MATHEMATICAL ARCHITECTURE FOR MODELS OF FLUID FLOW PHENOMENA

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This thesis is a study of several high accuracy numerical methods for fluid flow problems and turbulence modeling.

First we consider a stabilized finite element method for the Navier-Stokes equations which has second order temporal accuracy. The method requires only the solution of one linear system (arising from an Oseen problem) per time step.

We proceed by introducing a family of defect correction methods for the time dependent Navier-Stokes equations, aiming at higher Reynolds' number. The method presented is unconditionally stable, computationally cheap and gives an accurate approximation to the quantities sought.

Next, we present a defect correction method with increased time accuracy. The method is applied to the evolutionary transport problem, it is proven to be unconditionally stable, and the desired time accuracy is attained with no extra computational cost.

We then turn to the turbulence modeling in coupled Navier-Stokes systems - namely, MagnetoHydroDynamics. Magnetically conducting fluids arise in important applications including plasma physics, geophysics and astronomy. In many of these, turbulent MHD (magnetohydrodynamic) flows are typical. The difficulties of accurately modeling and simulating turbulent flows are magnified many times over in the MHD case.

We consider the mathematical properties of a model for the simulation of the large eddies in turbulent viscous, incompressible, electrically conducting flows. We prove existence, uniqueness and convergence of solutions for the simplest closed MHD model. Furthermore, we show that the model preserves the properties of the 3D MHD equations.

Lastly, we consider the family of approximate deconvolution models (ADM) for turbulent MHD flows. We prove existence, uniqueness and convergence of solutions, and derive a bound on the modeling error. We verify the physical properties of the models and provide the results of the computational tests.

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PREFACE

I would like to thank everyone who has helped me during these five exciting years.

I am (and will always be) very grateful to my advisor, Professor William Layton. His wise guidance, moral support and endless patience have helped me understand a lot about mathematics, creativity and life.

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1.0 INTRODUCTION

The accurate and reliable solution of fluid flow problems is important for many applications. In these one core problem is the Navier-Stokes equations, given by:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \text{ for } \mathbf{x} \in \Omega, 0 < t \leq T$$
$$\nabla \cdot \mathbf{u} = 0, \ \mathbf{x} \in \Omega, \text{ for } 0 \leq t \leq T.$$

In the numerical solution of higher Reynolds number flow problems some of the standard iterative methods fail. One common mode of failure is non-convergence of the iterative nonlinear and linear solvers used to compute the velocity and pressure at the new time levels. In Sections 2 and 3 we introduce two unconditionally stable methods designed to overcome this type of failure.

The method, introduced in Section 2, is the Crank-Nicolson Linear Extrapolation with Stabilization. The two main ingredients are the linear extrapolation of the velocity and the artificial viscosity stabilization. The method is unconditionally stable and second-order accurate. Most importantly, the method requires the solution of one *linear* system per time step, and this linear system is a discretized Oseen problem (with cell Reynolds number $O(1)$) plus an $O(h)$ artificial viscosity operator. Thus, the standard iterative solvers and well-tested preconditioners can be used successfully, independent of how small the viscosity coefficient is. We also show that the stabilization in the method alters the numerical method's kinetic energy rather than in its energy dissipation. We discuss the physical fidelity of the method and provide the results of numerical tests, that verify the accuracy and the convergence rates.

In Sections 3 and 4 we introduce the family of Defect Correction methods for time dependent fluid flow problems. There has been an extensive study and development of this approach for equilibrium flow problems, see e.g. Hemker[Hem82], Koren[K91], Heinrichs[Hei96], Layton, Lee, Peterson[LLP02], Ervin, Lee[EL06], and subsection 3.1.1 for a review of this work. Briefly, let a k^{th} order accurate discretization of the *equilibrium* Navier-Stokes equations (NSE) be written as

$$NSE^h(u^h) = f, \tag{1.0.1}$$

The DCM computes $u_1^h, ..., u_k^h$ as

$$\begin{aligned} -\alpha h \Delta^h u_1^h + NSE^h(u_1^h) &= f, \quad (1.0.2) \\ -\alpha h \Delta^h u_l^h + NSE^h(u_l^h) &= f - \alpha h \Delta^h u_{l-1}^h, \text{ for } l = 2, ..., k, \end{aligned}$$

where the velocity approximations u_i^h are sought in the finite element space of piecewise polynomials of degree k.

It has been proven under quite general conditions (see, e.g., [LLP02]) that for the intermediate approximations of the equilibrium NSE

$$\|u_{NSE} - u_l^h\|_{energy-norm} = O(h^k + h\|u_{NSE} - u_{l-1}^h\|_{energy-norm}) = O(h^k + h^l),$$

and thus, after $l = k$ steps,

$$\|u - u_k^h\|_{energy-norm} = O(h^k).$$

Note that (1.0.2) requires solving an AV approximation k times which is often cheaper and more reliable than solving (1.0.1) once.

For many years, it has been widely believed that the method could be directly imported into implicit time discretizations of flow problems in the obvious quasistatic way. Unfortunately, this natural idea doesn't seem to be even stable (see Section 3.7). We give a critically important modification of the natural extension to time dependent problems, that we prove to be unconditionally stable (Theorem 3.1) and convergent. Hence, we develop a method for which standard iterative solvers can be applied (for arbitrarily large Reynolds number); the method is unconditionally stable, computationally attractive and highly accurate: in order to obtain an accuracy of $O(h^k)$, one needs to solve an artificial viscosity approximation k times, which is often cheaper and more reliable (for high Reynolds number) than solving the NSE once. Section 4 presents a modification of this method, that allows to obtain extra time accuracy with almost no extra computational cost.

In Chapter 5 we consider the coupling between the porous media problem

$$-\nabla \cdot (k \nabla p) = g$$
$$\mathbf{u} = -k \nabla p,$$

and the convection diffusion problem

$$\phi_t - \epsilon \Delta \phi + \mathbf{u} \cdot \nabla \phi + c\phi = f.$$

This type of coupling is of great importance in a wide array of applications, including oil recovery and nuclear waste storage. The method introduced in this chapter is based on a consistent multiscale mixed method formulation, presented for the stationary convection diffusion problem by W. Layton [Layton02]. We couple the eddy viscosity discretization to the porous media problem, prove the stability of the method and track the velocity error estimate from Darcy's problem to the convection diffusion to prove the near optimal error bound.

In Sections 6 and 7 we consider turbulence modeling in MagnetoHydroDynamics. Even in the hydro-dynamic (flow governed by the Navier-Stokes equations) case the modern science does not yet have a good understanding of turbulent phenomena - due to the turbulence being diffusive, chaotic, irregular, highly dissipative. Another important characteristics of the turbulent flow is the continuum of scales (unsteady vortices can appear at different scales and interact with each other). From the critical length scale determined by Kolmogorov, the size of the smallest persistent eddy is $O(Re^{-3/4})$, where the Reynolds number Re could be described as the ratio of advection coefficient to the diffusion coefficient. Hence, in order to accurately capture all physical properties of the three-dimensional flow, one needs to resolve the flow with $O(Re^{9/4})$ meshpoints. However, the Reynolds number for air flow around a car is of the order 10^6, around an airplane - 10^7, and it can achieve $O(10^{20})$ for some atmospheric flows. Therefore, it is not computationally feasible to use direct numerical simulations for most of the turbulent flows. Hence - the modeling.

Magnetically conducting fluids arise in important applications including climate change forecasting, plasma confinement, controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD sea water propulsion. In many of these, turbulent MHD (magnetohydrodynamics [Alfv42]) flows are typical. The difficulties of accurately modeling and simulating turbulent flows are magnified many times over in the MHD case. They are evinced by the more complex dynamics of the flow due to the coupling of Navier-Stokes and Maxwell equations via the Lorentz force and Ohm's law.

The flow of an electrically conducting fluid is affected by Lorentz forces, induced by the interaction of electric currents and magnetic fields in the fluid. The Lorentz forces can be used to control the flow and to attain specific engineering design goals such as flow stabilization, suppression or delay of flow separation, reduction of near-wall turbulence and skin friction, drag reduction and thrust generation.

The mathematical description of the problem proceeds as follows. Assuming the fluid to be viscous and incompressible, the governing equations are the Navier- Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law (see e.g. [Sher65]). Let $\Omega = (0, L)^3$ be the flow domain, and $u(t, x), p(t, x), B(t, x)$ be the velocity, pressure, and the magnetic field of the flow, driven by the velocity body force f and magnetic field force $\operatorname{curl} g$. Then u, p, B satisfy the MHD equations:

$$\begin{aligned} &u_t + \nabla \cdot (uu) - \frac{1}{\mathrm{Re}}\Delta u + \frac{S}{2}\nabla(B^2) - S\nabla \cdot (BB) + \nabla p = f, \\ &B_t + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} B) + \operatorname{curl}(B \times u) = \operatorname{curl} g, \\ &\nabla \cdot u = 0, \nabla \cdot B = 0, \end{aligned} \tag{1.0.3}$$

in $Q = (0, T) \times \Omega$, with the initial data:

$$u(0, x) = u_0(x), \quad B(0, x) = B_0(x) \quad \text{in } \Omega, \tag{1.0.4}$$

and with periodic boundary conditions (with zero mean):

$$\Phi(t, x + Le_i) = \Phi(t, x), i = 1, 2, 3, \quad \int_\Omega \Phi(t, x)dx = 0, \tag{1.0.5}$$

for $\Phi = u, u_0, p, B, B_0, f, g$.

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively.

Direct numerical simulation of a $3d$ turbulent flow is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport. This led to various numerical regularizations; one of these is Large Eddy Simulation (LES) [S01], [J04], [BIL06]. It is based on the idea that the flow can be represented by a collection of scales with different sizes, and instead of trying to approximate all of them down to the smallest one, one defines a filter width $\delta > 0$ and computes only the scales of size bigger than δ (large scales), while the effect of the small scales on the large scales is modeled. This reduces the number of degrees of freedom in a simulation and represents accurately the large structures in the flow.

In Sections 6 and 7 we consider the problem of modeling the motion of large structures in a viscous, incompressible, electrically conducting, turbulent fluid. We introduce a family of approximate deconvolution models - referring to the family of models in [AS01]. Given the filtering widths δ_1 and δ_2, the model computes w, q, W - the approximations to $\overline{u}^{\delta_1}$,$\overline{p}^{\delta_1}$,$\overline{B}^{\delta_2}$. Here $^{-\delta_1}$, $^{-\delta_2}$ denote two local, spacing averaging operators that commute with the differentiation. The ADM for the MHD reads

$$w_t + \nabla \cdot \overline{(G_N^1 w)(G_N^1 w)}^{\delta_1} - \frac{1}{\mathrm{Re}}\Delta w - S\nabla \cdot \overline{(G_N^2 W)\,(G_N^2 W)}^{\delta_1} + \nabla q = \overline{f}^{\delta_1}, \tag{1.0.6a}$$

$$W_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\, W) + \nabla \cdot (\overline{(G_N^2 W)(G_N^1 w)}^{\delta_2}) - \nabla \cdot (\overline{(G_N^1 w)(G_N^2 W)}^{\delta_2}) \tag{1.0.6b}$$

$$= \mathrm{curl}\, \overline{g}^{\delta_2},$$

$$\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{1.0.6c}$$

subject to $w(0, x) = \overline{u}_0^{\delta_1}(x), W(0, x) = \overline{B}_0^{\delta_2}(x)$ and periodic boundary conditions (with zero means). Here G_N^1 and G_N^2 are the deconvolution operators, that will be defined in Section 7.2.

We begin by proving the existence and uniqueness of solutions to the equations of the model, and that the solutions to the model equations converge to the solution of the MHD equations in a weak sense as the averaging radii converge to zero. Then the physical fidelity of the models has to be established. For that we consider the conservation laws - and verify that the model's energy and helicities are also conserved, as they are for the MHD equations; the models also preserve the Alfvén waves - the unique property of the MHD flows. We perform the computational tests to verify the models' verifiability, and we also conclude that in the situations when the direct numerical simulation is no longer available (flows with high Reynolds and magnetic Reynolds numbers), the solution can still be obtained by the ADM approach.

2.0 THE STABILIZED, EXTRAPOLATED TRAPEZOIDAL FINITE ELEMENT METHOD FOR THE NAVIER-STOKES EQUATIONS

2.1 INTRODUCTION

The accurate and reliable solution of fluid flow problems is important for many applications. In these one core problem is the Navier-Stokes equations, given by: find $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d\,(d = 2, 3),\ p : \Omega \times (0, T] \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned}
&\mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f},\ \text{for } \mathbf{x}\ \in\ \Omega,\ 0 < t \leq T \\
&\nabla \cdot \mathbf{u} = 0,\ \mathbf{x}\ \in\ \Omega,\ \text{for } 0 \leq t \leq T, \\
&\mathbf{u} = 0,\ \text{on } \partial\Omega,\ \text{for } 0 < t \leq T, \\
&\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),\ \text{for } \mathbf{x}\ \in\ \Omega,
\end{aligned} \tag{2.1.1}$$

with the usual normalization condition that $\int_\Omega p(\mathbf{x}, t)\ d\mathbf{x} = 0$ for $0 < t \leq T$ when (2.1.1) is discretized by accepted, accurate and stable methods, such as the finite element method in space and Crank-Nicolson in time, the approximation can still fail for many reasons. One common mode of failure is non-convergence of the iterative nonlinear and linear solvers used to compute the velocity and pressure at the new time levels. We consider herein a simple, second order accurate, and unconditionally stable method which addresses these failure modes. The method requires the solution of one *linear* system per time step.

This linear system is a discretized Oseen problem plus an $O(h)$ artificial viscosity operator - so the standard iterative solvers and well-tested preconditioners can be used successfully (the preconditioners are described, e.g., in chapter 8 of [ESW05]). Suppressing the spatial discretization, the method can be written as (with time step $k = \Delta t$ and tuning parameter $\alpha = O(1)$)

$$\nabla \cdot \mathbf{u}_{n+1} = 0 \text{ and}$$

$$\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{k} + \mathbf{U}_{n+1/2} \cdot \nabla\left(\frac{\mathbf{u}_{n+1} + \mathbf{u}_n}{2}\right) - \nu\Delta\left(\frac{\mathbf{u}_{n+1} + \mathbf{u}_n}{2}\right) - \alpha h \Delta \mathbf{u}_{n+1}$$
$$+\nabla\left(\frac{p_{n+1} + p_n}{2}\right) = \mathbf{f}(t_{n+1/2}) - \alpha h \Delta \mathbf{u}_n. \qquad (2.1.2)$$

Here $\mathbf{U}_{n+1/2} := \frac{3}{2}\mathbf{u}_n - \frac{1}{2}\mathbf{u}_{n-1}$ is the linear extrapolation of the velocity to $t_{n+1/2}$ from previous time levels. Thus, (2.1.2) is an extension of Baker's [B76] extrapolated Crank-Nicolson method. Artificial viscosity stabilization is introduced into the linear system for $\mathbf{u}_{n+1}$ by adding $-\alpha h \Delta \mathbf{u}_{n+1}$ to the LHS and correcting for it by $-\alpha h \Delta \mathbf{u}_n$ (the previous time level) on the RHS. This is a known idea[1] in practical CFD, and likely has been used in practical computations with many different timestepping methods. To our knowledge however, it has only been proven unconditionally stable in combination with first order, backward Euler time discretizations, e.g. E and Liu [EL01], Anitescu, Layton and Pahlevani [ALP04], Pahlevani [P06] for related stabilizations and also He [He03] for a two-level method based on Baker's extrapolated Crank-Nicolson method.

The increase in accuracy from first order Backward Euler with stabilization to second order in (2.1.2) (extrapolated CN with stabilization) is important. There is also a quite simple proof that (2.1.2) is unconditionally stable. We give the stability proof in Proposition 2.3 and then explore the effect the stabilization (and correction) in (2.1.2) have on the rates of convergence for various flow quantities.

No discretization is perfect. However, simple and stable ones leading to easily solvable linear systems can be very useful. We therefore conclude with numerical tests which verify accuracy and decrease in complexity in the linear equation solver.

[1] William Layton first saw it used as a numerical regularization in 1980 and it seems to have been known well before that. It is related to the simple Kelvin-Voight model of viscoelasticity, Oskolkov [O80], Kalantarev and Titi [KT07].

Defining the method precisely requires a small amount of notation. The spatial part of (2.1.1) is naturally formulated in

$$\mathbf{X} := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega).$$

The finite element approximation begins by selecting conforming finite element spaces $\mathbf{X}^h \subset \mathbf{X}, Q^h \subset Q$ satisfying the usual discrete inf-sup condition (defined in Section 2). Denote the usual L^2 norm and inner product by $\|\cdot\|$ and $(\cdot,\cdot)$, and the space of discretely divergence free functions $\mathbf{V}^h$ by:

$$\mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{X}^h : (q^h, \nabla \cdot \mathbf{v}^h) = 0, \ \forall \ q^h \in Q^h\}.$$

Define the explicitly skew-symmetrized trilinear form

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}), \tag{2.1.3}$$

and the extrapolation to $t_{n+\frac{1}{2}} := \frac{t_n+t_{n+1}}{2}$ by

$$E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h] := \frac{3}{2}\mathbf{u}_n^h - \frac{1}{2}\mathbf{u}_{n-1}^h, \tag{2.1.4}$$

where $\mathbf{u}_j^h(x)$ is a *known* approximation to $\mathbf{u}(x,t_j)$.

The method studied is a 2-step method, so the initial condition and first step must be specified, but are not essential. We choose the Stokes Projection, defined in Section 2.2.

Algorithm 2.1 (Stabilized, extrapolated trapezoid rule)**.** *Let* $\mathbf{u}_0^h$ *be the Stokes Projection of* $\mathbf{u}_0(x)$ *into* $\mathbf{V}^h$*. At the first time level* $(\mathbf{u}_1^h, p_1^h) \in (\mathbf{X}^h, Q^h)$ *are sought, satisfying*

$$\begin{aligned}
(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{k}, \mathbf{v}^h) + \nu(\nabla(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2}), \nabla\mathbf{v}^h) + \alpha h(\nabla\mathbf{u}_1^h, \nabla\mathbf{v}^h)& \\
+b^*(\mathbf{u}_0^h, \frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2}, \mathbf{v}^h) \ - (\frac{1}{2}(p_1^h+p_0^h), \nabla\cdot\mathbf{v}^h)& \\
= (\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}^h) + \alpha h(\nabla\mathbf{u}_0^h, \nabla\mathbf{v}^h), \qquad \forall \ \mathbf{v}^h \in \mathbf{X}^h&, \\
(\nabla\cdot\mathbf{u}_1^h, q^h) = 0, \qquad \forall \ q^h \in Q^h&.
\end{aligned} \tag{2.1.5}$$

Given a time step $k > 0$ and an $O(1)$ constant α, the method computes $\mathbf{u}_2^h, \mathbf{u}_3^h, \cdots, p_2^h, p_3^h, \cdots$ where $t_j = jk$ and $\mathbf{u}_j^h(x) \cong \mathbf{u}(x, t_j), p_j^h(x) \cong p(x, t_j)$. For $n \geq 1$, given $(\mathbf{u}_n^h, p_n^h) \in (\mathbf{X}^h, Q^h)$ find $(\mathbf{u}_{n+1}^h, p_{n+1}^h) \in (\mathbf{X}^h, Q^h)$ satisfying

$$\begin{aligned}
&\left(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}, \mathbf{v}^h\right) + \nu\left(\nabla\left(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}\right), \nabla \mathbf{v}^h\right) + \alpha h(\nabla \mathbf{u}_{n+1}^h, \nabla \mathbf{v}^h) \\
&\quad + b^*\left(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbf{v}^h\right) - \left(\frac{1}{2}(p_{n+1}^h + p_n^h), \nabla \cdot \mathbf{v}^h\right) \\
&\quad = (\mathbf{f}(t_{n+\frac{1}{2}}), \mathbf{v}^h) + \alpha h(\nabla \mathbf{u}_n^h, \nabla \mathbf{v}^h), \qquad \forall\ \mathbf{v}^h \in \mathbf{X}^h, \\
&(\nabla \cdot \mathbf{u}_{n+1}^h, q^h) = 0, \qquad \forall\ q^h \in Q^h.
\end{aligned} \tag{2.1.6}$$

We will refer to Algorithm 2.1 as CNLEStab (Crank-Nicolson with Linear Extrapolation Stabilized). If $\alpha = 0$, i.e. if no stabilization is used, Algorithm 2.1 reduces to one studied by G. Baker in 1976 [B76] and others, that we will refer to as CNLE.

We shall show that Algorithm 2.1 (CNLEStab) is unconditionally stable and second order accurate, $O(k^2 + hk + \text{spatial error})$. The extra stabilization terms added are $O(hk)$ because

$$\alpha h(\nabla(\mathbf{u}_{n+1}^h - \mathbf{u}_n^h), \nabla \mathbf{v}^h) = \alpha hk\left(\nabla\left(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}\right), \nabla \mathbf{v}^h\right) \simeq hk(-\Delta \mathbf{u}_t) = O(hk).$$

As stated above, each time step of the method requires the solution of only one linear Oseen problem at cell Reynolds number $O(1)$.

Remark 2.1. *At the first time level, a nonlinear treatment of the trilinear term can be used instead of extrapolation: find $(\mathbf{u}_1^h, p_1^h) \in (\mathbf{X}^h, Q^h)$, satisfying*

$$\begin{aligned}
&\left(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{k}, \mathbf{v}^h\right) + \nu\left(\nabla\left(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}\right), \nabla \mathbf{v}^h\right) + \alpha h(\nabla \mathbf{u}_1^h, \nabla \mathbf{v}^h) \\
&\quad + b^*\left(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h\right) - \left(\frac{1}{2}(p_1^h + p_0^h), \nabla \cdot \mathbf{v}^h\right) \\
&\quad = (\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}^h) + \alpha h(\nabla \mathbf{u}_0^h, \nabla \mathbf{v}^h), \qquad \forall\ \mathbf{v}^h \in \mathbf{X}^h, \\
&(\nabla \cdot \mathbf{u}_1^h, q^h) = 0, \qquad \forall\ q^h \in Q^h.
\end{aligned} \tag{2.1.7}$$

We shall show that this modification affects neither the stability of the method nor the convergence rate of the velocity error approximation, but increases the convergence rate of pressure approximation.

The stabilization in the method alters the numerical method's kinetic energy rather than in its energy dissipation. Proposition 2.4 and Section 2.5 show that

$$\text{Kinetic Energy in CNLEStab} = \frac{1}{2L^3}[||\mathbf{u}_n^h||^2 + \alpha k h||\nabla \mathbf{u}_n^h||^2],$$

$$\text{Energy Dissipation in CNLEStab} = \frac{\nu}{L^3}||\nabla \mathbf{u}_n^h||^2.$$

We shall show in Sections 2.5 and 2.6 that this has several interesting consequences.

Section 2.2 collects some mathematical preliminaries for the analysis that follows. Sections 2.3 and 2.4 present a convergence analysis of the method (2.1.2). The modification of the method's kinetic energy influences the norm in which convergence is proven. A basic convergence analysis is fundamental to a numerical method's usefulness but there are many important questions it does not answer. We try to address some of these in Section 2.5 and onward. In Section 2.5 we consider physical fidelity of a simulation produced by the method (2.1.2). One aspect of physical fidelity is conservation of important integral invariants of the Euler equations ($\nu = 0$) and near conservation when ν is small. The conservation of the method's kinetic energy when $\nu = 0$ is clear from the stability proof in Section 2.3. The second important integral invariant of the Euler equations in 3d is helicity, [MT92],[DG01],[CCE03] and in 2d, enstrophy. Approximate conservation of these is explored in Section 2.5. Section 2.6 gives some insight into the predictions of (2.1.1) of flow statistics in turbulent flows. In Section 2.7 we present the results of the computational tests. These confirm the rates of convergence, predicted in Section 2.3.

2.2 MATHEMATICAL PRELIMINARIES

Recall that (2.1.1) is naturally formulated in

$$\mathbf{X} := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega).$$

The dual space of $\mathbf{X}$ is denoted by $\mathbf{X}^*$ (and its norm, by $||\cdot||_{-1}$), and $\mathbf{V} = \{\mathbf{v} \in \mathbf{X} : (q, \nabla\cdot\mathbf{v}) = 0,\ \forall\ q \in Q\}$ is the set of weakly divergence free functions in $\mathbf{X}$. Norms in the Sobolev spaces $H^k(\Omega)^d$ (or $W_2^k(\Omega)^d$) are denoted by $\|\cdot\|_k$, and seminorms by $|\cdot|_k$.

Later analysis will require upper bounds on the nonlinear term, given in the following lemma.

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^3$ *or* $\mathbb{R}^2$. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\Omega)\|\nabla \mathbf{u}\|\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|,$$

and

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\Omega)\sqrt{\|\mathbf{u}\|\|\nabla \mathbf{u}\|}\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|.$$

If, in addition, $\mathbf{v}, \nabla \mathbf{v} \in L^\infty(\Omega)$*,*

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\Omega)(\|\mathbf{v}\|_{L^\infty(\Omega)} + \|\nabla \mathbf{v}\|_{L^\infty(\Omega)})\|\mathbf{u}\|\|\nabla \mathbf{w}\|$$

and

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(\|\mathbf{u}\|\|\nabla \mathbf{v}\|_{L^\infty(\Omega)} + \|\nabla \mathbf{u}\|\|\mathbf{v}\|_{L^\infty(\Omega)})\|\mathbf{w}\|.$$

Proof. See Girault and Raviart [GR86] for a proof of the first inequality. The second inequality follows from Hölder's inequality, the Sobolev embedding theorem and an interpolation inequality, e.g., [LT98]. The third bound follows from the definition of the skew-symmetric form and Hölder's inequality

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{1}{2}\|\nabla \mathbf{v}\|_{L^\infty(\Omega)}\|\mathbf{u}\|\|\mathbf{w}\| + \frac{1}{2}\|\mathbf{v}\|_{L^\infty(\Omega)}\|\mathbf{u}\|\|\nabla \mathbf{w}\|,$$

and Poincare's inequality, since $\mathbf{w} \in \mathbf{X}$. The proof of the last inequality can be found, e.g., in [LT98]. □

Throughout the chapter, we shall assume that the velocity-pressure finite element spaces $\mathbf{X}^h \subset \mathbf{X}$ and $Q^h \subset Q$ are conforming, have approximation properties typical of finite element spaces commonly in use, and satisfy the discrete inf-sup, or LBB^h, condition

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|\nabla \mathbf{v}^h\|\|q^h\|} \geq \beta^h > 0, \tag{2.2.1}$$

where β^h is bounded away from zero uniformly in h. Examples of such spaces can be found in [GR79], [GR86], [G89]. In addition, we assume that an inverse inequality holds, i.e. there exists a constant C independent of h and k, such that

$$\|\nabla \mathbf{v}\| \leq Ch^{-1}\|\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{X}^h. \tag{2.2.2}$$

We assume that $(\mathbf{X}^h, Q^h)$ satisfy the following approximation properties typical of piecewise polynomials of degree $(m, m-1)$, [BS94]:

$$\inf_{\mathbf{v}\in\mathbf{X}^h} \|\mathbf{u}-\mathbf{v}\| \leq Ch^{m+1}|\mathbf{u}|_{m+1}, \quad \mathbf{u} \in H^{m+1}(\Omega), \tag{2.2.3}$$

$$\inf_{\mathbf{v}\in\mathbf{X}^h} \|\nabla(\mathbf{u}-\mathbf{v})\| \leq Ch^{m}|\mathbf{u}|_{m+1}, \quad \mathbf{u} \in H^{m+1}(\Omega), \tag{2.2.4}$$

$$\inf_{q\in Q^h} \|p-q\| \leq Ch^{m}|p|_{m}, \quad p \in H^{m}(\Omega). \tag{2.2.5}$$

We will also use the following inequality, which holds under (2.2.1) and for all $\mathbf{u} \in \mathbf{V}$:

$$\inf_{\mathbf{v}\in\mathbf{V}^h} \|\nabla(\mathbf{u}-\mathbf{v})\| \leq C(\Omega) \inf_{\mathbf{v}\in\mathbf{X}^h} \|\nabla(\mathbf{u}-\mathbf{v})\|. \tag{2.2.6}$$

The proof of (2.2.6) can be found, e.g., in [GR79] (p.60, inequality (1.2)).

Throughout the chapter we use the following Stokes Projection.

Definition 2.1 (Stokes Projection)**.** *The Stokes projection operator* P_S*:* $(\mathbf{X}, Q) \to (\mathbf{X}^h, Q^h)$*,* $P_S(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p})$*, satisfies*

$$\begin{aligned} \nu(\nabla(\mathbf{u}-\tilde{\mathbf{u}}), \nabla \mathbf{v}^h) - (p-\tilde{p}, \nabla\cdot\mathbf{v}^h) &= 0, \\ (\nabla\cdot(\mathbf{u}-\tilde{\mathbf{u}}), q^h) &= 0, \end{aligned} \tag{2.2.7}$$

for any $\mathbf{v}^h \in \mathbf{X}^h$*,* $q^h \in Q^h$*.*

In $(\mathbf{V}^h, Q^h)$ this formulation reads: given $(\mathbf{u}, p) \in (\mathbf{X}, Q)$, find $\tilde{\mathbf{u}} \in \mathbf{V}^h$ satisfying

$$\nu(\nabla(\mathbf{u}-\tilde{\mathbf{u}}), \nabla \mathbf{v}^h) - (p-q^h, \nabla\cdot\mathbf{v}^h) = 0, \tag{2.2.8}$$

for any $\mathbf{v}^h \in \mathbf{V}^h, q^h \in Q^h$. Under the discrete inf-sup condition (2.2.1), the Stokes projection is well defined.

Proposition 2.1 (Stability of the Stokes Projection)**.** *Let* $\mathbf{u}$, $\tilde{\mathbf{u}}$ *satisfy (2.2.8). The following bound holds*

$$\nu\|\nabla\tilde{\mathbf{u}}\|^2 \leq 2[\nu\|\nabla\mathbf{u}\|^2 + d\nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2], \tag{2.2.9}$$

where d *is the dimension,* $d = 2, 3$*.*

Proof. Take $\mathbf{v}^h = \tilde{\mathbf{u}} \in \mathbf{V}^h$ in (2.2.8). This gives

$$\nu\|\nabla\tilde{\mathbf{u}}\|^2 = \nu(\nabla\mathbf{u}, \nabla\tilde{\mathbf{u}}) - (p - q^h, \nabla\cdot\tilde{\mathbf{u}}). \tag{2.2.10}$$

Using the Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned} \nu\|\nabla\tilde{\mathbf{u}}\|^2 \ \leq \ & \nu\|\nabla\mathbf{u}\|^2 + \frac{\nu}{4}\|\nabla\tilde{\mathbf{u}}\|^2 \\ & + d\nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2 + \frac{\nu}{4d}\|\nabla\cdot\tilde{\mathbf{u}}\|^2. \end{aligned} \tag{2.2.11}$$

Next, use the obvious inequality $\|\nabla\cdot\tilde{\mathbf{u}}\|^2 \leq d\|\nabla\tilde{\mathbf{u}}\|^2$. Combining the like terms in (2.2.11) concludes the proof. □

In the error analysis we shall use the error estimate of the Stokes Projection (2.2.8).

Proposition 2.2 (Error estimate for the Stokes Projection)**.** *Suppose the discrete inf-sup condition (2.2.1) holds. Then the error in the Stokes Projection satisfies*

$$\nu\|\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^2 \leq C[\nu\inf_{\mathbf{v}^h\in\mathbf{X}^h}\|\nabla(\mathbf{u}-\mathbf{v}^h)\|^2 + \nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2], \tag{2.2.12}$$

where C *is a constant independent of* h *and* ν*.*

Proof. Decompose the projection error $\mathbf{e} = \mathbf{u} - \tilde{\mathbf{u}}$ into $\mathbf{e} = \mathbf{u} - I(\mathbf{u}) - (\tilde{\mathbf{u}} - I(\mathbf{u})) = \boldsymbol{\eta} - \boldsymbol{\phi}$, where $\boldsymbol{\eta} = \mathbf{u} - I(\mathbf{u})$, $\boldsymbol{\phi} = \tilde{\mathbf{u}} - I(\mathbf{u})$, and $I(\mathbf{u})$ approximates $\mathbf{u}$ in $\mathbf{V}^h$. Take $\mathbf{v}^h = \boldsymbol{\phi} \in \mathbf{V}^h$ in (2.2.8). This gives

$$\nu\|\nabla\boldsymbol{\phi}\|^2 = \nu(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\phi}) - (p - q^h, \nabla\cdot\boldsymbol{\phi}). \tag{2.2.13}$$

The Cauchy-Schwarz and Young inequalities lead to

$$\nu\|\nabla\boldsymbol{\phi}\|^2 \leq 2\nu\|\nabla\boldsymbol{\eta}\|^2 + C\nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2. \tag{2.2.14}$$

Since $I(\mathbf{u})$ is an approximation of $\mathbf{u}$ in $\mathbf{V}^h$, we can take infimum over $\mathbf{V}^h$. The proof is concluded by applying (2.2.6) and the triangle inequality. □

Remark 2.2. *Using the Aubin-Nitsche lift, one can obtain (see, e.g., [BDK82])*

$$\|\mathbf{u} - \tilde{\mathbf{u}}\| \le Ch\left(\inf_{\mathbf{v}^h \in \mathbf{X}^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\| + \inf_{q^h \in Q^h} \|p - q^h\|\right), \tag{2.2.15}$$

where $C = C(\nu, \Omega)$*.*

The following variation on the discrete Gronwall Lemma is given in [HR90] as a remark to Lemma 5.1. In this estimate, the first sum on the right hand side is only up to the next-to-last time step, which allows for an estimate with no smallness condition on k.

Lemma 2.2 (Discrete Gronwall)**.** *Let* k, B, a_n, b_n, c_n, d_n *for integers* $n \ge 0$ *be nonnegative numbers such that for* $N \ge 1$*, if*

$$a_N + k\sum_{n=0}^{N} b_n \le k\sum_{n=0}^{N-1} d_n a_n + k\sum_{n=0}^{N} c_n + B,$$

then for all $k > 0$*,*

$$a_N + k\sum_{n=0}^{N} b_n \le \exp(k\sum_{n=0}^{N-1} d_n)\left(k\sum_{n=0}^{N} c_n + B\right).$$

The following results are readily obtained by Taylor series expansion.

Lemma 2.3. *Let* $k = t_{n+1} - t_n$ *for all* i *and denote* $t_{n+1/2} = \frac{t_{n+1}+t_n}{2}$*. Let* $\psi(\cdot, t)$ *be a function such that* $\psi_t \in \mathcal{C}^0(0, T; L^2(\Omega))$*. Then there exists* $\theta \in (0, 1)$ *such that*

$$\|\frac{\psi(\cdot, t_{n+1}) - \psi(\cdot, t_n)}{k}\| \le C\|\psi_t(\cdot, t_{n+\theta})\|.$$

If $\psi_{tt} \in \mathcal{C}^0(0, T; L^2(\Omega))$*, then there exist* $\theta_1, \theta_2 \in (0, 1)$ *such that*

$$\|\frac{\psi(\cdot, t_{n+1}) + \psi(\cdot, t_n)}{2} - \psi(\cdot, t_{n+1/2})\| \le Ck^2\|\psi_{tt}(\cdot, t_{n+\theta_1})\|$$

and

$$\|\frac{3}{2}\psi(\cdot, t_n) - \frac{1}{2}\psi(\cdot, t_{n-1}) - \psi(\cdot, t_{n+1/2})\| \le Ck^2\|\psi_{tt}(\cdot, t_{n+\theta_2})\|.$$

If $\psi_{ttt} \in \mathcal{C}^0(0, T; L^2(\Omega))$*, then there exists* $\theta_3 \in (0, 1)$ *such that*

$$\|\frac{\psi(\cdot, t_{n+1}) - \psi(\cdot, t_n)}{k} - \psi_t(\cdot, t_{n+1/2})\| \le Ck^2\|\psi_{ttt}(\cdot, t_{n+\theta_3})\|.$$

We start with the proof of unconditional stability, which is the mathematical key to the good properties of the method, and motivates the more technical error analysis that follows.

The unconditional stability of Algorithm 2.1 is proven in the following proposition.

Proposition 2.3. *[Stability of extrapolated trapezoidal method] Let* $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega))$. *The stabilized, extrapolated trapezoid scheme (2.1.5)-(2.1.6) (and the scheme (2.1.6)-(2.1.7)) is unconditionally stable. For any* $h, k > 0$ *and* $\alpha \geq 0, n \geq 0$

$$||\mathbf{u}_{n+1}^h||^2 + \alpha kh||\nabla \mathbf{u}_{n+1}^h||^2 + \nu k \sum_{i=0}^{n} ||\nabla(\frac{\mathbf{u}_{i+1}^h + \mathbf{u}_i^h}{2})||^2$$
$$\leq ||\mathbf{u}_0^h||^2 + \alpha kh||\nabla \mathbf{u}_0^h||^2 + \nu^{-1} k \sum_{i=0}^{n} ||\mathbf{f}(t_{i+\frac{1}{2}})||_{-1}^2.$$

Proof. Taking $\mathbf{v}^h = \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \in \mathbf{V}^h$ in (2.1.5) (and in (2.1.7)) gives

$$(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{k}, \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}) + \nu\|\nabla(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2})\|^2 + \alpha hk(\nabla \frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{k}, \nabla \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2})$$
$$= (\mathbf{f}(t_{\frac{1}{2}}), \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}). \qquad (2.3.1)$$

Apply the Cauchy-Schwarz and Young inequalities. This gives

$$\frac{\|\mathbf{u}_1^h\|^2 - \|\mathbf{u}_0^h\|^2}{2k} + \nu\|\nabla(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2})\|^2 + \alpha hk \frac{\|\nabla \mathbf{u}_1^h\|^2 - \|\nabla \mathbf{u}_0^h\|^2}{2k}$$
$$\leq \frac{1}{2}\nu^{-1}\|\mathbf{f}(t_{\frac{1}{2}})\|_{-1}^2 + \frac{1}{2}\nu\|\nabla(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2})\|^2. \qquad (2.3.2)$$

Thus, on the first time level we obtain the stability bound

$$\|\mathbf{u}_1^h\|^2 + \nu k\|\nabla(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2})\|^2 + \alpha hk\|\nabla \mathbf{u}_1^h\|^2 \leq \|\mathbf{u}_0^h\|^2 + \alpha hk\|\nabla \mathbf{u}_0^h\|^2 + \nu^{-1}k\|\mathbf{f}(t_{\frac{1}{2}})\|_{-1}^2. \qquad (2.3.3)$$

Now consider (2.1.6) for $n \geq 1$; let $\mathbf{v}^h = \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \in \mathbf{V}^h$. This gives

$$(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}, \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}) + \nu\|\nabla(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2})\|^2 + \alpha hk(\nabla(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}), \nabla(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}))$$
$$= (\mathbf{f}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}). \qquad (2.3.4)$$

Applying Cauchy-Schwarz and Young inequalities leads to

$$\frac{\|\mathbf{u}^h_{n+1}\|^2 - \|\mathbf{u}^h_n\|^2}{2k} + \nu\|\nabla(\frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2})\|^2 + \alpha hk\frac{\|\nabla\mathbf{u}^h_{n+1}\|^2 - \|\nabla\mathbf{u}^h_n\|^2}{2k}$$
$$\leq \frac{1}{2}\nu^{-1}\|\mathbf{f}(t_{n+\frac{1}{2}})\|^2_{-1} + \frac{1}{2}\nu\|\nabla(\frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2})\|^2. \quad (2.3.5)$$

Simplifying (2.3.5) gives

$$(\|\mathbf{u}^h_{n+1}\|^2 - \|\mathbf{u}^h_n\|^2) + \nu k\|\nabla(\frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2})\|^2 + \alpha hk(\|\nabla\mathbf{u}^h_{n+1}\|^2 - \|\nabla\mathbf{u}^h_n\|^2)$$
$$\leq \nu^{-1}k\|\mathbf{f}(t_{n+\frac{1}{2}})\|^2_{-1}. \quad (2.3.6)$$

Summing (2.3.6) over the time levels gives

$$\|\mathbf{u}^h_{n+1}\|^2 + k\sum_{i=1}^{n}\nu\|\nabla(\frac{\mathbf{u}^h_{i+1} + \mathbf{u}^h_i}{2})\|^2 + \alpha hk\|\nabla\mathbf{u}^h_{n+1}\|^2$$
$$\leq \|\mathbf{u}^h_1\|^2 + \alpha hk\|\nabla\mathbf{u}^h_1\|^2 + k\sum_{i=1}^{n}\nu^{-1}\|\mathbf{f}(t_{i+\frac{1}{2}})\|^2_{-1}. \quad (2.3.7)$$

Finally, using the bound on $(\|\mathbf{u}^h_1\|^2 + \alpha hk\|\nabla\mathbf{u}^h_1\|^2)$ from (2.3.3), we obtain that for all $n \geq 1$

$$\|\mathbf{u}^h_{n+1}\|^2 + k\sum_{i=0}^{n}\nu\|\nabla(\frac{\mathbf{u}^h_{i+1} + \mathbf{u}^h_i}{2})\|^2 + \alpha hk\|\nabla\mathbf{u}^h_{n+1}\|^2$$
$$\leq \|\mathbf{u}^h_0\|^2 + \alpha hk\|\nabla\mathbf{u}^h_0\|^2 + k\sum_{i=0}^{n}\nu^{-1}\|\mathbf{f}(t_{i+\frac{1}{2}})\|^2_{-1}. \quad (2.3.8)$$

This result, combined with Proposition 2.1, proves the Proposition. □

Hence the method is unconditionally stable. The question remains: how fast does $\mathbf{u}^h$ converge to $\mathbf{u}$? To evaluate the rates of convergence as $h \to 0$, we must make a specific choice of $\mathbf{X}^h, Q^h$.

Theorem 2.3.1 (Velocity Convergence Rates). *Let the finite-element spaces* $(\mathbf{X}^h, Q^h)$ *include continuous piecewise polynomials of degree* m *and* $m-1$ *respectively* ($m \geq 2$), *and satisfy the discrete inf-sup condition* (2.2.1) *and approximation properties* (2.2.3)-(2.2.5). *Let* $C\|\mathbf{u}\|_{L^\infty(0,T;H^{m+1}(\Omega))} k h^{m-\frac{3}{2}} \leq 1/2$, *and*

$$\mathbf{u} \in L^\infty(0,T;H^{m+1}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \cap \mathcal{C}^0(0,T;H^1(\Omega)),$$

$$\nabla \mathbf{u} \in L^\infty(0,T;L^\infty(\Omega)),$$

$$\mathbf{u}_t \in L^2(0,T;H^{m+1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \nabla \mathbf{u}_{tt} \in L^2(0,T;H^1(\Omega)),$$

$$p_{tt} \in L^2(0,T;L^2(\Omega)).$$

Then there is a $C = C(\nu, \mathbf{u}, p, T) < \infty$ *such that* $\forall n \in \{0, 1, ..., N-1\}$ *the error in Algorithm 2.1 satisfies*

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}^h_{n+1}\| + \left(k \sum_{i=0}^{n} \nu \|\nabla (\frac{(\mathbf{u}(t_{i+1}) - \mathbf{u}^h_{i+1}) + (\mathbf{u}(t_i) - \mathbf{u}^h_i)}{2})\|^2 \right)^{\frac{1}{2}}$$
$$+\alpha^{\frac{1}{2}} h^{\frac{1}{2}} k^{\frac{1}{2}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}^h_{n+1})\| \leq C(\nu, \mathbf{u}, p) \left(h^m + \alpha h k + k^2\right).$$

The rest of this section will be devoted to proving this theorem.

Proof. Consider the variational formulation corresponding to the Navier-Stokes equations (2.1.1), for any time t, in $\mathbf{X}^h$,

$$(\mathbf{u}_t, \mathbf{v}^h) + b^*(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}^h) - (p, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{X}^h. \tag{2.3.9}$$

Then subtract (2.1.6) from (2.3.9), taken at $t = t_{n+\frac{1}{2}}$, to get

$$\begin{aligned}
&(\mathbf{u}_t(t_{n+\frac{1}{2}}) - \frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{k}, \mathbf{v}^h) + \nu(\nabla \mathbf{u}(t_{n+\frac{1}{2}}) - \nabla(\frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2}), \nabla \mathbf{v}^h) \\
-\;& \alpha h(\nabla(\mathbf{u}^h_{n+1} - \mathbf{u}^h_n), \nabla \mathbf{v}^h) + b^*(\mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{v}^h) \\
-\;& b^*(E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \frac{\mathbf{u}^h_{n+1} + \mathbf{u}^h_n}{2}, \mathbf{v}^h) - (p(t_{n+\frac{1}{2}}) - \frac{p(t_{n+1}) + p(t_n)}{2}, \nabla \cdot \mathbf{v}^h) = 0
\end{aligned} \tag{2.3.10}$$

Let the velocity error be decomposed as

$$\mathbf{e}_n := \mathbf{u}(t_n) - \mathbf{u}^h_n = (\mathbf{u}(t_n) - \mathbf{U}_n) - (\mathbf{u}^h_n - \mathbf{U}_n) =: \boldsymbol{\eta}_n - \boldsymbol{\phi}^h_n, \tag{2.3.11}$$

where $\mathbf{U}_n$ is the Stokes Projection of $\mathbf{u}_n$ into $\mathbf{V}^h$ (therefore $\boldsymbol{\phi}_n^h \in \mathbf{V}^h$, but $\boldsymbol{\eta}_n \notin \mathbf{V}^h$). For $\boldsymbol{\xi} = \mathbf{e}, \boldsymbol{\phi}^h$ or $\boldsymbol{\eta}$, define $\boldsymbol{\xi}_{n+\frac{1}{2}} := \frac{\boldsymbol{\xi}_{n+1}+\boldsymbol{\xi}_n}{2}$.

Add and subtract

$$
\begin{aligned}
&(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k}, \mathbf{v}^h) + \nu(\nabla(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{2}), \nabla \mathbf{v}^h) \\
&\quad + \alpha h(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), \nabla \mathbf{v}^h) - (\frac{p(t_{n+1}) + p(t_n)}{2}, \nabla \cdot \mathbf{v}^h) \\
&\quad + b^*(\mathbf{u}(t_{n+\frac{1}{2}}) + E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})] + E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h)
\end{aligned}
$$

to (2.3.10) to obtain the error equation (recall also that $(q^h, \nabla \cdot \mathbf{v}^h) = 0, \forall q^h \in Q^h$)

$$
\begin{aligned}
(\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k}, \mathbf{v}^h) \;\; &+ \;\; \nu(\nabla \mathbf{e}_{n+1/2}, \nabla \mathbf{v}^h) + \alpha h(\nabla(\mathbf{e}_{n+1} - \mathbf{e}_n), \nabla \mathbf{v}^h) \\
&= (\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \mathbf{v}^h) - b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \mathbf{e}_{n+1/2}, \mathbf{v}^h) \\
&\quad + b^*(E[\mathbf{e}_n, \mathbf{e}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h) + T(\mathbf{u}, p; \mathbf{v}^h), \qquad (2.3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
T(\mathbf{u}, p; \mathbf{v}^h) \;\; = \;\; & (\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k} - \mathbf{u}_t(t_{n+\frac{1}{2}}), \mathbf{v}^h) + \nu(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}) - \nabla \mathbf{u}(t_{n+\frac{1}{2}}), \nabla \mathbf{v}^h) \\
& - \alpha h(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), \nabla \mathbf{v}^h) - (\frac{p(t_{n+1}) + p(t_n)}{2} - p(t_{n+\frac{1}{2}}), \nabla \cdot \mathbf{v}^h) \\
& + b^*(\mathbf{u}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{v}^h) \\
& - b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})] - \mathbf{u}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h). \qquad (2.3.13)
\end{aligned}
$$

Using the error decomposition (2.3.11) and setting $\mathbf{v}^h = \boldsymbol{\phi}_{n+1/2}^h$ in (2.3.12) gives

$$
\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}_{n+1}^h\|^2 - \|\boldsymbol{\phi}_n^h\|^2) + \nu\|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + \frac{\alpha h}{2}(\|\nabla \boldsymbol{\phi}_{n+1}^h\|^2 - \|\nabla \boldsymbol{\phi}_n^h\|^2) \\
&\quad = (\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}, \boldsymbol{\phi}_{n+1/2}^h) + \nu(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}_{n+1/2}^h) \\
&\quad + \alpha hk(\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}), \nabla \boldsymbol{\phi}_{n+1/2}^h) - (\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \boldsymbol{\phi}_{n+1/2}^h) \\
&\quad + b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) + b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}_{n+1/2}^h) \\
&\quad + b^*(E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}_{n+1/2}^h) + T(\mathbf{u}, p; \boldsymbol{\phi}_{n+1/2}^h), \qquad (2.3.14)
\end{aligned}
$$

since $b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \boldsymbol{\phi}_{n+1/2}^h, \boldsymbol{\phi}_{n+1/2}^h) = 0$.

Also it follows from the choice of the projection $\mathbf{U}_n$ that

$$\nu(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}^h_{n+1/2}) - (\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \boldsymbol{\phi}^h_{n+1/2}) = 0.$$

Applying the Cauchy-Schwarz and Young's inequalities to the linear terms on the right hand side of (2.3.14) gives

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}^h_{n+1}\|^2 - \|\boldsymbol{\phi}^h_n\|^2) + \frac{3\nu}{4}\|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \frac{\alpha h}{2}(\|\nabla \boldsymbol{\phi}^h_{n+1}\|^2 - \|\nabla \boldsymbol{\phi}^h_n\|^2) \\
&\quad \le C\nu^{-1}\|\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}\|^2 + C\nu^{-1}\alpha h k\|\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k})\|^2 \\
&\quad + |\, b^*(E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2})\,| \\
&\quad + |\, b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^h_{n+1/2})\,| \\
&\quad + |\, b^*(E[\boldsymbol{\phi}^h_n, \boldsymbol{\phi}^h_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^h_{n+1/2})\,| + |\, T(\mathbf{u}, p; \boldsymbol{\phi}^h_{n+1/2})\,|, \qquad (2.3.15)
\end{aligned}$$

For clarity, we analyze each of the remaining nonlinear terms on the RHS of (2.3.15) individually. Here we use frequently Lemma 2.1 and the inverse estimate (2.2.2), together with Young's inequality.

We start with the first nonlinear term in (2.3.15). Adding and subtracting the quantity $b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2})$, and using Lemma 2.1, followed by Young's inequality, we get

$$\begin{aligned}
&\left| b^*(E[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \right| \\
&\qquad \le \quad \frac{\nu}{16}\|\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1}\|\nabla E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})]\|^2\, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
&\qquad + \quad C\nu^{-1}\|\nabla E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}]\|^2\, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
&\qquad + \quad C\|E[\boldsymbol{\phi}^h_n, \boldsymbol{\phi}^h_{n-1}]\|^{1/2}\|\nabla E[\boldsymbol{\phi}^h_n, \boldsymbol{\phi}^h_{n-1}]\|^{1/2}\|\nabla \boldsymbol{\eta}_{n+1/2}\|\, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|. \qquad (2.3.16)
\end{aligned}$$

The first two terms involving the operator $E[\cdot,\cdot]$ can be bounded by using its definition (2.1.4) and regularity assumptions on $\mathbf{u}$,

$$\|\nabla E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})]\| \le C \qquad \text{and} \qquad \|\nabla E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}]\| \le \frac{3}{2}\|\nabla \boldsymbol{\eta}_n\| + \frac{1}{2}\|\nabla \boldsymbol{\eta}_{n-1}\|. \qquad (2.3.17)$$

For the third and fourth terms, we also need the inverse estimate (2.2.2), resulting in

$$\begin{aligned}\|E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\| \, \|\nabla E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\| &\leq C(\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n-1}^h\|) \, (\|\nabla \boldsymbol{\phi}_n^h\| + \|\nabla \boldsymbol{\phi}_{n-1}^h\|), \\ &\leq Ch^{-1}(\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n-1}^h\|)^2,\end{aligned}$$

so that

$$\begin{aligned}&\|E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|^{1/2} \, \|\nabla E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\|^{1/2} \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}\| \\ &\qquad \leq Ch^{-3/2} \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, (\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n-1}^h\|) \, (\|\boldsymbol{\phi}_n^h\| + \|\boldsymbol{\phi}_{n+1}^h\|),\end{aligned} \tag{2.3.18}$$

Putting (2.3.17) and (2.3.18) back into (2.3.16), we have

$$\begin{aligned}&\left| b^*(E[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \right| \\ &\qquad \leq \frac{\nu}{16} \|\boldsymbol{\phi}_{n+1/2}^h\|^2 + C\nu^{-1} \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\ &\qquad\quad + C\nu^{-1}(\|\nabla \boldsymbol{\eta}_n\|^2 + \|\nabla \boldsymbol{\eta}_{n-1}\|^2) \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\ &\qquad\quad + Ch^{-3/2} \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, (\|\boldsymbol{\phi}_{n-1}^h\|^2 + \|\boldsymbol{\phi}_n^h\|^2 + \|\boldsymbol{\phi}_{n+1}^h\|^2).\end{aligned} \tag{2.3.19}$$

For the second trilinear term, use Lemma 2.1 and the assumption that $\|\nabla \mathbf{u}(t)\|$ is bounded for any $t \in [0, T]$. Then we apply Young's inequality and (2.3.17), resulting in

$$\begin{aligned}\left| b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}_{n+1/2}^h) \right| &\leq C \|\nabla E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}]\| \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\ &\leq C\nu^{-1}(\|\nabla \boldsymbol{\eta}_n\|^2 + \|\nabla \boldsymbol{\eta}_{n-1}\|^2) \\ &\quad + \frac{\nu}{16} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2.\end{aligned} \tag{2.3.20}$$

The third trilinear term is bounded with the help of the third inequality in Lemma 2.1 and the regularity assumptions on $\mathbf{u}$. As a result,

$$\begin{aligned}\left| b^*(E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}_{n+1/2}^h) \right| &\leq C \|E[\boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n-1}^h]\| \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\ &\leq C\nu^{-1}(\|\boldsymbol{\phi}_n^h\|^2 + \|\boldsymbol{\phi}_{n-1}^h\|^2) \\ &\quad + \frac{\nu}{16} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2,\end{aligned} \tag{2.3.21}$$

where the last step follows from Young's inequality.

Now, with (2.3.19), (2.3.20) and (2.3.21), the error equation (2.3.15) can be rewritten as

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}^h_{n+1}\|^2 - \|\boldsymbol{\phi}^h_n\|^2) + \frac{9\nu}{16}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + \frac{\alpha h}{2}(\|\nabla\boldsymbol{\phi}^h_{n+1}\|^2 - \|\nabla\boldsymbol{\phi}^h_n\|^2)\\
&\quad\le C\nu^{-1}\|\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k}\|^2 + C\nu^{-1}\alpha hk\|\nabla(\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k})\|^2 + C\nu^{-1}\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2\\
&\quad+C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2 + \|\nabla\boldsymbol{\eta}_{n-1}\|^2)\|\nabla\boldsymbol{\eta}_{n+1/2}\|^2\\
&\quad+C\nu^{-1}(\|\nabla\boldsymbol{\eta}_n\|^2 + \|\nabla\boldsymbol{\eta}_{n-1}\|^2) + C\nu^{-1}(\|\boldsymbol{\phi}^h_n\|^2 + \|\boldsymbol{\phi}^h_{n-1}\|^2)\\
&\quad+Ch^{-3/2}\|\nabla\boldsymbol{\eta}_{n+1/2}\|(\|\boldsymbol{\phi}^h_{n-1}\|^2 + \|\boldsymbol{\phi}^h_n\|^2 + \|\boldsymbol{\phi}^h_{n+1}\|^2) + |\,T(\mathbf{u},p;\boldsymbol{\phi}^h_{n+1/2})\,|, \qquad (2.3.22)
\end{aligned}$$

and what is left is to bound $|\,T(\mathbf{u},p;\boldsymbol{\phi}^h_{n+1/2})\,|$. Each of its four linear terms can be bounded by the Cauchy-Schwarz and Young's inequalities, together with the estimates in Lemma 2.3. We take care of one at a time below.

$$\begin{aligned}
|(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{k} - \mathbf{u}_t(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2})| \;&\le\; \frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2\\
&\quad+C\nu^{-1}k^4\|\mathbf{u}_{ttt}(t_{n+\theta_1})\|^2, \qquad (2.3.23)
\end{aligned}$$

$$\begin{aligned}
\nu|(\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+\frac{1}{2}})), \nabla\boldsymbol{\phi}^h_{n+1/2})| \;&\le\; \frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2\\
&\quad+C\nu k^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta_2})\|^2, \qquad (2.3.24)
\end{aligned}$$

$$\begin{aligned}
\alpha hk|(\nabla(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{k}), \nabla\boldsymbol{\phi}^h_{n+1/2})| \;&\le\; \frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2\\
&\quad+C\nu^{-1}\alpha^2h^2k^2\|\nabla\mathbf{u}_t(t_{n+\theta_3})\|^2, \qquad (2.3.25)
\end{aligned}$$

$$\begin{aligned}
|(\frac{p(t_{n+1})+p(t_n)}{2} - p(t_{n+1/2}), \nabla\cdot\boldsymbol{\phi}^h_{n+1/2})| \;&\le\; \frac{\nu}{80}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2\\
&\quad+C\nu^{-1}k^4\|p_{tt}(t_{n+\theta_4})\|^2, \qquad (2.3.26)
\end{aligned}$$

for some $\theta_1, \theta_2, \theta_3, \theta_4 \in (0,1)$.

For the two nonlinear terms in $|\,T(\mathbf{u}, p; \boldsymbol{\phi}^h_{n+1/2})\,|$, use Lemma 2.1, Lemma 2.3 and Young's inequality, together with $\|\nabla \mathbf{u}(t)\| \leq C$, for any $t \in [0, T]$. This gives

$$\begin{aligned}
&\left| b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})] - \mathbf{u}(t_{n+1/2}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^h_{n+1/2}) \right| \\
&\qquad + \left| b^*(\mathbf{u}(t_{n+1/2}), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2}) \right| \\
&\qquad \leq C(\Omega)\|\nabla(\frac{3}{2}\mathbf{u}(t_n) - \frac{1}{2}\mathbf{u}(t_{n-1}) - \mathbf{u}(t_{n+1/2}))\|\|\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2})\|\|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \\
&\qquad +C(\Omega)\|\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+1/2}))\|\|\nabla \mathbf{u}(t_{n+1/2})\|\|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \\
&\qquad \leq C\nu^{-1}k^4\|\nabla \mathbf{u}_{tt}(t_{n+\theta_5})\|^2 + \frac{\nu}{80}\|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2, \qquad (2.3.27)
\end{aligned}$$

for some $\theta_5 \in (0, 1)$.

Combining (2.3.23)-(2.3.27), we have

$$\begin{aligned}
|\,T(\mathbf{u}, p; \boldsymbol{\phi}^h_{n+1/2})\,| \;\leq\; & \frac{\nu}{16}\|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1}k^4(\|\mathbf{u}_{ttt}(t_{n+\theta_1})\|^2 + \|p_{tt}(t_{n+\theta_4})\|^2) \\
& + C\nu k^4\|\nabla \mathbf{u}_{tt}(t_{n+\theta_5})\|^2 + C\nu^{-1}\alpha^2 h^2 k^2\|\nabla \mathbf{u}_t(t_{n+\theta_3})\|^2, \qquad (2.3.28)
\end{aligned}$$

so that error equation (2.3.22) gives

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{\phi}^h_{n+1}\|^2 - \|\boldsymbol{\phi}^h_n\|^2) + \frac{\nu}{2}\|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \frac{\alpha h}{2}(\|\nabla \boldsymbol{\phi}^h_{n+1}\|^2 - \|\nabla \boldsymbol{\phi}^h_n\|^2) \\
&\qquad \leq C\nu^{-1}\|\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}\|^2 + C\nu^{-1}\alpha h k\|\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k})\|^2 + C\nu^{-1}\|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
&\qquad +C\nu^{-1}(\|\nabla \boldsymbol{\eta}_n\|^2 + \|\nabla \boldsymbol{\eta}_{n-1}\|^2)\|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
&\qquad +C\nu^{-1}(\|\nabla \boldsymbol{\eta}_n\|^2 + \|\nabla \boldsymbol{\eta}_{n-1}\|^2) + C\nu^{-1}(\|\boldsymbol{\phi}^h_n\|^2 + \|\boldsymbol{\phi}^h_{n-1}\|^2) \\
&\qquad +Ch^{-3/2}\|\nabla \boldsymbol{\eta}_{n+1/2}\|(\|\boldsymbol{\phi}^h_{n-1}\|^2 + \|\boldsymbol{\phi}^h_n\|^2 + \|\boldsymbol{\phi}^h_{n+1}\|^2) \\
&\qquad +C\nu^{-1}k^4(\|\mathbf{u}_{ttt}(t_{n+\theta_1})\|^2 + \|p_{tt}(t_{n+\theta_4})\|^2) \\
&\qquad +C\nu k^4\|\nabla \mathbf{u}_{tt}(t_{n+\theta_5})\|^2 + C\nu^{-1}\alpha^2 h^2 k^2\|\nabla \mathbf{u}_t(t_{n+\theta_3})\|^2. \qquad (2.3.29)
\end{aligned}$$

Multiply both sides of (2.3.29) by $2k$ and use (2.2.12),(2.2.15) together with the approximation properties (2.2.3)-(2.2.5) of the spaces $(\mathbf{X}^h, Q^h)$. Then sum over the time levels from 1 to n, choosing $\mathbf{U}_0 = \mathbf{u}_0^h$, which gives $\boldsymbol{\phi}_0^h = 0$, and

$$\begin{aligned}
&\|\boldsymbol{\phi}_{n+1}^h\|^2 + k\sum_{i=1}^{n} \nu\|\nabla\boldsymbol{\phi}_{i+1/2}^h\|^2 + \alpha hk\|\nabla\boldsymbol{\phi}_{n+1}^h\|^2 \\
&\leq \quad \|\boldsymbol{\phi}_1^h\|^2 + \alpha hk\|\nabla\boldsymbol{\phi}_1^h\|^2 + C\nu^{-1}h^{2m+2}\|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega))} \\
&\quad +C\nu^{-1}\alpha h^{2m+1}k\|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega))} + C\nu^{-1}h^{2m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega))} \\
&\quad +C\nu^{-1}h^{4m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega))} \\
&\quad +C\nu^{-1}h^{2m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega))} + C\nu^{-1}k^4(\|\mathbf{u}_{ttt}\|^2_{L^2(0,T;L^2(\Omega))} + \|p_{tt}\|^2_{L^2(0,T;L^2(\Omega))}) \\
&\quad +C\nu k^4\|\nabla\mathbf{u}_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + C\nu^{-1}\alpha^2h^2k^2\|\nabla\mathbf{u}_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad +C\nu^{-1}k\sum_{i=1}^{n}(\|\boldsymbol{\phi}_{i-1}^h\|^2 + \|\boldsymbol{\phi}_i^h\|^2) \\
&\quad +Ch^{m-3/2}k\sum_{i=1}^{n} |\,\mathbf{u}(t_{i+1/2})\,|_{m+1}(\|\boldsymbol{\phi}_{i-1}^h\|^2 + \|\boldsymbol{\phi}_i^h\|^2 + \|\boldsymbol{\phi}_{i+1}^h\|^2).
\end{aligned} \tag{2.3.30}$$

Since $\mathbf{u} \in L^\infty(0,T;H^{m+1}(\Omega))$, the last two sums in (2.3.30) can be combined as

$$C\|\mathbf{u}\|_{L^\infty(0,T;H^{m+1}(\Omega))}h^{m-3/2}k\|\boldsymbol{\phi}_{n+1}^h\|^2 + C(h^{m-3/2} + \nu^{-1})k\sum_{i=1}^{n}\|\boldsymbol{\phi}_i^h\|^2.$$

Using the regularity of $\mathbf{u}$ and p, and the assumption that $C\|\mathbf{u}\|_{L^\infty(0,T;H^{m+1}(\Omega))}h^{m-3/2}k \leq 1/2$, the error equation finally takes the form

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\phi}_{n+1}^h\|^2 + k\sum_{i=1}^{n} \nu\|\nabla\boldsymbol{\phi}_{i+1/2}^h\|^2 + \alpha hk\|\nabla\boldsymbol{\phi}_{n+1}^h\|^2 \\
&\quad \leq \|\boldsymbol{\phi}_1^h\|^2 + \alpha hk\|\nabla\boldsymbol{\phi}_1^h\|^2 + C\nu^{-1}(2 + h^2 + \alpha hk + h^{2m})h^{2m} \\
&\quad +C\nu^{-1}\alpha^2h^2k^2 + C(\nu^{-1} + \nu)k^4 \\
&\quad +Ck\sum_{i=1}^{n}(\nu^{-1} + h^{m-3/2})\|\boldsymbol{\phi}_i^h\|^2.
\end{aligned} \tag{2.3.31}$$

To complete the proof bounds are needed for $\boldsymbol{\phi}_1^h$ in the above estimates. These bounds depend upon the way the first time step is taken, and there are two possibilities (2.1.5) and (2.1.7); we shall analyze both. Both lead to an optimal velocity error estimate. The more expensive method (2.1.7) also leads to an optimal pressure error estimate (in Theorem 2.4.3

below). The error equation for $\boldsymbol{\phi}_1^h$ is the same as for $\boldsymbol{\phi}_n^h$ except for the nonlinear terms, and is treated in the same way, except for the nonlinear term. Therefore, we go directly to the treatment of the nonlinear term in both cases (2.1.5) and (2.1.7).

We start with formulation (2.1.5). Adding and subtracting $b^*(\mathbf{u}_0^h - \mathbf{u}(t_0), \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h)$ to the nonlinear terms in (2.3.10), we have

$$\begin{aligned} b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}^h) - b^*(\mathbf{u}_0^h, \frac{\mathbf{u}_0^h + \mathbf{u}_1^h}{2}, \mathbf{v}^h) &= b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}^h) \\ &+ b^*(\mathbf{u}_0^h, \mathbf{e}_{1/2}, \mathbf{v}^h) - b^*(\mathbf{e}_0, \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2}, \mathbf{v}^h) \\ &+ b^*(\mathbf{u}(t_0), \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2}, \mathbf{v}^h). \end{aligned} \tag{2.3.32}$$

Taking $\mathbf{v}^h = \boldsymbol{\phi}_{1/2}^h$, the second and third terms in (2.3.37) can be treated exactly as in (2.3.16), (2.3.20) and (2.3.21). The first and last are bounded as follows. Using Lemma 2.3 and the fact that there exists $t_\theta \in (0, k)$ such that $\mathbf{u}(t_{1/2}) - \mathbf{u}(t_0) = k\mathbf{u}_t(t_\theta)$, we obtain

$$\begin{aligned} &|b^*(\mathbf{u}(t_0), \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, \boldsymbol{\phi}_{1/2}^h) - b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h)| \\ &= |b^*(\mathbf{u}(t_0), \mathbf{u}(t_{1/2}) + Ck^2\mathbf{u}_{tt}(t_\theta), \boldsymbol{\phi}_{1/2}^h) - b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h)| \\ &\leq |b^*(\mathbf{u}(t_0) - \mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h)| + Ck^2|b^*(\mathbf{u}(t_0), \mathbf{u}_{tt}(t_\theta), \boldsymbol{\phi}_{1/2}^h)| \\ &\leq k|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h)| + Ck^2|b^*(\mathbf{u}(t_0), \mathbf{u}_{tt}(t_\theta), \boldsymbol{\phi}_{1/2}^h)| \\ &\leq k|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h)| + \epsilon\nu\|\nabla\boldsymbol{\phi}_{1/2}^h\|^2 + C\nu^{-1}k^4. \end{aligned} \tag{2.3.33}$$

In order to bound the first term in (2.3.33), we use integration by parts and Hölder's inequality to obtain

$$b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h) = (\mathbf{u}_t(t_\theta) \cdot \nabla\mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h) + \frac{1}{2}(\nabla \cdot \mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}) \cdot \boldsymbol{\phi}_{1/2}^h). \tag{2.3.34}$$

Thus,

$$\begin{aligned} k|b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \boldsymbol{\phi}_{1/2}^h)| &\leq Ck(\|\mathbf{u}_t(t_\theta)\|\|\nabla\mathbf{u}(t_{1/2})\|_{L^\infty(\Omega)} \\ &\quad +\|\nabla\mathbf{u}_t(t_\theta)\|\|\mathbf{u}(t_{1/2})\|_{L^\infty(\Omega)})\|\boldsymbol{\phi}_{1/2}^h\| \\ &\leq Ck^3 + \frac{1}{4k}\|\boldsymbol{\phi}_{1/2}^h\|^2. \end{aligned} \tag{2.3.35}$$

Now use the bounds (2.3.33) and (2.3.35) in the error analysis at the first time level (note that $\phi^h_{1/2} = \frac{1}{2}\phi^h_1$, since $\phi^h_0 = 0$) to get

$$\begin{aligned}\|\phi^h_1\|^2 + \nu k\|\nabla\phi^h_1\|^2 + \alpha hk\|\nabla\phi^h_1\|^2 \;\leq\; & C[\nu^{-1}kh^{2m} + \nu^{-1}kh^{2m} + \nu^{-1}kh^{2m+2} \\ & +\nu^{-1}\alpha^2h^2k^3 + \nu^{-1}\alpha h^{2m+1}k^2 + \nu^{-1}kh^{4m} \\ & +\nu^{-1}k^5 + \nu k^5 + k^4]. \qquad (2.3.36)\end{aligned}$$

If formulation (2.1.7) is used, then, instead of (2.3.37), we obtain, by adding and subtracting $b^*(\frac{\mathbf{u}^h_1+\mathbf{u}^h_0}{2} - \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2} + \mathbf{u}(t_{1/2}), \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h)$ to the nonlinear terms in first time level analog of (2.3.10), the following

$$\begin{aligned} & b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}^h) - b^*(\frac{\mathbf{u}^h_0 + \mathbf{u}^h_1}{2}, \frac{\mathbf{u}^h_0 + \mathbf{u}^h_1}{2}, \mathbf{v}^h) \\ &\quad = b^*(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, \mathbf{v}^h) \\ &\quad\quad +b^*(\frac{\mathbf{u}^h_0 + \mathbf{u}^h_1}{2}, \mathbf{e}_{1/2}, \mathbf{v}^h) - b^*(\mathbf{e}_0, \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2}, \mathbf{v}^h) \\ &\quad\quad +b^*(\frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2} - \mathbf{u}(t_{1/2}), \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2}, \mathbf{v}^h) \qquad (2.3.37)\end{aligned}$$

Taking $\mathbf{v}^h = \phi^h_{1/2}$, the second and third terms in (2.3.37) can be treated exactly as in (2.3.16), (2.3.20) and (2.3.21). The first and last are similar, since, after application of Lemma 2.1 and regularity assumptions on $\mathbf{u}$, both can be bounded as

$$C\|\nabla(\mathbf{u}(t_{1/2} - \frac{\mathbf{u}(t_0) + \mathbf{u}(t_1)}{2})\|\|\nabla\phi^h_{1/2}\| \leq \epsilon\nu\|\nabla\phi^h_{1/2}\|^2 + C\nu^{-1}k^4,$$

with the help of Lemma 2.3 and Young's inequality. This leads to the upper bound

$$\begin{aligned}\|\phi^h_1\|^2 + \nu k\|\nabla\phi^h_1\|^2 + \alpha hk\|\nabla\phi^h_1\|^2 \;\leq\; & C[\nu^{-1}kh^{2m} + \nu^{-1}kh^{2m} + \nu^{-1}kh^{2m+2} + \nu^{-1}\alpha^2h^2k^3 \\ & +\nu^{-1}\alpha h^{2m+1}k^2 + \nu^{-1}kh^{4m} + \nu^{-1}k^5 + \nu k^5]. \quad (2.3.38)\end{aligned}$$

This bound is sharper than (2.3.36), but it will not contribute to a higher order estimate. We thus insert the bound for $\|\phi^h_1\|^2 + \alpha hk\|\nabla\phi^h_1\|^2$, obtained in (2.3.36), into (2.3.31), which gives

$$\begin{aligned}\|\phi^h_{n+1}\|^2 \;+\; & 2k\sum_{i=0}^{n}\nu\|\nabla(\frac{\phi^h_{i+1} + \phi^h_i}{2})\|^2 + 2\alpha hk\|\nabla\phi^h_{n+1}\|^2 \\ & \leq C(\nu + \nu^{-1})\left(h^{2m} + \alpha^2h^2k^2 + k^4\right) + C\nu^{-1}(k\sum_{i=0}^{n}\|\phi^h_i\|^2). \quad (2.3.39)\end{aligned}$$

Hence, it follows from the discrete Gronwall Lemma, that there exists $C = C(\nu, \Omega, T, \mathbf{u}, p)$ such that for any $n \geq 0$

$$\|\boldsymbol{\phi}^h_{n+1}\|^2 + k\sum_{i=0}^{n} \nu\|\nabla(\frac{\boldsymbol{\phi}^h_{i+1} + \boldsymbol{\phi}^h_i}{2})\|^2 + \alpha hk\|\nabla \boldsymbol{\phi}^h_{n+1}\|^2 \leq C\left(h^{2m} + \alpha^2 h^2 k^2 + k^4\right). \tag{2.3.40}$$

Finally, the statement of the theorem follows from the triangle inequality. □

2.4 ERROR ESTIMATES FOR TIME DERIVATIVES AND PRESSURE

In order to prove pressure stability and convergence, we need to derive a bound on the time difference of the velocity error $\|\frac{\mathbf{e}_{n+1}-\mathbf{e}_n}{k}\|$.

Theorem 2.4.1. *Let the finite-element spaces $(\mathbf{X}^h, Q^h)$ include continuous piecewise polynomials of degree m and $m-1$ respectively ($m \geq 2$) and satisfy the discrete inf-sup condition. Let the assumptions of Theorem 2.3.1 be satisfied and*

$$\nabla \mathbf{u}_{tt} \in L^2(0,T; L^\infty(\Omega)), \Delta \mathbf{u}_{tt} \in L^2(0,T; L^2(\Omega)),$$

$$\mathbf{u}_{ttt} \in L^\infty(0,T; L^2(\Omega)),$$

$$\nabla p_{tt} \in L^2(0,T; L^2(\Omega)).$$

Then, if the finite element approximation $\mathbf{u}^h_n$ is defined by (2.1.5)-(2.1.6), there exists a constant $C = C(\nu, \mathbf{u}, p, T) < \infty$ such that

$$\nu k^2\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 + \nu\|\nabla(\frac{\mathbf{e}_n + \mathbf{e}_{n-1}}{2})\|^2 + k\sum_{i=0}^{n-1}\|\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k}\|^2 + \alpha hk \cdot k\sum_{i=0}^{n-1}\|\nabla(\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k})\|^2 \leq C(h^{2m} + \alpha^2 h^2 k^2 + h^{-3}k^8 + k^3). \tag{2.4.1}$$

If the finite element approximation $\mathbf{u}_n^h$ *is defined via (2.1.7)-(2.1.6), then there exists a* $C = C(\nu, \mathbf{u}, p, T) < \infty$ *such that*

$$\begin{aligned}
\nu k^2 \|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 + \nu \|\nabla(\frac{\mathbf{e}_n + \mathbf{e}_{n-1}}{2})\|^2 \\
+ \quad k \sum_{i=0}^{n-1} \|\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k}\|^2 + \alpha h k \cdot k \sum_{i=0}^{n-1} \|\nabla(\frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{k})\|^2 \\
\leq C(h^{2m} + \alpha^2 h^2 k^2 + h^{-3} k^8 + k^4). \qquad (2.4.2)
\end{aligned}$$

Proof. Consider the error decomposition (2.3.11). Take $\mathbf{v}^h = \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k} \in \mathbf{V}^h$ in (2.3.12),(2.3.13) to obtain

$$\begin{aligned}
\|\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}\|^2 \quad &+ \quad \nu \frac{\|\nabla \boldsymbol{\phi}_{n+1}^h\|^2 - \|\nabla \boldsymbol{\phi}_n^h\|^2}{2k} + \alpha h k \|\nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})\|^2 \\
&= \quad (\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) + \nu(\nabla(\frac{\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n}{k}), \nabla(\frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k})) \\
&\quad -(\frac{p(t_{n+1}) + p(t_n)}{2} - q^h, \nabla \cdot \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +b^*(E[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})], \mathbf{e}_{n+1/2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad -b^*(E[\boldsymbol{\eta}_n, \boldsymbol{\eta}_{n-1}], \mathbf{e}_{n+1/2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +b^*(E[\boldsymbol{\phi}_n, \boldsymbol{\phi}_{n-1}], \mathbf{e}_{n+1/2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +b^*(E[\mathbf{e}_n, \mathbf{e}_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}) \\
&\quad +T(\mathbf{u}, p; \frac{\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h}{k}), \qquad (2.4.3)
\end{aligned}$$

where, using Taylor expansion,

$$
\begin{aligned}
T(\mathbf{u},p;\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}) \;=\;& (\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{k}-\mathbf{u}_t(t_{n+1/2}),\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\\
&+\alpha hk(\nabla(\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k}),\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}))\\
&+Ck^2\nu(\nabla\mathbf{u}_{tt}(t_{n+\theta}),\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}))+Ck^2(\mathbf{u}_{ttt}(t_{n+\theta}),\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\\
&+Ck^2b^*(\mathbf{u}_{tt}(t_{n+\theta}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\\
&+Ck^2b^*(\mathbf{u}(t_{n+1/2}),\mathbf{u}_{tt}(t_{n+\theta}),\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\\
&+\alpha hk(\nabla(\frac{\mathbf{u}^h_{n+1}-\mathbf{u}^h_n}{k}),\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}))\\
&+Ck^2(p_{tt}(t_{n+\theta}),\nabla\cdot(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})),
\end{aligned}
\tag{2.4.4}
$$

for some $\theta\in(0,1)$ and $\forall q^h\in Q^h$.

Also it follows from the definition of Stokes Projection that

$$
\nu(\nabla(\frac{\boldsymbol{\eta}_{n+1}-\boldsymbol{\eta}_n}{k}),\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}))-(\frac{p(t_{n+1})+p(t_n)}{2}-q^h,\nabla\cdot\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})=0. \tag{2.4.5}
$$

We bound the four nonlinear terms on the right-hand side of (2.4.3), using Lemma 2.1 and Cauchy-Schwarz and Young's inequalities. For the first term integrating by parts and applying Hölder's inequality gives

$$
\begin{aligned}
&|b^*(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})],\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&= |(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]\cdot\nabla\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\\
&\quad+\frac{1}{2}(\nabla\cdot E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})],\mathbf{e}_{n+1/2}\cdot\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&= |(E[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]\cdot\nabla\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\
&\le C\|\nabla\mathbf{e}_{n+1/2}\|\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|\\
&\le \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2+C\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2.
\end{aligned}
\tag{2.4.6}
$$

Using the first bound from Lemma 2.1 and the inverse inequality (2.2.2), we obtain the bounds on the second and third nonlinear terms

$$\begin{aligned}|b^*(E[\boldsymbol{\eta}_n,\boldsymbol{\eta}_{n-1}],\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})| &\leq Ch^{-1}\|\nabla E[\boldsymbol{\eta}_n,\boldsymbol{\eta}_{n-1}]\|\|\nabla\mathbf{e}_{n+1/2}\|\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|\\ &\leq \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2\\ &+ Ch^{-2}\|\nabla(\frac{3}{2}\boldsymbol{\eta}_n-\frac{1}{2}\boldsymbol{\eta}_{n-1})\|^2\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2 \qquad (2.4.7)\end{aligned}$$

and, using also the intermediate result (2.3.40) of Theorem 2.3.1,

$$\begin{aligned}&|b^*(E[\boldsymbol{\phi}_n,\boldsymbol{\phi}_{n-1}],\mathbf{e}_{n+1/2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\ &\leq Ch^{-3/2}\|E[\boldsymbol{\phi}_n,\boldsymbol{\phi}_{n-1}]\|\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|\\ &\leq \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2+Ch^{-3}(h^{2m}+\alpha^2h^2k^2+k^4)\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2. \qquad (2.4.8)\end{aligned}$$

Finally, consider the fourth nonlinear term. Use the obvious identity $\frac{3}{2}\mathbf{e}_n-\frac{1}{2}\mathbf{e}_{n-1}=\frac{\mathbf{e}_n+\mathbf{e}_{n-1}}{2}+(\mathbf{e}_n-\mathbf{e}_{n-1})$ and the regularity of $\mathbf{u}$. It follows from the last inequality of Lemma 2.1 that

$$\begin{aligned}&|b^*(E[\mathbf{e}_n,\mathbf{e}_{n-1}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\ &\quad\leq |b^*(\frac{\mathbf{e}_n+\mathbf{e}_{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\ &\qquad+|b^*(\boldsymbol{\eta}_n-\boldsymbol{\eta}_{n-1},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\ &\qquad+|b^*(k\frac{\boldsymbol{\phi}^h_n-\boldsymbol{\phi}^h_{n-1}}{k},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|\\ &\quad\leq \epsilon\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2+C\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2\\ &\qquad+C(h^{2m}+\alpha^2h^2k^2+k^4)+Ck^2\|\nabla(\frac{\boldsymbol{\phi}^h_n-\boldsymbol{\phi}^h_{n-1}}{k})\|^2. \qquad (2.4.9)\end{aligned}$$

Insert these bounds in (2.4.3). The bound on $|T(\mathbf{u},p;\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})|$ is obtained as in the proof of Theorem 2.3.1. Choosing $\epsilon=\frac{1}{24}$ gives

$$\begin{aligned}\frac{1}{2}\|\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k}\|^2 &+ \nu\frac{\|\nabla\boldsymbol{\phi}^h_{n+1}\|^2-\|\nabla\boldsymbol{\phi}^h_n\|^2}{2k}+\frac{\alpha hk}{2}\|\nabla(\frac{\boldsymbol{\phi}^h_{n+1}-\boldsymbol{\phi}^h_n}{k})\|^2\\ &\leq C\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2+Ck^2\|\nabla(\frac{\boldsymbol{\phi}^h_n-\boldsymbol{\phi}^h_{n-1}}{k})\|^2+C(h^{2m}+\alpha^2h^2k^2+k^4)\\ &\quad+Ch^{-3}(h^{2m}+\alpha^2h^2k^2+k^4)\|\nabla(\frac{\mathbf{e}_{n+1}+\mathbf{e}_n}{2})\|^2. \qquad (2.4.10)\end{aligned}$$

At the first time level, take $\mathbf{v}^h = \frac{\phi_1^h - \phi_0^h}{k}$; taking $\mathbf{U}_0 = \mathbf{u}_0^h$ in the initial error decomposition gives $\phi_0^h = 0$. For the constant extrapolation (2.1.5) we obtain

$$\frac{1}{2}\|\frac{\phi_1^h - \phi_0^h}{k}\|^2 + \nu\frac{\|\nabla\phi_1^h\|^2 - \|\nabla\phi_0^h\|^2}{2k} + \frac{\alpha h k}{2}\|\nabla(\frac{\phi_1^h - \phi_0^h}{k})\|^2 \leq C(h^{2m} + \alpha^2 h^2 k^2 + k^4) + k b^*(\mathbf{u}_t(t_\theta), \mathbf{u}_{1/2}, \frac{\phi_1^h - \phi_0^h}{k}). \quad (2.4.11)$$

If we use (2.1.7) instead of (2.1.5) at the first time level, we have

$$\frac{1}{2}\|\frac{\phi_1^h - \phi_0^h}{k}\|^2 + \nu\frac{\|\nabla\phi_1^h\|^2 - \|\nabla\phi_0^h\|^2}{2k} + \frac{\alpha h k}{2}\|\nabla(\frac{\phi_1^h - \phi_0^h}{k})\|^2 \leq C(h^{2m} + \alpha^2 h^2 k^2 + k^4) + k^2 b^*(\mathbf{u}_t(t_\theta), \mathbf{u}_{1/2}, \frac{\phi_1^h - \phi_0^h}{k}). \quad (2.4.12)$$

Sum (2.4.10) over the time levels $n \geq 1$ and add to (2.4.11) (or to (2.4.12) in the case of linear extrapolation). Multiply by $2k$ to obtain

$$\begin{aligned} k\sum_{i=0}^{n}\|\frac{\phi_{i+1}^h - \phi_i^h}{k}\|^2 &+ \nu\|\nabla\phi_{n+1}^h\|^2 + \alpha h k \cdot k\sum_{i=0}^{n}\|\nabla(\frac{\phi_{i+1}^h - \phi_i^h}{k})\|^2 \\ &\leq Ck^2 \cdot k\sum_{i=0}^{n-1}\|\nabla(\frac{\phi_{i+1}^h - \phi_i^h}{k})\|^2 + C(h^{2m} + \alpha^2 h^2 k^2 + k^4 + h^{-3}k^8) \\ &\quad + k^{2+\sigma} b^*(\mathbf{u}_t(t_\theta), \mathbf{u}(t_{1/2}), \frac{\phi_1^h - \phi_0^h}{k}), \end{aligned} \quad (2.4.13)$$

where $\sigma = 0$ for the constant extrapolation (2.1.5) and $\sigma = 1$ for the linear extrapolation (2.1.7).

For any $n \geq 1$, add the inequalities (2.4.13) at the time levels $n+1$ and n. Use the identity

$$\|\nabla\phi_{n+1}^h\|^2 + \|\nabla\phi_n^h\|^2 = \frac{1}{2}k^2\|\nabla(\frac{\phi_{n+1}^h - \phi_n^h}{k})\|^2 + 2\|\nabla(\frac{\phi_{n+1}^h + \phi_n^h}{2})\|^2. \quad (2.4.14)$$

At any time level $n \geq 1$ we obtain

$$\begin{aligned}
\frac{1}{2}\nu k^2 \|\nabla(\frac{\phi^h_{n+1}-\phi^h_n}{k})\|^2 &+ 2\nu\|\nabla(\frac{\phi^h_{n+1}+\phi^h_n}{2})\|^2 \\
&+k\sum_{i=0}^{n}\|\frac{\phi^h_{i+1}-\phi^h_i}{k}\|^2 + \alpha hk\cdot k\sum_{i=0}^{n}\|\nabla(\frac{\phi^h_{i+1}-\phi^h_i}{k})\|^2 \\
&\leq C\nu^{-1}\cdot k\sum_{i=0}^{n-1}\frac{1}{2}\nu k^2\|\nabla(\frac{\phi^h_{i+1}-\phi^h_i}{k})\|^2 \\
&+C(h^{2m}+\alpha^2h^2k^2+k^4+h^{-3}k^8) \\
&+k^{2+\sigma}b^*(\mathbf{u}_t(t_\theta),\mathbf{u}(t_{1/2}),\frac{\phi^h_1-\phi^h_0}{k}). \qquad (2.4.15)
\end{aligned}$$

Next, decompose the last term in the right-hand side of (2.4.15), using Lemma 2.1 and Young's inequality. This yields

$$k^{2+\sigma}|b^*(\mathbf{u}_t(t_\theta),\mathbf{u}(t_{1/2}),\frac{\phi^h_1-\phi^h_0}{k})| \leq \frac{1}{2}k\|\frac{\phi^h_1-\phi^h_0}{k}\|^2 + Ck^{3+2\sigma}. \qquad (2.4.16)$$

Hence it follows from the discrete Gronwall Lemma that

$$\begin{aligned}
\nu k^2\|\nabla(\frac{\phi^h_{n+1}-\phi^h_n}{k})\|^2 + \nu\|\nabla(\frac{\phi^h_{n+1}+\phi^h_n}{2})\|^2 + k\sum_{i=0}^{n}\|\frac{\phi^h_{i+1}-\phi^h_i}{k}\|^2 \\
+\alpha hk\cdot k\sum_{i=0}^{n}\|\nabla(\frac{\phi^h_{i+1}-\phi^h_i}{k})\|^2 \leq C(h^{2m}+\alpha^2h^2k^2+k^4+k^{3+2\sigma}+h^{-3}k^8). (2.4.17)
\end{aligned}$$

The proof of the theorem is now concluded by the triangle inequality. □

For the stability of pressure we will need the following *á priori* bounds

Lemma 2.4. *Let the assumptions of Theorem 2.4.1 hold. Then there exists a constant $C = C(\nu, \mathbf{u}, p, T)$ such that for any n*

$$k\sum_{i=0}^{n}\|\frac{\mathbf{u}^h_{i+1}-\mathbf{u}^h_i}{k}\| \leq k\sum_{i=0}^{n}\|\frac{\mathbf{e}_{i+1}-\mathbf{e}_i}{k}\| + k\sum_{i=0}^{n}\|\frac{\mathbf{u}(t_{i+1})-\mathbf{u}(t_i)}{k}\| \leq C,$$

$$k^2\|\nabla(\frac{\mathbf{u}^h_{n+1}-\mathbf{u}^h_n}{k})\|^2 \leq k^2\|\nabla(\frac{\mathbf{e}_{n+1}-\mathbf{e}_n}{k})\|^2 + k^2\|\nabla(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{k})\|^2 \leq C.$$

Proof. Use the decomposition $\mathbf{u}^h_i = \mathbf{u}(t_i) - \mathbf{e}_i$. The triangle inequality completes the proof.

□

Theorem 2.4.2 (Pressure Stability)**.** *Let* $(\mathbf{u}_n^h, p_n^h)$ *satisfy (2.1.5)-(2.1.6) (or (2.1.7)-(2.1.6)). Let* $f \in L^2(0,T;H^{-1}(\Omega))$ *and let the assumptions of Theorem 2.4.1 be satisfied. Then,*

$$k\sum_{i=0}^{n-1} \| \frac{p_{i+1}^h + p_i^h}{2} \| \leq C(\mathbf{u}_0^h, \mathbf{f}, \beta^h),$$

where β^h *is the constant from the discrete* LBB^h *condition (2.2.1).*

Proof. Consider (2.1.6). Using the Cauchy-Schwarz inequality, the first bound from Lemma 2.1, the discrete LBB^h condition (2.2.1) and the identity $\frac{3}{2}\mathbf{u}_{n+1}^h - \frac{1}{2}\mathbf{u}_n^h = \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} + k\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}$, we obtain

$$\begin{aligned}
\beta^h \| \frac{p_{n+1}^h + p_n^h}{2} \| \quad \leq \quad & \| \frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k} \|_{-1} + \nu \| \nabla (\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}) \| \\
& + \alpha h k \| \nabla (\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}) \| + C \| \nabla (\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}) \|^2 \\
& + C k^2 \| \nabla (\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{k}) \|^2 + \| \mathbf{f}(t_{n+1/2}) \|_{-1}.
\end{aligned}$$

Sum over all time levels; the bounds of Lemma 2.4 complete the proof. □

We conclude this section by deriving the pressure error estimate.

Theorem 2.4.3 (Pressure Convergence)**.** *Let* $(\mathbf{u}_n^h, p_n^h)$ *satisfy (2.1.6) for* $n \geq 2$*. Let* $(\mathbf{u}_1^h, p_1^h)$ *satisfy the constant extrapolation (2.1.5) or the linear extrapolation (2.1.7). Then, under the assumptions of Theorem 2.4.1,*

$$k\sum_{i=0}^{n-1} \| p(t_{i+1/2}) - p_{i+1/2}^h \| \leq C(\nu, \mathbf{u}, p, T)(h^m + \alpha h k + h^{-3/2}k^4 + k^{3/2+\sigma/2}), \qquad (2.4.18)$$

where $\sigma = 0$ *for the constant extrapolation and* $\sigma = 1$ *for the linear extrapolation.*

Proof. Consider (2.3.10), which holds true for any $\mathbf{v}^h \in \mathbf{X}^h$. Decompose the pressure approximation error into

$$p(t_{n+1}) - p^h_{n+1} = (p(t_{n+1}) - I(p)) - (p^h_{n+1} - I(p)) = \tilde{\eta}_{n+1} - \tilde{\phi}^h_{n+1}, \tag{2.4.19}$$

where $\tilde{\phi}^h_{n+1} \in Q^h$, $I(p)$ is a projection of $p(t_{n+1})$ into Q^h.

Use the error decomposition (2.4.19) in (2.3.10) and apply the discrete LBB^h condition to obtain for any $n \geq 1$

$$\begin{aligned}\beta^h \|\frac{\tilde{\phi}^h_{n+1} + \tilde{\phi}^h_n}{2}\| \leq & \|\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k}\|_{-1} + C\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\| \\ & + C\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\|^2 + Ck^2\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 + Ck\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\| \\ & + \nu\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\| + \alpha hk\|\nabla(\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k})\| \\ & + \|\frac{\tilde{\eta}_{n+1} + \tilde{\eta}_n}{2}\| + C\nu k^2 + Ck^2 + C\alpha hk. \end{aligned} \tag{2.4.20}$$

Hence from the triangle inequality we get

$$\begin{aligned}\beta^h \|\frac{(p(t_{n+1}) - p^h_{n+1}) + (p(t_n) - p^h_n)}{2}\| \leq & \|\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k}\|_{-1} + C\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\| \\ & + C\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\|^2 + Ck^2\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\|^2 \\ & + Ck\|\nabla(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k})\| + \nu\|\nabla(\frac{\mathbf{e}_{n+1} + \mathbf{e}_n}{2})\| \\ & + \alpha hk\|\nabla(\frac{\mathbf{e}_{n+1} - \mathbf{e}_n}{k})\| + C\nu k^2 + Ck^2 + C\alpha hk \\ & + \inf_{q^h \in Q^h} \|\frac{p(t_{n+1}) + (p(t_n)}{2} - q^h\|. \end{aligned} \tag{2.4.21}$$

On the first time level consider the constant extrapolation (2.1.5). Using the discrete LBB^h condition and (2.3.36), we obtain the following bound (which can be improved in the case of linear extrapolation):

$$\beta^h k\|\frac{(p(t_1) - p^h_1) + (p(t_0) - p^h_0)}{2}\| \leq C(k^2 + h^m + \alpha hk). \tag{2.4.22}$$

Add the inequalities (2.4.21) for all $n \geq 1$, multiply by k and add to (2.4.22). The proof is concluded by applying the result of Theorem 2.4.1 □

2.5 PHYSICAL FIDELITY: CONSERVATION OF INTEGRAL INVARIANTS

We begin by proving that CNLEStab exactly conserves a modified kinetic energy.

Proposition 2.4. *Let the boundary conditions be periodic; assume also* $\mathbf{f} = \nu = 0$. *Define*

$$\textit{Kinetic energy in (2.1.6)} = KE(t_n) := \frac{1}{2L^3}[||\mathbf{u}_n^h||^2 + \alpha kh||\nabla \mathbf{u}_n^h||^2]$$

The method exactly conserves kinetic energy. Specifically, for all $t_n > 0$

$$KE(t_n) = KE(0).$$

Proof. Set $\mathbf{v}^h = \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}$ and $\nu = \mathbf{f} = 0$ in (2.1.5)-(2.1.6). □

Exact conservation of helicity likely does not hold for CNLEStab. We thus consider approximate helicity conservation experimentally by considering and inviscid ($\nu = 0$) fluid with no forcing term ($\mathbf{f} = 0$). A comparison of the CNLE and CNLEStab under these conditions, in Figure 1, shows that, for a fixed mesh size, CNLEStab nearly conserves helicity, while CNLE does not. Both conserve kinetic energies during these experiments.

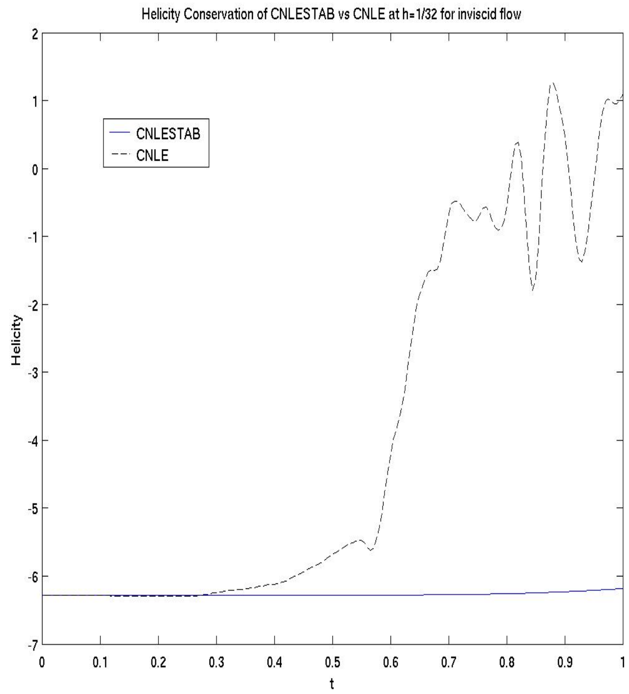

Figure 1: Conservation of helicity, CNLE ($\alpha = 0$) versus CNLEStab ($\alpha = 1$)

A comparison of the CNLEStab performance for different mesh sizes is shown in Figure 2, confirming that as the mesh is refined, conservation of helicity improves.

2.6 PHYSICAL FIDELITY: PREDICTIONS OF THE TURBULENT ENERGY CASCADE

We consider the energy cascade predicted by (1.1) in the case of homogeneous, isotropic turbulence. Motivated by the consistency error argument, we consider the modified equation of the method (1.1). Since $\alpha h(\nabla \mathbf{u}(t_{n+1}), \nabla \mathbf{v}) - \alpha h(\nabla \mathbf{u}(t_n), \nabla \mathbf{v}) = -\alpha hk(\Delta \mathbf{u}_t, \mathbf{v})$, we postulate a fluid with equations of motion given by: $w : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, $q : \Omega \times (0, T] \rightarrow \mathbb{R}$

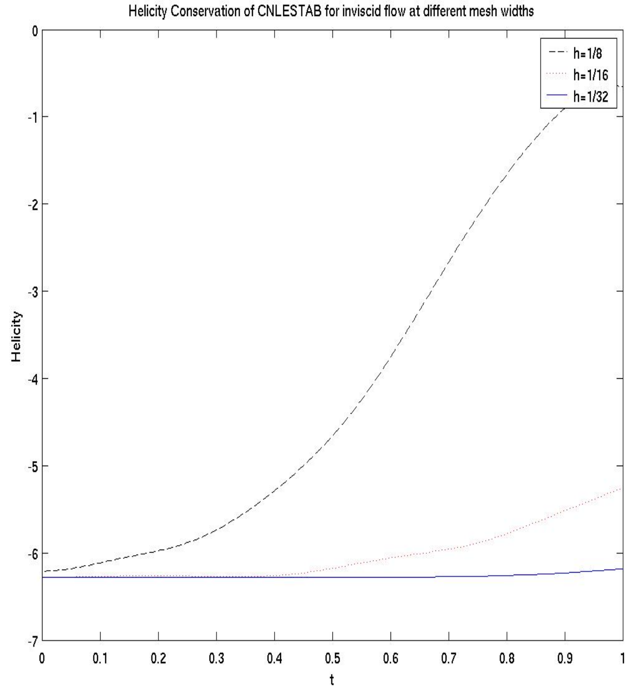

Figure 2: Conservation of helicity for different mesh sizes in CNLESTAB (with $\alpha = 1$): as h gets smaller, helicity is conserved longer.

satisfying:

$$\begin{aligned}
&[\mathbf{w} - \alpha hk \triangle \mathbf{w}]_t + \mathbf{w} \cdot \nabla \mathbf{w} - \nu \Delta \mathbf{w} + \nabla q = \mathbf{f}, \text{ for } \mathbf{x} \in \Omega, \, 0 < t \leq T \\
&\nabla \cdot \mathbf{w} = 0, \; \mathbf{x} \in \Omega, \text{ for } 0 < t \leq T, \\
&\text{periodic boundary conditions on } \partial\Omega, \text{ for } 0 < t \leq T, \qquad (2.6.1) \\
&\mathbf{w}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \text{ for } \mathbf{x} \in \Omega,
\end{aligned}$$

and the usual normalization condition in the periodic case that $\int_\Omega \phi(\mathbf{x}, t) \; d\mathbf{x} = 0$ on $\phi = \mathbf{w}, q, \mathbf{f}, \mathbf{u}_0$ for $0 < t \leq T$. Thus we explore more subtle effects of the stabilization in Algorithm 2.1 through its modified equation (2.6.1).

We multiply (2.6.1) by $\mathbf{w}$ and integrate over the domain and time to obtain its precise energy balance given by

$$\frac{1}{2}\{||\mathbf{w}(t)||^2 + \alpha hk||\nabla \mathbf{w}(t)||^2\} + \int_0^T \nu||\nabla \mathbf{w}(t)||^2 = \frac{1}{2}\{||\mathbf{w}(0)||^2 + \alpha hk||\nabla \mathbf{w}(0)||^2\} + (\mathbf{f}(t), \mathbf{w}(t)).$$

We can clearly identify three physical quantities of kinetic energy, energy dissipation rate and power input. Let L denote the global length scale, e.g., $L = vol(\Omega)^{1/3}$; then these are given by

$$\text{Modified equations kinetic energy: } E_{model}(\mathbf{w})(t) := \frac{1}{2L^3}\{||\mathbf{w}(t)||^2 + \alpha hk||\nabla \mathbf{w}(t)||^2\}, \quad (2.6.2)$$

$$\text{Modified equations dissipation rate: } \varepsilon_{model}(\mathbf{w})(t) := \frac{\nu}{L^3}||\nabla \mathbf{w}(t)||^2, \quad (2.6.3)$$

$$\text{Modified equations power input: } P_{model}(\mathbf{w})(t) := \frac{1}{L^3}(\mathbf{f}(t), \mathbf{w}(t)). \quad (2.6.4)$$

The kinetic energy has an extra term which reflects extraction of energy from resolved scales. The energy dissipation rate in the model (2.6.3) is the same as for NSE equations.

Equation (2.6.1) shares the common features of the Navier-Stokes equations which make existence of an energy cascade likely, e.g. [F95], [P00]. First, (2.6.1) has the same nonlinearity as the Navier-Stokes equations, which pumps energy from larger to smaller scales. Next, the solution of (2.6.1) satisfies an energy equality in which its kinetic energy and energy dissipation are readily discernible, and for $\nu = 0$ the kinetic energy is conserved through a large range of scales/wave-numbers. Since both conditions are satisfied we are to proceed to develop a quantitative theory of energy cascade of (2.6.1).

2.6.1 Kraichnan's Dynamic Analysis Applied to CNLEStab

The energy cascade will now be investigated more closely using the dynamical argument of Kraichnan, [K71]. Let $\Pi_{model}(\kappa)$ be defined as the total rate of energy transfer from all wave numbers $< \kappa$ to all wave numbers $> \kappa$ (not to confuse the wave number κ with the time step k). Following Kraichnan [K71] we assume that $\Pi_{model}(\kappa)$ is proportional to the total energy $(\,\kappa E_{model}(\kappa)\,)$ in wave numbers of the order κ and to some effective rate of shear $\sigma(\kappa)$ which acts to distort flow structures of scale $1/\kappa$. That is:

$$\Pi_{model}(\kappa) \simeq \sigma(\kappa)\,\kappa\,E_{model}(\kappa) \tag{2.6.5}$$

Furthermore, we expect

$$\sigma(\kappa)^2 \;\simeq\; \int_0^{\kappa} p^2 E_{model}(p)dp \tag{2.6.6}$$

The major contribution to (2.6.6) is from $p \simeq \kappa$, in accord with Kolmogorov's localness assumption, [Kol41]. This is because all wave numbers $\leq \kappa$ should contribute to the effective mean-square shear acting on wave numbers of order κ, while the effects of all wave numbers $\gg \kappa$ can plausibly be expected to average out over the scales of order $1/\kappa$ and over times the order of the characteristic distortion time $\sigma(\kappa)^{-1}$.

Let $E(\kappa) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T E(\kappa,t)$ is the distribution of the time averaged kinetic energy by wave number. Here, we have $E(\kappa,t) = \frac{L}{2\pi}\sum_{|\mathbf{k}|=\kappa} \frac{1}{2}|\hat{\mathbf{u}}(\mathbf{k},t)|^2$ where L - the reference length, $\mathbf{k},\kappa$ - the wave number vector and the wave number respectively, and $\hat{\mathbf{u}}(\mathbf{k},t)$ - the Fourier modes of the Navier-Stokes velocity.

We shall say that there is an energy cascade if in some "inertial" range, $\Pi_{model}(\kappa)$ is independent of the wave number, i.e., $\Pi_{model}(\kappa) = \varepsilon_{model}$. Using the equations (2.6.5) and (2.6.6) we get

$$E_{model}(\kappa) \simeq \varepsilon_{model}^{2/3}\kappa^{-5/3}$$

We have the relation

$$E_{\ model}(\kappa) \simeq (1+\alpha hk\kappa^2)E(\kappa). \tag{2.6.7}$$

Using (2.6.7) we obtain:

$$\begin{aligned}
\text{Model's cutoff lengthscale} \quad &: \quad \sqrt{\alpha h k},\\
E(\kappa) \quad &\simeq \quad \varepsilon_{model}^{2/3}\kappa^{-5/3}, \text{ for } \kappa \ll \frac{1}{(\alpha h k)^{1/2}},\\
E(\kappa) \quad &\simeq \quad \varepsilon_{model}^{2/3}(\alpha h k)^{-1}\kappa^{-11/3}, \text{ for } \kappa \gg \frac{1}{(\alpha h k)^{1/2}}.
\end{aligned}$$

Therefore, (2.6.1) possesses an energy cascade with an enhanced kinetic energy. The extra term in (2.6.1) triggers an accelerated energy decay of $O(\kappa^{-11/3})$ beyond the cutoff length scale. Above the cutoff length scale (2.6.1) predicts the correct energy cascade of $O(\kappa^{-5/3})$.

2.7 COMPUTATIONAL TESTS

We first test convergence rates for a problem with a known exact solution. The example is one for which the true solution is known,

$$\mathbf{u} = \begin{pmatrix} \cos(2\pi(z+t)) \\ \sin(2\pi(z+t)) \\ \sin(2\pi(x+t)) \end{pmatrix}, \qquad (2.7.1)$$

and then the right-hand side $\mathbf{f}$ and initial condition $\mathbf{u}_0$ are computed such that (2.7.1) satisfies (2.1.1). We selected this test problem because it is simple but already possesses complex rotational structures.

For $\alpha = 1$, $\nu = 1$ and final time $T = 0.5$, the calculated convergence rates in Table 1 confirm what is predicted by Theorem 2.3.1 for (P_2, P_1) discretization in space.

Next we give a simple test of the positive effects of the stabilization on the methods complexity. The linear solver used in the simulations was (unpreconditioned) Conjugate Gradient Squared (CGS). On a $h = 1/16$ mesh in $\mathbb{R}^3$, with $\nu = \frac{1}{500}$ and the same true solution (2.7.1), the number of CGS iterates needed for the first 8 solves of Crank-Nicolson with Linear Extrapolation (CNLE), i.e. $\alpha = 0$, and CNLE with stabilization (CNLEStab, $\alpha > 0$) are compared in Table 2.

The linear system to be solved at each time step is also better conditioned when $\alpha > 0$.

h	$\|\| \mathbf{u} - \mathbf{u}^h \|\|_{H^1(\Omega)}$	ratio	rate
1/8	0.6910	-	-
1/16	0.1772	3.8995	1.9633
1/32	0.0447	3.9642	1.9870

Table 1: Experimental convergence rates.

time level	CNLE	CNLEStab
1	349	193
2	350	199
3	347	200
4	372	212
5	348	206
6	347	206
7	351	205
8	365	192

Table 2: Number of CGS iterations for CNLE versus CNLEStab.

2.8 CONCLUSIONS

A simple second order time stepping algorithm for the Navier-Stokes equations was analyzed. It is a modification (by introduction of artificial viscosity stabilization and correction for the associated loss of accuracy) of the commonly used Crank-Nicolson scheme that requires the solution of only one linear system per time step. We not only proved that it is unconditionally stable and investigated how the rates of convergence for velocity and pressure behave, but we also went beyond error analysis. We showed that this scheme conserves kinetic energy exactly, and provided experimental numerical evidence that it nearly conserves helicity, an important integral invariant in three dimensional rotational flows. Dynamic analysis applied to their algorithm reveals the existence of an energy cascade with the correct statistics up to a cutoff length scale and with an accelerated energy decay above the cutoff length scale. Lastly, we presented more computational tests. The first confirms the velocity convergence rates obtained in the analysis in Section 3, and the second shows that even with a simple, unpreconditioned iterative method the linear system to be solved at each time step is better conditioned than the corresponding system without stabilization.

3.0 A DEFECT CORRECTION METHOD FOR THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS

3.1 INTRODUCTION

In the numerical solution of higher Reynolds number flow problems some of the standard iterative methods fail - see, e.g., [ES00] and remarks in [S03] (p. 48, section 2.1.2), [B96] (p.24), [C00] (project overview). Often "failure" means that the iterative method used to solve the linear and/or nonlinear system for the approximate solution at the new time level failed to converge within the time constraints of the problem or the resulting approximation had poor solution quality. The first type of failure can usually be overcome easily by using an upwind or artificial viscosity (AV) discretization at the expense of decreasing dramatically the accuracy of the method and possibly even altering the predictions of the simulation at the qualitative, O(1) level, therefore increasing the likelihood of the second type of failure.

One interesting approach to attaining (by a convergent method) an approximate solution of desired accuracy is the defect correction method (DCM). Briefly, let a k^{th} order accurate discretization of the *equilibrium* Navier-Stokes equations (NSE) be written as

$$NSE^h(u^h) = f, \tag{3.1.1}$$

The DCM computes $u_1^h, ..., u_k^h$ as

$$-\alpha h \Delta^h u_1^h + NSE^h(u_1^h) = f, \tag{3.1.2}$$
$$-\alpha h \Delta^h u_l^h + NSE^h(u_l^h) = f - \alpha h \Delta^h u_{l-1}^h, \text{ for } l = 2, ..., k,$$

where the velocity approximations u_i^h are sought in the finite element space of piecewise polynomials of degree k.

It has been proven under quite general conditions (see, e.g., [LLP02]) that for the intermediate approximations of the equilibrium NSE

$$\|u_{NSE} - u_l^h\|_{energy-norm} = O(h^k + h\|u_{NSE} - u_{l-1}^h\|_{energy-norm}) = O(h^k + h^l),$$

and thus, after $l = k$ steps,

$$\|u - u_k^h\|_{energy-norm} = O(h^k).$$

Note that (3.1.2) requires solving an AV approximation k times which is often cheaper and more reliable than solving (3.1.1) once.

In problems with high Reynolds number we may expect turbulence. In that case the DCM needs to be combined with appropriate turbulence models. These models tend to introduce extra nonlinearities (due to the closure of the model); it might be possible to incorporate them into the residual on the right-hand side, as was done in the quasistatic case by Ervin, Layton, Maubach [ELM00].

There has been an extensive study and development of this approach for equilibrium flow problems, see e.g. Hemker[Hem82], Koren[K91], Heinrichs[Hei96], Layton, Lee, Peterson[LLP02], Ervin, Lee[EL06], and subsection 3.1.1 for a review of this work.

For many years, it has been widely believed that (3.1.2) can be directly imported into implicit time discretizations of flow problems in the obvious way: discretize in time, given $u^h(t_{OLD})$, the quasistatic flow problem for $u^h(t_{NEW})$ is solved by applying (3.1.2) directly, resulting in

$$\begin{aligned} &-\alpha h \Delta^h u_1^h(t_{NEW}) + B(u_1^h(t_{NEW}), u_k^h(t_{OLD})) + NSE^h(u_1^h(t_{NEW})) = f, \qquad (3.1.3)\\ &-\alpha h \Delta^h u_l^h(t_{NEW}) + B(u_l^h(t_{NEW}), u_k^h(t_{OLD})) + NSE^h(u_l^h(t_{NEW}))\\ &\qquad = f - \alpha h \Delta^h u_{l-1}^h(t_{NEW}), \text{ for } l = 2, ..., k, \end{aligned}$$

where B is a time stepping operator (e.g., Backward Euler), and k is the degree of piecewise polynomials in the finite element space.

Unfortunately, this natural idea doesn't seem to be even stable (see Section 3.7).

On the other hand, there is a parallel development of DCM's, for initial value problems in which no spacial stabilization (such as $-\alpha h \Delta^h$ in (3.1.2)) is used, but DCM is used to increase the accuracy of the time discretization. This work contains no reports of instabilities: see, e.g., Heywood, Rannacher[HR90], Hemker, Shishkin[HSS], Lallemand, Koren[LK93], Minion[M04]. Yet, in spite of this parallel development and after 30+ years of studies of (3.1.2), there has yet to be a successful extension of (3.1.2) to time dependent flow problems.

This chapter will present this extension of (3.1.2) to the time dependent problem. We notice that the obvious extension, described above, is in fact unstable, see Section 3.7 . We give a small but critically important modification of the above natural extension to time dependent problems, that we prove to be unconditionally stable (Theorem 3.1) and convergent (Theorem 3.2). We complement the stability proof of the modified DCM by a complete error analysis, which confirms the expected error in the resulting method: $\|u(t_n) - u_l^h(t_n)\|_{energy-norm} = O(\Delta t^a + h^k + h^l), l = 1, ..., k$, where a is the order of accuracy of the (implicit) time stepping employed.

The error analysis is necessarily technical. To keep the details under some control, we study the backward Euler time discretization (It will be clear from our analysis that extension to more accurate time discretizations requires no new ideas and only more pages).

In subsection 3.1.1 we review important previous work on DCM in space and DCM in time. Section 3.2 begins with (the inevitable) notation and preliminaries. Section 3.3, the heart of the chapter, gives the stability proof. The error analysis is given in Sections 3.4, 3.5.2 and Section 3.7 gives a numerical illustration.

Consider the time dependent, incompressible Navier-Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} - Re^{-1}\Delta u + u\cdot\nabla u + \nabla p &= f, \text{ for } x \in \Omega, 0 < t \leq T, \\ \nabla \cdot u &= 0, x \in \Omega, 0 < t \leq T, \\ u(x,0) &= u_0(x), x \in \Omega, \\ u|_{\partial\Omega} &= 0, \text{ for } 0 < t \leq T, \end{aligned} \tag{3.1.4}$$

where $\Omega \subset \mathbb{R}^d, d = 2, 3.$

Before proceeding with the analysis we shall present carefully next the precise extension of (3.1.2) to the time dependent NSE that we study.

Let $X^h \subset X, Q^h \subset Q$ be finite-dimensional finite element spaces. Denote the finite-element discretization of the Navier-Stokes operator by

$$N^h_{Re}(u,p) \equiv \frac{\partial u}{\partial t} - Re^{-1}\Delta^h u + (u \cdot \nabla^h)u + \nabla^h p.$$

Adding an artificial viscosity parameter to the inverse Reynolds number leads to the modified Navier-Stokes operator

$$N^h_{\tilde{Re}}(u,p) \equiv \frac{\partial u}{\partial t} - (h + Re^{-1})\Delta^h u + (u \cdot \nabla^h)u + \nabla^h p.$$

The method proceeds as follows: first we compute the AV approximation $(u_1, p_1) \in (X^h, Q^h)$ via

$$N^h_{\tilde{Re}}(u_1, p_1) \;=\; f.$$

The accuracy of the approximation is then increased by the correction step: compute $(u_2, p_2) \in (X^h, Q^h)$, satisfying

$$N^h_{\tilde{Re}}(u_2, p_2) - N^h_{\tilde{Re}}(u_1, p_1) \;=\; f - N^h_{Re}(u_1, p_1).$$

The Backward Euler time discretization, combined with the two-step defect correction method in space leads to the following system of equations for $(u_1^{h,n+1}, p_1^{h,n+1})$, $(u_2^{h,n+1}, p_2^{h,n+1}) \in (X^h, Q^h), \forall v^h \in X^h$ at $t = t_{n+1}$, $n \geq 0$, with $k := \Delta t = t_{i+1} - t_i$

$$\begin{aligned}
(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v^h) + (h + Re^{-1})(\nabla u_1^{h,n+1}, \nabla v^h) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) \\
-(p_1^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h), \\
(\frac{u_2^{h,n+1} - u_2^{h,n}}{k}, v^h) + (h + Re^{-1})(\nabla u_2^{h,n+1}, \nabla v^h) + b^*(u_2^{h,n+1}, u_2^{h,n+1}, v^h) \\
-(p_2^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h) + h(\nabla u_1^{h,n+1}, \nabla v^h),
\end{aligned} \tag{3.1.5}$$

where $b^*(\cdot,\cdot,\cdot)$ is the explicitely skew-symmetrized trilinear form, defined below.

The initial value approximations are taken to be $u_1^{h,0} = u_2^{h,0} = u_0^s$, where u_0^s is the modified Stokes projection of u_0 onto the space V^h of discretely divergence-free functions (this projection and this space are defined in section 3.2). The stability and error estimate for the modified Stokes projection are proven in the sections 3.3 and 3.4.

3.1.1 Previous results

Many iterative methods can be written as a Defect Correction method, see e. g. Bohmer, Hemker, Stetter [BHS]. In the DCM we consider, no iterates occur; a small number of updates are calculated to increase the accuracy of the velocity and pressure approximations. Thus it is most similar to DCM's which are close to Richardson extrapolation (see, for example, Mathews, Fink [MF04]). In the late 1970's Hemker (Bohmer, Stetter, Heinrichs and others) discovered that DCM, properly interpreted, is good also for nearly singular problems. Examples for which this has been successful include equilibrium Euler equations (Koren, Lallemand [LK93]), high Reynolds number problems (Layton, Lee, Peterson [LLP02]), viscoelastic problems (Ervin, Lee [EL06]).

There has also been interesting work on Spectral Deferred Correction (SDC) for IVP's (e.g., Minion [M04], Bourlioux, Layton, Minion [BLM03], Kress, Gustafsson [KG02], Dutt, Greengard, Rokhlin [DGR00]). With the exception of the SDC methods for time stepping, the majority of the results has been obtained for the equilibrium problems - an odd fact, since, e.g., for the Euler equations the time-dependent problem is natural. For example, it has not been known apparently if the natural idea of time stepping combined with the DCM in space for the associated quasi-equilibrium problem is stable.

3.2 MATHEMATICAL PRELIMINARIES AND NOTATIONS

Throughout this chapter the norm $\|\cdot\|$ will denote the usual $L^2(\Omega)$-norm of scalars, vectors and tensors, induced by the usual L^2 inner-product, denoted by $(\cdot,\cdot)$. The space that velocity (at time t) belongs to, is

$$X = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d\times d} \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

with the norm $\|v\|_X = \|\nabla v\|$. The space dual to X, is equipped with the norm

$$\|f\|_{-1} = \sup_{v\in X} \frac{(f,v)}{\|\nabla v\|}.$$

The pressure (at time t) is sought in the space

$$Q = L_0^2(\Omega) = \{q : q \in L^2(\Omega), \int_\Omega q(x)dx = 0\}.$$

Also introduce the space of weakly divergence-free functions

$$X \supset V = \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For measurable $v : [0,T] \to X$, we define

$$\|v\|_{L^p(0,T;X)} = (\int_0^T \|v(t)\|_X^p dt)^{\frac{1}{p}},\ 1 \le p < \infty,$$

and

$$\|v\|_{L^\infty(0,T;X)} = ess \sup_{0\le t\le T} \|v(t)\|_X.$$

Define the trilinear form on $X \times X \times X$

$$b(u,v,w) = \int_\Omega u \cdot \nabla v \cdot w dx.$$

The following lemma is also necessary for the analysis

Lemma 3.1. *There exist finite constants $M = M(d)$ and $N = N(d)$ s.t. $M \ge N$ and*

$$M = \sup_{u,v,w\in X} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} < \infty \ ,\ N = \sup_{u,v,w\in V} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} < \infty.$$

The proof can be found, for example, in [GR79]. The corresponding constants M^h and N^h are defined by replacing X by the finite element space $X^h \subset X$ and V by $V^h \subset X$, which will be defined below. Note that $M \ge \max(M^h, N, N^h)$ and that as $h \to 0$, $N^h \to N$ and $M^h \to M$ (see [GR79]).

Throughout the chapter, we shall assume that the velocity-pressure finite element spaces $X^h \subset X$ and $Q^h \subset Q$ are conforming, have typical approximation properties of finite element spaces commonly in use, and satisfy the discrete inf-sup, or LBB^h, condition

$$\inf_{q^h\in Q^h} \sup_{v^h\in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|\nabla v^h\|\|q^h\|} \ge \beta^h > 0, \tag{3.2.1}$$

where β^h is bounded away from zero uniformly in h. Examples of such spaces can be found in [GR79]. We shall consider $X^h \subset X$, $Q^h \subset Q$ to be spaces of continuous piecewise polynomials of degree m and $m-1$, respectively, with $m \geq 2$. The case of $m=1$ is not considered, because the optimal error estimate (of the order h) is obtained after the first step of the method - and therefore the DCM in this case is reduced to the artificial viscosity approach.

The space of discretely divergence-free functions is defined as follows

$$V^h = \{v^h \in X^h : (q^h, \nabla \cdot v^h) = 0, \forall q^h \in Q^h\}.$$

In the analysis we use the properties of the following Modified Stokes Projection

Definition 3.1 (Modified Stokes Projection). *Define the Stokes projection operator P_S: $(X, Q) \to (X^h, Q^h)$, $P_S(u, p) = (\tilde{u}, \tilde{p})$, satisfying*

$$\begin{aligned}(h + Re^{-1})(\nabla(u - \tilde{u}), \nabla v^h) - (p - \tilde{p}, \nabla \cdot v^h) = 0, \\ (\nabla \cdot (u - \tilde{u}), q^h) = 0,\end{aligned} \tag{3.2.2}$$

for any $v^h \in V^h, q^h \in Q^h$.

In (V^h, Q^h) this formulation reads: given $(u, p) \in (X, Q)$, find $\tilde{u} \in V^h$ satisfying

$$(h + Re^{-1})(\nabla(u - \tilde{u}), \nabla v^h) - (p - q^h, \nabla \cdot v^h) = 0, \tag{3.2.3}$$

for any $v^h \in V^h, q^h \in Q^h$.

Define the explicitly skew-symmetrized trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

The following estimate is easy to prove (see, e.g., [GR79]): there exists a constant $C = C(\Omega)$ such that

$$|b^*(u, v, w)| \leq C(\Omega) \|\nabla u\| \|\nabla v\| \|\nabla w\|. \tag{3.2.4}$$

The proofs will require the sharper bound on the nonlinearity. This upper bound is improvable in $\mathbb{R}^2$.

Lemma 3.2 (The sharper bound on the nonlinear term). *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. For all $u, v, w \in X$*

$$|b^*(u, v, w)| \leq C(\Omega) \sqrt{\|u\| \|\nabla u\|} \|\nabla v\| \|\nabla w\|.$$

Proof. See [GR79]. □

We will also need the following inequalities: for any $u \in V$

$$\inf_{v \in V^h} \|\nabla(u - v)\| \leq C(\Omega) \inf_{v \in X^h} \|\nabla(u - v)\|, \tag{3.2.5}$$

$$\inf_{v \in V^h} \|u - v\| \leq C(\Omega) \inf_{v \in X^h} \|\nabla(u - v)\|. \tag{3.2.6}$$

The proof of (3.2.5) can be found, e.g., in [GR79], and (3.2.6) follows from the Poincare-Friedrich's inequality and (3.2.5).

Define also the number of time steps $N := \frac{T}{k}$.

We conclude the preliminaries by formulating the discrete Gronwall's lemma, see, e.g. [HR90]

Lemma 3.3. *Let k, B, and $a_\mu, b_\mu, c_\mu, \gamma_\mu$, for integers $\mu \geq 0$, be nonnegative numbers such that:*

$$a_n + k \sum_{\mu=0}^{n} b_\mu \leq k \sum_{\mu=0}^{n} \gamma_\mu a_\mu + k \sum_{\mu=0}^{n} c_\mu + B \text{ for } n \geq 0.$$

Suppose that $k\gamma_\mu < 1$ for all μ, and set $\sigma_\mu = (1 - k\gamma_\mu)^{-1}$. Then

$$a_n + k \sum_{\mu=0}^{n} b_\mu \leq e^{k \sum_{\mu=0}^{n} \sigma_\mu \gamma_\mu} \cdot [k \sum_{\mu=0}^{n} c_\mu + B].$$

3.3 STABILITY OF THE VELOCITY

In this section we prove the unconditional stability of the discrete artificial viscosity approximation u_1^h and use this result to prove stability of the higher order approximation u_2^h. Over $0 \leq t \leq T < \infty$ the approximations u_1^h and u_2^h are bounded uniformly in *Re*.

Hence, the formulation (3.1.5) gives the unconditionally stable extension of the defect correction method to the time-dependent Navier-Stokes equations. We start by proving stability of the modified Stokes Projection, that we use as the approximation $\tilde{u}^0$ to the initial velocity u_0.

Proposition 3.1 (Stability of the Stokes projection)**.** *Let u, $\tilde{u}$ satisfy (3.2.3). The following bound holds*

$$(h + Re^{-1})\|\nabla \tilde{u}\|^2 \leq 2(h + Re^{-1})\|\nabla u\|^2 \qquad (3.3.1)$$
$$+2d(h + Re^{-1})^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2,$$

where d is the dimension, $d = 2, 3$.

Proof. Take $v^h = \tilde{u} \in V^h$ in (3.2.3). This gives

$$(h + Re^{-1})\|\nabla \tilde{u}\|^2 = (h + Re^{-1})(\nabla u, \nabla \tilde{u}) \qquad (3.3.2)$$
$$-(p - q^h, \nabla \cdot \tilde{u}).$$

Using the Cauchy-Schwarz and Young's inequalities, we obtain

$$(h + Re^{-1})\|\nabla \tilde{u}\|^2 \leq (h + Re^{-1})\|\nabla u\|^2 + \frac{h + Re^{-1}}{4}\|\nabla \tilde{u}\|^2 \qquad (3.3.3)$$
$$+d(h + Re^{-1})^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2 + \frac{h + Re^{-1}}{4d}\|\nabla \cdot \tilde{u}\|^2.$$

Using the inequality $\|\nabla \cdot \tilde{u}\|^2 \leq d\|\nabla \tilde{u}\|^2$ and combining the like terms concludes the proof. $\square$

Now we prove the main results of this section - stability of the AV approximation u_1^h and the Correction Step approximation u_2^h.

Lemma 3.4 (Stability of the AV approximation)**.** *Let u_1^h satisfy the first equation of (3.1.5). Let $f \in L^2(0, T; H^{-1}(\Omega))$. Then for $n = 0, ..., N - 1$*

$$\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \leq \|u_0^s\|^2$$

$$+\frac{1}{h + Re^{-1}} k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.$$

Also, if $f \in L^2(0, T; L^2(\Omega))$ and the time constraint T is finite, then there exists a constant $C = C(T)$ such that

$$\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \qquad (3.3.4)$$
$$\leq C(\|u_0^s\|^2 + k\Sigma_{i=1}^{n+1}\|f(t_i)\|^2).$$

Proof. Let $v^h = u_1^{h,n+1} \in V^h$ in the first equation of (3.1.5). Since $b^*(u,v,v) = 0$, we obtain

$$\frac{\|u_1^{h,n+1}\|^2 - (u_1^{h,n}, u_1^{h,n+1})}{k} + (h + Re^{-1})\|\nabla u_1^{h,n+1}\|^2 - (p_1^{h,n+1}, \nabla \cdot u_1^{h,n+1}) \\ = (f(t_{n+1}), u_1^{h,n+1}).$$

Since $p_1^{h,n+1} \in Q^h$ and $u_1^{h,n+1} \in V^h$ it follows that $(p_1^{h,n+1}, \nabla \cdot u_1^{h,n+1}) = 0$. Applying Cauchy-Schwartz and Young's inequalities gives

$$\frac{\|u_1^{h,n+1}\|^2 - \|u_1^{h,n}\|^2}{2k} + (h + Re^{-1})\|\nabla u_1^{h,n+1}\|^2 \leq (f(t_{n+1}), u_1^{h,n+1}). \tag{3.3.5}$$

The definition of the dual norm and the Young's inequality, applied to the inner-product on the right-hand side, lead to

$$(f^{n+1}, u_1^{h,n+1}) \leq \|f^{n+1}\|_{-1}\|\nabla u_1^{h,n+1}\| \\ \leq \frac{h + Re^{-1}}{2}\|\nabla u_1^{h,n+1}\|^2 + \frac{1}{2(h + Re^{-1})}\|f(t_{n+1})\|_{-1}^2. \tag{3.3.6}$$

We obtain

$$\frac{\|u_1^{h,n+1}\|^2 - \|u_1^{h,n}\|^2}{2k} + \frac{h + Re^{-1}}{2}\|\nabla u_1^{h,n+1}\|^2 \leq \frac{1}{2(h + Re^{-1})}\|f(t_{n+1})\|_{-1}^2. \tag{3.3.7}$$

Summing (3.3.7) over all time levels and multiplying by $2k$ gives

$$\|u_1^{h,n+1}\|^2 + (h + Re^{-1})k\Sigma_{i=1}^{n+1}\|\nabla u_1^{h,i}\|^2 \leq \|u_0^s\|^2 \\ + \frac{1}{h + Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2. \tag{3.3.8}$$

This proves the first part of Lemma.

Consider (3.3.5). Apply the Cauchy-Schwarz and Young's inequalities to the right-hand side. Different choice of constants in the Young's inequality gives

$$(f(t_{n+1}), u_1^{h,n+1}) \leq \|f(t_{n+1})\|\|u_1^{h,n+1}\| \leq \frac{1}{2}\|u_1^{h,n+1}\|^2 + \frac{1}{2}\|f(t_{n+1})\|^2 \tag{3.3.9}$$

and

$$(f(t_{n+1}), u_1^{h,n+1}) \leq \|f(t_{n+1})\|\|u_1^{h,n+1}\| \leq \frac{1}{4k}\|u_1^{h,n+1}\|^2 + k\|f(t_{n+1})\|^2. \tag{3.3.10}$$

Sum (3.3.5) over all time levels, using (3.3.9) at the time levels $t_0, t_1, ..., t_n$ and (3.3.10) at $t = t_{n+1}$. We obtain

$$\frac{\|u_1^{h,n+1}\|^2 - \|u_0^s\|^2}{2k} + \Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \tag{3.3.11}$$
$$\leq \frac{1}{4k}\|u_1^{h,n+1}\|^2 + \frac{1}{2}\Sigma_{i=1}^{n}\|u_1^{h,i}\|^2 + k\|f(t_{n+1})\|^2 + \frac{1}{2}\Sigma_{i=1}^{n}\|f(t_i)\|^2.$$

Multiply by $4k$ and simplify to obtain

$$\|u_1^{h,n+1}\|^2 + 4k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \tag{3.3.12}$$
$$\leq 2\|u_0^s\|^2 + 4k^2\|f(t_{n+1})\|^2 + 2k\Sigma_{i=1}^{n}\|f(t_i)\|^2 + 2k\Sigma_{i=1}^{n}\|u_1^{h,i}\|^2.$$

For the finite time constraint T, the discrete Gronwall's lemma yields

$$\|u_1^{h,n+1}\|^2 + 4k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \tag{3.3.13}$$
$$\leq 2e^{(\frac{2T}{1-k})}(\|u_0^s\|^2 + k\Sigma_{i=1}^{n+1}\|f(t_i)\|^2).$$

□

We use the result of Lemma 3.4 in the following

Theorem 3.1 (Stability)**.** *Let u_1^h, u_2^h satisfy (3.1.5). Let $f \in L^2(0,T; H^{-1}(\Omega))$. Then for $n = 0, ..., N-1$: $u_1^{h,n+1}, u_2^{h,n+1}$ are bounded and*

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h + Re^{-1})^2}\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.3.14}$$
$$\leq (1 + \frac{2h^2}{(h + Re^{-1})^2})\|u_0^s\|^2$$
$$+(1 + \frac{h^2}{(h + Re^{-1})^2})\frac{2}{h + Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.$$

Also, if $f \in L^2(0,T; L^2(\Omega))$ and the time constraint T is finite, then there exists a constant $C = C(T)$ such that

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h + Re^{-1})^2}\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.3.15}$$
$$\leq C(\|u_0^s\|^2 + k\Sigma_{i=1}^{n+1}\|f(t_i)\|^2).$$

It follows from (3.3.15) that both approximations u_1^h and u_2^h are bounded at any time level and for any viscosity, provided that the initial approximation and the forcing term are L^2-integrable.

The rest of the section is devoted to the proof of Theorem 3.1.

Proof. Take $v^h = u_2^{h,n+1} \in V^h$ in the second equation of (3.1.5). This gives

$$\frac{1}{2k}(\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2) + (h + Re^{-1})\|\nabla u_2^{h,n+1}\|^2 \leq (f(t_{n+1}), u_2^{h,n+1}) + h(\nabla u_1^{h,n+1}, \nabla u_2^{h,n+1}). \tag{3.3.16}$$

The Cauchy-Schwarz and Young's inequalities give

$$\begin{aligned}&\frac{1}{2k}(\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2) + (h + Re^{-1})\|\nabla u_2^{h,n+1}\|^2 \\ &\leq \frac{1}{h + Re^{-1}}\|f(t_{n+1})\|_{-1}^2 + \frac{h + Re^{-1}}{4}\|\nabla u_2^{h,n+1}\|^2 \\ &+ \frac{h^2}{h + Re^{-1}}\|\nabla u_1^{h,n+1}\|^2 + \frac{h + Re^{-1}}{4}\|\nabla u_2^{h,n+1}\|^2.\end{aligned} \tag{3.3.17}$$

Multiply (3.3.17) by $2k$ and simplify to obtain

$$\begin{aligned}&\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2 + (h + Re^{-1})k\|\nabla u_2^{h,n+1}\|^2 \\ &\leq \frac{2}{h + Re^{-1}}k\|f(t_{n+1})\|_{-1}^2 + \frac{2h^2}{h + Re^{-1}}k\|\nabla u_1^{h,n+1}\|^2.\end{aligned} \tag{3.3.18}$$

Summing over all time levels leads to

$$\begin{aligned}&\|u_2^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_2^{h,i}\|^2 \\ &\leq \|u_0^s\|^2 + \frac{2}{h + Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2 \\ &+ \frac{2h^2}{(h + Re^{-1})^2}k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2.\end{aligned} \tag{3.3.19}$$

Inserting the bound on $k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2$ from the stability result (3.3.8) in (3.3.19) gives

$$\begin{aligned}&\|u_2^{h,n+1}\|^2 + (h + Re^{-1})k\Sigma_{i=1}^{n+1}\|\nabla u_2^{h,i}\|^2 \\ &\leq \|u_0^s\|^2 + \frac{2}{h + Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2 \\ &+ \frac{2h^2}{(h + Re^{-1})^2}(\|u_0^s\|^2 - \|u_1^{h,n+1}\|^2 + \frac{1}{h + Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2).\end{aligned} \tag{3.3.20}$$

Thus

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h+Re^{-1})^2}\|u_1^{h,n+1}\|^2 + (h+Re^{-1})k\Sigma_{i=1}^{n+1}\|\nabla u_2^{h,i}\|^2 \tag{3.3.21}$$
$$\leq (1+\frac{2h^2}{(h+Re^{-1})^2})\|u_0^s\|^2$$
$$+(1+\frac{h^2}{(h+Re^{-1})^2})\frac{2}{h+Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.$$

This proves the first statement of Theorem 3.1. To conclude, consider (3.3.16); as in (3.3.9)-(3.3.10), use the Young's inequalities differently at different time levels to obtain

$$\frac{\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2}{2k} + (h+Re^{-1})\|\nabla u_2^{h,n+1}\|^2 \tag{3.3.22}$$
$$\leq k\|f(t_{n+1})\|^2 + \frac{1}{4k}\|u_2^{h,n+1}\|^2$$
$$+\frac{h^2}{2(h+Re^{-1})}\|\nabla u_1^{h,n+1}\|^2 + \frac{h+Re^{-1}}{2}\|\nabla u_2^{h,n+1}\|^2,$$

and

$$\frac{\|u_2^{h,i+1}\|^2 - \|u_2^{h,i}\|^2}{2k} + (h+Re^{-1})\|\nabla u_2^{h,i+1}\|^2 \tag{3.3.23}$$
$$\leq \frac{1}{2}\|f(t_{i+1})\|^2 + \frac{1}{2}\|u_2^{h,i+1}\|^2$$
$$+\frac{h^2}{2(h+Re^{-1})}\|\nabla u_1^{h,i+1}\|^2 + \frac{h+Re^{-1}}{2}\|\nabla u_2^{h,i+1}\|^2,$$
$$\text{for } \forall i = 0, 1, .., n-1.$$

Sum (3.3.23) over all time levels and add to (3.3.22); multiply by $4k$ to obtain

$$\|u_2^{h,n+1}\|^2 - 2\|u_0^s\|^2 + 2k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.3.24}$$
$$\leq 2k\Sigma_{i=1}^{n}\|u_2^{h,i}\|^2 + 4k^2\|f(t_{n+1})\|^2$$
$$+2k\Sigma_{i=1}^{n}\|f(t_i)\|^2 + \frac{2h^2}{(h+Re^{-1})^2}k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_1^{h,i}\|^2.$$

Insert the bound on $k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_1^{h,i}\|^2$ from (3.3.13) into (3.3.24) and simplify. For the finite time constraint T, the discrete Gronwall's lemma yields

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h+Re^{-1})^2}\|u_1^{h,n+1}\|^2 + 2k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.3.25}$$
$$\leq (2e^{(\frac{2T}{1-k})} + 4e^{(\frac{4T}{1-k})}\frac{h^2}{(h+Re^{-1})^2})[\|u_0^s\|^2 + k\Sigma_{i=1}^{n}\|f(t_i)\|^2].$$

□

The result of Theorem 3.1, combined with the result of Proposition 3.1, proves the unconditional stability of both $u_1^{h,i}$ and $u_2^{h,i}$ for any $i \geq 0$.

3.4 ERROR ESTIMATES

In this section we explore the error estimates in approximating the NSE velocity u by the Artificial Viscosity approximation u_1 and the Correction Step approximation u_2. The results agree with the general theory of the defect correction methods: $\|u-u_1^h\|_{energy-norm} \leq C(h^m+h)$, $\|u-u_2^h\|_{energy-norm} \leq C(h^m+h^2)$, where the velocity approximations u_1^h and u_2^h are sought in the finite-element space of piecewise polynomials of degree m.

In the error analysis we shall use the error estimate of the Stokes projection (3.2.3).

Proposition 3.2 (Error estimate for Stokes Projection)**.** *Suppose the discrete inf-sup condition (3.2.1) holds. Then the error in the Stokes Projection satisfies*

$$
\begin{aligned}
(h + Re^{-1})\|\nabla(u-\tilde{u})\|^2 \leq C[(h+Re^{-1}) \inf_{v^h \in V^h} \|\nabla(u-v^h)\|^2 \\
+(h+Re^{-1})^{-1} \inf_{q^h \in Q^h} \|p-q^h\|^2],
\end{aligned}
\tag{3.4.1}
$$

where C is a constant independent of h and Re.

Proof. Decompose the projection error $e = u - \tilde{u}$ into $e = u - I(u) - (\tilde{u} - I(u)) = \eta - \phi$, where $\eta = u - I(u)$, $\phi = \tilde{u} - I(u)$, and $I(u)$ approximates u in V^h. Take $v^h = \phi \in V^h$ in (3.2.3). This gives

$$
\begin{aligned}
(h + Re^{-1})\|\nabla\phi\|^2 = (h+Re^{-1})(\nabla\eta, \nabla\phi) \\
-(p-q^h, \nabla\cdot\phi).
\end{aligned}
\tag{3.4.2}
$$

Since $\Omega \subset \mathbb{R}^d$, we have $\|\nabla\cdot\phi\|^2 \leq d\|\nabla\phi\|^2$.

Applying the Cauchy-Schwarz and Young's inequalities to (3.4.2) gives

$$
\begin{aligned}
(h + Re^{-1})\|\nabla\phi\|^2 \leq 2(h+Re^{-1})\|\nabla\eta\|^2 \\
+2d(h+Re^{-1})^{-1} \inf_{q^h \in Q^h} \|p-q^h\|^2.
\end{aligned}
\tag{3.4.3}
$$

Since $I(u)$ is an approximation of u in V^h, we can take infimum over V^h. The proof is concluded by applying the triangle inequality. □

The following constants (depending upon Ω and u) are introduced in order to simplify the notation.

Definition 3.2. *Let*

$$C_u := \|u(x,t)\|_{L^\infty(0,T;L^\infty(\Omega))},$$
$$C_{\nabla u} := \|\nabla u(x,t)\|_{L^\infty(0,T;L^\infty(\Omega))},$$

and introduce $\tilde{C}$, satisfying

$$\inf_{v\in V^h} \|\nabla(u-v)\| \leq C \inf_{v\in X^h} \|\nabla(u-v)\| \leq C_1 h^m \|u\|_{H^{m+1}} \leq \tilde{C} h^m.$$

Also, using the constant $C(\Omega)$ from Lemma 3.2, we define

$$\bar{C} := 1728 C^4(\Omega).$$

The main results of this section are presented in the following theorem:

Theorem 3.2 (Error estimates)**.** *Let $f \in L^2(0,T;H^{-1}(\Omega))$, let u_1^h, u_2^h satisfy (3.1.5),*

$$k \leq \frac{h+Re^{-1}}{4C_u^2 + 2(h+Re^{-1})C_{\nabla u} + 2\bar{C}\tilde{C}^4(h+Re^{-1})^{-2}h^{4m}},$$
$$u \in L^2(0,T;H^{m+1}(\Omega)) \bigcap L^\infty(0,T;L^\infty(\Omega)), \nabla u \in L^\infty(0,T;L^\infty(\Omega)),$$
$$u_t \in L^2(0,T;H^{m+1}(\Omega)), u_{tt} \in L^2(0,T;L^2(\Omega)), p \in L^2(0,T;H^m(\Omega)).$$

Then there exists a constant $C = C(\Omega,T,u,p,f,h+Re^{-1})$, such that

$$\max_{1\leq i\leq N} \|u(t_i) - u_1^{h,i}\| + \left(k\sum_{i=1}^{n+1}(h+Re^{-1})\|\nabla(u(t_i)-u_1^{h,i})\|^2\right)^{1/2} \leq C(h^m + h + k),$$

and

$$\max_{1\leq i\leq N} \|u(t_i) - u_2^{h,i}\| + \left(k\sum_{i=1}^{n+1}(h+Re^{-1})\|\nabla(u(t_i)-u_2^{h,i})\|^2\right)^{1/2} \leq C(h^m + h^2 + hk + k).$$

Hence, the second (Correction) step of the method gives an approximation of the true solution, that is improved by (roughly) an order of h compared to the first step (Artificial Viscosity) approximation.

The goal of this section is to prove Theorem 3.2 - that is, that the method is of the first order in time and that the order of the approximation in space depends upon the step of defect correction procedure.

Proof. By Taylor expansion, $\frac{u(t_{n+1})-u(t_n)}{k} = u_t(t_{n+1}) - k\rho^{n+1}$, where $\rho^{n+1} = u_{tt}(t_{n+\theta})$, for some $\theta \in [0,1]$. The variational formulation of the NSE, followed by the equations (3.1.5), gives for $u \in X, p \in Q, u_1, u_2 \in X^h, p_1, p_2 \in Q^h, \forall v \in V^h$

$$(\frac{u(t_{n+1}) - u(t_n)}{k}, v) + (h + Re^{-1})(\nabla u(t_{n+1}), \nabla v) + b^*(u(t_{n+1}), u(t_{n+1}), v) \tag{3.4.4}$$
$$-(p(t_{n+1}), \nabla \cdot v) = (f(t_{n+1}), v) + h(\nabla u(t_{n+1}), \nabla v) - k(\rho^{n+1}, v),$$
$$(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v) + (h + Re^{-1})(\nabla u_1^{h,n+1}, \nabla v) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v) \tag{3.4.5}$$
$$-(p_1^{h,n+1}, \nabla \cdot v) = (f(t_{n+1}), v),$$
$$(\frac{u_2^{h,n+1} - u_2^{h,n}}{k}, v) + (h + Re^{-1})(\nabla u_2^{h,n+1}, \nabla v) + b^*(u_2^{h,n+1}, u_2^{h,n+1}, v) \tag{3.4.6}$$
$$-(p_2^{h,n+1}, \nabla \cdot v) = (f(t_{n+1}), v) + h(\nabla u_1^{h,n+1}, \nabla v).$$

Subtract (3.4.5) from (3.4.4). Introduce the error in the AV approximation $e_1^i := u(t_i) - u_1^{h,i}, \forall i$. This gives

$$(\frac{e_1^{n+1} - e_1^n}{k}, v) + (h + Re^{-1})(\nabla e_1^{n+1}, \nabla v) \tag{3.4.7}$$
$$+[b^*(u(t_{n+1}), u(t_{n+1}), v) - b^*(u_1^{h,n+1}, u_1^{h,n+1}, v)]$$
$$-((p(t_{n+1}) - p_1^{h,n+1}), \nabla \cdot v) = h(\nabla u(t_{n+1}), \nabla v) - k(\rho^{n+1}, v).$$

Adding and subtracting $b^*(u_1^{h,n+1}, u(t_{n+1}), v)$ to the nonlinear terms in (3.4.7) gives

$$b^*(u(t_{n+1}), u(t_{n+1}), v) - b^*(u_1^{h,n+1}, u_1^{h,n+1}, v) \tag{3.4.8}$$
$$= b^*(e_1^{n+1}, u(t_{n+1}), v) + b^*(u_1^{h,n+1}, e_1^{n+1}, v).$$

Decompose the error

$$e_1^i = u(t_i) - u_1^{h,i} = u(t_i) - \tilde{u}^i + \tilde{u}^i - u_1^{h,i} = \eta_1^i - \phi_1^{h,i}, \tag{3.4.9}$$
$$\text{where } \tilde{u}^i \in V^h \text{ is some projection of } u(t_i) \text{ into } V^h,$$
$$\text{and } \eta_1^i = u(t_i) - \tilde{u}^i,\ \ \phi_1^{h,i} = u_1^{h,i} - \tilde{u}^i, \phi_1^{h,i} \in V^h, \forall i.$$

Take $v = \phi_1^{h,n+1} \in V^h$ in (3.4.7) and use (3.4.8). Using also $b^*(\cdot, \phi_1^{h,n+1}, \phi_1^{h,n+1}) = 0$ and $V^h \perp Q^h$, we obtain

$$\begin{aligned}
&(\frac{\eta_1^{n+1} - \eta_1^n}{k}, \phi_1^{h,n+1}) - (\frac{\phi_1^{h,n+1} - \phi_1^{h,n}}{k}, \phi_1^{h,n+1}) \\
&+(h + Re^{-1})(\nabla \eta_1^{n+1}, \nabla \phi_1^{h,n+1}) - (h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 \\
&+b^*(\eta_1^{n+1}, u(t_{n+1}), \phi_1^{h,n+1}) - b^*(\phi_1^{h,n+1}, u(t_{n+1}), \phi_1^{h,n+1}) \\
&+b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1}) - (p(t_{n+1}) - q^{h,n+1}, \nabla \cdot \phi_1^{h,n+1}) \\
&= h(\nabla u(t_{n+1}), \nabla \phi_1^{h,n+1}) - k(\rho^{n+1}, \phi_1^{h,n+1}).
\end{aligned} \tag{3.4.10}$$

Apply the Cauchy-Schwarz and Young's inequalities to (3.4.10). Since $\|\nabla \cdot \phi_1^{h,n+1}\|^2 \leq d\|\nabla \phi_1^{h,n+1}\|^2$ for $\forall \epsilon > 0$

$$\begin{aligned}
&\frac{\|\phi_1^{h,n+1}\|^2 - \|\phi_1^{h,n}\|^2}{2k} + (h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 \\
&\leq \epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + \frac{1}{4\epsilon(h + Re^{-1})}\|\frac{\eta_1^{n+1} - \eta_1^n}{k}\|_{-1}^2 \\
&+\epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + \frac{(h + Re^{-1})}{4\epsilon}\|\nabla \eta_1^{n+1}\|^2 \\
&+|b^*(\eta_1^{n+1}, u(t_{n+1}), \phi_1^{h,n+1})| + |b^*(\phi_1^{h,n+1}, u(t_{n+1}), \phi_1^{h,n+1})| \\
&+|b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| \\
&+\epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + \frac{d}{4\epsilon(h + Re^{-1})} \inf_{q^h \in Q^h} \|p(t_{n+1}) - q^{h,n+1}\|^2 \\
&+\epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + \frac{h^2}{4\epsilon(h + Re^{-1})}\|\nabla u(t_{n+1})\|^2 \\
&+\epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + \frac{1}{4\epsilon(h + Re^{-1})}k^2\|\rho^{n+1}\|_{-1}^2.
\end{aligned} \tag{3.4.11}$$

We bound the nonlinear terms on the right-hand side of (3.4.11), starting now with the first one. Use the bound (3.2.4), the regularity of u and Young's inequality to obtain

$$|b^*(\eta_1^{n+1}, u(t_{n+1}), \phi_1^{h,n+1})| \leq \epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + C\frac{1}{h + Re^{-1}}\|\nabla \eta_1^{n+1}\|^2. \tag{3.4.12}$$

The second nonlinear term can be bounded, using the definition of $b^*(\cdot, \cdot, \cdot)$ and the regularity of u. This gives

$$\begin{aligned}|b^*(\phi_1^{h,n+1}, u(t_{n+1}), \phi_1^{h,n+1})| &\leq \frac{C_{\nabla u}}{2}\|\phi_1^{h,n+1}\|^2 + \frac{C_u}{2}(|\phi_1^{h,n+1}|, |\nabla \phi_1^{h,n+1}|) \\ &\leq \frac{C_{\nabla u}}{2}\|\phi_1^{h,n+1}\|^2 + \epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 + \frac{C_u^2}{16\epsilon(h + Re^{-1})}\|\phi_1^{h,n+1}\|^2.\end{aligned} \tag{3.4.13}$$

For the third nonlinear term of (3.4.11), use the error decomposition to obtain

$$|b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| \leq |b^*(u(t_{n+1}), \eta_1^{n+1}, \phi_1^{h,n+1})| + |b^*(\eta_1^{n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| + |b^*(\phi_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})|. \tag{3.4.14}$$

Use the regularity of u and the inequality (3.2.4) to bound the first two terms on the right-hand side of (3.4.14). Applying Lemma 3.2 to the third term gives

$$|b^*(\phi_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| \leq C(\Omega)\|\nabla \phi_1^{h,n+1}\|^{3/2}\|\phi_1^{h,n+1}\|^{1/2}\|\eta_1^{n+1}\|. \tag{3.4.15}$$

We apply the Young's inequality to (3.4.15) with $p = \frac{4}{3}$ and $q = 4$. Finally it follows from (3.4.14) that

$$\begin{aligned}|b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| &\leq \epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 \\ &+ \frac{C}{h + Re^{-1}}(\|\nabla \eta_1^{n+1}\|^2 + \|\nabla \eta_1^{n+1}\|^4) \\ &+ \frac{27C^4(\Omega)}{64\epsilon^3(h + Re^{-1})^3}\|\nabla \eta_1^{n+1}\|^4\|\phi_1^{h,n+1}\|^2,\end{aligned} \tag{3.4.16}$$

where $C(\Omega)$ is the constant from Lemma 3.2 .

Take $\epsilon = \frac{1}{16}$ in (3.4.11). Using the bounds (3.4.12)-(3.4.16), we obtain

$$\begin{aligned}
&\frac{\|\phi_1^{h,n+1}\|^2 - \|\phi_1^{h,n}\|^2}{2k} + \frac{h + Re^{-1}}{2}\|\nabla\phi_1^{h,n+1}\|^2 \qquad (3.4.17)\\
&\le \frac{C}{h + Re^{-1}}\|\frac{\eta_1^{n+1} - \eta_1^n}{k}\|_{-1}^2 \\
&+C(h + Re^{-1})\|\nabla\eta_1^{n+1}\|^2 \\
&+\frac{C}{h + Re^{-1}}\inf_{q^h\in Q^h}\|p(t_{n+1}) - q^{h,n+1}\|^2 \\
&+\frac{C}{h + Re^{-1}}h^2\|\nabla u(t_{n+1})\|^2 \\
&+\frac{C}{h + Re^{-1}}k^2\|\rho^{n+1}\|_{-1}^2 + \frac{C}{h + Re^{-1}}(\|\nabla\eta_1^{n+1}\|^2 + \|\nabla\eta_1^{n+1}\|^4) \\
&+(\frac{1}{2}C_{\nabla u} + \frac{C_u^2}{h + Re^{-1}} + \frac{\bar{C}}{(h + Re^{-1})^3}\|\nabla\eta_1^{n+1}\|^4)\|\phi_1^{h,n+1}\|^2.
\end{aligned}$$

Sum (3.4.17) over all time levels and multiply by $2k$. It follows from the regularity assumptions of the theorem that

$$k\sum_{i=0}^{n}\|\rho^{i+1}\|_{-1}^2 \le Ck\sum_{i=0}^{n}\|\rho^{i+1}\|^2 \le C.$$

Therefore we obtain

$$\begin{aligned}
&\|\phi_1^{h,n+1}\|^2 + (h + Re^{-1})k\sum_{i=0}^{n}\|\nabla\phi_1^{h,i+1}\|^2 \le \|\phi_1^{h,0}\|^2 \qquad (3.4.18)\\
&+\frac{2C}{h + Re^{-1}}k\sum_{i=0}^{n}[\|\frac{\eta_1^{i+1} - \eta_1^i}{k}\|_{-1}^2 + (h + Re^{-1})^2\|\nabla\eta_1^i\|^2 \\
&+\|\nabla\eta_1^i\|^2 + \|\nabla\eta_1^i\|^4 + \inf_{q^h\in Q^h}\|p(t_i) - q^{h,i}\|^2 + h^2 + k^2] \\
&+k\sum_{i=0}^{n}(C_{\nabla u} + \frac{2C_u^2}{h + Re^{-1}} + \frac{2\bar{C}}{(h + Re^{-1})^3}\|\nabla\eta_1^{i+1}\|^4)\|\phi_1^{h,i+1}\|^2.
\end{aligned}$$

Take $\tilde{u}^i$ in the error decomposition (3.4.9) to be the L^2-projection of $u(t_i)$ into V^h, for $i \ge 1$. Take $\tilde{u}^0$ to be u_0^s. This gives $\phi_1^{h,0} = 0$ and $e_1^0 = \eta_1^0$. Also it follows from Proposition 3.2

that $\|\nabla \eta_1^0\| \leq Ch^m$; under the assumptions of the theorem the discrete Gronwall's lemma gives

$$\|\phi_1^{h,n+1}\|^2 + (h + Re^{-1})k \sum_{i=0}^{n} \|\nabla \phi_1^{h,i+1}\|^2 \tag{3.4.19}$$
$$\leq \frac{C}{h + Re^{-1}} k \sum_{i=0}^{n} [\|\frac{\eta_1^{i+1} - \eta_1^i}{k}\|_{-1}^2 + \|\nabla \eta_1^i\|^2$$
$$+\|\nabla \eta_1^i\|^4 + \inf_{q^h \in Q^h} \|p(t_i) - q^{h,i}\|^2 + h^2 + k^2].$$

Using the error decomposition and the triangle inequality, we obtain

$$\|e_1^{n+1}\| \leq \|\eta_1^{n+1}\| + \|\phi_1^{h,n+1}\|, \tag{3.4.20}$$
$$\|e_1^{n+1}\|^2 \leq 2\|\eta_1^{n+1}\|^2 + 2\|\phi_1^{h,n+1}\|^2,$$
$$\|\nabla e_1^{i+1}\|^2 \leq 2\|\nabla \eta_1^{i+1}\|^2 + 2\|\nabla \phi_1^{h,i+1}\|^2,$$
$$k \sum_{i=0}^{n} (h + Re^{-1}) \|\nabla e_1^{i+1}\|^2$$
$$\leq 2k \sum_{i=0}^{n} (h + Re^{-1}) \|\nabla \phi_1^{h,i+1}\|^2 + 2k \sum_{i=0}^{n} (h + Re^{-1}) \|\nabla \eta_1^{i+1}\|^2.$$

Then it follows from (3.4.19),(3.4.20) that

$$\|e_1^{n+1}\|^2 + k \sum_{i=0}^{n} (h + Re^{-1}) \|\nabla e_1^{i+1}\|^2 \tag{3.4.21}$$
$$\leq \frac{C}{h + Re^{-1}} k \sum_{i=0}^{n} [\|\frac{\eta_1^{i+1} - \eta_1^i}{k}\|_{-1}^2 + \|\nabla \eta_1^i\|^2$$
$$+\|\nabla \eta_1^i\|^4 + \inf_{q^h \in Q^h} \|p(t_i) - q^{h,i}\|^2 + h^2 + k^2].$$

Use the approximation properties of X^h, Q^h. Since the mesh nodes do not depend upon the time level, it follows from (3.2.5),(3.2.6) that

$$k \sum_{i=0}^{n} \|\frac{\eta_1^{i+1} - \eta_1^i}{k}\|_{-1}^2 \leq Ck \sum_{i=0}^{n} \|\frac{\eta_1^{i+1} - \eta_1^i}{k}\|^2 \leq Ch^{2m}, \tag{3.4.22}$$
$$k \sum_{i=0}^{n} \|\nabla \eta_1^i\|^2 \leq Ch^{2m},$$
$$k \sum_{i=0}^{n} \inf_{q^h \in Q^h} \|p(t_i) - q^{h,i}\|^2 \leq Ch^{2m}.$$

Hence, we obtain from (3.4.21),(3.4.22) that

$$\|u(t_{n+1}) - u_1^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h + Re^{-1})\|\nabla(u(t_{n+1}) - u_1^{h,n+1})\|^2 \le \frac{C}{h + Re^{-1}}[h^{2m} + h^2 + k^2], \quad \text{where } C = C(\Omega, T, u, p, f). \tag{3.4.23}$$

This proves the first statement of the theorem.

Now subtract (3.4.6) from (3.4.4). Introduce the error in the Correction Step approximation $e_2^i := u(t_i) - u_2^{h,i}, \forall i$. This gives

$$(\frac{e_2^{n+1} - e_2^n}{k}, v) + (h + Re^{-1})(\nabla e_2^{n+1}, \nabla v) + [b^*(u(t_{n+1}), u(t_{n+1}), v) - b^*(u_2^{h,n+1}, u_2^{h,n+1}, v)] - ((p(t_{n+1}) - p_2^{h,n+1}), \nabla \cdot v) = h(\nabla e_1^{n+1}, \nabla v) - k(\rho^{n+1}, v). \tag{3.4.24}$$

Note that (3.4.24) differs from (3.4.7) only in the first term on the right-hand side. Using the Cauchy-Schwarz and Young's inequality, we obtain that for any $\epsilon > 0$

$$|h(\nabla e_1^{n+1}, \nabla v)| \le \epsilon(h + Re^{-1})\|\nabla v\|^2 + \frac{1}{4\epsilon(h + Re^{-1})}h^2\|\nabla e_1^{n+1}\|^2. \tag{3.4.25}$$

Therefore,

$$k\sum_{i=0}^{n}|h(\nabla e_1^{n+1}, \nabla v)| \le k\sum_{i=0}^{n}\epsilon(h + Re^{-1})\|\nabla v\|^2 + \frac{1}{4\epsilon(h + Re^{-1})^2}h^2 k\sum_{i=0}^{n}(h + Re^{-1})\|\nabla e_1^{n+1}\|^2. \tag{3.4.26}$$

Using the bound on $k\sum_{i=0}^{n}(h + Re^{-1})\|\nabla e_1^{n+1}\|^2$ from (3.4.23), we obtain

$$k\sum_{i=0}^{n}|h(\nabla e_1^{n+1}, \nabla v)| \le k\sum_{i=0}^{n}\epsilon(h + Re^{-1})\|\nabla v\|^2 + \frac{C}{(h + Re^{-1})^3}[h^{2m+2} + h^4 + h^2 k^2]. \tag{3.4.27}$$

Decompose the error

$$e_2^i = u(t_i) - u_2^{h,i} = u(t_i) - \tilde{u}^i + \tilde{u}^i - u_2^{h,i} = \eta_2^i - \phi_2^{h,i}, \quad \text{where } \eta_2^i = u(t_i) - \tilde{u}^i,\ \phi_2^{h,i} = u_2^{h,i} - \tilde{u}^i, \phi_2^{h,i} \in V^h, \forall i. \tag{3.4.28}$$

To conclude, repeat the proof of the first statement of the theorem, replacing $u_1^h, e_1, \phi_1^h, \eta_1$ by $u_2^h, e_2, \phi_2^h, \eta_2$, respectively, and using (3.4.27). Note that the term $\frac{C}{h+Re^{-1}}h^2$ on the right-hand side of (3.4.23), which was obtained from the bound on $h(\nabla u(t_{n+1}), \nabla v)$, is now replaced by $\frac{C}{(h+Re^{-1})^3}[h^{2m+2} + h^4 + h^2k^2]$. Hence, we obtain

$$\|u(t_{n+1}) - u_2^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h + Re^{-1})\|\nabla(u(t_{n+1}) - u_2^{h,n+1})\|^2 \qquad (3.4.29)$$
$$\leq \frac{C}{(h + Re^{-1})^3}[h^{2m} + h^4 + h^2k^2 + k^2],$$
$$\text{where } C = C(\Omega, T, u, p, f).$$

□

This completes the proof of Theorem 3.2. Thus, we have derived the error estimates, that agree with the general theory of the defect correction methods. Namely, the Correction Step approximation u_2^h is improved by an order of h, compared to the Artificial Viscosity approximation u_1^h.

Next we shall prove stability and derive the error estimates for the pressure.

3.5 PRESSURE

This section gives the proof of stability and the convergence rates for pressure approximations p_1^h and p_2^h.

For the pressure analysis we shall need the bounds on discrete time derivatives $\|\frac{e_1^{n+1}-e_1^n}{k}\|$ and $\|\frac{e_2^{n+1}-e_2^n}{k}\|$. For pressure stability it is enough to bound these quantities by a constant, but a more subtle estimate is needed for proving the convergence rates. We start by proving this estimate as a theorem.

Throughout this section we use the error decomposition $e_j^i = u(t_i) - u_j^{h,i} = \eta_j^i - \phi_j^{h,i}$, $j = 1, 2,\ i = 1, ..., n$, introduced in (3.4.9),(3.4.28).

Also, taking $\tilde{u}^i = u_0^s$ on the initial time level gives $\phi_1^{h,0} = \phi_2^{h,0} = 0$ and $e_1^0 = \eta_1^0$, $e_2^0 = \eta_2^0$. It follows from Proposition 3.2 that $\|\nabla\eta_1^0\| \leq Ch^m$ and $\|\nabla\eta_2^0\| \leq Ch^m$.

Theorem 3.3. *Let the regularity assumptions of Theorem 3.2 be satisfied. Let*

$$p_t \in L^2(0,T;H^m(\Omega)), u_{ttt} \in L^2(0,T;L^2(\Omega)).$$

Also let $k \le \min(h,(h+Re^{-1})^3)$. *Then for any time level* $n \ge 0$

$$\|\frac{e_1^{n+1}-e_1^n}{k}\| + (k\sum_{i=1}^{n}(h+Re^{-1})\|\nabla(\frac{e_1^{i+1}-e_1^i}{k})\|^2)^{1/2} \le C(h^m+h+k),$$

and

$$\|\frac{e_2^{n+1}-e_2^n}{k}\| + (k\sum_{i=1}^{n}(h+Re^{-1})\|\nabla(\frac{e_2^{i+1}-e_2^i}{k})\|^2)^{1/2} \le C(h^m+h^2+hk+k).$$

Proof. Start with the proof of the bound for $\|\frac{\phi_1^{h,n+1}-\phi_1^{h,n}}{k}\|$. Consider (3.4.7) with (3.4.8) for $n \ge 1$

$$\begin{aligned}(\frac{e_1^{n+1}-e_1^n}{k},v) + (h+Re^{-1})(\nabla e_1^{n+1},\nabla v) \\ +b^*(e_1^{n+1},u(t_{n+1}),v) + b^*(u_1^{h,n+1},e_1^{n+1},v) \\ -((p(t_{n+1})-p_1^{h,n+1}),\nabla\cdot v) = h(\nabla u(t_{n+1}),\nabla v) - k(\rho^{n+1},v), \\ \text{where } k\rho^{n+1} = u_t(t_{n+1}) - \frac{u(t_{n+1})-u(t_n)}{k}.\end{aligned} \tag{3.5.1}$$

Take $v = \frac{\phi_1^{h,n+1}-\phi_1^{h,n}}{k} =: s^{h,n+1} \in V^h$ in (3.5.1). Then consider (3.5.1) at the previous time level and make exactly the same choice $v = s^{h,n+1} \in V^h$. Subtract the equations, using the Taylor expansion to simplify the last term on the right-hand side. We obtain

$$\begin{aligned}k(\frac{\eta_1^{n+1}-2\eta_1^n+\eta_1^{n-1}}{k^2},s^{h,n+1}) - (s^{h,n+1}-s^{h,n},s^{h,n+1}) \\ +(h+Re^{-1})k(\nabla(\frac{\eta_1^{n+1}-\eta_1^n}{k}),\nabla s^{h,n+1}) - (h+Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\ +b^*(e_1^{n+1},u(t_{n+1}),s^{h,n+1}) + b^*(u_1^{h,n+1},e_1^{n+1},s^{h,n+1}) \\ -b^*(e_1^n,u(t_n),s^{h,n+1}) - b^*(u_1^{h,n},e_1^n,s^{h,n+1}) \\ -k(\frac{(p(t_{n+1})-p_1^{h,n+1})-(p(t_n)-p_1^{h,n})}{k},\nabla\cdot s^{h,n+1}) \\ = hk(\nabla(\frac{u(t_{n+1})-u(t_{n+1})}{k}),\nabla s^{h,n+1}) - Ck^2(\rho_t^{n+1},s^{h,n+1}), \\ \text{where } \rho_t^{n+1} = u_{ttt}(t_{n+\theta}) \text{ for some } \theta\in[0,1].\end{aligned} \tag{3.5.2}$$

Consider the nonlinear terms of (3.5.2). Adding and subtracting $b^*(e_1^n, u(t_{n+1}), s^{h,n+1})$ and $b^*(u_1^{h,n+1}, e_1^n, s^{h,n+1})$ gives

$$\begin{aligned} &b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_n), s^{h,n+1}) \\ &+b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1}) \\ &= [b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_{n+1}), s^{h,n+1}) \\ &+b^*(e_1^n, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_n), s^{h,n+1})] \\ &+[b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) - b^*(u_1^{h,n+1}, e_1^n, s^{h,n+1}) \\ &+b^*(u_1^{h,n+1}, e_1^n, s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1})]. \end{aligned} \tag{3.5.3}$$

Use the error decomposition (3.4.9). Since $b^*(\cdot, s^{h,n+1}, s^{h,n+1}) = 0$, it follows from (3.5.3) that

$$\begin{aligned} &b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_n), s^{h,n+1}) \\ &+b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1}) \\ &= kb^*(\frac{\eta_1^{n+1} - \eta_1^n}{k}, u(t_{n+1}), s^{h,n+1}) - kb^*(s^{h,n+1}, u(t_{n+1}), s^{h,n+1}) \\ &+kb^*(e_1^{n+1}, \frac{u(t_{n+1}) - u(t_n)}{k}, s^{h,n+1}) + kb^*(u_1^{h,n+1}, \frac{\eta_1^{n+1} - \eta_1^n}{k}, s^{h,n+1}) \\ &+kb^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, e_1^n, s^{h,n+1}). \end{aligned} \tag{3.5.4}$$

Use the regularity of u and the Cauchy-Schwarz and Young's inequalities to obtain the bounds on the terms in (3.5.4). It follows from (3.2.4) that for any $\epsilon > 0$

$$\begin{aligned} &k|b^*(\frac{\eta_1^{n+1} - \eta_1^n}{k}, u(t_{n+1}), s^{h,n+1})| \\ &\leq \epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 + \frac{C}{h + Re^{-1}}k\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2. \end{aligned} \tag{3.5.5}$$

For the second term on the right-hand side of (3.5.4) use the regularity of u and the Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned} k|b^*(s^{h,n+1}, u(t_{n+1}), s^{h,n+1})| &\leq \epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\ &+\frac{C}{h + Re^{-1}}C_u^2 k\|s^{h,n+1}\|^2 + \frac{1}{2}C_{\nabla u}k\|s^{h,n+1}\|^2. \end{aligned} \tag{3.5.6}$$

The third nonlinear term on the right-hand side of (3.5.4) is bounded by

$$\begin{aligned} & k|b^*(e_1^{n+1}, \frac{u(t_{n+1}) - u(t_n)}{k}, s^{h,n+1})| \\ & \leq \epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 + \frac{C}{h + Re^{-1}}k\|\nabla e_1^{n+1}\|^2. \end{aligned} \tag{3.5.7}$$

For the fourth nonlinear term, add and subtract $u(t_{n+1})$ to the first term of the trilinear form. Using (3.2.4) and Lemma 3.2 leads to

$$\begin{aligned} & k|b^*(u_1^{h,n+1}, \frac{\eta_1^{n+1} - \eta_1^n}{k}, s^{h,n+1})| \leq 2\epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\ & + \frac{C}{h + Re^{-1}}k\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2 + Ck\|e_1^{n+1}\|\|\nabla e_1^{n+1}\|\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2. \end{aligned} \tag{3.5.8}$$

For the fifth term add and subtract $u(t_{n+1})$ to the first term of the trilinear form to obtain

$$\begin{aligned} & k|b^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, e_1^n, s^{h,n+1})| \leq k|b^*(\frac{u(t_{n+1}) - u(t_n)}{k}, e_1^n, s^{h,n+1})| \\ & + k|b^*(\frac{\eta_1^{n+1} - \eta_1^n}{k}, e_1^n, s^{h,n+1})| + k|b^*(s^{h,n+1}, e_1^n, s^{h,n+1})|. \end{aligned} \tag{3.5.9}$$

Apply the result of Lemma 3.2 to the last trilinear form in (3.5.9) and use the Young's inequality with $p = \frac{4}{3}$ and $q = 4$. This gives

$$\begin{aligned} & k|b^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, e_1^n, s^{h,n+1})| \\ & \leq 3\epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 + \frac{C}{h + Re^{-1}}k\|\nabla e_1^n\|^2 \\ & + \frac{C}{h + Re^{-1}}k\|\nabla e_1^n\|^2\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2 + \frac{C}{(h + Re^{-1})^3}k\|\nabla e_1^n\|^4\|s^{h,n+1}\|^2. \end{aligned} \tag{3.5.10}$$

Apply the Cauchy-Schwarz and Young's inequalities to (3.5.2), using the bounds (3.5.4)-(3.5.10) for the nonlinear terms. This gives

$$\begin{aligned}
&\frac{\|s^{h,n+1}\|^2 - \|s^{h,n}\|^2}{2} + (h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\
&\leq 13\epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\
&+\frac{C}{h + Re^{-1}}k\|\frac{\eta_1^{n+1} - 2\eta_1^n + \eta_1^{n-1}}{k^2}\|_{-1}^2 + C(h + Re^{-1})k\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2 \\
&+\frac{C}{h + Re^{-1}}k \inf_{q^h \in Q^h} \|\frac{p(t_{n+1}) - p(t_n)}{k} - \frac{q^{h,n+1} - q^{h,n}}{k}\|^2 \\
&+\frac{C}{h + Re^{-1}}k[\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2 + \|\nabla e_1^n\|^2 + \|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2\|\nabla e_1^n\|^2] \\
&+Ck\|e_1^n\|^2\|\nabla e_1^n\|^2 + Ck\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^4 \\
&+\frac{C}{h + Re^{-1}}k \cdot h^2\|\nabla(\frac{u(t_{n+1}) - u(t_n)}{k})\|^2 + \frac{C}{h + Re^{-1}}k \cdot k^2\|\rho_t^{n+1}\|_{-1}^2 \\
&+C(C_{\nabla u} + \frac{C_u^2}{h + Re^{-1}} + \frac{1}{(h + Re^{-1})^3}\|\nabla e_1^n\|^4)k\|s^{h,n+1}\|^2.
\end{aligned} \tag{3.5.11}$$

Since $u_{ttt} \in L^2(0, T; L^2(\Omega))$, we have

$$k\sum_{i=0}^{n} \|\rho_t^{i+1}\|_{-1}^2 \leq Ck\sum_{i=0}^{n} \|\rho_t^{i+1}\|^2 \leq C.$$

It follows from the assumption $k \leq h$ and the result of Theorem 3.2 that

$$\max_i \|\nabla e_1^i\| \leq C,$$
$$\max_i \|\nabla e_2^i\| \leq C.$$

Take $\epsilon = \frac{1}{26}$ in (3.5.11), simplify, multiply both sides of the inequality by 2 and sum over all time levels $n \geq 1$ to obtain

$$\|s^{h,n+1}\|^2 + (h + Re^{-1})k \sum_{i=1}^{n} \|\nabla s^{h,i+1}\|^2 \leq \|s^{h,1}\|^2 \tag{3.5.12}$$
$$+\frac{C}{h+Re^{-1}} k \sum_{i=1}^{n} [\|\frac{\eta_1^{i+1} - 2\eta_1^i + \eta_1^{i-1}}{k^2}\|_{-1}^2$$
$$+(h+Re^{-1})^2 \|\nabla(\frac{\eta_1^{i+1} - \eta_1^i}{k})\|^2 + \|\nabla(\frac{\eta_1^{i+1} - \eta_1^i}{k})\|^2$$
$$+(h+Re^{-1})\|\nabla(\frac{\eta_1^{i+1} - \eta_1^i}{k})\|^4$$
$$+\inf_{q^h \in Q^h} \|\frac{p(t_{i+1}) - p(t_i)}{k} - \frac{q^{h,i+1} - q^{h,i}}{k}\|^2 + h^2 + k^2]$$
$$+\frac{C}{(h+Re^{-1})^2} k \sum_{i=1}^{n} (h+Re^{-1}) \|\nabla e_1^i\|^2 + Ck \sum_{i=1}^{n} \|e_1^i\|^2$$
$$+Ck \sum_{i=1}^{n} (C_{\nabla u} + \frac{C_u^2}{h+Re^{-1}} + \frac{1}{(h+Re^{-1})^3} \|\nabla e_1^i\|^4) \|s^{h,i+1}\|^2.$$

Consider the error decomposition (3.4.9). Take $\tilde{u}^i$ to be the L^2 projection of $u(t_i)$ into V^h, for all $i \geq 1$. Since the mesh nodes do not depend upon the time level, it follows from the approximation properties of X^h, Q^h and the regularity of u, p that

$$k \sum_{i=1}^{n} \|\frac{\eta_1^{i+1} - 2\eta_1^i + \eta_1^{i-1}}{k^2}\|_{-1}^2 \leq Ck \sum_{i=1}^{n} \|\frac{\eta_1^{i+1} - 2\eta_1^i + \eta_1^{i-1}}{k^2}\|^2 \leq Ch^{2m}, \tag{3.5.13}$$
$$k \sum_{i=1}^{n} \|\nabla(\frac{\eta_1^{i+1} - \eta_1^i}{k})\|^2 \leq Ch^{2m},$$
$$k \sum_{i=1}^{n} \|\nabla(\frac{\eta_1^{i+1} - \eta_1^i}{k})\|^4 \leq Ch^{4m},$$
$$k \sum_{i=1}^{n} \inf_{q^h \in Q^h} \|\frac{p(t_{i+1}) - p(t_i)}{k} - \frac{q^{h,i+1} - q^{h,i}}{k}\|^2 \leq Ch^{2m}.$$

Using (3.5.13) and (3.4.23), we derive from (3.5.12) that

$$\|s^{h,n+1}\|^2 + (h + Re^{-1})k \sum_{i=1}^{n} \|\nabla s^{h,i+1}\|^2 \leq \|s^{h,1}\|^2 \tag{3.5.14}$$
$$+C[h^{2m} + h^2 + k^2]$$
$$+Ck \sum_{i=1}^{n} (C_{\nabla u} + \frac{C_u^2}{h+Re^{-1}} + \frac{1}{(h+Re^{-1})^3} \|\nabla e_1^i\|^4) \|s^{h,i+1}\|^2.$$

Take $\tilde{u}^0 = u_0^s$ on the initial time level. This gives $\phi_1^{h,0} = 0$ and $e_1^0 = \eta_1^0 = u_0 - u_0^s$.

For the bound on $\|s^{h,1}\|^2 = \|\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}\|^2$, consider (3.5.1) at $n = 0$ and take $v = \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}$. This gives

$$
\begin{aligned}
&(\frac{e_1^1 - e_1^0}{k}, \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}) + (h + Re^{-1})(\nabla e_1^1, \nabla(\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) \\
&+ b^*(e_1^1, u(t_1), \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}) + b^*(u_1^{h,1}, e_1^1, \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}) \\
&- ((p(t_1) - p_1^{h,1}), \nabla \cdot (\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) \\
&= h(\nabla u(t_1), \nabla(\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) - k(\rho^1, \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}).
\end{aligned} \tag{3.5.15}
$$

Rewrite the left-hand side of (3.5.15) so that we could use the properties of the modified Stokes projection (3.2.3)

$$
\begin{aligned}
&(\frac{e_1^1 - e_1^0}{k}, \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}) + (h + Re^{-1})k(\nabla(\frac{e_1^1 - e_1^0}{k}), \nabla(\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) \\
&+ b^*(e_1^1, u(t_1), \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}) + b^*(u_1^{h,1}, e_1^1, \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}) \\
&+ (h + Re^{-1})(\nabla e_1^0, \nabla(\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) - ((p(t_1) - p_1^{h,1}), \nabla \cdot (\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) \\
&= h(\nabla u(t_1), \nabla(\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) - k(\rho^1, \frac{\phi_1^{h,1} - \phi_1^{h,0}}{k}).
\end{aligned} \tag{3.5.16}
$$

Since $\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k} \in V^h$ and $p_1^{h,1} \in Q^h$, it follows from the choice of initial approximation $\tilde{u}^0$ and from (3.2.3) that

$$
(h + Re^{-1})(\nabla e_1^0, \nabla(\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) - ((p(t_1) - p_1^{h,1}), \nabla \cdot (\frac{\phi_1^{h,1} - \phi_1^{h,0}}{k})) = 0. \tag{3.5.17}
$$

Hence, using the Cauchy-Schwarz and Young's inequalities, we derive from (3.5.16) and (3.5.17) that for any ϵ, $\epsilon_1 > 0$

$$\begin{aligned}
&\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&\leq \epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C\|\frac{\eta_1^1-\eta_1^0}{k}\|^2 \\
&+\epsilon(h+Re^{-1})k\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C(h+Re^{-1})k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2 \\
&+\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + Ck^2\|\rho^1\|^2 + \epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&+\frac{C}{h+Re^{-1}}h^2k\|\nabla(\frac{u(t_1)-u_0}{k})\|^2 + \epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + Ch^2\|\Delta u_0\|^2 \\
&+kb^*(\frac{e_1^1-e_1^0}{k}, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(e_1^0, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) \\
&+b^*(u(t_1), e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(\phi_1^{h,1}, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) \\
&-b^*(\eta_1^1, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}).
\end{aligned} \tag{3.5.18}$$

Using the fact that $\phi_1^{h,0}=0$, we obtain $b^*(\cdot, \phi_1^{h,1}, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) = 0$. The nonlinear terms in (3.5.18) are bounded by applying Cauchy-Schwarz and Young's inequalities. We obtain

$$\begin{aligned}
&kb^*(\frac{e_1^1-e_1^0}{k}, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(e_1^0, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) \\
&+b^*(u(t_1), e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(\phi_1^{h,1}, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) \\
&-b^*(\eta_1^1, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) \\
&\leq \frac{C}{h+Re^{-1}}k\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + \epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&+\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 + \frac{C}{h+Re^{-1}}k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2 \\
&+\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 + \frac{C}{(h+Re^{-1})^3}k\|\nabla\eta_1^1\|^4\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 \\
&+2\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C\|\eta_1^0\|^2 + C\|\nabla\eta_1^0\|^2 \\
&+2\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C\|\eta_1^1\|^2 + C\|\nabla\eta_1^1\|^2 \\
&+\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + Ch^{-2}\|\nabla\eta_1^1\|^4.
\end{aligned} \tag{3.5.19}$$

The inequalities (3.5.18)-(3.5.19) give

$$
\begin{aligned}
&\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&\leq (8\epsilon_1 + \frac{C}{h+Re^{-1}}k + \frac{C}{(h+Re^{-1})^3}k\|\nabla\eta_1^1\|^4)\|\frac{\phi_1^{h,1}-\phi_1^0}{k}\|^2 \\
&\quad +5\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^1-\phi_1^0}{k})\|^2 \\
&\quad +C[\|\frac{\eta_1^1-\eta_1^0}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2 \\
&\quad +k^2\|\rho^1\|^2 + \frac{1}{h+Re^{-1}}h^2k\|\nabla(\frac{u(t_1)-u(t_0)}{k})\|^2 + h^2\|\Delta u_0\|^2 \\
&\quad +\frac{1}{h+Re^{-1}}k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2 + \|\eta_1^1\|^2 + \|\nabla\eta_1^1\|^2 + h^{-2}\|\nabla\eta_1^1\|^4].
\end{aligned}
\tag{3.5.20}
$$

It follows from the approximation properties of X^h, Q^h that

$$
\begin{aligned}
&\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&\leq C[h^{2m}+h^2+k^2],
\end{aligned}
\tag{3.5.21}
$$

and the triangle inequality gives

$$
\begin{aligned}
&\|\frac{e_1^1-e_1^0}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{e_1^1-e_1^0}{k})\|^2 \\
&\leq C[h^{2m}+h^2+k^2].
\end{aligned}
\tag{3.5.22}
$$

Insert the bound on $\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2$ into (3.5.14). The restriction on the time step k allows to apply discrete Gronwall's lemma. This leads to

$$
\begin{aligned}
&\|\frac{\phi_1^{h,n+1}-\phi_1^{h,n}}{k}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{\phi_1^{h,i+1}-\phi_1^{h,i}}{k})\|^2 \\
&\leq C[h^{2m}+h^2+k^2].
\end{aligned}
\tag{3.5.23}
$$

Using the triangle inequality we obtain

$$
\begin{aligned}
&\|\frac{e_1^{n+1}-e_1^n}{k}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{e_1^{i+1}-e_1^i}{k})\|^2 \\
&\leq C[h^{2m}+h^2+k^2].
\end{aligned}
\tag{3.5.24}
$$

This result proves the first statement of Theorem 3.3.

For the bound on $\|\frac{\phi_2^{h,n+1}-\phi_2^{h,n}}{k}\|$ consider (3.4.24), $n \geq 1$. Following the proof above, take $v = \frac{\phi_2^{h,n+1}-\phi_2^{h,n}}{k} =: s_2^{h,n+1}$, then consider (3.4.24) at the previous time level, make the same choice $v = s_2^{h,n+1}$ and subtract the two equations. This leads to

$$
\begin{aligned}
& k(\frac{\eta_2^{n+1} - 2\eta_2^n + \eta_2^{n-1}}{k^2}, s_2^{h,n+1}) - (s_2^{h,n+1} - s_2^{h,n}, s_2^{h,n+1}) \\
& + (h + Re^{-1})k(\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k}), \nabla s_2^{h,n+1}) - (h + Re^{-1})k\|\nabla s_2^{h,n+1}\|^2 \\
& + b^*(e_2^{n+1}, u(t_{n+1}), s_2^{h,n+1}) + b^*(u_2^{h,n+1}, e_2^{n+1}, s_2^{h,n+1}) \\
& - b^*(e_2^n, u(t_n), s_2^{h,n+1}) - b^*(u_2^{h,n}, e_2^n, s_2^{h,n+1}) \\
& - k(\frac{(p(t_{n+1}) - p_2^{h,n+1}) - (p(t_n) - p_2^{h,n})}{k}, \nabla \cdot s_2^{h,n+1}) \\
& = hk(\nabla(\frac{e_1^{n+1} - e_1^n}{k}), \nabla s_2^{h,n+1}) - Ck^2(\rho_t^{n+1}, s_2^{h,n+1}).
\end{aligned} \tag{3.5.25}
$$

The nonlinear terms in (3.5.25) are bounded in the same manner as those in (3.4.24), with s^h, ϕ_1^h and η_1 replaced by s_2^h, ϕ_2^h and η_2. Using these bounds and the Cauchy-Schwarz and Young's inequalities, we obtain from (3.5.25) that

$$
\begin{aligned}
& \frac{\|s_2^{h,n+1}\|^2 - \|s_2^{h,n}\|^2}{2} + (h + Re^{-1})k\|\nabla s_2^{h,n+1}\|^2 \\
& \leq 13\epsilon(h + Re^{-1})k\|\nabla s_2^{h,n+1}\|^2 \\
& + \frac{C}{h + Re^{-1}}k\|\frac{\eta_2^{n+1} - 2\eta_2^n + \eta_2^{n-1}}{k^2}\|_{-1}^2 + C(h + Re^{-1})k\|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^2 \\
& + \frac{C}{h + Re^{-1}}k \inf_{q^h \in Q^h} \|\frac{p(t_{n+1}) - p(t_n)}{k} - \frac{q^{h,n+1} - q^{h,n}}{k}\|^2 \\
& + \frac{C}{h + Re^{-1}}k[\|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^2 + \|\nabla e_2^n\|^2 + \|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^2\|\nabla e_2^n\|^2] \\
& + Ck\|e_2^n\|^2\|\nabla e_2^n\|^2 + Ck\|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^4 \\
& + \frac{C}{h + Re^{-1}}k \cdot h^2\|\nabla(\frac{e_1^{n+1} - e_1^n}{k})\|^2 + \frac{C}{h + Re^{-1}}k \cdot k^2\|\rho_t^{n+1}\|_{-1}^2 \\
& + C(C_{\nabla u} + \frac{C_u^2}{h + Re^{-1}} + \frac{1}{(h + Re^{-1})^3}\|\nabla e_2^n\|^4)k\|s_2^{h,n+1}\|^2.
\end{aligned} \tag{3.5.26}
$$

It follows from the assumption $k \leq h$ and the result of Theorem 3.2 that $\max_i \|\nabla e_2^i\| \leq C$.

Take $\epsilon = \frac{1}{26}$ in (3.5.26), simplify, multiply both sides of (3.5.26) by 2 and sum over all time levels $n \geq 1$. The bound on $(h + Re^{-1})k\sum_{i=1}^{n} \|\nabla(\frac{e_1^{i+1}-e_1^i}{k})\|^2$ is obtained from (3.5.24). Using the approximation properties of X^h, Q^h and the triangle inequality, we obtain

$$\|s_2^{h,n+1}\|^2 + (h + Re^{-1})k\sum_{i=1}^{n} \|\nabla s_2^{h,i+1}\|^2 \leq \|s_2^{h,1}\|^2 + C[h^{2m} + h^4 + h^2k^2 + k^2] + Ck\sum_{i=1}^{n}(C_{\nabla u} + \frac{C_u^2}{h + Re^{-1}} + \frac{1}{(h + Re^{-1})^3}\|\nabla e_2^i\|^4)\|s_2^{h,i+1}\|^2. \quad (3.5.27)$$

The bound on $\|s_2^{h,1}\| = \|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|$ is obtained in the same manner as the bound on $\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|$. Consider (3.4.24) at $n = 0$ and take $v = s_2^{h,1} = \frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}$. This leads to

$$\|\frac{\phi_2^{h,1} - \phi_2^{h,0}}{k}\|^2 + (h + Re^{-1})k\|\nabla(\frac{\phi_2^{h,1} - \phi_2^{h,0}}{k})\|^2 \leq (8\epsilon_1 + \frac{C}{h + Re^{-1}}k + \frac{C}{(h + Re^{-1})^3}k\|\nabla\eta_2^1\|^4)\|\frac{\phi_2^{h,1} - \phi_2^{h,0}}{k}\|^2 + 5\epsilon(h + Re^{-1})k\|\nabla(\frac{\phi_2^{h,1} - \phi_2^{h,0}}{k})\|^2 + C[\|\frac{\eta_2^1 - \eta_2^0}{k}\|^2 + (h + Re^{-1})k\|\nabla(\frac{\eta_2^1 - \eta_2^0}{k})\|^2 + k^2\|\rho^1\|^2 + \frac{1}{h + Re^{-1}}h^2k\|\nabla(\frac{e_1^1 - e_1^0}{k})\|^2 + h^2\|\Delta\eta_1^0\|^2 + \frac{1}{h + Re^{-1}}k\|\nabla(\frac{\eta_2^1 - \eta_2^0}{k})\|^2 + \|\eta_2^1\|^2 + \|\nabla\eta_2^1\|^2 + h^{-2}\|\nabla\eta_2^1\|^4]. \quad (3.5.28)$$

Use the bound on $(h + Re^{-1})k\|\nabla(\frac{e_1^1-e_1^0}{k})\|^2$ from (3.5.22). It follows from the approximation properties of X^h, Q^h and the triangle inequality, that

$$\|\frac{\phi_2^{h,1} - \phi_2^{h,0}}{k}\|^2 + (h + Re^{-1})k\|\nabla(\frac{\phi_2^{h,1} - \phi_2^{h,0}}{k})\|^2 \leq C[h^{2m} + h^4 + h^2k^2 + k^2] \quad (3.5.29)$$

and

$$\|\frac{e_2^1 - e_2^0}{k}\|^2 + (h + Re^{-1})k\|\nabla(\frac{e_2^1 - e_2^0}{k})\|^2 \leq C[h^{2m} + h^4 + h^2k^2 + k^2]. \quad (3.5.30)$$

Insert the bound on $\|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|^2$ into (3.5.27). The restriction on the time step k allows to apply discrete Gronwall's lemma. This leads to

$$\|\frac{\phi_2^{h,n+1}-\phi_2^{h,n}}{k}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{\phi_2^{h,i+1}-\phi_2^{h,i}}{k})\|^2 \\ \leq C[h^{2m}+h^4+h^2k^2+k^2]. \tag{3.5.31}$$

Using the triangle inequality we obtain

$$\|\frac{e_2^{n+1}-e_2^{n}}{k}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{e_2^{i+1}-e_2^{i}}{k})\|^2 \\ \leq C[h^{2m}+h^4+h^2k^2+k^2]. \tag{3.5.32}$$

This completes the proof of Theorem 3.3. □

3.5.1 Stability of the Pressure

The stability of the pressure approximations p_1^h and p_2^h follows from the discrete inf-sup condition (3.2.1). The required bound on the time derivative of velocity is obtained under the assumptions of Theorem 3.3.

Theorem 3.4. *Let $f \in L^2(0,T;H^{-1}(\Omega))$. Let p_1^h and p_2^h satisfy the equations (3.1.5) and let the assumptions of Theorem 3.3 be satisfied. Then there exists a constant $C = C(T, f, h + Re^{-1}, u_0^s)$ s.t.*

$$k\sum_{i=0}^{n}\|p_1^{h,i+1}\| \leq C$$

and

$$k\sum_{i=0}^{n}\|p_2^{h,i+1}\| \leq C$$

Proof. Consider the first equation of (3.1.5). It holds true for $\forall v^h \in X^h$. Apply the Cauchy-Schwarz inequality, divide both sides of the inequality by $\|\nabla v^h\|$ and regroup the terms, leaving only the pressure term on the left-hand side. Using Lemma 3.1 gives

$$\frac{(p_1^{h,n+1}, \nabla\cdot v^h)}{\|\nabla v^h\|} \leq \|\frac{u_1^{h,n+1}-u_1^{h,n}}{k}\|_{-1} + (h+Re^{-1})\|\nabla u_1^{h,n+1}\| \\ +M\|\nabla u_1^{h,n+1}\|^2 + \|f(t_{n+1})\|_{-1}. \tag{3.5.33}$$

It follows from (3.5.33) and the discrete LBB condition (3.2.1) that

$$\beta^h \|p_1^{h,n+1}\| \le \|\frac{u_1^{h,n+1}-u_1^{h,n}}{k}\|_{-1} + (h+Re^{-1})\|\nabla u_1^{h,n+1}\| \\ +M\|\nabla u_1^{h,n+1}\|^2 + \|f(t_{n+1})\|_{-1}. \tag{3.5.34}$$

Decompose the first term on the right-hand side of (3.5.34), using the error decomposition and the triangle inequality. This gives

$$\|\frac{u_1^{h,n+1}-u_1^{h,n}}{k}\|_{-1} \le \|\frac{u(t_{n+1})-u(t_n)}{k}\|_{-1} + \|\frac{e_1^{n+1}-e_1^n}{k}\|_{-1}. \tag{3.5.35}$$

Multiply both sides of (3.5.34) by k and sum over the time levels. Using (3.5.35), we obtain

$$\beta^h k\sum_{i=0}^n \|p_1^{h,i+1}\| \le k\sum_{i=0}^n \|\frac{u(t_{i+1})-u(t_i)}{k}\|_{-1} + k\sum_{i=0}^n \|\frac{e_1^{i+1}-e_1^i}{k}\|_{-1} \\ +(h+Re^{-1})k\sum_{i=0}^n \|\nabla u_1^{h,i+1}\| + Mk\sum_{i=0}^n \|\nabla u_1^{h,i+1}\|^2 + k\sum_{i=0}^n \|f(t_{i+1})\|_{-1}. \tag{3.5.36}$$

The discrete Hölder's inequality gives

$$k\sum_{i=0}^n \|\nabla u_1^{h,i+1}\| = k\sum_{i=0}^n \|\nabla u_1^{h,i+1}\| \cdot 1 \\ \le (k\sum_{i=0}^n \|\nabla u_1^{h,i+1}\|^2)^{\frac{1}{2}} \cdot (k\sum_{i=0}^n 1^2)^{\frac{1}{2}} = C(k\sum_{i=0}^n \|\nabla u_1^{h,i+1}\|^2)^{\frac{1}{2}}. \tag{3.5.37}$$

Similarly,

$$k\sum_{i=0}^n \|\frac{e_1^{i+1}-e_1^i}{k}\|_{-1} \le C(k\sum_{i=0}^n \|\frac{e_1^{i+1}-e_1^i}{k}\|_{-1}^2)^{\frac{1}{2}} \le C(k\sum_{i=0}^n \|\frac{e_1^{i+1}-e_1^i}{k}\|^2)^{\frac{1}{2}}. \tag{3.5.38}$$

The stability bound on $k\sum_{i=0}^n \|\nabla u_1^{h,i+1}\|^2$ is obtained from Lemma 3.4. Using (3.5.38) and Theorem 3.3, it follows from (3.5.36) that

$$\beta^h k\sum_{i=0}^n \|p_1^{h,i+1}\| \le C[\frac{1}{h+Re^{-1}}\|u_0^s\|^2 + \frac{1}{(h+Re^{-1})^2}k\sum_{i=0}^n \|f(t_{i+1})\|_{-1}^2]. \tag{3.5.39}$$

Hence, if the forcing term f is sufficiently smooth, the pressure approximation p_1^h is stable.

Next, consider the second equation of (3.1.5). Apply the Cauchy-Schwarz inequality, divide both sides of the inequality by $\|\nabla v^h\|$ and regroup the terms. Following the outline of the proof above, we obtain

$$\beta^h k \sum_{i=0}^{n} \|p_2^{h,i+1}\| \leq k \sum_{i=0}^{n} \|\frac{u(t_{i+1}) - u(t_i)}{k}\|_{-1} + k \sum_{i=0}^{n} \|\frac{e_2^{i+1} - e_2^i}{k}\|_{-1} \tag{3.5.40}$$
$$+(h + Re^{-1})k \sum_{i=0}^{n} \|\nabla u_2^{h,i+1}\| + Mk \sum_{i=0}^{n} \|\nabla u_2^{h,i+1}\|^2$$
$$+hk \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\| + k \sum_{i=0}^{n} \|f(t_{i+1})\|_{-1}.$$

Use the discrete Hölder's inequality as in (3.5.37). It follows from Theorem 3.3 and Theorem 3.1 that

$$\beta^h k \sum_{i=0}^{n} \|p_2^{h,i+1}\| \leq C[\frac{1}{h + Re^{-1}} \|u_0^s\|^2 + \frac{1}{(h + Re^{-1})^2} k \sum_{i=0}^{n} \|f(t_{i+1})\|_{-1}^2]. \tag{3.5.41}$$

□

3.5.2 Error estimates for the pressure

In this section we estimate the error in pressure approximations $\|p(t_i) - p_1^{h,i}\|$ and $\|p(t_i) - p_2^{h,i}\|$. The results are obtained by using the inf-sup condition (3.2.1) and the result of Theorem 3.3. The main result of the section is

Theorem 3.5 (Pressure Convergence Rates). *Let $u, p, u_1^h, p_1^h, u_2^h, p_2^h$ satisfy the equations (3.4.4)-(3.4.6). Let the assumptions of Theorem 3.3 be satisfied. Then, for $\forall n \geq 0$*

$$k \sum_{i=0}^{n} \|p(t_{i+1}) - p_1^{h,i+1}\| \leq C[h^m + h + k] \tag{3.5.42}$$

and

$$k \sum_{i=0}^{n} \|p(t_{i+1}) - p_2^{h,i+1}\| \leq C[h^m + h^2 + hk + k]. \tag{3.5.43}$$

Proof. Decompose the error in the pressure approximation

$$p(t_{n+1}) - p_1^{h,n+1} = (p(t_{n+1}) - q^{h,n+1}) - (p_1^{h,n+1} - q^{h,n+1}) \\ =: \gamma_1^{n+1} - \psi_1^{h,n+1}, \tag{3.5.44}$$

where $q^{h,n+1}$ is some projection of $p(t_{n+1})$ into Q^h. Thus, $\psi_1^{h,n+1} \in Q^h$.

Divide both sides of (3.5.1) by $\|\nabla v\|$ and regroup the terms. Use the result of Lemma 3.1 and the Cauchy-Schwarz inequality to obtain

$$\frac{(\psi_1^{h,n+1}, \nabla \cdot v)}{\|\nabla v\|} \leq \|\frac{e_1^{n+1} - e_1^n}{k}\|_{-1} \\ +(h + Re^{-1})\|\nabla e_1^{n+1}\| + M\|\nabla u(t_{n+1})\|\|\nabla e_1^{n+1}\| + M\|\nabla u_1^{h,n+1}\|\|\nabla e_1^{n+1}\| \\ + \inf_{q^h \in Q^h} \|p(t_{n+1}) - q^{h,n+1}\| + h\|\nabla u(t_{n+1})\| + k\|\rho^{n+1}\|_{-1}. \tag{3.5.45}$$

Apply the discrete inf-sup condition. Multiply both sides of (3.5.45) by k and sum over all time levels. Decomposing $u_1^{h,n+1} = u(t_{n+1}) - e_1^{n+1}$ gives

$$\beta^h k \sum_{i=0}^n \|\psi_1^{h,i+1}\| \leq k \sum_{i=0}^n \|\frac{e_1^{i+1} - e_1^i}{k}\|_{-1} \\ +(h + Re^{-1})k \sum_{i=0}^n \|\nabla e_1^{i+1}\| + 2M \max_{0 \leq i \leq n+1} \|\nabla u(t_i)\| k \sum_{i=0}^n \|\nabla e_1^{i+1}\| \\ +Mk \sum_{i=0}^n \|\nabla e_1^{i+1}\|^2 + hk \sum_{i=0}^n \|\nabla u(t_{i+1})\| \\ +k \sum_{i=0}^n \inf_{q^h \in Q^h} \|p(t_{i+1}) - q^{h,i+1}\| + k \cdot k \sum_{i=0}^n \|\rho^{i+1}\|_{-1}. \tag{3.5.46}$$

Applying the discrete Hölder's inequality and the triangle inequality and using Theorem 3.3 and Theorem 3.2 proves (3.5.42).

Next, subtract (3.4.6) from (3.4.4). This gives for any $v \in X^h$

$$(\frac{e_2^{n+1} - e_2^n}{k}, v) + (h + Re^{-1})(\nabla e_2^{n+1}, \nabla v) \\ +b^*(e_2^{n+1}, u(t_{n+1}), v) + b^*(u_2^{h,n+1}, e_2^{n+1}, v) \\ -((p(t_{n+1}) - p_2^{h,n+1}), \nabla \cdot v) = h(\nabla e_1^{n+1}, \nabla v) - k(\rho^{n+1}, v). \tag{3.5.47}$$

Following the proof above, we obtain

$$\beta^h k \sum_{i=0}^{n} \|\psi_2^{h,n+1}\| \le k \sum_{i=0}^{n} \|\frac{e_2^{i+1} - e_2^i}{k}\|_{-1} \tag{3.5.48}$$
$$+(h + Re^{-1})k \sum_{i=0}^{n} \|\nabla e_2^{i+1}\| + 2M \max_{0\le i\le n+1} \|\nabla u(t_i)\| k \sum_{i=0}^{n} \|\nabla e_2^{i+1}\|$$
$$+Mk \sum_{i=0}^{n} \|\nabla e_2^{i+1}\|^2 + hk \sum_{i=0}^{n} \|\nabla e_1^{i+1}\|$$
$$+k \sum_{i=0}^{n} \inf_{q^h\in Q^h} \|p(t_{i+1}) - q^{h,i+1}\| + k \cdot k \sum_{i=0}^{n} \|\rho^{i+1}\|_{-1}.$$

Applying the discrete Hölder's inequality and the triangle inequality and using Theorem 3.3 and Theorem 3.2 leads to (3.5.43). □

3.6 COMPUTATIONAL TESTS

We test the convergence rates for a two-dimensional problem with a known exact solution. Consider the Chorin's vortex decay problem in the unit square $\Omega = (0,1)^2$. Take

$$u = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) exp(-2\pi^2 t/Re) \\ \sin(\pi x)\cos(\pi y) exp(-2\pi^2 t/Re) \end{pmatrix}, \tag{3.6.1}$$
$$p = -\frac{1}{4}(\cos(2\pi x) + \cos(2\pi y)) exp(-4\pi^2 t/Re),$$

and then the right-hand side f and initial condition u_0 are computed such that (3.6.1) satisfies (3.1.4).

In order to reduce the influence of the time discretization error, the time step is taken to be very small: $\Delta t = O(h^3)$.

For $Re = 1$, $Re = 100000$ and final time $T = 1/320$, the calculated convergence rates in Tables 3-6 confirm what is predicted by Theorem 3.2 for (P_2, P_1) discretization in space.

The convergence rate of $\|u - u_2^h\|_{L^2(0,T;L^2(\Omega))}$, predicted by Theorem 3.2, appears to be improvable in the case of moderate Reynolds' number. However, for the flow with sufficiently large Reynolds' number, the computed rates agree with those predicted by the theorem.

Table 3: AV approximation. $Re = 1$.

h	$\|\| u - u_1^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_1^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.000295318	-	0.0111291	-
1/8	5.77794E-05	2.35	0.00280563	1.99
1/16	2.28146E-05	1.34	0.000756655	1.89
1/32	0.000011235	1.02	0.000244007	1.63

Table 4: Correction Step approximation. $Re = 1$.

h	$\|\| u - u_2^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_2^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.00027283	-	0.0110347	-
1/8	3.56252E-05	2.94	0.00271592	2.02
1/16	4.55025E-06	2.97	0.000665649	2.03
1/32	5.77583E-07	2.98	0.000164297	2.02

Table 5: AV approximation. $Re = 100000$.

h	$\|\| u - u_1^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_1^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.000339015	-	0.00534596	-
1/8	7.39569E-05	2.2	0.00104601	2.35
1/16	3.19763E-05	1.21	0.00025783	2.02
1/32	1.62156E-05	0.98	9.19028E-05	1.49

Table 6: Correction Step approximation. $Re = 100000$.

h	$\|\| u - u_2^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_2^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.000300427	-	0.00525358	-
1/8	0.000040032	2.91	0.000975526	2.43
1/16	5.94795E-06	2.75	0.000190267	2.36
1/32	1.37357E-06	2.11	4.26364E-05	2.16

3.7 COMPARISON OF THE APPROACHES

For many years, it has been widely believed that (3.1.2) can be directly imported into implicit time discretizations of flow problems in the obvious way: discretize in time, given $u^h(t_{OLD})$, the quasistatic flow problem for $u^h(t_{NEW})$ is solved by DCM of the form (3.1.2). In this section we compare this approach and our method (3.1.5), applied to the same problem.

Apply both methods to the one-dimensional singularly perturbed equation

$$u_t - \epsilon u_{xx} + u_x = f$$

Our method leads to the coupled pair of equations

$$\begin{aligned}\frac{u_{1,i}^{n+1} - u_{1,i}^n}{\Delta t} - (\epsilon + h)\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2} + \frac{u_{1,i+1}^{n+1} - u_{1,i-1}^{n+1}}{2h} \\ = f_i^{n+1}, \\ \frac{u_{2,i}^{n+1} - u_{2,i}^n}{\Delta t} - (\epsilon + h)\frac{u_{2,i-1}^{n+1} - 2u_{2,i}^{n+1} + u_{2,i+1}^{n+1}}{h^2} + \frac{u_{2,i+1}^{n+1} - u_{2,i-1}^{n+1}}{2h} \\ = f_i^{n+1} - h\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2},\end{aligned}$$

whereas the other method gives

$$\begin{aligned}\frac{u_{1,i}^{n+1} - u_{2,i}^n}{\Delta t} - (\epsilon + h)\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2} + \frac{u_{1,i+1}^{n+1} - u_{1,i-1}^{n+1}}{2h} = f_i^{n+1}, \\ \frac{u_{2,i}^{n+1} - u_{2,i}^n}{\Delta t} - (\epsilon + h)\frac{u_{2,i-1}^{n+1} - 2u_{2,i}^{n+1} + u_{2,i+1}^{n+1}}{h^2} + \frac{u_{2,i+1}^{n+1} - u_{2,i-1}^{n+1}}{2h} \\ = f_i^{n+1} - h\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2}.\end{aligned}$$

Consider $0 \leq x \leq 1$, $0 \leq t \leq 1$, $u(0,t) = 0$, $u(1,t) = 25$, $u(x,0) = 0$, $f(x,t) = tx^2 + 20x$, $\epsilon = 0.000001$.

As one could have predicted, if we let the time interval be fixed and reasonably big ($\Delta t = 0.1$) and decrease the space-interval, both methods give almost the same results, since they mainly differ in treating the time-derivative. But if we fix Δh and monotonically decrease Δt, we immediately see the oscillations of the solution, obtained by the alternative method.

Figures Fig.3-Fig.6 show the solution, obtained by our method (denoted by the solid line) and the solution, obtained by the alternative approach (dashed line on the graphs). The spacial mesh is fixed (with $\Delta h = 0.01$) and the time step Δt decreases to zero (see the captions).

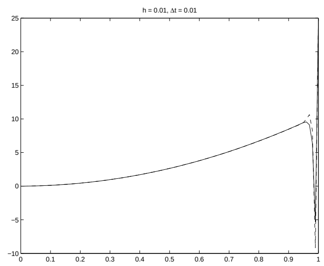

Figure 3: $\Delta t = 0.01$

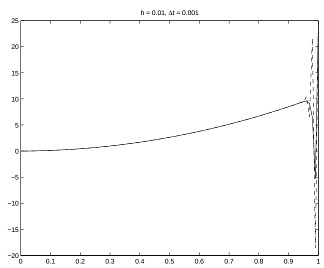

Figure 4: $\Delta t = 0.001$

As we see, although this alternative approach uses the Correction Step approximation of the true solution on each time level (instead of the AV approximation), the results are

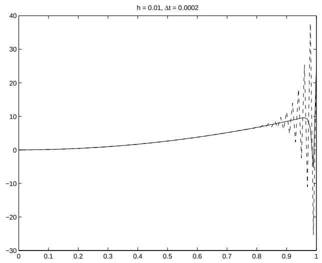

Figure 5: $\Delta t = 0.0002$

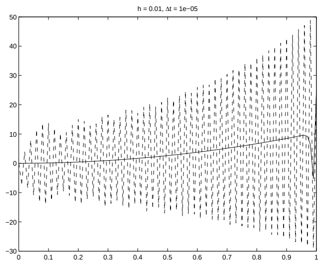

Figure 6: $\Delta t = 0.00001$

worse even for a simple one-dimensional problem with the bounded domain and bounded right-hand side.

We conclude the comparison of the methods by applying them to the Navier-Stokes equations in $\mathbb{R}^2$. Consider the Chorin's vortex decay problem in the square $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$ with

$$f(x,y,t) = \begin{pmatrix} \frac{1}{2}\pi \sin(2\pi x) exp(-4\pi^2 t/Re) \\ \frac{1}{2}\pi \sin(2\pi y) exp(-4\pi^2 t/Re) \end{pmatrix} \tag{3.7.1}$$

and $Re = 10^5$. The final time is taken to be $T = 1/10$ and the mesh diameter is fixed at $h = 1/4$. As the time step Δt is decreased, the error estimates, obtained by the DCM (3.1.5), do not change - see the following table.

At the same time, applying the alternative approach we obtain

Hence, in the alternative approach the error increases as Δt tends to zero.

Table 7: DCM. $Re = 100000$, $T = 1/10$, $h = 1/4$

Δt	$\|\| u - u_2^h \|\|_{L^2(0,T;L^2(\Omega))}$	$\|\| u - u_2^h \|\|_{L^2(0,T;H^1(\Omega))}$
10^{-3}	0.00682	0.0585
10^{-4}	0.00682	0.0585
10^{-5}	0.00682	0.0585

Table 8: ALTERNATIVE APPROACH. $Re = 100000$, $T = 1/10$, $h = 1/4$

Δt	$\|\| u - u_2^h \|\|_{L^2(0,T;L^2(\Omega))}$	$\|\| u - u_2^h \|\|_{L^2(0,T;H^1(\Omega))}$
10^{-3}	0.01019	0.1104
10^{-4}	0.01449	0.1759
10^{-5}	0.01582	0.2076

We have seen from Figures Fig.3-Fig.6 that the alternative approach gives worse results than the DCM, when solving the convection diffusion equation. Comparing the Tables 7-8, we conclude that the Defect Correction Method (3.1.5) also behaves better, when applied to a more difficult Navier-Stokes problem.

4.0 A DEFECT CORRECTION METHOD FOR THE EVOLUTIONARY CONVECTION DIFFUSION PROBLEM WITH INCREASED TIME ACCURACY

4.1 INTRODUCTION

Consider the evolutionary convection diffusion problem: find $\mathbf{u} : \Omega \times [0,T] \;\rightarrow \mathbb{R}^d\,(d = 2,3)$ such that

$$\mathbf{u}_t - \epsilon\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{u} + g\mathbf{u} = \mathbf{f},\;\text{for }\mathbf{x}\;\in\;\Omega,\,0 < t \leq T \tag{4.1.1}$$
$$\mathbf{u} = 0,\;\text{on }\partial\Omega,\,\text{for }0 < t \leq T,$$

where $\mathbf{u}$ is the velocity field, ϵ is a diffusion coefficient, g is an absorbtion/reaction coefficient, and $\mathbf{f}$ is a forcing function.

In the problems with high Peclet number (i.e. $\epsilon \ll 1$) some iterative solvers fail to converge to a solution of (4.1.1). We propose a certain Defect Correction Method (DCM), that is stable, computes a solution to (4.1.1) for any ϵ with high space and time accuracy, and is computationally attractive.

The general theory of Defect Correction Methods is presented, e.g., in Bohmer, Hemker, Stetter [BHS]. In the late 1970's Hemker (Bohmer, Stetter, Heinrichs and others) discovered that DCM, properly interpreted, is good also for nearly singular problems. Examples for which this has been successful include equilibrium Euler equations (Koren, Lallemand [LK93]), high Reynolds number problems (Layton, Lee, Peterson [LLP02]), viscoelastic problems (Ervin, Lee [EL06]).

There has been an extensive study and development of the DC methods for equilibrium flow problems, see e.g. Hemker[Hem82], Koren[K91], Heinrichs[Hei96], Layton, Lee, Peterson[LLP02], Ervin, Lee[EL06]. On the other hand, there is a parallel development of DCM's, for initial value problems in which no spacial stabilization is used, but DCM is used to increase the accuracy of the time discretization. This work contains no reports of instabilities: see, e.g., Heywood, Rannacher[HR90], Hemker, Shishkin[HSS], Minion[M04], Bourlioux, Layton, Minion [BLM03].

It was shown in [L07] that the natural idea of time stepping combined with the DCM in space for the associated quasi-equilibrium problem gives an oscillatory computed solution of poor quality. Another DC method was introduced for an evolutionary PDE, that was proven to be stable and accurate.

The method presented in this chapter, is the modification (aiming at higher accuracy in time) of the DCM for the evolutionary PDEs, presented in [L07]. Compared to the method in [L07], the right hand side of the system is modified in the correction step, resulting in higher time accuracy with no extra computational cost.

The method proceeds as follows: first we compute the AV approximation $\mathbf{u}_1^h \in X^h$ via

$$L^h_{\epsilon+h}(\mathbf{u}_1^h) = \mathbf{f},$$

where

$$L^h_{\epsilon+h}(\mathbf{u}^h) = \mathbf{u}_t^h - (h+\epsilon)\Delta \mathbf{u}^h + \mathbf{b}\cdot\nabla\mathbf{u}^h + g\mathbf{u}^h.$$

The accuracy of the approximation is then increased by the correction step: compute $\mathbf{u}_2^h \in X^h$, satisfying

$$L^h_{\epsilon+h}(\mathbf{u}_2^h) - L^h_{\epsilon+h}(\mathbf{u}_1^h) = \mathbf{f} - L^h_{\epsilon}(\mathbf{u}_1^h) + B(\mathbf{u}_1^h).$$

Here $B(\cdot)$ is the time difference operator, that increases the accuracy of the discrete time difference for $\mathbf{u}_t$.

The Crank-Nicolson time discretization, combined with the two-step defect correction method in space leads to the following system of equations for $\mathbf{u}_1^{h,n+1}, \mathbf{u}_2^{h,n+1} \in \mathbf{X}^h, \forall \mathbf{v}^h \in \mathbf{X}^h$ at $t = t_{n+1}$, $n \geq 0$, with $k := \Delta t = t_{i+1} - t_i$

$$(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}, \mathbf{v}^h) + (h+\epsilon)(\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}), \nabla \mathbf{v}^h) + (\mathbf{b} \cdot \nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}), \mathbf{v}^h) \tag{4.1.2a}$$
$$+g(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}, \mathbf{v}^h) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}^h),$$
$$(\frac{\mathbf{u}_2^{h,n+1} - \mathbf{u}_2^{h,n}}{k}, \mathbf{v}^h) + (h+\epsilon)(\nabla(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}), \nabla \mathbf{v}^h) + (\mathbf{b} \cdot \nabla(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}), \mathbf{v}^h) \tag{4.1.2b}$$
$$+g(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}, \mathbf{v}^h) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}^h) + h(\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}), \nabla \mathbf{v}^h) + B_n(\mathbf{u}_1^h, \mathbf{v}^h),$$

where

$$B_0(u,v) = \frac{1}{12}k^2\left(\frac{u^4 - 5u^3 + 9u^2 - 7u^1 + 2u^0}{k^3}, v\right) - \frac{1}{16}k^2\left(\frac{f(t_3) - 5f(t_2) + 7f(t_1) - 3f(t_0)}{k^2}, v\right), \tag{4.1.3a}$$

$$B_n(u,v) = -\frac{1}{12}k^2\left(\frac{u^{n+2} - 3u^{n+1} + 3u^n - u^{n-1}}{k^3}, v\right) + \frac{1}{16}k^2\left(\frac{f(t_{n+2}) - f(t_{n+1}) - f(t_n) + f(t_{n-1})}{k^2}, v\right), \text{ for } n = 1, ..., N-2, \tag{4.1.3b}$$

$$B_{N-1}(u,v) = -\frac{1}{12}k^2\left(\frac{2u^N - 7u^{N-1} + 9u^{N-2} - 5u^{N-3} + u^{N-4}}{k^3}, v\right) + \frac{1}{16}k^2\left(\frac{3f(t_N) - 7f(t_{N-1}) + 5f(t_{N-2}) - f(t_{N-3})}{k^2}, v\right). \tag{4.1.3c}$$

Depending on the current time level, we vary the templates - this demonstrates the resilience of the method. However, the condition $N \geq 4$ needs to be satisfied, where $N = T/k$ is the number of time levels.

Note that the operator B is chosen so that for any n the true solution $\mathbf{u}$ satisfies

$$(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k}, \mathbf{v}^h) + (h+\epsilon)(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}), \nabla \mathbf{v}^h) + (\mathbf{b} \cdot \nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}), \mathbf{v}^h)$$
$$+g(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}^h) + h(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}), \nabla \mathbf{v}^h) + B_n(\mathbf{u}, \mathbf{v}^h)$$
$$+k^4(\mathbf{u}^{(5)}(t_{n+\theta}), \mathbf{v}^h),$$

for some $\theta \in]0,1[$.

Also, the only extra computational cost for the correction step is due to the storage of a few vectors $\mathbf{u}_1^{h,i}$. The Crank-Nicolson scheme is used for computing both $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$, but the time accuracy of the approximate solution $\mathbf{u}_2^h$ is increased to be $O(k^4)$.

The method is proven to be unconditionally stable over the finite time; it is also stable over all time under the assumption $g - \frac{1}{2}\nabla \cdot \mathbf{b} \geq \beta > 0$.

In section 4.2 we briefly describe notation used and a few established results. Stability of the method is proven in Section 4.3. We conclude with the numerical results, proving the error estimates for the method - this is presented in Section 4.4.

4.2 NOTATION AND PRELIMINARIES

We begin with a few definitions, assumptions, and forms used, and conclude the section with a statement of the method to be studied. The variational formulation of (4.1.1) is naturally stated in

$$\mathbf{X} := H_0^1(\Omega)^d = \{\mathbf{v} : \Omega \to \mathbb{R}^d, \mathbf{v} \in L^2(\Omega)^d, \nabla \mathbf{v} \in L^2(\Omega)^d, \mathbf{v} = 0 \text{ on } \partial\Omega\}.$$

We use the standard L^2 norm, $\|\cdot\|$, and the usual norm on the Sobolev space H^k, namely $\|\cdot\|_k$.

We make several common assumptions.

Remark 4.1. *There exists a constant β such that*

$$g - \frac{1}{2}\nabla \cdot \mathbf{b} \geq \beta > 0.$$

The method is proven to be stable over a finite time even if the Assumption 4.1 doesn't hold. If it does, the method is stable over all time.

We shall assume that the velocity finite element spaces $\mathbf{X}^h \subset \mathbf{X}$ are conforming and have typical approximation properties of finite element spaces commonly in use. Namely, we take $\mathbf{X}^h$ to be spaces of continuous piecewise polynomials of degree k, with $k \geq 1$.

The interpolating properties of $\mathbf{X}^h$ are given by the following assumption.

Remark 4.2. *For any function* $\mathbf{u} \in \mathbf{X}$

$$\inf_{\chi \in \mathbf{X}^h} \left\{ \|\mathbf{u} - \chi\| + h\|\nabla(\mathbf{u} - \chi)\| \right\} \quad \leq \quad ch^{r+1}\|\mathbf{u}\|_{r+1}, \quad 1 \leq r \leq k.$$

We conclude the preliminaries by formulating the discrete Gronwall's lemma, see, e.g. [HR90]

Lemma 4.1. *Let* $k, B,$ *and* $a_\mu, b_\mu, c_\mu, \gamma_\mu$*, for integers* $\mu \geq 0$*, be nonnegative numbers such that:*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B \text{ for } n \geq 0.$$

Suppose that $k\gamma_\mu < 1$ *for all* μ*, and set* $\sigma_\mu = (1 - k\gamma_\mu)^{-1}$*. Then*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq e^{k\sum_{\mu=0}^{n}\sigma_\mu\gamma_\mu} \cdot [k\sum_{\mu=0}^{n} c_\mu + B].$$

4.3 STABILITY OF THE METHOD

In this section we prove the unconditional stability of the discrete artificial viscosity approximation $\mathbf{u}_1^h$ and use this result to prove stability of the higher order approximation $\mathbf{u}_2^h$. The approximations $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ are shown to be bounded uniformly in ϵ.

Theorem 4.1. *Let* $\mathbf{f} \in L^2(0, T; L^2(\Omega))$*. Let* $g - \frac{1}{2}\nabla \cdot \mathbf{b} \geq \beta > -\infty$*. If* $\beta < 0$*, let the length of the time step satisfy* $k|\beta| < 1/4$*. Then the approximation* $\mathbf{u}_1^h$*, satisfying (4.1.2a), is stable over the finite time* $T < \infty$*. Specifically, there exist positive constants* C_1, C_2 *such that for any* $n \leq N - 1$

$$\|\mathbf{u}_1^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h + \epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2})\|^2 \qquad (4.3.1)$$
$$\leq e^{C_1 T}\left(\|\mathbf{u}_1^{h,0}\|^2 + \frac{1}{C_2}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2\right).$$

If Assumption 4.1 is satisfied, then $\mathbf{u}_1^h$ *is stable over all time and*

$$\|\mathbf{u}_1^{h,n+1}\|^2 + \beta k \sum_{i=0}^{n} \|\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2}\|^2 + k \sum_{i=0}^{n} (h+\epsilon) \|\nabla(\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2})\|^2 \qquad (4.3.2)$$
$$\leq \|\mathbf{u}_1^{h,0}\|^2 + \frac{1}{\beta} k \sum_{i=0}^{n} \|\mathbf{f}(t_{i+1/2})\|^2.$$

Proof. Take $\mathbf{v}^h = \frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2} \in \mathbf{X}^h$ in (4.1.2a). Apply Green's theorem to the last two terms on the left hand side (with $\mathbf{u}_1^h = 0$ on $\partial\Omega$) and use the assumption $g - \frac{1}{2}\nabla \cdot \mathbf{b} \geq \beta$. The Cauchy-Schwartz and Young's inequalities, applied to the left hand side, yield

$$\frac{\|\mathbf{u}_1^{h,n+1}\|^2 - \|\mathbf{u}_1^{h,n}\|^2}{2k} + (h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2})\|^2 \qquad (4.3.3)$$
$$+\beta\|\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\|^2 \leq (\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}).$$

If Assumption 4.1 is satisfied, we bound the right hand side of (4.3.3) by

$$|(\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2})| \leq \frac{1}{2\beta}\|\mathbf{f}(t_{n+1/2})\|^2 + \frac{\beta}{2}\|\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\|^2. \qquad (4.3.4)$$

Multiply (4.3.3) by $2k$ and sum over the time levels $i = 0, .., n+1$. Using (4.3.4), we obtain

$$\|\mathbf{u}_1^{h,n+1}\|^2 + k \sum_{i=0}^{n} 2(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2})\|^2 \qquad (4.3.5)$$
$$+k \sum_{i=0}^{n} \beta \|\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2}\|^2 \leq \|\mathbf{u}_1^{h,0}\|^2 + k \sum_{i=0}^{n} \frac{1}{\beta}\|\mathbf{f}(t_{i+1/2})\|^2,$$

which proves stability over all time (provided that Assumption 4.1 is satisfied).

If $g - \frac{1}{2}\nabla \cdot \mathbf{b} \geq \beta$ with $-\infty < \beta < 0$, then we bound the right hand side of (4.3.3) by

$$|(\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2})| \leq \frac{1}{4|\beta|}\|\mathbf{f}(t_{n+1/2})\|^2 + |\beta|\|\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\|^2. \qquad (4.3.6)$$

It follows from (4.3.3),(4.3.6) and the triangle inequality that

$$\frac{\|\mathbf{u}_1^{h,n+1}\|^2 - \|\mathbf{u}_1^{h,n}\|^2}{2k} + (h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2})\|^2 \qquad (4.3.7)$$
$$\leq \frac{1}{4|\beta|}\|\mathbf{f}(t_{n+1/2})\|^2 + |\beta|(\|\mathbf{u}_1^{h,n+1}\|^2 + \|\mathbf{u}_1^{h,n}\|^2).$$

Multiply ([4.3.7]) by $2k$ and sum over the time levels $i = 0, .., n+1$. Under the condition $k|\beta| < 1/4$ the discrete Gronwall's lemma yields

$$\|\mathbf{u}_1^{h,n+1}\|^2 + k\sum_{i=0}^{n} 2(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1}+\mathbf{u}_1^{h,i}}{2})\|^2 \\ \leq e^{cT}\left(\|\mathbf{u}_1^{h,0}\|^2 + k\sum_{i=0}^{n}\frac{1}{2|\beta|}\|\mathbf{f}(t_{i+1/2})\|^2\right),$$

with $c > 0$.

□

We now proceed to the proof of stability of $\mathbf{u}_2^h$. It follows from (4.1.3) that $|B_i(\mathbf{u}_1^h, \mathbf{v}^h)| \leq C\|\mathbf{v}^h\|$ for any $\mathbf{v}^h \in \mathbf{X}^h$, provided that the time difference $\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i}}{k}$ is bounded for any $i = 0, ..., N-1$. Hence, we begin by establishing the bound for $\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i}}{k}$.

Lemma 4.2. *Let $\mathbf{u}_1^{h,0} \in H^2(\Omega)$, $\mathbf{f}_t \in L^2(0,T;L^2(\Omega))$. Let $g - \frac{1}{2}\nabla\cdot\mathbf{b} \geq \beta > -\infty$. If $\beta < 0$, let the length of the time step satisfy $k|\beta| < 1/4$. Then $\frac{\mathbf{u}_1^{h,n+1}-\mathbf{u}_1^{h,n}}{k}$ is bounded for the finite time $T < \infty$. Specifically, there exist positive constants c, $C = C(\mathbf{b}, g, \mathbf{f}, \mathbf{u}_1^{h,0})$ such that*

$$\|\frac{\mathbf{u}_1^{h,n+1}-\mathbf{u}_1^{h,n}}{k}\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i-1}}{k})\|^2 \\ \leq e^{cT}\left(C + k\sum_{i=1}^{n}\frac{1}{2\beta}\|\frac{\mathbf{f}(t_{i+1/2})-\mathbf{f}(t_{i-1/2})}{k}\|^2\right).$$

If Assumption 4.1 is satisfied, then $\frac{\mathbf{u}_1^{h,n+1}-\mathbf{u}_1^{h,n}}{k}$ is bounded over all time: there exists $C = C(\mathbf{b}, g, \mathbf{f}, \mathbf{u}_1^{h,0})$ such that

$$\|\frac{\mathbf{u}_1^{h,n+1}-\mathbf{u}_1^{h,n}}{k}\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i-1}}{k})\|^2 \\ +k\sum_{i=1}^{n}\frac{1}{4}\beta\|\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i-1}}{k}\|^2 \leq C + k\sum_{i=1}^{n}\frac{1}{\beta}\|\frac{\mathbf{f}(t_{i+1/2})-\mathbf{f}(t_{i-1/2})}{k}\|^2.$$

Proof. Consider (4.1.2a) at any time level $n \geq 1$. Take $\mathbf{v}^h = \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}$ and make the same choice for $\mathbf{v}^h$ in (4.1.2a) at the previous time level. Subtracting the resulting equations leads to

$$\begin{aligned}
&(\frac{\mathbf{u}_1^{h,n+1} - 2\mathbf{u}_1^{h,n} + \mathbf{u}_1^{h,n-1}}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}) + \frac{1}{2}(h+\epsilon)k\|\nabla(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k})\|^2 \\
&+\frac{1}{2}k(\mathbf{b}\cdot\nabla(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}), \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}) + \frac{1}{2}kg\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\|^2 \\
&= k(\frac{\mathbf{f}(t_{n+1/2}) - f(t_{n-1/2})}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}).
\end{aligned}$$

Apply Green's theorem to the last two terms on the left hand side (with $\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k} = 0$ on $\partial\Omega$) and use the assumption $g - \frac{1}{2}\nabla \cdot b \geq \beta$. Rewrite the first term on the left hand side, using the identity

$$(\frac{\mathbf{u}_1^{h,n+1} - 2\mathbf{u}_1^{h,n} + \mathbf{u}_1^{h,n-1}}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}) = \|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\|^2 - \|\frac{\mathbf{u}_1^{h,n} - \mathbf{u}_1^{h,n-1}}{k}\|^2.$$

This yields

$$\begin{aligned}
&\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\|^2 - \|\frac{\mathbf{u}_1^{h,n} - \mathbf{u}_1^{h,n-1}}{k}\|^2 + \frac{1}{2}(h+\epsilon)k\|\nabla(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k})\|^2 \\
&+\frac{1}{2}k\beta\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\|^2 \leq k(\frac{\mathbf{f}(t_{n+1/2}) - \mathbf{f}(t_{n-1/2})}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}).
\end{aligned} \tag{4.3.8}$$

Sum (4.3.8) over all time levels $i = 1, .., n$ and consider the cases $\beta > 0$ and $-\infty < \beta < 0$ separately, as in the proof of Theorem 4.1. If the Assumption 4.1 holds, this yields

$$\begin{aligned}
&\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i-1}}{k})\|^2 \\
&+k\sum_{i=1}^{n}\frac{1}{4}\beta\|\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i-1}}{k}\|^2 \leq \|\frac{\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0}}{k}\|^2 + k\sum_{i=1}^{n}\frac{1}{\beta}\|\frac{\mathbf{f}(t_{i+1/2}) - \mathbf{f}(t_{i-1/2})}{k}\|^2.
\end{aligned} \tag{4.3.9}$$

If $-\infty < \beta < 0$ and $k|\beta| < 1/4$, it follows from the discrete Gronwall's lemma that

$$\begin{aligned}
&\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i-1}}{k})\|^2 \\
&\leq e^{cT}\left(\|\frac{\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0}}{k}\|^2 + k\sum_{i=1}^{n}\frac{1}{2\beta}\|\frac{\mathbf{f}(t_{i+1/2}) - \mathbf{f}(t_{i-1/2})}{k}\|^2\right).
\end{aligned} \tag{4.3.10}$$

To complete the proof, we need a bound on $\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2$. Consider (4.1.2a) at $n = 0$ and take $v^h = \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}$. This gives

$$\begin{aligned}\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2 + (h+\epsilon)(\nabla(\frac{\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0}}{2}), \nabla(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}))\\ +(\mathbf{b}\cdot\nabla(\frac{\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0}}{2}), \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}) + g(\frac{\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0}}{2}, \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k})\\ = (\mathbf{f}(t_{1/2}), \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}).\end{aligned} \tag{4.3.11}$$

Using the identity $\frac{\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0}}{2} = \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{2} + \mathbf{u}_1^{h,0}$, we can rewrite the last three terms in the left hand side of (4.3.11). Applying Green's theorem as in the proofs above, yields

$$\begin{aligned}\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2 + (h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k})\|^2 + (h+\epsilon)(\Delta\mathbf{u}_1^{h,0}, \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k})\\ +\frac{1}{2}k\beta\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2 + (\mathbf{b}\cdot\nabla\mathbf{u}_1^{h,0}, \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}) + g(\mathbf{u}_1^{h,0}, \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k})\\ \leq (\mathbf{f}(t_{1/2}), \frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}).\end{aligned}$$

The Cauchy-Schwartz and Young's inequalities give

$$\begin{aligned}\frac{1}{2}\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2 + (h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k})\|^2 + \frac{1}{2}k\beta\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2\\ \leq C,\end{aligned}$$

where $C = \|\mathbf{b}\cdot\nabla\mathbf{u}_1^{h,0}\|^2 + 2g^2\|\mathbf{u}_1^{h,0}\|^2 + 2(h+\epsilon)^2\|\Delta\mathbf{u}_1^{h,0}\|^2 + \|\mathbf{f}(t_{1/2})\|^2 < \infty$. Hence, if Assumption 4.1 is satisfied, or if $-\infty < \beta < 0$ and $k|\beta| < 1/4$, we obtain the bound

$$\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2 + (h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k})\|^2 \leq C < \infty. \tag{4.3.12}$$

Inserting the bound on $\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\|^2$ into (4.3.9) and (4.3.10) completes the proof.

□

The unconditional stability of $\mathbf{u}_2^h$ (uniform in ϵ) follows from Theorem 4.1 and Lemma 4.2. The last term in the right hand side of (4.1.2b) is bounded by means of Lemma 4.2.

Theorem 4.2. *Let the assumptions of Theorem 4.1 and Lemma 4.2 be satisfied.*

Let $g-\frac{1}{2}\nabla\cdot\mathbf{b} \geq \beta > -\infty$. If $\beta < 0$, let the length of the time step satisfy $k|\beta| < 1/4$. Then the approximation $\mathbf{u}_2^h$, satisfying (4.1.2b), is stable over the finite time $T < \infty$. Specifically, there exist positive constants c_1, C_2, C_3 such that for any $n \leq N-1$

$$\|\mathbf{u}_2^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_2^{h,i+1}+\mathbf{u}_2^{h,i}}{2})\|^2$$
$$\leq e^{c_1 T}\left(C_3 + \|\mathbf{u}_2^{h,0}\|^2 + \|\mathbf{u}_1^{h,0}\|^2 + \frac{1}{C_2}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2\right).$$

If Assumption 4.1 is satisfied, then $\mathbf{u}_1^h$ is stable over all time and

$$\|\mathbf{u}_2^{h,n+1}\|^2 + \beta k\sum_{i=0}^{n}\|\frac{\mathbf{u}_2^{h,i+1}+\mathbf{u}_2^{h,i}}{2}\|^2 + k\sum_{i=0}^{n}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_2^{h,i+1}+\mathbf{u}_2^{h,i}}{2})\|^2$$
$$\leq C + \|\mathbf{u}_2^{h,0}\|^2 + \|\mathbf{u}_1^{h,0}\|^2 + \frac{1}{\beta}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2,$$

with $C = C(\mathbf{b}, g, \mathbf{f}, \mathbf{u}_1^{h,0})$.

Proof. The proof resembles the proof of Theorem 4.1. Take $\mathbf{v}^h = \frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2} \in X^h$ in (4.1.2b). Apply Green's theorem to the last two terms on the left hand side. This gives

$$\frac{\|\mathbf{u}_2^{h,n+1}\|^2 - \|\mathbf{u}_2^{h,n}\|^2}{2k} + (h+\epsilon)\|\nabla(\frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2})\|^2 + \beta\|\frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2}\|^2$$
$$\leq (\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2}) + h(\nabla(\frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2}), \nabla(\frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2}))$$
$$+B_n(\mathbf{u}_1^h, \frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2}). \tag{4.3.13}$$

It is easy to verify that for any n

$$|B_n(\mathbf{u}_1^h, \frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2})| \leq C + \frac{1}{2}|\beta|\|\frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2}\|^2, \tag{4.3.14}$$

where $C = \frac{1}{|\beta|}\max_{0\leq i\leq N-1}\|\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i}}{k}\|^2 + \frac{1}{|\beta|}k^2\max_{0\leq i\leq N-1}\|\frac{\mathbf{f}(t_{i+1})-\mathbf{f}(t_i)}{k}\|^2.$

It also follows from Lemma 4.2 that this constant is finite, $C < \infty$.

Using Cauchy-Schwartz and Young's inequalities gives

$$h|(\nabla(\frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2}),\nabla(\frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2}))| \le \frac{1}{2}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_2^{h,n+1}+\mathbf{u}_2^{h,n}}{2})\|^2 + \frac{h^2}{2(h+\epsilon)^2}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2})\|^2. \tag{4.3.15}$$

Multiply (4.3.13) by $2k$, sum over the time levels and use (4.3.14),(4.3.15). Theorem 4.1 gives the bound on $k\sum_{i=0}^{n}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1}+\mathbf{u}_1^{h,i}}{2})\|^2$. The cases when Assumption 4.1 holds and when $-\infty<\beta<0$, $k|\beta|<1/4$ are treated as in the proof of Theorem 4.1.

□

Thus, the method is unconditionally stable for all time, provided that the Assumption 4.1 is satisfied. The approximate solutions $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ are bounded uniformly in h and ϵ. If the condition $g-\frac{1}{2}\nabla\cdot\mathbf{b}\ge\beta$ is satisfied with $-\infty<\beta<0$, then the assumption $k|\beta|<1/4$ is needed to conclude stability and uniform boundedness of the approximate solutions over a finite time.

4.4 COMPUTATIONAL RESULTS

Based on the results of [L07] (and the general theory of Defect Correction Methods), we are expecting to obtain the following error estimates:

$$\|\mathbf{u}-\mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))} \le C(h+k^2),$$
$$\|\mathbf{u}-\mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))} \le C(h\|\mathbf{u}-\mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}+k^4) \le C(h^2+hk^2+k^4).$$

Consider the following transport problem in $\Omega=[0,1]\times[0,1]$: find u satisfying (4.1.1) with $\epsilon=10^{-5}$, $b=(1,1)^T$, $g=1$ and $f=[(2+2\epsilon\pi^2)\sin(\pi x)\sin(\pi y)+\pi\sin(\pi x+\pi y)]e^t$. This problem has a solution $u=\sin(\pi x)\sin(\pi y)e^t$.

The results presented in the following tables are obtained by using the software $FreeFEM++$. In order to draw conclusions about the convergence rate, we take $k=h$ and $k=\sqrt{h}$. Note that the method needs the number of time steps $N\ge 4$.

Table 9: Error estimates, $\epsilon = 10^{-5}$, $T = 1$, $k = h$

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/4	0.648482		0.36992	
1/8	0.406708	0.6731	0.159371	1.2148
1/16	0.233742	0.7991	0.0590029	1.4335
1/32	0.126373	0.8872	0.0202292	1.5443

Table 10: Error estimates, $\epsilon = 10^{-5}$, $T = 1$, $k = \sqrt{h}$

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/16	0.267117		0.059804	
1/64	0.0717964	0.9477	0.00712605	1.5345
1/256	0.0179559	0.9997	0.00076384	1.6109

The method doesn't resolve the problem of oscillations in the boundary layer, but the oscillations do not spread beyond the boundary layer. This is verified by the figure plots of the computed solution $\mathbf{u}_2^h$, as the mesh size and the time step are decreased.

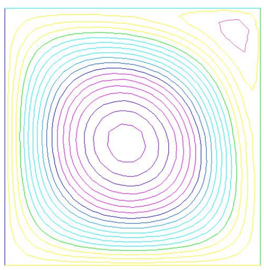

Figure 7: Computed solution $\mathbf{u}_2^h$, $k = h = 1/8$

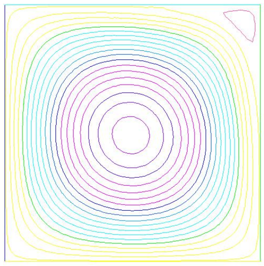

Figure 8: Computed solution $\mathbf{u}_2^h$, $k = h = 1/16$

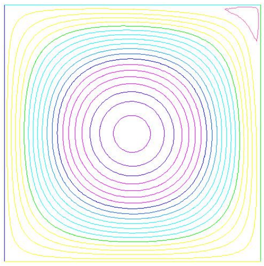

Figure 9: Computed solution $\mathbf{u}_2^h$, $k = h = 1/32$

Finally, we compute the approximation error away from the boundary layer, namely in $(0, 0.75) \times (0, 0.75)$.

Table 11: Error estimates in $(0, 0.75) \times (0, 0.75)$, $\epsilon = 10^{-5}$, $k = h$

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/4	0.545619		0.26598	
1/8	0.32111	0.7648	0.0844715	1.6548
1/16	0.172327	0.8979	0.02094	2.0122

Hence, the computational results verify the claimed accuracy of the method away from boundaries. Also, the oscillations of the computed solution do not spread outside of the boundary layer.

Table 12: Error estimates in $(0, 0.75) \times (0, 0.75)$, $\epsilon = 10^{-5}$, $k = \sqrt{h}$

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/16	0.197961		0.0186758	
1/64	0.0491873	1.0044	0.0016107	1.7677
1/256	0.0120236	1.0162	0.000112279	1.9213

5.0 NUMERICAL ANALYSIS OF A METHOD FOR HIGH PECLET NUMBER TRANSPORT IN POROUS MEDIA

5.1 INTRODUCTION

Consider the porous media problem (or Darcy's problem): find $(\mathbf{u}, p)$ such that

$$\begin{aligned} -\nabla \cdot (k\nabla p) &= g \\ \mathbf{u} &= -k\nabla p. \end{aligned} \tag{5.1.1}$$

In this equation, $\mathbf{u}$ represents the convection field of some fluid through a porous medium, p is the pressure, k represents the relative permeability, and g is a source term.

The associated evolutionary convection diffusion problem is: find ϕ such that

$$\phi_t - \epsilon\Delta\phi + \mathbf{u} \cdot \nabla\phi + c\phi = f \tag{5.1.2}$$

where ϕ is a scalar quantity modeling some characteristic of a fluid flow such as temperature or concentration level, ϵ is a diffusion coefficient, c is an absorbtion/reaction coefficient, and f is a forcing function.

The coupling of the porous media problem with the convection diffusion problem is of great importance in a wide array of applications. Any situation in which one is concerned with a scalar quantity associated with a flow through a porous medium is pertinent. Applications include oil recovery, the tracking of contaminants in groundwater flow, and nuclear waste storage among many others.

Numerically speaking, this coupling is rather straightforward. Darcy's problem is solved to obtain a convection field, $\mathbf{u}$. This computed convection field is then used in the convection diffusion problem to obtain a concentration level, ϕ. This coupling is unidirectional in that the computed value of ϕ plays no part in determining $\mathbf{u}$.

A full error analysis of the coupled system requires two components. First, we determine a bound on the error in the approximation, $\mathbf{u^h}$, of $\mathbf{u}$ through whatever numerical method is used to solve Darcy's problem. Next, we must follow this inherited error and its effects through the analysis of the error in approximating the concentration level, ϕ via the convection diffusion problem. For Darcy's problem we use the Galerkin approximation to obtain $\mathbf{u^h}$. For the convection diffusion problem we follow a thread of recent results of Guermond [guermond], Layton [Layton02], and Heitmann [heitmann].

The fundamental difficulty in this type of modeling arises from the interplay of convection, a large scale result of fluid velocity, and diffusion, which acts on small scales via Brownian motion. Specifically, the inequality $|\mathbf{u}|h/\epsilon >> 1$ frequently results in solutions displaying numerical instability around boundary or interior layers. A wide variety of stabilization methods have been developed in an effort to gain stability while maintaining solution quality (see Codina [codina] for a survey of some common methods). One branch of techniques have been the multiscale methods of Hughes et al. [hughes1, hughes2, hughes3, hughes4] and Guermond [guermond, guermond2] among others, which have added stability by augmenting the solution space with bubble functions.

In particular, the method of Guermond creates a composite space via the direct sum of a large scale finite element space, $\overline{X}$ and a space composed of bubble functions, X'. The resulting space $X^h = \overline{X} \oplus X'$ allows for every $v^h \in X^h$ to be decomposed as $v^h = \overline{v} + v'$ where $\overline{v}$ seeks to capture the large scale behavior and artificial viscosity is added only to the fine scales through v'.

In Layton [Layton02] a consistent multiscale mixed method formulation is presented for the stationary convection diffusion problem. The method decomposes the finite element space X^h into large scales represented by L^H and fine scales represented by $I - L^H$. Stability is added to the equation and then removed via the mixed variable. By clever selection of the space L^H that part of the stabilization term remaining acts only on fluctuations in $\nabla\phi^h$. A natural selection is the choice $L^H \equiv \nabla X^H$. This choice is analyzed by Layton with

- $X^h :=$ conforming, C^0 piecewise linears on a mesh of width h,

- $X^H :=$ conforming, C^0 piecewise linears on a coarser mesh of width H, and
- $L^H := \nabla X^H = L^2$ piecewise linears on the coarser mesh.

This results in near optimal error bounds. In Heitmann [heitmann] the method of Layton is extended to the evolutionary problem using a Crank-Nicholson time discretization. Employing the same choice of $L^H \equiv \nabla X^H$ bounds are obtained for the cases of conforming, C^0 piecewise linear, quadratic, and cubic elements.

This chapter seeks to move the efforts of Layton and Heitmann forward in a next logical step by coupling the eddy viscosity discretization of their work to the porous media problem. We turn to a backward Euler time discretization to simplify some of the computation and notation however these results would translate easily to the Crank-Nicholson method or any other higher order methods.

In section 5.2 we briefly describe notation used and a few established results. We also present the method to be studied. Section 5.3 states and proves results for Darcy's problem. In particular, stability is established and a bound on the approximation error for the convection field is determined. In section 5.4, a comprehensive analysis of stability and the error associated with finding ϕ is performed.

5.2 NOTATION AND PRELIMINARIES

We begin with a few definitions, assumptions, and forms used, and conclude the section with a statement of the method to be studied. The variational formulation of the coupled problem (5.1.1)-(5.1.2) is naturally stated in

$$\mathbf{X} := H_{0,div}(\Omega)^d, \quad S := H_0^1(\Omega), \quad Q := L_0^2(\Omega),$$

where $H_{div}(\Omega)^d = \{\mathbf{v} : \Omega \to \mathbb{R}^d, \mathbf{v} \in L^2(\Omega)^d, \nabla \cdot \mathbf{v} \in L^2(\Omega)^d\}$.

Definition 5.1. *For $v \in L^2$ let P be the orthogonal projection operator from $L^2(\Omega)$ to L^H. Let $P' = I - P$.*

We use the standard L^2 norm, $\|\cdot\|$, and the usual norm on the Sobolev space H^k, namely $\|\cdot\|_k$, as well as the H_{div} norm defined below.

Definition 5.2. *For* $\mathbf{u}$ *such that* $\mathbf{u} \in L^2$ *and* $\nabla \cdot \mathbf{u} \in L^2$ *we define the* H_{div} *norm*

$$\|\mathbf{u}\|_{H_{div}}^2 = \|\mathbf{u}\|^2 + \|\nabla \cdot \mathbf{u}\|^2$$

Definition 5.3. *For* $\phi \in H^1(\Omega)$*, the weighted norm of a function* $\phi : \Omega \to \mathbb{R}$ *is defined by*

$$\|\phi\|_{1,\epsilon,\alpha}^2 = \|\phi\|^2 + \epsilon\|\nabla\phi\|^2 + \alpha\|P'\nabla\phi\|^2$$

We make several common assumptions about the finite element spaces. The first is the discrete inf-sup condition.

Remark 5.1. *The finite dimensional spaces* $\mathbf{X}^h \subset \mathbf{X}$ *and* $Q^h \subset Q$ *satisfy the discrete inf-sup condition*

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in X^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{||q^h||\,||\mathbf{v}^h||_{H_{div}}} \geq \beta > 0 \tag{5.2.1}$$

Examples of such spaces are common in the literature. We shall consider $\mathbf{X}^h \subset \mathbf{X}$, $Q^h \subset Q$ to be spaces of continuous piecewise polynomials of degree k and $k-1$, respectively, with $k \geq 1$. The concentration finite element space S^h is the space of continuous piecewise polynomials of degree m, $m \geq 1$.

The interpolating properties of the finite element spaces $\mathbf{X}^h, S^h$ and Q^h are given by the following assumption.

Remark 5.2. *For the functions* $\mathbf{u} \in \mathbf{X}$*,* $\phi \in S$*,* $p \in Q$

$$\begin{aligned} \inf_{\chi \in \mathbf{X}^h, q^h \in Q^h} \left\{\|\mathbf{u} - \chi\| + h\|\nabla(\mathbf{u} - \chi)\| + h\|p - q^h\|\right\} &\leq ch^{r+1}(\|\mathbf{u}\|_{r+1} + \|p\|_r), \quad 1 \leq r \leq k, \\ \inf_{\xi \in S^h} \|\phi - \xi\| &\leq ch^{r+1}\|\phi\|_{r+1}, \quad 1 \leq r \leq m. \end{aligned}$$

The following assumption is known as the inverse estimate.

Remark 5.3. *For any* $v^\mu \in X^\mu$

$$\|\nabla v^\mu\| \leq C\mu^{-1}\|v^\mu\|,$$

holds in X^μ*, where* μ *is a characteristic length scale and* C *is of order one for typical finite element spaces.*

Remark 5.4. *Let P be the orthogonal projection from L^2 onto a given finite element space $L^H \subset X^h$, where h and H denote characteristic mesh widths ($h < H$). Then, for any $\phi^h \in X^h$*

$$\|P\nabla\phi^h\| \leq CH^{-1}\|\phi^h\|.$$

For more detail on this assumption see John, Kaya, and Layton [johnkayalayton] and Kaya [kaya].

We also define the following three forms for notational efficiency.

Definition 5.4. *For all $\phi, w \in S$ define*

$$b(\phi, w) \equiv (\mathbf{u} \cdot \nabla\phi, w) + (c\,\phi, w),$$

$$a(\phi, w) \equiv \epsilon(\nabla\phi, \nabla w) + b(\phi, w),$$

$$A(\phi, w) \equiv a(\phi, w) + \alpha(P'\nabla\phi, P'\nabla w).$$

The superscript h on any of these forms indicates that the concentration level and the velocity field to be used are the approximations $\mathbf{u}^h, \phi^h$. Thus, for instance we have

$$a^h(\phi^h, w) \equiv \epsilon(\nabla\phi^h, \nabla w) + b^h(\phi^h, w) \equiv \epsilon(\nabla\phi^h, \nabla w) + (\mathbf{u}^h \cdot \nabla\phi^h, w) + (c\,\phi^h, w).$$

We now turn toward the approximation method to be studied. The variational formulation of (5.1.1)-(5.1.2) is found as usual by multiplying (5.1.1) by a test function $\mathbf{v} \in \mathbf{X}$, multiplying (5.1.2) by a test function $w \in S$ and integrating over the spatial domain. The result is to find $(\mathbf{u}, p, \phi) \in (\mathbf{X}, Q, S)$ such that for all $(\mathbf{v}, w) \in (\mathbf{X}, S)$

$$k^{-1}(\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0 \tag{5.2.2}$$

$$\nabla \cdot \mathbf{u} = g \tag{5.2.3}$$

$$(\phi_t, w) + a(\phi, w) = (f, w) \tag{5.2.4}$$

Following the developments of Layton [Layton02] and Heitmann [heitmann] for the stationary and evolutionary convection diffusion problem respectively, the coupled method is to find $\mathbf{u}^h \in \mathbf{X}^h$, $\phi^h \in S^h$ and $p^h \in Q^h$ such that for all $\mathbf{v}^h \in X^h, w^h \in S^h$ and for all $q^h \in Q^h$

$$k^{-1}(\mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) = 0, \tag{5.2.5}$$

$$(\nabla \cdot \mathbf{u}^h, q^h) = (g, q^h), \tag{5.2.6}$$

$$(\phi_t^h, w^h) + \alpha(P'\nabla\phi^h, P'\nabla w^h) + a^h(\phi^h, w^h) = (f, w^h). \tag{5.2.7}$$

Recall, this system is only coupled in one direction (porous media to convection diffusion), thus we begin by proving stability and error estimates of the Galerkin approximation (5.2.5,5.2.6) of the Darcy's problem (5.2.2,5.2.3).

5.3 STABILITY AND ERROR ANALYSIS OF THE DARCY PROBLEM

Theorem 5.1. *The Galerkin approximation* $(\mathbf{u}^h, p^h)$ *to equations (5.2.5,5.2.6) is stable provided* $g \in L^2(\Omega)$.

Proof. Let $v^h = \mathbf{u}^h$ in (5.2.5) and $q^h = p^h$ in (5.2.6). This gives

$$k^{-1}||\mathbf{u}^h||^2 = (p^h, \nabla \cdot \mathbf{u}^h) \tag{5.3.1}$$

and

$$(\nabla \cdot \mathbf{u}^h, p^h) = (p^h, \nabla \cdot \mathbf{u}^h) = (g, p^h). \tag{5.3.2}$$

Inserting into the inf-sup condition (5.2.1) and rearranging gives

$$\beta||p^h|| \le \frac{(p^h, \nabla \cdot \mathbf{u}^h)}{||\mathbf{u}^h||_{H_{div}}} = \frac{(p^h, \nabla \cdot \mathbf{u}^h)}{\sqrt{||\mathbf{u}^h||^2 + ||\nabla \cdot \mathbf{u}^h||^2}} \le \frac{(p^h, \nabla \cdot \mathbf{u}^h)}{||\mathbf{u}^h||} = \frac{k^{-1}||\mathbf{u}^h||^2}{||\mathbf{u}^h||} = k^{-1}||\mathbf{u}^h||. \tag{5.3.3}$$

It follows from equations (5.3.1)-(5.3.3) that

$$k^{-1}||\mathbf{u}^h||^2 = (g, p^h) \le ||g||\,||p^h|| \le \frac{1}{\beta}k^{-1}||g||\,||\mathbf{u}^h||.$$

Thus,

$$||\mathbf{u}^h|| \leq \frac{1}{\beta}||g||$$

proving the stability of $\mathbf{u}^h$. Furthermore, equation (5.3.3) combined with the above gives

$$||p^h|| \leq \frac{1}{\beta^2}k^{-1}||g||$$

completing the proof. □

Convergence is established in the following theorem.

Theorem 5.2. *Let $(\mathbf{u}, p) \in (\mathbf{X}, Q)$ satisfying equations (5.2.2,5.2.3). Let $(\mathbf{u}^h, p^h) \in (\mathbf{X}^h, Q^h)$ be the Galerkin approximation of equations (5.2.5,5.2.6). Then*

$$||\mathbf{u} - \mathbf{u}^h||_{H_{div}} + ||p - p^h|| \leq C\Big(\inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||_{H_{div}} + \inf_{q^h \in Q^h} ||p - q^h||\Big).$$

For the proof, we will bound each norm on the left hand side individually. Beginning with the $\mathbf{u} - \mathbf{u}^h$ term, it follows from (5.2.3) that for all $q^h \in Q^h$

$$(\nabla \cdot \mathbf{u}, q^h) = (g, q^h).$$

Subtracting (5.2.6) from the above gives

$$(\nabla \cdot (\mathbf{u} - \mathbf{u}^h), q^h) = 0, \qquad \forall q^h \in Q^h. \tag{5.3.4}$$

Let $I(\mathbf{u})$ be some projection of $\mathbf{u}$ into $\mathbf{X}^h$. The error is then decomposed via $e_{\mathbf{u}} = \mathbf{u} - \mathbf{u}^h = (\mathbf{u} - I(\mathbf{u})) - (\mathbf{u}^h - I(\mathbf{u})) = \boldsymbol{\eta} - \boldsymbol{\psi}^h$, where $\boldsymbol{\psi}^h \in \mathbf{X}^h$ and $\boldsymbol{\eta} \notin \mathbf{X}^h$. Then (5.3.4) can be written as

$$(\nabla \cdot \boldsymbol{\psi}^h, q^h) = (\nabla \cdot \boldsymbol{\eta}^h, q^h)$$

Letting $\mathbf{v}^h = \boldsymbol{\psi}^h$ in the inf-sup condition (5.2.1) and using the above equation gives

$$\beta||q^h|| \leq \frac{(\nabla \cdot \boldsymbol{\psi}^h, q^h)}{||\boldsymbol{\psi}^h||_{H_{div}}} = \frac{(\nabla \cdot \boldsymbol{\eta}, q^h)}{||\boldsymbol{\psi}^h||_{H_{div}}}.$$

It follows then that

$$\beta||q^h||\,||\boldsymbol{\psi}^h||_{H_{div}} \leq ||\nabla \cdot \boldsymbol{\eta}||\,||q^h||,$$

which implies

$$||\boldsymbol{\psi}^h||_{H_{div}} \leq \frac{1}{\beta}||\nabla \cdot \boldsymbol{\eta}||.$$

Now, by the triangle inequality

$$||\mathbf{u}-\mathbf{u}^h||_{H_{div}} \le ||\boldsymbol{\eta}||_{H_{div}} + ||\boldsymbol{\psi}^h||_{H_{div}} \le \Big(1+\frac{1}{\beta}\Big)||\boldsymbol{\eta}||_{H_{div}}.$$

Since $I(\mathbf{u})$ is an arbitrary function in $\mathbf{X}^h$, we obtain

$$||\mathbf{u}-\mathbf{u}^h||_{H_{div}} \le \Big(1+\frac{1}{\beta}\Big)\inf_{\mathbf{v}^h\in\mathbf{X}^h}||\mathbf{u}-\mathbf{v}^h||_{H_{div}}, \tag{5.3.5}$$

completing the bound on $\mathbf{u}-\mathbf{u}^h$. The bound on $p-p^h$ is completed in similar fashion. Subtracting equations (5.2.2)-(5.2.3) gives

$$k^{-1}(\mathbf{u}-\mathbf{u}^h,\mathbf{v}^h)-(p-p^h,\nabla\cdot\mathbf{v}^h)=0, \qquad \forall\mathbf{v}^h\in\mathbf{X}^h.$$

The error in the pressure approximation is decomposed via $p-p^h=(p-I(p))-(p^h-I(p))=\eta_p-\psi_p^h$ where $\psi_p^h\in Q^h$, $\eta_p\notin Q^h$, and $I(p)$ is some projection of p into Q^h. The previous equation is then rewritten

$$(\psi_p^h,\nabla\cdot\mathbf{v}^h)=(\eta_p,\nabla\cdot\mathbf{v}^h)-k^{-1}(\mathbf{u}-\mathbf{u}^h,\mathbf{v}^h).$$

Using the inf-sup condition on the left hand side and the triangle inequality on the right hand side gives

$$\beta\,||\psi_p^h||\,||\mathbf{v}^h||_{H_{div}} \le ||\eta_p||\,||\nabla\cdot\mathbf{v}^h||+k^{-1}||\mathbf{u}-\mathbf{u}^h||\,||\mathbf{v}^h||.$$

Dividing by β and $||\mathbf{v}^h||_{H_{div}}$ which is greater than $||\nabla\cdot\mathbf{v}^h||$ and $||\mathbf{v}^h||$ gives

$$||\psi_p^h|| \le \frac{1}{\beta}||\eta_p||+\frac{1}{\beta}k^{-1}||\mathbf{u}-\mathbf{u}^h||.$$

The triangle inequality on the decomposition gives $||p-p^h||\le||\eta_p||+||\psi_p^h||$, which applied to the above gives

$$||p-p^h|| \le \Big(1+\frac{1}{\beta}\Big)||\eta_p||+\frac{1}{\beta}k^{-1}||\mathbf{u}-\mathbf{u}^h||.$$

Since $I(p)$ is an arbitrary function in Q^h we have

$$||p-p^h|| \le \Big(1+\frac{1}{\beta}\Big)\inf_{q^h\in Q^h}||p-q^h||+\frac{1}{\beta}k^{-1}||\mathbf{u}-\mathbf{u}^h|| \tag{5.3.6}$$

and combining (5.3.5) with (5.3.6) completes the proof. □

5.4 THE CONVECTION DIFFUSION PROBLEM

The backward Euler method is used to approximate the time derivative. The subscript n is used to denote the value of the function at the time level t_n. Thus (5.2.4) is rewritten

$$\Big(\frac{\phi_{n+1}-\phi_n}{\Delta t}, w^h\Big) + \epsilon(\nabla\phi_{n+1}, \nabla w^h) + (\mathbf{u}_{n+1}\cdot\nabla\phi_{n+1}, w^h) + (c\phi_{n+1}, w^h) \\ = (f_{n+1}, w^h) + (\rho_{n+1}, w^h), \tag{5.4.1}$$

where $\rho_{n+1} = (\phi_{n+1}-\phi_n)/\Delta t - \phi_t(t_{n+1})$ is the error in the approximation of the time derivative at t_{n+1}. The approximation method seeks to find $\phi^h \in S^h$ satisfying

$$\Big(\frac{\phi^h_{n+1}-\phi^h_n}{k}, w^h\Big) + (\mathbf{u}^h_{n+1}\cdot\nabla\phi^h_{n+1}, w^h) + \alpha(P'\nabla\phi^h_{n+1}, P'\nabla w^h) \\ + \epsilon(\nabla\phi^h_{n+1}, \nabla w^h) + (c\phi^h_{n+1}, w^h) = (f_{n+1}, w^h), \qquad \forall w^h \in S^h. \tag{5.4.2}$$

5.4.1 Stability of the method

Theorem 5.3. *For some $C = C(\Omega) = O(1)$ let*

$$\Delta t(H^{-3/2}\inf_{\mathbf{v}^h\in\mathbf{X}^h}||\mathbf{u}-\mathbf{v}^h||_{H_{div}} + \alpha^{-3}\inf_{\mathbf{v}^h\in\mathbf{X}^h}||\mathbf{u}-\mathbf{v}^h||^4_{H_{div}}) < C \tag{5.4.3}$$

Let $f \in L^2(0,T;L^2(\Omega))$. Then the solution of (5.4.2) is stable over any time $T < \infty$ and

$$||\phi^h_{n+1}||^2 + \Delta t\sum_{i=1}^{n+1}\Big[\epsilon||\nabla\phi^h_i||^2 + \beta_c||\phi^h_i||^2 + ||P'\nabla\phi^h_i||^2\Big] \leq C(||\phi^h_0||^2 + \Delta t\sum_{i=1}^{n+1}||f_i||^2),$$

where β_c is a constant satisfying the coercivity condition $C - \frac{1}{2}g \geq \beta_c > 0$.

Proof. Let $w^h = \phi^h_{n+1} \in S^h$ in (5.4.2). This gives

$$\frac{||\phi^h_{n+1}||^2 - ||\phi^h_n||^2}{2\Delta t} + \epsilon||\nabla\phi^h_{n+1}||^2 + \alpha||P'\nabla\phi^h_{n+1}||^2 + (c\phi^h_{n+1}, \phi^h_{n+1}) - \frac{1}{2}((\nabla\cdot\mathbf{u}^h_{n+1})\phi^h_{n+1}, \phi^h_{n+1}) \le (f_{n+1}, \phi^h_{n+1}).$$

Introduce the error term $e_{n+1} = \mathbf{u}_{n+1} - \mathbf{u}^h_{n+1}$. Adding and subtracting $\frac{1}{2}((\nabla\cdot\mathbf{u}_{n+1})\phi^h_{n+1}, \phi^h_{n+1})$ to the left hand side and rearranging gives

$$\frac{||\phi^h_{n+1}||^2 - ||\phi^h_n||^2}{2\Delta t} + \epsilon||\nabla\phi^h_{n+1}||^2 + \alpha||P'\nabla\phi^h_{n+1}||^2 + ((c - \frac{1}{2}g)\phi^h_{n+1}, \phi^h_{n+1}) \le (f_{n+1}, \phi^h_{n+1}) - \frac{1}{2}((\nabla\cdot e_{n+1})\phi^h_{n+1}, \phi^h_{n+1}). \quad (5.4.4)$$

Hölder's inequality is used for the last term on the right hand side giving

$$\frac{1}{2}((\nabla\cdot e_{n+1})\phi^h_{n+1}, \phi^h_{n+1}) = \int_\Omega (\nabla\cdot e_{n+1})\,\phi^h_{n+1}\,\phi^h_{n+1} \le ||\nabla\cdot e_{n+1}||_{L^2}\,||\phi^h_{n+1}||_{L^6}\,||\phi^h_{n+1}||_{L^3}.$$

It follows from the Sobolev embedding theorem that in either 2-d or 3-d, $H^{1/2} \hookrightarrow L^3$ and $H^1 \hookrightarrow L^6$, thus we have

$$\begin{aligned} |\tfrac{1}{2}((\nabla\cdot e_{n+1})\phi^h_{n+1}, \phi^h_{n+1})| &\le ||\nabla\cdot e_{n+1}||\,||\phi^h_{n+1}||_{H^{1/2}}\,||\phi^h_{n+1}||_{H^1} \\ &\le C||\nabla\cdot e_{n+1}||\,||\phi^h_{n+1}||^{1/2}\,||\nabla\phi^h_{n+1}||^{3/2}. \end{aligned}$$

Using the above estimate, the Cauchy-Schwarz inequality, and Young's inequality on (5.4.4) it follows that

$$\frac{||\phi^h_{n+1}||^2 - ||\phi^h_n||^2}{2\Delta t} + \epsilon||\nabla\phi^h_{n+1}||^2 + \alpha||P'\nabla\phi^h_{n+1}||^2 + \beta_c||\phi^h_{n+1}||^2 \le \frac{1}{2}\beta_c||\phi^h_{n+1}||^2 + \frac{1}{2\beta_c}||f_{n+1}||^2 + C||\nabla\cdot e_{n+1}||\,||\phi^h_{n+1}||^{1/2}\,||\nabla\phi^h_{n+1}||^{3/2}. \quad (5.4.5)$$

Noting that $\nabla\phi^h_{n+1} = P\nabla\phi^h_{n+1} + P'\nabla\phi^h_{n+1}$, the triangle inequality insures the last term on the left hand side is bounded by

$$C||\nabla\cdot e_{n+1}||\,||\phi^h_{n+1}||^{1/2}\,||P\nabla\phi^h_{n+1}||^{3/2} + C||\nabla\cdot e_{n+1}||\,||\phi^h_{n+1}||^{1/2}\,||P'\nabla\phi^h_{n+1}||^{3/2}.$$

For the first term we use the inverse inequality $||P\nabla\phi_{n+1}^h|| \leq CH^{-1}||\phi_{n+1}^h||$. The second term is bounded using the generalized Young's inequality (with $p = 4/3$, $q = 4$). Thus, (5.4.5) can be rewritten

$$\frac{||\phi_{n+1}^h||^2 - ||\phi_n^h||^2}{2\Delta t} + \epsilon||\nabla\phi_{n+1}^h||^2 + \alpha||P'\nabla\phi_{n+1}^h||^2 + \beta_c||\phi_{n+1}^h||^2 \leq \frac{1}{2\beta_c}||f_{n+1}||^2 + CH^{-3/2}||\nabla\cdot e_{n+1}||\,||\phi_{n+1}^h||^2 + \frac{\alpha}{2}||P'\nabla\phi_{n+1}^h||^2 + C\alpha^{-3}||\nabla\cdot e_{n+1}||^4\,||\phi_{n+1}^h||^2.$$

Rearranging, multiplying by $2\Delta t$, and then summing from $i = 0$ to n gives

$$||\phi_{n+1}^h||^2 + \Delta t\sum_{i=0}^{n}\Big[2\epsilon||\nabla\phi_i^h||^2 + \alpha||P'\nabla\phi_i^h||^2 + \beta_c||\phi_i^h||^2\Big] \leq ||\phi_0^h||^2 + \Delta t\sum_{i=0}^{n}\frac{1}{\beta_c}||f_i||^2 + C\Delta t\sum_{i=1}^{n+1}\Big[(H^{-3/2}||\nabla\cdot e_i|| + \alpha^{-3}||\nabla\cdot e_i||^4)\,||\phi_i^h||^2\Big].$$

Applying the discrete Gronwall's lemma and the hypothesis of the theorem completes the proof. □

Corollary 5.1. *Let the finite dimensional subspace $X^h \in X$ be a space of C^0 piecewise polynomials of degree k. Then the method is stable provided that*

$$\Delta t(H^{-3/2}h^k + \alpha^{-3}h^{4k}) < C(\Omega).$$

In addition, the solution is bounded uniformly in epsilon.

5.4.2 Error estimates for the method

Let $\chi^h \in S^h$ be the equilibrium projection of $\phi \in S$ into S^h satisfying

$$A(\phi, w^h) = A(\chi^h, w^h) - ((\mathbf{u} - \mathbf{u}^h) \cdot \nabla \chi^h, w^h), \qquad \forall w^h \in S^h \tag{5.4.6}$$

For the right hand side we define the bilinear form $B(\cdot,\cdot)$ given by

$$B(s, w) \equiv A(s, w) - (e_u \cdot \nabla s, w), \qquad \forall s, w \in S,$$

where $e_u = \mathbf{u} - \mathbf{u}^h$.

Lemma 5.1. *Let $\mathbf{u} \in L^\infty(\Omega)$. $B(\cdot,\cdot)$ is continuous. Furthermore if*

$$H^{-3/2} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||_{H_{div}} + \alpha^{-3} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||^4_{H_{div}} < C(\Omega, \beta_c) = O(1) \cdot \beta_c \tag{5.4.7}$$

is satisfied, then $B(\cdot,\cdot)$ is coercive. Specifically, there exists a constant $C = C(\Omega)$ such that for all $s, w \in S$

$$B(s, w) \leq C||s||_{1,1,\alpha}\, ||w||_{1,1,\alpha}, \text{ and}$$
$$B(s, s) \geq C||s||_{1,\epsilon,\alpha}$$

Proof. The Cauchy-Schwarz inequality implies that

$$\begin{aligned} B(s, w) &\leq \epsilon||\nabla s||\, ||\nabla w|| + ||\mathbf{u}||_{L^\infty(\Omega)}||s||\, ||\nabla w|| + ||c||_{L^\infty(\Omega)}||s||\, ||w|| + \alpha||P'\nabla s||\, ||P'\nabla w|| \\ &\quad + ||\nabla e_u||\, ||\nabla s||\, ||\nabla w|| \\ &\leq C\big(||s||^2 + ||\nabla s||^2 + \alpha||P'\nabla s||^2\big)^{1/2}\big(||w||^2 + ||\nabla w||^2 + \alpha||P'\nabla w||^2\big)^{1/2} \\ &= C||s||_{1,1,\alpha}||w||_{1,1,\alpha}. \end{aligned}$$

This proves continuity. For coercivity we have

$$\begin{aligned} B(s, s) &= \epsilon||\nabla s||^2 + \Big(\big(c - \frac{1}{2}\nabla \cdot \mathbf{u}\big)s, s\Big) + \alpha||P'\nabla s||^2 - (e_u \cdot \nabla s, s) \\ &\geq \epsilon||\nabla s||^2 + \beta_c||s||^2 + \alpha||P'\nabla s||^2 + \frac{1}{2}\big((\nabla \cdot e_u)s, s\big). \end{aligned}$$

The last term can be decomposed using Hölder's inequality and then the embeddings $H^{1/2} \hookrightarrow L^3$ and $H^1 \hookrightarrow L^6$ to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega (\nabla e_u)s \cdot s &\leq \frac{1}{2}||\nabla e_u||\,||s||_{L^3}||s||_{L^6} \\ &\leq C||\nabla e_u||\,||s||^{1/2}||\nabla s||^{3/2} \\ &= C||\nabla e_u||\,||s||^{1/2}\big(||P\nabla s||^{3/2} + ||P'\nabla s||^{3/2}\big).\end{aligned}$$

We use the inverse inequality on $||P\nabla s||$ and Young's inequality ($p = 4$, $q = 4/3$) to obtain

$$\begin{aligned}B(s,s) \geq \epsilon||\nabla s||^2 + \beta_c||s||^2 + \alpha||P'\nabla s||^2 - CH^{-3/2}||\nabla e_u||\,||s||^2 \\ -\frac{\alpha}{2}||P'\nabla s||^2 - C\alpha^{-3}||\nabla e_u||^4||s||^2 \\ = \epsilon||\nabla s||^2 + \big(\beta_c - CH^{-3/2}||\nabla e_u|| - C\alpha^{-3}||\nabla e_u||^4\big)||s||^2 + \frac{\alpha}{2}||P'\nabla s||^2\end{aligned}$$

Finally, we use the bound (5.4.7) to obtain that $B(s,s) \geq C||s||_{1,\epsilon,\alpha}$, which completes the proof. □We will assume throughout the rest of the chapter that the bound (5.4.7) holds, thus $B(\cdot,\cdot)$ is continuous and coercive.

Corollary 5.2. *Let X^h be a space of C^0 piecewise polynomials of degree k. Then* (5.4.7) *reads*

$$H^{-3/2}h^k + \alpha^{-3}h^{4k} < C(\Omega, \beta_c) = O(1)\cdot\beta_c.$$

Theorem 5.4. *Let $\phi \in S$. The equilibrium projection $\chi^h \in S^h$, given by equation* (5.4.6), *exists uniquely.*

Proof. Define $F(w) \equiv A(\phi, w)$ for any $\phi \in S$. Then equation (5.4.6) can be rewritten as $B(\chi^h, w^h) = F(w^h)$. F is a continuous, linear functional. In the finite dimensional space S^h all norms are equivalent. Thus, we need only one norm in which $B(\cdot,\cdot)$ is continuous and coercive. This was accomplished by lemma 5.1. Thus, the hypotheses of the Lax-Milgram theorem are satisfied and χ^h exists uniquely. □

We next seek an a priori error estimate in the approximation of ϕ by the equilibrium projection χ^h.

Theorem 5.5. *Let* $\mathbf{u} \in L^\infty(\Omega) \cap H^{k+1}(\Omega)$, $\phi \in H^{m+1}(\Omega)$, *and* $\nabla\phi \in L^\infty(\Omega)$. *Further, let*

$$H^{-3/2} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||_{H_{div}} + \alpha^{-3} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||^4_{H_{div}} < C(\Omega). \tag{5.4.8}$$

Then

$$||\phi - \chi^h||_{1,\epsilon,\alpha} \leq C(\Omega, m, k, g, f) \Big\{(1 + H^{-1} + \alpha^{-1/2})h^{m+1} + (\epsilon + \alpha)^{1/2} h^m + h^k + (H^{-1} + \alpha^{-1/2})h^{m+k-1/2}\Big\}$$

Proof. We define and decompose the error as $e = \phi - \chi^h = (\phi - I(\phi)) - (\chi^h - I(\phi)) = \eta - \gamma^h$ such that $\eta \notin S^h$ and $\gamma^h \in S^h$, and where $I(\phi)$ is some projection of ϕ into S^h. Using the triangle inequality we will complete the proof by finding bounds on $||\eta||_{1,\epsilon,\alpha}$ and $||\gamma^h||_{1,\epsilon,\alpha}$ separately. Beginning with the γ^h, it follows from equation (5.4.6) that

$$A(\gamma^h, w^h) = A(\eta, w^h) + (e_u \cdot \nabla\chi^h, w^h),$$

where $e_u \equiv \mathbf{u} - \mathbf{u}^h$. We choose $w^h = \gamma^h \in S^h$ and decompose $A(\cdot,\cdot)$ into its symmetric (A_s) and skew-symmetric (A_{ss}) parts. As $A_{ss}(\gamma^h, \gamma^h) = 0$ we have

$$A_s(\gamma^h, \gamma^h) = A_s(\eta, \gamma^h) + A_{ss}(\eta, \gamma^h) + (e_u \cdot \nabla\chi^h, \gamma^h).$$

Applying the Cauchy-Schwarz inequality and Green's theorem leads to

$$\begin{aligned} A_s(\gamma^h, \gamma^h) &\leq \sqrt{A_s(\gamma^h, \gamma^h)} + \sqrt{A_s(\eta, \eta)} + A_{ss}(\eta, \gamma^h) + (e_u \cdot \nabla\chi^h, \gamma^h) \\ &\leq \frac{1}{2} A_s(\gamma^h, \gamma^h) + \frac{1}{2} A_s(\eta, \eta) - (\mathbf{u}\eta, \nabla\gamma^h) - \frac{1}{2}((\nabla \cdot \mathbf{u})\eta, \gamma^h) + (e_u \cdot \nabla\chi^h, \gamma^h). \end{aligned}$$

Following the proof of Heitmann [heitmann] we obtain

$$||\gamma^h||^2_{1,\epsilon,\alpha} \leq C\Big\{||\eta||^2_{1,\epsilon,\alpha} + \frac{1}{\beta_C} H^{-2} ||P(\mathbf{u}\eta)||^2 + \frac{1}{2\alpha} ||P'(\mathbf{u}\eta)||^2 + \frac{1}{\beta_C} ||\eta||^2\Big\} + (e_u \cdot \nabla\chi^h, \gamma^h). \tag{5.4.9}$$

Using the error decomposition the last term on the right hand side can be bounded via

$$|(e_u \cdot \nabla\chi^h, \gamma^h)| \leq |(e_u \cdot \nabla\phi, \gamma^h)| + |(e_u \cdot \nabla\eta, \gamma^h)| + |(e_u \cdot \nabla\gamma^h, \gamma^h)|.$$

We proceed by separately bounding each of the terms on the right hand side. The regularity of ϕ gives

$$|(e_u \cdot \nabla \phi, \gamma^h)| \leq ||\nabla \phi||_{L^\infty(\Omega)} |(e_u, \gamma^h)| \leq \frac{1}{8}||\gamma^h||^2 + 2||\nabla \phi||^2_{L^\infty(\Omega)}||e_u||^2 \tag{5.4.10}$$

for the first term. For the second term a combination of Hölder's inequality with Sobolev embeddings, followed by the triangle inequality, inverse inequality, and Young's inequality gives

$$\begin{aligned}
|(e_u \cdot \nabla \eta, \gamma^h)| &\leq ||e_u||_{L^3}||\nabla \eta||_{L^2}||\gamma^h||_{L^6} \\
&\leq C||e_u||_{H^{1/2}}||\nabla \eta||\,||\gamma^h||_{H^1} \\
&\leq C||e_u||^{1/2}||\nabla e_u||^{1/2}||\nabla \eta||\,||\nabla \gamma^h|| \\
&\leq C||e_u||^{1/2}||\nabla e_u||^{1/2}||\nabla \eta||\big(||P\nabla \gamma^h|| + ||P'\nabla \gamma^h||\big) \\
&\leq \frac{1}{8}||\gamma^h||^2 + CH^{-2}||e_u||\,||\nabla e_u||\,||\nabla \eta||^2 + \frac{\alpha}{4}||P'\nabla \gamma^h||^2 \\
&\quad + \frac{C}{\alpha}||e_u||\,||\nabla e_u||\,||\nabla \eta||^2.
\end{aligned} \tag{5.4.11}$$

Finally, for the third term we obtain

$$\begin{aligned}
|(e_u \cdot \nabla \gamma^h, \gamma^h)| &\leq \frac{1}{2}\big|\big((\nabla \cdot e_u)\gamma^h, \gamma^h\big)\big| \\
&\leq C||\nabla \cdot e_u||\,||\gamma^h||^{1/2}||\nabla \gamma^h||^{3/2} \\
&\leq C||\nabla \cdot e_u||\,||\gamma^h||^{1/2}\big(||P\nabla \gamma^h||^{3/2} + ||P'\nabla \gamma^h||^{3/2}\big) \\
&\leq CH^{-3/2}||\nabla \cdot e_u||\,||\gamma^h||^2 + \frac{\alpha}{4}||P'\nabla \gamma^h||^2 + C\alpha^{-3}||\nabla \cdot e_u||^4||\gamma^h||^2.
\end{aligned} \tag{5.4.12}$$

The combination of equations (5.4.9) - (5.4.12) gives

$$\begin{aligned}
||\gamma^h||^2_{1,\epsilon,\alpha} \leq C\Big\{&||\eta||^2_{1,\epsilon,\alpha} + \big(1 + ||\mathbf{u}||^2_{L\infty(\Omega)}(H^{-2} + \alpha^{-1})\big)||\eta||^2 + ||\nabla \phi||^2_{L\infty(\Omega)} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||^2_{H_{div}} \\
&+ (H^{-2} + \alpha^{-1})h^{-1} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||^2_{H_{div}}||\nabla \eta||^2\Big\} \\
&+ C\big(H^{-3/2} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||_{H_{div}} + \alpha^{-3} \inf_{\mathbf{v}^h \in \mathbf{X}^h} ||\mathbf{u} - \mathbf{v}^h||^4_{H_{div}}\big)||\gamma^h||^2.
\end{aligned} \tag{5.4.13}$$

For the bound on $||\eta||_{1,\epsilon,\alpha}$ we choose $I(\phi)$ to be the L^2-projection of ϕ into S^h. This gives $||\eta|| \leq Ch^{m+1}||\phi||_{H^{m+1}}$ and $||\nabla\eta|| \leq Ch^m||\phi||_{H^{m+1}}$. Thus, we have

$$||\eta||_{1,\epsilon,\alpha} \leq C(h^{m+1} + (\epsilon+\alpha)^{1/2}h^m)||\phi||_{H^{m+1}}. \tag{5.4.14}$$

Now applying the hypothesis (5.4.8), we obtain from equations (5.4.13) and (5.4.14) the result

$$\begin{aligned}||\phi - \chi^h||_{1,\epsilon,\alpha} \leq C(\Omega, m, k, g, f)\Big\{&h^{m+1} + (\epsilon+\alpha)^{1/2}h^m + H^{-1}h^{m+1} + \alpha^{-1/2}h^{m+1} \\ &+ h^k + H^{-1}h^{m+k-1/2} + \alpha^{-1/2}h^{m+k-1/2}\Big\}\end{aligned}$$

□

Corollary 5.3. *Under the condition $H^{-3/2}h^m + h^{4m-3} < C(\Omega)$, the choice of $\alpha(h) = h$ gives*

$$||\phi - \chi^h||_{1,\epsilon,\alpha} \leq C(\Omega, m, k, g, f)\Big\{h^k + (1 + H^{-1}h^{1/2})(h^{m+1/2} + h^{m+k-1})\Big\}.$$

The next step is to find an a priori bound on the discrete time derivative.

Theorem 5.6. *Let the assumptions of theorem 5.5 be satisfied. Let $\mathbf{u}_t \in L^\infty(\Omega)^d$ and $\phi_t, \nabla\phi_t \in L^\infty(\Omega)$. Then for any $n \geq 0$*

$$\begin{aligned}\left\|\frac{(\phi_{n+1} - \chi^h_{n+1}) - (\phi_n - \chi^h_n)}{\Delta t}\right\| \leq\ & C(\Omega, m, k, g, f)(H^{-1} + \alpha^{-1/2}) \\ &\Big\{h^{m+1} + (\epsilon+\alpha)^{1/2}h^m + H^{-1}h^{m+1} + \alpha^{-1/2}h^{m+1} \\ &+ h^k + H^{-1}h^{m+k-1/2} + \alpha^{-1/2}h^{m+k-1/2}\Big\}.\end{aligned}$$

Proof. The proof is analagous to that of theorem 5.5. We introduce the bilinear form

$$\widetilde{A}(\phi, w) \equiv \epsilon(\nabla\phi, \nabla w) + (\mathbf{u}_{n+1} \cdot \nabla\phi, w) + (c\phi, w) + \alpha(P'\nabla\phi, P'\nabla w).$$

It is straightforward to show the same continuity and coercivity results hold for $\widetilde{A}$. At the $(n+1)$st time level we obtain

$$\widetilde{A}(\phi_{n+1} - \chi^h_{n+1}, w^h) = -(e_{\mathbf{u},n+1} \cdot \nabla\chi^h_{n+1}, w^h),$$

where $e_{\mathbf{u},n+1} \equiv \mathbf{u}_{n+1} - \mathbf{u}^h_{n+1}$. From the nth time level we have

$$\widetilde{A}(\phi_n - \chi^h_n, w^h) = \Delta t\Big(\Big(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}\Big) \cdot \nabla(\phi_n - \chi^h_n), w^h\Big) - (e_{\mathbf{u},n} \cdot \nabla\chi^h_n, w^h).$$

Subtracting these two equations yields

$$\widetilde{A}\Big(\frac{(\phi_{n+1} - \chi^h_{n+1}) - (\phi_n - \chi^h_n)}{\Delta t}\Big) = -\Big[\Big(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} \cdot \nabla(\phi_n - \chi^h_n), w^h\Big) + \frac{1}{\Delta t}(e_{\mathbf{u},n+1} \cdot \nabla\chi^h_{n+1} - e_{\mathbf{u},n} \cdot \nabla\chi^h_n, w^h)\Big]. \quad (5.4.15)$$

Decompose $\phi_{n+1} - \chi^h_{n+1} = (\phi_{n+1} - I(\phi_{n+1})) - (\chi^h_{n+1} - I(\phi_{n+1})) = \eta_{n+1} - \gamma^h_{n+1}$. Since $I(\phi_{n+1})$ is an arbitrary projection of ϕ_{n+1} into S^h, we have

$$\Big\|\frac{\eta_{n+1} - \eta_n}{\Delta t}\Big\| \leq Ch^{m+1}\|\phi_t\|_{H^{m+1}}.$$

Thus, we only need to bound $\|(\gamma^h_{n+1} - \gamma^h)/\Delta t\|$ and then apply the triangle inequality. Using the error decomposition, rewrite (5.4.2) as

$$\widetilde{A}\Big(\frac{\eta_{n+1} - \eta_n}{\Delta t}, w^h\Big) + \Big(\Big(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}\Big) \cdot \nabla(\phi_n - \chi^h_n), w^h\Big) + \frac{1}{\Delta t}(e_{\mathbf{u},n+1} \cdot \nabla\chi^h_{n+1} - e_{\mathbf{u},n+1} \cdot \nabla\chi^h_n + e_{\mathbf{u},n+1} \cdot \nabla\chi^h_n - e_{\mathbf{u},n} \cdot \nabla\chi^h_n, w^h) = \widetilde{A}\Big(\frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}, w^h\Big)$$

Take $w^h = \frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t} \in S^h$ above. This gives

$$\Big\|\frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}\Big\|^2_{1,\epsilon,\alpha} \leq \widetilde{A}_s\Big(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}\Big) + \widetilde{A}_{ss}\Big(\frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}\Big) + \Big(\Big(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}\Big) \cdot \nabla(\phi_n - \chi^h_n), \frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}\Big) + \Big(e_{\mathbf{u},n+1} \cdot \Big(\frac{\chi^h_{n+1} - \chi^h_n}{\Delta t}\Big), \frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}\Big) - \Big(\frac{e_{\mathbf{u},n+1} - e_{\mathbf{u},n}}{\Delta t} \cdot \nabla\chi^h_n, \frac{\gamma^h_{n+1} - \gamma^h_n}{\Delta t}\Big).$$

Following the proof of theorem 5.5, we obtain

$$
\begin{aligned}
\Big\|\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big\|_{1,\epsilon,\alpha}^2 \le C\Big\{\Big\|\frac{\eta_{n+1}-\eta_n}{\Delta t}\Big\|_{1,\epsilon,\alpha}^2 &+ (H^{-2}+\alpha^{-1})\Big\|\frac{\eta_{n+1}-\eta_n}{\Delta t}\Big\|^2\Big\} \\
&+\Big|\Big(\Big(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}\Big)\cdot\nabla(\phi_n-\chi_n^h),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&+\Big|\Big(e_{\mathbf{u},n+1}\cdot\nabla\Big(\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&+\Big|\Big(e_{\mathbf{u},n+1}\cdot\nabla\Big(\frac{\eta_{n+1}-\eta_n}{\Delta t}\Big),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&+\Big|\Big(e_{\mathbf{u},n+1}\cdot\nabla\Big(\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&+\Big|\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\cdot\nabla\phi_n,\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&+\Big|\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\cdot\nabla(\phi_n-\chi_n),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big|.
\end{aligned}
$$

We proceed by bounding each of the six as yet unbounded terms on the right hand side individually. For the first term, the regularity of $\mathbf{u}$ gives

$$
\begin{aligned}
&\Big|\Big(\Big(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}\Big)\cdot\nabla(\phi_n-\chi_n^h),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&\quad\le\Big\|\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}\Big\|_{L^\infty(\Omega)}\Big|\Big(\nabla(\phi_n-\chi_n^h),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&\quad\le CH^{-2}\|\phi_n-\chi_n^h\|^2+\frac{1}{8}\Big\|\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big\|^2+C\alpha^{-1}\|\phi_n-\chi_n^h\|^2+\frac{\alpha}{8}\Big\|P'\nabla\Big(\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big\|^2.
\end{aligned}
\tag{5.4.16}
$$

The second term uses the regularity of ϕ yielding

$$
\begin{aligned}
\Big|\Big(e_{\mathbf{u},n+1}\cdot\nabla\Big(\frac{\phi_{n+1}-\phi_n}{\Delta t}\Big),\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| &\le\Big\|\nabla\Big(\frac{\phi_{n+1}-\phi_n}{\Delta t}\Big)\Big\|_{L^\infty(\Omega)}\Big|\Big(e_{\mathbf{u},n+1},\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big)\Big| \\
&\le\frac{1}{8}\Big\|\frac{\gamma_{n+1}^h-\gamma_n^h}{\Delta t}\Big\|^2+C\|e_{\mathbf{u},n+1}\|^2.
\end{aligned}
\tag{5.4.17}
$$

The third term is bounded by using Hölder's inequality, followed by Sobolev embeddings and Young's inequality giving

$$\begin{aligned}
&\Big|\Big(e_{\mathbf{u},n+1}\cdot\nabla\Big(\frac{\eta_{n+1}-\eta_n}{\Delta t}\Big),\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big| \\
&\le C\|e_{\mathbf{u},n+1}\|^{1/2}\|\nabla e_{\mathbf{u},n+1}\|^{1/2}\Big\|\nabla\Big(\frac{\eta_{n+1}-\eta_n}{\Delta t}\Big)\Big\|\Big(\Big\|P\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|+\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|\Big) \\
&\le \frac{1}{8}\Big\|\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big\|^2+\frac{\alpha}{8}\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|^2 \\
&\quad +C(H^{-2}+\alpha^{-1})\|e_{\mathbf{u},n+1}\|\,\|\nabla e_{\mathbf{u},n+1}\|\,\Big\|\nabla\Big(\frac{\eta_{n+1}-\eta_n}{\Delta t}\Big)\Big\|^2.
\end{aligned} \tag{5.4.18}$$

The fourth term is bounded as follows

$$\begin{aligned}
&\Big|e_{\mathbf{u},n+1}\cdot\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big),\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big| \\
&\qquad =\frac{1}{2}\Big|\Big((\nabla\cdot e_{\mathbf{u},n+1})\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t},\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big| \\
&\qquad \le C\|\nabla\cdot e_{\mathbf{u},n+1}\|\Big\|\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big\|^{1/2}\Big(\Big\|P\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|^{3/2}+\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|^{3/2}\Big) \\
&\qquad \le \frac{\alpha}{8}\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|^2+C(H^{-3/2}\|\nabla\cdot e_{\mathbf{u},n+1}\|+\alpha^{-3}\|\nabla\cdot e_{\mathbf{u},n+1}\|^4)\Big\|\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big\|^2.
\end{aligned} \tag{5.4.19}$$

The fifth term uses the regularity of ϕ leading to

$$\begin{aligned}
\Big|\Big(\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big)\cdot\nabla\phi_n,\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big| &\le \|\nabla\phi_n\|_{L^\infty(\Omega)}\Big|\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t},\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big| \\
&\le \frac{1}{8}\Big\|\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big\|^2+C\Big\|\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big\|^2.
\end{aligned} \tag{5.4.20}$$

Finally, the sixth term is bounded by

$$\begin{aligned}
&\Big|\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\cdot\nabla(\phi_n-\chi^h_n),\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big| \\
&\quad \le C\Big\|\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big\|^{1/2}\Big\|\nabla\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big)\Big\|^{1/2} \\
&\qquad\quad \|\nabla(\phi_n-\chi^h_n)\|\Big(\Big\|P\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|+\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|\Big) \\
&\quad \le \frac{1}{8}\Big\|\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big\|^2+\frac{\alpha}{8}\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|^2+C(H^{-2}+\alpha^{-1})\Big\|\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big\| \\
&\qquad\quad \Big\|\nabla\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big)\Big\|(\|P\nabla(\phi_n-\chi^h_n)\|^2+\|P'\nabla(\phi_n-\chi^h_n)\|^2) \\
&\quad \le \frac{1}{8}\Big\|\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big\|^2+\frac{\alpha}{8}\Big\|P'\nabla\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}\Big)\Big\|^2+C(H^{-2}+\alpha^{-1})\Big\|\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big\| \\
&\qquad\quad \Big\|\nabla\Big(\frac{e_{\mathbf{u},n+1}-e_{\mathbf{u},n}}{\Delta t}\Big)\Big\|\|\phi_n-\chi^h_n\|^2_{1,\epsilon,\alpha}.
\end{aligned} \tag{5.4.21}$$

Applying the bounds of the inequalities (5.4.16) - (5.4.21) to the inequality (5.4.2) and using the triangle inequality gives

$$\left\| \frac{(\phi_{n+1} - \chi_{n+1}^h) - (\phi_n - \chi_n^h)}{\Delta t} \right\|^2$$
$$\leq C(\Omega, m, k, g, f) \Big\{ h^{2k} + \big[1 + (H^{-2} + \alpha^{-1}) + (H^{-2} + \alpha^{-1}) h^{2k-1} \big] \|\phi_n - \chi_n^h\|_{1,\epsilon,\alpha}^2 \Big\}. \quad (5.4.22)$$

Finally, the application of theorem 5.5 to $\|\phi_n - \chi_n^h\|_{1,\epsilon,\alpha}^2$ completes the proof. □

Remark 5.1. *The bound*

$$\left\| \frac{e_{\mathbf{u},n+1} - e_{\mathbf{u},n}}{\Delta t} \right\| \leq C h^k \|\mathbf{u}_t\|_{H^{k+1}}$$

is obtained by following the proof of theorem 5.2 with $\boldsymbol{\eta}$ and $\boldsymbol{\psi}^h$ replaced by $(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n)/\Delta t$ and $(\boldsymbol{\psi}_{n+1}^h - \boldsymbol{\psi}_n^h)/\Delta t$ respectively.

We conclude with the theorem that gives the a priori estimate for the approximation error of the method (5.4.2).

Theorem 5.7. *Let the assumptions of theorem 5.6 be satisfied and let $\phi_{tt} \in L^2(0, T; L^2(\Omega))$. Further, assume that the stability condition (5.4.3) of theorem 5.3 is satisfied. Then for any time level n*

$$\|\phi_n - \phi_n^h\|_{1,\epsilon,\alpha}^2 \leq \|\phi_0 - \phi_0^h\|^2 + C(\Omega, m, f, g)$$
$$\Big((H^{-2} + \alpha^{-1}) \Big\{ h^{2m+2} + (\epsilon + \alpha) h^{2m} + H^{-2} h^{2m+2} + \alpha^{-1} h^{2m+2}$$
$$+ h^{2k} + H^{-2} h^{2m+2k-1} + \alpha^{-1} h^{2m+2k-1} \Big\}$$
$$+ \alpha k \sum_{i=1}^{N} \|P' \nabla \phi_i\|^2 + \Delta t^2 \Big).$$

Proof. The model equation and the equation of the method can be written respectively as

$$\Big(\frac{\phi_{n+1}-\phi_n}{\Delta t}, w^h\Big) + \widetilde{A}(\phi_{n+1}, w^h) - \alpha(P'\nabla\phi_{n+1}, P'\nabla w^h)$$
$$= (f_{n+1}, w^h) + \Big(\frac{\phi_{n+1}-\phi_n}{\Delta t} - \phi_t(t_{n+1}), w^h\Big). \quad (5.4.23)$$

and

$$\Big(\frac{\phi^h_{n+1}-\phi^h_n}{\Delta t}, w^h\Big) + \widetilde{A}(\phi^h_{n+1}, w^h) - (e_{\mathbf{u},n+1}\cdot\nabla\phi^h_{n+1}) = (f_{n+1}, w^h). \quad (5.4.24)$$

We use the equilibrium projection χ^h in the error decomposition

$$\phi_{n+1} - \phi^h_{n+1} = (\phi_{n+1} - \chi^h_{n+1}) - (\phi^h_{n+1} - \chi^h_{n+1}) = \eta_{n+1} - \gamma^h_{n+1},$$

where $\eta_{n+1} \equiv \phi_{n+1} - \chi^h_{n+1} \notin S^h$ and $\gamma^h_{n+1} \equiv \phi^h_{n+1} - \chi^h_{n+1} \in S^h$. Note that we have obtained the bounds on $\|\eta_{n+1}\|_{1,\epsilon,\alpha}$ and $\left\|\frac{\eta_{n+1}-\eta_n}{\Delta t}\right\|_{1,\epsilon,\alpha}$ in theorems 5.5 and 5.6. Thus, we only need to bound $\|\gamma^h_{n+1}\|_{1,\epsilon,\alpha}$ and use the triangle inequality. We subtract (5.4.24) from (5.4.23) and use the error decomposition to obtain for all $w^h \in S^h$

$$\Big(\frac{\eta_{n+1}-\eta_n}{\Delta t}, w^h\Big) - \Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}, w^h\Big) + \widetilde{A}(\eta_{n+1}, w^h) - \widetilde{A}(\gamma^h_{n+1}, w^h)$$
$$- \alpha(P'\nabla\phi_{n+1}, P'\nabla w^h) + (e_{\mathbf{u},n+1}\cdot\nabla\phi^h_{n+1}, w^h) = \Delta t(\rho_{n+1}, w^h), \quad (5.4.25)$$

where $\rho_{n+1} = \phi_{tt}(t_{n+1-\theta})$ for some $\theta \in]0,1[$. Now, by definition of the equilibrium projection we have

$$\widetilde{A}(\eta_{n+1}, w^h) = \widetilde{A}(\phi_{n+1} - \chi^h_{n+1}, w^h) = -(e_{\mathbf{u},n+1}\cdot\nabla\chi^h_{n+1}, w^h).$$

By regrouping terms in (5.4.25) and using the above we have for all $w^h \in S^h$

$$\Big(\frac{\gamma^h_{n+1}-\gamma^h_n}{\Delta t}, w^h\Big) + \widetilde{A}(\gamma^h_{n+1}, w^h) =$$
$$\Big(\frac{\eta_{n+1}-\eta_n}{\Delta t}, w^h\Big) - \alpha(P'\nabla\phi_{n+1}, P'\nabla w^h) - \Delta t(\rho_{n+1}, w^h) + (e_{\mathbf{u},n+1}\cdot\nabla\gamma^h_{n+1}, w^h).$$

Now set $w^h = \gamma^h_{n+1} \in S^h$ above. This gives

$$\frac{\|\gamma^h_{n+1}\|^2 - \|\gamma^h_n\|^2}{2\Delta t} + \epsilon\|\nabla\gamma^h_{n+1}\|^2 + \beta_c\|\gamma^h_{n+1}\|^2 + \alpha\|P'\nabla\gamma^h_{n+1}\|^2$$
$$\leq \left\|\frac{\eta_{n+1}-\eta_n}{\Delta t}\right\|\|\gamma^h_{n+1}\| + \alpha\|P'\nabla\phi_{n+1}\|\,\|P'\nabla\gamma^h_{n+1}\| + \Delta t\|\rho_{n+1}\|\,\|\gamma^h_{n+1}\|$$
$$- \frac{1}{2}((\nabla\cdot e_{\mathbf{u},n+1})\gamma^h_{n+1}, \gamma^h_{n+1}).$$

Applying Young's inequality gives

$$\begin{aligned}\frac{\|\gamma^h_{n+1}\|^2-\|\gamma^h_n\|^2}{2\Delta t} &+ \epsilon\|\nabla\gamma^h_{n+1}\|^2+\beta_c\|\gamma^h_{n+1}\|^2+\alpha\|P'\nabla\gamma^h_{n+1}\|^2\\ &\le \frac{1}{4}\beta_c\|\gamma^h_{n+1}\|^2+\frac{1}{\beta_c}\left\|\frac{\eta_{n+1}-\eta_n}{\Delta t}\right\|^2+\frac{\alpha}{4}\|P'\nabla\gamma^h_{n+1}\|^2+\alpha\|P'\nabla\phi_{n+1}\|^2\\ &\quad+\frac{1}{4}\beta_c\|\gamma^h_{n+1}\|^2+\frac{1}{\beta_c}\Delta t^2\|\rho_{n+1}\|^2+\frac{\alpha}{4}\|P'\nabla\gamma^h_{n+1}\|^2\\ &\quad+C(H^{-3/2}\|\nabla\cdot e_{\mathbf{u},n+1}\|+\alpha^{-3}\|\nabla\cdot e_{\mathbf{u},n+1}\|^4)\|\gamma^h_{n+1}\|^2.\end{aligned}$$

Finally, summing over the time levels, multiplying by $2\Delta t$ and using the discrete Gronwall's lemma completes the proof. □

5.5 CONCLUSIONS

We have presented rigorous analysis of the coupling of the porous media problem with the evolutionary convection diffusion problem. In the case of the porous media problem we use the Galerkin approximation to obtain the velocity field, $\mathbf{u}^h$. The convection diffusion problem is then solved using this approximation in conjunction with the stabilization schemes presented by Layton [Layton02] and Heitmann [heitmann]. It is shown that the convergence rate is near optimal and independent of the diffusion coefficient, ϵ. Logical next steps in this direction would include coupling with the Navier-Stokes equations and computational experiments using this method.

6.0 LARGE EDDY SIMULATION FOR MHD FLOWS

6.1 INTRODUCTION

Magnetically conducting fluids arise in important applications including plasma physics, geophysics and astronomy. In many of these, turbulent MHD (magnetohydrodynamics [Alfv42]) flows are typical. The difficulties of accurately modeling and simulating turbulent flows are magnified many times over in the MHD case. They are evinced by the more complex dynamics of the flow due to the coupling of Navier-Stokes and Maxwell equations via the Lorentz force and Ohm's law.

In this chapter we consider the problem of modeling the motion of large structures in a viscous, incompressible, electrically conducting, turbulent fluid.

The MHD equations are related to engineering problems such as plasma confinement, controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD sea water propulsion.

The flow of an electrically conducting fluid is affected by Lorentz forces, induced by the interaction of electric currents and magnetic fields in the fluid. The Lorentz forces can be used to control the flow and to attain specific engineering design goals such as flow stabilization, suppression or delay of flow separation, reduction of near-wall turbulence and skin friction, drag reduction and thrust generation. There is a large body of literature dedicated to both experimental and theoretical investigations on the influence of electromagnetic force on flows (see e.g., [HeSt95, MeHeHr92, MeHeHr94, GPT, Tsin90, GL61, TS67, HS95, SB97, BKLL00]). The MHD effects arising from the macroscopic interaction of liquid metals with applied currents and magnetic fields are exploited in metallurgical processes to control the flow of metallic melts: the electromagnetic stirring of molten metals [MDRV84], electromagnetic turbulence control in induction furnaces [ViRi85], electromagnetic damping of buoyancy-driven flow during solidification [PrIn93], and the electromagnetic shaping of ingots in continuous casting [SaLiEv88].

The mathematical description of the problem proceeds as follows. Assuming the fluid to be viscous and incompressible, the governing equations are the Navier- Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law (see e.g. [Sher65]). Let $\Omega = (0, L)^3$ be the flow domain, and $u(t, x), p(t, x), B(t, x)$ be the velocity, pressure, and the magnetic field of the flow, driven by the velocity body force f and magnetic field force $\operatorname{curl} g$. Then u, p, B satisfy the MHD equations:

$$
\begin{aligned}
& u_t + \nabla \cdot (uu^T) - \frac{1}{\mathrm{Re}}\Delta u + \frac{S}{2}\nabla(B^2) - S\nabla \cdot (BB^T) + \nabla p = f, \\
& B_t + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} B) + \operatorname{curl}(B \times u) = \operatorname{curl} g, \\
& \nabla \cdot u = 0, \nabla \cdot B = 0,
\end{aligned}
\tag{6.1.1}
$$

in $Q = (0, T) \times \Omega$, with the initial data:

$$
u(0, x) = u_0(x), \quad B(0, x) = B_0(x) \quad \text{in } \Omega, \tag{6.1.2}
$$

and with periodic boundary conditions (with zero mean):

$$
\Phi(t, x + Le_i) = \Phi(t, x), i = 1, 2, 3, \qquad \int_\Omega \Phi(t, x)dx = 0, \tag{6.1.3}
$$

for $\Phi = u, u_0, p, B, B_0, f, g$.

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. For derivation of (6.1.1), physical interpretation and mathematical analysis, see [Cowl57, LL69, ST83, GMP91] and the references therein.

If $\overline{\cdot}^{\delta_1}$, $\overline{\cdot}^{\delta_2}$ denote two local, spacing averaging operators that commute with the differentiation, then averaging (6.1.1) gives the following non-closed equations for $\overline{u}^{\delta_1}$, $\overline{B}^{\delta_2}$, $\overline{p}^{\delta_1}$ in $(0,T)\times\Omega$:

$$\begin{aligned}
&\overline{u}_t^{\delta_1} + \nabla\cdot(\overline{uu^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta\overline{u}^{\delta_1} - S\nabla\cdot(\overline{BB^T}^{\delta_1}) + \nabla\Big(\frac{S}{2}\overline{B^2}^{\delta_1} + \overline{p}^{\delta_1}\Big) = \overline{f}^{\delta_1},\\
&\overline{B}_t^{\delta_2} + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\,\overline{B}^{\delta_2}) + \nabla\cdot(\overline{Bu^T}^{\delta_2}) - \nabla\cdot(\overline{uB^T}^{\delta_2}) = \mathrm{curl}\,\overline{g}^{\delta_2},\\
&\nabla\cdot\overline{u}^{\delta_2} = 0, \quad \nabla\cdot\overline{B}^{\delta_2} = 0.
\end{aligned} \tag{6.1.4}$$

The usual closure problem which we study here arises because $\overline{uu^T}^{\delta_1} \neq \overline{u}^{\delta_1}\,\overline{u}^{\delta_1}$, $\overline{BB^T}^{\delta_1} \neq \overline{B}^{\delta_1}\,\overline{B}^{\delta_1}$, $\overline{uB^T}^{\delta_2} \neq \overline{u}^{\delta_1}\,\overline{B^T}^{\delta_2}$. To isolate the turbulence closure problem from the difficult problem of wall laws for near wall turbulence, we study (6.1.1) hence (6.1.4) subject to (6.1.3). The closure problem is to replace the tensors $\overline{uu^T}^{\delta_1}$, $\overline{BB^T}^{\delta_1}$, $\overline{uB^T}^{\delta_2}$ with tensors $\mathscr{T}(\overline{u}^{\delta_1},\overline{u}^{\delta_1})$, $\mathscr{T}(\overline{B}^{\delta_2},\overline{B}^{\delta_2})$, $\mathscr{T}(\overline{u}^{\delta_1},\overline{B}^{\delta_2})$, respectively, depending only on $\overline{u}^{\delta_1},\overline{B}^{\delta_2}$ and not u,B. There are many closure models proposed in large eddy simulation reflecting the centrality of closure in turbulence simulation. Calling w,q,W the resulting approximations to $\overline{u}^{\delta_1},\overline{p}^{\delta_1},\overline{B}^{\delta_2}$, we are led to considering the following model

$$\begin{aligned}
&w_t + \nabla\cdot\mathscr{T}(w,w) - \frac{1}{\mathrm{Re}}\Delta w - S\mathscr{T}(W,W) + \nabla q = \overline{f}^{\delta_1}\\
&W_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\,W) + \nabla\cdot\mathscr{T}(w,W) - \nabla\cdot\mathscr{T}(W,w) = \mathrm{curl}\,\overline{g}^{\delta_2},\\
&\nabla\cdot w = 0, \quad \nabla\cdot W = 0.
\end{aligned} \tag{6.1.5}$$

With any reasonable averaging operator, the true averages $\overline{u}^{\delta_1},\overline{B}^{\delta_2},\overline{p}^{\delta_1}$ are smoother than u,B,p. We consider the simplest, accurate closure model that is exact on constant flows (i.e., $\overline{u}^{\delta_1} = u, \overline{B}^{\delta_2} = B$) is

$$\begin{aligned}
\overline{uu^T}^{\delta_1} &\approx \overline{\overline{u}^{\delta_1}\,\overline{u^T}^{\delta_1}}^{\delta_1} &&=: \mathscr{T}(\overline{u}^{\delta_1},\overline{u}^{\delta_1}),\\
\overline{BB^T}^{\delta_1} &\approx \overline{\overline{B}^{\delta_2}\,\overline{B^T}^{\delta_2}}^{\delta_1} &&=: \mathscr{T}(\overline{B}^{\delta_2},\overline{B}^{\delta_2}),\\
\overline{uB^T}^{\delta_2} &\approx \overline{\overline{u}^{\delta_1}\,\overline{B^T}^{\delta_2}}^{\delta_2} &&=: \mathscr{T}(\overline{u}^{\delta_1},\overline{B}^{\delta_2}),
\end{aligned} \tag{6.1.6}$$

leading to

$$w_t + \nabla \cdot (\overline{ww^T}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta w - S\nabla \cdot (\overline{W\,W^T}^{\delta_1}) + \nabla q = \overline{f}^{\delta_1}, \tag{6.1.7a}$$

$$W_T + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\, W) + \nabla \cdot (\overline{Ww^T}^{\delta_2}) - \nabla \cdot (\overline{wW^T}^{\delta_2}) = \mathrm{curl}\, \overline{g}^{\delta_2}, \tag{6.1.7b}$$

$$\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{6.1.7c}$$

subject to $w(x,0) = \overline{u}_0^{\delta_1}(x), W(x,0) = \overline{B}_0^{\delta_2}(x)$ and periodic boundary conditions (with zero means).

We shall show that the LES MHD model (6.1.7) has the mathematical properties which are expected of a model derived from the MHD equations by an averaging operation and which are important for practical computations using (6.1.7).

The model considered can be developed for quite general averaging operators, see e.g. [AS01]. The choice of averaging operator in (6.1.7) is a differential filter, defined as follows. Let the $\delta > 0$ denote the averaging radius, related to the finest computationally feasible mesh. (In this chapter we use different lengthscales for the Navier-Stokes and Maxwell equations). Given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^{\delta} \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$A_\delta \overline{\phi}^{\delta} := -\delta^2 \Delta \overline{\phi}^{\delta} + \overline{\phi}^{\delta} = \phi \quad \text{in } \Omega, \tag{6.1.8}$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation, and with this averaging operator, the model (6.1.6) has consistency $O(\delta^2)$, i.e.,

$$\begin{aligned}
\overline{uu^T}^{\delta_1} &= \overline{\overline{u}^{\delta_1}\, \overline{u^T}^{\delta_1}} + O({\delta_1}^2),\\
\overline{BB^T}^{\delta_1} &= \overline{\overline{B}^{\delta_2}\, \overline{B^T}^{\delta_2}}^{\delta_1} + O({\delta_2}^2),\\
\overline{uB^T}^{\delta_2} &= \overline{\overline{u}^{\delta_1}\, \overline{B^T}^{\delta_2}}^{\delta_2} + O({\delta_1}^2 + {\delta_2}^2),
\end{aligned}$$

for smooth u, B. We prove that the model (6.1.7) has a unique, weak solution w, W that converges in the appropriate sense $w \to u$, $W \to B$, as $\delta_1, \delta_2 \to 0$.

In Section 6.2 we prove the global existence and uniqueness of the solution for the closed MHD model, after giving the notations and a definition. Section 6.3 treats the questions of limit consistency of the model and verifiability. The conservation of the kinetic energy and helicity for the approximate deconvolution model is presented in Section 6.4. Section 6.5 shows that the model preserves the Alfén waves, with the velocity tending to the velocity of Alfvén waves in the MHD, as the radii δ_1, δ_2 tend to zero.

6.2 EXISTENCE AND UNIQUENESS FOR THE MHD LES EQUATIONS

6.2.1 Notations and preliminaries

We shall use the standard notations for function spaces in the space periodic case (see [Tema95]). Let $H_p^m(\Omega)$ denote the space of functions (and their vector valued counterparts also) that are locally in $H^m(\mathbb{R}^3)$, are periodic of period L and have zero mean, i.e. satisfy (6.1.3). We recall the solenoidal space

$$\mathscr{D}(\Omega) = \{\phi \in C^\infty(\Omega) : \phi \text{ periodic with zero mean}, \nabla \cdot \phi = 0\},$$

and the closures of $\mathscr{D}(\Omega)$ in the usual $L^2(\Omega)$ and $H^1(\Omega)$ norms :

$$H = \{\phi \in H_2^0(\Omega), \nabla \cdot \phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2,$$
$$V = \{\phi \in H_2^1(\Omega), \nabla \cdot \phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2.$$

We define the operator $\mathscr{A} \in \mathscr{L}(V, V')$ by setting

$$\langle \mathscr{A}(w_1, W_1), (w_2, W_2)\rangle = \int_\Omega \left(\frac{1}{\mathrm{Re}} \nabla w_1 \cdot \nabla w_2 + \frac{S}{\mathrm{Re_m}} \mathrm{curl}\, W_1 \mathrm{curl}\, W_2 \right) dx, \tag{6.2.1}$$

for all $(w_i, W_i) \in V$. The operator $\mathscr{A}$ is an unbounded operator on H, with the domain $D(\mathscr{A}) = \{(w, W) \in V; (\Delta w, \Delta W) \in H\}$ and we denote again by $\mathscr{A}$ its restriction to H. We define also a continuous tri-linear form $\mathscr{B}_0$ on $V \times V \times V$ by setting

$$\begin{aligned} \mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3)) = \int_\Omega \Big(\nabla \cdot (\overline{w_2 w_1^T}^{\delta_1})\, w_3 \qquad\qquad (6.2.2) \\ - S\nabla \cdot (\overline{W_2 W_1^T}^{\delta_1})\, w_3 + \nabla \cdot (\overline{W_2 w_1^T}^{\delta_2})\, W_3 - \nabla \cdot (\overline{w_2 W_1^T}^{\delta_2})\, W_3 \Big)\, dx \end{aligned}$$

and a continuous bilinear operator $\mathscr{B}(\cdot) : V \to V$ with

$$\langle \mathscr{B}(w_1, W_1), (w_2, W_2) \rangle = \mathscr{B}_0((w_1, W_1), (w_1, W_1), (w_2, W_2))$$

for all $(w_i, W_i) \in V$.

The following properties of the trilinear form $\mathscr{B}_0$ hold (see [JLL69, ST83, Gris80, Furs00])

$$\begin{aligned}
&\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} w_2, SA_{\delta_2} W_2)) = 0, \\
&\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} w_3, SA_{\delta_2} W_3)) \\
&\qquad = -\mathscr{B}_0((w_1, W_1), (w_3, W_3), (A_{\delta_1} w_2, SA_{\delta_2} W_2)),
\end{aligned} \tag{6.2.3}$$

for all $(w_i, W_i) \in V$. Also

$$\begin{aligned}
&|\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3))| \\
&\qquad \leq C \|(w_1, W_1)\|_{m_1} \|(w_2, W_2)\|_{m_2+1} \|(\overline{w_3}^{\delta_1}, \overline{W_3}^{\delta_2})\|_{m_3}
\end{aligned} \tag{6.2.4}$$

for all $(w_1, W_1) \in H^{m_1}(\Omega), (w_2, W_2) \in H^{m_2+1}(\Omega), (w_3, W_3) \in H^{m_3}(\Omega)$ and

$$\begin{aligned}
m_1 + m_2 + m_3 &\geq \frac{d}{2}, \quad \text{if } m_i \neq \frac{d}{2} \text{ for all } i = 1, \ldots, d, \\
m_1 + m_2 + m_3 &> \frac{d}{2}, \quad \text{if } m_i = \frac{d}{2} \text{ for any of } i = 1, \ldots, d.
\end{aligned}$$

In terms of $V, H, \mathscr{A}, \mathscr{B}(\cdot)$ we can rewrite (6.1.7) as

$$\begin{aligned}
&\frac{d}{dt}(w, W) + \mathscr{A}(w, W)(t) + \mathscr{B}((w, W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}), t \in (0, T), \\
&(w, W)(0) = (\overline{u}_0^{\delta_1}, \overline{B}_0^{\delta_2}),
\end{aligned} \tag{6.2.5}$$

where $(\boldsymbol{f}, \operatorname{curl} \boldsymbol{g}) = P(f, \operatorname{curl} g)$, and $P : L^2(\Omega) \to H$ is the Hodge projection.

Definition 6.1. *Let* $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$, $\overline{f}^{\delta_1}$, $curl\overline{g}^{\delta_2} \in L^2(0,T;V')$. *The measurable functions* $w, W : [0,T] \times \Omega \to \mathbb{R}^3$ *are the weak solutions of (6.2.5) if* $w, W \in L^2(0,T;V) \cap L^\infty(0,T;H)$, *and* w, W *satisfy*

$$
\begin{aligned}
&\int_\Omega w(t)\phi dx + \int_0^t \int_\Omega \frac{1}{\mathrm{Re}} \nabla w(\tau)\nabla\phi + \overline{w(\tau)\cdot\nabla w(\tau)}^{\delta_1}\phi - S\overline{W(\tau)\cdot\nabla W(\tau)}^{\delta_1}\phi \, dx d\tau \\
&\qquad = \int_\Omega \overline{u_0}^{\delta_1}\phi dx + \int_0^t \int_\Omega \overline{f(\tau)}^{\delta_1}\phi \, dx d\tau, \\
&\int_\Omega W(t)\psi dx + \int_0^t \int_\Omega \frac{1}{\mathrm{Re_m}} \nabla W(\tau)\nabla\psi + \overline{w(\tau)\cdot\nabla W(\tau)}^{\delta_2}\psi - \overline{W(\tau)\cdot\nabla w(\tau)}^{\delta_2}\psi \, dx d\tau \\
&\qquad = \int_\Omega \overline{B_0}^{\delta_2}\psi dx + \int_0^t \int_\Omega curl\,\overline{g(\tau)}^{\delta_2}\psi \, dx d\tau,
\end{aligned}
\tag{6.2.6}
$$

$\forall t \in [0,T), \phi, \psi \in \mathscr{D}(\Omega)$.

Also, it is easy to show that for any $u, v \in H^1(\Omega)$ with $\nabla \cdot u = \nabla \cdot v = 0$, the following identity holds

$$\nabla \times (u \times v) = v \cdot \nabla u - u \cdot \nabla v. \tag{6.2.7}$$

6.2.2 Stability and existence for the model

The first result states that the weak solution of the MHD LES model (6.1.7) exists globally in time, for large data and general $\mathrm{Re}, \mathrm{Re}_m > 0$ and that it satisfies an energy equality while initial data and the source terms are smooth enough.

Theorem 6.1. *Let* $\delta_1, \delta_2 > 0$ *be fixed. For any* $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in V$ *and* $(\overline{f}^{\delta_1}, curl\overline{g}^{\delta_2}) \in L^2(0,T;H)$, *there exists a unique weak solution* w, W *to (6.1.7). The weak solution also belongs to* $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ *and* $w_t, W_t \in L^2((0,T)\times\Omega)$. *Moreover, the following energy equality holds for* $t \in [0,T]$*:*

$$\mathscr{M}(t) + \int_0^t \mathscr{N}(\tau)d\tau = \mathscr{M}(0) + \int_0^t \mathscr{P}(\tau)d\tau, \tag{6.2.8}$$

where

$$
\begin{aligned}
\mathscr{M}(t) &= \frac{{\delta_1}^2}{2}\|\nabla w(t,\cdot)\|_0^2 + \frac{1}{2}\|w(t,\cdot)\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\nabla W(t,\cdot)\|_0^2 + \frac{S}{2}\|W(t,\cdot)\|_0^2, \\
\mathscr{N}(t) &= \frac{{\delta_1}^2}{\mathrm{Re}}\|\Delta w(t,\cdot)\|_0^2 + \frac{1}{\mathrm{Re}}\|\nabla w(t,\cdot)\|_0^2 + \frac{{\delta_2}^2 S}{\mathrm{Re_m}}\|\Delta W(t,\cdot)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\nabla W(t,\cdot)\|_0^2, \\
\mathscr{P}(t) &= (f(t), w(t)) + S(curl\,g(t), W(t)).
\end{aligned}
\tag{6.2.9}
$$

We shall use the semigroup approach proposed in [BS01] for the Navier-Stokes equations, based on the machinery of nonlinear differential equations of accretive type in Banach spaces.

Let us define the modified nonlinearity $\mathscr{B}_N(\cdot) : V \to V$ by setting

$$\mathscr{B}_N(w, W) = \begin{cases} \mathscr{B}(w, W) & \text{if } \|(w, W)\|_1 \le N, \\ \left(\frac{N}{\|(w,W)\|_1}\right)^2 \mathscr{B}(w, W) & \text{if } \|(w, W)\|_1 > N. \end{cases} \tag{6.2.10}$$

By (6.2.4) we have for the case of $\|(w_1, W_1)\|_1, \|(w_2, W_2)\|_1 \le N$

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_1 - w_2, W_1 - W_2)\rangle| \\ &= |\mathscr{B}_0((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_1 - w_2, W_1 - W_2)| \\ &\quad + \mathscr{B}_0((w_2, W_2), (w_1 - w_2, W_1 - W_2), (w_1 - w_2, W_1 - W_2)| \\ &\le C\|(w_1 - w_2, W_1 - W_2)\|_{1/2}\|(w_1, W_1)\|_1\|(\overline{w_1 - w_2}^{\delta_1}, \overline{W_1 - W_2}^{\delta_2})\|_1 \\ &\le \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2 + C_N\|(w_1 - w_2, W_1 - W_2)\|_0^2, \end{aligned}$$

where $\nu = \inf\{1/\text{Re}, S/\text{Re}_m\}$.

In the case of $\|(w_i, W_i)\|_1 > N$ we have

$$\begin{aligned} &|\langle \mathscr{B}_N(w_1, W_1) - \mathscr{B}_N(w_2, W_2), (w_1 - w_2, W_1 - W_2)\rangle| \\ &= \frac{N^2}{\|(w_1, W_1)\|_1^2}\mathscr{B}_0((w_1 - w_2, W_1 - W_2), (w_1, W_1), (w_1 - w_2, W_1 - W_2)) \\ &\quad + \left(\frac{N^2}{\|(w_1, W_1)\|_1^2} - \frac{N^2}{\|(w_2, W_2)\|_1^2}\right)\mathscr{B}_0((w_2, W_2), (w_2, W_2), (w_1 - w_2, W_1 - W_2)) \\ &\le CN\|(w_1 - w_2, W_1 - W_2)\|_1^{3/2}\|(w_1 - w_2, W_1 - W_2)\|_0^{1/2} \\ &\quad + CN\|(w_1 - w_2, W_1 - W_2)\|_1^2 \\ &\le \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2 + C_N\|(w_1 - w_2, W_1 - W_2)\|_0^2. \end{aligned}$$

For the case of $\|(w_1,W_1)\|_1 > N, \|(w_2,W_2)\|_1 \le N$ (similar estimates are obtained when $\|(w_1,W_1)\|_1 \le N, \|(w_2,W_2)\|_1 > N$) we have

$$\begin{aligned}
&|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2),(w_1-w_2,W_1-W_2)\rangle| \\
&= \frac{N^2}{\|(w_1,W_1)\|_1^2}\mathscr{B}_0((w_1-w_2,W_1-W_2),(w_1,W_1),(w_1-w_2,W_1-W_2)) \\
&\quad - \left(1-\frac{N^2}{\|(w_1,W_1)\|_1^2}\right)\mathscr{B}_0((w_2,W_2),(w_2,W_2),(w_1-w_2,W_1-W_2)) \\
&\le CN\|(w_1-w_2,W_1-W_2)\|_1^{3/2}\|(w_1-w_2,W_1-W_2)\|_0^{1/2} \\
&\quad + CN\|(w_1-w_2,W_1-W_2)\|_1\|(w_1-w_2,W_1-W_2)\|_{1/2} \\
&\le \frac{\nu}{2}\|(w_1-w_2,W_1-W_2)\|_1^2 + C_N\|(w_1-w_2,W_1-W_2)\|_0^2.
\end{aligned}$$

Combining all the cases above we conclude that

$$\begin{aligned}
&|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2),(w_1-w_2,W_1-W_2)\rangle| \\
&\quad \le \frac{\nu}{2}\|(w_1-w_2,W_1-W_2)\|_1^2 + C_N\|(w_1-w_2,W_1-W_2)\|_0^2.
\end{aligned} \tag{6.2.11}$$

The operator $\mathscr{B}_N$ is continuous from V to V'. Indeed, as above we have (using (6.2.4) with $m_1 = 1, m_2 = 0, m_3 = 1$)

$$\begin{aligned}
&|\langle \mathscr{B}_N(w_1,W_1) - \mathscr{B}_N(w_2,W_2),(w_3,W_3)\rangle| \\
&\le |\mathscr{B}_0\left((w_1-w_2,W_1-W_2),(w_1,W_1),(w_3,W_3)\right)| \\
&\quad + |\mathscr{B}_0\left((w_2,W_2),(w_1-w_2,W_1-W_2),(w_3,W_3)\right)| \\
&\le C_N\|(w_1-w_2,W_1-W_2)\|_1\|(w_3,W_3)\|_1.
\end{aligned} \tag{6.2.12}$$

Now consider the operator $\Gamma_N : D(\Gamma_N) \to H$ defined by

$$\Gamma_N = \mathscr{A} + \mathscr{B}_N, \quad D(\Gamma_N) = D(\mathscr{A}).$$

Here we used (6.2.4) with $m_1 = 1, m_2 = 1/2, m_3 = 0$ and interpolation results (see e.g. [GR86, Tema79, Furs00]) to show that

$$\|\mathscr{B}_N(w,W)\|_0 \le C\|(w,W)\|_1^{3/2}\|\mathscr{A}(w,W)\|_0^{1/2} \le C_N\|\mathscr{A}(w,W)\|_0^{1/2}. \tag{6.2.13}$$

Lemma 6.1. *There exists $\alpha_N > 0$ such that $\Gamma_N + \alpha_N I$ is m-accretive (maximal monotone) in $H \times H$.*

Proof. By (6.2.11) we have that

$$((\Gamma_N + \lambda)(w_1, W_1) - (\Gamma_N + \lambda)(w_2, W_2), (w_1 - w_2, W_1 - W_2)) \geq \frac{\nu}{2}\|(w_1 - w_2, W_1 - W_2)\|_1^2, \text{ for all } (w_i, W_i) \in D(\Gamma_N), \tag{6.2.14}$$

for $\lambda \geq C_N$. Next we consider the operator

$$\mathscr{F}_N(w, W) = \mathscr{A}(w, W) + \mathscr{B}_N(w, W) + \alpha_N(w, W), \quad \text{for all } (w, W) \in D(\mathscr{F}_N),$$

with

$$D(\mathscr{F}_N) = \{(w, W) \in V; \mathscr{A}(w, W) + \mathscr{B}_N(w, W) \in H\}.$$

By (6.2.12) and (6.2.14) we see that $\mathscr{F}_N$ is monotone, coercive and continuous from V to V'. We infer that $\mathscr{F}_N$ is maximal monotone from V to V' and the restriction to H is maximal monotone on H with the domain $D(\mathscr{F}_N) \supseteq D(\mathscr{A})$ (see e.g. [Brez73, Barb76]).
Moreover, we have $D(\mathscr{F}_N) = D(\mathscr{A})$. For this we use the perturbation theorem for nonlinear m-accretive operators and split $\mathscr{F}_N$ into a continuous and a ω-m-accretive operator on H

$$\mathscr{F}_N^1 = (1 - \frac{\varepsilon}{2})\mathscr{A}, \quad D(\mathscr{F}_N^1) = D(\mathscr{A}),$$
$$\mathscr{F}_N^2 = \frac{\varepsilon}{2}\mathscr{A} + \mathscr{B}_N(\cdot) + \alpha_N I, \quad D(\mathscr{F}_N^2) = \{(w, W) \in V, \mathscr{F}_N^2(w, W) \in H\}.$$

As seen above by (6.2.13) we have

$$\left\|\mathscr{F}_N^2(w, W)\right\|_0 \leq \frac{\varepsilon}{2}\|\mathscr{A}(w, W)\|_0 + \|\mathscr{B}_N(w, W)\|_0 + \alpha_N\|(w, W)\|_0$$
$$\leq \varepsilon\|\mathscr{A}(w, W)\|_0 + \alpha_N\|(w, W)\|_0 + \frac{C_N^2}{2\varepsilon}, \quad \text{for all } (w, W) \in D(\mathscr{F}_N^1) = D(\mathscr{A}),$$

where $0 < \varepsilon < 1$.
Since $\mathscr{F}_N^1 + \mathscr{F}_N^2 = \Gamma_N + \alpha_N I$ we infer that $\Gamma_N + \alpha_N I$ with domain $D(\mathscr{A})$ is m-accretive in H as claimed. □

Proof of Theorem 6.1. As a consequence of Lemma 6.1 (see, e.g., [Barb76, Barb93]) we have that for $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in D(\mathscr{A})$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in W^{1,1}([0,T], H)$ the equation

$$\begin{aligned} &\frac{d}{dt}(w,W) + \mathscr{A}(w,W)(t) + \mathscr{B}_N((w,W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}),\ t \in (0,T), \\ &(w,W)(0) = (\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}), \end{aligned} \tag{6.2.15}$$

has a unique strong solution $(w_N, W_N) \in W^{1,\infty}([0,T]; H) \cap L^\infty(0,T; D(\mathscr{A}))$.
By a density argument (see, e.g., [Barb93, JLL69]) it can be shown that if $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in L^2(0,T,V')$ then there exist absolute continuous functions $(w_N, W_N) : [0,T] \to V'$ that satisfy $(w_N, W_N) \in C([0,T]; H) \cap L^2(0,T:V) \cap W^{1,2}([0,T], V')$ and (6.2.15) a.e. in $(0,T)$, where d/dt is considered in the strong topology of V'.

First, we show that $D(\mathscr{A})$ is dense in H. Indeed, if $(w,W) \in H$ we set $(w_\varepsilon, W\varepsilon) = (I + \varepsilon\Gamma_N)^{-1}(w,W)$, where I is the unity operator in H. Multiplying the equation

$$(w_\varepsilon, W_\varepsilon) + \varepsilon\Gamma_N(w_\varepsilon, W_\varepsilon) = (w,W)$$

by $(w_\varepsilon, W_\varepsilon)$ it follows by (6.2.3), (6.2.11) that

$$\|(w_\varepsilon, W_\varepsilon)\|_0^2 + 2\varepsilon\nu\|(w_\varepsilon, W_\varepsilon)\|_1^2 \le \|(w,W)\|_0^2$$

and by (6.2.10)

$$\|(w_\varepsilon - w, W_\varepsilon - W)\|_{-1} = \varepsilon\|\Gamma_\varepsilon(w_\varepsilon, W_\varepsilon)\|_{-1} \le \varepsilon N\|(w_\varepsilon, W_\varepsilon)\|_0^{1/2}\|(w_\varepsilon, W_\varepsilon)\|_1^{1/2}.$$

Hence, $\{(w_\varepsilon, W_\varepsilon)\}$ is bounded in H and $(w_\varepsilon, W_\varepsilon) \to (w,W)$ in V' as $\varepsilon \to 0$. Therefore, $(w_\varepsilon, W_\varepsilon) \rightharpoonup (w,W)$ in H as $\varepsilon \to 0$, which implies that $D(\Gamma_N)$ is dense in H.

Secondly, let $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in H$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \in L^2(0,T,V')$. Then there are sequences $\{(\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2})\} \subset D(\Gamma_N)$, $\{(\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2})\} \subset W^{1,1}([0,T]; H)$ such that

$$\begin{aligned} &(\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2}) \to (\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \quad \text{in } H, \\ &(\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2}) \to (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) \quad \text{in } L^2(0,T;V'), \end{aligned}$$

as $n \to \infty$. Let $(w_N^n, W_N^n) \in W^{1,\infty}([0,T];H)$ be the solution to problem (6.2.15) where $(w, W)(0) = (\overline{u_0}_n^{\delta_1}, \overline{B_0}_n^{\delta_2})$ and $(\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}^{\delta_2}) = (\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2})$. By (6.2.14) we have

$$\begin{aligned}
&\frac{d}{dt}\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_0^2 + \frac{\nu}{2}\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_1^2 \\
&\quad \le 2C_N\|(w_N^n - w_N^m, W_N^n - W_N^m)\|_0^2 + \frac{2}{\nu}\|(\overline{\boldsymbol{f}}_n^{\delta_1} - \overline{\boldsymbol{f}}_m^{\delta_1}, \operatorname{curl}(\overline{\boldsymbol{g}}_n^{\delta_2} - \overline{\boldsymbol{g}}_m^{\delta_2}))\|_{-1}^2,
\end{aligned}$$

for a.e. $t \in (0,T)$. By the Gronwall inequality we obtain

$$\begin{aligned}
&\|(w_N^n - w_N^m, W_N^n - W_N^m)(t)\|_0^2 \le e^{2C_N t}\|(\overline{u_0}_n^{\delta_1} - \overline{u_0}_m^{\delta_1}, \overline{B_0}_n^{\delta_2} - \overline{B_0}_m^{\delta_2})\|_0^2 \\
&\quad + \frac{2e^{2C_N t}}{\nu}\int_0^t \|(\overline{\boldsymbol{f}}_n^{\delta_1} - \overline{\boldsymbol{f}}_m^{\delta_1}, \operatorname{curl}(\overline{\boldsymbol{g}}_n^{\delta_2} - \overline{\boldsymbol{g}}_m^{\delta_2}))(\tau)\|_{-1}^2 d\tau.
\end{aligned}$$

Hence

$$(w_N(t), W_N(t)) = \lim_{n\to\infty}(w_N^n(t), W_N^n(t))$$

exists in H uniformly in t on $[0,T]$. Similarly we obtain

$$\begin{aligned}
&\|w_N^n(t)\|_0^2 + \|W_N^n(t)\|_0^2 + \int_0^t \left(\frac{1}{\mathrm{Re}}(\|\nabla w_N^n(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}(\|\operatorname{curl} W_N^n(s)\|_0^2\right) ds \\
&\quad \le C_N\left[\|\overline{u_0}_n^{\delta_1}\|_0^2 + \|\overline{B_0}_n^{\delta_2}\|_0^2 + \int_0^t \left(\|\overline{\boldsymbol{f}}_n^{\delta_1}(s)\|_{-1}^2 + \|\operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2}(s)\|_{-1}^2\right) ds\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \left\|\frac{d}{dt}(w_N^n, W_N^n)(t)\right\|_{-1}^2 dt \\
&\quad \le C_N\left[\|\overline{u_0}_n^{\delta_1}\|_0^2 + \|\overline{B_0}_n^{\delta_2}\|_0^2 + \int_0^t \left(\|\overline{\boldsymbol{f}}_n^{\delta_1}(s)\|_{-1}^2 + \|\operatorname{curl} \overline{\boldsymbol{g}}_n^{\delta_2}(s)\|_{-1}^2\right) ds\right].
\end{aligned}$$

Hence on a sequence we have

$$\begin{aligned}
&(w_N^n, W_N^n) \to (w_N, W_N) \qquad \text{weakly in } L^2(0,T;V), \\
&\frac{d}{dt}(w_N^n, W_N^n) \to \frac{d}{dt}(w_N, W_N) \qquad \text{weakly in } L^2(0,T;V'),
\end{aligned}$$

where $d(w_N, W_N)/dt$ is considered in the sense of V'-valued distributions on $(0,T)$. We proved that $(w_N, W_N) \in C([0,T];H) \cap L^2(0,T;V) \cap W^{1,2}([0,T];V')$.

It remains to prove that (w_N, W_N) satisfies the equation (6.2.15) a.e. on $(0,T)$. Let $(w,W) \in V$ be arbitrary but fixed. We multiply the equation

$$\frac{d}{dt}(w_N^n, W_N^n) + \Gamma_N(w_N^n, W_N^n) = (\overline{\boldsymbol{f}}_n^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2}), \quad \text{a.e. } t \in (0,T),$$

by $(w_N^n - w, W_N^n - W)$, integrate on (s,t) and get

$$\begin{aligned}
&\frac{1}{2}\Big(\|(w_N^n(t), W_N^n(t)) - (w,W)\|_0^2 - \|(w_N^n(s), W_N^n(s)) - (w,W)\|_0^2\Big) \\
&\quad \le \int_s^t \langle (\overline{\boldsymbol{f}}_n^{\delta_1}(\tau), \operatorname{curl}\overline{\boldsymbol{g}}_n^{\delta_2}(\tau)) - \Gamma_N(w,W), (w_N^n(\tau), W_N^n(\tau)) - (w,W)\rangle d\tau.
\end{aligned}$$

After we let $n \to \infty$ we get

$$\begin{aligned}
&\left\langle \frac{(w_N(t), W_N(t)) - (w_N(s), W_N(s))}{t-s}, (w_N(s), W_N(s)) - (w,W) \right\rangle \\
&\quad \le \frac{1}{t-s}\int_s^t \langle (\overline{\boldsymbol{f}}^{\delta_1}(\tau), \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}(\tau)) - \Gamma_N(w,W), (w_N(\tau), W_N(\tau)) - (w,W)\rangle d\tau.
\end{aligned} \tag{6.2.16}$$

Let t_0 denote a point at which (w_N, W_N) is differentiable and

$$(\overline{\boldsymbol{f}}^{\delta_1}(t_0), \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}(t_0)) = \lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h} (\overline{\boldsymbol{f}}^{\delta_1}(h), \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2}(h))dh.$$

Then by (6.2.16) we have

$$\left\langle \frac{d(w_N, W_N)}{dt}(t_0) - (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2})(t_0) + \Gamma_N(w,W), (w_N, W_N)(t_0) - (w,W) \right\rangle \le 0.$$

Since (w,W) is arbitrary in V and Γ_N is maximal monotone in $V \times V'$ we conclude that

$$\frac{d(w_N, W_N)}{dt}(t_0) + \Gamma_N(w_N, W_N)(t_0) = (\overline{\boldsymbol{f}}^{\delta_1}, \operatorname{curl}\overline{\boldsymbol{g}}^{\delta_2})(t_0).$$

If we multiply (6.2.15) by $(A_{\delta_1} w_N, S A_{\delta_2} W_N)$, use (6.2.3) and integrate in time we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|w_N(t)\|_0^2 + S\|W_N(t)\|_0^2\right) + \frac{{\delta_1}^2}{2}\|\nabla w_N(t)\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\operatorname{curl} W_N(t)\|_0^2 \\
&\quad + \int_0^t \bigg(\frac{1}{\mathrm{Re}}(\|\nabla w_N(s)\|_0^2 + {\delta_1}^2\|\Delta w_N(s)\|_0^2) \\
&\quad + \frac{S}{\mathrm{Re_m}}(\|\operatorname{curl} W_N(s)\|_0^2 + {\delta_2}^2\|\operatorname{curl}\operatorname{curl} W_N(s)\|_0^2)\bigg) ds \\
&= \frac{1}{2}\left(\|\overline{u_0}^{\delta_1}\|_0^2 + S\|\overline{B_0}^{\delta_2}\|_0^2\right) + \frac{{\delta_1}^2}{2}\|\nabla \overline{u_0}^{\delta_1}\|_0^2 + \frac{{\delta_2}^2 S}{2}\|\operatorname{curl}\overline{B_0}^{\delta_2}\|_0^2 \\
&\quad + \int_0^t \left(\|\overline{f}^{\delta_1}(s)\|_{-1}\|w_N(s)\|_1 + S\|\operatorname{curl}\overline{g}^{\delta_2}(s)\|_{-1}\|W_N(s)\|_1\right) ds.
\end{aligned}$$

Using the Cauchy-Schwarz and Gronwall inequalities this implies

$$\|(w_N, W_N)(t)\|_1 \leq C_{\delta_1,\delta_2} \quad \text{for all } t \in (0,T),$$

where C_{δ_1,δ_2} is independent of N. In particular, for N sufficiently large it follows from (6.2.10) that $\mathscr{B}_N = \mathscr{B}$ and $(w_N, W_N) = (w, W)$ is a solution to (6.1.7).

In the following we prove the uniqueness of the weak solution. Let (w_1, W_1) and (w_2, W_2) be two solutions of the system (6.2.5) and set $\varphi = w_1 - w_2$, $\Phi = B_1 - B_2$. Thus (φ, Φ) is a solution to the problem

$$\begin{aligned}
&\frac{d}{dt}(\varphi, \Phi) + \mathscr{A}(\varphi, \Phi)(t) = -\mathscr{B}((w_1, W_1)(t)) + \mathscr{B}((w_2, W_2)(t)), \quad t \in (0,T),\\
&(\varphi, \Phi)(0) = (0,0).
\end{aligned}$$

We take $(A_{\delta_1}\varphi, SA_{\delta_2}\Phi)$ as test function, integrate in space, use the incompressibility condition (6.2.3) and the estimate (6.2.4) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_0^2 + {\delta_1}^2\|\nabla\varphi\|_0^2 + S\|\Phi\|_0^2 + S{\delta_2}^2\|\nabla\Phi\|_0^2\right)\\
&\qquad + \frac{1}{\mathrm{Re}}\left(\|\nabla\varphi\|_0^2 + \delta_1^2\|\Delta\varphi\|_0^2\right) + \frac{S}{\mathrm{Re_m}}\left(\|\nabla\Phi\|_0^2 + \delta_2^2\|\Delta\Phi\|_0^2\right)\\
&= \mathscr{B}_0((\varphi,\Phi),(w_1,W_1),(A_{\delta_1}\varphi, SA_{\delta_2}\Phi))\\
&\leq C\|(w_1,W_1)\|_0\|(\varphi,\Phi)\|_0^{1/2}\|(\nabla\varphi,\nabla\Phi)\|_0^{3/2}\\
&\leq C_{\delta_1,\delta_2}\|(w_1,W_1)\|_0\left(\|\varphi\|_0^2 + {\delta_1}^2\|\nabla\varphi\|_0^2 + S\|\Phi\|_0^2 + S{\delta_2}^2\|\nabla\Phi\|_0^2\right).
\end{aligned}$$

Applying the Gronwall's lemma we deduce that (φ, Φ) vanishes for all $t \in [0,T]$, and hence the uniqueness of the solution. □

Remark 6.1. *The pressure is recovered from the weak solution via the classical DeRham theorem (see [Lera34]).*

6.2.3 Regularity

Theorem 6.2. *Let $m \in \mathbb{N}$, $(u_0, B_0) \in V \cap H^{m-1}(\Omega)$ and $(f, curl\, g) \in L^2(0,T;\ H^{m-1}(\Omega))$. Then there exists a unique solution w, W, q to the equation (6.1.7) such that*

$$(w, W) \in L^\infty(0,T; H^{m+1}(\Omega)) \cap L^2(0,T; H^{m+2}(\Omega)),$$
$$q \in L^2(0,T; H^m(\Omega)).$$

Proof. The result is already proved when $m = 0$ in Theorem 6.1. For any $m \in \mathbb{N}^*$, we assume that

$$(w, W) \in L^\infty(0,T; H^m(\Omega)) \cap L^2(0,T; H^{m+1}(\Omega)) \tag{6.2.17}$$

so it remains to prove

$$(D^m w, D^m W) \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

where D^m denotes any partial derivative of total order m. We take the m^{th} derivative of (6.1.7) and have

$$\begin{aligned}
&(D^m w)_t - \frac{1}{\mathrm{Re}}\Delta(D^m w) + \overline{D^m(w\cdot\nabla w)}^{\delta_1} - S\overline{D^m(W\cdot\nabla W)}^{\delta_1} = \overline{D^m f}^{\delta_1},\\
&(D^m W)_t + \frac{1}{\mathrm{Re_m}}\nabla\times\nabla\times(D^m W) + \overline{D^m(w\cdot\nabla W)}^{\delta_2} - \overline{D^m(W\cdot\nabla w)}^{\delta_2} = \nabla\times\overline{D^m g}^{\delta_2},\\
&\nabla\cdot(D^m w) = 0, \nabla\cdot(D^m W) = 0,\\
&D^m w(0,\cdot) = D^m\overline{u_0}^{\delta_1}, D^m W(0,\cdot) = D^m\overline{B_0}^{\delta_2},
\end{aligned}$$

with periodic boundary conditions and zero mean, and the initial conditions with zero divergence and mean. Taking $A_{\delta_1}D^m w, A_{\delta_1}D^m W$ as test functions we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|D^m w\|_0^2 + {\delta_1}^2\|\nabla D^m w\|_0^2 + S\|D^m W\|_0^2 + S{\delta_2}^2\|\nabla D^m W\|_0^2\right)\\
&\qquad + \frac{1}{\mathrm{Re}}\left(\|\nabla D^m w\|_0^2 + \delta_1^2\|\Delta D^m w\|_0^2\right) + \frac{1}{\mathrm{Re_m}}\left(\|\nabla D^m W\|_0^2 + \delta_2^2\|\Delta D^m W\|_0^2\right)\\
&= \int_\Omega (D^m f D^m w + \nabla\times g D^m W)\, dx - \mathscr{X},
\end{aligned} \tag{6.2.18}$$

where

$$\mathscr{X}=\int_\Omega\Big(D^m(w\cdot\nabla w)-SD^m(W\cdot\nabla W)\Big)D^m w+\Big(D^m(w\cdot\nabla W)-D^m(W\cdot\nabla w)\Big)D^m W\,dx.$$

Now we apply (6.2.4) and use the induction assumption (6.2.17)

$$\begin{aligned}\mathscr{X}&=\sum_{|\alpha|\le m}\begin{pmatrix} m\\ \alpha\end{pmatrix}\sum_{i,j=1}^{3}\int_\Omega D^\alpha w_i D^{m-\alpha}D_i w_j D^m w_j-SD^\alpha W_i D^{m-\alpha}D_i W_j D^m w_j\\&\qquad-D^\alpha w_i D^{m-\alpha}D_i W_j D^m W_j-D^\alpha W_i D^{m-\alpha}D_i w_j D^m W_j\\&\le\|w\|_{m+1}^{3/2}\|w\|_{m+2}^{1/2}\|w\|_m+\|W\|_{m+1}^{3/2}\|W\|_{m+2}^{1/2}\|w\|_m\\&\qquad+\|w\|_{m+1}\|W\|_{m+1}^{1/2}\|W\|_{m+2}^{1/2}\|W\|_m+\|W\|_{m+1}^{3/2}\|W\|_{m+2}^{1/2}\|W\|_m.\end{aligned}$$

Integrating (6.2.18) on $(0,T)$, using the Cauchy-Schwarz and Hölder inequalities, and the assumption (6.2.17) we obtain the desired result for w, W. We conclude the proof mentioning that the regularity of the pressure term q is obtained via classical methods, see e.g. [Tart78, AmGi94]. □

6.3 ACCURACY OF THE MODEL

We will address first the question of consistency error, i.e., we show in Theorem 6.3 that the solution of the closed model (6.1.7) converges to a weak solution of the MHD equations (6.1.1) when δ_1,δ_2 go to zero. This proves that the model is consistent as $\delta_1,\delta_2\to 0$.
Let τ_u,τ_B,τ_{Bu} denote the model's consistency errors

$$\tau_u=\overline{u}^{\delta_1}\overline{u}^{\delta_1}-uu,\quad \tau_B=\overline{B}^{\delta_2}\overline{B}^{\delta_2}-BB,\quad \tau_{Bu}=\overline{B}^{\delta_2}\overline{u}^{\delta_1}-Bu, \tag{6.3.1}$$

where u, B is a solution of the MHD equations obtained as a limit of a subsequence of the sequence $w_{\delta_1}, W_{\delta_2}$.
We will also prove in Theorem 6.4 that $\|\overline{u}^{\delta_1}-w\|_{L^\infty(0,T;L^2(Q))}$, $\|\overline{B}^{\delta_2}-W\|_{L^\infty(0,T;L^2(Q))}$ are bounded by $\|\tau_u\|_{L^2(Q_T)}$, $\|\tau_B\|_{L^2(Q_T)}$, $\|\tau_{Bu}\|_{L^2(Q_T)}$.

6.3.1 Limit consistency of the model

Theorem 6.3. *There exist two sequences $\delta_1^n, \delta_2^n \to 0$ as $n \to 0$ such that*

$$(w_{\delta_1^n}, W_{\delta_2^n}, q_{\delta_1^n}) \to (u, B, p) \quad \text{as } \delta_1^n, \delta_2^n \to 0,$$

where $(u, B, p) \in L^\infty(0,T;H) \cap L^2(0,t;V) \times L^{\frac{4}{3}}(0,T;L^2(\Omega))$ is a weak solution of the MHD equations (6.1.1). The sequences $\{w_{\delta_1^n}\}_{n\in\mathbb{N}}, \{W_{\delta_2^n}\}_{n\in\mathbb{N}}$ converge strongly to u, B in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$, respectively, while $\{q_{\delta_1^n}\}_{n\in\mathbb{N}}$ converges weakly to p in $L^{\frac{4}{3}}(0,T;L^2(\Omega))$.

Proof. The proof follows that of Theorem 3.1 in [LaLe04], and is an easy consequence of Theorem 6.4 and Proposition 6.2; we will sketch it for the reader's convenience. □

6.3.2 Verifiability of the model

Theorem 6.4. *Suppose that the true solution of (6.1.1) satisfies the regularity condition $(u, B) \in L^4(0,T;V)$. Then $e = \overline{u}^{\delta_1} - w$, $E = \overline{B}^{\delta_2} - W$ satisfy*

$$\begin{aligned} &\|e(t)\|_0^2 + S\|E(t)\|_0^2 + \int_0^t \left(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|curl\,E(s)\|_0^2\right) ds \\ &\quad \le C\Phi(t)\int_0^t \left(\mathrm{Re}\|\tau_u(s) + S\tau_B(s)\|_0^2 + \mathrm{Re_m}\|\tau_{Bu}(s) - {\tau_{Bu}}^T(s)\|_0^2\right) ds, \end{aligned} \tag{6.3.2}$$

where $\Phi(t) = \exp\left\{\mathrm{Re}^3 \int_0^t \|\nabla u\|_0^4 ds, {\mathrm{Re_m}}^3 \int_0^t \|\nabla u\|_0^4 ds + \mathrm{Re_m}\mathrm{Re}^2 \int_0^t \|\nabla B\|_0^4\right\}$.

Proof. The errors $e = \overline{u}^{\delta_1} - w, E = \overline{B}^{\delta_2} - W$ satisfy in variational sense

$$\begin{aligned} &e_t + \nabla\cdot(\overline{\overline{u}^{\delta_1}\overline{u}^{\delta_1} - ww}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta e + S\nabla\cdot(\overline{\overline{B}^{\delta_2}\overline{B}^{\delta_2} - WW}^{\delta_1}) + \nabla(\overline{p}^{\delta_1} - q) \\ &\quad = \nabla\cdot(\overline{\tau}_u^{\delta_1} + S\overline{\tau}_B^{\delta_1}), \\ &E_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}\,\mathrm{curl}\,E + \nabla\cdot(\overline{\overline{B}^{\delta_2}\overline{u}^{\delta_1} - Ww}^{\delta_2}) - \nabla\cdot(\overline{\overline{u}^{\delta_1}\overline{B}^{\delta_2} - wW}^{\delta_2}) \\ &\quad = \nabla\cdot(\overline{\tau}_{Bu}^{\delta_2} - {\overline{\tau}_{Bu}^{\delta_2}}^T), \end{aligned}$$

and $\nabla \cdot e = \nabla \cdot E = 0$, $e(0) = E(0) = 0$. Taking the inner product with $(A_{\delta_1} e, SA_{\delta_2} E)$ we get as in (6.2.8) the energy estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|e\|_0^2 + S\|E\|_0^2 + \delta_1^2\|\nabla e\|_0^2 + \delta_2^2 S\|\operatorname{curl} E\|_0^2\right) \\
&\quad + \frac{1}{\mathrm{Re}}\|\nabla e\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\operatorname{curl}\ \mathrm{E}\|_0^2 + \frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2 + \frac{\delta_2^2 S}{\mathrm{Re_m}}\|\operatorname{curl}\operatorname{curl} E\|_0^2 \\
&\quad + \int_\Omega \Big(\nabla \cdot (\overline{u}^{\delta_1}\overline{u}^{\delta_1} - ww)e + S\nabla \cdot (\overline{B}^{\delta_2}\overline{B}^{\delta_2} - WW)e \\
&\qquad + S\nabla \cdot (\overline{B}^{\delta_2}\overline{u}^{\delta_1} - Ww)E - S\nabla \cdot (\overline{u}^{\delta_1}\overline{B}^{\delta_2} - wW)E\Big)dx \\
&= -\int_\Omega \Big((\tau_u + S\tau_B)\cdot \nabla e + S(\tau_{Bu} - {\tau_{Bu}}^T)\cdot \nabla E\Big)dx \\
&\le \frac{1}{2\mathrm{Re}}\|\nabla e\|_0^2 + \frac{S}{2\mathrm{Re_m}}\|\operatorname{curl} E\|_0^2 + \frac{\mathrm{Re}}{2}\|\tau_u + S\tau_B\|_0^2 + \frac{\mathrm{Re_m}}{2S}\|\tau_{Bu} - {\tau_{Bu}}^T\|_0^2.
\end{aligned}
$$

Since $\overline{u}^{\delta_1}\overline{u}^{\delta_1} - ww = e\overline{u}^{\delta_1} + we$, $\overline{B}^{\delta_2}\overline{B}^{\delta_2} - WW = E\overline{B}^{\delta_2} + WE$, $\overline{B}^{\delta_2}\overline{u}^{\delta_1} - Ww = E\overline{u}^{\delta_1} + We$, $\overline{u}^{\delta_1}\overline{B}^{\delta_2} - wW = e\overline{B}^{\delta_1} + wE$, and $\int_\Omega \nabla \cdot (we)edx = \int_\Omega \nabla \cdot (WE)Edx = 0$ we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2 + S\|E\|_0^2 + \delta_1^2\|\nabla e\|_0^2 + S\delta_2^2\|\operatorname{curl} E\|_0^2\right) \\
&\quad + \frac{1}{\mathrm{Re}}\|\nabla e\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\operatorname{curl}\ \mathrm{E}\|_0^2 + \frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2 + \frac{\delta_2^2 S}{\mathrm{Re_m}}\|\operatorname{curl}\operatorname{curl} E\|_0^2 \\
&\le \int_\Omega \Big(-e\cdot \nabla\overline{u}^{\delta_1} e - S\nabla \cdot (E\overline{B}^{\delta_2})e - S\nabla \cdot (E\overline{u}^{\delta_1})E + Se\cdot \nabla \overline{B}^{\delta_2}E\Big)dx \\
&\quad + \mathrm{Re}\|\tau_u + S\tau_B\|_0^2 + \mathrm{Re_m}\|\tau_{Bu} - {\tau_{Bu}}^T\|_0^2 \\
&\le C\Big(\|\nabla e\|_0^{3/2}\|e\|_0^{1/2}\|\nabla \overline{u}^{\delta_1}\|_0 + 2S\|E\|_0^{1/2}\|\nabla E\|_0^{1/2}\|\nabla \overline{B}^{\delta_2}\|_0\|\nabla e\|_0 \\
&\qquad + S\|E\|_0^{1/2}\|\nabla E\|_0^{3/2}\|\nabla\overline{u}^{\delta_1}\|_0\Big) + \mathrm{Re}\|\tau_u + S\tau_B\|_0^2 + \mathrm{Re_m}\|\tau_{Bu} - {\tau_{Bu}}^T\|_0^2.
\end{aligned}
$$

Using $ab \le \varepsilon a^{4/3} + C\varepsilon^{-3}b^4$ we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2 + S\|E\|_0^2 + \delta_1^2\|\nabla e\|_0^2 + S\delta_2^2\|\operatorname{curl} E\|_0^2\right) \\
&\quad + \frac{1}{\mathrm{Re}}\|\nabla e\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\operatorname{curl}\ \mathrm{E}\|_0^2 + \frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2 + \frac{\delta_2^2 S}{\mathrm{Re_m}}\|\operatorname{curl}\operatorname{curl} E\|_0^2 \\
&\le C\Big(\mathrm{Re}^3\|e\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4 + \mathrm{Re_m}\mathrm{Re}^2\|E\|_0^2\|\nabla\overline{B}^{\delta_2}\|_0^4 + {\mathrm{Re_m}}^3\|E\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4\Big) \\
&\quad + \mathrm{Re}\|\tau_u + S\tau_B\|_0^2 + \mathrm{Re_m}\|\tau_{Bu} - {\tau_{Bu}}^T\|_0^2
\end{aligned}
$$

and by the Gronwall inequality we deduce

$$
\begin{aligned}
&\|e(t)\|_0^2 + S\|E(t)\|_0^2 + \int_0^t \left(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\, E(s)\|_0^2\right) ds \\
&\leq C\Psi(t)\int_0^t \left(\mathrm{Re}\|\tau_u(s) + S\tau_B(s)\|_0^2 + \mathrm{Re_m}\|\tau_{Bu}(s) - {\tau_{Bu}}^T(s)\|_0^2\right) ds,
\end{aligned}
$$

where

$$
\Psi(t) = \exp\left\{\mathrm{Re}^3\int_0^t \|\nabla\overline{u}^{\delta_1}\|_0^4 ds, {\mathrm{Re_m}}^3\int_0^t \|\nabla\overline{u}^{\delta_1}\|_0^4 ds + \mathrm{Re_m}\mathrm{Re}^2\int_0^t \|\nabla\overline{B}^{\delta_2}\|_0^4 ds\right\}.
$$

Using the stability bounds $\|\nabla\overline{u}^{\delta_1}\|_0 \leq \|\nabla u\|_0$, $\|\nabla\overline{B}^{\delta_2}\|_0 \leq \|\nabla B\|_0$ we conclude the proof. □

6.3.3 Consistency error estimate

Here we shall give bounds on the consistency errors (6.3.1) as $\delta_1, \delta_2 \to 0$ in $L^1((0,T)\times\Omega)$ and $L^2((0,T)\times\Omega)$.

Proposition 6.1. *Let us assume that $(f, curl\, g) \in L^2(0,T;V')$. Then the following holds*

$$
\begin{aligned}
&\|\tau_u\|_{L^1(0,T;L^1(\Omega))} \leq 2^{3/2}\delta_1 T^{1/2}\mathrm{Re}^{1/2}\mathscr{E}(T), \\
&\|\tau_B\|_{L^1(0,T;L^1(\Omega))} \leq 2^{3/2}\delta_2 T^{1/2}\frac{{\mathrm{Re_m}}^{1/2}}{S}\mathscr{E}(T), \\
&\|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} \leq 2^{1/2}T^{1/2}\frac{1}{S}(\delta_1\mathrm{Re}^{1/2} + \delta_2{\mathrm{Re_m}}^{1/2})\mathscr{E}(T),
\end{aligned}
\tag{6.3.3}
$$

where

$$
\mathscr{E}(T) = \left(\|u_0\|_0^2 + S\|B_0\|_0^2 + \mathrm{Re}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{\mathrm{Re_m}}{S}\|\mathrm{curl}\, g\|^2_{L^2(0,T;H^{-1}(\Omega))}\right).
$$

Proof. Using the stability bounds we have

$$\begin{aligned}\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\leq \|u+\overline{u}^{\delta_1}\|_{L^2(0,T;L^2(\Omega))}\|\overline{u}^{\delta_1}-u\|_{L^2(0,T;L^2(\Omega))}\\ &\leq 2\|u\|_{L^2(0,T;L^2(\Omega))}\sqrt{2}\delta_1\|\nabla u\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Similarly

$$\begin{aligned}\|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\leq \|B+\overline{B}^{\delta_2}\|_{L^2(0,T;L^2(\Omega))}\|\overline{B}^{\delta_2}-B\|_{L^2(0,T;L^2(\Omega))}\\ &\leq 2\|B\|_{L^2(0,T;L^2(\Omega))}\sqrt{2}\delta_2\|\nabla B\|_{L^2(0,T;L^2(\Omega))},\\ \|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\leq \|\overline{B}^{\delta_2}-B\|_{L^2(Q)}\|\overline{u}^{\delta_1}\|_{L^2(Q)}+\|B\|_{L^2(Q)}\|\overline{u}^{\delta_1}-u\|_{L^2(Q)}\\ &\leq \sqrt{2}\delta_2\|\nabla B\|_{L^2(Q)}\|u\|_{L^2(Q)}+\sqrt{2}\delta_1\|\nabla u\|_{L^2(Q)}\|B\|_{L^2(Q)}.\end{aligned}$$

The classical energy estimates for the MHD system (6.1.1) will yield now (6.3.3). □

Assuming more regularity on (u, B) leads to the sharper bounds on the consistency errors.

Remark 6.2. *Let* $(u, B) \in L^2(0, T; H^2(\Omega))$. *Then*

$$\begin{aligned}\|\tau_u\|_{L^1(0,T;L^1(\Omega))} &\leq C\delta_1^2,\\ \|\tau_B\|_{L^1(0,T;L^1(\Omega))} &\leq C\delta_2^2,\\ \|\tau_{Bu}\|_{L^1(0,T;L^1(\Omega))} &\leq C(\delta_1^2+\delta_2^2),\end{aligned}$$

where $C = C(T, Re, Re_m, \|(u, B)\|_{L^2(0,T;L^2(\Omega))}, \|(u, B)\|_{L^2(0,T;H^2(\Omega))})$.

Proof. The result is obtained by following the proof of Proposition 6.1 and using the bounds

$$\begin{aligned}\|\overline{u}^{\delta_1}-u\|_{L^2(0,T;L^2(\Omega))} &\leq \delta_1^2\|\Delta u\|_{L^2(0,T;L^2(\Omega))},\\ \|\overline{B}^{\delta_2}-B\|_{L^2(0,T;L^2(\Omega))} &\leq \delta_2^2\|\Delta B\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

□

Next we estimate the L^2-norms of the consistency errors $\tau_u, \tau_B, \tau_{Bu}$, which were used in Theorem 6.4 to estimate the filtering errors e, E.

Proposition 6.2. *Let u, B be a solution of the MHD equations (6.1.1) and assume that*

$$(u, B) \in L^4((0,T)\times\Omega) \cap L^2(0,T;H^2(\Omega)).$$

Then we have

$$\begin{aligned}
\|\tau_u\|_{L^2(Q)} &\leq C\delta_1, \\
\|\tau_B\|_{L^2(Q)} &\leq C\delta_2, \\
\|\tau_{Bu}\|_{L^2(Q)} &\leq C(\delta_1+\delta_2),
\end{aligned}$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^2(0,T;H^2(\Omega))})$.

Proof. As in the proof of Proposition 6.1, using the stability bounds we have

$$\begin{aligned}
\|\tau_u\|_{L^2(Q)} &\leq 2\|u\|_{L^4(Q)}\|\overline{u}^{\delta_1} - u\|_{L^4(Q)} \\
&\leq 2^{3/2}\|u\|_{L^4(Q)}\left(\int_0^T \|\overline{u}^{\delta_1} - u\|_{L^2(\Omega)}\|\nabla(\overline{u}^{\delta_1} - u)\|^3_{L^2(\Omega)}dt\right)^{1/4} \\
&\leq 2^{3/2}\|u\|_{L^4(Q)}\left(\int_0^T 4\delta_1^4\|\nabla u\|_{L^2(\Omega)}\|\Delta u\|^3_{L^2(\Omega)}dt\right)^{1/4} \\
&\leq 4\delta_1\|u\|_{L^4(Q)}\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

Similarly we deduce

$$\|\tau_B\|_{L^2(Q)} \leq 4\delta_2\|B\|_{L^4(Q)}\|B\|_{L^2(0,T;H^1(\Omega))}\|B\|_{L^2(0,T;H^2(\Omega))},$$

and

$$\begin{aligned}
\|\tau_{Bu}\|_{L^2(Q)} &\leq \|u\|_{L^4(Q)}\|\overline{B}^{\delta_2} - B\|_{L^4(Q)} + \|B\|_{L^4(Q)}\|\overline{u}^{\delta_2} - u\|_{L^4(Q)} \\
&\leq 2\delta_2\|u\|_{L^4(Q)}\|B\|_{L^2(0,T;H^1(\Omega))}\|B\|_{L^2(0,T;H^2(\Omega))} \\
&\quad + 2\delta_1\|B\|_{L^4(Q)}\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

□

As in Remark 6.2, assuming extra regularity on (u, B) leads to the sharper bounds.

Remark 6.3. *Let*

$$(u, B) \in L^4((0,T)\times\Omega) \cap L^4(0,T;H^2(\Omega)).$$

Then

$$\|\tau_u\|_{L^2(Q)} \le C\delta_1^2,$$
$$\|\tau_B\|_{L^2(Q)} \le C\delta_2^2,$$
$$\|\tau_{Bu}\|_{L^2(Q)} \le C(\delta_1^2 + \delta_2^2),$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^4(0,T;H^2(\Omega))})$.

The proof repeats the one of Remark 6.2.

6.4 CONSERVATION LAWS

As our model is some sort of a regularizing numerical scheme, we would like to make sure that the model inherits some of the original properties of the 3D MHD equations.

It is well known that kinetic energy and helicity are critical in the organization of the flow.

The energy $E = \frac{1}{2}\int_\Omega (u(x)\cdot u(x) + SB(x)\cdot B(x))dx$, the cross helicity $H_C = \frac{1}{2}\int_\Omega (u(x)\cdot B(x))dx$ and the magnetic helicity $H_M = \frac{1}{2}\int_\Omega (\mathbb{A}(x)\cdot B(x))dx$ (where $\mathbb{A}$ is the vector potential, $B = \nabla\times\mathbb{A}$) are the three invariants of the MHD equations (6.1.1) in the absence of kinematic viscosity and magnetic diffusivity ($\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$).

Introduce the characteristic quantities of the model

$$E_{LES} = \frac{1}{2}[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)],$$

$$H_{C,LES} = \frac{1}{2}(A_{\delta_1}w, A_{\delta_2}W),$$

and

$$H_{M,LES} = \frac{1}{2}(A_{\delta_2}W, \overline{\mathbb{A}}^{\delta_2}), \text{ where } \overline{\mathbb{A}}^{\delta_2} = A_{\delta_2}^{-1}\mathbb{A}.$$

This section is devoted to proving that these quantities are conserved by (6.1.7) with the periodic boundary conditions and $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Also, note that

$$E_{LES} \to E, \ H_{C,LES} \to H_C, \ H_{M,LES} \to H_M, \qquad \text{as } \delta_{1,2} \to 0.$$

Theorem 6.5 (Conservation Laws). *The following conservation laws hold,* $\forall T > 0$

$$E_{LES}(T) = E_{LES}(0), \tag{6.4.1}$$

$$H_{C,LES}(T) = H_{C,LES}(0) + C(T) \max_{i=1,2} \delta_i^2, \tag{6.4.2}$$

and

$$H_{M,LES}(T) = H_{M,LES}(0). \tag{6.4.3}$$

Note that the cross helicity $H_{C,LES}$ of the model is not conserved exactly, but it possesses two important properties:

$$H_{C,LES} \to H_C \text{ as } \delta_{1,2} \to 0,$$

and

$$H_{C,LES}(T) \to H_{C,LES}(0) \text{ as } N \text{ increases}.$$

Proof. Start by proving (6.4.1). Consider (6.1.7) with $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Multiply (6.1.7a) by $A_{\delta_1} w$, and multiply (6.1.7b) by $SA_{\delta_2}W$. Integrating both equations over Ω gives

$$\frac{1}{2}\frac{d}{dt}(A_{\delta_1}w, w) = S((\nabla \times W) \times W, w), \tag{6.4.4}$$

$$\frac{1}{2}S\frac{d}{dt}(A_{\delta_2}W, W) - S(W \cdot \nabla w, W) = 0. \tag{6.4.5}$$

Use the identity

$$((\nabla \times v) \times u, w) = (u \cdot \nabla v, w) - (w \cdot \nabla v, u). \tag{6.4.6}$$

Add (6.4.4) and (6.4.5). Using (6.4.6) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)\right]\\ &\quad = S(W \cdot \nabla W, w) - S(w \cdot \nabla W, W) + S(W \cdot \nabla w, W).\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}[(A_{\delta_1}w, w) + S(A_{\delta_2}W, W)] = 0, \tag{6.4.7}$$

which proves (6.4.1).

To prove (6.4.2), multiply (6.1.7a) by $A_{\delta_1}W$, and multiply (6.1.7b) by $A_{\delta_2}w$. Integrating both equations over Ω gives

$$(\frac{\partial A_{\delta_1}w}{\partial t}, W) + (w \cdot \nabla w, W) = 0, \tag{6.4.8}$$

$$(\frac{\partial A_{\delta_2}W}{\partial t}, w) + (w \cdot \nabla W, w) = 0. \tag{6.4.9}$$

Add (6.4.8) and (6.4.9); the identity $(u \cdot \nabla v, w) = -(u \cdot \nabla w, v)$ implies

$$(\frac{\partial A_{\delta_1}w}{\partial t}, W) + (\frac{\partial A_{\delta_2}W}{\partial t}, w) = 0. \tag{6.4.10}$$

It follows from (6.1.8) that

$$w = A_{\delta_1} w + \delta_1^2 \Delta w, \tag{6.4.11}$$
$$W = A_{\delta_2} W + \delta_2^2 \Delta W.$$

Then (6.4.10) gives

$$\begin{aligned}(\frac{\partial A_{\delta_1} w}{\partial t}, A_{\delta_2} W) + (\frac{\partial A_{\delta_2} W}{\partial t}, A_{\delta_1} w)\\ = (\frac{\partial A_{\delta_1} w}{\partial t}, \delta_2^2 \Delta W) + (\frac{\partial A_{\delta_2} W}{\partial t}, \delta_1^2 \Delta w).\end{aligned} \tag{6.4.12}$$

Hence,

$$\begin{aligned}\frac{d}{dt}(A_{\delta_1} w, A_{\delta_2} W) = \delta_2^2 (\frac{\partial A_{\delta_1} w}{\partial t}, \Delta W)\\ + \delta_1^2 (\frac{\partial A_{\delta_2} W}{\partial t}, \Delta w),\end{aligned} \tag{6.4.13}$$

which proves (6.4.2).

Next, we prove (6.4.3) by multiplying (6.1.7b) by $A_{\delta_2} \overline{\mathbb{A}}^{\delta_2}$, and integrating over Ω. This gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\nabla \times A_{\delta_2} \overline{\mathbb{A}}^{\delta_2}, \overline{\mathbb{A}}^{\delta_2})\\ + (w \cdot \nabla W, \overline{\mathbb{A}}^{\delta_2}) - (W \cdot \nabla w, \overline{\mathbb{A}}^{\delta_2}) = 0.\end{aligned} \tag{6.4.14}$$

Since the cross-product of two vectors is orthogonal to each of them,

$$((\nabla \times \overline{\mathbb{A}}^{\delta_2}) \times w, \nabla \times \overline{\mathbb{A}}^{\delta_2}) = 0.$$

It follows from (6.4.15) and (6.4.6) that

$$(w \cdot \nabla \overline{\mathbb{A}}^{\delta_2}, \nabla \times \overline{\mathbb{A}}^{\delta_2}) = ((\nabla \times \overline{\mathbb{A}}^{\delta_2}) \cdot \nabla \overline{\mathbb{A}}^{\delta_2}, w). \tag{6.4.15}$$

Since $W = \nabla \times \overline{\mathbb{A}}^{\delta_2}$, we obtain from (6.4.14) and (6.4.15) that (6.4.3) holds. $\square$

6.5 ALFVÉN WAVES

In this section we prove that our model possesses a very important property of the MHD: the ability of the magnetic field to transmit transverse inertial waves - Alfvén waves. We follow the argument typically used to prove the existence of Alfvén waves in MHD, see, e.g., [Davi01].

Using the density ρ and permeability μ, we write the equations of the model (6.1.7) in the form

$$w_t + \nabla \cdot (\overline{ww^T}^{\delta_1}) + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu} \overline{(\nabla \times W) \times W}^{\delta_1} - \nu \nabla \times (\nabla \times w), \tag{6.5.1a}$$

$$\frac{\partial W}{\partial t} = \overline{\nabla \times (w \times W)}^{\delta_2} - \eta \nabla \times (\nabla \times W), \tag{6.5.1b}$$

$$\nabla \cdot w = 0, \quad \nabla \cdot W = 0, \tag{6.5.1c}$$

where $\nu = \frac{1}{\mathrm{Re}}$, $\eta = \frac{1}{\mathrm{Re_m}}$.

Assume a uniform, steady magnetic field W_0, perturbed by a small velocity field w. We denote the perturbations in current density and magnetic field by j_{model} and W_p, with

$$\nabla \times W_p = \mu j_{model}. \tag{6.5.2}$$

Also, the vorticity of the model is

$$\omega_{model} = \nabla \times w. \tag{6.5.3}$$

Since $w \cdot \nabla w$ is quadratic in the small quantity w, it can be neglected in the Navier-Stokes equation (6.5.1a), and therefore

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho\mu} \overline{(\nabla \times W_p) \times W_0}^{\delta_1} - \nu \nabla \times (\nabla \times w). \tag{6.5.4}$$

The leading order terms in the induction equation (6.5.1b) are

$$\frac{\partial W_p}{\partial t} = \overline{\nabla \times (w \times W_0)}^{\delta_2} - \eta \nabla \times (\nabla \times W_p). \tag{6.5.5}$$

Using (6.5.2), we rewrite (6.5.4) as

$$\frac{\partial w}{\partial t} + \nabla \overline{p}^{\delta_1} = \frac{1}{\rho}\overline{j_{model} \times W_0}^{\delta_1} + \nu \Delta w. \tag{6.5.6}$$

Take *curl* of (6.5.6) and use the identity (6.2.7). Since $\nabla W_0 = 0$, we obtain from (6.5.3) that

$$\frac{\partial \omega_{model}}{\partial t} = \frac{1}{\rho}\overline{W_0 \cdot \nabla j_{model}}^{\delta_1} + \nu \Delta \omega_{model}. \tag{6.5.7}$$

Taking *curl* of (6.5.5) and using (6.5.2),(6.5.3) yields

$$\mu \frac{\partial j_{model}}{\partial t} = \overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta\mu \Delta j_{model}. \tag{6.5.8}$$

Divide (6.5.8) by μ to obtain

$$\frac{\partial j_{model}}{\partial t} = \frac{1}{\mu}\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta \Delta j_{model}. \tag{6.5.9}$$

We now eliminate j_{model} from (6.5.7) by taking the time derivative of (6.5.7) and substituting for $\frac{\partial j_{model}}{\partial t}$ using (6.5.9). This yields

$$\frac{\partial^2 \omega_{model}}{\partial t^2} = \frac{1}{\rho}\overline{W_0 \cdot \nabla \Big(\frac{1}{\mu}\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2} + \eta \Delta j_{model}\Big)}^{\delta_1} + \nu \Delta \frac{\partial \omega_{model}}{\partial t}. \tag{6.5.10}$$

The linearity of $A_{\delta_1}^{-1}$ implies

$$\begin{aligned} \frac{\partial^2 \omega_{model}}{\partial t^2} &= \frac{1}{\rho\mu}\overline{W_0 \cdot \nabla (\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2})}^{\delta_1} \\ &+ \frac{\eta}{\rho}\overline{W_0 \cdot \nabla (\Delta j_{model})}^{\delta_1} + \nu \Delta \frac{\partial \omega_{model}}{\partial t}. \end{aligned} \tag{6.5.11}$$

In order to eliminate the term containing Δj_{model} from (6.5.11), we take the Laplacian of (6.5.7):

$$\Delta \frac{\partial \omega_{model}}{\partial t} = \frac{1}{\rho}\overline{W_0 \cdot \nabla (\Delta j_{model})}^{\delta_1} + \nu \Delta^2 \omega_{model}. \tag{6.5.12}$$

It follows from (6.5.11)-(6.5.12) that

$$\begin{aligned} \frac{\partial^2 \omega_{model}}{\partial t^2} &= \frac{1}{\rho\mu}\overline{W_0 \cdot \nabla (\overline{W_0 \cdot \nabla \omega_{model}}^{\delta_2})}^{\delta_1} \\ &+ (\eta + \nu)\Delta \frac{\partial \omega_{model}}{\partial t} - \eta\nu \Delta^2 \omega_{model}. \end{aligned} \tag{6.5.13}$$

Next we look for the plane-wave solutions of the form

$$\omega_{model} \sim \omega_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\theta t)}, \tag{6.5.14}$$

where $\mathbf{k}$ is the wavenumber. It immediately follows from (6.5.14) that

$$\begin{aligned}\frac{\partial \omega_{model}}{\partial t} &= -i\theta\omega_{model},\\ \frac{\partial^2 \omega_{model}}{\partial t^2} &= -\theta^2\omega_{model},\\ \Delta\frac{\partial \omega_{model}}{\partial t} &= i\theta k^2\omega_{model},\\ \Delta^2(\omega_{model}) &= k^4\omega_{model}.\end{aligned} \tag{6.5.15}$$

Substitute (6.5.14) into the wave equation (6.5.13). Using (6.5.15) gives

$$\begin{aligned}-\theta^2\omega_{model} = \frac{1}{\rho\mu}&\overline{W_0\cdot\nabla(\overline{W_0\cdot\nabla\omega_{model}}^{\delta_2})}^{\delta_1}\\ &+(\eta+\nu)i\theta k^2\omega_{model} - \eta\nu k^4\omega_{model}.\end{aligned} \tag{6.5.16}$$

It follows from (6.1.8) that

$$\begin{aligned}\overline{W_0\cdot\nabla\omega_{model}}^{\delta_2} &= W_0\cdot\nabla\omega_{model} + O(\delta_2^2),\\ \overline{W_0\cdot\nabla(\overline{W_0\cdot\nabla\omega_{model}}^{\delta_2})}^{\delta_1} &= (W_0\cdot\nabla)^2\omega_{model} + O(\delta_1^2) + O(\delta_2^2).\end{aligned} \tag{6.5.17}$$

Thus we obtain from (6.5.16),(6.5.17) that

$$\begin{aligned}-\theta^2\omega_{model} = \frac{1}{\rho\mu}&(W_0\cdot\nabla)^2\omega_{model} + (\eta+\nu)i\theta k^2\omega_{model}\\ &- \eta\nu k^4\omega_{model} + O(\delta_1^2+\delta_2^2).\end{aligned} \tag{6.5.18}$$

It follows from (6.5.14) that

$$(W_0\cdot\nabla)^2\omega_{model} = -W_0^2 k_{||}^2\omega_{model}, \tag{6.5.19}$$

where $k_{||}$ is the component of $\mathbf{k}$ parallel to W_0. Hence, (6.5.18),(6.5.19) imply

$$\begin{aligned}-\theta^2\omega_{model} = -\frac{W_0^2k_{||}^2}{\rho\mu}&\omega_{model} + (\eta+\nu)i\theta k^2\omega_{model}\\ &- \eta\nu k^4\omega_{model} + O(\delta_1^2+\delta_2^2).\end{aligned} \tag{6.5.20}$$

This gives

$$-\theta^2 = -\frac{W_0^2 k_{||}^2}{\rho\mu} + (\eta+\nu)i\theta k^2 - \eta\nu k^4 + O(\delta_1^2+\delta_2^2). \tag{6.5.21}$$

Solving this quadratic equation for θ gives the dispersion relationship

$$\theta = -\frac{(\eta+\nu)k^2}{2}i \pm \Big(\sqrt{\frac{W_0^2 k_{||}^2}{\rho\mu} - \frac{(\nu-\eta)^2 k^4}{4}} + O(\delta_1^2+\delta_2^2)\Big). \tag{6.5.22}$$

Hence, for a perfect fluid ($\nu = \eta = 0$) we obtain

$$\theta = \pm\tilde{v}_a k_{||},$$
$$\tilde{v}_a = v_a + O(\delta_1^2+\delta_2^2),$$

where v_a is the Alfvén velocity $W_0/\sqrt{\rho\mu}$.

When $\nu = 0$ and η is small (i.e. for high Re_m) we have

$$\theta = \pm\tilde{v}_a k_{||} - \frac{\eta k^2}{2}i,$$

which represents a transverse wave with a group velocity equal to $\pm v_a + O(\delta_1^2+\delta_2^2)$.

We conclude that our model (6.1.7) preserves the Alfvén waves and the group velocity of the waves $\tilde{v}_a$ tends to the true Alfvén velocity v_a as the radii tend to zero.

7.0 APPROXIMATE DECONVOLUTION MODELS FOR MAGNETOHYDRODYNAMICS

7.1 INTRODUCTION

Magnetically conducting fluids arise in important applications including climate change forecasting, plasma confinement, controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD sea water propulsion. In many of these, turbulent MHD (magnetohydrodynamics [Alfv42]) flows are typical. The difficulties of accurately modeling and simulating turbulent flows are magnified many times over in the MHD case. They are evinced by the more complex dynamics of the flow due to the coupling of Navier-Stokes and Maxwell equations via the Lorentz force and Ohm's law.

The flow of an electrically conducting fluid is affected by Lorentz forces, induced by the interaction of electric currents and magnetic fields in the fluid. The Lorentz forces can be used to control the flow and to attain specific engineering design goals such as flow stabilization, suppression or delay of flow separation, reduction of near-wall turbulence and skin friction, drag reduction and thrust generation. There is a large body of literature dedicated to both experimental and theoretical investigations on the influence of electromagnetic force on flows (see e.g., [HeSt95, MeHeHr92, MeHeHr94, GPT, GT05, Tsin90, GL61, TS67, HS95, SB97, BKLL00, GK06]). The MHD effects arising from the macroscopic interaction of liquid metals with applied currents and magnetic fields are exploited in metallurgical processes to control the flow of metallic melts: the electromagnetic stirring of molten metals [MDRV84], electromagnetic turbulence control in induction furnaces [ViRi85], electromagnetic damping of buoyancy-driven flow during solidification [PrIn93], and the electromagnetic shaping of ingots in continuous casting [SaLiEv88].

In Section 6 we considered the problem of modeling the motion of large structures in a viscous, incompressible, electrically conducting, turbulent fluid. We introduced a simple closed LES model, and performed full numerical analysis. This model can be also addressed as zeroth order Approximate Deconvolution Model - referring to the family of models in [AS01]. In this chapter we consider the family of the Approximate Deconvolution Models for MagnetoHydroDynamics (ADM for MHD); we perform the numerical analysis of the models and also verify their physical fidelity.

The mathematical description of the problem proceeds as follows. Assuming the fluid to be viscous and incompressible, the governing equations are the Navier- Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law (see e.g. [Sher65]). Let $\Omega = (0, L)^3$ be the flow domain, and $u(t, x), p(t, x), B(t, x)$ be the velocity, pressure, and the magnetic field of the flow, driven by the velocity body force f and magnetic field force $\operatorname{curl} g$. Then u, p, B satisfy the MHD equations:

$$\begin{aligned} & u_t + \nabla \cdot (uu) - \frac{1}{\mathrm{Re}}\Delta u + \frac{S}{2}\nabla(B^2) - S\nabla \cdot (BB) + \nabla p = f, \\ & B_t + \frac{1}{\mathrm{Re_m}}\operatorname{curl}(\operatorname{curl} B) + \operatorname{curl}(B \times u) = \operatorname{curl} g, \\ & \nabla \cdot u = 0, \nabla \cdot B = 0, \end{aligned} \tag{7.1.1}$$

in $Q = (0, T) \times \Omega$, with the initial data:

$$u(0, x) = u_0(x), \quad B(0, x) = B_0(x) \quad \text{in } \Omega, \tag{7.1.2}$$

and with periodic boundary conditions (with zero mean):

$$\Phi(t, x + Le_i) = \Phi(t, x), i = 1, 2, 3, \quad \int_\Omega \Phi(t, x)dx = 0, \tag{7.1.3}$$

for $\Phi = u, u_0, p, B, B_0, f, g$.

Here Re, $\mathrm{Re_m}$, and S are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. For derivation of (7.1.1), physical interpretation and mathematical analysis, see [Cowl57, LL69, ST83, GMP91] and the references therein.

If $\overline{}^{\delta_1}$, $\overline{}^{\delta_2}$ denote two local, spacing averaging operators that commute with the differentiation, then averaging (7.1.1) gives the following non-closed equations for $\overline{u}^{\delta_1}$, $\overline{B}^{\delta_2}$, $\overline{p}^{\delta_1}$ in $(0,T)\times\Omega$:

$$
\begin{aligned}
&\overline{u}^{\delta_1}_t + \nabla\cdot(\overline{uu}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta\overline{u}^{\delta_1} - S\nabla\cdot(\overline{BB}^{\delta_1}) + \nabla\Big(\frac{S}{2}\overline{B^2}^{\delta_1} + \overline{p}^{\delta_1}\Big) = \overline{f}^{\delta_1},\\
&\overline{B}^{\delta_2}_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\,\overline{B}^{\delta_2}) + \nabla\cdot(\overline{Bu}^{\delta_2}) - \nabla\cdot(\overline{uB}^{\delta_2}) = \mathrm{curl}\,\overline{g}^{\delta_2}, \qquad (7.1.4)\\
&\nabla\cdot\overline{u}^{\delta_1} = 0, \quad \nabla\cdot\overline{B}^{\delta_2} = 0.
\end{aligned}
$$

The usual closure problem which we study here arises because $\overline{uu}^{\delta_1} \neq \overline{u}^{\delta_1}\,\overline{u}^{\delta_1}$, $\overline{BB}^{\delta_1} \neq \overline{B}^{\delta_1}\,\overline{B}^{\delta_1}$, $\overline{uB}^{\delta_2} \neq \overline{u}^{\delta_1}\,\overline{B}^{\delta_2}$. To isolate the turbulence closure problem from the difficult problem of wall laws for near wall turbulence, we study (7.1.1) hence (7.1.4) subject to (7.1.3). The closure problem is to replace the tensors $\overline{uu}^{\delta_1}$, $\overline{BB}^{\delta_1}$, $\overline{uB}^{\delta_2}$ with tensors $\mathscr{T}(\overline{u}^{\delta_1},\overline{u}^{\delta_1})$, $\mathscr{T}(\overline{B}^{\delta_2},\overline{B}^{\delta_2})$, $\mathscr{T}(\overline{u}^{\delta_1},\overline{B}^{\delta_2})$, respectively, depending only on $\overline{u}^{\delta_1},\overline{B}^{\delta_2}$ and not u, B. There are many closure models proposed in large eddy simulation reflecting the centrality of closure in turbulence simulation. Calling w, q, W the resulting approximations to $\overline{u}^{\delta_1},\overline{p}^{\delta_1},\overline{B}^{\delta_2}$, we are led to considering the following model

$$
\begin{aligned}
&w_t + \nabla\cdot\mathscr{T}(w,w) - \frac{1}{\mathrm{Re}}\Delta w - S\mathscr{T}(W,W) + \nabla q = \overline{f}^{\delta_1}\\
&W_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\,W) + \nabla\cdot\mathscr{T}(w,W) - \nabla\cdot\mathscr{T}(W,w) = \mathrm{curl}\,\overline{g}^{\delta_2},\\
&\nabla\cdot w = 0, \quad \nabla\cdot W = 0.
\end{aligned}
$$

With any reasonable averaging operator, the true averages $\overline{u}^{\delta_1},\overline{B}^{\delta_2},\overline{p}^{\delta_1}$ are smoother than u, B, p. We consider the family of closure models, pioneered by Stolz and Adams [AS01]. These Approximate Deconvolution Models (ADM) use the deconvolution operators G_N^1 and G_N^2, that will be defined in Section 7.2. The ADM for the MHD reads

$$
w_t + \nabla\cdot\overline{(G_N^1 w)(G_N^1 w)}^{\delta_1} - \frac{1}{\mathrm{Re}}\Delta w - S\nabla\cdot\overline{(G_N^2 W)\,(G_N^2 W)}^{\delta_1} + \nabla q = \overline{f}^{\delta_1}, \qquad (7.1.5a)
$$

$$
\begin{aligned}
W_t + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}\,W) + \nabla\cdot(\overline{(G_N^2 W)(G_N^1 w)}^{\delta_2}) - \nabla\cdot(\overline{(G_N^1 w)(G_N^2 W)}^{\delta_2}) \qquad (7.1.5b)\\
= \mathrm{curl}\,\overline{g}^{\delta_2},
\end{aligned}
$$

$$
\nabla\cdot w = 0, \quad \nabla\cdot W = 0, \qquad (7.1.5c)
$$

subject to $w(0,x) = \overline{u}_0^{\delta_1}(x), W(0,x) = \overline{B}_0^{\delta_2}(x)$ and periodic boundary conditions (with zero means).

We shall show that the ADM MHD model (7.1.5) has the mathematical properties expected of a model derived from the MHD equations by an averaging operation and which are important for practical computations. Note that $N = 0$ in (7.1.5) leads to the model discussed in [LaTr07, LLMNRST07].

The model considered can be developed for quite general averaging operators, see e.g. [AS01]. The choice of averaging operator in (7.1.5) is a differential filter, defined as follows. Let the $\delta > 0$ denote the averaging radius, related to the finest computationally feasible mesh. (In this chapter we use different lengthscales for the Navier-Stokes and Maxwell equations). Given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^\delta \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$A_\delta \overline{\phi}^\delta := -\delta^2 \Delta \overline{\phi}^\delta + \overline{\phi}^\delta = \phi \quad \text{in } \Omega, \tag{7.1.6}$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation, and with this averaging operator, the model (7.1.5) has consistency $O(\delta^{2N+2})$, i.e.,

$$\begin{aligned}
\overline{uu}^{\delta_1} &= \overline{G_N^1 \overline{u}^{\delta_1}\, G_N^1 \overline{u}^{\delta_1}}^{\delta_1} + O({\delta_1}^{2N+2}),\\
\overline{BB}^{\delta_1} &= \overline{G_N^2 \overline{B}^{\delta_2}\, G_N^2 \overline{B}^{\delta_2}}^{\delta_1} + O({\delta_2}^{2N+2}),\\
\overline{uB}^{\delta_2} &= \overline{G_N^1 \overline{u}^{\delta_1}\, G_N^2 \overline{B}^{\delta_2}}^{\delta_2} + O({\delta_1}^{2N+2} + {\delta_2}^{2N+2}),
\end{aligned}$$

for smooth u, B. We prove that the model (7.1.5) has a unique, strong solution w, W that converges in the appropriate sense $w \to u$, $W \to B$, as $\delta_1, \delta_2 \to 0$.

In Section 7.2 we address the global existence and uniqueness of the solution for the closed MHD model. Section 7.3 treats the questions of limit consistency of the model and verifiability. The conservation of the kinetic energy and helicity for the approximate deconvolution model is presented in Section 7.4. Section 7.5 shows that the model preserves the Alfén waves, with the velocity tending to the velocity of Alfvén waves in the MHD, as the radii δ_1, δ_2 tend to zero. The computational results in Section 7.6 confirm the accuracy and the physical fidelity of the models.

7.2 EXISTENCE AND UNIQUENESS FOR THE ADM MHD EQUATIONS

Introduce the family of the approximate deconvolution operators G_N^1, G_N^2, that are used in the ADM models (7.1.5).

Definition 7.1 (Approximate Deconvolution Operator)**.** *For a fixed finite N, define the Nth approximate deconvolution operators G_N^1 and G_N^2 by*

$$G_N^i \phi = \sum_{n=0}^{N} (I - A_{\delta_i}^{-1})^n \phi, \text{ for } i = 1, 2.$$

Note that since the differential filter A_{δ_i} is self adjoint, G_N^i is also. G_N^i was shown to be an $O(\delta_i^{2N+2})$ approximate inverse to the filter operator $A_{\delta_i}^{-1}$ (see [DE06]). Finally, it is easy to show that since A_{δ_i} commutes with differentiation, so does G_N^i.

Lemma 7.1. *The operator G_N^i is compact, positive, and is an asymptotic inverse to the filter $A_{\delta_i}^{-1}$, i.e., for very smooth ϕ and as $\delta_i \to 0$ satisfies*

$$\begin{aligned} \phi &= G_N^1 \overline{\phi}^{\delta_1} + (-1)^{N+1} \delta_1^{2N+2} \Delta^{N+1} A_{\delta_1}^{-(N+1)} \phi, \\ \phi &= G_N^2 \overline{\phi}^{\delta_2} + (-1)^{N+1} \delta_2^{2N+2} \Delta^{N+1} A_{\delta_2}^{-(N+1)} \phi. \end{aligned} \tag{7.2.1}$$

The proof of Lemma 7.1 can be found in [DE06].

Lemma 7.2. *$\| \cdot \|_{G_N^i}$ defined by $\|v\|_{G_N^i} = (v, G_N^i v)$ is a norm on Ω, equivalent to the $L^2(\Omega)$ norm, and $(\cdot, \cdot)_{G_N^i}$ defined by $(v, w)_{G_N^i} = (v, G_N^i w)$ is an inner product on Ω.*

For the proof see [BIL06].

We shall use the standard notations for function spaces in the space periodic case (see [Tema95]). Let $H_p^m(\Omega)$ denote the space of functions (and their vector valued counterparts also) that are locally in $H^m(\mathbb{R}^3)$, are periodic of period L and have zero mean, i.e. satisfy (7.1.3). We recall the solenoidal spaces

$$H = \{\phi \in H_2^0(\Omega), \nabla \cdot \phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2,$$

$$V = \{\phi \in H_2^1(\Omega), \nabla \cdot \phi = 0 \text{ in } \mathscr{D}(\Omega)'\}^2.$$

We define the operator $\mathscr{A} \in \mathscr{L}(V, V')$ by setting

$$\langle \mathscr{A}(w_1, W_1), (w_2, W_2)\rangle = \int_\Omega \left(\frac{1}{\mathrm{Re}} \nabla w_1 \cdot \nabla w_2 + \frac{1}{\mathrm{Re_m}} \mathrm{curl}\, W_1 \mathrm{curl}\, W_2 \right) dx, \tag{7.2.2}$$

for all $(w_i, W_i) \in V$. The operator $\mathscr{A}$ is an unbounded operator on H, with the domain $D(\mathscr{A}) = \{(w, W) \in V; (\Delta w, \Delta W) \in H\}$ and we denote again by $\mathscr{A}$ its restriction to H. We define also a continuous tri-linear form $\mathscr{B}_0$ on $V \times V \times V$ by setting

$$\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3)) = \int_\Omega \Big(\nabla \cdot \overline{(G_N^1 w_2)(G_N^1 w_1)}^{\delta_1} w_3 \tag{7.2.3}$$
$$-S\nabla \cdot \overline{(G_N^2 W_2)(G_N^2 W_1)}^{\delta_1} w_3 + \nabla \cdot \overline{(G_N^2 W_2)(G_N^1 w_1)}^{\delta_2} W_3 - \nabla \cdot \overline{(G_N^1 w_2)(G_N^2 W_1)}^{\delta_2} W_3 dx$$

and a continuous bilinear operator $\mathscr{B}(\cdot) : V \to V$ with

$$\langle \mathscr{B}(w_1, W_1), (w_2, W_2)\rangle = \mathscr{B}_0((w_1, W_1), (w_1, W_1), (w_2, W_2))$$

for all $(w_i, W_i) \in V$.

The following properties of the trilinear form $\mathscr{B}_0$ hold (see [JLL69, ST83, Gris80, Furs00])

$$\begin{aligned} &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} G_N^1 w_2, S A_{\delta_2} G_N^2 W_2)) = 0, \\ &\mathscr{B}_0((w_1, W_1), (w_2, W_2), (A_{\delta_1} G_N^1 w_3, S A_{\delta_2} G_N^2 W_3)) \\ &\qquad = -\mathscr{B}_0((w_1, W_1), (w_3, W_3), (A_{\delta_1} G_N^1 w_2, S A_{\delta_2} G_N^2 W_2)), \end{aligned} \tag{7.2.4}$$

for all $(w_i, W_i) \in V$. Also

$$\begin{aligned} &|\mathscr{B}_0((w_1, W_1), (w_2, W_2), (w_3, W_3))| \\ &\quad \le C \|(G_N^1 w_1, G_N^2 W_1)\|_{m_1} \|(G_N^1 w_2, G_N^2 W_2)\|_{m_2+1} \|(\overline{w_3}^{\delta_1}, \overline{W_3}^{\delta_2})\|_{m_3} \end{aligned} \tag{7.2.5}$$

for all $(w_1, W_1) \in H^{m_1}(\Omega), (w_2, W_2) \in H^{m_2+1}(\Omega), (w_3, W_3) \in H^{m_3}(\Omega)$ and

$$\begin{aligned} m_1 + m_2 + m_3 \ge \frac{d}{2}, &\quad \text{if } m_i \ne \frac{d}{2} \text{ for all } i = 1, \ldots, d, \\ m_1 + m_2 + m_3 > \frac{d}{2}, &\quad \text{if } m_i = \frac{d}{2} \text{ for any of } i = 1, \ldots, d. \end{aligned}$$

In terms of $V, H, \mathscr{A}, \mathscr{B}(\cdot)$ we can rewrite (7.1.5) as

$$\begin{aligned} &\frac{d}{dt}(w, W) + \mathscr{A}(w, W)(t) + \mathscr{B}((w, W)(t)) = (\overline{\boldsymbol{f}}^{\delta_1}, \mathrm{curl}\, \overline{\boldsymbol{g}}^{\delta_2}), t \in (0, T), \\ &(w, W)(0) = (\overline{u}_0^{\delta_1}, \overline{B}_0^{\delta_2}), \end{aligned} \tag{7.2.6}$$

where $(\boldsymbol{f}, \mathrm{curl}\, \boldsymbol{g}) = P(f, \mathrm{curl}\, g)$, and $P : L^2(\Omega) \to H$ is the Hodge projection.

Theorem 7.1. *For any* $(\overline{u_0}^{\delta_1}, \overline{B_0}^{\delta_2}) \in V$ *and* $(\overline{f}^{\delta_1}, curl\,\overline{g}^{\delta_2}) \in L^2(0,T;H)$ *there exists a unique strong solution to (7.1.5)* $(w,W) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ *and* $w_t, W_t \in L^2((0,T)\times\Omega)$*. Moreover, the following energy equality holds:*

$$\mathscr{E}(t) + \int_0^t \varepsilon(\tau)d\tau = \mathscr{E}(0) + \int_0^t \mathscr{P}(\tau)d\tau, \qquad t \in [0,T], \tag{7.2.7}$$

where

$$\begin{aligned}
\mathscr{E}(t) &= \frac{{\delta_1}^2}{2}\|\nabla w(t,\cdot)\|^2_{G_N^1} + \frac{1}{2}\|w(t,\cdot)\|^2_{G_N^1} + \frac{{\delta_2}^2 S}{2}\|\nabla W(t,\cdot)\|^2_{G_N^2} + \frac{S}{2}\|W(t,\cdot)\|^2_{G_N^2},\\
\varepsilon(t) &= \frac{{\delta_1}^2}{\mathrm{Re}}\|\Delta w(t,\cdot)\|^2_{G_N^1} + \frac{1}{\mathrm{Re}}\|\nabla w(t,\cdot)\|^2_{G_N^1} + \frac{{\delta_2}^2 S}{\mathrm{Re_m}}\|\Delta W(t,\cdot)\|^2_{G_N^2} + \frac{S}{\mathrm{Re_m}}\|\nabla W(t,\cdot)\|^2_{G_N^2}, \quad (7.2.8)\\
\mathscr{P}(t) &= (f(t), G_N^1 w(t)) + S(curl\,g(t), G_N^2 W(t)).
\end{aligned}$$

Proof. (Sketch) The proof follows from [LaTr07], using a semigroup approach and the machinery of nonlinear differential equations of accretive type in Banach spaces. The key to the model, as in MHD, is to make the nonlinear terms to vanish by an appropriate choice of test function. We observe that by (7.2.4)

$$\mathscr{B}_0((w,W),(w,W),(A_{\delta_1}G_N^1 w, SA_{\delta_2}G_N^2 W)) = 0,$$

thus taking the inner product of (7.2.6) with $(A_{\delta_1}G_N^1 w, SA_{\delta_2}G_N^2 W)$ and integrating by parts we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|w\|^2_{G_N^1} + \delta_1^2\|\nabla w\|^2_{G_N^1} + S\|W\|^2_{G_N^2} + \delta_2^2 S\|\nabla W\|^2_{G_N^2}\Big)\\
&\quad + \frac{1}{\mathrm{Re}}\Big(\|\nabla w\|^2_{G_N^1} + \delta_1^2\|\Delta w\|^2_{G_N^1}\Big) + \frac{S}{\mathrm{Re_m}}\Big(\|\nabla W\|^2_{G_N^2} + \delta_2^2 S\|\Delta W\|^2_{G_N^2}\Big)\\
&= (f, G_N^1 w) + S(\mathrm{curl}\,g, G_N^2 W).
\end{aligned}$$

□

The pressure is recovered from the weak solution via the classical DeRham theorem (see [Lera34]).

Theorem 7.2. *Let $m \in \mathbb{N}$, $(u_0, B_0) \in V \cap H^{m-1}(\Omega)$ and $(f, curl\, g) \in L^2(0,T;\ H^{m-1}(\Omega))$. Then there exists a unique solution w, W, q to the equation (7.1.5) such that*

$$(w, W) \in L^\infty(0,T; H^{m+1}(\Omega)) \cap L^2(0,T; H^{m+2}(\Omega)),$$
$$q \in L^2(0,T; H^m(\Omega)).$$

Proof. The result is already proved when $m = 0$ in Theorem 7.1. For any $m \in \mathbb{N}^*$, we assume that

$$(w, W) \in L^\infty(0,T; H^m(\Omega)) \cap L^2(0,T; H^{m+1}(\Omega)) \tag{7.2.9}$$

so it remains to prove

$$(D^m w, D^m W) \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

where D^m denotes any partial derivative of total order m. We take the m^{th} derivative of (7.1.5) and have

$$\begin{aligned}
&D^m w_t - \frac{1}{\text{Re}}\Delta D^m w + D^m\overline{(G_N^1 w \cdot \nabla G_N^1 w)}^{\delta_1} - SD^m\overline{(G_N^2 W \cdot \nabla G_N^2 W)}^{\delta_1} + \nabla D^m q = D^m\overline{f}^{\delta_1},\\
&D^m W_t + \frac{1}{\text{Re}_\text{m}}\nabla\times\nabla\times D^m W + D^m\overline{(G_N^1 w \cdot \nabla G_N^2 W)}^{\delta_2} - D^m\overline{(G_N^2 W \cdot \nabla G_N^1 w)}^{\delta_2}\\
&\qquad = \nabla \times D^m\overline{g}^{\delta_2},\\
&\nabla\cdot D^m w = 0, \nabla\cdot D^m W = 0,\\
&D^m w(0,\cdot) = D^m\overline{u_0}^{\delta_1}, D^m W(0,\cdot) = D^m\overline{B_0}^{\delta_2},
\end{aligned}$$

with periodic boundary conditions and zero mean, and the initial conditions with zero divergence and mean. Taking $A_{\delta_1} D^m w, A_{\delta_2} D^m W$ as test functions we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|D^m w\|_0^2 + {\delta_1}^2\|\nabla D^m w\|_0^2 + S\|D^m W\|_0^2 + S{\delta_2}^2\|\nabla D^m W\|_0^2\right)\\
&\qquad + \frac{1}{\text{Re}}\left(\|\nabla D^m w\|_0^2 + \delta_1^2\|\Delta D^m w\|_0^2\right) + \frac{1}{\text{Re}_\text{m}}\left(\|\nabla D^m W\|_0^2 + \delta_2^2\|\Delta D^m W\|_0^2\right)\\
&= \int_\Omega (D^m f D^m w + \nabla\times D^m g D^m W)\, dx - \mathscr{X},
\end{aligned} \tag{7.2.10}$$

where

$$\mathscr{X} = \int_\Omega \Big(D^m (G_N^1 w \cdot \nabla G_N^1 w) - S D^m (G_N^2 W \cdot \nabla G_N^2 W) \Big) D^m w$$
$$+ \Big(D^m (G_N^1 w \cdot \nabla G_N^2 W) - D^m (G_N^2 W \cdot \nabla G_N^1 w) \Big) D^m W \, dx.$$

Now we apply (7.2.5) and use the induction assumption (7.2.9)

$$\begin{aligned} \mathscr{X} = & \sum_{|\alpha|\le m} \binom{m}{\alpha} \sum_{i,j=1}^{3} \int_\Omega \Big(D^\alpha G_N^1 w_i D^{m-\alpha} D_i G_N^1 w_j - S D^\alpha G_N^2 W_i D^{m-\alpha} D_i G_N^2 W_j \Big) D^m w_j \\ & + \Big(D^\alpha G_N^1 w_i D^{m-\alpha} D_i G_N^2 W_j - D^\alpha G_N^2 W_i D^{m-\alpha} D_i G_N^1 w_j \Big) D^m W_j \, dx \\ \le & \, C(m) \Big(\|G_N^1 w\|_m^{3/2} \|G_N^1 w\|_{m+1}^{1/2} + \|G_N^2 W\|_m^{3/2} \|G_N^2 W\|_{m+1}^{1/2} \Big) \|w\|_m \\ & + \Big(\|G_N^1 w\|_m \|G_N^2 W\|_m^{1/2} \|G_N^2 W\|_{m+1}^{1/2} + \|G_N^2 W\|_m \|G_N^1 w\|_m^{1/2} \|G_N^1 w\|_{m+1}^{1/2} \Big) \|W\|_m. \end{aligned}$$

Integrating (7.2.10) on $(0, T)$, using the Cauchy-Schwarz and Hölder inequalities, Lemma 7.1, 7.2 and the assumption (7.2.9) we obtain the desired result for w, W. We conclude the proof mentioning that the regularity of the pressure term q is obtained via classical methods, see e.g. [Tart78, AmGi94]. □

7.3 ACCURACY OF THE MODEL

We address first the question of consistency, i.e., we show that the solution of the closed model (7.1.5) converges to a solution of the MHD equations (7.1.1) when δ_1, δ_2 tend zero. Let $\tau_u, \tau_B, \tau_{Bu}$ denote

$$\tau_u = G_N^1 \overline{u}^{\delta_1} G_N^1 \overline{u}^{\delta_1} - uu, \;\; \tau_B = G_N^2 \overline{B}^{\delta_2} G_N^2 \overline{B}^{\delta_2} - BB, \;\; \tau_{Bu} = G_N^2 \overline{B}^{\delta_2} G_N^1 \overline{u}^{\delta_1} - Bu, \qquad (7.3.1)$$

where u, B is a solution of the MHD equations obtained as a limit of a subsequence of the sequence $w_{\delta_1}, W_{\delta_2}$.

We prove in Theorem 7.4 that the model's consistency errors $\|\overline{u}^{\delta_1} - w\|_{L^\infty(0,T;L^2(Q))}$, $\|\overline{B}^{\delta_2} - W\|_{L^\infty(0,T;L^2(Q))}$ are bounded by $\|\tau_u\|_{L^2(Q_T)}$, $\|\tau_B\|_{L^2(Q_T)}$, $\|\tau_{Bu}\|_{L^2(Q_T)}$.

7.3.1 Limit consistency of the model

Theorem 7.3. *There exist two sequences* $\delta_1^n, \delta_2^n \to 0$ *as* $n \to 0$ *such that*

$$(w_{\delta_1^n}, W_{\delta_2^n}, q_{\delta_1^n}) \to (u, B, p) \quad \text{as } \delta_1^n, \delta_2^n \to 0,$$

where $(u, B, p) \in L^\infty(0, T; H) \cap L^2(0, T; V) \times L^{\frac{4}{3}}(0, T; L^2(\Omega))$ *is a solution of the MHD equations (7.1.1). The sequences* $\{w_{\delta_1^n}\}_{n\in\mathbb{N}}, \{W_{\delta_2^n}\}_{n\in\mathbb{N}}$ *converge strongly to* u, B *in* $L^{\frac{4}{3}}(0, T; L^2(\Omega))$ *and weakly in* $L^2(0, T; H^1(\Omega))$*, respectively, while* $\{q_{\delta_1^n}\}_{n\in\mathbb{N}}$ *converges weakly to* p *in* $L^{\frac{4}{3}}(0, T; L^2(\Omega))$.

Proof. The proof follows that of Theorem 3.1 in [LaTr07], and is an easy consequence of Theorem 7.4 and Proposition 7.1. □

7.3.2 Verifiability of the model

Theorem 7.4. *Suppose that the true solution of (7.1.1) satisfies the regularity condition* $(u, B) \in L^4(0, T; V)$*. Then the consistency errors* $e = \overline{u}^{\delta_1} - w$, $E = \overline{B}^{\delta_2} - W$ *satisfy*

$$\begin{aligned} &\|e(t)\|_0^2 + S\|E(t)\|_0^2 + \int_0^t \Big(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|curl\, E(s)\|_0^2\Big) ds \\ &\quad\le C\Phi(t)\int_0^t \big(\mathrm{Re}\|\tau_u(s) + S\tau_B(s)\|_0^2 + \mathrm{Re_m}\|\tau_{Bu}(s) - \tau_{Bu}(s)\|_0^2\big)\, ds, \end{aligned} \tag{7.3.2}$$

where $\Phi(t) = \exp\Big\{\mathrm{Re}^3 \int_0^t \|\nabla u\|_0^4 ds, \mathrm{Re_m}^3 \int_0^t \|\nabla u\|_0^4 ds + \mathrm{Re_m}\mathrm{Re}^2 \int_0^t \|\nabla B\|_0^4\Big\}$.

Proof. The errors $e = \overline{u}^{\delta_1} - w, E = \overline{B}^{\delta_2} - W$ satisfy in variational sense

$$\begin{aligned} &e_t + \nabla\cdot(\overline{G_N^1\overline{u}^{\delta_1}G_N^1\overline{u}^{\delta_1} - G_N^1 w G_N^1 w}^{\delta_1}) - \frac{1}{\mathrm{Re}}\Delta e + S\nabla\cdot(\overline{G_N^2\overline{B}^{\delta_2}G_N^2\overline{B}^{\delta_2} - G_N^2 W G_N^2 W}^{\delta_1}) \\ &\quad + \nabla(\overline{p}^{\delta_1} - q) = \nabla\cdot(\overline{\tau}_u^{\delta_1} + S\overline{\tau}_B^{\delta_1}), \\ &E_t + \frac{1}{\mathrm{Re_m}}\nabla\times\nabla\times E + \nabla\cdot\overline{G_N^2\overline{B}^{\delta_2}G_N^1\overline{u}^{\delta_1} - G_N^2 W G_N^1 w}^{\delta_2} - \nabla\cdot\overline{G_N^1\overline{u}^{\delta_1}G_N^2\overline{B}^{\delta_2} - G_N^1 w G_N^2 W}^{\delta_2} \\ &\quad = \nabla\cdot(\overline{\tau}_{Bu}^{\delta_2} - \overline{\tau}_{uB}^{\delta_2}), \end{aligned}$$

and $\nabla\cdot e=\nabla\cdot E=0$, $e(0)=E(0)=0$. Taking the inner product with $(A_{\delta_1}G_N^1 e,\ SA_{\delta_2}G_N^2 E)$ we get as for (7.2.7) the following

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|e\|^2_{G_N^1}+S\|E\|^2_{G_N^2}+\delta_1^2\|\nabla e\|^2_{G_N^1}+\delta_2^2 S\|\mathrm{curl}\,E\|^2_{G_N^2}\Big)\\
&+\frac{1}{\mathrm{Re}}\|\nabla e\|^2_{G_N^1}+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl\ \ E}\|^2_{G_N^2}+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|^2_{G_N^1}+\frac{\delta_2^2 S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|^2_{G_N^2}\\
&+\int_\Omega\!\Big(\nabla\cdot(G_N^1\overline{u}^{\delta_1}G_N^1\overline{u}^{\delta_1}-G_N^1wG_N^1w)G_N^1e+S\nabla\cdot(G_N^2\overline{B}^{\delta_2}G_N^2\overline{B}^{\delta_2}-G_N^2WG_N^2W)G_N^1e\\
&\;+S\nabla\cdot(G_N^2\overline{B}^{\delta_2}G_N^1\overline{u}^{\delta_1}-G_N^2WG_N^1w)G_N^2E-S\nabla\cdot(G_N^1\overline{u}^{\delta_1}G_N^2\overline{B}^{\delta_2}-G_N^1wG_N^2W)G_N^2E\Big)dx\\
&=-\int_\Omega\Big((\tau_u+S\tau_B)\cdot\nabla G_N^1e+S(\tau_{Bu}-\tau_{uB})\cdot\nabla G_N^2E\Big)dx\\
&\le\frac{1}{2\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{2\mathrm{Re_m}}\|\mathrm{curl}\,E\|_0^2+\frac{\mathrm{Re}}{2}\|\tau_u+S\tau_B\|_0^2+\frac{\mathrm{Re_m}}{2S}\|\tau_{Bu}-\tau_{uB}\|_0^2.
\end{aligned}$$

Using the identity $G_N^1\overline{u}^{\delta_1}G_N^1\overline{u}^{\delta_1}-G_N^1wG_N^1w=G_N^1eG_N^1\overline{u}^{\delta_1}+G_N^1wG_N^1e$, Lemmas 7.1, 7.2, the divergence free condition and (7.2.5) we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl\ \ E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le\int_\Omega\Big(-G_N^1e\cdot\nabla G_N^1\overline{u}^{\delta_1}G_N^1e-S\nabla\cdot(G_N^2EG_N^2\overline{B}^{\delta_2})G_N^1e-S\nabla\cdot(G_N^2EG_N^1\overline{u}^{\delta_1})G_N^2E\\
&\qquad+SG_N^1e\cdot\nabla G_N^2\overline{B}^{\delta_2}G_N^2E\Big)dx+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-\tau_{uB}\|_0^2\\
&\le C\Big(\|\nabla e\|_0^{3/2}\|e\|_0^{1/2}\|\nabla\overline{u}^{\delta_1}\|_0+2S\|E\|_0^{1/2}\|\nabla E\|_0^{1/2}\|\nabla\overline{B}^{\delta_2}\|_0\|\nabla e\|_0\\
&\qquad+S\|E\|_0^{1/2}\|\nabla E\|_0^{3/2}\|\nabla\overline{u}^{\delta_1}\|_0\Big)+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-\tau_{uB}\|_0^2.
\end{aligned}$$

Using $ab\le\varepsilon a^{4/3}+C\varepsilon^{-3}b^4$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|e\|_0^2+S\|E\|_0^2+\delta_1^2\|\nabla e\|_0^2+S\delta_2^2\|\mathrm{curl}\,E\|_0^2\right)\\
&\quad+\frac{1}{\mathrm{Re}}\|\nabla e\|_0^2+\frac{S}{\mathrm{Re_m}}\|\mathrm{curl\ \ E}\|_0^2+\frac{\delta_1^2}{\mathrm{Re}}\|\Delta e\|_0^2+\frac{\delta_2^2S}{\mathrm{Re_m}}\|\mathrm{curl}\,\mathrm{curl}\,E\|_0^2\\
&\le C\Big(\mathrm{Re}^3\|e\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4+\mathrm{Re_m}\mathrm{Re}^2\|E\|_0^2\|\nabla\overline{B}^{\delta_2}\|_0^4+\mathrm{Re_m}^3\|E\|_0^2\|\nabla\overline{u}^{\delta_1}\|_0^4\Big)\\
&\quad+\mathrm{Re}\|\tau_u+S\tau_B\|_0^2+\mathrm{Re_m}\|\tau_{Bu}-\tau_{uB}\|_0^2
\end{aligned}$$

and by the Gronwall inequality we deduce

$$\|e(t)\|_0^2 + S\|E(t)\|_0^2 + \int_0^t \Big(\frac{1}{\mathrm{Re}}\|\nabla e(s)\|_0^2 + \frac{S}{\mathrm{Re_m}}\|\mathrm{curl}\, E(s)\|_0^2\Big) ds$$
$$\leq C\Psi(t)\int_0^t \big(\mathrm{Re}\|\tau_u(s) + S\tau_B(s)\|_0^2 + \mathrm{Re_m}\|\tau_{Bu}(s) - \tau_{uB}(s)\|_0^2\big)\, ds,$$

where

$$\Psi(t) = \exp\left\{\mathrm{Re}^3\int_0^t \|\nabla\overline{u}^{\delta_1}\|_0^4 ds, {\mathrm{Re_m}}^3\int_0^t \|\nabla\overline{u}^{\delta_1}\|_0^4 ds + \mathrm{Re_m}\mathrm{Re}^2\int_0^t \|\nabla\overline{B}^{\delta_2}\|_0^4 ds\right\}.$$

Using the stability bounds $\|\nabla\overline{u}^{\delta_1}\|_0 \leq \|\nabla u\|_0$, $\|\nabla\overline{B}^{\delta_2}\|_0 \leq \|\nabla B\|_0$ we conclude the proof. □

7.3.3 Consistency error estimate

The bounds on the errors (7.3.1) are given in the following proposition.

Proposition 7.1. *Let*

$$(u, B) \in L^4((0,T)\times\Omega) \cap L^4(0,T;H^{2N+2}(\Omega)), N \geq 0.$$

Then

$$\|\tau_u\|_{L^2(Q)} \leq C\delta_1^{2N+2},$$
$$\|\tau_B\|_{L^2(Q)} \leq C\delta_2^{2N+2},$$
$$\|\tau_{Bu}\|_{L^2(Q)} \leq C(\delta_1^{2N+2} + \delta_2^{2N+2}),$$

where $C = C(\|(u,B)\|_{L^4((0,T)\times\Omega)}, \|(u,B)\|_{L^4(0,T;H^{2N+2}(\Omega))})$.

The proof uses Lemma 7.1 and follows the outline of the proofs in Section 3.3 of [LaTr07].

7.4 CONSERVATION LAWS

As our model is some sort of a regularizing numerical scheme, we would like to make sure that the model inherits some of the original properties of the 3D MHD equations.

It is well known that kinetic energy and helicity are critical in the organization of the flow.

The energy $E = \frac{1}{2}\int_\Omega (v(x)\cdot v(x) + B(x)\cdot B(x))dx$, the cross helicity $H_C = \frac{1}{2}\int_\Omega (v(x)\cdot B(x))dx$ and the magnetic helicity $H_M = \frac{1}{2}\int_\Omega (\mathbb{A}(x)\cdot B(x))dx$ (where $\mathbb{A}$ is the vector potential, $B = \nabla\times\mathbb{A}$) are the three invariants of the MHD equations (7.1.1) in the absence of kinematic viscosity and magnetic diffusivity ($\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$).

Introduce the characteristic quantities of the model (7.1.5)

$$
\begin{aligned}
E_{ADM} &= \frac{1}{2}[(A_{\delta_1}w, w)_{G_N^1} + (A_{\delta_2}W, W)_{G_N^2}],\\
H_{C,ADM} &= \frac{1}{2}(A_{\delta_1}w, A_{\delta_2}W), \text{ and}\\
H_{M,ADM} &= \frac{1}{2}(A_{\delta_2}W, \overline{\mathbb{A}}^{\delta_2})_{G_N^2}, \text{ where } \overline{\mathbb{A}}^{\delta_2} = A_{\delta_2}^{-1}\mathbb{A}.
\end{aligned}
$$

This section is devoted to proving that these quantities are conserved by (7.1.5) with the periodic boundary conditions and $\frac{1}{\mathrm{Re}} = \frac{1}{\mathrm{Re_m}} = 0$. Also, note that

$$
E_{ADM} \to E,\ H_{C,ADM} \to H_C,\ H_{M,ADM} \to H_M, \qquad \text{as } \delta_{1,2} \to 0.
$$

Theorem 7.5. *The following conservation laws hold,* $\forall T > 0$

$$
E_{ADM}(T) = E_{ADM}(0), \tag{7.4.1}
$$

$$
H_{C,ADM}(T) = H_{C,ADM}(0) + C(T)\max_{i=1,2}\delta_i^{2N+2}, \tag{7.4.2}
$$

and

$$
H_{M,ADM}(T) = H_{M,ADM}(0). \tag{7.4.3}
$$

Remark 7.1. *Note that the cross helicity $H_{C,ADM}$ of the model is not conserved exactly, but it possesses two important properties:*

$$H_{C,ADM} \to H_C \text{ as } \delta_{1,2} \to 0,$$

and

$$H_{C,ADM}(T) \to H_{C,ADM}(0) \text{ as } N \text{ increases.}$$

In the case of equal radii, $\delta_1 = \delta_2$, the following cross helicity is exactly conserved:

$$H_{\times,ADM}(w,W)(t) = \frac{1}{2}\left((w,W)_N + \delta^2(\nabla w, \nabla W)_N\right).$$

Proof. The proof follows the outline of the corresponding proof in [LaTr07]. Consider (7.1.5) with $\frac{1}{\text{Re}} = \frac{1}{\text{Re}_\text{m}} = 0$.

Start by proving (7.4.1). Multiply (7.1.5a) by $A_{\delta_1} G_N^1 w$, and multiply (7.1.5b) by $A_{\delta_2} G_N^2 W$. Integrating both equations over Ω gives

$$\frac{1}{2}\frac{d}{dt}(A_{\delta_1} w, w)_{G_N^1} = ((\nabla \times G_N^2 W) \times G_N^2 W, w)_{G_N^1}, \tag{7.4.4}$$

$$\frac{1}{2}\frac{d}{dt}(A_{\delta_2} W, W)_{G_N^2} - (G_N^2 W \cdot \nabla G_N^1 w, W)_{G_N^2} = 0. \tag{7.4.5}$$

Adding (7.4.4)-(7.4.5) and using the identity

$$((\nabla \times v) \times u, w) = (u \cdot \nabla v, w) - (w \cdot \nabla v, u) \tag{7.4.6}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\left[(A_{\delta_1} w, w)_{G_N^1} + (A_{\delta_2} W, W)_{G_N^2}\right]$$
$$= (G_N^2 W \cdot \nabla G_N^2 W, G_N^1 w) - (G_N^1 w \cdot \nabla G_N^2 W, G_N^2 W) + (G_N^2 W \cdot \nabla G_N^1 w, G_N^2 W) = 0,$$

which yields (7.4.1).

To prove (7.4.2), multiply (7.1.5a)-(7.1.5b) by $A_{\delta_1} G_N^2 W$ and $A_{\delta_2} G_N^1 w$, respectively, and integrate over Ω to get

$$(\frac{\partial A_{\delta_1} w}{\partial t}, W)_{G_N^2} + (G_N^1 w \cdot \nabla G_N^1 w, W)_{G_N^2} = 0, \tag{7.4.7}$$

$$(\frac{\partial A_{\delta_2} W}{\partial t}, w)_{G_N^1} + (G_N^1 w \cdot \nabla G_N^2 W, w)_{G_N^1} = 0. \tag{7.4.8}$$

Adding (7.4.7) and (7.4.8), we obtain

$$(\frac{\partial A_{\delta_1} w}{\partial t}, G_N^2 W) + (\frac{\partial A_{\delta_2} W}{\partial t}, G_N^1 w) = 0. \tag{7.4.9}$$

From Corollary 7.1 it follows that

$$\begin{aligned} G_N^1 w &= A_{\delta_1} w + (-1)^N \delta_1^{2N+2} \Delta^{N+1} A_{\delta_1}^{-N} w, \\ G_N^2 W &= A_{\delta_2} W + (-1)^N \delta_2^{2N+2} \Delta^{N+1} A_{\delta_2}^{-N} W. \end{aligned} \tag{7.4.10}$$

Then (7.4.9) gives

$$\begin{aligned} &\frac{d}{dt}(A_{\delta_1} w, A_{\delta_2} W) = (\frac{\partial A_{\delta_1} w}{\partial t}, A_{\delta_2} W) + (\frac{\partial A_{\delta_2} W}{\partial t}, A_{\delta_1} w) \\ &= (\frac{\partial A_{\delta_1} w}{\partial t}, (-1)^{N+1} \delta_2^{2N+2} \Delta^{N+1} A_{\delta_2}^{-N} W) + (\frac{\partial A_{\delta_2} W}{\partial t}, (-1)^{N+1} \delta_1^{2N+2} \Delta^{N+1} A_{\delta_1}^{-N} w). \\ &= (-1)^{N+1} \delta_2^{2N+2} (\frac{\partial A_{\delta_1} w}{\partial t}, \Delta^{N+1} A_{\delta_2}^{-N} W) + (-1)^{N+1} \delta_1^{2N+2} (\frac{\partial A_{\delta_2} W}{\partial t}, \Delta^{N+1} A_{\delta_1}^{-N} w), \end{aligned} \tag{7.4.11}$$

which proves (7.4.2).

Next, we prove (7.4.3). By multiplying (7.1.5b) by $A_{\delta_2} G_N^2 \overline{\mathbb{A}}^{\delta_2}$, and integrating over Ω we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\nabla \times A_{\delta_2} \overline{\mathbb{A}}^{\delta_2}, G_N^2 \overline{\mathbb{A}}^{\delta_2}) \\ &\quad + (G_N^1 w \cdot \nabla G_N^2 W, G_N^2 \overline{\mathbb{A}}^{\delta_2}) - (G_N^2 W \cdot \nabla G_N^1 w, G_N^2 \overline{\mathbb{A}}^{\delta_2}) = 0. \end{aligned} \tag{7.4.12}$$

Since the cross-product of two vectors is orthogonal to each of them

$$((\nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) \times G_N^1 w, \nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) = 0, \tag{7.4.13}$$

it follows from (7.4.13) and (7.4.6) that

$$(G_N^1 w \cdot \nabla G_N^2 \overline{\mathbb{A}}^{\delta_2}, \nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) = ((\nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}) \cdot \nabla G_N^2 \overline{\mathbb{A}}^{\delta_2}, G_N^1 w). \tag{7.4.14}$$

Since $G_N^2 W = \nabla \times G_N^2 \overline{\mathbb{A}}^{\delta_2}$, we obtain from (7.4.12) and (7.4.14) that (7.4.3) holds. □

7.5 ALFVÉN WAVES

In this section we prove that our model possesses a very important property of the MHD: the ability of the magnetic field to transmit transverse inertial waves - Alfvén waves. We follow the argument typically used to prove the existence of Alfvén waves in MHD, see, e.g., [Davi01].

Using the density ρ and permeability μ, we write the equations of the model (7.1.5) in the form

$$w_t + \nabla\cdot(\overline{(G_N^1 w)(G_N^1 w)}^{\delta_1}) + \nabla\overline{p}^{\delta_1} = \frac{1}{\rho\mu}\overline{(\nabla\times G_N^2 W)\times G_N^2 W}^{\delta_1} - \nu\nabla\times(\nabla\times w), \tag{7.5.1a}$$

$$\frac{\partial W}{\partial t} = \overline{\nabla\times((G_N^1 w)\times(G_N^2 W))}^{\delta_2} - \eta\nabla\times(\nabla\times W), \tag{7.5.1b}$$

$$\nabla\cdot w = 0, \quad \nabla\cdot W = 0, \tag{7.5.1c}$$

where $\nu = \frac{1}{\mathrm{Re}}$, $\eta = \frac{1}{\mathrm{Re_m}}$.

Assume a uniform, steady magnetic field W_0, perturbed by a small velocity field w. We denote the perturbations in current density and magnetic field by j_{model} and W_p, with

$$\nabla\times W_p = \mu j_{model}. \tag{7.5.2}$$

Also, the vorticity of the model is

$$\omega_{model} = \nabla\times w. \tag{7.5.3}$$

Since $G_N^1 w\cdot\nabla G_N^1 w$ is quadratic in the small quantity w, it can be neglected in the Navier-Stokes equation (7.5.1a), and therefore

$$\frac{\partial w}{\partial t} + \nabla\overline{p}^{\delta_1} = \frac{1}{\rho\mu}\overline{(\nabla\times G_N^2 W_p)\times G_N^2 W_0}^{\delta_1} - \nu\nabla\times(\nabla\times w). \tag{7.5.4}$$

The leading order terms in the induction equation (7.5.1b) are

$$\frac{\partial W_p}{\partial t} = \overline{\nabla\times(G_N^1 w\times G_N^2 W_0)}^{\delta_2} - \eta\nabla\times(\nabla\times W_p). \tag{7.5.5}$$

Following the argument of [LaTr07] and using the approximating result of Corollary 7.1, we obtain that in the case of a perfect fluid ($\nu = \eta = 0$) and in the case $\nu = 0$, $\eta \gg 1$ a transverse wave is recovered. The group velocity of the wave is equal to

$$\tilde{v}_a = v_a + O(\delta_1^{2N+2} + \delta_2^{2N+2}),$$

where v_a is the Alfvén velocity $W_0/\sqrt{\rho\mu}$.

We conclude that our model (7.1.5) preserves the Alfvén waves and the group velocity of the waves $\tilde{v}_a$ tends to the true Alfvén velocity v_a as the radii tend to zero.

7.6 COMPUTATIONAL RESULTS

In this section we present computational results for the ADM models of zeroth, first and second order. The convergence rates are presented and the fidelity of the models is verified by comparing the quantities, which are conserved in the ideal inviscid case. The computations are made for the two-dimensional problem, where the energy and enstrophy of the models are compared to those of the averaged MHD.

Consider the MHD flow in $\Omega = (0.5, 1.5) \times (0.5, 1.5)$. The Reynolds number and magnetic Reynolds number are $Re = 10^5, Re_m = 10^5$, the final time is $T = 1/4$, and the averaging radii are $\delta_1 = \delta_2 = h$.

Take

$$f = \begin{pmatrix} \frac{1}{2}\pi \sin(2\pi x) e^{-4\pi^2 t/Re} - x e^{2t} \\ \frac{1}{2}\pi \sin(2\pi y) e^{-4\pi^2 t/Re} - y e^{2t} \end{pmatrix},$$

$$\nabla \times g = \begin{pmatrix} e^t(x - (\cos \pi x \sin \pi y + \pi x \sin \pi x \sin \pi y + \pi y \cos \pi x \cos \pi y) e^{-2\pi^2 t/Re}) \\ e^t(-y - (\sin \pi x \cos \pi y + \pi x \cos \pi x \cos \pi y + \pi y \sin \pi x \sin \pi y) e^{-2\pi^2 t/Re}) \end{pmatrix}.$$

The solution to this problem is

$$u = \begin{pmatrix} -\cos(\pi x)\sin(\pi y)e^{-2\pi^2 t/Re} \\ \sin(\pi x)\cos(\pi y)e^{-2\pi^2 t/Re} \end{pmatrix},$$

$$p = -\frac{1}{2}(\cos(2\pi x) + \cos(2\pi y))e^{-4\pi^2 t/Re},$$

$$B = \begin{pmatrix} xe \\ -ye \end{pmatrix}.$$

Hence, although the theoretical results were obtained only for the periodic boundary conditions, we apply the family of ADMs to the problem with Dirichlet boundary conditions.

The results presented in the following tables are obtained by using the software $FreeFEM+$ $+$. The velocity and magnetic field are sought in the finite element space of piecewise quadratic polynomials, and the pressure in the space of piecewise linears. In order to draw conclusions about the convergence rate, we take the time step $k = h^2$. We compare the solutions (w, W), obtained by the ADM models, to the true solution (u, B) and the average of the true solution $(\bar{u}, \bar{B})$. The second order accuracy in approximating the true solution (u, B) is expected for ADM models of any order, whereas the accuracy in approximating the averaged solution $(\bar{u}, \bar{B})$ should increase as the order of the model increases.

The solution, computed by the zeroth order ADM, approximates both the true solution (u, B) and the average of the true solution ($\bar{u} = (-\delta_1^2\Delta + I)^{-1}u, \bar{B} = (-\delta_2^2\Delta + I)^{-1}B$ with the second order accuracy. The accuracy in approximating the averaged solution increases as the order of the model is increased.

Hence, the computational results verify the claimed accuracy of the model.

Since the flow is not ideal (nonzero power input, nonzero viscosity/magnetic diffusivity, non-periodic boundary conditions), the energy and enstrophy are not conserved. But we expect the energy and enstrophy of the models to approximate the energy and enstrophy of the averaged MHD.

The enstrophy of the first and second order models approximates the enstrophy of the averaged MHD better than the zeroth order model's enstrophy, see Figure 10.

Figure 11 shows that the graph of the models energy is hardly distinguishable from that of the averaged MHD.

Table 13: Approximating the true solution, $Re = 10^5$, $Re_m = 10^5$, Zeroth Order ADM

h	$\|w-u\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-B\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.0862904		0.0253257	
1/8	0.0515562	0.7431	0.0268628	-0.085
1/16	0.0204763	1.3322	0.0132399	1.0207
1/32	0.00611337	1.7439	0.00412013	1.6841
1/64	0.00163356	1.9039	0.001116	1.8844

Table 14: Approximating the true solution, $Re = 10^5$, $Re_m = 10^5$, First Order ADM

h	$\|w-u\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|W-B\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/4	0.086748		0.0219869	
1/8	0.0504853	0.781	0.0146218	0.5885
1/16	0.0196045	1.3647	0.00401043	1.8663
1/32	0.00589278	1.7342	0.00078723	2.3489
1/64	0.00159084	1.8892	0.000170555	2.2065

Zooming in at the final time $t = 0.25$ we verify that the ADM energy approximates the averaged MHD energy better as the model's order increases, see Figure 12.

Table 15: Approximating the true solution, $Re = 10^5$, $Re_m = 10^5$, Second Order ADM

h	$\|w - u\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|W - B\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/4	0.0854318		0.0229699	
1/8	0.0500093	0.7726	0.0170217	0.4324
1/16	0.0194169	1.3649	0.00472331	1.8495
1/32	0.00587995	1.7234	0.000856363	2.4635
1/64	0.00159835	1.8792	0.000167472	2.3543

Table 16: Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$, Zeroth Order ADM

h	$\|w - \bar{u}\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|W - \bar{B}\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/4	0.0247837		0.0253257	
1/8	0.0245241	0.0152	0.0268628	-0.085
1/16	0.0131042	0.9042	0.0132399	1.0207
1/32	0.00434599	1.5923	0.00412013	1.6841
1/64	0.00120907	1.8458	0.001116	1.8844

Table 17: Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$, First Order ADM

h	$\|w - \bar{u}\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|W - \bar{B}\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/4	0.0228254		0.0219869	
1/8	0.015202	0.5864	0.0146218	0.5885
1/16	0.0043297	1.8119	0.00401043	1.8663
1/32	0.000867986	2.3185	0.00078723	2.3489
1/64	0.000192121	2.1757	0.000170555	2.2065

Table 18: Approximating the average solution, $Re = 10^5$, $Re_m = 10^5$, Second Order ADM

h	$\|w - \bar{u}\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|W - \bar{B}\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/4	0.0236209		0.0229699	
1/8	0.0172027	0.4574	0.0170217	0.4324
1/16	0.00506669	1.7635	0.00472331	1.8495
1/32	0.000956194	2.4057	0.000856363	2.4635
1/64	0.000194768	2.2955	0.000167472	2.3543

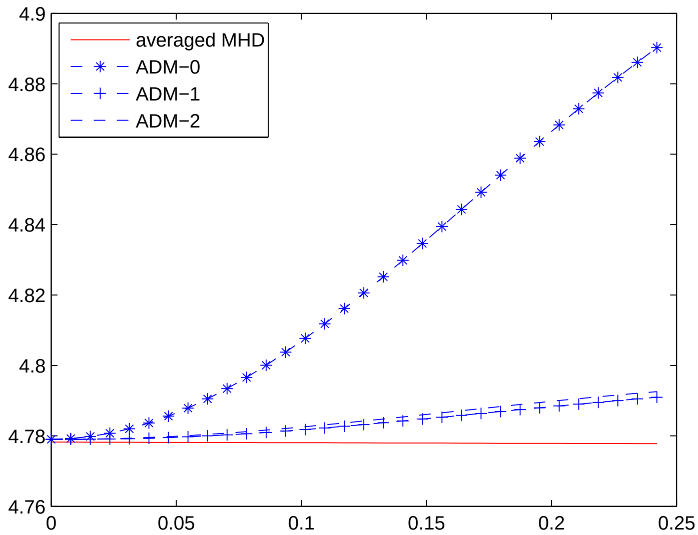

Figure 10: ADM Enstrophy vs. averaged MHD

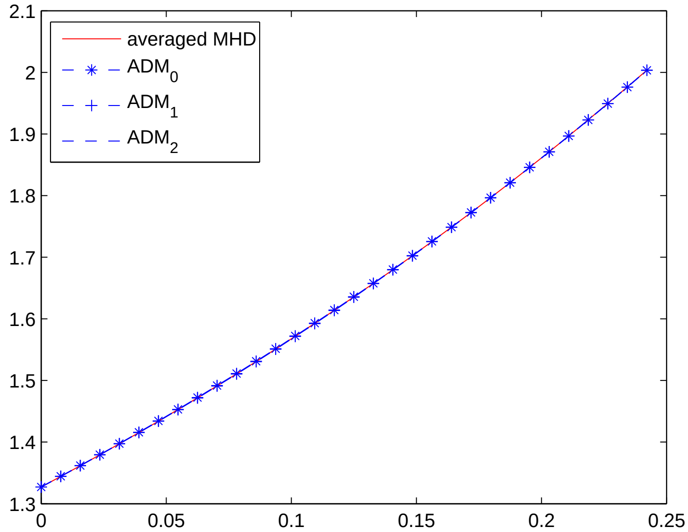

Figure 11: ADM Energy vs. averaged MHD

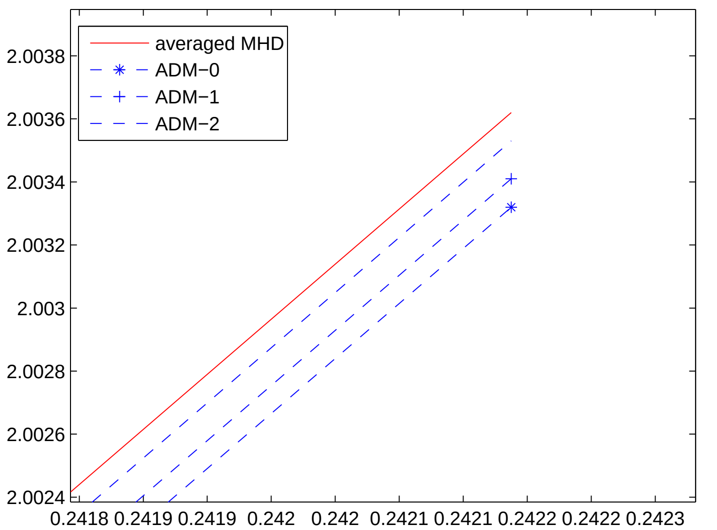

Figure 12: ADM Energy vs. averaged MHD: zoom in

8.0 CONCLUSIONS AND FUTURE RESEARCH

8.1 CONCLUSIONS

We have considered three numerical methods (Chapters 2, 3 and 4) for Navier Stokes equations, aiming at higher Reynolds number. Many iterative methods fail when applied to this type of problems. Often "failure" means that the iterative method used to solve the linear and/or nonlinear system for the approximate solution at the new time level failed to converge within the time constraints of the problem or the resulting approximation had poor solution quality. However, all three of the methods introduced in this work have been shown to overcome both of these types of failure. We proved their stability, performed full numerical analysis of these methods and discussed their physical fidelity. The results of computational tests were provided, proving the effectiveness of these methods.

A Large Eddy Simulation approach to the MagnetoHydroDynamic Turbulence was considered in Chapters 6 and 7. The Approximate Deconvolution Models were introduced for the incompressible MHD equations, and this family of models was analyzed. We proved the existence and uniqueness of solutions, and their convergence in the weak sense to a solution of the MHD equations, as the filtering widths are decreased to zero. We proved the accuracy of the model both theoretically (by establishing an á priori bound on the model's consistency error) and numerically.

Also, all models in the family of the ADMs were proven to possess the physical properties of the MHD - the energy and helicity of the models are conserved, and the models were also proven to preserve the Alfvén waves, a unique feature of the MHD equations. The physical fidelity of the models was also verified computationally. The test results prove that both the solution and the energy of the averaged MHD equations are approximated better, as one increases the models' order N (from zeroth ADM to the first ADM, and from the first to the second ADM). This gives a freedom of choosing the model's order N, based on the desired accuracy of approximation and the available computational power. Finally, the tests demonstrate that in the situations when the direct numerical simulation is no longer available (flows with high Reynolds and magnetic Reynolds numbers), the solution can still be obtained by the ADM approach.

8.2 FUTURE RESEARCH

This thesis can be extended into the following projects.

Defect Correction:

- Extend this idea to turbulent flows. Does the DCM have to be combined with any turbulent models?
- If it is combined, does it improve the results obtained by that turbulence model? Should the DCm be used as a preconditioner?
- DCM near boundaries? Can the higher nonlinearity be embedded into the DCMs so that the boundary layer oscillations could be controlled?

Convection diffusion coupled with porous media:

- Consider the idea of natural convection: coupling the Navier-Stokes equations with convection diffusion.

Turbulence modeling:

- Perform full numerical analysis of the MHD ADMs - fully discrete methods, stability and error analysis. Verify the convergence rates computationally - using either the test space of higher order polynomials, or a spectral (Fourier) code.
- Investigate (theoretically and numerically) the possibility of choosing the averaging radii so that the consistency error of the model is minimized. In any given application we are provided with the empirical data, and our goal is to choose the filtering widths for velocity

and magnetic fields so that the balance is kept between approximating the empirical data and reducing the computational cost.

- Explore the cascades of the conserved quantities - model's energy and magnetic helicity. First we should restrict ourselves to a given application. For instance, one can consider the case of isotropic magnetic field.
- Investigate the pressure in the ADMs. This is related to an idea of drag reduction by the means of magnetic field. It is known (and proven by experiment) that applying the same magnetic field could reduce drag in one region of the flow and at the same time increase the drag in another region. There is a theory that this is related to the pressure.
- Models for compressible turbulence (HD flows). Time relaxation; Large Eddy Simulation. VAst variety of applications are concerned with compressible turbulent flows. There are lots of open questions in this area: how should the turbulence be modeled? Will the LES approach work? Is it going to be dissipative enough? How should the models be modified in order to be applicable in the compressible case? One starting point could be an idea of time relaxation - addition of a lower order term, that drives the fluctuations to zero exponentially fast.

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