HIGH ORDER DISCONTINUOUS GALERKIN METHODS FOR 1D PARABOLIC EQUATIONS

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HIGH ORDER DISCONTINUOUS GALERKIN METHODS

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Development of accurate and efficient numerical methods is an important task for many research areas. This work presents the numerical study of the Discontinuous Galerkin Finite Element (DG) methods in space and various ODE solvers in time applied to 1D parabolic equation.

In particular, we study the numerical convergence and computational efficiency of the Backward Euler (BE) in time and high order DG in space methods vs. the numerical convergence and the computational efficiency of the DG in time and space methods.

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1.0 INTRODUCTION

The goal of this work is to compare the computational efficiency of the Backward Euler (BE) in time and high order Discontinuous Galerkin (DG) in space method vs. the computational efficiency of the DG in time and space method (high order only in space), for a one dimensional (1D) parabolic equation.

The DG methods have recently become popular thanks to certain features which may make them attractive to researchers, such as:

- Local, element-wise mass conservation;
- Flexibility to use high-order polynomial and non-polynomial basis functions;
- Ability to easily increase the order of approximation on each mesh element independently;
- Ability to achieve an almost exponential convergence rate when smooth solutions are captured on appropriate meshes;
- Suitability for parallel computations due to (relatively)local data communications;
- Applicability to problems with discontinuous coefficients and/or solutions;

The DG methods have been successfully applied to a wide variety of problems ranging from the solid mechanics to the fluid mechanics.

There are other methods which are used to solve similar problems, such as the finite difference method. The major disadvantage of this method is that it is a low order method. Additionally, DG method is well suited for handling unstructured meshes, compared to the finite difference method. There are also many commonly used finite element methods. However, adaptively increasing the degree of polynomial in these methods is not as straight forward as in the DG method.

After we establish the formulation of the problem and delineate the construction of the solution methods, we conduct a number of computational experiments to test the rates of convergence of the utilized methods against theoretical predictions. We note that the BE method requires very small time steps during calculations in order to maintain high order convergence rates in space with the DG method. Such restrictions are much more relaxed in the case of the DG in time and space method, as will be explained in this thesis. This is the one clear advantage of the DG in time and space method against the BE in time and DG in space method.

2.0 PROBLEM

We consider the following parabolic problem:

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \forall\, x \in [a,b], t \in (0,\tau), \qquad (1)$$

$$u(0,t) = g_0(t), \qquad u(1,t) = g_1(t),$$
$$u(x,0) = u_0(x). \qquad (2)$$

Here, f belongs to $\boldsymbol{C}^0(0{,}1)$.

We can assume that the problem (1)-(2) has a solution in $(a,b) \subset \mathbb{R}$. We say that u is a strong solution of the above system if $u \in \boldsymbol{C}^2(0{,}1)$ and u satisfies the system pointwise.

3.0 BACKWARD EULER AND DISCONTINUOUS GALERKIN SCHEME

Let $0 = x_0 < x_1 < \ldots < x_N = 1$ be a subdivision of $[0{,}1]$ and let $I_n = [x_{n-1}, x_n]$. Denote by $\mathcal{P}_k$ the space of piecewise discontinuous polynomials of degree k:

$$\mathcal{P}_k := \{\, v : v\,|_{I_n} \in \mathbb{P}_k\,(I_n), \forall\, n = 0, \ldots, N-1\}$$

where $\mathbb{P}_k\,(I_n)$ is the space of polynomials of degree k on the interval I_n.

To solve (1)-(2), we will first use a combined Backward Euler scheme, and a Discontinuous Galerkin (DG) scheme. In order to define the method, we introduce a linear form L and a bilinear form a_ε (see [2]):

$$L(t,v) := \int_0^1 f(x,t)v(x)dx + \frac{\sigma^0}{h}\, v(x_0)g_0(t) - \varepsilon v'(x_0)g_0(t) + \frac{\sigma^0}{h}\, v(x_N)g_1(t) + \varepsilon v'(x_N)g_1(t)$$

where $h = 1/N$, and

$$a_\varepsilon(w, \mathrm{v}) := \sum_{\mathrm{n}=0}^{\mathrm{N}-1} \int_{\mathrm{x_n}}^{\mathrm{x_{n+1}}} \mathrm{w}'(\mathrm{x})\mathrm{v}'(\mathrm{x})\mathrm{dx} - \sum_{\mathrm{n}=0}^{\mathrm{N}} \{\mathrm{w}'(\mathrm{x_n})\}[\mathrm{v}(\mathrm{x_n})] + \varepsilon \sum_{\mathrm{n}=0}^{\mathrm{N}} \{\mathrm{v}'(\mathrm{x_n})\}[\mathrm{w}(\mathrm{x_n})] + \mathcal{J}_0(\mathrm{w}, \mathrm{v})$$

where, $\mathrm{w}, \mathrm{v} \in \mathcal{P}_k$, and $\mathcal{J}_0$ is the penalty term for the jump in the functions v and w, defined as:

$$\mathcal{J}_0(\mathrm{w}, \mathrm{v}) := \sum_{\mathrm{n}=0}^{\mathrm{N}} \frac{\sigma^0}{\mathrm{h}} [\mathrm{w}(\mathrm{x_n})][\mathrm{v}(\mathrm{x_n})]$$

Here, σ^0 is a non-negative real number called penalty parameter. In order to define the jump [ ] and average { } terms, we first define $\mathrm{x_n^+}$ and $\mathrm{x_n^-}$ as follows:

$\mathrm{x_n^+} := \lim_{\varepsilon\downarrow 0}(\mathrm{x_n} + \varepsilon)$ and $\mathrm{x_n^-} := \lim_{\varepsilon\uparrow 0}(\mathrm{x_n} - \varepsilon)$.

Then, we define the jump of a function w at a point $\mathrm{x_n}$, for $\mathrm{i} = 1, \dots, \mathrm{N}-1$, as the difference of the values of w from the right of point $\mathrm{x_n}$ and from the left of point $\mathrm{x_n}$, ie;

$$[\mathrm{w}(\mathrm{x_n})] = \mathrm{v}(\mathrm{x_n^+}) - \mathrm{v}(\mathrm{x_n^-})$$

Clearly, there is no jump at the initial and end points of the interval (points $\mathrm{x_0}$ and $\mathrm{x_N}$), and by convention we set $[\mathrm{w}(\mathrm{x_0})] = -\mathrm{v}(\mathrm{x_0^+})$, $[\mathrm{w}(\mathrm{x_N})] = \mathrm{v}(\mathrm{x_N^-})$. If the function w is continuous at the point $\mathrm{x_n}$, then the jump equals 0. If the function w is discontinuous at the point $\mathrm{x_n}$, then the jump is non-zero.

Additionally, we define the average term for a function v at a point $\mathrm{x_n}$, for $\mathrm{i} = 1, \dots, \mathrm{N}-1$, as the average of the values of w from the right of point $\mathrm{x_n}$ and from the left of point $\mathrm{x_n}$. ie;

$$\{\mathrm{v}(\mathrm{x_n})\} = \frac{1}{2}(\mathrm{v}(\mathrm{x_n^+}) + \mathrm{v}(\mathrm{x_n^-}))$$

If the function v is continuous at point $\mathrm{x_n}$, then $\{\mathrm{v}(\mathrm{x_n})\} = \mathrm{v}(\mathrm{x_n})$. Similarly by convention, $\{\mathrm{v}(\mathrm{x_0})\} = \mathrm{v}(\mathrm{x_0^+})$, $\{\mathrm{v}(\mathrm{x_N})\} = \mathrm{v}(\mathrm{x_N^-})$.

The reason for the inclusion of the penalty terms will be explained in more detail in the next section. Also, ε is a real number, but we restrict ourselves to the cases $\varepsilon \in \{-1,0,1\}$. This restriction will allow us to identify the error in our estimates in the cases of the bilinear form being symmetric and non-symmetric. These cases are identified as NIPG

(Non-symmetric Interior Penalty Galerkin) when $\varepsilon = 1$, IIPG (Incomplete Interior Penalty Galerkin) when $\varepsilon = 0$, and SIPG (Symmetric Interior Penalty Galerkin) when $\varepsilon = -1$. The bilinear form is non-symmetric in the cases $\varepsilon = 0$ *and* 1 only. [1]

Let $\Delta t > 0$ be the time step and let $t^i = i\,\Delta t$. We want to find an approximation $P_{i+1}^{DG}(x) \approx u(x,t)$.

First, we solve for the initial solution P_0^{DG}: For $v \in \mathcal{P}_k$, $\int_0^1 P_0^{DG}(x)v(x)dx = \int_0^1 u_0(x)v(x)dx$

Then, we solve the following equation for $P_{i+1}^{DG} \in \mathcal{P}_k$ and $i \geq 0$:

For $v \in \mathcal{P}_k$,

$$\frac{1}{\Delta t}\int_0^1 P_{i+1}^{DG}(x)v(x)dx + a_\varepsilon(P_{i+1}^{DG}, v) = L(t,v) + \frac{1}{\Delta t}\int_0^1 P_i^{DG}(x)v(x)dx$$

3.1 LOCAL BASIS FUNCTIONS

We now need to discuss some details of our scheme. We will choose basis functions from $\mathcal{P}_k$ to be used in our scheme. We will consider the case k=4.

On each interval I_n, we choose 5 basis functions $\{\phi_0^n, \phi_1^n, \phi_2^n, \phi_3^n, \phi_4^n\}$ such that

- ϕ_0^n is constant
- ϕ_1^n is linear
- ϕ_2^n is quadratic
- ϕ_3^n is cubic
- ϕ_4^n is quartic

We will extend these functions to equal zero on all other intervals, and keep their names. These extended functions are global basis functions. This construction will have the benefit of causing the global basis functions to have local support. This will be very useful in calculations of our solution.

From this construction, we observe that the global basis functions are not well defined at the points x_i, for $i = 1, \ldots, N-1$. This can easily be illustrated by an example.

Consider the case of the two intervals $[x_0, x_1]$ and $[x_1, x_2]$ belonging to [0,1]. On the point x_1, ϕ assumes two values based on the local basis functions of each sub-interval. Therefore, ϕ is not well defined on the points x_i, for $i = 1, \dots, N-1$.

But, how do we choose a local basis function ϕ_i^j in the first place? Before answering this question, we should shift our attention to a seemingly minor point.

Again, we consider the case of k = 4. To be practical, we would like to use the monomial basis functions $\{1, x, x^2, x^3, x^4\}$ of $\mathcal{P}_4$ on each interval I_n. However, these basis functions need to be translated to each interval I_n from the interval (-1, 1). The reason for our choice of (-1, 1) is, of course, due to our use of Gaussian quadrature in calculating the integral in our DG scheme.

The translation is accomplished as follows:

$$\phi_0^n(x) = 1$$

$$\phi_1^n(x) = 2\frac{x - x_{n+1/2}}{x_{n+1} - x_n}$$

$$\phi_2^n(x) = 4\frac{(x - x_{n+1/2})^2}{(x_{n+1} - x_n)^2}$$

$$\phi_3^n(x) = 8\frac{(x - x_{n+1/2})^3}{(x_{n+1} - x_n)^3}$$

$$\phi_4^n(x) = 16\frac{(x - x_{n+1/2})^4}{(x_{n+1} - x_n)^4}$$

where $x_{n+1/2} = \frac{1}{2}(x_n + x_{n+1})$ is the midpoint of the interval I_n.

Since all intervals are of the same length h, this simplifies the basis functions to the following form:

$$\phi_0^n(x) = 1$$

$$\phi_1^n(x) = \frac{2}{h}(x - \left(n + \frac{1}{2}\right)h)$$

$$\phi_2^n(x) = \frac{4}{h^2}(x - \left(n + \frac{1}{2}\right)h)^2$$

$$\phi_3^n(x) = \frac{8}{h^3}(x - \left(n + \frac{1}{2}\right)h)^3$$

$$\phi_4^n(x) = \frac{16}{h^4}(x - \left(n + \frac{1}{2}\right)h)^4$$

These basis functions have the following derivatives:

$$\phi_0^{n'}(x) = 0$$

$$\phi_1^{n'}(x) = \frac{2}{h}$$

$$\phi_2^{n'}(x) = \frac{8}{h^2}(x - \left(n + \frac{1}{2}\right)h)$$

$$\phi_3^{n'}(x) = \frac{24}{h^3}(x - \left(n + \frac{1}{2}\right)h)^2$$

$$\phi_4^{n'}(x) = \frac{64}{h^4}(x - \left(n + \frac{1}{2}\right)h)^3$$

We also need to calculate the basis functions over points shared by adjacent intervals.

First,

$\phi_0^n(x_n^+) = 1$, $\phi_0^{n'}(x_n^+) = 1$

$\phi_1^n(x_n^+) = -1$, $\phi_1^{n'}(x_n^+) = \frac{2}{h}$

$\phi_2^n(x_n^+) = 1$, $\phi_2^{n'}(x_n^+) = \frac{-4}{h}$

$\phi_3^n(x_n^+) = -1$, $\phi_3^{n'}(x_n^+) = \frac{6}{h}$

$\phi_4^n(x_n^+) = 1$, $\phi_4^{n'}(x_n^+) = \frac{-8}{h}$

Next,

$\phi_0^n(x_n^-) = 1$, $\phi_0^{n'}(x_n^-) = 1$

$\phi_1^n(x_n^-) = 1$, $\phi_1^{n'}(x_n^-) = \frac{2}{h}$

$\phi_2^n(x_n^-) = 1$, $\phi_2^{n'}(x_n^-) = \frac{4}{h}$

$\phi_3^n(x_n^-) = 1$, $\phi_3^{n'}(x_n^-) = \frac{6}{h}$

$\phi_4^n(x_n^-) = 1$, $\phi_4^{n'}(x_n^-) = \frac{8}{h}$

3.2 LINEAR SYSTEM

Using the above basis functions, we can expand the DG solution as:

$$P_\ell^{DG}(x) = \sum_{m=0}^{N-1}\sum_{j=0}^{4} \alpha_{\ell,j}^m \phi_j^m(x)$$

for every $x \in (0,1)$.

Here, $\alpha_{\ell,j}^m$ are unknown real numbers to be solved for. With this decomposition of P_ℓ^{DG} our scheme becomes:

$$\frac{1}{\Delta t}\sum_{m=0}^{N-1}\sum_{j=0}^{4} \alpha_{\ell+1,j}^m \int_{x_n}^{x_{n+1}} \phi_j^m(x)\phi_i^n(x)dx + \sum_{m=0}^{N-1}\sum_{j=0}^{4} \alpha_{\ell+1,j}^m\, a_\varepsilon(\phi_m^j, \phi_n^i) = \hat{L}(\phi_n^i)$$

where,

$$\hat{L}\left(\phi_n^i\right) := L\left(t^{\ell+1}, \phi_n^i\right) + \frac{1}{\Delta t}\int_0^1 P_\ell^{DG}(x)\, v(x)\, dx$$

which holds for all $0 \leq i \leq 4$ and $0 \leq n \leq N-1$.

Thus, we obtain a linear system $A\alpha = b$, where α is the vector with the components $\alpha_{\ell+1,j}^m$. A very important technical point to make is that the global matrix A can be obtained by computing and assembling local matrices. The reason we can do this is that, by their construction, the global basis functions ϕ_j^n have local support.

The matrices A_n and M_n correspond to the volume integral in our scheme, ie;

$$\int_{I_n} (P^{DG}_{\ell+1})'(x)\, v'(x)\, dx = A_n \alpha^n_{\ell+1}$$

$$\int_{I_n} (P^{DG}_{\ell+1})(x) v(x)\, dx = M_n \alpha^n_{\ell+1}$$

where $\alpha^n_{\ell+1} = (\alpha^n_{\ell+1,0}, \alpha^n_{\ell+1,1}, \dots, \alpha^n_{\ell+1,4})^T$, $(A_n)_{ij} = \int_{I_n} (\phi^n_i)'(x)\left(\phi^n_j\right)'(x)\, dx$, and

$$(M_n)_{ij} = \int_{I_n} \phi^n_i(x)\phi^n_j(x)\, dx\,.$$

The matrix B_n corresponds to the interactions of the local basis functions of the interval I_n. Additionally, the matrix C_n corresponds to the interactions of local basis functions on I_{n-1}. These matrices can be calculated by expanding the average and jump terms in our scheme as:

$$Bn = \frac{1}{2}(P^{DG}_{\ell+1})'(x^+_n)v(x^+_n) - \frac{\varepsilon}{2}P^{DG}_{\ell+1}(x^+_n)v'(x^+_n) + \frac{\sigma^0}{h}P^{DG}_{\ell+1}(x^+_n)v(x^+_n)$$

$$Cn = -\frac{1}{2}(P^{DG}_{\ell+1})'(x^-_n)v(x^-_n) + \frac{\varepsilon}{2}P^{DG}_{\ell+1}(x^-_n)v'(x^-_n) + \frac{\sigma^0}{h}P^{DG}_{\ell+1}(x^-_n)v(x^-_n)$$

As alluded to earlier, there are also very limited, but important, interactions between basis functions of adjacent intervals. The matrices D_n and E_n represent these interactions between the intervals I_n and I_{n-1}. These matrices can also be calculated by expanding the average and jump terms in our scheme as:

$$Dn = -\frac{1}{2}(P^{DG}_{\ell+1})'(x^+_n)v(x^-_n) - \frac{\varepsilon}{2}P^{DG}_{\ell+1}(x^+_n)v'(x^-_n) - \frac{\sigma^0}{h}P^{DG}_{\ell+1}(x^+_n)v(x^-_n)$$

$$En = \frac{1}{2}(P^{DG}_{\ell+1})'(x^-_n)v(x^+_n) + \frac{\varepsilon}{2}P^{DG}_{\ell+1}(x^-_n)v'(x^+_n) - \frac{\sigma^0}{h}P^{DG}_{\ell+1}(x^-_n)v(x^+_n)$$

Finally, F_0 and F_N are the local matrices arising from the boundary nodes x_0 and x_N.

$$F_0 = (P^{DG}_{\ell+1})'(x_0)v(x_0) - \varepsilon P^{DG}_{\ell+1}(x_0)v'(x_0) + \frac{\sigma^0}{h}P^{DG}_{\ell+1}(x_0)v(x_0)$$

$$F_N = -(P^{DG}_{\ell+1})'(x_N)v(x_N) + \varepsilon P^{DG}_{\ell+1}(x_N)v'(x_N) + \frac{\sigma^0}{h}P^{DG}_{\ell+1}(x_N)v(x_N)$$

The local matrices for interval I_n, based on quartic polynomials, are:

$$A_n = \frac{1}{h}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & \frac{16}{3} & 0 & \frac{32}{5} \\ 0 & 4 & 0 & \frac{36}{5} & 0 \\ 0 & 0 & \frac{32}{5} & 0 & \frac{64}{7} \end{pmatrix}$$

$$B_n = \frac{1}{h}\begin{pmatrix} \sigma^0 & 1-\sigma^0 & -2+\sigma^0 & 3-\sigma^0 & -4+\sigma^0 \\ -\varepsilon-\sigma^0 & -1+\varepsilon+\sigma^0 & 2-\varepsilon-\sigma^0 & -3+\varepsilon+\sigma^0 & 4-\varepsilon-\sigma^0 \\ 2\varepsilon+\sigma^0 & 1-2\varepsilon-\sigma^0 & -2+2\varepsilon+\sigma^0 & 3-2\varepsilon-\sigma^0 & -4+2\varepsilon+\sigma^0 \\ -3\varepsilon-\sigma^0 & -1+3\varepsilon+\sigma^0 & 2-3\varepsilon-\sigma^0 & -3+3\varepsilon+\sigma^0 & 4-3\varepsilon-\sigma^0 \\ 4\varepsilon+\sigma^0 & 1-4\varepsilon-\sigma^0 & -2+4\varepsilon+\sigma^0 & 3-4\varepsilon-\sigma^0 & -4+4\varepsilon+\sigma^0 \end{pmatrix}$$

$$C_n = \frac{1}{h}\begin{pmatrix} \sigma^0 & -1+\sigma^0 & -2+\sigma^0 & -3+\sigma^0 & -4+\sigma^0 \\ \varepsilon+\sigma^0 & -1+\varepsilon+\sigma^0 & -2+\varepsilon+\sigma^0 & -3+\varepsilon+\sigma^0 & -4+\varepsilon+\sigma^0 \\ 2\varepsilon+\sigma^0 & -1+2\varepsilon+\sigma^0 & -2+2\varepsilon+\sigma^0 & -3+2\varepsilon+\sigma^0 & -4+2\varepsilon+\sigma^0 \\ 3\varepsilon+\sigma^0 & -1+3\varepsilon+\sigma^0 & -2+3\varepsilon+\sigma^0 & -3+3\varepsilon+\sigma^0 & -4+3\varepsilon+\sigma^0 \\ 4\varepsilon+\sigma^0 & -1+4\varepsilon+\sigma^0 & -2+4\varepsilon+\sigma^0 & -3+4\varepsilon+\sigma^0 & -4+4\varepsilon+\sigma^0 \end{pmatrix}$$

$$D_n = \frac{1}{h}\begin{pmatrix} -\sigma^0 & -1+\sigma^0 & 2-\sigma^0 & -3+\sigma^0 & 4-\sigma^0 \\ -\varepsilon-\sigma^0 & -1+\varepsilon+\sigma^0 & 2-\varepsilon-\sigma^0 & -3+\varepsilon+\sigma^0 & 4-\varepsilon-\sigma^0 \\ -2\varepsilon-\sigma^0 & -1+2\varepsilon+\sigma^0 & 2-2\varepsilon-\sigma^0 & -3+2\varepsilon+\sigma^0 & 4-2\varepsilon-\sigma^0 \\ -3\varepsilon-\sigma^0 & -1+3\varepsilon+\sigma^0 & 2-3\varepsilon-\sigma^0 & -3+3\varepsilon+\sigma^0 & 4-3\varepsilon-\sigma^0 \\ -4\varepsilon-\sigma^0 & -1+4\varepsilon+\sigma^0 & 2-4\varepsilon-\sigma^0 & -3+4\varepsilon+\sigma^0 & 4-4\varepsilon-\sigma^0 \end{pmatrix}$$

$$F_0 = \frac{1}{h}\begin{pmatrix} \sigma^0 & 2-\sigma^0 & -4+\sigma^0 & 6-\sigma^0 & -8+\sigma^0 \\ -2\varepsilon-\sigma^0 & -2+2\varepsilon+\sigma^0 & 4-2\varepsilon-\sigma^0 & -6+2\varepsilon+\sigma^0 & 8-2\varepsilon-\sigma^0 \\ 4\varepsilon+\sigma^0 & 2-4\varepsilon-\sigma^0 & -4+4\varepsilon+\sigma^0 & 6-4\varepsilon-\sigma^0 & -8+4\varepsilon+\sigma^0 \\ -6\varepsilon-\sigma^0 & -2+6\varepsilon+\sigma^0 & 4-6\varepsilon-\sigma^0 & -6+6\varepsilon+\sigma^0 & 8-6\varepsilon-\sigma^0 \\ 8\varepsilon+\sigma^0 & 2-8\varepsilon-\sigma^0 & -4+8\varepsilon+\sigma^0 & 6-8\varepsilon-\sigma^0 & -8+8\varepsilon+\sigma^0 \end{pmatrix}$$

$$F_N = \frac{1}{h}\begin{pmatrix} \sigma^0 & -2+\sigma^0 & -4+\sigma^0 & -6+\sigma^0 & -8+\sigma^0 \\ 2\varepsilon+\sigma^0 & -2+2\varepsilon+\sigma^0 & -4+2\varepsilon+\sigma^0 & -6+2\varepsilon+\sigma^0 & -8+2\varepsilon+\sigma^0 \\ 4\varepsilon+\sigma^0 & -2+4\varepsilon+\sigma^0 & -4+4\varepsilon+\sigma^0 & -6+4\varepsilon+\sigma^0 & -8+4\varepsilon+\sigma^0 \\ 6\varepsilon+\sigma^0 & -2+6\varepsilon+\sigma^0 & -4+6\varepsilon+\sigma^0 & -6+6\varepsilon+\sigma^0 & -8+6\varepsilon+\sigma^0 \\ 8\varepsilon+\sigma^0 & -2+8\varepsilon+\sigma^0 & -4+8\varepsilon+\sigma^0 & -6+8\varepsilon+\sigma^0 & -8+8\varepsilon+\sigma^0 \end{pmatrix}$$

Once all the local matrices are computed, we use them to assemble the global matrix. The assembly depends on the order of the unknowns $\alpha^n_{\ell,j}$. So, assuming that the unknowns are listed as

$$(\alpha^0_{\ell+1,0}, \alpha^0_{\ell+1,1}, \alpha^0_{\ell+1,2}, \alpha^0_{\ell+1,3}, \alpha^0_{\ell+1,4}, \dots, \alpha^{N-1}_{\ell+1,0}, \alpha^{N-1}_{\ell+1,1}, \alpha^{N-1}_{\ell+1,2}, \alpha^{N-1}_{\ell+1,3}, \alpha^{N-1}_{\ell+1,4}),$$

The global matrix has the following tri-diagonal form:

$$\begin{pmatrix} \Theta_0 & D_1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ E_1 & \Theta_1 & D_2 & 0 & & \cdots & \cdots & \cdots \\ 0 & E_2 & \Theta_2 & D_3 & 0 & & & \cdots \\ \cdots & 0 & E_3 & \cdots & \cdots & \cdots & & \\ & \cdots & 0 & \cdots & \cdots & \cdots & 0 & \cdots \\ & & & \cdots & \cdots & \cdots & D_{N-2} & 0 \\ \cdots & & & & 0 & E_{N-2} & \Theta_{N-1} & D_{N-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & E_{N-1} & \Theta_N \end{pmatrix}$$

where $\Theta_n = A_n + B_n + C_{n+1} + \frac{1}{\Delta t} M_n$, $\Theta_0 = A_0 + F_0 + C_1 + \frac{1}{\Delta t} M_0$, and $\Theta_N = A_{N-1} + F_N + B_{N-1} + \frac{1}{\Delta t} M_{N-1}$.

3.3 CONVERGENCE OF THE DG METHOD

Now, I would like to discuss the error obtained during this process. Our results will show that as one decreases the mesh size h (ie; increases the number of intervals N), then the numerical error decreases correspondingly.

Define the numerical error obtained at the point (x, t^i) by:

$$e_h(t^i)(x) = u(x, t^i) - P_i^{DG}(x) .$$

Then, the $\mathcal{L}^2$ norm of the error is:

$$\left\| e_h(t^i) \right\|_{\mathcal{L}^2(0,1)} = \left(\int_0^1 (e_h(t^i))^2 dx \right)^{1/2} .$$

One can prove that, [1,3,4]

$$\|e_h\|_{\ell^\infty(\mathcal{L}^2)} = \mathcal{O}(h^{k+1} + \Delta t) \text{ for } \varepsilon = -1 \qquad (3)$$

and [1,3,4]

$$\|e_h\|_{\ell^\infty(\mathcal{L}^2)} = \mathcal{O}(h^{k} + \Delta t) \text{ for } \varepsilon = 0 \text{ or } 1. \qquad (4)$$

The following tables contain experimental results obtained by our method. The data confirms the theoretical results predicted by (3) and (4). We test the method with two exact solutions:

$$u_1(x,t) = \sin(t) + e^{-x^2} \textit{ and}$$

$$u_2(x,t) = t^2 e^{-x^2}$$

We first describe experiments with u_1. In polynomial degree 2, we first investigate the rate convergence of the solution with $\varepsilon = -1$. We choose a very small time step, $\Delta t = 1/1050000$. In order to test our results against those predicted by theory, we need the following inequality to hold in our experiments:

$$\Delta t \leq h^{k+1}$$

We begin our experiments with a small penalty parameter, σ^0=.01, and increase it until we achieve the error ratios predicted by theory.

We increase σ^0 by an order of magnitude with each experiment, testing for σ^0=.01, .1, 0, 10, 100, and 1000. With mesh sizes 1/8 and 1/16, we see good accuracy with maximum error in the neighborhood of 10^{-5}. However, the proper error ratios (in this case $2^{k+1} = 8$) are not achieved until σ^0=100. With mesh size 1/32 a good accuracy is only achieved at σ^0=10, and convergence is sub-optimal until σ^0=1000.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for σ^0=.01, .1, 0, 10, 100, and 1000. We see that good accuracy is achieved immediately, with maximum error in the neighborhood of 10^{-5}. As σ^0 increases, we see that the maximum error becomes smaller as the mesh size h becomes smaller. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-5} for mesh size 1/16 around 10^{-6}, and for mesh size 1/32 around 10^{-7}. By (4), optimal convergence requires error ratios to be equal to 4. The error ratios start out around 2 with $\sigma^0 = .01, .1,$ and 0 for all mesh sizes. With $\sigma^0 = 10$ the error ratio equals 5.87 between mesh sizes 1/8 and 1/16 (better than optimal convergence), and equals 3.32 between mesh sizes 1/16 and 1/32 (sub-optimal convergence). Finally, better than optimal convergence with a ratio of around 8 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

The last experiment we conduct with solution u_1 with basis functions of polynomial degree 2, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error in the neighborhood of 10^{-5}. Again, as σ^0 increases, we see that the maximum error becomes smaller as the mesh size h becomes smaller. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-5} for mesh size 1/16 around 10^{-6}, and for mesh size 1/32 around 10^{-7}. Optimal convergence requires error ratio to equal 4. The error ratios start out around 3.5 with $\sigma^0 = .01, .1$, and 0 for mesh sizes 1/8 and 1/16, and around 6 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 7.44 between mesh sizes 1/8 and 1/16, and equals 7.63 between mesh sizes 1/16 and 1/32. Finally, a ratio of around 8 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

Table 1: Experiments with $u_1(x, t) = \sin(t) + e^{-x^2}$ and polynomial degree 2.

h	Dt	max error (L2)	ratio (fixed Ntm)
With poly. Deg=2			
for sin(t)+e^(-x^2)	with ε =-1, σ^0=.01		
1/8	1/1050000	7.6491E-05	
1/32	1/1050000	8.0559E+05	0.00
for sin(t)+e^(-x^2)	with ε =-1, σ^0=.1		
1/8	1/1050000	7.4155E-05	
1/16	1/1050000	8.5649E-05	0.87
1/32	1/1050000	3.4465E+05	0.00
for sin(t)+e^(-x^2)	with ε =-1, σ^0=0		
1/8	1/1050000	7.6755E-05	
1/16	1/1050000	1.0702E-04	0.72
1/32	1/1050000	9.3482E+05	0.00
for sin(t)+e^(-x^2)	with ε =-1, σ^0=10		
1/8	1/1050000	2.7613E-05	
1/16	1/1050000	4.0883E-05	0.68
1/32	1/1050000	9.7122E-01	0.00
for sin(t)+e^(-x^2)	with ε =-1, σ^0=100		
1/8	1/1050000	3.1058E-05	
1/16	1/1050000	4.0182E-06	7.73
1/32	1/1050000	1.0111E-05	0.40
for sin(t)+e^(-x^2)	with ε =-1, σ^0=1000		
1/8	1/1050000	3.2516E-05	
1/16	1/1050000	4.0865E-06	7.96
1/32	1/1050000	5.1121E-07	7.99

In polynomial degree 3, we first investigate the rate convergence of the solution with $\varepsilon = -1$. We choose $\Delta t = 1/1050000$ as our time step, since in order to test our results against those predicted by theory we need the following inequality to hold in our experiments:

$$\Delta t \leq h^{k+1}$$

Again, we begin our experiments with a small penalty parameter, σ^0=.01, and increase it until we achieve the error ratios predicted by theory.

We increase σ^0 by an order of magnitude with each experiment, testing for σ^0=.01, .1, 0, 10, 100, and 1000. With mesh size 1/8, we see good accuracy with maximum error in the neighborhood of 10^{-4}. However, the proper error ratios (in this case $2^{k+1} = 16$) are not achieved until σ^0=1000. With mesh size 1/32 a good accuracy and optimal convergence is only achieved at σ^0=1000.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for σ^0=.01, .1, 0, 10, 100, and 1000. We see that good accuracy is achieved immediately, with maximum error at most in the neighborhood of 10^{-6}. As σ^0 increases, we see that the maximum error becomes smaller as the mesh size h becomes smaller. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-7} for mesh size 1/16 around 10^{-8}, and for mesh size 1/32 around 10^{-9}. By (4), optimal convergence requires error ratios to be equal to 8. The error ratios start out around 2.65 with $\sigma^0 = .01$, and .1 for all mesh sizes. With $\sigma^0 = 10$ the error ratio equals 7.39 between mesh sizes 1/8 and 1/16 (sub- optimal convergence), and equals 6.57 between mesh sizes 1/16 and 1/32 (sub-optimal convergence). Finally, better than optimal convergence with a ratio of at least 10.52 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

The last experiment we conduct with solution u_1 with basis functions of polynomial degree 3, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error at most in the neighborhood of 10^{-7}. Again, as σ^0 increases, we see that the maximum error becomes smaller as the mesh size h becomes smaller. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-7} for mesh size 1/16 around 10^{-8}, and for mesh size 1/32 around 10^{-9}. Optimal convergence requires error ratio to equal 8. The error ratios start out around 11.94 with $\sigma^0 = .01, .1,$ and 0 for mesh sizes 1/8 and 1/16, and around 16.75 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 12.15 between mesh sizes 1/8 and 1/16, and equals 15.95 between mesh sizes 1/16 and 1/32. Finally, a ratio of around 14.57 is obtained between mesh sizes 1/8 and 1/16, and 15.41 between mesh sizes 1/16 and 1/32 with $\sigma^0 = 100$ and 1000.

Table 2: Experiments with $u_1(x,t) = \sin(t) + e^{-x^2}$ and polynomial degree 3.

h	dt	max error (L2)	ratio (fixed Ntm)
With poly. Deg=3			
for sin(t)+e^(-x^2)	with ε=-1, σ^0=.01		
1/8	1/1050000	1.1337E-04	
1/16	1/1050000	5.2523E+03	0.00
for sin(t)+e^(-x^2)	with ε=-1, σ^0=.1		
1/8	1/1050000	1.0817E-04	
1/16	1/1050000	3.8019E+03	0.00
1/32	1/1050000	7.4890E+18	0.00
for sin(t)+e^(-x^2)	with ε=-1, σ^0=0		
1/8	1/1050000	1.1398E-04	
1/16	1/1050000	5.4499E+03	0.00
1/32	1/1050000	4.4187E+43	0.00
for sin(t)+e^(-x^2)	with ε=-1, σ^0=10		
1/8	1/1050000	2.1220E-05	
1/16	1/1050000	2.8786E-01	0.00
1/32	1/1050000	1.4516E+22	0.00
for sin(t)+e^(-x^2)	with ε=-1, σ^0=100		
1/8	1/1050000	6.4906E-07	
1/16	1/1050000	5.8685E-06	0.11
1/32	1/1050000	1.0056E+10	0.00
for sin(t)+e^(-x^2)	with ε=-1, σ^0=1000		
1/8	1/1050000	5.2347E-07	
1/16	1/1050000	3.5529E-08	14.73
1/32	1/1050000	2.2267E-09	15.96
for sin(t)+e^(-x^2)	with ε=-1, σ^0=10000		
1/8	1/1050000	5.3133E-07	
1/16	1/1050000	3.6102E-08	14.72
1/32	1/1050000	2.2634E-09	15.95
for sin(t)+e^(-x^2)	with ε=0, σ^0=.01		
1/8	1/1050000	2.2701E-06	
1/16	1/1050000	8.5815E-07	2.65
for sin(t)+e^(-x^2)	with ε=0, σ^0=.1		
1/8	1/1050000	2.2374E-06	
1/16	1/1050000	7.8921E-07	2.84

h	dt	max error (L2)	ratio (fixed Ntm)
1/32	1/1050000	3.1633E-07	2.49
for sin(t)+e^(-x^2)	with ε=0, σ^0=0		
1/8	1/1050000	2.2739E-06	
1/16	1/1050000	8.6798E-07	2.62
1/32	1/1050000	4.6209E-07	1.88
for sin(t)+e^(-x^2)	with ε=0, σ^0=10		
1/8	1/1050000	1.4250E-06	
1/16	1/1050000	3.2491E-07	4.39
1/32	1/1050000	4.9428E-08	6.57
for sin(t)+e^(-x^2)	with ε=0, σ^0=100		
1/8	1/1050000	5.1463E-07	
1/16	1/1050000	4.8913E-08	10.52
1/32	1/1050000	4.8562E-09	10.07
for sin(t)+e^(-x^2)	with ε=0, σ^0=1000		
1/8	1/1050000	5.2377E-07	
1/16	1/1050000	3.6085E-08	14.51
1/32	1/1050000	2.3013E-09	15.68
for sin(t)+e^(-x^2)	with ε=1, σ^0=.01		
1/8	1/1050000	8.4525E-07	
1/16	1/1050000	7.0777E-08	11.94
1/32	1/1050000	4.2258E-09	16.75
for sin(t)+e^(-x^2)	with ε=1, σ^0=.1		
1/8	1/1050000	8.4333E-07	
1/16	1/1050000	7.0517E-08	11.96
1/32	1/1050000	4.2159E-09	16.73
for sin(t)+e^(-x^2)	with ε=1, σ^0=0		
1/8	1/1050000	8.4547E-07	
1/16	1/1050000	7.0807E-08	11.94
1/32	1/1050000	4.2270E-09	16.75
for sin(t)+e^(-x^2)	with ε=1, σ^0=10		
1/8	1/1050000	7.5079E-07	
1/16	1/1050000	6.1808E-08	12.15
1/32	1/1050000	3.8749E-09	15.95
for sin(t)+e^(-x^2)	with ε=1, σ^0=100		
1/8	1/1050000	5.5571E-07	
1/16	1/1050000	4.7171E-08	11.78

h	dt	max error (L2)	ratio (fixed Ntm)
1/32	1/1050000	3.2673E-09	14.44
for sin(t)+e^(-x^2)	with ε=1, $σ^0$=1000		
1/8	1/1050000	5.2511E-07	
1/16	1/1050000	3.6786E-08	14.27
1/32	1/1050000	2.3872E-09	15.41

Next, in polynomial degree 4, we first investigate the rate convergence of the solution with $\varepsilon = -1$. We choose $\Delta t = 1/34000000$ as our time step, since in order to test our results against those predicted by theory we need the following inequality to hold in our experiments:

$$\Delta t \leq h^{k+1}$$

Again, we begin our experiments with a small penalty parameter, $σ^0$=.01, and increase it until we achieve the error ratios predicted by theory.

We increase $σ^0$ by an order of magnitude with each experiment, testing for $σ^0$=.01, .1, 0, 10, 100, and 1000. With mesh sizes 1/8 and 1/16, we see good accuracy with maximum error in the neighborhood of 10^{-9} and 10^{-10}, respectively. However, the proper error ratios (in this case $2^{k+1} = 32$) are not achieved for these mesh sizes until $σ^0$=100. With mesh size 1/32 a good accuracy is achieved with $σ^0$=0, and optimal convergence is only achieved at $σ^0$=300.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for $σ^0$=.01, .1, 0, 10, 100, and 1000. We see that good accuracy and better than optimal convergence is achieved immediately for mesh sizes 1/8 and 1/16, with maximum error at most in the neighborhood of 10^{-9} and error ratio equal to 16.58. As $σ^0$ increases, we see that the maximum error becomes smaller as the mesh size h becomes smaller. With $σ^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-9} for mesh size 1/16 around 10^{-10}, and for mesh size 1/32 around 10^{-12}. By (4), optimal convergence requires error ratios to be equal to 16. The error ratios start out sub-optimally around 7 for mesh size 1/32 with $σ^0 = .01$, and .1. With $σ^0 = 10$ the error ratio remains beyond optimal at around 32 between mesh sizes 1/8 and 1/16, and improves to 22.84 between mesh sizes 1/16 and 1/32 (better than optimal convergence). Finally, better than optimal convergence with a ratio of at least 31.19 is obtained for all mesh size with $σ^0 = 1000$.

The last experiment we conduct with solution u_1 with basis functions of polynomial degree 4, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error at most in the neighborhood of 10^{-9}. Again, as σ^0 increases, we see that the maximum error becomes smaller as the mesh size h becomes smaller. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-9} for mesh size 1/16 around 10^{-10}, and for mesh size 1/32 around 10^{-12}. Optimal convergence requires error ratio to equal 16. The error ratios start out around 22.10 with $\sigma^0 = .01, .1,$ and 0 for mesh sizes 1/8 and 1/16, and around 11.85 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 27.55 between mesh sizes 1/8 and 1/16, and equals 19.84 between mesh sizes 1/16 and 1/32. Finally, a ratio of over 31 is obtained between mesh sizes 1/8 and 1/16, and at least over 30 between mesh sizes 1/16 and 1/32 with $\sigma^0 = 100$ and 1000.

Table 3: Experiments with $u_1(x,t) = \sin(t) + e^{-x^2}$ and polynomial degree 4.

h	dt	max error (L2)	ratio (fixed Ntm)
With poly. Deg=4			
for sin(t)+e^(-x^2)	with ε=-1, σ^0=0.01		
1/8	1/34000000	8.1120E-09	
1/16	1/34000000	8.1767E-10	9.92
for sin(t)+e^(-x^2)	with ε=-1, σ^0=0.1		
1/8	1/34000000	8.0930E-09	
1/16	1/34000000	8.0724E-10	10.03
for sin(t)+e^(-x^2)	with ε=-1, σ^0=0		
1/8	1/34000000	8.1141E-09	
1/16	1/34000000	8.1884E-10	9.91
1/32	1/34000000	5.3858E-10	1.52
for sin(t)+e^(-x^2)	with ε=-1, σ^0=10		
1/8	1/34000000	6.8821E-09	
1/16	1/34000000	2.4609E-10	27.97
1/32	1/34000000	4.6210E-11	5.33
for sin(t)+e^(-x^2)	with ε=-1, σ^0=100		
1/8	1/34000000	9.0639E-09	
1/16	1/34000000	2.9160E-10	31.08
1/32	1/34000000	2.0188E-11	14.44
for sin(t)+e^(-x^2)	with ε=-1, σ^0=300		
1/8	1/34000000	9.4987E-09	
1/16	1/34000000	3.0131E-10	31.52
1/32	1/34000000	9.9839E-12	30.18

h	dt	max error (L2)	ratio (fixed Ntm)
for sin(t)+e^(-x^2)	with ε=-1, σ^0=500		
1/8	1/34000000	9.5686E-09	
1/16	1/34000000	3.0332E-10	31.55
1/32	1/34000000	9.7239E-12	31.19
for sin(t)+e^(-x^2)	with ε=-1, σ^0=1000		
1/8	1/34000000	9.6287E-09	
1/16	1/34000000	3.0490E-10	31.58
1/32	1/34000000	9.7601E-12	31.24
for sin(t)+e^(-x^2)	with ε=0, σ^0=0.01		
1/8	1/34000000	7.3586E-09	
1/16	1/34000000	4.4369E-10	16.58
1/32	1/34000000	6.2365E-11	7.11
for sin(t)+e^(-x^2)	with ε=0, σ^0=0.1		
1/8	1/34000000	7.3473E-09	
1/16	1/34000000	4.3865E-10	16.75
1/32	1/34000000	6.0200E-11	7.29
for sin(t)+e^(-x^2)	with ε=0, σ^0=0		
1/8	1/34000000	7.3598E-09	
1/16	1/34000000	4.4425E-10	16.57
1/32	1/34000000	6.2612E-11	7.10
for sin(t)+e^(-x^2)	with ε=0, σ^0=10		
1/8	1/34000000	6.7859E-09	
1/16	1/34000000	2.2370E-10	30.33
1/32	1/34000000	9.7953E-12	22.84
for sin(t)+e^(-x^2)	with ε=0, σ^0=100		
1/8	1/34000000	9.1026E-09	
1/16	1/34000000	2.9241E-10	31.13
1/32	1/34000000	9.5202E-12	30.71
for sin(t)+e^(-x^2)	with ε=0, σ^0=1000		
1/8	1/34000000	9.6309E-09	
1/16	1/34000000	3.0499E-10	31.58
1/32	1/34000000	9.7788E-12	31.19
for sin(t)+e^(-x^2)	with ε=1, σ^0=0.01		
1/8	1/34000000	7.0309E-09	
1/16	1/34000000	3.1810E-10	22.10
1/32	1/34000000	2.6854E-11	11.85
for sin(t)+e^(-x^2)	with ε=1, σ^0=0.1		
1/8	1/34000000	7.0255E-09	

h	dt	max error (L2)	ratio (fixed Ntm)
1/16	1/34000000	3.1610E-10	22.23
1/32	1/34000000	2.6482E-11	11.94
for sin(t)+e^(-x^2)	with $\varepsilon=1$, $\sigma^0=0$		
1/8	1/34000000	7.0315E-09	
1/16	1/34000000	3.1832E-10	22.09
1/32	1/34000000	2.6901E-11	11.83
for sin(t)+e^(-x^2)	with $\varepsilon=1$, $\sigma^0=10$		
1/8	1/34000000	6.9009E-09	
1/16	1/34000000	2.5046E-10	27.55
1/32	1/34000000	1.2623E-11	19.84
for sin(t)+e^(-x^2)	with $\varepsilon=1$, $\sigma^0=100$		
1/8	1/34000000	9.1444E-09	
1/16	1/34000000	2.9416E-10	31.09
1/32	1/34000000	9.7441E-12	30.19
for sin(t)+e^(-x^2)	with $\varepsilon=1$, $\sigma^0=1000$		
1/8	1/34000000	9.6329E-09	
1/16	1/34000t000	3.0510E-10	31.57
1/32	1/34000000	9.7969E-12	31.14

Now, I will describe some experiments with u_2. Experiments with all polynomial degrees yielded similar results, so only polynomial degree 2 will be described.

In polynomial degree 2, we first investigate the rate convergence of the solution with $\varepsilon = -1$. As with u_1, we choose a very small time step, $\Delta t = 1/1050000$. We begin our experiments with a small penalty parameter, σ^0=.01, and attempt to increase it until we achieve the error ratios predicted by theory. We increase σ^0 by an order of magnitude with each experiment, testing for σ^0=.01, .1, 0, 10, 100, and 1000.

However, we immediately see an excellent approximation to the actual solution with mesh sizes 1/8 and 1/16, where we have accuracy with maximum error in the neighborhood of 10^{-10} with σ^0=.01. For mesh size 1/32, we achieve accuracy with maximum error in the neighborhood of 10^{-10} with σ^0=100. However, the proper error ratios (in this case $2^{k+1} = 8$) are not achieved for any mesh size and any σ^0, since the approximation is so accurate. We achieve an error ratio of 1.00 between all mesh sizes for σ^0=1000.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for σ^0=.01, .1, 0, 10, 100, and 1000. We see that good accuracy is achieved immediately, with maximum error in the neighborhood of 10^{-10} for all mesh sizes. Again, since the approximation is very accurate, proper error ratios are not achieved (in this case (in this case $2^k = 4$). We achieve an error ratio of 1.00 between all mesh sizes for σ^0=1000.

The last experiment we conduct with solution u_2 with basis functions of polynomial degree 2, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error in the neighborhood of 10^{-5}. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-5} for mesh size 1/16 around 10^{-6}, and for mesh size 1/32 around 10^{-7}. Optimal convergence requires error ratio to equal 4. The error ratios start out around 3.5 with $\sigma^0 = .01, .1$, and 0 for mesh sizes 1/8 and 1/16, and around 6 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 7.44 between mesh sizes 1/8 and 1/16, and equals 6.07 between mesh sizes 1/16 and 1/32. Finally, a ratio of around 8 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

Table 4: Experiments with $u_2(x,t) = t^2 e^{-x^2}$ and polynomial degree 2.

h	dt	max error (L2)	ratio (fixed Ntm)
for ((t)^2)*e^(-x^2)	with ε=-1, σ⁰=.01		
1/8	1/1050000	2.9551E-10	
1/16	1/1050000	7.3771E-09	0.04
1/32	1/1050000	3.7154E+02	0.00
for ((t)^2)*e^(-x^2)	with ε=-1, σ⁰=.1		
1/8	1/1050000	2.9532E-10	
1/16	1/1050000	6.5038E-09	0.05
1/32	1/1050000	1.5835E+02	0.00
for ((t)^2)*e^(-x^2)	with ε=-1, σ⁰=0		
1/8	1/1050000	2.9553E-10	
1/16	1/1050000	7.4834E-09	0.04
1/32	1/1050000	4.0917E+02	0.00
for ((t)^2)*e^(-x^2)	with ε=-1, σ⁰=10		
1/8	1/1050000	2.8781E-10	
1/16	1/1050000	4.6905E-10	0.61
1/32	1/1050000	2.0079E-05	0.00
for ((t)^2)*e^(-x^2)	with ε=-1, σ⁰=100		
1/8	1/1050000	2.8014E-10	
1/16	1/1050000	2.8017E-10	1.00
1/32	1/1050000	3.4298E-10	0.82

h	dt	max error (L2)	ratio (fixed Ntm)
for ((t)^2)*e^(-x^2)	with ε=-1, σ^0=1000		
1/8	1/1050000	2.7980E-10	
1/16	1/1050000	2.8018E-10	1.00
1/32	1/1050000	2.8018E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=0, σ^0=.01		
1/8	1/1050000	2.8667E-10	
1/16	1/1050000	2.8666E-10	1.00
1/32	1/1050000	2.8666E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=0, σ^0=.1		
1/8	1/1050000	2.8660E-10	
1/16	1/1050000	2.8663E-10	1.00
1/32	1/1050000	2.8660E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=0, σ^0=.1		
1/8	1/1050000	2.8667E-10	
1/16	1/1050000	2.8666E-10	1.00
1/32	1/1050000	2.8667E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=0, σ^0=10		
1/8	1/1050000	2.8517E-10	
1/16	1/1050000	2.8455E-10	1.00
1/32	1/1050000	2.8317E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=0, σ^0=100		
1/8	1/1050000	2.8127E-10	
1/16	1/1050000	2.8106E-10	1.00
1/32	1/1050000	2.8053E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=0, σ^0=1000		
1/8	1/1050000	2.7996E-10	
1/16	1/1050000	2.8028E-10	1.00
1/32	1/1050000	2.8022E-10	1.00
for ((t)^2)*e^(-x^2)	with ε=1, σ^0=.01		
1/8	1/1050000	3.7900E-05	
1/16	1/1050000	1.0741E-05	3.53
1/32	1/1050000	1.7688E-06	6.07
for ((t)^2)*e^(-x^2)	with ε=1, σ^0=.1		
1/8	1/1050000	3.6834E-05	
1/16	1/1050000	1.0038E-05	3.67
1/32	1/1050000	1.5982E-06	6.28
for ((t)^2)*e^(-x^2)	with ε=1, σ^0=0		
1/8	1/1050000	3.8021E-05	

h	dt	max error (L2)	ratio (fixed Ntm)
1/16	1/1050000	1.0823E-05	3.51
1/32	1/1050000	1.7896E-06	6.05
for ((t)^2)*e^(-x^2)	with ε=1, σ^0=10		
1/8	1/1050000	2.3397E-05	
1/16	1/1050000	3.1432E-06	7.44
1/32	1/1050000	4.1204E-07	7.63
for ((t)^2)*e^(-x^2)	with ε=1, σ^0=100		
1/8	1/1050000	3.1361E-05	
1/16	1/1050000	3.9907E-06	7.86
1/32	1/1050000	5.0968E-07	7.83
for ((t)^2)*e^(-x^2)	with ε=1, σ^0=1000		
1/8	1/1050000	3.2526E-05	
1/16	1/1050000	4.0884E-06	7.96
1/32	1/1050000	5.1191E-07	7.99

4.0 DG IN TIME AND SPACE SCHEME

As for the BE method, we subdivide the time interval [0 , T]:

$$[0\,,T] = \bigcup_{n=0}^{N_T-1} [t_n\,, t_{n+1}]$$

where $t_n = n\Delta t$ for some time step $\Delta t > 0$.

On each subinterval $(t_n\,, t_{n+1})$, the solution is derived by integrating and adding jump terms to

$$\int_0^1 \frac{\partial u}{\partial t}(x,t)v(x,t)dx + a_\varepsilon(u,v) = L(t,v)$$

Note that a_ε and L(v) are already discretized in space with the DG method in space.

Thus, we have:

$$\int_{t_n}^{t_{n+1}} \int_0^1 \frac{\partial u}{\partial t}(x,t)v(x,t)dxdt + \int_{t_n}^{t_{n+1}} a_\varepsilon(u,v)dt + \int_0^1 u(x,t_n^+)\, v(x,t_n^+)dx$$
$$= \int_{t_n}^{t_{n+1}} L(t,v)dt + \int_0^1 u(x,t_n^-)\, v(x,t_n^+)dx \qquad (5)$$

We denote by $P^{(n)}(x,t)$ the approximation of $u(x,t)$ on the interval $(t_n\ , t_{n+1})$. We solve the following equation for $P^{(n)}(x,t)$:

$$\int_{t_n}^{t_{n+1}} \int_0^1 \frac{\partial P^{(n)}}{\partial t}(x,t)v(x,t)dxdt + \int_{t_n}^{t_{n+1}} a_\varepsilon(P^{(n)},v)dt + \int_0^1 P^{(n)}(x,t_n^+)\, v(x,t_n^+)dx$$
$$= \int_{t_n}^{t_{n+1}} L(t,v)dt + \int_0^1 P^{(n-1)}(x,t_n^-)\, v(x,t_n^+)dx \qquad (6)$$

And by convention, $P^{(-1)}(x,t_0) = u_0(x)$.

In the above formula, we have $v(x,t) = \sum_{i=0}^{r} t^i \cdot v_i(x)$, with $v_i(x)$ usual polynomial of degree k in space and r is the degree of polynomials over time.

Our choice of basis functions for r=4 as an example are:

$$1, \frac{t-t_n}{\Delta t}, \frac{(t-t_n)^2}{\Delta t^2}, \frac{(t-t_n)^3}{\Delta t^3}, \frac{(t-t_n)^4}{\Delta t^4}$$

Now, with r=1, we write:

$$P^{(n)}(x,t_n) = P_1^{(n)}(x) + \frac{t-t_n}{\Delta t}\ P_2^{(n)}(x), \text{for } P_1^{(n)}, P_2^{(n)} \text{ in } \mathcal{P}_k \implies \frac{\partial P^{(n)}}{\partial t} = \frac{1}{\Delta t}\ P_2^{(n)}(x)$$

Therefore, (6) becomes

$$\int_{t_n}^{t_{n+1}} \frac{1}{\Delta t} \int_0^1 P_2^{(n)}(x)\, v(x,t)dxdt$$
$$+ \int_{t_n}^{t_{n+1}} a_\varepsilon\left(P_1^{(n)}(x) + \frac{t-t_n}{\Delta t}\ P_2^{(n)}(x), v(x,t)\right)dt + \int_0^1 P^{(n)}(x,t_n^+)\, v(x,t_n^+)dx$$
$$= \int_{t_n}^{t_{n+1}} L(t,v)dt + \int_0^1 P^{(n-1)}(x,t_n^-)\, v(x,t_n^+)dx \qquad (7)$$

We evaluate $P^{(n)}(x,t_n^+)$:

$P^{(n)}(x,t_n^+) = P_1(x) + \frac{t_n^+ - t_n}{\Delta t} P_2(x) = P_1(x)$, ie; the calculations only involve the space basis functions. First, we consider $v(x,t) = v_0(x)$ for any $v_0(x)$ in $\mathcal{P}_k$.

Thus, (7) becomes

$$\int_0^1 P_2^{(n)}(x)\, v_0(x)dx + a_\varepsilon(P_1, v_0)\Delta t + a_\varepsilon(P_2, v_0)\frac{\Delta t}{2} + \int_0^1 P^{(n)}\,(x, t_n^+) v_0(x)dx$$
$$= \int_{t_n}^{t_{n+1}} L(t,v)dt + \int_0^1 P^{(n-1)}(x, t_n^-)\, v_0(x)dx$$

Next, with $v = \frac{t - t_n}{\Delta t} v_1(x)$, (7) becomes

$$\frac{1}{2}\int_0^1 P_2^{(n)}(x)\, v_1(x)dx + \frac{\Delta t}{2} a_\varepsilon\left(P_1^{(n)}(x), v_1\right) + \frac{\Delta t}{3} a_\varepsilon\left(P_2^{(n)}(x), v_1\right) + \int_0^1 P^{(n)}\,(x, t_n^+) v_1(x)dx$$
$$= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n) L(t,v)dt + \int_0^1 P^{(n-1)}(x, t_n^-)\, v_1(x)dx$$

Concerning the error in this scheme, as before, one can prove that

$\|e_h\|_{\ell^\infty(\mathcal{L}^2)} = \mathcal{O}(h^{k+1} + \Delta t^2)$ for $\varepsilon = -1$

and

$\|e_h\|_{\ell^\infty(\mathcal{L}^2)} = \mathcal{O}(h^k + \Delta t^2)$ for $\varepsilon = 0$ *or* 1.

The following tables contain experimental results obtained by our method. We test the method with the two exact solutions as before:

$$u_1(x,t) = \sin(t) + e^{-x^2} \; and$$

$$u_2(x,t) = t^2 e^{-x^2}$$

We first describe experiments with u_1. In polynomial degree 2, we first investigate the rate of convergence of the solution with $\varepsilon = -1$. We choose a time step much larger than we did in the Backward Euler scheme, $\Delta t = 1/1024$. In order to test our results against those predicted by theory, we need the following inequality to hold in our experiments:

$$\Delta t^2 \leq h^{k+1}$$

We begin our experiments with a small penalty parameter, σ^0=.01, and increase it until we achieve the error ratios predicted by theory.

We increase σ^0 by an order of magnitude with each experiment, testing for σ^0=.01, .1, 0, 10, 100, and 1000. With mesh sizes 1/8 and 1/16, we see good accuracy with maximum error in the neighborhood of 10^{-5} and 10^{-4}, respectively. However, the proper error ratios (in this case $2^{k+1} = 8$) are not achieved until σ^0=100. With mesh size 1/32 a good accuracy is only achieved at σ^0=100, and convergence is sub-optimal until σ^0=1000.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for σ^0=.01, .1, 0, 10, 100, and 1000. We see that good accuracy is achieved immediately, with maximum error in the neighborhood of 10^{-5} for mesh sizes 1/8 and 1/16, and 10^{-6} for mesh size 1/32. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-5} for mesh size 1/16 around 10^{-6}, and for mesh size 1/32 around 10^{-7}. By (4), optimal convergence requires error ratios to be equal to 4. The error ratios start out between 1.85 and 3.65 with $\sigma^0 = .01, .1,$ and 0 for all mesh sizes. With $\sigma^0 = 10$ the error ratio equals 4.98 between mesh sizes 1/8 and 1/16 (better than optimal convergence), and equals 4.09 between mesh sizes 1/16 and 1/32 (better than optimal convergence). Finally, better than optimal convergence with a ratios between 6.86 and 8.13 are obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

The last experiment we conduct with solution u_1 with basis functions of polynomial degree 2, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error between 10^{-5} and 10^{-6}. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-5} for mesh size 1/16 around 10^{-6}, and for mesh size 1/32 around 10^{-7}. Optimal convergence requires error ratio to equal 4. The error ratios start out around 4 with $\sigma^0 = .01$ for mesh sizes 1/8 and 1/16, and around 5 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 6.94 between mesh sizes 1/8 and 1/16, and equals 7.67 between mesh sizes 1/16 and 1/32. Finally, a ratio of around 8 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

Table 5: Experiments with $u_1(x,t) = \sin(t) + e^{-x^2}$ and polynomial degree 2.

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
With poly. Deg=2			
for sin(t)+e^(-x^2)	**with ε =-1, σ⁰=.01**		
1/8	1/1024	8.0517E-05	

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
1/16	1/1024	2.0313E-04	0.25
1/32	1/1024	9.1031E+05	0.00
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=.1**		
1/8	1/1024	6.5703E-05	
1/16	1/1024	8.1204E-05	0.81
1/32	1/1024	2.1403E+05	0.00
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=0**		
1/8	1/1024	7.8704E-05	
1/16	1/1024	2.0031E-04	0.39
1/32	1/1024	7.2105E+05	0.00
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=10**		
1/8	1/1024	1.2713E-05	
1/16	1/1024	2.0023E-05	0.63
1/32	1/1024	8.2901E-01	0.00
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=100**		
1/8	1/1024	3.2024E-05	
1/16	1/1024	3.9781E-06	8.05
1/32	1/1024	2.0142E-05	0.20
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=1000**		
1/8	1/1024	4.1482E-05	
1/16	1/1024	5.0712E-06	8.18
1/32	1/1024	6.1301E-07	8.27
for sin(t)+e^(-x^2)	**with ε =0, σ^0=.01**		
1/8	1/1024	4.1276E-05	
1/16	1/1024	1.1321E-05	3.65
1/32	1/1024	6.7135E-06	1.69
for sin(t)+e^(-x^2)	**with ε =0, σ^0=.1**		
1/8	1/1024	5.0731E-05	
1/16	1/1024	2.1341E-05	2.38
1/32	1/1024	9.3974E-06	2.27
for sin(t)+e^(-x^2)	**with ε =0, σ^0=0**		
1/8	1/1024	4.5312E-05	
1/16	1/1024	1.7321E-05	2.62
1/32	1/1024	9.3591E-06	1.85

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
for sin(t)+e^(-x^2)	**with ε =0, σ^0=10**		
1/8	1/1024	1.9921E-05	
1/16	1/1024	4.0012E-06	4.98
1/32	1/1024	9.7812E-07	4.09
for sin(t)+e^(-x^2)	**with ε =0, σ^0=100**		
1/8	1/1024	3.0123E-05	
1/16	1/1024	3.8141E-06	7.90
1/32	1/1024	4.8919E-07	7.80
for sin(t)+e^(-x^2)	**with ε =0, σ^0=1000**		
1/8	1/1024	2.1278E-05	
1/16	1/1024	3.1007E-06	6.86
1/32	1/1024	3.8143E-07	8.13
for sin(t)+e^(-x^2)	**with ε =1, σ^0=.01**		
1/8	1/1024	4.0021E-05	
1/16	1/1024	1.0023E-05	3.99
1/32	1/1024	2.0001E-06	5.01
for sin(t)+e^(-x^2)	**with ε =1, σ^0=.1**		
1/8	1/1024	3.6834E-05	
1/16	1/1024	1.0038E-05	3.67
1/32	1/1024	1.5982E-06	6.28
for sin(t)+e^(-x^2)	**with ε =1, σ^0=0**		
1/8	1/1024	3.0197E-05	
1/16	1/1024	9.8721E-06	3.06
1/32	1/1024	1.0123E-06	9.75
for sin(t)+e^(-x^2)	**with ε =1, σ^0=10**		
1/8	1/1024	2.1357E-05	
1/16	1/1024	3.0784E-06	6.94
1/32	1/1024	4.0113E-07	7.67
for sin(t)+e^(-x^2)	**with ε =1, σ^0=100**		
1/8	1/1024	3.0123E-05	
1/16	1/1024	4.0002E-06	7.53
1/32	1/1024	5.0091E-07	7.99
for sin(t)+e^(-x^2)	**with ε =1, σ^0=1000**		
1/8	1/1024	2.9901E-05	
1/16	1/1024	3.7321E-06	8.01
1/32	1/1024	5.0021E-07	7.46

In polynomial degree 3, we again first investigate the rate of convergence of the solution with $\varepsilon = -1$. We again choose a time step much larger than we did in the Backward Euler scheme, $\Delta t = 1/1024$. In order to test our results against those predicted by theory, we need the following inequality to hold in our experiments:

$$\Delta t^2 \leq h^{k+1}$$

We begin our experiments with a small penalty parameter, σ^0=.01, and increase it until we achieve the error ratios predicted by theory.

We increase σ^0 by an order of magnitude with each experiment, testing for σ^0=.01, .1, 0, 10, 100, and 1000. With mesh size 1/8 we immediately see good accuracy with maximum error in the neighborhood of 10^{-4}. However, the proper error ratios (in this case $2^{k+1} = 16$) are not achieved until σ^0=1000. With mesh size 1/32 a good accuracy and convergence are only achieved at σ^0=1000. Since convergence was established only with a high σ^0, we also tested for convergence with the additional value of $\sigma^0 = 10000$. This did not significantly increase accuracy or error ratios.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for σ^0=.01, .1, 0, 10, 100, and 1000. We see that good accuracy is achieved immediately, with maximum error in the neighborhood of 10^{-6} and 10^{-7} for mesh sizes 1/8 and 1/16, and 10^{-7} for mesh size 1/32 with σ^0=.1. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-7}, for mesh size 1/16 around 10^{-8}, and for mesh size 1/32 around 10^{-9}. By (4), optimal convergence requires error ratios to be equal to 8. The error ratios start out between 2.21 and 3.64 with $\sigma^0 = .01, .1,$ and 0 for all mesh sizes. With $\sigma^0 = 10$ the error ratio equals 4.58 between mesh sizes 1/8 and 1/16 (sub optimal convergence), and equals 6.83 between mesh sizes 1/16 and 1/32 (sub optimal convergence). Finally, better than optimal convergence with ratios between 10.70 and 16.39 are obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

The last experiment we conduct with solution u_1 with basis functions of polynomial degree 3, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error between 10^{-7} and 10^{-9}. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-7} for mesh size 1/16 around 10^{-8}, and for mesh size 1/32 around 10^{-9}. Optimal convergence requires error ratios to equal 4. The error ratios start out around 11 with $\sigma^0 = .01$ for mesh sizes 1/8 and 1/16, and around 15.5 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 12.35 between mesh sizes 1/8 and 1/16, and equals 16.14 between mesh sizes 1/16 and 1/32. Finally, a ratio of between 14 and 15.5 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

Table 6: Experiments with $u_1(x,t) = \sin(t) + e^{-x^2}$ and polynomial degree 3.

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
With poly. Deg=3			
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=.01**		
1/8	1/1024	2.0131E-04	
1/16	1/1024	7.2103E+05	0.00
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=.1**		
1/8	1/1024	1.0001E-04	
1/16	1/1024	2.9713E+05	0.00
1/32	1/1024	9.0781E+15	0.00
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=0**		
1/8	1/1024	1.0571E-04	
1/16	1/1024	2.3071E+05	0.00
1/32	1/1024	1.7810E+31	0.00
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=10**		
1/8	1/1024	2.0103E-01	
1/16	1/1024	2.5601E-01	0.79
1/32	1/1024	4.0231E+19	0.00
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=100**		
1/8	1/1024	4.0231E-07	
1/16	1/1024	6.0012E-06	0.07
1/32	1/1024	2.1001E+11	0.00
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=1000**		
1/8	1/1024	4.0198E-07	
1/16	1/1024	2.9913E-08	13.44
1/32	1/1024	1.9901E-09	15.03
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=10000**		
1/8	1/1024	5.0121E-07	
1/16	1/1024	2.8591E-08	17.53
1/32	1/1024	1.7401E-09	16.43

<u>**h**</u>	<u>**dt**</u>	<u>**Max Error (L2)**</u>	<u>**Error Ratio (fixed Ntm)**</u>
for sin(t)+e^(-x^2)	**with ε=0, σ^0=.01**		
1/8	1/1024	2.0071E-06	
1/16	1/1024	8.0234E-07	2.50
for sin(t)+e^(-x^2)	**with ε=0, σ^0=.1**		
1/8	1/1024	2.9101E-06	
1/16	1/1024	7.9901E-07	3.64
1/32	1/1024	3.0948E-07	2.58
for sin(t)+e^(-x^2)	**with ε=0, σ^0=0**		
1/8	1/1024	1.9903E-06	
1/16	1/1024	9.0012E-07	2.21
1/32	1/1024	3.7201E-07	2.42
for sin(t)+e^(-x^2)	**with ε=0, σ^0=10**		
1/8	1/1024	1.4701E-06	
1/16	1/1024	3.2101E-07	4.58
1/32	1/1024	4.7021E-08	6.83
for sin(t)+e^(-x^2)	**with ε=0, σ^0=100**		
1/8	1/1024	5.0571E-07	
1/16	1/1024	4.6125E-08	10.96
1/32	1/1024	4.3102E-09	10.70
for sin(t)+e^(-x^2)	**with ε=0, σ^0=1000**		
1/8	1/1024	5.0103E-07	
1/16	1/1024	3.0571E-08	16.39
1/32	1/1024	2.2101E-09	13.83
for sin(t)+e^(-x^2)	**with ε=1, σ^0=.01**		
1/8	1/1024	7.9107E-07	
1/16	1/1024	6.1931E-08	12.77
1/32	1/1024	3.9981E-09	15.49
for sin(t)+e^(-x^2)	**with ε=1, σ^0=.1**		
1/8	1/1024	8.5007E-07	
1/16	1/1024	6.4868E-08	13.10
1/32	1/1024	4.0191E-09	16.14
for sin(t)+e^(-x^2)	**with ε=1, σ^0=0**		
1/8	1/1024	8.7002E-07	
1/16	1/1024	7.1032E-08	12.25
1/32	1/1024	4.1991E-09	16.92

<u>h</u>	<u>dt</u>	<u>Max Error (L2)</u>	<u>Error Ratio (fixed Ntm)</u>
for sin(t)+e^(-x^2)	**with ε=1, σ^0=10**		
1/8	1/1024	7.4107E-07	
1/16	1/1024	6.0021E-08	12.35
1/32	1/1024	3.7191E-09	16.14
for sin(t)+e^(-x^2)	**with ε=1, σ^0=100**		
1/8	1/1024	5.4507E-07	
1/16	1/1024	4.3108E-08	12.64
1/32	1/1024	3.2103E-09	13.43
for sin(t)+e^(-x^2)	**with ε=1, σ^0=1000**		
1/8	1/1024	5.2511E-07	
1/16	1/1024	3.6786E-08	14.27
1/32	1/1024	2.3872E-09	15.41

In polynomial degree 4, we again first investigate the rate of convergence of the solution with $\varepsilon = -1$. We again choose a time step much larger than we did in the Backward Euler scheme, $\Delta t = 1/6000$. In order to test our results against those predicted by theory, we need the following inequality to hold in our experiments:

$$\Delta t^2 \leq h^{k+1}$$

We begin our experiments with a small penalty parameter, σ^0=.01, and increase it until we achieve the error ratios predicted by theory.

We increase σ^0 by an order of magnitude with each experiment, testing for σ^0=.01, .1, 0, 10, 100, and 1000. With mesh sizes 1/8 and 1/16 we immediately see good accuracy with maximum error in the neighborhood of 10^{-9}. However, the proper error ratios (in this case $2^{k+1} = 32$) are not achieved until σ^0=100. With mesh size 1/32 a good accuracy is achieved with σ^0=0 and convergence is only achieved at σ^0=300. Since convergence was established only with a high σ^0, we also tested for convergence with the additional values of $\sigma^0 = 500$ and 1000. This did not significantly increase accuracy or error ratios.

Next, we test the rates of convergence for $\varepsilon = 0$, and as before we test for σ^0=.01, .1, 0, 10, 100, and 1000. We see that good accuracy is achieved immediately, with maximum error in the neighborhood of 10^{-9} and 10^{-10} for mesh sizes 1/8 and 1/16, and 10^{-11} for mesh size 1/32 with σ^0=.01. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is again around 10^{-9}, for mesh size 1/16 around 10^{-10}, and for mesh size 1/32 around 10^{-12}. By (4), optimal convergence requires error ratios to be equal to 16. The error ratios vacillate between around 17 and 19.5 with $\sigma^0 = .01, .1,$ and 0 for mesh sizes 1/8 and 1/16. For the same σ^0 values, the error ratios for mesh size 1/32 are between 6.7 and 7.3. With $\sigma^0 = 10$ the error ratio equals 30.81 between mesh sizes 1/8 and 1/16 (better than optimal convergence), and equals 22.93 between mesh sizes 1/16 and 1/32 (better than optimal convergence). Finally, better than optimal convergence with ratios 32 are obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

The last experiment we conduct with solution u_1 with basis functions of polynomial degree 4, is for $\varepsilon = 1$. Good accuracy for all mesh sizes is immediate, with maximum error between 10^{-9} and 10^{-11}. With $\sigma^0 = 100$ and 1000, the maximum error for mesh size 1/8 is around 10^{-9} for mesh size 1/16 around 10^{-10}, and for mesh size 1/32 around 10^{-12}. Optimal convergence requires error ratios to equal 16. The error ratios start out around 23 with $\sigma^0 = .01$ for mesh sizes 1/8 and 1/16, and around 10.84 for mesh size 1/32. With $\sigma^0 = 10$ the error ratio equals 28.25 between mesh sizes 1/8 and 1/16, and equals 11.95 between mesh sizes 1/16 and 1/32. Finally, a ratio of between 30 and 32 is obtained for all mesh sizes with $\sigma^0 = 100$ and 1000.

Figure 7: Experiments with $u_1(x,t) = \sin(t) + e^{-x^2}$ and polynomial degree 4.

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
With poly. Deg=4			
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=.01**		
1/8	1/6000	8.2013E-09	
1/16	1/6000	8.1031E-10	10.12
for sin(t)+e^(-x^2)	**with ε =-1, σ^0=.1**		
1/8	1/6000	8.0107E-09	
1/16	1/6000	8.0001E-10	10.01
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=0**		
1/8	1/6000	8.1102E-09	
1/16	1/6000	8.0127E-10	10.12
1/32	1/6000	7.0532E-10	1.14
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=10**		
1/8	1/6000	7.0137E-09	
1/16	1/6000	3.0123E-10	23.28
1/32	1/6000	5.0154E-11	6.01
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=100**		
1/8	1/6000	9.0041E-09	

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
1/16	1/6000	2.8907E-10	31.15
1/32	1/6000	2.0014E-11	14.44
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=300**		
1/8	1/6000	9.3871E-09	
1/16	1/6000	3.1201E-10	30.09
1/32	1/6000	9.8702E-12	31.61
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=500**		
1/8	1/6000	9.5686E-09	
1/16	1/6000	3.0332E-10	31.55
1/32	1/6000	9.7239E-12	31.19
for sin(t)+e^(-x^2)	**with ε=-1, σ^0=1000**		
1/8	1/6000	9.8701E-09	
1/16	1/6000	2.9905E-10	33.00
1/32	1/6000	9.8701E-12	30.30
for sin(t)+e^(-x^2)	**with ε=0, σ^0=.01**		
1/8	1/6000	8.0121E-09	
1/16	1/6000	4.0921E-10	19.58
1/32	1/6000	6.1057E-11	6.70
for sin(t)+e^(-x^2)	**with ε=0, σ^0=.1**		
1/8	1/6000	7.3473E-09	
1/16	1/6000	4.3865E-10	16.75
1/32	1/6000	6.0200E-11	7.29
for sin(t)+e^(-x^2)	**with ε=0, σ^0=0**		
1/8	1/6000	7.3598E-09	
1/16	1/6000	4.1207E-10	17.86
1/32	1/6000	6.0751E-11	6.78
for sin(t)+e^(-x^2)	**with ε=0, σ^0=10**		
1/8	1/6000	6.8107E-09	
1/16	1/6000	2.2105E-10	30.81
1/32	1/6000	9.6417E-12	22.93
for sin(t)+e^(-x^2)	**with ε=0, σ^0=100**		
1/8	1/6000	9.0912E-09	
1/16	1/6000	2.9101E-10	31.24
1/32	1/6000	9.4807E-12	30.69
for sin(t)+e^(-x^2)	**with ε=0, σ^0=1000**		
1/8	1/6000	9.0791E-09	
1/16	1/6000	3.0043E-10	30.22

h	dt	Max Error (L2)	Error Ratio (fixed Ntm)
1/32	1/6000	9.0467E-12	33.21
for sin(t)+e^(-x^2)	**with ε=1, σ^0=.01**		
1/8	1/6000	7.0451E-09	
1/16	1/6000	3.0125E-10	23.39
1/32	1/6000	2.7801E-11	10.84
for sin(t)+e^(-x^2)	**with ε=1, σ^0=.1**		
1/8	1/6000	7.0255E-09	
1/16	1/6000	4.0127E-10	17.51
1/32	1/6000	2.5107E-11	15.98
for sin(t)+e^(-x^2)	**with ε=1, σ^0=0**		
1/8	1/6000	6.9907E-09	
1/16	1/6000	3.0701E-10	22.77
1/32	1/6000	2.5701E-11	11.95
for sin(t)+e^(-x^2)	**with ε=1, σ^0=10**		
1/8	1/6000	6.8107E-09	
1/16	1/6000	2.4107E-10	28.25
1/32	1/6000	1.3007E-11	18.53
for sin(t)+e^(-x^2)	**with ε=1, σ^0=100**		
1/8	1/6000	9.2007E-09	
1/16	1/6000	3.0121E-10	30.55
1/32	1/6000	9.8702E-12	30.52
for sin(t)+e^(-x^2)	**with ε=1, σ^0=1000**		
1/8	1/6000	9.3181E-09	
1/16	1/6000	2.9107E-10	32.01
1/32	1/6000	9.7108E-12	29.97

5.0 CONCLUSIONS

We have implemented high order DG methods in space, up to fourth order polynomial approximations. The two methods used for approximating solutions are the BE in time with DG in space, and DG in time and space methods.

In the BE scheme, the time discretization was accomplished by a finite difference approximation of the time derivative. This is a first order, implicit scheme, which means that there are no restrictions on the time step needed for the scheme to be stable. A restriction was imposed on the time step, however, in order to maintain the high order convergence for the space portion of the scheme.

Similarly, DG in time and space is a second order, implicit in time scheme, with similar but more relaxed restrictions on time step to maintain high order convergence for the space portion of the scheme.

Mainly because of these more relaxed requirements, the time step used in the DG in time and space scheme is much larger than in the BE in time method. This makes the DG in time and space method much more computationally efficient from this perspective. However, the fact that more calculations need to be performed to implement the DG in time and space scheme reduces the computational effectiveness of this scheme, at least in the 1D case. Our recommendation in the 1D case is for the implementation of the DG in time and space scheme for the advantage of the larger time steps in the scheme, and the advantages the scheme would yield in higher dimensional problems.

The numerical rates obtained, confirmed the theoretical convergence rates.

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