ANALYSIS OF BERNOULLI'S EQUATION IN LES TURBULENCE MODELS

by

Sara Marashinia

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This thesis was presented

by

Sara Marashinia

It was defended on

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and approved by

Dr. William Layton, University of Pittsburgh, Department of Mathematics

Dr. Mike Sussman, University of Pittsburgh, Department of Mathematics

Dr. Anna Vainchtein, University of Pittsburgh, Department of Mathematics

Dr. Leo Rebholz, Bechtel Bettis Atomic Power Labratory

Thesis Advisor: Dr. William Layton, University of Pittsburgh, Department of Mathematics

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Sara Marashinia, M.S.

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In this thesis, we present an analysis of Bernoulli's equation in LES turbulence models. This analysis can be employed for prediction of Lift. Our approach is based on studying the variation of the Bernoulli pressure in an infinite, inviscid, and incompressible flow along streamlines in different LES models. We prove Bernoulli pressure is constant for Zeroth Order Model (ZOM) and α model. However for Leray Regularization and Bardina Model, the Bernoulli pressure is likely not constant. We show that Zeroth Order Model and α model conserve Bernoulli pressure and by our simulation we demonstrate how a very small frictional force in the fluid can have a major effect on the flow properties i.e. in a different setting of the kinematic viscosity parameter, we observe an approximate conservation of the Bernoulli pressure.

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1.0 INTRODUCTION

The Bernoulli pressure $P = p + \frac{1}{2}|u|^2$ is constant along streamlines for solutions of the steady Euler equation. It is thus a fundamental quantity in flow. We study when several LES models possess an analog of the Bernoulli pressure which is conserved when $\nu = 0$.

The d'Alembert paradox states "There is no drag on a finite body at rest in an infinite, incompressible, inviscid fluid otherwise in uniform motion"[1]. Actually, for a finite solid object in a steady, incompressible, inviscid fluid free from vorticity it can be shown that there would be no lift and no drag acting on the object, [2]. The d'Alembert paradox is related to Bernoulli's theorem, which states that

$$p + \frac{1}{2}u^2 = \text{constant} \tag{1.0.1}$$

where p is the pressure, and u the local velocity. The question of lift can be studied using Navier-Stokes equation in a sense of a mathematical problem,

$$-u \times \operatorname{curl} u = -\nabla(p + \frac{1}{2}u^2) + \nu\Delta u \tag{1.0.2}$$

The d'Alembert paradox leads to the absurd conclusion that a solid object moving through a fluid with a uniform velocity doesn't experience any resistance. Thus numerous studies have been conducted over the past two hundred years to propose alternative theories, [1]. These theories have shown how a very small frictional force in the fluid can have a major effect on the flow properties. To solve this problem a three-way approach has been chosen, a combination of experimental observation, computation, and study of the asymptotic form of the solution as the friction approaches zero.

Lift is generated by the interaction of any fluid and a solid object. Lift is a mechanical force. To generate lift a solid body and a fluid have to be in contact. For example, there is no lift on the space shuttle's wings in space because of lack of fluid. The space shuttle stays in space because of orbital mechanics related to its speed. Figure 1 shows the interaction of fluid and a solid body [3].

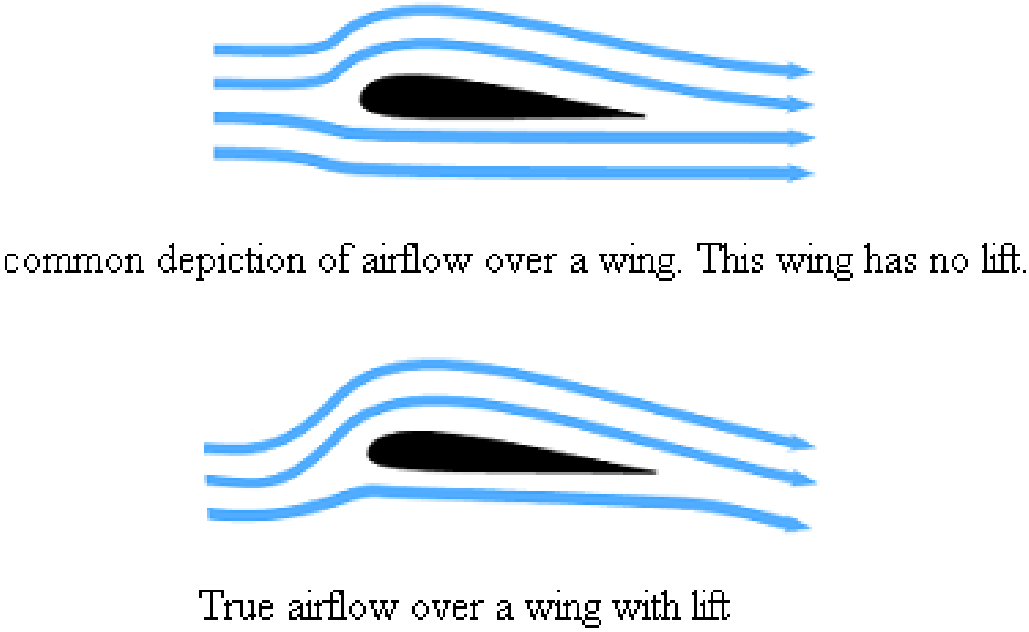

Figure 1: The interaction of fluid and a solid body.

As you may notice in Figure 1, the direction of the flow in the first part is forward but in the other part the direction of the flow is downward. Thus it means there exists a force upward on this wing, and its reaction makes the flow change direction. The component of the force acting vertically on the wing is lift. When there is a motion between the fluid and the solid body, the difference between their velocities causes the generation of the lift. There is no difference if the solid object moves in the fluid or the fluid moves towards the static object.

Different explanations of lift generation can be found in physics literature. There are many incorrect explanations that have become a source of controversy and heated arguments for many years. There are two main theories about the creation of lift: (1) Bernoulli's theory says that lift is generated by a pressure difference across the airfoi; and, (2) Newton's theory which says: Lift is the reaction force on a body caused by deflecting a flow of gas.

Bernoulli's equations is the relationship between the pressure and the local velocity of the fluid; so if the velocity changes around a solid body then the pressure will change as well. By integrating the pressure variations along a streamline, we can find the aerodynamic force on the body.

$$L + D = \oint_{\partial\Omega} pn d\partial\Omega, \tag{1.0.3}$$

where p is the pressure, n is the normal vector and $\partial\Omega$ is the boundary of the domain[9]. The lift is the component of the aerodynamic force which is perpendicular to the direction of the flow.

There is also another way to determine the aerodynamic force on the body. That is integrating the normal stress around the solid body. This generate a circulation in the fluid flow. By Newton's third law, this circulation will cause a reaction (aerodynamic force) to the object, [9].

$$L = \rho u \times \Gamma, \tag{1.0.4}$$

where ρ is the density of the fluid, u is the velocity and Γ is the circulation.

Therefore both Bernoulli's theory and Newton's theory are correct. However, different interpretations of these two theories have been the subject of debates in the research communities. Integrating the variation of pressure or velocity both determines the aerodynamic force on a solid object.

In this study we will concentrate on the Bernoulli's equation and we will consider it as a criteria to predict the lift in four different large eddy simulation (LES) models: (1) Zeroth Order Method, (2) Leray Regularization, (3) Alpha Model, and (4) Bardina Model. Thus in each of these models we simplify the equations and find the analog of the Bernoulli pressure for the model and then integrate the model's Bernoulli pressure along a streamline. We show that the value of this integral is zero along a streamline for some of these models and it would be non zero for some other.

This thesis is organized as follows: chapter 2 describes the notation and preliminaries.

In chapter 3 analyzes the Bernoulli's equation for different LES models. Chapter 4 presents numerical simulations. Chapter 5 concludes the thesis and provides ideas for future work.

2.0 NOTATION AND PRELIMINARIES

The domain Ω used throughout this study is considered in $\mathbb{R}^2$ or $\mathbb{R}^3$ with periodic boundary conditions. We have assumed that solutions are smooth enough to justify our work. Also the time interval that we are considering in chapters 2 and 3 is $[0, T]$.

Definition 2.1. *(Navier-Stokes Equations)If u stands for the velocity of the flow, p for the pressure, ν for the kinematic viscosity, and f for other body forces then the motion of a wide class of incompressible fluids can be described by:*

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f \tag{2.0.1}$$

$$\nabla \cdot u = 0 \tag{2.0.2}$$

Definition 2.2. *(Steady flow) A steady flow has a constant velocity in time at any point occupied by fluid, [7]. In other words, if u stands for the velocity of the fluid then*

$$\frac{\partial u}{\partial t} = 0. \tag{2.0.3}$$

Definition 2.3. *(Streamline)These are curves, $X = x(t)$ such that the tangent to a streamline at any point gives the direction of the velocity, $\frac{dx}{dt} = u(x(t), t)$ at that point. In steady flow the streamlines do not vary with time, and coincide with the paths of the fluid particles, [8].*

Definition 2.4. *(Reynolds Number) The primary parameter correlating the viscous behavior of all newtonian fluids is the dimensionless Reynolds number*

$$\mathrm{Re} = \frac{VL}{\nu} \tag{2.0.4}$$

where V and L represent the characteristic velocity and length scales of the flow and ν is called the kinematic viscosity,[8].

The next lemma give useful vector identities which we will use throughout next chapter.

Lemma 2.1. *Let $w = \nabla \times u$ be the vorticity then for sufficiently smooth u,*

$$u \cdot \nabla u = w \times u + \nabla(\frac{|u|^2}{2}). \tag{2.0.5}$$

Proof. Suppose $u = (u_1, u_2, u_3)^T$, then

$$w = \operatorname{curl} u = \begin{pmatrix} u_{3,2} - u_{2,3} \\ u_{1,3} - u_{3,1} \\ u_{2,1} - u_{1,2} \end{pmatrix}. \tag{2.0.6}$$

We calculate

$$w \times u = \begin{pmatrix} u_3(u_{1,3} - u_{3,1}) - u_2(u_{2,1} - u_{1,2}) \\ u_1(u_{2,1} - u_{1,2}) - u_3(u_{3,2} - u_{2,3}) \\ u_2(u_{3,2} - u_{2,3}) - u_1(u_{1,3} - u_{3,1}) \end{pmatrix}. \tag{2.0.7}$$

We can also write:

$$1/2\nabla(|u|^2) = 1/2\nabla(u_1^2 + u_2^2 + u_3^2) = \begin{pmatrix} u_1 u_{1,1} + u_2 u_{2,1} + u_3 u_{3,1} \\ u_1 u_{1,2} + u_2 u_{2,2} + u_3 u_{3,2} \\ u_1 u_{1,3} + u_2 u_{2,3} + u_3 u_{3,3} \end{pmatrix}. \tag{2.0.8}$$

Thus,

$$w \times u + 1/2\nabla(|u|^2) = \begin{pmatrix} u_1 u_{1,1} + u_2 u_{1,2} + u_3 u_{1,3} \\ u_1 u_{2,1} + u_2 u_{2,2} + u_3 u_{2,3} \\ u_1 u_{3,1} + u_2 u_{3,2} + u_3 u_{3,3} \end{pmatrix}. \tag{2.0.9}$$

So we can write this as:

$$w \times u + 1/2\nabla(|u|^2) = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \cdot \begin{pmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & u_{3,3} \end{pmatrix}, \tag{2.0.10}$$

and since

$$\begin{pmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & u_{3,3} \end{pmatrix} = \begin{pmatrix} \nabla u_1 & \nabla u_2 & \nabla u_3 \end{pmatrix} = \nabla u,$$

we can rewrite it as:

$$w \times u + 1/2\nabla(|u|^2) = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \cdot \begin{pmatrix} \nabla u_1 & \nabla u_2 & \nabla u_3 \end{pmatrix} = u \cdot \nabla u. \tag{2.0.11}$$

∎

Lemma 2.2. *Suppose* v and w are sufficiently smooth and $\nabla \cdot w = 0$ then,

$$v \cdot \nabla w = (\nabla \times v) \times w + \nabla(w \cdot v) - \frac{1}{2}\nabla \cdot (w \otimes v). \tag{2.0.12}$$

Proof. Suppose $v = (v_1\ v_2\ v_3)^T$ and $w = (w_1\ w_2\ w_3)^T$, then by the definition of curl we have:

$$\nabla \times v = \begin{pmatrix} v_{3,2} - v_{2,3} \\ v_{1,3} - v_{3,1} \\ v_{2,1} - v_{1,2} \end{pmatrix}. \tag{2.0.13}$$

We calculate

$$(\nabla \times v) \times w = \begin{pmatrix} w_3(v_{1,3} - v_{3,1}) - w_2(v_{2,1} - v_{1,2}) \\ w_1(v_{2,1} - v_{1,2}) - w_3(v_{3,2} - v_{2,3}) \\ w_2(v_{3,2} - v_{2,3}) - w_1(v_{1,3} - v_{3,1}) \end{pmatrix}, \tag{2.0.14}$$

and

$$\nabla(w.v) = \begin{pmatrix} w_{1,1}v_1 + w_1 v_{1,1} + w_{2,1}v_2 + w_2 v_{2,1} + w_{3,1}v_3 + w_3 v_{3,1} \\ w_{1,2}v_1 + w_1 v_{1,2} + w_{2,2}v_2 + w_2 v_{2,2} + w_{3,2}v_3 + w_3 v_{3,2} \\ w_{1,3}v_1 + w_1 v_{1,3} + w_{2,3}v_2 + w_2 v_{2,3} + w_{3,3}v_3 + w_3 v_{3,3} \end{pmatrix}. \tag{2.0.15}$$

We can easily show:

$$\nabla \cdot (w \otimes v) = (\nabla \cdot w)v + w \cdot \nabla v, \tag{2.0.16}$$

$$w \otimes v = \begin{pmatrix} w_1 v_1 & w_1 v_2 & w_1 v_3 \\ w_2 v_1 & w_2 v_2 & w_2 v_3 \\ w_3 v_1 & w_3 v_2 & w_3 v_3 \end{pmatrix}, \tag{2.0.17}$$

thus

$$\nabla \cdot (w \otimes v) = \begin{pmatrix} (w_{1,1} + w_{2,2} + w_{3,3})v_1 + w_1 v_{1,1} + w_2 v_{1,2} + w_3 v_{1,3} \\ (w_{1,1} + w_{2,2} + w_{3,3})v_2 + w_1 v_{2,1} + w_2 v_{2,2} + w_3 v_{2,3} \\ (w_{1,1} + w_{2,2} + w_{3,3})v_3 + w_1 v_{3,1} + w_2 v_{3,2} + w_3 v_{3,3} \end{pmatrix} \tag{2.0.18}$$

$$= (w_{1,1} + w_{2,2} + w_{3,3}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix} \tag{2.0.19}$$

$$= (\nabla \cdot w)v + w \cdot \nabla v. \tag{2.0.20}$$

But as we have assumed

$$\nabla \cdot w = 0. \tag{2.0.21}$$

Thus

$$\nabla \cdot (w \otimes v) = w \cdot \nabla v = \begin{pmatrix} w_1 v_{1,1} + w_2 v_{2,1} + w_3 v_{3,1} \\ w_1 v_{1,2} + w_2 v_{2,2} + w_3 v_{3,2} \\ w_1 v_{1,3} + w_2 v_{2,3} + w_3 v_{3,3} \end{pmatrix}. \tag{2.0.22}$$

Finally we can write:

$$(\nabla \times v) \times w + \nabla(w \cdot v) - \frac{1}{2}\nabla \cdot (w \otimes v) = \begin{pmatrix} w_{1,1}v_1 + w_{2,1}v_2 + w_{3,1}v_3 \\ w_{1,2}v_1 + w_{2,2}v_2 + w_{3,2}v_3 \\ w_{1,3}v_1 + w_{2,3}v_2 + w_{3,3}v_3 \end{pmatrix} \tag{2.0.23}$$

$$= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \cdot \begin{pmatrix} w_{1,1} & w_{1,2} & w_{3,1} \\ w_{2,1} & w_{2,2} & w_{3,2} \\ w_{3,1} & w_{2,3} & w_{3,3} \end{pmatrix} \tag{2.0.24}$$

$$= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \cdot \begin{pmatrix} \nabla w_1 \\ \nabla w_2 \\ \nabla w_3 \end{pmatrix} \tag{2.0.25}$$

Thus,

$$(\nabla \times v) \times w + \nabla(w \cdot v) - \frac{1}{2}\nabla \cdot (w \otimes v) = v \cdot \nabla w \tag{2.0.26}$$

∎

Definition 2.5. *(Bernoulli's Pressure) Let p denote the pressure and u the velocity of the fluid, then Bernoulli's pressure is defined by*

$$P = p + \frac{1}{2}|u|^2 . \tag{2.0.27}$$

Theorem 2.1. *(Bernoulli's Equation)Let u be the velocity of the flow and p the pressure. Suppose the flow is steady, $\frac{\partial u}{\partial t} = 0$; inviscid, $\nu = 0$ and not forced, $f = 0$ along a streamlines*

$$p + 1/2\,|u|^2 = constant \tag{2.0.28}$$

Proof. In this proof we have assumed Using Lemma 2.1. we can write equation (2.0.2) in this new form:

$$\begin{gathered} (\nabla \times u) \times u + \nabla P = 0 \\ \nabla \cdot u = 0 \\ P = p + 1/2\,|u|^2 \;\; \text{(Bernoulli's pressure)} \end{gathered} \tag{2.0.29}$$

Now consider the flow around an obstacle (i.e. an airfoil) that enters from L and is uniform upstream.

Integrate (2.0.29) along the streamline from Q_0 to Q_{Top} and along the streamline from $Q_1$to Q_{Bottom}. Notice that Q_0, Q_{Top} are supposed to be in the same streamline and the same property holds for Q_1, Q_{Bottom}, also we can choose Q_0 and Q_{Top} independent of Q_1 and Q_{Bottom}.

$$0 = \int_{Q_0}^{Q_{Top}} ((\nabla \times u) \times u + \nabla P) \cdot \frac{dx}{dt} dt. \tag{2.0.30}$$

Since $\frac{dx}{dt} = u$:

$$0 = \int_0^T ((\nabla \times u) \times u) \cdot u dt + \int_{Q_0}^{Q_{Top}} \nabla P(x(t)) \cdot x'(t) dt. \tag{2.0.31}$$

Using the fact that $((\nabla \times u) \times u) \perp u$, indeed $((\nabla \times u) \times u) \cdot u = 0$, thus

$$0 = 0 + \int_{Q_0}^{Q_{Top}} \nabla P(x(t)) \cdot x'(t) dt, \tag{2.0.32}$$

and finally

$$0 = P(Q_{Top}) - P(Q_0). \tag{2.0.33}$$

Using the same ideas, we can prove that:

$$0 = P(Q_{Bottom}) - P(Q_1) \tag{2.0.34}$$

By assuming uniform in flows :

$$P(Q_0) = P(Q_1), \tag{2.0.35}$$

thus

$$P(Q_{Top}) = P(Q_{Bottom}). \tag{2.0.36}$$

Now by using the definition of Bernoulli's pressure, $P = p + 1/2\,|u|^2$, we can say:

$$p(Q_{Top}) + 1/2\,|u(Q_{Top})|^2 = p(Q_{Bottom}) + 1/2\,|u(Q_{Bottom})|^2. \tag{2.0.37}$$

Which means the Bernoulli's pressure is constant over the whole field. ∎

Averaging Operators: Now we will talk about a smoothing averaging process which has an important role in the proofs of the third chapter.

Definition 2.6. Let δ to be a length scale selected by the user. Define g_δ to be

$$g_\delta(x) = \begin{cases} \frac{1}{2\delta} & |x| \le \delta \\ 0 & |x| > \delta \end{cases}, \tag{2.0.38}$$

such that, $\int_{-\infty}^{+\infty} g_\delta(x)dx = 1$.

Now let $\Omega = \{y \mid |x-y| \le \delta\}$, then we can define:

$$\bar{u}(x) = \frac{1}{vol(\Omega)} \int_\Omega u(y)dy, \tag{2.0.39}$$

and $\quad u' = u - \bar{u}.$

A smoothing averaging process is often depicted as decomposing a wiggly function, u, into its mean, $\bar{u}$, and its fluctuation about the mean,

$u' = u - \bar{u}$, as it is shown in Figure 4, [4].

Definition 2.7. (The differential filter) Given u and a filtering radius δ, define the average of u, $\overline{u}$, to be the solution of

$$-\delta^2 \Delta \overline{u} + \overline{u} = u \tag{2.0.40}$$

2.1 DEFINITION OF LES MODELS

Large Eddy Simulation is numerical method to solve partial differential equations related to turbulent flow (by solving for $\overline{u}$ rather than u.) Comparing to the Direct Numerical Simulation (DNS) i.e., solving for u itself, LES requires less computational effort. As another feature of LES we can mention that LES is able to calculate instantaneous flow features and determine turbulent flow structures.

After the above preliminaries, we now introduce the four considered LES models.

Definition 2.8. *(Zeroth Order Method)Let* $\bar{u}$ denote a spacial average of u where the operator $(\bar{\cdot})$ is a differential filter, [10]. The ZOM for $\bar{u}$ is:

$$\overline{u}_t + \overline{\bar{u} \cdot \nabla \bar{u}} - \nu \Delta \bar{u} + \nabla \overline{p} = \overline{f} \tag{2.1.1}$$

$$\nabla \cdot \bar{u} = 0 \tag{2.1.2}$$

Definition 2.9. *(Leray Regularization)The Leray Regularization, [10], is defined to be*

$$\overline{u}_t + \bar{\overline{u}} \cdot \nabla \overline{u} - \nu \Delta \overline{u} + \nabla \overline{p} = f \tag{2.1.3}$$

$$\nabla \cdot \overline{u} = 0 \tag{2.1.4}$$

Definition 2.10. *(The α Model)This model ([6]) is given by*

$$\bar{u}_t + (\nabla \times \bar{u}) \times \bar{\bar{u}} - \nu \Delta \bar{u} + \nabla \tilde{\bar{p}} = \bar{f} \tag{2.1.5}$$

$$\nabla \cdot \bar{u} = 0 \tag{2.1.6}$$

Definition 2.11. *(Bardina Model)This model is defined to be, [11],*

$$\bar{u}_t + \bar{u} \cdot \nabla \bar{u} + \nabla \cdot \left\{ \overline{\bar{u} \otimes \bar{u}} - \bar{\bar{u}} \otimes \bar{\bar{u}} \right\} - \nu \Delta \bar{u} + \nabla \bar{p} = \bar{f} \tag{2.1.7}$$

$$\nabla \cdot \bar{u} = 0 \tag{2.1.8}$$

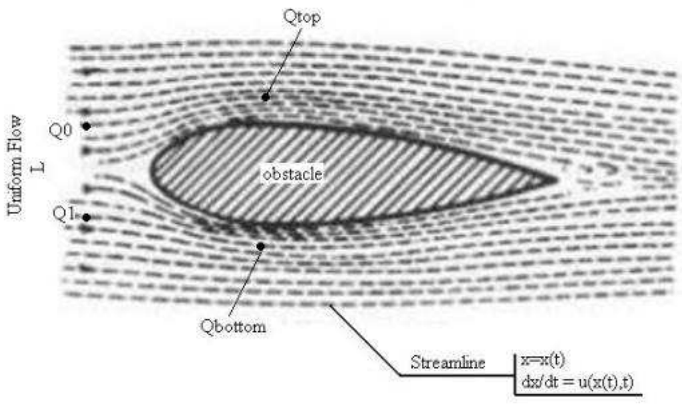

Figure 2: Flow passing around an obstacle, showing the streamlines.

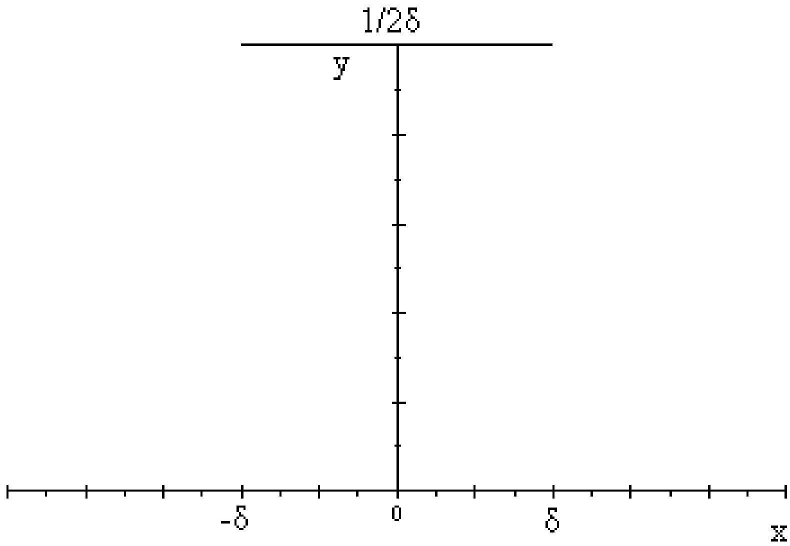

Figure 3: A typical filter

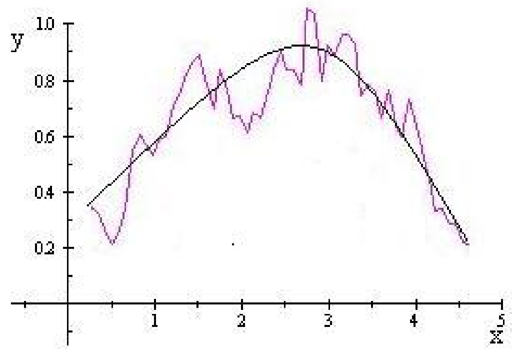

Figure 4: A curve and its mean (black line).

3.0 ANALYSIS OF THE BERNOULLI EQUATION

In the Navier-Stokes equations, a high Reynolds number is the cause of instability. We can say "$Re \sim \frac{|nonlinearity|}{|viscosity|}$" so the nonlinearity causes instability and the viscous term is the cause of stability. Thus when Re $\gg 1$ the problem is nonlinearity, $u \cdot \nabla u$, versus diffusion, $\nu \Delta u$. To solve this problem we have two different approaches: increasing diffusion (eddy viscosity) and decreasing nonlinearity. In this study we concentrate on the second approach.

3.1 DECREASING NONLINEARITY

The models which are used to decrease the nonlinearity are called *Dispersive or Reversible models.* In this chapter the models that we use are those that we defined in the second chapter. In all the four models we have assumed that there is no body force acting on the fluid, the kinematic viscosity is zero, and we have steady flow which means $\frac{\partial u}{\partial t} = 0$. Also note that throughout the analysis in this Chapter, we extensively use the fact that under periodic boundary conditions, differential operators commute.

3.2 ZEROTH ORDER MODEL

As we defined in the second chapter the spacially filtered NSE (2.8)are:

$$\begin{aligned} \bar{u}_t + \overline{\bar{u} \cdot \nabla \bar{u}} - \nu \Delta \bar{u} + \nabla \overline{p} &= \bar{f} \\ \nabla \cdot \bar{u} &= 0 \end{aligned} \tag{3.2.1}$$

Suppose $\bar{u} = w$. Since $\overline{\bar{u} \cdot \nabla \bar{u}} \simeq \overline{u \cdot \nabla u}$ we can define the zeroth order method as:

$$\begin{aligned} w_t + \overline{w \cdot \nabla w} - \nu \Delta w + \nabla \overline{p} &= \bar{f} \\ \nabla \cdot w &= 0 \end{aligned} \tag{3.2.2}$$

Considering the assumption that we made at the beginning of the chapter we have

$$\begin{aligned} \overline{w \cdot \nabla w} + \nabla \overline{p} &= 0 \\ \nabla \cdot w &= 0 \end{aligned} \tag{3.2.3}$$

Apply $(-\delta^2 \Delta + 1)$ to equation (3.2):

$$(-\delta^2 \Delta(\overline{w \cdot \nabla w}) + \overline{w \cdot \nabla w}) + \nabla(-\delta^2 \Delta \overline{p} + \overline{p}) = 0. \tag{3.2.4}$$

This is equal to: $\quad w \cdot \nabla w + p = 0.$

Thus the equation (3.2) will change to:

$$\begin{aligned} w \cdot \nabla w + p &= 0 \\ \nabla \cdot w &= 0 \end{aligned} \tag{3.2.5}$$

By using Lemma 2.1, we have

$$w.\nabla w = (\nabla \times w) \times w + \nabla(\frac{|w|^2}{2}). \tag{3.2.6}$$

Now we can rewrite (3.2) in this new form :

$$\begin{aligned} (\nabla \times w) \times w + \nabla(\tfrac{|w|^2}{2} + p) &= 0 \\ \nabla \cdot w &= 0 \end{aligned} \tag{3.2.7}$$

Definition 3.1. *(Zeroth Order Model's Bernoulli's Pressure) Let p denote the pressure and w the approximation of the velocity of the fluid, $w = \bar{u}$ then the Zeroth Order Model's Bernoulli's pressure is given by*

$$P_{ZOM} = p + \frac{1}{2} |w|^2 . \tag{3.2.8}$$

In the next theorem by using Theorem 2.1 we will show that the Bernoulli's pressure is constant in Zeroth Order Model.

Theorem 3.1. *Consider the Zeroth Order Model, then we can show that the Bernoulli's pressure, P_{ZOM}, is constant along streamlines.*

Proof. We will use the same argument that we used in the proof of Theorem 2.1. Consider the flow around an obstacle (i.e. an airfoil) that enters from L and is uniform upstream. Also assume $\frac{dx}{dt} = w(x(t),t) \simeq u(x(t),t)$. Integrate (3.3) along the streamline from Q_0 to Q_{Top} and along the streamline from $Q_1$to Q_{Bottom} :

$$(\nabla \times w) \times w + \nabla P_{ZOM} = 0 \tag{3.2.9}$$

Integrate this equation with respect to space and we get:

$$0 = \int_{Q_0}^{Q_{Top}} ((\nabla \times w) \times w + \nabla P_{ZOM}) \cdot \frac{dx}{dt}\ dt. \tag{3.2.10}$$

Since $\frac{dx}{dt} = w$:

$$0 = \int_0^T ((\nabla \times w) \times w) \cdot w\ dt + \int_{Q_0}^{Q_{Top}} \nabla P_{ZOM}(x(t)) \cdot x'(t)\ dt. \tag{3.2.11}$$

Using the fact that $((\nabla \times w) \times w) \perp w$, indeed $((\nabla \times w) \times w) \cdot w = 0$, thus

$$0 = 0 + \int_{Q_0}^{Q_{Top}} \nabla P_{ZOMX}(x(t)) \cdot x'(t)dt, \tag{3.2.12}$$

and finally

$$0 = P_{ZOM}(Q_{Top}) - P_{ZOM}(Q_0). \tag{3.2.13}$$

Using the same ideas, we can prove that:

$$0 = P_{ZOM}(Q_{Bottom}) - P_{ZOM}(Q_1). \tag{3.2.14}$$

By assuming uniform incoming flow:

$$P_{ZOM}(Q_0) = P_{ZOM}(Q_1), \tag{3.2.15}$$

thus

$$P_{ZOM}(Q_{Top}) = P_{ZOM}(Q_{Bottom}). \tag{3.2.16}$$

Now by using the definition of Bernoulli's pressure, $P_{ZOM} = p + 1/2\left|w\right|^2$, we can say:

$$p(Q_{Top}) + 1/2\left|w(Q_{Top})\right|^2 = p(Q_{Bottom}) + 1/2\left|w(Q_{Bottom})\right|^2. \tag{3.2.17}$$

This means the Zeroth Order Model Bernoulli's pressure is constant over the whole field. ∎

3.3 LERAY REGULARIZATION

As we defined in the second chapter the Leray Regularization is given by

$$\begin{aligned} \bar{u}_t + \bar{\bar{u}} \cdot \nabla \bar{u} - \nu \Delta \bar{u} + \nabla \bar{p} = f \\ \nabla \cdot \bar{u} = 0 \end{aligned} . \tag{3.3.1}$$

Suppose $\bar{u} = w$ then we can rewrite the Leray Regularization as:

$$\begin{aligned} w_t + \bar{w} \cdot \nabla w - \nu \Delta w + \nabla \bar{p} = f \\ \nabla \cdot w = 0 \end{aligned} \tag{3.3.2}$$

Considering the assumption that we made at the beginning of the chapter we have:

$$\begin{aligned} \bar{w} \cdot \nabla w + \nabla \bar{p} = 0 \\ \nabla \cdot w = 0 \end{aligned} \tag{3.3.3}$$

Now assume $\bar{w} = v$ then by using Lemma 2.2 we can rewrite this equation as:

$$\begin{aligned} (\nabla \times v) \times w + \nabla(w \cdot v) - \tfrac{1}{2}\nabla \cdot (w \otimes v) + \nabla \bar{p} = 0 \\ \nabla \cdot w = 0 \end{aligned} , \tag{3.3.4}$$

which is equivalent to:

$$\begin{aligned} (\nabla \times v) \times w - \tfrac{1}{2}\nabla \cdot (w \otimes v) + \nabla(\bar{p} + w \cdot v) = 0 \\ \nabla \cdot w = 0 \end{aligned} \tag{3.3.5}$$

Definition 3.2 (Leray Regularization Bernoulli's Pressure). *Let $\bar{p}$ denote the approximation of the pressure and w the approximation of the velocity of the fluid, $w = \bar{u}$ and $\bar{w} = v$, then we can define the Leray Regularization's Bernoulli pressure by*

$$P_{LR} = \bar{p} + w \cdot v. \tag{3.3.6}$$

Thus we can rewrite the NSE using Leray Regularization as :

$$\begin{aligned} (\nabla \times v) \times w - \tfrac{1}{2}\nabla \cdot (w \otimes v) + \nabla(P_{LR}) = 0 \\ \nabla \cdot w = 0 \end{aligned} \tag{3.3.7}$$

In the next remark, by using Theorem 2.2, we will show that the Bernoulli's pressure, P_{LR}, is constant in Leray Regularization.

Remark 3.1. *Consider the Leray Regularization. The Bernoulli pressure, P_{LR}, is not constant along streamlines in general.*

Indeed, we will use the same argument that we used in the proof of Theorem 2.1. Consider the flow around an obstacle (i.e. an airfoil) that enters from L and is uniform upstream, also assume $\frac{dx}{dt} = w(x(t), t) \simeq u(x(t), t)$. The same situation that we had in Figure 2. Integrate (3.3.7) along the streamline from Q_0 to Q_{Top} and along the streamline from $Q_1$to Q_{Bottom} :

$$0 = \int_{Q_0}^{Q_{Top}} ((\nabla \times v) \times w - \frac{1}{2}\nabla \cdot (w \otimes v) + \nabla(P_{LR})) \cdot \frac{dx}{dt} dt. \tag{3.3.8}$$

Since $\frac{dx}{dt} = w$,

$$0 = \int_0^T ((\nabla \times v) \times w) \cdot w dt - \frac{1}{2}\int_{Q_0}^{Q_{Top}} \nabla \cdot (w \otimes v) \cdot w \; dt + \int_{Q_0}^{Q_{Top}} \nabla(P_{LR}(x(t))) \cdot x'(t) \; dt. \tag{3.3.9}$$

Using the fact that $((\nabla \times v) \times w) \perp w$, indeed $((\nabla \times v) \times w) \cdot w = 0$, thus

$$0 = 0 - \frac{1}{2}\int_{Q_0}^{Q_{Top}} \nabla \cdot (w \otimes v) \cdot x'(t) \; dt + \int_{Q_0}^{Q_{Top}} \nabla(P_{LR}(x(t))) \cdot x'(t) \; dt, \tag{3.3.10}$$

and finally

$$\frac{1}{2}\int_{Q_0}^{Q_{Top}} \nabla \cdot (w \otimes v) \cdot x'(t)dt = P_{LR}(Q_{Top}) - P_{LR}(Q_0). \tag{3.3.11}$$

Using the same ideas, we can prove that

$$\frac{1}{2}\int_{Q_0}^{Q_{Top}} \nabla \cdot (w \otimes v) \cdot x'(t)dt = P_{LR}(Q_{Bottom}) - P_{LR}(Q_1). \tag{3.3.12}$$

This means the Leray Regularization Bernoulli's pressure is constant along a streamline if and only if $w \otimes v : \nabla w = 0$ along streamlines. This not generally to be expected. Thus, P_{LR} can not be expected to be constant along streamlines.

3.4 THE α MODEL

As we defined in the second chapter the α Model is given by

$$\begin{gathered} \bar{u}_t + (\nabla \times \bar{u}) \times \bar{\bar{u}} - \nu \Delta \bar{u} + \nabla \tilde{p} = \bar{f} \\ \nabla \cdot \bar{u} = 0 \end{gathered} \tag{3.4.1}$$

Suppose $\bar{u} = w$ then we can rewrite the α Model as:

$$\begin{gathered} w_t + (\nabla \times w) \times \bar{w} - \nu \Delta w + \nabla \tilde{p} = \bar{f} \\ \nabla \cdot w = 0 \end{gathered} \tag{3.4.2}$$

Considering the assumption that we assumed at the beginning of the chapter we have:

$$\begin{gathered} (\nabla \times w) \times \bar{w} + \nabla \tilde{p} = 0 \\ \nabla \cdot w = 0 \end{gathered} . \tag{3.4.3}$$

Now assume $\overline{w} = v$. Then we can rewrite this equation as:

$$\begin{gathered} (\nabla \times w) \times v + \nabla \tilde{p} = 0 \\ \nabla \cdot w = 0 \end{gathered} \tag{3.4.4}$$

In this model the Bernoulli pressure is $\tilde{p}$, and it is equal to $\tilde{p} = \bar{p} + w \cdot v$.

In the next theorem by using Theorem 2.1 we are looking at the variation of the averaged pressure in α Model .

Theorem 3.2. *Consider the α Model. Then the Bernoulli pressure is constant along streamlines defined by the averaged velocity $\bar{w}$.*

Proof. We will use the same argument that we used in the proof of Theorem 2.1. Consider the flow around an obstacle (i.e. an airfoil) that enters from L and is uniform upstream. Also we have $\frac{dx}{dt} = \bar{w}(x(t), t)$. The situation is the same as Figure 2. Integrate (3.4.4) along the streamline from Q_0 to Q_{Top} and along the streamline from $Q_1$to Q_{Bottom} :

$$0 = \int_{Q_0}^{Q_{Top}} ((\nabla \times w) \times v + \nabla \tilde{p}) \cdot \frac{dx}{dt}\, dt. \tag{3.4.5}$$

Since $\frac{dx}{dt} = \bar{w} = v$,

$$0 = \int_0^T ((\nabla \times w) \times v) \cdot v dt + \int_{Q_0}^{Q_{Top}} \nabla \tilde{p}(x(t)) \cdot x'(t) dt. \tag{3.4.6}$$

Using the fact that $((\nabla \times w) \times v) \perp v$, indeed $((\nabla \times w) \times v) \cdot v = 0$,, and thus we have

$$0 = 0 + \int_{Q_0}^{Q_{Top}} \nabla \tilde{p}(x(t)) \cdot x'(t)\ dt, \tag{3.4.7}$$

and finally

$$0 = \tilde{p}(Q_{Top}) - \tilde{p}(Q_0). \tag{3.4.8}$$

Using the same ideas, we can prove that:

$$0 = \tilde{p}(Q_{Bottom}) - \tilde{p}(Q_1) \tag{3.4.9}$$

By assuming uniform in flows :

$$\tilde{p}(Q_0) = \tilde{p}(Q_1), \tag{3.4.10}$$

thus

$$\tilde{p}(Q_{Top}) = \tilde{p}(Q_{Bottom}). \tag{3.4.11}$$

Which means the α Model Bernoulli pressure is constant along a streamline. ∎

3.5 THE BARDINA MODEL

As we defined in the second chapter Bardina Model is given by

$$\begin{gathered}\bar{u}_t + \bar{u}\cdot\nabla\bar{u} + \nabla\cdot\left\{\overline{\bar{u}\otimes\bar{u}} - \bar{\bar{u}}\otimes\bar{\bar{u}}\right\} - \nu\Delta\bar{u} + \nabla\bar{p} = \bar{f}\\ \nabla\cdot\bar{u} = 0\end{gathered} \tag{3.5.1}$$

Considering the assumption that we assumed at the beginning of the chapter we have:

$$\begin{gathered}\bar{u}\cdot\nabla\bar{u} + \nabla\cdot\left\{\overline{\bar{u}\otimes\bar{u}} - \bar{\bar{u}}\otimes\bar{\bar{u}}\right\} + \nabla\bar{p} = 0\\ \nabla\cdot\bar{u} = 0\end{gathered} \tag{3.5.2}$$

Suppose $\bar{u} = w$ then we can rewrite the Bardina Model as:

$$\begin{gathered}w\cdot\nabla w + \nabla\cdot\{\overline{w\otimes w} - \bar{w}\otimes\bar{w}\} + \nabla\bar{p} = 0\\ \nabla\cdot w = 0\end{gathered} \tag{3.5.3}$$

Assuming $\bar{w} = v$, we can rewrite this equation as:

$$\begin{gathered}w\cdot\nabla w + \nabla\cdot\{\overline{w\otimes w} - v\otimes v\} + \nabla\bar{p} = 0\\ \nabla\cdot w = 0\end{gathered} \tag{3.5.4}$$

By using Lemma 2.1, we have

$$\begin{gathered}(\nabla\times w)\times w + \nabla(\tfrac{|w|^2}{2} + \bar{p}) + \nabla\cdot\{\overline{w\otimes w} - v\otimes v\} = 0\\ \nabla\cdot w = 0\end{gathered} \tag{3.5.5}$$

Definition 3.3 (Bardina Model Bernoulli's Pressure). *Let $\bar{p}$ denote the approximation of the pressure and w the approximation of the velocity of the fluid, $w = \overline{u}$, then we can define the Bardina Model Bernoulli's pressure by*

$$P_{BM} = \bar{p} + \frac{1}{2}|w|^2. \tag{3.5.6}$$

Thus we can rewrite the Bardina Model as :

$$(\nabla\times w)\times w + \nabla(P_{BM}) + \nabla\cdot\{\overline{w\otimes w} - v\otimes v\} = 0 \tag{3.5.7}$$

$$\nabla\cdot w = 0 \tag{3.5.8}$$

We will indicate that the Bernoulli's pressure is not likely constant along a streamline.

Remark 3.2. *Consider Bardina Model. Then the Bernoulli's pressure is likely not constant along a streamline.*

Indeed, we use here the same argument as in the proof of Theorem 2.1. Consider the flow around an obstacle (i.e. an airfoil) that enters from L and is uniform upstream. Assume $\frac{dx}{dt} = w(x(t), t) \simeq u(x(t), t)$ *(e.g. the same situation as in Figure 2). Integrate (3.5.7) along the streamline from* Q_0 *to* Q_{Top} *and along the streamline from* Q_1 *to* Q_{Bottom} *to get*

$$0 = \int_{Q_0}^{Q_{Top}} ((\nabla \times w) \times w + \nabla(P_{BM}) + \nabla \cdot \{\overline{w \otimes w} - v \otimes v\}) \cdot \frac{dx}{dt}\, dt. \tag{3.5.9}$$

Since $\frac{dx}{dt} = w$,

$$0 = \int_{0}^{T} ((\nabla \times w) \times w) \cdot w\ dt - \int_{Q_0}^{Q_{Top}} \nabla \cdot (\{\overline{w \otimes w} - v \otimes v\}) \cdot w\ dt + \int_{Q_0}^{Q_{Top}} \nabla(P_{BM}(x(t))) \cdot x'(t) dt. \tag{3.5.10}$$

Using the fact that $((\nabla \times w) \times w) \perp w$, *it follows that* $((\nabla \times w) \times w) \cdot w = 0$, *and thus*

$$0 = 0 - \int_{Q_0}^{Q_{Top}} \nabla \cdot (\{\overline{w \otimes w} - v \otimes v\}) \cdot w\ dt + \int_{Q_0}^{Q_{Top}} \nabla(P_{BM}(x(t))) \cdot x'(t)\ dt, \tag{3.5.11}$$

and finally

$$\int_{Q_0}^{Q_{Top}} \nabla \cdot (\{\overline{w \otimes w} - v \otimes v\}) dx = P_{BM}(Q_{Top}) - P_{BM}(Q_0). \tag{3.5.12}$$

Using the same technique, we can prove that:

$$\int_{Q_0}^{Q_{Top}} \nabla \cdot (\{\overline{w \otimes w} - v \otimes v\}) dx = P_{BM}(Q_{Bottom}) - P_{BM}(Q_1), \tag{3.5.13}$$

which suggests that, as above, the Bernoulli pressure in the Bardina Model is not constant along a streamline.

4.0 NUMERICAL EXPERIMENTS

In this chapter, we present the numerical simulations of a 2D flow around a cylinder. In chapter 3 we proved for a steady, inviscid, incompressible flow and no external force, the Bernoulli pressure is constant along a streamline. Due to following limitations in the FreeFEM++ software we could not follow the same assumptions as what we had in our theoretical proof. First, we could not model a steady flow which requires $\frac{\partial u}{\partial t} = 0$, because to solve different LES models we have to change the time in each iteration. Thus the solution (u, p) changes over time and it is not constant. Second, to model an inviscid flow we have to have a very high Reynolds number because the viscosity is defined by $\nu = \mathrm{Re}^{-1}$, but FreeFEM++ is not able to solve an equation with a very low viscosity. So in this part we assume that we have a nonsteady flow, with low viscosity.

We choose a rectangle as the domain with a cylinder inside it. The boundary condition for the inflow velocity is constant. At the outflow we also imposed the "do nothing" boundary condition: this is a relatively new outflow boundary condition in CFD that is equivalent to a zero traction (no normal stress) boundary condition except that the Laplacian form of the viscous term is used instead of the stress tensor form, [5]. The velocity on other boundaries are zero.

Here we consider two different tests for Navier-Stokes equation, Zeroth Order Model, and Leray Regularization. The first test is a flow with Reynolds number 1000 and the second test has Reynolds number 200.

4.1 NAVIER-STOKES EQUATIONS

For the Navier-Stokes equation we proved in Theorem 2.1 that $p + \frac{1}{2}\left|u\right|^2$ is constant along a streamline for no viscosity and external force, where p is the pressure and u is the velocity. We expect to observe the same pattern changes in our simulations.

The results for Reynolds number 1000

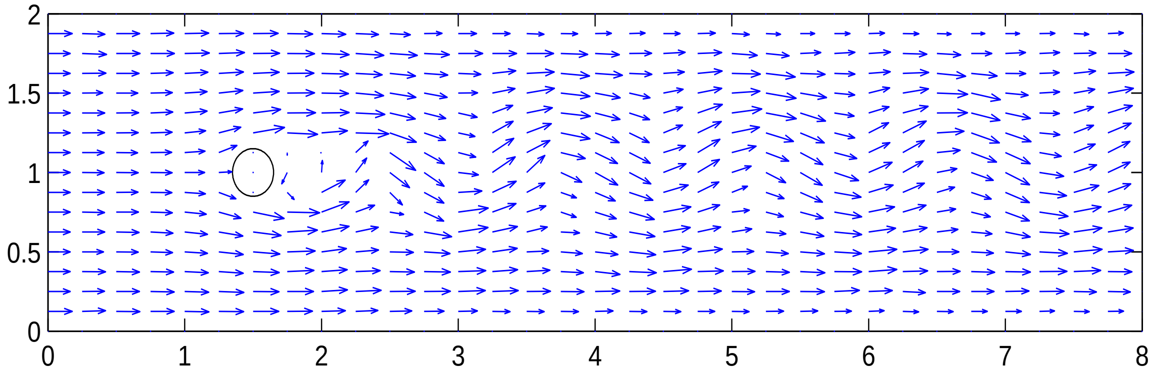

Figure 5: Shown above is the velocity field for the Navier-Stokes equation at Re=1000

For the Navier-Stokes equation we proved in Theorem 2.1 that $p + \frac{1}{2}\left|u\right|^2$ is constant along a streamline for no viscosity and external force, where p is the pressure and u is the velocity. We expect to observe the same pattern changes in our simulations.

As you may notice from figure 5 and figure 7 the quantity $\frac{1}{2}\left|u\right|^2$ and the pressure are constant before the obstacle so the Bernoulli pressure as you can see in figure 8 is also constant before the obstacle. But as soon as we can observe the wakes, we can not conclude anything from the pictures and the Bernoulli's pressure has a very varying value along a streamline.

Figure 9 shows that the vorticity before the obstacle is zero and after the obstacle it starts to change.

The results for Reynolds number 200

We can notice from figure 10 and figure12 that as long as $\frac{1}{2}\left|u\right|^2$ gets larger along a streamline pressure gets smaller which leads us to conclude the Bernoulli's pressure is almost constant along the same streamline.

Even though the viscosity in this test is not zero as you may notice from figure 13, the Bernoulli's pressure is almost constant along a streamline!

Figure14 shows us that the vorticity is almost zero over the whole field.

4.2 ZEROTH ORDER MODEL

$$w_t + \overline{w \cdot \nabla w} - \nu \Delta w + \nabla \overline{p} = 0 \quad (4.2.1)$$

$$\nabla \cdot w = 0 \quad (4.2.2)$$

In this model as we defined before in Definition (2.8). $w = \bar{u}$. So first we have to solve the differential filter problem $(-\delta^2 \Delta + 1)\bar{u} = u$. To solve this problem the only assumption that we changed is the boundary condition of the velocity on the inflow, which is zero. We have used the same assumptions as we had in NSE.

We proved for this model in Theorem 3.1 that the Bernoulli's pressure which is defined by $P_{ZOM} = p + \frac{1}{2}|w|^2$, is constant along a streamline. In the following we present the results that we got from our simulations:

The Results for Reynolds number 1000

In this case p and $\frac{1}{2}|u|^2$ also follow the same pattern as they do in NSE, so we expect to see that the Bernoulli pressure stays constant along a streamline.

Before the cylinder, we can see that the Bernoulli pressure is constant along a streamline. However, after the obstacle it starts to change along some streamlines and it remains almost constant along some other streamlines, thus we can say that this picture confirms our results from theoretical part.

As it is shown in figure 19 the vorticity of the fluid is almost zero before the obstacle and it starts to change after it.

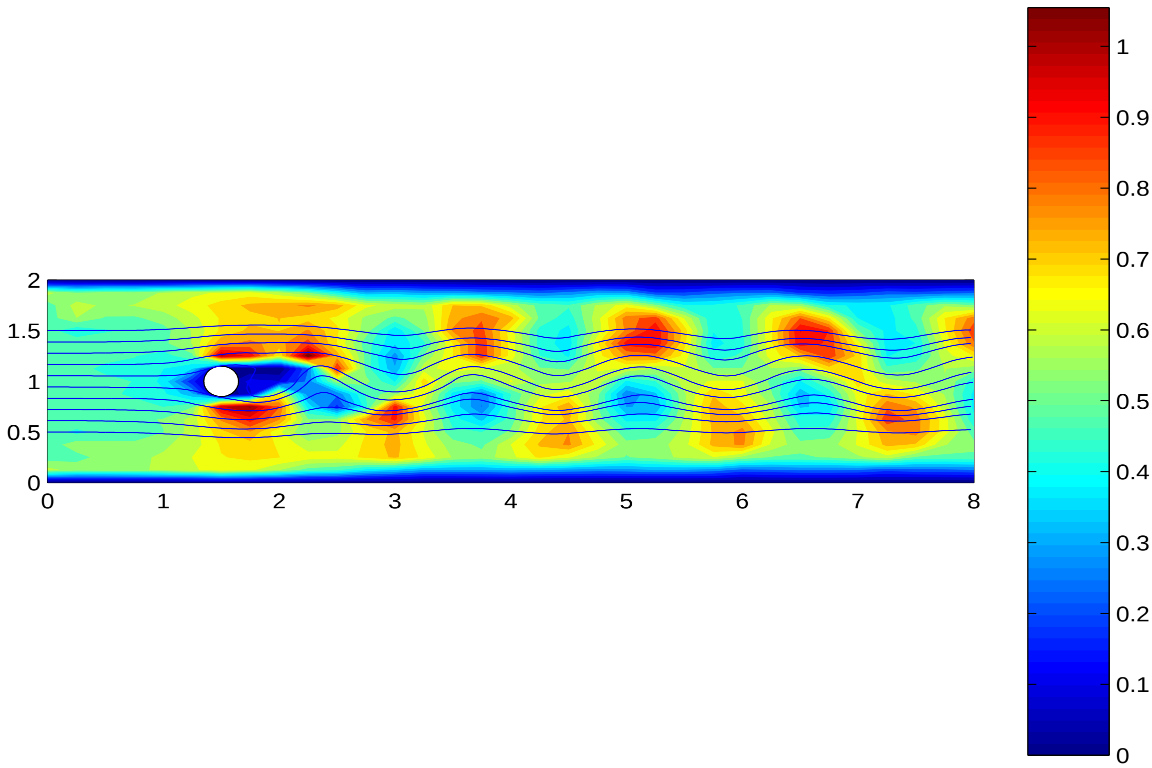

Figure 6: Shown above is $\frac{1}{2}\left|u\right|^2$ field for the Navier-Stokes equation at Re=200

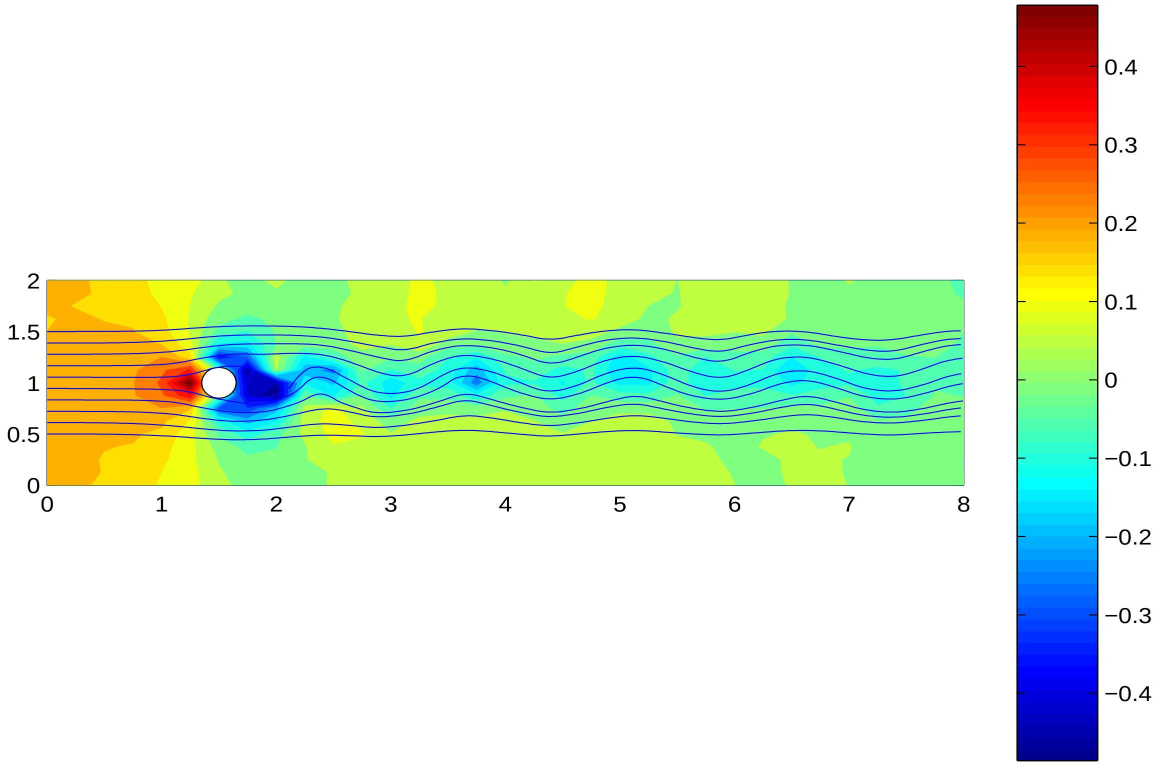

Figure 7: Shown above is the pressure field for the Navier-Stokes equation at Re=1000

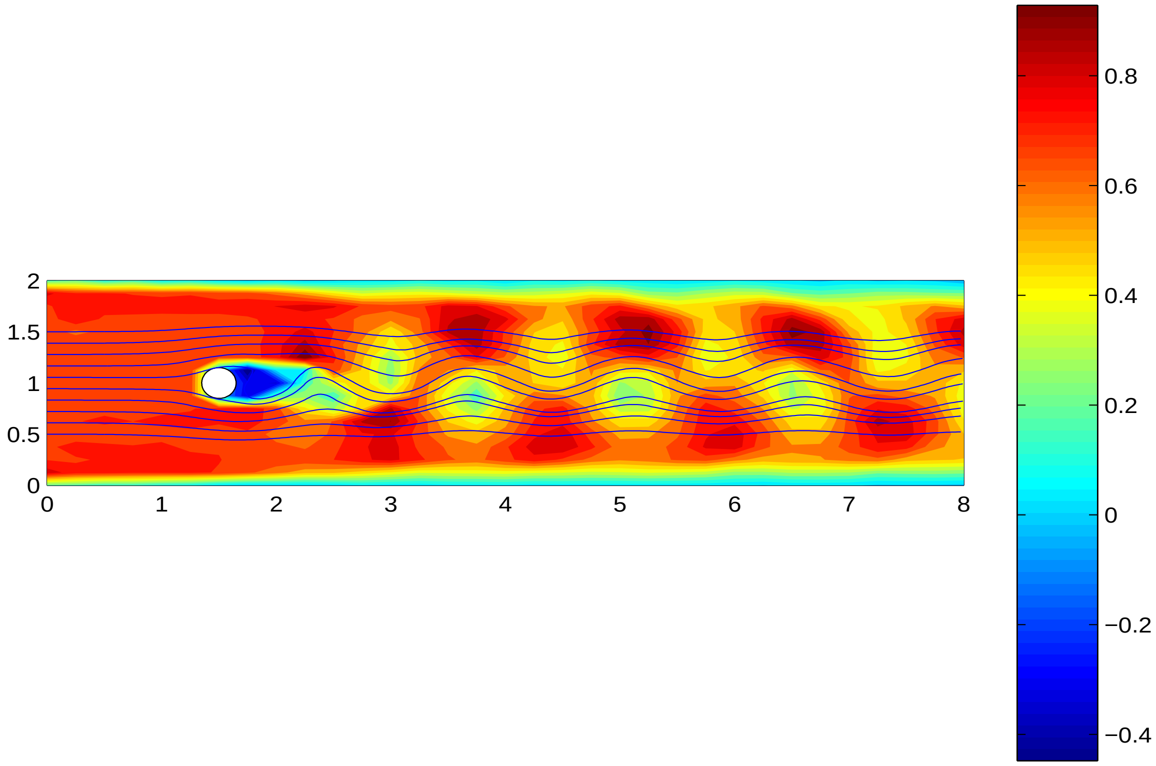

Figure 8: Shown above is the Bernoulli pressure field for the Navier-Stokes equation at Re=1000

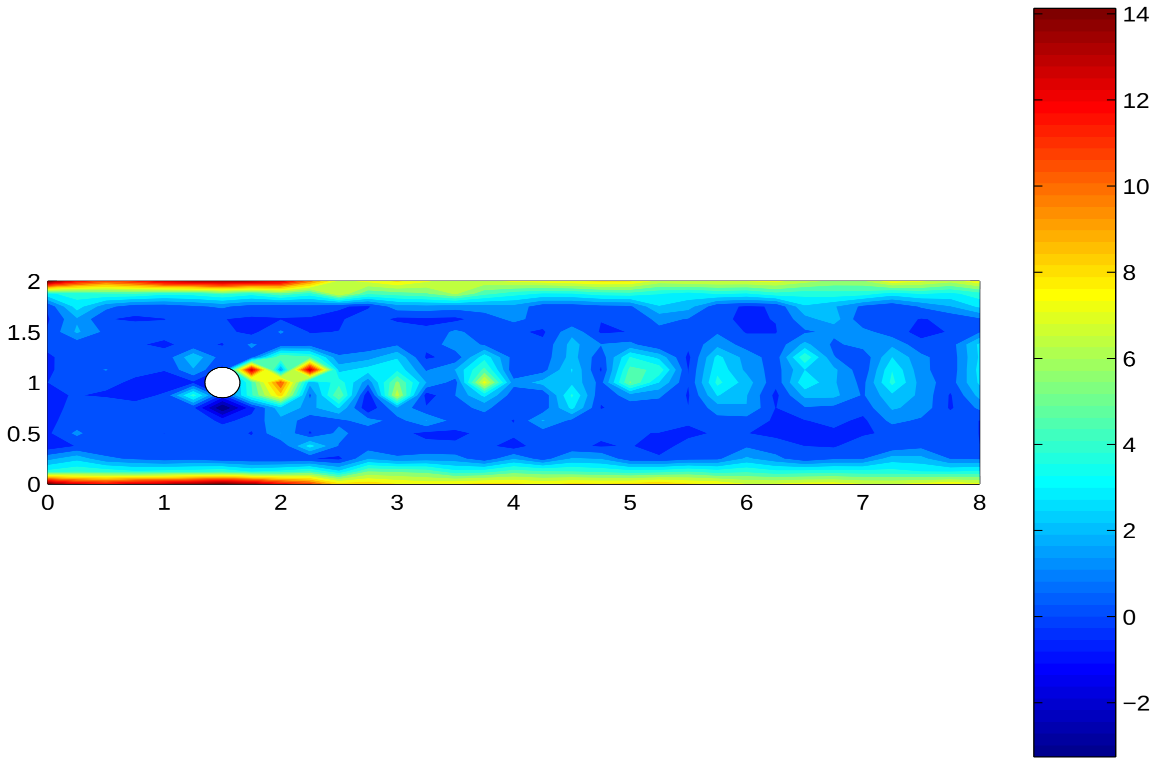

Figure 9: Shown above is the vorticity field for the Navier-Stokes equation at Re=1000

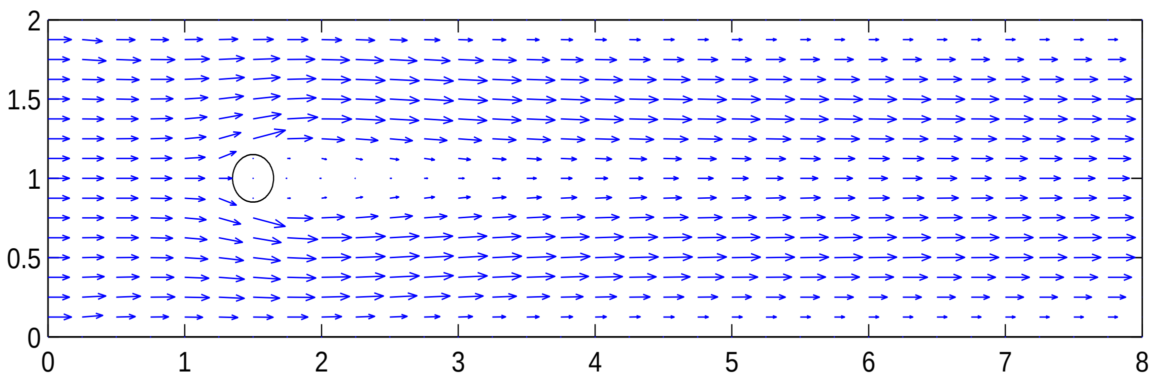

Figure 10: Shown above is the velocity field for the Navier-Stokes equation at Re=200

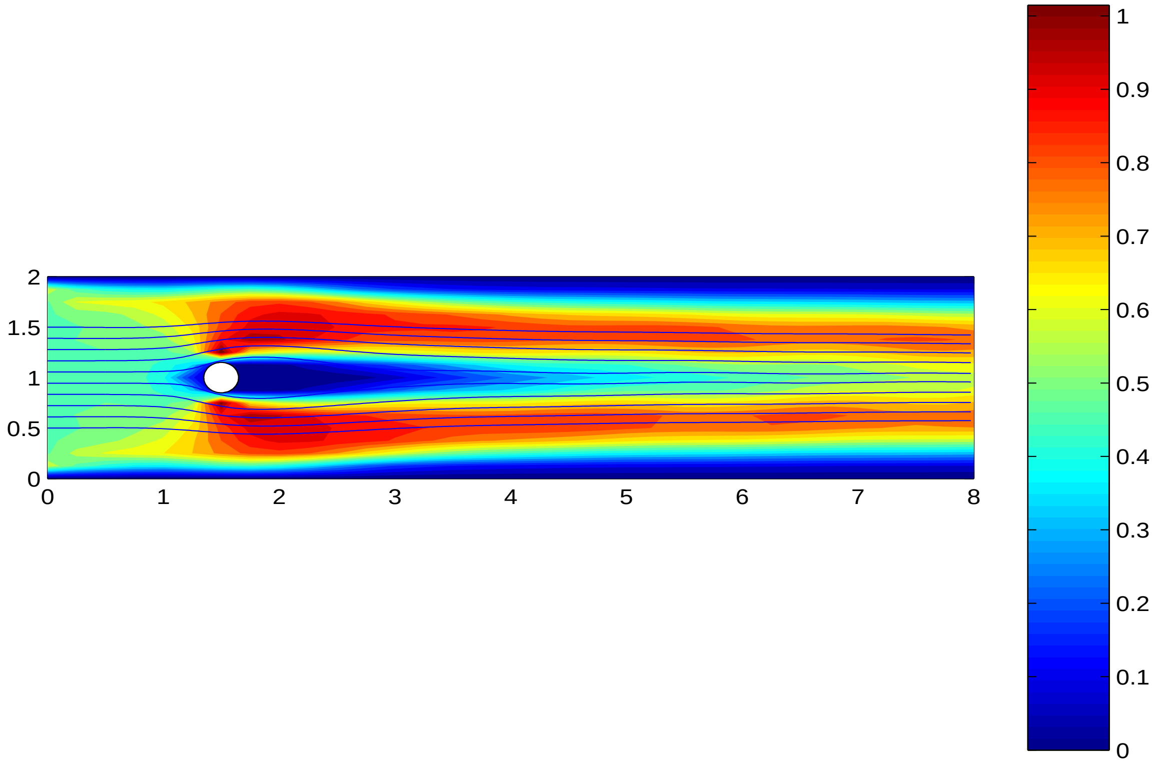

Figure 11: Shown above is $\frac{1}{2}|u|^2$ field for the Navier-Stokes equation at Re=200

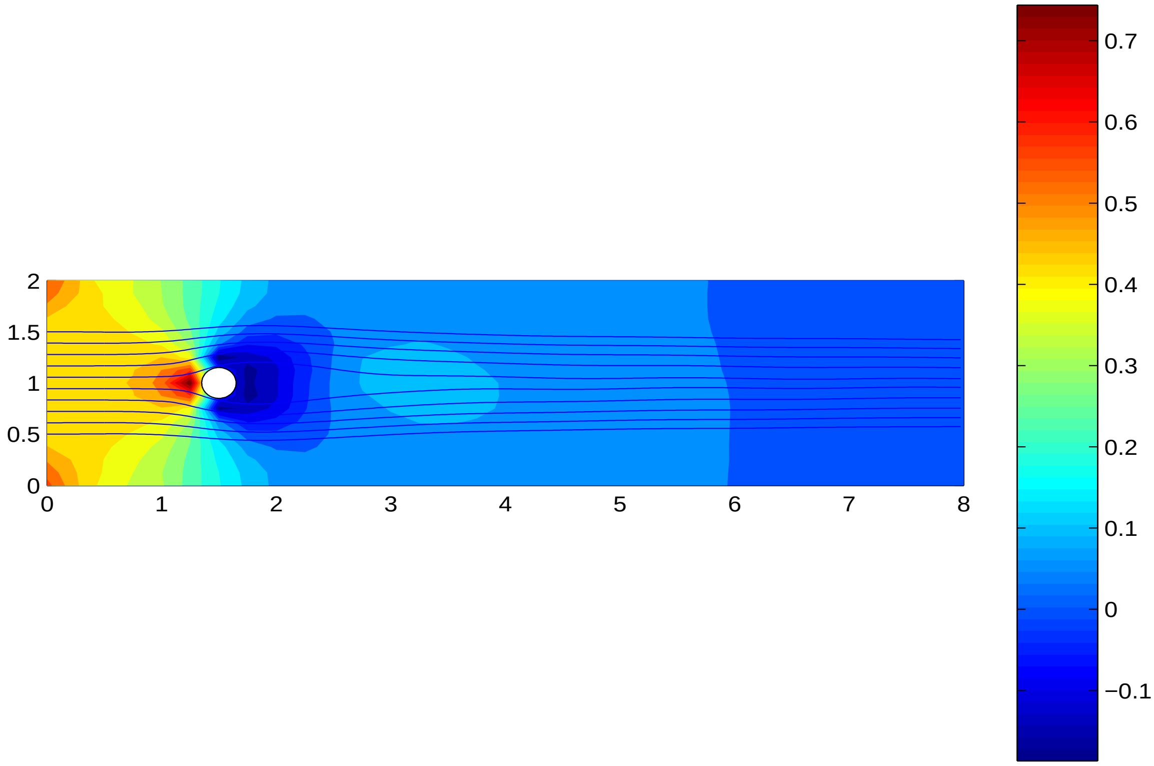

Figure 12: Shown above is the pressure field for the Navier-Stokes equation at Re=200

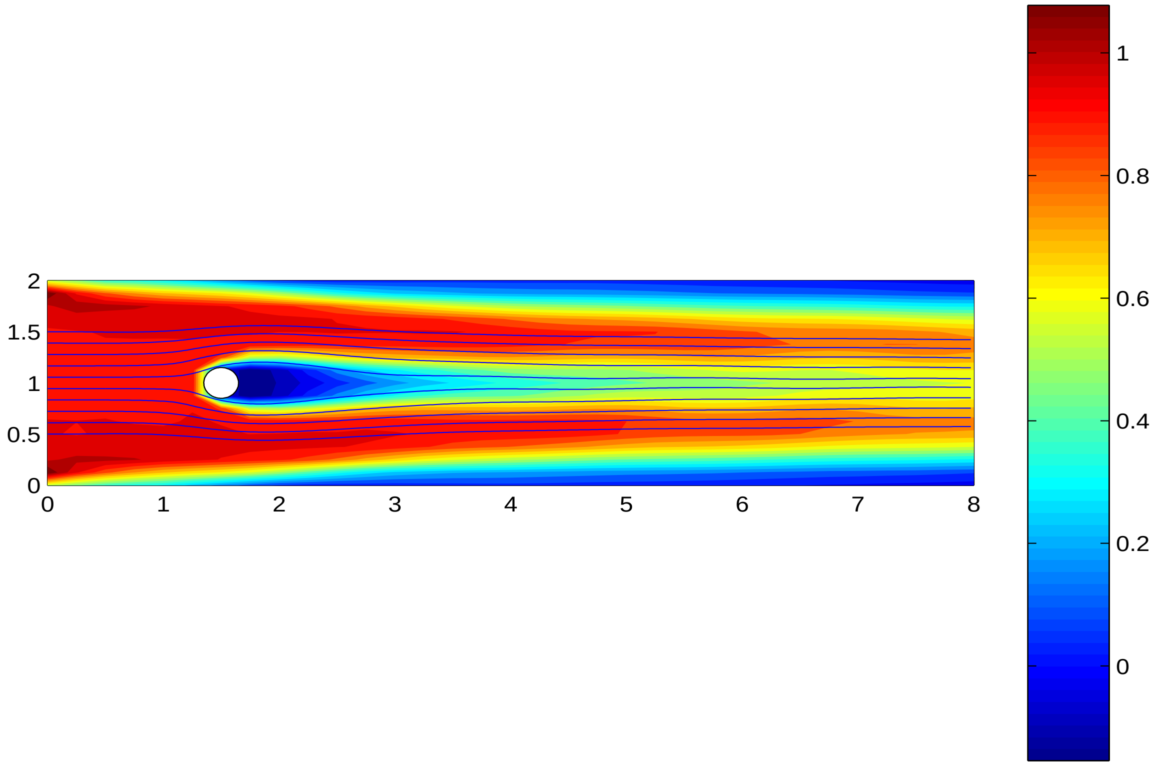

Figure 13: Shown above is the Bernoulli pressure field for the Navier-Stokes equation at Re=200

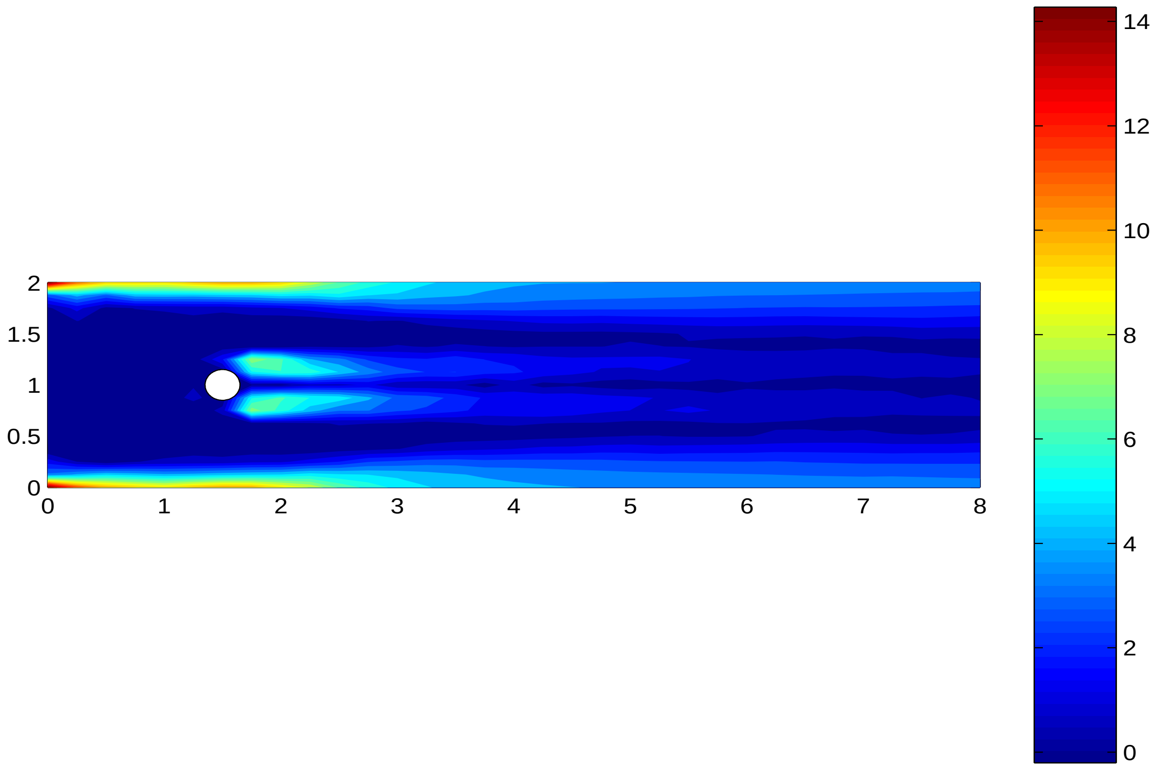

Figure 14: Shown above is the vorticity field for the Navier-Stokes equation at Re=200

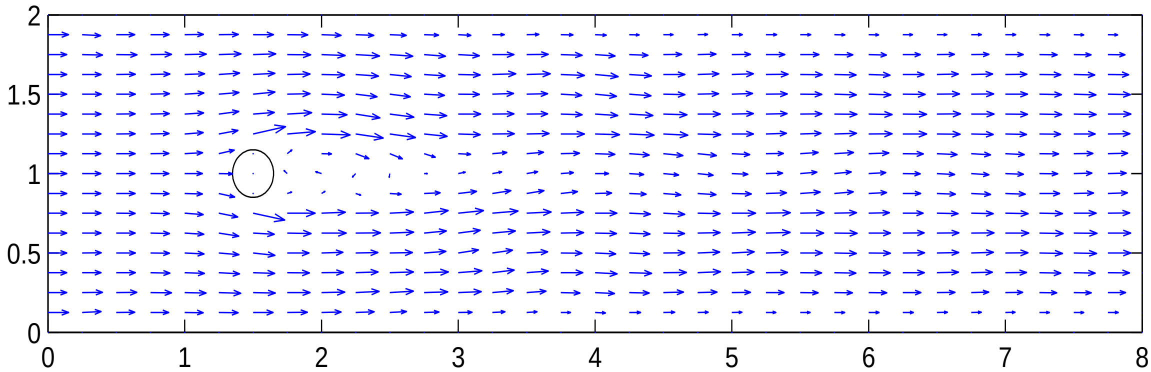

Figure 15: Shown above is the velocity field for the Zeroth Order Model at Re=1000

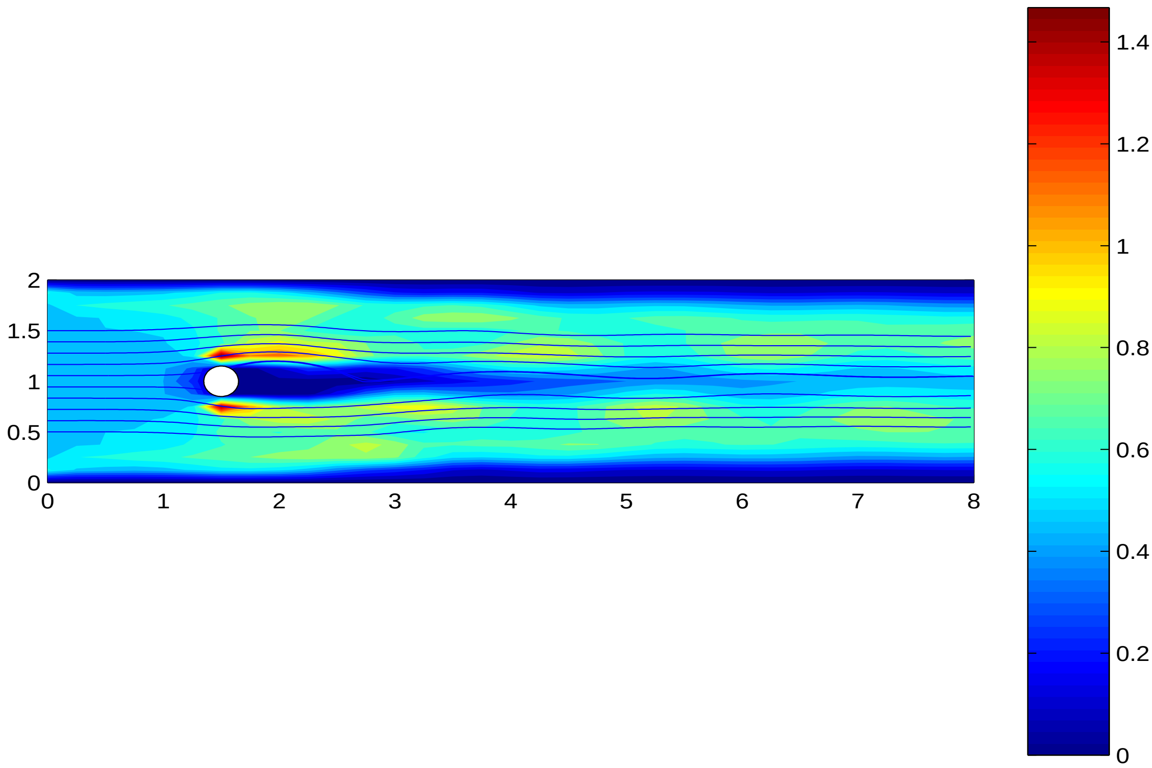

Figure 16: Shown above is $\frac{1}{2}|u|^2$ field for the Zeroth Order Model at Re=1000

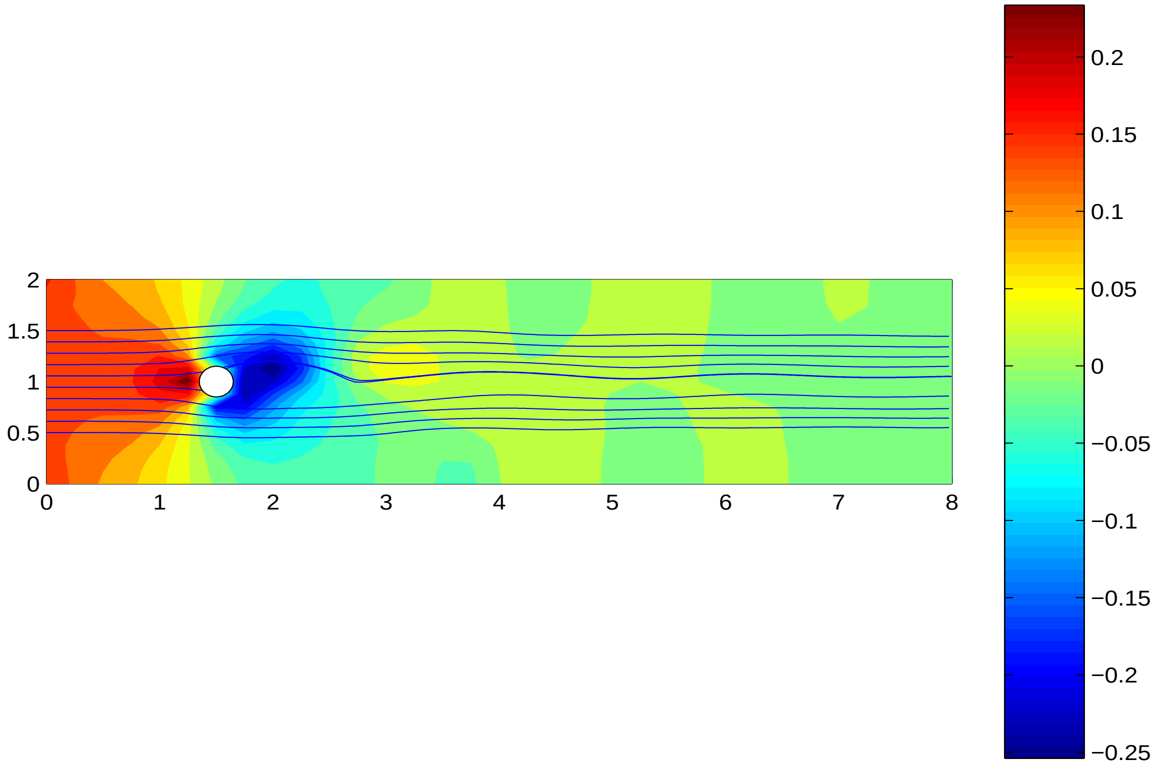

Figure 17: Shown above is the pressure field for Zeroth Order Model at Re=1000

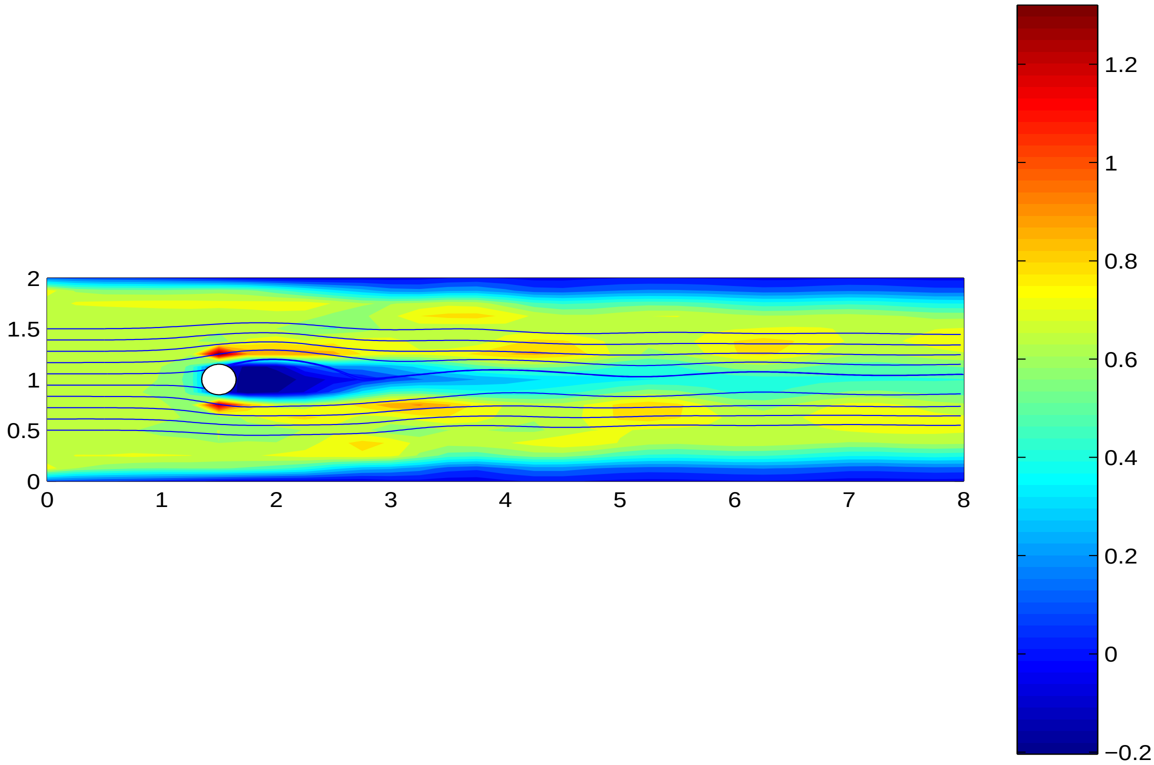

Figure 18: Shown above is the Bernoulli pressure field for Zeroth Order Model at Re=1000

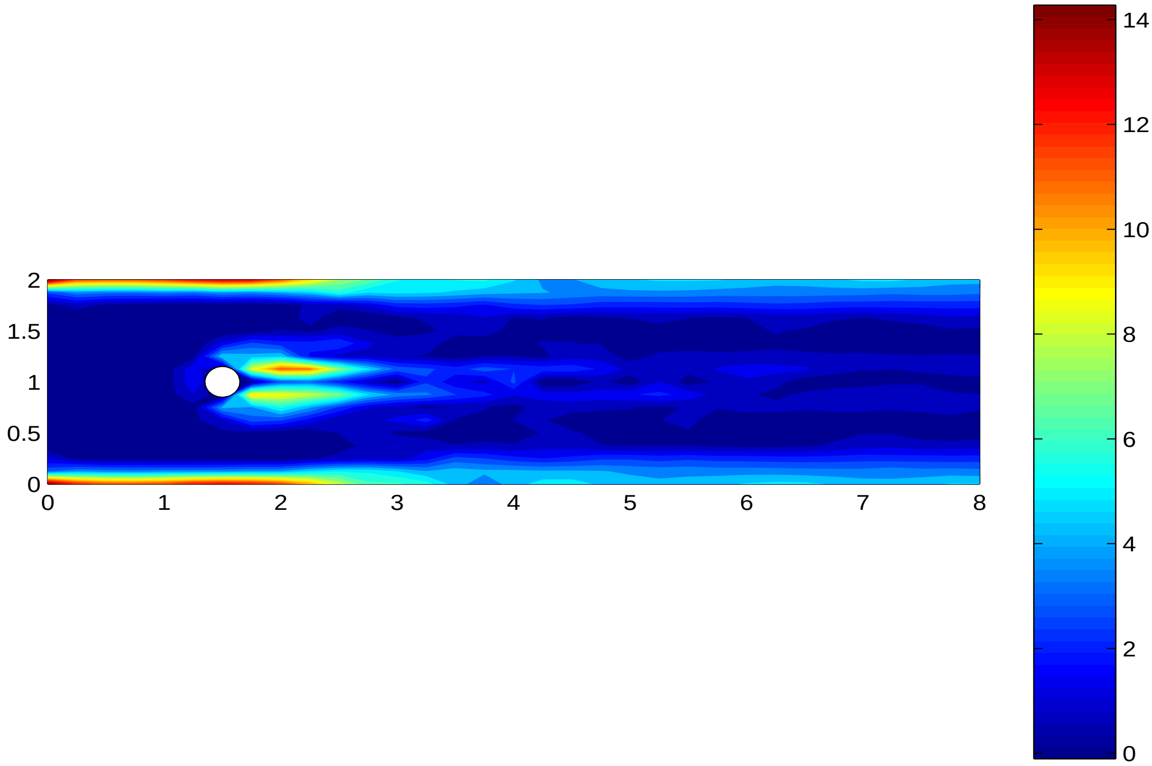

Figure 19: Shown above is the vorticity field for Zeroth Order Model at Re=1000

The results for Reynolds number 200

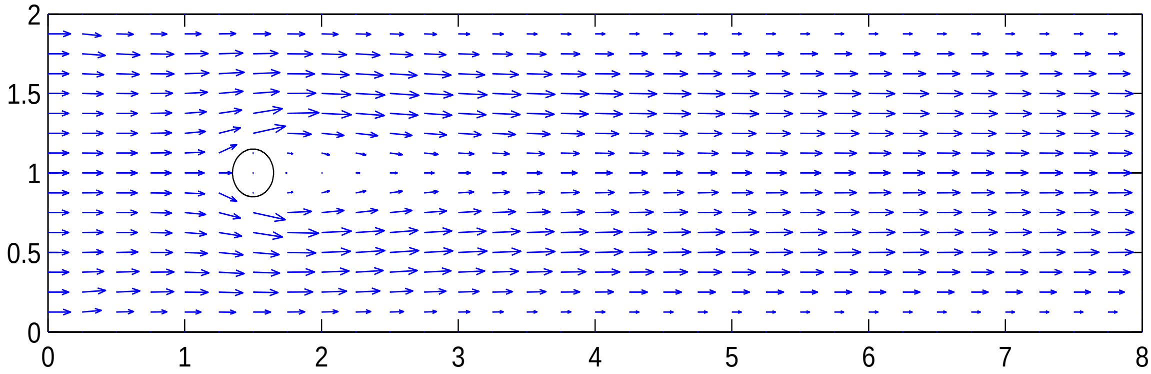

Figure 20: Shown above is the velocity field for the Zeroth Order Model at Re=200

We obtains the same result as we got in the case of Re=200 in NSE. So the pressure gets smaller along a streamline as long as $\frac{1}{2}|u|^2$ becomes larger and the Bernoulli pressure is almost constant!

Figure24 shows the vorticity of the flow which is almost zero over the field.

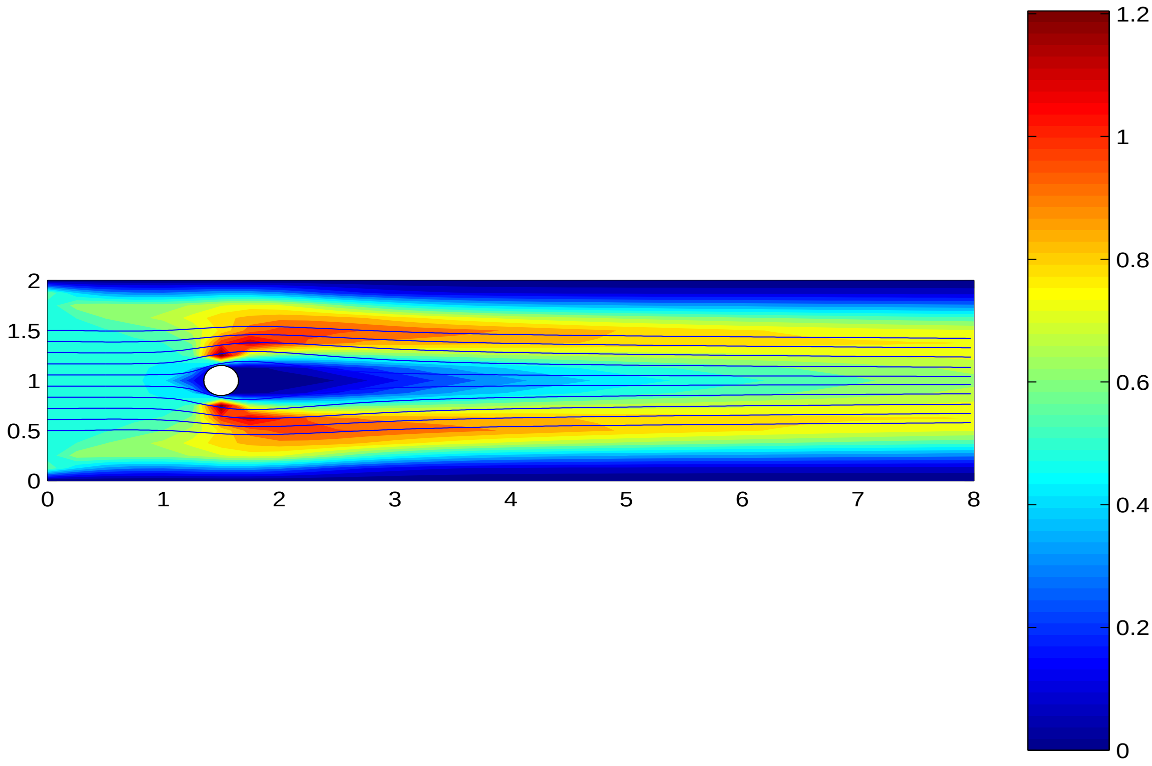

Figure 21: Shown above is $\frac{1}{2}|u|^2$ field for the Zeroth Order Model at Re=200

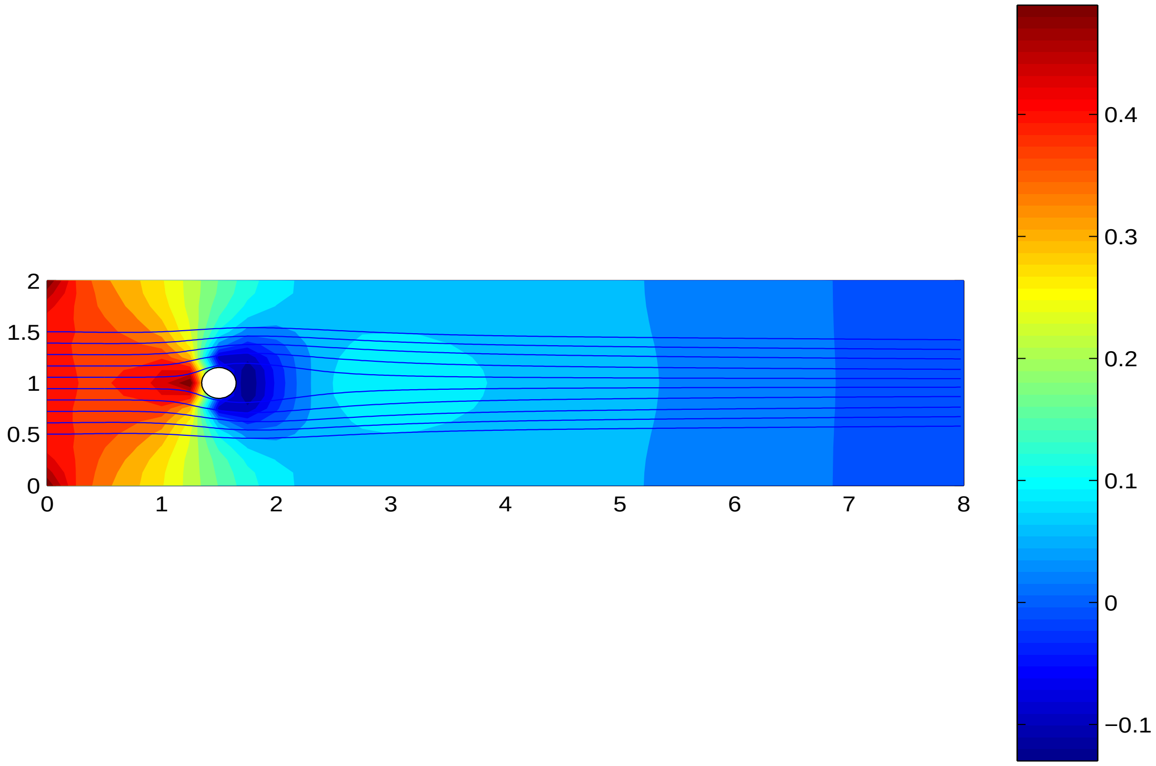

Figure 22: Shown above is the pressure field for Zeroth Order Model at Re=200

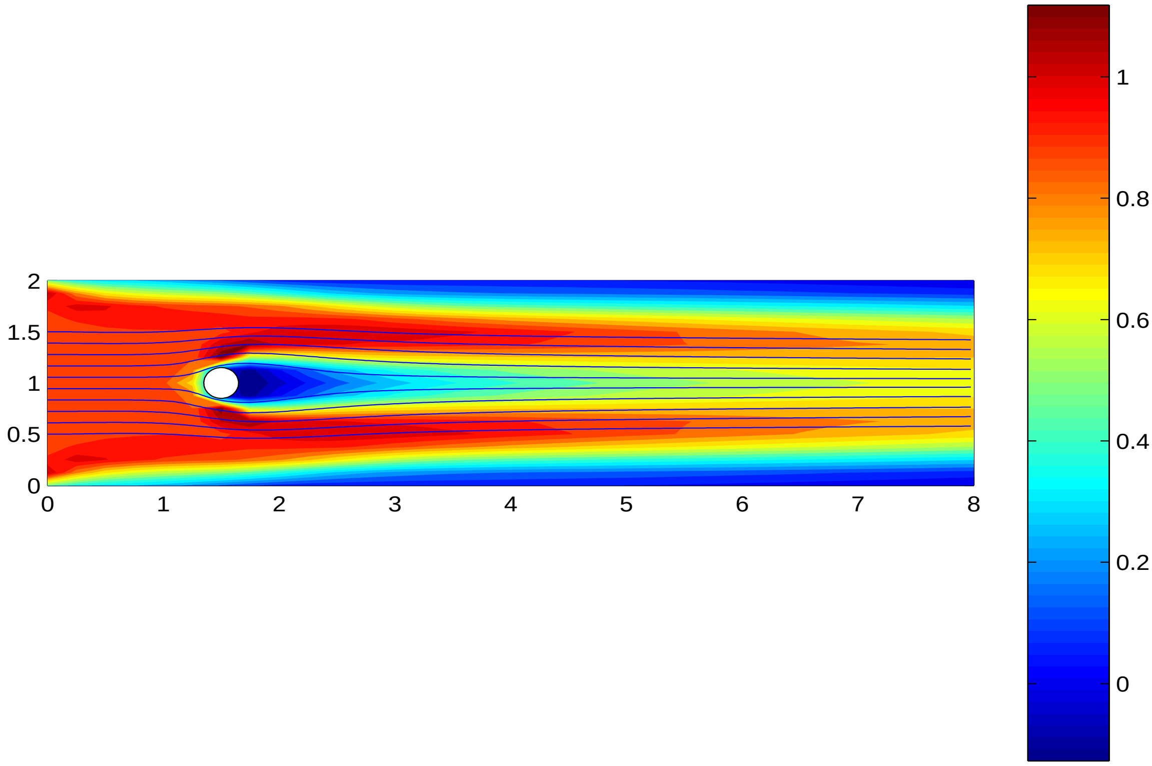

Figure 23: Shown above is the Bernoulli pressure field for Zeroth Order Model at Re=200

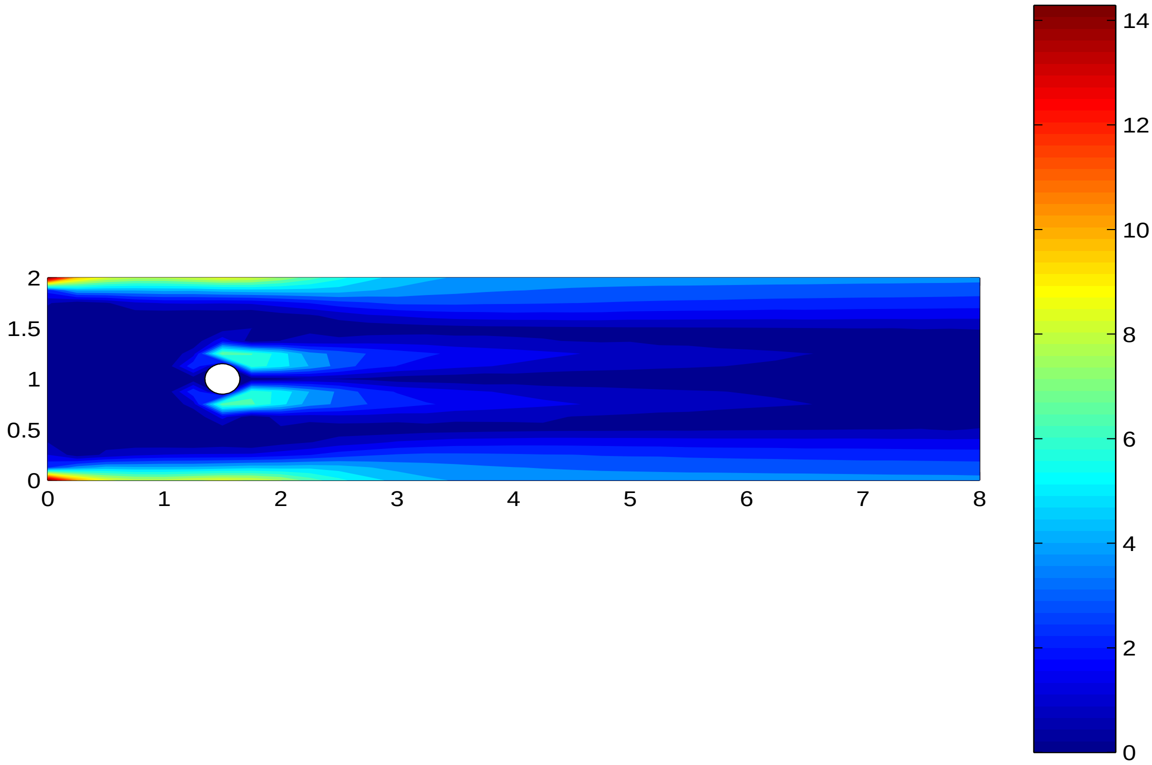

Figure 24: Shown above is the vorticity field for Zeroth Order Model at Re=200

4.3 LERAY REGULARIZATION

As we defined in Definition (2.9)

$$u_t + w \cdot \nabla u - \nu \Delta u + \nabla \overline{p} = 0 \quad (4.3.1)$$

$$\nabla \cdot u = 0 \quad (4.3.2)$$

where $w = \bar{u}$. We have used the same assumptions as ZOM to solve the differential filter. We proved for this method in Theorem 3.2. that the Bernoulli's pressure which is defined by $P_{LR} = \bar{p} + w \cdot v$, is constant along a streamline. These are the result that we got from our simulation:

The results for Reynolds number 1000

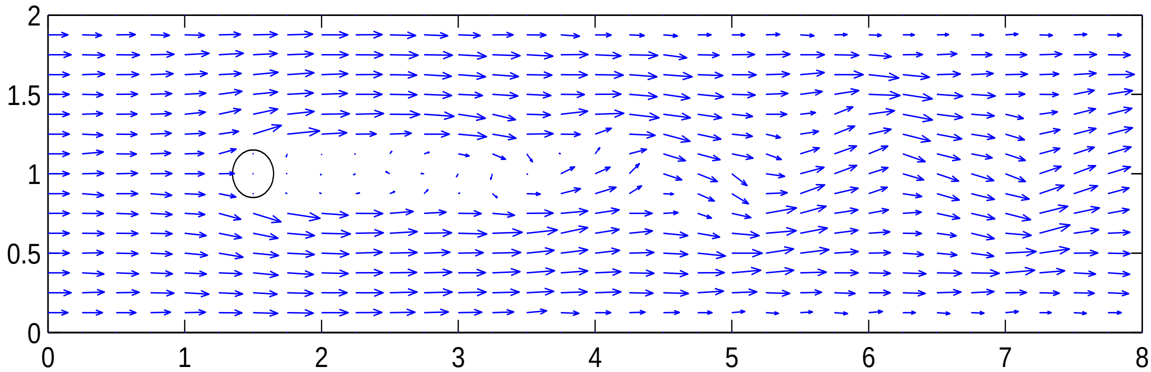

Figure 25: Shown above is the velocity field for the Leray Regularization at Re=1000

Like the two other models as long as the pressure gets smaller along a streamline $\frac{1}{2}|u|^2$ becomes larger. Thus we expect that the Bernoulli pressure stays constant.

Like two other models with Re=1000 the Bernoulli pressure stays constant along some streamlines and it varies along some other streamlines. Thus this result is not good enough to conclude that Bernoulli's pressure is not constant along a streamline!

In Figure 29 we can see the variation of the vorticity in Leray Regularization.

The results for Reynold number 200

Surprisingly we got the same result as we got for the case Re=200 in NSE and ZOM. So the pressure gets smaller along a streamline as long as $\frac{1}{2}\left|u\right|^2$ becomes larger. Unlike the theory part we can observe almost the conservation of Bernoulli pressure along a streamline!

Figure 34 shows the vorticity of the flow which is almost zero over the field.

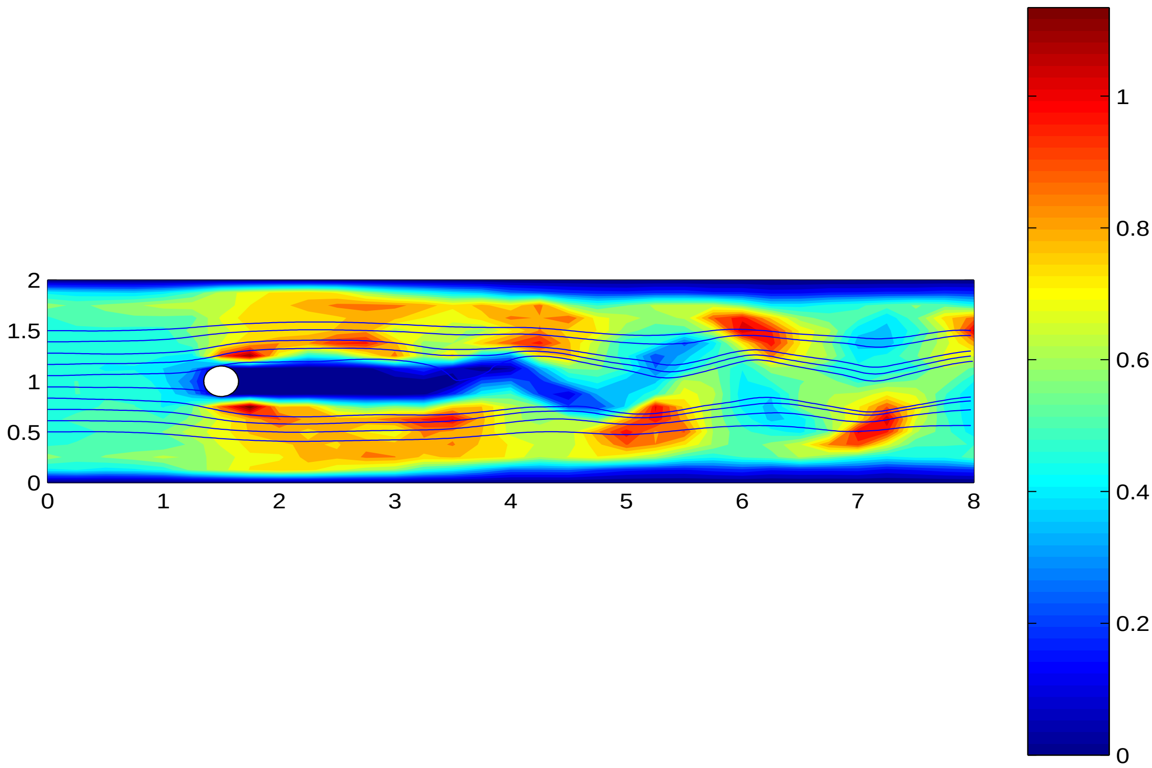

Figure 26: Shown above is $\frac{1}{2}|u|^2$ field for the Leray Regularization at Re=1000

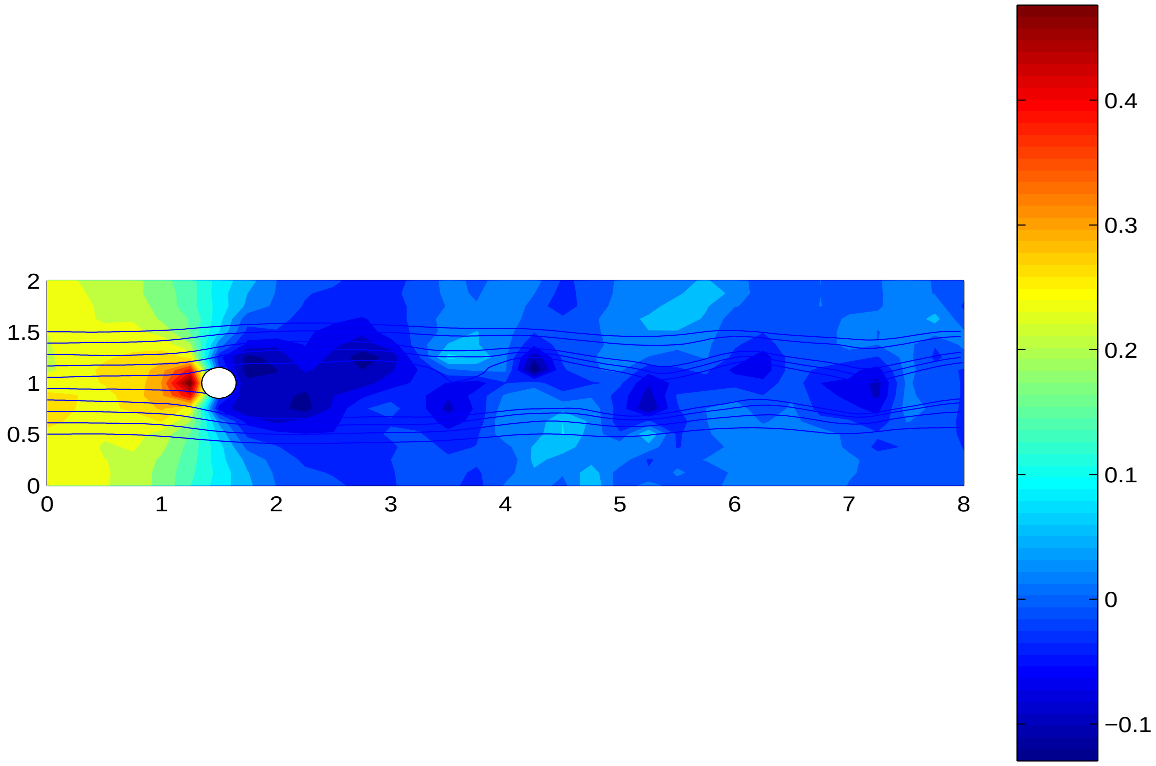

Figure 27: Shown above is the pressure field for Leray Regularization at Re=1000

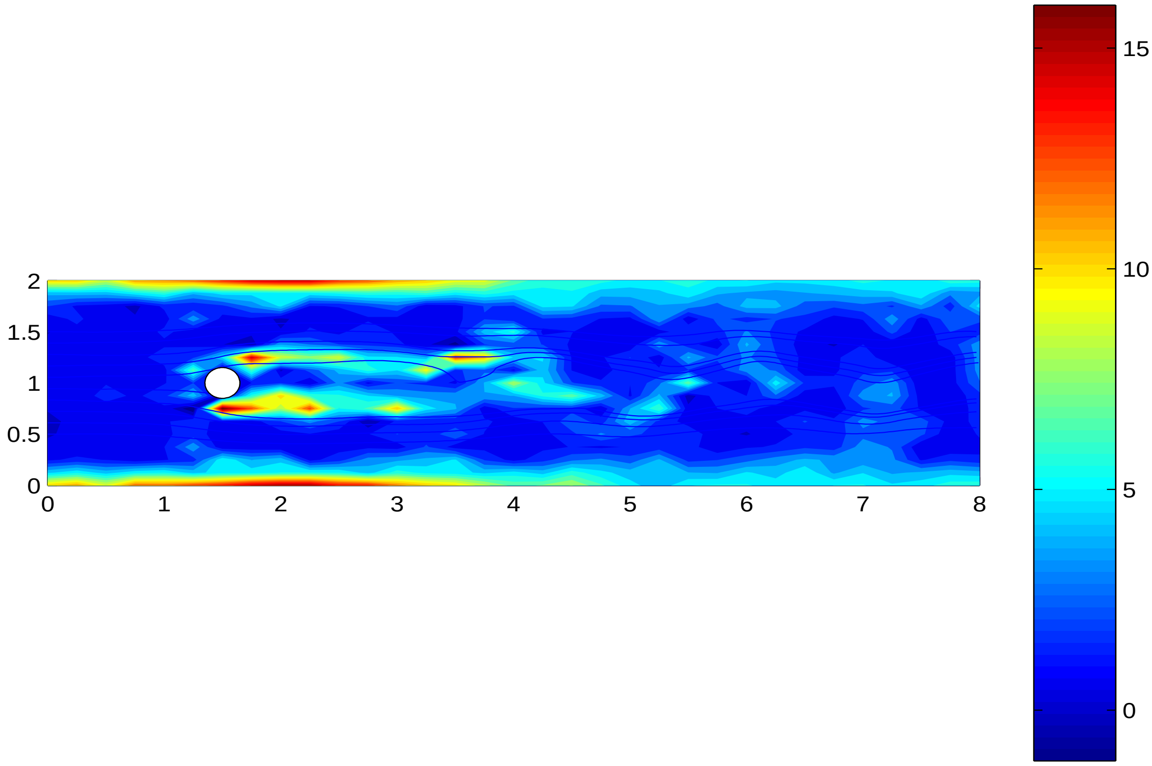

Figure 28: Shown above is the Bernoulli pressure field for Leray Regularization at Re=1000

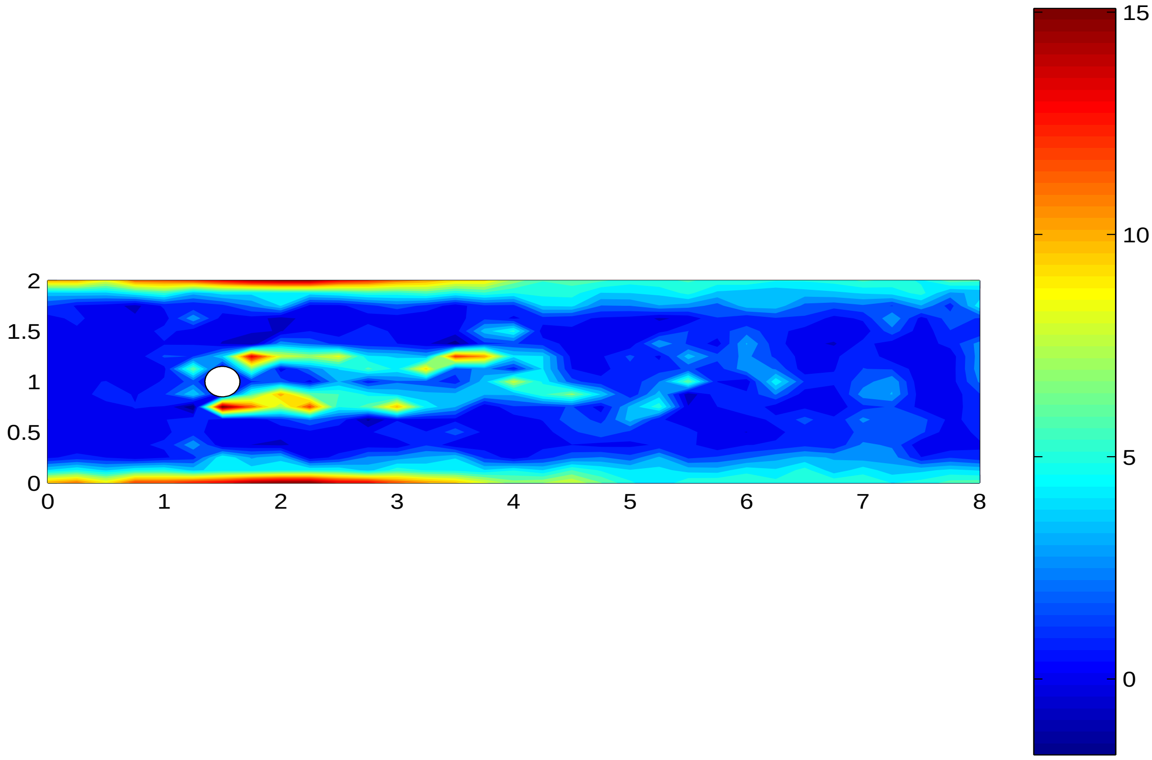

Figure 29: Shown above is the vorticity field for Leray Regularization at Re=1000

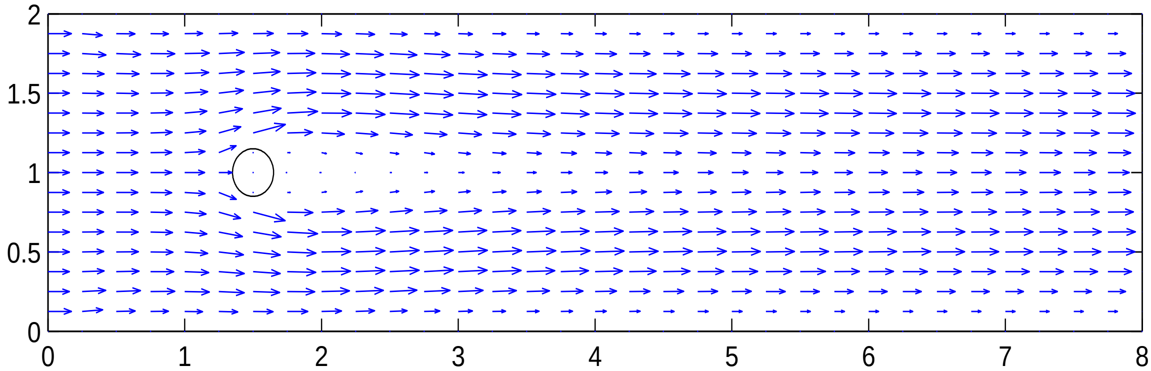

Figure 30: Shown above is the velocity field for the Leray Regularization at Re=200

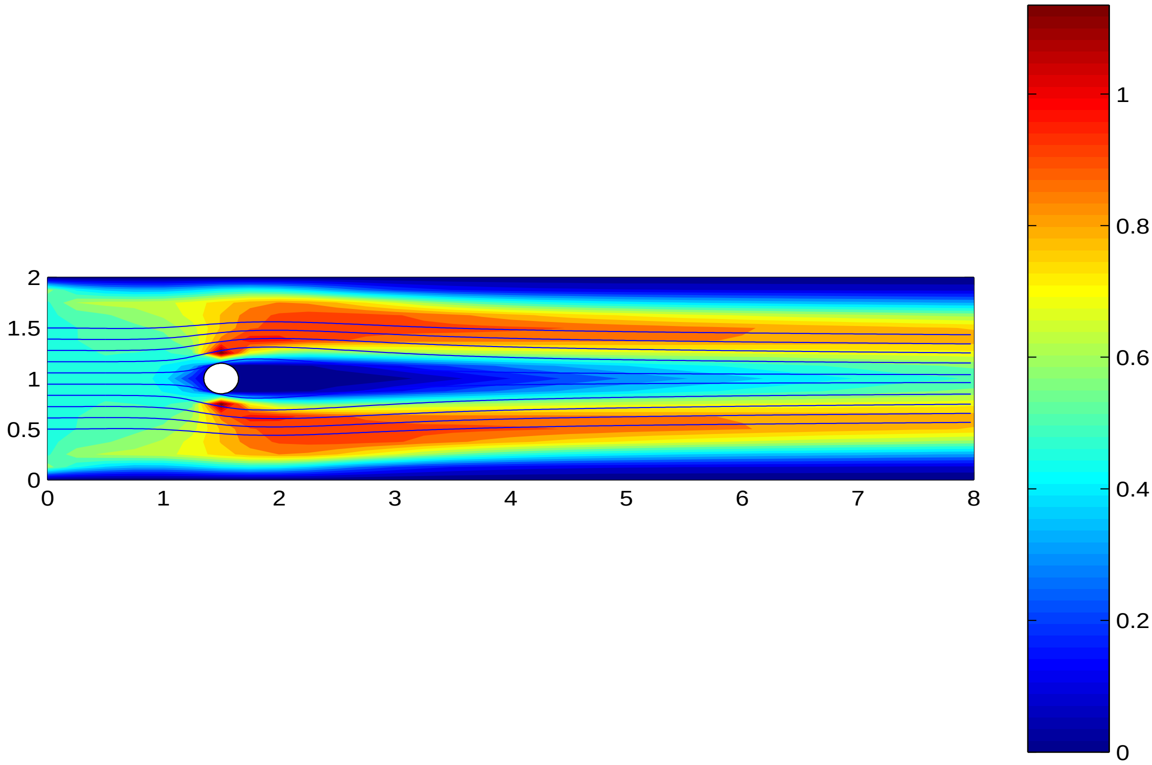

Figure 31: Shown above is $\frac{1}{2}|u|^2$ field for the Leray Regularization at Re=200

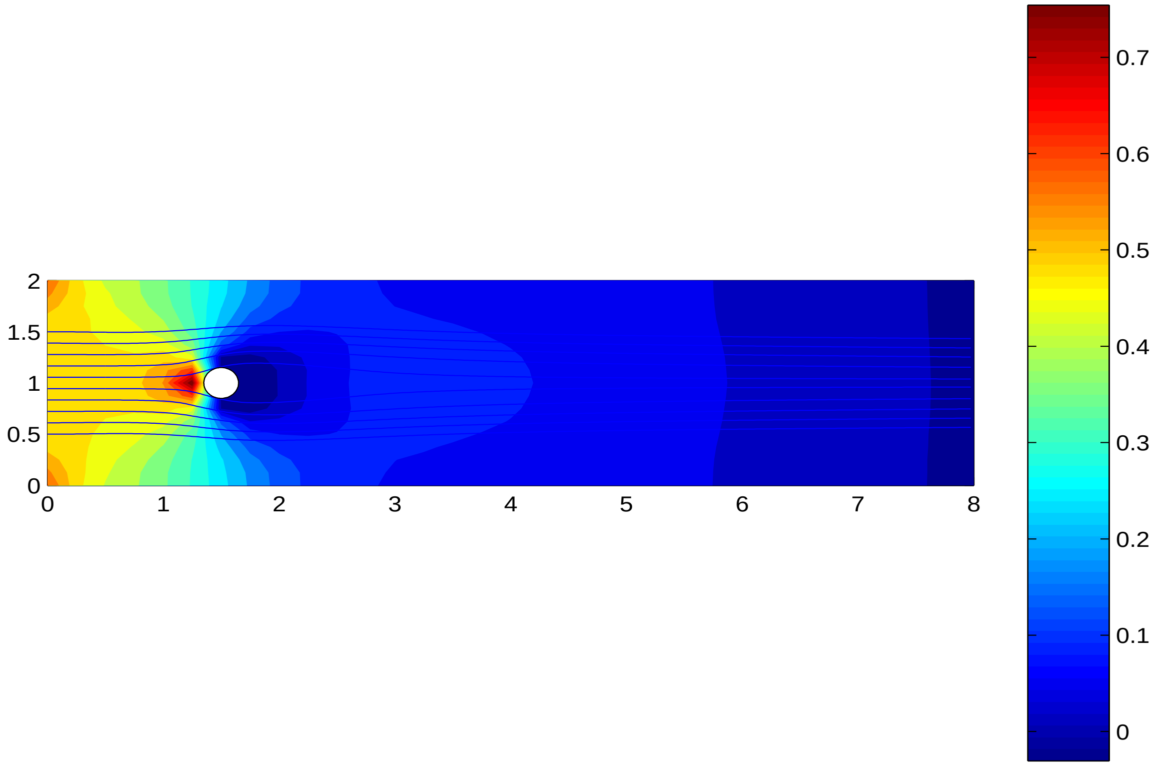

Figure 32: Shown above is the pressure field for Leray Regularization at Re=200

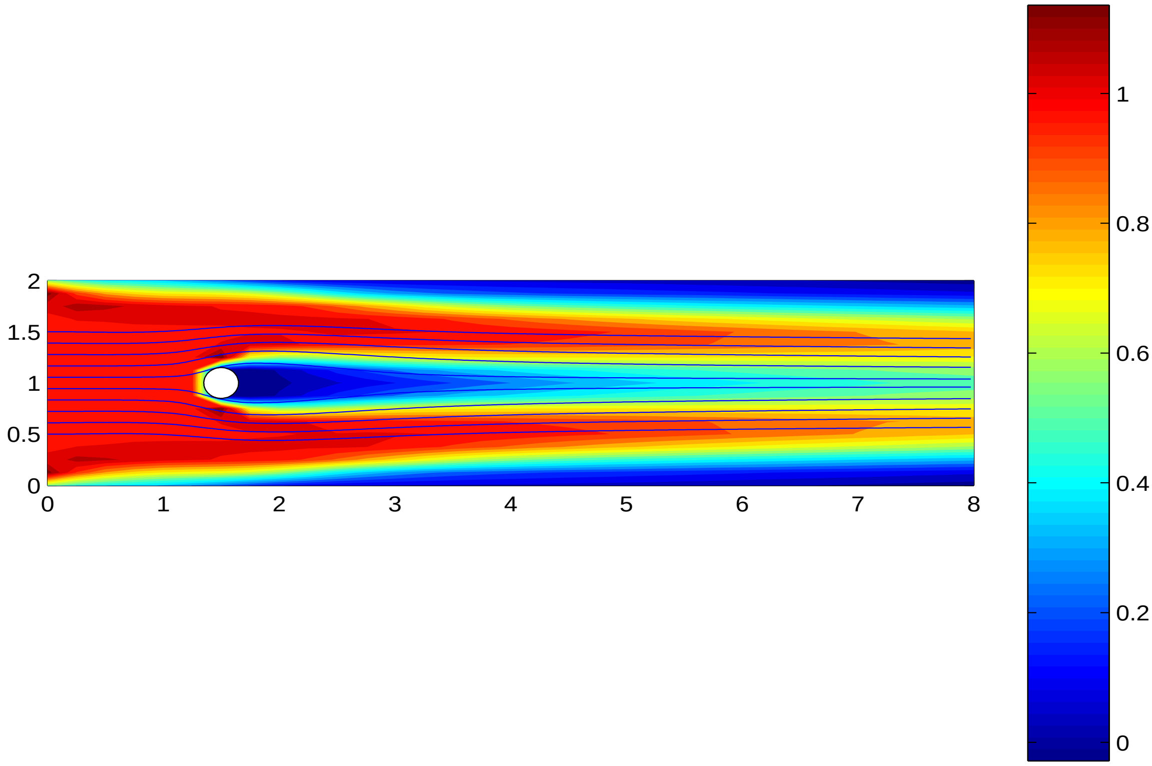

Figure 33: Shown above is the Bernoulli pressure field for Leray Regularization at Re=200

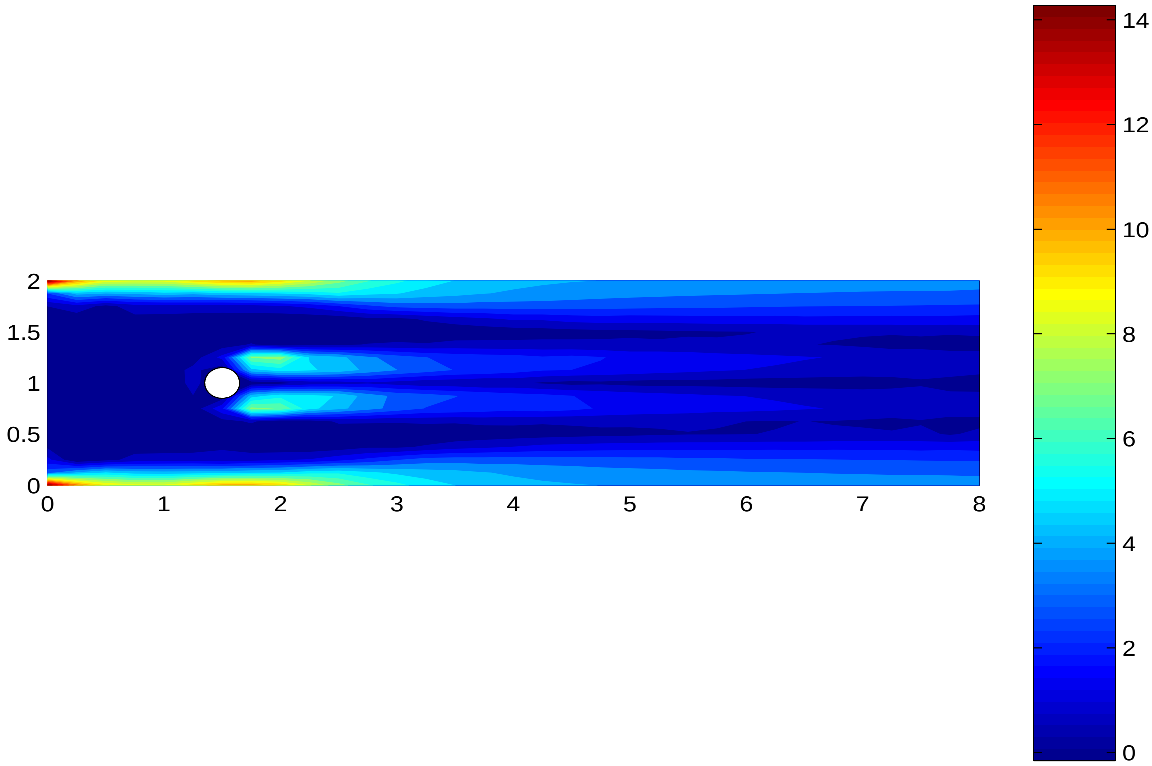

Figure 34: Shown above is the vorticity field for Leray Regularization at Re=200

5.0 CONCLUSIONS AND FUTURE WORK

5.1 CONCLUSION

The main conclusions of this thesis are: Although we proved the conservation of the Bernoulli pressure in some LES models but our numerical experiments are not sufficient enough to confirm our results in the theoretical part. We proved in chapter 3 that Zeroth Order Model and α model conserve Bernoulli pressure. We also showed that Leray Regularization and Bardina models are likely not constant and thus they are possibly not good to predict lift accurately. In chapter 4 we implemented two different models. Zeroth Order Model and Leray Regularization. We expected to observe the conservation of the Bernoulli pressure in ZOM with a small enough viscosity. First we chose $\nu = 0.001$ and the result of this modeling was not consistent with what we expected in the theoretical part. Then we chose $\nu = 0.005$ and surprisingly we obtained a better result: we observed an approximate conservation of Bernoulli pressure. For the Leray Regularization we used the same ν values as in ZOM. From the results that we got from ZOM (For $\nu = 0.001$) , we could not conclude anything specific from our numerical results. However for $\nu = 0.005$ we got completely different result from our theoretical part result. In Leray Regularization when $\nu = 0.005$ we could see an approximate conservation of Bernoulli pressure.

5.2 FUTURE WORK

There are several changes in what we studied in this thesis which could lead us to a better understanding about the Conservation of Bernoulli pressure:

- Implementation of the LES model for steady flows.
- Studying the role of viscosity in theoretical part and simulation part.
- Modification of the solid obstacle from a symmetric shape to nonsymmetric.

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