Renormalization Group Methods in Applied Mathematical Problems

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RENORMALIZATION GROUP METHODS IN APPLIED MATHEMATICAL PROBLEMS

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This work presents the application of the methods known as renormalization group (RG) and scaling in the physics literature to applied mathematics problems after a brief review of the methodology.

The first part of the thesis involves an application to a class of nonlinear parabolic differential equations. We consider equations of the form $u_t = \frac{1}{2}u_{xx} + \varepsilon N(x, u, u_x, u_{xx})$ where ε is a small positive number and N is dimensionally consistent without additional dimensional constants. First, RG methods are described for determining the key exponents related to the decay of solutions to these equations. The determination of decay exponents is viewed as an asymptotically self-similar process that facilitates an RG approach. These methods are extended to higher order in the small coefficient of the nonlinearity. The RG calculations lead to the result that for large space and time, the solution is characterized by $u(x,t) \sim t^{-\frac{1}{2}-\alpha}u^*(xt^{-1/2}, 1)$ where the exponent α is a simple function of the exponents of the terms in N. Finally, the RG results are verified in some cases by rigorous proofs and other calculations.

In the second part, the application of renormalization technique to systems of equations describing interface problems are presented. The temporal evaluation of an interface separating two phases is analyzed for large time. We study the standard sharp interface problem in the quasi-static regime. The characteristic length, $R(t)$, of a self-similar system that is the time dependent length scale characterizing the pattern growth is calculated by implementing a renormalization procedure. It behaves as t^β where β has values in the continuous spectrum $[1/3, 1/2]$ when the dynamical undercooling is non-zero, and β in $[1/3, \infty)$ when the undercooling is set at zero. The single value

of $\beta = 1$ is extracted from this continuous spectrum as a consequence of boundary conditions that impose a plane wave. It is also shown that in almost all of these cases, the capillarity length (arising from surface tension) is irrelevant for the large time behavior even though it has a crucial role at the early stage evolution of an interface.

To my parents: Hatice and Salih Merdan

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Introduction

Many of the important and challenging problems in Applied Mathematics involve describing a complex system evolving in time (such as systems of equations describing the temporal evolution of an unstable interface). A stochastic element in these problems enhances the importance of a global understanding in addition to a complete and detailed large scale computation. A key goal is the development of a calculation technique that is comparable to asymptotic analysis and linear stability theory.

For many physical quantities the time evolution is described by nonlinear differential equations. For most of these equations, obtaining a closed form solution is clearly a hopeless task. Therefore, a deeper perspective is needed in order to understand the nature of solutions to these equations. In particular, one tries to determine certain qualitative properties of the solution, such as its existence and regularity for all times and then investigate its long-time asymptotics. It turns out that, for certain equations, the long-time behavior can be predicted because the solution becomes asymptotically scale-invariant; a simple example is given by the usual heat equation $u_t = u_{xx}$.

Many of the problems in applied mathematics are philosophically similar to those encountered by physicists in the study of critical phenomena, where renormalization and scaling theory promote an understanding of physical singularities and facilitate the computation of the relevant exponents. A focus of recent research has been the adapting of this theory in order to understand large time behavior as an asymptotic fixed point.

Renormalization group (RG) methods were originally developed for quantum field theory and statistical mechanics and have provided a powerful tool for calculation of key exponents that are otherwise extremely difficult to evaluate [22]. These methods have been broadened in recent years to include a spectrum of problems such as fractals, random walk and difference equations [17] and have evolved into a broad philosophy rather than a single technique, as each new application often involves different methods. The application of RG to these problems illustrates the central themes that provide insight into new problems [17]. The diversity of the problems that have been understood through RG suggests that it has the potential to become a systematic tool of applied mathematics. Since differential equations are central to much of applied mathematics, it is important to examine RG in this context, particularly within classes of equations for which we can verify some of the results independently.

There are several aspects of differential equations in which self-similarity is exhibited at an asymptotic fixed point. These include (i) decay of solutions for large time and space, (ii) finite time blow-up of solutions; and (iii) finite time extinction of solutions. In particular a key question involves the exponent that characterizes decay, blow-up or extinction. For systems of equations describing interface problems an interesting issue is (iv) the large time evolution of the interface, and, for example, the exponent that characterizes the length of the interface as a function of time.

The purpose of this work is to adapt this methodology to problems in applied mathematics, to make it a tool that could be used for a broad spectrum of problems, and to show that it yields new mathematical results.

Thesis description

The outline of the thesis is as follows.

Chapter 1 is dedicated to a brief review of renormalization group (RG) methods and scaling. First, we begin introducing the general notion of these methods in Physics. Second, the methods are briefly described for the two examples in Applied Mathematics: random walk and fractals. It is followed by introducing the preliminary applications to nonlinear differential equations.

In **Chapter 2** using RG methods we calculate the anomalous exponent related to decay of solutions to nonlinear heat equation of the form $u_t = \frac{1}{2}u_{xx} + \varepsilon N(x, u, u_x, u_{xx})$ where ε is a small parameter and N is dimensionally consistent without additional dimensional constants. We begin describing RG methods and compute the exponent and scaling form of the leading term of the decay arising from a narrowly peaked Gaussian (of width l) as an initial condition, and extend these methods for the higher order (in ε). The methodology involves two basic parts. First, we obtain an asymptotic expression for leading order behavior in l^{-1} of the form

$$u(x, t'; \varepsilon, l) \sim u_0(x, t'; l) + \varepsilon u_1(x, t'; l) + \varepsilon^2 u_2(x, t'; l) + \varepsilon^3 u_3(x, t'; l) + \cdots \tag{1}$$

where t' is the original time variable (which is rescaled as $t := 2Dt'$ in the analysis). The asymptotic expression (1) is not an approximation to entire function but rather to most singular term in l. Thus, for example, u_2 is the part of the $O(\varepsilon^2)$ term that will dominate simultaneous expansions involving small values of $\varepsilon, l, 1/t$ and $1/x$.

Second, the information in (1) is utilized along the lines of renormalization group approach by writing $u(x, t'; \varepsilon, l)$ (suppressing l and ε dependence) as

$$\begin{aligned} u(x, t') &= Z_r(b)^{-1} u(b^{1/2}x, bt') \\ &=: R_{b,1/2} u(x, t') \end{aligned} \tag{2}$$

for a suitable function Z_r, for all $b > 1$. Iterating this transformation n times and taking the limit as $n \to \infty$ yields a fixed point (if it exists) only if

$$u_r^*(x, t') := \lim_{n\to\infty} Z_r(b)^{-n} u(b^{n/2}x, b^n t') \tag{3}$$

is well defined. This relation then leads to the anomalous exponent α through the relation

$$\lim_{n\to\infty} \left[Z_r \left(\left(\frac{b^n t'}{Q_1^2/D} \right)^{1/n} \right) \right]^n \sim \text{constant} \times (b^n t')^{-\frac{1}{2}-\alpha} \tag{4}$$

and also u_r^*.

Finally, we resolve rigorously and exactly the exponent for some nonlinearities verifying RG results and present alternative methods in order to calculate the exponent. In addition we produce an iterative expansion leading the solution involving close form integrals. Using *shooting methods* we also prove a theorem that confirms RG results in [12] in Appendix A.

The work presented in this chapter resulted in two papers that have recently been published [38], [37].

Chapter 3 addresses the application of RG methods to interface problems. The chapter is started presenting the preliminary results that were obtained by adapting these methods. It follows the study of temporal evolution of an interface separating two phases for its large time behavior by (once again) implementing the RG methods. The late stage growth issue is examined in the context of a general geometry and more general conditions on the degree of undercooling.

In this chapter, we consider a sharp interface model in the quasi-static regime, i.e. the heat equation $u_t = \Delta u$ is replaced by Laplace's equation $\Delta u = 0$, in order to calculate the characteristic length, $R(t)$, that is the time dependent length scale characterizing the evolution of the pattern. The calculations are based upon very general assumption on the initial conditions. The main physical assumption is that the pattern evolves self-similarly with a single scale. The asymptotic large scale growth is within the context of the statistical set of interfaces that evolve from a set of initial condition.

The methodology basically involves rewriting the basic equations in terms of a Green's function identity after introducing a phase function. The RG analysis then proceeds in several steps as the equations are transformed and then converted them back into their original form with renormalized physical parameters.

We examine two cases, namely that the dynamical undercooling is (i) nonzero and (ii) zero, i.e. $\alpha \neq 0$ and $\alpha = 0$. The main result is that without reference to a plane wave the characteristic length, $R(t)$, varies as $t^{-1/\lambda}$, where $\lambda \in [-3, -2]$, when the dynamical undercooling is nonzero, i.e. $\alpha \neq 0$. For the case $\alpha = 0$, the spectrum is $[-3, 0)$. The single value of $\lambda = -1$ that was obtained in [31] is selected by imposing a plane wave through boundary conditions.

The analysis also indicates that in almost all of these cases, the capillarity length, which is the length scale associated with the surface tension, is not relevant for the large time behavior of an interface. This is in sharp contrast to its role in the linear stability theory for short time.

The work above resulted in the papers [39], [40].

Chapter 1

Renormalization Group Methods and Scaling

1.1 Renormalization and Scaling in Physics

Renormalization group and scaling methods originated as part of an effort to understand the cooperative behavior of a large number of molecules or spins that leads to the divergence of some measurable quantity with a characteristic exponent (see for example Wilson and Kogut [55], and Fisher [20] and references contained therein) . Pioneered by Kenneth Wilson in the 1970's, the basic ansatz of the methodology is that *averaging* the detailed interactions between the individual members of a complex system cannot change the basic characteristics of measurables near a critical point, which is a point on the phase plane where the length scales diverge. If one performs some type of *averaging* repeatedly, the resulting physical quantity is likely to be zero or infinity unless one also transforms the magnitude of the interactions properly. On the other hand if one does understand how these should transform, then the nature of the transformation should permit the calculation of the *critical* exponent directly. The successful implementation of this ansatz allowed simple calculation of these divergence exponents that had previously been prohibitively difficult, and led to Wilson's Nobel Prize, as well as thousands of papers in statistical mechanics (see [17] and [22]).

Before we present its application to the problems in Applied Mathematics, we first try to formulate a general notion of an RG below. Although RG transformations on parameter space do not actually comprise a mathematical group, there is often a type of invariance principle at work. For example, in the example of Random Walk (RW) to be considered presently, the RG is constructed in such a way as to keep the root mean square (RMS) length of the RW invariant.

RG calculations basically involves two steps. In the first step, which is called *coarse graining*, one averages out a subset of the degrees of freedom of the system that vary on small scales. The motivation behind of it is that at the critical point the behavior of the system is dominated by the fluctuations on very large scales. The second step of the RG transformation is called *rescaling* and involves the redefinition of the unit of length. The scale factor, which is generally denoted by b, is the ratio of the coarse grained unit of length to the original unit of length.

As a result of these two operations, the parameters of the system will be renormalized and one will obtain the RG equations that yield the relation between the new renormalized parameters, θ'_n, and the old parameters, θ_n, and may be formally expressed, for $n = 1, 2, \cdots$, as

$$\theta'_n = R_n(\theta_n).$$

Under the repeated application of the RG, the renormalized parameters tend to a fixed point ${\theta_n}^*$ at which the system and its renormalized copy are identical. Recaling that the unit of length has been rescaled by a factor b in the RG one sees that the system is in fact self-similar at a fixed point of the RG (see [17] for further details).

After this short review of the RG notion in Physics, we now review below the application of RG methods to applied mathematics problems.

1.2 Renormalization Group Methods in Applied Mathematics

The RG methods that were very successful in resolving delicate issues of statistical mechanics, such as critical exponents, have been applied to a broad spectrum of problems in applied mathematics. In particular, in the subsequent decades after 70's, interest in problems exhibiting self-similarity has increased dramatically and posed the question of whether RG can be applied to such problems. A particularly illuminating text by Creswick, Farach and Poole [17] describes a number of such applications, such as self-avoiding walk, fractals, etc.,. For example, suppose a random walk consists of n steps each in a random direction. Typically, we are interested in the large-scale properties, so that it makes sense to average out the small-scale details, or *coarse grain*. One can accomplish this by starting with the probability distribution, $p(r)$, for each step, given by a Gaussian, for example, as

$$p(r) := (2\pi\sigma_0^2)^{-d/2} \exp\{-|r|^2/(2\sigma_0^2)\} \tag{1.1}$$

where σ_0 is the width of the Gaussian. Defining $r' = \sum_{i=1}^{n} r_i$ as the rescaled step, one has a new random walk with the new random variable r'. By performing the integration over the old variables, one can show (p. 15 of [17]) that the new coarse-grained probability distribution is

$$P(r') := (2\pi n\sigma_0^2)^{-d/2} exp\{-|r'|^2/(2n\sigma_0^2)\}. \tag{1.2}$$

We now observe that the two expressions above differ only in that the parameter σ_0 has changed, i.e., the coarse-grained σ is $\sigma' := n^{1/2}\sigma_0$. The second step in the RG procedure is to *rescale* by redefining the unit of length by defining $r' = n^{1/2}r''$, so that the original form of the probability distribution (1.1) is recovered in terms of r''. Now we would like to calculate the RMS distance $R(M)$ covered by a walk of M steps of average length σ. Since σ is the only length scale in the problem, dimensional analysis implies that $R(M) = \sigma M^{\nu}$ where ν is the scaling exponent. By rescaling in terms of n steps we can write

$$\sigma M^{\nu} = n^{1/2}\sigma(M/n)^{\nu} \tag{1.3}$$

and conclude that $\nu = 1/2$. Obtaining this classical result through the modern RG methodology is an illustration of this perspective that can be appreciated without extensive statistical mechanics or quantum field theory.

Using similar methods one can calculate the fractal or Hausdorf dimension of geometric structures, defined as the number D such that $N(a)$ is the minimum number of (d-dimensional) balls needed to cover object, where $N(a) \sim a^{-D}$ as $a \to 0$. Applying RG to the Cantor set, defined by taking a line segment $[0, 1]$ and removing the middle third at each step, we can use $N(a) = 2^n$ line segments of length $a = 3^{-n}$. With some algebraic simplification this leads to

$$N(a) = a^{-\ln 2/\ln 3}. \tag{1.4}$$

Note that this relationship can be obtained from a RG perspective by examining the ratio of $N(a)$ for different magnitudes. In particular, the geometry implies that the transformation must have the form $N(a) = 2N(3a)$ for all $a > 0$. So substituting $N(a) \sim a^{-D}$ implies the same relation 1.4, thereby establishing the fractal dimension $D = \ln 2/\ln 3 \approx 0.63$.

The application of RG methods to differential equations is particularly important since so many applications can be addressed. Recently, the RG philosophy has been directed toward understanding some basic aspects of nonlinear differential equations. As with critical exponents in statistical mechanics, the potential of this research direction lies in the capability to determine a characteristic scaling exponent with a relatively simple calculation upon understanding a transformation that relates two parameters. In the case of critical exponents the dependent parameter is the thermodynamic quantity which diverges while the independent parameter, e.g. temperature, is a measure of the distance from the singularity at the critical temperature, i.e. T_c. In the case of blow-up in differential equations (see Berger and Kohn [4], Giga and Kohn [27], Galaktionov and Posashkov [26] and references within) the solution $u(x,t)$ diverges as $t \to t_c$ where t_c is the critical value. Bricmont and Kupianen [9] have provided rigorous proofs of the existence of infinitely many profiles around the blow-up point using related methods. Bertozzi, Brenner, Dupont and Kadanoff [5] have applied the concept of similarity solutions for the onset of singularities in problems involving flow through thin films. Topological transformation and singularities in viscous flows have been studied by Goldstein at al [28].

The justification for the renormalization group method in both problems can be made in terms of the asymptotic self-similarity of the solution (or thermodynamic variable) as the critical value is approached. More explicitly the profile of the physical quantity u appears to be nearly identical as one zooms in on the critical value t_c, provided that u (and x in the blow-up case) is scaled appropriately. In this case of critical phenomena, real-space renormalization has its origin in the intuition of the underlying physical interactions (see [55]). However, in the case of parabolic differential equations, one can obtain the leading behavior of the singularity by observing simple scaling rules that govern the key transformations, and then applying a methodology similar to asymptotic analysis. In other words, the differential equations, by virtue of their scaling properties, already incorporate the essential information on the cooperative behavior in the physical system.

A problem that is seemingly unrelated to blow-up is the large time delay in a nonlinear parabolic equation and the associated nonclassical exponents. However, the renormalization group techniques apply in a similar way to these problems due to the asymptotic self-similarity as the horizontal axis is approached. In terms of analytic geometry this problem is analogous to the inverse of the blow-up problem. An asymptotic delay of the form

$$u(x,t) \sim t^{-\frac{1+\alpha}{2}} u^*(xt^{-1/2}, 1) \tag{1.5}$$

(for large time and space, see [12]) can be regarded as an expression for time in terms of u and $x/\sqrt{t}$,

$$t \sim \left[\frac{u(x,t)}{u^*(xt^{-1/2}, 1)} \right]^{-2/(1+\alpha)}, \tag{1.6}$$

so that t exhibits a divergence as u goes to zero with x scaled appropriately. Thus the self-similarity arises in much the same way as in blow-up problems. Given a profile of u as a function of t with x scaled appropriately, one can rescale t and the profile would look almost identical provided the size of u is reduced by appropriate factor. The self-similarity is asymptotic in that the transformation is only approximate for any finite t, but the error vanishes in the limit $t \to \infty$. It is in this sense that the classical applied mathematical techniques of asymptotic methods (involving small 1/t) can be used in conjunction with modern physical methods of renormalization to provide a powerful tool for analytic computation.

Calculation of nonclassical exponents in the absence of stochastic using RG methods was first done for the porous medium equation (also called Barenblatt's Equation)

$$u_t - \frac{1}{2} u_{xx} = \frac{\varepsilon}{2} H(-u_{xx}) u_{xx},$$

$$H(\theta) := \frac{1}{2} \left\{ \frac{\theta + |\theta|}{|\theta|} \right\} \quad \varepsilon > 0 \tag{1.7}$$

by Goldenfeld et al. [23]. These methods were extended by Caginalp [12] in order to study the large time behavior of solutions to nonlinear parabolic differential equations ,which include (i) decay of solutions for large time and space, (ii) finite time blow-up of solutions; and (iii) finite time extinction of solutions [13]. These methods were also applied to systems of parabolic equations. The higher order extension of RG calculations including proofs that confirm these calculations for some special cases were studied by Merdan and Caginalp [37]. Some existence proof has also been obtained for nonlinear parabolic equations using the RG methods by Bricmont, Kupianen and Lin [10].

RG methods have also been applied to stochastic differential equations by Glimm, Zhang and

Sharp for chaotic, mixing of interfaces [25], Zhang for random velocity field [57], Avellaneda and Majda for Stochastic and turbulent transport [2] and references contained therein.

Another application of RG methods involves the understanding of large time behavior of the systems of equations describing interface problems. Using the RG methods the study of these equations has been considered by Jasnow and Vinals [31], Caginalp [15], Merdan and Caginalp [39] and references contained therein.

Using renormalization and scaling techniques structural stability problems of propagating fronts have been investigated by Paquette and Oono [49] and Paquette at al [48].

Renormalization group techniques have also been utilized in other dynamical differential equation problems (see, for example, [32], [35], [41] and references therein).

Chapter 2

Decay of Solutions to Nonlinear Parabolic Equations: Renormalization and Rigorous Results

2.1 Introduction

Since differential equations are central to much of applied mathematics, it is important to examine RG in this context, particularly within classes of equations for which we can verify some of the results independently. There are several aspects of differential equations in which self-similarity is exhibited at an asymptotic fixed point. These include (i) decay of solutions for large time and space, (ii) finite time blow-up of solutions; and (iii) finite time extinction of solutions. In particular a key question involves the exponent that characterizes decay, blow-up or extinction.

Decay problems using renormalization group techniques were studied by Goldenfeld, Martin, Oono and Liu [23], Bricmont, Kupiainen and Lin [10], Caginalp [12], and Merdan and Caginalp [38] [37](see other references therein). In particular, Goldenfeld et. al. used RG to calculate the decay exponent for the porous medium equation having a small nonlinear term, and showed that it differed from the classical heat equation.

An important set of goals has been to (a) render RG methods more systematic within the context of applied mathematical methods, (b) define large classes of differential equations for which these methods lead to simple rules for asymptotic decay of solutions, (c) understand these classes of equations in terms of *universality classes* whereby different equations have similar behavior, (d) determine whether the methods can be implemented for higher order in ε, (e) verify the exponent results of RG methods through different types of calculations, (f) prove the RG results rigorously, (g) verify the exponents numerically.

For the goals above (particularly (a)-(c)), a first step was undertaken in [12] where the equation

$$u_t = \frac{1}{2}u_{xx} + \varepsilon F(x, u, u_x, u_{xx}) \tag{2.1}$$

in an infinite domain, where $F(x, u, u_x, u_{xx})$ is of the form $x^m u^n u_x^p u_{xx}^q$, ε is a small parameter. The parameters (m, n, p, q) are constrained by dimensionality so that the nonlinear term has the same physical dimensions as u_{xx}, i.e. they satisfy the following constraints:

$$n + p + q = 1 \text{ and } p + 2q - m = 2. \tag{2.2}$$

The large time decay (up to $O(\varepsilon)$) was found to be of the following form:

$$u(x,t) \sim t^{-\frac{1}{2}-\alpha} u^*(xt^{-1/2}, 1)$$

where $\alpha = \varepsilon A$ is a simple function of the powers of x, u, u_x, and u_{xx} in F. Note that standard dimensional analysis cannot be used to calculate the exponent α.

In many cases such exponents arise as a result of a limit of vanishing length (or other) scale that is a singular rather that a regular perturbation (see [3]). Above the dimensionless small number ε appears to provide the correction to the classical exponent. These results were also generalized to systems of parabolic equations [13].

Equations of the form (2.1) arise in a broad range of diffusion problems in detailed physics is taken into account (see [47]). The discussion about the limitations of the linear theory of diffusion and also derivation a number of key nonlinearities of the form of (2.1) from basic thermodynamics can be found in [50]. From a macroscopic perspective, a basic source of nonlinearities involves inhomogeneities in the diffusion coefficient in the flux or variable dependent potentials in Fick's laws (see p.5 and p.25 in [53]). Particular examples involve (1) temperature dependent heat conduction, (2) compressible fluid flow equations [36], (3) phase transitions involving alloys [11], (4) magnetic fields with permeability depending upon field strength [29], (5) heat diffusion and phase transition problems in which $(temperature)^{-1}$ dependence is considered [50] and many other applications [33].

In the first chapter, we focus on some of the key issues outlined above (particularly (d)-(f)) with two general goals for the case $q = 0$. First, we want to extend the RG analysis, particularly to examine terms of $O(\varepsilon^2)$ and higher. We also show that the RG process can be used to establish upper bounds for decay exponents. In particular, we determine the transform operators and perform an RG calculation that yields higher order terms beyond $O(\varepsilon)$. In fact this is an infinite series that can be summed to yield exact exponents in some cases. In other cases, if the $O(\varepsilon^2)$ term is negative, one can bound the exponent from above.

Second, we want to resolve rigorously and exactly the exponents for some nonlinearities described above. As part of this process we prove in some cases that the exponents obtained in Caginalp [12] above are, in fact, the first terms of a convergent expansion in ε. In addition to the renormalization and rigorous calculations, we produce an iterative expansion. In particular, we transform the equations so that the exponent can be calculated exactly by solving iteratively a set of ordinary differential equations. The solution involves closed form integrals that can be evaluated in terms of error functions.

A subset of the exponents obtained in [12] using RG methods are proved rigorously using shooting methods. The rigorous and exact calculations confirming the results further bolster the observation that dimensionality expressed in (2.2) above is a key feature that governs the decay of solutions. The dimensionality criteria establishes large classes of equations with similar decay properties.

This chapter is organized as follows. In section 2.2 we rewrite the equation (2.1) in terms of the fundamental solution, treating the nonlinear term as a source term. We apply asymptotic analysis in order to write the solution in terms of an integral that is in the appropriate form for the RG treatment. In section 2.3 we write the RG transformation for arbitrary order in ε. In section 2.4 we present alternative methods for calculating exponents that demonstrate agreement with the RG methods. The result are summarized in the conclusion (Section 2.5). A proof of a theorem that confirms earlier RG results is presented in Appendix A.

The methodology presented in this chapter is useful not only for exact calculation of large time profiles, but also in establishing equivalence classes in nonlinearities, since the exponents are determined by a simple formula. This also makes possible additional criteria for deciding on models that agree with experiment.

Decay of exponents to solutions of nonlinear equations have also been studied by related self-similarity methods in [6]-[8], [34], [54].

2.2 Renormalization group calculations

Let ε be a small, positive, dimensionless number and consider the diffusion equation with the nonlinearity of the form

$$C_p u_{t'} = K\{u_{xx} + 2\varepsilon F[x, u, u_x, u_{xx}]\} \tag{2.3}$$

where C_p and K are constants (with $D := K/C_p$) and the nonlinear term, F, is a linear sum of the terms of the form $x^m u^n u_x^p u_{xx}^q$ where the integers m, n, p, q satisfy (2.2). Defining $t := 2Dt'$ (which has units of $(length)^2 = area$) we simplify notation and use (2.3) of the form

$$u_t = \frac{1}{2}u_{xx} + \varepsilon F[x, u, u_x, u_{xx}] \tag{2.4}$$

for the remainder of the paper. We consider

$$u(x, 0; l) := g(x, l) = \frac{Q_0}{(2\pi l^2)^{1/2}} \exp\left(\frac{-x^2}{2l^2}\right) \tag{2.5}$$

as the initial condition in which l is a small parameter in order to study the decay from a sharply peaked Gaussian and $Q_0 := T_0 Q_1$ with T_0 having temperature units and Q_1 length units. Our procedure is to extract, for each order in ε, the leading order behavior in l^{-1}, so that only positive contributions to the decay are significant in the $O(\varepsilon^2)$ and higher. A key step in this process is to obtain a transformation that rescales variables. While RG methods usually involve an identity in this transformation, we utilize the basic ideas by using an identity up to a particular order in ε.

Asymptotics of the heat equation with small nonlinearity. In the following we obtain a basic solution for the equation (2.4) with the initial condition (2.5). Using the Green's Function

$$G(x, t) := \frac{1}{(2\pi t)^{1/2}} \exp\left(\frac{-x^2}{2t}\right) \tag{2.6}$$

and taking the nonlinearity F as a source term one can express the solution of (2.4) and (2.5) as

$$u(x, t) = \int_{-\infty}^{\infty} G(x - y, t)g(y)dy + \varepsilon \int_0^t \int_{-\infty}^{\infty} G(x - y, t - s)F[y, u(y, s), ...]dyds. \tag{2.7}$$

We solve (2.7) using an asymptotic expansion for small ε and write the formal sum as

$$u(x,t;\varepsilon,l) = u_0(x,t;l) + \varepsilon u_1(x,t;l) + \varepsilon^2 u_2(x,t;l) + \varepsilon^3 u_3(x,t;l) + \cdots \tag{2.8}$$

so that l is not yet treated as a small number in comparison with ε here. Following [12] we write

$$u_0(x,t;l) = \frac{Q_0}{[2\pi(t+l^2)]^{1/2}} \exp\left(\frac{-x^2}{2(t+l^2)}\right), \tag{2.9}$$

$$\frac{\partial u_0}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_0, \tag{2.10}$$

$$u_1(x,t;l) = \frac{Q_0}{(2\pi)^{1/2}} t^{-1/2} e^{-x^2/(2t)} \{(-1)^p (1\cdot 3\cdots|2p-3|)\} \log\left(\frac{t+l^2}{l^2}\right) \tag{2.11}$$

for $p \geq 1$ and $q := 0$, and

$$u_1(x,t;l) = \frac{Q_0}{(2\pi)^{1/2}} t^{-1/2} e^{-x^2/(2t)}$$
$$\times \left\{ \sum_{j=0}^{q} (-1)^{j+p} (1\cdot 3\cdots|2p+4q-2j-3|) \right\} \log\left(\frac{t+l^2}{l^2}\right) \tag{2.12}$$

for $q \neq 0$. The derivative of u_1 is given by $\frac{\partial u_1}{\partial x} = \left(\frac{-x}{t}\right) u_1$ and we rewrite it as

$$\frac{\partial u_1}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_1 + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right) u_1 \tag{2.13}$$

using the equality

$$\left(\frac{x}{t+l^2}\right) - \left(\frac{x}{t}\right) = \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right). \tag{2.14}$$

We proceed by using u_0 and u_1 to generate the next term of (2.8), namely u_2, and by using u_0, u_1 and u_2 to generate the u_3 term, and so on. The nonlinear term is taken as $x^m u^n u_x^p u_{xx}^q$ subject to (2.2) for the simplicity of the calculations as in [12]. In this work, we consider the case $q = 0$ only, so that the nonlinearity will be completely specified by $p \geq 1$, as $n = 1-p$ and $m = p-2$ (see (2.2)), and the nonlinear term is given by $F[x,u,u_x,u_{xx}] := x^{p-2}u^{1-p}u_x^p$. We should also mention here that p will be taken as $p \geq 1$ and $p \in Z^+$ throughout the paper. In addition, in the subsequent analysis the calculations are formally valid for

$$\varepsilon A \log(t/l^2) << 1.$$

In other words, for a given small ε the expansion is valid in some intermediate region of large t. Continuity arguments may be used to conjecture that the decay exponents obtained remain valid for arbitrarily large time. This has been confirmed in terms of the examples for which we have rigorous theorems or exact calculations. For the problem under consideration the result can be stated as follows:

Proposition 2.2.1. *Consider the equation (2.4) with the initial condition (2.5). One has to leading order in l within $O(\varepsilon^r)$ the solution*

$$u(x,t;\varepsilon,l) = \frac{Q_0}{\sqrt{2\pi}} t^{-1/2} e^{-x^2/(2t)} \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log(t/l^2)\right]^j \tag{2.15}$$

for $p \geq 1$ and $q := 0$, where $A := A(p) := (-1)^p(1 \cdot 3 \cdots |2p-3|)$, where only non-negative terms contribute for $O\left(\varepsilon^2\right)$ and beyond.

Verification. The derivation is done by induction. We first calculate the term u_2 and then use induction (on k for $k \geq 3$) in order the calculate the remaining terms in (2.8), i.e. u_3,...,u_k etc. In the calculation of each term, we first state the estimates, then calculate the term, and finally prove the lemmas.

2.2.1 The calculation of the u_2 term.

I. The estimates. We have the following:

Lemma 2.2.1. *We state the result (see [1], p. 302, 7.4.4) as follow. Let*

$$L := \int_{-\infty}^{\infty} y^{2(p-1)} e^{-y^2/(2t)} dy. \tag{2.16}$$

It follows that $L = \sqrt{2\pi}\{1 \cdot 3 \cdots |2p-3|\} t^{p-\frac{1}{2}}$.

Lemma 2.2.2. *Let*

$$L_{2,1} := \int_0^t (s+l^2)^{-p} s^{p-1} \log\left(\frac{s+l^2}{l^2}\right) ds. \tag{2.17}$$

We then have the following bounds

$$C_{2,1}^{(1)}\left(\frac{t}{l^2}\right)^{p+1} \leq L_{2,1} < C_{2,1}^{(2)}\left(\frac{t}{l^2}\right)^{p+1} \quad \textit{for} \quad t \leq l^2 \tag{2.18}$$

$$\frac{1}{2!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + C_{2,1}^{(3)} < L_{2,1}$$
$$< \frac{1}{2!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^2 + C_{2,1}^{(4)} \quad \textit{for} \quad t > l^2 \tag{2.19}$$

where $C_{2,1}^{(i)}$ *is a constant depending on* p *for* $i = 1, 2, 3, 4$.

Lemma 2.2.3. *Let*

$$L_{2,2} := \int_0^t pl^2(s+l^2)^{-p}s^{p-2}\log\left(\frac{s+l^2}{l^2}\right) ds. \tag{2.20}$$

We then have the following bounds

$$0 \leq L_{2,2} < C_{2,2}^{(1)}\left(\frac{t}{l^2}\right)^{p} \quad \textit{for} \quad t \leq l^2 \tag{2.21}$$

$$0 < L_{2,2} < C_{2,2}^{(2)} \quad \textit{for} \quad t > l^2 \tag{2.22}$$

where $C_{2,2}^{(i)}$ *is a constant depending on* p *for* $i = 1, 2$.

We prove Lemma 2.2.2 and Lemma 2.2.3 after the calculation of u_2 (see III. Proofs of lemmas).

II. The calculation of the u_2 term. Using u_0 and u_1 we will calculate the u_2 term in (2.8). In order to do this, we need to substitute $u := u_0 + \varepsilon u_1$ into (2.7) so that we first need to find $(u_0 + \varepsilon u_1)^{(1-p)}(u_0 + \varepsilon u_1)_x^p$. Using (2.10) and (2.13) one has

$$(u_0 + \varepsilon u_1)_x = \left(\frac{-x}{t+l^2}\right)(u_0 + \varepsilon u_1) + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right)(\varepsilon u_1). \tag{2.23}$$

Applying now the Binomial Formula to this we obtain

$$(u_0 + \varepsilon u_1)^{(1-p)}(u_0 + \varepsilon u_1)_x^p = \left(\frac{-x}{t+l^2}\right)^p (u_0 + \varepsilon u_1) + \lambda_1 \left(\frac{-x}{t+l^2}\right)^p \left(\frac{l^2}{t}\right)(\varepsilon u_1)$$
$$+ \sum_{n=2}^{p} \lambda_n \left(\frac{-x}{t+l^2}\right)^p \left(\frac{l^2}{t}\right)^n \frac{(\varepsilon u_1)^n}{(u_0 + \varepsilon u_1)^{n-1}}$$

$$\cong \left(\frac{-x}{t+l^2}\right)^p (u_0 + \varepsilon u_1) + \lambda_1 \left(\frac{-x}{t+l^2}\right)^p \left(\frac{l^2}{t}\right) (\varepsilon u_1)$$

$$+ \sum_{n=2}^{p} \lambda_n \left(\frac{-x}{t+l^2}\right)^n \left(\frac{l^2}{t}\right)^n \frac{\varepsilon^n {u_1}^n}{{u_0}^{n-1}} \tag{2.24}$$

where $\lambda_n := \binom{p}{n} = \frac{p!}{(p-n)!n!}$. Substituting (2.24) into (2.8) and retaining up to $O(\varepsilon^2)$ terms lead to the expression

$$u_2(x, t; l) := I + J \tag{2.25}$$

where

$$I := \int_0^t \int_{-\infty}^{\infty} G(x-y, t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} u_1(y, s) dy ds \tag{2.26}$$

$$J := \int_0^t \int_{-\infty}^{\infty} G(x-y, t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \lambda_1 l^2 s^{-1} u_1(y, s) dy ds. \tag{2.27}$$

i. Evaluation of I integral. Using (2.6) and (2.11) we write (2.26) as

$$\begin{aligned} I \quad := \quad & \int_0^t \int_{-\infty}^{\infty} \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \\ & \times \frac{Q_0}{(2\pi)^{1/2}} s^{-1/2} e^{-y^2/(2s)} \{(-1)^p (1 \cdot 3 \cdots |2p-3|)\} \log\left(\frac{s+l^2}{l^2}\right) dy ds \end{aligned}$$

$$\begin{aligned} \cong \quad & \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{2p} (1 \cdot 3 \cdots |2p-3|)\} \\ & \times \int_0^t ds (s+l^2)^{-p} \log\left(\frac{s+l^2}{l^2}\right) s^{-1/2} \int_{-\infty}^{\infty} y^{2(p-1)} e^{-y^2/(2s)} dy. \end{aligned} \tag{2.28}$$

The approximations involve replacing $t-s$ by t, and $x-y$ by x to obtain the $t^{-1/2} e^{-x^2/(2t)}$ term above. The justification (see [12], p. 9-12) is based on the Laplace's method for integrals, since the main contribution to the integral must arise from the regions near $y = 0$ and $s = 0$ for small l. Letting now

$$\gamma := \gamma(x, t) := \frac{Q_0}{\sqrt{2\pi}} t^{-1/2} e^{-x^2/(2t)} \tag{2.29}$$

$$A := A(p) := (-1)^p (1 \cdot 3 \cdots |2p-3|) \tag{2.30}$$

and applying Lemma 2.2.1 to (2.28) one has

$$I \cong \gamma A^2 \int_0^t (s+l^2)^{-p} s^{p-1} \log\left(\frac{s+l^2}{l^2}\right) ds =: \gamma A^2 L_{2,1}(s;l,p). \tag{2.31}$$

Now using Lemma 2.2.2 we have the following bounds for the first integral:

$$\gamma A^2 \left\{\frac{1}{2!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + C_{2,1}^{(3)}\right\} < I < \gamma A^2 \left\{\frac{1}{2!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^2 + C_{2,1}^{(4)}\right\}. \tag{2.32}$$

ii. Evaluation of J integral. Similarly, following the procedure in the previous calculation one writes (2.27) as

$$J \cong \gamma A^2 \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-2} \log\left(\frac{s+l^2}{l^2}\right) ds =: \gamma A^2 L_{2,2}(s;l,p). \tag{2.33}$$

Using now Lemma 2.2.3 one has the following bounds for the integral J

$$0 < J < \gamma A^2 C(p), \tag{2.34}$$

where $C(p)$ is a constant depending on p. Combining (2.25), (2.32) and (2.34) one has

$$\gamma A^2 \left\{\frac{1}{2!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + C_1\right\} < u_2(x,t;l) < \gamma A^2 \left\{\frac{1}{2!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^2 + C_2\right\} \tag{2.35}$$

where C_1 and C_2 are constants. Furthermore, combining (2.25), (2.31) and (2.34) one can express $u_2(x,t;l)$ as

$$u_2(x,t;l) \cong \gamma A^2 [L_{2,1}(s;l,p) + L_{2,2}(s;l,p)]. \tag{2.36}$$

so that using (2.14) one obtains the derivative of u_2 of the form:

$$\frac{\partial u_2}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_2 + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right) u_2. \tag{2.37}$$

III. Proofs of lemmas.

Proof of Lemma 2.2.2. We begin by labelling some basic inequalities:

$$0 < log(1+z) < z \quad \text{for} \quad z > 0 \tag{2.38}$$

$$\frac{z}{2c} \leq \log(1+z) \quad \text{for} \quad 0 \leq z < 1 \leq c \tag{2.39}$$

$$log(z) < log(1+z) \quad \text{for} \quad z > 0. \tag{2.40}$$

Letting $z := s/l^2$ we write (2.17) as

$$\begin{aligned} L_{2,1} &:= \int_0^t l^{-2p}(1+\frac{s}{l^2})^{-p} s^{p-1} \log\left(\frac{s+l^2}{l^2}\right) ds \\ &= \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \log(1+z) dz. \end{aligned} \tag{2.41}$$

Upper and lower bounds for $t \leq l^2$: Using the inequality $(1+z)^{-p} \leq 1$ and (2.38) one has

$$L_{2,1} < \int_0^{t/l^2} z^{p-1} z dz = \frac{1}{p+1}\left(\frac{t}{l^2}\right)^{p+1}. \tag{2.42}$$

Note that $(1+z)^{-p} \geq 2^{-p}$ for $z \leq 1$ (since $t \leq l^2$). Using then this inequality and (2.39) one has

$$L_{2,1} \geq \int_0^{t/l^2} 2^{-p} z^{p-1} \frac{z}{2} dz = \frac{1}{2^{p+1}(p+1)}\left(\frac{t}{l^2}\right)^{p+1}. \tag{2.43}$$

Upper and lower bounds for $t > l^2$: In this case, we split the integral (2.41) into two parts as follows:

$$\begin{aligned} L_{2,1} &= \int_0^1 (1+z)^{-p} z^{p-1} \log(1+z) dz + \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \log(1+z) dz \\ &=: L_{2,1}^{(1)} + L_{2,1}^{(2)} \end{aligned} \tag{2.44}$$

so that using (2.42) and (2.43) we have

$$0 < \frac{1}{2^{p+1}(p+1)} \leq L_{2,1}^{(1)} < \frac{1}{p+1}. \tag{2.45}$$

Next we obtain an upper bound and a lower bound for $\mathrm{L}_{2,1}^{(2)}$. Using the inequality $z^{p-1} \leq (1+z)^{p-1}$

for $z \geq 0$ we have

$$
\begin{aligned}
L_{2,1}^{(2)} &\leq \int_1^{t/l^2} (1+z)^{-1} \log(1+z) dz \\
&= \frac{1}{2!} \left[\log \left(\frac{t+l^2}{l^2} \right) \right]^2 + \text{constant}.
\end{aligned} \tag{2.46}
$$

Using the infinite series

$$
\frac{1}{1+z} = \frac{1}{z} \frac{1}{1+\frac{1}{z}} = \frac{1}{z}(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots) \quad \text{for} \quad z > 1 \tag{2.47}
$$

one obtains

$$
(1+z)^{-p} = z^{-p}(1 + \frac{\nu_1(p)}{z} + \frac{\nu_2(p)}{z^2} + \cdots). \tag{2.48}
$$

Using now (2.48) and (2.40) one has

$$
\begin{aligned}
L_{2,1}^{(2)} &> \int_1^{t/l^2} z^{-1}(1 + \frac{\nu_1(p)}{z} + \frac{\nu_2(p)}{z^2} + \cdots) \log(z) dz \\
&= \int_1^{t/l^2} z^{-1} \log(z) dz + \sum_{j=1}^{\infty} \int_1^{t/l^2} \nu_j(p) z^{-1-j} \log(z) dz \\
&> \frac{1}{2!} \left[\log \left(\frac{t}{l^2} \right) \right]^2 + C(p),
\end{aligned} \tag{2.49}
$$

where $C(p)$ is a constant. □

Proof of Lemma 2.2.3. Following the proof of Lemma 2.2.2 one writes (2.20) as

$$
L_{2,2} = \int_0^{t/l^2} p(1+z)^{-p} z^{p-2} \log(1+z) dz \tag{2.50}
$$

so that for $t \leq l^2$ one has

$$
0 \leq L_{2,2} < \int_0^{t/l^2} z^{p-2} z dz = \left(\frac{t}{l^2} \right)^p. \tag{2.51}
$$

For the large t/l^2, one similarly splits the integral into two parts as follows:

$$
\begin{aligned}
L_{2,2} &= \int_0^1 (1+z)^{-p} z^{p-2} \log(1+z) dz + \int_1^{t/l^2} (1+z)^{-p} z^{p-2} \log(1+z) dz \\
&=: L_{2,2}^{(1)} + L_{2,2}^{(2)}
\end{aligned}
$$

so that utilizing (2.51) one has

$$0 \leq L_{2,2}^{(1)} < 1. \tag{2.52}$$

For the second part of the integral, namely $L_{2,2}^{(2)}$, using the inequality $(1+z)^{-p} < z^{-p}$ for $z > 1$ (in order to obtain the following upper bound) one has

$$\begin{aligned} 0 < L_{2,2}^{(2)} &= \int_1^{t/l^2} z^{-2} \log(1+z) dz \\ &< \int_1^{\infty} z^{-2} \log(1+z) dz < \text{constant}. \end{aligned} \tag{2.53}$$

□

We use induction (on k for $k \geq 3$) below to calculate the rest of terms in (2.8).

2.2.2 The calculation of the u_3 term

I. The estimates. We have the following:

Lemma 2.2.4. *Let*

$$L_{3,1} := \int_0^t (s + l^2)^{-p} s^{p-1} \{L_{2,1} + L_{2,2}\} ds \tag{2.54}$$

where $L_{2,1}$ *and* $L_{2,2}$ *are as defined in Lemma 2.2.2 and in Lemma 2.2.3, respectively. We then have the following bounds*

$$C_{3,1}^{(1)} \left(\frac{t}{l^2}\right)^{2p+1} \leq L_{3,1} < C_{3,1}^{(2)} \left(\frac{t}{l^2}\right)^{2p+1} + C_{3,1}^{(3)} \left(\frac{t}{l^2}\right)^{2p} \quad \textit{for} \quad t \leq l^2 \tag{2.55}$$

$$\begin{aligned} &\frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t}{l^2}\right)\right) < L_{2,1} \\ &< \frac{1}{3!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right) \quad \textit{for} \quad t > l^2 \end{aligned} \tag{2.56}$$

where $C_{3,1}^{(j)}$ *is a constant depending on* p *for* $j = 1, 2, 3$.

Lemma 2.2.5. *Let*

$$L_{3,2} := \int_0^t \lambda_1 l^2 (s + l^2)^{-p} s^{p-1} \{L_{2,1} + L_{2,2}\} ds \tag{2.57}$$

where $L_{2,1}$ and $L_{2,2}$ are as in Lemma 2.2.4. We then have the following bounds

$$0 \leq L_{3,2} < C_{3,2}^{(1)} \left(\frac{t}{l^2}\right)^{2p} + C_{3,2}^{(2)} \left(\frac{t}{l^2}\right)^{2p-1} \quad \textit{for} \quad t \leq l^2 \tag{2.58}$$

$$0 \leq L_{3,2} < C_{3,2}^{(3)}(p) \quad \textit{for} \quad t > l^2 \tag{2.59}$$

where $C_{3,2}^{(i)}$ is a constant depending on p for $i = 1, 2, 3$.

Lemma 2.2.6. *Let*

$$L_k := \int_0^t \lambda_n l^{2n} (s + l^2)^{-p+(\frac{n-1}{2})} s^{p-(\frac{3n+1}{2})} \left[\log\left(\frac{s + l^2}{l^2}\right)\right]^n ds \tag{2.60}$$

for $p \geq 2$, $n \geq 2$, $p \geq n$ and p, $n \in Z^+$. We then have the following bounds

$$0 \leq L_k < C_k^{(1)}(p, n) \left(\frac{t}{l^2}\right)^{p-\frac{n}{2}+\frac{1}{2}} \quad \textit{for} \quad t \leq l^2 \tag{2.61}$$

$$0 \leq L_k < C_k^{(2)}(p, n) \quad \textit{for} \quad t > l^2 \tag{2.62}$$

where $C_k^{(1)}$ and $C_k^{(2)}$ are constants depending on p and n.

The lemmas above are proved in Appendix B.

II. The calculation of the u_3 term. Following a similar procedure in the calculation of the u_2 term we calculate u_3 so that we first need to find $(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)^{(1-p)}(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)_x^p$. Using (2.23) with together (2.37) we write $(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)_x$ as

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)_x = \left(\frac{-x}{t + l^2}\right)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2) + \left(\frac{-x}{t + l^2}\right)\left(\frac{l^2}{t}\right)(\varepsilon u_1 + \varepsilon^2 u_2). \tag{2.63}$$

Similarly, applying the Binomial Formula we obtain

$$\begin{aligned}(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)^{(1-p)}(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)_x^p \;\cong\; & \left(\frac{-x}{t + l^2}\right)^p (u_0 + \varepsilon u_1 + \varepsilon^2 u_2) \\ & + \lambda_1 \left(\frac{-x}{t + l^2}\right)^p \left(\frac{l^2}{t}\right)(\varepsilon u_1 + \varepsilon^2 u_2) \\ & + \sum_{n=2}^{p} \lambda_n \left(\frac{-x}{t + l^2}\right)^p \left(\frac{l^2}{t}\right)^n \frac{\varepsilon^n {u_1}^n}{{u_0}^{n-1}}.\end{aligned} \tag{2.64}$$

Substituting (2.64) into (2.8) and retaining up to $O(\varepsilon^3)$ terms yield the expression

$$u_3(x,t;l) := M + N + Q \tag{2.65}$$

where

$$M := \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} u_2(y,s) dyds \tag{2.66}$$

$$N := \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \lambda_1 l^2 s^{-1} u_2(y,s) dyds \tag{2.67}$$

$$Q := \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \lambda_2 l^4 s^{-2} \frac{(u_1(y,s))^2}{u_0(y,s)} dyds. \tag{2.68}$$

i. Evaluation of M integral. Using (2.6) and (2.36) we write (2.66) as

$$\begin{aligned} M &:= \int_0^t \int_{-\infty}^{\infty} \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right) y^{2(p-1)} (-1)^p (s+l^2)^{-p} \\ &\quad \times \frac{Q_0}{(2\pi)^{1/2}} s^{-1/2} e^{-y^2/(2s)} \{(-1)^p (1\cdot 3 \cdots |2p-3|)\}^2 [L_{2,1}(s;l,p) + L_{2,2}(s;l,p)] dyds \end{aligned}$$

$$\begin{aligned} &\cong \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{3p} (1\cdot 3 \cdots |2p-3|)^2\} \\ &\quad \times \int_0^t ds (s+l^2)^{-p} [L_{2,1}(s;l,p) + L_{2,2}(s;l,p)] s^{-1/2} \int_{-\infty}^{\infty} y^{2(p-1)} e^{-y^2/(2s)} dy. \end{aligned} \tag{2.69}$$

As done in the previous calculation, we approximate $t^{-1/2}e^{-x^2/(2t)}$ by replacing $t-s$ by t, and $x-y$ by x (see (2.28)). Applying now Lemma 2.2.1 to (2.69) and using (2.29) and (2.30) we have

$$M \cong \gamma A^3 \int_0^t (s+l^2)^{-p} s^{p-1} [L_{2,1}(s;l,p) + L_{2,2}(s;l,p)] ds =: \gamma A^3 L_{3,1}(s;l,p). \tag{2.70}$$

Using now Lemma 2.2.4 we have the following bounds for the first integral:

$$\gamma A^3 \left\{ \frac{1}{3!} \left[\log\left(\frac{t}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t}{l^2}\right)\right)\right\} < M$$

$$< \gamma A^3 \left\{ \frac{1}{3!} \left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right)\right\}. \tag{2.71}$$

ii. Evaluation of N integral. Following a similar procedure in (i) one writes (2.67) as

$$N \cong \gamma A^3 \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-2}[L_{2,1}(s;l,p) + L_{2,2}(s;l,p)]ds =: \gamma A^3 L_{3,2}(s;l,p) \tag{2.72}$$

so that applying Lemma 2.2.5 yields the following bounds for the second integral

$$0 \leq N < \gamma A^3 C(p), \tag{2.73}$$

where $C(p)$ is a constant depending on p.

iii. Evaluation of Q integral. Using (2.6), (2.9) and (2.11) we first write (2.68) as

$$\begin{aligned} Q \;&:=\; \int_0^t \int_{-\infty}^{\infty} \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right) y^{2(p-1)}(-1)^p (s+l^2)^{-p} \lambda_2 l^4 s^{-2} \\ &\times \frac{Q_0}{(2\pi)^{1/2}} \{(-1)^p (1 \cdot 3 \cdots |2p-3|)\}^2 s^{-1}(s+l^2)^{1/2} \\ &\times \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 e^{-2y^2/(2s)} e^{y^2/(2(s+l^2))} dyds \end{aligned} \tag{2.74}$$

and applying then Laplace's method for integrals (see (2.28)) we obtain

$$\begin{aligned} Q \;&\cong\; \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{3p}(1 \cdot 3 \cdots |2p-3|)^2\} \\ &\times \int_0^t ds \lambda_2 l^4 (s+l^2)^{-p+\frac{1}{2}} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 s^{-3} \\ &\times \int_{-\infty}^{\infty} y^{2(p-1)} e^{-2y^2/(2s)} e^{y^2/(2(s+l^2))} dy. \end{aligned} \tag{2.75}$$

Letting now, for $n \geq 2$,

$$\begin{aligned} \widehat{L}_n \;&:=\; \int_0^t ds \lambda_n l^{2n} (s+l^2)^{-p+(\frac{n-1}{2})} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 s^{-3n/2} \\ &\times \int_{-\infty}^{\infty} y^{2(p-1)} \exp(-ny^2/(2s)) \exp((n-1)y^2/(2(s+l^2)))dy \end{aligned} \tag{2.76}$$

and using (2.29) and (2.30) we rewrite (2.75) as

$$Q \cong: \gamma A^2 (-1)^p (2\pi)^{-1/2} \widehat{L}_3(s,l;p). \tag{2.77}$$

Now consider, for $n \geq 2$,

$$L_k^{(n)} := \int_{-\infty}^{\infty} y^{2(p-1)} \exp\left(\frac{-ny^2}{2s}\right) \exp\left(\frac{(n-1)y^2}{2(s+l^2)}\right) dy$$
$$= \int_{-\infty}^{\infty} y^{2(p-1)} \exp\left(\frac{-y^2}{2s}\right) \exp\left(\frac{-(n-1)y^2}{2s}\right) \exp\left(\frac{(n-1)y^2}{2(s+l^2)}\right) dy \qquad (2.78)$$

so that

$$L_k^{(n)} \leq \int_{-\infty}^{\infty} y^{2(p-1)} \exp\left(\frac{-y^2}{2s}\right) dy$$

since $\exp\left(\frac{-(n-1)y^2}{2s}\right) \exp\left(\frac{(n-1)y^2}{2(s+l^2)}\right) \leq 1$. Thus, applying Lemma 2.2.1 one has

$$L_k^{(n)} \leq (2\pi)^{1/2}\{1 \cdot 3 \cdots |2p-3|\} s^{p-\frac{1}{2}}. \qquad (2.79)$$

Using (2.78) and (2.79) (with n=2), and (2.29) and (2.30) we rewrite (2.75) as

$$Q \leq \gamma A^3 \int_0^t \lambda_2 l^4 (s+l^2)^{-p+\frac{1}{2}} s^{p-\frac{7}{2}} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^2 ds =: \gamma A^3 L_3(s;l,p) \qquad (2.80)$$

so that applying Lemma 2.2.6 ((2.62) with $n = 2$) we obtain the following bounds for the third integral

$$0 \leq Q < \gamma A^3 C(p,n), \qquad (2.81)$$

where C is a constant depending on p and n. Once again, combining (2.65), (2.71), (2.73), and (2.81) yields the following bounds for u_3:

$$\gamma A^3 \{\left\{\frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^2 + O\left(\log\left(\frac{t}{l^2}\right)\right)\right\} < u_3(x,t;l)$$
$$< \gamma A^3 \left\{\frac{1}{3!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right)\right\}. \qquad (2.82)$$

As done before (see (2.36)), one can also express u_3 as

$$u_3(x,t;l) := \gamma A^3 L_{3,1}(s;l,p) + \gamma A^3 L_{3,2}(s;l,p) + \gamma A^2 (-1)^p (2\pi)^{-1/2} \widehat{L}_3(s,l;p) \qquad (2.83)$$

that can be used to obtain the derivative of u_3 in order to calculate the next term in (2.8). Thus, the derivative of u_3 has of the form

$$\frac{\partial u_3}{\partial x} = \left(\frac{-x}{t+l^2}\right) u_3 + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right) u_3. \tag{2.84}$$

Suppose that the claim is true for $k-1$. We next prove the claim for k as follows. To calculate the general term u_k in (2.8) we will follow a similar procedure in the calculation of the terms above.

2.2.3 The calculation of the u_k term

I. The estimates. In the following lemmas $L_{k-1,v}$ is as in the induction hypothesis for $v = 1, 2, 3, 4$, respectively, and $\widehat{L}_{k-1}$ is given by (2.76). In addition, $L_{k-1,3}$ and $L_{k-1,4}$ are taken as $L_{k-1,3} = L_{k-1,4} = 0$ when $k \leq 3$ for the notational convenience.

Lemma 2.2.7. *Let*

$$L_{k,1} := \int_0^t (s+l^2)^{-p} s^{p-1} \{L_{k-1,1} + L_{k-1,2}\} ds. \tag{2.85}$$

We then have the following bounds

$$C_{k,1}^{(1)} \left(\frac{t}{l^2}\right)^{(k-1)p+1} \leq L_{k,1} < C_{k,1}^{(2)} \left(\frac{t}{l^2}\right)^{(k-1)p+1} + O\left(\frac{t}{l^2}\right) \quad \textit{for} \quad t \leq l^2 \tag{2.86}$$

$$\begin{aligned} &\frac{1}{k!}\left[\log\left(\frac{t}{l^2}\right)\right]^k + O\left(\log\left(\frac{t}{l^2}\right)\right) < L_{k,1} \\ &< \frac{1}{k!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^k + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right) \quad \textit{for} \quad t > l^2 \end{aligned} \tag{2.87}$$

where $C_{k,1}^{(1)}$ and $C_{k,1}^{(2)}$ are constants depending on p.

Lemma 2.2.8. *Let*

$$L_{k,2} := \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-2} \{L_{k-1,1} + L_{k-1,2}\} ds. \tag{2.88}$$

We then have the following bounds

$$0 \leq L_{k,2} < C_{k,2}^{(1)} \left(\frac{t}{l^2}\right)^{(k-1)p} + O\left(\frac{t}{l^2}\right) \quad \textit{for} \quad t \leq l^2 \tag{2.89}$$

$$0 \le L_{k,2} < C^{(2)}_{k,2}(p) \quad for \quad t > l^2 \tag{2.90}$$

where $C^{(1)}_{k,2}$ *and* $C^{(2)}_{k,2}$ *are constants depending on* p.

Lemma 2.2.9. *Let*

$$L_{k,3} := \int_0^t (s+l^2)^{-p} s^{p-1} \{L_{k-1,3} + L_{k-1,4} + \widehat{L}_{k-1}\} ds. \tag{2.91}$$

We then have the following bounds

$$0 \le L_{k,3} < C^{(1)}_{k,3} \left(\frac{t}{l^2}\right)^{(k-2)p-\frac{1}{2}} + O\left(\frac{t}{l^2}\right) \quad for \quad t \le l^2 \tag{2.92}$$

$$0 < L_{k,3} < C^{(2)}_{k,3} \left[\log\left(\frac{t+l^2}{l^2}\right)\right]^{k-3} + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right) \quad for \quad t > l^2 \tag{2.93}$$

where $C^{(1)}_{k,3}$ *and* $C^{(2)}_{k,3}$ *are constants depending on* p.

Lemma 2.2.10. *Let*

$$L_{k,4} := \int_0^t \lambda_1 l^2 (s+l^2)^{-p} s^{p-2} \{L_{k-1,3} + L_{k-1,4} + \widehat{L}_{k-1}\} ds. \tag{2.94}$$

We then have the following bounds

$$0 \le L_{k,4} < C^{(1)}_{k,4} \left(\frac{t}{l^2}\right)^{(k-2)p-\frac{3}{2}} + O\left(\frac{t}{l^2}\right) \quad for \quad t \le l^2 \tag{2.95}$$

$$0 < L_{k,4} < C^{(2)}_{k,4} \quad for \quad t > l^2 \tag{2.96}$$

where $C^{(1)}_{k,4}$ *and* $C^{(2)}_{k,4}$ *are constants depending on* p.

The proofs of lemmas are given in Appendix C.

II. The calculation of the u_k term. Using $u_0, \ldots, u_{k-1}$ we will calculate u_k so that we should first begin calculating the term $(u_0 + \varepsilon u_1 + \cdots + \varepsilon^{k-1} u_{k-1})^{(1-p)} (u_0 + \varepsilon u_1 + \cdots + \varepsilon^{k-1} u_{k-1})^p_x$. By the induction hypothesis $(u_0+\varepsilon u_1+\cdots+\varepsilon^{k-1}u_{k-1})_x$ is expressed as

$$
\begin{aligned}
(u_0+\varepsilon u_1+\cdots+\varepsilon^{k-1}u_{k-1})_x &= \left(\frac{-x}{t+l^2}\right)(u_0+\varepsilon u_1+\cdots+\varepsilon^{k-1}u_{k-1}) \\
&\quad + \left(\frac{-x}{t+l^2}\right)\left(\frac{l^2}{t}\right)(\varepsilon u_1+\varepsilon^2 u_2\cdots+\varepsilon^{k-1}u_{k-1}). \qquad (2.97)
\end{aligned}
$$

Let $(\cdots)$ and $[\cdots]$ denote $(u_0+\varepsilon u_1+\cdots+\varepsilon^{k-1}u_{k-1})$ and $[\varepsilon u_1+\cdots+\varepsilon^{k-1}u_{k-1}]$, respectively. Applying the Binomial Formula to (2.97) one obtains

$$
\begin{aligned}
(\cdots)^{(1-p)}[\cdots]_x^p &\cong \left(\frac{-x}{t+l^2}\right)^p(\cdots)+\lambda_1\left(\frac{-x}{t+l^2}\right)^p\left(\frac{l^2}{t}\right)[\cdots] \\
&\quad +\sum_{n=2}^{p}\lambda_n\left(\frac{-x}{t+l^2}\right)^p\left(\frac{l^2}{t}\right)^n\frac{\varepsilon^n {u_1}^n}{{u_0}^{n-1}}. \qquad (2.98)
\end{aligned}
$$

Substituting (2.98) into (2.8) and retaining up to $O(\varepsilon^k)$ terms leads to the expression

$$
u_k(x,t;l) := V+W+Z \qquad (2.99)
$$

where

$$
V := \int_0^t ds\int_{-\infty}^{\infty} dy G(x-y,t-s)y^{2(p-1)}(-1)^p(s+l^2)^{-p}u_{k-1}(y,s) \qquad (2.100)
$$

$$
W := \int_0^t ds\int_{-\infty}^{\infty} dy G(x-y,t-s)y^{2(p-1)}(-1)^p(s+l^2)^{-p}\lambda_1 l^2 s^{-1}u_{k-1}(y,s) \qquad (2.101)
$$

$$
Z := \int_0^t ds\int_{-\infty}^{\infty} dy G(x-y,t-s)y^{2(p-1)}(-1)^p(s+l^2)^{-p}\lambda_{k-1}l^{2(k-1)}s^{-(k-1)}\frac{u_1(y,s)^{k-1}}{u_0(y,s)^{k-2}}. \qquad (2.102)
$$

i. Evaluation of V integral. Using (2.6) and the induction hypothesis we write (2.100) as

$$
\begin{aligned}
V \;:=\; & \int_0^t ds\int_{-\infty}^{\infty} dy\frac{(t-s)^{-1/2}}{(2\pi)^{1/2}}\exp\left(\frac{-(x-y)^2}{2(t-s)}\right)y^{2(p-1)}(-1)^p(s+l^2)^{-p} \\
& \times\left\{
\begin{array}{c}
\left[\frac{Q_0}{(2\pi)^{1/2}}s^{-1/2}e^{-y^2/(2s)}\{(-1)^p(1\cdot 3\cdots|2p-3|)\}^{k-1}[L_{k-1,1}(s;l,p)+L_{k-1,2}(s;l,p)]\right] \\
+\left[\begin{array}{c}\frac{Q_0}{2\pi}s^{-1/2}e^{-y^2/(2s)}(-1)^p\{(-1)^p(1\cdot 3\cdots|2p-3|)\}^{k-2} \\ \times[L_{k-1,3}(s;l,p)+L_{k-1,4}(s;l,p)+\widehat{L}_{k-1}(s;l,p)]\end{array}\right]
\end{array}
\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\cong \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{kp}(1\cdot 3 \cdots |2p-3|)^{k-1}\} \\
&\times \left[\int_0^t ds (s+l^2)^{-p} [L_{k-1,1}(s;l,p) + L_{k-1,2}(s;l,p)] s^{-1/2} \int_{-\infty}^{\infty} dy y^{2(p-1)} e^{-y^2/(2s)} \right] \qquad (2.103) \\
&+ \frac{Q_0}{(2\pi)^{3/2}} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{kp}(1\cdot 3 \cdots |2p-3|)^{k-2}\} \\
&\times \left[\int_0^t ds (s+l^2)^{-p} [L_{k-1,3}(s;l,p) + L_{k-1,4}(s;l,p) + \widehat{L}_{k-1}(s;l,p)] s^{-1/2} \int_{-\infty}^{\infty} dy y^{2(p-1)} e^{-y^2/(2s)} \right].
\end{aligned}
$$

The approximations involve replacing $t-s$ by t, and $x-y$ by x to obtain the $t^{-1/2}e^{-x^2/(2t)}$ term using the Laplace's method for integrals (as in (2.28)) so that applying Lemma 2.2.1 to (2.103) and using (2.29) and (2.30) we have

$$
\begin{aligned}
V \quad &\cong \quad \gamma A^k \int_0^t ds (s+l^2)^{-p} s^{p-1} [L_{k-1,1}(s;l,p) + L_{k-1,2}(s;l,p)] \\
&\quad + \gamma A^{k-1}(-1)^p (2\pi)^{-1/2} \int_0^t ds (s+l^2)^{-p} s^{p-1} [L_{k-1,3}(s;l,p) + L_{k-1,4}(s;l,p) + \widehat{L}_{k-1}(s;l,p)] \\
&=: \quad \gamma A^k L_{k,1}(s;l,p) + \gamma A^{k-1}(-1)^p (2\pi)^{-1/2} L_{k,3}(s;l,p). \qquad (2.104)
\end{aligned}
$$

Now using Lemma 2.2.7 and Lemma 2.2.8 we have the following bounds for the integral V:

$$
\begin{aligned}
&\gamma A^k \left\{ \frac{1}{k!} \left[\log\left(\frac{t}{l^2}\right) \right]^k + O\left(\log\left(\frac{t}{l^2}\right) \right) \right\} < V \\
&< \gamma A^k \left\{ \frac{1}{k!} \left[\log\left(\frac{t+l^2}{l^2}\right) \right]^k + O\left(\log\left(\frac{t+l^2}{l^2}\right) \right) \right\}. \qquad (2.105)
\end{aligned}
$$

ii. Evaluation of W integral. Following the procedure in the previous calculation one writes (2.101) as

$$
\begin{aligned}
W \quad &\cong \quad \gamma A^k \int_0^t ds (s+l^2)^{-p} s^{p-2} [L_{k-1,1}(s;l,p) + L_{k-1,2}(s;l,p)] \\
&\quad + \gamma A^{k-1}(-1)^p (2\pi)^{-1/2} \int_0^t ds (s+l^2)^{-p} s^{p-2} [L_{k-1,3}(s;l,p) + L_{k-1,4}(s;l,p) + \widehat{L}_{k-1}(s;l,p)] \\
&=: \quad \gamma A^k L_{k,2}(s;l,p) + \gamma A^{k-1}(-1)^p (2\pi)^{-1/2} L_{k,4}(s;l,p). \qquad (2.106)
\end{aligned}
$$

Applying Lemma 2.2.9 and Lemma 2.2.10 one has the following bounds for the second integral

$$
0 \le W < \gamma A^k C(p) \qquad (2.107)
$$

where $C(p)$ is a constant depending on p.

iii. Evaluation of Z integral. Using (2.6), (2.9) and (2.11) we write (2.102) as

$$
\begin{aligned}
Z \;:=\; & \int_0^t ds \int_{-\infty}^{\infty} dy \frac{(t-s)^{-1/2}}{(2\pi)^{1/2}} \exp\left(\frac{-(x-y)^2}{2(t-s)}\right) y^{2(p-1)}(-1)^p (s+l^2)^{-p}\lambda_{k-1} l^{2(k-1)} s^{-(k-1)} \\
& \times \frac{Q_0}{(2\pi)^{1/2}} \{(-1)^p (1\cdot 3\cdots|2p-3|)\}^{k-1} s^{-(k-1)/2}(s+l^2)^{(k-2)/2}\left[\log\left(\frac{s+l^2}{l^2}\right)\right]^{k-1} \\
& \times \exp\left(\frac{-(k-1)y^2}{2s}\right)\exp\left(\frac{(k-2)y^2}{2(s+l^2)}\right). \qquad (2.108)
\end{aligned}
$$

Applying Laplace's methods for integrals to this and using (2.78) one can show

$$
\begin{aligned}
Z \;\cong\; & \frac{Q_0}{2\pi} t^{-1/2} e^{-x^2/(2t)} \{(-1)^{kp}(1\cdot 3\cdots|2p-3|)^{k-1}\} \\
& \times \int_0^t ds \lambda_{k-1} l^{2(k-1)} (s+l^2)^{-p+\frac{(k-2)}{2}} \left[\log\left(\frac{s+l^2}{l^2}\right)\right]^{k-1} s^{\frac{-3(k-1)}{2}} L_k^{(k-1)} \qquad (2.109)
\end{aligned}
$$

so that using (2.76) Z can be expressed as

$$
Z \cong: \gamma A^{k-1}(-1)^p (2\pi)^{-1/2} \widehat{L}_k(s,l;p). \qquad (2.110)
$$

Now using (2.79) one has

$$
\begin{aligned}
Z &\le \gamma A^k \int_0^t ds \lambda_{k-1} l^{2(k-1)} (s+l^2)^{-p+\frac{(k-2)}{2}} s^{p-(\frac{3k-2}{2})} \left[\log\left(\frac{t+l^2}{l^2}\right)\right]^{k-1} \\
&=: \gamma A^k L_k(s;l,p). \qquad (2.111)
\end{aligned}
$$

We then have the following bounds for the third integral

$$
0 \le Z < \gamma A^k C(p,k) \qquad (2.112)
$$

applying Lemma 2.2.6 (for $n = k-1$), where C is a constant depending on p and n. Thus, combining (2.99), (2.105), (2.107) and (2.112) one has

$$
\gamma A^k \left\{ \frac{1}{k!}\left[\log\left(\frac{t}{l^2}\right)\right]^k + O\left(\log\left(\frac{t}{l^2}\right)\right)\right\} < u_k(x,t;l)
$$

$$< \gamma A^k \left\{ \frac{1}{k!} \left[\log \left(\frac{t+l^2}{l^2} \right) \right]^k + O \left(\log \left(\frac{t+l^2}{l^2} \right) \right) \right\}. \tag{2.113}$$

Note that using (2.99), (2.104), (2.106) and (2.110) the term u_k can be expressed as

$$\begin{aligned} u_k(x,t;l) \;&:=\; \gamma A^k [L_{k,1}(s;l,p) + L_{k,2}(s;l,p)] \\ &\quad + \gamma A^{k-1} (-1)^p (2\pi)^{-1/2} [L_{k,3}(s;l,p) + L_{k,4}(s;l,p) + \widehat{L}_k(s;l,p)] \end{aligned} \tag{2.114}$$

so that using (2.14) one obtains the derivative of u_k as

$$\frac{\partial u_k}{\partial x} = \left(\frac{-x}{t+l^2} \right) u_k + \left(\frac{-x}{t+l^2} \right) \left(\frac{l^2}{t} \right) u_k. \tag{2.115}$$

Combining now (2.113) with (2.9), (2.11), (2.35), (2.42) in (2.8) one has to leading order in ε and to leading order in l within $O\left(\varepsilon^k\right)$ the solution (2.15).

2.3 The renormalization group transformations

If one has an asymptotic relation such as (2.15), one can then calculate the anomalous exponent explicitly and obtain the similarity solution for large time and space. The arguments below are within the context of formal applied analysis without reference to numerical procedures or physical analogies. We follow the methodology in [12] and [38] that was motivated by [23]. For the problem under consideration, we state the result as follows using true dimensions:

Proposition 2.3.1. *Suppose u can be expressed as*

$$u(x,t';\varepsilon,l) = \frac{T_0}{2\pi^{1/2}} \left(\frac{t'}{Q_1^2/D} \right)^{-1/2} e^{-x^2/(4Dt')} \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log(2Dt'/l^2) \right]^j \tag{2.116}$$

where A is independent of x, t', ε and l and only positive terms contribute to singularity. Then, to leading order in ε^r, u can be expressed as

$$u(x,t';\varepsilon,l) = \left(\frac{t'}{Q_1^2/D} \right)^{-\frac{1}{2}+\varepsilon A} u_r^* \left(\frac{x}{\left(Dt'/Q_1^2\right)^{1/2}}, \frac{Q_1^2}{D} \right) \tag{2.117}$$

so that the anomalous exponent is given by only εA. The fixed point function u_r^* has the following *form*

$$u_r^*(\xi,\tau) = \frac{T_0}{2\pi^{1/2}} \exp\left(-\frac{\xi^2}{4D\tau}\right) \sum_{j=0}^{r} \frac{1}{j!} \left[\varepsilon A \log\left(\frac{2D}{l^2}\tau\right)\right]^j . \tag{2.118}$$

VERIFICATION. We first verify the claim for $r = 2$, and then generalize it for arbitrary $r \in Z^+$ below by dividing the derivation into five stages. Particular, the verification was done for $r = 1$ in [12].

2.3.1 The verification for $r = 2$

Stage 1. One needs to obtain an identity (up to $O(\varepsilon^2)$) of the form

$$u(b^\phi x, bt') = Z_2(b)u(x,t') \tag{2.119}$$

which is valid for a particular choice of Z_2 and ϕ and all $b > 1$. Notice that the exponential term in (2.116) forces $\phi = \frac{1}{2}$ for each r. Rewriting (2.116) up to $O(\varepsilon^2)$ one has

$$\begin{aligned} u(b^{1/2}x, bt') &= \frac{T_0}{2\pi^{1/2}} \left(\frac{t'}{Q_1^2/D}\right)^{-1/2} e^{-\frac{x^2}{4Dt'}} \left\{1 + \varepsilon A \log(2Dt'/l^2) + \frac{1}{2!}\left[\varepsilon A \log(2Dt'/l^2)\right]^2\right\} \\ &\quad \times b^{-1/2}\left\{1 + \varepsilon A \log(b) + \frac{1}{2!}\left[\varepsilon A \log(b)\right]^2\right\} \end{aligned} \tag{2.120}$$

so that (2.117) is satisfied with $\phi = \frac{1}{2}$ and

$$Z_2(b) := b^{-1/2}\left\{1 + \varepsilon A \log(b) + \frac{1}{2!}\left[\varepsilon A \log(b)\right]^2\right\}. \tag{2.121}$$

One then defines the operator (see [17] and [12])

$$R_{b,\phi}u(x,t') := \frac{1}{Z_2(b)} u(b^{1/2}x, bt'). \tag{2.122}$$

Stage 2. By iteration one obtains (suppressing ε and l and ignoring $O(\varepsilon^3)$ terms)

$$u(b^{k/2}x, b^k t') = Z_2(b)^k u(x,t') \tag{2.123}$$

and a fixed point of this iteration will exist only if

$$u_2^*(x,t') := \lim_{k\to\infty} Z_2(b)^{-k} u(b^{k/2}x, b^k t') \tag{2.124}$$

is well defined. Now assuming the existence of a fixed point in this formal derivation we rewrite (2.124) for large but finite k as

$$u(b^{k/2}x, b^k t') \cong Z_2(b)^k u_2^*(x,t'). \tag{2.125}$$

Note that $b > 1$ was necessary for considering large time and space, and in fact for the assumption of approximate self-similarity that underlies the existence of the fixed point u_2^*. Letting $\bar{x} := b^{k/2}x$ and $\bar{t} := b^k t'$ one has (for large k)

$$u(\bar{x},\bar{t}) \cong [Z_2(b)]^k u_2^*\left(\bar{x}b^{-k/2}, \bar{t}b^{-k}\right). \tag{2.126}$$

This means that for any large $\bar{t}$ one can determine the u profile by setting $b^k := \bar{t}/(Q_1^2/D)$, so that the second argument remains unchanged as one examines different values of $\bar{t}$. Letting $t_1 := D\bar{t}/Q_1^2$ we can then write (2.126) as

$$u(\bar{x},t_1) \cong \left[Z_2\left({t_1}^{1/k}\right)\right]^k u_2^*\left(\bar{x}t_1^{-1/2}, Q_1^2/D\right). \tag{2.127}$$

Stage 3. The scaling exponent will be determined by the limit

$$\lim_{k\to\infty}\left[Z_2\left({t_1}^{1/k}\right)\right]^k = \lim_{k\to\infty} t_1^{-1/2}\left\{1 + \frac{\varepsilon A\log(t_1)}{k} + \frac{1}{2!}\left[\frac{\varepsilon A\log(t_1)}{k}\right]^2\right\}^k \tag{2.128}$$

if it exists. To calculate this we first let

$$y := \frac{\varepsilon A\log(t_1)}{k} \tag{2.129}$$

so that

$$1 + \frac{\varepsilon A\log(t_1)}{k} + \frac{1}{2!}\left[\frac{\varepsilon A\log(t_1)}{k}\right]^2 = \left[1 + \frac{1}{(1+i)}y\right]\left[1 + \frac{1}{(1-i)}y\right]. \tag{2.130}$$

Utilizing now the asymptotic expansion $e^\delta \cong 1+\delta$ for small δ one has

$$\left\{1+\frac{\varepsilon A\log(t_1)}{k}+\frac{1}{2!}\left[\frac{\varepsilon A\log(t_1)}{k}\right]^2\right\}^k \cong \exp\left\{\frac{\varepsilon A}{(1+i)}\log(t_1)\right\}\exp\left\{\frac{\varepsilon A}{(1-i)}\log(t_1)\right\} = t_1^{\varepsilon A}. \tag{2.131}$$

We then have, from (2.128),

$$\lim_{k\to\infty}\left[Z_2\left({t_1}^{1/k}\right)\right]^k = t_1^{-\frac{1}{2}+\varepsilon A}. \tag{2.132}$$

Stage 4. Using (2.132) in (2.127) and dropping superbar since (2.127) is valid for arbitrary large $\bar{t}$ we have

$$u(x,t') = \left(Dt'/Q_1^2\right)^{-\frac{1}{2}+\varepsilon A} u_2^*\left(x\left(Dt'/Q_1^2\right)^{-1/2}, Q_1^2/D\right) \tag{2.133}$$

so that the anomalous exponent or "dimension" is $\alpha = -\varepsilon A$.

Stage 5. Explicit evaluation of u_2^* is possible by writing (2.116) as

$$\begin{aligned} u(x,t';\varepsilon,l) &= \frac{T_0}{2\pi^{1/2}}\left(\frac{t'}{Q_1^2/D}\right)^{-1/2} e^{-x^2/(4Dt')}\left\{1+\varepsilon A\log\left(\frac{Dt'}{Q_1^2}\right)+\frac{1}{2!}\left[\varepsilon A\log\left(\frac{Dt'}{Q_1^2}\right)\right]^2\right\} \\ &\quad\times\left\{1+\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)+\frac{1}{2!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^2\right\}. \end{aligned} \tag{2.134}$$

Utilizing (2.131) again we can show that

$$\begin{aligned} \left\{1+\varepsilon A\log\left(\frac{Dt'}{Q_1^2}\right)+\frac{1}{2!}\left[\varepsilon A\log\left(\frac{Dt'}{Q_1^2}\right)\right]^2\right\} &\cong \exp\left\{\frac{\varepsilon A}{(1+i)}\log\left(\frac{Dt'}{Q_1^2}\right)\right\}\exp\left\{\frac{\varepsilon A}{(1-i)}\log\left(\frac{Dt'}{Q_1^2}\right)\right\} \\ &= \left(\frac{Dt'}{Q_1^2}\right)^{\varepsilon A} = \left(\frac{t'}{Q_1^2/D}\right)^{\varepsilon A}. \end{aligned} \tag{2.135}$$

Using now this we rewrite (2.134) as

$$u(x,t';\varepsilon,l) = \frac{T_0}{2\pi^{1/2}}\left(\frac{Dt'}{Q_1^2}\right)^{-\frac{1}{2}+\varepsilon A} e^{-x^2/(4Dt')}\left\{1+\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)+\frac{1}{2!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^2\right\}. \tag{2.136}$$

Comparison of (2.136) with (2.133) leads to the evaluation of u_2^* as

$$u_2^*\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D}\right) = \frac{T_0}{2\pi^{1/2}} \exp\left(-\frac{\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}\right)^2}{4D\left(Q_1^2/D\right)}\right) \times\left\{1+\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)+\frac{1}{2!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^2\right\}. \quad (2.137)$$

2.3.2 The verification for arbitrary $r \in Z^+$

Stage 1. We first need to find an identity (up to $O(\varepsilon^r)$) of the form

$$u(b^\phi x, bt') = Z_r(b)u(x,t') \quad (2.138)$$

which is valid for a particular choice of Z_r and ϕ and all $b > 1$. Once again, one can easily see that the exponential term in (2.116) yields $\phi = \frac{1}{2}$. We next rewrite (2.116) up to $O(\varepsilon^r)$ as

$$u(b^{1/2}x, bt') = \frac{T_0}{2\pi^{1/2}}\left(\frac{t'}{Q_1^2/D}\right)^{-1/2} e^{-\frac{x^2}{4Dt'}}\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log(2Dt'/l^2)\right]^j\right\} \times b^{-1/2}\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log(b)\right]^j\right\} \quad (2.139)$$

that leads to the expression

$$Z_r(b) := b^{-1/2}\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log(b)\right]^j\right\}. \quad (2.140)$$

Note that Z_r does not depend upon l.

Following [38] and [12] we now define the operator as

$$R_{b,\phi}u(x,t') := \frac{1}{Z_r(b)}u(b^{1/2}x, bt'). \quad (2.141)$$

Stage 2. By iteration we have (suppressing ε and l and ignoring $O(\varepsilon^{r+1})$ terms)

$$u(b^{k/2}x, b^k t') = Z_r(b)^k u(x,t'). \quad (2.142)$$

A fixed point of this iteration will exist only if

$$u_r^*(x,t') := \lim_{k\to\infty} Z_r(b)^{-k} u(b^{k/2}x, b^k t') \tag{2.143}$$

is well defined. Under the assumption of the existence of a fixed point in this formal derivation, we rewrite this for large but finite k as

$$u(b^{k/2}x, b^k t') \cong Z_r(b)^k u_r^*(x,t'). \tag{2.144}$$

Letting now $\bar{x} := b^{k/2}x$ and $\bar{t} := b^k t'$ one rewrites the last equation so that one has (for large k)

$$u(\bar{x},\bar{t}) \cong [Z_r(b)]^k \, u_r^* \left(\bar{x}b^{-k/2}, \bar{t}b^{-k}\right). \tag{2.145}$$

This means that for any large $\bar{t}$ we can determine the u profile by setting $b^k := \bar{t}/(Q_1^2/D)$ (so that the second argument does not change as $\bar{t}$ varies), and write (2.145) as

$$u(\bar{x},t_1) \cong \left[Z_r\left({t_1}^{1/k}\right)\right]^k u_r^* \left(\bar{x}t_1^{-1/2}, Q_1^2/D\right) \tag{2.146}$$

by letting $t_1 := D\bar{t}/Q_1^2$.

Stage 3. The limit below (if it exists)

$$\lim_{k\to\infty} \left[Z_r\left({t_1}^{1/k}\right)\right]^k = \lim_{k\to\infty} t_1^{-1/2} \left\{ \sum_{j=0}^{r} \frac{1}{j!} \left[\frac{\varepsilon A \log(t_1)}{k}\right]^j \right\}^k \tag{2.147}$$

will determine the scaling exponent. Letting $y := \frac{\varepsilon A \log(t_1)}{k}$ one can show that

$$\begin{aligned} \sum_{j=0}^{r} \frac{1}{j!} \left[\frac{\varepsilon A \log(t_1)}{k}\right]^j &= \frac{1}{r!} \sum_{j=0}^{r} \frac{r!}{(r-j)!} y^{r-j} \\ &= \frac{1}{r!} \prod_{j=1}^{r} (y+\beta_j) = \frac{\beta_1\beta_2\cdots\beta_r}{r!} \prod_{j=1}^{r} \left(1+\frac{y}{\beta_j}\right) \end{aligned} \tag{2.148}$$

where $\beta_1, \beta_2, \cdots, \beta_r$ are roots of the polynomial $\prod_{j=1}^{r} (y+\beta_j)$ such that $\frac{\beta_1\beta_2\cdots\beta_r}{r!} = 1$, and

$\sum_{j=1}^{r}(1/\beta_j) = 1$ (see [38] for $r = 2$). Utilizing now the asymptotic expansion $e^x \cong 1 + x$ for small x we have

$$\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\frac{\varepsilon A\log(t_1)}{k}\right]^j\right\}^k \cong \prod_{j=1}^{r}\exp\left[\frac{\varepsilon A}{\beta_j}\log(t_1)\right] = t_1^{\varepsilon A}. \tag{2.149}$$

that yields the result

$$\lim_{k\to\infty}\left[Z_r\left({t_1}^{1/k}\right)\right]^k = t_1^{-\frac{1}{2}+\varepsilon A}. \tag{2.150}$$

Stage 4. We first substitute (2.150) into (2.146), and then drop the superbar since (2.146) is valid for arbitrary large $\bar{t}$. This yields the identity

$$u(x,t') = \left(Dt'/Q_1^2\right)^{-\frac{1}{2}+\varepsilon A} u_r^*\left(x\left(Dt'/Q_1^2\right)^{-1/2}, Q_1^2/D\right) \tag{2.151}$$

so that the anomalous exponent is $\alpha = -\varepsilon A$.

Stage 5. To obtain u_r^* we first rewrite (2.116) as

$$\begin{aligned} u(x,t';\varepsilon,l) \quad = \quad & \frac{T_0}{2\pi^{1/2}}\left(\frac{t'}{Q_1^2/D}\right)^{-1/2} e^{-\frac{x^2}{4Dt'}}\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{Dt'}{Q_1^2}\right)\right]^j\right\} \\ & \times\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^j\right\}. \end{aligned} \tag{2.152}$$

Following a procedure similar to (2.148) - (2.149) one can show

$$\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{Dt'}{Q_1^2}\right)\right]^j \cong \prod_{j=1}^{r}\exp\left[\frac{\varepsilon A}{\beta_j}\log\left(\frac{Dt'}{Q_1^2}\right)\right] = \left(\frac{Dt'}{Q_1^2}\right)^{\varepsilon A} \tag{2.153}$$

so that one has

$$u(x,t';\varepsilon,l) = \frac{T_0}{2\pi^{1/2}}\left(\frac{Dt'}{Q_1^2}\right)^{-1/2+\varepsilon A}\exp\left(-\frac{x^2}{4Dt'}\right)\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^j\right\}. \tag{2.154}$$

Comparison of (2.154) with (2.151) yields u_r^* as

$$u_r^*\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}, \frac{Q_1^2}{D}\right) = \frac{T_0}{2\pi^{1/2}}\exp\left(-\frac{\left(\frac{x}{(Dt'/Q_1^2)^{1/2}}\right)^2}{4D\left(Q_1^2/D\right)}\right) \times\left\{\sum_{j=0}^{r}\frac{1}{j!}\left[\varepsilon A\log\left(\frac{2Q_1^2}{l^2}\right)\right]^j\right\}$$

which is (2.118).

2.4 Exact results

In this section we consider exact solutions to some special cases of (2.4) subject to constraints (2.2) with the aim of checking the RG calculations. In each of the calculations below we transform (2.4) into

$$\varphi_\tau = \frac{1}{2}\left[\varphi_{\xi\xi} + \xi\varphi_\xi + \varphi_\xi^2\right] + \varepsilon F\left[\xi, 1, \varphi_\xi, \varphi_{\xi\xi} + \varphi_\xi^2\right] \tag{2.155}$$

using the change of variables

$$u(x,t) := e^{\varphi(\xi,\tau)}, \quad \tau := \log(t+t_0) \quad \text{and} \quad \xi := x(t+t_0)^{-1/2} \tag{2.156}$$

where F is taken as in section 2.2. This transformation will be utilized in order to determine exact solutions for the following two cases:

2.4.1 Examples

Example 1. Using the transformation above we rewrite the equation

$$u_t = \frac{1}{2}u_{xx} + \varepsilon x^{-1}u_x \tag{2.157}$$

as

$$\varphi_\tau = \frac{1}{2}\left[\varphi_{\xi\xi} + \varphi_\xi^2 + \xi\varphi_\xi\right] + \varepsilon\xi^{-1}\varphi_\xi. \tag{2.158}$$

We seek a non-negative exact solution to the equation (2.158) of the form

$$\varphi(\xi,\tau)=\sigma\xi^2+\alpha\tau, \tag{2.159}$$

where $\sigma,\alpha\in R$, so that the solution to the equation (2.157) has the form

$$u\left(x,t-t_0\right)=t^{-\alpha}\exp\left(\frac{\sigma x^2}{t}\right). \tag{2.160}$$

The substitutions yield that the equation (2.157) has the exact (non-negative) solution

$$u\left(x,t-t_0\right)=t^{-\frac{1}{2}-\varepsilon}e^{\frac{-x^2}{2t}}. \tag{2.161}$$

Examples 2. Following a similar procedure in Example 1 we find that the non-linear equation

$$u_t=\frac{1}{2}u_{xx}+\varepsilon u^{-1}u_x^2 \tag{2.162}$$

has the exact (non-negative) solution

$$u\left(x,t-t_0\right)=t^{\frac{-1}{2(1-2\varepsilon)}}\exp\left(\frac{-x^2}{(1-2\varepsilon)2t}\right). \tag{2.163}$$

Remark 2.1. *Letting $u(x,t)=[w(x,t)]^{\frac{1}{(1-2\varepsilon)}}$, where $\varepsilon\neq\frac{1}{2}$, one can transform the equation (2.162) into linear (diffusion equation) form, i.e. $w_t=\frac{1}{2}w_{xx}$. Hence, the exact (non-negative) solution (2.163) can be obtained by using the fundamental solution to linear equation, namely $\Gamma(x,t)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$.*

These examples confirm the RG results (see [12] and Section 2.3).

2.4.2 Series-integral solutions

We describe briefly an additional asymptotic method for equations of the form (2.155) in order to calculate the decay exponents [38]. This procedure involves using the formal expansion for the special variables,

$$\varphi\left(\xi,\tau;\varepsilon\right)=\phi_0\left(\xi,\tau\right)+\varepsilon\phi_1\left(\xi,\tau\right)+\varepsilon^2\phi_2\left(\xi,\tau\right)+\cdots \tag{2.164}$$

in equation (2.155). Substituting this expansion into equation (2.155) we obtain a hierarchy of equations in terms of order in ε. The zeroth order equation is the nonlinear equation

$$\phi_{0\tau} = \frac{1}{2}\left[\phi_{0\xi\xi} + \xi\phi_{0\xi} + \phi_{0\xi}^2\right] \tag{2.165}$$

while the remaining equations involve the linear operator

$$\mathbf{L}\phi = \frac{1}{2}\left[\phi_{\xi\xi} - \xi\phi_{\xi}\right]. \tag{2.166}$$

In particular, the first order equation is

$$\phi_{1\tau} - \mathbf{L}\phi_1 = \Omega_0(\xi) \tag{2.167}$$

where

$$\Omega_0(\xi) := F\left[\xi, 1, \phi_{0\xi}, \phi_{0\xi\xi} + \phi_{0\xi}^2\right]. \tag{2.168}$$

One can verify that the zeroth order nonlinear equation has the solution

$$\phi_o(\xi,\tau) = -\frac{1}{2}\xi^2 - \frac{1}{2}\tau. \tag{2.169}$$

The coefficient of τ yields the leading (classical) exponent $\alpha_0 = 1/2$, i.e., $t^{-1/2}$. The linear equations can be solved with the constraints imposed by the boundary conditions. The analysis yields the next term in the exponent

$$\alpha_1 = -\frac{\int_0^\infty \Omega_0(\eta)\, e^{-\eta^2/2} d\eta}{\int_0^\infty e^{-\eta^2/2} d\eta} \tag{2.170}$$

that provides the correction to the classical decay, i.e., $t^{-1/2}$ and agrees with the RG calculations.

Similarly, higher order corrections can be generated by analyzing successive linear equations.

2.4.3 Self-similar solutions

Another method for calculating these exponents involves self-similarity methods. We consider the case $q = 0$ so that we write (2.155) as

$$\varphi_\tau = \frac{1}{2}\left[\varphi_{\xi\xi} + \varphi_\xi^2 + \xi\varphi_\xi\right] + \varepsilon\xi^{p-2}\varphi_\xi^p. \tag{2.171}$$

One seeks an exact solution to (2.171) of the form

$$\varphi(\xi, \tau; \varepsilon) = \phi(\xi; \varepsilon) - \alpha(\varepsilon)\tau, \tag{2.172}$$

where $(\alpha, \phi) \in R^1 \times C^2(R)$ is the unknown, so that

$$u\left(x, t - t_0\right) = t^{-\alpha} e^{\phi\left(x/\sqrt{t}\right)}. \tag{2.173}$$

It is then equivalent to solve

$$\ddot{\phi} + \dot{\phi}^2 + \xi\dot{\phi} + 2\varepsilon\xi^{p-2}\dot{\phi}^p + 2\alpha = 0 \quad \text{on } (-\Xi, \Xi)$$

$$\lim_{\xi\to\pm\Xi} e^{\phi(\xi;\varepsilon)} = 0, \qquad \lim_{\xi\to\pm\Xi} \dot{\phi}e^{\phi} = 0 \tag{2.174}$$

where Ξ is either finite or infinite and $\cdot := \frac{d}{d\xi}$. Note that if Ξ is finite, then $u = u_x \equiv 0$ for $|x| \geq \Xi\sqrt{t}$ and we have a compactly supported self similar solution to (2.4).

Suppose we are looking for an even solution, i.e., $\phi(-\xi) = \phi(\xi)$. Setting $w(\xi; \varepsilon) = \dot{\phi}(\xi; \varepsilon)$ one now has

$$\dot{w} + \xi w + w^2 + 2\varepsilon\xi^{p-2}w^p + 2\alpha = 0 \quad \text{on } (0, \Xi)$$

$$w(0) = 0, \qquad \lim_{\xi\to\Xi} w = -\infty, \qquad \lim_{\xi\to\Xi} we^{\int_0^\xi w(\eta)d\eta} = 0. \tag{2.175}$$

Using shooting methods one can show the existence of a unique solution (α, ϕ) (see Appendix A). Introducing the expansion

$$\alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots \tag{2.176}$$

$$w(\xi; \varepsilon) = w_0(\xi) + \varepsilon w_1(\xi) + \varepsilon^2 w_2(\xi) + \cdots \tag{2.177}$$

one has a sequence of IVP's:

$$\dot{w}_0 = -2\alpha_0 - \xi w_0 - w_0^2, \qquad w_0(0) = 0 \tag{2.178}$$

$$\dot{w}_1 = -2\alpha_1 - \xi w_1 - 2w_0 w_1 - 2\xi^{p-2} w_0^p, \qquad w_1(0) = 0 \tag{2.179}$$

$$\dot{w}_2 = -2\alpha_2 - \xi w_2 - (w_1^2 + 2w_0 w_2) - 2p\xi^{p-2} w_0^{p-1} w_1, \qquad w_2(0) = 0 \tag{2.180}$$

$$\dot{w}_k = -2\alpha_k + \xi w_k + \theta\left(\xi, w_0, \cdots, w_{k-1}; p\right), \qquad w_k(0) = 0 \quad \text{for} \quad k \geq 3 \tag{2.181}$$

where θ is a known function of these variables.

Note that the initial conditions in (2.178)-(2.181) ensure (through the expansion (2.177) for w) the first of the three conditions in (2.175). The second condition can be guaranteed by imposing it on w_0 alone, since the remaining terms are of the lower order in ε. The third condition in (2.175) can be written as

$$(w_0(\xi) + \varepsilon w_1(\xi) + \cdots)e^{\int_0^\xi w_0(\eta)d\eta} e^{\varepsilon \int_0^\xi w_1(\eta)d\eta + \varepsilon^2 \int_0^\xi w_2(\eta)d\eta + \cdots} \to 0 \tag{2.182}$$

and can be ensured by imposing the condition

$$\lim_{\xi \to \Xi} w_i e^{\int_0^\xi w(\eta)d\eta} = 0 \quad \text{for} \quad i = 0, 1, 2, \ldots. \tag{2.183}$$

With this additional condition we proceed to solve (2.178)-(2.181) for w_i and α_i. Note that the first of these is nonlinear while all of the others are linear in terms of the differentiated function. Accordingly the treatment differs in the two cases. One can easily verify that equation (2.178), subject to the limiting condition above, has a solution

$$\alpha_0 = \frac{1}{2} \text{ and } w_0(\xi) = -\xi \quad \Rightarrow \quad \phi_0(\xi) = -\frac{1}{2}\xi^2. \tag{2.184}$$

Further discussion of nonlinear equations of this type can be found in Appendix A.

The remaining equations (2.179)-(2.181) can be solved successively by multiplying by the integrating factor. In particular, upon multiplication by $e^{-\xi^2/2}$, (2.179) has the form

$$\frac{d}{d\xi}\left[e^{-\xi^2/2}w_1(\xi)\right] = e^{-\xi^2/2}\left[-2\alpha_1 - 2(-1)^p\xi^{p-2}\right] \tag{2.185}$$

so that integration on $[0,\infty)$ and utilizing the condition (2.183)($\lim_{\xi\to\Xi} w_1 e^{-\xi^2/2} = 0$) yields the value

$$\alpha_1 = \frac{(-1)^{p+1}\int_0^\infty \eta^{2p-2}e^{-\eta^2/2}d\eta}{\int_0^\infty e^{-\eta^2/2}d\eta} = (-1)^{p+1}\left(1\cdot 3\cdot\cdots\cdot|2p-3|\right). \tag{2.186}$$

Thus, this result yields the same exponent as the formal RG calculation (Caginalp [12]).

One needs to find a solution of (2.185) to obtain the next term α_2, which is important in determining the most singular term in the anomalous exponent for nonlinear diffusion, and also to compare the results obtained by the RG methods. Substituting the value above for α_1 (as well as $w_0(\xi) = -\xi$) into (2.179) we obtain a first order linear IVP with variable coefficients that can be solved by standard methods. Thus , a solution to (2.185) is given by

$$w_1(\xi) = 0 \quad \text{for} \quad p = 1 \tag{2.187}$$

$$w_1(\xi) = \sum_{k=1}^{p-1}\frac{(-1)^p 2\left(1\cdot 3\cdot\cdots\cdot|2p-3|\right)}{\left(1\cdot 3\cdot\cdots\cdot|2k-1|\right)}\xi^{2k-1} \quad \text{for} \quad p \geq 2. \tag{2.188}$$

Next, we substitute these expressions for w_0 and w_1 into (2.180), and utilize the same methods to obtain the value of α_2 as

$$\alpha_2 = \frac{(-1)^p\, p\int_0^\infty \eta^{2p-3}w_1(\eta)e^{-\eta^2/2}d\eta - \frac{1}{2}\int_0^\infty w_1^2(\eta)e^{-\eta^2/2}d\eta}{\int_0^\infty e^{-\eta^2/2}d\eta}. \tag{2.189}$$

Evaluating these integrals yields the following values:

$$\alpha_2 = 0 \quad \text{for} \quad p = 1 \tag{2.190}$$

$$
\begin{aligned}
\alpha_2 &= 2p\,(1\cdot 3\cdot \cdots \cdot |2p-3|)\sum_{k=1}^{p-1}\frac{(1\cdot 3\cdot \cdots \cdot |2(k+p)-5|)}{(1\cdot 3\cdot \cdots \cdot |2k-1|)} \\
&\quad -2\,(1\cdot 3\cdot \cdots \cdot |2p-3|)^2 \\
&\quad \times\sum_{r=1}^{p-1}\sum_{j=1}^{p-1}\frac{(1\cdot 3\cdot \cdots \cdot |2(r+j)-3|)}{(1\cdot 3\cdot \cdots \cdot |2r-1|)\,(1\cdot 3\cdot \cdots \cdot |2j-1|)} \quad \text{for} \quad p\geq 2
\end{aligned}
\tag{2.191}
$$

Note that the sign of α_2 would depend on that of the nonlinear term F. In addition, following the similar procedure above one can calculate α_k for $k \geq 3$.

Remark 2.2. *Note that for $p = 1$ one has $\alpha_1 = 1$ and $\alpha_2 = 0$. These yield the nonclassical exponent as $\alpha(\varepsilon) = \frac{1}{2} + \varepsilon$ that agrees with (2.161).*

2.5 Conclusions

We have developed the renormalization group ideas to higher order in ε by deriving the operator Z in (2.138) that allows us to write expressions such as (2.117). Our procedure is to extract, for each order in ε, the leading order behavior in l^{-1}, in a large but finite interval, so that only positive contributions to the decay are significant in $O(\varepsilon^2)$ and higher. A key step in this process is to obtain a transformation that rescales variables. While RG methods usually involve an identity in this transformation, we utilize the basic ideas by using an identity up to a particular order in ε.

For example, the equation

$$u_t = \frac{1}{2}u_{xx} + \varepsilon x^{-1}u_x$$

is characterized by the large time behavior

$$u(x,t) \sim t^{-\frac{1}{2}-\varepsilon}$$

since we can characterize all of the higher order terms as an exponential that is the sum of a convergent infinite series. The exponential with logarithmic terms as arguments can then be written as $t^{-\varepsilon}$.

The methodology presented in this paper can be expected to be useful in other problems in which there is an asymptotically self-similar structure. Examples of other such situations are finite-time blow up and extinction of solutions to nonlinear differential equations.

The rigorous results presented in this paper confirm that some of the earlier RG calculations are in fact valid for arbitrarily large time and space.

Chapter 3

Renormalization Group Methods and Regimes in Interface Problems

3.1 Introduction

Spatial pattern formation arising from a variety of non-equilibrium growth phenomena has attracted much attention. A number of mathematical methods such as analytical methods, linear stability theory and large scale computations have been used to study these problems. In many cases, the pattern arises through the motion of an interface separating two phases or liquids [18]. Early studies of pattern formation generally focused on the existence, the nature of steady states and their stability, etc., for example, the onset of stability for fluids, alloys, and Stefan-like supercooled solidification (see [52], [43], [44], and [45]). Analytical methods and linear stability have been a valuable tool to answer such key questions. In particular, the latter has been used to study the evolution of the interface for its small time behavior.

From a practical standpoint, the large time evolution has been of great interest in a number of problems such as dendritic growth, directional solidification in binary alloys, and fluids. In many problems, this is the key issue in the interfacial phenomena and merits analysis. For example, in solidification of a binary alloy, a key issue involves the deposition of the impurities as the late-stage dendrites involve into fully solidified material. The pattern of impurities in the solid has a strong bearing on the mechanical properties such as brittleness.

A first aim along these lines has been the development of an analytical method, analogous to linear stability theory, that can be used to determine the characteristic length, $R(t)$, as a function of time if the pattern is self-similar. Progress toward this goal was made by Jasnow and Vinals [30], [31], and Caginalp [14], [15] who adapt renormalization group (RG) and scaling theory to study the large time behavior of an interface.

The characteristic length, $R(t)$, is the time dependent length scale governing the morphology of late stage pattern growth. For example, it may be the radius of a circle which contains the

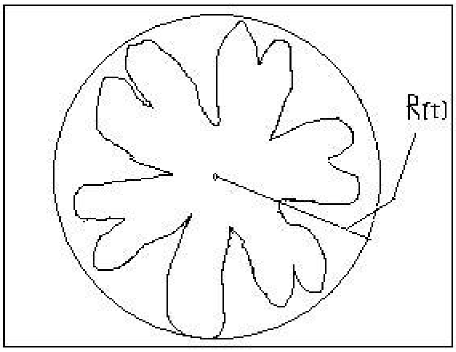


Figure 3.1. The viscous-fingering pattern.

pattern evolving self-similarly in time [see Figure 3.1]. Jasnow and Yeung [32] describe a possible characteristic length as the radius of gyration of pattern in the study of asymptotic behavior of viscous-fingering patterns in a circular Hele-Shaw cell. In their work, $R(t)$ characterizes the fractal pattern growth which evolves linearly in time in the constant flux mode [see FIG.1 in [32]]. A key issue is whether $R(t) \sim A(t)^{1/2}$, where $R(t) := A(t)^{1/D}$, D is the fractal dimension, in which case the pattern is compact (non-fractal). In other words, the pattern "fills space" as $t \to \infty$. If the exponent is greater than 1/2 (in two dimensional space) then pattern is fractal. While some experiments [51] have suggested $D \simeq 1.79$, Jasnow and Yeung's numerical computations lead to their conclusion.

We summarize now some results obtained for complex interface problems that arise frequently in applications. Using the terminology of thermal problems we write the sharp interface problem as follows. We consider a material occupying a spatial region, Ω, in d-dimensional space that can be in either of two phases, which we call liquid and solid. Mathematical model consists of determining the temperature, $T(x,t)$, and the interface, $\Gamma(t)$, in the system of equations:

$$CT_t = K\Delta T \qquad \text{in} \quad \Omega \tag{3.1}$$

$$lv_n = -K[\nabla T \cdot \widehat{n}]_-^+ \qquad \text{on} \quad \Gamma \tag{3.2}$$

$$T = \frac{-\sigma_0}{[s]_{eq}}(\kappa + \alpha v_n) \qquad \text{on} \quad \Gamma. \tag{3.3}$$

Here, C is the specific heat per unit volume, K is the thermal conductivity (so that one defines that $D := K/C$), l is the latent heat per unit volume, σ_0 is the surface tension, $[s]_{eq}$ is the entropy difference per unit volume between phases, α is the dynamical undercooling and $[...]_-^+$ is the difference in the limiting values between the two sides of the interface. The variables v_n and κ denote the (normal) velocity and the sum of the principle curvatures at a point on the interface, respectively. In addition, + denotes the phase with the higher internal energy, i.e., liquid and − denotes the phase with the lower internal energy, i.e., solid.

Jasnow and Vinals utilized the following conditions: (i) the dynamical undercooling was set to zero ($\alpha = 0$); (ii) one of the two phases was suppressed so that the equations involved one of the phases; (iii) the quasi-static limit was considered by suppressing the time dependence in (3.1) (i.e., $CT_t = 0$); (iv) a plane wave solution was utilized (through flux conditions) and subtracted from the solutions. Under these conditions they found that the characteristic length, $R(t)$, of a system with single scale self-similarity must have the large time behavior $R(t) \sim t$.

Subsequently, Caginalp examined the problem above under the conditions (i) $\alpha \neq 0$; (ii) two-phases are present; (iii) the fully-dynamic case is considered ($CT_t \neq 0$); (iv) both with a particular plane wave and without. Under these conditions RG led to the conclusion that $R(t) \sim t^{1/2}$ assuming R is nonsingular as d_0 approaches zero.

This suggests the following questions. (a) What feature of the equations is responsible for the difference in the scaling exponents? (b) How is the transition made between the different regimes? (c) What insights can we obtain for other problems involving nonlinear dynamics?

There is also another issue that is illuminated by the RG process. As we will see in the next section the RG methodology also distinguishes between those physical parameters that are "relevant" and those that are "irrelevant." An irrelevant variable (in terms of large time behavior) is one whose value can be set to some value (zero in this case) without influencing the scaling exponent (i.e., 1 or 1/2 in the two cases above). Both of the analyses above, as well as the analysis we present show that the capillarity length (which is the length scale associated with the surface tension, σ_0) is irrelevant for large time in most cases even though the difference in the exponents of the characteristic length suggests that there is an important difference between in the two regimes.. This is in sharp contrast with the crucial role it has for stability in short time where it essentially determines the nature of the interface evolution. A large capillarity length means that the interface seeks to minimize the curvature, making the interface more rounded. Consequently, the irrelevance of the capillarity length is one of the key surprises presented by the RG analysis.

In this chapter, we consider the full two-phase interface problem in the quasi-static regime in a d-dimensional space where $d > 2$. The growth process for sufficiently long time is analyzed in the context of a general geometry and more general conditions on the degree of undercooling. The quasi-static regime is important since it is a good approximation for many materials with common boundary conditions, as the temperature quickly approaches a solution to Laplace's equation. Thus, a key difference in long term behavior between quasi-static and fully dynamic would be significant in theoretical and practical terms. The main result of our work is that without reference to a plane wave the characteristic length, $R(t)$, varies as $t^{-1/\lambda}$ where $\lambda \in [-3, -2]$, under the single scale self similarity assumption when the dynamical undercooling is non-zero ($\alpha \neq 0$). For $\alpha = 0$ the spectrum is $[-3, 0)$ so that the single value of $\lambda = -1$ is selected by the plane wave imposed through the boundary conditions by Jasnow and Vinals [31]. This work indicates that the pattern evolves with different forms as λ varies in a continuous spectrum and extends the earlier results and those of Jasnow and Vinals who obtained a single growth form for the interface problem in the same regime [see Figure 3.2]. Analogous results are also obtained for other interface models [see Figure 3.2 and Figure 3.3].

Furthermore, the results confirm, as in [31] and [15], that for almost all values of λ the capillarity length, d_0, which is the length scale associated with the surface tension, is not relevant to the scaling of the large scale behavior of an interface. This is an intriguing consequence since we know that the surface tension plays the stabilizing role in the early stage growth so that it has a critical influence for short time. Indeed, while large capillarity length tends to suppress instabilities small capillarity length permits it. The only exception arises at the value $\lambda = -3$ (so that $R(t) \sim t^{1/3}$) for which the capillarity length, d_0, is invariant.

The methodology we use is similar to that in Caginalp [15], and basically utilizes RG transformations that are identities involving Green's function representation for Poisson's equation. The outline of the chapter is as follows: In Section 3.2, we first rewrite, for the case $\alpha \neq 0$, the basic equations in terms of a Green's function representation by introducing a phase parameter (in Section 3.2.1). In Section 3.2.2, the RG analysis is implemented in several steps so that the equations are first transformed and then converted back into the original form by renormalizing physical parameters. In Section 3.3, the case $\alpha = 0$ is studied.

3.2 The RG analysis in the quasi-static regime

We address the question of isolating the factors behind the different scaling regimes. In particular we begin by using (3.1)-(3.3) with $CT_t = 0$, i.e., the quasi-static regime. Without reference to a plane wave we find that the scaling exponent in $R(t) \sim t^{-1/\lambda}$ is given by $\lambda \in [-3, -2]$ if $\alpha \neq 0$, and $\lambda \in [-3, 0)$ if $\alpha = 0$. In the latter case, the continuous spectrum includes the value $\lambda = -1$ obtained by Jasnow and Vinals [31] for the plane wave subtraction. Hence the plane wave selects the particular value from the spectrum.

The basic steps in the RG process can be presented as follows. In this section, the dynamical undercooling is nonzero, i.e. $\alpha \neq 0$.

3.2.1 The model and Green's function representation

We begin the calculations by writing equations (3.1) and (3.2) as a single equation. In order to do this, we utilize the heat equation

$$CT_t = K\Delta T \tag{3.4}$$

where C is the specific heat per unit volume and K is the thermal conductivity and $D := K/C$. Following Caginalp [15], the equations (3.4) and (3.2) can be reformulated and written as a single equation by defining locally a signed distance, r (defined a sufficiently small distance from the interface), which is positive on the liquid phase, and introducing a phase variable $\varphi(r, t)$ that is a step function having the value –1 in the solid phase and +1 in the liquid phase. One then has

$$CT_t - K\Delta T = -\frac{l}{2}\varphi_t. \tag{3.5}$$

This formulation is known as Oleinik formulation that is related to a continuously varying function φ in the phase field equation (see Caginalp [14] [15] and Oleinik [46] for more details). Multiplying the two sides of the equation (3.5) by $1/K$ and setting $CT_t = 0$ we obtain

$$\Delta T = \frac{l}{2K}\varphi_t. \tag{3.6}$$

Treating the phase change as a source term with support along the interface, $\Gamma(t)$, and using the Green's formulation one can express the solution of (3.6) as

$$T\left(x\right) = \int_{\Omega} d^d y \; G(\vec{x}-\vec{y}) \left(\frac{l}{2K}\varphi_t(\vec{y},t)\right) + \int_{\partial\Omega} \left(T\left(\vec{y}\right)\frac{\partial G}{\partial \nu}(\vec{x}-\vec{y}) + G(\vec{x}-\vec{y})\frac{\partial T}{\partial \nu}(\vec{y})\right) d^{d-1}\sigma_y \tag{3.7}$$

where the Green's function that we use is

$$G(\vec{x}-\vec{y}) = \begin{cases} \frac{1}{2(2-d)\omega_d}\left|\vec{x}-\vec{y}\right|^{2-d} & \text{if } d > 2, \\ \frac{1}{2\pi}\log\left|\vec{x}-\vec{y}\right| & \text{if } d = 2. \end{cases} \tag{3.8}$$

Here, the simplest Green's function for infinite domains is implemented. Since we are interested in very large domains, this is a good approximation.

We now examine the region near the interface to evaluate the first integral $\left(\text{i.e } \int_{\Omega} ...\right)$ in (3.7). Following Caginalp [15], let $z = h(X,t)$, $X \in R^{d-1}$, denote the displacement of the interface from $\vec{z} = 0$ which is in the original stationary units. Then dh/dt will be the velocity in the $\widehat{k}$, or z, direction where $\widehat{k}$ is the unit normal. Thus, assuming the interface is sufficiently smooth, one can write the normal velocity of the interface as

$$\upsilon_n = \widehat{k}\cdot\hat{n}\frac{dh}{dt} \tag{3.9}$$

where $\widehat{n}$ is the unit normal in direction from solid to liquid.

Let $\widetilde{\varphi}\left(x,t\right)$ be a smoothing of the step function $\varphi\left(x,t\right)$ where the transition from –1 to +1 appears on a small distance scale. In order to perform the integral, we use local coordinates $(\vec{r}, \vec{\sigma})$ which are the signed normal and the tangential to the interface, respectively. We consider a suitable local neighborhood of the interface in order to eliminate problems such as the uniqueness of the signed normal. To leading order, the smoothing function $\widetilde{\varphi}(x,t)$ and its derivatives are functions of $r - \upsilon_n t$. By defining a variable ϕ as a function $r - \upsilon_n t$, to leading order we have

$$\varphi(x,t) = \widetilde{\varphi}(x,t) = \phi(r - \upsilon_n t). \tag{3.10}$$

If the interface is sufficiently smooth and the thickness of the interface is sufficiently small, then the transition region will be in this local region and this approximation can be used in order to compute the integration across the interface in (3.7). In particular, one has

$$\varphi_t(x,t) = -\upsilon_n \phi_r(r - \upsilon_n t) \tag{3.11}$$

and also, for enough small δ,

$$\int_{-\delta}^{\delta} \phi_r(r - \upsilon_n t) dr = 2. \tag{3.12}$$

Using then these new definitions, (3.7) is rewritten as

$$T(x) = \int_{\Omega} d^d y \; G(\vec{x} - \vec{y}) \left(\frac{l}{2K}\right) \left(-\left(\hat{k} \cdot \hat{n} \frac{dh}{dt}\right) \phi_{r_{\vec{y}}} \left(r - \left(\hat{k} \cdot \hat{n} \frac{dh}{dt}\right) t\right)\right) + \mathrm{BI} \tag{3.13}$$

where BI denotes the integral that is taken over the boundary of the domain in (3.7). Since the derivatives of ϕ vanish just outside of the interfacial region, we can perform the integral in the normal direction thereby reducing the integral over Ω to one over Γ, with the result,

$$T(x) = \int_{\Gamma(t)} d^{d-1}\sigma_y \; G(\vec{x} - \vec{y}) \left(\frac{l}{2K}\right) \left(-2\hat{k} \cdot \hat{n} \frac{dh}{dt}\right) + \mathrm{BI}. \tag{3.14}$$

For the points on the interface, one can combine (3.14) and (3.3). Recalling $D = K/C$ and neglecting BI term since it is far away from the interface one then has

$$\frac{-\sigma_0}{[s]_{eq}} \left(\kappa + \alpha \hat{k} \cdot \hat{n} \frac{dh(\vec{x},t)}{dt}\right) = \frac{l}{C} \frac{1}{D} \int_{\Gamma(t)} d^{d-1}\sigma_y \; G(\vec{x} - \vec{y}) \left(-\hat{k} \cdot \hat{n} \frac{dh(\vec{y},t)}{dt}\right). \tag{3.15}$$

Following [15] we now define the standard capillarity length to be

$$d_0 := \frac{\sigma_0 / [s]_{eq}}{l/C} \tag{3.16}$$

and rewrite (3.15) as

$$d_0 \left(\kappa + \alpha \hat{k} \cdot \hat{n} \frac{dh(\vec{x}, t)}{dt} \right) = \frac{1}{D} \int_{\Gamma(t)} d^{d-1} \sigma_y \; G(\vec{x} - \vec{y}) \left(\hat{k} \cdot \hat{n} \frac{dh(\vec{y}, t)}{dt} \right). \tag{3.17}$$

Dividing the variables in the equation above by appropriate reference length, L_0, and time, T_0, scales etc., we convert all constants and variables in (3.17) to their dimensionless counterparts (see [15] for further details), and write the equation entirely in dimensionless variables in order to compare pure numbers after a RG procedure. Using the dimensionless units, replacing $\vec{\eta}$ in the place of $\vec{x}$ for the points on the interface and recalling also that $\upsilon_n = \widehat{k} \cdot \hat{n} \cdot (dh/dt)$ (see (3.9)) one writes the equation (3.17) as

$$d_0 \left\{ \kappa(\vec{\eta}, t) + \alpha \upsilon_n (\vec{\eta}, t) \right\} = \frac{1}{D} \int_{\Gamma(t)} d^{d-1} \sigma_y \; G(\vec{\eta} - \vec{y}) \upsilon_n (\vec{y}, t) \,. \tag{3.18}$$

3.2.2 Renormalization group analysis of the interface equation

We now implement a renormalization procedure as follows [see [31], [14] and [15]].

Step 1. The first step is to make the algebraic substitutions

$$b\vec{\eta} \quad \text{for} \quad \vec{\eta} \qquad \text{and} \qquad b^{-\lambda} t \quad \text{for} \quad t \tag{3.19}$$

into (3.18) for any $b > 0$ and $\lambda \in R$, which will be determined later, so that one has

$$d_0 \left\{ \kappa(b\vec{\eta}, b^{-\lambda} t) + \alpha \upsilon_n (b\vec{\eta}, b^{-\lambda} t) \right\} = \frac{1}{D} \int_{\Gamma(b^{-\lambda} t)} d^{d-1} \sigma_y \; G(b\vec{\eta} - \vec{y}) \upsilon_n (\vec{y}, b^{-\lambda} t). \tag{3.20}$$

Next we define new variables

$$\vec{y'} = y/b \qquad \text{and} \qquad \sigma_{y'} = \sigma_y / b \tag{3.21}$$

in order to rescale space. These two substitutions into (3.20) yield

$$d_0 \left\{ \kappa(b\vec{\eta}, b^{-\lambda}t) + \alpha\ \upsilon_n(b\vec{\eta}, b^{-\lambda}t) \right\} = \frac{1}{D} \int_{by' \in \Gamma(b^{-\lambda}t)} b^{d-1} d^{d-1}\sigma_{y'}\ G(b\vec{\eta} - b\vec{y'})\upsilon_n(b\vec{y'}, b^{-\lambda}t). \quad (3.22)$$

Note that the surface integral in (3.22) is over those points for which $y \in \Gamma(b^{-\lambda}t)$ which is identical (algebraically) to $by' \in \Gamma(b^{-\lambda}t)$. The latter will be equivalent to $y' \in \Gamma(b^{-\lambda}t)$ upon assuming single scale self similarity in (3.24) below.

Step 2. The second step involves the examination of the scaling of individual terms. Purely algebraic transformation for the Green's function, for $d \geq 3$, leads to the result

$$G(b\vec{\eta} - b\vec{y'}) = b^{2-d} G(\vec{\eta} - \vec{y'}). \quad (3.23)$$

We assume the *single scale self similarity* for the scaling of the physical quantities involving length in this work. That is, it is assumed that all physical lengths and all physical time measurements in the problem scale as

$$\xi(b\vec{\eta}, b^{-\lambda}t) = b\xi(\vec{\eta}, t); \quad (3.24)$$

$$\mathrm{T}\left(b\vec{\eta}, b^{-\lambda}t\right) = b^{-\lambda}\mathrm{T}\left(\vec{\eta}, t\right), \quad (3.25)$$

respectively, (see [15] and [31]). Note that (3.24) implies $by' \in \Gamma(b^{-\lambda}t) \Leftrightarrow y' \in \Gamma(t)$. One interpretation of the first relation, for example, is that if one rescales the position on the interface, $\Gamma(t)$, by b, and the time by $b^{-\lambda}$, then the position in the z-direction, which is $\xi := h/L_0$ in the calculation above, will change by a factor of b. In other words if $\xi(\vec{\eta}, t)$ is the value for the height at time t, then that at time $b^{-\lambda}t$ can be obtained by multiplying it by b.

As a result of these assumptions, i.e. (3.24) and (3.25), one can obtain the scaling relations for the (normal) velocity, υ_n, and the curvature, κ, which has units of $1/length$, as

$$\upsilon_n\left(b\vec{\eta}, b^{-\lambda}t\right) = b^{1+\lambda}\upsilon_n\left(\vec{\eta}, t\right), \quad (3.26)$$

$$b\kappa\left(\vec{\eta}, t\right) = \kappa(b\vec{\eta}, b^{-\lambda}t). \quad (3.27)$$

Substituting the relations above into (3.22) and simplifying the terms lead to the new interface equation below that can be compared with the original equation (3.17), after the physical parameters are renormalized

$$\frac{d_0}{b^{3+\lambda}}\left\{\kappa(\vec{\eta},t)+\frac{\alpha}{b^{-2-\lambda}}\,\upsilon_n\left(\vec{\eta},t\right)\right\}=\frac{1}{D}\int_{y'\in\Gamma(t)} d^{d-1}\sigma_{y'}\;G(\vec{\eta}-\vec{y'})\upsilon_n\left(\vec{y'},t\right). \tag{3.28}$$

Step 3. At this stage in the renormalization process, we rescale the physical parameters in order that the new equation has the same form as the original equation. The key observation here is that the new equation (3.28) is identical to the original (3.17) upon replacing

$$d_0\rightarrow\frac{d_0}{b^{3+\lambda}}\qquad\text{and}\qquad\alpha\rightarrow\frac{\alpha}{b^{-2-\lambda}}. \tag{3.29}$$

In summary , the process of algebraic substitutions (i.e. $b\vec{\eta}\rightarrow\vec{\eta}$ and $b^{-\lambda}t\rightarrow t$), as done in Step 1, the (single scale) self-similarity assumption together with the scaling of the physical parameters (3.29) allow us transform the new interface equation back into its original form.

Since it is assumed that the system evolves in a self-similar manner with a single length scale, all physical quantities having units of length must grow at a rate proportional to a characteristic length, $R\left(t\right)$, that depends on $(t;\alpha,d_0)$. Hence, R must satisfy the same relationship as the length ξ does (see (3.24)) so that one has the following self-similarity relation

$$R\left(b^{-\lambda}t;\alpha,d_0\right)=bR\left(t;\alpha/b^{-2-\lambda},d_0/b^{3+\lambda}\right). \tag{3.30}$$

The equality (3.30) expresses the relation between the characteristic lengths of two systems with different parameters and at different times and also describes the necessary changes in the physical parameters, (α,d_0). The algebraic substitution $t=b^{\lambda}t_1$ into scaling equation (3.30) yields

$$R\left(t_1;\alpha,d_0\right)=bR\left(b^{\lambda}t_1;\alpha/b^{-2-\lambda},d_0/b^{3+\lambda}\right). \tag{3.31}$$

Step 4. Recalling that the calculations are valid for any $b>0$ and any real valued parameter λ that was to be determined (see Step 1). One then chooses $b=t_1^{-1/\lambda}$, (so that $b^{\lambda}t_1=1$), and rewrites the identity (3.31), omitting the subscript 1 on t_1, as

$$R\left(t;\alpha,d_0\right) = t^{-1/\lambda} R\left(1;\alpha/t^{(2+\lambda)/\lambda}, d_0/t^{-(3+\lambda)/\lambda}\right). \tag{3.32}$$

We now examine this new identity in terms of its implications for the parameter λ.

Analysis of the parameter λ. The value of λ clearly determines the long time asymptotics of the characteristic length, $R(t)$ assuming that R is not singular in d_0 as $d_0 \to 0$. Both the cases $\lambda < -3$ and $\lambda > 0$ lead to the result that d_0 approaches ∞ as $t \to \infty$. These, however, yield fixed points that are physically not meaningful. Similarly, if $\lambda \in (-2, 0)$, then α approaches ∞ for large t, while d_0 approaches 0 which yield also the nonphysical fixed points. Hence, any possible value for λ which yields the nontrivial fixed point lies in the interval $[-3, -2]$. This indicates that the characteristic length, $R(t)$, increases as $t^{-1/\lambda}$ as λ varies in the continuous spectrum $[-3, -2]$.

The result also confirms, once again, for $\lambda \in (-3, -2]$ that the capillarity length, d_0, is essentially irrelevant for large time, which is sharp contrast with its stabilizing role for short times (see references [43], [44] and [45]). The only exception is the value $\lambda = -3$ (i.e. $R(t) \sim t^{1/3}$) for which the capillarity length, d_0, is invariant for large time. In this case, the scaling does not depend on the non-singularity of R as a function of d_0.

3.3 The case $\alpha = 0$

In this section, we set $\alpha = 0$ in the equation (3.3) so that it becomes

$$T = \frac{-\sigma_0}{[s]_{eq}} \kappa \tag{3.33}$$

and examine the large time characteristic of the characteristic length, $R(t)$, corresponding to the system of equations (3.1), (3.2) and (3.33). Following Section 3.2.1 one rewrites the equations of the form

$$d_0 \kappa = \frac{1}{D} \int_{\Gamma(t)} d^{d-1}\sigma_y \; G(\vec{x} - \vec{y}) \upsilon_n(\vec{y}, t). \tag{3.34}$$

The RG analysis, following Section 3.2.2, yields the identity

$$R\left(t;\alpha,d_0\right)=t^{-1/\lambda}R\left(1;d_0/t^{-(3+\lambda)/\lambda}\right). \tag{3.35}$$

Analysis of the parameter λ, once again, determines the large time characteristic of the characteristic length, $R(t)$. Since we expect a positive growth, λ must be negative, i.e $\lambda<0$. On the other hand, if $\lambda<-3$, then d_0 approaches ∞ as $t\to\infty$ that leads to the physically irrelevant fixed points. The values of λ which are physically relevant stay in the continuous spectrum, $[-3,0)$. The single value of $\lambda=-1$ is selected from this spectrum by the plane wave imposed by Jasnow and Vinals [31].

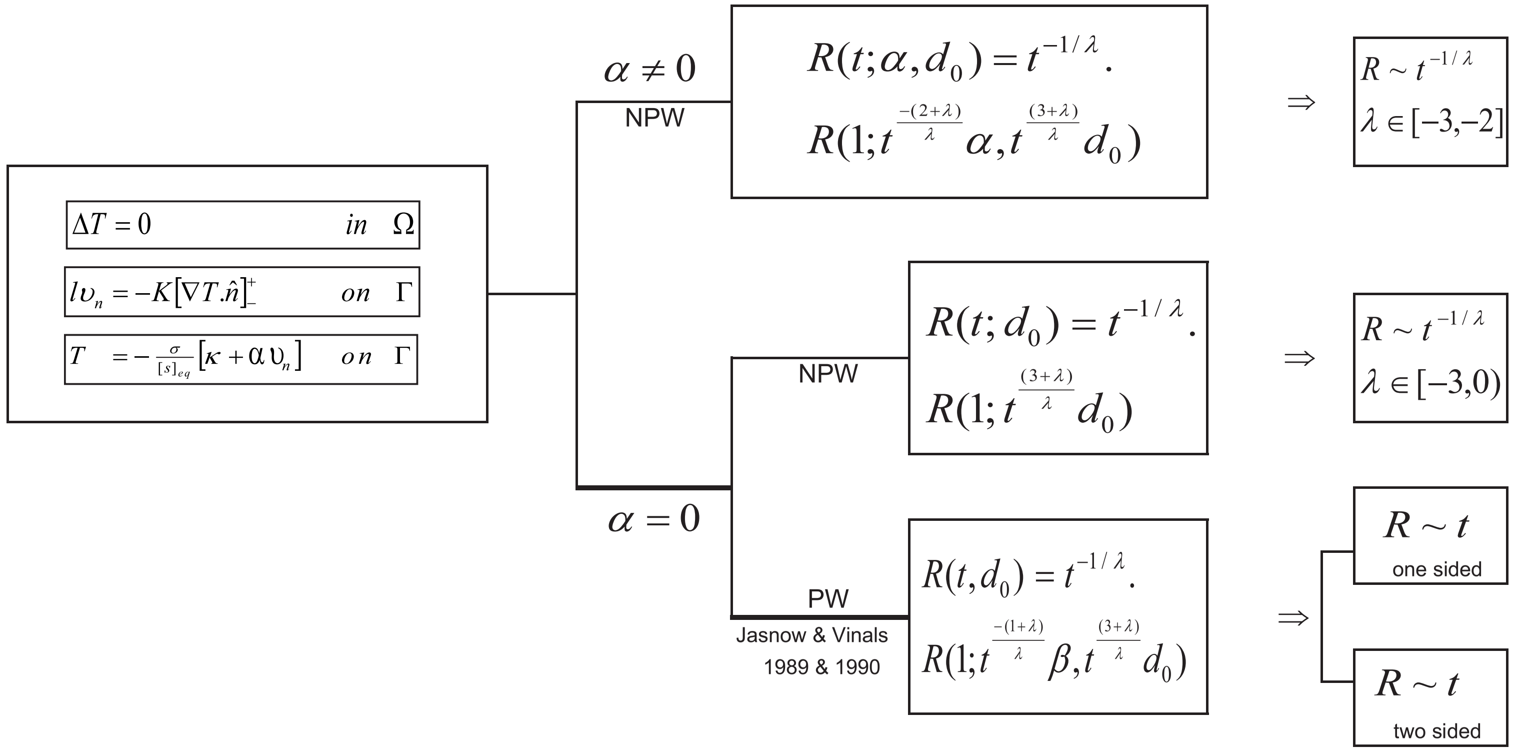
Similarly, the result also shows, for $\lambda\in(-3,0)$, that the capillarity length, d_0, is essentially irrelevant for large time. Once again, the exceptional case arises for the value of $\lambda=-3$ at which the capillarity length, d_0, is unchanged during the scaling of the large scale behavior. This yields $R(t)\sim t^{1/3}$ with no assumption of non-singularity of R. Moreover, the only scaling which d_0 is invariant has $t^{1/3}$ behavior.

3.4 Conclusions

We have performed a RG analysis for the large scale dynamical behavior of the full-two phase interface problem, defined by the system of equations (3.1)-(3.3) in the quasi-static regime. The calculations involve the implementation of renormalization group methods once a Green's function representation is introduced for the equations. Two cases were considered for the coefficient of the dynamical undercooling: $\alpha=0$ and $\alpha\neq0$. The latter condition includes the effect of a lower temperature on the interface that is associated with motion. We assume that the system evolves self-similarly with a single length scale and find that the characteristic length $R\left(t\right)$, evolves as $t^{-1/\lambda}$ without reference a plane wave. For the case $\alpha\neq0$ we find that a continuous spectrum of λ is possible, namely, $\lambda\in[-3,-2]$. The case $\alpha=0$ corresponds to the completely quasi-static case as the dynamical undercooling effect is suppressed. Here the single length scale self-similarity implies the continuous spectrum $\lambda\in[-3,0)$. When a particular plane wave is imposed [31] a single value, namely, $\lambda=-1$, is selected from this spectrum. The difference in the exponents of the characteristic length suggests that there is an important difference between the fully dynamic and static regimes.

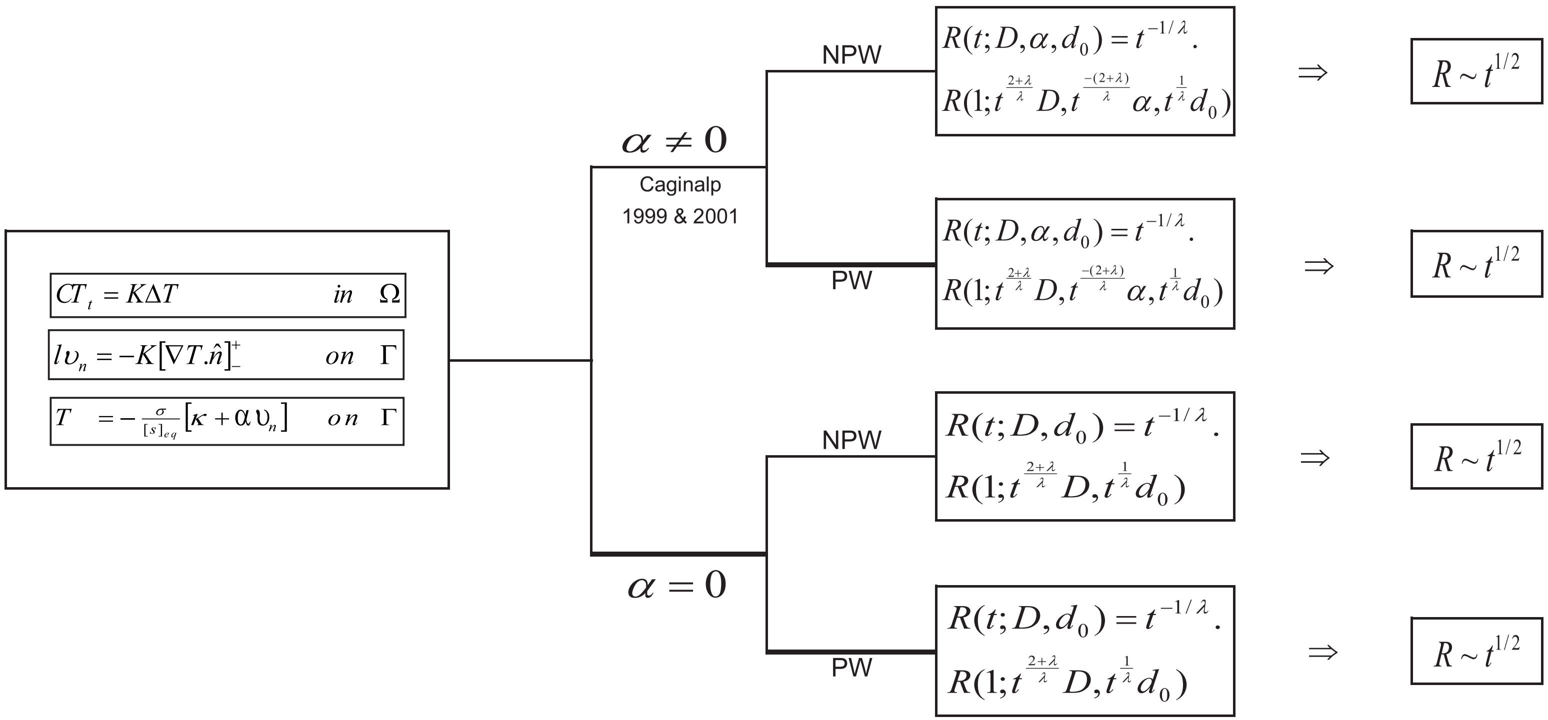
Another important conclusion resulting from this and prior works is that in almost all of these cases, the capillarity length, d_0, associated with the surface tension is irrelevant for the large time behavior. This is a very important consequence since we know that the capillarity length is a crucial factor for the initial velocity and the linear stability of an interface. An interesting question is how the role of the capillarity length changes from the early stage growth to the late stage growth.

The study of the late stage interface behavior using RG provides a complement to the systematic approach provided by linear stability theory. As methodology is developed for these two regimes, the most challenging problem may be the understanding of the transition between the short term and the long term asymptotics and crossover behaviors.



NPW : There is not a plane wave imposed　　　**PW** : There is a plane wave imposed.

Figure 3.2. The quasi-static model



NPW : There is not a plane wave imposed

PW : There is a plane wave imposed.

Figure 3.3. The fully dynamic model

Chapter 4

Conclusions, Discussion and Future Direction

The scaling and renormalization group (RG) methods originally introduced for statistical mechanics and quantum field theory can be adapted to applied mathematical problems such as random walk and fractals [17]. Application of these methods to differential equations is an important facet that can lead to an understanding of large time behavior in many applied problems.

Historically, RG methods were developed and understood in the context of equilibrium problems such as the divergence of exponents of physical measurables in statistical mechanics. The generalization of this methodology to dynamic problems would be of significance in a broad spectrum of applied mathematical problems. A focus of recent research has been the adapting of these methods in order to understand large time behavior as an asymptotic fixed point.

In this thesis we have implemented these methods in order the study two applied mathematics problems: decay of solutions to a class of nonlinear parabolic equations and the large time behavior of an unstable interface.

Chapter 2 has studied parabolic equations with a small nonlinear term and calculated the decay exponents adapting scaling and RG methods. The determination of decay exponents is viewed as an asymptotically self similar process that facilitates an RG approach. These RG methods are extended to higher order in the small coefficient of the nonlinearity. The RG results were verified in some cases by rigorous proofs and other calculational methods.

Chapter 3 has shown that the prototype sharp interface model (3.1)-(3.3) can be analyzed for large time behavior using RG methods (see Figure 3.2 and Figure 3.3). The use of RG techniques has led to a calculation of the characteristic length of an interface between two phases (with surface tension and kinetics) evolving self-similarly. One of the surprising conclusions that have arisen from almost all of these analyses is that the capillarity length associated with surface tension is essentially irrelevant for large time behavior of the interface. This differs significantly from its role for short time, where linear stability stipulates a smoothing and stabilizing effect due to larger capillarity lengths. It would be interesting to investigate quantitatively how the role role of the surface tension evolves from a crucial role at the initial stage to an irrelevant one at the late stage.

Many problems in materials science have been simplified through the use of quasi-static formalisms such as replacing the heat equation, $u_t = \Delta u$, by Laplace's equation, $\Delta u = 0$. In many cases this appears to be justified due to the very rapid heat conduction (particularly in metals such as aluminium) that leads to a very small u_t term shortly after the introduction of a constant heat source. These ideas have potential application to other materials science and applied mathematical problems in which the quasi-static approximation is of practical and/or theoretical importance.

The variety of models for phase transitions offers an opportunity to examine the connection between the static (for which most of renormalization group theory in physics has been developed) and the dynamic. Understanding renormalization group calculations for dynamic and static problems and their relationships remains a central issue for the theory as a whole.

Our RG analysis indicates that the large time behavior of the quasi static solution may differ significantly from the fully dynamical system. The different scaling regimes exhibited by the quasi-static and the dynamic pose an interesting question about the nature of the transition between the two regimes. This problem can be studied by using expansion techniques in conjunction with RG methods. The crux of the transition in the exponent of the large time behavior of the interface can be understood in terms of the distinguished limit of the Green's function for the elliptic equation through the fundamental solution to the parabolic equation.

The central aspect of the problem in mathematical terms is the relationship between the basic solutions of the parabolic problem to those of the elliptic problem (specifically in this case, the heat equation and Laplace's equation). In particular the key is to understand the relationship between the RG scalings of these formalisms.

As methodology is developed for these two regimes, the most challenging problem may be the understanding of the transition between the short term asymptotics that have been treated by linear stability theory and the long term asymptotics that have been studied through a RG approach and crossover behaviors.

Beyond interface problems, these RG methods offer the possibility to perform manageable calculations on complex systems of equations. Large scale computer calculations in three dimensions are still difficult for realistic systems of equations with many variables. The RG techniques can be utilized on such systems in a manner similar to those discussed in this thesis. Furthermore, there is an additional salient feature in that RG methodology yields answers that are very easy to interpret. The stochastic element involved in many large scale computations means that the resulting interface behavior could be difficult to reduce to simpler terms. In this way, the RG methods can be an important complement to large scale computations and linear stability in problems exhibiting self-similarity in either a stochastic or asymptotic form. The latter is the case for the decay, blow-up and extinction of solutions to nonlinear problems.

Appendix A

Theorem A.1. *Suppose that $F(x, u, p, q)$ is independent of q, that F/p^2 is smooth such that it and its first derivative are uniformly bounded. Then there exists a unique positive number, $\alpha(\varepsilon)$, such that the equation (2.4) has a solution of the form*

$$u(x, t, \varepsilon) = t^{-\alpha(\varepsilon)} e^{\phi(xt^{-1/2}, \varepsilon)}, \tag{A.1}$$

where α and ϕ have the limiting properties

$$\lim_{\varepsilon \to 0} \alpha(\varepsilon) = \frac{1}{2}, \qquad \lim_{\varepsilon \to 0} \phi(\xi, \varepsilon) = -\frac{1}{4}\xi^2. \tag{A.2}$$

Proof: We let Ψ be defined by

$$F(1, 1, p, q) = \Psi(p)p^2 \tag{A.3}$$

and assume that the Ψ and Ψ' are bounded by 1 in absolute value, since the larger constants can be absorbed into ε in (2.4). We use a shooting argument to prove the existence of a solution of self-similar type, i.e., (A.1 above). We look for even solutions ϕ, i.e., $\phi(-\xi) = \phi(\xi)$, and set $w(\xi) = \phi'(\xi)$. Hence it is equivalent to the equation,

$$w' + w^2 + \frac{1}{2}\xi w + \varepsilon F(\xi, 1, w, w' + w^2) + \alpha = 0 \tag{A.4}$$

for all $\xi \in (0, \Xi)$, where $(0, \Xi)$ is the maximal existence interval, subject to [see (2.175)]

$$w(0) = 0, \qquad \lim_{\xi \to \Xi} w = -\infty, \qquad \lim_{\xi \to \Xi} w e^{\int_0^\xi w(\eta) d\eta} = 0. \tag{A.5}$$

Proposition A.0.1. *For all $\alpha \in \mathbf{R}$ and $\xi \in [0, \Xi(\alpha))$ one has the inequality*

$$\frac{\partial w}{\partial \alpha} < 0. \tag{A.6}$$

Proof: The ODE for w is an initial value problem with smooth coefficients and so there is a unique solution for each α. Suppose that $\alpha_1 < \alpha_2$. We show that the corresponding solutions, w_1 and w_2, cannot intersect. Initially, the right hand side of (A.4) implies $w_1 > w_2$ since $w_1(0) = w_2(0) = 0$. Suppose for the purpose of contradiction that for some ξ_0 one has $w_1(\xi_0) = w_2(\xi_0)$. Then w_1 and w_2 satisfy the respective equations

$$w_1' = -\alpha_1 - w_1^2 (1 + \varepsilon \Psi(\xi_0 w_1) - \frac{1}{2} \xi w_1, \tag{A.7}$$

$$w_2' = -\alpha_2 - w_2^2 (1 + \varepsilon \Psi(\xi_0 w_2) - \frac{1}{2} \xi w_2. \tag{A.8}$$

Hence, $w_1'(\xi_0) - w_2'(\xi_0) = -\alpha_1 + \alpha_2 > 0$, so that w_1' dominates w_2' at this point so that w_1 cannot cross below w_2. We can write this inequality for any α and α^* as

$$\frac{w(\xi; \alpha) - w(\xi; \alpha^*)}{\alpha - \alpha^*} \leq 0, \tag{A.9}$$

so that taking the limit as $\alpha \to \alpha^*$ we obtain the bound for the derivative

$$\frac{\partial w}{\partial \alpha} = \lim_{\alpha \to \alpha^*} \frac{w(\xi; \alpha) - w(\xi; \alpha^*)}{\alpha - \alpha^*} \leq 0. \tag{A.10}$$

Proposition A.0.2. *If $\alpha > \frac{1}{2} + C^2 \varepsilon$ for sufficiently large $C \in \mathbf{R}^+$ then $\Xi(\alpha) < \infty$. Furthermore, $lim_{\xi \to \Xi(\alpha)} w e^{\int_0^\xi w(\eta) d\eta} = -1$.*

Proof: Suppose $\alpha \geq \frac{1}{2} + C^2 \varepsilon$ for some large $C \in \mathbf{R}^+$.. Let $z = -w$. Then for any ξ we can write

$$\left\{ z - \left(\frac{1}{2} + \frac{C^2}{2} \varepsilon \right) \xi \right\}' \geq \frac{C^2 \varepsilon}{2} + (1 - \varepsilon) z \left\{ z - \frac{1}{1 - \varepsilon} \frac{\xi}{2} \right\}. \tag{A.11}$$

Using $1 + C^2 \varepsilon > (1 - \varepsilon)^{-1}$ in the left hand side one can rewrite this as

$$\left\{ z - \frac{1}{1 - \varepsilon} \frac{\xi}{2} \right\}' \geq \frac{C^2}{2} \varepsilon + (1 - \varepsilon) z \left\{ z - \frac{1}{1 - \varepsilon} \frac{\xi}{2} \right\}. \tag{A.12}$$

Let $Z(\xi) := z - \frac{1}{1-\varepsilon}\frac{\xi}{2}$. Since the initial condition implies $Z(0) = 0$, then clearly Z is initially positive in the equation above. For comparison we consider the equation

$$Y' = \frac{C^2}{2}\varepsilon + (1-\varepsilon)Y^2 \tag{A.13}$$

which has solutions

$$Y(\xi) = C\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2} \tan\left\{C\sqrt{\varepsilon(1-\varepsilon)}(\xi + C_1)\right\}. \tag{A.14}$$

Since tan diverges for finite values of its argument, $Y(\xi)$ diverges for finite ξ. Comparing $Y(\xi)$ with $Z(\xi)$ for the same initial conditions we see that Z (and hence z) also diverge for finite ξ.

Proposition A.0.3. *If $\alpha < \frac{1}{2} - C^2\varepsilon$ for sufficiently large $C \in \mathbf{R}$ then $\Xi(\alpha) = \infty$ and $lim_{\xi\to\infty} w = 0$. Furthermore, one has $lim_{\xi\to\infty} we^{\int_0^\xi w(\eta)d\eta} = 0$.*

Proof: Again using $z = -w$ one has

$$\left(z - \frac{1}{2}\xi\right)' = -C^2\varepsilon + z(z - \frac{1}{2}\xi) + \varepsilon z^2\Psi(-\xi z). \tag{A.15}$$

As a consequence of the initial condition $z(0) = 0$ and the inequality $z'(0) = \frac{1}{2} - C^2\varepsilon > 0$ one has $z(\xi) > 0$ at least for some interval $(0, \xi_0)$. If at some point ξ_1, one has $z(\xi_1) = 0$ then the middle terms vanish and one obtains

$$z'(\xi_1) = \frac{1}{2} - C^2\varepsilon > 0. \tag{A.16}$$

Consequently, one has the result that $z(\xi) > 0$ for all ξ. Next, we prove that $z(\xi) < \xi/2$ for all ξ. We first prove that this is the case at least when $\xi < 2C$. Initially, $(z - \frac{1}{2}\xi) \mid_{\xi=0} = 0$ and $(z - \frac{1}{2}\xi)' < 0$ so that $z - \frac{1}{2}\xi < 0$ at least for some maximal interval $(0, \xi_1)$. Suppose that $z(\xi_1) = \frac{1}{2}\xi_1$. Then we have

$$(z - \frac{1}{2}\xi)' \mid_{\xi=\xi_1} = -C^2\varepsilon + \varepsilon(\frac{1}{2}\xi_1)^2\Psi < 0 \tag{A.17}$$

if $\xi_1 < 2C$. Consequently, $z - \frac{1}{2}\xi < 0$ on this interval.

Proposition A.0.4. *Suppose ξ_0 is such that $z(\xi_0) < (2\varepsilon)^{-\frac{1}{2}}$. Then one cannot have a neighborhood of ξ_0 such that in this neighborhood, $\xi < \xi_0$ implies $z'(\xi_0) < 0$ while $\xi > \xi_0$ implies $z'(\xi_0) > 0$.*

Proof: From (A.15) we have for all ξ,

$$z'' = z'(z - \frac{1}{2}\xi) + z(z' - \frac{1}{2}) + \\ 2\varepsilon z z' \Psi + \varepsilon z^2 \Psi'(-z - \xi z'). \tag{A.18}$$

If $z' = 0$ then this becomes

$$z'' = -\frac{1}{2}z - \varepsilon z^3 \Psi'. \tag{A.19}$$

Since $| \Psi' | \leq 1$ one has that $z'' < 0$ so long as $\varepsilon z^3 < z/2$, i.e.,

$$z < (2\varepsilon)^{-1/2}. \tag{A.20}$$

Hence, z' cannot change sign from negative to positive so long as z satisfies this inequality.

Proposition A.0.5. *At* $\xi = C/2$ *one has*

$$z(C/2) - \frac{1}{2}(C/2) \leq -(\frac{1}{2} - \frac{1}{96})C^3\varepsilon. \tag{A.21}$$

Proof: Consider the interval $0 \leq \xi \leq C/2$ in which $z < \xi/2$. Then the original ODE for z implies

$$(z - \frac{1}{2}\xi)' \leq -C^2\varepsilon + z(z - \frac{1}{2}\xi) + \varepsilon z^2 \\ \leq -C^2\varepsilon + \varepsilon z^2 \tag{A.22}$$

so that integrating this expression results in the inequality

$$\int_0^{C/2} (z - \frac{1}{2}\xi)' d\xi \leq \int_0^{C/2} \{-C^2\varepsilon + \varepsilon z^2\} \\ \leq -(\frac{1}{2} - \frac{1}{96})C^3\varepsilon \tag{A.23}$$

from which the conclusion follows.

In view of the estimates obtained above, we can write, for some small $\mid \gamma \mid$, the inequality

$$\begin{aligned}(z - \frac{1}{2}\xi)' &= -C^2\varepsilon + z\left\{(1+\varepsilon\Psi)z - \frac{1}{2}\xi\right\} \\ &\leq -C^2\varepsilon - \frac{\xi z}{2+\gamma}.\end{aligned} \tag{A.24}$$

We compare z satisfying this inequality with solutions of the equation for y below:

$$y' = C_0 - \frac{\xi y}{2+\gamma} \tag{A.25}$$

subject to the initial condition $y(C/2) := z(C/2)$ with $C_0 := \frac{1}{2} - C^2\varepsilon$. Solutions to this equation have the form

$$y(\xi) = C_0 e^{-\xi^2/(2+\gamma)^2} \int e^{s^2/(2+\gamma)^2} ds + C_1 e^{-\xi^2/(2+\gamma)^2}. \tag{A.26}$$

Note that both terms on the right hand side approach zero with the first term dominating as it diminishes as $1/\xi$. Hence one has the bound

$$0 < z(\xi) < y(\xi) \tag{A.27}$$

so that $z(\xi) \to 0$ as $\xi \to \infty$. By standard degree theory arguments we obtain the conclusion that for some $\alpha = \alpha(\varepsilon)$ satisfies

$$\frac{1}{2} - C^2\varepsilon < \alpha(\varepsilon) < \frac{1}{2} + C^2\varepsilon \tag{A.28}$$

and the boundary conditions. The conclusion of Theorem A.1 follows.

Theorem A.1 thus proves rigorously (for the subset of nonlinearities defined by the hypothesis) the RG calculations of the decay exponents. The results are compatible with the decay exponents obtained using RG.

Appendix B

Proof of lemmas in section 2.2.2

Proof of Lemma 2.2.4. We first find an upper and a lower bound for the case $t \le l^2$. We first set $z := s/l^2$. Using (2.18) and (2.21) we write (2.54) as

$$\begin{aligned} L_{3,1} &< \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{C_{2,1}^{(2)} z^{p+1} + C_{2,2}^{(1)} z^p\} dz \\ &= \frac{C_{2,1}^{(2)}}{2p+1} \left(\frac{t}{l^2}\right)^{2p+1} + \frac{C_{2,2}^{(1)}}{2p} \left(\frac{t}{l^2}\right)^{2p} \end{aligned} \tag{B.1}$$

since $(1+z)^{-p} \le 1$. Similarly, using (2.19), (2.22), and also the inequality $(1+z)^{-p} \ge 2^{-p}$ for $z \le 1$ we obtain the following lower bound

$$\begin{aligned} L_{3,1} &\ge \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{C_{2,1}^{(1)} z^{p+1}\} dz \\ &= \frac{C_{2,1}^{(1)}}{2^p(2p+1)} \left(\frac{t}{l^2}\right)^{2p+1}. \end{aligned} \tag{B.2}$$

For the case $t > l^2$, we first split the integral (2.54) into two parts as follows:

$$\begin{aligned} L_{3,1} &:= \int_0^{l^2} (s+l^2)^{-p} s^{p-1} \{ L_{2,1} + L_{2,2}\} ds + \int_{l^2}^{t} (s+l^2)^{-p} s^{p-1} \{L_{2,1} + L_{2,2}\} ds \\ &=: L_{3,1}^{(1)} + L_{3,1}^{(2)} \end{aligned} \tag{B.3}$$

so that one has, from (B.1) and (B.2),

$$\frac{C_{2,1}^{(1)}}{2^p(2p+1)} \le L_{3,1}^{(1)} < \frac{C_{2,1}^{(2)}}{2p+1} + \frac{C_{2,2}^{(1)}}{2p}. \tag{B.4}$$

To find an upper bound and a lower bound for $\mathrm{L}_{3,1}^{(2)}$ we apply Lemma 2.2.2 and Lemma 2.2.3 so that $\mathrm{L}_{3,1}^{(2)}$ is bounded above by

$$L_{3,1}^{(2)} < \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \{\frac{1}{2!}[\log(1+z)]^2\} dz + (C_{2,1}^{(4)} + C_{2,2}^{(2)}) \int_1^{t/l^2} (1+z)^{-p} z^{p-1} dz$$
$$= \frac{1}{3!}\left[\log\left(\frac{t+l^2}{l^2}\right)\right]^3 + (C_{2,1}^{(4)} + C_{2,2}^{(2)})\left[\log\left(\frac{t+l^2}{l^2}\right)\right] + C(p) \quad \text{(B.5)}$$

by using the inequality $z^{p-1} \leq (1+z)^{p-1}$ for $z > 1$. Similarly, utilizing (2.47) and (2.48) $\mathrm{L}_{3,1}^{(2)}$ is bounded below by

$$L_{3,1}^{(2)} > \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \{\frac{1}{2!}[\log(z)]^2\} dz + C_{2,1}^{(3)} \int_1^{t/l^2} (1+z)^{-p} z^{p-1} dz$$
$$= \frac{1}{3!}\left[\log\left(\frac{t}{l^2}\right)\right]^3 + C_{2,1}^{(3)}\left[\log\left(\frac{t}{l^2}\right)\right] + C(p), \quad \text{(B.6)}$$

where $C\,(p)$ is a constant depending on p. □

Proof of Lemma 2.2.5. Following the previous proof, for the case $t \leq l^2$ one can show

$$L_{3,2} < \int_0^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \{C_{2,1}^{(2)} z^{p+1} + C_{2,2}^{(1)} z^p\} dz$$
$$= \frac{\lambda_1 C_{2,1}^{(2)}}{2p}\left(\frac{t}{l^2}\right)^{2p} + \frac{\lambda_1 C_{2,2}^{(1)}}{2p-1}\left(\frac{t}{l^2}\right)^{2p-1} \quad \text{(B.7)}$$

and

$$L_{3,2} \geq \int_0^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \{C_{2,1}^{(1)} z^{p+1}\} dz$$
$$= \frac{\lambda_1 C_{2,1}^{(1)}}{2^{p+1} p}\left(\frac{t}{l^2}\right)^{2p} \geq 0. \quad \text{(B.8)}$$

Similarly, for the case $t > l^2$ one first splits the integral (2.57) into two parts as follows:

$$L_{3,2} := \int_0^{l^2} \lambda_1 (s+l^2)^{-p} s^{p-2} \{L_{2,1} + L_{2,2}\} ds + \int_{l^2}^{t} \lambda_1 (s+l^2)^{-p} s^{p-2} \{\, L_{2,1} + L_{2,2}\} ds$$
$$=: L_{3,2}^{(1)} + L_{3,2}^{(2)}. \quad \text{(B.9)}$$

A similar procedure above yields the following bounds:

$$\frac{\lambda_1 C_{2,1}^{(1)}}{2^{p+1}p} \leq L_{3,2}^{(1)} < \frac{\lambda_1 C_{2,1}^{(2)}}{2p} + \frac{\lambda_1 C_{2,2}^{(1)}}{2p-1}, \tag{B.10}$$

$$\begin{aligned} L_{3,2}^{(2)} &< \lambda_1 \int_1^{t/l^2} z^{-2}\{\frac{1}{2!}[\log(1+z)]^2\}dz \\ &\quad +(C_{2,1}^{(4)} + C_{2,2}^{(2)})\lambda_1 \int_1^{t/l^2} z^{-2}dz \\ &< \text{constant} \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} L_{3,2}^{(2)} &> \int_1^{t/l^2} \lambda_1(1+z)^{-p}z^{p-2}\{\frac{1}{2!}[\log(z)]^2\}dz \\ &\quad +C_{2,1}^{(3)} \int_1^{t/l^2} \lambda_1(1+z)^{-p}z^{p-2}dz \\ &\geq C(p) \end{aligned} \tag{B.12}$$

where $C(p)$ is a constant depending on p. □

Proof of Lemma 2.2.6. We first write (2.60) as

$$L_k := \int_0^{t/l^2} \lambda_n(1+z)^{-p+(\frac{n-1}{2})}z^{p-(\frac{3n+1}{2})}[\log(1+z)]^n dz. \tag{B.13}$$

We use (2.38) with the inequality $(1+z)^{-p+(\frac{n-1}{2})} \leq 1$ (since $-p+\frac{n}{2}-\frac{1}{2} < -p+n-\frac{1}{2} < 0$ for $p \geq 2$, $n \geq 2$, $p \geq n$ and $n, p \in Z^+$) so that for $t \leq l^2$ we have the following bounds for L_k

$$0 \leq L_k < \int_0^{t/l^2} \lambda_n z^{p-(\frac{3n+1}{2})}z^n dz = \frac{\lambda_n}{p-\frac{n}{2}+\frac{1}{2}}\left(\frac{t}{l^2}\right)^{p-\frac{n}{2}+\frac{1}{2}}. \tag{B.14}$$

Note that $p-(\frac{3n+1}{2})+n = \frac{p}{2}+\frac{p}{2}-\frac{n}{2}-\frac{1}{2} > 0$ for $p \geq 2$, $n \geq 2$, $p \geq n$ and n, $p \in Z^+$. Thus, for the large t/l^2 we first split the integral into two parts as

$$\begin{aligned} L_k &= \int_0^1 \lambda_n (1+z)^{-p+(\frac{n-1}{2})} z^{p-(\frac{3n+1}{2})} [\log(1+z)]^n dz \\ &\quad + \int_1^{t/l^2} \lambda_n (1+z)^{-p+(\frac{n-1}{2})} z^{p-(\frac{3n+1}{2})} [\log(1+z)]^n dz \\ &=: L_k^{(1)} + L_k^{(2)} \end{aligned} \tag{B.15}$$

so that utilizing (B.4) we have

$$0 \le L_k^{(1)} < \frac{\lambda_n}{p - \frac{n}{2} + \frac{1}{2}}. \tag{B.16}$$

For the second part of the integral, namely $L_k^{(2)}$, we use the inequality $(1+z)^{-p+(\frac{n-1}{2})} < z^{-p+(\frac{n-1}{2})}$ in order to obtain the following upper bound

$$\begin{aligned} 0 < L_k^{(2)} &= \int_1^{t/l^2} z^{-n-1} [\log(1+z)]^n dz \\ &< \int_1^{\infty} z^{-n-1} [\log(1+z)]^n dz < \text{constant}. \end{aligned} \tag{B.17}$$

□

Appendix C

Proof of lemmas in section 2.2.3

Proof of Lemma 2.2.7.

Upper and lower bounds for $t \leq l^2$: Using the induction hypothesis with together (2.41) and the inequality $(1+z)^{-p} \leq 1$ we write (2.85) as

$$\begin{aligned} L_{k,1} &< \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{C^{(2)}_{k-1,1} z^{(k-2)p+1} + O(z)\} dz \\ &= \frac{C^{(2)}_{k-1,1}}{(k-1)p+1} \left(\frac{t}{l^2}\right)^{(k-1)p+1} + O\left(\frac{t}{l^2}\right). \end{aligned} \tag{C.1}$$

Similarly, using the induction hypothesis with together (2.41) and the inequality $(1+z)^{-p} \geq 2^{-p}$ (for $z \leq 1$) one obtains the following lower bound:

$$\begin{aligned} L_{k,1} &\geq \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{C^{(1)}_{k-1,1} z^{(k-2)p+1}\} dz \\ &= \frac{C^{(1)}_{k-1,1}}{2^p((k-1)p+1)} \left(\frac{t}{l^2}\right)^{(k-1)p+1}. \end{aligned} \tag{C.2}$$

Upper and lower bounds for $t > l^2$: As done before, we first split the integral in (2.85) into two parts as follows:

$$\begin{aligned} L_{k,1} &:= \int_0^{l^2} (s+l^2)^{-p} s^{p-1} \{L_{k-1,1} + L_{k-1,2}\} ds + \int_{l^2}^{t} (s+l^2)^{-p} s^{p-1} \{L_{k-1,1} + L_{k-1,2}\} ds \\ &=: L^{(1)}_{k,1} + L^{(2)}_{k,1} \end{aligned} \tag{C.3}$$

so that using first (2.41) and then using (C.1) with together (C.2), we have

$$\frac{C^{(1)}_{k-1,1}}{2^p((k-1)p+1)} \leq L^{(1)}_{k,1} < \frac{C^{(2)}_{k-1,1}}{2^p((k-1)p+1)}. \tag{C.4}$$

To obtain an upper bound and a lower bound for $\mathrm{L}_{k,1}^{(2)}$, one can use induction hypothesis with together (2.41) again so that $\mathrm{L}_{k,1}^{(2)}$ is bounded above by

$$\begin{aligned} L_{k,1}^{(2)} &< \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \left\{ \frac{1}{(k-1)!} [\log(1+z)]^{k-1} + O(\log(1+z) \right\} dz \\ &= \frac{1}{k!} \left[\log\left(\frac{t+l^2}{l^2}\right)\right]^k + O\left(\log\left(\frac{t+l^2}{l^2}\right)\right) \end{aligned} \tag{C.5}$$

by using the inequality $z^{p-1} \leq (1+z)^{p-1}$and is bounded below by

$$\begin{aligned} L_{k,1}^{(2)} &> \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \left\{ \frac{1}{(k-1)!} [\log(z)]^{k-1} + O(\log(z)) \right\} dz \\ &= \frac{1}{k!} \left[\log\left(\frac{t}{l^2}\right)\right]^k + O\left(\log\left(\frac{t}{l^2}\right)\right) \end{aligned} \tag{C.6}$$

by using (2.47) and (2.48). □

Proof of Lemma 2.2.8.

Following the proof of Lemma 2.2.7, one has, for $t \leq l^2$, the following bounds:

$$\begin{aligned} L_{k,2} &< \int_0^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \{ C_{k-1,1}^{(2)} z^{(k-2)p+1} + O(z) \} dz \\ &= \frac{\lambda_1 C_{k-1,1}^{(2)}}{(k-1)p} \left(\frac{t}{l^2}\right)^{(k-1)p} + O\left(\frac{t}{l^2}\right) \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} L_{k,2} &\geq \int_0^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \{ C_{k-1,1}^{(1)} z^{(k-2)p+1} \} dz \\ &= \frac{\lambda_1 C_{k-1,1}^{(1)}}{2^p (k-1)p} \left(\frac{t}{l^2}\right)^{(k-1)p} \geq 0. \end{aligned} \tag{C.8}$$

In the case $t > l^2$, one first splits the integral in (2.88) into two parts as follow:

$$\begin{aligned} L_{k,2} \quad &:= \quad \int_0^{l^2} \lambda_1 (s+l^2)^{-p} s^{p-2} \{ L_{k-1,1} + L_{k-1,2} \} ds \\ &\quad + \int_{l^2}^{t} \lambda_1 (s+l^2)^{-p} s^{p-2} \{ L_{k-1,1} + L_{k-1,2} \} ds \\ &=: \quad L_{k,2}^{(1)} + L_{k,2}^{(2)} \end{aligned} \tag{C.9}$$

so that, following the same procedure as done before, one has

$$0 \leq L_{k,2}^{(1)} < \frac{\lambda_1 C_{k-1,1}^{(2)}}{(k-1)p}, \tag{C.10}$$

$$\begin{aligned} L_{k,2}^{(2)} &< \int_1^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \left\{ \frac{1}{(k-1)!} [\log(1+z)]^{k-1} + O(\log(1+z) \right\} dz \\ &< \lambda_1 \int_1^{\infty} z^{-2} \left\{ \frac{1}{(k-1)!} [\log(1+z)]^{k-1} + O(\log(1+z) \right\} \\ &< \text{constant} \end{aligned} \tag{C.11}$$

and

$$\begin{aligned} L_{k,2}^{(2)} &> \int_1^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \left\{ \frac{1}{(k-1)!} [\log(z)]^{k-1} + O(\log(z)) \right\} \\ &\geq C(p,k) \end{aligned} \tag{C.12}$$

where C is a constant depending on p and k. □

Proof of Lemma 2.2.9.

Upper and lower bounds for $t \leq l^2$: Using induction hypothesis and (2.41), $L_{k,3}$ is bounded by

$$\begin{aligned} 0 \leq L_{k,3} &< \int_0^{t/l^2} (1+z)^{-p} z^{p-1} \{ C_{k-1,3}^{(1)} z^{(k-3)p-\frac{1}{2}} + O(z) \} dz \\ &= \frac{C_{k-1,3}^{(1)}}{(k-2)p - \frac{1}{2}} \left(\frac{t}{l^2} \right)^{(k-2)p-\frac{1}{2}} + O\left(\frac{t}{l^2} \right) \end{aligned} \tag{C.13}$$

since $(1+z)^{-p} \leq 1$.

Upper and lower bounds for $t > l^2$: As in the proof of Lemma 2.2.7, we first split the integral (2.91) into two parts as follow

$$\begin{aligned} L_{k,3} &:= \int_0^{l^2} (s+l^2)^{-p} s^{p-1} \{ L_{k-1,3} + L_{k-1,4} + \widehat{L}_{k-1} \} ds \\ &\quad + \int_{l^2}^{t} (s+l^2)^{-p} s^{p-1} \{ L_{k-1,3} + L_{k-1,4} + \widehat{L}_{k-1} \} ds \\ &=: L_{k,3}^{(1)} + L_{k,3}^{(2)} \end{aligned} \tag{C.14}$$

so that using (2.41) and (C.13) we have

$$0 \leq L_{k,3}^{(1)} < \frac{C_{k-1,3}^{(1)}}{(k-2)p - \frac{1}{2}}. \tag{C.15}$$

Using again the induction hypothesis with together (2.41) one can show that $\mathrm{L}_{k,3}^{(2)}$ is bounded by

$$\begin{aligned} 0 < L_{k,3}^{(2)} &< \int_1^{t/l^2} (1+z)^{-p} z^{p-1} \left\{ C_{k-1,3}^{(1)} [\log(1+z)]^{k-4} + O(\log(1+z) \right\} dz \\ &= \frac{C_{k-1,3}^{(1)}}{(k-3)!} \left[\log\left(\frac{t+l^2}{l^2} \right) \right]^{k-3} + O\left(\log\left(\frac{t+l^2}{l^2} \right) \right) \end{aligned} \tag{C.16}$$

by using the inequality $z^{p-1} \leq (1+z)^{p-1}$for $z > 1$. □

Proof of Lemma 2.2.10.

Following a similar procedure one has, for $t \leq l^2$,

$$\begin{aligned} 0 \leq L_{k,4} &< \int_0^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \{ C_{k-1,3}^{(1)} z^{(k-3)p - \frac{1}{2}} + O(z) \} dz \\ &= \frac{\lambda_1 C_{k-1,3}^{(1)}}{(k-2)p - \frac{3}{2}} \left(\frac{t}{l^2} \right)^{(k-2)p - \frac{3}{2}} + O\left(\frac{t}{l^2} \right). \end{aligned} \tag{C.17}$$

For the large t/l^2, one first splits the integral in (2.94) into two parts as follow

$$\begin{aligned} L_{k,4} &:= \int_0^{l^2} \lambda_1 (s+l^2)^{-p} s^{p-2} \{ L_{k-1,3} + L_{k-1,4} + \widehat{L}_{k-1} \} ds \\ &\quad + \int_{l^2}^{t} \lambda_1 (s+l^2)^{-p} s^{p-2} \{ L_{k-1,3} + L_{k-1,4} + \widehat{L}_{k-1} \} ds \\ &=: L_{k,4}^{(1)} + L_{k,4}^{(2)} \end{aligned} \tag{C.18}$$

and one has

$$0 \leq L_{k,4}^{(1)} < \frac{\lambda_1 C_{k-1,3}^{(1)}}{((k-2)p - \frac{3}{2})}, \tag{C.19}$$

and

$$
\begin{aligned}
0 < L_{k,4}^{(2)} &< \int_1^{t/l^2} \lambda_1 (1+z)^{-p} z^{p-2} \left\{ C_{k-1,3}^{(1)} [\log(1+z)]^{k-4} + O(\log(1+z) \right\} dz \\
&= \lambda_1 \int_1^{\infty} z^{-2} \left\{ C_{k-1,3}^{(1)} [\log(1+z)]^{k-4} + O(\log(1+z) \right\} dz \qquad \text{(C.20)} \\
&\leq \text{constant.}
\end{aligned}
$$

□

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