COMPACT COMPOSITION OPERATORS ON THE HARDY AND BERGMAN SPACES

by

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ABSTRACT

COMPACT COMPOSITION OPERATORS ON THE HARDY AND BERGMAN SPACES

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The thesis consists of three pieces of results on compact composition operators on the Hardy and Bergman spaces. In the first part, chapter 2, we re-formulate Lotto's conjecture on the weighted Bergman space A^2_α,$(-1 < \alpha < \infty)$, setting. We used the result of D. H. Luecking and K. H. Zhu [3] to extend Zhu's solution (on the Hardy space H^2) to the weighted Bergman space A^2_α. The results of this chapter has been published in [18].

In the second part of the thesis, chapter 3, we investigate compact composition operators which are not Hilbert–Schmidt. We consider the class of examples (see B. Lotto [2]) of composition operators C_ϕ whose symbol ϕ is a Riemann map from the unit disk D onto the semi–disk with center $(\frac{1}{2}, 0)$, radius $\frac{1}{2}$ and, in general, onto a "crescent" shaped regions constructed based on this semi-disk (see also [2].) We use the R.Riedel [8] characterization of β–boundedness/compactness on H^2 to determine the range of values of $\beta \in \mathbb{R}$ for which C_ϕ is β–bounded/compact. Similar result also extends to composition operators acting on the weighted Bergman spaces A^2_α $(\alpha \geq -1)$ based on W.Smith ([5]) characterization of β–boundednes/compactness on these spaces. In particular, we show that the class of Riemann maps under consideration gives example(s) of β –bounded composition operators C_ϕ which fail to be β compact $(0 < \beta < \infty.)$ This was an open question raised by Hunziker and Jarchaw [6](Section 5.2). Our second result arises from our attempt to generalize these observations to relate Hilbert–Schmidt classes with β–bounded/compact operators. We prove a necessary condition for C_ϕ to be Hilbert–Schmidt in terms of β–boundedness. Extending

this result to the Schatten classes, we proved a necessary condition relating β–bounded composition operators with those that belong to the Schatten ideals. The results of this chapter has been presented at the January 2005 AMS joint meeting in Atlanta, Georgia, and they are under preparation for publication.

In the last part of the thesis, Chapter 4, we characterized compact composition operators on the Hardy–Smirnov spaces over a simply connected domain. In the end, we gave an explicit example demonstrating the main results of this chapter for a simple geometry where an explicit and simplified expression for the Riemann map is known. The results of this chapter has been presented at the January 2006 AMS joint meeting in San Antonio, Texas, at the Analysis conference in honor of Prof. Vladmir Gurariy at Kent State University, March, 2006 and at the Banack Space conference in honor of Prof. Negel Kalton at the University of Miami, Oxford, Ohio, April, 2006. It is also under preparation for publication.

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PREFACE

I would like to dedicate this thesis in honor of my advisor the late Prof. T. A Metzger for his unconditional love, inspiration and unreserved help on the formulation of my thesis topic/problems and subsequently to the solutions development.

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1.0 INTRODUCTION

The study of composition operators is a recent development which links the mathematical fields of operator theory and geometric function theory.

Given a space of functions acting on a common domain and a function ϕ mapping the domain to itself, the action of a composition operator, usually denoted by C_ϕ, defines an operator from the given space to itself.

In operator theory one wants to know how "simple" an operator is by looking at how close it is in norm to an operator whose range has finite dimension. More specifically, the main theme is to discover a connection between operator theoretic properties of C_ϕ (say boundedness, compactness, closed range, Schatten classes etc.) and function theoretic properties of the defining symbol ϕ (typically, the geometry of the image of ϕ.) This leads to such classes of operators as compact operators, operators of Schatten class, closed range operators etc. In my work to date, these questions have been considered by looking at the geometry of the image of the function defining the operator. My problem is to extend the original works of J. Shapiro, Zhu, B. Lotto, and W. Smith to both the Hardy and the Bergmann spaces of plannar multiply connected domains and possibly to Riemann surfaces.

In chapter 2, we re-formulate Lotto's conjecture on the weighted Bergmann space A^2_α setting and extend Zhu's solution (on the Hardy space H^2) to the space A^2_α. The results of this chapter has been published in [18].

In Chapter 3, we investigate compact composition operators which are not Hilbert–Schmidt. We consider the class of examples (see B. Lotto [2]) of composition operators C_ϕ whose symbol ϕ is a Riemann map from the unit disk D onto the semi–disk with center $(\frac{1}{2}, 0)$, radius $\frac{1}{2}$ and, in general, onto a "crescent" shaped regions constructed based on this semi-disk (see also [2].) We use the R.Riedel [8] characterization of β–boundedness/compactness on

H^2 to determine the range of values of $\beta \in \mathbb{R}$ for which C_ϕ is β–bounded/compact. Similar result also extends to composition operators acting on the weighted Bergmann spaces A^2_α ($\alpha \geq -1$) based on W.Smith ([5]) characterization of β–boundednes/compactness on these spaces. In particular, as our first main result, we show that the class of Riemann maps under consideration gives example(s) of β –bounded composition operators C_ϕ which fail to be β compact ($0 < \beta < \infty$.) This was an open question raised by Hunziker and Jarchaw [6](Section 5.2). Our second result arises from our attempt to generalize these observations to relate Hilbert–Schmidt classes with β–bounded/compact operators. We prove a necessary condition for C_ϕ to be Hilbert–Schmidt in terms of β–boundedness. Finally, we state a conjecture relating β–bounded composition operators with those that belong to the Schatten ideals. The results of this chapter has been presented at the January 2005 AMS joint meeting in Atlanta, Georgia, and they are under preparation for publication.

In chapter 4,we characterized compact composition operators on the Hardy–Smirnov spaces over a simply connected domain. In the end, we gave an explicit example demonstrating the main results of this chapter for a simple geometry where an explicit and simplified expression for the Riemann map is known. The results of this chapter has been presented at the January 2006 AMS joint meeting in San Antonio, Texas, at the Analysis conference in honor of Prof. Vladmir Gurariy at Kent State University, March, 2006, and at the Banack Space conference in honor of Prof. Negel Kalton at Miami University, Ohio, April 2006. It is also under preparation for publication.

2.0 EXTENSION OF LOTTO'S CONJECTURE ON THE WEIGHTED BERGMAN SPACES

In this chapter we re-formulate Lotto's conjecture on the weighted Bergman space A^2_α setting and extend Zhu's solution (on the Hardy space H^2) to the space A^2_α. In the first section we present some background information and introduce the terminologies we need for the subsequent sections.

2.1 BACKGROUND AND TERMINOLOGY

Let H denote the space of analytic maps on the unit disk D. For $0 < p < \infty$ the Hardy space H^p is the subspace of H consisting of functions f that satisfy

$$\|f\|^p_{H^p} = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

The weighted Bergman space A^2_α is defined (for $\alpha > -1$) as

$$A^2_\alpha = \{f \in H \colon \iint_D |f(z)|^2 (1-|z|^2)^\alpha \, dxdy < \infty\}.$$

Given $\phi \in H$ with $\phi(D) \subset D$, the composition operator C_ϕ on A^2_α is defined by

$$C_\phi(f)(z) = f(\phi(z)), \quad z \in D.$$

The following facts are well–known.

- A^2_α is a Hilbert space (with the norm $\|f\| = \left(\iint_D |f(z)|^2(1-|z|^2)^\alpha \, dxdy\right)^{\frac{1}{2}}$). ([14], Lemma, Page 36)

- C_ϕ is a bounded linear operator on A^2_α. (A consequence of Littlewood's Subordination Theorem ([14],Theorem 1.7)

The compactness of C_ϕ is characterized in B. D. MacCluer and J. H. Shapiro [4] in terms of the angular derivative of the symbol ϕ. We say the angular derivative of ϕ exists at a point $\eta \in \partial U$ if there exists $\omega \in \partial U$ such that the difference quotient

$$\frac{\phi(\eta) - \omega}{z - \eta}$$

has a finite limit as z tends non-tangentially to η through U. The theorem is stated as follows

Theorem 2.1.1. *Suppose $0 < p < \infty$ and $\alpha > -1$ are given. Then C_ϕ is compact on A^p_α if and only if ϕ has no angular derivative at any point of ∂D.*

Another important result we need is Fatou's Radial Limit Theorem ([14],Theorem 2.2,2.6) which is stated as follows:

Theorem 2.1.2. *If $0 < p \leq \infty$ and $f \in H^p$ then the radial limit $f^*(\eta) = lim_{r\to 1^-} f(r\eta)$ exists for almost every $\eta \in \partial U$ and the map $f \to f^*$ is a linear isometry of H^p onto a closed subspace of $L^p(\partial U)$.*

The Schatten p-class $\mathcal{S}_p(A^2_\alpha)$ is defined as

$$\mathcal{S}_p(A^2_\alpha) = \left\{ T \in \mathcal{L}(A^2_\alpha) : \sum_{n=0}^{\infty} s_n(T)^p < \infty \right\},$$

where $s_n(T)$ are the singular numbers for T given by

$$s_n(T) = \inf\{\|T - K\| \colon K \text{ has rank } \leq n\},$$

and $\mathcal{L}(A^2_\alpha)$ denotes the space of bounded linear operators on A^2_α. Note that in general the above definition of Schatten p-class holds on any infinite dimensional Hilbert space H. The classes $\mathcal{S}_1(A^2_\alpha)$ (the trace class) and $\mathcal{S}_2(A^2_\alpha)$ (the Hilbert-Schmidt class) are the best-known examples.

It is known that $\mathcal{S}_2(H)$ is a two sided ideal in $\mathcal{B}(H)$ (see [3]), where $\mathcal{B}(A^2_\alpha)$ is the space of bounded composition operators on A^2_α. Indeed, this follows from the identities $s_n(TS) \leq$

$|T|s_n(S)$ and $s_n(ST) \leq |S|s_n(T)$ for $T \in S_p(H)$ and $S \in B(H)$ which intern follows from the definition of the singular numbers s_n. As a consequence of this the following important comparison properties hold which are used to construct compact but non-Schatten ideals in A^2_α.

Let $\Omega_0 \subset \Omega_1 \subset D$ be simply connected domains and $\phi_n(n = 0, 1)$ be univalent maps from D onto Ω_n, respectively.

Lemma 2.1.1. *If $C_{\phi_1} \in \mathcal{S}_p(A^2_\alpha)$ then $C_{\phi_0} \in \mathcal{S}_p(A^2_\alpha)$*

Indeed, let $\phi = \phi_1^{-1} \circ \phi_0$. It is easy to check that ϕ is a well–defined slef–map of D into itself and C_ϕ is a bounded linear operator on H^2 (i.e,$C_\phi \in B(A^2_\alpha)$) which implies that $C_{\phi_0} = C_\phi C_{\phi_1} \in S_p(A^2_\alpha)$ whenever $C_{\phi_1} \in S_p(A^2_\alpha)$ as $S_p(A^2_\alpha)$ is an ideal of $B(A^2_\alpha)$. Analogous argument yields the following:

Lemma 2.1.2. *suppose that Ω is the image of Ω_1 under an automorphism of the unit disk D and ϕ is a univalent of D onto Ω. Then $C_{\phi_1} \in \mathcal{S}_p(A^2_\alpha)$ if and only if $C_\phi \in \mathcal{S}_p(A^2_\alpha)$.*

B. A. Lotto [2] began the investigation of the connection between the geometry of $\phi(D)$ and the membership of C_ϕ in $\mathcal{S}_p(H^2)$. He considered the Riemann map ϕ from D onto the semi–disk

$$G = \{z : \text{Im}(z) > 0 \text{ and } |z - \frac{1}{2}| < \frac{1}{2}\}$$

which fixes the point 1 (see figure 1.) Lotto computed an explicit formula for ϕ given by

$$\phi(z) = \frac{1}{1 - ig(z)}, \text{ where } g(z) = \sqrt{i\frac{1 - z}{1 + z}}$$

and proved that C_ϕ is a compact composition operator on H^2 but not Hilbert Schmidt $(C_\phi \notin \mathcal{S}_p(A^2_\alpha)$ and came up with the following conjectures:

Conjecture 2.1.1. *The composition operator C_ϕ belongs to the Schatten p-ideal $\mathcal{S}_p(H^2)$ for $p > 2$.*

Conjecture 2.1.2. *Given p, $0 < p < \infty$, there exists a simple example of a domain G_p with $G_p \subseteq D$, or there are easily verifiable geometric conditions on G_p, such that the Riemann map from D onto G_p induces a compact operator that is not in $\mathcal{S}_p(H^2)$.*

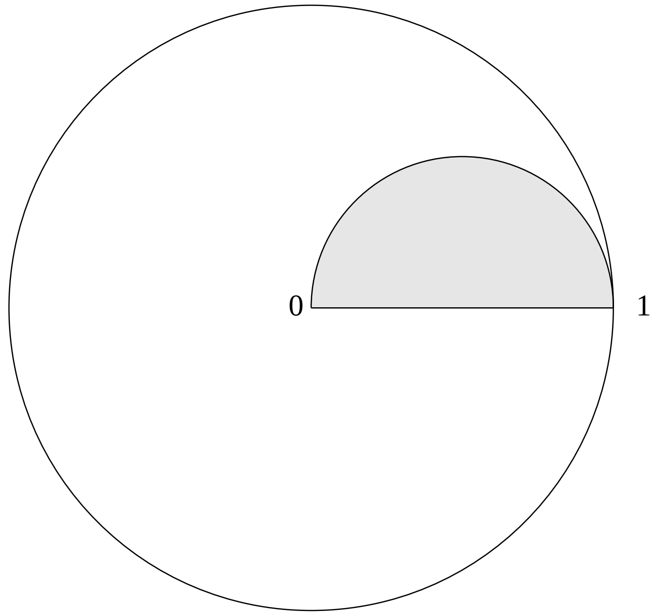

Figure 1: The half disk

Y.Zhu [1] proved both Lotto's conjectures and constructed a Riemann map that induces a compact composition operator which is not in any of the the Schatten ideals on H^2. The main result of Y.Zhu [1] reads as follows:

Theorem 2.1.3. *Let ϕ be the Riemann map from D onto the semi–disk G described above. Then the composition operator C_ϕ induced by ϕ belongs to the Schatten ideals S_p for all $p > 2$*

The goal of this chapter is to extend Zhu's solution of Lotto's Conjectures to the weighted Bergman space $\mathcal{S}_p(A^2_\alpha)$. In the $\mathcal{S}_p(A^2_\alpha)$ setting, Lotto's question can be summarized as follows:

Consider the Riemann map ϕ described above

1. Find a p, $0 < p < \infty$, such that $C_\phi \notin \mathcal{S}_p(A^2_\alpha)$.

2. Given p, $0 < p < \infty$, look for an analogous geometric conditions on $G_p \subseteq D$ such that the Riemann map $\phi_p \colon D \mapsto G_p$ induces a compact composition operator that is not in $\mathcal{S}_p(A^2_\alpha)$, and use this fact to construct C_ϕ which is compact but not in any $\mathcal{S}_p(A^2_\alpha)$ for all $0 < p < \infty$.

The compactness criterion (Theorem 2.1.1) assures us that C_ϕ is compact on A^2_α. Note here that the compactness of C_ϕ is independent of α. In the next section, we address both of these questions. For $\alpha = 0$, we extend Zhu's solution ([1]) to prove that $C_\phi \in \mathcal{S}_p(A^2_0)$ if and only if $p > 1$, showing that the trace class $\mathcal{S}_1(A^2_0)$ "draws" the "borderline" of membership of the C_ϕ's in the Schatten ideals on $\mathcal{S}_p(A^2_0)$. Likewise, we extend Zhu's results on Conjecture 2 first in $\mathcal{S}_p(A^2_0)$ and then for the general $\mathcal{S}_p(A^2_\alpha)$, $\alpha > -1$.

2.2 EXTENSION OF LOTTO'S CONJECTURE ON THE WEIGHTED BERGMAN SPACES

To answer the first question, we need the Luecking-Zhu's Theorem (see[3]) to characterize membership in $\mathcal{S}_p(A^2_\alpha)$. This theorem reads

$$C_\phi \in \mathcal{S}_p(A^2_\alpha) \text{ if and only if } N_{\phi,\alpha+2}(z)\left(\log\left(\frac{1}{|z|}\right)\right)^{-\alpha-2} \in \mathcal{L}^{\frac{p}{2}}(d\lambda),$$

where

$$N_{\phi,\beta}(\omega) = \sum_{z\in\phi^{-1}(\omega)} \log\left(\frac{1}{|z|}\right)^{\beta}$$

is the generalized Nevanlinna counting function, and $d\lambda(z) = (1-|z|^2)^{-2}dxdy$ is the Möbius invariant measure on D.

For ϕ univalent self map of D into itself,

$$N_{\phi,\beta}(z) = \left(\log(\frac{1}{|\phi^{-1}(z)|})\right)^{\beta} \approx (1-|\phi^{-1}(z)|)^{\beta}, \quad \text{for} \quad |\phi^{-1}(z)| \to 1.$$

Thus, we have

Lemma 2.2.1.

For ϕ univalent self map of D into itself, it holds

$$C_\phi \in \mathcal{S}_p(A^2_\alpha) \textit{ if and only if } \quad \chi_{\phi(D)}\cdot\left(\frac{1-|\phi^{-1}(\omega)|}{1-|\omega|}\right)^{\alpha+2} \in \mathcal{L}^{\frac{p}{2}}(d\lambda).$$

Therefore, we conclude

Corollary 2.2.1.

Let $\alpha > -1$ and ϕ a univalent self map of D into itself, We have:

$$C_\phi \in \mathcal{S}_p(A^2_\alpha) \text{ if and only if } C_\phi \in \mathcal{S}_{(\alpha+2)p}(H^2)$$

We use Corollary 2.2.1 to update Theorem 3.1 of [1] on the setting of $\mathcal{S}_p(A^2_\alpha)$ spaces. We first consider the standard case $\alpha = 0$. The analogue of Theorem 2.1.1 reads:

Theorem 2.2.1.

Let ϕ be a Riemann mapping from D onto the semi-disk

$$G = \{z : Im(z) > 0 \text{ and } |z - \frac{1}{2}| < \frac{1}{2}\},$$

such that $\phi(1) = 1$. Then the composition operator C_ϕ belongs to the Schatten ideals $\mathcal{S}_p(A^2_0)$ if and only if $p > 1$.

Remark 2.2.1.

It's interesting to compare (Theorem 2.2.1) with the corresponding result in the H^2 case (see Theorem 3.1 in [1]) which holds for $p > 2$ showing here that the trace class $\mathcal{S}_1(A^2_0)$ is the "borderline" case for membership of C_ϕ in the Schatten-p ideals. For the proof, see the general case next.

Let us now consider the general case when $-1 < \alpha$ is arbitrary. Corollary 2.2.1 and Theorem 2.1.2 at once implies the following

Theorem 2.2.2.

For $-1 < \alpha$, under the assumptions of Theorem 2.2.1, we have

$C_\phi \in \mathcal{S}_p(A^2_\alpha)$ if and only if $p > \frac{2}{\alpha+2}$.

In the following, we address the second question. For $0 < \beta < 1$, we let G_β be the crescent shaped region bounded by

$$G = \{z : Im(z) > 0 \text{ and } |z - \frac{1}{2}| = \frac{1}{2}\}$$

and a circular arc in D joining 0 to 1 with the two arcs forming an angle $\beta\pi$ at 0 and 1 (see Figure 2 and Figure 3 and Figure 4 for three different values of β)

Let ϕ_β be the Riemann map of D onto G_β with $\phi_\beta(1) = 1$. The second result of Y.Zhu [1] for the Hardy space reads:

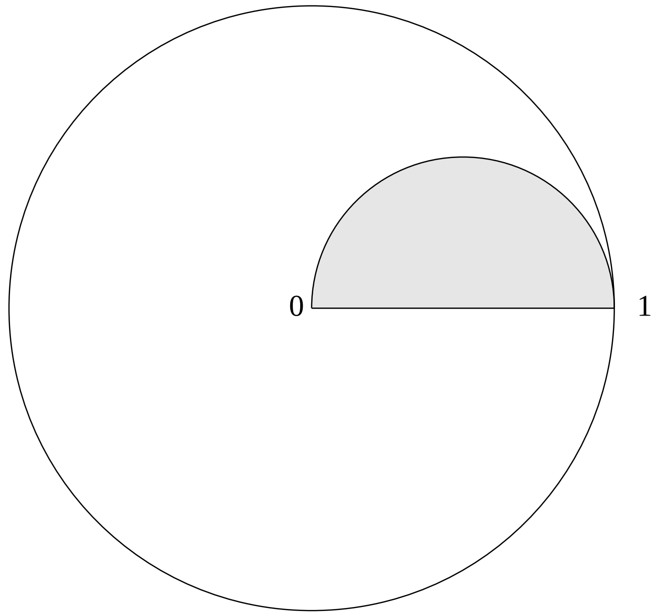

Figure 2: The half disk ($\beta = 1/2$)

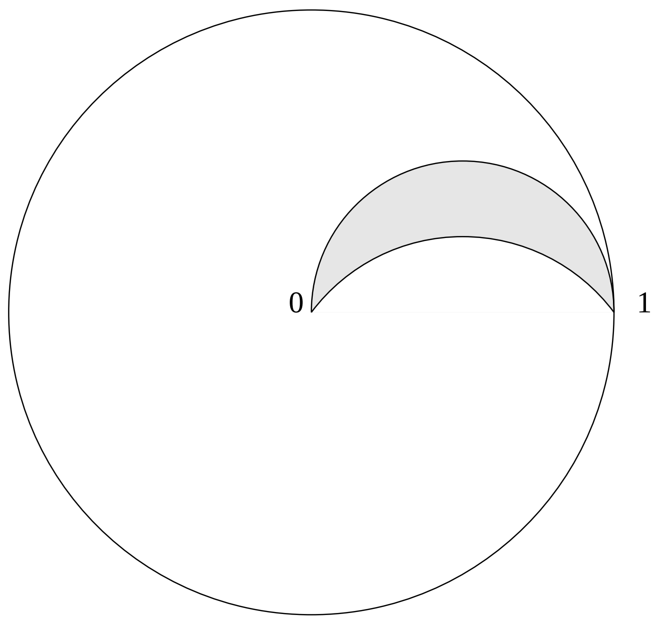

Figure 3: Crescent shape region ($0 < \beta < 1/2$)

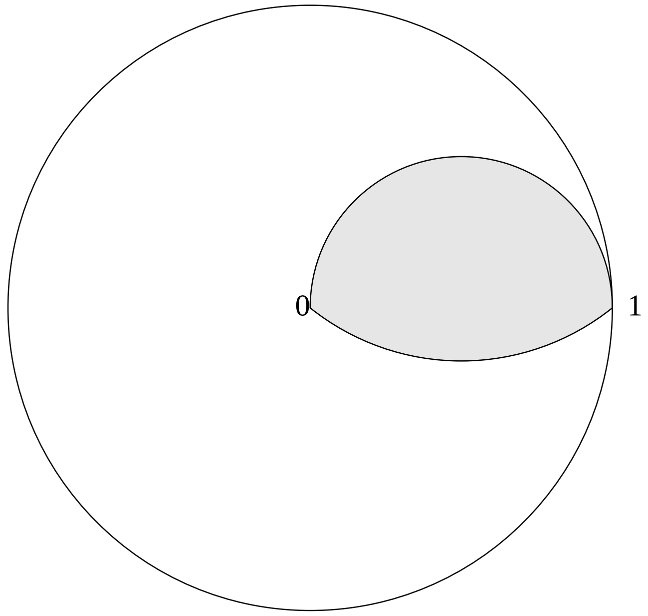

Figure 4: Lens shape region $(1/2 < \beta < 1)$

Theorem 2.2.3. *Suppose that C_{ϕ_β} is the composition operator induced by ϕ_β. Then*

C_{ϕ_β} does not belong to the Schatten ideal $\mathcal{S}_{\frac{2\beta}{(1-\beta)}(H^2)}$.

$C_{\phi_\beta} \in \mathcal{S}_p(H^2)$ for all $p > \frac{2\beta}{(1-\beta)}$.

Applying Corollary 2.2.1 and Theorem 2.2.3 we obtain

Theorem 2.2.4.

- *a) $C_{\phi_\beta} \notin \mathcal{S}_{\frac{2\beta}{(1-\beta)(\alpha+2)}}(A^2_\alpha)$.*
- *b) $C_{\phi_\beta} \in \mathcal{S}_p(A^2_\alpha)$ for all $p > \frac{2\beta}{(1-\beta)(\alpha+2)}$.*

Remark 2.2.2.

Note that here β characterizes the geometry of $\phi_\beta(D)$.It is also interesting to note that the geometry for $\beta = 1/2$ (the half disk for which the associated composition operators do not belong to $\mathcal{S}_2(H^2)$ and $\mathcal{S}_1(A^2_\alpha)$ respectively) is a borderline, in the sense that if the half disk is shrunk slightly so that it is bounded by circular arcs meeting at an angle less than $\pi/2$, then the associated composition operator is Hilbert-Schmidt($\mathcal{S}_2(H^2)$) for the Hardy space H^2 (

Theorem 2.2.3) and is in $\mathcal{S}_1(A^2_{\frac{\alpha}{\alpha+2}})$ (the trace class for $\alpha = 0$) and (theorem 2.3.4). See [2] for the Hardy space case.

In the following , addressing Lotto's second question, We present Zhu's construction of compact composition operators that is not in any of the Schatten-p ideals (see Section 5 ([1]) and consequently Corollary 2.2.1 is used to extend Zhu's result to the Bergman space A^2_α.

The construction is read as follows:

Let $\theta_n = \frac{\pi}{n+1}$, $z_n = e^{i\theta_n}$, $r_n = \left(\frac{1}{2}\right)\sin\theta_n$ and $c_n = (1 - r_n)z_n$ where $n = 1, 2, ...$ Define Ω_n to be the region bounded by the semi-circle

$$\{z : Im(z) \geq 0 \text{ and } |z - |c_n|| = r_n\}$$

and a circular arc that is inside D joining $1 - 2r_n$ to 1 forming an angle of $\frac{n+1}{(n+3)}\pi$ at 1.

Let

$$\Omega'_n = \{ze^{i\theta_n} : z \in \Omega_n\}$$

and

$$\Omega = \cup_{n=1}^{\infty}\Omega'_n \tag{2.1}$$

Zhu's result is summarized in the following Theorem (see [1]):

Theorem 2.2.5.

Suppose Ω is as defined in (2.3), then

- *Ω is a simply connected domain contained in the upper-half of D.*
- *Any Riemann map ϕ that maps D onto Ω induces a compact composition operator C_ϕ that does not belong to any of the Schatten-p ideals $\mathcal{S}_p(H^2)$, $p > 0$.*

The outline of Zhu's ([1]) proof goes as follows:

First he showed that Ω is simply connected by estimating the distance between the centers c_{n-1} and c_n of Ω'_{n-1} and $\Omega'_n(n \geq 2)$ as $O(\frac{1}{n^2})$. On the other hand the radius r_n of Ω'_n is $\frac{1}{2}\sin(\frac{\pi}{n+1}) \geq \frac{1}{n+1}$ hence showing that Ω'_{n-1} and Ω'_n overlap and hence Ω is simply connected. Since $Im(c_n) = (1 - r_n)Im(z_n) = (1 - r_n)\sin(\frac{\pi}{n+1}) \geq \frac{1}{2}\sin(\frac{\pi}{n+1}) = r_n$. Thus, Ω'_n lies in the upper half of D.Consequently, Ω is in the upper half of D. By the construction of Ω, we know that Ω touches the boundary of D at $z_n, n = 1, 2, 3, ...$ and at 1.One can see that ϕ is not conformal at z_n and hence has no angular derivative at z_n. Note that Ω is in the upper half of D and $z_n \leftarrow 1$ as $n \leftarrow \infty$, thus ϕ is not conformal at 1 either.By the angular derivative criterion for compactness (the analogue of Theorem 2.1.1 for the Hardy spaces and for ϕ univalent see [9]), We know that C_ϕ is compact. Let ϕ_n be a Riemann map that maps D onto Ω_n' and c_{ϕ_n} be the induced composition operator. Let G_β be the region defined in Theorem 2.2.3 and ψ_β be a Riemann map from D onto G_α. By theorem 2.2.3, We know that the composition operator induced by $\psi_{\frac{n+1}{n+3}}$ does not belong to the Schatten ideal $\mathcal{S}_{n+1}$. Let

$$\eta_n(z) = \frac{z + (1 - 2r_n)}{1 + (1 - 2r_n)z} e^{i\theta_n}, z \in D.$$

Then η_n is an automorphism of D with $\eta_n(G_{\frac{n+1}{n+3}}) = \Omega_n'$(see figure..). Thus , by lemma 2.1.2 $C_{\phi_n} \notin \mathcal{S}_{n+1}(H^2)$. Moreover, since $\Omega_n' \subset \Omega$ for any positive integer n, $C_\phi \notin \mathcal{S}_{n+1}(H^2)$ by lemma 2.1.1 for any n. Since $\mathcal{S}_p(H^2) \subset \mathcal{S}_q(H^2)$ for $p < q$, it follows that C_ϕ does not belong to any Schatten classes. This completes the proof of Theorem 2.2.4. Theorem 2.2.4 easily extends to the Bergman space A^2_α setting by applying corollary 2.2.1. Thus Theorem 2.2.4 holds when H^2 is replaced by A^2_α with no modification of the region Ω.

In the next chapter we shall be using these same class of examples ([2]) to explore β–boundedness/compactness of composition operators on the Hardy spaces and their relationships to the Hilbert–Schmidt class.

3.0 BETA – BOUNDED AND SCHATTEN CLASS COMPOSITION OPERATORS ON THE HARDY AND BERGMAN SPACES

In this chapter we investigate β–boundedness on the the class of examples (see B.Lotto[2]) of composition operators C_ϕ whose symbols ϕ are Riemann maps from the unit disk D onto the semi–disk with center $(\frac{1}{2}, 0)$, radius $\frac{1}{2}$ and onto a "crescent" shaped regions based on this semi-disk (see also [2]). We use the R.Riedel [8] characterization of β–boundedness and compactness on H^2 to determine a range of values of $\beta \in \mathbb{R}$ for which C_ϕ is β–bounded/compact. A similar result also extends to composition operators acting on the weighted Bergman spaces A^2_α ($\alpha \geq -1$) based on W.Smith ([5]) characterization of β–boundedness and compactness on these spaces. We also prove necessary condition for C_ϕ to be in the Schatten–p classes in terms of β–boundedness. In the first section we give some background material.

3.1 BACKGROUND

The problem of characterizing composition operators $C_\phi : H^p \to H^q$, $(0 < p \leq q)$, has been considered by several authors, beginning with H. Hunziker and H. Jarchow [6] (see also [7].) In this paper they observed that for $\beta \geq 1$, $C_\phi : H^p \to H^{\beta p}$ is bounded for some $p > 0$ if and only if it is bounded (i.e β–bounded) for all $p > 0$. Then they characterized those ϕ that induce such composition operators as those for which m_ϕ satisfies a β-Carlson measure condition (see [6]),

$$\sup\left\{\frac{m_\phi(D(\eta,\delta))}{(\delta)^\beta}, \delta > 0, |\eta| = 1\right\} < \infty$$

if and only if C_ϕ is β–bounded and

$$\lim_{\delta\to 0}\sup_{|\eta|=1}\left\{\frac{m_\phi(D(\eta,\delta))}{(\delta)^\beta}\right\} = 0$$

if and only if C_ϕ is β–compact.

where

$$D(\eta, \delta) = \{z \in D : |z - \eta| < \delta\},$$

and m_ϕ is a measure canonically associated with the symbol ϕ given by

$$m_\phi(A) = m((\phi^*)^{-1}(A))$$

for all Borel sets $A \subseteq \overline{D}$, m is the Lebesgue measure on ∂D and $\phi^* : \partial D \to \overline{D}$ is the radial limit function. Then ,R.Riedl ([8]) , applying this result, proved the following result in his dissertation. We will be using this theorem and it's extension to the Bergman spaces by W.Smith [5] to derive our main results.

Theorem 3.1.1.

Let $\beta \geq 1, 0 < p < \infty$ and suppose ϕ is an analytic self-map of D. Then $C_\phi : H^p \to H^{\beta p}$ is bounded if and only if

$$N_\phi(\omega) = O\left(\left[\log\left(\frac{1}{|\omega|}\right)\right]^\beta\right), \quad (as|\omega| \to 1)$$

compact if and only if

$$N_\phi(\omega) = o\left(\left[\log\left(\frac{1}{|\omega|}\right)\right]^\beta\right), \quad (as|\omega| \to 1)$$

where N_ϕ is the classical Nevanlinna counting function for ϕ.

We also need the following generalization due W.Smith ([5])

Theorem 3.1.2. *[Corollary 4.4,Theorem 5.1 [5]]*

Let $0 < p < \infty, \eta \geq 1$, and let ϕ be an analytic self–map of D. Let $\alpha \geq -1, \beta \geq -1$. Then $C_\phi : A^p_\alpha \to A^{\eta p}_\beta$ is bounded if and only if

$$N_{\phi,\beta+2}(\omega) = O\left(\left[\log\left(\frac{1}{|\omega|}\right)\right]^{(2+\alpha)\eta}\right), \quad as|\omega| \to 1,$$

compact if and only if

$$N_{\phi,\beta+2}(\omega) = o\left(\left[\log\left(\frac{1}{|\omega|}\right)\right]^{(2+\alpha)\eta}\right), \quad as|\omega| \to 1$$

where $N_{\phi,\beta}$ is the generalized Nevanlinna counting function for ϕ.

Remark 3.1.1.

Note here that the case $\alpha = -1$ and $\beta = -1$ represent the limiting case ($A^p_{-1} = H^p$) thus Theorem 3.1.2 reduces to Theorem 3.1.1.

In this chapter we are interested for the case where ϕ is univalent. In this case, the condition

$$N_\phi(\omega) = O\left(\left[\log\left(\frac{1}{|\omega|}\right)\right]^\beta\right), \quad \text{as}|\omega| \to 1$$

says, for $|\omega| \approx 1$, there exists such that

$$1 - |\phi^{-1}(\omega)| \leq M(1 - |\omega|)^\beta.$$

Setting $\phi^{-1}(\omega) = z$, this reduces to saying that for $|\omega| \approx \delta$, $\delta \to 0$, there exists $M > 0$ such that

$$\phi^{-1}(\{\omega : 1 - \delta < |\omega| < 1\}) \subseteq \{z : 1 - M\delta^\beta < |z| < 1\}$$

Thus the operator $C_\phi : H^p \to H^{\beta p}$ is β–bounded, if points δ–close to the boundary of D are taken on by points $M\delta^\beta$–close to the boundary of D. A simple example illustrating this condition could be $\phi(z) = \frac{z}{2}$ for which the above condition obviously holds for all $\beta \geq 1$, which means C_ϕ is β–bounded for all $\beta \geq 1$. (Note here that this also directly follows from Theorem 1.1.) Indeed this condition holds for any univalent self maps ϕ with $||\phi||_\infty < 1$. From the other extreme, for $\phi(z) = z$, the condition is satisfied only for $\beta = 1$ and, consequently, C_ϕ is 1–bounded (i.e bounded) and not β–bounded for $\beta > 1$.

On the other hand, the corresponding little-oh condition

$$N_\phi(\omega) = o\left(\left[\log\left(\frac{1}{|\omega|}\right)\right]^\beta\right), \quad |\omega| \to 1$$

says given $|\omega| \approx \delta$, $\delta \to 0$, we have for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\phi^{-1}(\{\omega : 1 - \delta < |\omega| < 1\}) \subseteq \{z : 1 - \epsilon\delta_\epsilon^\beta < |z| < 1\}$$

which, geometrically, means that points δ–close to the boundary of D are taken on by pre-image points $\epsilon\delta_\epsilon^\beta$ –close to the boundary of D for $\epsilon > 0$ arbitrary. Examining the above simple examples, we can easily check that for $\phi(z) = \frac{z}{2}$ (in fact, this is true for any univalent self map ϕ with $||\phi||_\infty < 1$), C_ϕ is β–compact for $\beta \geq 1$, where as $\phi(z) = z$ induces a non

β–compact operator for all $\beta \geq 1$. W. Smith[5], has constructed a necessary condition for a Riemann map $\phi : D \to G($ G is simply connected), that induces a β–bounded composition operator $C^p : A^p \to H^{\beta p}$ for all $\beta \geq 1$.The condition is stated in the next

Theorem 3.1.3.

Let $G \subseteq D$ be a simply connected domain such that

$$\lim_{|\omega| \to 1} \frac{\delta_G(\omega)}{1 - |\omega|} = 0$$

where $\delta_G(\omega)$ is the distance from ω to $\mathbb{C} \setminus G$ for $\omega \in G$,and $\delta_G(\omega) = 0$ if $\omega \in \mathbb{C} \setminus G$.
If $\phi : D \to G$ is a Riemann map, then $C_\phi : A^p \to H^{\beta p}$ is bounded for all $\beta \geq 1$

Theorem 1.3 certainly covers a large class of examples including the family of composition operators $C_\phi : H^p \to H^{\beta p}$ for which $||\phi||_\infty < 1$. However, a simple geometric consideration shows that the hypothesis of Theorem 3.1.3 is too strong to obtain polygonal β–boundedness and compactness results.(see 6.7 Theorem in W.Smith [5]– for polygonal β–boundedness/compactness result). We shall see in Section 3.2 that Theorem 3.1.3 is not also applicable to our class of examples.

In the next section, we will primarily be dealing with the [2] class of geometric examples discussed in chapter 2 and it's derived class "induced" by "crescent–shaped" regions (to be described later in the next section) to investigate β–boundedness and compactness and receive further insight on the connection between β –boundedness and the Hilbert–Schmidt classes (on the subsequent section.)

3.2 BETA–BOUNDEDNESS ON THE HARDY AND BERGMAN SPACES

Let $1 \leq \beta < \infty$ and consider

$C_\phi : H^2 \to H^{2\beta}$

We begin by recalling that C_ϕ is β–bounded (resp. compact) if and only if $C_\phi : H^2 \to H^{2\beta}$ is bounded (resp. compact)

Let us re–consider the Riemann $\phi : D \to G$ (onto) which fixes 1.(described in section 1), where G is the semi-disk

$$G = \{z : Im(z) > 0 \quad and \quad |z - \frac{1}{2}| < \frac{1}{2}\}$$

Firstly, we observe that, for ϕ univalent and for $|\omega| \to 1$(i.e $\omega \to 1$), we have

$$N_\phi(\omega) \approx 1 - |\phi^{-1}(\omega)|$$

and

$$\log\left(\frac{1}{\omega}\right) \approx 1 - |\omega|$$

A simple geometric consideration shows that $\delta_G(\omega) \approx (1 - |\omega|)$, where $\delta_G(\omega)$ is as defined in Theorem 3.1.3. Thus, we have $\delta_G(\omega) \approx 1 - |\omega|$ showing that Theorem 3.1.3 is not applicable in this case.

In the following we investigate β–boundedness/Compactness for an extended class of composition operators induced by the modified "crescent" shaped regions which are already considered in chapter1. For simplicity, we restrict to the H^2 case.

For $0 < \alpha < 1$, let G_α represent the region bounded by the semi–circle

$$\{z : Im(z) \geq 0 \quad and \quad |z - 1/2| = 2\}$$

and a circular arc that is inside of D joining 0 to 1 (see Figure(4.1) in [1]). These two arcs form an angle $\alpha\pi$ at 0 and 1. Let $\phi_\alpha : D \to G_\alpha$ (onto) be a Riemann map with $\phi_\alpha(0) = 1$. To derive our results, we apply a sequence of conformal maps starting with $\tau(z) = i(1/z - 1)$ which takes G_α onto a sector A_α, where the two sides of the sector forming an angle of $\alpha\pi$ with initial side the +ve real axis. (see Figure(4.2)in [1]) and subsequently $z^{1/\alpha}$ takes A_α to the upper half plane H^+ and finally the map $\eta(\sigma) = \frac{1+i\sigma}{1-i\sigma}$ takes H^+ back onto D. Thus we have

$$\phi_\alpha^{-1} = \eta \circ \tau^{1/\alpha}.$$

Moreover, writing $\tau(\omega) = \rho e^{i\theta}$ we estimate

$$1 - |\omega|^2 \approx \rho^2 + 2\rho \sin\theta, \quad as\ \rho \to 0.$$

Indeed,

$$\begin{aligned}
1-|\omega|^2 &= 1-\left(\left|\frac{1}{1-i\tau}\right|\right)^2 \\
&= 1-\left(\left|\frac{1}{1-i\rho e^{i\theta}}\right|\right)^2 \\
&= 1-\frac{1}{1+2\rho\sin(\theta)+\rho^2} \\
&= \frac{1+2\rho\sin(\theta)+\rho^2-1}{1+2\rho\sin(\theta)+\rho^2} \\
&\approx \rho^2+2\rho\sin(\theta) \quad \textit{as} \quad \rho\to 0.
\end{aligned}$$

Similarly, writing $\eta(\sigma)=re^{i\theta}$, it's not hard to get the estimate

$$1-|\eta|\approx Im(\sigma) \quad as\ \sigma\to 0 \quad (\ or\ \eta\to 1).$$

Indeed,

$$\sigma=\frac{i(1-re^{i\theta})}{1+re^{i\theta}}=\frac{i(1+2ri\sin(\theta)-r^2)}{1+2r\cos(\theta)+r^2}$$

from which we conclude that

$$Im(\sigma)\approx 1-r \quad as \quad r\to 1$$

On the other hand,

$$1-|\eta|^2=1-|re^{i\theta}|^2=1-r^2\approx 1-r\approx Im(\sigma) \quad as \quad \sigma\to 0 \quad (\ or\ \eta\to 1, r\to 1)$$

But then,

$$1-|\phi^{-1}(\omega)|=1-|\eta|\approx Im\sigma\approx \rho^{1/\alpha}\sin(\theta/\alpha) \quad as\ \rho\to 0 \quad (\ or\ \eta\to 1)$$

(where the last estimates comes from expressing $\sigma=\tau^{1/\alpha}=(\rho e^{i\theta})^{1/\alpha}$)

Now we have all the ingredients to establish our next result. Indeed, we write

$$\begin{aligned}
\frac{1-|\phi^{-1}(\omega)|}{(1-|\omega|)^\beta} &\approx \frac{Im\sigma}{(\rho^2+2\rho\sin\theta)^\beta} \\
&\approx \frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{(\rho^2+2\rho\sin\theta)^\beta} \\
&\leq \frac{\rho^{1/\alpha+1}}{\rho^{2\beta}} \quad as \quad \rho\to 0
\end{aligned}$$

where the last estimate is justified considering two cases:

Note that $0 < \theta < \infty$ and since $\sin(\theta)$ is symmetric with the line $y = \pi/2$ for the following arguments we may assume that $0 < \theta < \pi/2$

Case a: θ is "large":

In this case we have $LHS \approx \frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{(\rho^2+2\rho\sin\theta)^\beta} \leq \frac{\rho^{1/\alpha}}{\rho^\beta} \leq \frac{\rho^{1/\alpha}}{\rho^\beta}\frac{\rho}{\rho^\beta} = \frac{\rho^{1/\alpha+1}}{\rho^{2\beta}}$.

Case b: θ is "small":

In this case we have $\sin(\theta) \approx \theta$.

If $\theta \leq \rho$ then $\rho\theta \leq \rho^2$, thus we obtain

$$\begin{aligned}
LHS &\approx \frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{(\rho^2+2\rho\sin\theta)^\beta} \\
&\leq \frac{\rho^{1/\alpha}\rho}{((\rho)^2)^\beta} \\
&\leq \frac{\rho^{1/\alpha+1}}{\rho^{2\beta}}
\end{aligned}$$

If $\rho \leq \theta$ then $\rho^2 \leq \rho\theta$, thus we get

$$\begin{aligned}
LHS &\approx \frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{(\rho^2+2\rho\sin\theta)^\beta} \\
&\leq \frac{\rho^{1/\alpha}\theta}{(\rho\theta)^\beta} \\
&= \frac{\rho^{1/\alpha}}{\rho^\beta\theta^{\beta-1}} \\
&\leq \frac{\rho^{1/\alpha}}{\rho^\beta\rho^{\beta-1}} \\
&= \frac{\rho^{1/\alpha+1}}{\rho^{2\beta}}
\end{aligned}$$

and it is clear that the estimate is sharp asymptotically and optimality is obtained along the line $\theta = \rho$.

Thus we write $LHS \approx \rho^{1/\rho+1-2\beta}$ as $\rho \to 0$

Applying Theorem 3.1.1, we read

Theorem 3.2.1. *For $0 < \alpha < 1$, ϕ_α as defined above,*

$$C_{\phi_\alpha} : H^2 \to H^{2\beta}$$

is

a) *bounded if and only if $\beta \leq \frac{1}{2\alpha} + \frac{1}{2}$ and*

b) *compact if and only if $\beta < \frac{1}{2\alpha} + \frac{1}{2}$*

Remark 3.2.1.

1.) It's interesting to note that Theorem 3.2.1 gives an affirmative answer to the open question posed by Hunziker and Jarchow (see 5.2 in [6]) which asks: Find an example of a β –bounded $(1 < \beta < \infty)$ composition operator C_ϕ which fails to be β–compact. Theorem 3.2.1 gives an example of a β–bounded composition operator C_ϕ which fails to be β–compact($\alpha = 1/2$ i.e the half disk,$\beta = 3/2$ in Theorem 3.2.1). Thus,

$$\begin{aligned}
\beta(\phi) &= \sup\{\beta \geq 1 : C_\phi(H^1) \subseteq H^\beta\} \\
&= \sup\{\beta \geq 1 : C_\phi(H^1) \subseteq H^{p\beta}\} \\
&= \beta = 3/2 \quad \textit{(in our case,)}
\end{aligned}$$

where $0 < p < \infty$. Note here that, $\beta(\phi)$ tells how much a composition operator C_ϕ improves integrability properties of functions to which it is applied.

2.) What about the case $\beta = \infty$? i.e Is $C_{\phi_\alpha} : H^2 \to H^\infty$ (for ϕ_α as in Theorem 2.1) bounded?compact? Note here that Theorem 2.1 cannot be applied here. But then, since $||\phi_\alpha||_\infty \not< 1$ (since $\phi_\alpha(1) = 1$), applying Nzar and Jaoua[10] characterization we conclude that $C_{\phi_\alpha}(H^2) \nsubseteq H^\infty$, which means $C_{\phi_\alpha} : H^2 \to H^\infty$ is not bounded (and hence not compact.).

The above argument can easily be reproduced Theorem 3.2.1 on the weighted Bergman space setting. Indeed, for $0 < \gamma < 1$, $1 \leq \beta < \infty$ where ϕ_γ as in Theorem 2.3, $\alpha \geq -1, \eta \geq -1$, we consider $C_{\phi_\gamma} : A^2_\alpha \to A^{2\beta}_\eta$
Reproducing the same chains of estimates leading Theorem 3.2.1 we obtain

$$\frac{N_{\phi_\gamma.\eta+2}(\omega)}{(\log(1/|\omega|))^{(\alpha+2)\beta}} \leq \frac{\rho^{1/\gamma(\eta+2)+1}}{\rho^{2(\alpha+2)\beta}}$$

and the estimate is asymptotically optimal along the line $\rho = \theta$.Consequently. we obtain the analogue of Theorem 3.2.1 stated as

Theorem 3.2.2.
Let $0 < \gamma < 1$, $1 \leq \beta < \infty$ *where* ϕ_γ *as in Theorem 2.3,* $\alpha \geq -1, \eta \geq -1$.
$C_{\phi_\gamma} : A^2_\alpha \to A^{2\beta}_\eta$ *is*
bounded if and only if $\beta \leq \frac{1/\gamma(\eta+2)+1}{2(\alpha+2)}$ *and*
compact if and only if $\beta < \frac{1/\gamma(\eta+2)+1}{2(\alpha+2)}$

Remark 3.2.2.

1.) *Note that the case* $\alpha = \beta = 0$ *yields the condition* $\beta \leq \frac{1}{2\gamma} + 1/4$ *(res.* $\beta < \frac{1}{2\gamma} + 1/4$*) for beta–boundedness (res.* β*–compactness) for the classical Bergman spaces and we recover Theorem 3.2.1 for* $\alpha = -1, \beta = -1$. *In particular for the half disk geometry (i.e* $\gamma = 1/2$ *) we obtain an example of a* $5/4$*–bounded composition operator which is not* $5/4$*–compact on the standard Bergman space.*

2.) *It's also interesting to compare Theorem 3.2.2 with the result on polygonal compactness Theorem (6.7) in W.Smith [5]. which asserts that composition operators induced by polygonal self–maps are both* β *-bounded and* β*–compact, for all* $1 \leq \beta < \infty$.

3.) *The case* $\beta = \infty$ *is not included in Theorem 3.2.2 and it is also interesting to ask if the analogous result also holds as in Remark 3.2.1(2)*

In the next section we investigate the connection between β–boundedness and Hilbert–schmidt operators on H^2

3.3 BETA–BOUNDEDNESS VS. HILBERT–SCHMIDT/SCHATTEN CLASS OPERATORS

Based on the observation of the results of Section 2 and the fact that the C_ϕ's (ϕ is the Riemann map taking the unit disk$D$ onto the semi-disk described in Section 2) are *not Hilbert-Schmidt*(see [2]), it is natural to ask the following:

Given $0 < p < \infty$ and ϕ a univalent self–maps of the Unit disk D which induces a compact composition operator C_ϕ on H^2, for which values of $\beta \geq 1$, the statement

C_ϕ is β–bounded implies $C_\phi \in \mathcal{S}_p(H^2)$ holds? Under what extra assumptions on ϕ?

We investigate this on the the general Schatten–p ideals ($0 < p < \infty$), for this, once more, we need the Luecking-Zhu's Theorem ([11]) to characterize membership in $\mathcal{S}_p(H^2)$ which reads:

$$C_\phi \in \mathcal{S}_p(H^2) \text{ if and only if} N_\phi(z)\left(\log\left(\frac{1}{|z|}\right)\right) \in \mathcal{L}^{\frac{p}{2}}(d\lambda),$$

where $N_\phi(z)$ is the Nevanlinna counting function and $d\lambda(z) = (1 - |z|^2)^{-2}dxdy$ the Möbius invariant measure on D.

For ϕ univalent self map of D into itself,

$$N_\phi(z) = \left(\log(\frac{1}{|\phi^{-1}(z)|}\right) \approx (1 - |\phi^{-1}(z)|), \quad \text{for} \quad |\phi^{-1}(z)| \to 1.$$

Thus, we have

$$C_\phi \in \mathcal{S}_p(H^2) \leftrightarrow \int\int_{\phi(D)} \left(\frac{1 - |\phi^{-1}(\omega)|}{1 - |\omega|}\right)^{\frac{p}{2}} (1 - |w|^2)^{-2} dA(\omega) < \infty$$

which can be re-written as

$$C_\phi \in \mathcal{S}_p(H^2) \leftrightarrow \int\int_{\phi(D)} \left(\frac{1 - |\phi^{-1}(\omega)|}{(1 - |\omega|)^{1+\frac{4}{p}}}\right)^{\frac{p}{2}} dA(\omega) < \infty$$

which certainly holds if C_ϕ is β–bounded for $\beta \leq 1 + \frac{4}{p}$, in particular if $\beta = 1 + \frac{4}{p}$.

Theorem 3.3.1.

Let ϕ be a univalent self–map of D, with $\phi(1) = 1$, which induces a compact composition operator C_ϕ on H^2, and $\phi(D)$ is contained in the stolz angle at the boundary point $1 \in \partial D$, then

$C_\phi : H^2 \to H^{2\beta}$ *bounded for* $\beta \leq 1 + \frac{4}{p}$ *implies* $C_\phi \in \mathcal{S}_p(H^2(D))$.

The above (Theorem 3.3.1) result easily extends to the weighted Bergman spaces using the corresponding Luecking–Zhu's Characterization(See Lemma 2.2.1) which reads as

Theorem 3.3.2.

Let $\alpha > -1$

Let ϕ be a univalent self–map of D, with $\phi(1) = 1$ which induces a compact composition operator C_ϕ on A^2_α, and $\phi(D)$ is contained in the stolz angle at the boundary point $1 \in \partial D$, then

$C_\phi : A^2_\alpha \to A^{2\beta}_\alpha$ *bounded for* $\beta \leq (\alpha + 2) + \frac{4}{p}$ *implies* $C_\phi \in \mathcal{S}_p(A^2_\alpha(D))$.

In the following we use the Hilbert–Schmidt condition to derive a weaker criterion for membership to the Hilbert Schmidt class on H^2

We start with the following fact:

$$\frac{1}{(1-z)^t} \in H^2(D) \qquad \textit{for } 0 < t < 1/2$$

Setting $\beta = 1/t$, we have $0 < t < 1/2 \iff \beta > 2$

Assuming: $C_\phi : H^2 \to H^{2\beta}$ bounded for $\beta \leq 2$, we obtain

$$1/(1 - \phi(z))^t \in H^{2\beta}(D)$$

which implies

$$1/(1 - \phi(z))^{2t\beta} \in H^1(D)$$

Hence, putting $\beta = \frac{1}{2}$, we get

$$1/(1 - \phi(z))^2 \in H^1(D)$$

At this point we need to assume that $\phi(1) = 1$, with this, we estimate

$$\begin{aligned} 1 - |\phi(z)|^2 &\approx 1 - |\phi(z)| \\ &\approx |1 - \phi(z)|, \qquad \textit{for } z \to 1 \\ &\geq |1 - \phi(z)|^2 \end{aligned}$$

where for the middle estimate we require that $\phi(D)$ has to be contained in the stolz domain at the point 1. Now applying the well–known Hilbert–Schmidt criterion (see [12]), we obtain

$$\int_{-\pi}^{\pi} \frac{1}{(1 - |\phi(e^{i\theta})|^2)} d\theta \leq \int_{-\pi}^{\pi} \frac{1}{|1 - \phi(e^{i\theta})|^2} d\theta < \infty.$$

Consequently, we get

Theorem 3.3.3.

let ϕ be a univalent self–map of D, with $\phi(1) = 1$ which induces a compact composition operator C_ϕ on H^2, and $\phi(D)$ is contained in the stolz angle at the boundary point $1 \in \partial D$, then

$C_\phi : H^2 \to H^{2\beta}$ *bounded for* $\beta \leq 2 \Longrightarrow C_\phi \in \mathcal{S}_2(H^2(D))$.

In the next chapter, we investigate the existence of compact composition operators on the Bergmann spaces on multiply connected domains based on the recent result of [12] and [5] on simply connected domains.

4.0 CHARACTERIZATION OF COMPACT COMPOSITION OPERATORS ON THE HARDY–SMIRNOV SPACES.

In this chapter, we characterize boundedness and compactness of composition operators on the 'Hardy–Simirnov' spaces over simply connected domains.

4.1 PRELIMINARIES

For G simply connected domain properly contained in (C), we used the recent result of Contreras, Manuel D.Hernandez-Diaz, and Alfredo on Weighted composition operators between different Hardy spaces [17] and the recent result of J. H. Shapiro and W. Smith [12] to give a β–Carlson characterization of boundedness and compactness of composition operators on the–Simirnov spaces $E^p(G)$ over simply–connected domains.
Let η be a Riemann map that takes the open unit disk D univalently onto G. For $0 < p < \infty$ we define the 'Hardy–Simirnov' Spaces: $E^p(G)$ to be the collection of functions F holomorphic on G such that

$$\sup_{0<r<1} \int_{\eta(z:|z|=r)} |f(w)|^p |dw| < \infty.$$

When G is a Jordan domain with rectifiable boundary, $E^p(G)$ coincides with $H^p(G)$ up to an isometric isomorphism.([14]) In particular, $E^p(D) = H^p$.

However, if the region G is an interior of a Jordan curve which is analytic except at one point, where it has a corner with interior angle α, then $E^p(G)$ is properly contained in G if $0 < \alpha < \pi$ while $H^p(G)$ properly contained in $E^p(G)$ if $\pi < \alpha < 2\pi$.([14])

Let D be the unit disk, $b \in \partial(G)$, and define

$$S(b,r) = \{z \in D : |z-b| \leq r\} \subset D.$$

For ϕ , ψ Holomorphic maps on D with $\phi(D) \subset D$ $0 < p < \infty$

A Weighted composition operator $W_{\phi,\psi} : H^p \leftarrow H^p$ defined as

$$W_{\phi,\psi}(f) = \psi(f \circ \phi), \quad f \in H^p$$

Naturally, composition operators are special cases when $f = 1$. We need the following result of [17] on weighted composition operators on the Hardy spaces. Let $W_{\phi,\psi} : H^p \to H^p$ denote a weighted composition operator on the Hardy space H^p defined by

$$W_{\phi,\psi}(f) = \psi(f \circ \phi)$$

, here ϕ and ψ denotes holomorphic maps with $\phi(D) \subset D$.

Theorem 4.1.1. *(* **A. Tadesse)**

Let $1 \leq p$.If $\psi \in H^p$, then

a) $W_{\phi,\psi} : H^p \to H^p$ is bounded if and only if $\exists M > 0$ such that

$$\int_{\varphi^{-1}(S(b,r))\cap\partial(D)} |\psi(z)|^p dm \leq Mr$$

for all $b \in \partial(D)$, $0 < r < 1$

b) $W_{\phi,\psi} : H^p \to H^p$ is compact if and only if

$$\lim_{r\to 0} \sup_{b\in\partial D} \frac{\int_{\varphi^{-1}(S(b,r))\cap\partial(D)} |\psi(z)|^p dm}{r} = 0$$

where m denote Lebesgue arc–length measure on $\partial(D)$; normalized to have total mass one.

4.2 MAIN RESULTS

In the following, we consider composition operators $C_\phi : E^p(G) \leftarrow E^p(G)$ for $0 < p < \infty$. Using change of variable formula (see e.g [14],Corollary, page 169), it can be verified that

$$f \in E^p(G) \leftrightarrow f(\eta(\omega))(\eta'(\omega))^{1/p} \in H^p$$

Associated with C_ϕ we define a weighted composition operator:

$$W_{\varphi,p} : H^p \to H^p$$

defined by

$$W_{\varphi,p} = V_p \circ C_\phi \circ V_p^{-1}$$

where $V_p f = (\eta')^{1/p}(f \circ \eta)$, $f \in Hol(G)$. it can be easily verified that

$$(W_{\varphi,p})(f)(z) = (Q_\varphi(z))^{1/p}(f(\varphi(z))), z \in G$$

where $Q_\varphi(z) = \frac{\eta'(z)}{\eta'(\varphi(z))}, z \in \psi(G)$

The following facts are extracted from the recent paper of J.H.Shapiro and W. smith (see [12])

Remark 4.2.1. *a) V_p defines isometric similarity between $C_\phi : E^p(G) \to E^p(G)$ and $W_{\varphi,p} : H^p \to H^p$. Thus, the two are unitarily equivalent.*

b) C_ϕ bounded/compact if and only if $W_{\varphi,p}$ is bounded/compact.(a direct consequence of b))

c) Boundedness and compactness of C_ϕ is independent of p. (i.e if these properties hold for some p, $0 < p < \infty$, it holds for all p)

d) Both η' and $\frac{1}{\eta'}$ are bounded on D if and only if every composition operator on $E^p(G)$ is bounded.

e) $E^p(G)$ supports compact composition operators if and only if $\eta' \in H^1$ which can be rephrased as $\partial(G)$ having finite dimensional Hausdorff measure.(see [15]). Theorem 10.11, pp. 320-321) In the case G is a Jordan domain, this condition is in turn equivalent to saying G is rectifiable. (see also [15], Lemma 10.7, page 319)

Thus, given $C_\phi : E^p(G) \to E^p(G)$ where G is simply connected, it can be viewed as a weighted composition operator on H^p with weight $(\frac{\eta'(z)}{\eta'(\varphi(z))})^{1/p}$ (s ee [12]) where η is the Riemann map from D onto G. Applying Theorem 4.6.1 using weight $\psi(z) = \{\frac{\eta'(z)}{\eta'(\varphi(z))}\}^{1/p}$ we read the following result.

Theorem 4.2.1.

Let G be simply connected.

$$\eta : D \to G$$

be the Riemann map

Let $C_\phi : E^p(G) \to E^p(G)$ and define $\varphi = \eta^{-1} \circ \phi \circ \eta : D \to D$. Let $Q_\varphi(z) = \frac{\eta'(z)}{\eta'(\varphi(z))}$, for all $z \in \psi(G)$

Suppose that $Q_\varphi(z)^{\frac{1}{p}} \in H^p$.

Then, the following statements are equivalent.

a) $C_\phi : E^p(G) \to E^p(G)$ is bounded if and only if $\exists M > 0$ such that

$$\int_{\varphi^{-1}(S(b,r))\cap\partial(D)} |Q_\varphi(z)|dm \leq Mr$$

for all $b \in \partial(D)$, $0 < r < 1$

b) $C_\phi : E^p(G) \to E^p(G)$ is compact if and only if

$$\lim_{r\to 0} \sup_{b\in\partial D} \frac{\int_{\varphi^{-1}(S(b,r))\cap\partial(D)} |Q_\varphi(z)|dm}{r} = 0$$

Remark 4.2.2.

Note that for the standard composition operators on the unit disk D ($G = D$ and hence $\eta' = 1$) Theorem 4.1.1 a) gives boundedness of composition operators on the Hardy Spaces of the unit disk for free and Theorem 4.1.1 b) reduces to the Carlson characterization of compactness as expected.(see [17])

Furthermore, since not all Hardy–Smirnov spaces support compact composition operators [12], Theorem 4.1.1 applies only if $\eta' \in H^1$ (i.e $E^p(G)$ supports compact composition operators)

In the case both η' and $\frac{1}{\eta'}$ are bounded (which means every composition operator is bounded in $E^p(G)$ [12]) a weaker condition can be obtained in terms of the classical Carlson condition. We state this result as a corollary.

Corollary 4.2.1. (A.Tadesse) *Suppose that both η' and $\frac{1}{\eta'}$ bounded, then $C_\phi : E^p(G) \to E^p(G)$ is compact if*

$$\lim_{r\to 0} \sup_{b\in\partial D} \frac{m(\varphi^{-1}(S(b,r)) \cap \partial(D))}{r} = 0$$

4.3 EXAMPLES

In the following we give an example(adopted from [12]) verifying Theorem 4.1.1 for a simple geometry where an explicit and simplified expression for the Riemann map is known.

Example 4.3.1. *For reasons which comes shortly we consider now the case $p = 1$. The remaining values of p is taken care of by remark 4.2.1*

As usual let D represent the unit disk. Let $\eta(z) = 1 - (1 - z)^{1/2}$, so that $\eta(D)$ is a "teardrop" shaped domain symmetric about the real axis, whose boundary meets the unit circle at the point 1, where it makes an angle of $\pi/4$ radians with the unit interval. Let $G = \eta(D)$. It follows from the elementary inequality

$$|1 - \omega^{1/2}| < |1 - \omega| \quad (Re(\omega) > 0)$$

Let $\phi = \eta/G$ (i.e the restriction of η to G), and so $\phi(G) = \eta(G) = \eta(\eta(D)) \subset \eta(D) = G$, i.e ϕ is a holomorphic self map of G. The disk map that corresponds to ϕ is $\varphi = \eta^{-1} \circ \phi \circ \eta =$

$\eta^{-1}\circ\eta\circ\eta=\eta$ Now $\eta'(z)=(1/2)(1-z)^{-1/2}$, so $Q_\varphi(z)=(1-z)^{-1/4}$, an unbounded function on the unit disk. We show that, nevertheless, C_ϕ is compact. Since $\frac{1}{(1-z)^{1/4}}\in H^1(D)$(and hence the choice of p $=1$), the hypothesis of Theorem 4.1.1 is satisfied. Since the boundary of G touches the unit disk D only at the point 1 with the boundary of $\phi(G)=\eta(G)$ forming a stolz angle at this point, the only value of interest for b is 1.

Thus, suffices to show that $\int_{\varphi^{-1}(S(1,r))\cap\partial(D)}|Q_\varphi(z)|=\int_{\varphi^{-1}(\overline{S(1,r)})\cap\partial(D)}\frac{1}{|1-z|^{1/4}}\to 0$ *as* $r\to 0$

A simple algebraic manipulation shows that $z\in\varphi^{-1}(\overline{S(1,r)})\cap\partial(D)$ if and only if $(1-z)(1-\overline{z})=r^4$ and $z\in\partial(D)$. Parameterizing this with $z=e^{i\theta}$ shows that this is indeed equivalent to $\theta=\arccos(1-r^2/2)\approx r^2$*, where the last approximation follows from the identity*

$$\arccos(1-r^4/2)=r^2+O(r^6)$$

Thus we have

$$\begin{aligned}
&\int_{\varphi^{-1}(\overline{S(1,r)})\cap\partial(D)}\frac{1}{|1-z|^{1/4}}dm\\
&\qquad= \int_0^{\arccos(1-r4/2)}\frac{1}{(1-\cos(\theta))^{1/8}}d\theta\\
&\qquad\approx \int_0^{r^2}\frac{1}{(1-\cos(\theta))^{1/8}}d\theta\\
&\qquad\approx \int_0^{r^2}\left(\frac{2^{1/8}}{\theta^{1/4}}+\frac{2^{1/8}\theta^{7/4}}{96}\right)d\theta\\
&\qquad\approx O(r^{3/2})
\end{aligned}$$

where the estimate second to last comes from the identity

$$\frac{1}{(1-\cos(\theta))^{1/8}}=\frac{2^{1/8}}{\theta^{1/4}}+\frac{2^{1/8}\theta^{7/4}}{96}+O(\theta^{15/4})$$

thus showing that

$$\frac{\int_{\varphi^{-1}(\overline{S(1,r)})\cap\partial(D)}\frac{1}{|1-z|^{1/4}}dm}{r}=O(r^{1/2})\to 0$$

as $r\to 0$

Thus, by Theorem 4.1.1 b) C_ϕ is compact on $E^1(G)$ and hence on any $E^p(G)$, for $0<p<\infty$ *as expected.*

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