ON THE BÁEZ-DUARTE APPROACH TO THE NYMAN-BEURLING CRITERION FOR THE RIEMANN HYPOTHESIS

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The Nyman-Beurling Criterion paraphrases the Riemann Hypothesis as a closure problem in a Hilbert space. The simplest version of this, due to Baez-Duarte, states that RH is equivalent to one particular element in a Hilbert space being in the closure of a span of countably many other elements. We investigate this numerically and analytically. In particular, we establish new formulas for the inner products of the vectors involved.

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PREFACE

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PRELIMINARIES

In 1859, Bernhard Riemann posited a theory regarding the distribution of prime numbers. To this day, his proposal remains what many consider to be the most important open problem in the whole of pure mathematics. It is one of the Clay Mathematics Institute's Millennium Prize Problems - the unsolved remnants of the twenty-three problems proffered in 1900 by David Hilbert. Hilbert believed these problems to be the most impacting and influential problems in mathematics, and that they should occupy the attention of mathematicians throughout the following century.

There have been several different reformulations of Riemann's conjecture cultivated throughout that 20th century, and even some reformation as late as 2006. We will investigate the hypothesis in the vein of some of the most recent developments that have been made. Let us begin with some general definitions, as well as pertinent functions and theorems. First and foremost, we define the function and conjecture that constitute the crux of this paper:

Definition: (Riemann Zeta Function)

The Reimann Zeta function is defined on the half-plane $s \in \mathbb{C}$ *with* $\Re\mathfrak{e}(s) > 1$ *by*

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Zeta function has long been a subject of great interest due to its many interesting qualities. Chief among these qualities is its relation to the distribution of prime numbers. Using the formula for a geometric series, and the convergence of $\zeta(s)$ as a Dirichlet series, we can obtain the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-p^{-s}}, \qquad \text{for } \Re\mathfrak{e}(s) > 1 \tag{1}$$

over all primes, p. This identity yields another noteworthy property of ζ in its relation to the following function:

Definition: (Möbius Function)

For any $n \in \mathbb{N}$ *with prime factorizatiom* $= \ p_1^{a_1} p_2^{a_2} \ldots\ p_k^{a_k}$, *then*

$$\mu(n) \coloneqq \begin{cases} (-1)^k, & \text{if } a_1 = a_2 = \cdots = a_k = 1 \\ 0, & otherwise \end{cases}$$

From (1), we have $\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right) \ldots \ = \left(1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{6^s}\right)\left(1 - \frac{1}{5^s}\right) \ldots$. From this it becomes clear that we can derive the identity

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\,, \qquad \text{for } \Re\mathrm{e}(s) > 1 \tag{2}$$

Conjecture: (Riemann Hypothesis)

All non-trivial zeros of ζ(s) have real part equal to $\frac{1}{2}$. (RH)

It is important to note that zeros of the zeta function are not limited to those of the function as it is defined above. Here, we are referring to the zeros of the zeta function as depicted by its *analytic continuation*. This is a technique used to extend the domain of an analytic function defined on an open subset of the complex plane to a larger open subset. When the analytic continuation, F, of a function, f, is defined over connected subset, $\Omega \subseteq \mathbb{C}$, then F is unique to f on Ω. Let us define a few key functions involved in the analytic continuation of ζ.

Definition: (Mellin Transform)

For any $f: [0, \infty) \to \mathbb{R}$, *the Mellin transform of* f *is given by*

$$\{\mathcal{M}f\}(s) \coloneqq \varphi(s) = \int_0^{\infty} \frac{t^s f(t)}{t}\, dt$$

The inverse transform is then given by

$$\{\mathcal{M}^{-1}\varphi\}(x) \coloneqq f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\varphi(s)}{x^s}\, ds$$

Definition: (Gamma Function)

The Gamma function is defined on $s \in \mathbb{C}$ *with* $\Re\mathrm{e}(s) > 0$ *by*

$$\Gamma(s) := \int_0^\infty \frac{e^{-t}t^s}{t}\,dt.$$

The Gamma function has several interesting properties that are pertinent to our investigation. It satisfies the functional equations:

$$\Gamma(s+1) = s\Gamma(s)\,, \tag{3}$$

and

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{4}$$

It is well-known that the function $Z(s) := \pi^{\frac{-s}{t}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$is the analytic continuation of ζ onto the whole complex plane, except at 1 where ζ has a simple pole. One method Riemann used to show this was via the Mellin transform of the Jacobi theta function $\theta(z) := \sum_{-\infty}^{\infty} e^{i\pi n^2 z}$, defined on the upper half of the complex plane. Using the Poussin summation formula, it is straightforward to show that θ satisfies the functional equation $\theta\left(-\frac{1}{z}\right) = \theta(z)\sqrt{iz}$. Then, we can employ the change of variables $t \rightarrow n^2\pi t$ in the gamma function, yielding $n^{-2s}\pi^{-s}\Gamma(s) = \int_0^\infty e^{-n^2\pi t}\,\frac{t^s}{t}dt$. Summing both sides over $\mathbb{N}$, we can switch the summand with the integral, since both sides converge uniformly, and we achieve

$$\frac{\Gamma(s)}{\pi^s}\sum_{n=1}^{\infty}\frac{1}{n^{2s}} = \int_0^\infty t^s \sum_{n=1}^{\infty} e^{-n^2\pi t}\,\frac{dt}{t}.$$

Since $\theta(iz) = \sum_{n=-\infty}^{-1} e^{i\pi n^2 z} + 1 + \sum_{n=1}^{\infty} e^{i\pi n^2 z}$, then we have

$$\pi^{-s}\Gamma(\,s\,)\zeta(2s) = \int_0^\infty \frac{1}{2}(\theta(it)-1)t^s\,\frac{dt}{t}$$

$$= \int_0^1 \frac{1}{2}(\theta(it)-1)t^s\,\frac{dt}{t} + \int_1^\infty \frac{1}{2}(\theta(it)-1)t^s\,\frac{dt}{t}$$

From another change of variables $t \rightarrow \frac{1}{t}$ in the portion of the integral from 0 to 1, we have

$$= \int_1^\infty \frac{1}{2}\left(\theta\left(-\frac{1}{it}\right)-1\right)t^{-s}\,\frac{dt}{t} + \int_1^\infty \frac{1}{2}(\theta(it)-1)t^s\,\frac{dt}{t}$$

Since $\theta\left(-\frac{1}{z}\right) = \theta(z)\sqrt{iz}$, then

$$= \int_1^\infty \frac{1}{2}\left(\sqrt{t}\theta(it) - 1\right)t^{-s}\frac{dt}{t} + \int_1^\infty \frac{1}{2}(\theta(it) - 1)t^{s}\frac{dt}{t}$$

$$= \int_1^\infty \frac{1}{2}(\theta(it) - 1)\left(t^{\frac{1}{2}-s} + t^{s}\right)\frac{dt}{t} + \int_1^\infty \frac{1}{2}\left(-t^{\frac{1}{2}-s} - t^{-s}\right)\frac{dt}{t}$$

$$= -\frac{1}{1-2s} - \frac{1}{2s} - \int_1^\infty \frac{1}{2}(\theta(it) - 1)\left(t^{\frac{1}{2}-s} + t^{s}\right)\frac{dt}{t}$$

It is clear now that the right hand side is analytic and defined on the whole complex plane except at the simple poles $s = 0$ and $s = \frac{1}{2}$. However, this function is symmetric about the change of variables $s \to \frac{1}{2} - s$. So, changing from $2s$ to s, we have that $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is the analytic continuation onto $\mathbb{C} \setminus \{1\}$.

Now, we concern ourselves with the zeros of the zeta function as defined by its analytic continuation, $Z(s)$; i.e. $\zeta(s) = \frac{Z(s)\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}$. From (3) and (4), we can derive that $\Gamma(-z) = -\frac{\pi}{z\Gamma(z)\sin(\pi z)}$. Given this identity, it becomes clear that $\frac{1}{\Gamma\left(\frac{s}{2}\right)} = 0$ for all $s \in \{-2, -4, \ldots\}$. This set is commonly referred to as the trivial zeros of the Riemann zeta function.

Let us define a few more number-theoretical functions, and then we will rigorously prove a very important theorem.

Definition: (Prime Counting Function)

The prime counting function $\pi : [1, \infty) \to \mathbb{N}$ *is given explicitly by*

$$\pi(x) \coloneqq \textit{the number of primes less than or equal to } x$$

Definition: (Mangoldt Function)

For any integer $n \geq 1$, *then* $\Lambda : \mathbb{N} \to \mathbb{R}$ *by*

$$\Lambda(n) \coloneqq \begin{cases} \ln(p), & \textit{if } n = p^m \textit{ for some prime } p \textit{ and some } m \geq 1 \\ 0, & \textit{otherwise} \end{cases}$$

Definition: (Chebychev ψ-function)

The Chebychev function $\psi : [1, \infty) \to \mathbb{N}$ *is given by*

$$\psi(x) \coloneqq \sum_{n \leq x} \Lambda(n)$$

Theorem: (Prime Number Theorem)

$$\lim_{x\to\infty} \frac{x\,\pi(x)}{\ln(x)} \;=\; 1 \qquad \text{(PNT)}$$

Proof: Taking the logarithm of (1), we have $\ln\big(\zeta(s)\big) = \ln\left(\prod_p \left(1-\frac{1}{p^s}\right)^{-1}\right) = -\sum_p \ln\left(1-\frac{1}{p^s}\right)$;

$$\Rightarrow \quad \frac{d}{ds}\ln\big(\zeta(s)\big) = -\sum_p \frac{d}{ds}\ln\left(1-\frac{1}{p^s}\right) = \sum_p \frac{\ln(p)}{p^s-1}$$

Next, fix a prime, p. Then

$$\sum_{k=1}^{\infty} \frac{\Lambda((p^s)^k)}{(p^s)^k} = \sum_{k=1}^{\infty} \frac{\ln(p)}{(p^s)^k} = \ln(p)\sum_{k=1}^{\infty}\frac{1}{(p^s)^k} = \ln(p)\frac{p^s}{1-p^s}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = \sum_p \frac{1}{p^s}\ln(p)\frac{p^s}{1-p^s} = -\frac{d}{ds}\ln\big(\zeta(s)\big) = -\frac{\zeta'(s)}{\zeta(s)}$$

Since $\Lambda(n)$ is arithmetic, then Perron's formula, which employs the Mellin transform, yields

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \quad \Rightarrow \sum_{n\le x}\Lambda(n) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

So, we have

$$\psi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

We know that at any x, the integral of this contour is evaluated as a sum of the opposites of the residues of each singularity of the integrand. We know that ζ, as defined by its analytic continuation over the complex plane, has a simple pole at $s=1$, a set of trivial zeros at all even negative integers, and a set of nontrivial zeros. Combining these singularities of ζ with the singularity $s=0$ of the integrand, we achieve

$$\psi(x) = x - \sum_{n=1}^{\infty}\frac{x^{-2n}}{2n} - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}$$

Since the convergence of both series is uniform for $x>1$, then we have

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_{n=1}^{\infty}\frac{x^{-2n+1}}{2n(2n-1)} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)}x + C$$

$$\Rightarrow \quad \frac{\int_0^x \psi(t)dt}{\frac{x^2}{2}} = 1 - \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{x^{-2n}}{n(2n-1)} - 2\sum_{\rho}\frac{x^{\rho}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{x}C\right)$$

$$\Rightarrow \frac{\int_0^x \psi(t)dt}{\frac{x^2}{2}} \to 1 \text{ as } x \to \infty$$

So we have $\int_0^x \psi(t)dt \sim \frac{x^2}{2}$. Fix $\lambda > 1$. Then for $\varepsilon > 0$, $\exists\, X$ such that $\forall\, x \geq X$

$$(1-\varepsilon)\frac{x^2}{2} < \int_0^x \psi(t)dt < (1+\varepsilon)\frac{x^2}{2} \text{ and } (1-\varepsilon)\frac{(\lambda x)^2}{2} < \int_0^{\lambda x} \psi(t)dt < (1+\varepsilon)\frac{(\lambda x)^2}{2}$$

$$\Rightarrow \quad (1-\varepsilon)\frac{(\lambda x)^2 - x^2}{2} - 2\varepsilon x^2 < \int_0^{\lambda x} \psi(t)dt - \int_0^x \psi(t)dt < (1+\varepsilon)\frac{(\lambda x)^2 - x^2}{2} + 2\varepsilon x^2 \quad (5)$$

Since ψ is increasing, we have

$$(\lambda x - x)\psi(x) \leq \int_x^{\lambda x} \psi(t)dt \leq (\lambda x - x)\psi(\lambda x) \quad (6)$$

Combining (5) and (6) yields inequalities independent of the integral:

$$\frac{\psi(x)}{x} \leq (1+\varepsilon)\frac{\lambda+1}{2} + \frac{\varepsilon}{\lambda-1}\text{, and}$$

$$(1-\varepsilon)\frac{\lambda+1}{2\lambda} - \frac{\varepsilon}{\lambda(\lambda-1)} \leq \frac{\psi(\lambda x)}{\lambda x}\ .$$

We can choose λ close enough to 1 and ε close enough to zero that for sufficiently large X we have $\frac{\psi(x)}{x}$ is within any arbitrarily small neighborhood of 1 for all $x \geq X$; i.e. $\psi(x) \sim x$.

Finally, we have

$$\psi(x) = \sum_{n \leq x}\Lambda(n) = \sum_{p \leq x}\ln(p)\left[\frac{\ln(x)}{\ln(p)}\right] \leq \sum_{p \leq x}\ln(x) = \pi(x)\ln(x)\ .$$

Also, for $1 < y < x$,

$$\pi(x) = \pi(y) + \sum_{y<p\leq x} 1 \leq \pi(y) + \sum_{y<p\leq x}\frac{\ln(p)}{\ln(y)} < y + \frac{1}{\ln(y)}\sum_{y<p\leq x}\ln(p) \leq y + \frac{1}{\ln(y)}\psi(x)$$

Then, for $y = \frac{x}{(\ln(x))^2}$, we have

$$\pi(x) \leq \frac{x}{\left(\ln(x)\right)^2} + \frac{1}{\ln(x) - 2\ln(\ln(x))}\psi(x)$$

$$\Rightarrow \frac{\pi(x)\ln(x)}{x} \leq \frac{1}{\ln(x)} + \frac{\ln(x)}{\ln(x) - 2\ln(\ln(x))}\frac{\psi(x)}{x}$$

Thus

$$\frac{\psi(x)}{x} \leq \frac{\pi(x)\ln(x)}{x} \leq \frac{1}{\ln(x)} + \left(\frac{1}{1 - \frac{2\ln(\ln(x))}{x}}\right)\frac{\psi(x)}{x}$$

So, since $\frac{\psi(x)}{x} \to 1$ as $x \to \infty$, then these estimates reveal that $\frac{\pi(x)\ln(x)}{x} \to 1$ as $x \to \infty$ as well.

■

Currently, the Prime Number Theorem is the best estimate of the distribution of primes that we have. And while it is an impressive postulate that sheds light on the behavior of prime numbers, it is still not that great an estimation. This brings us to the main consequence of the Riemann Hypothesis. If RH is true, this would provide a stronger estimate of the prime counting function:

$$\pi(x) = \int_0^x \frac{dt}{\log t} + O\left(\sqrt{x}\log x\right).$$

In his 1950 thesis, Bertil Nyman made an astounding breakthrough. He proved that RH is equivalent to a problem concerning closure in a Hilbert space. Let us define the space of functionals $L^p(0,1) \coloneqq \left\{f\colon (0,1) \to \mathbb{C} \,\middle|\, \sqrt[p]{\int_0^1 |f(x)|^p\, dx} < \infty\right\}$. Subsequently, we define a subspace

of $L^2(0,1)$ as $N_{(0,1)} \coloneqq \left\{g \in L^2(0,1) \,\middle|\, g(x) = \sum_{k=1}^n c_k \left\{\frac{\theta_k}{x}\right\},\ 0 < \theta_k \leq 1 \text{ and } \sum_{k=1}^n c_k\theta_k = 0\right\}$,

where, for all $x \in \mathbb{R}$, the fractional part of x is given by $\{x\}$, and $[x]$ denotes the integer part of x (i.e. $x = [x] + \{x\}$).

Theorem: (Nyman):

RH is true if and only if $N_{(0,1)}$ *is dense in* $L^2(0,1)$.

Five years later, Arne Beurling formulated a generalization of Nyman's theorem. Today, it is generally referred to as the Nyman-Beurling Criterion for the Riemann Hypothesis.

Theorem: (Beurling):

For $1 < p < \infty$, *the following are equivalent:* (NB)

i. $\zeta(s)$ *has no zeros in the half-plane* $\mathfrak{Re}(s) > \frac{1}{p}$

ii. $N_{(0,1)}$ *is dense in* $L^p(0,1)$

iii. $\chi_{(0,1)} \in \overline{N_{(0,1)}}$, *where* $\chi_{(0,1)}$ *is the characteristic function on* $(0,1)$.

In 2002, Luis Báez -Duarte modified NB into a problem regarding a countable set. Let us define $\varrho_a(x) := \left\{\frac{1}{ax}\right\}$, and $\mathfrak{B}$ be the space of all linear combinations of the set $\{\varrho_a | a \in \mathbb{N}\}$. Then we have

Theorem: (Báez-Duarte)

RH is true if and only if $\chi_{(0,1)} \in \overline{\mathfrak{B}}$. (BD)

RESULTS AND DISCUSSION

Let $\mathcal{V}$ denote the ℓ^2-space of sequences $\boldsymbol{x} = (x_i)_{i=1}^{\infty}$ such that $\boldsymbol{x} \in \mathbb{R}$ and $\sum_{i=1}^{\infty} \frac{|x_i|^2}{i(i+1)}$ converges. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, we define the inner-product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{\infty} \frac{x_i y_i}{i(i+1)}$. *(Note: Generally, this space is defined for sequences over the complex field with the inner-product* $\sum_{i=1}^{\infty} \frac{x_i \overline{y_i}}{i(i+1)}$*, but we restricted our investigation to the real numbers.)* We then can define $\mathcal{U}$, an orthogonal basis for $\mathcal{V}$, as $\mathcal{U} \coloneqq \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots\}$, where $\boldsymbol{u_i} = (0,..,0,1,0,..)$, the 1 in the i^{th} position. Then, for each $j \in \mathbb{N}$, we define the element $\boldsymbol{v}^{(j)} \in \mathcal{V}$ as $\boldsymbol{v}^{(j)} \coloneqq \sum_{i=1}^{\infty} i^{(mod\ j)} \boldsymbol{u_n}$, and a constant vector $\boldsymbol{v} \in \mathcal{V}$ as $\boldsymbol{v} \coloneqq \sum_{i=1}^{\infty} \boldsymbol{u_i}$. Finally, let $\mathcal{V}^{(n)}$ be the span of the set $\{\boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(n)}\}$. Under the framework of this particular notation, we can rephrase Báez-Duarte's criterion:

The Riemann Hypothesis is true if and only if $\boldsymbol{v}$ is in the closure of $\bigcup_{n=1}^{\infty} \mathcal{V}^{(n)}$. (7)

It is evident that the constant vector $\boldsymbol{v}$ cannot lie in any $\mathcal{V}^{(n)}$. However, since $\mathcal{V}$ is an inner-product space, then at each n, there is a non-negative minimal distance from $\boldsymbol{v}$ to $\mathcal{V}^{(n)}$. In other words, at each n, there is an "error" vector $\boldsymbol{e}^{(\boldsymbol{n})}$ that is orthogonal to $\mathcal{V}^{(n)}$ and that the difference $\boldsymbol{e}^{(\boldsymbol{n})} - \boldsymbol{v}$ is in $\mathcal{V}^{(n)}$.

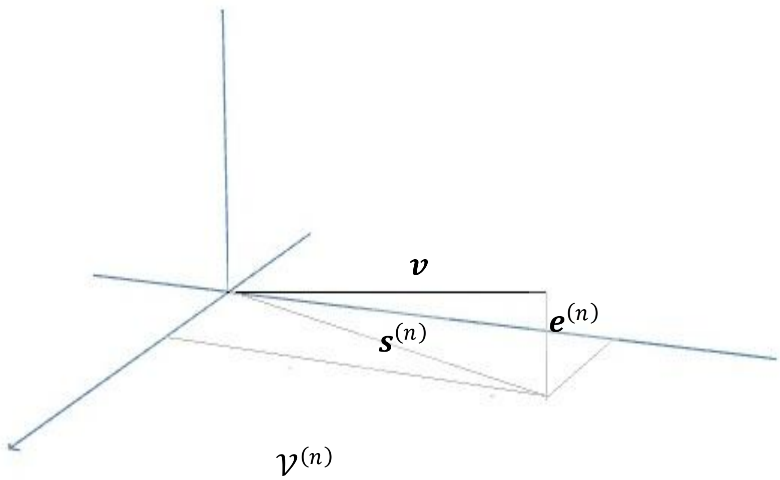

Figure 1.

Subsequently, at each n, there is a unique set of n scalars $\mathcal{A}^{(n)} = \left\{\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}\right\}$ that serve as coefficients to the spanning set $\left\{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}\right\}$ for which $\boldsymbol{e}^{(\boldsymbol{n})} - \boldsymbol{v}$ is a linear combination. Hence $\langle \boldsymbol{e}^{(\boldsymbol{n})}, \boldsymbol{v}^{(\boldsymbol{j})} \rangle = 0$ for each $\boldsymbol{v}^{(\boldsymbol{j})} \in \left\{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(n)}\right\}$, and this yields the n-dimensional system $\langle \boldsymbol{v}, \boldsymbol{v}^{(j)} \rangle = \sum_{k=1}^{n} \alpha_k^{(n)} \left(\boldsymbol{v}^{(\boldsymbol{k})}, \boldsymbol{v}^{(j)}\right)$. Assuming the norm, $\|.\|$, induced from the inner-product, we can once again restate BD:

$$\textit{The Riemann Hypothesis is true if and only if } \left\|\boldsymbol{e}^{(n)}\right\| \to 0 \textit{ as } n \to \infty. \tag{8}$$

We can now rely on some basic linear manipulation to solve for the set of scalars $\mathcal{A}^{(n)}$. Let's call $C^{(n)}$ the $n \times n$ matrix given by $C^{(n)} = \left[c_{j,k}\right]$, where $c_{j,k} = \langle \boldsymbol{v}^{(\boldsymbol{j})}, \boldsymbol{v}^{(k)} \rangle$. It is evident that $C^{(n)}$ is both real and symmetric. Then $\langle \boldsymbol{v}, \boldsymbol{v}^{(j)} \rangle = \sum_{k=1}^{n} \alpha_k^{(n)} c_{j,k}$. Also, let $B^{(n)}$ be the $n \times n$ inverse of $C^{(n)}$, with entries $b_{k,j}^{(n)}$ (*Note:* The superscript (n) is necessary in $b_{k,j}^{(n)}$, since the k, j^{th} entry of the inverse is dependent on n , as opposed to $c_{j,k}$). Then we have

$$\alpha_k^{(n)} = \sum_{j=1}^{n} b_{k,j}^{(n)} \langle \boldsymbol{v}^{(j)}, \boldsymbol{v}^{(k)} \rangle.$$

Now, we will employ a function that will let us express $\langle \boldsymbol{v}, \boldsymbol{v}^{(j)} \rangle$ and $\langle \boldsymbol{v}^{(j)}, \boldsymbol{v}^{(k)} \rangle$ as finite sums.

Definition: (Digamma Function)

The Digamma function is defined on $\mathbb{C} \backslash \{-n | \; n \in \mathbb{N}\}$ *by*

$$\Psi(x) := \frac{d}{dx}\ln\big(\Gamma(x)\big) = \frac{\Gamma'(x)}{\Gamma(x)}$$

Note: Though the notation for this function, Ψ*, is similar to the Chebychev function,* ψ*, besides the subtleties of italics and lower-case, understand that these signify two separate functions; this is the common nomenclature.* With this function, we can show the following:

Lemma1: $\langle \boldsymbol{v}, \boldsymbol{v}^{(j)} \rangle = ln(j)$

Proof: We have the identity $\Psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$, where $\Psi(1) = \gamma$ is the Euler constant. Through some manipulation, we achieve the following:

$$\sum_{\text{n}=0}^{\infty} \frac{1}{(nr+d+1)(nr+d)} = \frac{\Psi\left(\frac{d+1}{r}\right) - \Psi\left(\frac{d}{r}\right)}{r} \tag{9}$$

Then, since $\left(\boldsymbol{v}, \boldsymbol{v}^{(j)}\right) = \sum_{i=1}^{\infty} \frac{1 \cdot i^{(mod\ j)}}{i(i+1)}$, and since $(nr+d)^{(mod\ r)} = d^{(mod\ r)}, \forall\, n$ (i.e. the numerators have period j), then, by rearranging terms, this sum can be expressed as

$$\sum_{t=1}^{j-1} t \sum_{n=0}^{\infty} \frac{1}{(nj+t+1)(nj+t)}$$

By (9), we have

$$\langle \boldsymbol{v}, \boldsymbol{v}^{(j)} \rangle = \sum_{t=1}^{j-1} t \left(\frac{\Psi\left(\frac{t+1}{j}\right) - \Psi\left(\frac{t}{j}\right)}{j} \right)$$

$$= \frac{-1}{j}\left(\Psi\left(\frac{1}{j}\right) - \Psi\left(\frac{2}{j}\right) + 2\Psi\left(\frac{2}{j}\right) - 2\Psi\left(\frac{3}{j}\right) + \ldots + (j-1)\Psi\left(\frac{j-1}{j}\right) - (j-1)\Psi\left(\frac{j}{j}\right)\right)$$

$$= \frac{-1}{j}\left(\Psi\left(\frac{1}{j}\right) + \Psi\left(\frac{2}{j}\right) \ldots + \Psi\left(\frac{j-1}{j}\right) + \Psi\left(\frac{j}{j}\right) - \Psi\left(\frac{j}{j}\right) - (j-1)\Psi(1)\right)$$

$$= \frac{-1}{j}\left(\sum_{t=1}^{j} \Psi\left(\frac{t}{j}\right)\right) + \Psi(1)$$

It is well-known that $\sum_{t=1}^{j} \Psi\left(\frac{t}{j}\right) = -j\big(\gamma + \ln(j)\big)$, and $\Psi(1) = -\gamma.$ Hence

$$\langle \boldsymbol{v}, \boldsymbol{v}^{(j)} \rangle = \frac{-1}{j}\big(-j(\gamma + \ln(j)\big) - \gamma = \ln(j)$$

■

Now at each k, $\alpha_k^{(n)} = \sum_{j=1}^{n} b_{k,j}^{(n)} \ln(j)$. Next, we examine $\langle \boldsymbol{v}^{(k)}, \boldsymbol{v}^{(j)} \rangle = \sum_{i=1}^{\infty} \frac{i^{(mod\ j)} i^{(mod\ k)}}{i(i+1)}$. Here, the period of the numerators is jk. So, again, by rearranging terms, and by (9), we have

$$\langle \boldsymbol{v}^{(k)}, \boldsymbol{v}^{(j)} \rangle = \frac{1}{jk} \sum_{t=1}^{jk-1} t^{(mod\ j)} \cdot t^{(mod\ k)} \cdot \left(\Psi\left(\frac{t+1}{jk}\right) - \Psi\left(\frac{t}{jk}\right) \right)$$

$$\Rightarrow\ c_{jk} = \sum_{t=1}^{jk-1} \left\{\frac{t}{j}\right\} \cdot \left\{\frac{t}{k}\right\} \cdot \left(\Psi\left(\frac{t+1}{jk}\right) - \Psi\left(\frac{t}{jk}\right) \right)$$

Let us now refer to $\boldsymbol{v} - \boldsymbol{e}^{(\boldsymbol{n})}$, the orthogonal projection of $\boldsymbol{v}$ onto $\mathcal{V}^{(n)}$, as $\boldsymbol{s}^{(\boldsymbol{n})}$. Since $\boldsymbol{s}^{(\boldsymbol{n})} = \sum_{k=1}^{n} \alpha_k^{(n)} \boldsymbol{v}^{(k)}$, then the i^{th} component of $\boldsymbol{s}^{(\boldsymbol{n})}$ is $s_i^{(n)} = \sum_{k=1}^{n} i^{(mod\ k)} \alpha_k^{(n)}$; that is $s_i^{(n)} = 1 - e_i^{(n)}$. By Lemma1, we also have for each k, $\alpha_k^{(n)} = \sum_{j=1}^{n} b_{kj}^{(n)} \ln(j)$. Since RH is equivalent to $\boldsymbol{e}^{(\boldsymbol{n})}$ tending to zero, then clearly if there is any i such that $s_i^{(n)}$ does not converge to 1 as $n \to \infty$, then RH does not hold. It is natural, then, to look for a limit α_k to which each $\alpha_k^{(n)}$ converges, so that $s_i^{(n)} \to 1$ at each i.

Lemma2: *$lim_{n\to\infty}$ $\boldsymbol{s}^{(\boldsymbol{n})} = \boldsymbol{v}$ if and only if $lim_{n\to\infty}$ $\alpha_k^{(n)} = -\frac{\mu(k)}{k}$, $\forall\ k$.*

Proof: Assume $\alpha_k^{(n)} \to -\frac{\mu(k)}{k}$ as $n \to \infty$. Then at each i , we have

$$s_i^{(n)} = \sum_{k=1}^{n} \alpha_k^{(n)} \cdot i^{(mod\ k)} = \sum_{k=2}^{n} \alpha_k^{(n)} \cdot k \left\{\frac{i}{k}\right\}$$

$$= \sum_{k=2}^{i} \alpha_k^{(n)} \cdot k \left\{\frac{i}{k}\right\} + \sum_{k=i+1}^{n} \alpha_k^{(n)} \cdot k \left(\frac{i}{k}\right) = \sum_{k=2}^{i} \alpha_k^{(n)} \cdot k \left\{\frac{i}{k}\right\} + \sum_{k=i+1}^{n} \alpha_k^{(n)} \cdot i$$

$$= -\sum_{k=2}^{i} \alpha_k^{(n)} \cdot k \left[\frac{i}{k}\right] + \sum_{k=2}^{i} \alpha_k^{(n)} \cdot k \left(\frac{i}{k}\right) \sum_{k=i+1}^{n} \alpha_k^{(n)} \cdot i$$

$$= -\sum_{k=2}^{i} \alpha_k^{(n)} \cdot k \left[\frac{i}{k}\right] + \sum_{k=2}^{n} \alpha_k^{(n)} \cdot i$$

$$\Rightarrow \lim_{n\to\infty} s_i^{(n)} = -\sum_{k=2}^{i} \lim_{n\to\infty} \alpha_k^{(n)} \cdot k \left[\frac{i}{k}\right] + \lim_{n\to\infty} \sum_{k=2}^{n} \alpha_k^{(n)} \cdot i$$

$$\Rightarrow \qquad s_i = \sum_{k=2}^{i} \frac{\mu(k)}{k} \cdot k \left[\frac{i}{k}\right] + i \lim_{n\to\infty} \sum_{k=2}^{n} \alpha_k^{(n)}$$

Since $\sum_{k=1}^{i} \frac{\mu(k)}{k} \cdot k \left[\frac{i}{k}\right] = 1$, $\forall\, i \geq 1$, we have

$s_i = 1 + (s_1 - 1)\, i$

If $s_1 \neq 1$, then s_i diverges. Hence $s_1 = 1$, and, subsequently $s_i = 1$, $\forall\, i$;

i.e. $\lim_{n\to\infty} \boldsymbol{s^{(n)}} = \boldsymbol{v}$.

Conversely, assume $\lim_{n\to\infty} \boldsymbol{s^{(n)}} = \boldsymbol{v}$; i.e. $\lim_{n\to\infty} s_i^{(n)} \to 1$, $\forall\, i$. We have

$s_1^{(n)} = \alpha_1^{(n)} 1^{(mod\ 1)} + \alpha_2^{(n)} 1^{(mod\ 2)} + \ \dots\ + \alpha_n^{(n)} 1^{(mod\ n)} = \alpha_2^{(n)} + \ \dots\ + \alpha_n^{(n)}$, and

$s_2^{(n)} = \alpha_1^{(n)} 2^{(mod\ 1)} + \alpha_2^{(n)} 2^{(mod\ 2)} + \ \dots\ + \alpha_n^{(n)} 2^{(mod\ n)} = 2\left(\alpha_3^{(n)} + \ \dots\ + \alpha_n^{(n)}\right)$

$\Rightarrow \quad 2s_1^{(n)} - s_2^{(n)} = \alpha_2^{(n)}$

$\Rightarrow \quad 2s_1 - s_2 = 2 - 1 = 2\alpha_2 \qquad\qquad \Rightarrow \quad \alpha_2 = \frac{1}{2} = -\frac{\mu(2)}{2}$

Now we employ strong induction on k; assume $\alpha_k^{(n)} \to \alpha_k = -\frac{\mu(k)}{k}$, for $k \in \{1, \dots, i-1\}$.

Then

$$s_i^{(n)} = \alpha_1^{(n)} i^{(mod\ 1)} + \alpha_2^{(n)} i^{(mod\ 2)} + \ \dots\ + \alpha_n^{(n)} i^{(mod\ n)}$$

$$= \sum_{k=2}^{i-1} \alpha_k^{(n)} i^{(mod\ k)} + \sum_{k=2}^{i-1} \alpha_k^{(n)} i$$

$$\Rightarrow \quad 1 = \lim_{n\to\infty} \sum_{k=2}^{i-1} \alpha_k^{(n)} i^{(mod\ k)} + \lim_{n\to\infty} \sum_{k=i+1}^{n} \alpha_k^{(n)} i \text{ , and}$$

$$1 = \lim_{n\to\infty} \sum_{k=2}^{n} \alpha_k^{(n)} 1^{(mod\ k)} = \lim_{n\to\infty} \sum_{k=2}^{n} \alpha_k^{(n)}$$

$$\Rightarrow \quad i = \lim_{n\to\infty} \sum_{k=2}^{n} \alpha_k^{(n)} i$$

$$(i-1) = \lim_{n\to\infty} \sum_{k=2}^{n} \alpha_k^{(n)} i - \lim_{n\to\infty} \sum_{k=2}^{i-1} \alpha_k^{(n)} i^{(mod\ k)} - \lim_{n\to\infty} \sum_{k=i+1}^{n} \alpha_k^{(n)} i$$

$$= \lim_{n\to\infty} \sum_{k=2}^{i-1} \alpha_k^{(n)} \left(i - i^{(mod\ k)}\right) - \lim_{n\to\infty} \sum_{k=2}^{i-1} \alpha_k^{(n)} i + \alpha_i^{(n)} i$$

$$= -\lim_{n\to\infty}\sum_{k=2}^{i-1}\frac{\mu(k)}{k}\left[\frac{i}{k}\right] + i\lim_{n\to\infty}\alpha_i^{(n)} = -\big(1 - i - \mu(i)\big) + i\lim_{n\to\infty}\alpha_i^{(n)}$$

$$\Rightarrow \quad \alpha_i^{(n)} \to \frac{\mu(i)}{i} \text{ as } n \to \infty.$$

■

Thus, our final rephrasing of BD:

$$\textit{For } \alpha_k^{(n)} = \sum_{j=1}^{n} b_{kj}^{(n)} \ln(j), \textit{ then RH is true if and only if } \alpha_k^{(n)} \to -\frac{\mu(k)}{k}, \forall\, k \in \mathbb{N}.$$

Conjecture:

$$\textit{RH is true if and only if } \sum_{k=1}^{\infty} c_{jk} \cdot \left(-\frac{\mu(k)}{k}\right) = \ln(j) \qquad (10)$$

The following is a graph of n vs. $\sum_{k=1}^{n} c_{5k} \cdot \left(-\frac{\mu(k)}{k}\right) - \ln(5)$.

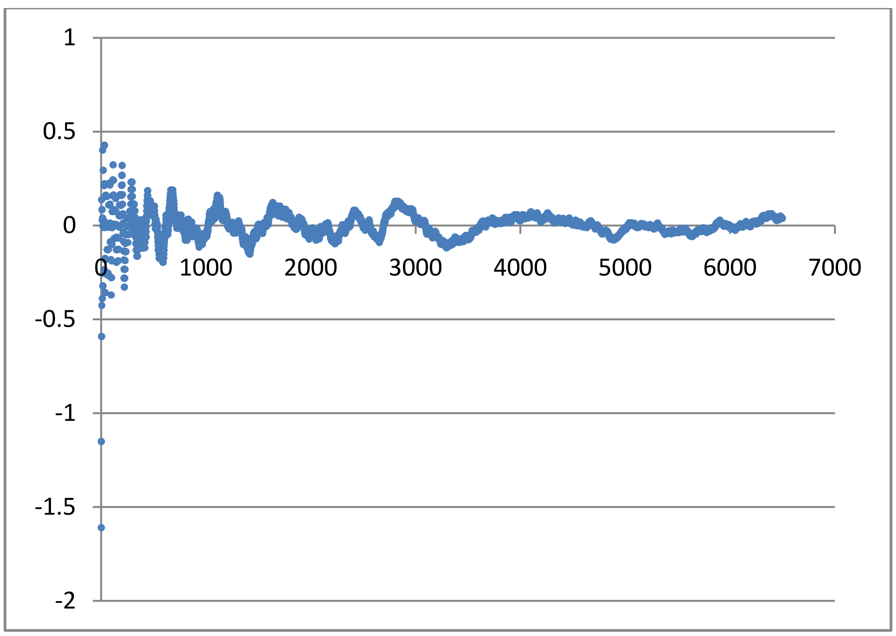

Figure 2.

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