [q_2 M_0 + N_0 P_a (1 - q_2)]^2} \right] \\
&= \frac{N_0 P_a - M_0 + \frac{1}{2} (N_0^2 P_a^2)^{-1/2} (4M_0 N_0 P_a - 2N_0^2 P_a^2 + 2M_0 N_0 P_a)}{2N_0} \\
&= \frac{1}{2} [P_a - L + 3L - P_a] \\
&= L.
\end{aligned}$$

Therefore, we have shown that this expression goes to P_a as $q_2 \rightarrow \infty$ and goes to L as $q_2 \rightarrow 0$.

Next, we consider the limits of the expressions $\frac{P_a(q_2-1)}{q_2}$ (for Case 1) and $\frac{P_a(q_2+1)}{q_2}$ (for Case 2). Indeed,

$$\lim_{q_2 \rightarrow \infty} \frac{P_a (q_2 - 1)}{q_2} = P_a \tag{C.3}$$

and

$$\lim_{q_2 \rightarrow 0} \frac{P_a (q_2 - 1)}{q_2} = -\infty. \tag{C.4}$$

Similarly,

$$\lim_{q_2 \rightarrow \infty} \frac{P_a (q_2 + 1)}{q_2} = P_a \tag{C.5}$$

and

$$\lim_{q_2 \rightarrow 0} \frac{P_a (q_2 + 1)}{q_2} = \infty. \tag{C.6}$$

We now consider Case 1. From expression C.1 we have

$$\max \left[L, \frac{P_a (q_2 - 1)}{q_2} \right] < P_{eq} \leq E. \quad (\text{C.7})$$

Suppose $q_2 \rightarrow \infty$ and note for large q_2 , as $P_a > L$, we have $\max \left[L, \frac{P_a (q_2 - 1)}{q_2} \right] = \frac{P_a (q_2 - 1)}{q_2}$. From the limit C.3 we have that for fixed (arbitrary and small) $\varepsilon > 0$, there exists a $q'_2 > 0$ such that for all $q_2 > q'_2$ we have

$$P_a - \frac{P_a (q_2 - 1)}{q_2} = \frac{P_a}{q_2} < \varepsilon.$$

Note we choose q'_2 large enough so that $\max \left[L, \frac{P_a (q_2 - 1)}{q_2} \right] = \frac{P_a (q_2 - 1)}{q_2}$. Also, note the absolute value in the definition of the limit is not required as $P_a > \frac{P_a (q_2 - 1)}{q_2}$ for $q_2 > 0$. From expression C.7 for $q_2 > q'_2$ we have

$$\begin{aligned} P_{eq} &> \frac{P_a (q_2 - 1)}{q_2} \Leftrightarrow \\ 0 &< P_a - P_{eq} < P_a - \frac{P_a (q_2 - 1)}{q_2} < \varepsilon. \end{aligned}$$

Thus, as $q_2 \rightarrow \infty$, P_{eq} approaches P_a .

Next, for Case 1 suppose $q_2 \rightarrow 0$. From the limit C.4 and expression C.7 we see that $\max \left[L, \frac{P_a (q_2 - 1)}{q_2} \right] = L$ for q_2 small enough. So, we use the $\lim_{q_2 \rightarrow 0} E = L$ to squeeze P_{eq} . Indeed, for fixed (arbitrary and small) $\varepsilon > 0$, there exists a $q'_2 > 0$ such that for all $0 < q_2 < q'_2$ we have

$$0 < E - L < \varepsilon.$$

Again, the absolute value in the limit definition is not required as $L < P_{eq} \leq E$ from expression C.8. For $q_2 < q'_2$ (q'_2 chosen small enough that $\max \left[L, \frac{P_a (q_2 - 1)}{q_2} \right] = L$) we have

$$\begin{aligned} L &< P_{eq} \leq E \Leftrightarrow \\ 0 &< P_{eq} - L \leq E - L < \varepsilon. \end{aligned}$$

Thus, as $q_2 \rightarrow 0$, P_{eq} approaches L .

Next, we repeat similar arguments for Case 2, $P_a < P_{eq} < L$. Indeed, consider expression C.2

$$E \leq P_{eq} < \min \left[L, \frac{P_a (1 + q_2)}{q_2} \right]. \quad (\text{C.8})$$

Suppose $q_2 \rightarrow \infty$ and note for large q_2 , as $P_a < L$, we have $\min \left[L, \frac{P_a(1+q_2)}{q_2} \right] = \frac{P_a(1+q_2)}{q_2}$. From the limit C.5 we have that for fixed (arbitrary and small) $\varepsilon > 0$, there exists a $q'_2 > 0$ such that for all $q_2 > q'_2$ we have

$$\frac{P_a(1+q_2)}{q_2} - P_a = \frac{P_a}{q_2} < \varepsilon.$$

Note we choose q'_2 large enough so that $\min \left[L, \frac{P_a(1+q_2)}{q_2} \right] = \frac{P_a(1+q_2)}{q_2}$. Also, the absolute value is not required as $\frac{P_a(1+q_2)}{q_2} > P_a$ for $q_2 > 0$. From expression C.8 for $q_2 > q'_2$ we have

$$\begin{aligned} P_{eq} &< \frac{P_a(1+q_2)}{q_2} \Leftrightarrow \\ 0 &< P_{eq} - P_a < \frac{P_a(1+q_2)}{q_2} - P_a < \varepsilon. \end{aligned}$$

Thus, as $q_2 \rightarrow \infty$, P_{eq} approaches P_a .

Next, for Case 2 suppose $q_2 \rightarrow 0$. From the limit C.6 and expression C.8 we see that $\min \left[L, \frac{P_a(1+q_2)}{q_2} \right] = L$ for q_2 small enough. So, we use the $\lim_{q_2 \rightarrow 0} E = L$ to squeeze P_{eq} . Indeed, for fixed (arbitrary and small) $\varepsilon > 0$, there exists a $q'_2 > 0$ such that for all $0 < q_2 < q'_2$ we have

$$0 < L - E < \varepsilon.$$

Again, the absolute value is not required as $E \leq P_{eq} < L$ from expression C.7 for q_2 small enough. For $q_2 < q'_2$ (q'_2 chosen small enough that $\min \left[L, \frac{P_a(1+q_2)}{q_2} \right] = L$) we have

$$\begin{aligned} E &\leq P_{eq} < L \Leftrightarrow \\ E - L &\leq P_{eq} - L < 0 \Leftrightarrow \\ \varepsilon &> L - E \geq L - P_{eq} > 0. \end{aligned}$$

Thus, as $q_2 \rightarrow 0$, P_{eq} approaches L . □

APPENDIX D

ROUTH-HURWITZ CRITERION: SUMMARY AND APPLICATION

Two necessary conditions for the real parts of all roots of a polynomial (with real coefficients) to lie in the left half of the complex plane are: (i) all coefficients must be nonzero and (ii) all coefficients must have the same sign. Thus, if either condition (i) or (ii) is not satisfied, then the equation has a root(s) with real part greater than or equal to zero. The Routh-Hurwitz criterion, however, gives a necessary and sufficient condition for the real parts of the roots of a polynomial to be negative. As such, this method is utilized to determine the stability, i.e. real parts of all eigenvalues strictly negative, of the equilibrium points considered in Sections 3.4 and 3.5.

We consider the specific case of a cubic polynomial $p(\lambda) = \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$. The Routh Array is:

$$\begin{array}{c|cc} \lambda^3 & \alpha_3 & \alpha_1 \\ \lambda^2 & \alpha_2 & \alpha_0 \\ \lambda^1 & \frac{\alpha_2\alpha_1 - \alpha_3\alpha_0}{\alpha_2} & 0 \\ \lambda^0 & \alpha_0 & 0 \end{array} .$$

The Routh-Hurwitz criterion states that the number of roots with positive real parts is given by the number of sign changes in the middle column. In general there are four scenarios to consider:

- (i) There are no zero elements in the middle column.
- (ii) A zero appears in the middle column but the other element in the same row (rightmost column) is non-zero, i.e. $\alpha_3 = 0$ or $\alpha_2 = 0$. In this case the zero should be replaced

with a very small $\varepsilon > 0$. For instance, consider the scenario $\alpha_3 = 1$, $\alpha_2 = 0$, $\alpha_1 > 0$, and $\alpha_0 > 0$. The second entry in the middle column would be $\alpha_2 = 0$. We replace this with $\varepsilon > 0$. The Routh Array becomes

$$\begin{array}{c|cc} \lambda^3 & 1 & \alpha_1 \\ \lambda^2 & \varepsilon & \alpha_0 \\ \lambda^1 & \frac{\varepsilon\alpha_1 - \alpha_0}{\varepsilon} & 0 \\ \lambda^0 & a_0 & 0 \end{array} .$$

As $\varepsilon \rightarrow 0$, the sign of the term $\frac{\varepsilon\alpha_1 - \alpha_0}{\varepsilon}$ becomes negative as $\alpha_0 > 0$ in this scenario. Thus, there are 2 sign changes in the middle column indicating two roots with positive real parts.

(iii) There is a row of zeros in the array. This means that the polynomial has roots that are located symmetrically about the origin of the complex plane, i.e. the roots are of the form $(\lambda - a)(\lambda + a)$, indicating instability, or $(\lambda - ia)(\lambda + ia)$, $a \in \mathbb{R}$, indicating a pair of pure imaginary roots. These roots are solutions of the auxiliary polynomial [20]. The coefficients of this auxiliary polynomial are taken as the elements in the row immediately above the row of zeros.

For example, suppose $\alpha_2\alpha_1 = \alpha_3\alpha_0$, $\alpha_i \neq 0$, $i = 0, 1, 2, 3$. The auxiliary polynomial has the form $a(\lambda) = \alpha_2\lambda^2 + \alpha_0 = 0$. Solving $a(\lambda) = 0$ yields $\lambda_{1,2} = \pm\sqrt{-\frac{\alpha_0}{\alpha_2}}$. So, if α_2 and α_0 are of the same sign, then the roots will be pure imaginary. Otherwise, they will be real roots with opposite sign.

To determine the sign of the remaining eigenvalue, one may use long division, i.e. divide the cubic polynomial, $\alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$, by the auxiliary polynomial, $\alpha_2\lambda^2 + \alpha_0$. This divides evenly since the roots of the auxiliary polynomial are roots of the original cubic polynomial. In this case the quotient is $\frac{\alpha_3}{\alpha_2}\lambda + 1$. Thus, the third root of the cubic polynomial is $\lambda = -\alpha_2/\alpha_3$. If α_3 and α_2 are of the same sign, then this root is negative. Otherwise, this root is positive.

(iv) Same as (iii) but in this case there are repeated roots on the imaginary axis. This cannot occur for the cubic polynomial case because it only has three roots.

For a cubic polynomial the following is a necessary and sufficient condition for stability: all roots of the polynomial have negative real parts if and only if the coefficients are positive and $\alpha_2\alpha_1 - \alpha_3\alpha_0 > 0$ [20].

This criterion was utilized to determine the stable and non-stable regions in Sections 3.4 and 3.5. Specifically, the figures in Section 3.5 and Appendix F were produced by utilizing this criterion to determine the stable regions.

Identification of the stable region boundary curves in Figures 3.2, 3.4, and 3.6 that are not included within the actual stable regions was determined numerically via Scenario (iii) above with $\alpha_i > 0$, $i = 0, 1, 2, 3$ and $\alpha_2\alpha_1 - \alpha_3\alpha_0 = 0$. Indeed, each equilibrium point along these curves admits a pair of pure imaginary eigenvalues with a third, negative eigenvalue. We term this case to be Marginally Stable.

For the Cases in Sections 3.4 and 3.5 the cubic factor of the characteristic polynomial has the form

$$\alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$$

where we assume $\alpha_3 = 1$. As the Routh-Hurwitz criterion is based upon the sign of the coefficients, i.e. $\alpha_2 > 0$, $\alpha_2 < 0$, or $\alpha_2 = 0$, there are 27 cases to consider. We utilized the Mathematica Reduce¹ command to determine which of the 27 scenarios are applicable, i.e. which scenarios correspond to a valid equilibrium - meaning the assumptions in Section 3.4 are satisfied. For example, to determine if (and when) Scenario A (see Table D1) is applicable for Case 2 ($8 = P_a < P_{eq} < L = 10$) with $q_1 = 10$, the Reduce command was run for the following conditions:

- (i) $\alpha_2 > 0$ and $\alpha_1 > 0$ and $\alpha_0 > 0$ and $\alpha_2\alpha_1 - \alpha_0 > 0$ (for stable case) and
- (ii) $0 \leq \hat{M}^{(1)} \leq M_0$ and $0 \leq \hat{N}^{(1)} \leq N_0$ and $-1 < \hat{\zeta}^{(2)} < 1$ and
- (iii) $P_a = 8$ and $M_0 = 30000$ and $N_0 = 3000$ and $P_a < P_{eq} < M_0/N_0$ and
- (iv) $q_1 = 10$ and $q_2 > 0$ and $c_1 = 1/10$ and $c_2 = 1$.

If the Reduce command returns a region in the parameter space, then these criteria are met and the scenario is applicable. Specifically, for this example any equilibrium point(s) in the region returned by the Reduce command would be deemed stable. This region

¹The Reduce command attempts to simplify equations and inequalities by replacing them with simpler expressions. For example, Reduce[$x > 0$ and $-1 < x < 1$] returns the inequalities [$x > 0$ and $x < 1$]. If the inequalities and/or equations are not compatible, then the Reduce command returns "False." For example, Reduce[$x > 0$ and $-1 < x < 1$ and $x = 5$] returns "False." While the first two expressions are compatible, they are certainly not in agreement with the third.

corresponds to the union of the Red and Yellow areas in Figure 3.4. The scenarios applicable to the 12 parameter regimes considered in Section 3.5 are listed in Table D1. This table indicates that scenarios B through I yield unstable equilibrium points, i.e. at least one root of the characteristic polynomial has positive real part. Scenario A may yield stable, unstable, or marginally stable equilibrium points depending upon the sign of the $\alpha_2\alpha_1 - \alpha_0$ term.

Table D1: Of the 27 scenarios resulting from the signs of α_i , $i = 0, 1, 2$ only nine are applicable to the 12 parameter regimes from Section 3.5. Note that only Scenario A may yield stable equilibrium points provided $\alpha_2\alpha_1 - \alpha_0 > 0$.

Scenario	α_3	α_2	α_1	α_0	$\alpha_2\alpha_1 - \alpha_0$	Stability
A	1	+	+	+	?	Stable, Unstable, Marginally stable. See notes.
B	1	+	-	+	-	Unstable
C	1	-	+	+	-	Unstable
D	1	-	-	+	?	Unstable
E	1	0	+	+	-	Unstable. Used ε argument.
F	1	0	-	+	-	Unstable. Used ε argument.
G	1	0	0	+	-	Unstable. See notes.
H	1	+	0	+	-	Unstable
I	1	-	0	+	-	Unstable

Notes for selected scenarios in Table D1.

Scenario A. If $\alpha_2\alpha_1 - \alpha_0 > 0$, then the system is stable. If $\alpha_2\alpha_1 - \alpha_0 < 0$, then the system is unstable as there will be two sign changes in the middle column. If $\alpha_2\alpha_1 - \alpha_0 = 0$, then according to the above criteria the system will be marginally stable with a pair pure imaginary eigenvalues as α_2 and α_0 are both positive. As α_2 and $\alpha_3 = 1$ are both positive, the remaining eigenvalue will be negative in this case.

Scenario G. The characteristic equation is given by

$$\begin{aligned}\lambda^3 + \alpha_0 &= 0 \Leftrightarrow \\ \lambda &= \sqrt[3]{-\alpha_0}.\end{aligned}$$

The other roots are a complex conjugate pair. As $\alpha_0 > 0$, we have $\lambda < 0$. Then we can compute the remaining roots by first dividing $\lambda^3 + \alpha_0$ by $(\lambda - \sqrt[3]{-\alpha_0})$. This yields a quotient of

$$\lambda^2 + \sqrt[3]{-\alpha_0}\lambda + (-\alpha_0)^{2/3}.$$

Using the quadratic formula gives the roots

$$\begin{aligned}\lambda_{+,-} &= \frac{-\sqrt[3]{-\alpha_0} \pm \sqrt{(-\alpha_0)^{2/3} - 4(-\alpha_0)^{2/3}}}{2} \\ &= \frac{-\sqrt[3]{-\alpha_0} \pm \sqrt{-3(-\alpha_0)^{2/3}}}{2}.\end{aligned}$$

The real part of $\lambda_{+,-}$ is positive as $\alpha_0 > 0$. Therefore, the equilibrium is unstable.

The parameter regimes considered in Section 3.5 are:

- (1) $8 = P_a < P_{eq} < L = 10; q_1 = 0.447$
- (2) $8 = P_a < P_{eq} < L = 10; q_1 = 10$
- (3) $8 = P_a < P_{eq} < L = 10; q_1 = 20$
- (4) $10 = L < P_{eq} < P_a = 12; q_1 = 0.447$
- (5) $10 = L < P_{eq} < P_a = 12; q_1 = 10$
- (6) $10 = L < P_{eq} < P_a = 12; q_1 = 20$
- (7) $10 = L = P_{eq} = P_a; q_1 = 0.447$
- (8) $10 = L = P_{eq} = P_a; q_1 = 10$
- (9) $10 = L = P_{eq} = P_a; q_1 = 20$
- (10) $10 = L = P_{eq} = P_a; q_2 = 0.073$
- (11) $10 = L = P_{eq} = P_a; q_2 = 10$
- (12) $10 = L = P_{eq} = P_a; q_2 = 20$

Table D2 lists these regimes and the corresponding scenarios from Table D1. A "Yes" indicates the scenario is applicable to the corresponding parameter regime. Note that Scenario A may result in a stable, unstable, or marginally stable equilibrium depending upon the sign of the $\alpha_2\alpha_1 - \alpha_0$ term. Thus, "Yes - all 3" indicates that there exists regions consisting of stable, unstable, and marginally stable equilibrium points.

Table D2: The nine scenarios from Table D1 are listed along with the 12 parameter regimes from Section 3.5. A "Yes" indicates the scenario is applicable for the corresponding parameter regime. Only Scenario A may result in stable equilibrium points. All other scenarios yield instability. For parameter regimes 1, 4, and 7 all equilibrium points are stable.

Scenario	Regime 1	Regime 2	Regime 3	Regime 4	Regime 5	Regime 6
A	Stable	Yes - all 3	Yes - all 3	Stable	Yes - all 3	Yes - all 3
B	No	Yes	Yes	No	Yes	Yes
C	No	No	Yes	No	No	Yes
D	No	No	Yes	No	No	Yes
E	No	No	Yes	No	No	Yes
F	No	No	Yes	No	No	Yes
G	No	No	Yes	No	No	Yes
H	No	Yes	Yes	No	Yes	Yes
I	No	No	Yes	No	No	Yes
Scenario	Regime 7	Regime 8	Regime 9	Regime 10	Regime 11	Regime 12
A	Stable	Yes - all 3				
B	No	Yes	Yes	Yes	Yes	Yes
C	No	No	Yes	No	Yes	Yes
D	No	No	Yes	Yes	Yes	Yes
E	No	No	Yes	No	Yes	Yes
F	No	No	Yes	Yes	Yes	Yes
G	No	No	Yes	No	Yes	Yes
H	No	Yes	Yes	Yes	Yes	Yes
I	No	No	Yes	No	Yes	Yes

APPENDIX E

JACOBIAN MATRICES

Linearization about the arbitrary equilibrium point identified by equation 3.49 for Cases 1 and 2, i.e. $L < P_{eq} < P_a$ and $P_a < P_{eq} < L$, yields a Jacobian matrix that, when evaluated at this equilibrium point, is given by

$$\begin{pmatrix} -\frac{N_0 P_{eq} C1}{C4} & \frac{M_0 C2}{P_{eq} C4} & 0 & \frac{(M_0 - N_0 P_{eq}) C3 C5}{2C0^2 P_{eq} q_2^2 C4} & \frac{P_a^2 C6 C3}{2C0^2 P_{eq} q_2^2 C4} \\ \frac{N_0 P_{eq}^2 C1}{C4} & -\frac{M_0 C2}{C4} & 0 & \frac{(M_0 - N_0 P_{eq})(-C3) C5}{2C0^2 q_2^2 C4} & -\frac{P_a^2 C6 C3}{2C0^2 q_2^2 C4} \\ -\frac{2P_{eq}^2 C0 q_2}{C4} & -\frac{2P_{eq} C0 q_2}{C4} & -1 & \frac{2P_a P_{eq} C3}{C0 q_2 C4} & -\frac{2P_a^2 P_{eq} C6}{C0 q_2 C4} \\ -\frac{2c_1 P_{eq} C0 q_1 q_2}{C4} & -\frac{2c_1 C0 q_1 q_2}{C4} & -\frac{c_1 q_1}{P_{eq}} & \frac{c_1 (M_0 C2 C8 + N_0 P_{eq} C1 C7)}{C0 q_2 C4} & -\frac{2c_1 P_a^2 C6 q_1}{C0 q_2 C4} \\ 0 & 0 & -\frac{c_2 q_2}{P_a} & 0 & -c_2 \end{pmatrix}$$

where

$$C0 := P_a - P_{eq},$$

$$C1 := P_a - q_2 C0,$$

$$C2 := P_a + q_2 C0,$$

$$C3 := M_0 C2 - N_0 P_{eq} C1,$$

$$C4 := M_0 C2 + N_0 P_{eq} C1,$$

$$C5 := -2P_a P_{eq} q_2^2 + P_{eq}^2 q_2^2 + P_a^2 (-1 + q_2^2),$$

$$C6 := M_0 - N_0 P_{eq},$$

$$C7 := P_{eq} q_2 - P_a (2q_1 + q_2), \text{ and}$$

$$C8 := 2P_a q_1 + C1.$$

For Case 3 in Section 3.4 where $P_{eq} = P_a = L$ an arbitrary equilibrium point is given by equation 3.52. Linearization about this equilibrium point yields the following Jacobian matrix (evaluated at the equilibrium point):

$$\begin{pmatrix} -\frac{1}{2} & \frac{N_0}{2M_0} & 0 & \hat{N}^{(1)} - \frac{\hat{N}^{(1)2}}{N_0} & \frac{\hat{N}^{(1)}(-N_0 + \hat{N}^{(1)})}{N_0} \\ \frac{M_0}{2N_0} & -\frac{1}{2} & 0 & \frac{M_0 \hat{N}^{(1)}(-N_0 + \hat{N}^{(1)})}{N_0^2} & \frac{M_0(N_0 - \hat{N}^{(1)})\hat{N}^{(1)}}{N_0^2} \\ 0 & 0 & -1 & \frac{2M_0 \hat{N}^{(1)}}{N_0^2} & \frac{2M_0(N_0 - \hat{N}^{(1)})}{N_0^2} \\ 0 & 0 & -\frac{c_1 N_0 q_1}{M_0} & c_1 \left(-1 + \frac{2\hat{N}^{(1)} q_1}{N_0} \right) & \frac{2c_1(N_0 - \hat{N}^{(1)})q_1}{N_0} \\ 0 & 0 & -\frac{c_2 N_0 q_2}{M_0} & 0 & -c_2 \end{pmatrix}.$$

APPENDIX F

STABILITY ANALYSIS FOR VARYING C_I PARAMETER VALUES

In Section 3.5 we considered equilibrium and stability for equilibrium points of the system 3.39 - 3.43. Recalling that equilibrium is independent of the c_i parameters, this section considers stability for this system as the c_1 and c_2 parameters vary. As in Section 3.5, we fix the following parameter values: $M_0 = 30,000$, $N_0 = 3,000$ and $P_a = 12$ or 10 depending upon the case. The q_1 and q_2 parameters will be set to $1/5$, 1 , or 5 , respectively.

For Cases 1-3 in Section 3.5, we consider diagrams demonstrating the stability of the system for q_1 equal q_2 ($q_1 = q_2 = 1$), q_1 much larger than q_2 ($q_1 = 5, q_2 = 1/5$), and q_1 much less than q_2 ($q_1 = 1/5, q_2 = 5$). The regions were determined via the Mathematica Reduce command and plotted with the Mathematica RegionPlot utility.

Case 1. $10 = L < P_{eq} < P_a = 12$

Although any $P_{eq} \in (10, 12)$ corresponds to an equilibrium of the system, it is evident from Figure 3.1 that there exists a smaller range of permissible P_{eq} , i.e. those for which all assumptions are satisfied. Utilizing the expressions in the Remark to Corollary 7, the maximum permissible P_{eq} is 11.2788 if $q_2 = 1$ and 10.5095 for $q_2 = 1/5$. The diagram on the left in Figure F1 represents stable regions for $P_{eq} = 10.05, 10.25, 10.45, 10.65, 10.85, 11.05,$ and 11.25 . Note that the regions are nested within each other. For example, the stable region for $P_{eq} = 10.65$ is given by the union of the Red, Orange, Yellow, and Green regions. Similarly, the diagram on the right in Figure F1 represents stable regions for $P_{eq} = 10.05, 10.15, 10.25, 10.35, 10.45,$ and 10.5 . Again, the regions are nested so that the stable region for $P_{eq} = 10.35$ corresponds to the union of the Red, Orange, Yellow, and Green regions.

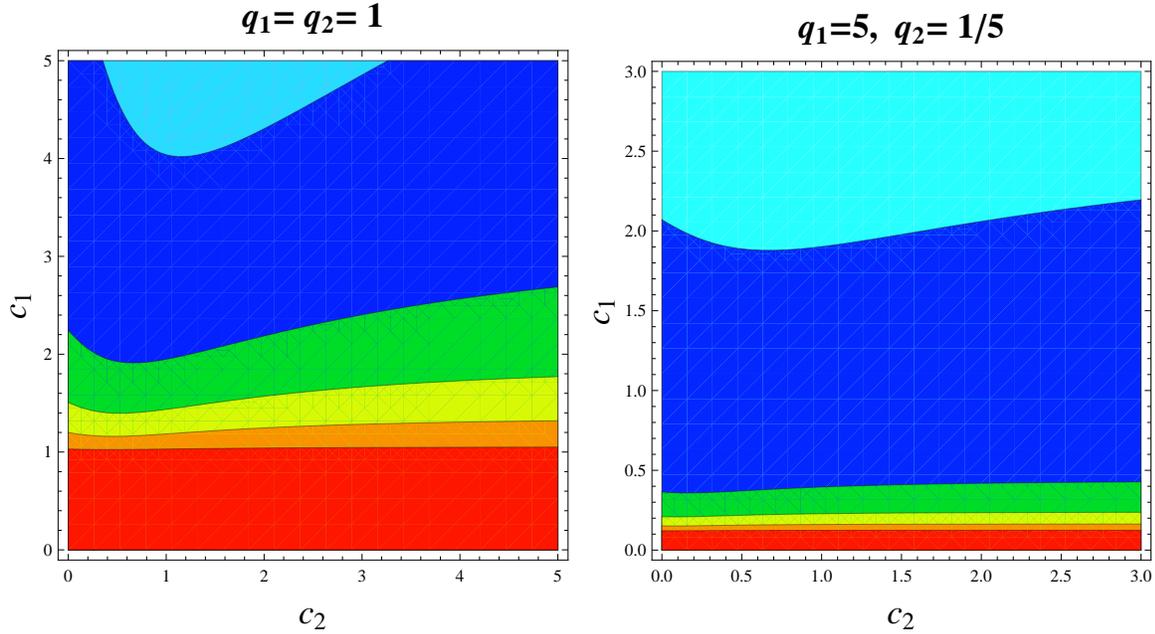


Figure F1: Regions in (c_2, c_1) space for $10 = L < P_{eq} < P_a = 12$ with $q_1 = q_2 = 1$ (left) and $q_1 = 5, q_2 = 1/5$ (right). For the diagram on the left stable regions correspond to the following equilibrium prices: $P_{eq} = 10.05$ (Red), 10.25 (Red and Orange), 10.45 (Red, Orange, and Yellow), 10.65 (Red, Orange, Yellow, and Green), 10.85 (Red, Orange, Yellow, Green, and Blue), 11.05 (All colors), and 11.25 (All colors). For the diagram on the right stable regions correspond to the following equilibrium prices: $P_{eq} = 10.05$ (Red), 10.15 (Red and Orange), 10.25 (Red, Orange, and Yellow), 10.35 (Red, Orange, Yellow, and Green), 10.45 (Red, Orange, Yellow, Green, and Blue), and 10.5 (All colors). Note that any point within the plotted region yields an equilibrium point of the system as equilibrium is independent of the c_i parameters. Areas outside of the designated colored regions correspond to non-stability for the specified P_{eq} .

Comments on the diagrams in Figure F1:

(1) When q_1 is much greater than q_2 , the regions are similarly shaped, but compressed vertically and shifted down on the c_1 axis. Thus, when there is a much greater emphasis

on the trend than the valuation, the timescale for the trend ($1/c_1$) must be longer for the equilibrium to be stable. Alternately, when q_1 is much less than q_2 ($q_1 = 1/5$ and $q_2 = 5$), all equilibrium are stable (utilized Mathematica's Reduce command to verify via the Routh-Hurwitz criterion).

(2) As P_{eq} moves toward the fundamental value, P_a , the stable region increases in size. For example, in Figure F1 with $P_a = 12$ and $q_1 = q_2 = 1$ the stable region for $P_{eq} = 11.05$ and 11.25 consists of the entire region, whereas for $P_{eq} = 10.05$ only the Red region admits a stable equilibrium. The scenario with $q_1 = 5$ and $q_2 = 1/5$ is similar.

(3) With c_1 fixed, increasing c_2 may impact stability for larger values of c_1 . However, for $c_1 < 1$ (Figure F1 with $q_1 = q_2 = 1$) or $c_1 < 0.1$ (Figure F1 with $q_1 = 5$ and $q_2 = 1/5$), all P_{eq} appear to be stable. Thus, increasing c_2 has no effect. Indeed, the claim that all P_{eq} are stable for $c_1 < 1$ for $q_1 = q_2 = 1$ (alternately, $c_1 < 0.1$ for $q_1 = 5$ and $q_2 = 1/5$) is verified by the Routh-Hurwitz criterion via Mathematica's Reduce command.

Case 2. $8 = P_a < P_{eq} < L = 10$

Although any $P_{eq} \in (8, 10)$ corresponds to an equilibrium of the system, similar to Case 1 the minimum permissible P_{eq} (as derived from the Remark to Corollary 7) is 8.60147 for $q_2 = 1$ and 9.34798 for $q_2 = 1/5$. The diagram on the left in Figure F2 represents stable regions for $P_{eq} = 8.65, 8.95, 9.25, 9.55, 9.85,$ and 9.99 , while the diagram on the right corresponds to stable regions for $P_{eq} = 9.35, 9.475, 9.6, 9.775, 9.9,$ and 9.99 . Note that the regions are nested within each other. For example, the stable region for $P_{eq} = 9.55$ in the diagram on the left in Figure F2 is given by the union of the Green, Blue, and Cyan regions.

Comments on the diagrams in Figure F2:

(1) When q_1 is much greater than q_2 , the regions are similarly shaped, but compressed vertically and shifted down on the c_1 axis. Thus, when there is a much greater emphasis on the trend than the valuation, the timescale for the trend ($1/c_1$) must be longer for the equilibrium to be stable. Alternately, when q_1 is much less than q_2 ($q_1 = 1/5$ and $q_2 = 5$), all equilibrium are stable (utilized Mathematica's Reduce command to verify via the Routh-Hurwitz criterion).

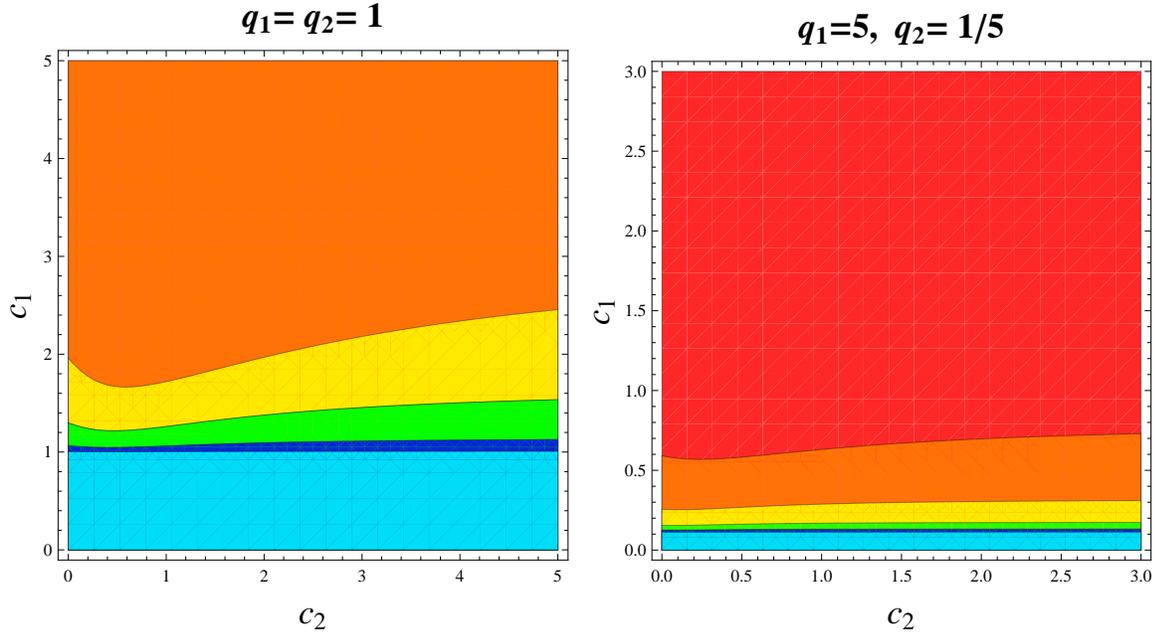


Figure F2: Regions in (c_2, c_1) space for $8 = P_a < P_{eq} < L = 10$ with $q_1 = q_2 = 1$ (left) and $q_1 = 5, q_2 = 1/5$ (right). For the diagram on the left stable regions correspond to the following equilibrium prices: $P_{eq} = 8.65$ (All colors), 8.95 (All colors), 9.25 (Yellow, Green, Blue, and Cyan), 9.55 (Green, Blue, and Cyan), 9.85 (Blue and Cyan), and 9.99 (Cyan). For the diagram on the right stable regions correspond to the following equilibrium prices: $P_{eq} = 9.35$ (All colors), 9.475 (Orange, Yellow, Green, Blue, and Cyan), 9.6 (Yellow, Green, Blue, and Cyan), 9.775 (Green, Blue, and Cyan), 9.9 (Blue and Cyan), and 9.99 (Cyan). Note that any point within the plotted region yields an equilibrium point of the system as equilibrium is independent of the c_i parameters. Areas outside of the designated colored regions correspond to non-stability for the specified P_{eq} .

(2) As P_{eq} moves toward the fundamental value, P_a , the stable region increases in size. For example, in Figure F2 with $P_a = 8$ and $q_1 = q_2 = 1$ the stable region for $P_{eq} = 8.65$ and 8.95 consists of the entire region, whereas for $P_{eq} = 9.99$ only the Cyan region admits a stable equilibrium. The scenario with $q_1 = 5$ and $q_2 = 1/5$ is similar.

(3) With c_1 fixed, increasing c_2 may impact stability for larger values of c_1 . However, for $c_1 < 1$ (Figure F2 with $q_1 = q_2 = 1$) or $c_1 < 0.1$ (Figure F2 with $q_1 = 5$ and $q_2 = 1/5$), all P_{eq} appear to be stable. Thus, increasing c_2 has no effect. Indeed, the claim that all P_{eq} are stable for $c_1 < 1$ for $q_1 = q_2 = 1$ (alternately, $c_1 < 0.1$ for $q_1 = 5$ and $q_2 = 1/5$) is verified by the Routh-Hurwitz criterion via Mathematica's Reduce command.

Case 3. $P_a = P_{eq} = L = 10$

In this scenario the curve of equilibrium points is parameterized by the $\hat{N}^{(1)}$ variable with no limitations on $\hat{N}^{(1)}$ other than $0 \leq \hat{N}^{(1)} \leq N_0$. The following diagrams represent stable regions for $\hat{N}^{(1)} = 0, 500, 1000, 1500, 2000, 2500,$ and 3000 . Note that the regions are nested within each other. For example, the stable region for $\hat{N}^{(1)} = 2000$ in the diagram on the left in Figure F3 is given by the union of the Blue, Cyan, and Magenta regions.

Note the following regarding the diagrams in Figure F3:

(1) When q_1 is much greater than q_2 , the regions are similarly shaped, but compressed vertically and shifted down on the c_1 axis. Thus, when there is a much greater emphasis on the trend than the valuation, the timescale for the trend ($1/c_1$) must be longer for the equilibrium to be stable. Alternately, when q_1 is much less than q_2 , all equilibrium were stable (utilized Mathematica's Reduce command to verify via the Routh-Hurwitz criterion).

(2) As $\hat{N}^{(1)}$ moves toward the total number of shares in the system, N_0 , the stable region decreases in size. Alternately, as the number of shares owned by investor group 1 decreases to 0, the stable region increases in size. For example, in Figure F3 with $N_0 = 3000$ and $q_1 = q_2 = 1$ the stable region for $\hat{N}^{(1)} = 0$ consists of the entire region, whereas for $\hat{N}^{(1)} = 3000$ only the Magenta region admits a stable equilibrium. The scenario with $q_1 = 5$ and $q_2 = 1/5$ is similar.

(3) With c_1 fixed, increasing c_2 may impact stability for larger values of c_1 . However, for $c_1 < 1$ (Figure F3 with $q_1 = q_2 = 1$) or $c_1 < 0.1$ (Figure F3 with $q_1 = 5$ and $q_2 = 1/5$), all P_{eq} appear to be stable. Thus, increasing c_2 has no effect. Indeed, the claim that all P_{eq} are stable for $c_1 < 1$ for $q_1 = q_2 = 1$ (alternately, $c_1 < 0.1$ for $q_1 = 5$ and $q_2 = 1/5$) is verified by the Routh-Hurwitz criterion via Mathematica's Reduce command.

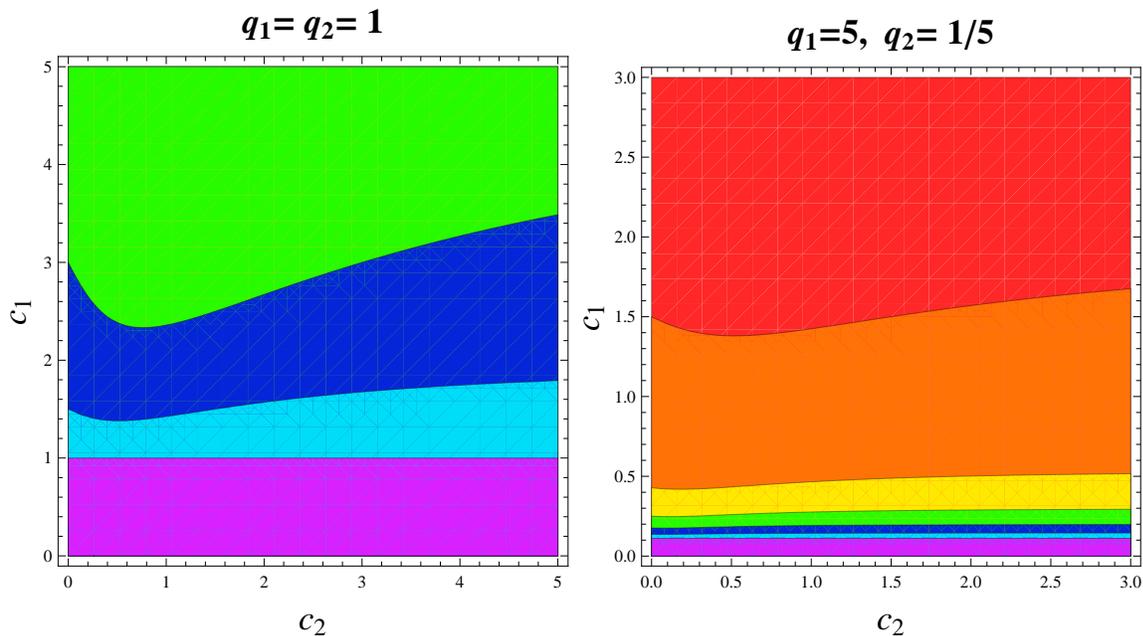


Figure F3: Regions in (c_2, c_1) space for $L = P_{eq} = P_a = 10$ with $q_1 = q_2 = 1$ (left) and $q_1 = 5, q_2 = 1/5$. For the diagram on the left stable regions correspond to the following equilibrium values: $\hat{N}^{(1)} = 0$ (All colors), 500 (All colors), 1000 (All colors), 1500 (Green, Blue, Cyan, and Magenta), 2000 (Blue, Cyan, and Magenta), 2500 (Cyan and Magenta), and 3000 (Magenta). For the diagram on the right stable regions correspond to the following equilibrium values: $\hat{N}^{(1)} = 0$ (All colors), 500 (Orange, Yellow, Green, Blue, Cyan, and Magenta), 1000 (Yellow, Green, Blue, Cyan, and Magenta), 1500 (Green, Blue, Cyan, and Magenta), 2000 (Blue, Cyan, and Magenta), 2500 (Cyan and Magenta), and 3000 (Magenta). Note that any point within the plotted region yields an equilibrium point of the system as equilibrium is independent of the c_i parameters. Areas outside of the designated colored regions correspond to non-stability for the specified $\hat{N}^{(1)}$.

Case 4. $P_{eq} = L = 10 \neq P_a$

Rather than consider any diagrams for this case, recall that criteria for the stability (instability) of an arbitrary equilibrium point is given in Theorem 10, i.e. the equilibrium

point is stable provided $q_1 < \frac{1+c_1}{2c_1}$ and unstable for $q_1 > \frac{1+c_1}{2c_1}$. Solving for c_1 yields the following:

$$\begin{aligned}
 \text{Stable} &\Leftrightarrow \left[c_1 < \frac{1}{-1 + 2q_1} \text{ and } q_1 > 1/2 \right] & (\text{F.1}) \\
 &\text{or } \left[c_1 > \frac{1}{-1 + 2q_1} \text{ and } q_1 < 1/2 \right] \\
 &\text{or } [q_1 = 1/2].
 \end{aligned}$$

Remark 14. For $c_1 = 1/10$ this criteria indicates stability for $q_1 < 11/2$. This agrees with the result for Case 4 in Section 3.5.

Remark 15. The c_2 parameter does not impact stability provided it is always positive.

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