S AND L SPACES

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An S-space is any topological space which is hereditarily separable but not Lindelöf. An L-space, on the other hand, is hereditarily Lindelöf but not separable. For almost a century, determining the necessary and sufficient conditions for the existence of these two kinds of spaces has been a fruitful area of research at the boundary of topology and axiomatic set theory. For most of that time, the two problems were imagined to be dual; that is, it was believed that the same sets of conditions that required or precluded one type would suffice for the other as well. This, however, is not the case. When Todorčević proved in 1981 that it is consistent, under ZFC, for no S-spaces to exist, everyone expected a similar result to follow for L-spaces as well. Justin Tatch Moore surprised everyone when, in 2005, he constructed an L-space in ZFC. This paper summarizes and contextualizes that result, along with several others in the field.

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CHAPTER 1

INTRODUCTION TO S AND L SPACES

The joy of the set-theoretic topologist lies in walking a tightrope between what must be and what can be– constructing a precise yet fantastical ontology. All of what follows tries to find a balance between our lust for a diversely populated universe full of the strangely wonderful things that befuddle our intuition and the need for an orderly and controllable universe within the grasp of comprehension. What makes the *S*-spaces & *L*-spaces we will discuss in this paper so very interesting is that they sit just at this boundary.

A space is called an $\mathbf{L}-$space if it is hereditarily Lindelöf, but not hereditarily separable, and an $\mathbf{S}-$space if it is hereditarily separable but not hereditarily Lindelöf. Intuitively, the two notions are in some sense dual; we expect the existence of a regular S space to guarantee that of a regular *L*-space and vice-versa. The explicit existence of these types of spaces and their relation to each other was first conjectured by Hajnal and Juhasz, and became known as the *S* and *L* problems, respectively.

It is quite easy to construct non-regular, T_2 S & L spaces, we do so in 2.1 . Like the examples that follow, they are constructed on an arbitrary cardinality ω_1 subset of the real line. However, the process used destroys the regularity of $\mathbb{R}$.

In 1976 Ostaszewski's *S* space was developed; a similar method soon led to the discovery of the Kunen Line, a more sophisticated version of Ostaszewski's Space. Section 2.3 of this paper contains an explicit and thorough development of the Kunen Line, while Section 2.4 offers a glimpse at the similar Ostaszewski's Space. More recently, an easier construction of an S space was found using the set theoretic assumption $\mathfrak{b} = \omega_1$. It is presented in 2.2. Examples of regular *S* & *L* spaces were thought for many decades to require set theoretic assumptions in excess of the standard ZFC model.

In the early 1980's, Todorcevic proved that, indeed, some assumption(s) in excess of ZFC is necessary to create an *S*-space; by using the method of Proper Forcing, one can create a model of ZFC where every *HS* space is *HL* (i.e., in which no *S*-space can exist). This result solved the *S*- space problem, but the *L*-space problem remained open until very recently. The belief was that the *L*-space problem would have a similar solution; that in some sense the ability of an *S*-space to exist should coincide with that of an *L* space. Stunningly, in 2005 Justin Moore constructed an *L* space without assumptions in excess of ZFC, solving the *L* space problem. We will explore this proof in the final section.

1.1. Cardinal Invariants

1.1.1. Definitions.

w(X):: The weight of a space X is $w(X) = [\min\{\#\mathcal{B} : \mathcal{B} \text{ is a basis for } X\}] \cup \omega$

2^0:: A space is called 2^0 or "second countable" if it has countable weight.

d (X):: The density of a space X is $d(X) = \min\left\{\#S : S \subseteq X \,\&\, \overline{S} = X\right\}$

Separable:: A space with countable density is called "separable".

Cellular:: A pairwise disjoint collection of non-empty open sets in X is called a "cellular family".

c(X):: The cellularity of a space X is the maximal size of a cellular family.

CCC:: A space, X, is said to be CCC (or to have the "*Countable Chain Condition*") if and only if it has countable cellularity.

s (X):: The spread of a space X is the maximal size of a discrete set.

L (X):: The Lindelöf degree of a set X, $L(X)$, is the smallest κ so that every open cover has a sub-cover of size κ.

Lindelöf:: A space is Lindelöf when $L(X) = \omega$.

Hereditary:: For any of the above cardinal invariants, the number $h\Phi = \max\{\kappa : (\forall Y \subseteq X)\,(\Phi(Y) = \kappa)\}$

Hereditarily:: For any of the above properties, a space, X, is said to have property Φ hereditarily, or to be hereditarily Φ, or HΦ whenever every subset of X is Φ.

HL:: Hereditarily Lindelöf

HS:: Hereditarily Separable

HCCC:: Hereditarily CCC

1.1.2. Relations between cardinal invariants.

1.1.2.1. *Hereditary cellularity is less than or equal to hereditary Lindelöf degree.*

Theorem 1.1.1. $hc(X) \leq hL(X)$

PROOF. Assume X is so that $\lambda = hL(X)$ and $\kappa = hc(X)$. Then, $\exists C = \{C_\alpha\}_{\alpha \epsilon \kappa}$ open and pairwise disjoint in X. Let $Y = \bigcup_{\alpha \epsilon \kappa} C_\alpha$. These C_α's form an open cover of Y, so there must be a (at most) λ-sized sub-cover of them. However, the C_α's are pairwise disjoint, so no proper sub-cover can work, and so $\kappa \leq \lambda$ as desired. □

1.1.2.2. *Hereditary cellularity is less than or equal to hereditary density.*

Theorem 1.1.2. $hc(X) \leq hd(X)$

PROOF. Assume every subset, Y, of a space X has a dense subset D_Y with $\#D_Y = \delta$ (and that δ is the minimal such δ). Fix some particular $Y \subseteq X$. Let C be a pairwise disjoint collection of open sets in Y, call them $C = \{C_\alpha\}_{\alpha \epsilon A}$. Each C_α contains some $d_\alpha \epsilon D_Y$, but there are only δ many of the d_α's, so $\#C \leq \delta$. So, $hc(X) \leq hd(X)$ as desired. □

1.1.2.3. *hd(X) vs. hL(X).* The relationship between $hd(X)$ and $hL(X)$ is an interesting and complex one. Even in the countable case, the relationship between hereditary separability and hereditary Lindelöfness is deep. These characteristics seem closely related to one another, and much research has been done into their exact relationship. In particular, a fruitful area of topological research has been the existence of spaces which are hereditarily separable, but not Lindelöf and vice-versa. These are the S and L spaces, respectively.

1.2. Definitions of S and L Spaces

- **Right-Separated::** A space, X, is called right separated (RS) if and only if it can be well ordered in such a way that every initial segment is open.
- **Left-Separated::** A space, X, is called left separated (LS) if and only if it can be well ordered in such a way that every final segment is open.
- **GO-Space::** A GO-space ("*generalized order space*") is a triple $(X, \tau, <)$ where $<$ is a linear ordering of X and τ is a Hausdorff topology on X that has a base of $<$-convex sets.
- **L-Space::** A space is called an L-space if and only if it is HL but not HS.
- **S:** space: A space is called an S-space if and only if it is HS but not HL.

1.3. Lemmas Concerning Right and Left Separability

Lemma 1.3.1. *A space is HCCC if and only if it does not have a uncountable discrete subset.*

PROOF. Assume there is an uncountable discrete set, $D = \{d_a\}_{a \in A}$. Then, surely, for every $d_a \in D$ there is a U_a open in D around each d_a, so that the U_a are pairwise disjoint. Likewise, if we assume X is not HCCC, then there is a $Y \subseteq X$ with an uncountable family of pairwise disjoint open sets. From this, we can choose a parliament (that is, a set containing one representative element from each set). This Parliament is clearly an uncountable discrete set. From this, it is clear that the possession of an uncountable discrete set is equivalent to the possession of an uncountable collection of disjoint subsets. That is to say, a set is *HCCC* if and only if it does not have an uncountable discrete subset, as desired. □

Lemma 1.3.2. *A space is HL if and only if it does not have an uncountable RS set.*

PROOF. Assume that a space X is HL and also assume that $R = \{r_a\}_{a \in A}$is a right separated subset of maximal size in X. Suppose, for a contradiction, that A is uncountable. Then the well ordered set A has an initial segment of order type ω_1. So, without loss of generality, assume that A itself has order type ω_1. From this, we see that for every $a \in A$ there is a U_a open around r_a so that $r_b \in U_a \implies b \leq a$ (the U_a's nest backwards), and that each U_a is countable because each initial segment of A is countable (recall, A has order type ω_1). Now, let $\mathcal{U} = \{U_a a \in A\}$. Clearly, $\mathcal{U}$ is an open cover of R, with no countable subcover (a countable union of countable sets is countable). So, a HL space cannot have an uncountable RS set.

Conversely, if we assume that X is not HL, then there is some subset, Y, with some open cover which has no countable sub-cover. Let $\mathcal{U} = \{U_a\}_{a \in \kappa}$ be that cover. Now, we chose any $r_0 \in U_0$. Next, we can choose an $r_1 \in Y \setminus U_0$. Likewise, for any countable α, we can choose $r_\alpha \in Y \setminus \bigcup_{\beta<\alpha} U_\beta$, because otherwise $\bigcup_{\beta<\alpha} U_\beta$ would be a countable open cover of $\mathcal{U}$. So, we can make an uncountable set $R = \{r_a\}_{a \in \kappa}$ with $\alpha \leq \beta \implies r_\alpha \notin U_\beta$. Equivalently, $r_\beta \in U_\alpha \implies \beta \leq \alpha$, which is the very definition of a right separated set. □

Lemma 1.3.3. *A space is HS if and only if it does not contain an uncountable LS set.*

PROOF. Let $L \subseteq S$ be an uncountable left-ordered set. Let $L = \{l_\alpha\}_{\alpha \in \kappa}$. Since A is well-ordered, it has an initial segment of type ω_1. Without loss of generality, we can assume A itself is of type ω_1. Now, $\forall \alpha \in A\ \exists U_\alpha$ open around $l_\alpha$with $l_\beta \epsilon U_\alpha$ whenever $\beta \geq \alpha$. That is to say, the U_α's look forwards. Further assume that $X \subseteq L$ is HS, so that $\exists D \subseteq L$ with $D = \{l_{\alpha_n}\}_{n \in \omega}$ dense in L. However, that means that $\forall \alpha \forall U_\alpha$open around l_α, $\exists l_{\alpha_n} \in U_\alpha$ so that $l_{\alpha_n} \geq l_\alpha$ (since that's the criteria for openness in LS spaces). However, that means that there are only $\aleph_0$ α's, so indeed X cannot be HS and have an uncountable LS subset.

Conversely, assume that X is not HS, so that there is some subset $Y \subseteq X$ which is separable. Since Y isn't separable, we know that no countable collection can be dense, i.e.: $\forall \alpha < \omega_1$, $\overline{\{x_\alpha\}_{\alpha<\kappa}} \neq Y$. So, we know that we can choose $x_\beta \in Y \setminus \overline{\{x_\alpha\}_{\alpha<\beta}}$ so long as β is countable. That is, we can construct a sequence $\{x_\alpha\}_{\alpha<\omega}$ so that $(\forall\beta)\left(x_\beta \notin \overline{\{x_\alpha\}_{\alpha<\beta}}\right)$ so, since Y is T_3, we can form, $(\forall\alpha < \beta)$, a U_α open around $x_\alpha$which misses x_β. So, $x_\gamma \in U_\alpha$ implies $\gamma < \alpha$. And so, $\{x_\beta\}_{\beta<\omega_1}$ is left separated and uncountable as desired. □

Lemma 1.3.4. *A locally countable space is HS if and only if it contains no uncountable discrete subset.*

PROOF. Surely an uncountable discrete subset is not separable, so one direction is immediate. Conversely, imagine Y were a non-separable subspace of a locally countable space X. Since Y is not separable, we can find and order a $Z = \{z_\alpha : \alpha < \omega_1\} \subseteq Y$ each $z_\alpha \in U_\alpha$ a countable open neighborhood which contains no earlier z's. That is, $z_\beta \in U_\alpha \implies \beta > \alpha$. Then, Z is uncountable and discrete. □

Lemma 1.3.5. *If there is any S space, there is one of the form* $X = \{x_\alpha\}_{\alpha<\omega_1}$ *which is RS.*

PROOF. Recall that, by the above, a space S is an S space if and only if it has an uncountable RS subset but no uncountable LS subset. Fix S an S space. $\exists X = \{x_\alpha\}_{\alpha<\omega_1}$ which is RS in S. So, $\exists U_\alpha$ open around x_α with $U_\alpha \cap X = \{x_\gamma\}_{\gamma<\alpha}$. So, $\{x_\gamma\}_{\gamma<\alpha}$ is open in X. Now, S is hereditarily separable, so X must be as well, and X was consturcted to be of the desired form. From this point forward, we will, without loss of generality, assume every S space can be ordered so that its initial intervals are open. □

Lemma 1.3.6. *If there is any L space, there is one of the form* $X = \{x_\alpha\}_{\alpha<\omega_1}$ *which is LS.*

PROOF. Likewise, recall that L is an L space if and only if it has an uncountable left separated space, but no uncountable RS space. That is, $\exists X \subseteq L$ with $X = \{x_\alpha\}_{\alpha<\omega_1}$ so that $\forall\alpha\exists U_\alpha$ open around x_α with $x_\beta \in U_\alpha = \{x_\gamma\}_{\gamma\geq\alpha}$. This implies that, in X, $\{x_\gamma\}_{\gamma\geq\alpha}$is open. Now, recall that L is hereditarily Lindelöf, so X is also. So, X is LS and HL, and so it is itself an L space, and is of the desired form. From this point on, we will, without loss of generality, assume that every L-space can be ordered so that its tail intervals are open. □

1.4. Small Cardinals and Set Theoretic Axioms

1.4.1. Definitions of Small Cardinals. A *small cardinal* is any cardinal number κ, for which $\aleph_1 \leq \kappa \leq \#\mathbb{R}$. In practice, small cardinals are the cardinalities of sets somehow related to $\mathbb{N}$.

"$\exists^\infty$n $\in \mathbb{N}$": means: "there are infinitely many natural numbers n such that..."

"$\forall^\infty$n $\in\mathbb{N}$": means "for all except finitely many natural numbers n we have...".

$f <^* g$ :: For any two functions, $f, g \in \omega^\omega$, we say $f <^* g$ if and only if $f(n) < g(n)\ \forall^\infty n \in \mathbb{N}$

Dominating:: A sub-collection D of ω^ω is *dominating* or *cofinal* if and only if $\forall f \in \omega^\omega$ there is an $g \in D$ such that $f <^* g$.

Bounded:: A sub-collection B of ω^ω is *bounded* if and only if $\exists f \epsilon \omega^\omega$such that $\forall g \epsilon B$ we have $g <^* f$.

Unbounded:: As expected, B is called *unbounded* if and only if it is not bounded.

$\mathfrak{d} = \min\{\#D : D$**is dominating**$\}$**:** OR

$\mathfrak{d} = \min\{\#F : F \subseteq \omega^\omega \ \wedge \ (\forall g \epsilon \omega^\omega)(\exists f \epsilon F)(\forall^\infty n \epsilon \omega)(g(n) < f(n))\}$**,:**

$\mathfrak{b} = \min\{\#B : B$**is unbounded**$\}$**:** OR

$\mathfrak{b} = \min\{\#F : F \subseteq \omega^\omega \ \wedge \ (\forall g \in \omega^\omega)(\exists f \in F)(\exists^\infty n \in \omega)(g(n) < f(n))\}$**.:**

1.4.2. V=L*:* Gödel's Axiom of Constructibility. V is the class of all sets. L is the set of all constructable sets. That is to say, L is the smallest possible model of ZFC set theory; it includes only those sets specifically guaranteed by the axioms. Godel's Axiom of Constructibility says that $V = L$, or that all sets are constructable. $V = L$ is the strongest requirement that the universe be "controllable" in the way we spoke of above.

1.4.3. CH: The Continuum Hypothesis. The continuum hypothesis, first posited by Georg Cantor, states that there are no cardinals between ω and $\mathfrak{c}$. Alternatively, it can be thought of as saying that $2^{\aleph_0} = \aleph_1$. In 1900, Hilbert named the Continuum Hypothesis to be the first of his centennial problems. In 1939, Kurt Gödel proved that it was impossible to disprove CH with only the ZFC axioms. It took another three decades for Paul Cohen (using forcing) to show it likewise impossible to prove CH within ZFC.

1.4.4. MA: Martin's Axiom. Martin's Axiom, attributed to D.A. Martin, says that *no compact CCC Hausdorff space is the union of* $<\mathfrak{c}$ *nowhere dense sets*. Martin observed that several models of set theory which contained Suslin Lines (see) contained a similar assumption. That common feature is now called Martin's Axiom.

Notice the similarity between Martin's Axiom and the Baire Category Theorem. In point of fact, assuming CH, the Baire Category Theorem is simply a stronger version of Martin's Axiom. Indeed, assume CH. In this case, the Baire Category Theorem tells us that no locally compact Hausdorff space is the union of $<\mathfrak{c}$ nowhere dense sets.

There are other versions of Martin's Axiom. In particular, for any cardinal κ, $MA(\kappa)$ says that for any CCC space, X, and any family, D of dense sets in X, with $\#D \leq \kappa$, there is a filter F on X such that $F \bigcap d$ is non-empty for every $d \in D$. The usual MA is equivalent to $(\forall \kappa < \mathfrak{c})\,(MA\,(\kappa))$. Of particular interest is $MA\,(\aleph_1)$.

1.4.5. $\diamondsuit$: Diamond. Recall that a subset of ω_1 is stationary if it meets every closed, unbounded subset of ω_1. $\diamondsuit$ can be defined in many alternative ways:

(1) $\diamondsuit$: There is a family of functions, $\{f_\alpha\}_{\alpha\in\omega_1}$ so that
   (a) for each α, f_α maps α into α
   (b) If f maps ω_1 into ω_1, then $\{\alpha : f \restriction \alpha = f_\alpha\}$ is stationary.
(2) $\diamondsuit_1$: There is a family of subsets of ω_1, $\{S_\alpha\}_{\alpha\in\omega_1}$ so that
   (a) $S_\alpha \subset \alpha$
   (b) If $S \subset \omega_1$then $\{\alpha : S \cap \alpha = S_\alpha\}$ is stationary.
(3) $\diamondsuit_2$: There is a family of subsets of $\omega_1 \times \omega_1$, $\{M_\alpha\}_{\alpha\in\omega_1}$ so that
   (a) $M_\alpha \subset \alpha \times \alpha$
   (b) If $M \subset \omega_1 \times \omega_1$, then $\{\alpha : M \cap (\alpha \times \alpha) = M_\alpha\}$is stationary.
(4) $\clubsuit$: Let $\{\lambda_\alpha\}_{\alpha\in\omega_1}$be an order-preserving index of the limit ordinals in ω_1. Then $\clubsuit$ says that: There is a family of subsets of ω_1, $\{S_\alpha\}_{\alpha\in\omega_1}$ so that
   (a) S_α is cofinal in λ_α
   (b) If S is an uncountable subset of ω_1, then there is an $\alpha \in \omega_1$ with $S_\alpha \subset S$.

$\clubsuit$ is also sometimes called "Ostaszewski's Principle", because it was first formulated by Ostaszewski to construct his space (see Section 2.4).

It is a fact that these three formulations of $\diamondsuit$ are equivalent, but that $\clubsuit$ is weaker than $\diamondsuit$ [Rudin, p. 32-33]

1.4.6. PFA: Proper Forcing Axiom. A partially ordered set (*poset*), X, is called *proper* if, for all regular cardinals $\lambda > \aleph_0$, forcing with P preserves stationary subsets of $[\lambda]^\omega$. The Proper Forcing Axiom asserts that if X is proper and (for each $\alpha < \omega_1$) D_α is dense in X, then there is a filter $G \subseteq P$ such that $D_\alpha \cap G \neq \emptyset$ for all $\alpha < \omega_1$.

1.4.7. Relationships between these assumptions.

(1) V=L implies $\diamondsuit$.
(2) $\diamondsuit$ implies CH.
(3) $\diamondsuit$ implies $\clubsuit$.
(4) $\clubsuit$ and (MA + $\neg$CH) are contradictory.
(5) $\clubsuit$ + CH is equivalent to $\diamondsuit$
(6) CH implies MA.
(7) PFA implies MA.
(8) PFA implies $2^{\aleph_0} = \aleph_2$ (and so PFA and CH are contradictory).

CHAPTER 2

EXAMPLE SPACES

In this section, I will present several examples of S and L-spaces which are T_3. Each of them require assumptions in excess of ZFC. Some will be constructed in detail, while others will be only briefly sketched. The details of those constructions can be found in the works cited.

2.1. Non-Regular S & L spaces

Theorem 2.1.1. *There is a T_2 (non-regular) S space.*

PROOF. Fix X, an arbitrary subset of $\mathbb{R}$ with cardinality ω_1. Order it so that $X = \{x_\alpha\}_{\alpha<\omega_1}$. Define a topology on X so that a basic set around $x_\alpha$is the α^{th} initial interval intersected with a standard epsilon-ball. That is, let the basic open sets around x_α be $U_{\alpha,\varepsilon} = B(x_\alpha, \varepsilon) \cap \{x_\beta\}_{\beta<\alpha}$ for each $0 < \varepsilon \leq \infty$. Call this new topology τ. We have constructed τ to be RS, so its not HL.

However, it is HS. Indeed, suppose Y is an arbitrary subset of $X \subseteq \mathbb{R}$. Because $\mathbb{R}$ is HS under its usual topology, there's some subset, $D \subseteq Y$, which is countable and (metrically) dense in Y. Without loss of generality, $D = Y \cap \{x_\alpha\}_{\alpha<\delta}$ for some countable δ.

Now D is also τ-dense in Y because for any $x_\alpha \in Y$ and for any $\varepsilon > 0$:

if $\alpha < \delta$, then $x_\alpha \in D$ itself

if $\alpha \geq \delta$ then (since D is metric-dense in Y) $B(x_\alpha, \varepsilon)$ contains an $x_\beta \epsilon D$ which implies that $\beta < \delta \leq \alpha$, so $x_\beta \in \{x_\beta\}_{\beta<\alpha}$as well, and so $x_\beta \epsilon U_\alpha$as desired. □

When, in Section 2.3, we construct the Kunen line as our archetypal example of a regular S-space, we will modify the method of the above proof in order to ensure that the neighborhoods are clopen, thus ensuring zero-dimensionality, and thus regularity.

Theorem 2.1.2. *There is a T_2 (non-regular) L space.*

PROOF. As in the above S space example, we will construct new neighborhoods of the real line in in order to destroy separability, and then show that Lindelöfness was not destroyed.

Let (X, τ) be an arbitrary subset of the reals, well-ordered so that $X = \{x_\alpha\}_{\alpha\in\omega_1}$. Further, let each τ-open neighborhood of each x_α be a set of the form $B(x_\alpha, \varepsilon) \cap \{x_\beta\}_{\beta>\alpha}$. By defining intervals in this way, we have certainly made X to be LS, and so surely not HS.

To ensure that X is HL, let $Y \subseteq X$ and $\mathcal{C} = \{C_\alpha\}_{\alpha \in A}$ a cover of Y by basic open sets. We shall construct a countable sub-cover of $\mathcal{C}$. Recall that since the $C_\alpha$are basic open sets, we have, for each α, $C_\alpha = Y \cap (I_\alpha \cap U_\alpha)$ where $I_\alpha$is an open interval in $\mathbb{R}$ and $U_\alpha = \{x_\beta\}_{\beta > \gamma_\alpha}$ for some $\gamma_\alpha \in \omega_1$.

Notice that $I = \{I_\alpha\}_{\alpha \in A}$ is an open cover of Y in the metric topology, so it must have a countable sub-cover, call it $I^* = \{I_\beta\}_{\beta \in B}$ for some countable $B \subseteq A$. Now, $(\exists \gamma < \omega_1)\,(\forall \beta \in B)\,(\beta < \gamma)$. Let $\mathcal{C}^* = \{C_\alpha\}_{\alpha \epsilon B}$. Notice that C^*now covers $\{x_\delta \in Y : \delta > \gamma\}$, but, since γ is countable, there are only countably many $\delta \not> \gamma$. We can add in the C_α's which cover those, and acquire a countable subset of $\mathcal{C}$ which covers all of Y. That is: $\mathcal{C}^* \cup \{C_\delta\}_{\delta \leq \beta}$ is a countable sub-cover of $\mathcal{C}$ which covers all of Y. So, X is HL as desired, and so an L-space, as claimed. $\square$

2.2. A T_3 S space using the boundedness number

2.2.1. Notation.

$X[\leq f]$**::** For any set X and any $f \in X$, let $X[\leq f] = \{g \in X : g \leq f\}$.

$B_n(f)$**::** For any $f \in X$ and any $n \in \mathbb{N}$, let$B_n(f) = \{g \in X : (\forall m < n)\, g(m) = f(m)\}$.

$B_n^{\leq}(f)$ **::** For any $f \in X$ and any $n \in \mathbb{N}$, let $B_n^{\leq}(f) = B_n(f) \cap X[\leq f]$

2.2.2. The existence of a regular S space.

Theorem 2.2.1. *There exists a T_3 S space under $\mathfrak{b} = \omega_1$.*

Assume that $\mathfrak{b} = \omega_1$. Let $X = \{f_\alpha\}_{\alpha < \omega_1}$ be any unbounded subset of ω^ω wherein all the f_α's are strictly increasing functions. To begin, we think of X as having the (Baire) product topology it inherits as a subset of ω^ω. We want to construct a new topology on X which will make it an S space. To do so, for each $f \in X$, we add $X[\leq f]$ as a new open set. We will call this new space $X[\leq]$.

Proposition 2.2.2. *$X[\leq]$ is T_3, not Lindelöf, but is hereditarily separable, that is, $X[\leq]$ is a T_3 S space.*

2.2.3. Proof.

Claim 2.2.3. $X[\leq]$ is regular because it has a basis of clopen sets.

PROOF. Observe that the nth basic neighborhood of $f \in X[\leq]$ is $B_n^{\leq}(f)$. In other words, the n^{th} basic open neighborhood of f consists of all those sequences in X which agree with f for the first n places, and thereafter remain smaller than f.

Now, each $X[\leq f]$ is closed in the original (Baire) topology, because $X \setminus X[\leq f] = \{g \in X : (\exists n)\,(g(n) > f(n))\}$. Now, choose any $g \in X / X[\leq f]$. $B_{n+1}(g) = \{h : (\forall m < n+1)\,(h(m) = g(m))\}$is a Baire neighborhood around g. Surely, if $h \in B_{n+1}(g)$ then $h(n) = g(n) > f(n)$ and so $h \in X \setminus X[\leq f]$ which implies (since h

was arbitrary) that $B_{n+1}(g) \subseteq X \setminus X[\leq f]$ and so $X / X[\leq f]$ is Baire-open and so $X[\leq f]$ is Baire-closed as desired.

Let $\{T_n\}_{n\in\omega}$ be a countable base for ω^ω (we know this exists because ω^ω is surely second countable). For every $f \in X$, the collection $\{T_n \cap X[\leq f] : f \in T_n\}$ is a local base for f in $X[\leq]$ because, given any $B_n^{\leq}(f)$, we know $(\exists k)(f \in T_k \subseteq B_n(f))$ and so $f \in T_k \cap X[\leq f] \subseteq B_n^{\leq}(f)$. □

Claim 2.2.4. $X[\leq]$ is not Hereditarily Lindelöf.

PROOF. In order to show that $X[\leq]$ is not HL, we show that it is right separated. To do this, we show that every initial segment under the $<^*$well-ordering is open in $X[\leq]$. Let $I(f) = \{g \in X : g <^* f\}$ be such an initial segment, and fix $h \in I(f)$. We need to find an $n \in \omega$ so that $B_n^{\leq}(h) \subseteq I(f)$. Recall that $h \in I(f)$ implies $h <^* f$, which means $(\exists n \in \omega)(m > n \implies h(m) < f(m))$. So, for that particular n, $B_n^{\leq}(h) \subseteq I(f)$. Recall that the fact that initial segments are open under any topology implies that a set is RS, and that an uncountable RS set cannot be HL. □

Claim 2.2.5. $X[\leq]$ is Hereditarily Separable.

PROOF. Assume that $X[\leq]$ is not HS. Then it must have an uncountable LS subspace, which we will enumerate $Y = \{y_\alpha\}_{\alpha<\omega_1}$. Now, notice that for every α we have U_α open around y_α so that $y_\beta \in U_\alpha \Rightarrow \beta \geq \alpha$; in fact, this is the very definition of LS. Since we have a local basis (above) at each f, we need only consider U_α of the form $T_{n_{y_\alpha}} \cap X[\leq y_\alpha]$. Now, there are only countably many T_n, but there are an uncountable number of $y'_\alpha s$, so the vast majority of the $T_{n_{y_\alpha}}$ must all be the same.

Without loss of generality, we assume that all of them are, say $(\forall\alpha)\left(T_{n_{y_\alpha}} = T_N\right)$. Now, recall that $\beta < \alpha \implies y_\beta \not\leq y_\alpha$, so, if we fix an $n \in \mathbb{N}$, and let the $\alpha' s$ increase through ω_1, we find that we have a strictly increasing, uncountable set $\{y_\alpha(n)\}_{\alpha\in\omega_1}$, but each of the $y_\alpha(n)$ has to be a natural number, and there just aren't enough of them! Having arrived at a contradiction, we see that there cannot be an uncountable LS subspace of X, and so X must be HS as desired. □

2.3. The Kunen Line: A regular S space under CH

The Kunen Line is a classic example of a T_3 S space, with a usefully generalizable construction technique. It is similar to the T_2 example of an S space constructed in Theorem 5.1. The Kunen Line, like the examples above, is built out of the real line, however, it is important to note that, despite the name, the Kunen Line is NOT linearly ordered.

Most texts, including [Kunen&Vaughn] and [Just&Weese] from which this treatment is adapted, present the complicated and technical construction of the Kunen Line without much motivational exposition. It is my hope here to provide a readable explanation of the reasons why the Kunen Line is constructed as it is.

2.3.1. Definition of the Kunen Line. The Kunen Line is a zero-dimensional, first countable, locally compact, regular S space.

Theorem 2.3.1. *Let* $\mathbb{R} = \{x_\alpha\}_{\alpha \in \omega_1}$ *be a well ordering of the real numbers, which assumes* $\mathfrak{c} = \omega_1$ *(CH). There is some topology* τ*, so that:*

(1) Each initial segment $X_\beta = \{x_\alpha : \alpha < \beta\}$ is open. That is to say, $(\mathbb{R}, \tau)$ is RS, and so not HL.
(2) The difference between the usual metric topology and the new one is small. More formally, $\#\left(\overline{A}/\overline{A^\tau}\right) \leq \aleph_0$ for every countable $A \subseteq \mathbb{R}$. (where μ is the usual topology on $\mathbb{R}$).
(3) $(\mathbb{R}, \tau)$ is zero-dimensional; it has a basis of clopen sets.
(4) $(\mathbb{R}, \tau)$ is first countable; every point has a countable local basis.
(5) $(\mathbb{R}, \tau)$ is locally compact; every point has a compact neighborhood.

Proposition 2.3.2. $(\mathbb{R}, \tau)$ *as described above is a* T_3 *S space.*

PROOF. Notice that for any X, a subspace of $\mathbb{R}$, X is surely μ-separable, so there is a $D \subseteq X$ which is countable and μ-dense in X. By (2) above, we know that $X/\overline{D^\tau}$ is countable. So $D \bigcup X/\overline{D^\tau}$ is countable and $\tau - dense$ in X. So, $(\mathbb{R}, \tau)$ is hereditarily separable. Moreover, we know that zero-dimensional T_2 spaces are regular, and so, since (1) requires that it not be hereditarily Lindelöf, $(\mathbb{R}, \tau)$is a regular S space as desired. □

2.3.2. Motivation for the Construction. Recall (1.3.2) that a space is hereditarily Lindelöf if and only if it does not contain an uncountable set which can be right separated. As in the non-regular example in Theorem 5.1, we want to modify the usual topology on some cardinality ω_1 subset of $\mathbb{R}$, which we call X, so that every initial segment is open, in order to prevent hereditary Lindelöfness.

At the same time, we want to "build in" regularity. By examining the example in 5.1, we see that what precluded the regularity of that space was that the closures were "too small". That is, too many sets were closed, which means too many sets were open. We want to make the least number of new sets we can which will still allow the set to be right separated (so not hereditarily Lindelöf) without messing up hereditary separability.

To do this, we will construct our new topology on the real numbers so that, for every countable set, $A \subseteq X$, $\overline{A^\mu} \setminus \overline{A^\tau}$ is only countable. If we assure ourselves of this, then, given any particular subset of X, say Y, we will be able to find a countable set A which is metric dense in Y, and then we can let $D^* = D \cup \left(\overline{D^\mu} \setminus \overline{D^\tau}\right)$, and then $\overline{D^{*\tau}} = Y$

In order to ensure that we don't add too many open sets, we will add our open neighborhoods "one point at a time" along X's order. That is, we will construct a sequence of topologies $\left(X_\beta, \tau_\beta\right)$ where at step β we will determine the open neighborhoods of the point x_β and add them to the topology we have constructed up to that point. Because we need to be careful to maintain the hereditary separability inherited from $\mathbb{R}$ as we go, we need to construct these neighborhoods in such a way as to preserve the density of those metric-dense sets. To do this, we will list all the countable sets which contain x_β in their closure, and then construct the neighborhoods of x_β by stepping along that order, making sure each neighborhood of x_β hits each of those sets at least once.

For little extra effort, we can, in fact, make a new countable basis of open/compact sets, which garners us local compactness and first countability in addition to regularity.

2.3.3. Proof of Existence.

2.3.3.1. *Constructing the topology.*

Notation. We label everything as follows:

(1) $X_\beta = \{x_\alpha : \alpha < \beta\}$ is the β^{th} initial segment.
(2) Let $[\mathbb{R}]^{\aleph_0} = \{A_\gamma\}_{\gamma\epsilon\omega_1}$ be the set of all countable subsets of $\mathbb{R}$, ordered arbitrarily.
(3) For each β, let $\mathcal{A}_\beta$ be the set of all countable sets which contain x_β in their metric-closure.
(4) $C_\beta = \left\{A_\gamma : \left(x_\beta \in \overline{A_\gamma}\right) \& \left(A_\gamma \subseteq X_\beta\right) \& \left(\gamma < \beta\right)\right\}$ That is, C_β is all sets in $\mathcal{A}_\beta$ which lie entirely within the β^{th} initial segment and whose place in the arbitrary order from (2) is less that β. Notice that, since β is countable, C_β is as well.
(5) For each β, we have a sequence $S_\beta = \{A_{\gamma_n}\}_{n\epsilon\omega}$wherein each $A_\gamma \in C_\beta$ occurs infinitely many times. This S_β can be chosen arbitrarily, or it can "cycle through" all the $A_\gamma \in C_\beta$ "in order".
(6) For each β, let $y^\beta = \left\{y_n^\beta\right\}_{n\epsilon\omega}$ be a sequence where each $y_n \epsilon A_{\gamma_n}$ is chosen so than $\lim \left(y_n^\beta\right) \xrightarrow{\mu} x_\beta$. In particular, we choose y^β so that $\left\|y_n^\beta - x_\beta\right\| \longrightarrow 0$. Notice that since every A_γ appears infinitely many times in S_β, y^βcontains a sub-sequence $y^\gamma \subseteq A_\gamma$ which converges to x_β. We will construct the neighborhoods of x_β in such a way that they each contain a tail of y^β, and therefore a tail of each y^γ. In this way, we ensure that each neighborhood of x_β intersects each $A_\gamma \epsilon \mathcal{A}_\beta$ and thus doesn't spoil the hereditary density inherited from $\mathbb{R}$.

Set Up. Our goal is to construct, for each $\beta \in \omega_1$ a topology τ_β on $X_\beta = \{x_\alpha : \alpha < \beta\}$ with the following properties:

- a) τ_0 is the usual metric topology and each τ_β contains all the metric-open intervals.
- b) If $\gamma < \beta$ then $\tau_\gamma = \tau_\beta$ on X_γ. That is, we only add new sets to the topology when we add new elements to the underlying space.
- c) If β is a limit ordinal then $\bigcup_{\alpha<\beta}(\tau_\alpha)$ is a basis for τ_β.

- d) Every countable set which was metric-dense remains τ_β-dense. To do so, we'll to construct the neighborhoods of x_β in such a way that hit each $A_\gamma \in C_\beta$. Formally: $A \in C_\beta = \Big\{A_\gamma : (\gamma < \beta) \,\&\, (A_\gamma \subseteq X_\beta) \,\&\, \Big(x_\beta \in$ $x_\beta \in \overline{A^{\tau_{\beta+1}}}$.
- e) (X_β, τ_β) is zero dimensional; it has a basis of clopen sets.
- f) (X_β, τ_β) is first countable; it has a countable local basis at each point.
- g) (X_β, τ_β) is locally compact; every point has a compact neighborhood.

Once we have these topologies, the union of all of them will be a basis for our desired topology, τ. Notice that (a)-(c) above satisfy (1) in our description of τ in Section 2.3.3.1. Further, it is clear that $\tau \upharpoonright X_\beta = \tau_\beta$, and so any properties concerning local bases carry over directly from the $X_\beta' s$ to, and so (e) implies (3), (f) implies (4) and (g) implies (5).

Finally, we see that (d) implies that $A \in C_\beta \Longrightarrow x_\beta \in \overline{A^\tau}$ because on X_β, $\tau_\beta = \tau$. From this, we see that, for any countable set, the difference between our new closure and the old metric one is at most countable:

Fix some countable $A_\gamma = \{a_n\}_{n<\omega} \subseteq (\mathbb{R}, \tau)$. Now, if $x_\beta \in \overline{A_\gamma}/\overline{A_\gamma^\tau}$, then $A_\gamma \notin C_\beta$, which means that one of the following cases holds:

(1) $\beta < \gamma \Longrightarrow \beta$ is countable.

(2) $A_\gamma \nsubseteq X_\beta \Longrightarrow (\exists n)\,(\alpha_n) \geq \beta \Longrightarrow \beta$ is countable.

And so, it is clear that there can be only countably many $x_\beta \in \overline{A_\gamma}/\overline{A_\gamma^\tau}$, which is exactly the condition desired in 7.1.

The Construction of the nested topologies. Now, we need only to construct the sequence of topologies, $\{\tau_\beta\}_{\beta<\omega_1}$(perhaps easier said than done).

Countable β. For the first countably many β, we have $\beta < \omega$ and so X_β is finite, and therefor we can simply consider it to be discrete. In this case, the usual metric topology satisfies all our criteria.

Limit Ordinals. As we approach each limit ordinal, λ, (c) above tells us we must set τ_λ to be the topology generated by the basis $\bigcup_{\alpha<\lambda} \tau_\lambda$.

Successor Ordinals. Now, for $\beta > \omega$, we must define how to construct $\tau_{\beta+1}$from the preceding τ_β's. Since we need our topologies to nest, the $\tau_{\beta+1}$ open neighborhoods of x_α are the same as those under τ_β whenever $\alpha < \beta$, so we need only determine neighborhoods for x_β. (d) above tells us how to choose them. Fix a β.

Recall that $x_\beta \in \overline{A_{\gamma_n}}$ for every $A_\gamma \in C_\beta$ (by definition). Because of this, we constructed a sequence of reals $y^\beta = \left\{y_n^\beta\right\}_{n\in\mathbb{N}} \subseteq X_\beta$ with each $y_n^\beta \in A_{\gamma_n}$ so that $\left|y_n^\beta - x_\beta\right|$ decreases to zero. Notice that, for every $A_\gamma \subseteq C_\beta$, y^βcontains a sub-sequence $y^\gamma \subseteq A_\gamma$.

Now, we want to construct the neighborhoods of x_β so that they each include the n-tail of y^β, and so intersect each $A_{\gamma.}$ To get all the features we want from our space, we want to do this in such a way that each neighborhood is compact. In essence, we will fatten each y_n into a compact interval, and then let unions of those intervals be neighborhoods of x_β.

For each n, we choose a sequence of closed, pairwise-disjoint intervals $\{I_n\}_{n\in\mathbb{N}}$ so that y_n is in the interior of I_n for each $n \in \mathbb{N}$. Next, we pick an inward-nesting sequence, of τ_β-compact neighborhoods $U_{\beta_n} \subseteq I_n$ around y_n. Finally, we choose a sequence $\{V_{\beta_n}\}_{n\in\mathbb{N}}$ of subsets of $X_{\beta+1}$ to add to τ_β to form a basis for $\tau_{\beta+1}$ according to the following criteria:

(1) For each $k \in \mathbb{N}$, $\left\{y_n^\beta : n \geq k\right\} \subseteq V_{\beta_k}$, that is, V_{β_k}contains the k-tail of y^β (and, therefore, the k-tail of each y^γ).
(2) For each $k \in \mathbb{N}$, $V_{\beta_k} \bigcap X_\beta$is τ_β-open. (That is, it was already open.)
(3) $\bigcap_{k\in\mathbb{N}} V_{\beta_k}$=$\{x_\beta\}$. (That is, the $V's$ nest down onto x_β.)
(4) $V_{\beta_k} = \{x_\beta\} \cup \bigcup_{n\geq k} U_{\beta_n}$

Having thus chosen our $\tau_\beta's$, it remains to check that they do in fact fulfill all the necessary criteria.

2.3.3.2. *Proof (by induction) that the nested topologies are as desired.*

The induction hypotheses. At every step $\alpha < \omega_1$, we assume (a) through (g) above for every $\beta < \alpha$.

Finite β. In the finite case, all 7 properties are obvious.

Limit Ordinals. Assuming (a)-(e) for all $\alpha < \beta$, (where β is a limit ordinal), we can infer they hold for (X_β, τ_β) because we built in (a)-(c) and $\tau_\beta = \tau_\alpha$ on $X_\alpha = X_\beta \setminus \{x_\beta\}$ and so local properties like (e)-(g) hold in τ_β if and only if they hold in all the preceding $\tau_\alpha's$. Since β is a limit ordinal, property (d) does not apply.

Successor Ordinals. Now, assuming (a)-(c) and (e)-(g) hold at stage $\beta \geq \omega$, we need to show (d) holds at level β and (a)-(c) and (e)-(g) hold at $\beta + 1$. To prove (d) at level β, fix $A_\gamma \in C_\beta$. We want to show that $x_\beta \in \overline{A^{\tau_{\beta+1}}}$. By the definition of C_β, we know that $x_\beta \in \overline{A_\gamma^\mu}$. We know, likewise that, for any $\alpha < \gamma < \beta$, $x_\alpha \in A_\gamma$. Now, we constructed the $\tau_{\beta+1}$ neighborhoods of x_β to include the k-tail of each $y^\gamma \subseteq A_\gamma$ which each converges to x_β and so $x_\beta \epsilon \overline{A_\gamma^{\tau_{\beta+1}}}$, which is exactly condition (d).

To show (e), that $(X_{\beta+1}, \tau_{\beta+1})$ is zero-dimensional, we need only prove that the $V_{\beta k}$ are each β-closed. Let U be a clopen basis for τ_β which doesn't have any sets which contain an infinite number of V_{β_k}'s. Let $U' = U \cup \{u \cap V_{\beta_n} : u \epsilon U \cup \{X_{\beta+1}\}\}$ be a basis for $\tau_{\beta+1}$.

Fix $x \in X_{\beta+1}$ and $V \in U'$ with $x \notin V$. To show that V_{β_k} is closed, we need to show that there is a neighborhood of x disjoint from V. There are three cases to consider:

(1) If $x = x_\beta$, then some V_{β_k} misses x_β by hypothesis, and so we're done.
(2) $x \in X_\beta$ and $V \in U$ then V is clopen, since U is a basis of τ_β which is zero dimensional.
(3) $x \in X_\beta$ and $V = u \cap V_{\beta_k}$for some u and some k. If $x < x_\beta$ (in the usual number-order of $\mathbb{R}$) then we can pick an n so that $x \neq \inf \bigcup_{j<n} I_j$. Now, since τ_β refines the usual metric topology, we can

find a $w \in \tau_\beta$ open around x so that $w \cap \left\{u \cap \bigcup_{m<j<n} U_{j,\beta}\right\} = \emptyset$. Now, $w < \inf \bigcup_{j<n} I_j$ and so we are done. The case where $x > x_\beta$ is identical.

2.4. Ostaszewski's Space: an S space under CH + CLUB

2.4.1. Ostaszewski's Space.

Theorem 2.4.1. *$CH + \clubsuit$ implies the existence of a topology, τ, on ω_1 which is:*

(1) Hausdorff
(2) perfectly normal and hereditarily normal
(3) countably compact
(4) hereditarily separable
(5) first countable
(6) possessed of a basis of compact, countable, open and closed sets
(7) so that all open sets are countable or co-countable
(8) so that every $\alpha < \omega_1$ is open.

Notice that this implies that (ω_1, τ) is neither compact nor Lindelöf, since there is an open cover of sets which are each countable, and so no countable collection of them can cover ω_1. A summary of the construction (as per [Rudin]) follows. This construction is very similar to that of Kunen's Line.

2.4.2. Summary Proof of Existence. Assume S is a family satisfying $\clubsuit$. Use CH to index the set of all countably infinite subsets of ω_1 as $\{X_\alpha\}_{\alpha<\omega_1}$ with $X_\alpha \subset \lambda_\alpha$ (where λ_α is the α^{th} limit ordinal in ω_1).

For each $\beta < \omega_1$, and for each $n \epsilon \omega$ we define a set $U_{\beta,n}$ by induction:

Define $U_{k,n} = k$ for all $k, n\epsilon\omega$

If $0 < \gamma < \omega_1$ and for all $\alpha < \gamma$ and all $\beta < \lambda_\alpha$ then $U_{\beta,n}$ has been defined so that

(1) $\{U_{\beta,n} : \beta < \lambda_\alpha \& n\epsilon\omega\}$ is a basis for a Hausdorff topology τ_α on λ_α.
(2) Each $U_{\beta,n}$ with $\beta < \lambda_\alpha$ and $n\epsilon\omega$ is compact in τ_α.
(3) $(\beta + 1) \supset U_{\beta,0} \supset U_{\beta,1} \supset ...$ for any $\beta < \lambda_\alpha$ and, moreover, $\{U_{\beta,n}\}_{n\epsilon\omega}$ is a local basis for β in τ_α.

If γ is a successor ordinal, say $\gamma = \alpha + 1$, we define $U_{\beta,n}$ for all $\lambda_\alpha \le \beta < \lambda_\gamma$ in two cases:

Case 1: Suppose X_α has no limit point in $(\lambda_\alpha, \tau_\alpha)$. Choose disjoint subsets $X = (x_0 < x_1 < x_2 < ...)$ of X_α and $S = (s_0 < s_1 < ...)$ of S_α so that S is co-final with λ_α. Since $X \cap S = \emptyset$ and $X \cup S$ is discrete and since$(\lambda_\alpha, \tau_\alpha)$is countable, Hausdorff, and has a basis of open/compact sets there are disjoint families $\{V_k\}_{k\epsilon\omega}$ and $\{W_k\}_{k\epsilon\omega}$ of disjoint basic open/compact sets with $x_k\epsilon V_k$, $s_k\epsilon W_k$ where $\bigcup_{k\epsilon\omega} (V_k \cup W_k)$ is closed.

Partition ω into infinitely many infinite disjoint subsets $N_0, N_{1...}$ Then, for each $i, n \epsilon \omega$, define $U_{(\lambda_\alpha+1),n} = \{\lambda_\alpha + i\} \cup \bigcup_{k\epsilon N_i}^{k>n} \{V_k \cup W_k\}$

Since, for $i \neq j$, $U_{\lambda_\alpha+1,0} = \emptyset$ it will turn out that (1), (2), and (3) will all hold for γ.

Case 2: If X_α does have a limit point in (λ_α, τ) we make the same construction except we leave out all the X's and V's. (Since the whole function of the X's was to make sure the space was countable compact....every countable infinite set needs some limit point. In this case we already have those limit points, there is no need to add them artificially).

Now, $\{U_{\beta,n}\}_{\beta\epsilon\omega_1}^{n\epsilon\omega}$ is a basis for a topology τ on ω_1. Ostaszewski's space is now (ω_1, τ). Clearly, (1), (2), and (3) are satisfied with $\alpha = \omega_1$ and, moreover, τ is clearly countably compact by construction. ♣ will now ensure that open sets are either countable or co-countable and that closed sets are G_δ's. Hereditary normality and separability will follow therefrom.

2.4.3. General Ostaszewski Spaces.

Definition 2.4.2. A Hausdorff space X is called "sub-Ostaszewski" if it is uncountable, but every closed set is ether countable or co-countable. A sub-Ostaszewski space which is also regular, countably compact, but no compact is called an Ostaszewski space.

Theorem 2.4.3. *Every sub-Ostaszewski space is hereditarily separable.*

PROOF. Let X be a sub-Ostaszewski space. Since X is Hausdorff, given any two points, there are disjoint neighborhoods each containing one of the points, call them U and V. But, now $M = X \setminus U$ and $N = X \setminus V$ are both closed. That means either U or M is countable, and either V or N is countable. But, $U \subseteq N$ and $V \subseteq M$, so either U or V is itself countable. Now, since this argument will work for any two points, there can be only one point in X which doesn't have a countable neighborhood. By removing that point, we can assume that X is locally countable. Now, recall 1.3.4 tells us that a locally countable space is hereditarily separable if and only if it does not contain an uncountable discrete space. So, we assume Y were an uncountable discrete subspace of X. We can split Y up into two uncountable sets, $Y_1$and Y_2 with $\overline{Y_1} \cap Y_2 = \emptyset$. Now, Y_1 is uncountable, but so is $Y \setminus Y_1$, which is a contradiction, since X is sub-Ostaszewski. Therefor, X must be hereditarily separable as desired. □

Theorem 2.4.4. *There are models of ZFC+CH which do not allow any Ostaszewski spaces. (Eisworth, 1999)*

2.5. Suslin (Souslin) Lines and Trees:

2.5.1. Trees.

2.5.1.1. *Definitions and Preliminary Theorems.*

Tree:: A tree is a partially ordered set $(T,<)$ such that every initial segment, $[x]=\{y\in T: y<x\}$ is well ordered.

Level:: The α^{th} level of a tree T is $T(\alpha)=T_\alpha=\{t\in T:[t]\simeq\alpha\}$, that is, the set of all elements the set of whose predecessors is order isomorphic to α.

Height:: The height of a tree T is the smallest α so that the α^{th} level of T is empty. We write $ht(T)$ for the height of T.

Branch:: see chain

Chain:: A chain (or branch) is a linearly ordered subset of T.

Antichain:: An antichain is a subset of a tree all of whose members are incomparable. More formally, it is a subset of the tree such that no node in the set is the descendant of any other node. The size of the largest antichain gives a rough measure of the width of the tree. Antichains are also called "levels".

Everywhere-splitting:: A tree is called an everywhere-splitting tree if above every element are (at least) two incomparable elements. That is, $(\forall t\epsilon T)\,(\exists r,s)\,(r,s>t\,\&\,r\not< s\,\&\,s\not< r)$

Tree-topology:: The tree topology is that whose basis is the open intervals of T. A tree topology of this type is always regular and Hausdorff.

Partial-order-topology:: The partial order topology is that which has $\{x\in T: x\geq y\}_{y\in T}$ as its basis. This topology is rarely even Hausdorff.

König's-Lemma:: If T is an infinite tree, and if each antichain T_n is finite, then T has an infinite branch.

Canonical Extension:: Every tree extends to a canonical linear order as follow:

Suppose T is a tree under $\leq$. Let $\prec$ be an arbitrary linear order on $T(\alpha)$, the α^{th} level of T. For $x\in T(\alpha)$ and $\beta<\alpha$, we define $x(\beta)$ to be the unique $z\in T(\beta)$ with $z<x$. We define the linear order on T by $x<y\Longrightarrow x(\alpha)\prec y(\beta)$ and if α is the least ordinal with $x(\alpha)\neq y(\alpha)$ then $x(\alpha)\prec y(\alpha)$.

2.5.1.2. *Cantor Trees.* A Cantor Tree, T, is developed from the usual middle-third Cantor set in the following way: Let the first level of the tree be $[0,1]$, the second $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$ and so forth. The ω level of this tree is C, the usual middle-thirds Cantor set. The partial order on this tree, as expected, is $\subseteq$, set inclusion.

Now, given any cardinal κ with $\omega < \kappa \leq \mathfrak{c}$ and any B with $\#B = \kappa$ $T \setminus C \subseteq B \subseteq T$, then we call B a κ-Cantor tree.

2.5.1.3. *Aronszajn Trees.* An Aronszajn tree is an uncountable tree with no uncountable branches and no uncountable antichains. More generally, for any cardinal $\varkappa$, a $\varkappa$-Aronszajn tree is a tree of height $\varkappa$ wherein all the antichains and branches have size less than $\varkappa$. Notice that a usual Aronszajn tree is the same as $\aleph_1$-Aronszajn trees. It is a fact (König's Lemma) that there is no such thing as an $\aleph_0$-Aronszajn tree. It is likewise known (proved by Aronszajn) that Aronszajn trees exist in ZFC. The existence of $\aleph_2$-Aronszajn trees is known to be undecidable. They exist when CH is true, but it is consistent for them not to exist under certain other assumptions.

2.5.2. Suslin Lines: The real line is characterized as the unique unbounded, dense, complete, separable, linear order. What if we try to replace separability by CCC? Is the resulting space still unique? A *Suslin line* is any non-empty, totally ordered set S with the following five properties:

(1) S has neither a least nor a greatest element.
(2) S is order-dense.
(3) S is order-complete.
(4) S is CCC.
(5) S is NOT order isomorphic to the real line.

First proposed by Mikhail Yakovlevich Suslin in the early 1920' s the *Suslin Hypothesis* conjectures that there are no Suslin lines. Equivalently, since $\mathbb{R}$ surely satisfies #1 through 4, the Suslin Hypothesis can be phrased as, "*Every CCC, dense, complete linear order without endpoints is isomorphic to the real line.*" Alternatively, a Suslin Line can be defined as "*linear CCC connected non-separable space*". For many years, the existential status of Suslin lines was unclear. It is now known to be independent of ZFC.

2.5.3. Suslin Trees: A tree is a partially ordered set, $(S, \leq)$ with a smallest element, so that every initial segment is well ordered by $\leq$. A Suslin tree is a tree of height ω_1 such that every branch and every antichain is at most countable. Notice that every Suslin tree is an Aronszajn tree; in fact, under CH, Aronszajn and Suslin trees are identical.

Definition 2.5.1. (regular) A tree with the following two properties is called regular.

(1) For each node on the tree, the set of all immediate successors is countable.
(2) If $x, y \in T_\alpha$ for a limit ordinal $\alpha < ht(T)$ and If $\{z \in T | z < x\} = \{z \in T | z < y\}$ then $x = y$.

Theorem 2.5.2. *The existence of a Suslin Tree guarantees the existence of an everywhere-splitting Suslin Tree.*

PROOF. Let T be a Suslin tree. We need to get rid of all those elements which do not split. Let $S = \{t \in T : (r, s > t) \Longrightarrow ((r < s) \vee (s < r))\}$, That is, S is the set of all t whose predecessors are all comparable (places where no branches split off). Let A be the set of all minimal elements of S (that is, $A = \{t \in T : (\forall s \in S)\,(s \nless t)\}$.

A is an antichain, so (since T is Suslin) A must be countable. Now, for any $t\epsilon A$, $R = \{r\epsilon T : r > t\}$ is countable, since it's a chain in a Suslin tree. It is a chain becasue $t \in A \subseteq S$ and so everything bigger than t is comparable (this is the defintion of S).

So, $T^* = T / \bigcup_{t\epsilon A} \{r \in T : r > t\}$ is uncountable, and a subset of T, so it must have no uncountable chains or antichains, and is therefore Suslin. [Roitman] □

Theorem 2.5.3. *The existence of an everywhere-splitting Suslin Tree guarantees the existence of a Suslin Line.*

PROOF. Assume $(T, \leq_T)$ is an everywhere-splitting Suslin tree. Let B be the set of all branches of T, and let $T^* = T \cup B$. We order T^* by $\lessdot$ as follows:

$t, s \in T \Longrightarrow (t \lessdot s \Leftrightarrow t <_T s)$

$t \in T \& b\epsilon B \Longrightarrow (t \lessdot b \Leftrightarrow t \in b)$

$t \in T^* \& \, b \in B \Longrightarrow (b \lessdot t \Rightarrow b = t)$

Extend $\lessdot$ linearly in the canonical way. Call this extension $\prec$.

I claim that $(B, \prec)$is a Suslin Line. Since it is obviously linearly ordered and connected, we need check only that it is non-separable and CCC. □

Claim 2.5.4. $(B, \prec)$ is not separable.

PROOF. Imagine B had a countable dense set. Without loss of generality, we can extend that countable set to a countable initial segment (since every branch is countable). Call the dense initial segment T^*_α. If $t\epsilon T$ and the height of t is greater that α (so $t \notin T^*_\alpha$), then (since T splits everywhere) there is some $s\epsilon T$ so that $s \geq t$ so that the interval $(t, s) = \left\{x\epsilon T^* : t \prec x \,\&\, s \nprec x\right\}$ has at least three branches in it. Therefore $B \cap (t, s)$ contains a whole interval of B. Now, fix any $b\epsilon B \cap (t, s)$. Since $ht(b) > ht(t) > \alpha$, b cannot be in T^*_α, and so T^*_αcannot be dense in B (since it misses a whole interval of B). □

Claim 2.5.5. $(B, \prec)$ is CCC.

PROOF. Imagine C was a pairwise disjoint collection of non-empty $\prec$-intervals in B (that is, intervals of branches). Since the endpoints of each interval in C are in B, the endpoints of each interval in C are not $\prec$-comparable. For each interval $I = (r_I, s_I) \in C$, let $I^* = \{t\epsilon T : r_I < t < s_I\}$. Pick a branch $b(I) \in I$ and $t_I \neq s_I \in b(I)$. Then $t_I\epsilon I^*$and if $t_I < t_j$ then $t_j\epsilon I^*$. But then $b(J)\epsilon I$, even though J and I were supposed to be disjoint. So, the t_I's must be incomparable in T, and so, since T is Suslin, C must be countable, and so B must be CCC as desired. this completes the proof. □

Theorem 2.5.6. *The existence of a Suslin Line guarantees the existence of a Suslin Tree.*

Let Y be a Suslin line. Since Y is not separable, no countable subset of Y is dense in Y. We can assume, without loss of generality, (by possibly replacing Y with a small enough subinterval) that Y is everywhere non-separable, i.e., that every countable set is nowhere dense.

We will define, by induction, a tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$, where T_α is the set of elements on level α. Each T_α will be a maximal collection of non-empty open intervals. T will be ordered by reverse inclusion.

(1) $T_0 = \{Y\}$, the whole Suslin line.
(2) Given T_α, $T_{\alpha+1}$ is chosen so that:
  (a) every interval in $T_{\alpha+1}$ is open
  (b) every interval is properly contained in some interval of T_α
  (c) the intervals are disjoint.
  (d) T_α is maximal. (i.e., $\bigcup T_\alpha$is dense.)
(3) If γ is a limit ordinal, T_γ is defined to be the set of non-empty intervals of the form $\left(\bigcap_{\epsilon<\gamma} \bigcup T_\epsilon\right)$.

Now, we need to show that the $T'_\gamma s$ defined like this are maximal (i.e., $\bigcup T_\gamma$ is dense). Recall that Y is a Suslin Line, so that it is CCC. Therefore, T_γ, being a collection of pairwise disjoint open intervals, is at most countable. Therefore T_γ has at most countably many endpoints. Since Y is everywhere-nonseparable, this collection, E_γ, of endpoints, must be nowhere dense. Since at each stage T_a was maximal, there are no interval "gaps", that is $Y = T_a \cup E_a$

Now, letting $T = \bigcup_{\alpha<\omega_1} T_\alpha$, it is clear that T has height ω_1, has branches/chains which are countable, and that the antichains are exactly the levels which we just showed to be countable collections of intervals, since Y is CCC, and so T is a Suslin Tree.

Theorem 2.5.7. *V=L implies the existence of a κ-Suslin Tree for every infinite successor cardinal κ. (Jensen, 1972)*

The existence of a Suslin Tree is to the existence of a certain uncountable collection of real sets.

The existence of a Suslin Tree is equivalent to the existence of an uncountable collection of real sets, C, so that:

(1) any two sets in the collection are either disjoint or one of them is a subset of the other
(2) if G is any uncountable sub-collection of C, then G has at least two disjoint members and at least two members one of whom is a subset of the other

PROOF. Assume there is a Suslin tree, S. Let f be a one-to-one function from some uncountable subset of S into R. For each $x \epsilon S$, let $U(x) = \{y : x \leq y\}$ and let $F = \{f(U(x)) : x \in S\}$. Then F has the desired properties. Conversely, if there is a collection F, let $A \leq B$ mean $B \subseteq A$, for any $A, B \epsilon F$. Then, F is a Suslin Tree. □

Theorem 2.5.8. *The existence of a Suslin Line guarantees the existence of a regular S space.*

2.5.3.1. *Notation.*

T_α	Let T_α be the α^{th} level of T.
$ht(t)$	Let $ht(t)$ be the level of the tree to which t belongs.
R	Suppose that $H \subset T$. Let $R = \{r \in T : r < h \in H\}$. That is, let R be the set of all predecessors of H.

Let $(T, \leq)$ be a Suslin Tree. Let Q be a maximal antichain in $T \setminus R$ and choose $\alpha \in \omega_1$ with $\alpha > ht(t)$ for all $t \in Q$. If $t \in R \cap T_\alpha$, then any maximal antichain in $\{x \in H : x > t\}$ is also a maximal antichain in $\{x \in T : x > t\}$. Assume there were a $y \epsilon T \setminus H$ so that $y > t$ and y incomparable with all other $x > t$. Since $t \in R \cap T_\alpha$and $y > t$, we know that $ht(y) = \alpha$, and so $ht(y) > \alpha$, which means that it must be in H, since R is the set of predecessors of H.

If $\gamma > \alpha$and $t \epsilon R \cap T_\gamma$, define $\gamma = \gamma_0 < \gamma_1 < ...$ and define antichains $A_0, A_1, ...$ by induction so that A_n is a maximal antichain in $\left\{x \epsilon H \cap \left(\bigcup_{\beta > \gamma_n} T_\beta\right) : x > t\right\}$ and $\gamma_{n+1} > l(x)$ for all $x \epsilon A_n$. Then, if γ' is the limit of $\gamma_{0,}\gamma_1 ...$, the tail of every chain running through the tree from t to level γ' hits infinitely many of the A_n.

Now, to make T an S space, for each $t \epsilon T$, let $\mathfrak{A} = \{(n, \alpha, t) \in \omega \times \omega_1 \times T : \alpha < l(t)\}$where α is a limit level. If α is a limit ordinal in ω_1, then select an increasing sequence $\alpha^0 < \alpha^1 < ...$ having α as a limit.

For each $A = (n, \alpha, t) \epsilon \mathfrak{A}$, we choose a chain Z(A) in T running from the predecessor of t in T_{α^n} to T_α such that $Z(A)$ is not contained in $\{p < r\}$ for any $r \epsilon T_{\alpha.}$ We make the choice in such a way that, for any $\gamma < w_1$, there is a tail $Z_\gamma(A)$ of the chain $Z(A)$ so that $Z_\gamma(A) \cap Z\gamma(m, \beta, r) = \varnothing$ unless the same term of T_γ precedes both t and r.

Topologize T by declaring a $V \subseteq T$ to be open if, for each $t \in V$ and for each limit $\alpha < l(t)$, there is a $k \in \omega$ so that, for all $n > k$, a tail of $Z(n, \alpha, t)$ is contained in V.

Now, T with this topology is:

(1) not Lindelöf, because: $\left\{\bigcup_{\beta < a} T_\beta\right\}_{\alpha < \omega_1}$ is an open cover with no countable sub-cover.
(2) hereditarily separable, because of the argument at the beginning.
(3) T_1
(4) Normal, and therefore regular, because:

(a) Imagine that H and K are disjoint closed sets. By the argument at the beginning, there is a $\gamma < \omega_1$ so that, for every $t \epsilon T_\gamma$, either $\{x > t\} \subset H$ or $\{x > t\} \subset K$ or $(H \cap K) = \emptyset$. Because of this, we can assume that $\bigcup_{\delta > \gamma} T_\delta \subset (H \cap K)$ and also that γ is not a limit ordinal.

(b) If $t \in T$ and $\alpha < l(t)$is a limit, let $A_n = (n, \alpha, t)$ for each $n \in \omega$. Now, there is a $k \in \omega$ and, for each n, a tail $Y(A_n)$ of $Z(A_n)$ which intersects H only if $t \in H$ and intersects K only if $t \in K$. Take $Y(A_n) \subset Z_\gamma(A_n)$.

(c) Let $Y_t = \bigcup_{n>k} Y(A_n)$. Then, $\bigcup_{t \epsilon H} Y_t$ and $\bigcup_{t \epsilon K} Y_t$ are disjoint closed sets containing H and K, respectively; hence T is normal (and regular).

And so T is the regular S space we predicted.

2.5.4. The existence of a Suslin Tree guarantees a regular L-Space. Suppose S contains a Suslin tree. Then S contains an everywhere-splitting Suslin tree T. Since $T_x = \{y \in T : y < x\}$is completely ordered for each x, we can associate each node of T with an ordinal number. Since we know that the branches and antichains of T are at most countable, we order each branch to make it a copy of $\mathbb{Z}$ (or a subset of $\mathbb{Z}$). Call the resulting branch-space X, and topologize this set of branches so that $[t] = \{B \in X : t \in B\}$ is open for each $t \epsilon T$. This is the standard branch-space topology.

Now, for each $t \epsilon T$, $[t]$ is actually clopen in X. To see that $[t]$ is closed, consider $Y = X \setminus [t]$ =all branches which do not contain t. Fix $B \epsilon Y$. We need to find an open set containing B which is inside Y. Since B is a branch of T, which splits-everywhere, there is a minimal place where B and $[t]$ disagree; call that element s. Now, $[s]$ surely cannot contain t, and so $B \epsilon [s] \subseteq Y$ as desired. So, Y is open, and so $[t]$ is closed, and therefore clopen, as desired. Since X now has a basis of clopen sets, it is regular.

Let Y be any collection of branches in X, and let $\mathcal{U}$ be any open cover of Y by basic open sets, i.e. $\mathcal{U} = \{[u]\}$. We will call $t \in T$ "$\mathcal{U}$"-minimal if

(1) There is a $U \in \mathcal{U}$ with $[t] \cap Y \subseteq U$.

(2) If $s < t$, then there is NOT a $U \in U$ with $[s] \cap Y \subseteq U$.

Then the set $M(\mathcal{U})$ consisting of all $\mathcal{U}$-minimal points of T is an anti-chain of T, so (since T is Suslin) $M(\mathcal{U})$ is countable.

Define $r(\mathcal{U}) = \{[t] \epsilon Y : t \in M(\mathcal{U})\}$. This is clearly countable, but it's not quite a sub-cover of $\mathcal{U}$. For that, we simply choose, for each $t \epsilon M(\mathcal{U})$, we can choose a $U_t \epsilon \mathcal{U}$ so that $[t] \subseteq U_t$. Call the collection of those U_t's $\mathcal{U}'$. $\mathcal{U}'$ is a countable sub-cover of Y. To show this, it suffices to show that every branch has a $\mathcal{U}$-minimal element. Assume it did not, then for every $a < b$, there would be a $U \in \mathcal{U}$ with $[a] \cap Y \subseteq U$. However, since U is basic, it is clopen, and so, if it contains every $[a]$with $a < b$, it must also contain $[b]$ and thus B.

So Y is Lindelöf, and therefore X is hereditarily Lindelöf.

Now, X is not hereditarily separable. If it were, then it would be an unbounded, dense, complete, separable, linear order, but $\mathbb{R}$ is the only such space.

2.5.5. Diamond Guarantees a Suslin Tree.

2.5.5.1. *Summary.* We construct a tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$ by transfinite induction on α. We need to do this in such a way that T has no uncountable antichains. Since $\#T = \aleph_1$ there are $2^{\aleph_1} > \aleph_1$ subsets of T. So there are $> \aleph_1$ uncountable subsets, none of which can be allowed to be antichains. That is a lot of things to keep track of at each step! It is for this that we will use $\diamondsuit$.

We will set up our recursion in such a way that any $A \in W_\alpha$ which was a maximal antichain in T_α constructed before stage α stays maximal from then on. $\diamondsuit$ implies that when X is a maximal antichain in T, then $X \cap T_\alpha$is a maximal antichain in T_α and that $X \cap T_\alpha \in W_\alpha$for some $\alpha < \omega_1$.

Since the antichains can't grow, $X \cap T_\alpha = X$ and so the antichain X must be countable (since $T_\alpha$is).

2.5.5.2. *A tiny problem.* The power of $\diamondsuit$ applies only to subsets of ω_1, but we want to construct our tree on subsets of $[\omega]^{<\omega_1}$. To solve this, we prove the following lemma:

Lemma 2.5.9. *$\diamondsuit$ implies that there exists a sequence of sets, $\{Z_\alpha\}_{\alpha<\omega_1}$ so that, for each $\alpha < \omega_1$:*

(1) $Z_\alpha \subseteq [\omega]^{<\alpha}$
(2) *$Z_\alpha$is at most countable*
(3) *If $X \subseteq [\omega]^{<\omega_1}$ then $\left\{\alpha < \omega_1 | X \cap [\omega]^{<\alpha} \in Z_\alpha\right\}$ is stationary.*

2.5.5.3. *Proof of 2.5.9.* Notice that $\#[\omega]^{<\omega_1} = \#\bigcup_{\alpha<\omega_1} [\omega]^\alpha = \sum_{\alpha<\omega_1} 2^{\aleph_0} = \aleph_1$. Now, let F be a 1:1 mapping of $[\omega]^{<\omega_1}$ onto ω_1.

Claim 2.5.10. $S_F = \left\{\alpha < \omega_1 | \sup F[\omega]^\alpha = \alpha\right\}$ is closed and unbounded.

It is clear that S_F is closed. Indeed, given any $\beta < \omega_1$, we can recursively define: $\alpha_0 = \beta$ and $\alpha_{n+1} = \sup F[\omega]^{\alpha_n}$ and then let $\alpha = \sup\{\alpha_n | n \in \mathbb{N}\}$. Then surely $\beta \leq \alpha < \omega_1$ and so $\alpha \in S_F$

Now, to see that it is unbounded, set $Z_\alpha = \left\{F^{-1}[A] \cap [\omega]^{<\alpha} | A \in W_\alpha\right\}$ for any $\alpha \in S_F$ and $Z_\alpha = \varnothing$ otherwise. Surely Z_α is countable (it's a subset of $[\omega]^{<\alpha}$. If $X \subseteq [\omega]^{\omega_1}$ then $F[X] \subseteq \omega_1$ and so $S = \{\alpha < \omega_1 | F[X] \cap \alpha \in W_\alpha\}$ is stationary. This means that $S \cap S_F$ is also stationary and so $\alpha \in S \cap S_f$ implies that $X \cap [\omega]^{<\alpha} = F^{-1}[F[X] \cap \alpha] \in Z_\alpha$.

2.5.5.4. *The Suslin Tree Construction.* In this section we construct a Suslin Tree. Before doing so, we will need a few definitions and

Definition 2.5.11. *(normal)* A regular tree $(T,\leq)$ is normal if, for all $\alpha<\beta<h(T)$ and for all $x\in T_\alpha$ we have $y\in T_\beta$ so that $x<y$.

We will construct T_α by transfinite recursion so that each $\#T_\alpha\leq\aleph_0$ and so that $T_{\alpha+1}=\bigcup_{\beta\leq\alpha}T_\beta$ is normal. We put $T_0=\{\emptyset\}$.

Successor ordinals: Given that $T_\alpha\subseteq\omega^\alpha$ so that the conditions above are fulfilled for α, we let $T_{\alpha+1}=\left\{f\cup\{(\alpha,n)\}_{n\in\mathbb{N}}\ \&\ f\in T_\alpha\right\}$. Notice that now $\#T_{\alpha+1}\leq\aleph_0$ and that $T_{\alpha+2}$ is normal as desired.

Limit ordinals: By the induction hypothesis, every T_β is normal whenever $\beta<\alpha$, so $T_\alpha=\bigcup_{\beta<\alpha}T_\beta$is also normal. Additionally, $\#T_\alpha\leq\sum_{\beta<\alpha}\#T_\alpha\leq\aleph_0$.

Now, let $\{C_n\}_{n\in\mathbb{N}}$ be the (at most countable) collection of all elements of Z_α which are maximal antichains in T_α.

Claim 2.5.12. For each $f\in T^\alpha$ there is a branch b of length α such that $f\subseteq b$ and, for every $n\in\mathbb{N}$, there is some $g\in C_n$with $g\subseteq b$.

PROOF. We fix an increasing sequence of ordinals, $\{\alpha_n\}_{n\in\omega}$ so that $\alpha_0=\operatorname{dom}(f)$ and $\sup_{n\in\omega}\alpha_n=\alpha$. We will construct the required b by recursion. Let $b_0=f$. Given that $\operatorname{dom}b_n\geq\alpha_n$, we know that there is a $g\in C_n$ comparable with g_n. If there wasn't, then $C_n\cup\{b_n\}$ would be an antichain in T^α, but C_n is supposed to be maximal.

Now, if $\alpha_{n+1}\leq\operatorname{dom}(g)$, then we let $b_{n+1}=g$. If not, then we can let $b_{n+1}\in T_{\alpha_{n+1}}$with $b_{n+1}\supseteq g\cup b_n$ (We know this exists because $T_{\alpha_{n+1}}$is normal.) If we now let $b=\bigcup_{n\in\mathbb{N}}b_n$, we have that $\operatorname{dom}(b)=\sup\{\alpha_n|n\in\mathbb{N}\}$. A quick examination shows that all the other promised properties of b are as they should be, and so the claim is proved. □

For every $f\in T^\alpha$ we choose one branch b_f, as in the above claim. we set $T_\alpha=\left\{b_f|f\in T^\alpha\right\}$. Surely this is a normal tree, and $\#T^{\alpha+1}\leq\aleph_0$/ Moreover, each C_n remains a maximal antichain in $T^{\alpha+1}$/ This is because each $b\in T_\alpha$was chosen to be comparable with some $g\in C_n$.

This completes the recursive construction. From here, we let $T=\bigcup_{\alpha<\omega_1}T_\alpha$and notice that T is a normal tree of height ω_1. Now, we need to show that T has no antichains of cardinality $\aleph_1$. Assume it did. Call that antichain X, without loss of generality X can be maximal.

Claim 2.5.13. $S_X=\{\alpha<\omega_1|X\cap T^\alpha\text{is a maximal antichain in }T^\alpha\}$

is closed and unbounded.

PROOF. Let $\beta < \omega_1$be any. We construct a sequence $\{a_n\}_{n\in\mathbb{N}}$ by recursion: $a_0 = \beta$, and, with a induction hypothesis that $T^{\alpha_n}$is at most countable, and for every $f \in T^{a_n}$ there is some $g_f \in X$ comparable with f. (If there wasn't, then X would not be a maximal antichain.) By letting $a_{n+1} = \sup \left[\left\{\left(\mathrm{dom}\left(g_f\right)+1 | f \in T^{a_n}\right)\right\} \cup \{a_n\}\right]$. If $a = \sup\{a_n | n \in \omega\}$, then we have $\beta \leq \alpha < \omega_1$ and each $f \in T^a$ is maximal antichain T^a. This shows that S_X is unbounded. That it is closed is easy to see. □

We now finish off the proof by using , which provides $a \in S_X$ as a limit so that $X \cap T^a$ is a maximal antichain in T. indeed, if $f \in T \setminus T^{a+1}$then $f \restriction \alpha \in T^{a+1} \subseteq g$ and so $f \in g$ for some $g \in X \cap T^\alpha$ and in particular, $\#X \leq \#T^\alpha \leq \aleph_0$. This completes the proof.

2.5.6. A Second Proof that Diamond guarantees a Suslin Tree.

2.5.6.1. *Summary.* We construct a tree $(T, <)$ on ω_1 so that the interval $I_\alpha = [\omega \cdot \alpha, \omega \cdot (\alpha + 1))$ is the α^{th} level of T. Notice that $I_0 = [0, \omega)$ is totally unordered. If $\alpha = \beta + 1$, $I_\alpha = I_{\beta+1}$ is unordered, but it does have two direct$<$-successors for each $x \epsilon I_\beta$.

If, on the other hand, α is a limit ordinal and A_α is a maximal antichain in the ordering on $[0, \omega \cdot \alpha)$ constructed thus far, then we choose countably many branches that cover the set and such that each intersects A_α. We top each of these branches with a point of I_α; this ensures that $A_\alpha$is maximal in T.

Finally, if A is a maximal antichain in $(T, <)$, then the set of all α's so that $A \cap \alpha$ is maximal in $(\alpha, <)$ is closed and unbounded. Then, $\exists \alpha\, (A \cap \alpha = A_\alpha)$ and that $A_\alpha$is maximal, i.e. $A = A_\alpha$, and so A is countable, as desired.

2.5.6.2. *Notation:* If X is an infinite countable set, choose an infinite subset X_0 of X and construct the tree $T_X = (X, \leq)$ of height ωso that:

(1) X_0 is the first level of T_x
(2) If x belongs to the n^{th} level of T_x, then there are exactly two elements, y and z, in the $n + 1^{th}$ level of T_x with $x < y$ and$x < z$.

If $(A, <_A)$and $(B, <_B)$ are both trees, then let $(A + B, <) = (A \cup B, <)$ so that $a < b$ whenever $\exists x \epsilon A \cup B$ so that $a <_B x$ and $x <_B b$. If $A \subset B$ or $B \subset A$, or if the last level of one is the first level of the other, then $A + B$ is a tree. In general, however, it is not even a partial order. (Since it is possible that $a < b$ and $b < a$).

2.5.6.3. *Proof:* Let $\{\lambda_\alpha\}_{\alpha\in\omega_1}$be an order-preserving index of the limit ordinals in ω_1 and let $\{S_\alpha\}_{\alpha\in\omega_1}$ be a family of subsets of ω_1 witnessing $\diamondsuit_1$, so that (1) $S_\alpha \subset \alpha$ and (2) if $S \subset \omega_1$then $\{\alpha : S \cap \alpha = S_\alpha\}$is stationary. We will construct a Suslin Tree by induction; for each $\alpha \in \omega_1$,we construct a tree $(\lambda_\alpha, \leq)$of height λ_α so that:

(1) If $\beta < \alpha$ and $\delta < \lambda_\alpha$ then the δ^{th} level of λ_β is the δ^{th} level of λ_α and if $x < y$ in λ_β then $x < y$ inλ_α.

(2) If $\beta < \lambda < \alpha$ and x is in the β^{th} level of λ_α, then there are at least two elements of λ_α which follow x.

Define $(\lambda_0, \leq) = T_\omega$. Suppose that $\gamma < \omega_1$ and that $(\lambda_\alpha, \leq)$ has been defined for all $\alpha < \gamma$ so that it satisfies the two conditions above.

Fact 2.5.14. *If γ is a limit ordinal, then $(\lambda_\gamma, \leq) = \sum_{\alpha<\gamma} (\lambda_{\alpha,\leq})$ is a tree with all the desired properties.*

If $\gamma = \alpha + 1$ let $X_\gamma = \lambda_{\alpha+1}/\lambda_a$. Our plan is to add X_γ to $(\lambda_\alpha, \leq)$ as a λ_α^{th} level in a "special way", and then add T_x to this tree to get $(\lambda_{a+1}, \leq)$. Let $g : X_\gamma \to \lambda_\alpha$ be a 1:1 correspondence and choose an increasing sequence $\delta_0 < \delta_1 < ...$ having $\lambda_\alpha$as a limit.

Case 1: Suppose that $S_{\lambda_\alpha}$is a maximal anti-chain in $(\lambda_\alpha, \leq)$. For each xϵX^0,choose $x_0 < x_1 < ...$ in $(\lambda_\alpha, \leq)$such that x_n belongs to the $\delta_{n^{th}}$level of $(\lambda_\alpha, \leq)$ and, for some $k \epsilon \omega$, both $g(x)$ and some term of S_{λ_α}precede x_k in $(\lambda_\alpha, \leq)$.

Case 2: If $S_{\lambda_\alpha}$is not maximal, then follow the above procedure without the requirement that some term of S_{λ_α}precede x_k in $(\lambda_\alpha, \leq)$.

Now, let $\left(\lambda_\alpha \cup X^0, \leq\right)$be the tree where $y < x$ if and only if:

(1) y, $x \in \lambda_\alpha$and y$\leq x$ in $(\lambda_\alpha, \leq)$

(2) $y \in \lambda_\alpha$and $x \in X_\gamma$and $y \leq x_n$for some $n \in \omega$.

Define $(\lambda_{\alpha+1}, \leq) = \left(\lambda_\alpha \cup X^0, \leq\right) + T_X$

The Suslin tree $(\omega_1, \leq) = \sum_{\alpha<\omega_1} (\lambda_{\alpha,\leq})$.

We only need to check that there are no uncountable antichains in $(\omega_1, \leq)$.

Assume that S is a maximal anti-chain in $(\omega_1, \leq)$. Then $A = \{\lambda_\alpha : S \cap \lambda_\alpha \textit{ is maximal in } (\lambda_\alpha, \leq)\}$is a closed unbounded set in ω_1. Thus by $\diamondsuit_1$,there is a $\lambda_\alpha \epsilon A$ such that $S_{\lambda_\alpha} = S \cap \lambda_\alpha$. This is our Case 1 from above. Hence, if $x \epsilon X^0$ which is the λ_α^{th}level of $(\omega_1, \leq)$, x is preceded by a member of S. Therefore, S$\subseteq$ $(\lambda_\alpha, \leq)$ and S is countable.

[Jech p.234]

2.5.7. When Suslin Lines Don't Exist.

2.5.7.1. *It is consistent with ♣ that no Suslin Tree exists. [Džamonja & Shelah].*

2.5.7.2. *$MA(\aleph_1)$ implies that a product of CCC topological spaces is CCC (this in turn implies there are no Suslin lines). [Rudin].*

2.6. S&L Spaces Known to Exist

Theorem 2.6.1. *CH guarantees the existence of a regular S space of cardinality $2^{\mathfrak{c}}$. (Rudin, p25 #7)*

Lemma 2.6.2. *CH guarantees the existence of an L-space of cardinality $\mathfrak{c}$ wherein every countable subset is discrete and whose every uncountable subset has weight $2^{\mathfrak{c}}$. (Rudin p 25 #8)*

Theorem 2.6.3. *CH guarantees a first countable S space. (Juhasz & Hajnal) (see Rudin, p 31-35)*

The next theorem gives very tightly sufficient conditions under which an S space exists.

Theorem 2.6.4. *Theorem: There is a first countable S space if and only if:*

(1) There is a family $\{A_{a,n} : a < \omega_q \,\&\, n \in \omega\}$ of nonempty subsets of ω with the following properties:
   (a) $\alpha < \omega_1$ implies $A_{\alpha,0} \supset A_{\alpha,1} \supset ...$
   (b) $\beta < \alpha < \omega_1$implies there exists $n \in \omega$so that $A_{\beta,0} \cap A_{a,n} = \emptyset$.
   (c) $\beta < \alpha < \omega_1$and $n \in \omega$imply there is a $k \in \omega$so that either $A_{\beta,k} \subset A_{\alpha,n}$or $A_{\beta,k} \cap A_{\alpha,n} = \emptyset$.
   (d) $M \subset \omega_1$ is uncountable and $n \in \omega$imply there are $\beta < \alpha \in M$ and $k \in \omega$ such that $A_{\beta,k} \subset A_{\alpha,n}$.

PROOF. If (1) holds, then a first countable S space is obtained by topologizing $\omega_1$by the rule: for any$U \subset \omega_1$ open and any $\alpha \in U$ then there is an $n \in \omega$so that $U \supset \left\{\beta \in \omega_1 : (\exists k \in \omega)\, A_{\beta,k} \subset A_{\alpha,n}\right\}$. In the other direction, supposed X is a first countable S space. Since X is not Lindelöf, there is a subset $Y = \{x_\alpha\}_{\alpha<\omega}$of X so that $\{x_\beta\}_{\beta\leq\alpha}$is open in Y for all α. Since X is hereditarily separable, Y is separable. Let $\{x_n\}_{n\in\omega}$ be dense in Y. Now, for each $\alpha < \omega_1$,we choose a nested sequence $U_{\alpha,0} \supset U_{\alpha,1} \supset$ ...of open sets in Y to form a basis of for the topology of Y at x_α, so that $U_{\alpha,0} \cap \{x_\beta\}_{\beta>\alpha} = \emptyset$. Define $A_{\alpha,n} = \{i \in \omega : x_i \in U_{\alpha,n}\}$now, $\{A_{\alpha,n} : \alpha < \omega_1 \& n \in \omega\}$has the desired properties for (1).

From here, Juhasz & Hajnal use CH to index the real numbers (and subsets of the real order isomorphic to the rationals) by countable ordinals; then, by induction, they choose sets to satisfy (1). ☐

2.6.1. Coming Attractions, The Tatch-Moore L Space.

Theorem 2.6.5. *In ZFC, there exists an L Space.*

A guide through the proof of this landmark theorem is provided in Part IV below.

2.7. Conditions Precluding the Existence of S & L Spaces

Theorem 2.7.1. *$MA + \neg CH$ guarantees that no compact L space exists (Juhasz)*

.

Theorem 2.7.2. *$MA + \neg CH$ guarantees that no compact S space exists. (Szentmiklossy)*

.

Theorem 2.7.3. *It is consistent under $MA + \neg CH$ that S spaces exist. (Szentmiklossy)*

.

Theorem 2.7.4. *It is consistent under $MA + \neg CH$ that L spaces exist. (Abraham & Todorcevic)*

.

Theorem 2.7.5. *It is consistent under ZFC that no S spaces exist. (Todorcevic, 1981)*

This last theorem led everyone to believe that a similar theorem concerning L spaces would soon emerge. However, over the next 25 years, no such theorem could be proved. Finally, just Tatch Moore showed that, in fact, such a theorem would NOT be true; that ZFC always contains an L space. That space is constructed in the next chapter.

CHAPTER 3

THE TATCH-MOORE L SPACE

What follows is intended to be an explanation of the L space constructed in Justin Tatch Moore's landmark paper "A SOLUTION TO THE L SPACE PROBLEM AND RELATED ZFC CONSTRUCTIONS". Think of this as a study guide for that paper, and you won't go wrong. In addition to presenting an unexpected solution to an interesting problem, Moore's solution is important becasue it combines several purely set theoretical techniques to solve what, on the surface, appears to a quintessentially topological problem. The first of these methods is partition theory and colorings on partitions. In Section 3.1, below, I will present a very brief overview of the theory and show its application in a simple example. The second method, walks on ordinals, is realtively new. It was pioneered by Stevo Todorcevic, and will likely find much use in the coming years. Sections 3.2 and 3.3 below provide a glimpse at how it works.

3.1. Colorings & Partion Theory

Among the classic examples of work usign partions is **Ramsey Theory**, a branch of mathematics which examines how structure arises out of disorder. More formally, given a space and a partition of that space, Ramsey Theory attempts to determine how large the original space needed to have been in order to ensure that (at least) one element of the partion has some desired property. For example, imagine that you are having a dinner party. As we all know, getting the right mix of people who know each other and people who don't is among the hardest elements of party planning. How many people do you need to invite in order to ensure that there is some group of three who either all know each other or are all strangers to each other? The famous **Friends and Strangers Theorem** tells us that the answer is six.

We can also interpret the above result as a statement about colorings on a hexagon. Take a hexagon and assign each vertex to one party guest. Join every pair of vertices with an edge. Color each of these edges red if the two people at the ends know each other and blue if they do not. The Friends and Strangers Theorem tells us that there must be either a red triangle or a blue triangle. For example, the the picture below, Ann, Bryan, and David all know each other.

The next image shows every possible coloring; examination will reveal a triangle in each.

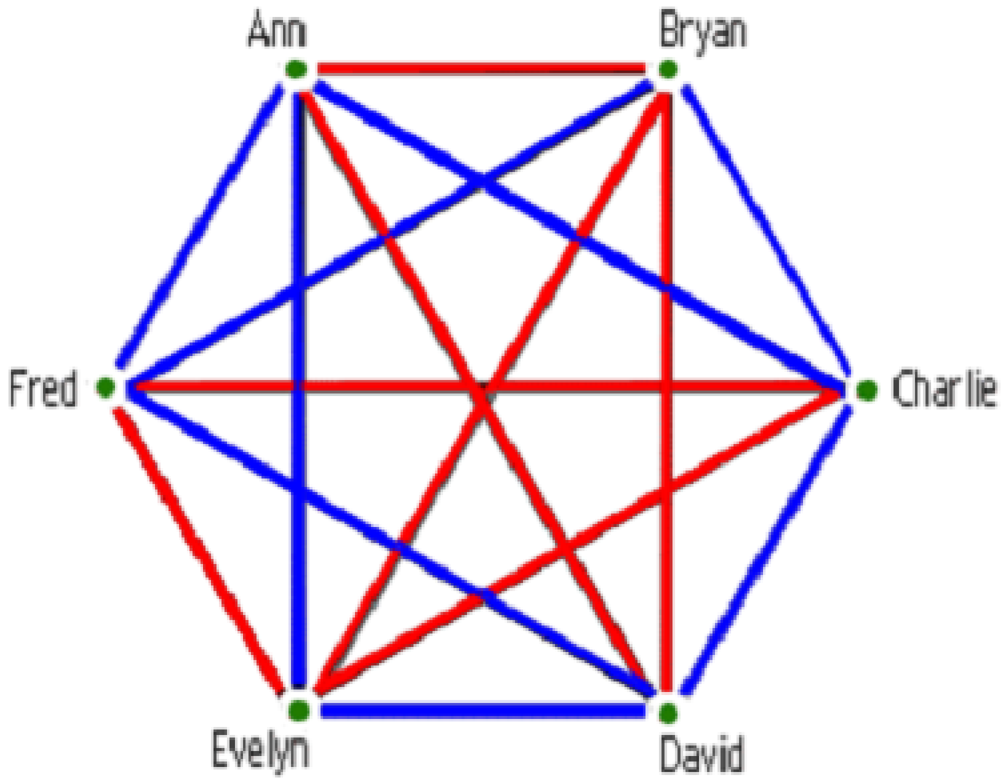


FIGURE 3.1.1. Who Knows Who

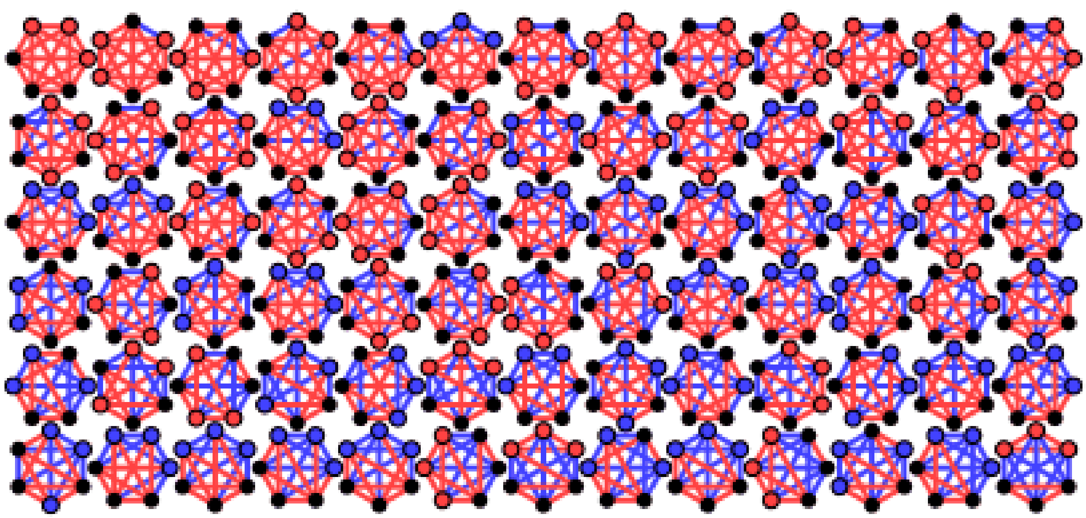


FIGURE 3.1.2. All Possible Colorings

3.1.1. Ramsey's Theorem. The Friends and Strangers Theorem is a special case of **Ramsey's Theorem**, which assures us that in any colouring of the edges of a sufficiently large complete graph (that is, a polygon in which an edge connects every pair of vertices), one will find monochromatic complete subgraphs. Given any two integers, r and s, the Ramsey number $R(r, s)$ is the least number so that a complete graph on $R(r, s))$ has either a completly blue complete r-grpah or a completely red s-graph. The Friends and Strangers Theorem, then, says that $R(3, 3) = 6$. The value of Ramsey numbers quickly becomes very difficult to compute. $R(4, 4) = 18$. The exact value for $R(5, 5)$ is unknown, but we know that $42 < R(5, 5) < 50$. Joel Spencer relays the following anecdote to highlight the difficulty in computing Ramsey numbers: "*Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of* $R(5, 5)$ *or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for* $R(6, 6)$. *In that case, he believes, we should attempt to destroy the aliens.*" [Spencer]

3.2. Walks on Ordinals

3.2.1. C Sequences. This paper makes use of a method of constructing mathematical structures which are subsets of some ordinal. We start off with a single transformation κ which assigns to any ordinal α a closed, unbounded (in α) set of smaller ordinals, C_α. Such a set is called a **C-sequence**. C-sequences are also sometimes called **ladder systems**, a more evocative name.

These C-sequences can then be used as a path from one ordinal to a larger one. The upper trace, defined below, records the sequence of "hops" from one ordinal to a smaller one. They can also be used as the "steps" in a recursive construction.

Our ultimate goal is to prove the following theorem, from which the result about L spaces will follow.

Theorem 3.2.1. *Suppose that* $\{e_\beta : \beta \to \omega\}_{\beta<\omega_1}$ *is a coherent sequence of finite-to-one functions. Further suppose that L satisfies the properties of a lower trace function. If* $A, B \subseteq \omega_1$ *are uncountable, then the set of integers* $\{O(\alpha, \beta) \,|\alpha \in A,\ \beta \in B,\ \alpha < \beta\}$ *contains arbitrarily long intervals. Here,* $O(\alpha, \beta) = osc\left(e_\alpha \upharpoonright L(\alpha, \beta), e_\beta \upharpoonright L(\alpha, \beta)\right)$*(see 3.4.1).*

3.3. The Trace

Lemma 3.3.1. *There are only countable many continuous functions from* 2^ω *into* ω.

PROOF. Notice that 2^ω is homeomorphic to the Cantor set, which is compact. Now, fix $f \in \mathcal{C}$. Let $U = \{f^{-1}(n)\}_{n\in\omega}$ U is an open cover of 2^ω, and so it must have a finite sub-cover. That means that, in fact, any continuous function on 2^ω attains at most finitely many discrete values. So, the total number of such functions is the sum of the number of possibilities of n-many-valued functions. That is, $\aleph_0 + \aleph_0^2 + \aleph_0^3 + \ldots = \aleph_0 \cdot \aleph_0 = \aleph_0$ □

Definition 3.3.2. (*coherence*) A set of functions on overlapping domains is *coherent* if, where there domains agree, the functions agree almost everywhere, that is, they disagree at only finitely many points.

(C-sequence) $C = \langle C_\alpha | \alpha < \omega_1 \rangle$ is a sequence of sets of ordinals so that:

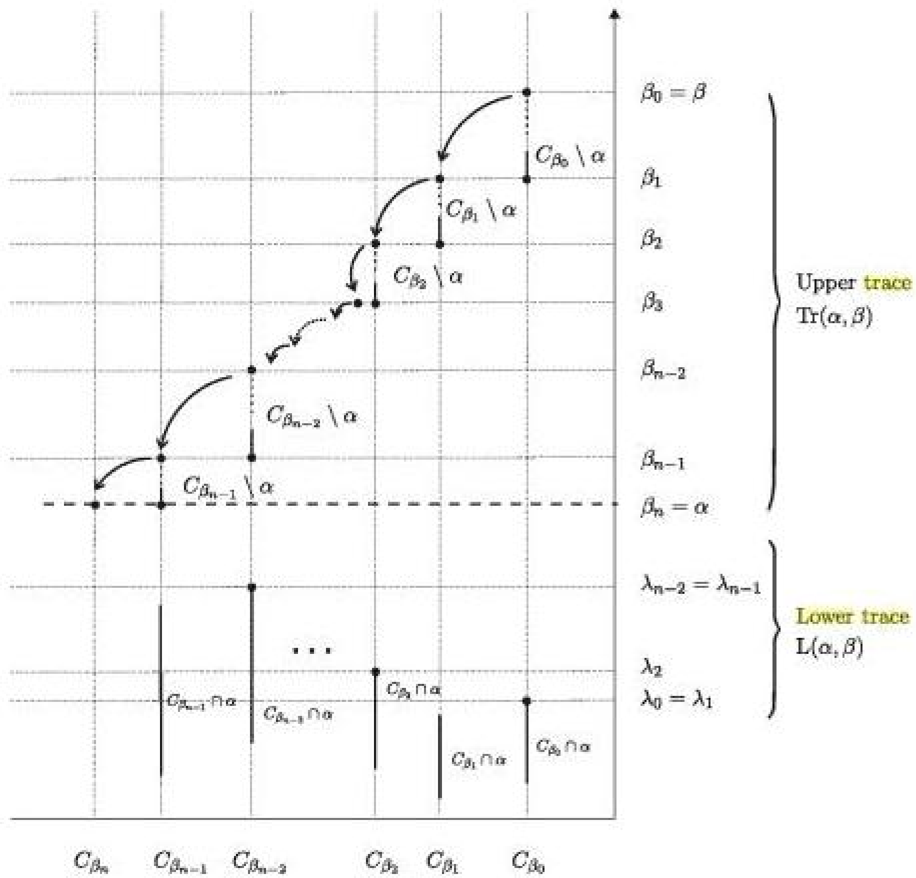
(1) Each $C_\alpha$is co-final in α
(2) Whenever $\gamma < \alpha$ then $C_\alpha \cap \gamma$ is finite
(3) $0 \epsilon C_\alpha$ for every α.

For what follows, we will fix all of the following:

- an arbitrary sequence $z = \{z_\alpha\}_{\alpha<\omega_1}$ of distinct elements in 2^ω
- an enumeration of $\mathcal{C}$, the continuous functions from 2^ω into ω, namely $\mathcal{C} = \{f_\alpha\}_{\alpha<\omega_1}$. (see 3.3.1)
- a coherent sequence of functions, $e = \{e_\beta\}_{\beta<\omega_1}$ with each e_β a finite-to-one function from that β into ω. Notice that the domains of these functions nest outwards, and so, for them to be coherent requires that $e_\beta \restriction \alpha = e_\alpha$ almost everywhere whenever $\alpha < \beta$.
- An arbitrary C-sequence $C = \langle C_\alpha | \alpha < \omega_1 \rangle$

Definition 3.3.3. *(Minimal Walk)* The minimal walk from a countable ordinal β to $\alpha < \beta$ along the fixed C-sequence C is a finite, decreasing sequence of "steps" $\beta = \beta_0 > \beta_1 > ... > \beta_n = \alpha$ where at each step, $\beta_{k+1} = \min \{C_{\beta_k} \setminus \alpha\}$.

Before we define the upper and lower trace, the following diagram (taken from [Todorcevic p21]) is a helpful explanation of what we're aiming for:



Definition 3.3.4. *(Upper-Trace)* The upper trace from α to β is the set of steps along the minimal walk from β to α along our fixed C, that is, $Tr(\alpha, \beta) = \{\beta_k\}_{\kappa=0}^{n}$ with each $\beta_{k+1} = \min\left\{C_{\beta_k} \setminus \alpha\right\}$.

The following lemma characterizes the properties of the upper trace.

Lemma 3.3.5. *For any uncountable subset Γ of ω_1, the union of all the $Tr(\alpha, \beta)$ for $\alpha < \beta$ in Γ, that is $\bigcup_{\alpha<\beta\epsilon\Gamma}\{Tr(\alpha,\beta)\}$, contains a closed, unbounded set of ω_1.*

(see Todorcevic)

For our purposes, we are interested in the upper trace only as a conceptual motivator for the more complicated lower trace.

Definition 3.3.6. (*Lower Trace*) If $\alpha < \beta$ then the lower trace from α to β is $L(\alpha, \beta) = \{\lambda(\xi, \beta) \,|\xi \in Tr(\alpha, \beta) \;\&\, \xi \neq \beta\}$ where:

$\lambda(\xi, \beta) = \max\left[\max\left\{C_\eta \cap \alpha | \eta \in Tr(\alpha, \beta) \;\&\, \eta \neq \beta\right\}\right]$

That is to say, $L(\alpha, \beta) = \{\lambda_0, \lambda_1 ... \lambda_{n-1}\}$where, at each step k, $\lambda_k = \max\left[\bigcup_{j=0}^{k}\left(C_{\beta_j} \cap \alpha\right)\right]$. That is, at each step, we jump to the "end-point" of the next C_β which is bigger than where we are now.

Remark 3.3.7. There is also something called the "full lower trace", which is more commonly used. This is not that. (see [Todorcevic, p. 20])

Definition 3.3.8. The lower trace is characterized by the following lemmas:

Lemma 3.3.9. *(Colinearity) If $\alpha \leq \beta \leq \gamma$ and $L(\beta,\gamma) < L(\alpha,\beta)$, then $L(\alpha,\gamma) = L(\alpha,\beta) \cup L(\beta,\gamma)$.*

This lemma relies on the linear ordering of ordinals. It says that to get from α to β, you have to go through every γ between them. Essentially, it can be understood as saying that the lower trace admits no "shortcuts".

Lemma 3.3.10. *(Continuity) If δ is a limit ordinal, then $\lim\limits_{\xi\to\delta}(\min L(\xi,\delta)) = \lim\limits_{\xi\to\delta}(\max(C_\delta \cap \xi)) = \delta$.*

This lemma can be understood as saying that the lower trace's minimum (it's starting point) continuously approaches it's endpoint.

Now, we are ready to define the μ function.

Definition 3.3.11. As above, let $\beta_{k+1} = \min\left\{C_{\beta_k} \setminus \alpha\right\}$ and $\xi_k = \bigcup_{j=0}^{k} C_{\beta_j} \cap \alpha$. Define $\mu_{(\alpha,\beta)} : L(\alpha,\beta) \to \mathcal{C}$ by $\mu_{(\alpha,\beta)}(\xi_k) = \omega_{\beta_k}$whenever $k = 0$ or $\xi_k \neq \xi_{k-1}$.

Like the lower trace, the μ function is both "colinear" and "continuous". That is:

Lemma 3.3.12. *If $\alpha < \beta < \gamma$ and $L(\beta,\gamma) < L(\alpha,\beta)$ then $\mu_{(\alpha,\gamma)} = \mu_{(\alpha,\beta)} \cup \mu_{(\beta,\gamma)}$.*

This is the colinearity mentioned above.

Lemma 3.3.13. *If $\xi < \delta$ then $\mu_{(\xi,\delta)}(\min L(\xi,\delta)) = \omega_\delta$*

This is the continuity.

3.4. Oscillations of the Trace

Definition 3.4.1. (*oscilation*) Suppose that s and t are two functions whose domain is D. Let $Osc(s,t,D)$ count the number of times s surpasses t on D. That is, $Osc(s,t,D) = \{x\epsilon D | s(x^-) \leq t(x^-) \ \& \ s(x) \geq t(x)\}$ where $s(x^-) = \lim \varepsilon \to 0^- s(x-\varepsilon)$. Further, let $osc(s,t,D) = \#Osc(s,t,D)$.

Example 3.4.2. Let $s(x) = \sin(x)$ and $t(x) = 0$ and $D = \mathbb{R}$, then $Osc(s,t,D) = \{2k\pi\}_{k\in\mathbb{N}}$.

In what follows, we will only be interested in oscillations of functions with finite domains.

Definition 3.4.3. If $\alpha < \beta < \gamma$, let $Osc(\alpha,\beta) = Osc\left(e_\alpha, e_\beta, L(\alpha,\beta)\right)$ and let $osc(\alpha,\beta) = \#Osc(\alpha,\beta)$.

Lemma 3.4.4. *For every uncountable, pairwise disjoint $A \subseteq [\omega_1]^k = \{X \subseteq \omega_1 | \#X = k\}$ and $B \subseteq [\omega_1]^\ell$ and for every $f \epsilon \mathcal{C}(2^\omega, \omega)$, there is $\{b_m\}_{m \epsilon \omega} \subseteq B$ so that, for every $n \epsilon \mathbb{N}$ there's an $a = \{a_1, a_2 ... a_k\} \in A$ and $\{\xi_1, \xi_2 ... \xi_n\}$ so that for any $m < n$, $i < k$, and $j < \ell$ we have:*

(1) $a < b_m$ for every $m < n$ (i.e., $a_i < b_{m_j}$ for every $i = 1...k$ and $j = 1...\ell$.)
(2) $Osc(a_i, b_{m_j})$ is the disjoint union of $Osc(a_i, b_{0_j})$ and $\{\xi_{m'} | m' < m\}$.
  (a) That is to say, a_i surpasses b_m whenever it surpasses b_0 and also at $\xi_0, \xi_2 ... \xi_{m-1}$.
(3) $\mu_{(a_i, b_{0_j})} = \mu_{(a_i, b_{m_j})} \restriction L(a_i, b_{0_j})$
  (a) The μ function of a_i over b_{0_j} is just the μ function of a_i over b_{0_j} restricted to the lower trace of a_i onto b_{0_j}
(4) $\mu_{(a_i, b_{m_j})}(\xi_{m'}) = f$ whenever $m' < m$

This lemma is a direct result of the following theorem, for proof of this, see [Moore, p13].

Definition 3.4.5. We denote $\{a \in A | a > \delta\} = A_{>\delta}$ whenever A is a collection of finite subsets of $\omega_1$and $\delta < \omega_1$

Theorem 3.4.6. *Let $A \subseteq [\omega_1]^k$and $B \subseteq [\omega_1]^\ell$ be both uncountable and pairwise disjoint. Then there is a closed, unbounded set of $\delta < \omega_1$ so that if $a \in A_{>\delta}$ and $b \in B_{>\delta}$ then there is an $a^+ \in A_{>\delta}$ and $b^+ \in B_{>\delta}$ so that, for any $i < k$ and $j < \ell$ we have:*

(1) $\max \left[L\left(\delta, b_j\right)\right] < \min \left[\Delta\left(e_{a_i}, e_{a_i^+}\right), \Delta\left(e_{b_j}, e_{b_j^+}\right)\right]$
(2) There is a non-empty L^+ so that, for every j, $L\left(\delta, b_j\right) < L^+$and $L\left(\delta, b_j^+\right) = L\left(\delta, b_j\right) \cup L^+$
(3) If $\xi \in L^+$ then $e_{a_i^+}\left(\xi\right) = e_{b_j^+}\left(\xi\right)$
(4) $\mu\left(\delta, b_j\right) = \mu\left(\delta, b_j^+\right) \restriction L\left(\delta, b_j\right)$
(5) $\mu\left(\delta, b_j^+, \min L^+\right) = f_\delta$

For the proof of this theorem, see [Moore, p11]

3.5. Coloring

Definition 3.5.1. The function $* : \mathbb{N} \to \mathbb{N}$ is defined by letting $*0 = 0$, and thereafter setting $*m = n$ where n is the least prime which does not divide m.

For example, $*360 = 7$, $*8 = 3$, and $*p = 2$ whenever p is prime and > 2.

Remark 3.5.2. We write osc* to mean the composition of osc, followed by *.

Fact 3.5.3. *If $X \subseteq \mathbb{N}$ contains arbitrarily long intervals, that is if $\forall n \in \mathbb{N}\ \exists k \in \mathbb{N}$ so that $\{k,\ k+1,\ k+2 \ldots k+n\} \subseteq X$, then $*X = \mathbb{N}$.*

Applying this fact to 3.4.4 gives us:

Theorem 3.5.4. *If $A, B \subseteq \omega_1$ are uncountable and $n < \omega$, then there are α in A and β in B with $\alpha < \beta$ and $(osc * (\alpha, \beta)) = n$.*

Remark 3.5.5. We will use the symbol ♡ as shorthand for the statement $\omega_1 \rightarrow \left(\omega_1, \left([\omega_1]^{<\aleph_0}; \omega_1\right)\right)^2$, which means that, for any function $c : [\omega_1]^2 \rightarrow \{0, 1\}$, either:

(1) There is an uncountable $X \subseteq \omega_1$ so that c is zero everywhere on $[X]^2$ OR
(2) There is an uncountable $A \subseteq [\omega_1]^{<\aleph_0}$ and an uncountable, pairwise-disjoint $B \subseteq \omega_1$ so that, for any $a \in A$ and any $b \in B$ with $a < b$, there is a $\beta \in b$ so that $c(a, \beta) = 1$

Fact 3.5.6. ♡ *is relatively consistent in ZFC. (Todorcevic)*

For ♡to be false, there needs to be a function, $f : [\omega_1]^2 \rightarrow \{0, 1\}$ which is zero only countably many times, and so that, for every uncountable $A \subseteq [\omega_1]^{<\aleph_0}$ and every uncountable, pairwise-disjoint $B \subseteq \omega_1$ there is an $a \in A$ and there is a $b \in B$ with $a < b$, so that, for every $\beta \in b$, $c(\alpha, \beta) = 0$

Definition 3.5.7. Let $o(\alpha, \beta) = \sum_{q \neq 0} \left(\#\mu(\alpha, \beta; \alpha)^{-1}(q)\,[mod\ q]\right)$

Theorem 3.5.8. *(Moore 5.3) Let $A \subseteq [\omega_1]^j$ and let $B \subseteq [\omega_1]^2$ each be uncountable, pairwise disjoint families of functions. for every $\chi : j \rightarrow 2$ and any $\pi : j \rightarrow 2$, there is an $a = \{a_1, a_2 ... a_j\} \in A$ and there is $b = \{b_1, b_2\} \in B$ so that:*

(1) $a < b$ ($a_i < b_i$ for every $i \in \omega_1$)
(2) for every $i < j$, $*o\left(a_i, b_{\pi(i)}\right) = \chi(i)$.

This theorem follows from theorem 3.4 , the Chinese Remainder Theorem, and a great deal of cleverness. (Moore p15-16)

Definition 3.5.9. Definition. Let $c(\alpha, \beta)$ denote $o(\alpha, \beta)\ mod\ 2$ whenever $\alpha < \beta$.

Remark 3.5.10. It will be helpful to keep in mind the following interpretation of this theorem: Given integers k & j, and $D \subseteq k \times j$ then we can think of a function $\chi : D \rightarrow \omega$ as some sort of pattern, like a color-by-number picture. A coloring $c : [\omega_1]^2 \rightarrow \omega$ tells us which of the ω-many colors goes in each "piece" of the picture.

A coloring, c, is like a large table that assigns every ordered pair in $\omega_1 \times \omega_1$ (including each one in D) a particular color. c is said to "realize the pattern χ" if and only if, whenever $A \subseteq [\omega_1]^k$ and $B \subseteq [\omega_1]^\ell$ are each pairwise disjoint and uncountable, there is an $a \in A$ and a $b \in B$ such that $a < b$ and $c\left(a_i, b_j\right) = \chi(i.j)$. That is, if, when we color by the rule c, somewhere on the large plane, the picture χemerges.

Fact 3.5.11. *The c defined in 3.5.9 is a coloring.*

Remark 3.5.12. We can interpret theorem 4.3 (5.9) above as saying that, if D is a function, and χ is a binary pattern on D, then $*o$ realizes χ. (A binary pattern assigns only two colors. Think of it as $\chi : D \to \{black, white\}$)

3.6. The L Space

3.6.1. Definitions.

point-countable:

point-separating:

countably-tight:

$\triangle - system$**:** A $\triangle - system$ is a collection of sets whose pairwise intersection is constant. The *delta-lemma* is an extra set-theoretical assumption which tells us that every uncountable collection of finite sets contains an uncountable $\triangle$-system.

semi-norm: A semi-norm on the space X is any $m : X \to \mathbb{R}^+$ which is scalable and subadditive. That is, a semi-norm would be a norm, except some things which aren't zero have norm zero.

Frechet: A topological vector space is called "Frechet" if and only if it is (1) complete as a uniform space, (2) Hausdorff, and (3) its topology can be induced by a countable family of seminorms.

3.6.2. The Definition of the L Space.

Definition 3.6.1. For every $\alpha < \omega_1$, set $W_\alpha = \{\alpha\} \cup \{\beta > \alpha : c(\alpha, \beta) = 1\}$

Now, in order to construct our L space, we will choose an uncountable subset $X \subseteq \omega_1$and then define a topology on it which is non-separable, but hereditarily Lindelöf. This topology, $\tau[X]$, will be constructed by declaring $W_\xi \cap X$ to be clopen for every $\xi \in X$. Since X is uncountable, we have now created an uncountable collection of sets, each of which is it's own closure, and so surely no countable set can be dense. Finally, the following theorem will be used to ensure that $\tau[X]$ is hereditarily Lindelöf.

Theorem 3.6.2. *If X, $Y \subseteq \omega_1$ are disjoint, then there is no continuous injection from any uncountable subspace of $(X, \tau[X])$ into $(Y, \tau[Y])$.*

Corollary 3.6.3. *For every X, $(X, \tau[X])$ is hereditarily Lindelöf.*

PROOF. If X weren't Lindelöf, then there would be an uncountable discreet subspace, call it D, and so it contains two disjoint, uncountable discreet spaces, D_1 and D_2. Now, every function from one discreet space to another is continuous, but that directly contradicts theorem 17.2, which says that there can be no such continuous function. □

3.6.3. Proof of Theorem 17.3.

PROOF. Imagine such an injection existed; call it $f : X_0 \to Y$ where X_0 is some uncountable subset of X. Let $B = \{\{\beta, f(\beta)\}\}_{\beta \in X_0}$. For each $\beta \in X_0$ we can pick U_β open around β so that $U \subseteq f^{-1}\left(W_{f(\beta)}\right)$.

By refining X_0 (if necessary) we know there is some k and some $\chi_0 : k \to \{0,1\}$ so that there is a finite $F \subseteq X_0$ and a$\triangle$-system $F_\beta : \beta \in X_0$ with root F with each $\left|F_\beta\right| = |F| + k$:

1) If $F_\alpha < \beta$, then $\beta \in U_i$ if and only if for all $i < |F| + k$, $c(F_\alpha(i), \beta) = \chi_0(i)$.

2) $m = |F_\alpha \cap f(\alpha)|$is a constant, independent of α.

3) max (F) is less than both α and min $(F_\alpha \setminus F)$. (that is, F is an initial segment F_α)

Let $A = \{\{f(\alpha)\} \cup F_\alpha \setminus F\}_{\alpha \in X_0}$ Since X is disjoint from Y , $f(\alpha) \notin F_\alpha$ (for any $\alpha \in X_0$). Define coloring$\chi : k+1 \to 2$ by $\chi(i) = \chi_0(|F| + i)$ if $i < m$, $\chi(m) = 0$, and $\chi(i) = \chi_0(|F| + i - 1)$ thereafter. Next, define $\pi : k+1 \to 2$ by $\pi(i) = 0$ if $i \neq m$ and $\pi(m) = 1$.

Applying Theorem 4.8, (1) there is an $a = \{a_1, a_2 ... a_{k+1}\} \in A$ and a $b = \{\beta, f(\beta)\} \in B$ such that$a < b$ and (2) for all $i < k+1$, $c\left(a_i . b_{\pi(i)}\right) = \chi(i)$. Now, fix $\alpha, \beta \in X_0$ be such that $a = \{f(\alpha)\} \cup (F_\alpha \setminus F)$ and $b = \{\beta, f(\beta)\}$.

To derive a contradiction (to f's continuity), it suffices to show that (1)$\beta \in U_\alpha$ but that (2) $f(\beta)$ is not in $W(f(\beta))$.

(1): To show $\beta \in U_\alpha$ we need to prove that, for every $i < |F| + k$ we have $c\left(F(i)_\alpha, \beta\right) = \chi_0(i)$.

Case 1: If $i < |F|$, then $F_\alpha(i) = F_\beta(i)$ and so, since $\beta \in U_\beta$ and $F < \beta$, we have that $c(F_\alpha(i), \beta) = c\left(F_\beta(i), \beta\right) = \chi_0(i)$

Case 2a: If $0 \leq i - |F| < m$, then $c(F_\alpha(i), \beta) = c\left(a_{i-|F|}, \beta\right) = \chi(i - |F|) = \chi_0(i)$.

Case 2b: If $m < i - |F|$, then $c(F_\alpha(i), \beta) = c\left(a_{i-|F|+1}, \beta\right) = \chi(i - |F| + 1) = \chi_0(i)$

(2): Now, recall that $W(f(\beta))$ means exactly that $c\left(a_m, b_{\pi(m)}\right) = c(f(\alpha), f(\beta)) = \chi(m) = 0$

And, so we see that (1) and (2) are surely both true. □

CHAPTER 4

CONCLUDING REMARKS

The startlingly counterintuitive solutions to the S and L space problems are not only fascinating as mathematical curiosities, but also highlight how approaches from separate fields can come together to solve deep problems.

4.1. Open Problems

Some realted problems, open at the time of the writing, can be found below. [Pearl]

(1) Is there an L space whose square is also an L space?

(2) If X is compact and X^2 is hered. normal, must X be separable?

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