EXISTENCE AND ASYMPTOTIC ANALYSIS OF SOLUTIONS OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

by

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Submitted to the Graduate Faculty of

the Department of Mathematics in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2011

UNIVERSITY OF PITTSBURGH

MATHEMATICS DEPARTMENT

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We study existence and uniform asymptotic expansions of solutions of two different classes of singularly perturbed boundary value problems. The first boundary value problem that we consider is

$$\varepsilon y'' + 2y' + f(y) = 0, \quad y(0) = y(A) = 0,$$

where f is a smooth, positive increasing function satisfying certain properties and $A > 0$. We will show that the problem has two solutions for certain values of A. We will also derive and prove a uniform asymptotic expansion of the smaller solution when $f(y) = e^y$ and $A = 1$. The second boundary value problem that we consider is

$$\varepsilon^2 y'' = y(q(x,\varepsilon) - y), \quad y(-1) = \alpha_-, \quad y(1) = \alpha_+,$$

where $q(x,\varepsilon)$ is a smooth function with uniformly bounded derivatives and is uniformly bounded from below by a positive constant $q_\star$ for ε sufficiently small. The boundary values $\alpha_\pm$ are specified positive numbers bounded from above by $q_\star$. We will derive uniform asymptotic expansion of solutions to this problem that have 3 or fewer critical points.

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PREFACE

It gives me a very nostalgic feeling to write this part of my thesis. It would be too small an attempt, if I try to mention and thank all those people for what I am today. Starting from the day when I was born in a government hospital, through all my school days where I grew up in a small industrial town, till my journey to the United States, I owe my gratitude to the hospital nurses, school-bus drivers, school teachers, neighborhood locality and many other people who have directly or indirectly contributed to my upbringing. I wouldn't have chosen this field if it were not for my college teachers who have been a great source of inspiration for me. Coming to the United States to pursue graduate studies didn't occur to me till I went for my Masters at the Indian Institute of Technology, where I got examples from my seniors and alumni members who had their Ph.Ds from the United States. I sincerely thank my professors at the Indian Institute of Technology for their recommendations, without which I wouldn't have been here.

My deepest gratitude goes to my advisor Professor Stuart Hastings, who has not only been supportive over all these years, but has also raised my confidence and outlook towards research. Professor Hastings is an excellent mentor and is very approachable. I often discussed with him various problems or issues that I ran into, all of which were not necessarily related to my research and he always listened to me patiently and expressed his concerns towards me. I owe my sincere thanks to him for all the time he has invested in me and for all the efforts he has taken, without which I cannot even imagine to have a dissertation. I would sincerely like to thank Professor Xinfu Chen, whose inspiring classes in differential equations motivated me to step my foot in door. During my early graduate years, I got the privilege of doing a directed study with him. He has been always accessible whenever I needed any guidance. I have had lots of fruitful research discussions with him and I would

always cherish them. My sincere thanks also goes to Professor Bryce McLeod, who has also contributed a lot to my knowledge in differential equations. I would always treasure the experience that I had in collaborating with him. Thanks to Professor Ferdinand Verhulst for sharing his thoughts on one of the problems that I was working on for my dissertation. I would also like to thank all my committee members Professors Anna Vainchtein, William Troy and Shangbing Ai for being very kind to serve on my committee. Special thanks to Professor Ai who has agreed to fly to Pittsburgh from Alabama to attend my defense.

My stay in Pittsburgh would have been very difficult without my friends and colleagues. I would like to thank Mark Desantis, Justin Dunmyre, Greg Francos, Lucy Spardy, Alex Sviridov and others for all the good times that we have spent together during the last six years. Special thanks to Justin for supporting me during very stressful days of my graduate career, and to Alex, for sharing his thesis with me and helping me with various technical stuffs. I would also thank all the staff members of the Mathematics Department who have always been very helpful.

Finally I would like to thank my parents who have been always supportive and caring through all these years, and have allowed me to be as ambitious as I am by fuelling my enthusiasm. I would also like to thank my friends and other family members back in India. I would end by thanking my husband, Saikat, for his support, encouragement, patience and unwavering love for over all these years. I got an unending strength from him to recover from the worst frustrating days of my life and surviving the challenges to meet my loved one's expectations.

1.0 INTRODUCTION

Singularly perturbed boundary value problems arise in many physical phenomena like fluid dynamics, aerodynamic flows, magneto-hydrodynamic flows, diffusion reactions etc. These problems often have solutions with at least one "boundary layer". A "boundary layer" generally refers to the edge of a physical region where a rapid transition in the structure of a solution occurs over a very short length scale. This presents interesting mathematical challenges.

To cite a few examples, in fluid mechanics, a boundary layer is the layer of fluid that is in the immediate vicinity of a bounding surface. On an aircraft wing the boundary layer is the part of the flow close to the wing. In earth's atmosphere, the "planetary boundary layer" is the air layer near the ground affected by daily heat, moisture, momentum transfer to or from the surface.

Classical methods usually fail to give exact analytic solutions for nonlinear boundary value problems and hence one tries to find approximate solutions. Matched asymptotic expansion is a traditional method for finding an approximate solution of a singularly perturbed boundary value problem. Sometimes the method could give rise to "spurious" solutions, namely it could give existence of formal solutions which do not correspond to actual ones. The well-known Carrier-Pearson's autonomous equation

$$\varepsilon^2 y'' + y^2 = 1, \quad y(-1) = y(1) = 0$$

is an example of such a case. In this problem, the method of matched asymptotic expansion gives rise to single/multi-spiked solutions with spikes that can cluster at any arbitrary $x_0 \in (-1, 1)$ as $\varepsilon \to 0$. However, a phase plane analysis tells us that $x_0 = 0$ is the only possible point where the spikes can coalesce, a fact that perturbation theory could not detect (see [10],

[11], [12], [13], [18]). Hence, it is essential to check first that a solution exists analytically and then verify that the approximate solution obtained by heuristic methods actually converges to the exact solution.

In my thesis, I have considered the two model BVPs, the solutions of both of which exhibit boundary layers. The main goal is to rigorously prove that formal approximation of a solution obtained by matched asymptotics is correct i.e. the approximate solution converges uniformly to the exact solution as the parameter goes to zero.

My thesis will consist of two parts. One part will be devoted to proving existence (and uniqueness) of solutions of two different classes of singularly perturbed boundary value problems and the other part will be devoted to rigorously proving uniform asymptotic expansions of bounded solutions to these problems with three or fewer critical points.

2.0 AN EXAMPLE OF A SINGULARLY PERTURBED BVP

We consider the following singularly perturbed BVP:

$$\varepsilon y'' + 2y' + e^y = 0 \tag{2.1a}$$

$$y(0) = 0,\ y(1) = 0. \tag{2.1b}$$

Here ε is a small parameter and $' = \frac{d}{dx}$. This problem appears in [3],[21] as an example to show that the techniques used to approximate linear boundary-layer problems can apply equally well for this nonlinear problem. Numerical evidence in [3] suggested the existence of one solution and a uniform approximation was given for that solution using matched asymptotic expansions. In this chapter we will prove that this problem has at least two solutions each having a boundary layer at $x = 0$. Both solutions have the same outer solution on $(0, 1]$, but have different boundary behavior near $x = 0$. The second solution has a spike at the edge that is unbounded as $\varepsilon \to 0$. Problems of this type with two or more solutions are known, for example in [5] (Chapter 18), but we are not aware of examples in which one of the boundary layers is unbounded this way.

In [3] and [21], a uniform approximation of the "smaller" solution is given by

$$y_u(x) = \ln 2(1 - e^{-2x/\varepsilon}) - \ln(x + 1). \tag{2.2}$$

In addition to proving the existence of two solutions we will also prove rigorously that the asymptotic formula given by (2.2) approximates the smaller solution of (2.1a)-(2.1b) correct up to $O(\varepsilon)$ as $\varepsilon \to 0$.

If we set $t = x/\varepsilon$ and $z(t) = y(x)$ then (2.1*a*)-(2.1*b*) transform to

$$z'' + 2z' + \varepsilon e^z = 0 \tag{2.3a}$$

$$z(0) = 0,\ \ z(1/\varepsilon) = 0. \tag{2.3b}$$

Since (2.1a)-(2.1*b*) and (2.3*a*)-(2.3*b*) are equivalent, we shall prove existence of solutions to (2.3*a*)-(2.3*b*).

Theorem 1. *For $0 < \varepsilon \leq 19/100$ there are at least two solutions to (2.3a)-(2.3b).*

Remark 2.0.1. *It will follow from the existence proof of the larger solution of (2.3a)-(2.3b) that the initial velocity of the larger solution is bounded from above by $100/\varepsilon$ for all $\varepsilon \in (0, 19/100]$. However, numerical evidence suggests that the initial velocity of the larger solution is $O(|\log \varepsilon|)$ as $\varepsilon \to 0$.*

Theorem 2. *The smaller solution y of (2.1a)-(2.1b) is given uniformly by $y = y_u + O(\varepsilon)$ as $\varepsilon \to 0$, where y_u is given by (2.2).*

It will follow from the existence proof of the smaller solution of (2.3*a*)-(2.3*b*) that it is bounded from above by 1 for all $\varepsilon \in (0, 19/100]$. We will prove that the smaller solution exists uniquely in the rectangle $[0, 1] \times [0, \ln(\varepsilon^{-1})]$ as $\varepsilon \to 0$. This would imply that the larger solution is unbounded as $\varepsilon \to 0$.

Thus we have the result:

Theorem 3. *The boundary value problem (2.1a)-(2.1b) has at most one solution on the rectangle $[0, 1] \times [0, \ln(\varepsilon^{-1})]$ for $\varepsilon > 0$ sufficiently small.*

To prove Theorem 1 we need some basic concepts presented in the next section.

2.1 PRELIMINARIES

Consider a boundary value problem given by

$$Ly = f(t, y) \tag{2.4a}$$

$$y(a) = \alpha,\ \ y(b) = \beta \tag{2.4b}$$

where

$$Ly = (py')' + qy$$

and f is a function of t and y.

Definition 1. *A C^1 function u is a lower solution for (2.4a)-(2.4b) if $Lu \geq f(t, u)$ and $u(a) \leq \alpha$ and $u(b) \leq \beta$.*

Definition 2. *A C^1 function v is an upper solution for (2.4a)-(2.4b) if $Lv \leq f(t, v)$ and $v(a) \geq \alpha$ and $v(b) \geq \beta$.*

Theorem 4. *Assume that $p \in C^1[a, b]$, $q \in C^0[a, b]$, $p(t) > 0$ in $[a, b]$ and that u is a lower solution and v is an upper solution for (2.4a)-(2.4b) with $u \leq v$. If $f(t, y)$ is continuous in the region $K = \{(t, y) : a \leq t \leq b, u(t) \leq y(t) \leq v(t)\}$, then there exists a solution to (2.4a)-(2.4b) between u and v.*

The proof can be obtained in [22], page 264.

Theorem 5. *Let*

$$(p_1(x)y')' + q_1(x)y = 0 \tag{2.5}$$

$$(p_2(x)y')' + q_2(x)y = 0 \tag{2.6}$$

be two homogeneous linear second order differential equations in self-adjoint form with

$$0 < p_2(x) \leq p_1(x)$$

and

$$q_1(x) \leq q_2(x).$$

Let u be a non-trivial solution of (2.5) with successive roots at z_1 and z_2 and let v be a non-trivial solution of (2.6). Then one of the following holds:
(i) there exists an $x \in [z_1, z_2]$ such that $v(x) = 0$; or
(ii) there exists a λ such that $v(x) = \lambda u(x)$.

The proof can be found in [22], page 273.

2.2 EXISTENCE OF THE SMALLER SOLUTION

In this section we will prove that the BVP (2.3a)-(2.3b) has a solution for $\varepsilon \in (0, 19/100]$.

Remark 2.2.1. *The method that we use in this section to prove existence of a solution to* (2.3a)-(2.3b) *applies even for a larger range of* ε*, namely for* $\varepsilon \in (0, 27/100]$*. However, we will only consider* ε *in* $(0, 19/100]$*.*

Multiplying (2.3a) by the integrating factor e^{2t}, (2.3a)-(2.3b) can be re-written as

$$Lz = f(t, z) \tag{2.7a}$$

$$z(0) = 0, \;\; z(1/\varepsilon) = 0 \tag{2.7b}$$

where $Lz = (e^{2t}z)'$ and $f(t, z) = -\varepsilon e^{2t}e^{z}$.

To find a solution to (2.7a)- (2.7b) we will first find a lower and an upper solution to (2.7a)- (2.7b). Clearly $u = 0$ is a lower solution to (2.3a)-(2.3b).

For an upper solution, consider the BVP

$$v'' + 2v' + \varepsilon(1 + (e - 1)v) = 0 \tag{2.8a}$$

$$v(0) = 0, \;\; v(1/\varepsilon) = 0. \tag{2.8b}$$

Solving (2.8a)-(2.8b) we obtain

$$v(t) = e^{-t}\left(Ae^{\alpha t} + Be^{-\alpha t}\right) - \frac{1}{e-1},$$

where

$$A = \frac{e^{\frac{1}{\varepsilon}} - e^{\frac{-\alpha}{\varepsilon}}}{2(e-1)\sinh\left(\frac{\alpha}{\varepsilon}\right)}, \quad B = \frac{e^{\frac{\alpha}{\varepsilon}} - e^{\frac{1}{\varepsilon}}}{2(e-1)\sinh\left(\frac{\alpha}{\varepsilon}\right)}$$

and

$$\alpha = \sqrt{1 - \varepsilon(e-1)}.$$

It can be checked that v attains its maximum at $t = \tilde{t}$, where

$$\tilde{t} = \frac{\varepsilon}{2\alpha}\ln\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)$$

and

$$v(\tilde{t}) = \left(A\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{\frac{1}{2}-\frac{1}{2\alpha}} + B\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{-\frac{1}{2}-\frac{1}{2\alpha}}\right) - \frac{1}{e-1}. \tag{2.9}$$

Note that $v(\tilde{t})$ is a function of ε and using L' Hospital's rule we can show that

$$\lim_{\varepsilon\to 0} v(\tilde{t}) = \frac{e^{\frac{e-1}{2}}-1}{e-1} < 0.8. \tag{2.10}$$

We will prove that that $v(\tilde{t}) < 1$ for all $0 < \varepsilon \le 19/100$.

Lemma 2.2.1. *For every* $\varepsilon \in (0, 19/100]$, $v(\tilde{t}) < 1$.

Proof. First of all, we note that

$$\frac{B\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{-\frac{1}{2}-\frac{1}{2\alpha}}}{A\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{\frac{1}{2}-\frac{1}{2\alpha}}} = \frac{\alpha-1}{\alpha+1}.$$

Since $A > 0$, hence for every $\varepsilon \in (0, 1/4]$ we have from (2.9) that

$$\begin{aligned} v(\tilde{t}) &= A\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{\frac{1}{2}-\frac{1}{2\alpha}}\left(1+\frac{\alpha-1}{\alpha+1}\right) - \frac{1}{e-1} \\ &= A\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{\frac{1}{2}-\frac{1}{2\alpha}}\left(1-\frac{\varepsilon(e-1)}{(1+\sqrt{1-\varepsilon(e-1)})^2}\right) - \frac{1}{e-1} \\ &< A\left(\frac{\alpha+1}{\alpha-1}\frac{B}{A}\right)^{\frac{1}{2}-\frac{1}{2\alpha}} - \frac{1}{e-1}. \end{aligned} \tag{2.11}$$

First, we will find an upper bound on A. Note that

$$A < \frac{e^{\frac{1}{\varepsilon}}}{2(e-1)\sinh\left(\frac{\alpha}{\varepsilon}\right)} =: f_1(\varepsilon)$$

Hence using the fact that $\coth(\alpha/\varepsilon) > 1$, we have

$$\begin{aligned} f_1'(\varepsilon) &= \frac{e^{\frac{1}{\varepsilon^2}}}{\varepsilon^2}\operatorname{csch}\left(\frac{\alpha}{\varepsilon}\right)\left(-1+\coth\left(\frac{\alpha}{\varepsilon}\right)\left(\frac{2-\varepsilon(e-1)}{2\sqrt{1-\varepsilon(e-1)}}\right)\right) \\ &> \frac{e^{\frac{1}{\varepsilon^2}}}{\varepsilon^2}\operatorname{csch}\left(\frac{\alpha}{\varepsilon}\right)\left(-1+\frac{2-\varepsilon(e-1)}{2\sqrt{1-\varepsilon(e-1)}}\right) \\ &= \frac{e^{\frac{1}{\varepsilon^2}}}{\varepsilon^2}\operatorname{csch}\left(\frac{\alpha}{\varepsilon}\right)\left(\frac{-2\sqrt{1-\varepsilon(e-1)}+2-\varepsilon(e-1)}{2\sqrt{1-\varepsilon(e-1)}}\right). \end{aligned} \tag{2.12}$$

Define

$$f_2(\varepsilon) = -2\sqrt{1-\varepsilon(e-1)} + 2 - \varepsilon(e-1).$$

Then for $\varepsilon \in (0, 19/100]$, we have

$$f_2'(\varepsilon) = (e-1)\left(\frac{1}{\sqrt{1-\varepsilon(e-1)}} - 1\right) > 0.$$

Note that $f_2(0) = 0$. Hence $f_2(\varepsilon) > 0$ for ε in that range. Using this fact, from (2.12) we obtain that $f_1'(\varepsilon) > 0$ and thus $f_1(\varepsilon) < f_1(19/100)$. Hence

$$A < f_1(19/100). \tag{2.13}$$

Next, we define

$$f_3(\varepsilon) = \frac{\alpha+1}{\alpha-1}\frac{B}{A}. \tag{2.14}$$

Note that

$$\frac{-B}{A} = \frac{1-e^{\frac{\alpha-1}{\varepsilon}}}{1-e^{\frac{-\alpha-1}{\varepsilon}}} > 1 - e^{\frac{\alpha-1}{\varepsilon}}.$$

Using this fact, we obtain from (2.14) that

$$f_3(\varepsilon) > \frac{\alpha+1}{1-\alpha}\left(1 - e^{\frac{\alpha-1}{\varepsilon}}\right) \tag{2.15}$$

Note that

$$\begin{aligned}
\frac{\alpha-1}{\varepsilon} &= \frac{\sqrt{1-\varepsilon(e-1)}-1}{\varepsilon} \\
&= \frac{-(e-1)}{1+\sqrt{1-\varepsilon(e-1)}} < \frac{-(e-1)}{2}.
\end{aligned} \tag{2.16}$$

Hence from (2.15) and (2.16) we obtain that

$$f_3(\varepsilon) > \frac{\alpha+1}{1-\alpha}\left(1 - e^{\frac{-e+1}{2}}\right). \tag{2.17}$$

The right hand side of (2.17) > 1 if and only if

$$\alpha > \frac{e^{\frac{1-e}{2}}}{2 - e^{\frac{1-e}{2}}},$$

which implies that

$$\varepsilon < \frac{1}{e-1}\left(1-\left(\frac{e^{\frac{1-e}{2}}}{2-e^{\frac{1-e}{2}}}\right)^2\right) \approx 0.54$$

Thus, in particular for $\varepsilon \in (0, 19/100]$, we obtain that $f_3(\varepsilon) > 1$ and thus

$$(f_3(\varepsilon))^{\frac{1}{2}-\frac{1}{2\alpha}} < 1, \tag{2.18}$$

since $1/2 - 1/(2\alpha) < 0$. Thus, from (2.11), (2.13) and (2.18) we obtain that

$$v(\tilde{t}) < A - \frac{1}{e-1} < f_1\left(\frac{19}{100}\right) - \frac{1}{e-1} < 0.92 < 1.$$

□

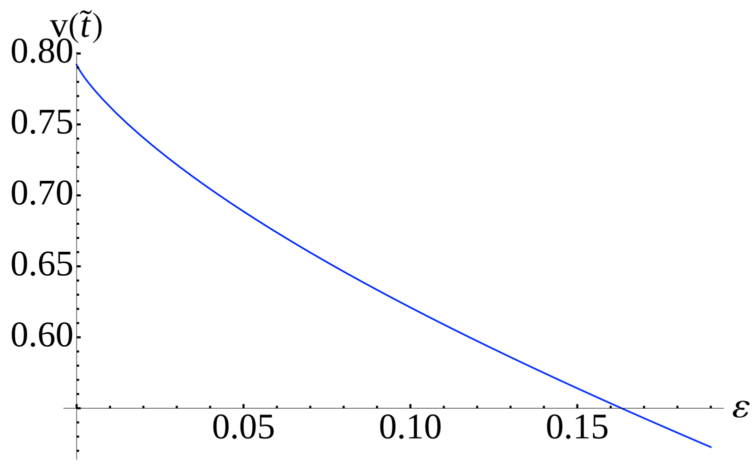

Figure 1: $v(\tilde{t})$ against $\varepsilon \in (0, 19/100]$. Note that $v(\tilde{t}) < 1$.

Note that the initial velocity of v is given by

$$v'(0) = A(\alpha - 1) - B(\alpha + 1), \tag{2.19}$$

and using L' Hospital's rule we can show that

$$\lim_{\varepsilon \to 0} v'(0) = \frac{2(e^{\frac{e-1}{2}} - 1)}{e-1} < 1.6. \tag{2.20}$$

In fact, we will show that $v'(0) < 1.83$ for $0 < \varepsilon \leq 19/100$, a fact that will be used later. Note that since $\alpha < 1$, $A > 0$ and $B < 0$, we have from (2.19) that

$$v'(0) < -B(\alpha+1) < -2B = \frac{2\left(e^{\frac{1-\alpha}{\varepsilon}} - 1\right)}{(e-1)(1-e^{\frac{-2\alpha}{\varepsilon}})}.$$

If we define

$$f_4(\varepsilon) = \frac{1-\alpha}{\varepsilon},$$

then

$$f_4'(\varepsilon) = \frac{-2\sqrt{1-\varepsilon(e-1)} + 2 - \varepsilon(e-1)}{2\varepsilon^2\sqrt{1-\varepsilon(e-1)}} = \frac{f_2(\varepsilon)}{2\varepsilon^2\sqrt{1-\varepsilon(e-1)}} > 0.$$

Hence $f_4(\varepsilon) < f_4(19/100)$ for $\varepsilon \in (0, 19/100]$. Hence

$$v'(0) < \frac{2\left(e^{f_4(19/100)} - 1\right)}{(e-1)(1-e^{\frac{-2\alpha}{\varepsilon}})}. \tag{2.21}$$

Since $-\alpha(\varepsilon)/\varepsilon$ increases with ε, so

$$\frac{-\alpha(\varepsilon)}{\varepsilon} < -4\alpha(19/100).$$

Hence (2.21) is bounded above by

$$\frac{2\left(e^{f_4(19/100)} - 1\right)}{(e-1)(1-e^{-8\alpha(19/100)})} < 1.83,$$

and thus $v'(0) < 1.83$ for all $\varepsilon \in (0, 19/100]$.

Since v has exactly one critical point, and that corresponds to the point of maximum, and v satisfies the boundary conditions (2.8b), we conclude that $v \geq 0$. Thus for $0 < \varepsilon \leq 19/100$, we obtain that

$$0 \leq v < 1$$

and hence

$$e^v \leq 1 + (e-1)v. \tag{2.22}$$

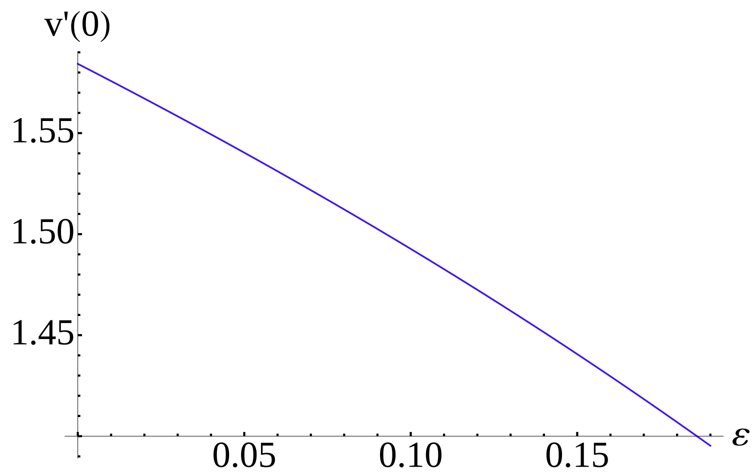

Figure 2: $v'(0)$ against ε for $0 < \varepsilon \leq 19/100$.

To show that v is an upper solution to (2.3a)-(2.3b), consider Lv. From (2.22) we have

$$Lv = -\varepsilon e^{2t}(1 + (e-1)v) \leq -\varepsilon e^{2t} e^{v} = f(t, v).$$

Moreover

$$v(0) = 0,\ v(1/\varepsilon) = 0,$$

thus making v an upper solution to (2.3a)-(2.3b).

Note that $u \leq v$. To prove this, set $w = v - u$ and consider

$$w'' + 2w' + \varepsilon(e-1)v = 0$$

$$w(0) = 0,\ w(1/\varepsilon) = 0.$$

Since $v > 0$ on $(0, \varepsilon^{-1})$, $w'' < 0$ whenever $w' = 0$, hence w has no minimum in $(0, \varepsilon^{-1})$. Thus $w \geq 0$. Now we can appeal to Theorem 4 and thereby conclude that there exists a solution z to (2.3a)-(2.3b) such that $u \leq z \leq v$. One should also note that

$$u'(0) \leq z'(0) \leq v'(0). \tag{2.24}$$

Thus the existence of the smaller solution is proved. □

Let us denote the smaller solution by z_s.

Remark 2.2.2. *The method of lower and upper solutions that we considered in this section can be applied to prove existence of solutions to a bigger class of smooth functions $f(y)$ satisfying $0 < f(y) \leq e^y$. Note that such an f need not satisfy $f' \geq 0$ and $f'' \geq 0$. The equations that we considered for lower and upper solutions would work for this class of functions as well. The method would work for all $A \in (0, 1]$ and also for some values of A larger than 1 but not for $A \geq 2$. We will not address this issue here. The method would also work for $f(y) = e^{ky}$ for different values of k, but A might have to be decreased from 1.*

2.3 EXISTENCE OF THE LARGER SOLUTION

To find another solution we transform (2.3*a*)-(2.3*b*) into an initial value problem:

$$z'' + 2z' + \varepsilon e^z = 0 \tag{2.25a}$$

$$z(0) = 0, \; z'(0) = \alpha. \tag{2.25b}$$

Denote the solution of (2.25*a*)-(2.25*b*) by z_α. From the previous section, we know that the BVP (2.3*a*)-(2.3*b*) admits at least one solution for $\varepsilon \in (0, 19/100]$. We will show that it also admits another solution for the same range of ε. Fix $\varepsilon \leq 19/100$. Let

$$\alpha_0 = \min \; \{z'(0) > 0 : z \text{ satisfies the BVP } (2.3a) - (2.3b)\}.$$

Our goal is to find α_1, $\alpha_2 > \alpha_0$ such that $z_{\alpha_1}(1/\varepsilon) > 0$ and $z_{\alpha_2}(1/\varepsilon) < 0$. Since $z_\alpha(1/\varepsilon)$ is a continuous function of α, by the Intermediate Value theorem there exists an $\alpha \in (\alpha_1, \alpha_2)$ such that $z_\alpha(1/\varepsilon) = 0$ which will then prove the existence of the other solution.

To prove the existence of α_1, we consider the BVP (2.26*a*)-(2.26*b*) given below:

$$z'' + 2z' + \varepsilon e^z = 0 \tag{2.26a}$$

$$z(0) = 0, \; z(11/(10\varepsilon)) = 0. \tag{2.26b}$$

To prove (2.26*a*)-(2.26*b*) has a solution, we use the same method as we did to prove the existence of z_s. The equations that we considered for lower and upper solutions of z_s will also work in this case with boundary conditions as in (2.26b). Moreover, the solution to

(2.26a)-(2.26b) will be positive at $1/\varepsilon$. We will show that the initial velocity β of a solution of (2.26a)-(2.26b) will be greater than α_0 and will be a candidate for α_1.

If possible, let $\beta < \alpha_0$. Denote $t_\alpha > 0$ to be the point where $z_\alpha(t_\alpha) = 0$. Then by our assumption $t_\beta = 11/(10\varepsilon)$. Also note that $t_\alpha \to 0$ as $\alpha \to 0$. Since t_α is continuous in α, it follows that there exists $\alpha \in (0, \beta)$ such that $t_\alpha = 1/\varepsilon$, a contradiction to the definition of α_0. Hence $\beta > \alpha_0$ and thus, a candidate for α_1.

To find α_2 we need a series of lemmas which we present in the next two sections. In all of these lemmas we will assume that $\varepsilon \leq 19/100$.

2.3.1 Finding α_2

Multiplying (2.25a) by the integrating factor e^{2t} and integrating we get

$$z'(t) = \alpha e^{-2t} - \varepsilon e^{-2t} \int_0^t e^{2s} e^{z(s)} ds. \tag{2.27}$$

Integrating once again we obtain

$$z(t) = \frac{\alpha}{2}(1 - e^{-2t}) + \frac{\varepsilon}{2} e^{-2t} \int_0^t e^{2s} e^{z(s)} - \frac{\varepsilon}{2} \int_0^t e^{z(s)} ds.$$

Hence

$$z(1/\varepsilon) = \frac{\alpha}{2}(1 - e^{-\frac{2}{\varepsilon}}) + \frac{\varepsilon}{2} e^{-\frac{2}{\varepsilon}} \int_0^{\frac{1}{\varepsilon}} e^{2s} e^{z(s)} - \frac{\varepsilon}{2} \int_0^{\frac{1}{\varepsilon}} e^{z(s)} ds. \tag{2.28}$$

Multiplying (2.25a) by z' and integrating we get

$$\frac{1}{2} z'^2(t) + 2 \int_0^t z'^2(s) ds + \varepsilon e^{z(t)} = \frac{\alpha^2}{2} + \varepsilon. \tag{2.29}$$

Integrating (2.25a) we obtain

$$z'(t) + 2z(t)) + \varepsilon \int_0^t e^{z(s)} ds = \alpha. \tag{2.30}$$

Lemma 2.3.1. *If* $\alpha = \frac{100}{\varepsilon}$ *and* $0 < \varepsilon \leq \frac{19}{100}$, *then* $z_\alpha\left(\frac{1}{\varepsilon}\right) < 0$.

If we prove Lemma 2.3.1 then we obtain α_2. Note that $\alpha_2 > 1.83$ for our choice of ε. Hence by (2.20), (2.24) and by the definition of α_0, we conclude that $\alpha_2 > \alpha_0$.

To prove Lemma 2.3.1 we need a few more lemmas as presented in the next section. In the next section we will replace z_α by z.

2.3.2 Some Important Lemmas

Lemma 2.3.2. *There is a δ with $0 < \delta < 3/4$ such that $z'(\delta) = \alpha/2$, and moreover, δ tends to zero as α tends to ∞.*

Proof. Since $z'(0) = \alpha > 0$, by continuity there exists a $\delta_1 > 0$ such that $|z'(t) - z'(0)| < \alpha/2$ for $t \in (0, \delta_1)$. This in turn implies that $z'(t) > \alpha/2$ on $(0, \delta_1)$. From (2.25a) we note that $z''(t) < -\alpha - \varepsilon$ as long as $z'(t) > \alpha/2$, and hence there exists t for which $z'(t) = \alpha/2$. Therefore one can choose $\delta > 0$ such that $z'(\delta) = \alpha/2$ with $z'(t) > \alpha/2$ on $(0, \delta)$. We will show that $\delta \in (0, \varepsilon^{-1})$. From (2.29) we have

$$\begin{aligned} \frac{\alpha^2}{2} + \varepsilon &= \frac{1}{2} z'^2(\delta) + 2\int_0^\delta z'^2(s)ds + \varepsilon e^{z(\delta)} \\ &> \frac{\alpha^2}{8} + \frac{2\alpha^2}{4}\delta + \varepsilon e^{\frac{\alpha}{2}\delta} \end{aligned}$$

The above inequality implies that

$$(3 - 4\delta)\frac{\alpha^2}{8} > \varepsilon(e^{\frac{\alpha}{2}\delta} - 1). \tag{2.31}$$

which in turn implies that $\delta < 3/4$ and hence $\delta \in (0, \varepsilon^{-1})$. Clearly from (2.31), one can see $\delta \to 0$ as $\alpha \to \infty$ and the lemma is proved. □

Now set $\alpha = \frac{100}{\varepsilon}$ and henceforth we work with this α.

Lemma 2.3.3. *There exists some $t \in (0, \varepsilon^{-1})$ for which $\varepsilon e^{z(t)} > \alpha$.*

Proof. Choose $\delta > 0$ as in the proof of Lemma 2.3.2 such that $z'(\delta) = \alpha/2$. Then $z(t) > \frac{\alpha}{2}t$ for $t \in (0, \delta)$. We shall prove that $\delta < \frac{1}{16}$. If not, then from (2.31) we must have

$$\frac{11}{32}\alpha^2 > \varepsilon(e^{\frac{\alpha}{32}} - 1). \tag{2.32}$$

With $\alpha = \frac{100}{\varepsilon}$, (2.32) can be written as

$$\frac{6875}{2\varepsilon^2} - \varepsilon(e^{\frac{25}{8\varepsilon}} - 1) > 0. \tag{2.33}$$

Define

$$g(\varepsilon) = \frac{6875}{2\varepsilon^2} - \varepsilon(e^{\frac{25}{8\varepsilon}} - 1),$$

then for $\varepsilon \in (0, 19/100]$, we have

$$\begin{aligned} g'(\varepsilon) &= \frac{-6875}{\varepsilon^3} + \left(\frac{25}{8\varepsilon} - 1\right)\varepsilon e^{\frac{25}{8\varepsilon}} + 1 \\ &> \frac{-6875}{\varepsilon^3} + \left(\frac{625}{38} - 1\right)\varepsilon e^{\frac{25}{8\varepsilon}} > 0. \end{aligned}$$

Note that $g(19/100) < 0$, hence we must have $g(\varepsilon) < 0$ for all $\varepsilon \in (0, 19/100]$. This contradicts (2.33) and therefore we must have $\delta < 1/16$. For this range of δ, from (2.29) we have

$$\begin{aligned} \varepsilon e^{z(\delta)} &= \frac{\alpha^2}{2} + \varepsilon - \frac{1}{2}z'^2(\delta) - 2\int_0^\delta z'^2(s)ds \\ &> \frac{\alpha^2}{2} + \varepsilon - \frac{\alpha^2}{8} - 2\alpha^2\delta \\ &= \left(\frac{3}{8} - 2\delta\right)\alpha^2 + \varepsilon \\ &> \frac{\alpha^2}{4} + \varepsilon. \end{aligned}$$

The inequality

$$\varepsilon e^{z(\delta)} > \frac{\alpha^2}{4} + \varepsilon. \tag{2.34}$$

proves Lemma 2.3.3. □

Since $z''(t) < -\varepsilon$ if $z'(t) > 0$, $z'(t)$ decreases at least until it is equal to zero. Let $z'(t_0) = 0$. Observe that $z''(t_0) < 0$ and hence z attains its maximum at $t = t_0$.

Lemma 2.3.4. *With t_0 defined as above, $t_0 < \frac{1}{2}$.*

Proof. Evaluating (2.27) at t_0 it follows that

$$\alpha = \varepsilon \int_0^{t_0} e^{2s} e^{z(s)} ds > \varepsilon \int_\delta^{t_0} e^{2s} e^{z(s)} ds \tag{2.35}$$

where δ is such that $\varepsilon e^{z(\delta)} > \frac{\alpha^2}{4}$ (from (2.34)) and $\delta < \frac{1}{16}$. Since $y(t)$ is increasing on the interval $(0, t_0)$,

$$\varepsilon e^{z(t)} > \frac{\alpha^2}{4}$$

for $t \in (\delta, t_0)$. Therefore we have

$$\alpha \quad > \quad \varepsilon e^{z(\delta)} \int_{\delta}^{t_0} e^{2s} ds > \frac{\alpha^2}{4} \int_{\delta}^{t_0} e^{2s} ds = \frac{\alpha^2}{8} \left(e^{2t_0} - e^{2\delta} \right).$$

Thus

$$\alpha > \frac{\alpha^2}{8} \left(e^{2t_0} - e^{\frac{1}{8}} \right). \tag{2.36}$$

If $t_0 > \frac{1}{2}$ then from (2.36) we have $8 > \alpha(e - e^{\frac{1}{8}})$ which is not true for our choice of α and ε. Therefore we conclude that $t_0 < \frac{1}{2}$. $\square$

Note from (2.29) that

$$\varepsilon e^{z(t_0)} < \frac{\alpha^2}{2} + \varepsilon. \tag{2.37}$$

Since $z'' < 0$ if $z' = 0$, y cannot have a minimum. Hence, $z'(t) < 0$ for $t > t_0$. Differentiating (2.25a) gives us

$$z''' = -2z'' - \varepsilon e^z z' \tag{2.38}$$

and (2.38) implies that $z''' > 0$ if $z'' < 0$. Hence z'' increases to the right of t_0. Also since $z''' > -\varepsilon y'$ if $z'' < 0$ and $z' < 0$, so z'' increases at least until for some t, $z''(t) = 0$. Let $z''(t_2) = 0$. Let us denote the interval (t_0, t_2) by I. Note that on I, z' is decreasing (since $z'' < 0$), hence, there exists $t_1 \in I$ such that $z'(t_1) = z''(t_1)$. From (2.25a) we have

$$\varepsilon e^{z(t_1)} = -z''(t_1) - 2z'(t_1) = -3z'(t_1). \tag{2.39}$$

We will prove that $t_1 < \varepsilon^{-1}$ and this proof is independent of the length of the interval I. We will prove later that $I \subset (0, \varepsilon^{-1})$, but at present we focus on t_1.

Lemma 2.3.5. *Let t_1 and t_0 be as above. Then $t_1 - t_0 < \frac{1}{2} \ln 3$.*

Proof. Using the first relation in (2.35), (2.27) and the fact that $z(t)$ is decreasing on the interval (t_0, t_1) we get that

$$\begin{aligned} z'(t_1) &= e^{-2t_1}\left(\alpha - \varepsilon\int_0^{t_1} e^{2s}e^{z(s)}ds\right) \\ &= -\varepsilon e^{-2t_1}\int_{t_0}^{t_1} e^{2s}e^{z(s)}ds \\ &< -\varepsilon e^{-2t_1}e^{z(t_1)}\int_{t_0}^{t_1} e^{2s}ds \\ &= -\frac{\varepsilon}{2}e^{z(t_1)}\left(1 - e^{2(t_0-t_1)}\right) \end{aligned}$$

Hence from (2.39) and the above inequality we obtain that

$$-\frac{\varepsilon}{3}e^{z(t_1)} < -\frac{\varepsilon}{2}e^{z(t_1)}\left(1 - e^{2(t_0-t_1)}\right)$$

which in turn implies that

$$e^{2(t_0-t_1)} > \frac{1}{3}$$

and hence

$$t_1 - t_0 < \frac{1}{2}\ln 3.$$

□

Thus from Lemma 2.3.5 and Lemma 2.3.4 we conclude that $t_1 < \frac{1}{2} + \frac{1}{2}\ln 3$ and therefore t_1 is in $(0, \varepsilon^{-1})$. Also from the choice of ε, we have $t_1 + \frac{1}{2} < \frac{1}{\varepsilon}$.

Lemma 2.3.6. $z'(t_1) < -\frac{\alpha}{4}$.

Proof. Suppose that $-\frac{\alpha}{4} < z'(t_1) < 0$. Then from (2.39) $\varepsilon e^{z(t_1)} < \frac{3\alpha}{4}$. Using (2.29) and Lemma 2.3.5 we have

$$\begin{aligned} \varepsilon e^{z(t_0)} &= \frac{1}{2}z'^2(t_1) + 2\int_{t_0}^{t_1} z'^2(s)ds + \varepsilon e^{z(t_1)} \\ &< \frac{\alpha^2}{32} + \frac{2\alpha^2}{16}(t_1 - t_0) + \frac{3\alpha}{4} \\ &< \frac{\alpha^2}{32} + \frac{\alpha^2}{16}\ln 3 + \frac{3\alpha}{4} \\ &< \frac{\alpha^2}{4} \end{aligned}$$

The last inequality is true for our choice of α and ε. Hence $\varepsilon e^{z(t_0)} < \frac{\alpha^2}{4}$, but that contradicts (2.34). Thus the lemma is established. □

Getting back to the the interval $I = (t_0, t_2)$, we note that z' decreases on I (since $z'' < 0$ on I) and increases after t_2. If $z'(t) < -\frac{\alpha}{4}$ for all $t > t_1$ then one can easily see that $z(1/\varepsilon) < 0$ for our choice of α and that will prove Lemma 2.3.1. Hence we assume z' is not less than $-\frac{\alpha}{4}$ for all $t > t_1$. This implies there exists $t_3 > t_2$ such that $z'(t_3) = -\frac{\alpha}{4}$ and $z'(t) < -\frac{\alpha}{4}$ for all $t \in (t_1, t_3)$. As pointed out earlier, $t_1 + \frac{1}{2} < \frac{1}{\varepsilon}$. We will prove either $z\left(t_1 + \frac{1}{2}\right) < 0$, in which case we are done, or $t_3 - t_1 < \frac{1}{2}$.

If $t_3 - t_1 > \frac{1}{2}$ then by the fact that

$$z'(t) < -\frac{\alpha}{4}$$

we have for all $t \in (t_1, t_3)$

$$z(t) - z(t_1) < -\frac{\alpha}{4}(t - t_1).$$

Hence

$$\begin{aligned} z\left(t_1 + \frac{1}{2}\right) < z(t_1) - \frac{\alpha}{4}\left(t_1 + \frac{1}{2} - t_1\right) &< z(t_0) - \frac{\alpha}{8} \\ &< \ln\left(\frac{\alpha^2}{2\varepsilon} + 1\right) - \frac{\alpha}{8} < 0. \end{aligned}$$

Here the second last inequality followed from (2.37).

Now assume $t_3 - t_1 < \frac{1}{2}$. From Lemma 2.3.5 we have

$$t_3 - t_0 = (t_3 - t_1) + (t_1 - t_0) < \frac{1}{2} + \frac{1}{2}\ln 3. \tag{2.40}$$

From (2.40) and Lemma 2.3.4 we obtain $t_3 < 1 + \frac{1}{2}\ln 3$ and hence $t_2 < 1 + \frac{1}{2}\ln 3$ and therefore t_2 and t_3 both are in $(0, \varepsilon^{-1})$.

2.3.3 Proof of Lemma 2.3.1.

Consider the integral

$$\varepsilon e^{-\frac{2}{\varepsilon}}\int_0^{t_3} e^{2s}e^{z(s)}ds.$$

From the first relation in (2.35), (2.37), (2.40) and Lemma 2.3.4 we have

$$\begin{aligned}
\varepsilon e^{-\frac{2}{\varepsilon}}\int_0^{t_3} e^{2s}e^{z(s)}ds &= \varepsilon e^{-\frac{2}{\varepsilon}}\left(\int_0^{t_0} e^{2s}e^{z(s)}ds + \int_{t_0}^{t_3} e^{2s}e^{z(s)}ds\right)\\
&= \alpha e^{-\frac{2}{\varepsilon}} + \varepsilon e^{-\frac{2}{\varepsilon}}\int_{t_0}^{t_3} e^{2s}e^{z(s)}ds\\
&< \alpha e^{-\frac{2}{\varepsilon}} + \varepsilon e^{-\frac{2}{\varepsilon}}e^{z(t_0)}\int_{t_0}^{t_3} e^{2s}ds\\
&< \alpha e^{-\frac{2}{\varepsilon}} + e^{-\frac{2}{\varepsilon}}\left(\frac{\alpha^2}{2}+\varepsilon\right)e^{2t_0}\left(e^{2(t_3-t_0)}-1\right)\\
&< \alpha e^{-2/\varepsilon} + \left(\frac{\alpha^2}{2}+\varepsilon\right)e^{-2/\varepsilon}e\left(e^{(1+\ln 3)}-1\right)\\
&= \alpha e^{-\frac{2}{\varepsilon}} + \left(\frac{\alpha^2}{2}+\varepsilon\right)e^{-\frac{2}{\varepsilon}}e(3e-1).
\end{aligned}$$

Thus we have

$$\varepsilon e^{-\frac{2}{\varepsilon}}\int_0^{t_3} e^{2s}e^{z(s)}ds < \alpha e^{-\frac{2}{\varepsilon}} + \left(\frac{\alpha^2}{2}+\varepsilon\right)e^{-\frac{2}{\varepsilon}}e(3e-1). \tag{2.41}$$

From (2.30) and definition of t_3 we have

$$\varepsilon\int_0^{t_3} e^{z(s)}ds = \alpha - z'(t_3) - 2z(t_3) > \alpha + \frac{\alpha}{4} - 2z(t_0)$$

Thus from the above inequality and (2.37) we get

$$\varepsilon\int_0^{t_3} e^{z(s)}ds > \frac{5\alpha}{4} - 2\ln\left(\frac{\alpha^2}{2\varepsilon}+1\right) \tag{2.42}$$

Now consider

$$\begin{aligned}
\varepsilon\int_{t_3}^{\frac{1}{\varepsilon}} e^{z(s)}ds - \varepsilon e^{-2/\varepsilon}\int_{t_3}^{\frac{1}{\varepsilon}} e^{2s}e^{z(s)}ds &= \varepsilon\int_{t_3}^{\frac{1}{\varepsilon}}(1-e^{-\frac{2}{\varepsilon}}e^{2s})e^{z(s)}ds\\
&> 0.
\end{aligned} \tag{2.43}$$

Finally from (2.28), (2.37), (2.41), (2.42) and (2.43) we obtain

$$
\begin{aligned}
z\left(\frac{1}{\varepsilon}\right) &= \frac{\alpha}{2}(1-e^{-\frac{2}{\varepsilon}}) + \frac{\varepsilon}{2}e^{-\frac{2}{\varepsilon}}\int_0^{\frac{1}{\varepsilon}} e^{2s}e^{z(s)} - \frac{\varepsilon}{2}\int_0^{\frac{1}{\varepsilon}} e^{z(s)}ds \\
&= \frac{\alpha}{2}(1-e^{-\frac{2}{\varepsilon}}) + \frac{\varepsilon}{2}e^{-\frac{2}{\varepsilon}}\int_0^{t_3} e^{2s}e^{z(s)} - \frac{\varepsilon}{2}\int_0^{t_3} e^{z(s)}ds \\
&- \frac{\varepsilon}{2}\left(\int_{t_3}^{\frac{1}{\varepsilon}} e^{z(s)}ds - e^{-2/\varepsilon}\int_{t_3}^{\frac{1}{\varepsilon}} e^{2s}e^{z(s)}ds\right) \\
&< \frac{\alpha}{2}(1-e^{-\frac{2}{\varepsilon}}) + \frac{\alpha}{2}e^{-\frac{2}{\varepsilon}} + \frac{1}{2}\left(\frac{\alpha^2}{2}+\varepsilon\right)e^{-\frac{2}{\varepsilon}}e(3e-1) \\
&- \frac{5\alpha}{8} + \ln\left(\frac{\alpha^2}{2\varepsilon}+1\right) \\
&= -\frac{\alpha}{8} + \left(\frac{\alpha^2}{4}+\frac{\varepsilon}{2}\right)e^{-\frac{2}{\varepsilon}}e(3e-1) + \ln\left(\frac{\alpha^2+2\varepsilon}{2\varepsilon}\right) =: h(\varepsilon)(\text{say})
\end{aligned}
$$

With $\alpha = 100/\varepsilon$ and $\varepsilon \in (0, 19/100]$, we have

$$
\begin{aligned}
h'(\varepsilon) &= e^{(1-2/\varepsilon)}(-1+3e)\left(\frac{1}{2}+\frac{1}{\varepsilon}+\frac{5000}{\varepsilon^3}\left(\frac{1}{\varepsilon}-1\right)\right) + \frac{25}{2\varepsilon^2} - \frac{15000}{5000\varepsilon+\varepsilon^4} \\
&> \frac{25}{2\varepsilon^2} - \frac{15000}{5000\varepsilon} > 0.
\end{aligned}
$$

Note that $h(19/100) < 0$, hence $h(\varepsilon) < 0$ for $\varepsilon \in (0, 19/100]$ and thus $z(1/\varepsilon) < 0$. $\square$

Thus we obtain α_2 and hence obtain a second solution to (2.3a)-(2.3b). This proves Theorem 1. $\square$

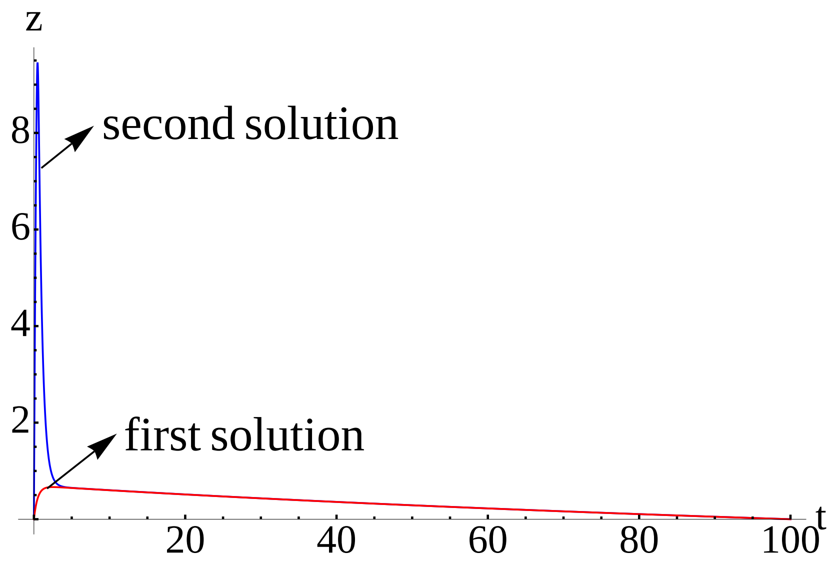

Figure 3: Two solutions of the BVP (2.3a)-(2.3b). Here $\varepsilon = 0.01$.

2.4 ASYMPTOTIC BEHAVIOR

As mentioned earlier (see eq. (2.2)), a uniform approximation of the "smaller" solution using matched asymptotic expansions is given by (see [3] and [21])

$$y_u(x) = \ln 2(1 - e^{-2x/\varepsilon}) - \ln(x+1).$$

We will prove that y_u approximates the smaller solution y of (2.1a)-(2.1b) correct up to $O(\varepsilon)$. This will prove Theorem 2.

Remark 2.4.1. *In the next chapter, we will consider a more generalized boundary value problem, where we will rigorously prove a uniform expansion of the smaller solution by considering the difference between the actual solution and the conjectured asymptotic expansion and showing that the error is $O(\varepsilon)$. In this section, we will do a similar thing. A crucial part of the proof lies in the knowledge of a priori bounds on the maximum value and initial velocity of the smaller solution of* (2.3a)-(2.3b). *These facts motivated us to work with bounded solutions that have uniformly bounded velocities for small ε of the generalized boundary value problem considered in the next chapter.*

The next section will be devoted to the proof of Theorem 2.

2.4.1 Proof of Theorem 2

Proof. Suppose that y and u satisfy the equations

$$\begin{aligned} \varepsilon y'' + 2y' + e^y &= 0,\ y(0) = 0,\ y(1) = 0, \\ \varepsilon u'' + 2u' + e^u &= \varepsilon u'',\ u(1) = 0 \end{aligned}$$

respectively. If $y'(0) = C$, where y is the smaller solution of (2.1a)-(2.1b) then

$$y(x) - u(x) = h_1(x) - \frac{1}{2}\varepsilon C e^{\frac{-2x}{\varepsilon}} + \frac{1}{\varepsilon}\int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \left(e^{y(s)} - e^{u(s)}\right) dsdt, \tag{2.44}$$

where

$$h_1(x) = \frac{\varepsilon C}{2} e^{-\frac{2}{\varepsilon}} - \frac{\varepsilon}{2} u'(0) \left(e^{-\frac{2}{\varepsilon}} - e^{-\frac{2x}{\varepsilon}}\right) + \frac{1}{\varepsilon}\int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} u''(s)\, dsdt.$$

Since u' and u'' are bounded and $\varepsilon C \leq K$ for some $k \leq 1.6$, (follows from (2.20) and (2.24)), it follows that $h_1(x) = O(\varepsilon)$ uniformly as $\varepsilon \to 0$.

Define

$$g(x) = y(x) - u(x) + \frac{\varepsilon C}{2} e^{-2x/\varepsilon}. \tag{2.45}$$

Then from (2.44), it follows that g satisfies the integral equation

$$\begin{aligned} g(x) &= h_1(x) + \frac{1}{\varepsilon}\int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \left(e^y - e^u\right)\, dsdt. \\ &= h_1(x) + \frac{1}{\varepsilon}\int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \left(e^{u+g-\frac{\varepsilon C}{2}e^{-\frac{2s}{\varepsilon}}} - e^u\right)\, dsdt. \end{aligned} \tag{2.46}$$

Also note that since u satisfies

$$2u' + e^u = 0,\ u(1) = 0,$$

it implies that

$$u(x) = \ln 2 - \ln(x+1). \tag{2.47}$$

In Section 2.2, we found out that an upper solution v of (2.1a)-(2.1b) is bounded from above by 0.8 for $\varepsilon > 0$ sufficiently small (see (2.10)). Hence from (2.47) and the fact that $\varepsilon C \leq 1.6$, we have from (2.45) that

$$g(x) \leq v(x) - \ln\left(\frac{2}{1+x}\right) + \frac{\varepsilon C}{2} e^{-2x/\varepsilon} \leq v(x) + 0.8e^{-2x/\varepsilon} \leq 0.8 \tag{2.48}$$

since $v(x) + 0.8e^{-2x/\varepsilon}$ is decreasing in x. Also since $y > 0$,

$$g(x) \geq \ln(1+x) - \ln 2 \geq -\ln 2. \tag{2.49}$$

Hence on combining (2.48) and (2.49) we have

$$\|g\| < 1 \tag{2.50}$$

on $[0, \varepsilon^{-1}]$ for all sufficiently small ε. Since εC is bounded and $\|g\| < 1$, by Taylor's theorem, we have that

$$e^{\left(u+g-\frac{\varepsilon C}{2}e^{-\frac{2x}{\varepsilon}}\right)} - e^{u} = e^{u+g} - e^{u} + O\left(\varepsilon C e^{-\frac{2x}{\varepsilon}}\right) \tag{2.51}$$

as $\varepsilon \to 0$.

Remark 2.4.2. *Note that in (2.51), $O\left(\varepsilon C e^{-2x/\varepsilon}\right)$ is a function that depends on g as well, but its bound is independent of g since $\|g\| < 1$.*

The contribution from the O-term in (2.51) to the integral in (2.46) is

$$O\left(\frac{1}{\varepsilon}\int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \varepsilon C e^{-\frac{2s}{\varepsilon}}\, dsdt\right) = O\left(C\varepsilon^2\right) = O(\varepsilon),$$

uniformly as $\varepsilon \to 0$, since $C\varepsilon \leq 1.6$. Hence (2.46) can be written as

$$g(x) = h_3 + \frac{1}{\varepsilon}\int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \left(e^{u+g} - e^{u}\right)\, dsdt, \tag{2.52}$$

where $h_3 = h_1(x,\varepsilon) + h_2(x,\varepsilon,\varepsilon C, g)$, $h_2 = O(C\varepsilon^2) = O(\varepsilon)$ and hence $h_3 = O(\varepsilon)$ uniformly as $\varepsilon \to 0$. Since $\|g\| < 1$ and u is bounded,

$$\left|e^{u+g} - e^{u}\right| \leq K|g|,$$

where $K = \max\limits_{0 \leq w \leq \|u\|+1} e^w$. Using this estimate, (2.52) can be estimated by

$$
\begin{aligned}
|g(x)| &\leq |h_3| + \frac{K}{\varepsilon} \int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} |g(s)| \, dsdt \qquad (2.53) \\
&\leq |h_3| + \frac{K}{\varepsilon} \|g\| \int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \, dsdt \\
&\leq |h_3| + \frac{K}{2} \|g\| (1-x) \qquad (2.54)
\end{aligned}
$$

Substituting the estimate given by (2.54) into (2.53), we obtain that

$$
\begin{aligned}
|g(x)| &\leq |h_4| + \frac{K}{\varepsilon} \int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \frac{K}{2} \|g\| (1-s) \, dsdt \\
&= |h_4| + \frac{K^2}{2\varepsilon} \|g\| \int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} (1-s) \, dsdt \\
&\leq |h_4| + \left(\frac{K}{2}\right)^2 \|g\| \left(\frac{(1-x)^2}{2!} + a_1^2 \varepsilon (1-x) \right), \qquad (2.55)
\end{aligned}
$$

where

$$h_4 = \frac{K}{\varepsilon} \int_x^1 e^{\frac{-2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} |h_3| \, dsdt = O(h_3) = O(\varepsilon)$$

and a_1^2 is some positive constant. Similarly substituting the estimate (2.55) in (2.53), we obtain that

$$|g(x)| \leq |h_5| + \left(\frac{K}{2}\right)^3 \|g\| \left(\frac{(1-x)^3}{3!} + a_2^3 \varepsilon \frac{(1-x)^2}{2!} + a_1^3 \varepsilon^2 (1-x) \right),$$

with $h_5 = O(\varepsilon)$ and a_2^3, a_1^3 being some constants that can be calculated explicitly. Proceeding iteratively, we obtain that

$$
\begin{aligned}
|g(x)| \leq & \quad |h_{n+2}| + \left(\tfrac{K}{2}\right)^n \|g\| \\
& \left(\tfrac{(1-x)^n}{n!} + a_{n-1}^n \varepsilon \tfrac{(1-x)^{n-1}}{(n-1)!} + a_{n-2}^n \varepsilon^2 \tfrac{(1-x)^{n-2}}{(n-2)!} + \ldots + a_1^n \varepsilon^n (1-x) \right), \qquad (2.56)
\end{aligned}
$$

where a_i^n are constants for $1 = 1, 2, \ldots n-1$ and $h_{n+2} = O(\varepsilon)$.

Now we fix some r with $0 < r < 1$. Choose n large enough such that

$$\left(\frac{K}{2}\right)^n \frac{1}{n!} < r. \qquad (2.57)$$

Now for this n there are finitely many terms $a_1^n, a_2^n, \ldots, a_{n-1}^n$, hence we can choose ε small enough such that

$$\left(\frac{K}{2}\right)^n \left(\frac{a_{n-1}^n \varepsilon}{(n-1)!} + \ldots + \frac{a_1^n \varepsilon^n}{(1)!}\right) < \frac{(1-r)}{2}. \tag{2.58}$$

Combining (2.57) and (2.58), we obtain from (2.56) that

$$|g(x)| \leq |h_{n+2}| + \frac{1+r}{2}\|g\|,$$

which would imply that $\left(\frac{1-r}{2}\right)\|g\| \leq |h_{n+2}|$, and so $\|g\| = O(\varepsilon)$ uniformly as $\varepsilon \to 0$.

Substituting that $\|g\| = O(\varepsilon)$ in (2.45), we obtain that

$$y(x) = u(x) - \frac{\varepsilon C}{2} e^{-2x/\varepsilon} + O(\varepsilon).$$

Since $y(0) = 0$, we obtain that $\varepsilon C = 2u(0) + O(\varepsilon) = 2\ln 2 + O(\varepsilon)$ and the correct uniform approximation is established.

□

□

2.5 UNIQUENESS OF THE SMALLER SOLUTION

Since (2.1a)-(2.1b) and (2.3a)-(2.3b) are equivalent, we shall prove the result for (2.3a)-(2.3b).

Suppose for a contradiction that there are two solutions z_1 and z_2. Set $\phi = z_1 - z_2$. Then $\phi(0) = \phi(\varepsilon^{-1}) = 0$. Then by subtraction

$$\phi'' + 2\phi' + \varepsilon e^{\xi}\phi = 0,$$

where ξ lies between z_1 and z_2. Setting $\phi = e^{-t}\psi$, we have

$$\psi'' + (\varepsilon e^{\xi} - 1)\psi = 0. \tag{2.59}$$

By our assumption, $e^{\xi} \leq \varepsilon^{-1}$, and so $\varepsilon e^{\xi} - 1 \leq 0$. Hence (2.59) has at most one zero (for a proof see Corollary 5.2, page 346-347 in [8]). Thus ψ (and so ϕ) has at most one zero, contradicting

$$\phi(0) = \phi(\varepsilon^{-1}) = 0.$$

Hence, we must have $\phi = 0$. □

Remark 2.5.1. *To see that the smaller solution z_s of (2.3a)-(2.3b) satisfies the conditions of Theorem 3, we note from (2.10) that $z_s(t) < 0.8$ for all $t \in [0, \varepsilon^{-1}]$ and for ε sufficiently small. For $\varepsilon > 0$ sufficiently small, $-\ln(\varepsilon) > 0.8$. Hence z_s lies in the rectangle $[0, \varepsilon^{-1}] \times [0, -\ln(\varepsilon)]$ and thus we have the uniqueness of the smaller solution.*

Remark 2.5.2. *For the larger solution it is clear from Theorem 3 that $z(t) > -\ln(\varepsilon)$ for some t and hence it gets unbounded as $\varepsilon \to 0$. This is an interesting feature of the second solution. The asymptotics for the second solution is still an open question. The method employed to find the asymptotic expansion for the smaller solution fails in this case.*

Remark 2.5.3. *The BVP (2.1a)-(2.1b) has exactly two solutions for $\varepsilon \in (0, 19/100]$. The proof will be discussed in the next chapter, where we deal with a BVP of a more general form.*

3.0 A CLASS OF SINGULARLY PERTURBED BVPS

In this chapter we will consider a generalization of the example considered in the previous chapter. This chapter is based on the paper [16], which is joint with Professor J.B. McLeod. We consider the BVP:

$$\varepsilon v'' + 2v' + f(v) = 0 \tag{3.1a}$$

$$v(0) = 0,\ v(A) = 0, \tag{3.1b}$$

where $A > 0$.

Here ε is a positive parameter and $' = \frac{d}{dx}$. Assume that $f \in C^2[0,\infty)$ with the following properties:

(a) $f(0) > 0,\ f' \geq 0,\ f'' \geq 0,$

(b) $\frac{f(v)}{v} \to \infty$ as $v \to \infty$ and

(c) if $F(v) = \int_0^v f(s)ds$, then

$$\int_0^v \frac{ds}{\sqrt{F(v) - F(s)}} \to 0$$

as $v \to \infty$.

These conditions are satisfied by $(v+2)\log(v+2)$, $(v+1)^p$ $(p > 1)$, e^v. Under these conditions on f, we will show that the BVP (3.1a)-(3.1b) (which of course depends on A) has at most two solutions.

In this chapter we will find conditions on A such that (3.1a)-(3.1b) has two solutions and also find an asymptotic expansion for the the first solution to (3.1a)-(3.1b) and prove rigorously that the formula is correct up to $O(\varepsilon)$ as $\varepsilon \to 0$.

If we set $t = x/\varepsilon$ and $y(t) = v(x)$ then (3.1*a*)-(3.1*b*) transform to

$$y'' + 2y' + \varepsilon f(y) = 0 \tag{3.2a}$$

$$y(0) = 0, \ y(A/\varepsilon) = 0. \tag{3.2b}$$

Since (3.1*a*)-(3.1*b*) and (3.2*a*)-(3.2*b*) are equivalent, we shall prove existence of solutions to (3.2*a*)-(3.2*b*).

3.1 EXISTENCE AND MULTIPLICITY

Theorem 6. *For each $\varepsilon > 0$ there exists $A(\varepsilon) > 0$ such that (3.2a)-(3.2b) has exactly two solutions, if $0 < A/\varepsilon < A(\varepsilon)$, exactly one solution, if $A/\varepsilon = A(\varepsilon)$, and no solutions if $A/\varepsilon > A(\varepsilon)$.*

Proof. To prove the existence of solutions we consider the initial value problem:

$$y'' + 2y' + \varepsilon f(y) = 0 \tag{3.3a}$$

$$y(0) = 0, \ y'(0) = \alpha \tag{3.3b}$$

where $\alpha > 0$. Write (3.3*a*) as

$$(y'e^{2t})' = -\varepsilon f(y)e^{2t} < -\varepsilon f(0)e^{2t}.$$

From this we see that if y satisfies (3.3*a*)-(3.3*b*) then y increases first, attains a maximum say at some t_0, and then decreases to 0. Also observe that since $f > 0$, y' has exactly one zero and is negative for $t > t_0$, and that $y'' < 0$ for $t \in [0, t_0]$. Differentiating (3.3*a*)-(3.3*b*), we obtain

$$y''' + 2y'' + \varepsilon f'(y)y' = 0$$

which implies that $y''' > 0$ whenever $y'' = 0$ and $t > t_0$. Hence y'' has at most one zero. Also consider the energy equation

$$\frac{y'^2(t)}{2} + 2\int_0^t y'^2(s)ds + \varepsilon F(y) = \frac{\alpha^2}{2}. \tag{3.4}$$

Denote the maximum of y by y_0.

Lemma 3.1.1. *$y_0 \to \infty$ as $\alpha \to \infty$.*

Proof. Suppose y_0 is bounded. Then there exists a constant $k > 0$ such that $y_0 < k$ for all α. Since f is continuous, there exists some $M > 0$ such that $f(y) < M$ for all $y \in [0, k]$. Hence from (3.3a) we have,

$$(y'e^{2t})' > -\varepsilon M e^{2t}.$$

Hence

$$y'(t) > \alpha e^{-2t} - \frac{\varepsilon M}{2}(1 - e^{-2t})$$

which in turn implies that for all $t > 0$

$$y(t) > \frac{\alpha}{2}(1 - e^{-2t}) - \frac{\varepsilon M t}{2} + \frac{\varepsilon M}{4}(1 - e^{-2t}).$$

In particular one can make $y(1) > k$ by choosing α sufficiently large, contradicting our assumption. □

Remark 3.1.1. *By Lemma 3.1.1, if we choose α large, then y_0 is large, and similarly, we see from (3.4) that if we choose y_0 large, then we can find α large such that the maximum of $y(t)$ is y_0.*

Lemma 3.1.2. *$t_0 \to 0$ as $y_0 \to \infty$, where t_0 is the location of the maximum of y.*

Proof. Multiplying (3.3a) by y' and integrating it over (t, t_0), $0 \le t < t_0$, we obtain

$$-y'^2(t) + 4\int_t^{t_0} y'^2(s)ds + 2\varepsilon(F(y_0) - F(y)) = 0,$$

and so

$$y'^2(t) > 2\varepsilon(F(y_0) - F(y)).$$

Since $y' > 0$ on $[0, t_0)$, we have from the above

$$y'(t) > \sqrt{2\varepsilon}\sqrt{F(y_0) - F(y)}.$$

Hence,

$$\sqrt{2\varepsilon}\int_0^{t_0} dt < \int_0^{y_0} \frac{dy}{\sqrt{F(y_0) - F(y)}} \to 0$$

as $y_0 \to \infty$ by assumption (c) on f. Thus we have proved that $t_0 \to 0$ as $\alpha \to \infty$. □

It is easy to see, since $y' < 0$ for $t > t_0$ and $f(0) > 0$, that y must have a second zero, say at $t = t^\star$.

Lemma 3.1.3. *As* $\alpha \to 0$, $t^\star \to 0$.

Proof. It is immediate that if $\alpha = 0$, then $y < 0$ for $t > 0$. Hence continuity in α proves the lemma. ☐

Lemma 3.1.4. $t^\star \to 0$ *as* $\alpha \to \infty$.

Proof. First assume that $y'' < 0$ for all $t \in [0, t^\star]$. Then y' is negative decreasing for $t > t_0$. Hence

$$\int_{t_0}^{t} y'^2(s)ds < (t - t_0)y'^2(t) \tag{3.5}$$

for $t > t_0$. Multiplying (3.3*a*) by y' and integrating it over (t_0, t), we obtain

$$y'^2(t) + 4\int_{t_0}^{t} y'^2(s)ds = 2\varepsilon(F(y_0) - F(y)). \tag{3.6}$$

As long as $t - t_0 < 1$, we have from (3.5) and (3.6) that

$$5y'^2(t) > 2\varepsilon(F(y_0) - F(y)).$$

Since $y' < 0$, we have

$$\int_{t_0}^{t} dt < \sqrt{\frac{5}{2\varepsilon}} \int_{y}^{y_0} \frac{ds}{\sqrt{F(y_0) - F(s)}} \to 0 \tag{3.7}$$

as $y_0 \to \infty$. By choosing y_0 large, and hence t_0 small (by Lemma 3.1.2), and setting $t = 1$, we have $t - t_0 < 1$ and yet (3.7) does not hold. Hence, for large y_0, y'' cannot always be negative for $t > t_0$, and so there exists t_1 such that $y'' = 0$. From (3.7) we see that $t_1 - t_0 \to 0$ as $\alpha \to \infty$. Let $y(t_1) = y_1$. Then y_1 satisfies $2y_1' = -\varepsilon f(y_1)$. Choose α large enough such that $t_1 - t_0 < 1/4$. Then from (3.5) and (3.6) we obtain

$$y_1'^2 > \varepsilon(F(y_0) - F(y_1)).$$

Therefore

$$\begin{aligned} y_1' &< -\sqrt{\varepsilon}\sqrt{(F(y_0)-F(y_1))} \\ &= -\sqrt{\varepsilon}\sqrt{F(y_0)}\sqrt{1-\frac{F(y_1)}{F(y_0)}} \end{aligned}$$

which implies that

$$\varepsilon f(y_1) > 2\sqrt{\varepsilon}\sqrt{F(y_0)}\sqrt{1-\frac{F(y_1)}{F(y_0)}}. \tag{3.8}$$

If $\frac{F(y_1)}{F(y_0)} < \frac{1}{2}$ then we see from (3.8) that

$$f(y_1) > \sqrt{\frac{2}{\varepsilon}}\sqrt{F(y_0)} \to \infty$$

as $\alpha \to \infty$. If $\frac{F(y_1)}{F(y_0)} \geq \frac{1}{2}$, then

$$f(y_1)y_1 \geq F(y_1) \geq \frac{F(y_0)}{2} \to \infty$$

as $\alpha \to \infty$. In either case, we must have that $y_1 \to \infty$ as $\alpha \to \infty$. Multiplying (3.3a) by e^{2t} we have, for all $t \geq t_1$,

$$(e^{2t}y')' \quad < \quad 0,$$

so that

$$e^{2(t-t_1)}y'(t) \quad < \quad y_1' = -\varepsilon f(y_1)/2$$

which implies

$$y'(t) < -\frac{\varepsilon}{2}f(y_1)e^{-2(t-t_1)}.$$

Integrating over $(t_1, t^\star)$, where $y(t^\star) = 0$, we obtain

$$-y_1 < -\frac{\varepsilon}{4}f(y_1)(1-e^{-2(t^\star-t_1)})$$

and hence

$$1-e^{-2(t^\star-t_1)} < \frac{4\varepsilon y_1}{f(y_1)} \to 0$$

as $\alpha \to \infty$. The last step is true by assumption (b) on f since $y_1 \to \infty$ as $\alpha \to \infty$. Thus we obtain $t^\star - t_1 \to 0$ as $\alpha \to \infty$, and so $t^\star \to 0$ as $\alpha \to \infty$. This proves the lemma. $\square$

Lemma 3.1.5. *The problem (3.3a)-(3.3b) has at most two solutions.*

Proof. Let $\phi = \frac{\partial y}{\partial \alpha}$. Then for each α, ϕ satisfies

$$\phi'' + 2\phi' + \varepsilon f'(y)\phi = 0, \tag{3.9a}$$

$$\phi(0) = 0,\ \ \phi'(0) = 1 \tag{3.9b}$$

and y' satisfies

$$(y')'' + 2(y')' + \varepsilon f'(y)y' = 0, \tag{3.10a}$$

$$y'(0) = \alpha,\ \ (y')'(0) = -2\alpha - \varepsilon f(0). \tag{3.10b}$$

We know that y' has precisely one zero at t_0. Comparing (3.9*a*) and (3.10*a*), we have by Sturm-Liouville applied to the self adjoint form of (3.9a) and (3.10a) that ϕ has no zero in $(0, t_0)$ and at most one zero in $(t_0, t^\star)$. Implicitly differentiating

$$y(t^\star(\alpha), \alpha) = 0$$

we have

$$y'(t^\star(\alpha), \alpha))\frac{dt^\star}{d\alpha} + \phi(t^\star) = 0. \tag{3.11}$$

Note from (3.11) that $\phi(t^\star) = 0$ for some α if and only if $\frac{dt^\star}{d\alpha} = 0$ since $y'(t^\star(\alpha), \alpha)) < 0$. Moreover for that α, $\phi > 0$ on $(t_0, t^\star)$ since ϕ can have at most one zero on that interval.

Remark 3.1.2. *Since $t^\star(\alpha)$ is a continuous function of α, Lemma 3.1.3 and Lemma 3.1.4 imply that $t^\star$ has at least one local maximum. Hence there exists at least one α for which $\frac{dt^\star}{d\alpha} = 0$.*

Our goal is to show that $t^\star$ has exactly one maximum and no local minima. To prove this we will show that $\frac{d^2t^\star}{d\alpha^2} < 0$ whenever $\frac{dt^\star}{d\alpha} = 0$.

Differentiating (3.11) with respect to α, we obtain

$$y''(t^\star)\left(\frac{dt^\star}{d\alpha}\right)^2 + y'(t^\star)\frac{d^2t^\star}{d\alpha^2} + \phi'(t^\star)\frac{dt^\star}{d\alpha} + \frac{\partial^2 y}{\partial\alpha^2}(t^\star) = 0.$$

If $\frac{dt^\star}{d\alpha} = 0$, then the above equality reduces to

$$y'(t^\star)\frac{d^2t^\star}{d\alpha^2} + \frac{\partial^2 y}{\partial\alpha^2}(t^\star) = 0. \tag{3.12}$$

Let $\psi = \frac{\partial^2 y}{\partial\alpha^2}$. Then

$$\psi'' + 2\psi' + \varepsilon f'(y)\psi = -\varepsilon f''(y)\phi^2, \tag{3.13a}$$

$$\psi(0) = 0,\ \psi'(0) = 0. \tag{3.13b}$$

Multiplying ($3.13a$) by ϕ and ($3.9a$) by ψ and subtracting we obtain

$$\frac{d}{dt}(\phi\psi' - \phi'\psi) + 2(\phi\psi' - \phi'\psi) = -\varepsilon f''(y)\phi^3. \tag{3.14}$$

Integrating (3.14), we obtain

$$(\phi\psi' - \phi'\psi)e^{2t} = -\varepsilon\int_0^t f''(y(s))\phi^3(s)e^{2s}ds \tag{3.15}$$

since $\phi = 0$ and $\psi = 0$ at $t = 0$.

Now we are interested in evaluating (3.15) at $t^\star$ for all those α for which $\frac{dt^\star}{d\alpha} = 0$ and for that we need a small lemma.

Lemma 3.1.6. *$f''(y(t)) > 0$ for some $t \in [0, t^\star]$, where y satisfies ($3.3a$)- ($3.3b$) and α is such that $\frac{dt^\star}{d\alpha} = 0$.*

Proof. Let y satisfy (3.3*a*)- (3.3*b*) for some α. Define $\alpha^\star = \inf\ B$, where

$$B = \{\alpha > 0 : f''(y(t)) > 0 \text{ for some t} \in [0, \mathrm{t}_0]\}.$$

Since $y(t_0) \to \infty$ as $\alpha \to \infty$ and f satisfies condition (b), we conclude that f'' cannot always remain 0. Thus the set B contains at least large values of α and is therefore nonempty. Hence $\alpha^\star$ exists. If $\alpha^\star = 0$, then for every $\alpha > 0$, we have $f''(t) > 0$ for some $t \in [0, t_0]$. In particular $f''(t) > 0$ for some $t \in [0, t_0]$ for all those α for which $\frac{dt^\star}{d\alpha} = 0$. Hence assume that $\alpha^\star > 0$. Note that for all $\alpha \leq \alpha^\star$, $f''(y(t)) = 0$ for all $t \in [0, t_0]$ and hence, it follows from (3.13*a*)-(3.13*b*) that $\psi = 0$. In particular $\psi(t^\star) = 0$, hence, $\frac{d^2t^\star}{d\alpha^2} = 0$ (this follows from (3.12)). We know that $\frac{dt^\star}{d\alpha}_{|\alpha=0} > 0$ (see Lemma 3.1.3), and so, $\frac{d^2t^\star}{d\alpha^2} = 0$ would imply $\frac{dt^\star}{d\alpha} > 0$ for all $\alpha \leq \alpha^\star$. Thus, $\frac{dt^\star}{d\alpha}$ cannot be equal to 0 for any $\alpha \leq \alpha^\star$. Hence, if $\frac{dt^\star}{d\alpha} = 0$ for some α, then clearly $\alpha > \alpha^\star$, and thus, the lemma is proved. □

Going back to (3.15), by Lemma 3.1.6, it also follows that $(\phi\psi' - \phi'\psi) < 0$ whenever $\frac{dt^\star}{d\alpha} = 0$. At $t^\star$, $\phi = 0$, $\phi' < 0$, hence we have $\psi(t^\star) < 0$. This in turn implies that $\frac{d^2t^\star}{d\alpha^2} < 0$ (follows from (3.12)). Thus, we proved that $\frac{d^2t^\star}{d\alpha^2} < 0$ whenever $\frac{dt^\star}{d\alpha} = 0$. Therefore $t^\star$ takes any value at most twice and Lemma 3.1.5 is proved. □

Thus, we establish the existence of at most two solutions to the problem (3.3*a*)-(3.3*b*).

Remark 3.1.3. *For each fixed $\varepsilon > 0$, $A(\varepsilon)$ is the maximum value of $t^\star$. The boundary value problem (3.2a)-(3.2b) will have either no solutions, or one solution or two solutions depending on whether $t^\star > A(\varepsilon)$, or $t^\star = A(\varepsilon)$ or $t^\star < A(\varepsilon)$ respectively.*

Proof of Theorem 6: Combining Lemma 3.1.3, Lemma 3.1.4, Lemma 3.1.5 and Remark 3.1.3 we have proved Theorem 6. □

Remark 3.1.4. *(i)The BVP (2.1a)-(2.1b) that we considered in Chapter 2 is a particular example of the BVP (3.1a)-(3.1b), with $f(y) = e^y$ and $A = 1$. However, note that the proof in Theorem 5 only gives us an existence of $A(\varepsilon)$, and we have no estimate on $A(\varepsilon)$. Hence, we cannot apply Theorem 5 to conclude that (2.1a)-(2.1b) has two solutions for any ε.*
(ii) Moreover, we have no idea how $A(\varepsilon)$ behaves as ε changes. For the specific case when $f(y) = e^y$, if $A(\varepsilon)$ were constant, say equal to 1, over a certain range of ε, then Theorem 5

would give us the existence of only one solution for all those values of ε*. However, at least we now know from the existence proof in Section 2.3 that* $A(\varepsilon) > 1$*, for* $\varepsilon \in (0, 19/100]$*. Hence the existence proofs discussed in the previous chapter are important on their own right.*

3.2 A UNIFORM EXPANSION OF THE SMALLER SOLUTION

Going back to the original variable we have the BVP:

$$\varepsilon v'' + 2v' + f(v) = 0 \tag{3.16a}$$

$$v(0) = 0, \ v(A) = 0. \tag{3.16b}$$

Note here $' = d/dx$. Define

$$\Phi(z) = \int_0^z \frac{dt}{f(t)},$$

so that the solution of

$$2u' + f(u) = 0, \quad u(A) = 0$$

is given by

$$\Phi(u) = -\frac{1}{2}(x - A),$$

and so

$$u = \Phi^{-1}\left(\frac{1}{2}(A - x)\right). \tag{3.17}$$

We choose A such that $A < 2\int_0^\infty dt/f(t)$ and then u is defined on $[0, A]$. (The condition is of course no restriction on A if the integral is divergent.) We will prove that for sufficiently small ε the asymptotic expansion of the smaller solution up to the first order is given by

$$v_1(x) = -\Phi^{-1}\left(\frac{A}{2}\right)e^{-\frac{2x}{\varepsilon}} + \Phi^{-1}\left(\frac{1}{2}(A - x)\right). \tag{3.18}$$

3.2.1 Theorem on the Asymptotic Expansion

Theorem 7. *Assume that $A < 2\int_0^\infty dt/f(t)$. Suppose that there exist constants $M > 0$ and $K > 0$ and an $\varepsilon_0 > 0$ such that (3.16a)-(3.16b) has a solution v for every $\varepsilon \in (0, \varepsilon_0]$ with the property that $\|v\| \leq M$ and $|\varepsilon v'(0)| \leq K$. Then $\|v - v_1\| = O(\varepsilon)$ as $\varepsilon \to 0$, where v_1 is given by (3.18).*

Remark 3.2.1. *In the special case when $f(y) = e^y$ and $A = 1$, it follows from the previous chapter that the smaller solution satisfies all the conditions mentioned in Theorem 7. However, in general, we haven't yet investigated to what classes of functions do these conditions apply.*

3.2.2 Proof of Theorem 7

Proof. Assume that the smaller solution v of (3.16a)-(3.16b) exists such that $\|v\| \leq M$ and $|\varepsilon v'(0)| \leq K$, for some $M, K > 0$. Also, let u be given by (3.17). Then v and u satisfy

$$\begin{aligned} \varepsilon v'' + 2v' + f(v) &= 0, \quad v(0) = 0,\ v(A) = 0, \\ \varepsilon u'' + 2u' + f(u) &= \varepsilon u'', \quad u(A) = 0, \end{aligned}$$

where $u'' = f(u)f'(u)/4$.

If $v'(0) = C$, then

$$\begin{aligned} v - u = h_1(x) - \frac{1}{2}\varepsilon C\left(e^{-\frac{2x}{\varepsilon}} - e^{-\frac{2A}{\varepsilon}}\right) - \\ \frac{1}{\varepsilon}\int_A^x e^{-\frac{2t}{\varepsilon}}dt\int_0^t e^{\frac{2s}{\varepsilon}}(f(v) - f(u))ds, \end{aligned} \tag{3.20}$$

where

$$h_1(x) = \int_x^A e^{-\frac{2s}{\varepsilon}}\int_0^s u'' e^{\frac{2\sigma}{\varepsilon}}d\sigma ds + \frac{\varepsilon u'(0)}{2}\left(e^{-\frac{2x}{\varepsilon}} - e^{-\frac{2A}{\varepsilon}}\right).$$

Here h_1 depends on ε, though we do not indicate this in our notation. Since u is defined and bounded on $[0, A]$, along with its derivatives, one can easily check that $h_1 = O(\varepsilon)$ uniformly in $[0, A]$ as $\varepsilon \to 0$. Set

$$g = v - u + \frac{\varepsilon C}{2}e^{-\frac{2x}{\varepsilon}}. \tag{3.21}$$

Then g satisfies

$$g = h_2(x) - \frac{1}{\varepsilon}\int_A^x e^{-\frac{2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \left(f\left(u + g - \frac{\varepsilon C}{2} e^{-\frac{2s}{\varepsilon}} \right) - f(u) \right) \, dsdt, \tag{3.22}$$

where

$$h_2(x) = h_1(x) + \frac{\varepsilon}{2} C e^{-\frac{2A}{\varepsilon}}$$

and $h_2 = O(\varepsilon)$ as $\varepsilon \to 0$. Note that since u is bounded and v is bounded by our assumption, g is uniformly bounded by R (say) for $\varepsilon \in (0, \varepsilon_0]$.

Since εC and g and u are bounded, we are dealing with a situation where f and f' are bounded. Thus

$$f\left(u + g - \frac{\varepsilon C}{2} e^{-\frac{2x}{\varepsilon}} \right) - f(u) = f(u+g) - f(u) + O(\varepsilon C e^{-\frac{2x}{\varepsilon}}),$$

where the constants implied in the O-term depend only on K and R, and certainly not on ε. Since $C\varepsilon \le K$ and $\|g\| \le R$, the contribution to the integral in (3.22) from the O-term is

$$\begin{aligned} O\left(\frac{1}{\varepsilon} \int_x^A e^{-\frac{2t}{\varepsilon}} dt \int_0^t e^{\frac{2s}{\varepsilon}} \varepsilon C e^{-\frac{2s}{\varepsilon}} ds \right) &= O\left(C \int_x^A t e^{-\frac{2t}{\varepsilon}} dt \right) \\ &= O(C\varepsilon^2) = O(\varepsilon) \end{aligned}$$

uniformly as $\varepsilon \to 0$. Hence (3.22) becomes

$$g = h_3(x, \varepsilon C, g) - \frac{1}{\varepsilon} \int_A^x e^{-\frac{2t}{\varepsilon}} dt \int_0^t e^{\frac{2s}{\varepsilon}} (f(u+g) - f(u)) ds, \tag{3.23}$$

where $h_3 = h_2 + h_c$, where $h_c = O(C\varepsilon^2)$ and thus $h_3 = O(\varepsilon)$ uniformly as $\varepsilon \to 0$.

Let

$$L = \max_{0 \le w \le u(0)+R} f'(w)$$

Then for $x \in [0, A]$, we have from (3.23) that

$$\begin{aligned} |g(x)| &\le |h_3| + \frac{L}{\varepsilon} \int_x^A e^{-\frac{2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} |g(s)| \, dsdt, && (3.24) \\ &\le |h_3| + \frac{L}{2} \|g\| (A - x). && (3.25) \end{aligned}$$

Substituting the estimate for $|g(x)|$ given by (3.25) into (3.24), we have

$$\begin{aligned} |g(x)| &\leq |h_4| + \frac{L}{\varepsilon}\int_x^A e^{-\frac{2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} \frac{L}{2}\|g\|(A-s)\; dsdt, &(3.26)\\ &\leq |h_4| + \left(\frac{L}{2}\right)^2 \|g\| \left(\frac{(A-x)^2}{2!} + a_1^2\varepsilon(A-x)\right), &(3.27) \end{aligned}$$

where

$$h_4 = \frac{L}{\varepsilon}\int_x^A e^{-\frac{2t}{\varepsilon}} \int_0^t e^{\frac{2s}{\varepsilon}} |h_3(s)|\; dsdt = O(h_3) = O(\varepsilon)$$

uniformly as $\varepsilon \to 0$. Again, substituting the estimate for $|g(x)|$ given by (3.27) into (3.24), we have

$$|g(x)| \leq |h_5| + \left(\frac{L}{2}\right)^3 \|g\| \left(\frac{(A-x)^3}{3!} + a_2^3\varepsilon\frac{(A-x)^2}{2!} + a_1^3\varepsilon^2(A-x)\right),$$

where a_2^3 and a_1^3 are positive constants and $h_5 = O(\varepsilon)$. After a finite number of steps, we obtain

$$\begin{aligned} |g(x)| &\leq |h_{n+2}| \\ &+ \left(\frac{L}{2}\right)^n \|g\| \left(\frac{(A-x)^n}{n!} + a_{n-1}^n\varepsilon\frac{(A-x)^{n-1}}{(n-1)!} + \ldots + a_1^n\varepsilon^{n-1}(A-x)\right), \quad (3.28) \end{aligned}$$

where a_i^n are positive constants for $i = 1, 2, \ldots n-1$ and $h_{n+2} = O(\varepsilon)$ uniformly as $\varepsilon \to 0$.

Fix some $r \in (0,1)$. Then we can choose n large enough such that

$$\left(\frac{L}{2}\right)^n \frac{A^n}{n!} < r. \tag{3.29}$$

For this n, there exist finitely many terms a_i^n, $i = 1, 2, \ldots, n-1$ and hence we can choose ε sufficiently small such that

$$\left(\frac{L}{2}\right)^n \left(a_{n-1}^n\varepsilon\frac{A^{n-1}}{(n-1)!} + a_{n-2}^n\varepsilon^2\frac{A^{n-2}}{(n-2)!} + \ldots + a_1^n\varepsilon^{n-1}A\right) < \frac{1-r}{2}. \tag{3.30}$$

Using (3.29) and (3.30), (3.28) can be written as

$$|g(x)| \leq |h_{n+2}| + \left(r + \frac{1-r}{2}\right)\|g\|,$$

hence

$$\|g\| = O(h_{n+2}) = O(\varepsilon)$$

as $\varepsilon \to 0$. Thus, from (3.21) it follows that

$$v(x) = u(x) - \frac{\varepsilon C}{2} e^{-\frac{2x}{\varepsilon}} + O(\varepsilon).$$

uniformly as $\varepsilon \to 0$. Since $v(0) = 0$, it follows that $\varepsilon C = u(0) + O(\varepsilon)$, and thus the conjectured uniform approximation is established.

$\square$

4.0 AN EXAMPLE OF ANOTHER SINGULARLY PERTURBED BVP

In this chapter, we will consider a well-known BVP introduced by G.F Carrier.

$$\varepsilon^2 y'' + 2(1-x^2)y + y^2 = 1 \tag{4.1}$$

$$y(-1) = 0,\ y(1) = 0. \tag{4.2}$$

Carrier introduced this problem in a survey paper [4] as an example to demonstrate that matched asymptotic expansion could give rise to spurious solutions, i.e to apparent approximate solutions that do not correspond to actual solutions! An autonomous version of this problem was considered by Carrier and Pearson in [5], where they considered

$$\varepsilon^2 y'' + y^2 = 1, \qquad y(-1) = y(1) = 0. \tag{4.3}$$

They showed that adding exponentially small terms in asymptotic expansions of solutions was consistent with matching and apparently produced valid approximate solutions (see below), which in fact were false solutions, something which perturbation theory failed to detect. More precisely, an approximation to an actual solution is given by

$$y = -1 + 3\ \mathrm{sech}^2\Big(\frac{1+x}{\sqrt{2}\varepsilon} + \ln(\sqrt{2}+\sqrt{3})\Big) + 3\ \mathrm{sech}^2\Big(\frac{1-x}{\sqrt{2}\varepsilon} + \ln(\sqrt{2}+\sqrt{3})\Big). \tag{4.4}$$

However, on adding a fourth term to the right side of (4.4) namely,

$$p(\xi) = \frac{12e^{\xi}}{(1+e^{\xi})^2} = 3\ \mathrm{sech}^2\frac{\xi}{\sqrt{2}},$$

where $\xi = (x-x_0)/\varepsilon$ and $1-|x_0| >> \varepsilon$ we obtain an approximate solution

$$\begin{aligned} y &= -1 + 3\ \mathrm{sech}^2\Big(\frac{1+x}{\sqrt{2}\varepsilon} + \ln(\sqrt{2}+\sqrt{3})\Big) + 3\ \mathrm{sech}^2\Big(\frac{1-x}{\sqrt{2}\varepsilon} + \ln(\sqrt{2}+\sqrt{3})\Big) \qquad (4.5)\\ &+ 3\ \mathrm{sech}^2\Big(\frac{x-x_0}{\sqrt{2}\varepsilon}\Big), \end{aligned}$$

which can be shown to satisfy the boundary value problem (4.3) except for exponentially small remainder terms. However, since the exact solution can be expressed in terms of elliptic functions, which is periodic, only $x_0 = 0$ corresponds to an actual solution as $\varepsilon \to 0$. O'Malley [17], using a phase plane analysis argued that any spike layers for autonomous problems of this type must be evenly-spaced. A lot of work has been done on the problem (4.3) (see [10], [11], [12], [13], [14], [17], [18]).

The method of matched asymptotic expansion also produces spurious solutions for the non-autonomous problem (4.1)-(4.2). Similar to the autonomous problem, even in this case, placing a spike at any position is consistent with matching. However such approximations do not correspond to true solutions. Since this problem is non-autonomous, it appears more interesting than the autonomous case. Bender and Orszag treated this problem in their book [3], where they produced many numerically generated solutions and discussed the application of the boundary layer theory to this example. The problem is known to have many solutions (see [1], [23]). Ai in [1] proved that the problem admits solutions that have internal spikes coalescing near $x = 0$ and that the spikes are separated by an amount $O(\varepsilon|\ln \varepsilon|)$, while single spikes can occur only near the end points. We are interested in four types of solutions, each of them having boundary layers at the end points.

First we seek for even solutions. Set $t = (1 + x)/\varepsilon$ and $z(t) = y((1 + x)/\varepsilon)$. Then (4.1)-(4.2) changes to

$$z'' + (z - \varepsilon t(\varepsilon t - 2))^2 = 1 + \varepsilon^2 t^2 (\varepsilon t - 2)^2$$

$$z(0) = 0, \; z'(1/\varepsilon) = 0.$$

Setting

$$u(t) = z(t) - \varepsilon t(\varepsilon t - 2) \tag{4.6}$$

we obtain the BVP:

$$u'' + u^2 = 1 + \varepsilon^2 t^2 (\varepsilon t - 2)^2 - 2\varepsilon^2 \tag{4.7}$$

$$u(0) = 0, \; u'(1/\varepsilon) = 0. \tag{4.8}$$

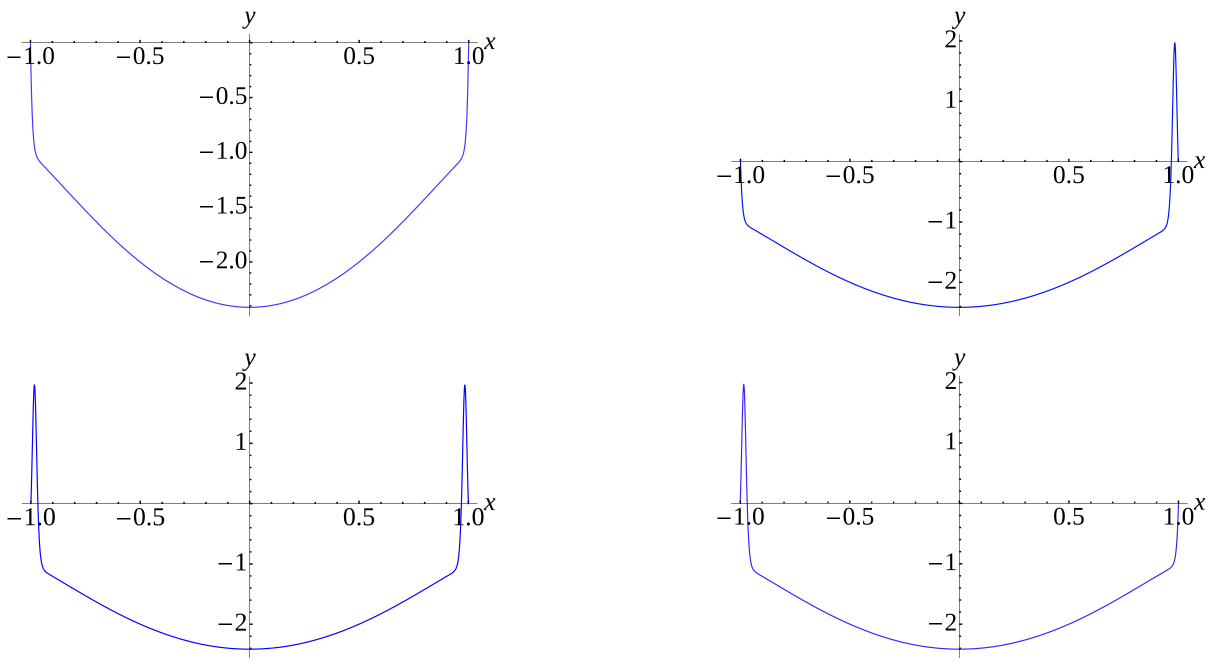

Figure 4: Some typical solutions of the BVP (4.1)-(4.2)

4.1 EXISTENCE THEOREMS

In this section we will prove existence of four different solutions of the BVP (4.7)-(4.8).

4.1.1 Existence theorem for the negative solution.

Theorem 8. *For $\varepsilon > 0$ sufficiently small,* (4.7)-(4.8) *has a solution that is always negative.*

Proof. To find one such solution we shall find a lower solution and an upper solution to (4.7)-(4.8).

We write (4.7) as

$$Lu = f(t, u)$$

$$u(0) = 0, \ u'(1/\varepsilon) = 0.$$

where

$$Lu = u'', \quad f(t,u) = 1 + \varepsilon^2 t^2 (2 - \varepsilon t)^2 - 2\varepsilon^2 - u^2.$$

For an upper solution consider

$$u_1'' = 0 \text{ with } \ u_1(0) = 0, \ u_1'(1/\varepsilon) = 0.$$

$u_1 = 0$ is the solution to the above problem and is an upper solution to (4.7)-(4.8) since

$$Lu_1 = 0 < 1 - 2\varepsilon^2 + \varepsilon^2 t^2 (2 - \varepsilon t)^2 - u_1^2 = f(t, u_1).$$

For a lower solution consider

$$u_2'' = 2 \text{ with } u_2(0) = 0, \ u_2'(1/\varepsilon) = -\sqrt{2}\varepsilon.$$

This can be solved exactly and if we denote the solution by u_2 then

$$Lu_2 = 2 \geq 1 + \varepsilon^2 t^2 (2 - \varepsilon t)^2 \geq 1 + \varepsilon^2 t^2 (2 - \varepsilon t)^2 - 2\varepsilon^2 - u_2^2 = f(t, u_2).$$

Hence u_2 is a lower solution.

By Theorem 4 we have at least one solution u between u_1 and u_2 satisfying $u(0) = 0, \ u'(1/\varepsilon) = 0.$ □

4.1.2 Existence theorem for the solution that has spikes at each end points.

Theorem 9. *For $\varepsilon > 0$ sufficiently small, (4.1)-(4.2) has a solution that has a spike at each end points.*

Proof. Again we are seeking for an even solution. So it is enough to consider the interval $[-1, 0]$ and we are looking for a solution y satisfying (4.1) with boundary conditions $y(-1) = 0$ and $y'(0) = 0$. We will use a shooting argument. Let $t = (1 + x)/\varepsilon$ and set $z(t) = y(x)$. Then (4.1)-(4.2) transforms to

$$\ddot{z} + z^2 = 1 - 2\varepsilon t(2 - \varepsilon t)z, \qquad z(0) = 0, \quad \dot{z}(\varepsilon^{-1}) = 0, \tag{4.9}$$

where $\dot{z} = dz/dt$. We consider an initial value problem:

$$\ddot{z} + z^2 = 1 - 2\varepsilon t(2 - \varepsilon t)z \tag{4.10}$$

$$z(0) = 0, \ \dot{z}(0) = \beta, \tag{4.11}$$

where $\beta \geq 0$. Our goal is to prove that there exists a β with $\dot{z}(0) = \beta$ such that z satisfies (4.9).

Set $\varepsilon = 0$ in (4.10) and consider

$$\ddot{v} + v^2 = 1 \tag{4.12}$$

$$v(0) = 0, \ \dot{v}(\infty) = 0. \tag{4.13}$$

The solution v of (4.12)-(4.13) is a part of the homoclinic orbit γ_0 with $\dot{v}(0) = 2/\sqrt{3}$. Any solution v of (4.12) with $v(0) = 0$ and $0 \leq \dot{v}(0) < 2/\sqrt{3}$ remains inside γ_0, and so it oscillates. In fact if v satisfes (4.12) with initial conditions $v(0) = 0$, $\dot{v}(0) = 0$, then from the phase plane analysis we note that v is periodic and always remains non-negative and has 0 as its minimum value. Denote this solution by v_0. Let $\tilde{t}$ be the time taken by v_0 to attain its first minimum. Then clearly $\tilde{t}$ is independent of ε. If t_1 is the time taken by v_0 to attain its first maximum, then since v_0 is symmetric, $\tilde{t} = 2t_1$. Similarly the time taken by v_0 to attain its second maximum is equal to $3t_1$, which is clearly independent of ε.

Let us denote the solution of (4.10)-(4.11) by z_0 for $\beta = 0$. For $\varepsilon > 0$ sufficiently small, by continuity, z_0 remains close to v_0 on the finite interval $[0, 3t_1]$. Hence, z_0 has a minimum in $(0, 3t_1]$. Denoting the location of the first minimum of z_0 by t_0^0, then $0 < t_0^0 < \varepsilon^{-1}$ for ε sufficiently small.

Now for each $\beta > 0$ denote the location of the first minimum of z_β by t_0^β (if it exists), where z_β is a solution of (4.10)-(4.11).

Let

$$\alpha = \sup\{\beta \geq 0 : \mathrm{t}_0^\beta \text{ exists and } 0 < \mathrm{t}_0^\beta < \varepsilon^{-1}\}.$$

We will show that z_β does not have a minimum for large β and that will prove that $\alpha < \infty$.

Suppose that v_β satisfies (4.12) with initial conditions $v_\beta(0) = 0$ and $v_\beta'(0) = \beta$. Note that for every $\beta \geq 0$, v_β has a maximum. Let v_β^M denote the maximum value of v_β attained at the point t_M. It follows from a phase plane analysis that $v_\beta^M \to \infty$ as $\beta \to \infty$. For the sake of completion, we will prove this as well as show that $t_M \to 0$ as $\beta \to \infty$.

Lemma 4.1.1. *$v_\beta^M \to \infty$ and $t_M \to 0$ as $\beta \to \infty$.*

Proof. The energy equation for v_β is

$$\dot{v}_\beta^2 = 2v_\beta - \frac{2}{3}v_\beta^3 + 2\beta^2. \tag{4.14}$$

Evaluating (4.14) at t_M, we obtain that

$$v_\beta^M\Big(\frac{1}{3}(v_\beta^M)^2 - 1\Big) = \beta^2,$$

which implies that $v_\beta^M \to \infty$ as $\beta \to \infty$. Thus we obtain that

$$\dot{v}_\beta^2 = 2(v_\beta - v_\beta^M) + \frac{2}{3}((v_\beta^M)^3 - v_\beta^3)$$

Choose β large enough such that $v_\beta^M > 3$. Integrating the above inequality over $(0, t_M)$, we obtain that

$$\begin{aligned}
\int_0^{t_M} ds &= \sqrt{\frac{2}{3}}\int_0^{v_\beta^M} \frac{du}{\sqrt{(v_\beta^M - u)((v_\beta^M)^2 + v_\beta^M u + u^2 - 3)}} \\
&< \sqrt{\frac{2}{3}}\int_0^{v_\beta^M} \frac{du}{\sqrt{(v_\beta^M - u)((v_\beta^M)^2 - 3)}} = \sqrt{\frac{2}{3}}\frac{\sqrt{v_\beta^M}}{\sqrt{(v_\beta^M)^2 - 3}} \to 0
\end{aligned}$$

as $\beta \to \infty$.

□

By an argument similar to that in Lemma 4.1.1, we can conclude that if t_m^β is the first $t > 0$ where $v_\beta(t_m^\beta) = 0$, then $t_m^\beta \to 0$ as $\beta \to \infty$, with $v'_\beta(t_m^\beta) = -\beta$. Choose β large enough such that $t_m^\beta < 1/2$. For ε sufficiently small, by continuity, we know that z_β is close to v_β on $[0,1]$. Hence there exist t_β^M and $t_\beta \in (0,1]$ such that $z_\beta(t_\beta^M) = z_\beta^M$ and $z_\beta(t_\beta) = 0$, where z_β^M is the maximum of z_β. Fix ε sufficiently small such that $t_\beta < 1$ and in the rest of the proof we will work with this fixed ε.

Now for all $\beta > 0$, multiplying (4.10) by $\dot{z}_\beta$ and integrating over $(0, t_\beta^M)$, we obtain

$$\begin{aligned} z_\beta^M\Big(\frac{z_\beta^{M^2}}{3} - 1\Big) &= \frac{\beta^2}{2} - 2\int_0^{t_\beta^M} \varepsilon s(2-\varepsilon s) z_\beta \dot{z}_\beta ds & (4.15) \\ &> \frac{\beta^2}{2} - \varepsilon t_\beta^M (2 - \varepsilon t_\beta^M) z_\beta^{M^2}. & (4.16) \end{aligned}$$

Similarly multiplying (4.10) by $\dot{z}_\beta$ and integrating over (t_β^M, t_β), we obtain

$$\begin{aligned} \frac{\dot{z}_\beta^2(t_\beta)}{2} &= z_\beta^M\Big(\frac{z_\beta^{M^2}}{3} - 1\Big) - 2\int_{t_\beta^M}^{t_\beta} \varepsilon s(2-\varepsilon s) z_\beta \dot{z}_\beta ds \\ &> z_\beta^M\Big(\frac{z_\beta^{M^2}}{3} - 1\Big) + \varepsilon t_\beta^M(2-\varepsilon t_\beta^M) z_\beta^{M^2} > \frac{\beta^2}{2}, \end{aligned}$$

where the last inequality follows from (4.16). Thus, we obtain $\dot{z}_\beta(t_\beta) < -\beta$. Let $\delta > 0$ be such that $\dot{z}_\beta(t_\beta + \delta) = -\beta/2$. If no such δ exists, then $\dot{z}_\beta < -\beta/2$ for all $t > t_\beta$, and so z_β has no minimum. Assume that such a δ exists. If $\delta < 1/2$, then by the Mean Value Theorem, there exists some $\zeta \in (t_\beta, t_\beta + \delta)$ such that

$$\ddot{z}_\beta(\zeta) = \frac{-\beta/2 - \dot{z}_\beta(t_\beta)}{\delta} > \frac{\beta}{2\delta} > \beta. \tag{4.17}$$

However, from (4.10), we note that

$$\ddot{z}_\beta < 1 - z_\beta^2 - z_\beta < 5/4$$

if $z_\beta < 0$. In particular, $\ddot{z}_\beta(\zeta) < 5/4$ which contradicts (4.17) for large β. Hence, we must have $\delta > 1/2$. However, if $\delta > 1/2$, then from the fact that $\dot{z}_\beta(t) < -\beta/2$ for $t \in (t_\beta, t_\beta + \delta)$, it follows that $z_\beta(t_\beta + \delta) < -\beta\delta/2 < -\beta/4$. Hence from (4.10), we again note that for all large β, $\ddot{z}_\beta(t_\beta + \delta) < 0$. Thus $\dot{z}_\beta(t) < 0$ for all $t > t_\beta + \delta$ and hence z_β never attains its minimum for large β.

Thus we proved that α exists and $\alpha < \infty$. Now for $\beta = \alpha$, it follows from the above argument that if $t_\alpha > 0$ is the first time that $z_\alpha(t_\alpha) = 0$, then $\dot{z}_\alpha(t_\alpha) < -\alpha$. Hence, if z_α has a minimum, then the minimum must be negative.

Let $\tau > t_\alpha$ be the first time such that $\dot{z}_\alpha(\tau) = 0$. We will show that τ exists. By the existence theory, we know that z_α has a maximum interval of existence. If τ does not exist, then we can have two possibilities, namely z_α goes unbounded negatively at or before $1/\varepsilon$ or z_α exists on $[0, 1/\varepsilon]$ with $\dot{z}_\alpha(t) < 0$ for all $t > t_\alpha^M$.
(i) If the first possibility occurs, then there exists some $\hat{t} < 1/\varepsilon$ such that $z_\alpha(t) < -3$ with $\dot{z}_\alpha(t) < 0$ for all $t > \hat{t}$, as long as z_α exists. By continuous dependence of solutions on initial conditions, we must have a $\beta < \alpha$ such that z_β behaves like z_α on $[0, \hat{t}]$. But if z_β crosses $-5/2$, then z_β cannot have a minimum, a contradiction.
(ii) If the second possibility occurs, then again by continuous dependence of solutions on initial conditions, we must have a $\beta < \alpha$ such that z_β behaves like z_α on $[0, 1/\varepsilon]$. But, z_β has a minimum in $[0, 1/\varepsilon]$. Hence the second possibility cannot occur as well.
Hence τ must exist. If $\tau < 1/\varepsilon$, then by the continuous dependence of solutions on initial conditions, we must have a $\beta > \alpha$ such that t_0^β exists, and that would contradict the definition of α. Hence $\tau = 1/\varepsilon$.

By our assumption z_α is even, so τ must be the location of the minimum for z_α, i.e $\tau = t_0^\alpha$. Hence we get an existence of a solution of (4.1) - (4.2) that has spikes at each end points. $\square$

4.1.3 Existence theorem for the solution that has a spike at the left end point.

Theorem 10. *For $\varepsilon > 0$ sufficiently small, (4.1)-(4.2) has a solution that has a spike at the left but not at the right end point.*

Proof. As in the proof of Theorem 9 we write (4.1)-(4.2) as an initial value problem:

$$\varepsilon^2 y'' + 2(1 - x^2)y + y^2 = 1 \tag{4.18}$$

$$y(-1) = 0,\ y'(-1) = \beta, \tag{4.19}$$

with $\beta > 0$. Denote the solution of (4.18)-(4.19) by y_β. Theorem 9 gives us the existence of a solution $y_\alpha(x)$ that has spikes at both the ends with $y'_\alpha(-1) = \alpha > 0$. Note that $y_\alpha(x_\alpha) = 0$ for some $0 < x_\alpha < 1$. Moreover y_α attains it only minimum at the point $x^\alpha_m = 0$. Let x^β_m be the location of the first minimum of y_β, where y_β satisfies (4.18)-(4.19) with $\beta > \alpha$ and close to α. Let x_β be the first time such that $y_\beta(x_\beta) = 0$, where $x^\beta_m < x_\beta < 1$, provided x_β exists.

Let

$$\gamma = \sup\{\beta \geq \alpha : \mathrm{x}_\beta \text{ exists with } \mathrm{x}^\beta_\mathrm{m} < \mathrm{x}_\beta < 1\}.$$

Then $\gamma > \alpha$, where $\alpha > 0$ is the initial velocity of y_α. In the proof of Theorem 9 we showed that for $\beta > 0$ large enough y_β has no minimum. This in turn implies that x_β does not exist for large β. Hence $\gamma < \infty$ and thus x^γ_m exists. We will prove that x_γ exists. If x_γ did not exist, then either y_γ would have attained a negative maximum at some point $x_M \in (x^\gamma_m, 1)$ with $y_\gamma(x) < 0$ for $x \in [x_M, 1]$ or y_γ would be increasing with $y_\gamma(x) < 0$ for $x \in [x^\beta_m, 1]$. In either of these two cases, by continuous dependence on initial conditions, we will find a $\beta < \gamma$ with y_β behaving like y_γ. But this is not possible, since x_β exists. Hence x_γ must exist. Now from the definition of γ, we conclude that $x_\gamma = 1$.

□

4.1.4 Existence theorem for the solution that has a spike at the right end point.

Theorem 11. *For $\varepsilon > 0$ sufficiently small, (4.1)-(4.2) has a solution that has a spike at the right but not at the left end point.*

Proof. From Theorem 10 we know that there is a solution $y(x)$ that has a spike at the left end point with $y(-1) = 0$ and $y(1) = 0$. Now set $\tau = -x$ and let $z(\tau) = y(-x)$. Then z satisfies

$$\varepsilon^2 z'' + 2(1 - \tau^2)z + z^2 = 1 \tag{4.20}$$

$$z(-1) = 0, \;\; z(1) = 0 \tag{4.21}$$

and thus z has a spike at the right end point.

□

4.2 ASYMPTOTIC EXPANSION OF THE NEGATIVE SOLUTION

Consider the BVP:

$$u'' + u^2 = 1 \tag{4.22}$$

$$u(0) = 0,\ \ u(\infty) = -1. \tag{4.23}$$

A solution u_1 to the above problem is a part of the homoclinic orbit based at $(-1, 0)$ and hence would satisfy $u_1'(\infty) = 0$. Set $w_1 = u_1 + 1$. Then we have

$$w_1'' + w_1^2 - 2w_1 = 0 \tag{4.24}$$

$$w_1(0) = 1,\ \ w_1(\infty) = 0. \tag{4.25}$$

Solving, we obtain

$$\frac{w_1'^2(t)}{2} + \frac{w_1^3(t)}{3} - w_1^2(t) = 0.$$

Thus

$$w_1'(t) = -\sqrt{2}w_1(t)\sqrt{1 - w_1(t)/3}.$$

Using the boundary conditions, we obtain

$$w_1(t) = 3\ \mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right).$$

Hence

$$u_1(t) = 3\ \mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) - 1. \tag{4.26}$$

Set

$$g(t) = 1 - \sqrt{1 + \varepsilon^2 t^2 (2 - \varepsilon t)^2}.$$

Theorem 12. *If u is a negative solution of (4.7)-(4.8) then $v = g + u_1$ is a uniform approximation of u, namely, there exists a constant $A > 0$ such that $|u(t) - v(t)| < A\varepsilon^2$ for all $t \in [0, \varepsilon^{-1}]$, provided $\varepsilon > 0$ is sufficiently small.*

Remark 4.2.1. *In the next chapter, we will prove asymptotic expansions for solutions with three or fewer critical points and that would also include this solution as well, but the estimates will only be to order $O(\varepsilon)$.*

Proof. To prove Theorem 12, we first note that $g'(\varepsilon^{-1}) = 0$. Hence v satisfies

$$v'' + v^2 = 1 + g^2 + 2u_1 g + g'' \tag{4.27}$$

$$v(0) = 0,$$

$$v'(1/\varepsilon) = -3\sqrt{2}\text{sech}^2\left(\tfrac{1}{\sqrt{2}\varepsilon} + \text{arctanh}\sqrt{\tfrac{2}{3}}\right)\tanh\left(\tfrac{1}{\sqrt{2}\varepsilon} + \text{arctanh}\sqrt{\tfrac{2}{3}}\right).$$

One can prove that $v'(1/\varepsilon) = O(e^{-\sqrt{2}/\varepsilon})$ as $\varepsilon \to 0$. v meets the same first boundary condition as u but fails to meet the second boundary condition. Set $w = u - v$. Then from (4.7) and (4.27), we obtain

$$w'' + (u+v)w = -2\varepsilon^2 - 2(u_1+1)g - g'' \tag{4.28}$$

$$w(0) = 0,\ \ w'(1/\varepsilon) = -v'(1/\varepsilon). \tag{4.29}$$

Denote the right hand side of (4.28) by $h(t)$.

Lemma 4.2.1. *$h(t)$ changes sign (positive to negative) exactly once on the interval $[0, \varepsilon^{-1}]$. Denote that point by t_ε. Then $t_\varepsilon < 1/(2\varepsilon)$ and $v(t_\varepsilon) < -1$.*

Proof. To prove this, one can check that $h(1/(8\varepsilon)) > 0.37\varepsilon^2$ and $h(1/(2\varepsilon)) < -211\varepsilon^2/125$. This shows that h has at least one zero on the interval $(1/(8\varepsilon), 1/(2\varepsilon))$. To show that h has exactly one zero on $[0, \varepsilon^{-1}]$, first consider

$$\begin{aligned}
2\varepsilon^2 t^2 + g(t) &= 2\varepsilon^2 t^2 + 1 - \sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2} \\
&= 2\varepsilon^2 t^2 - \frac{\varepsilon^2 t^2(2-\varepsilon t)^2}{1+\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} \\
&= \frac{\varepsilon^2 t^2\left(2\left(1+\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}\right) - (2-\varepsilon t)^2\right)}{1+\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} \\
&= \frac{\varepsilon^2 t^2(4-(2-\varepsilon t)^2)}{1+\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} + f(t,\varepsilon)\varepsilon^2 \\
&= \frac{\varepsilon^2 t^2(4\varepsilon t - \varepsilon^2 t^2)}{1+\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} + f(t,\varepsilon)\varepsilon^2,
\end{aligned}$$

where $f(t,\varepsilon) > 0$ and $f(t,\varepsilon),\ f'(t,\varepsilon) \to 0$ uniformly as $\varepsilon \to 0$ for all $t \in [0,6]$. Hence for all $t \in [0,6]$

$$\frac{-g(t)}{\varepsilon^2} = 2t^2 - s(t,\varepsilon), \tag{4.30}$$

where $s(t,\varepsilon) > 0$ and $s(t,\varepsilon),\ s'(t,\varepsilon) \to 0$ uniformly as $\varepsilon \to 0$. Thus we have

$$-(g(t)(1+u_1(t))/\varepsilon^2)' = 2t^2u_1'(t) + 4t(1+u_1(t)) - s(t,\varepsilon)u_1'(t) - s'(t,\varepsilon)(1+u_1(t))$$

on $[0,6]$. Note that the last two terms are small since $u_1'(t)$ and $1+u_1(t)$ are bounded. Let

$$\begin{aligned}\rho(t) &= 2t^2u_1'(t) + 4t(1+u_1(t)) \\ &= 6\sqrt{2}t \ \mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right)\left(\sqrt{2} - t \ \tanh\left(\frac{t}{\sqrt{2}} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right)\right)\end{aligned}$$

Clearly for $t \in (0,6]$, $\rho(t) = 0$ if and only if

$$\sigma(t) = \sqrt{2} - t \ \tanh\left(\frac{t}{\sqrt{2}} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) = 0$$

We can easily check that $\sigma' < 0$ in $(0,6]$ and that $\sigma(1) > 0$ while $\sigma(2) < 0$. This implies that σ has exactly one zero in $(1,2)$ and that $\sigma(t) < 0$ for $t \in [2,6]$. Hence, we conclude that $\rho(t) > 0$ in $(0,1]$ and ρ has exactly one root in the interval $(1,2)$. Figure 5 is the graph of $\rho(t)$ on $[0,6]$.

Hence for sufficiently small ε, we proved that $-(g(t)(1+u_1(t)))'$ has exactly one zero $t_0 \in (0,6]$. One can prove that $-g'''(t) < 0$ for all $t \in [0,\varepsilon^{-1}]$ by setting $\varepsilon t = x$ and considering $\tilde{g}(x) = 1 - \sqrt{1+x^2(2-x)^2}$ for $x \in [0,1]$. In Figure 6, we give the graph of $\tilde{g}'''(x)$ on $[0,1]$.

Hence

$$\begin{aligned}h'(t)/\varepsilon^2 &= -2(g(t)(1+u_1(t)))'/\varepsilon^2 - g'''(t)/\varepsilon^2 \\ &< 0\end{aligned} \tag{4.31}$$

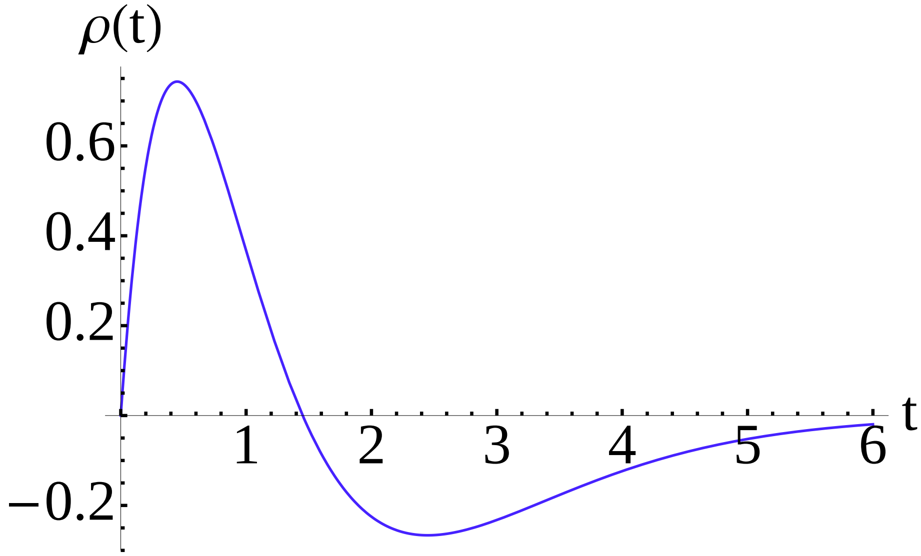

Figure 5: $\rho(t), t \in [0, 6]$

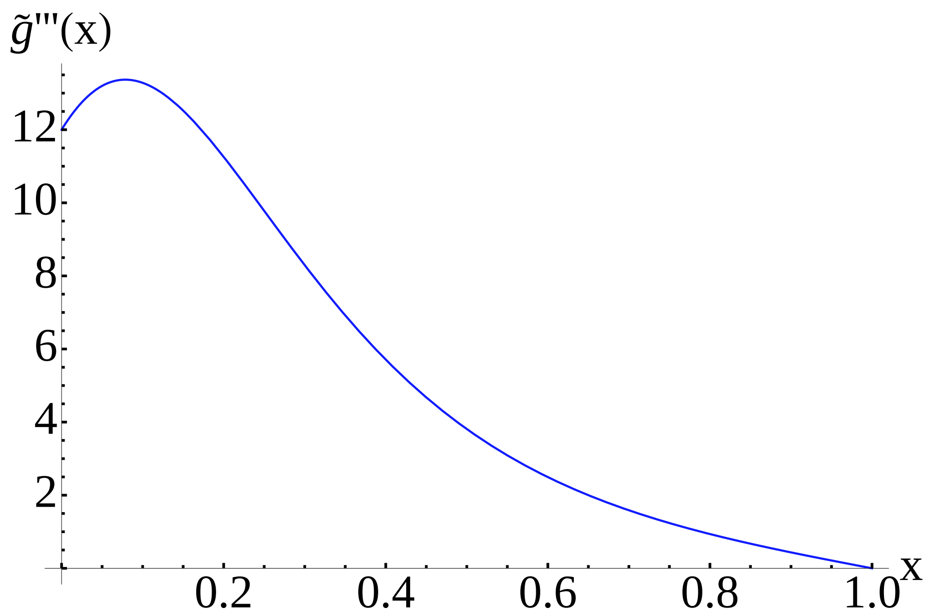

Figure 6: $\tilde{g}'''(x)$

for $t \in (t_0, 6]$. Also note that

$$-g(t) - \varepsilon^2 t^2 = \frac{\varepsilon^2 t^2((2-\varepsilon t)^2}{\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} - \varepsilon^2 t^2 > 0$$

on $[0, (2\varepsilon)^{-1}]$. In other words, $-g(t)/\varepsilon^2 > t^2$ on $[0, (2\varepsilon)^{-1}]$.

Also

$$-\frac{g'(t)}{\varepsilon^2} = \frac{2t(1-\varepsilon t)(2-\varepsilon t)}{\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} \leq 4t.$$

Using the above two facts and $u_1' < 0$, we obtain

$$\begin{aligned} -\left(\frac{g(t)(1+u_1(t))}{\varepsilon^2}\right)' &= \frac{-g(t)u_1'(t) - g'(t)(1+u_1(t))}{\varepsilon^2} \\ &< t^2 u_1'(t) + 4t(1+u_1(t)). \end{aligned} \tag{4.32}$$

On the interval $[6, (2\varepsilon)^{-1}]$, we claim that (4.32) < 0.

To see that (4.32) < 0, we compute it.

$$\begin{aligned} (4.32) &= -3\sqrt{2}t^2 \mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right)\tanh\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right) \\ &+ 12t\mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right) \\ &= t\mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right)\left(-3t\sqrt{2}\tanh\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right) + 12\right) \\ &\leq t\mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right)\left(-18\sqrt{2}\tanh\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right) + 12\right) \\ &< t\mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right)\left(-27\frac{\sqrt{2}}{2} + 12\right) < 0. \end{aligned}$$

In the second last step we have used the fact that $\tanh\left(\frac{t}{\sqrt{2}} + \operatorname{arctanh}\sqrt{\frac{2}{3}}\right) > 3/4$ on $[6, (2\varepsilon)^{-1}]$ for $\varepsilon > 0$ sufficiently small.

Thus, we obtain

$$h'(t)/\varepsilon^2 = -2(g(t)(1+u_1(t)))'/\varepsilon^2 - g'''(t)/\varepsilon^2 < 0$$

for all $t \in [6, (2\varepsilon)^{-1}]$ and hence $h'(t) < 0$ on $(t_0, (2\varepsilon)^{-1}]$. For $t \in [1/(2\varepsilon), 4/(5\varepsilon)]$, note that $h'(t) = -g'''(t) + O(e^{-1/\varepsilon})$. We can check that $g'''(t) > 9/10$ on that interval. Hence for

ε sufficiently small, $h'(t) < 0$ for $t \in [1/(2\varepsilon), 4/(5\varepsilon)]$. Now we will consider the interval $(4/(5\varepsilon), 1/\varepsilon]$. Note that

$$h'(t) = 12\text{sech}^2\left(\frac{t}{\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right)$$
$$\left(\tanh\left(\frac{t}{\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right)\frac{g(t)}{\sqrt{2}} + \frac{\varepsilon^2 t(1-\varepsilon t)(2-\varepsilon t)}{\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}}\right) - g'''(t)$$

Note that since $g < 0$, we have

$$\tanh\left(\frac{t}{\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right)\frac{g(t)}{\sqrt{2}} < 0.$$

Moreover,

$$\tanh\left(\frac{t}{\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right)\frac{g(t)}{\sqrt{2}} = O(1)$$

uniformly on $(4/(5\varepsilon), 1/\varepsilon]$, while

$$\frac{\varepsilon^2 t(1-\varepsilon t)(2-\varepsilon t)}{\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} = O(\varepsilon)$$

uniformly as $\varepsilon \to 0$. Hence for sufficiently small ε,

$$h'(t) < -g'''(t) < 0$$

on $(4/(5\varepsilon), 1/\varepsilon]$.

Choose ε small such that $(8\varepsilon)^{-1} > t_0$, then $h'(t) < 0$ for $t \in [(8\varepsilon)^{-1}, \varepsilon^{-1}]$. We already know that h has a zero in the interval $[(8\varepsilon)^{-1}, (2\varepsilon)^{-1}]$ and $h(t) > 0$ for $t \in [0, (8\varepsilon)^{-1}]$ (recall $h((8\varepsilon)^{-1}) > 0$ and $h' > 0$ on $(0, t_0)$). Since $h' < 0$ for $t \geq (8\varepsilon)^{-1}$, h has only zero. This gives us the existence of $t_\varepsilon \in ((8\varepsilon)^{-1}, (2\varepsilon)^{-1})$.

To prove $v(t_\varepsilon) < -1$, note that

$$\begin{aligned} v'(t) &= -3\sqrt{2}\text{sech}^2\left(\frac{t}{\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right)\tanh\left(\frac{t}{\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right) \\ &- \frac{2\varepsilon^2 t(1-\varepsilon t)(2-\varepsilon t)}{\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} < 0 \end{aligned}$$

on the interval $[0, \varepsilon^{-1}]$. Moreover

$$v(1/8\varepsilon) = 3\text{sech}^2(1/8\sqrt{2}\varepsilon + \text{arctanh}\sqrt{2/3}) - \sqrt{1 + 15^2/64^2} < -1$$

for ε sufficiently small. Since $v(t)$ is decreasing and $v(1/8\varepsilon) < -1$, we conclude that $v(t_\varepsilon) < -1$. This completes the proof of the lemma. □

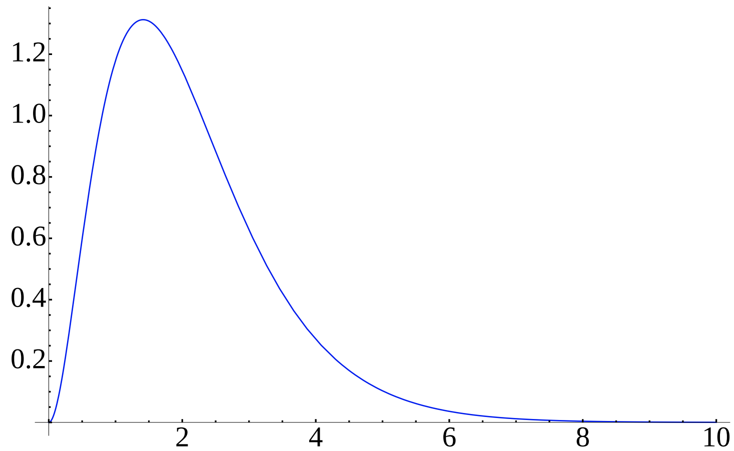

Figure 7: $48t^2\exp(-2(t/\sqrt{2}+\operatorname{arctanh}\sqrt{2/3}))$

Lemma 4.2.2. *$h(t)$ can be bounded by some function of ε^2. More precisely,*

$$|h(t)| < 3.5\varepsilon^2 \tag{4.33}$$

uniformly in $t \in [0, \varepsilon^{-1}]$ for small ε.

Proof. To prove this, first observe that we have already proved $-g(t)/\varepsilon^2 \leq 2t^2$ for all $t \in [0, \varepsilon^{-1}]$ (see (4.30)). Now

$$\begin{aligned} -2(u_1(t)+1)g(t)/\varepsilon^2 &\leq 12t^2\operatorname{sech}^2(t/\sqrt{2}+\operatorname{arctanh}\sqrt{2/3}) \\ &< 48t^2\exp(-2(t/\sqrt{2}+\operatorname{arctanh}\sqrt{2/3})) < 1.5. \end{aligned}$$

(see Figure 7).

Moreover,

$$\begin{aligned} \frac{-2\varepsilon^2 - g''(t)}{\varepsilon^2} &= -2 - \frac{4\varepsilon^2t^2(1-\varepsilon t)^2(2-\varepsilon t)^2}{(1+\varepsilon^2t^2(2-\varepsilon t)^2)^{3/2}} + \frac{4(1-\varepsilon t)^2}{\sqrt{1+\varepsilon^2t^2(2-\varepsilon t)^2}} \\ &- \frac{2\varepsilon t(2-\varepsilon t)}{\sqrt{1+\varepsilon^2t^2(2-\varepsilon t)^2}} \\ &< 2 \end{aligned}$$

for all $t \in [0, \varepsilon^{-1}]$. Hence

$$\frac{h(t)}{\varepsilon^2} = \frac{-2\varepsilon^2 - g''(t) - 2(u_1(t)+1)g(t)}{\varepsilon^2} < 3.5$$

on $[0, \varepsilon^{-1}]$. Since $-2\varepsilon^2 - g''(t)$ is decreasing in t, so

$$\frac{-2\varepsilon^2 - g''(t)}{\varepsilon^2} > \frac{-2\varepsilon^2 - g''(\varepsilon^{-1})}{\varepsilon^2} = -2 - \sqrt{2} > -3.5.$$

Also since $-2(u_1(t)+1)g(t)/\varepsilon^2 \geq 0$ for all $t \in [0, \varepsilon^{-1}]$, we have

$$\frac{h(t)}{\varepsilon^2} = \frac{-2\varepsilon^2 - g''(t) - 2(u_1(t)+1)g(t)}{\varepsilon^2} > -3.5$$

and thus we obtain (4.33). □

Remark 4.2.2. *Note from (4.29) that*

$$w'\left(\frac{1}{\varepsilon}\right) = \frac{3\sqrt{2}Ce^{\frac{-2}{\sqrt{2}\varepsilon}}\left(1 - Ce^{\frac{-2}{\sqrt{2}\varepsilon}}\right)}{\left(1 + Ce^{\frac{-2}{\sqrt{2}\varepsilon}}\right)^3}$$

where $C = e^{-2 arctanh \sqrt{\frac{2}{3}}}$.

Clearly $w'(1/\varepsilon) > 0$. *Also since* $3\sqrt{2}C < 1/2$, *we have*

$$0 < w'(1/\varepsilon) < \frac{1}{2}e^{-2/\sqrt{2}\varepsilon}. \tag{4.34}$$

Lemma 4.2.3. $|w(t)| < (K+2)\varepsilon^2$ *for all* $t \in [0, 1/\varepsilon]$ *for sufficiently small* ε *where* $K = 3.5$.

Proof. We shall prove this by contradiction. The proof will be divided into four cases. We will be using the fact that $-v(t)$ is an increasing function of t and $v(t) < -1$ for $t \geq t_\varepsilon$, where t_ε has been defined in Remark 4.2.1.

Case 1: $w(t) > (K+1)\varepsilon^2$ for some $t \in (0, t_\varepsilon]$.

Since $w(0) = 0$, for this case to hold there would exist some $t_1 \in (0, t_\varepsilon)$ such that $w(t_1) = (K+1)\varepsilon^2$ and $w'(t_1) \geq 0$. Since $h(t)$ is nonnegative on $(0, t_\varepsilon]$ and $u \leq 0$ we have from (4.28)

$$w''(t_1) \geq -(u+v)(t_1)w(t_1) \geq -v(t_1)(K+1)\varepsilon^2 > 0.$$

Hence there exists some maximal $\delta > 0$ with $t_1 + \delta \leq \varepsilon^{-1}$ such that $w'(t)$ is increasing on $(t_1, t_1 + \delta)$.

Case (i) $t_1 + \delta < \varepsilon^{-1}$.

Note that $w'(t) > 0$ for $t \in (t_1, t_1 + \delta)$, hence $w(t) > (K+1)\varepsilon^2$ on $(t_1, t_1 + \delta]$. If $t_1 + \delta < t_\varepsilon$ then since $-v(t)$ is an increasing function of t and $u \leq 0$, we have

$$w''(t) \geq -(u+v)(t)w(t) > -v(t_1)(K+1)\varepsilon^2$$

for all $t \in (t_1, t_1 + \delta]$. In particular $w''(t_1 + \delta) > 0$ which implies that $w'(t)$ is increasing to the right of $t_1 + \delta$. This contradicts the maximality of δ.

Hence $t_1 + \delta \geq t_\varepsilon$. Thus we proved $w'(t_\varepsilon) > 0$ and hence $w(t_\varepsilon) > (K+1)\varepsilon^2$.

Since $v(t) < -1$ for $t \in (t_\varepsilon, \varepsilon^{-1})$ and $h(t)$ is bounded below by $-K\varepsilon^2$, we obtain

$$w''(t_\varepsilon) > -K\varepsilon^2 - (u(t_\varepsilon) - 1)(K+1)\varepsilon^2 > \varepsilon^2.$$

Note that if $w(t) > (K+1)\varepsilon^2$ for any $t \in (t_\varepsilon, \varepsilon^{-1})$ then

$$w''(t) > -K\varepsilon^2 - (u(t) - 1)(K+1)\varepsilon^2 \geq \varepsilon^2. \tag{4.35}$$

Hence $w'(t)$ always increases on $(t_\varepsilon, \varepsilon^{-1})$ and thus $w(t) > (K+1)\varepsilon^2$. Hence (4.35) holds on $(t_\varepsilon, \varepsilon^{-1})$. This in turn implies that

$$w'(t) > \varepsilon^2(t - t_\varepsilon) + w'(t_\varepsilon)$$

for $t \in (t_\varepsilon, \varepsilon^{-1})$ and therefore

$$w'(1/\varepsilon) > \varepsilon^2(1/\varepsilon - t_\varepsilon) + w'(t_\varepsilon).$$

Since $t_\varepsilon < 1/2\varepsilon$, we obtain $w'(1/\varepsilon) > \varepsilon/2$ contradicting (4.34) for sufficiently small ε.

Case (ii) $t_1 + \delta = \varepsilon^{-1}$.

Since $w'(t) > 0$ for $t \in (t_1, t_1 + \delta)$ and $t_\varepsilon < 1/2\varepsilon$, we have $w(t_\varepsilon) > w(t_1) = (K+1)\varepsilon^2$. Then exactly like Case (i) we will have $w'(1/\varepsilon) > \varepsilon/2$ contradicting (4.34) for sufficiently small ε.

Case 2: $w(t) > (K+1)\varepsilon^2$ for some $t \in (t_\varepsilon, 1/\varepsilon)$.

Let $t_1 \in (t_\varepsilon, 1/\varepsilon)$ be the first time such that $w(t_1) = (K+1)\varepsilon^2$. Then $w'(t_1) \geq 0$.

Using the fact that $v(t_1) < -1$ and $h(t)$ is bounded below by $-K\varepsilon^2$ note that from (4.28) we have

$$\begin{aligned} w''(t_1) &> -K\varepsilon^2 - (u(t_1) + v(t_1))(K+1)\varepsilon^2 \\ &> -K\varepsilon^2 - (u(t_1) - 1)(K+1)\varepsilon^2 \geq \varepsilon^2. \end{aligned}$$

Hence $w(t) > (K+1)\varepsilon^2$ on a small interval to the right of t_1. Note that if $w(t) > (K+1)\varepsilon^2$ for any $t \in (t_1, \varepsilon^{-1})$, then

$$w''(t) > -K\varepsilon^2 - (u(t) - 1)(K+1)\varepsilon^2 \geq \varepsilon^2.$$

The above inequality shows that w' always increases on (t_1, ε^{-1}) and hence $w''(t) > \varepsilon^2$ for all $t \in (t_1, \varepsilon^{-1})$

If $1/\varepsilon - t_1 > 1$, then we have $w'(1/\varepsilon) > \varepsilon^2(1/\varepsilon - t_1) + w'(t_1) > \varepsilon^2$, a contradiction to (4.34) for sufficiently small $\varepsilon > 0$..

Hence assume $1/\varepsilon - t_1 \leq 1$. Then since $w''(t) > 0$ for $t > t_1$, we have $w'(t) < w'(1/\varepsilon)$ for all $t_1 < t < 1/\varepsilon$. Thus

$$\begin{aligned} w(1/\varepsilon) &< w'(1/\varepsilon)(1/\varepsilon - t_1) + (K+1)\varepsilon^2 \\ &\leq w'(1/\varepsilon) + (K+1)\varepsilon^2. \end{aligned}$$

Using (4.34) we obtain $w(1/\varepsilon) < (K+2)\varepsilon^2$ for sufficiently small $\varepsilon > 0$.

Since $w' > 0$ on (t_1, ε^{-1}), we obtain $w(t) < w(\varepsilon^{-1})$ on that same interval. Also by our assumption, $w(t) \leq (K+1)\varepsilon^2$ for $t \in (0, t_1]$, hence $w(t) < (K+2)\varepsilon^2$ for all $t \in [0, \varepsilon^{-1}]$.

Case 3: $w(t) < -K\varepsilon^2$ for some $t \in [t_\varepsilon, 1/\varepsilon]$.

Let $t_1 \in [t_\varepsilon, 1/\varepsilon)$ be the first time such that $w(t_1) = -K\varepsilon^2$.Then $w'(t_1) \leq 0$. Since $h(t_1) \leq 0$ and $v(t_1) < -1$ we have

$$w''(t_1) \leq -(u+v)(t_1)w(t_1) = K\varepsilon^2(u+v)(t_1) < -K\varepsilon^2.$$

Hence $w(t) < -K\varepsilon^2$ on a small interval to the right of t_1. Moreover, if $w(t) < -K\varepsilon^2$ for any $t \in (t_1, \varepsilon^{-1})$ then

$$w''(t) < -v(t)w(t) < -v(t_1)(-K\varepsilon^2) < -K\varepsilon^2. \tag{4.36}$$

(4.36) implies that w' is decreasing on $(t_1, \varepsilon^{-1}]$ and hence $w(t) < -K\varepsilon^2$ on that interval. Hence (4.36) holds on $(t_1, \varepsilon^{-1}]$. This indicates that $w'(1/\varepsilon) < -K\varepsilon^2(1/\varepsilon - t_1) + w'(t_1) < 0$ contradicting the sign of $w'(1/\varepsilon)$ [see (4.34)].

Case 4: $w(t) < -(K+1)\varepsilon^2$ for some $t \in (0, t_\varepsilon)$.

Let $w(t_1) = -(K+1)\varepsilon^2$ and $w'(t_1) \leq 0$ for some $t_1 \in (0, t_\varepsilon)$. Here we will have two cases:

Case a) $t_1 < 1/2$.

Case b) $t_1 \geq 1/2$.

Case b): First of all, note that since $K = 3.5$,

$$K + 1 > 5K/4. \tag{4.37}$$

Now

$$\begin{aligned} v\left(\frac{1}{2}\right) &= 3\text{sech}^2\left(\frac{1}{2\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right) - \sqrt{1 + \varepsilon^2 t^2 (2 - \varepsilon t)^2} \\ &< 3\text{sech}^2\left(\frac{1}{2\sqrt{2}} + \text{arctanh}\sqrt{\frac{2}{3}}\right) - 1 \\ &< -0.45. \end{aligned} \tag{4.38}$$

Hence from (4.37) and (4.38), we obtain

$$h(1/2) + 2(K+1)\varepsilon^2 v(1/2) < K\varepsilon^2 + \frac{5}{2}K\varepsilon^2 v(1/2) < -0.125K\varepsilon^2 < 0. \tag{4.39}$$

Since $h'(t) < 0$ for $t > t_0$ and $v'(t) < 0$, $h(t) + 2(K+1)\varepsilon^2 v(t)$ decreases for $t \geq t_0$. ($t_0 \approx 1.45$; see fig 1.) We wish to prove that $h(t) + 2(K+1)\varepsilon^2 v(t)$ decreases for $t \geq 1/2$. To prove that, from (4.31) and (4.32) note that

$$\begin{aligned} h'(t) + 2(K+1)\varepsilon^2 v'(t) &< 2\varepsilon^2 t^2 u_1'(t) + 8\varepsilon^2 t(1 + u_1(t)) \\ &\quad -g'''(t) + 2(K+1)\varepsilon^2(g'(t) + u_1'(t)). \end{aligned} \tag{4.40}$$

For $t \in [1/2, t_0]$ and ε sufficiently small, $(1-\varepsilon t)(2-\varepsilon t) > 1$, and so

$$\begin{aligned} g'(t) &= -\frac{2\varepsilon^2 t(1-\varepsilon t)(2-\varepsilon t)}{\sqrt{1+\varepsilon^2 t^2(2-\varepsilon t)^2}} < -\frac{2\varepsilon^2 t}{\sqrt{2}}(1-\varepsilon t_0)(2-\varepsilon t_0) \\ &< -\frac{2\varepsilon^2 t}{\sqrt{2}}. \end{aligned} \tag{4.41}$$

From (4.40), (4.41) and using the fact that $K = 3.5$, one can check that

$$\begin{aligned} h'(t) + 2(K+1)\varepsilon^2 v'(t) &< \varepsilon^2(2t^2+9)u_1'(t) + \varepsilon^2 t(8(1+u_1(t)) - 18/\sqrt{2}\varepsilon^2) - g'''(t) \\ &< 0 \end{aligned} \tag{4.42}$$

on $[1/2, t_0]$.

Thus, $h(t) + 2(K+1)\varepsilon^2 v(t)$ is decreasing in t for $t \geq 1/2$. Now

$$(u+v)(t_1) = w(t_1) + 2v(t_1) = -(K+1)\varepsilon^2 + 2v(t_1).$$

Hence

$$\begin{aligned} w''(t_1) &= h(t_1) + (K+1)\varepsilon^2(2v(t_1) - (K+1)\varepsilon^2) \\ &< -(K+1)^2\varepsilon^4 \end{aligned}$$

and moreover using (4.39) and (4.42) we have

$$\begin{aligned} w''(t) &= h(t) - 2v(t)w(t) - w^2(t) \\ &< h(t_1) + (K+1)\varepsilon^2 2v(t_1) - (K+1)^2\varepsilon^4 \\ &< -(K+1)^2\varepsilon^4 \end{aligned}$$

whenever $w(t) < -(K+1)\varepsilon^2$.

Arguing as in Case 3, we proved $w''(t) < -(K+1)^2\varepsilon^4$ for $t \geq t_1$. But this implies $w'(1/\varepsilon) < 0$ and that contradicts the sign of $w'(1/\varepsilon)$ [see (4.34)]. Hence, we conclude that $w(t)$ cannot cross $-(K+1)\varepsilon^2$ for any $t \geq 1/2$. A similar argument as above will also show that $w(t) < -(K+1)\varepsilon^2$ for $t \geq 1/2$ is impossible.

Case a) Let $t_2 > t_1$ be such that $w(t_2) = -(K+2)\varepsilon^2$. If there exists no such t_2 then we obtain $|w(t)| < (K+2)\varepsilon^2$ and we are done. Assume such a t_2 exists and $w(t) < -(K+1)\varepsilon^2$ for $t \in [t_1, t_2]$.

If $t_2 - t_1 > 1/2$ then clearly $t_2 > 1/2$ and $w(t_2) < -(K+1)\varepsilon^2$. But this is impossible, since we proved that $w(t) \nleq -(K+1)\varepsilon^2$ for $t \geq 1/2$ in Case b). Assume $t_2 - t_1 < 1/2$. By the Mean Value Theorem, there exists $\eta \in (t_1, t_2)$ such that

$$w'(\eta) = -\frac{\varepsilon^2}{t_2 - t_1}.$$

Then $w'(\eta) < -2\varepsilon^2$. Note that if $w(t) < 0$ then from (4.33) it follows that

$$\begin{aligned} w''(t) &= h(t) - (u+v)(t)w(t) < h(t) \\ &< K\varepsilon^2. \end{aligned} \tag{4.43}$$

In particular $w''(\eta) < K\varepsilon^2$. Let $\delta > 0$ be the maximum length of the interval on which $w''(t) < K\varepsilon^2$ and $t \in [\eta, \eta+\delta]$. Then $\eta + \delta \leq \varepsilon^{-1}$.

Case i): $\eta + \delta < \varepsilon^{-1}$.

Since $w''(t) < K\varepsilon^2$ for $t \in [\eta, \eta+\delta]$, we have $w'(t) < K\varepsilon^2(t-\eta) - 2\varepsilon^2$ on that interval. If $\delta < 1/2$ then

$$w'(\eta+\delta) < K\varepsilon^2\delta - 2\varepsilon^2 < K\varepsilon^2/2 - 2\varepsilon^2 < 0 \tag{4.44}$$

since $K = 3.5$. By a similar argument $w'(\eta+s) < 0$ for all $s < \delta$. Hence $w(t) \leq -(K+1)\varepsilon^2$ for all $t \in [\eta, \eta+\delta]$. From (4.44), we can say that there exists some $\tilde{t} > \eta+\delta$ such that $w(t) < w(\eta+\delta) < 0$ for all $t \in (\eta+\delta, \tilde{t})$. Hence from (4.43) we conclude that $w''(t) < K\varepsilon^2$ for $t \in [\eta+\delta, \tilde{t})$, contradicting the maximality of δ.

Hence $\delta > 1/2$. But then we have $w'(\eta+s) < K\varepsilon^2/2 - 2\varepsilon^2 < 0$ for all $s \in [0, 1/2]$. Hence $w(\eta+1/2) < -(K+1)\varepsilon^2$, but this is impossible as $\eta + 1/2 > 1/2$.

Case ii): $\eta + \delta = \varepsilon^{-1}$.

Choosing ε sufficiently small, we will have $\eta + \delta > 1/2$. Then as in Case i), we have

$$w'(\eta + s) < K\varepsilon^2 s - 2\varepsilon^2 < 0$$

for all $s \in [0, 1/2]$. Hence $w(\eta + 1/2) < w(\eta) < -(K+1)\varepsilon^2$, a contradiction.

Thus we proved that $|w(t)| < (K+2)\varepsilon^2$ for all $t \in [0, \varepsilon^{-1}]$. $\square$

Since $|w(t)| < (K+2)\varepsilon^2$ for all $t \in [0, \varepsilon^{-1}]$, we have proved Theorem 12.

$\square$

Thus we have

$$v(t) = 3\mathrm{sech}^2\left(\frac{t}{\sqrt{2}} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) - \sqrt{1 + \varepsilon^2 t^2 (2 - \varepsilon t)^2}$$

approximating u uniformly on $[0, \varepsilon^{-1}]$. Referring to (4.6), we obtain that $y(t)$ can be approximated uniformly by $z(t) = v(t) + \varepsilon t(\varepsilon t - 2)$. Transforming back to original variable x we have

$$z(x) = 3\mathrm{sech}^2\left(\frac{1+x}{\sqrt{2}\varepsilon} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) + x^2 - 1 - \sqrt{1 + (1 - x^2)^2}$$

approximating $y(x)$ on $(-1, 0)$. Since we are looking for a symmetric solution about $x = 0$

$$\tilde{z}(x) = 3\mathrm{sech}^2\left(\frac{1-x}{\sqrt{2}\varepsilon} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) + x^2 - 1 - \sqrt{1 + (1 - x^2)^2}$$

would approximate y uniformly on the interval $(0, 1)$. Hence

$$\begin{aligned} x^2 - 1 - \sqrt{1 + (1 - x^2)^2} + 3\mathrm{sech}^2\left(\frac{1+x}{\sqrt{2}\varepsilon} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) \\ + 3\mathrm{sech}^2\left(\frac{1-x}{\sqrt{2}\varepsilon} + \mathrm{arctanh}\sqrt{\frac{2}{3}}\right) \end{aligned}$$

is a uniform approximation of y on $(-1, 1)$; thus proving that the asymptotics given in Bender and Orzag is correct.

5.0 ASYMPTOTIC EXPANSIONS OF SOLUTIONS TO AN INHOMOGENEOUS EQUATION

In this chapter we consider a class of problems which also includes (after a transformation) the boundary value problem (4.1)-(4.2) that we considered in the previous chapter. This chapter is based on the paper [6], which is joint with Professor X. Chen. We consider the asymptotic behavior of solutions of

$$\varepsilon^2 u''(x) = u(x)(q(x,\varepsilon) - u(x)) \tag{5.1}$$

for all $x \in I = [-1,1]$ as $\varepsilon \searrow 0$. We assume that $q(x,\varepsilon)$ is a smooth function such that

$$\begin{cases} 0 < q_\star = \inf\limits_{\varepsilon\in(0,\varepsilon_0]} \min\limits_{x\in I} q(x,\varepsilon) \leqslant \sup\limits_{\varepsilon\in(0,\varepsilon_0]} \max\limits_{x\in I} q(x,\varepsilon) = q^\star < \infty, \\ \\ q_1 = \sup\limits_{\varepsilon\in(0,\varepsilon_0]} \max\limits_{x\in I} |q'(x,\varepsilon)| < \infty, \qquad q_2 = \sup\limits_{\varepsilon\in(0,\varepsilon_0]} \max\limits_{x\in I} |q''(x,\varepsilon)| < \infty \end{cases} \tag{5.2}$$

for some $\varepsilon_0 > 0$.

The motivation comes from the boundary value problem

$$\varepsilon^2 y'' + 2b(1-x^2)y + y^2 = 1, \quad y(-1) = y(1) = 0. \tag{5.3}$$

We studied this problem in the previous chapter when $b = 1$. The case $b = 0$ corresponds to an autonomous system. As discussed in the previous chapter, Carrier used this example to show that matched asymptotic expansions (MAE) could produce spurious solutions. He pointed out that the approximate solutions of (5.3) obtained by the MAE method that displayed spikes at arbitrary points did not correspond to true solutions. In this chapter, we will study this problem in a different setting. We will show that (5.3) can be transformed to (5.1) with different boundary conditions. We will do a rigorous asymptotic analysis of solutions satisfying (5.1).

We will deal with $b > 0$. To show that (5.3) can be transformed into (5.1), let us first define

$$N[y] := -y'' + f(x, y, \varepsilon),$$

where $f(x, y, \varepsilon) = (-2b(1 - x^2)y - y^2 + 1)/\varepsilon^2$. Let

$$y_\pm \quad = \quad \phi_0 + \varepsilon^2\phi_1 \pm \kappa\varepsilon^4,$$

where

$$\phi_0 \quad = \quad b(x^2 - 1) - \sqrt{1 + b^2(1 - x^2)^2}, \quad \phi_1 = \frac{\phi_0''(x)}{2\sqrt{1 + b^2(1 - x^2)^2}}$$

and

$$\kappa \quad = \quad 1 + \max_{x\in[-2,2]} \left| \frac{\phi_1^2(x) + \phi_1''(x)}{2\sqrt{1 + b^2(1 - x^2)^2}} \right|.$$

Note that ϕ_0 is a root of the algebraic equation

$$1 - 2b(1 - x^2)y - y^2 = 0.$$

It can be checked that $N[y_-] \le 0 \le N[y_+]$ for ε sufficiently small and that $y_\pm(-1) = y_\pm(1) = -1 - \varepsilon^2 \pm \kappa\varepsilon^4$. Hence by the method of upper and lower solutions (see the "Existence Theorem" on page 264 in [22]), the boundary value problem

$$N[y] = 0, \quad y(-1) = y(1) = -1 - \varepsilon^2$$

has a solution y_g which we call the "ground state" such that $y_- \le y_g \le y_+$. Moreover, $y_g = \phi_0 + \varepsilon^2\phi_1 + \varepsilon^4\psi(x, \varepsilon)$, where $|\psi(x, \varepsilon)| \le \kappa$. Here although we can expand y_g to arbitrary high orders of ε, the above expansion is sufficient, since $q(\cdot, 0)$ is not degenerate, in the sense that $|q'(\cdot, 0)| + |q''(\cdot, 0)| > 0$. Note that ϕ_0 and all its derivatives up to second order are bounded in I, hence ϕ_1 and κ are bounded as well. On setting $u = y - y_g$, where y satisfies (5.3), the equation for u becomes

$$\begin{aligned}\varepsilon^2 u'' &= -u^2 - 2y_g u - 2b(1-x^2)u \\ &= u(q-u), \quad u(-1) = u(1) = 1 + \varepsilon^2,\end{aligned}$$

where

$$q(x,\varepsilon) = -2b(1-x^2) - 2y_g = 2\sqrt{1 + b^2(1-x^2)^2} - 2\varepsilon^2\phi_1 - 2\varepsilon^4\psi. \tag{5.4}$$

Note that for $\varepsilon > 0$ sufficiently small, q satisfies (5.2) with $q_\star = 2$ and is symmetric around $x = 0$. The results that we obtain by analyzing (5.1) can be applied to Carrier's equation for $b > 0$ and they agree with the asymptotic formulas that have been obtained formally by Bender and Orszag in [3] for $b = 1$. It is worthwhile to mention that Carrier's autonomous case $b = 0$ relates to the constant function $q = 2$.

In this chapter, $O(1)$ will represent a function of x and ε that is bounded by a constant K, which depends only on $q_\star$, $q^\star$, q_1 and q_2. We define $O(\beta) = O(1)\beta$ for every $\beta \in (0,\infty)$, and hence $O(\beta)$ will represent a function of β, x and ε that is bounded by $K\beta$. Often we will denote q by $q(x)$ bearing in mind that q depends on ε as well.

5.1 ASYMPTOTIC EXPANSION ON A MONOTONIC INTERVAL

The main result that we prove is the following:

Theorem 13. *Suppose that u is a solution of (5.1) and that q satisfies (5.2). Let $m \in (0, q_\star/2]$ and $M > m$ be positive constants. Consider an interval $(x_m, x_M) \subseteq I$ such that*

$$u(x_m) = m, \quad u'(x_m) = 0, \quad u' > 0 \ \textit{in} \ (x_m, x_M), \quad u'(x_M) = 0 \ \textit{and} \ u(x_M) = M. \tag{5.5}$$

Then

$$M = \frac{3q(x_M)}{2} - \frac{m^2}{M+m} - \frac{[2+O(m)]\varepsilon q'(x_M)}{\sqrt{q(x_M)}} + O(\varepsilon^2) \tag{5.6}$$

and for every $x \in [x_m, x_M]$,

$$\ln \frac{1+\sqrt{1-u(x)/M}}{1-\sqrt{1-u(x)/M}} + \ln \frac{2}{1+\sqrt{1-m^2/u^2(x)}} = \ln \frac{8M}{m} - \int_{x_m}^{x} \left(\frac{\sqrt{q(y)}}{\varepsilon} - \frac{q'(y)}{4q(y)} \right) dy + O(\varepsilon + m). \quad (5.7)$$

Remark 5.1.1. *Note that there is no requirement in Theorem 13 that* ε *and* m *are small. With the definition of* O *given in the previous section, the theorem makes sense for all positive* ε *for which (5.2) holds. However it tells us nothing significant about the solutions unless* ε *and* m *are small. The "degree of smallness" does not depend on the particular solution, since the bounds in* $O(\varepsilon^2)$ *and* $O(\varepsilon + m)$ *depend only on the bounds on* q*,* q'*, and* q''*.*

There are several application of Theorem 13. To mention a few, note that the expression (5.7) gives the asymptotic expansion of u on an interval where it is monotonic. This is the key formula that helps us in finding out the location of a minimum of u, say x_m, on any interval (a, b), such that $u' < 0$ on (a, x_m) and $u' > 0$ on (x_m, b). We can also prove that the minimum is exponentially small if $1/(b-a) = O(1)$. Further, we can find the relation between two successive minima of u, even if both of the minima are of exponentially small order. We will discuss the details in the Section 5.2.

Before we proceed to the proof of Theorem 13, let us recall a few basic facts about hyperbolic functions:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{arctanh } (x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \quad \text{sech}^2(\text{arctanh}\sqrt{1-x}) = x.$$

The rest of this section is devoted to the proof of Theorem 13.

5.1.1 A New Technique of Variation of Constants.

First we derive a key identity which essentially transfers (5.1) into a first order separable ordinary differential equation. We obtain a new type of **variation of constants formula** that works for such nonlinear quadratic equations, but unfortunately it doesn't apply to cubic equations, as in [2]. With the help of this formula, we will then derive the asymptotic expansion (5.7) of u on a monotonic interval.

Lemma 5.1.1. *Suppose the assumptions in Theorem 13 hold. Then for every $x \in (x_m, x_M)$,*

$$\frac{\varepsilon u'(x)}{\rho(u(x), m, M)} \quad = \quad \sqrt{q(x) - \delta_1(x) - \delta_2(x)}, \tag{5.8}$$

where

$$\begin{aligned} \rho(v, m, M) \quad &:= \quad \sqrt{\frac{(v-m)(M-v)(Mv+mv+Mm)}{M^2+Mm+m^2}}, \\ \delta_1(x) \quad &:= \quad \frac{M^2+Mu(x)+u^2(x)}{[u(x)-m)][u(x)+\frac{Mm}{M+m}]} \int_{x_m}^{x} \frac{u^2(y)-m^2}{M^2-m^2} q'(y)dy \end{aligned}$$

and

$$\delta_2(x) \quad := \quad \frac{m^2+mu(x)+u^2(x)}{[M-u(x)][u(x)+\frac{Mm}{M+m}]} \int_{x}^{x_M} \frac{M^2-u^2(y)}{M^2-m^2} q'(y)dy.$$

Remark 5.1.2. *When q is a constant, $\delta_1 = \delta_2 \equiv 0$, so (5.8) is indeed a first integral of the autonomous ode (5.1), where M and m are related by the algebraic equation $(M^3 - m^3) = 3q(M^2 - m^2)/2$.*

Proof of Lemma 5.1.1. Integrating $2\varepsilon^2 u'u'' = 2uu'(q-u)$ over $[x_m, x]$ for each $x \in (x_m, x_M]$ we obtain

$$\begin{aligned} \varepsilon^2 u'(x)^2 \quad &= \quad 2\int_{x_m}^{x} u(y)u'(y)(q(y)-u(y))dy \\ &= \quad \frac{2}{3}(m^3-u^3) + (u^2-m^2)q - \int_{x_m}^{x}(u^2-m^2)q'dy \\ &= \quad \Big\{(u^2-m^2) - (u^3-m^3)\frac{M^2-m^2}{M^3-m^3}\Big\}q + J, \end{aligned} \tag{5.9}$$

where $u = u(x),\ q = q(x)$ and

$$J = J(x) := (u^3-m^3)\Big\{\frac{M^2-m^2}{M^3-m^3}q - \frac{2}{3}\Big\} - \int_{x_m}^{x}[u^2-m^2]q'dy.$$

Evaluating (5.9) at $x = x_M$ we obtain $J(x_M) = 0$ which implies that

$$\frac{M^2-m^2}{M^3-m^3}q(x_M) - \frac{1}{M^3-m^3}\int_{x_m}^{x_M}[u^2-m^2]q' = \frac{2}{3}. \tag{5.10}$$

Substituting the left-hand side of (5.10) for the constant $2/3$ in the definition of J we obtain

$$\begin{aligned} J &= (u^3-m^3)\Big\{\frac{M^2-m^2}{M^3-m^3}[q-q(x_M)] + \frac{1}{M^3-m^3}\int_{x_m}^{x_M}[u^2-m^2]q'\Big\} - \int_{x_m}^{x}[u^2-m^2]q' \\ &= \frac{u^3-M^3}{M^3-m^3}\int_{x_m}^{x}[u^2-m^2]q' + \frac{u^3-m^3}{M^3-m^3}\int_{x}^{x_M}[u^2-M^2]q' \end{aligned} \tag{5.11}$$

by the identity $q(x)-q(x_M) = -\int_x^{x_M} q'(y)dy$. The assertion (5.8) then follows from (5.9), (5.11), and the identity

$$(u^2-m^2)-(u^3-m^3)\frac{M^2-m^2}{M^3-m^3} = \frac{(u-m)(M-u)(Mu+mu+Mm)}{M^2+Mm+m^2} =: \rho^2(u,m,M).$$

5.1.2 The size of δ_1 and δ_2

In this subsection we prove the following:

Lemma 5.1.2. *With δ_1 and δ_2 defined as in Lemma 5.1.1, we have*

$$|\delta_1(x)| + |\delta_2(x)| = O(\varepsilon) \tag{5.12}$$

for all $x \in (x_m, x_M)$. Consequently (5.8) can be wrtten as

$$\varepsilon\frac{du}{dx} = [1+O(\varepsilon)]\sqrt{q}\rho(u,m,M). \tag{5.13}$$

Proof. We will first establish an upper bound on M and $\varepsilon u'$. For each $x \in (x_m, x_M)$, integrating $2\varepsilon^2 u'u'' = 2uu'(q-u)$ over $[x_m, x]$ and applying the mean value theorem gives

$$\begin{aligned} \varepsilon^2 u'(x)^2 &= \int_{x_m}^{x} 2uu'(q-u) = q(\hat{x})(u^2-m^2) - \frac{2}{3}(u^3-m^3) \\ &= (u^2-m^2)\left\{q(\hat{x}) - \frac{2}{3}u - \frac{2}{3}\frac{m^2}{u+m}\right\} \end{aligned}$$

for some $\hat{x} \in (x_m, x)$. We could apply the mean value theorem because uu' does not change sign in (x_m, x_M). Thus, for all $x \in (x_m, x_M)$, we have

$$M = \frac{3}{2}q(\hat{x}) - \frac{m^2}{M+m} < \frac{3q^\star}{2}, \qquad \varepsilon^2 u'^2(x) \leqslant \max_{s>0} s^2\Big\{\Big(q^\star - \frac{2}{3}s\Big)\Big\} = \frac{q^{\star 3}}{3}.$$

We also note from the above expression that $M > q_\star$. Next we define

$$x_1 = \min\{x \in [x_m, X_M] \mid u(x) \geqslant q_\star/2\}.$$

Note that $u''(x_M) \leqslant 0$, which implies that $u(x_M)(q(x_M) - u(x_M)) \leqslant 0$, so that $u(x_M) \geqslant q(x_M) \geqslant q_\star$. Hence, x_1 is well-defined, $u(x_1) = q_\star/2$ and $m \leqslant u \leqslant q_\star/2$ in $[x_m, x_1]$. Consequently, for $x \in [x_m, x_1]$,

$$\varepsilon^2 u'(x)^2 = (u^2 - m^2)\Big\{q(\hat{x}) - \frac{2u}{3} - \frac{2m}{3}\frac{m}{u+m}\Big\} \geqslant \frac{q_\star}{3}(u^2 - m^2).$$

It then follows that $\varepsilon u' \geqslant \sqrt{\frac{q_\star}{3}}\sqrt{u^2 - m^2}$ and

$$\int_{x_m}^{x}(u^2 - m^2) \leqslant \sqrt{\frac{3}{q_\star}}\int_{x_m}^{x}(u^2 - m^2)\frac{\varepsilon u'}{\sqrt{u^2 - m^2}} \leqslant \sqrt{\frac{3}{q_\star}}\varepsilon\int_m^u v dv = \frac{\sqrt{3}\varepsilon}{2\sqrt{q_\star}}(u^2 - m^2).$$

Since $m \leqslant q_\star/2$ and $M > q_\star$, we must have $\delta_1(x) = O(1)\varepsilon$ for $x \in (x_m, x_1]$. For the interval $[x_1, x_M]$, we consider the function $U(X) = u(x_1 + \varepsilon X)$

$$\ddot{U} = U(q(x_1) - U) + O(\varepsilon X), \quad U(0) = q_\star/2, \quad 0 \leqslant \dot{U}(0) \leqslant \frac{q^{\star 3}}{3},$$

where " . " represents $\frac{d}{dX}$. Since $q(x_1) \geqslant q_\star$, it follows from a regular perturbation that $x_M - x_1 = O(\varepsilon)$. Hence, for $x \in [x_1, x_M]$,

$$\begin{aligned}\int_{x_m}^{x}(u^2 - m^2) &\leqslant \int_{x_m}^{x_1}(u^2 - m^2) + \int_{x_1}^{x}(u^2 - m^2)\\ &\leqslant \frac{\sqrt{3}\varepsilon}{2\sqrt{q_\star}}(u^2(x_1) - m^2) + (u^2(x) - m^2)(x_M - x_1) = O(1)\varepsilon(u^2 - m^2),\end{aligned}$$

and so $\delta_1(x) = O(1)\varepsilon$ for all $x \in [x_1, x_M]$. Thus for all $x \in (x_m, x_M)$, $\delta_1(x) = O(1)\varepsilon$.

Now we shall estimate δ_2. If $x \in [x_1, x_M]$, then

$$\int_x^{x_M}(M^2 - u^2(y))dy \leqslant (M^2 - u^2(x))(x_M - x_1) = O(1)\varepsilon\ (M^2 - u^2(x)),$$

and so $\delta_2(x) = O(1)\varepsilon$. When $x \in [x_m, x_1]$, we have $M - u(x) \geqslant M - q_\star/2 > q_\star/2$, and hence

$$\begin{aligned}\delta_2(x) &\leqslant 3(M + u(x))u(x)\int_x^{x_M} 1dy \leqslant 6M\int_x^{x_M} u(y)dy\\ &\leqslant 6M\Big\{M(x_M - x_1) + \sqrt{\frac{3}{q_\star}}\int_{x_m}^{x_1}\frac{u\varepsilon u'}{\sqrt{u^2 - m^2}}\Big\}\\ &= O(1)\Big\{\varepsilon + \varepsilon\int_m^{q_\star/2}\frac{vdv}{\sqrt{v^2 - m^2}}\Big\} = O(1)\varepsilon.\end{aligned}$$

Thus, (5.12) holds. Finally (5.13) follows from (5.8) since $q \geqslant q_\star$. This completes the proof of the Lemma. □

5.1.3 An Integral Representation

Integrating (5.8) over $[z, x] \subseteq [x_m, x_M]$ and using

$$\sqrt{q - \delta_1 - \delta_2} = \sqrt{q}\Big\{1 - \frac{\delta_1 + \delta_2}{2q} + O(1)\frac{(\delta_1 + \delta_2)^2}{q^2}\Big\} = \sqrt{q} - \frac{\delta_1 + \delta_2}{2\sqrt{q}} + O(\varepsilon^2)$$

we obtain

$$\int_{u(z)}^{u(x)} \frac{ds}{\rho(s, m, M)} = \frac{1}{\varepsilon}\int_z^x \Big\{\sqrt{q(y)} - \frac{\delta_1(y) + \delta_2(y)}{2\sqrt{q(y)}} + O(\varepsilon^2)\Big\}dy. \tag{5.14}$$

We shall use (5.13) to estimate the integral in (5.14) that is related to the functions δ_1 and δ_2. The expansion to be derived is aimed at situations where $(x_M - x_m)/\varepsilon$ is very large, i.e., to the cases where m is very small.

1. First we investigate the integral in the definition of δ_1. For this, we introduce the function

$$R(v, m, M) := \int_m^v \frac{(s^2 - m^2)\, ds}{\rho(s, m, M)} = O(1)(v^2 - m^2), \tag{5.15}$$

since $1/\rho(s, m, M)$ is integrable on $[m, M]$. Using (5.13) we find that

$$\int_{x_m}^x (u^2 - m^2) = O(1)\int_{x_m}^x \frac{(u^2 - m^2)\varepsilon u'}{\rho(u, m, M)} = O(1)\, \varepsilon\, R(u(x), m, M) = O(\varepsilon)(u^2(x) - m^2).$$

Consequently, writing (5.13) as $1 = \varepsilon u'/(\sqrt{q}\rho) + O(\varepsilon)$ we have

$$\begin{aligned}
\int_{x_m}^x (u^2 - m^2)q' dy &= \int_{x_m}^x q'(u^2 - m^2)\Big\{\frac{\varepsilon u'}{\sqrt{q}\rho(u, m, M)} + O(\varepsilon)\Big\}dy \\
&= \int_{x_m}^x \Big\{\frac{\varepsilon q'}{\sqrt{q}}\Big(R(u(y), u, M)\Big)' + O(\varepsilon)(u^2 - m^2)q' dy\Big\} \\
&= \frac{\varepsilon q'(x)}{\sqrt{q(x)}}R(u(x), m, M)) \\
&\quad - \int_{x_m}^x \Big(\varepsilon\Big(\frac{q'}{\sqrt{q}}\Big)' R + O(\varepsilon)(u^2 - m^2)q'\Big) \qquad (5.16) \\
&= \frac{\varepsilon q'(x)}{\sqrt{q(x)}}R(u(x), m, M) + O(\varepsilon^2)(u^2(x) - m^2) \qquad (5.17)
\end{aligned}$$

where in the third equation, we have used integration by parts.

2. Next we consider the integral in (5.14) that involves δ_1. Notice that

$$\int_{x_m}^{x_M}\Big(u+\frac{m}{u}\Big)=O(\varepsilon)\int_m^M\frac{[v+\frac{m}{v}]dv}{\rho(v,m,M)}=O(\varepsilon).$$

We can write

$$\begin{aligned}R(u,m,M) &= \Big(1+O\Big(\frac{m^2}{u}\Big)\Big)\int_m^u\frac{(s^2-m^2)}{\sqrt{s^2-m^2)(1-s/M)}}ds\\ &= (u^2-m^2)\{\frac{1}{2}+O(u)+O(\frac{m}{u})\},\end{aligned}$$

hence, from the definition of δ_1 and (5.17) we obtain

$$\begin{aligned}\delta_1(x) &= \frac{\varepsilon q'(x)}{\sqrt{q(x)}}\frac{(M^2+mu+u^2)R(u,m,M)}{(u-m)(u+\frac{Mm}{m+M})(M^2-m^2)}+O(\varepsilon^2)\\ &= \frac{\varepsilon q'(x)}{\sqrt{q(x)}}\frac{(M^2+mu+u^2)R(u,m,M)}{(u^2-m^2)(1-\frac{m^2}{(u+m)(m+M)})(M^2-m^2)}+O(\varepsilon^2)\\ &= \frac{\varepsilon q'(x)}{\sqrt{q(x)}}\Big\{\frac{1}{2}+O\Big(u+\frac{m}{u}\Big)\Big\}+O(\varepsilon^2).\end{aligned}$$

Thus, for $[z,x]\subseteq[x_m,x_M]\subseteq[-1,1]$, we have

$$\int_z^x\frac{\delta_1}{2\sqrt{q}}=\frac{\varepsilon}{4}\int_z^x\frac{q'}{q}+O(\varepsilon)\int_{x_m}^{x_M}\Big(u+\frac{m}{u}+\varepsilon\Big)=\frac{\varepsilon}{4}\int_z^x\frac{q'}{q}dq+O(\varepsilon^2).$$

3. Finally, we estimate the integral of δ_2. From the definition of δ_2 we observe that

$$|\delta_2|\leqslant\frac{3u^2}{(M-u)u}\int_x^{x_M}\frac{q_1(M^2-u^2(y))}{M^2-m^2}dy=O(1)u\int_x^{x_M}1dy.$$

Thus,

$$\begin{aligned}\int_{x_m}^{x_M}|\delta_2|dx &= O(1)\int_{x_m}^{x_M}\int_x^{x_M}u(x)dydx\\ &= O(1)\int_{x_m}^{x_M}dy\int_{x_m}^{y}u(x)dx=O(\varepsilon)\int_{x_M}^{x_m}dy\int_m^u\frac{s}{\rho(s,m,M)}ds\\ &= O(\varepsilon)\int_{x_M}^{x_m}\sqrt{u^2(y)-m^2}dy=O(\varepsilon^2)\end{aligned}$$

where in the third equation, we have used the technique to bound $u(x)\ dx$ by $O(\varepsilon)udu/\rho(u,m,M)$. Substituting the above estimate in (5.14), we have the following:

Lemma 5.1.3. *For every* $[z, x] \subseteq [x_m, x_M]$,

$$L(u(z), m, M) - L(u(x), m, M) = \int_z^x \Big\{ \frac{\sqrt{q(y)}}{\varepsilon} - \frac{q'(y)}{4q(y)} \Big\} dy + O(\varepsilon) \tag{5.18}$$

where

$$L(v, m, M) := \int_v^M \frac{ds}{\rho(s, m, M)}, \tag{5.19}$$

for all $v \in [m, M]$.

5.1.4 The Function L

The function L defined in (5.19) is an elliptic function which is not so easy to use. Here we derive asymptotic expansions of L for small m. When $m = 0$, it is easy to find that

$$L(v, 0, M) = \int_v^M \frac{ds}{s\sqrt{1 - s/M}} = \ln \frac{1 + \sqrt{1 - v/M}}{1 - \sqrt{1 - v/M}} = 2 \operatorname{arctanh} \sqrt{1 - \frac{v}{M}}.$$

Thus, $L(u, 0, M) = A$ if and only if $u = M\Big(1 - \tanh^2 \frac{A}{2}\Big) = M \operatorname{sech}^2 \frac{A}{2}$.

Next we estimate the difference between $L(u, m, M)$ and $L(u, 0, M)$. For $s \in (m, M)$, we have

$$\begin{aligned}
&\frac{1}{\rho(s, m, M)} - \frac{1}{\rho(s, 0, M)} = \frac{1 + \frac{O(m^2)}{s}}{\sqrt{(1 - s/M)(s^2 - m^2)}} - \frac{1}{s\sqrt{(1 - s/M)}} \\
= \;& \Big(\frac{1}{\sqrt{s^2 - m^2}} - \frac{1}{s}\Big) + \Big(\frac{1}{\sqrt{1 - s/M}} - 1\Big)\Big(\frac{1}{\sqrt{s^2 - m^2}} - \frac{1}{s}\Big) + \frac{O(m^2)}{s\sqrt{1 - s/M}\sqrt{s^2 - m^2}} \\
= \;& \Big(\frac{1}{\sqrt{s^2 - m^2}} - \frac{1}{s}\Big) + \frac{O(m^2)}{s\sqrt{1 - s/M}\sqrt{s^2 - m^2}}
\end{aligned}$$

by using $1/\sqrt{a} - 1/\sqrt{b} = (b-a)/(\sqrt{a}\sqrt{b}[\sqrt{a}+\sqrt{b}])$ for the second term on the right-hand side of the second equation. Integrating the last equation over $[v, M]$ we then obtain

$$\begin{aligned}
L(v,m,M) &= L(v,0,M) + \ln\Big(\frac{1+\sqrt{1-m^2/M^2}}{1+\sqrt{1-m^2/v^2}}\Big) + O\Big(\frac{m^2}{v}\Big) \\
&= \ln\frac{1+\sqrt{1-v/M}}{1-\sqrt{1-v/M}} + \ln\Big(\frac{2}{1+\sqrt{1-m^2/v^2}}\Big) + O\Big(\frac{m^2}{v}\Big) \qquad (5.20) \\
&= \ln\frac{(1+\sqrt{1-v/M})^2}{4} + \ln\Big(\frac{8M}{v+\sqrt{v^2-m^2}}\Big) + O\Big(\frac{m^2}{v}\Big), \\
L(m,m,M) &= \ln\frac{8M}{m} + O(m). \qquad (5.21)
\end{aligned}$$

5.1.5 Completion of the Proof of Theorem 13

Evaluating (5.18) at $z = x_m$ and substituting (5.20) and (5.21) for $L(v,m,M)$ and $L(m,m,M)$ respectively, we obtain

$$\begin{aligned}
\ln\frac{1+\sqrt{1-v/M}}{1-\sqrt{1-v/M}} + \ln\Big(\frac{2}{1+\sqrt{1-m^2/v^2}}\Big) &= \ln\frac{8M}{m} - \int_z^x \Big\{\frac{\sqrt{q(y)}}{\varepsilon} - \frac{q'(y)}{4q(y)}\Big\} dy + O(\varepsilon) \\
&+ O\Big(\frac{m^2}{v}\Big).
\end{aligned}$$

Replacing $O(m^2/u(x))$ by $O(m)$ we obtain (5.7).

To find the relation between m and M, we multiply (5.10) by $3(M^3-m^3)/2(M^2-m^2)$ and use (5.17) to derive

$$\begin{aligned}
M + \frac{m^2}{M+m} - \frac{3q(x_M)}{2} &= -\frac{3}{2(M^2-m^2)}\int_{x_m}^{x_M}[u^2-m^2]q' = -\frac{3\varepsilon q'(x_M)}{2\sqrt{q(x_M)}}\frac{R(M,m,M)}{(M^2-m^2)} \\
&+ O(\varepsilon^2).
\end{aligned}$$

The exact value of $R(M,m,M)$ defined in (5.15) involves an elliptic integral and we do not want to use it here, nevertheless we can derive that $R(M,m,M) = 4M^2/3 + O(m)$, from which, we obtain (5.6). This completes the proof of Theorem 13. □

5.2 REMARKS AND APPLICATIONS OF THE MAIN RESULT

As mentioned in the previous section, we will now discuss some remarks and applications of Theorem 13. We will derive an important corollary which will later help us to analyze the solutions of (5.1) with certain specified boundary conditions. In all of this, we will assume the statement of Theorem 13.

Remark 5.2.1. **(1)** *The condition $m \in (0, q_\star/2]$ in Theorem 13 can be replaced by $m \in (0, q(x_m) - \eta]$ for any fixed small positive constant η. Indeed, if $m \in [\eta, q(x_m) - \eta]$, then we can use a regular perturbation for the function $U(X) := u(x_m + \varepsilon X)$ to obtain estimates that are more accurate than those stated in Theorem 13, see [23].*

(2) *If $x_M > 1$ then we can extend our equation to a slightly larger interval till u attains its first maximum and therefore (5.7) would hold for every $x \in [x_m, 1]$.*

(3) *If u attains its maximum at x_M and its minimum at x_m with $u' < 0$ in $(x_M, x_m) \subseteq I$, then by setting $y = 2x_M - x$ and extending our equation outside I if necessary, we can apply (5.6) to the functions $\tilde{u}(y) = u(x)$, $\tilde{q}(y) = q(x)$ on the interval $[2x_M - x_m, x_M]$ to obtain*

$$M = \frac{3}{2}q(x_M) - \frac{m^2}{M+m} + \frac{[2 + O(m)]\varepsilon q'(x_M)}{\sqrt{q(x_M)}} + O(\varepsilon^2). \tag{5.22}$$

Also by setting $y = 2x_m - x$ and extending our equation if necessary, we apply (5.7) to the functions $\tilde{u}(y) = u(x)$, $\tilde{q}(y) = q(x)$ on the interval $[x_m, 2x_m - x_M]$ to obtain

$$\ln \tfrac{1+\sqrt{1-u(x)/M}}{1-\sqrt{1-u(x)/M}} + \ln \tfrac{2}{1+\sqrt{1-m^2/u^2(x)}} = \ln \tfrac{8M}{m} - \textstyle\int_x^{x_m} \left(\tfrac{\sqrt{q(y)}}{\varepsilon} + \tfrac{q'(y)}{4q(y)} \right) dy + O(\varepsilon + m). \tag{5.23}$$

(4) *Using the technique in the next subsection, one can derive from (5.7) that*

$$u(x) = \frac{m}{1 + O(u(x))} \Big(\frac{q(x_m)}{q(x)} \Big)^{1/4} \cosh \Big(\frac{1}{\varepsilon} \int_{x_m}^{x} \sqrt{q(y)} dy + O(\varepsilon) \Big) \quad \forall\, x \in [x_m, x_M].$$

This estimate can be regarded as an extension of the WKB (Wentzel-Kramers-Brillouin) approximation method applied to the linear equation $\varepsilon^2 w'' = qw$ with initial value $w(x_m) = m$, $w'(x_m) = 0$.

Corollary 5.2.1. *Let u be a solution of (5.1) and let q satisfy (5.2). Then the following holds:*

1. **Asymptotic Formula.** *If* $u(x_m) = m \in (0, q_\star/2]$, $u'(x_m) = 0$ *and* $u' > 0$ *in* $(x_m, z) \subset I$, *then*

$$u(x) = \frac{3q(z)}{2}\operatorname{sech}^2\Big(\frac{\sqrt{q(z)}}{2\varepsilon}(z-x) + \operatorname{arctanh}\sqrt{1 - \frac{2u(z)}{3q(z)}}\Big) + O(\varepsilon + m) \quad (5.24)$$

for all $x \in [x_m, z]$.

2. **Neighboring Local Minima.** *Suppose that* x_{m_L}, x_M, x_{m_R} *are points in* I *satisfying*

$$u'(x_{m_L}) = 0, \quad u' > 0 \;\; in \; (x_{m_L}, x_M), \quad u' < 0 \;\; in \;\; (x_M, x_{m_R}), \quad u'(x_{m_R}) = 0,$$
$$u(x_{m_L}) = m_L \leqslant q_\star/2, \qquad u(x_M) = M, \qquad u(x_{m_R}) = m_R,$$

then

$$m_R^2 = m_L^2 + \varepsilon\, q'(x_M)\sqrt{q(x_M)}\,(6 + O(m_L + m_R)) + O(\varepsilon^2) \quad (5.25)$$

and at least one of the following holds:

$$\text{(i)} \;\; x_M = x_{m_R} + O(\varepsilon|\ln\varepsilon|), \quad \text{(ii)} \;\; x_M = x_{m_L} + O(\varepsilon|\ln\varepsilon|), \quad \text{(iii)} \;\; q'(x_M) = O(\varepsilon). \quad (5.26)$$

3. **Local Valley.** *Suppose that* a, x_m, b *are points in* I *satisfying*

$$u' < 0 \;\; in \;\; (a, x_m), \quad u' > 0 \;\; in \;\; (x_m, b), \qquad u(x_m) = m < q_\star/2 \leqslant \min\{u(a), u(b)\}.$$

Then

$$x_m = X_0(a,b) + \varepsilon X_1(a, b, u(a), u(b)) + O(\varepsilon^2 + \varepsilon m), \quad (5.27)$$

$$m = \frac{m_0(a, b, u(a), u(b))}{1 + O(\varepsilon)} \exp\Big(-\frac{1}{2\varepsilon}\int_a^b \sqrt{q(y)}dy\Big), \quad (5.28)$$

where $X_0(a,b)$ *is the middle point of* a *and* b *weighted by* $\sqrt{q}$*, in the sense that*

$$\int_a^{X_0(a,b)} \sqrt{q(y)}dy = \frac{1}{2}\int_a^b \sqrt{q(y)}dy,$$

and

$$X_1(a, b, \alpha, \beta) := \frac{\Big(\operatorname{arctanh}\sqrt{1 - \frac{2\beta}{3q(b)}} - \operatorname{arctanh}\sqrt{1 - \frac{2\alpha}{3q(a)}} + \frac{5}{8}\ln\frac{q(a)}{q(b)}\Big)}{\sqrt{q(X_0(a,b))}},$$

$$m_0(a, b, \alpha, \beta) := \frac{12(q(a)q(b))^{\frac{5}{8}}}{(q(X_0(a,b)))^{\frac{1}{4}}}\sqrt{\frac{(1 - \sqrt{1 - \frac{2\alpha}{3q(a)}})(1 - \sqrt{1 - \frac{2\beta}{3q(b)}})}{(1 + \sqrt{1 - \frac{2\alpha}{3q(a)}})(1 + \sqrt{1 - \frac{2\beta}{3q(b)}})}}.$$

Remark 5.2.2. **(1)** By **(2)** of Remark 5.2.1, we note that $z = 1$ is allowed in (1) of Corollary (5.2.1).

(2) If m is exponentially small as given by (5.28), then (5.24) approximates u up to $O(\varepsilon)$. If $z \in [x_m + 2\sqrt{\varepsilon}, 1]$, then (5.24) is very precise for every x such that $z - x = O(\varepsilon)$, however if x lies in the interval $(x_m, z - \sqrt{\varepsilon})$, then (5.24) does not give us much information.

(3) If $q'(z) = O(\varepsilon)$, then $O(\varepsilon + m)$ in (5.24) can be replaced by $O(\varepsilon^2 + m)$.

(4) If $u(x_m) = m$, $u'(x_m) = 0$ and $u' < 0$ in (z, x_m), then as in (1) of Corollary 5.2.1, we can derive an analogous formula for u by setting $y = 2x_m - x$ and defining $\tilde{u}(y) = u(x)$, $\tilde{q}(y) = q(x)$. Note that $y \in (x_m, 2x_m - z)$ and $\tilde{u} > 0$ on this interval and that $\tilde{u}'(x_m) = 0$. Define $\tilde{z} = 2x_m - z$. Then on applying (5.24) to $\tilde{u}$ and $\tilde{q}$ on $(x_m, \tilde{z})$ we obtain

$$u(x) = \frac{3q(z)}{2}\operatorname{sech}^2\left(\frac{\sqrt{q(z)}}{2\varepsilon}(x - z) + \operatorname{arctanh}\sqrt{1 - \frac{2u(z)}{3q(z)}}\right) + O(\varepsilon + m) \tag{5.29}$$

for all $x \in [z, x_m]$.

(5) We can relax the assumptions of (3) of Corollary 5.2.1 by taking $\min\{u(a), u(b)\} \geq \eta$, for some fixed $\eta > 0$ as long as $m < \eta$. Moreover, $u'(a) = 0$ or $u'(b) = 0$ are also allowed in the assumptions of (3) of Corollary 5.2.1.

5.2.1 Proof of Corollary 5.2.1 using Theorem 13

Without loss of generality, we assume that q is defined on $\mathbb{R}$ and satisfies (5.2) with I replaced by $\mathbb{R}$.

1. First we prove (5.24), assuming that $u(x_m) = m \in (0, 1]$, $u'(x_m) = 0$ and $u' > 0$ in $(x_m, z) \subset I$.

If $u(z) \leqslant \varepsilon$, then $u = O(\varepsilon)$ on $[x_m, z]$ and so (5.24) is trivially true. Hence, we only need to consider the case when $u(z) > \varepsilon$. By extending q to be constant outside I, if necessary, we notice that if u satisfies (5.1) then there exist x_M and M such that $u'(x_M) = 0$, $u' > 0$ in (x_m, x_M) and $u(x_M) = M$. In the proof of Theorem 13, we have shown that $M \in (q_\star, \frac{3q^\star}{2})$ and that $\varepsilon u' \geqslant \sqrt{q_\star/3}\sqrt{u^2 - m^2}$ if $u < q_\star/2$. Let x_1 be the point such that $u(x_1) = q_\star/2$. It then follows that $x_1 = x_m + O(\varepsilon|\ln m|)$. Moreover by a regular perturbation around x_M, we can show that $x_M = x_1 + O(\varepsilon)$. Hence $x_M - x_m = O(\varepsilon|\ln m|)$. Since $M > u(z) > \varepsilon$, we note from (5.7) that $m > K\varepsilon(\exp(-1/\varepsilon))$ and hence $x_M - x_m = O(1)$. Theorem 13 can be applied here since the length of the location of the maximum and the minimum is bounded.

Now for each $z \in (x_m, x_M]$, taking the difference of (5.7) evaluated at $x \in [x_m, z]$ and the same equation with x replaced by z we obtain

$$\ln \frac{1+\sqrt{1-u(x)/M}}{1-\sqrt{1-u(x)/M}} = 2\text{arctanh}\sqrt{1-\frac{u(x)}{M}} = A + B, \tag{5.30}$$

where

$$\begin{aligned}
A &:= \int_x^z \Big(\frac{\sqrt{q(y)}}{\varepsilon} - \frac{q'(y)}{4q(y)}\Big)dy + \ln\frac{1+\sqrt{1-u(z)/M}}{1-\sqrt{1-u(z)/M}} + O(\varepsilon+m),\\
B &:= \ln\frac{1+\sqrt{1-m^2/u^2(x)}}{1+\sqrt{1-m^2/u^2(z)}} = \ln\frac{1-m^2/4u^2(x)+O(m^4/u^4(x))}{1-m^2/4u^2(z)+O(m^4/u^4(z))}\\
&= O\Big(\frac{m^2}{u^2(x)}+\frac{m^2}{u^2(z)}\Big) = O\Big(\frac{m^2}{u^2(x)}\Big).
\end{aligned}$$

Thus we have,

$$\begin{aligned}
u(x) &= M\ \text{sech}^2\frac{A+B}{2} = M\ [1+O(B)]\ \text{sech}^2\frac{A}{2}\\
&= M\text{sech}^2\frac{A}{2} + O(B)M\text{sech}^2\frac{A}{2} = M\text{sech}^2\frac{A}{2} + O(B)u(x)\frac{M\text{sech}^2\frac{A}{2}}{M\text{sech}^2\frac{A+B}{2}}\\
&= M\text{sech}^2\frac{A}{2} + O(B)u(x) = M\text{sech}^2\frac{A}{2} + O\Big(\frac{m^2}{u(x)}\Big) = M\text{sech}^2\frac{A}{2} + O(m).
\end{aligned}$$

We will simplify the last expression to obtain an $O(\varepsilon+m)$ approximation. Evaluating A at $z = x_M$, we first note that

$$u(x) = O(1)\text{sech}^2\Big(\int_x^{x_M}\frac{\sqrt{q(y)}}{2\varepsilon}dy\Big) = O(1)\exp\Big(-\frac{(x_M-x)}{\varepsilon}\Big),$$

so that

$$\sup_{x\in[x_m,x_M]}\Big(\frac{(x_M-x)^2}{\varepsilon} + (x_M-x)\Big)\Big(m+u(x)\Big) = O(\varepsilon). \tag{5.31}$$

If $x_M - x \geqslant \sqrt{\varepsilon}$, then $u(x),\ m = O(e^{-1/\sqrt{\varepsilon}})$ and (5.24) is trivially true. On the other hand, if $x_M - x \leqslant \sqrt{\varepsilon}$, then substituting

$$\begin{aligned} M &= \frac{3q(x_M)}{2} + O(m^2+\varepsilon) = \frac{3q(z)}{2} + O(m^2+\varepsilon+|x_M-z|), \\ A &= \frac{\sqrt{q(z)}(z-x)}{\varepsilon} + 2\mathrm{arctanh}\sqrt{1-\frac{2u(z)}{3q(z)}} + O\Big(\varepsilon + m + \frac{|x_M-x|^2}{\varepsilon} + |x_M-x|\Big) \end{aligned}$$

into $u(x) = M\mathrm{sech}^2\frac{A+B}{2}$ we obtain

$$u(x) = M\mathrm{sech}^2 C\Big(1 + O\Big(\varepsilon + m + \frac{|x_M-x|^2}{\varepsilon} + |x_M-x|\Big)\Big) + O(m), \tag{5.32}$$

where

$$C := \frac{\sqrt{q(z)}(z-x)}{\varepsilon} + 2\mathrm{arctanh}\sqrt{1-\frac{2u(z)}{3q(z)}}.$$

Then from (5.31) and (5.32) we have

$$\begin{aligned} u(x) &= M\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O\Big(\varepsilon + m + \frac{|x_M-x|^2}{\varepsilon} + |x_M-x|\Big) M u(x) \frac{\mathrm{sech}^2\big(\frac{C}{2}\big)}{M\mathrm{sech}^2\big(\frac{A+B}{2}\big)} + O(m) \\ &= M\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O\Big(\varepsilon + m + \frac{|x_M-x|^2}{\varepsilon} + |x_M-x|\Big) O(1) u(x) + O(m) \\ &= M\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O(\varepsilon) + O(m) \\ &= \frac{3q(z)}{2}\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O(m^2+\varepsilon+|x_M-z|)\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O(\varepsilon+m) \\ &= \frac{3q(z)}{2}\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O(m^2+\varepsilon+|x_M-z|)O(1)\exp\Big(-\frac{\sqrt{q(z)}(z-x)}{\varepsilon}\Big) + O(\varepsilon+m) \\ &= \frac{3q(z)}{2}\mathrm{sech}^2\Big(\frac{C}{2}\Big) + O(\varepsilon+m). \end{aligned}$$

This proves the first assertion of the Corollary.

2. Next we prove the second assertion of the Corollary. Applying (5.6) to the function $u(x)$ on $[x_{m_L}, x_M]$ and (5.22) to the function $u(x)$ on $[x_M, x_{m_R}]$, we obtain

$$\begin{aligned} M &= \frac{3}{2}q(x_M) - \frac{m_L^2}{M+m_L} - \frac{[2+O(m_L)]\varepsilon q'(x_M)}{\sqrt{q(x_M)}} + O(\varepsilon^2) \\ &= \frac{3}{2}q(x_M) - \frac{m_R^2}{M+m_R} + \frac{[2+O(m_R)]\varepsilon q'(x_M)}{\sqrt{q(x_M)}} + O(\varepsilon^2). \end{aligned}$$

Setting them equal, we obtain

$$\frac{m_R^2}{M+m_R}-\frac{m_L^2}{M+m_L}=\frac{\varepsilon q'(x_M)}{\sqrt{q(x_M)}}\Big(4+O(m_L+\tilde{m}_R)\Big)+O(\varepsilon^2),$$

which implies (5.25) since the left-hand side can be written as

$$(m_R^2-m_L^2)\frac{M+\frac{m_Rm_L}{m_R+m_L}}{(M+m_R)(M+m_L)}=\frac{m_R^2-m_L^2}{M+O(m_R+m_L)}=\frac{m_R^2-m_L^2}{\frac{3}{2}q(x_M)+O(m_R+m_L+\varepsilon)}.$$

Note that (5.25) implies that (i) $m_R \geqslant \varepsilon$, or (ii) $m_L \geqslant \varepsilon$, or (iii) $q'(x_M)=O(\varepsilon)$. Suppose $m_L \geqslant \varepsilon$. Then evaluating (5.7) at $x=x_M$, we obtain

$$\ln\frac{2}{1+\sqrt{1-m_L^2/M^2}} \;=\; \ln\frac{8M}{m_L}-\frac{1}{\varepsilon}\int_{x_{m_L}}^{x_M}\sqrt{q(y)}dy+\frac{1}{4}\ln\frac{q(x_M)}{q(x_{m_L})}+O(\varepsilon+m_L).$$

Hence

$$\frac{1}{\varepsilon}\int_{x_{m_L}}^{x_M}\sqrt{q(y)}dy \;=\; O(1)\ln\frac{(1+\sqrt{1-m_L^2/M^2})}{m_L}=O(1)\ln\frac{1}{m_L}=O(|\ln\varepsilon|),$$

and thus

$$x_M-x_{m_L}=O(\varepsilon|\ln\varepsilon|).$$

Similarly, if $m_R \geqslant \varepsilon$, then one can prove that $x_{m_R}-x_M=O(\varepsilon|\ln\varepsilon|)$ and thus we proved that at least one of (i), (ii), and (iii) in (5.26) holds. This proves the second assertion of the Corollary.

3. Finally we prove the third assertion of the Corollary. As before, we extend our equation to a slightly larger interval till u attains it maxima at x_{M_L} and x_{M_R}, where $x_{M_L} < x_m < x_{M_R}$ with values M_L and M_R respectively. Since $u(a)$, $u(b) \geq q_\star/2$, by a regular perturbation, we can prove that $|x_{M_L}-a|=O(\varepsilon)$ and $|x_{M_R}-b|=O(\varepsilon)$. Evaluating (5.7) at $x=b$ with $M_R=3q(b)/2+O(\varepsilon+m)$ and (5.23) at $x=a$ with $M_L=3q(a)/2+O(\varepsilon+m)$ respectively, we obtain

$$\begin{aligned}&\tfrac{1}{\varepsilon}\textstyle\int_{x_m}^{b}\sqrt{q(y)}dy+2\mathrm{arctanh}\sqrt{1-\frac{2u(b)}{3q(b)}}+\frac{1}{4}\ln q(x_m)-\frac{5}{4}\ln q(b)+O(\varepsilon+m)= \qquad (5.33)\\ \ln\frac{12}{m}=&\tfrac{1}{\varepsilon}\textstyle\int_{a}^{x_m}\sqrt{q(y)}dy+2\mathrm{arctanh}\sqrt{1-\frac{2u(a)}{3q(a)}}+\frac{1}{4}\ln q(x_m)-\frac{5}{4}\ln q(a)+O(\varepsilon+m).\end{aligned}$$

Multiplying the second equation by ε and adding $\int_a^{x_m} \sqrt{q(y)}dy$ to both the equations, we then obtain

$$\begin{aligned}\int_a^{x_m} \sqrt{q(y)}dy &= \tfrac{1}{2}\int_a^b \sqrt{q(y)}dy \\ &+ \varepsilon\Big(\operatorname{arctanh}\sqrt{1-\tfrac{2u(b)}{3q(b)}} - \operatorname{arctanh}\sqrt{1-\tfrac{2u(a)}{3q(a)}} + \tfrac{5}{8}\ln\tfrac{q(a)}{q(b)}\Big) + O(\varepsilon^2+\varepsilon m).\end{aligned}$$

This equation, together with the definition of $X_0(a,b)$ and

$$\int_a^{x_m} \sqrt{q(y)}dy = \int_a^{X_0(a,b)} \sqrt{q(y)}dy + \sqrt{q(X_0(a,b)}(x_m - X_0(a,b)) + O((x_m - X_0(a,b))^2),$$

gives

$$\begin{aligned}\sqrt{q(X_0(a,b)}(x_m - X_0(a,b)) &= \varepsilon\Big(\operatorname{arctanh}\sqrt{1-\tfrac{2u(b)}{3q(b)}} - \operatorname{arctanh}\sqrt{1-\tfrac{2u(a)}{3q(a)}} + \tfrac{5}{8}\ln\tfrac{q(a)}{q(b)}\Big) \\ &+ O(\varepsilon^2+\varepsilon m)\end{aligned}$$

which gives us (5.27). Finally, adding the two equations in (5.33), we obtain

$$\begin{aligned}\ln\frac{12}{m} &= \frac{1}{2\varepsilon}\int_a^b \sqrt{q(y)}dy + \operatorname{arctanh}\sqrt{1-\frac{2u(b)}{3q(b)}} + \operatorname{arctanh}\sqrt{1-\frac{2u(a)}{3q(a)}} + \frac{1}{4}\ln q(x_m) \\ &- \frac{5}{8}\ln q(a)q(b) + O(\varepsilon+m),\end{aligned}$$

which implies

$$\ln\frac{mq(x_m)^{1/4}}{12(q(a)q(b))^{5/8}} + \operatorname{arctanh}\sqrt{1-\frac{2u(b)}{3q(b)}} + \operatorname{arctanh}\sqrt{1-\frac{2u(a)}{3q(a)}} = -\frac{1}{2\varepsilon}\int_a^b \sqrt{q(y)}dy.$$

Using (5.27) and one of the properties of hyperbolic functions, we have

$$\ln\frac{mq(X_0)^{1/4}}{12(q(a)q(b))^{5/8}} + \frac{1}{2}\ln\frac{\Big(1+\sqrt{1-\frac{2u(b)}{3q(b)}}\Big)\Big(1+\sqrt{1-\frac{2u(a)}{3q(a)}}\Big)}{\Big(1-\sqrt{1-\frac{2u(b)}{3q(b)}}\Big)\Big(1-\sqrt{1-\frac{2u(a)}{3q(a)}}\Big)} = -\frac{1}{2\varepsilon}\int_a^b \sqrt{q(y)}dy + O(\varepsilon)$$

and thus

$$\begin{aligned}m = \frac{12(q(a)q(b))^{5/8}}{q(X_0)^{1/4}}&\sqrt{\frac{(1-\sqrt{1-2u(a)/3q(a)})(1-\sqrt{1-2u(b)/3q(b)})}{(1+\sqrt{1+2u(a)/3q(a)})(1+\sqrt{1-2u(b)/3q(b)})}} \\ &\exp\Big(-\frac{1}{2\varepsilon}\int_a^b \sqrt{q(y)}dy\Big)\end{aligned}$$

which is (5.28). This completes the proof of Corollary 5.2.1. □

5.3 ASYMPTOTIC EXPANSIONS OF A FEW SPECIAL SOLUTIONS.

Here we investigate asymptotic expansion of solutions of the boundary value problem

$$\varepsilon^2 u''(x) = u(x)(q(x,\varepsilon) - u(x)) \quad \text{in } (-1,1), \qquad u(-1) = \alpha_-, \quad u(1) = \alpha_+, \tag{5.34}$$

where $q(x,\varepsilon)$ is a C^2 function satisfying (5.2) and $\alpha_\pm \in [\eta, q_\star)$, for some fixed η such that $0 < \eta < q_\star$. A **critical point** is a root of the equation $u'(\cdot) = 0$.

5.3.1 Existence

The boundary value problem (5.34) has many solutions for sufficiently small ε. In this section, we will only outline a general existence proof using a shooting argument. More specifically, suppose that u satisfies

$$\varepsilon^2 u'' = u(q-u) \quad \text{in } (-1,1), \qquad u(-1) = \alpha_-, \quad u'(-1) = c.$$

We will show that as c varies, $u(1)$ takes all values between η and $q_\star$, and in particular, attains the value α_+, giving us a solution of (5.34). Moreover, by choosing ε sufficiently small, we can obtain any pre-determined number of oscillations. We will also present a different existence proof using the method of lower and upper solutions to show that (5.34) has a solution with exactly one critical point.

1. Outline of an existence proof. We first define $v(t) = u(x)$ and $\tilde{q}(t) = q(x)$, where $t = (1+x)/\varepsilon$. Then (5.34) transforms to

$$\ddot{v} = v(\tilde{q} - v), \qquad v(0) = \alpha_-, \quad v(2/\varepsilon) = \alpha_+. \tag{5.35}$$

We will consider an initial value problem

$$\ddot{v} = v(\tilde{q} - v), \qquad v(0) = \alpha_-, \quad \dot{v}(0) = \beta, \tag{5.36}$$

where β will be specified later. Without loss of generality, assume that $\beta > 0$. A similar argument holds when $\beta < 0$. We will show that for a given N, (5.35) has a solution with exactly N oscillations for sufficiently small ε.

Consider the autonomous problem

$$\ddot{V} = V(\tilde{q}(0) - V), \qquad V(0) = \alpha_-, \quad \dot{V}(0) = \beta, \tag{5.37}$$

where

$$\beta < \beta^\star = \alpha_- \sqrt{\tilde{q}(0) - \frac{2}{3}\alpha_-}.$$

Fix $\beta = \beta_0 < \beta^\star$, where β_0 will be chosen later. Since $\alpha_- < q_\star$, (5.37) has a solution V with at least $(N+1)$ oscillations on $[0, 2/\varepsilon]$ for sufficiently small ε. Hence, by continuity (5.36) also has a solution v_{β_0} with at least $(N+1)$ oscillations on $[0, 2/\varepsilon]$, for small enough ε. Moreover, the first $(N+1)$ oscillations of v_{β_0} will be close to the oscillations of V, and hence they will have amplitudes bounded below by some constant K independent of ε.

We will show that there exists β_1, such that if $\beta = \beta_1$, then (5.36) has a solution v_{β_1} with exactly N oscillations, and in the process as β sweeps over from β_0 to β_1, $v_\beta(1)$ takes all values between $q_\star$ and η, where v_β satisfies (5.36) and $0 < \eta < q_\star$ is a fixed number as defined before. Hence we can conclude that there exists a $\beta \in (\beta_0, \beta_1)$ such that $v_\beta(1) = \alpha_+$.

Choose β_0 such that the first minimum of V is less than $\eta/4$. This is possible, because as β gets closer to $\beta^\star$, the periodic solutions of (5.37) get closer to the homoclinic orbit of the autonomous system given by (5.37), based at $(0, 0)$. Hence the minima of the solutions of (5.37) get closer to 0 as we raise β. By continuity, we can say that the first two minima of v_{β_0} are close to the minima of V and hence are less than equal to $\eta/4$. Let us denote the ith minimum of v_{β_0} by m_i and let m_N denote the minimum of the $(N+1)$st oscillation of v_{β_0}. Then by (5.25), we note that for $1 \le i \le N$

$$m_i^2 \le m_{i-1}^2 + K\varepsilon, \tag{5.38}$$

where $K > 0$ is independent of ε. Thus,

$$m_i \le \sqrt{m_1^2 + K(i-1)\varepsilon} < \eta/2$$

for sufficiently small ε. Hence the first N minima of v_{β_0} lie below $\eta/2$. Clearly, every maximum of v_β lies above q and hence above $q_\star$, and so, the first N maxima and minima do not merge as $\varepsilon \to 0$. Note that the location of all the minima of v_β are continuous functions of β. Hence for a fixed ε, as we increase β from β_0, they move across $t = 2/\varepsilon$. By a similar reasoning as above and using (5.38), we can show that the first N minima of v_β still stay below $\eta/2$ as we raise β from β_0. In particular, the Nth minimum always stays below $\eta/2$.

Let $\beta_1 > \beta_0$ be such that the location of the Nth minimum of v_{β_1} has crossed $t = 2/\varepsilon$. The $(N+1)$st maximum of v_β is always above $q_\star$ and the Nth minimum of v_β is below $\eta/2$. Hence as β varies between β_0 and β_1, $v_\beta(1)$ takes all values between η and $q_\star$. Moreover, we have lost an oscillation as β increases from β_0 to β_1. Thus, we have a $\tilde{\beta} \in (\beta_0, \beta_1)$ such that if $\beta = \tilde{\beta}$, (5.35) has a solution with exactly N oscillations.

2. Existence of a solution with exactly one critical point. We will show that there exists at least one solution of (5.34) that has only one critical point. We will use the method of upper and lower solutions to prove this. Let us first consider the boundary value problem

$$y'' = \frac{y}{\varepsilon^2}(q_\star - y), \quad y(-1) = \alpha_-, \quad y(1) = \alpha_+. \tag{5.39}$$

We will show that (5.39) has a solution with exactly one critical point. Let

$$y_1 = \alpha_+ \frac{\cosh \sqrt{\frac{q_\star}{\varepsilon}}(x - \beta)}{\cosh \sqrt{\frac{q_\star}{\varepsilon}}(1 - \beta)},$$

where β is chosen such that $y_1(-1) = \alpha_-$. Note that y_1 has exactly one critical point, the point of local minimum and we can check that y_1 is a lower solution of (5.39). Let

$$y_2 = \alpha_- + \frac{(\alpha_+ - \alpha_-)}{2}(x + 1).$$

Then y_2 is an upper solution of (5.39) and clearly $y_2 < q_\star$ by our choices of α_+ and α_-. We can easily check that $y_1 \le y_2$. By a well-known existence theorem (see page 264 in [22]), there exists a solution y of (5.39) such that $y_1 \le y \le y_2$. Moreover, y has exactly one critical point, the point of minimum, since $0 < y < q_\star$. Now, we consider our boundary value problem (5.34). Suppose that

$$z = \alpha_+ \frac{\cosh\sqrt{\frac{q^\star}{\varepsilon}}(x-\gamma)}{\cosh\sqrt{\frac{q^\star}{\varepsilon}}(1-\gamma)},$$

where γ is chosen, so that $z(-1) = \alpha_-$. We can check that z is lower solution of (5.34) Also if y satisfies (5.39), then by the definition of $q_\star$, we can show that y is an upper solution of (5.34). Hence by the existence theorem in [22], there exists a solution u of (5.34) such that $z \le u \le y$. Also, u has exactly one critical point since $u < q_\star$.

5.3.2 Solutions with one critical point.

Asymptotic Expansion.

Let u be a solution of (5.34) that has only one critical point. Then the critical point must be a point of local minimum. Denoting the critical value by m and the point by x_m, we have

$$u(x_m) = m, \quad u'(x_m) = 0, \quad u'(x) > 0 \text{ in } (x_m, 1), \quad u'(x) < 0 \text{ in } (-1, x_m).$$

Applying (3) of Corollary 5.2.1 we obtain,

$$m = \frac{m_0(-1,1,\alpha_-,\alpha_+)}{1+O(\varepsilon)} \exp\Big(-\frac{1}{2\varepsilon}\int_{-1}^{1}\sqrt{q(y)}dy\Big)$$

and

$$x_m = X_0(-1,1) + \varepsilon X_1(-1,1,\alpha_-,\alpha_+) + O(\varepsilon^2).$$

Also by (5.29) and (5.24) respectively, we have an asymptotic expansion for u given below:

$$u(x) = \left\{ \begin{array}{ll} \frac{3}{2}q(-1)\mathrm{sech}^2\Big(\frac{\sqrt{q(-1)}}{2\varepsilon}(1+x) + \mathrm{arctanh}\sqrt{1-\frac{2\alpha_-}{3q(-1)}}\Big) + O(\varepsilon) & \text{if } x \in [-1,0], \\ \frac{3}{2}q(1)\mathrm{sech}^2\Big(\frac{\sqrt{q(1)}}{2\varepsilon}(1-x) + \mathrm{arctanh}\sqrt{1-\frac{2\alpha_+}{3q(1)}}\Big) + O(\varepsilon) & \text{if } x \in [0,1]. \end{array} \right\}$$

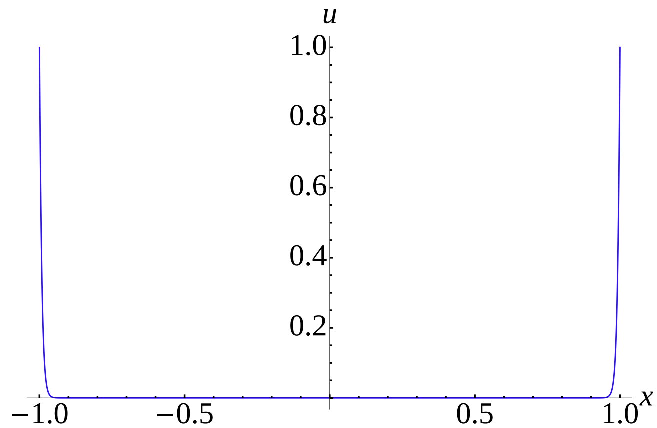

Figure 8: A solution of the transformed carrier's BVP. Here $\alpha_- = \alpha_+ = 1 + \varepsilon^2$.

The figure above represents an asymptotic solution of the transformed Carrier's equation (b=1) with one critical point and with boundary values $\alpha_- = \alpha_+ = 1 + \varepsilon^2$ for sufficiently small ε. Here $x_m = O(\varepsilon^2)$ since $X_0(-1, 1) = X_1(-1, 1, \alpha_-, \alpha_+) = 0$.

Remark 5.3.1. *In [2], Morse index was studied for solutions with finite number of oscillations. Analysis in [2] implies that solutions with one critical point have index one, and are therefore locally unique. We omit the details.*

5.3.3 Solutions with two critical points.

Such a solution will have a global maximum, say, M and a global minimum, say, m attained at points x_M and $x_m \in (-1, 1)$ respectively. There are two possibilities, (i) $x_m < x_M$ and (ii) $x_m > x_M$.

Suppose $x_m > x_M$. To obtain an asymptotic expansion of u on the interval $[x_m, 1]$, we apply (5.24) to obtain

$$u(x) = \frac{3}{2} q(1) \operatorname{sech}^2\Big(\frac{\sqrt{q(1)}}{2\varepsilon}(1 - x) + \operatorname{arctanh}\sqrt{1 - \frac{2\alpha_+}{3q(1)}}\Big) + O(\varepsilon + m). \tag{5.40}$$

On the interval $[x_M, x_m]$, we apply (5.29) with $z = x_M$ and use (5.22) to obtain

$$u(x) = \frac{3}{2}q(x_M)\text{sech}^2\Big(\frac{\sqrt{q(x_M)}}{2\varepsilon}(-x_M + x)\Big) + O(\varepsilon + m). \tag{5.41}$$

To obtain an expansion on the interval $[-1, x_M]$, we will do a regular perturbation around $x = x_M$. Define $U(z) = u(x_M + \varepsilon z)$. Then U satisfies

$$\ddot{U} = U(q(x_M) - U) + O(\varepsilon z), \quad U(0) = M,\ \dot{U}(0) = 0.$$

If $U\Big((-1 - x_M)/\varepsilon\Big) = \alpha_-$ then

$$U(z) = M\text{sech}^2\Big(\sqrt{q(x_M)}\Big(\frac{z}{2} + \frac{1 + x_M}{2\varepsilon}\Big) - \text{arctanh}\sqrt{1 - \frac{\alpha_-}{M}}\Big) + O(\varepsilon)$$

and hence

$$u(x) = M\text{sech}^2\Big(\frac{\sqrt{q(x_M)}}{2\varepsilon}(1 + x) - \text{arctanh}\sqrt{1 - \frac{\alpha_-}{M}}\Big) + O(\varepsilon), \quad x \in [-1, x_M]. \tag{5.42}$$

Substituting $x = x_M$ in (5.42) we observe that

$$1 + x_M = \frac{2\varepsilon}{\sqrt{q(x_M)}}\text{arctanh}\sqrt{1 - \frac{\alpha_-}{M}} + O(\varepsilon^2),$$

and hence we can write $x_M = -1 + \varepsilon\ell_- + O(\varepsilon^2)$ where

$$l_\pm := \frac{2d_\pm}{\sqrt{q(\pm 1)}}, \quad d_\pm := \text{arctanh}\sqrt{1 - \frac{\alpha_\pm}{M_\pm}}, \qquad M_\pm := \frac{3}{2}q(\pm 1).$$

To obtain the value of the local minimum, we apply the proof of the third part of Corollary 5.2.1 to $a = -1 + 2\varepsilon\ell_- + O(\varepsilon^2)$ and $b = 1$. Note that $u(-1 + 2\varepsilon\ell_- + O(\varepsilon^2)) = \alpha_- + O(\varepsilon)$. Just as in (5.33), and obtain

$$\begin{aligned} \ln\tfrac{12}{m} &= \tfrac{1}{\varepsilon}\textstyle\int_{x_m}^{1}\sqrt{q(y)}dy + 2\text{arctanh}\sqrt{1 - \tfrac{2\alpha_+}{3q(1)}} + \tfrac{1}{4}\ln q(x_m) - \tfrac{5}{4}\ln q(1) + O(\varepsilon + m) \\ &= \tfrac{1}{\varepsilon}\textstyle\int_{-1}^{x_m}\sqrt{q(y)}dy + 2\text{arctanh}\sqrt{1 - \tfrac{2\alpha_-}{3q(-1)}} + \tfrac{1}{4}\ln q(x_m) - \tfrac{5}{4}\ln q(-1) \\ &- \tfrac{1}{\varepsilon}\textstyle\int_{-1}^{-1+2\varepsilon\ell_-}\sqrt{q(y)}dy + O(\varepsilon + m). \end{aligned} \tag{5.43}$$

Using the definition of ℓ_-, the second equation can be written as

$$\ln\frac{12}{m} = \frac{1}{\varepsilon}\int_{-1}^{x_m}\sqrt{q(y)}dy - 2d_- + \frac{1}{4}\ln q(x_m) - \frac{5}{4}\ln q(-1) + O(\varepsilon + m). \tag{5.44}$$

Adding up (5.43) and (5.44) and using the definition of m_0, we obtain

$$m = m_0(-1,1,\alpha_-,\alpha_+)\exp\Big(-\frac{1}{2\varepsilon}\int_{-1}^{1}\sqrt{q(y)}dy + 2\text{arctanh}\sqrt{1-\frac{\alpha_-}{M_-}} + O(\varepsilon)\Big).$$

Setting the first equation of (5.43) equal to (5.44) and using the definition of X_0 and X_1 and the fact that m is exponentially small, we obtain

$$\begin{aligned} x_m &= X_0 + \frac{\varepsilon}{\sqrt{q(X_0)}}\Big(d_+ + d_- + \frac{5}{8}\ln\frac{q(-1)}{q(1)}\Big) + O(\varepsilon^2) \\ &= X_0 + \varepsilon X_1 + \varepsilon\frac{2\text{arctanh}\sqrt{1-\frac{2\alpha_-}{3q(-1)}}}{\sqrt{q(X_0)}} + O(\varepsilon^2). \end{aligned}$$

Thus on combining (5.40), (5.42) and substituting the value of x_M in (5.41), we obtain that the solution has the expansion

$$u(x) = \begin{cases} \frac{3}{2}q(-1)\text{sech}^2\Big(\frac{\sqrt{q(-1)}}{2\varepsilon}(1+x) - \text{arctanh}\sqrt{1-\frac{2\alpha_-}{3q(-1)}}\Big) + O(\varepsilon) & \text{if } x\in[-1,0], \\ \frac{3}{2}q(1)\text{sech}^2\Big(\frac{\sqrt{q(1)}}{2\varepsilon}(1-x) + \text{arctanh}\sqrt{1-\frac{2\alpha_+}{3q(1)}}\Big) + O(\varepsilon) & \text{if } x\in[0,1]. \end{cases}$$

If $x_M > x_m$, then $x_M = 1 - \varepsilon\ell_+ + O(\varepsilon^2)$, and we have the analogous expansion. The figure below represents two asymmetric solutions of the transformed Carrier's equation with $\alpha_- = \alpha_+ = 1+\varepsilon^2$ for sufficiently small ε. Since X_0 and X_1 are zero, hence in the first situation we have $x_m = \frac{\varepsilon}{\sqrt{2}}\text{arctanh}\sqrt{\frac{2}{3}} + O(\varepsilon^2)$ and in the second situation, $x_m = -\frac{\varepsilon}{\sqrt{2}}\text{arctanh}\sqrt{\frac{2}{3}} + O(\varepsilon^2)$.

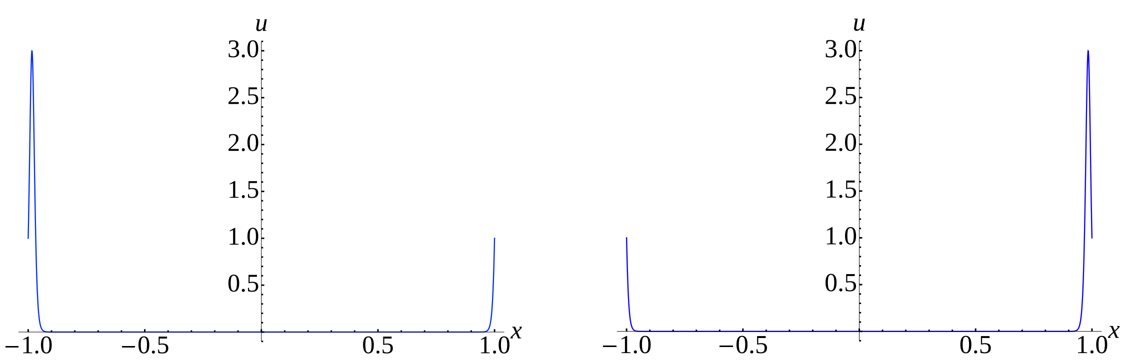

Figure 9: Asymmetric solutions of the transformed Carrier's problem with $b = 1$.

5.3.4 Solutions with Three Critical Points.

There are two cases to consider:

(i) Two interior local maxima and one interior local minimum.

(ii) Two local minima and one local maximum.

Case **(i)** Two interior local maxima and one interior local minimum. Similar to the previous analysis, it can be shown that the solution has the expansion

$$u(x) = \begin{cases} \frac{3}{2}q(-1)\mathrm{sech}^2\Big(\frac{\sqrt{q(-1)}}{2\varepsilon}(1+x) - \mathrm{arctanh}\sqrt{1-\frac{2\alpha_-}{3q(-1)}}\Big) + O(\varepsilon) & \text{if } x \in [-1,0], \\ \frac{3}{2}q(1)\mathrm{sech}^2\Big(\frac{\sqrt{q(1)}}{2\varepsilon}(1-x) - \mathrm{arctanh}\sqrt{1-\frac{2\alpha_+}{3q(1)}}\Big) + O(\varepsilon) & \text{if } x \in [0,1]. \end{cases}$$

Also, the two local maxima are attained at $x_{M_-} = -1+\varepsilon\ell_- + O(\varepsilon^2)$ and $x_{M_+} = 1-\varepsilon\ell_+ + O(\varepsilon^2)$ respectively. Again applying the proof of (3) of Corollary 5.2.1 to $a = -1 + 2\varepsilon\ell_- + O(\varepsilon^2)$ and $b = 1 - 2\varepsilon\ell_+ + O(\varepsilon^2)$, we can check that the interior minimum is attained at

$$x_m = X_0 + \varepsilon X_1 + \varepsilon\frac{2\mathrm{arctanh}\sqrt{1-\frac{2\alpha_-}{3q(-1)}} - 2\mathrm{arctanh}\sqrt{1-\frac{2\alpha_+}{3q(1)}}}{\sqrt{q(X_0)}} + O(\varepsilon^2),$$

with critical value

$$u(x_m) = m_0(-1,1,\alpha_-,\alpha_+)$$
$$\exp\Big(\int_{-1}^{1} -\frac{\sqrt{q(y)}}{2\varepsilon}dy + 2\mathrm{arctanh}\sqrt{1-\frac{2\alpha_-}{3q(-1)}} + 2\mathrm{arctanh}\sqrt{1-\frac{2\alpha_+}{3q(1)}} + O(\varepsilon)\Big).$$

Later (see Figure 10) we have a figure representing this case where we have an asymptotic symmetric solution of the transformed Carrier's equation for sufficiently small ε. Here $x_m = O(\varepsilon^2)$ and $\alpha_- = \alpha_+ = 1 + \varepsilon^2$.

Case **(ii)** Two interior local minima and one interior local maximum. Denote the locations of the interior minima by x_m, $x_{\tilde{m}}$, where $x_m < x_{\tilde{m}}$, and the location of the maximum by x_M with values m, $\tilde{m}$ and M respectively. Without loss of generality, we assume that $x_M \geqslant 0$. Then applying (3) of Corollary 5.2.1 with $a = -1$ and $b = x_M$, we must have $m = O(e^{-1/(2\varepsilon)})$. Consequently from (5.25) we can write

$$\tilde{m}^2 = 6\varepsilon q'(x_M)[1 + O(\tilde{m})] + O(\varepsilon^2). \tag{5.45}$$

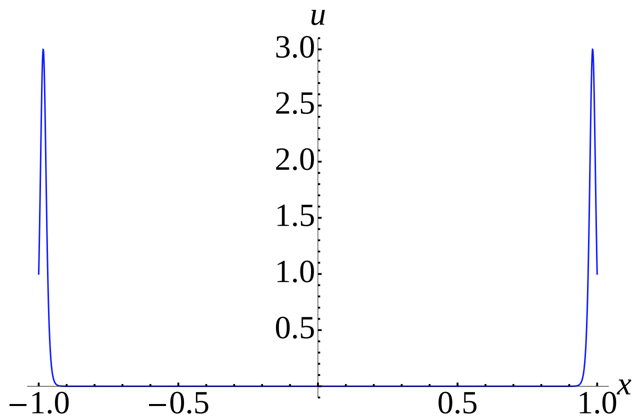

Figure 10: Another symmetric solution of the transformed Carrier's problem with $b = 1$.

Again applying (3) of Corollary 5.2.1 with $a = x_M$ and $b = 1$, we obtain

$$\tilde{m} = O(1) \exp\Big(- \frac{1}{2\varepsilon} \int_{x_M}^{1} \sqrt{q(y)} dy \Big). \tag{5.46}$$

For a further detailed discussion, we consider a special case where q satisfies (5.2) and is given by

$$\begin{cases} q(x, \varepsilon) = q_0(x) + \varepsilon^2 q_1(x) + O(\varepsilon^4), \\ \{z \in [-1, 1] \mid {q_0}'(z) = 0\} = \{-1, 0, 1\}, \qquad {q_0}''(\pm 1) > 0,\ {q_0}''(0) < 0. \end{cases} \tag{5.47}$$

Note that the q in Carrier's equation satisfies these properties for sufficiently small ε.

Consider the two situations:

(1) x_M is bounded away from 1 for all $\varepsilon \in (0, \varepsilon_0)$. Then $\tilde{m} = O(e^{-(1-x_M)/\varepsilon})$ and $\tilde{m} < \varepsilon$ for sufficiently small ε. Hence, it follows from (5.45) that $q'(x_M) = O(\varepsilon)$. Thus, the only possible location of x_M is near the point x_0 where $q'(x_0) = 0$. Since $x_M \geq 0$ and is bounded away from 1, we conclude that the only possible location of x_M is near 0. Additional analysis shows us that $q'(x_0) = 0$ and $q''(x_0) \leqslant 0$ are sufficient conditions for this case to occur. Indeed, when $q'(x_M) = O(\varepsilon)$, the next order expansion is

$$\tilde{m}^2 = 6\varepsilon q'(x_M) - c_2 \varepsilon^2 q''(x_M) + O(\varepsilon^3), \tag{5.48}$$

where c_2 is a positive constant. (see Appendix for a proof).

(2) x_M is close to 1. Since $q'(1) = O(\varepsilon^2)$, we have $q'(x_M) = -(1+o(1))q''(1)(1-x_M) + O(\varepsilon^2)$. Hence (5.45) can be written as

$$\tilde{m}^2 = -6\varepsilon q''(1)(1-x_M)(1+o(1))[1+O(\tilde{m})] + O(\varepsilon^2). \tag{5.49}$$

Note that since $q''(1) > 0$ and $1/q''(1) = O(1)$, (5.49) will give us a contradiction if $1 - x_M > 2K\varepsilon/6q''(1)$, where $K > 0$ has been defined at the end of Section 1. However if $1 - x_M \leq 2K\varepsilon/6q''(1)$, then $1/\tilde{m} = O(1)$ (follows from (5.46)) which would imply that our solution has many oscillations for small ε, contradicting the existence of only three critical points. Thus, such a solution does not exist.

Below we have a figure representing asymptotic solution of the transformed Carrier's equation with the same boundary conditions as mentioned earlier. The only possibility is that $x_M = O(\varepsilon^2)$, $x_m = -1 + \varepsilon\ell_- + O(\varepsilon^2)$ and $x_{\tilde{m}} = 1 - \varepsilon\ell_- + O(\varepsilon^2)$, where $\ell_\pm$ have been defined before.

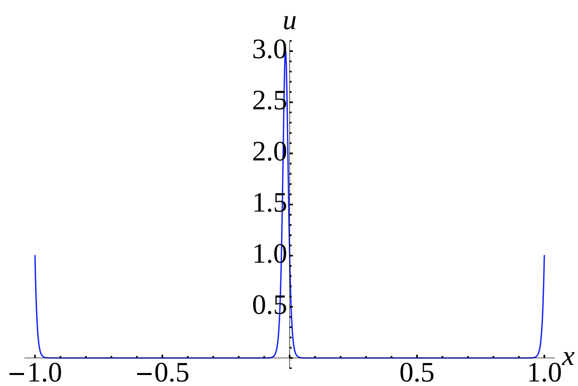

Figure 11: Another solution of the transformed Carrier's problem with three critical points.

5.3.5 Solutions with Four Critical Points.

Suppose that q is given by (5.47). Then one of the interior local maximum must exist near $x_M = O(\varepsilon^3)$. The solution has two pulses, one centered near the origin, the other centered either at $-1 + \varepsilon\ell_- + O(\varepsilon^2)$ or at $1 - \varepsilon\ell_+ + O(\varepsilon^2)$. The two local minima are in an $O(\varepsilon)$

neighborhood of z_L and z_R, where z_L and z_R are weighted averages of $\sqrt{q}$. In other words, z_L and z_R satisfy

$$\int_{-1}^{z_L} \sqrt{q(y)} = \tfrac{1}{2} \int_{-1}^{0} \sqrt{q(y)}$$

and

$$\int_{0}^{z_R} \sqrt{q(y)} dy = \frac{1}{2} \int_{0}^{1} \sqrt{q(y)}$$

respectively. The two possibilities that can occur in the transformed Carrier's equation for ε small in this situation are shown below.

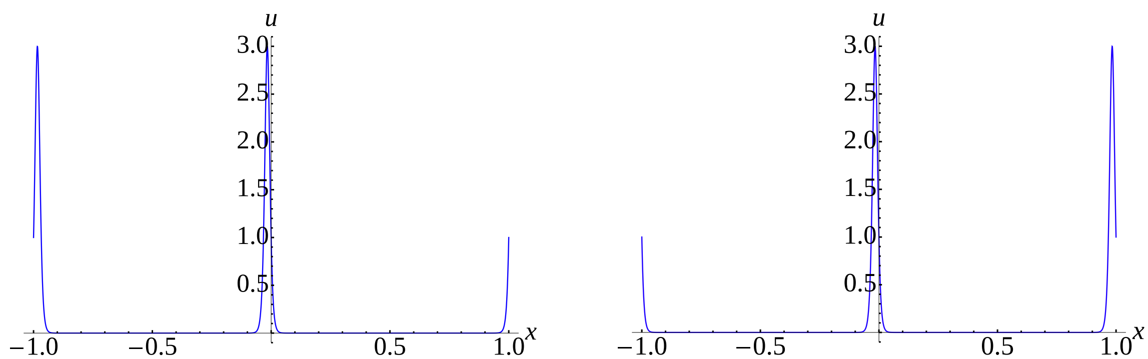

Figure 12: Solutions of the transformed Carrier's problem with four critical points.

5.3.6 Solutions with u possessing N spikes for a given N independent of ε

Let q satisfy (5.47). Ai studies this case in [1] for Carrier's equation. He proved that the solutions can have at most one oscillation near $x = \pm 1$ and the others are clustered near $x = 0$. We will not deal with this case here.

5.4 APPENDIX.

The proof of (5.48) will follow from this appendix. Equation (5.25) gives us a relation between the two successive local minima. We will find the next order term in (5.25) if $q'(x_M) = O(\varepsilon)$. First note that writing $1 = \varepsilon u'/(\sqrt{q}\rho) + O(\varepsilon)$, we have

$$\begin{aligned}\int_{x_m}^{x_M} (x_M - x)(u^2 - m^2) &= \varepsilon \int_{x_m}^{x_M} [1 + O(\varepsilon)] \frac{(x_M - x)}{\sqrt{q(x)}} \frac{(u^2 - m^2)u'dx}{\rho(u, m, M)} \\ &= \varepsilon[1 + O(\varepsilon)] \int_{x_m}^{x_M} \frac{x_M - x}{\sqrt{q(x)}} (R(u(x), m, M))'dx \\ &= -\varepsilon(1 + O(\varepsilon)) \int_{x_m}^{x_M} \Big(\frac{x_M - x}{\sqrt{q}}\Big)' R(u, m, M)dx = O(\varepsilon^2).\end{aligned}$$

Here in the second equation, the quantity $[1 + O(\varepsilon)]$ can be taken outside the integral since the integrand is non-negative. Thus, if $q'(x_M) = O(\varepsilon)$, then

$$|q'(x)| \leqslant |q'(x_M)| + q_2(x_M - x) = O(\varepsilon) + q_2(x_M - x).$$

Consequently,

$$\Big|\int_{x_m}^{x_M} O(\varepsilon)q'(u^2 - m^2)\Big| \leqslant \int_{x_m}^{x_M} O(\varepsilon)\Big\{O(\varepsilon) + q_2(x_M - x)\Big\}(u^2 - m^2) = O(\varepsilon^3).$$

Thus, from (5.16), we have

$$\begin{aligned}\int_{x_m}^{x_M} q'(u^2 - m^2) &= \frac{\varepsilon q'(x_M)R(M, m, M)}{\sqrt{q(x_M)}} \\ &- \varepsilon \int_{x_m}^{x_M} \Big(\frac{q'}{\sqrt{q}}\Big)' R(u, m, M)\Big\{\frac{\varepsilon u'}{\sqrt{q}\rho(u, m, M)} + O(\varepsilon)\Big\} + O(\varepsilon^3) \\ &= \frac{\varepsilon q'(x_M)R(M, m, M)}{\sqrt{q(x_M)}} - \varepsilon^2\Big(\Big(\frac{q'}{\sqrt{q}}\Big)' \frac{1}{\sqrt{q}}\Big)'\Big|_{x=x_M} R_1(M, m, M) \\ &+ O(\varepsilon^2) \int_{x_m}^{x_M} R_1 dx + O(\varepsilon^3) \\ &= \frac{\varepsilon q'(x_M)R(M, m, M)}{\sqrt{q(x_M)}} - \varepsilon^2\Big(\Big(\frac{q'}{\sqrt{q}}\Big)' \frac{1}{\sqrt{q}}\Big)'\Big|_{x=x_M} R_1(M, m, M) \\ &+ O(\varepsilon^3),\end{aligned}$$

where

$$R_1(v, m, M) := \int_m^v \frac{R(s, m, M)ds}{\rho(s, m, M)} = O(v^2 - m^2), \quad R_1(M, m, M) = \frac{2}{3}M^2(\ln 16 - 1) + O(m).$$

Noting that

$$\Big(\Big(\frac{q'}{\sqrt{q}}\Big)'\frac{1}{\sqrt{q}}\Big)'\Big|_{x=x_M} = \frac{2q''(x_M)}{q(x_M)} - \frac{q'(x_M)^2}{q^2(x_M)} = \frac{2q''(x_M)}{q(x_M)} + O(\varepsilon^2)$$

and $R(M,m,M) = 4M^2/3 + O(m)$, we then obtain

$$\frac{1}{M^2-m^2}\int_{x_m}^{x_M} q'[u^2-m^2] = \frac{4+O(m)}{3}\frac{\varepsilon q'(x_M)}{\sqrt{q(x_M)}} - \frac{4\varepsilon^2}{3}(\ln 16 - 1 + O(m))\frac{q''(x_M)}{q(x_M)} + O(\varepsilon^3).$$

Thus, when $q'(x_M) = O(\varepsilon)$, equation (5.6) can be refined as

$$M = \frac{3}{2}q(x_M) - \frac{m^2}{M+m} - \frac{2\varepsilon[1+O(m)]q'(x_M)}{\sqrt{q(x_M)}} + 2\varepsilon^2(\ln 16 - 1 + O(m))\frac{q''(x_M)}{q(x_M)} + O(\varepsilon^3),$$

and then equation (5.25) will be refined as

$$\tilde{m}^2 - m^2 \;\; = \;\; 6\varepsilon\sqrt{q(x_M)}q'(x_M)[1+O(m+\tilde{m})] - 6\varepsilon^2(\ln 16 - 1 + O(m))q''(x_M) + O(\varepsilon^3).$$

6.0 CONCLUSIONS

The main theme of my dissertation is to prove uniform asymptotic expansions of solutions of singularly perturbed boundary value problems. There are still some open questions and hence scopes of doing further research on the problems that I have considered. I will outline some specific questions below:

Monotonicity of $A(\varepsilon)$: In Chapter 3, we showed that for every $\varepsilon > 0$, there exists $A(\varepsilon)$ such that if $0 < A < A(\varepsilon)$, then the problem (3.1a)-(3.1b) has exactly two solutions. It would be interesting to study the limiting behavior of $A(\varepsilon)$ as $\varepsilon \to 0$. As a part of the proof of Theorem 5, we proved that for ε sufficiently small, the BVP (3.1a)-(3.1b) has a solution for every $A < \int_0^\infty \frac{2}{f(y)}\, dy$. This in turn implies that $A(\varepsilon) \geq \int_0^\infty \frac{2}{f(y)}\, dy$ for ε sufficiently small.

Conjecture: $A(\varepsilon)$ increases monotonically as $\varepsilon \to 0$. Moreover,

$$\lim_{\varepsilon \to 0} A(\varepsilon) = \int_0^\infty \frac{2}{f(y)}\, dy.$$

The significance of the above conjecture is that it implies that the asymptotic formula (3.18) applies to every smaller solution of (3.1a)-(3.1b).

A general theory A well-known problem by Lagerstrom was studied in [9], where an elementary approach was used to derive an asymptotic expansion of the solution of the boundary value problem

$$y'' + \frac{n-1}{x} y' + \varepsilon y y' = 0, \quad y(1) = 0,\ y(\infty) = 1, \tag{6.1}$$

where $n \geq 2$. The solution was expressed as an infinite series, uniformly convergent on $1 \leq x < \infty$, and the terms of the series can be evaluated recursively, leading to a unique asymptotic expansion as $\varepsilon \to 0$.

In Theorem 8 , we derived the asymptotic expansion of the smaller solution by finding a fixed point of an integral equation. As a by-product, we also obtained a new existence proof of the smaller solution of (3.1a)-(3.1b). It will be interesting to study whether it is possible to build a theory that would treat both the BVPs (3.1a)-(3.1b) and (6.1) together. More generally,

Question: Is it possible to develop a general theory that would unify problems of the types (3.1a)-(3.1b) and (6.1)?

Asymptotic Expansions: In Chapter 4, we proved that the asymptotic expansion of the solution to the BVP (4.1)-(4.2) with no spikes is correct up to $O(\varepsilon^2)$. In Chapter 5, we used a different method to obtain asymptotic expansions of solutions to the same BVP, that have three or fewer critical points (this of course includes the solution with no spikes). We obtained formulas that are correct up to $O(\varepsilon)$. A possible question is to check whether the asymptotic formulas for the spiked solutions are correct up to $O(\varepsilon^2)$? On the other hand, it is very possible that the spike approximation is not to this order.

Extension of the method of Variation of Constants In Chapter 5, we derived a new technique of "variation of constants" that works for quadratic equations $u(x)(q(x,\varepsilon)-u(x))$, but unfortunately does not work for cubic equations as in [2]. One possible area of study is to find out whether this method can be applied to functions with exactly two roots. In other words, I would like to study the following:

Question: Check whether the method of variation of constants can be applied to derive asymptotic expansions of solutions

$$u'' = f(x,u,\varepsilon),$$

where f has two roots.

Applications in real physical phenomena: All the problems that have been discussed so far are model BVPs that help us in studying boundary layer theory in details. I am looking forward to working on boundary layer problems that model real physical phenomena. Some possible problems that I would like to consider are studying final steady flow near a stagnation point on a vertical surface in a porous medium (see [19], [20]), understanding the dynamics of climate patterns (see [15]) and the boundary layer phenomenon occurring along thin edges of glaciers, coastlines etc. I am hoping to apply my techniques to some of these problems.

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