# TURBULENCE MODELING AS AN ILL-POSED PROBLEM

by

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This thesis is concerned with the derivation and mathematical analysis of new turbulence models, based on methods for solving ill-posed problems.

Turbulence causes the formation of eddies of many different length scales. Small, unresolved scales have deterministic roles in the statistics of the resolved scales. The main problem of computational turbulence is to accurately represent the effect of the unknown small scales upon the observable large scales. This is really just another ill-posed problem and the work in this thesis shows that excellent turbulence models do come from standard methods for ill-posed problems. Large Eddy Simulation (LES) exploits this decoupling of scales in a turbulent flow: the larger unsteady turbulent motions are directly represented, while the effects of the smaller scale motions are modeled. This is achieved by introducing a filtering operation, which depends on a chosen averaging radius. Once an averaging radius and a filtering process is selected, an LES model can be developed and then solved numerically. One of the most interesting approaches to generate LES models is via approximate deconvolution or approximate/asymptotic inverse of the filtering operator.

Herein, we develop an abstract approach to modeling the motion of large eddies in a turbulent flow and postulate conditions on a general deconvolution operator that guarantee the existence and uniqueness of strong solutions of Approximate Deconvolution Models. We also introduce new deconvolution operators which fit in this abstract theory. The Accelerated van Cittert algorithm and the Tikhonov regularization process are two methods for solving ill-posed problems that we adapt to turbulence. We study the mathematical properties of the resulting deconvolution operators.

We also study a new family of turbulence models, the Leray-Tikhonov deconvolution models, which is based on a modification (consistent with the large scales) of the Tikhonov regularization process. We perform rigorous numerical analysis of a computational attractive algorithm for the considered family of models. Numerical experiments that support our theoretical results are presented.

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*To my Mom and Dad*

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# 1.0 INTRODUCTION

### 1.0.1 Turbulence Ahead!

Turbulence is everywhere and all the time. Turbulence and fluid motion play an important role in our existence. Basically, almost every aspect of our everyday life (work, relaxation, body care, nourishment) involves fluid motion in one way or another. Air flows into our lungs and blood moves through our vessels in our bodies. Equally important for our existence is the fluid motion in many engineering applications. The complex flow in furnaces, chemical reactors, heat exchangers, as well as fluid flows around cars and airplanes are turbulent. An understanding of turbulence is also necessary to understand (and predict) meteorological phenomena or environmental hazards. All these are just very few examples of how turbulence affects our life, makes it possible, easier or sometimes harder. Because of these broad range of applications, turbulence has been the object of study of many great scientists. Unfortunately (or fortunately), the study of turbulence proved to be equally difficult and challenging as it is important.

But, what exactly is turbulence? While the above examples from our everyday life may be illuminating, scientifically there is not a broadly accepted definition of turbulence. However, the literature agrees with the description of turbulence by listing its characteristic features. Between them, chaotic and irregular are the keywords, [5]. From mathematical point of view, the Navier-Stokes equations (NSE) probably contain all of turbulence. Derived directly from Newton's laws of motion, the NSE, defined precisely below, are considered an exact model for the flow of a viscous, incompressible fluid, [24]. To this point, except for a few simple flows, an analytic solution of NSE is not known. Thus, the theory of NSE is also of great interest in a purely mathematical sense. The Clay Mathematics Institute considers the NSE to be one of the seven most important open problems in mathematics. *"Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations."*

Computational Fluid Dynamics (CFD) aims to provide both a qualitative and quantitative prediction of fluid flows. In CFD, computers are used to solve the NSE (or simplified version of the NSE) and perform all the calculations (often millions) that are necessary to simulate the fluid flow. In many cases, only approximate solutions can be achieved. "*It must be admitted that the problems are too vast to solve by a direct computational attack!*" These are the words of J. von Neumann and, 60 years later, they are still so very true. They also provide a strong motivation for the development of turbulence models.

The challenge in turbulence modeling is to accurately predict averages of flow variables (velocity and pressure) and obtain tractable theory that can be used to calculate quantities of practical importance, [57]. A *good* turbulence model is one which:

- has a mathematical justification: its solution will be close in some sense to a solution of NSE.
- is inexpensive: since numerical methods are used eventually to solve numerically the model, this process should be doable and require reasonable computer time and power.
- has a large range of applicability: a model is applicable to a flow if the model equations are well-posed.
- is accurate: this is the most desirable attribute of any model and it is determined by small discrepancies in the boundary conditions and small (numerical, measurement) errors.

Large Eddy Simulation (LES) has emerged as one of the most promising approaches in simulation of turbulent flows. Even though LES is a highly developed field in the engineering and geophysics communities, its mathematical foundations are still to be strengthened. Compared to Direct Numerical Simulations (DNS, which the direct approach in solving numerically the NSE), LES is less expensive and the results are not much worse. In fact, LES is considered a "logical compromise" by providing accurate solutions at affordable cost.

In LES, the larger unsteady turbulent motions are directly represented, while the effects of the smaller scale motions are modeled. This is achieved by applying a space filtering/averaging operation to the NSE. Typically, these averages are defined by convolution, $\overline{\mathbf{u}}(\mathbf{x},t) := (g_\delta \star \mathbf{u})(\mathbf{x},t)$, or through the differential filter $\overline{\mathbf{u}} := (-\delta^2\triangle + 1)^{-1}\mathbf{u}$. The famous closure problem in LES arises because the average of a product is not the product of the averages, i.e. $\overline{\mathbf{u}\mathbf{u}} \neq \overline{\mathbf{u}}\ \overline{\mathbf{u}}$. Solving the closure problem means finding useful approximations of $\mathbf{u}$. One way to find such approximation is via deconvolution and is developed in detail in the body of this thesis.

### 1.0.2 The Navier-Stokes Equations

The Navier-Stokes equations (NSE) are the "governing laws" of turbulence. The NSE describe the motion of *any* incompressible, newtonian fluid in a bounded domain. Let the velocity $\mathbf{u}(\mathbf{x},t)=u_j(x_1,x_2,x_3,t)$, (j=1,2,3) and pressure $p(\mathbf{x},t)=p(x_1,x_2,x_3,t)$ be solutions of the underlying NSE:

$$\begin{aligned} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\triangle\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \end{aligned} \tag{1.0.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\mathbf{f}$ is the body force, and $\Omega$ is a bounded and regular flow domain in $\mathbb{R}^n$ ($n = 2$ or 3). The NSE are supplemented by the initial condition and the usual pressure normalization condition (to ensure uniqueness of pressure)

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ and } \int_\Omega p\, d\mathbf{x} = 0.$$

The Reynolds number, $Re$, is the inverse of the viscosity and it represents the only control parameter that makes the difference between laminar flows (low $Re$) and turbulent flows (high $Re$). A beautiful and complete mathematical description of the NSE is presented in Galdi [24].

### 1.0.3 Is Turbulence just another ill-posed problem?

Turbulence causes the formation of eddies of many different length scales. Small, unresolved scales have deterministic roles in the statistics of the resolved scales. The main problem of computational turbulence is to accurately represent the effect of the unknown small scales upon the observable large scales. This is really just another ill-posed problem and excellent turbulence models do come from standard methods for ill-posed problems. However the challenges of turbulence are complex and an understanding of Fluid Mechanics is still indispensable in validating analytically model prediction.

LES exploits this decoupling of scales. One of the most interesting approaches to generate LES models is via approximate deconvolution or approximate/asymptotic inverse of the filtering operator. Approximate Deconvolution Models (ADM) have remarkable mathematical properties and perform well in computations. The approximate deconvolution framework can be thought of as viewing turbulence as an ill-posed problem and adapting to flow problems the extensive parallel development of methods for the approximate solution of ill-posed problems.

The forward problem is: *given* $\mathbf{u}$ *find* $\overline{\mathbf{u}}$. This is a well-posed problem. The inverse is *given* $\overline{\mathbf{u}}$ *find* $\mathbf{u}$. This is the ill-posed deconvolution problem. An approximate deconvolution operator $D$ is a bounded linear operator which is an approximate filter inverse

$$D(\overline{\mathbf{u}}) = \text{ approximation of } \mathbf{u},$$

where $\overline{\mathbf{u}}$ represents the filtered quantity of $\mathbf{u}$. This is an ill posed problem, for which there are many methods. Any deconvolution operator yields a closure approximation to the filtered nonlinear term in the NSE by

$$\overline{\mathbf{u}\mathbf{u}} \simeq \overline{D(\overline{\mathbf{u}})D(\overline{\mathbf{u}})}.$$

This closure approximation results in a large eddy structure whose solutions are intended to approximate the true flow averages, $\mathbf{w} \approx \overline{\mathbf{u}}$ and $q \approx \overline{p}$.

In Chapter 2, we present an abstract approach to modeling the motion of large eddies in a turbulent flow and answer the question: *Which methods yield a good model of turbulence?* Developing this idea, we adapt methods for ill-posed to turbulence to obtain new, more accurate turbulence models. In Chapters 3 and 4 we study two new deconvolution operators. The Accelerated van Cittert algorithm and the Tikhonov regularization process are two methods for solving ill-posed problems that we adapt to turbulence. We study the mathematical properties of the resulting deconvolution operators.

### 1.0.4 Consistency Error of Turbulence Models

In general, suppose $\mathbf{u}$ satisfies $N_{true}(\mathbf{u}) = \mathbf{f}$ and $\mathbf{w}$, an approximation to $\overline{\mathbf{u}}$, satisfies the approximate reduced model

$$N_{Reduced}(\mathbf{w}) = \overline{\mathbf{f}}. \tag{1.0.2}$$

The true equation can be rewritten as $\overline{N_{true}(\mathbf{u})} = \overline{\mathbf{f}}$ or

$$N_{Reduced}(\overline{\mathbf{u}}) = \overline{\mathbf{f}} - \left[\overline{N_{true}(\mathbf{u})} - N_{Reduced}(\overline{\mathbf{u}})\right]. \tag{1.0.3}$$

If the bracketed term on the RHS is exactly zero, this reduces to (1.0.2). Thus, the size of the bracketed term is a rough measure of the deviation of the solution of (1.0.2) from the flow averages.

**Definition 1.0.1.** *The modeling error is $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$ while the reduced model's consistency error or residual stress is the residual of $\overline{\mathbf{u}}$ in the approximate reduced model:*

$$\boldsymbol{\tau}(\mathbf{u}) = \overline{N_{true}(\mathbf{u})} - N_{Reduced}(\overline{\mathbf{u}}).$$

We wish to have small errors $(\overline{\mathbf{u}} - \mathbf{w})$, but can only act on residuals $\boldsymbol{\tau}(\mathbf{u})$ in deriving models. The coupling between errors and residuals is thus central. Comparing (1.0.3) to (1.0.2), the *deviation* of $\overline{\mathbf{u}}$ from $\mathbf{w}$ is driven by the consistency error/residual stress $\boldsymbol{\tau}(\mathbf{u})$. If an appropriate setting is selected for (1.0.2) and (1.0.3) under which the operators involved are $C^1$ and $N'$ is the Fréchet derivative of $N$, the error $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$ satisfies

$$\int_0^1 N'_{Reduced}(s\overline{\mathbf{u}} + (1-s)\mathbf{w})\mathbf{e}\; ds = \boldsymbol{\tau}(\mathbf{u}).$$

The error is thus driven by the turbulence model's consistency error and the error's size is related to the stability properties of the linearization of the reduced model. From either point of view, a small modeling error depends on a reduced model (1.0.2) with

(i) small consistency error, and

(ii)a sufficiently stable linearization.

When this framework is specialized to LES models of turbulence, the consistency error is often called the *residual stress*, [47], and, for LES-ADM, is derived next.

Given an approximate deconvolution operator, the associated *base* ADM is

$$\mathbf{w}_t + \nabla \cdot \overline{D(\mathbf{w})\ D(\mathbf{w})} - \nu \triangle \mathbf{w} + \nabla q = \overline{\mathbf{f}} \text{ and } \nabla \cdot \mathbf{w} = 0. \tag{1.0.4}$$

The model's error $\overline{\mathbf{u}} - \mathbf{w}$ is driven by the error in the deconvolution process itself. Indeed, the exact SFNSE can be rewritten as:

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{D(\overline{\mathbf{u}})\ D(\overline{\mathbf{u}})}) - \nu \triangle \overline{\mathbf{u}} + \nabla \overline{p} = \overline{\mathbf{f}} + \overline{\nabla \cdot \boldsymbol{\tau}}. \tag{1.0.5}$$

**Definition 1.0.2.** *The error in the model (1.0.4) is* $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$. *The consistency error of this model,* $\boldsymbol{\tau}(\mathbf{u})$*, and the deconvolution error,* $\mathbf{e}(\mathbf{u})$*, are defined as:*

$$\boldsymbol{\tau}(\mathbf{u}) = D(\overline{\mathbf{u}})\ D(\overline{\mathbf{u}}) - \mathbf{u}\ \mathbf{u},$$

$$\mathbf{e}(\mathbf{u}) = \mathbf{u} - D(\overline{\mathbf{u}}).$$

Comparing the exact SFNSE (1.0.5) to the LES model (1.0.4), exactly as in (1.0.2) to (1.0.3), the model's consistency error $\boldsymbol{\tau}$ drives the deviation of the true flow averages from the model's solution. Further, the model's error, $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$ satisfies

$$\mathbf{e}_t + \nabla \cdot \left( \overline{D(\overline{\mathbf{u}})\ D(\mathbf{e})} + \overline{D(\mathbf{e})\ D(\mathbf{w})} \right) - \nu \triangle \mathbf{e} + \nabla(\overline{p} - q) = \overline{\nabla \cdot \boldsymbol{\tau}}, \tag{1.0.6}$$

which gives a direct link between $\mathbf{e}$ and $\boldsymbol{\tau}$. Consider therefore $\boldsymbol{\tau}$. By rearrangement, $\boldsymbol{\tau}$ satisfies

$$\boldsymbol{\tau} = -D(\overline{\mathbf{u}})\mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{u})\mathbf{u}. \tag{1.0.7}$$

By (1.0.6) minimizing the error in an LES-ADM depends on minimizing the model's consistency error $\boldsymbol{\tau}(\mathbf{u})$. By (1.0.7), minimizing a model's consistency error hinges upon minimizing the deconvolution error $\mathbf{e}(\mathbf{u}) = \mathbf{u} - D(\overline{\mathbf{u}})$. In Chapter 3 we address and solve this optimization problem, which leads to a considerable increase in models' accuracy.

### 1.0.5 NSE vs. Regularizations of NSE

Numerical simulation of complex flows present many challenges. Often, simulations are based on various regularizations of the NSE, rather than the NSE themselves, [28], [39], [59]. The resulting models have remarkable and positive effects on computation results: errors are observed to be much better over much larger time intervals and the transition from one type of flow to another is not retarded.

Two critical features of any regularization model are

(i) its solutions must faithfully represent the qualitative properties of solutions of the NSE, and

(ii) it must be amenable to efficient numerical simulation with robust methods.

The oldest example was proposed by Leray in 1934, [54]:

$$\mathbf{v}_t + \overline{\mathbf{v}} \cdot \nabla \mathbf{v} - \nu \triangle \mathbf{v} + \nabla p = \mathbf{f}, \text{ in } \Omega$$
$$\nabla \cdot \mathbf{v} = 0, \text{ in } \Omega \tag{1.0.8}$$

where $\overline{\mathbf{v}}$ is a smoothed/averaged velocity. This combination is sometimes called the Leray-alpha regularization, [10], [8], [29]. The Leray regularization's solution is smoother, more stable, and possesses (marginally) fewer scales than the NSE's solution. Still, the resulting error, even with a high accuracy numerical method, cannot be better than the error committed in the first step, replacing $\mathbf{v}$ by $\overline{\mathbf{v}}$ in (1.0.8). With the differential filter, $\overline{\mathbf{v}} := (-\delta^2 \triangle + 1)^{-1}\mathbf{v}$, the error is $\mathbf{v} - \overline{\mathbf{v}} = O(\delta^2)$ at best. Experiments in [49] have shown that, due to its low accuracy, (1.0.8) with the above filter can have catastrophic error growth and not adequately conserve physically important integral invariants. The experiments in [49] also indicate that the increase in accuracy resulting from using deconvolution models (replacing $\overline{\mathbf{v}}$ with $D\overline{\mathbf{v}}$) decreases error growth and improves conservation properties.

Continuing Leray's idea, new regularization models can be derived every time a suitable regularization operator is chosen. In Chapter 4, we develop and study a new family of turbulence models: Leray-Tikhonov deconvolution models. This family of models is based on a modification (consistent with the large scales) of the Tikhonov regularization method. The Leray-Tikhonov deconvolution models have high accuracy, appropriate conservation of properties and physical fidelity.

### 1.0.6 Chapter Description

In Chapter 2 we study an abstract approach to modeling the motion of large eddies in a turbulent flow. If the Navier-Stokes equations are averaged with a local, spacial convolution type filter, $\overline{\boldsymbol{\phi}} = g_\delta * \boldsymbol{\phi}$, the resulting system is not closed due to the filtered nonlinear term $\overline{\mathbf{u}\mathbf{u}}$. An approximate deconvolution operator $D$ is a bounded linear operator which satisfies

$$\mathbf{u} = D(\overline{\mathbf{u}}) + O(\delta^\alpha), \tag{1.0.9}$$

where $\delta$ is the filter width and $\alpha \geq 2$. Using a deconvolution operator as an approximate filter inverse, yields the closure

$$\overline{\mathbf{u}\mathbf{u}} = \overline{D(\overline{\mathbf{u}})D(\overline{\mathbf{u}})} + O(\delta^\alpha).$$

Averaging the Navier-Stokes equations using the above closure, possible including a time relaxation term to damp unresolved scales, yields the ADM

$$\mathbf{w}_t + \nabla \cdot \overline{D(\mathbf{w})\ D(\mathbf{w})} - \nu \triangle \mathbf{w} + \nabla q + \chi \mathbf{w}^* = \overline{\mathbf{f}} \text{ and } \nabla \cdot \mathbf{w} = 0. \tag{1.0.10}$$

Here $\mathbf{w} \simeq \overline{\mathbf{u}}$, $\chi \geq 0$, and $\mathbf{w}^*$ is a generalized fluctuation, defined by a positive semi-definite operator. Assuming periodic boundary conditions, we develop an abstract theory of the LES model (1.0.10).

**Theorem 1.0.1.** *Let $T > 0$ and $D$ be a deconvolution operator satisfying the properties:*

***P**1. $D$ is a bounded linear operator on $L^2(Q)$ that is one-to-one and onto,*

***P**2. $D$ is self-adjoint and positive definite,*

***P**3. $D$ commutes with differentiation.*

*For $\mathbf{w}_0 \in H^1(Q) \cap L^2(Q)$ and $\mathbf{f} \in L^2(0,T;L^2(Q))$, there exists a weak solution $\mathbf{w} \in L^2(0,T;H^2(Q)) \cap L^\infty(0,T;L^2(Q))$ of (1.0.10). Moreover, for any $t \in (0,T]$, the following stability bound holds:*

$$\begin{aligned}
&\frac{1}{2}||\mathbf{w}(t)||^2 \ + \ \frac{1}{2}\delta^2||\nabla \mathbf{w}(t)||^2 \ + \ \nu \int_0^t \big(\ ||\nabla \mathbf{w}(\tau)||^2 + \delta^2 ||\triangle \mathbf{w}(\tau)||^2\ \big)\, d\tau \\
&\leq \ C\ \left( \left| \int_0^t (\ \mathbf{f}(\tau), D(\mathbf{w}(\tau))\ ) d\tau \right| + ||\mathbf{w}(0)||^2 + \delta^2 ||\nabla \mathbf{w}(0)||^2\ \right).
\end{aligned}$$

Using techniques from functional analysis and the theory of weak solutions of PDEs, [23], we show that the weak solution is a unique strong solution and satisfies an energy equality: (2.3.6) of Theorem 2.3.2. Moreover, this energy equality implies the stability bound given in Theorem 1.0.1 above..

In Chapter 3, we derive new methods to solve the ill-posed problem (1.0.9). This method satisfies the necessary conditions developed in Chapter 2 and minimizes the consistency error/residual stress of many turbulence models such as Leray deconvolution model, [54], [49], the time relaxation regularization model, [62], [50], and ADM, [3], [19], [5].

The van Cittert deconvolution algorithm approximates $\mathbf{u}$ using N steps of fixed point iteration. Its main cost is the filtering step. Thus, relaxation parameters can be introduced into the van Cittert algorithm with negligible increase in computational effort. Doing this, we obtain the Accelerated van Cittert algorithm: *choosing* $\mathbf{u}_0 = \overline{\mathbf{u}}$ *and* $\omega_0, \omega_1, ...., \omega_{N-1}$, *for* $n = 0, 1, 2, ..., N-1$, *perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega_n \{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

Let $D_N^{\omega}$ be the resulting deconvolution operator. The goal is to find the values of relaxation parameters $\omega_i$ which minimize the deconvolution error $\mathbf{u} - D_N^{\omega}(\overline{\mathbf{u}})$. We are led to the optimization problem: *Find* $\min_{\omega_i} \max_{\mathbf{u}} F_N(\omega_0, ..., \omega_{N-1})$, *where*

$$F_N(\omega_0, ..., \omega_{N-1}) = \sum_k \sum_{|\mathbf{k}|=k} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1})^2 (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2.$$

One immediate question is: *What is the right optimization problem?* It is natural either to optimize over a general velocity field or over a velocity field with the typical energy spectrum of $E(k) \sim \alpha \varepsilon^{2/3} k^{-5/3}$ of homogeneous, isotropic turbulence. Both cases are interesting and we consider them in Chapter 3.

We introduce the Accelerated van Cittert deconvolution operator $D_N^{\omega}$ and study its mathematical properties. For general velocity fields, we solve the problem: *Find* $\omega_i$ *to minimize*

$$\max_{\mathbf{u} \in L^2(Q)} F_N(\omega_0, ..., \omega_{N-1}).$$

Using Chebyshev polynomials, we solve the above optimization problem. We derive:

- the values of this optimal parameters and
- the reduction in the model consistency error that results in their use. This gives an *exponential increase in accuracy for zero additional cost*:

$$\|\mathbf{u} - D_N^{\omega}(\overline{\mathbf{u}})\| \ \le \ 2.16\ \alpha\ \epsilon^{2/3}\ \delta^{2/3}\ \frac{\pi}{L}\ \frac{4}{e^{1.24N}}.$$

For homogeneous, isotropic turbulence, we also compute the values of the relaxation parameters $\omega_i$ by solving the $N \times N$ system (N=1,...,6):

$$\left(\frac{\partial F_N}{\partial \omega_0}, ..., \frac{\partial F_N}{\partial \omega_{N-1}}\right) = 0.$$

With both sets of parameters, we perform extensive computational tests. First, we choose a known velocity and calculate $\|\mathbf{u} - D_N^{\omega}(\overline{\mathbf{u}})\|$. The computations are in accord with the theory: when the velocity is smooth, the regular van Cittert is more accurate, when the velocity oscillates faster, both versions of the Accelerated van Cittert are more competitive. For qualitative results, we consider a second example. We study an under-resolved flow with recirculation, the flow across a step with $N = 1$. Behind the step, the flow simulation, using both the optimal parameters and the usual van Cittert ($\omega_i = 1$), correctly develops vortices separate from the step (see Figure 5 below).

In Chapter 4 we develop and study a Tikhonov regularization based Leray Regularization of the NSE. This family of models is based on a modification (consistent with the large scales) of Tikhonov-Lavrentiev deconvolution. Without this modification, the Tikhonov-Lavrentiev process leads to a less accurate solution of (1.0.9). One way to see this is by plotting the transfer functions of exact and Tikhonov-Lavrentiev deconvolution (see Figure 6). The figure shows a separation between the graphs for wave numbers near 0 (which means that the large scales are not accurately recovered).

With the modified Tikhonov-Lavrentiev deconvolution process, we obtain an approximation of the unfiltered solution by one filtering step. *Let* $0 < \mu < 1$. *Given* $\overline{\mathbf{u}}$, *an approximate solution of the deconvolution problem (1.0.9) is given by*

$$\mathbf{u}_\mu = (\ (1-\mu)G + \mu I\ )^{-1}\ \overline{\mathbf{u}}. \qquad (1.0.11)$$

The operator $D_\mu$, $0 \leq \mu \leq 1$, is a Tikhonov-Lavrentiev regularization of the formal filter inverse *adapted to turbulence*, i.e. designed to accurately capture the large scales of a flow, while modeling the small (or under-resolved) scales (and truncating).

We prove that the operator $D_\mu$ satisfies the conditions we derived in Chapter 2 and has consistency error $\mathbf{u} = D_\mu(\overline{\mathbf{u}}) + O(\mu\delta^2)$. We are presenting computational comparisons and rigorous numerical analysis of a computational attractive algorithm for the family of Tikhonov-Leray regularization with time relaxation models

$$\mathbf{w}_t + D_\mu(\overline{\mathbf{w}}) \cdot \nabla \mathbf{w} - \nu \triangle \mathbf{w} + \nabla p + \chi(\mathbf{w} - D_\mu(\overline{\mathbf{w}})) = \mathbf{f} \text{ and } \nabla \cdot \mathbf{w} = 0.$$

**Algorithm 1.0.2.** *Let $\Delta t > 0$, $(\mathbf{w}_0, q_0) \in (X_h, Q_h)$, $\mathbf{f} \in X^*$ and $T := M\,\Delta t$ as $M$ is an integer. For $n = 0, 1, 2, \cdots, M-1$, find $(\mathbf{w}^h_{n+1}, q^h_{n+1}) \in (X_h, Q_h)$ satisfying*

$$\begin{aligned}\frac{1}{\Delta t}(\mathbf{w}^h_{n+1} - \mathbf{w}^h_n, \mathbf{v}^h) + b^*(D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{w}^h_{n+1/2}, \mathbf{v}^h) - (q^h_{n+1/2}, \nabla \cdot \mathbf{v}^h) + \nu(\nabla \mathbf{w}^h_{n+1/2}, \nabla \mathbf{v}^h) \\ + \chi(\mathbf{w}^h_{n+1/2} - D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h),\ \forall\ \mathbf{v}^h \in X_h \\ (\nabla \cdot \mathbf{w}^h_{n+1}, \boldsymbol{\phi}^h) = 0,\ \forall \boldsymbol{\phi}^h \in Q_h\end{aligned}$$

We prove that, at each time step, there exists a solution of the above scheme, the scheme is unconditionally stable and satisfies an á priori bound. We also perform a convergence analysis of the scheme, when $\delta$, $\mu$, $h \to 0$. Numerical experiments that support our theoretical results are presented.

# 2.0 EXISTENCE THEORY OF ABSTRACT APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

Many approaches have been used to simulate turbulent flows. In LES the evolution of averages is sought. These averages are defined through a local spatial averaging process associated with an averaging radius $\delta$. Once an averaging radius and a filtering process is selected, an LES model can be developed and then solved numerically. One of the most interesting approaches to generate LES models is via approximate deconvolution or approximate/asymptotic inverse of the filtering operator. Approximate deconvolution models are systematic (rather than ad hoc). They can achieve high theoretical accuracy and shine in practical tests; they contain few or no fitting/tuning parameters. The ADM approach has thus proven itself to be very promising. However, among the very many known approximate deconvolution operators from image processing, e.g. [4], so far only two have been studied for LES modeling, the van Cittert deconvolution operator and Geurts' approximate inverse filter. Their success suggests that it is time to develop a general theory of LES-ADM as a guide to development of models based on other, possibly better, deconvolution operators and refinement of existing ones.

Two basic requirements of an acceptable ADM are that a unique, strong solution exists and that the model's global energy balance be close in some sense to that of the NSE. In this chapter, we consider these two important questions. *We find conditions on the approximate deconvolution operator $D$ that guarantee that the ADM has a unique strong solution.* We also derive the model's energy balance and show that under these conditions on the deconvolution operator *the model correctly captures the global energy balance of the large scales.*

Recall the Navier-Stokes equations:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^n,$$

where $\mathbb{R}^n$ ( $n = 2$ or 3) is the flow domain. We consider $L$-periodic boundary conditions to separate the interior closure problem from other important problems associated with filtering through a boundary, [16], and finding boundary conditions for local averages (near wall laws), [41],

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t),\ j = 1, ..., n.$$

The NSE are supplemented by the initial condition, the usual normalization condition in the periodic case of zero mean velocity and pressure, and the assumption that all data are square integrable with zero mean

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \text{ and } \int_Q \mathbf{u}\ d\mathbf{x} = \int_Q p\ d\mathbf{x} = 0,$$
$$\int_Q |\mathbf{u}_0(\mathbf{x}, t)|^2 d\mathbf{x} < \infty,\ \int_Q |\mathbf{f}(\mathbf{x}, t)|^2 d\mathbf{x} < \infty,\ \text{and } \int_Q \mathbf{f}(\mathbf{x}, t) d\mathbf{x} = 0,\ \text{for } \ 0 \le t, \tag{2.0.1}$$

where $Q = (0, L)^n$. Let overbar denote a local, spatial averaging operator, such as averaging by convolution, that is linear and commutes with differentiation. Averaging the NSE, the (non-closed) equations for $\overline{\mathbf{u}}$ and $\overline{p}$, known as the Space Filtered Navier-Stokes equations (SFNSE), are

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u}\,\mathbf{u}}) - \nu \triangle \overline{\mathbf{u}} + \nabla \overline{p} = \overline{\mathbf{f}} \text{ and } \ \nabla \cdot \overline{\mathbf{u}} = 0. \tag{2.0.2}$$

Since in general $\overline{\mathbf{u}\mathbf{u}} \neq \overline{\mathbf{u}}\ \overline{\mathbf{u}}$, the closure problem is to replace $\overline{\mathbf{u}\mathbf{u}}$ by a tensor $S(\overline{\mathbf{u}}, \overline{\mathbf{u}})$ depending only on $\overline{\mathbf{u}}$, not on $\mathbf{u}$. If we denote filtering by $\overline{\mathbf{u}} = G\mathbf{u}$ and $D$ is an approximate inverse of $G$ (so $\mathbf{u} \cong D\overline{\mathbf{u}}$), then the variant of approximate deconvolution model we consider herein approximates

$$\overline{\mathbf{u}\mathbf{u}} \cong \overline{D(\overline{\mathbf{u}})D(\overline{\mathbf{u}})} =: S(\overline{\mathbf{u}}, \overline{\mathbf{u}}).$$

The deconvolution problem is central in image processing, [4]. Thus many algorithms can be adapted to give a possible LES closure model. The goal of deconvolution in LES is to recover accurately the resolved scales *asymptotically* as $\delta \to 0$. The resulting LES model should have a lucid [1] energy balance and favorable properties for its approximate solution. As an example, the $N^{th}$ van Cittert approximate deconvolution operator $D_N$, defined precisely in Section 2.4, see also [3], [19], [45], is an easy-to-construct bounded linear operator on $L^2(Q)$ satisfying

$$\boldsymbol{\phi} = D_N(\overline{\boldsymbol{\phi}}) + O(\delta^{2N+2}), \text{ for smooth } \boldsymbol{\phi}.$$

In other words, $D_N$ is an asymptotic (as $\delta \to 0$) approximate inverse of $G$. With the van Cittert approximate deconvolution operator, the closure problem in (2.0.2) can be solved approximately, but systematically, by:

$$\overline{\mathbf{u}\,\mathbf{u}} \simeq \overline{D_N(\overline{\mathbf{u}})\ D_N(\overline{\mathbf{u}})} + O(\delta^{2N+2}), \text{ for smooth } \mathbf{u}.$$

More generally and beyond the van Cittert operator, there is little analytical guidance as to the properties needed of a deconvolution operator to produce a reliable LES model. Since inverting a filter is an ill-posed problem, a deconvolution operator can also be generated by any method for solving approximately ill-posed problems. Thus, there are many possible choices of $D$ available. Once $D$ is selected, we define the higher order fluctuation $\mathbf{w}^* = \mathbf{w} - D(\overline{\mathbf{w}})$, if $I - DG$ is symmetric positive semi-definite and $\mathbf{w}^* = (I - DG)^*(I - DG)\mathbf{w}$ otherwise.

In general, *any* approximate deconvolution operator $D$, used as a closure approximation, leads to the approximate deconvolution LES model

$$\begin{aligned} \mathbf{w}_t + \nabla \cdot \overline{D(\mathbf{w})\ D(\mathbf{w})} - \nu \triangle \mathbf{w} + \nabla q + \chi \mathbf{w}^* &= \overline{\mathbf{f}} \\ \nabla \cdot \mathbf{w} &= 0 \\ \mathbf{w}|_{t=0} &= \overline{\mathbf{u}}_0 \\ \int_Q q dx &= 0. \end{aligned} \tag{2.0.3}$$

[1] "Lucid" here is used to mean physically lucid. The energy balance should connect the mathematical theory to the physics of turbulence.

**Remark 2.0.1.** [*On the definition of the time relaxation term* $\mathbf{w}^*$] *The time relaxation term* $\chi\mathbf{w}^*$, *where* $\chi \geq 0$ *is a model parameter, is often included in the ADM to damp marginally unresolved scales, see [62] and [50], so we include it in our analysis. The time relaxation term's intent is to dissipate energy around the length scale* $l \simeq \delta$. *Energy is only dissipated if the term* $\chi(\mathbf{w}^*, \mathbf{w})$ *is positive. Thus, when* $I - DG$ *is not symmetric positive definite, the definition of* $\mathbf{w}^*$ *is modified to ensure this. The modification does alter the scaling of* $\mathbf{w}^*$.

If $D$ is a symmetric positive definite operator then it induces a deconvolution weighted $L^2$ inner product and norm defined by $(\mathbf{u}, \mathbf{v})_D := \int_Q \mathbf{u} \cdot D(\mathbf{v}) d\mathbf{x}$ and $||\mathbf{v}||_D^2 = (\mathbf{v}, \mathbf{v})_D$. For specificity, we select the filtering operation $\overline{\mathbf{v}} := (I - \delta^2 \triangle)^{-1}\mathbf{v}$.

In Section 2.2 we show that if the deconvolution operator $D : L^2(Q) \to L^2(Q)$ is a one-to-one and onto, bounded, self adjoint and positive definite operator that commutes with differentiation, then a weak solution exists for the deconvolution model (2.0.3). In Section 2.3 we show that the weak solution is a unique strong solution and satisfies the energy equality:

$$\frac{1}{2}||\mathbf{w}(t)||_D^2 + \frac{\delta^2}{2}||\nabla \mathbf{w}(t)||_D^2 + \int_0^t \left( \nu||\nabla \mathbf{w}||_D^2 + \nu\delta^2||\triangle \mathbf{w}||_D^2 + \chi(\mathbf{w}^*, (I - \delta^2\triangle)\mathbf{w})_D \right) d\tau$$
$$= \int_0^t (\mathbf{f}, \mathbf{w})_D d\tau + \frac{1}{2}\left(||\mathbf{w}(0)||_D^2 + \delta^2||\nabla \mathbf{w}(0)||_D^2\right). \quad (2.0.4)$$

**Remark 2.0.2.** [ *An Important Difference Between Deconvolution Models* ] *The SFNSE can be rewritten as*

$$\overline{\mathbf{u}}_t + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} - \nu \triangle \overline{\mathbf{u}} + \nabla \overline{p} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\ \overline{\mathbf{u}}) = \overline{\mathbf{f}}.$$

*Another possible deconvolution LES model approximates*

$$\overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\ \overline{\mathbf{u}} = \overline{D(\overline{\mathbf{u}})\ D(\overline{\mathbf{u}})} - \overline{D(\overline{\mathbf{u}})}\ \overline{D(\overline{\mathbf{u}})}.$$

*In the simplest case* $(D = I)$ *this is the Bardina model*

$$\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} - \nu \triangle \mathbf{w} + \nabla q + \nabla \cdot (\overline{\mathbf{w}\mathbf{w}} - \overline{\mathbf{w}}\ \overline{\mathbf{w}}) = \overline{\mathbf{f}}. \quad (2.0.5)$$

*This common approach is NOT covered by the theory herein. In fact, we believe (but cannot prove yet) that this approach can be unstable, unless sufficient ad hoc eddy viscosity is added (thereby destroying the high accuracy of the ADM approach). We therefore have a strong preference for the simpler, accurate, and unconditionally stable model (2.0.3) over the similarity type deconvolution model (2.0.5).*

## 2.1 PRELIMINARIES, NOTATIONS, AND FUNCTION SPACES

We begin by briefly reviewing the concept of averaging/filtering in LES and define the function spaces and the norms needed for the variational formulation of the scale similarity model.

### 2.1.1 Averaging operators used in LES

The idea of LES is to split the velocity into $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ a local, spatial average $\overline{\mathbf{u}}$ and a fluctuation about the mean $\mathbf{u}'$. It is widely believed that given the random and chaotic character of the fluctuations, their average effects on the mean motion can successfully be modeled and thus the mean can be predicted accurately. The mean is defined by filtering or mollification (convolution with an approximate identity). The goal is to predict the mean accurately. Let $g(\cdot)$ denote a filter, such as the Gaussian filter $g(\mathbf{x}) = \frac{6}{\pi}e^{-6|\mathbf{x}|^2}$, and let $\delta$ denote the selected averaging radius. Define $g_\delta(\mathbf{x}) := \delta^{-3}g(\mathbf{x}/\delta)$. Averages are defined by convolution with the kernel $g(\cdot)$. Given a velocity $\mathbf{u}$, its mean $\overline{\mathbf{u}}$ and fluctuation $\mathbf{u}'$ are defined by $\overline{\mathbf{u}} = G\mathbf{u}$ and $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$ when

$$\overline{\mathbf{u}}(\mathbf{x}) := \int_{R^3} g_\delta(\mathbf{x} - \mathbf{x}')\mathbf{u}(\mathbf{x}')d\mathbf{x}'. \tag{2.1.1}$$

Typical choices used in LES include the top-hat filter, sharp spectral cut-off, the Gaussian filter and differential filters, defined next, which also fit into the convolution formalism.

**Definition 2.1.1.** [ *Differential Filter* ] *Let $A := -\delta^2\triangle + I$ and let $\overline{\boldsymbol{\varphi}}$ denote the unique L-periodic solution in $H^2(Q)$ of:*

$$A\overline{\boldsymbol{\varphi}} := -\delta^2\triangle\overline{\boldsymbol{\varphi}} + \overline{\boldsymbol{\varphi}} = \boldsymbol{\varphi}. \tag{2.1.2}$$

**Remark 2.1.1.** *Under periodic boundary conditions and for constant $\delta$ both (2.1.1) and (2.1.2) commute with differentiation, preserve incompressibility, and can be written as convolution with approximate identity.*

Differential filters are well-established in LES, starting with the work of Germano [27] and continuing with [26], [59]. They have many connections to regularization processes such as the Yoshida regularization of semigroups and the very interesting work of Foias, Holm, Titi [22] (and others) on Lagrange averaging of the Navier-Stokes equations.

The mean $\overline{\mathbf{u}}(\mathbf{x})$ is the weighted average of $\mathbf{u}$ about the point $\mathbf{x}$. As $\delta \to 0$, the points near $\mathbf{x}$ are weighted more and more heavily, so intuitively we expect that $\overline{\mathbf{u}} \to \mathbf{u}$ as $\delta \to 0$. This is known for convolution filters. For differential filters, it is also not difficult to show.

**Remark 2.1.2.** *Expanding the velocity* $\mathbf{u}$ *in Fourier series we obtain*

$$\mathbf{u}(\mathbf{x}) = \sum_{\boldsymbol{k}} \widehat{\mathbf{u}}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\mathbf{x}},$$

*where* $\boldsymbol{k}$ *is the wave number vector and* $\boldsymbol{k} = \frac{2\pi \boldsymbol{n}}{L}$, *for* $\boldsymbol{n} \in \mathbb{Z}^3$. *Let* $k = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ *be its magnitude. The Fourier coefficients are*

$$\widehat{\mathbf{u}}(\boldsymbol{k}) = \frac{1}{L^3} \int_Q \mathbf{u}(\mathbf{x}) e^{-i\boldsymbol{k}\cdot\mathbf{x}} d\mathbf{x}.$$

*Parseval's equality leads to*

$$\frac{1}{L^3} \int_Q |\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} = \sum_{\boldsymbol{k}} |\widehat{\mathbf{u}}(\boldsymbol{k})|^2 = \sum_{k} \sum_{|\boldsymbol{k}|=k} |\widehat{\mathbf{u}}(\boldsymbol{k})|^2.$$

**Lemma 2.1.1.** *Let* $\overline{\mathbf{u}} = (-\delta^2 \triangle + I)^{-1}\mathbf{u}$. *Then, for any* $\mathbf{u} \in L^2(Q)$, *we have*

$$\overline{\mathbf{u}} \to \mathbf{u} \ \textit{in } L^2(Q) \ \textit{as } \delta \to 0.$$

*Proof.* Given $\varepsilon > 0$, we show that $||\overline{\mathbf{u}} - \mathbf{u}|| < \varepsilon$, for $\delta$ small enough. Indeed, using Parseval's equality and (2.1.2), we obtain

$$||\overline{\mathbf{u}} - \mathbf{u}||^2 = \sum_{k} \sum_{|\mathbf{k}|=k} \left( \frac{\delta^2 k^2}{1 + \delta^2 k^2} \right) |\widehat{\mathbf{u}}(\mathbf{k})|^2.$$

For any positive integer $M$ we can write

$$||\overline{\mathbf{u}} - \mathbf{u}||^2 = \sum_{0<k\leq M} \sum_{|\mathbf{k}|=k} \left( \frac{\delta^2 k^2}{1 + \delta^2 k^2} \right) |\widehat{\mathbf{u}}(\mathbf{k})|^2 + \sum_{k>M} \sum_{|\mathbf{k}|=k} \left( \frac{\delta^2 k^2}{1 + \delta^2 k^2} \right) |\widehat{\mathbf{u}}(\mathbf{k})|^2.$$

We have

$$\sum_{|\mathbf{k}|=k}\left(\frac{\delta^2 k^2}{1+\delta^2 k^2}\right)|\widehat{\mathbf{u}}(\mathbf{k})|^2 \leq \sum_{|\mathbf{k}|=k}|\widehat{\mathbf{u}}(\mathbf{k})|^2.$$

Thus, for $M$ large enough

$$\sum_{k>M}\sum_{|\mathbf{k}|=k}\left(\frac{\delta^2 k^2}{1+\delta^2 k^2}\right)|\widehat{\mathbf{u}}(\mathbf{k})|^2 < \frac{\varepsilon}{2} \tag{2.1.3}$$

and for this $M$ we have

$$\sum_{0<k\leq M}\sum_{|\mathbf{k}|=k}\left(\frac{\delta^2 k^2}{1+\delta^2 k^2}\right)|\widehat{\mathbf{u}}(\mathbf{k})|^2 \quad\leq\quad \frac{\delta^2 M^2}{1+\delta^2 M^2}||\mathbf{u}||^2 \quad<\quad \frac{\varepsilon}{2}, \tag{2.1.4}$$

for some $\delta = \delta(M)$. From (2.1.3) and (2.1.4) the conclusion follows. $\square$

### 2.1.2 Function Spaces, Weak and Strong Solution

The notation we are using follows Temam [64] and Layton and Lewandowski [46]. Let $||\cdot||$ and $(\cdot,\cdot)$ be the $L^2$ norm and inner product respectively. The space $H^k$ represents the Sobolev space $W_2^k(Q)$ and $||\cdot||_k$ denotes the norm in $H^k$. For $k \geq 1$, let $V_{\#}^k(Q)$ denote the space of all $[0,L]^3$-periodic functions with restriction on the cell $Q = (0,L)^3$ in the space $H^k(Q)$. Thus

$$V_{\#}^k(Q) = \left\{\ \mathbf{u} \in H_{loc}^k(Q)|\ \mathbf{u} \text{ is L-periodic }\right\}.$$

On $V_{\#}^k(Q)$ we define the associated norm $||\cdot||_{V_{\#}^k(Q)}$ given by:

$$||\mathbf{u}||_{V_{\#}^k}(Q) = \sum_{j=0}^{k}\left(\int_Q |\nabla^j \mathbf{u}(x)|^2 dx\right)^{1/2}, \text{ for all } \mathbf{u} \in V_{\#}^k(Q).$$

For the variational formulation of the approximate deconvolution model (following, for example, Galdi [23]), we consider the spaces of periodic, divergence-free functions:

$$\overline{V}_{\#}^k(Q) = \left\{\ \mathbf{u} \in V_{\#}^k(Q)|\ \nabla\cdot\mathbf{u} = 0\ \right\},$$

$$H = \left\{\ \mathbf{u} \in L^2(Q)|\ \mathbf{u} \text{ is L-periodic and } \nabla\cdot\mathbf{u} = 0\ \right\},$$

and

$$V = \{ \ \mathbf{u} \in H^1(Q) | \ \nabla \cdot \mathbf{u} = 0 \ \} .$$

Further, let $Q_T = Q \times [0, T)$ and define

$$\begin{aligned} D(Q) = \{ \ \boldsymbol{\psi} \in C^\infty(Q) | \ \boldsymbol{\psi} \text{ is L-periodic, has compact support,} \\ \nabla \cdot \boldsymbol{\psi} = 0 \text{ and } \int_Q \boldsymbol{\psi} d\mathbf{x} = 0 \ \} \\ D(Q_T) = \{ \ \boldsymbol{\psi} \in C^\infty \left( Q_T \right) | \ \boldsymbol{\psi}(\cdot, t) \text{ is L-periodic, has compact support,} \\ \nabla \cdot \boldsymbol{\psi} = 0 \text{ and } \int_Q \boldsymbol{\psi} d\mathbf{x} = 0 \ \}. \end{aligned}$$

**Definition 2.1.2.** *Let $D$ be a symmetric, positive definite and bounded operator on $L^2(Q)$. The $D$ inner product and norm on $L^2(Q)$ are $(\boldsymbol{\phi}, \boldsymbol{\psi})_D := \int_Q \boldsymbol{\phi} \cdot D(\boldsymbol{\psi}) d\mathbf{x}$ and*

$$||\boldsymbol{\phi}||_D^2 = (\boldsymbol{\phi}, \boldsymbol{\phi})_D, \ \textit{for every } \boldsymbol{\phi} \in L^2(Q).$$

**Definition 2.1.3.** *Let $D$ and $G$ be bounded operators on $L^2(Q)$. We define*

$$\boldsymbol{\phi}^* := \begin{cases} (I - DG) \ \boldsymbol{\phi}, & \textit{if } I - DG \textit{ is symmetric positive semi-definite} \\ (I - DG)^*(I - DG) \ \boldsymbol{\phi}, & \textit{otherwise.} \end{cases}$$

*The $*$ semi-inner product and semi-norm on $L^2(Q)$ are $(\boldsymbol{\phi}, \boldsymbol{\psi})_* := (\boldsymbol{\phi}^*, \boldsymbol{\psi})$ and*

$$||\boldsymbol{\phi}||_*^2 := (\boldsymbol{\phi}, \boldsymbol{\phi})_*, \ \textit{for every } \boldsymbol{\phi} \in L^2(Q).$$

To make progress in the mathematical understanding of an LES model the key idea is the notion of weak solution. For the NSE, the notion of weak solution was introduced by Leray, [54]. We now introduce the notion of weak solution for the ADM (2.0.3).

**Definition 2.1.4.** *Let $\mathbf{f} \in L^2(0, T; L^2(Q))$ and $\mathbf{w}_0 \in \overline{V}^1_\#(Q)$. A function $\mathbf{w} : Q \times [0, T] \to \mathbb{R}^n$ is a weak solution of (2.0.3) if $\mathbf{w} \in L^2 \left(0, T; \overline{V}^2_\#(Q)\right) \cap L^\infty \left(0, T; H\right)$ and*

$$\begin{aligned} ( \ \mathbf{w}(T), \boldsymbol{\varphi}(T) \ ) - \int_0^T \ \left( \ (\mathbf{w}, \frac{\partial \boldsymbol{\varphi}}{\partial t}) \ - \nu \ (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}) - (\nabla \cdot \overline{D(\mathbf{w}) D(\mathbf{w})}, \boldsymbol{\varphi}) \ - \chi(\mathbf{w}^*, \boldsymbol{\varphi}) \ \right) \ dt \\ = \int_0^T ( \ \overline{\mathbf{f}}, \boldsymbol{\varphi} \ ) dt + (\mathbf{w}_0, \boldsymbol{\varphi}(0)) \, , \end{aligned} \tag{2.1.5}$$

*for all $\boldsymbol{\varphi} \in D(Q_T)$.*

To obtain the weak formulation (2.1.5), we multiply the first equation of (2.0.3) by $\boldsymbol{\varphi} \in D(Q_T)$ and integrate over $Q$. Integration by parts gives that $(\nabla q, \boldsymbol{\varphi}) = 0$. Now, integrating between 0 and $T$ we obtain the claimed formulation. Following Galdi [23], we define below a strong solution of the ADM (2.0.3).

**Definition 2.1.5.** *The pair $(\mathbf{w}, q)$ is a strong solution of the deconvolution model (2.0.3) if $\mathbf{w}$ and $q$ satisfy all the equations of (2.0.3), where*

$$\begin{aligned}
&\mathbf{w} \in \overline{V}^2_{\#}(\mathrm{Q}) \cap H, \ \textit{for a.e. } t \in [0, T] \\
&\mathbf{w} \in H^1([0, T]), \ \textit{for a.e. } \mathbf{x} \in Q \\
&q \in \overline{V}^1_{\#}(\mathrm{Q}) \cap L^2_0(Q), \ \textit{for a.e. } t \in (0, T].
\end{aligned} \tag{2.1.6}$$

### 2.1.3 The Stokes Operator

Recall the Stokes problem

$$-\nu\triangle\mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{u} = 0 \text{ in } Q, \tag{2.1.7}$$

where $\mathbf{f} \in L^2(Q)$ and $\nu > 0$. We say that $\mathbf{u} \in V$ is a weak solution of the Stokes problem if

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in D(Q).$$

Since $\triangle\mathbf{u} \in L^2(Q)$, by the Helmholtz-Weyl decomposition (see for example Galdi [23] pag. 6), we have that $\triangle\mathbf{u} = \boldsymbol{\psi} + \boldsymbol{\psi}^{\perp}$, for unique $\boldsymbol{\psi} \in H$ and $\boldsymbol{\psi}^{\perp} \in H^{\perp}$. Since $(\boldsymbol{\psi}^{\perp}, \mathbf{v}) = 0$ for all $\mathbf{v} \in D(Q)$, it means that $(\triangle\mathbf{u}, \mathbf{v}) = (\boldsymbol{\psi}, \mathbf{v})$, for all $\mathbf{v} \in D(Q)$. Thus, (2.1.7) is equivalent with

$$-\nu(P(\triangle\mathbf{u}), \mathbf{v}) = (P\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in D(Q),$$

where $P$ is the orthogonal projection $P : L^2(Q) \to H$.

**Definition 2.1.6.** *The Stokes operator $A : H^2(Q) \cap H \to H$ is defined by*

$$A = -P\triangle, \ i. \ e. \ A\mathbf{u} = P(-\triangle\mathbf{u}), \ \forall \mathbf{u} \in H^2(Q) \cap H.$$

**Theorem 2.1.1.**

*(i) The Stokes operator is symmetric and self-adjoint.*

*(ii) The inverse of the Stokes operator, $A^{-1}$, is a bounded compact operator in $H$.*

*Proof.* For the proof see [13] (pag.31-32). $\square$

Since $A^{-1}$ is compact and self-adjoint (since $A$ is self-adjoint), there exists an orthonormal basis of $H$ consisting of eigenfunctions of the Stokes operator. Let $\{\psi_j\}_j$ be this basis. Following [13] we have that, under periodic boundary conditions, $P$ and $\triangle$ commute (Remark 4.13, pag. 43). Proposition 4.6 in [13] also gives that $\{\psi_j\}_j \subset D(Q) \subset H$.

## 2.2 EXISTENCE OF WEAK SOLUTIONS OF ADM

Due to the nonlinearity in (2.0.3) small changes in the deconvolution operator can yield significant (positive or negative) changes in the solution of the induced model. The averaging/convolution operator $G$ is bounded, self-adjoint, positive definite and commutes with differentiation. We thus postulate that the deconvolution operator $D$, its approximate inverse, is also bounded, self-adjoint, positive definite and commutes with differentiation. We postulate the following properties of the deconvolution operator $D$:

**P**1. *$D$ is a bounded linear operator on $L^2(Q)$ that is one-to-one and onto,*

**P**2. *$D$ is self-adjoint and positive definite,*

**P**3. *$D$ commutes with differentiation.*

Note that the Open Mapping Theorem, **P**1, and **P**2 imply, that there exist positive constants $C_1$ and $C_2$ such that

$$C_1||\boldsymbol{\varphi}|| \leq ||\boldsymbol{\varphi}||_D \leq C_2||\boldsymbol{\varphi}||, \text{ for all } \boldsymbol{\varphi} \in L^2(Q).$$

There are many operators $D$ satisfying conditions **P**1 through **P**3; e.g., see Propositions 2.4.2 and 4.1.1.

**Remark 2.2.1.** *Let $\{\boldsymbol{\psi}_j\}$ be eigenfunctions of the Stokes operator in $H$. Then, due to periodic boundary conditions, $span\{\boldsymbol{\psi}_j\}_j$ is invariant under the Stokes operator and any other constant coefficient differential operators. By $\boldsymbol{P}3$, it is also invariant under $D$.*

Under postulates **P**1, **P**2, and **P**3 on the deconvolution operator $D$, we prove that the ADM model (2.0.3) admits weak solutions, in the sense of the Definition 2.1.4. We follow the exposition on the existence of the weak solutions of the NSE, in Galdi [23]. The analysis is for the case $\chi = 0$ of no time relaxation. It extends immediately to $\chi > 0$, when $\mathbf{w}^*$ is defined as in either cases of Remark 2.0.1.

**Theorem 2.2.1.** *Let $T > 0$, $\chi = 0$, and $D$ be a deconvolution operator satisfying the properties $\boldsymbol{P}1$, $\boldsymbol{P}2$, and $\boldsymbol{P}3$. For $\mathbf{w}_0 \in \overline{V}^1_{\#}(Q) \cap H$ and $\mathbf{f} \in L^2(0, T; L^2(Q))$, there exists a weak solution $\mathbf{w} \in L^2\left(0, T; \overline{V}^2_{\#}(Q)\right) \cap L^\infty(0, T; H)$ of (2.0.3) in the sense of the Definition 2.1.4. Moreover, for any $t \in (0, T]$, the following stability bound holds:*

$$\frac{1}{2}\|\mathbf{w}(t)\|^2 + \frac{1}{2}\delta^2\|\nabla\mathbf{w}(t)\|^2 + \nu\int_0^t (\|\nabla\mathbf{w}(\tau)\|^2 + \delta^2\|\triangle\mathbf{w}(\tau)\|^2)\,d\tau \\ \leq C\left(\left|\int_0^t (\,\mathbf{f}(\tau), D(\mathbf{w}(\tau))\,)\,d\tau\right| + \|\mathbf{u}_0\|^2\right). \tag{2.2.1}$$

*Existence also holds if $\chi > 0$, $(\mathbf{w}, D\ A\mathbf{w}^*) \geq 0$, and $\mathbf{w} \to \mathbf{w}^*$ is a bounded operator on $L^2(Q)$.*

*Proof.* Let $\{\boldsymbol{\psi}_j\} \in D(Q)$ be an orthonormal basis of $H$ consisting of eigenfunctions of the Stokes operator. We are looking for Galerkin approximate solutions of the form:

$$\mathbf{w}_k(\mathbf{x}, t) = \sum_{r=1}^{k} \eta_{kr}(t)\ \boldsymbol{\psi}_r(\mathbf{x}), \quad \text{for } k \in \mathbb{N}. \tag{2.2.2}$$

We require that the functions $\{\mathbf{w}_k\}_k$ satisfy the system of ordinary differential equations

$$(\ \frac{\partial\mathbf{w}_k}{\partial t}, \boldsymbol{\psi}_j\ ) + \nu(\ \nabla\mathbf{w}_k, \nabla\boldsymbol{\psi}_j\ ) + (\ \nabla\cdot\overline{D(\mathbf{w}_k)D(\mathbf{w}_k)}, \boldsymbol{\psi}_j\ ) = \left(\overline{\mathbf{f}}, \boldsymbol{\psi}_j\right) \tag{2.2.3}$$

$$(\ \mathbf{w}_k(0), \boldsymbol{\psi}_j\ ) = (\ \mathbf{w}_0, \boldsymbol{\psi}_j\ ), \tag{2.2.4}$$

for all $j = 1, ..., k$. Using (2.2.2) in (2.2.3) and (2.2.4), we obtain

$$\sum_{r=1}^{k} \frac{\partial \eta_{kr}}{\partial t} \ (\boldsymbol{\psi}_r, \boldsymbol{\psi}_j) + \nu \sum_{r=1}^{k} \eta_{kr} \ (\nabla \boldsymbol{\psi}_r, \nabla \boldsymbol{\psi}_j)$$

$$+ \sum_{r,i=1}^{k} \eta_{kr}\eta_{ki} \ ( \ \nabla \cdot \overline{D(\boldsymbol{\psi}_r) D(\boldsymbol{\psi}_i)}, \boldsymbol{\psi}_j \ ) = (\overline{\mathbf{f}}, \boldsymbol{\psi}_j) \tag{2.2.5}$$

$$\sum_{r=1}^{k} \eta_{kr}(0) \ (\boldsymbol{\psi}_r, \boldsymbol{\psi}_j) = (\mathbf{w}_0, \boldsymbol{\psi}_j), \tag{2.2.6}$$

for all $j = 1, ..., k$. For simplification, let $a_{rj} = (\nabla \boldsymbol{\psi}_r, \nabla \boldsymbol{\psi}_j)$, $a_{rij} = (\nabla \cdot \overline{D(\boldsymbol{\psi}_r) D(\boldsymbol{\psi}_i)}, \boldsymbol{\psi}_j)$, $f_j = ( \ \overline{\mathbf{f}}, \boldsymbol{\psi}_j)$ and $c_{0j} = (\mathbf{w}_0, \boldsymbol{\psi}_j)$. Since $(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j) = \delta_{ij}$, equations (2.2.5) and (2.2.6) become

$$\frac{\partial \eta_{kj}}{\partial t} + \nu \sum_{r=1}^{k} a_{rj}\eta_{kr} + \sum_{r,i=1}^{k} a_{rij}\eta_{kr}\eta_{ki} = f_j \tag{2.2.7}$$

$$\eta_{kj}(0) = c_{0j}, \tag{2.2.8}$$

for $j = 1, ..., k$. Since $f_j \in L^2(0, T)$, for all j, from the theory of ODEs we know that the problem (2.2.7)–(2.2.8) has a unique solution $\eta_{kj} \in W^{1,2}(0, T_k)$, for a small enough time $T_k \leq T$.

Because $\mathbf{w}_0 \in \overline{V}^1_{\#}(Q) \cap H$, there exists $\mathbf{u}_0 \in H$ such that $\overline{\mathbf{u}}_0 = \mathbf{w}_0$ and:

$$(\mathbf{w}_0, \boldsymbol{\psi}_j) = (\overline{\mathbf{u}}_0, \boldsymbol{\psi}_j), \quad \text{for } j = 1, ..., k. \tag{2.2.9}$$

From Remark 2.2.1, span$\{\boldsymbol{\psi}_j\}_j$ is invariant under the Stokes and differential operators. In particular, span$\{\boldsymbol{\psi}_j\}_j$ is invariant under the deconvolution operator $D$ and $(-\delta^2\triangle + I)$. Since $D(\mathbf{w}_k)$ is a linear combination of $\{\boldsymbol{\psi}_j\}$ we deduce that $D(\mathbf{w}_k) \in$ span $\{\boldsymbol{\psi}_j\}_j$ and $(-\delta^2\triangle + I)\mathbf{w}_k(0) \in$ span$\{\boldsymbol{\psi}_j\}_j$. So, we can use $(-\delta^2\triangle + I)$ $\mathbf{w}_k(0)$ as test function in (2.2.9). We have

$$\big(\mathbf{w}_k(0), (-\delta^2\triangle + I)\mathbf{w}_k(0)\big) = \big(\overline{\mathbf{u}}_0, (-\delta^2\triangle + I)\mathbf{w}_k(0)\big) = (\mathbf{u}_0, \mathbf{w}_k(0)). \tag{2.2.10}$$

Cauchy-Schwarz and Young's inequalities in the right hand side lead to the following à priori estimate

$$\frac{1}{2}||\mathbf{w}_k(0)||^2 + \delta^2||\nabla \mathbf{w}_k(0)||^2 \ \leq \ \frac{1}{2}||\mathbf{u}_0||^2. \tag{2.2.11}$$

Further, in (2.2.3) we can replace $\boldsymbol{\psi}_j$ by $(-\delta^2\triangle + I)D(\mathbf{w}_k) \in \text{span}\{\boldsymbol{\psi}_j\}_j$. Since $(-\delta^2\triangle + I)$ is symmetric, the nonlinear term vanishes:

$$( \nabla \cdot \overline{D(\mathbf{w}_k)D(\mathbf{w}_k)}, \left(-\delta^2\triangle + I\right) D(\mathbf{w}_k) ) = ( \nabla \cdot D(\mathbf{w}_k)D(\mathbf{w}_k), \; D(\mathbf{w}_k) ) = 0.$$

Thus, (2.2.3) becomes

$$( \frac{\partial \mathbf{w}_k}{\partial t}, (-\delta^2\triangle + I)D(\mathbf{w}_k)) + \nu \, (\nabla \mathbf{w}_k, \nabla(-\delta^2\triangle + I)D(\mathbf{w}_k)) = (\mathbf{f}, D(\mathbf{w}_k)).$$

Applying properties **P**1, **P**2, and **P**3 of $D$, and integrating between 0 and $t$ we obtain

$$\begin{aligned}\frac{1}{2}||\mathbf{w}_k(t)||_D^2 + \frac{1}{2}\delta^2||\nabla \mathbf{w}_k(t)||_D^2 + \nu \int_0^t ( \, ||\nabla \mathbf{w}_k(\tau)||_D^2 + \delta^2||\triangle \mathbf{w}_k(\tau)||_D^2 \, )d\tau \\ = \int_0^t (\mathbf{f}, D(\mathbf{w}_k(\tau)))d\tau + \frac{1}{2}||\mathbf{w}_k(0)||_D^2 + \frac{1}{2}\delta^2||\nabla \mathbf{w}_k(0)||_D^2. \end{aligned} \tag{2.2.12}$$

Moreover, since the $D$-norm and the $L^2$-norm are equivalent on $L^2(Q)$ there exists a constant $C$ such that

$$\begin{aligned}\frac{1}{2}||\mathbf{w}_k(t)||^2 + \frac{1}{2}\delta^2||\nabla \mathbf{w}_k(t)||^2 + \nu \int_0^t \left(||\nabla \mathbf{w}_k(\tau)||^2 + \delta^2||\triangle \mathbf{w}_k(\tau)||^2\right) d\tau \\ \leq C \; ( \int_0^t ||\mathbf{f}||^2 d\tau + \int_0^t ||\mathbf{w}_k(\tau)||^2 d\tau + ||\mathbf{w}_k(0)||^2 + \delta^2||\nabla \mathbf{w}_k(0)||^2 \; ). \end{aligned} \tag{2.2.13}$$

Gronwall's inequality in (2.2.13) implies that for all $t \in (0, T]$

$$\begin{aligned}\frac{1}{2}||\mathbf{w}_k(t)||^2 + \frac{1}{2}\delta^2||\nabla \mathbf{w}_k(t)||^2 + \nu \int_0^t ( \, ||\nabla \mathbf{w}_k(\tau)||^2 + \delta^2||\triangle \mathbf{w}_k(\tau)||^2 \, )d\tau \\ \leq Ce^{Ct} \; ( \int_0^t ||\mathbf{f}||^2 d\tau + ||\mathbf{w}_k(0)||^2 + \delta^2||\nabla \mathbf{w}_k(0)||^2 \; ). \end{aligned} \tag{2.2.14}$$

Also, for all $t \in (0, T]$ we have

$$Ce^{Ct}( \int_0^t ||\mathbf{f}||^2 d\tau + ||\mathbf{w}_k(0)||^2 + \delta^2||\nabla \mathbf{w}_k(0)||^2 \, ) \leq Ce^{CT} \, ( \int_0^T ||\mathbf{f}||^2 d\tau + ||\mathbf{w}_k(0)||^2 + \delta^2||\nabla \mathbf{w}_k(0)||^2 \, ).$$

Let $M := \max\{2, 2/\delta\} \, e^{C\,T}(\int_0^T \|\mathbf{f}\|^2 \, d\tau + \|\mathbf{u}_0\|^2)$, so that M is a constant independent of $t$ and $k$. In particular, from (2.2.14) we deduce

$$||\mathbf{w}_k(t)|| \leq M^{1/2} \;\; \text{and} \;\; ||\nabla \mathbf{w}_k(t)|| \leq M^{1/2}. \tag{2.2.15}$$

We also get $||\mathbf{w}_k(t)||^2 = (\mathbf{w}_k(t), \mathbf{w}_k(t)) = \sum_{r=1}^{k} \eta_{kr}^2(t)$ and thus $|\eta_{kr}(t)|$ is bounded,

$$|\eta_{kr}(t)| \leq ( \sum_{r=1}^{k} \eta_{kr}^2(t) )^{1/2} \leq M^{1/2}.$$

We already know that $T_k \leq T$. The à priori bound (2.2.15) and standard results on ordinary differential equations, [14], imply that, in fact, $T_k = T$, for all $k \in \mathbb{N}$.

We also note that estimate (2.2.14) implies that for $\delta > 0$

$$\begin{aligned} &\triangle \mathbf{w}_k \in L^2(0,T; L^2(Q)) \\ &\nabla \mathbf{w}_k \in L^\infty(0,T; L^2(Q)), \end{aligned} \tag{2.2.16}$$

uniformly in $k$, i.e., the indicated norms are bounded uniformly in $k$.

Next, we investigate the properties of convergence of $\{\mathbf{w}_k\}$ as $k \to \infty$. To begin, let $j$ be fixed, but arbitrary. Define the sequence $\left\{ N_k^{(j)}(t) \right\}_k$, where

$$N_k^{(j)}(t) = (\mathbf{w}_k(t), \boldsymbol{\psi}_j), \text{ for } t \in (0,T].$$

We shall first show that the sequence so defined satisfies the properties

**1.** $\left\{ N_k^{(j)}(t) \right\}_k$ is uniformly bounded,

**2.** $\left\{ N_k^{(j)}(t) \right\}_k$ is equicontinuous.

The first property follows from $||\boldsymbol{\psi}_j|| = 1$, the Cauchy- Schwarz inequality, and (2.2.15). Indeed,

$$|N_k^{(j)}(t)| \;\leq\; ||\mathbf{w}_k(t)||\; ||\boldsymbol{\psi}_j|| \;\leq\; M^{1/2}.$$

To prove the second, let us note that for every $t$ and $s$ in $(0,T)$ we have

$$|N_k^{(j)}(t) - N_k^{(j)}(s)| = |\ (\mathbf{w}_k(t) - \mathbf{w}_k(s), \boldsymbol{\psi}_j)\ | = |\sum_{r=1}^{k} (\eta_{kr}(t) - \eta_{kr}(s))(\boldsymbol{\psi}_r, \boldsymbol{\psi}_j)|.$$

We may suppose that $t \geq s$. Integrating between 0 and $t$ in (2.2.7) we first obtain that the coefficients $\eta_{kr}$ satisfy the equation

$$\eta_{kj}(t) = \eta_{kj}(0) + \int_0^t (f_j - \nu \sum_{r=1}^{k} a_{rj}\eta_{kr} - \sum_{r,i=1}^{k} a_{rij}\eta_{kr}\eta_{ki})\ d\tau \tag{2.2.17}$$

and thus

$$|N_k^{(j)}(t) - N_k^{(j)}(s)| \leq \int_s^t |(\,\overline{\mathbf{f}}, \boldsymbol{\psi}_j)| d\tau + \nu \int_s^t |(\nabla \mathbf{w}_k, \nabla \boldsymbol{\psi}_j)| d\tau + \int_s^t |(\nabla \cdot \overline{D(\mathbf{w}_k) D(\mathbf{w}_k)}, \boldsymbol{\psi}_j)| d\tau. \tag{2.2.18}$$

Definition 2.1.1 leads to $||\overline{\mathbf{f}}|| \leq ||\mathbf{f}||$. Using the Cauchy-Schwarz inequality and the orthogonality of $\{\boldsymbol{\psi}_j\}$, we can bound each term in (3.2.23)

$$\begin{aligned}
|(\overline{\mathbf{f}}, \boldsymbol{\psi}_j)| \leq ||\overline{\mathbf{f}}||\ ||\boldsymbol{\psi}_j|| \ &\leq\ ||\mathbf{f}|| \\
|(\nabla \mathbf{w}_k, \nabla \boldsymbol{\psi}_j)| \ &\leq\ M^{1/2}\ ||\nabla \boldsymbol{\psi}_j|| \qquad \text{and} \\
|(\nabla \cdot \overline{D(\mathbf{w}_k) D(\mathbf{w}_k)}, \boldsymbol{\psi}_j)| \ &\leq\ C(||D||)\ \|\boldsymbol{\psi}_j\|_{L^\infty(Q)}\ ||\mathbf{w}_k||\ ||\nabla \mathbf{w}_k|| \\
&\leq\ C(||D||)\ \ \|\boldsymbol{\psi}_j\|_{L^\infty(Q)}\ M^{1/2} M^{1/2}.
\end{aligned}$$

Putting everything together and letting $K_1 = \nu ||\nabla \boldsymbol{\psi}_j||$ and $K_2 = C(||D||) \|\boldsymbol{\psi}_j\|_{L^\infty(Q)}$ we have

$$|N_k^{(j)}(t) - N_k^{(j)}(s)| \ \leq\ \int_s^t (\ ||\mathbf{f}|| + K_1 M^{1/2} + K_2 M\ )\ d\tau. \tag{2.2.19}$$

Furthermore, the Cauchy-Schwarz inequality in time gives:

$$|N_k^{(j)}(t) - N_k^{(j)}(s)| \ \leq\ \sqrt{|t-s|}\ ||\mathbf{f}||_{L^2(s,t;L^2(Q))} + (K_1\, M^{1/2} + K_2\, M)\ |t-s|.$$

Finally, since $\mathbf{f} \in L^2(0,T;L^2(Q))$, our argument is over and $\left\{N_k^{(j)}(t)\right\}_k$ is equicontinuous.

By the Arzela-Ascoli Theorem, from $\{N_k^{(j)}(t)\}$ we may select a subsequence, which we redenote by $\{N_k^{(j)}(t)\}$, uniformly convergent to a continuous function $N^{(j)}(t)$, i.e.:

$$(\mathbf{w}_k(t), \boldsymbol{\psi}_j) \underset{k}{\longrightarrow} N^{(j)}(t) \text{ uniformly in } t. \tag{2.2.20}$$

We remark that the selected sequence (and subsequence) may depend on $j$. Using the Cantor diagonalization method, we end up with a sequence redenoted by $\{N_k^{(j)}(t)\}$, convergent to $N^{(j)}(t)$, for all $j$, uniformly in $t$.

To proceed, fix $t \in (0,T]$. From estimate (2.2.15), we obtain that $\{\mathbf{w}_k(t)\}$ is a bounded sequence in $H$. The weak compactness of closed bounded subsets of $H$ together with (2.2.20) give that there exists a subsequence of $\{\mathbf{w}_k(t)\}$, again denoted by $\{\mathbf{w}_k(t)\}$, weakly convergent to $\mathbf{w}(t) \in H$. Since $t$ was arbitrarily fixed, applying (2.2.20) we obtain

$$(\mathbf{w}_k(t), \boldsymbol{\psi}_j) \underset{k}{\longrightarrow} (\mathbf{w}(t), \boldsymbol{\psi}_j) \text{ uniformly in } t. \tag{2.2.21}$$

Now, fix $t \in (0, T]$ and let $\mathbf{v} \in H$, where $\mathbf{v} = \sum_{i=1}^{\infty} v_i \boldsymbol{\psi}_i$. We have

$$|(\mathbf{w}_k(t) - \mathbf{w}(t), \mathbf{v})| \leq |(\mathbf{w}_k(t) - \mathbf{w}(t), \sum_{i=1}^{N} v_i \boldsymbol{\psi}_i)| + |(\mathbf{w}_k(t) - \mathbf{w}(t), \sum_{i=N}^{\infty} v_i \boldsymbol{\psi}_i)|,$$

for all $N \geq 1$. We note that the RHS of the above inequality approaches 0 as $k \to \infty$. Indeed, let $\mathbf{v} \in H$ be fixed but arbitrary. Then, as above, $\mathbf{v} = \sum_{i=1}^{\infty} v_i \boldsymbol{\psi}_i$ with $v_i := (\mathbf{v}, \boldsymbol{\psi}_i)$. Since $H \subset L^2(Q)$ is a Hilbert space, $\sum_{i=1}^{\infty} v_i \boldsymbol{\psi}_i$ converges. Thus by the Cauchy-Schwarz inequality and (2.2.15), for any $\epsilon > 0$, we can choose $N = N(t)$ independent of $k$ and large enough so that $|(\mathbf{w}_k(t) - \mathbf{w}(t), \sum_{i=N}^{\infty} v_i \boldsymbol{\psi}_i)| < \epsilon/2$. By (2.2.21) we can choose $k(\epsilon)$ so that $|(\mathbf{w}_k(t) - \mathbf{w}(t), \sum_{i=1}^{N} v_i \boldsymbol{\psi}_i)| < \epsilon/2$, for all $k \geq k(\epsilon)$. With these, we obtain that, in fact,

$$(\mathbf{w}_k(t), \mathbf{v}) \underset{k}{\longrightarrow} (\mathbf{w}(t), \mathbf{v}), \ \forall \ \mathbf{v} \in H. \tag{2.2.22}$$

For all $\mathbf{v} \in H$, for each $k$, the scalar-valued function $(\mathbf{w}_k(t), \mathbf{v})$ is measurable on $[0, T]$. Hence, for all $\mathbf{v} \in H$, the function $(\mathbf{w}(t), \mathbf{v})$ is measurable on $[0, T]$. Consequently, applying [15] (Chapter 2, Theorem 2: Pettis measurability theorem), $\mathbf{w}$ is measurable (in the usual sense) on [0,T]. Estimate (2.2.15) gives:

$$\|\mathbf{w}(t)\| \leq \liminf_{k \to \infty} \|\mathbf{w}_k(t)\| \leq M^{1/2}.$$

The above estimate is independent of $t$; and thus, we can conclude that $\mathbf{w} \in L^\infty(0, T; H)$.

By (2.2.16), $\{\mathbf{w}_k\}$ is a bounded sequence in $L^2(0, T; \overline{V}^2_{\#}(Q))$. The weak compactness of closed bounded subsets of $L^2(0, T; \overline{V}^2_{\#}(Q))$ implies that there exists a convergent subsequence of $\{\mathbf{w}_k\}$ in $L^2(0, T; \overline{V}^2_{\#}(Q))$. Again, we redenote the convergent subsequence by $\{\mathbf{w}_k\}$. Then there exists $\mathbf{w}'(t) \in L^2(0, T; \overline{V}^2_{\#}(Q))$ such that

$$\int_0^T < \mathbf{w}_k(t), \mathbf{v}^\star(t) > d\tau \underset{k}{\longrightarrow} \int_0^T < \mathbf{w}'(t), \mathbf{v}^\star(t) > d\tau,$$

for all $\mathbf{v}^\star \in L^2(0, T; (\overline{V}^2_{\#}(Q))^\star)$. However, $\overline{V}^2_{\#}(Q) \subset H \equiv H^\star \subset (\overline{V}^2_{\#}(Q))^\star$.

Following Temam [64] (pag. 168), we have that the scalar product in $H$ of $\boldsymbol{\chi} \in H$ and $\mathbf{u} \in H$ is the same as the scalar product of $\boldsymbol{\chi}$ and $\mathbf{u}$ in the duality between $\overline{V}^2_{\#}(Q)$ and $(\overline{V}^2_{\#}(Q))^{\star}$, i.e.,

$$< \boldsymbol{\chi}, \mathbf{u} >= (\boldsymbol{\chi}, \mathbf{u}) = \int_Q \boldsymbol{\chi}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, d\mathbf{x} \;, \forall \boldsymbol{\chi} \in \overline{V}^2_{\#}(Q) \text{ and } \mathbf{u} \in H.$$

Thus, we can conclude that

$$\int_0^T (\mathbf{w}_k(t), \mathbf{v}(t)) d\tau \underset{k}{\longrightarrow} \int_0^T (\mathbf{w}'(t), \mathbf{v}(t)) d\tau, \tag{2.2.23}$$

for all $\mathbf{v} \in L^2(0,T; L^2(Q))$.

Re-examining our derivation of (2.2.22) above, and the paragraph immediately following (2.2.22), we see that $k(\epsilon)$ can be chosen independently of $t \in [0,T]$ and $\|\mathbf{w}(t)\| \leq M^{1/2}$, for all $t \in [0,T]$. Consequently, for all $\mathbf{v} \in H$,

$$(\mathbf{w}_k(t), \mathbf{v}) \underset{k}{\longrightarrow} (\mathbf{w}(t), \mathbf{v}), \text{ uniformly in } t.$$

Now, fix an arbitrary $\mathbf{v} = \mathbf{v}(\mathbf{x}) \in H$. Next, fix an arbitrary scalar-valued function $q = q(t) \in L^2(0,T)$. We define the $H$-valued function $\mathbf{u} = \mathbf{u}(t)$ by

$$(\mathbf{u}(t))(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t) := q(t)\, \mathbf{v}(\mathbf{x}), \text{ for all } \mathbf{x} \in Q \text{ and } \forall\, t \in [0,T].$$

It is straightforward to check that $\mathbf{u} \in L^2(0,T; H)$, with

$$\|\mathbf{u}\|_{L^2(0,T;H)} = \|q\|_{L^2(0,T)}, \|\mathbf{v}\|_{L^2(Q)}.$$

Replacing $\mathbf{v}(t)$ in (2.2.23) by $\mathbf{u}(t)$, we see that

$$\int_0^T (\mathbf{w}_k(t), \mathbf{v})\, q(t)\, dt \underset{k}{\longrightarrow} \int_0^T (\mathbf{w}'(t), \mathbf{v})\, q(t)\, dt.$$

By (2.2.15) and its consequences, the scalar-valued function $(\mathbf{w}(t), \mathbf{v}) \in L^2(0,T)$, and $(\mathbf{w}_k(t), \mathbf{v}) \in L^2(0,T)$, for all $k \in \mathbb{N}$. Since $q \in L^2(0,T) \subseteq L^1(0,T)$, the uniform convergence on $[0,T]$ discussed above implies that

$$\int_0^T (\mathbf{w}_k(t), \mathbf{v})\, q(t)\, dt \underset{k}{\longrightarrow} \int_0^T (\mathbf{w}(t), \mathbf{v})\, q(t)\, dt.$$

It is also easy to check that $(\mathbf{w}'(t), \mathbf{v}) \in L^2(0,T)$. Since $q \in L^2(0,T)$ is arbitrary, we see that

$$(\mathbf{w}_k(t), \mathbf{v}) \underset{k}{\longrightarrow} (\mathbf{w}'(t), \mathbf{v}) \text{ weakly in } L^2(0,T), \text{ and}$$

$$(\mathbf{w}_k(t), \mathbf{v}) \underset{k}{\longrightarrow} (\mathbf{w}(t), \mathbf{v}) \text{ weakly in } L^2(0,T).$$

By uniqueness of weak limits,

$$(\mathbf{w}'(t), \mathbf{v}) = (\mathbf{w}(t), \mathbf{v}).$$

But $\mathbf{v} \in H$ is arbitrary. Hence,

$$\mathbf{w}'(t) = \mathbf{w}(t) \text{ on } [0,T]; \text{ i.e., } \mathbf{w}' = \mathbf{w}.$$

Consequently, $\mathbf{w} \in L^2(0,T; \overline{V}^2_{\#}(Q)) \cap L^\infty(0,T;H)$. Moreover, along a subsequence (also denoted by $\{\mathbf{w}_k\}$, as usual), we have

$$\mathbf{w}_k \underset{k}{\longrightarrow} \mathbf{w} \text{ weakly in } L^2(0,T; L^2(Q));$$

i.e., for all $\mathbf{u} = \mathbf{u}(t) \in L^2(0,T; L^2(Q))$,

$$\int_0^T (\mathbf{w}_k(t), \mathbf{u}(t))\, dt \underset{k}{\longrightarrow} \int_0^T (\mathbf{w}(t), \mathbf{u}(t))\, dt.$$

From above, we also have

$$\mathbf{w}_k \underset{k}{\longrightarrow} \mathbf{w} \text{ weakly in } L^2(0,T; \overline{V}^2_{\#}(Q));$$

i.e., for all $\mathbf{u}^\star = \mathbf{u}^\star(t) \in L^2(0,T; (\overline{V}^2_{\#}(Q))^\star)$,

$$\int_0^T < \mathbf{w}_k(t), \mathbf{u}^\star(t) > \, dt \underset{k}{\longrightarrow} \int_0^T < \mathbf{w}(t), \mathbf{u}^\star(t) > \, dt.$$

We next wish to complete our proof that $\mathbf{w}$ is a *weak solution* of (2.0.3) by showing that it satisfies (2.1.5) of Definition 2.1.4 (with $\chi = 0$). Fix an arbitrary function $\boldsymbol{\varphi} = \boldsymbol{\varphi}(t) \in D(Q_T)$. Note that for all $t \in [0,T]$ and for all $\mathbf{x} \in Q$,

$$(\boldsymbol{\varphi}(t))(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}, t) = \sum_{j=1}^{\infty} \gamma_j(t)\, \boldsymbol{\psi}_j(\mathbf{x}),$$

where for fixed $t$, the convergence is in the norm of $L^2(Q)$. In particular, for all $t \in [0,T]$,

$$\sum_{j=1}^{\infty} |\gamma_j(t)|^2 < \infty.$$

Also, $\dfrac{\partial \boldsymbol{\varphi}}{\partial t} \in D(Q_T)$, and for all $t \in [0,T]$ and for all $\mathbf{x} \in Q$,

$$\frac{\partial \boldsymbol{\varphi}}{\partial t}(\mathbf{x},t) = \sum_{j=1}^{\infty} \gamma_j'(t)\, \boldsymbol{\psi}_j(\mathbf{x}),$$

and, for all $t \in [0,T]$,

$$\sum_{j=1}^{\infty} |\gamma_j'(t)|^2 < \infty.$$

For all $m \in \mathbb{N}$, we define the function $\boldsymbol{\varphi}_m \in D(Q_T)$ by

$$\boldsymbol{\varphi}_m(\mathbf{x},t) := \sum_{j=1}^{m} \gamma_j(t)\, \boldsymbol{\psi}_j(\mathbf{x}), \text{ for all } \mathbf{x} \in Q \text{ and } t \in [0,T].$$

Fix an arbitrary $m \in \mathbb{N}$. Note that

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_m\|^2_{L^2(0,T;L^2(Q))} \int_0^T \|\boldsymbol{\varphi}(\cdot,t) - \boldsymbol{\varphi}_m(\cdot,t)\|^2_{L^2(Q)}\, dt.$$

Now, for each $t \in [0,T]$,

$$\eta_m(t) := \|\boldsymbol{\varphi}(\cdot,t) - \boldsymbol{\varphi}_m(\cdot,t)\|^2_{L^2(Q)} = \sum_{j=m+1}^{\infty} |\gamma_j(t)|^2 \xrightarrow[m]{} 0.$$

Further, we have that for all $m \in \mathbb{N}$ and for each $t \in [0,T]$,

$$|\eta_m(t)| = \sum_{j=m+1}^{\infty} |\gamma_j(t)|^2 \leq \sum_{j=1}^{\infty} |\gamma_j(t)|^2 = \|\boldsymbol{\varphi}(\cdot,t)\|^2_{L^2(Q)} =: \eta(t).$$

But the function $\boldsymbol{\varphi} \in D(Q_T)$, and so $\boldsymbol{\varphi} : Q_T \longrightarrow \mathbb{R}^n$ ($n = 2$ or $3$) is continuous. Therefore, $\eta \in L^1(0,T)$. Consequently, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_m\|^2_{L^2(0,T;L^2(Q))} &= \int_0^T \|\boldsymbol{\varphi}(\cdot,t) - \boldsymbol{\varphi}_m(\cdot,t)\|^2_{L^2(Q)}\, dt \\ &= \int_0^T \eta_m(t)\, dt \xrightarrow[m]{} 0. \end{aligned}$$

Similarly, we can show that

$$\left\| \frac{\partial \boldsymbol{\varphi}}{\partial t} - \frac{\partial \boldsymbol{\varphi}_m}{\partial t} \right\|^2_{L^2(0,T;L^2(Q))} \xrightarrow[m]{} 0.$$

In order to show that $\mathbf{w}$ satisfies (2.1.5) of Definition 2.1.4, with $\chi = 0$, for our fixed but arbitrary function $\boldsymbol{\varphi} \in D(Q_T)$, we claim it is sufficient to show that for all $m \in \mathbb{N}$,

$$(\mathbf{w}(T), \boldsymbol{\varphi}_m(T)) - \int_0^T \left( (\mathbf{w}, \frac{\partial \boldsymbol{\varphi}_m}{\partial t}) \ - \nu \, (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}_m) - (\nabla \cdot \overline{D(\mathbf{w}) D(\mathbf{w})}, \boldsymbol{\varphi}_m) \right) dt$$
$$= \int_0^T (\overline{\mathbf{f}}, \boldsymbol{\varphi}_m) \, dt + (\mathbf{w}_0, \boldsymbol{\varphi}_m(0)) \, .$$

Indeed, suppose we have done this. Fix $\varepsilon > 0$. Let $\boldsymbol{\delta}_m := \boldsymbol{\varphi} - \boldsymbol{\varphi}_m$, for all $m \in \mathbb{N}$. Choose $m \in \mathbb{N}$ so large that

$$\|\boldsymbol{\delta}_m\|_{L^2(0,T;L^2(Q))} < \varepsilon, \ \left\| \frac{\partial \boldsymbol{\delta}_m}{\partial t} \right\|_{L^2(0,T;L^2(Q))} < \varepsilon, \ \|\boldsymbol{\delta}_m(T)\| < \varepsilon \ \text{ and } \|\boldsymbol{\delta}_m(0)\| < \varepsilon.$$

Then

$$|(\mathbf{w}(T), \boldsymbol{\delta}_m(T))| \leq \|\mathbf{w}(T)\| \, \|\boldsymbol{\delta}_m(T)\| < M^{1/2} \, \varepsilon \quad \text{and}$$
$$|(\mathbf{w}_0, \boldsymbol{\delta}_m(0))| \leq \|\mathbf{w}_0\| \, \|\boldsymbol{\delta}_m(0)\| < \|\mathbf{w}_0\| \, \varepsilon.$$

Further,

$$\left| \int_0^T \left( \mathbf{w}, \frac{\partial \boldsymbol{\delta}_m}{\partial t} \right) \, dt \right| \leq \|\mathbf{w}\|_{L^2(0,T;L^2(Q))} \left\| \frac{\partial \boldsymbol{\delta}_m}{\partial t} \right\|_{L^2(0,T;L^2(Q))} < M^{1/2} \, T^{1/2} \, \varepsilon,$$

$$\begin{aligned} \left| \int_0^T (\overline{\mathbf{f}}, \boldsymbol{\delta}_m) \, dt \right| \ &\leq \ \|\overline{\mathbf{f}}\|_{L^2(0,T;L^2(Q))} \, \|\boldsymbol{\delta}_m\|_{L^2(0,T;L^2(Q))} \\ &\leq \ \|\mathbf{f}\|_{L^2(0,T;L^2(Q))} \, \varepsilon, \ \text{ and} \end{aligned}$$

$$\begin{aligned} \left| \int_0^T \nu \, (\nabla \mathbf{w}, \nabla \boldsymbol{\delta}_m) \, dt \right| \ &= \ \left| \int_0^T \nu \, (\triangle \mathbf{w}, \boldsymbol{\delta}_m) \, dt \right| \\ &\leq \ \nu \, \|\triangle \mathbf{w}\|_{L^2(0,T;L^2(Q))} \, \|\boldsymbol{\delta}_m\|_{L^2(0,T;L^2(Q))} \\ &< \ \nu \, \|\triangle \mathbf{w}\|_{L^2(0,T;L^2(Q))} \, \varepsilon. \end{aligned}$$

Fix an arbitrary $t \in [0, T]$. Then by the argument above, from (2.2.12) through (2.2.15), (by passing to a subsequence if necessary) we can show that $\{\mathbf{w}_k(t)\}_k$ converges weakly in

$\overline{V}^1_{\#}(Q)$, this weak limit must be $\mathbf{w}(t)$ and (by the weak lower semicontinuity of the semi-norm $\|\nabla(\cdot)\|$ on $\overline{V}^1_{\#}(Q)$)

$$\|\nabla \mathbf{w}(t)\| \leq M^{1/2}.$$

Lastly, by the Ladyzhenskaya inequalities [43],

$$\begin{aligned}
&\left|\int_0^T (\nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, \boldsymbol{\delta}_m)\, dt\right| \\
&\quad = \left|\int_0^T (\nabla \cdot D(\mathbf{w})D(\mathbf{w}), \overline{\boldsymbol{\delta}_m})\, dt\right| \\
&\quad = \left|\int_0^T (D(\mathbf{w}) \cdot \nabla D(\mathbf{w}), \overline{\boldsymbol{\delta}_m})\, dt\right| \\
&\quad \leq \int_0^T \left|(D(\mathbf{w}) \cdot \nabla D(\mathbf{w}), \overline{\boldsymbol{\delta}_m})\right|\, dt \\
&\quad \leq C_1 \int_0^T \|D(\mathbf{w})\|_{L^3(Q)}\, \|\nabla D(\mathbf{w})\|_{L^2(Q)}\, \|\overline{\boldsymbol{\delta}_m}\|_{L^6(Q)}\, dt \\
&\quad \leq C_2 \int_0^T \|D(\mathbf{w})\|^{1/2}_{L^2(Q)}\, \|\nabla D(\mathbf{w})\|^{1/2}_{L^2(Q)}\, \|D(\nabla \mathbf{w})\|_{L^2(Q)}\, \|\nabla\overline{\boldsymbol{\delta}_m}\|_{L^2(Q)}\, dt \\
&\quad \leq C_2\, \|D\|^2 \int_0^T \|\mathbf{w}\|^{1/2}_{L^2(Q)}\, \|\nabla \mathbf{w}\|^{1/2}_{L^2(Q)}\, \|\nabla \mathbf{w}\|_{L^2(Q)}\, C_3(\delta)\, \|\boldsymbol{\delta}_m\|_{L^2(Q)}\, dt \\
&\quad \leq C_2\, \|D\|^2\, M\, C_3(\delta) \int_0^T \|\boldsymbol{\delta}_m(t)\|_{L^2(Q)}\, dt \\
&\quad \leq C_2\, \|D\|^2\, M\, C_3(\delta)\, T^{1/2}\, \|\boldsymbol{\delta}_m\|_{L^2(0,T;L^2(Q))} \\
&\quad < C_2\, \|D\|^2\, M\, C_3(\delta)\, T^{1/2}\, \varepsilon.
\end{aligned}$$

Consequently, since $\varepsilon > 0$ is arbitrary, it follows that

$$\begin{aligned}
(\mathbf{w}(T), \boldsymbol{\varphi}(T)) - \int_0^T \left(\left(\mathbf{w}, \frac{\partial \boldsymbol{\varphi}}{\partial t}\right) - \nu\, (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}) - (\nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, \boldsymbol{\varphi})\right) dt \\
= \int_0^T (\overline{\mathbf{f}}, \boldsymbol{\varphi})\, dt + (\mathbf{w}_0, \boldsymbol{\varphi}(0))\,.
\end{aligned}$$

It remains to show that for all $m \in \mathbb{N}$,

$$\begin{aligned}
(\mathbf{w}(T), \boldsymbol{\varphi}_m(T)) - \int_0^T \left(\left(\mathbf{w}, \frac{\partial \boldsymbol{\varphi}_m}{\partial t}\right) - \nu\, (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}_m) - (\nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, \boldsymbol{\varphi}_m)\right) dt \\
= \int_0^T (\overline{\mathbf{f}}, \boldsymbol{\varphi}_m)\, dt + (\mathbf{w}_0, \boldsymbol{\varphi}_m(0))\,.
\end{aligned}$$

Again fix an arbitrary $m \in \mathbb{N}$. Recall that for each $k \in \mathbb{N}$, $\{\mathbf{w}_k\}$ has been constructed in such a way that it satisfies equations (2.2.3) and (2.2.4), for all $j = 1, \ldots, k$. Fix $k \geq m$. For every $j \in \{1, \ldots, k\}$, multiply equation (2.2.3) through by $\gamma_j(t)$; and then sum both sides from $j = 1$ to $m$. Finally, integrate both sides from 0 to $T$, with respect to $t$. From this we get that for all $k \geq m$,

$$
\begin{aligned}
(\mathbf{w}_k(T), \boldsymbol{\varphi}_m(T)) - \int_0^T \left(\mathbf{w}_k, \frac{\partial \boldsymbol{\varphi}_m}{\partial t}\right) dt + \nu \int_0^T (\nabla \mathbf{w}_k, \nabla \boldsymbol{\varphi}_m)\, dt \\
+ \int_0^T (\nabla \cdot \overline{D(\mathbf{w}_k) D(\mathbf{w}_k)}, \boldsymbol{\varphi}_m)\, dt = \int_0^T (\overline{\mathbf{f}}, \boldsymbol{\varphi}_m)\, dt + (\mathbf{w}_0, \boldsymbol{\varphi}_m(0))\,.
\end{aligned}
$$

From the convergence properties of $\{\mathbf{w}_k\}_k$ that we have discovered so far,

$$
(\mathbf{w}_k(T), \boldsymbol{\varphi}_m(T)) \underset{k}{\longrightarrow} (\mathbf{w}(T), \boldsymbol{\varphi}_m(T)), \text{ and}
$$

$$
\int_0^T \left(\mathbf{w}_k, \frac{\partial \boldsymbol{\varphi}_m}{\partial t}\right) dt \underset{k}{\longrightarrow} \int_0^T \left(\mathbf{w}, \frac{\partial \boldsymbol{\varphi}_m}{\partial t}\right) dt.
$$

Also, for all $\mathbf{h} = (\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ we define $\widetilde{\mathbf{h}} \in (\overline{V}^2_{\#}(Q))^{\star}$, by

$$
\widetilde{\mathbf{h}}(\mathbf{q}) := < \mathbf{q}, \widetilde{\mathbf{h}} > := (\nabla \mathbf{q}, \mathbf{h}) := \sum_{l=1}^{n} (\nabla q_l, \mathbf{h}_l), \text{ for all } \mathbf{q} \in \overline{V}^2_{\#}(Q).
$$

Next, we define $\mathbf{u}^{\star} : [0, T] \longrightarrow (\overline{V}^2_{\#}(Q))^{\star}$ by

$$
\mathbf{u}^{\star}(t) := (\nabla \widetilde{\boldsymbol{\varphi}_m(\cdot, t)}), \text{ for all } t \in [0, T].
$$

It is easy to check that $\mathbf{u}^{\star} \in L^2(0, T; (\overline{V}^2_{\#}(Q))^{\star})$. Hence,

$$
\begin{aligned}
\int_0^T (\nabla \mathbf{w}_k, \nabla \boldsymbol{\varphi}_m)\, dt &= \int_0^T (\nabla \mathbf{w}_k(\cdot, t), \nabla \boldsymbol{\varphi}_m(\cdot, t))\, dt \\
&= \int_0^T < \mathbf{w}_k(\cdot, t), \mathbf{u}^{\star}(t) > \, dt \\
&\underset{k}{\longrightarrow} \int_0^T < \mathbf{w}(\cdot, t), \mathbf{u}^{\star}(t) > \, dt \\
&= \int_0^T (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}_m)\, dt.
\end{aligned}
$$

Further,

$$
\begin{aligned}
& \left|\int_0^T (\nabla \cdot \overline{D(\mathbf{w}_k)D(\mathbf{w}_k)}, \boldsymbol{\varphi}_m)\,dt - \int_0^T (\nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, \boldsymbol{\varphi}_m)\,dt\right| \\
= & \left|\int_0^T (\nabla \cdot D(\mathbf{w}_k)D(\mathbf{w}_k), \overline{\boldsymbol{\varphi}_m})\,dt - \int_0^T (\nabla \cdot D(\mathbf{w})D(\mathbf{w}), \overline{\boldsymbol{\varphi}_m})\,dt\right| \\
= & \left|\int_0^T (D(\mathbf{w}_k) \cdot \nabla D(\mathbf{w}_k), \overline{\boldsymbol{\varphi}_m})\,dt - \int_0^T (D(\mathbf{w}) \cdot \nabla D(\mathbf{w}), \overline{\boldsymbol{\varphi}_m})\,dt\right| \\
= & \left|\int_0^T (D(\mathbf{w}_k) \cdot \nabla D(\mathbf{w}_k) - D(\mathbf{w}) \cdot \nabla D(\mathbf{w}), \overline{\boldsymbol{\varphi}_m})\,dt\right| \\
= & \left|\int_0^T ((D(\mathbf{w}_k) - D(\mathbf{w})) \cdot \nabla D(\mathbf{w}_k) + D(\mathbf{w}) \cdot (\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})\,dt\right| \\
\leq & \int_0^T |((D(\mathbf{w}_k) - D(\mathbf{w})) \cdot \nabla D(\mathbf{w}_k), \overline{\boldsymbol{\varphi}_m})|\,dt \\
& \quad + \left|\int_0^T (D(\mathbf{w}) \cdot (\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})\,dt\right|.
\end{aligned}
$$

Applying the Ladyzhenskaya inequalities [43] again, we get that

$$
\begin{aligned}
& \int_0^T |((D(\mathbf{w}_k) - D(\mathbf{w})) \cdot \nabla D(\mathbf{w}_k), \overline{\boldsymbol{\varphi}_m})|\,dt \\
\leq & \; C_1 \int_0^T \|D(\mathbf{w}_k) - D(\mathbf{w})\|^{1/2} \, \|\nabla(D(\mathbf{w}_k) - D(\mathbf{w}))\|^{1/2} \, \|\nabla D(\mathbf{w}_k)\| \, \|\nabla \overline{\boldsymbol{\varphi}_m}\|\,dt \\
\leq & \; C_2 \|D\|^2 \int_0^T \|\mathbf{w}_k(t) - \mathbf{w}(t)\|^{1/2} \, \|\nabla \mathbf{w}_k(t) - \nabla \mathbf{w}(t)\|^{1/2} \, \|\nabla \mathbf{w}_k(t)\| \, \|\boldsymbol{\varphi}_m(\cdot, t)\|\,dt \\
\leq & \; C_2 \|D\|^2 M \, \|\mathbf{w}_k - \mathbf{w}\|^{1/2}_{L^2(0,T;L^2(Q))} \, \|\nabla \mathbf{w}_k - \nabla \mathbf{w}\|^{1/2}_{L^2(0,T;L^2(Q))} \, \|\boldsymbol{\varphi}_m\|_{L^2(0,T;L^2(Q))} \\
\leq & \; C_3 \|\mathbf{w}_k - \mathbf{w}\|^{1/2}_{L^2(0,T;L^2(Q))} \, (M^{1/2} T^{1/2} + \|\nabla \mathbf{w}\|_{L^2(0,T;L^2(Q))})^{1/2} \, \|\boldsymbol{\varphi}_m\|_{L^2(0,T;L^2(Q))}.
\end{aligned}
$$

So, in order to show that $\mathbf{w}$ is a *weak solution* it remains for us to verify that

$$
\|\mathbf{w}_k - \mathbf{w}\|_{L^2(0,T;L^2(Q))} \underset{k}{\longrightarrow} 0, \text{ and}
$$

$$
\left|\int_0^T (D(\mathbf{w}) \cdot (\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})\,dt\right| \underset{k}{\longrightarrow} 0.
$$

Here $n = 2$ or 3, the space dimension of our fluid flow and recall that $Q = [0, L]^n$. We introduce a key lemma of Friedrichs (1933), following Galdi [25], Lemma 4.2, page 53. We subdivide $Q$ into $N$ smaller cubes $C_i$ of equal size, each having side-length $\alpha = L/N^{1/n}$, so that $N = (L/\alpha)^n$, for some $\alpha > 0$ such that $N^{1/n} = L/\alpha \in \mathbb{N}$. For all $i \in \{1, \dots, N\}$, define the function $\gamma_i$ by

$$\gamma_i(\mathbf{x}) := 2 \left( \frac{L}{N^{1/n}} \right)^{-n/2} \chi_{C_i}(\mathbf{x}), \text{ for all } \mathbf{x} \in Q,$$

where $\chi_{C_i}$ denotes *the characteristic function* of the sub-cube $C_i$. Then, by Friedrichs' Lemma, we have that for all scalar-valued functions $u : Q \longrightarrow \mathbb{R}$ with $u \in W^{1,2}(Q)$,

$$\|u\|^2_{L^2(Q)} \leq \sum_{i=1}^{N} \left| \int_Q u(\mathbf{x})\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 + \frac{(2\,n\,L)^2}{N^{2/n}} \|\nabla u\|^2_{L^2(Q)}.$$

Fix $k \in \mathbb{N}$ with $k \geq m$, and fix $N \in \mathbb{N}$ as described above. Let $\boldsymbol{\eta}_k := \mathbf{w}_k - \mathbf{w}$. Then $\boldsymbol{\eta}_k = (\eta_{k,l})_{l=1}^n$, where each scalar-valued coordinate function $\eta_{k,l}$ is such that for all $t \in [0, T]$, $\eta_{k,l}(\cdot, t) \in W^{2,2}(Q) \subseteq W^{1,2}(Q)$. Thus,

$$\begin{aligned}
&\|\mathbf{w}_k - \mathbf{w}\|_{L^2(0,T;L^2(Q))} = \|\boldsymbol{\eta}_k\|_{L^2(0,T;L^2(Q))} \\
&= \int_0^T \|\boldsymbol{\eta}_k(\cdot, t)\|^2_{L^2(Q)}\, dt \\
&= \sum_{l=1}^{n} \int_0^T \|\eta_{k,l}(\cdot, t)\|^2_{L^2(Q)}\, dt \\
&\leq \sum_{l=1}^{n} \int_0^T \left( \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 + \frac{(2\,n\,L)^2}{N^{2/n}} \|\nabla \eta_{k,l}(\cdot, t)\|^2_{L^2(Q)} \right) dt \\
&= \sum_{l=1}^{n} \int_0^T \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 dt + \frac{(2\,n\,L)^2}{N^{2/n}} \int_0^T \sum_{l=1}^{n} \|\nabla \eta_{k,l}(\cdot, t)\|^2_{L^2(Q)}\, dt \\
&= \sum_{l=1}^{n} \int_0^T \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 dt + \frac{(2\,n\,L)^2}{N^{2/n}} \int_0^T \|\nabla \boldsymbol{\eta}_k(\cdot, t)\|^2_{L^2(Q)}\, dt \\
&\leq \sum_{l=1}^{n} \int_0^T \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 dt \\
&\qquad + \frac{(2\,n\,L)^2}{N^{2/n}} \int_0^T (\|\nabla \mathbf{w}_k(\cdot, t)\|_{L^2(Q)} + \|\nabla \mathbf{w}(\cdot, t)\|_{L^2(Q)})^2\, dt \\
&\leq \sum_{l=1}^{n} \int_0^T \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 dt + \frac{(2\,n\,L)^2}{N^{2/n}}\, 2\, (M\,T + \|\mathbf{w}\|^2_{L^2(0,T;L^2(Q))}).
\end{aligned}$$

Now, fix $\varepsilon > 0$. Clearly, we can choose an $N$ that is independent of $k$ and so large that

$$\|\mathbf{w}_k - \mathbf{w}\|_{L^2(0,T;L^2(Q))} \leq \sum_{l=1}^{n} \int_0^T \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 dt + \frac{\varepsilon}{2}.$$

For definiteness (and without loss of generality), we will henceforth suppose that $n = 3$. For all $i \in \{1, \ldots, N\}$, for every $\mathbf{x} \in Q$, define

$$\boldsymbol{\gamma}_{i,1}(\mathbf{x}) := (\gamma_i(\mathbf{x}), 0, 0)\ ,\ \boldsymbol{\gamma}_{i,2}(\mathbf{x}) := (0, \gamma_i(\mathbf{x}), 0)\ \text{ and } \boldsymbol{\gamma}_{i,3}(\mathbf{x}) := (0, 0, \gamma_i(\mathbf{x})).$$

Each function $\boldsymbol{\gamma}_{i,l} : Q \longrightarrow \mathbb{R}^3$ belongs to $L^2(Q)$. Moreover,

$$\begin{aligned}
\|\mathbf{w}_k - \mathbf{w}\|_{L^2(0,T;L^2(Q))} &\leq \sum_{l=1}^{n} \int_0^T \sum_{i=1}^{N} \left| \int_Q \eta_{k,l}(\mathbf{x}, t)\, \gamma_i(\mathbf{x})\, d\mathbf{x} \right|^2 dt + \frac{\varepsilon}{2} \\
&= \sum_{l=1}^{3} \sum_{i=1}^{N} \int_0^T \left| \int_Q \boldsymbol{\eta}_k(\mathbf{x}, t) \cdot \boldsymbol{\gamma}_{i,l}(\mathbf{x})\, d\mathbf{x} \right|^2 dt + \frac{\varepsilon}{2} \\
&= \sum_{l=1}^{3} \sum_{i=1}^{N} \int_0^T \left| (\boldsymbol{\eta}_k(\cdot, t), \boldsymbol{\gamma}_{i,l}) \right|^2 dt + \frac{\varepsilon}{2}.
\end{aligned}$$

Recall that for all $\mathbf{v} \in H$,

$$(\mathbf{w}_k(t), \mathbf{v}) \underset{k}{\longrightarrow} (\mathbf{w}(t), \mathbf{v}),\ \text{ uniformly in } t \in [0, T].$$

But, each $\mathbf{w}_k(t) \in H$ and $\mathbf{w}(t) \in H$; and so, for all $\mathbf{v} \in L^2(Q)$,

$$(\boldsymbol{\eta}_k(\cdot, t), \mathbf{v})(\mathbf{w}_k(\cdot, t) - \mathbf{w}(\cdot, t), \mathbf{v}) \underset{k}{\longrightarrow} 0,\ \text{ uniformly in } t \in [0, T].$$

When we apply this fact to each of the finitely many $\mathbf{v} \in \{\boldsymbol{\gamma}_{i,l} : 1 \leq i \leq N \text{ and } 1 \leq l \leq 3\}$, we see that there exists $k(\varepsilon) \in \mathbb{N}$ so that for all $k \geq k(\varepsilon)$,

$$\|\mathbf{w}_k - \mathbf{w}\|_{L^2(0,T;L^2(Q))} \leq \sum_{l=1}^{3} \sum_{i=1}^{N} \int_0^T \left| (\boldsymbol{\eta}_k(\cdot, t), \boldsymbol{\gamma}_{i,l}) \right|^2 dt + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we now know that

$$\|\mathbf{w}_k - \mathbf{w}\|_{L^2(0,T;L^2(Q))} \underset{k}{\longrightarrow} 0.$$

The last fact we will establish in detail is that

$$\left| \int_0^T (D(\mathbf{w}) \cdot (\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})\, dt \right| \underset{k}{\longrightarrow} 0.$$

Fix $t \in [0,T]$. Let $\boldsymbol{\alpha} := D(\mathbf{w})$, $\boldsymbol{\beta} := D(\mathbf{w}_k) - D(\mathbf{w}) = D\boldsymbol{\eta}_k$ and $\boldsymbol{\gamma} := \overline{\boldsymbol{\varphi}_m}$. Then

$$
\begin{aligned}
&(D(\mathbf{w}) \cdot (\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})(t) \\
&= \int_Q (D(\mathbf{w}) \cdot \nabla(D(\mathbf{w}_k) - D(\mathbf{w})))(\mathbf{x},t) \cdot \overline{\boldsymbol{\varphi}_m}(\mathbf{x},t)\, d\mathbf{x} \\
&= \int_Q (\boldsymbol{\alpha} \cdot \nabla \boldsymbol{\beta})(\mathbf{x},t) \cdot \boldsymbol{\gamma}(\mathbf{x},t)\, d\mathbf{x} = \int_Q (\boldsymbol{\alpha} \cdot \nabla \boldsymbol{\beta}) \cdot \boldsymbol{\gamma}\, d\mathbf{x} \\
&= \int_Q \left( \sum_{l=1}^{3} \alpha_l \frac{\partial \beta_1}{\partial x_l}, \sum_{l=1}^{3} \alpha_l \frac{\partial \beta_2}{\partial x_l}, \sum_{l=1}^{3} \alpha_l \frac{\partial \beta_3}{\partial x_l} \right) \cdot (\gamma_1, \gamma_2, \gamma_3)\, d\mathbf{x} \\
&= \int_Q \sum_{s=1}^{3} \sum_{l=1}^{3} \alpha_l \frac{\partial \beta_s}{\partial x_l} \gamma_s\, d\mathbf{x} = \sum_{l=1}^{3} \int_Q \sum_{s=1}^{3} \frac{\partial \beta_s}{\partial x_l} \alpha_l \gamma_s\, d\mathbf{x} \\
&= \sum_{l=1}^{3} \int_Q \left( \frac{\partial \boldsymbol{\beta}}{\partial x_l} \right) (\mathbf{x},t) \cdot (\alpha_l \boldsymbol{\gamma})(\mathbf{x},t)\, d\mathbf{x} \\
&= \sum_{l=1}^{3} \left( \frac{\partial \boldsymbol{\beta}}{\partial x_l}(\cdot,t), \mathbf{v}_l(\cdot,t) \right),
\end{aligned}
$$

where

$$
\mathbf{v}_l(\mathbf{x},t) := \alpha_l(\mathbf{x},t)\, \boldsymbol{\gamma}(\mathbf{x},t) = (D(\mathbf{w}))_l(\mathbf{x},t)\, \overline{\boldsymbol{\varphi}}_m(\mathbf{x},t)\ , \text{ for all } (\mathbf{x},t) \in Q_T;
$$

for all $l \in \{1,2,3\}$. Now, $\overline{\boldsymbol{\varphi}}_m \in \mathcal{D}(Q_T) \subseteq L^\infty(Q_T)$ and $D(\mathbf{w}) \in L^2(0,T;L^2(Q))$. It is therefore easy to check that $\mathbf{v}_l \in L^2(0,T;L^2(Q))$, for all $l \in \{1,2,3\}$.

Consequently,

$$
\begin{aligned}
&(D(\mathbf{w}) \cdot (\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})(t) \\
&= \sum_{l=1}^{3} \left( \frac{\partial \boldsymbol{\beta}}{\partial x_l}(\cdot,t), \mathbf{v}_l(\cdot,t) \right) = \sum_{l=1}^{3} \left( \frac{\partial (D\boldsymbol{\eta}_k)}{\partial x_l}(\cdot,t), \mathbf{v}_l(\cdot,t) \right) \\
&= \sum_{l=1}^{3} \left( D\, \frac{\partial \boldsymbol{\eta}_k}{\partial x_l}(\cdot,t), \mathbf{v}_l(\cdot,t) \right) = \sum_{l=1}^{3} \left( \frac{\partial \boldsymbol{\eta}_k}{\partial x_l}(\cdot,t), D\, \mathbf{v}_l(\cdot,t) \right) \\
&= \sum_{l=1}^{3} \left( \frac{\partial \boldsymbol{\eta}_k}{\partial x_l}(\cdot,t), \boldsymbol{\sigma}_l(\cdot,t) \right),
\end{aligned}
$$

where $\boldsymbol{\sigma}_l := D(\mathbf{v}_l) \in L^2(0,T;L^2(Q))$, for all $l \in \{1,2,3\}$.

Fix an arbitrary $l \in \{1,2,3\}$. For every $\mathbf{h} \in L^2(Q)$, we define $\mathbf{h}^{(l)} \in (\overline{V}^2_{\#}(Q))^\star$, by

$$
\mathbf{h}^{(l)}(\mathbf{q}) := < \mathbf{q}, \mathbf{h}^{(l)} > := \left( \frac{\partial \mathbf{q}}{\partial x_l}, \mathbf{h} \right), \text{ for all } \mathbf{q} \in \overline{V}^2_{\#}(Q).
$$

Next, we define $\mathbf{u}^{\star(l)} : [0,T] \longrightarrow (\overline{V}^2_{\#}(Q))^{\star}$ by

$$\mathbf{u}^{\star(l)}(t) := (\boldsymbol{\sigma}_l(\cdot,t))^{(l)}, \text{ for all } t \in [0,T].$$

It is easy to check that $\mathbf{u}^{\star(l)} \in L^2(0,T;(\overline{V}^2_{\#}(Q))^{\star})$. Also, recall that each $\boldsymbol{\eta}_k := \mathbf{w}_k - \mathbf{w} \in L^2(0,T;\overline{V}^2_{\#}(Q))$. Hence,

$$\begin{aligned}
&\int_0^T (D(\mathbf{w})\cdot(\nabla D(\mathbf{w}_k) - \nabla D(\mathbf{w})), \overline{\boldsymbol{\varphi}_m})(t)\,dt \\
&\quad = \sum_{l=1}^{3} \int_0^T \left( \frac{\partial \boldsymbol{\eta}_k}{\partial x_l}(\cdot,t), \boldsymbol{\sigma}_l(\cdot,t)\right) dt \\
&\quad = \sum_{l=1}^{3} \int_0^T < \boldsymbol{\eta}_k(\cdot,t), \mathbf{u}^{\star(l)}(t) > dt \\
&\quad = \sum_{l=1}^{3} \int_0^T < (\mathbf{w}_k - \mathbf{w})(\cdot,t), \mathbf{u}^{\star(l)}(t) > dt \\
&\quad = \sum_{l=1}^{3} \left( \int_0^T < \mathbf{w}_k(\cdot,t), \mathbf{u}^{\star(l)}(t) > dt - \int_0^T < \mathbf{w}(\cdot,t), \mathbf{u}^{\star(l)}(t) > dt \right) \underset{k}{\longrightarrow} 0.
\end{aligned}$$

So, at last, we have completed our proof that $\mathbf{w}$ is a *weak solution* of (2.0.3); i.e., $\mathbf{w}$ satisfies (2.1.5) of Definition 2.1.4 (with $\chi = 0$), for all $\boldsymbol{\varphi} \in D(Q_T)$.

We now prove the stability bound (2.2.1). Fix an arbitrary $t \in [0,T]$. Then by the argument above, from (2.2.12) through (2.2.15), (by passing to a subsequence if necessary) we can show that $\{\mathbf{w}_k(t)\}_k$ converges weakly in $\overline{V}^1_{\#}(Q)$, this weak limit must be $\mathbf{w}(t)$ and

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{w}_k(t)\|^2 + \frac{1}{2}\delta^2\|\nabla \mathbf{w}_k(t)\|^2 + \nu \int_0^t (\|\nabla \mathbf{w}_k(\tau)\|^2 + \delta^2\|\triangle \mathbf{w}_k(\tau)\|^2)\,d\tau \\
&\qquad \le C\left( \left| \int_0^t (\mathbf{f}(\tau), D(\mathbf{w}_k(\tau)))\,d\tau \right| + \|\mathbf{u}_0\|^2 \right),
\end{aligned}$$

for all $k \in \mathbb{N}$. Moreover, $\{\mathbf{w}_k\}_k$ converges weakly to $\mathbf{w}$ in $L^2(0,t;\overline{V}^2_{\#}(Q))$. By the weak lower semicontinuity of the norms in $\overline{V}^1_{\#}(Q)$ and $L^2(0,t;\overline{V}^2_{\#}(Q))$, it follows that

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{w}(t)\|^2 + \frac{1}{2}\delta^2\|\nabla \mathbf{w}(t)\|^2 + \nu \int_0^t (\|\nabla \mathbf{w}(\tau)\|^2 + \delta^2\|\triangle \mathbf{w}(\tau)\|^2)\,d\tau \\
&\qquad \le C\left( \left| \int_0^t (\mathbf{f}(\tau), D(\mathbf{w}(\tau)))\,d\tau \right| + \|\mathbf{u}_0\|^2 \right).
\end{aligned}$$

This concludes our proof when $\chi = 0$. The proof for $\chi > 0$ follows in the same way. Indeed, the á priori bound is uneffected since $\chi(\mathbf{w}^*, A\ D\mathbf{w}) = \chi(A\ D\mathbf{w}^*, \mathbf{w}) \geq 0$. Further, extracting the limit as $k \to \infty$ follows by an analogous argument.

□

**Proposition 2.2.1.** *Let* $\mathbf{w}$ *be a weak solution of the model (2.0.3); i.e.,* $\mathbf{w}$ *satisfies Definition 2.1.4. Let's define the linear functional* $\Lambda$ *on* $D(Q_T)$ *by*

$$\Lambda(\boldsymbol{\varphi}) := (\mathbf{w}(T), \boldsymbol{\varphi}(T)) - (\mathbf{w}_0, \boldsymbol{\varphi}(0)) - \int_0^T \left(\mathbf{w}, \frac{\partial \boldsymbol{\varphi}}{\partial t}\right) dt,$$

*for all* $\boldsymbol{\varphi} \in D(Q_T)$*. Then* $\Lambda$ *is a continuous linear functional on* $D(Q_T)$ *endowed with the* $L^2(0,T;L^2(Q))$*-norm. Consequently,* $\Lambda$ *extends uniquely to a continuous linear functional on* $L^2(0,T;H)$ *and there exists a unique* $\mathbf{u} \in L^2(0,T;H)$ *such that* $\Lambda(\mathbf{v}) =< \mathbf{u}, \mathbf{v} >_{L^2(0,T;L^2(Q))} := \int_0^T (\mathbf{u}(\tau), \mathbf{v}(\tau))\, d\tau$*, for all* $\mathbf{v} \in L^2(0,T;H)$*. We denote* $\mathbf{u}$ *by*

$$\mathbf{u} = \mathbf{w}_t = \frac{\partial \mathbf{w}}{\partial t} \in L^2(0,T;H). \tag{2.2.24}$$

*Proof.* Fix an arbitrary function $\boldsymbol{\varphi} \in D(Q_T)$. By equation (2.1.5) of Definition 2.1.4 and our definition of $\Lambda$,

$$\Lambda(\boldsymbol{\varphi}) = \int_0^T \Big(-\nu\, (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi}) - (\nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, \boldsymbol{\varphi}) - \chi\, (\mathbf{w}^*, \boldsymbol{\varphi})\Big)\, dt + \int_0^T (\overline{\mathbf{f}}, \boldsymbol{\varphi})\, dt.$$

In an analogous manner to part of the proof of Theorem 2.2.1 above, we can show that there exists a positive real constant $C = C(\delta, \chi, \|I - DG\|, \|D\|, M, \|\triangle \mathbf{w}\|_{L^2(0,T;L^2(Q))})$, independent of $\boldsymbol{\varphi}$, such that

$$|\Lambda(\boldsymbol{\varphi})| \leq C\, \|\boldsymbol{\varphi}\|_{L^2(0,T;L^2(Q))}.$$

Thus, $\Lambda$ is a continuous linear functional on $D(Q_T)$ endowed with the $L^2(0,T;L^2(Q))$-norm. But $D(Q_T)$ is norm dense in $L^2(0,T;H)$. Consequently, $\Lambda$ extends uniquely to a continuous linear functional, also denoted by $\Lambda$, on $L^2(0,T;H)$. By the Riesz Representation Theorem in the Hilbert space $L^2(0,T;H)$, there exists a unique $\mathbf{u} \in L^2(0,T;H)$ such that

$$\Lambda(\mathbf{v}) =< \mathbf{u}, \mathbf{v} >_{L^2(0,T;L^2(Q))} := \int_0^T (\mathbf{u}(\tau), \mathbf{v}(\tau))\, d\tau,$$

for all $\mathbf{v} \in L^2(0,T;H)$; which we denote by $\mathbf{u} = \mathbf{w}_t$. □

**Remark 2.2.2.** *Let* $\mathbf{w}$ *be a weak solution of the model (2.0.3); i.e.,* $\mathbf{w}$ *satisfies Definition 2.1.4. Let* $<\cdot,\cdot>$ *denote the usual inner product on* $L^2(0,T;L^2(Q))$.

*An important consequence of Proposition 2.2.1 is that*

$$< \mathbf{w}_t + \nabla\cdot\overline{D(\mathbf{w})D(\mathbf{w})} - \nu\,\triangle\mathbf{w} + \nabla q + \chi\,\mathbf{w}^* - \overline{\mathbf{f}}, \mathbf{v} >= 0,$$

*for all* $\mathbf{v}\in L^2(0,T;H)$. *Therefore, in the Hilbert space* $L^2(0,T;H)$,

$$\mathbf{w}_t + \nabla\cdot\overline{D(\mathbf{w})D(\mathbf{w})} - \nu\,\triangle\mathbf{w} + \nabla q + \chi\,\mathbf{w}^* - \overline{\mathbf{f}} = 0.$$

*It follows that* $(\mathbf{w}, q)$ *is a strong solution of the deconvolution model (2.0.3) (as described in Definition 2.1.5).*

## 2.3 ADM ENERGY BALANCE AND UNIQUENESS

In this section we prove that the weak solution of the ADM (2.0.3) is a unique strong solution. We also show that the model satisfies an energy equality rather than inequality. In our proofs we include the case when $I-DG$ is symmetric positive semi-definite and thus $\boldsymbol{\phi}^* := (I-DG)\boldsymbol{\phi}$. First we prove uniqueness.

**Theorem 2.3.1.** *Assume that* $\mathbf{w}_0\in\overline{V}^1_{\#}(Q)\cap H$, $\mathbf{f}\in L^2(0,T;V')$, *and* $\chi\geq 0$. *The strong solution of (2.0.3) is unique.*

*Proof.* By contradiction, assume that there exist two solutions $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$ of (2.0.3). Let $\boldsymbol{\phi} := \mathbf{w}_2 - \mathbf{w}_1$ (thus $\boldsymbol{\phi}^* := \mathbf{w}_2^* - \mathbf{w}_1^*$) and $r := q_2 - q_1$. Then, subtracting the weak formulations of $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$, it follows that $(\boldsymbol{\phi}, r)$ is a weak solution of the problem

$$\begin{aligned} \boldsymbol{\phi}_t + \nabla\cdot(\overline{D(\mathbf{w}_2)D(\mathbf{w}_2) - D(\mathbf{w}_1)D(\mathbf{w}_1)}) - \nu\triangle\boldsymbol{\phi} + \nabla r + \chi\boldsymbol{\phi}^* &= 0 \\ \nabla\cdot\boldsymbol{\phi} &= 0 \\ \boldsymbol{\phi}(0) &= 0, \end{aligned} \tag{2.3.1}$$

subject to periodic boundary condition and zero mean. From Remark 2.2.2, we can multiply the first equation of (2.3.1) by $(I-\delta^2\triangle)D(\boldsymbol{\phi}) \in L^2(0,T;L^2(Q))$. After algebraic manipulation and integration by parts, the nonlinear term becomes:

$$(\nabla \cdot (\overline{D(\mathbf{w}_2)D(\mathbf{w}_2) - D(\mathbf{w}_1)D(\mathbf{w}_1)}), (I-\delta^2\triangle)D(\boldsymbol{\phi}))d\mathbf{x} \ = - \int_Q D(\boldsymbol{\phi}) \cdot \nabla D(\boldsymbol{\phi}) \cdot D(\mathbf{w}_1) d\mathbf{x}.$$

Since both $I-DG$ and $D$ are symmetric, positive semi-definite and positive definite respectively, and all operators commute, the higher order fluctuation term is non-negative

$$\chi\ (\boldsymbol{\phi}^*, (I-\delta^2\triangle)D(\boldsymbol{\phi})) \ = \ \chi\ (\boldsymbol{\psi}^*, \boldsymbol{\psi}) + \chi\ ((\nabla\boldsymbol{\psi})^*, (\nabla\boldsymbol{\psi})) \ \geq \ 0, \tag{2.3.2}$$

where $\boldsymbol{\psi} = D^{1/2}(\boldsymbol{\phi})$. The other terms are

$$(\boldsymbol{\phi}_t, (I-\delta^2\triangle)D(\boldsymbol{\phi})\ ) = \frac{1}{2}\frac{d}{dt}\ (\ ||\boldsymbol{\phi}||_D^2 + \delta^2||\nabla\boldsymbol{\phi}||_D^2\ )$$

$$-\nu\ (\triangle\boldsymbol{\phi}, (I-\delta^2\triangle)D(\boldsymbol{\phi})\ ) = \nu\ (\ ||\nabla\boldsymbol{\phi}||_D^2 + \delta^2||\triangle\boldsymbol{\phi}||_D^2\ ) \tag{2.3.3}$$

$$(\nabla r, (I-\delta^2\triangle)D(\boldsymbol{\phi})\ ) = \ -(r, (I-\delta^2\triangle)D(\nabla\cdot\boldsymbol{\phi})\ ) = 0.$$

Thus, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\ (\ ||\boldsymbol{\phi}||_D^2 + \delta^2||\nabla\boldsymbol{\phi}||_D^2\ ) + \nu\ (\ ||\nabla\boldsymbol{\phi}||_D^2 + \delta^2||\triangle\boldsymbol{\phi}||_D^2\ ) \\ &\qquad \leq \ \int_Q D(\boldsymbol{\phi}) \cdot \nabla(D\boldsymbol{\phi}) \cdot D(\mathbf{w}_1) d\mathbf{x}. \end{aligned} \tag{2.3.4}$$

Next, we apply the Cauchy-Schwarz inequality in the right hand side

$$|\int_Q D(\boldsymbol{\phi}) \cdot \nabla D(\boldsymbol{\phi}) \cdot D(\mathbf{w}_1) d\mathbf{x}\ | \ \leq \ ||D(\mathbf{w}_1)||_{L^4(Q)}\ ||D(\boldsymbol{\phi})||_{L^4(Q)}\ ||\nabla D(\boldsymbol{\phi})||.$$

The Sobolev embedding theorem and an inequality of Ladyzhenskaya give that there exists $C = C(Q, \mathbf{f}, \mathbf{w}_1, ||D||)$ such that

$$|\ \int_Q D(\boldsymbol{\phi}) \cdot \nabla D(\boldsymbol{\phi}) \cdot D(\mathbf{w}_1) d\mathbf{x}\ | \ \leq C\ ||\nabla\boldsymbol{\phi}||^2.$$

Moreover, using the equivalence of the $D$-norm and the $L^2$-norm, there is a constant $C' = C'(\delta)$ such that

$$||\nabla\boldsymbol{\phi}||^2 \ \leq C' \ ( \ ||\boldsymbol{\phi}||^2 + \delta^2||\nabla\boldsymbol{\phi}||^2 \ ).$$

Putting everything together, (2.3.4) implies

$$\frac{1}{2}\frac{d}{dt} \ ( \ ||\boldsymbol{\phi}||_D^2 + \delta^2||\nabla\boldsymbol{\phi}||_D^2 \ ) \ \leq C' \ ( \ ||\boldsymbol{\phi}||^2 + \delta^2||\nabla\boldsymbol{\phi}||^2 \ ). \tag{2.3.5}$$

But, $\boldsymbol{\phi}(0) = 0$. Gronwall's Lemma implies that $\boldsymbol{\phi}$ vanishes everywhere for all $t$. Hence, uniqueness follows. □

**Lemma 2.3.1.** *The LES-ADM model (2.0.3) has a unique strong solution.*

*Proof.* This follows from Theorem 2.2.1, Remark 2.2.2 and Theorem 2.3.1. □

**Theorem 2.3.2.** *Let* $\mathbf{w}$ *be the unique strong solution of (2.0.3) and let* $\chi \geq 0$*. Then* $\mathbf{w}$ *satisfies the energy equality:*

$$\begin{aligned}
&\frac{1}{2}\left(||\mathbf{w}(t)||_D^2 + \delta^2||\nabla\mathbf{w}(t)||_D^2\right) + \nu\int_0^t \left(||\nabla\mathbf{w}(\tau)||_D^2 + \delta^2||\triangle\mathbf{w}(\tau)||_D^2\right) d\tau \\
&\qquad + \chi\int_0^t \left(\mathbf{w}^*(\tau), (-\delta^2\triangle + I)D\mathbf{w}(\tau)\right) d\tau = \int_0^t (\mathbf{f}, \mathbf{w}(\tau))_D d\tau \\
&\qquad\qquad + \frac{1}{2}\left(||\mathbf{w}(0)||_D^2 + \delta^2||\nabla\mathbf{w}(0)||_D^2\right),
\end{aligned} \tag{2.3.6}$$

*for all* $t \in (0, T]$.

*Proof.* Multiply the first equation of (2.0.3) by the test function $(-\delta^2\triangle + I)D(\mathbf{w})$. The nonlinear term vanishes because:

$$( \ \nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, (-\delta^2\triangle + I)D(\mathbf{w}) \ ) = ( \ \nabla \cdot D(\mathbf{w})D(\mathbf{w}), D(\mathbf{w}) \ ) = 0$$

Rewriting, we obtain:

$$\begin{aligned}
&( \ \mathbf{w}_t, (-\delta^2\triangle + I)D(\mathbf{w}) \ ) - \nu \ ( \ \triangle\mathbf{w}, (-\delta^2\triangle + I)D(\mathbf{w}) \ ) + \chi \ ( \ \mathbf{w}^*, (-\delta^2\triangle + I)D(\mathbf{w}) \ ) \\
&\qquad = ( \ \overline{\mathbf{f}}, (-\delta^2\triangle + I)D(\mathbf{w}) \ ).
\end{aligned} \tag{2.3.7}$$

The first two terms in the right hand side are handled as in (2.3.3). We also have

$$( \ \overline{\mathbf{f}}, (-\delta^2\triangle + I)D(\mathbf{w}) \ ) = (\mathbf{f}, D(\mathbf{w}) \ ). \tag{2.3.8}$$

Finally, integration between 0 and $t$ leads to (2.3.6), for all $t \in (0, T]$. □

**Remark 2.3.1.** *In Theorem 2.3.2,*

$$
\begin{aligned}
&\textit{the kinetic energy of the model } = \frac{1}{2}\ (\ ||\mathbf{w}(t)||_D^2 + \delta^2||\nabla\mathbf{w}(t)||_D^2\ ),\\
&\textit{total energy dissipation } =\\
&\qquad\int_0^t \Big(\ \nu(\ ||\nabla\mathbf{w}(\tau)||_D^2 + \delta^2||\triangle\mathbf{w}(\tau)||_D^2\ ) + \chi\ (\ \mathbf{w}^*(\tau), (-\delta^2\triangle + I)\mathbf{w}(\tau)\ )_D\Big)\, d\tau,\\
&\textit{the initial kinetic energy of the model } = \frac{1}{2}\ (\ ||\mathbf{w}(0)||_D^2 + \delta^2||\nabla\mathbf{w}(0)||_D^2\ ),\\
&\textit{total energy input by the body force } \mathbf{f} = \int_0^t (\mathbf{f}, \mathbf{w}(\tau))_D d\tau.
\end{aligned}
$$

*Thus Theorem 2.3.2 means that the kinetic energy of the model + total energy dissipation = the initial kinetic energy of the model + total energy input by the body force* $\mathbf{f}$*.*

## 2.4 EXAMPLES OF DECONVOLUTION OPERATORS

The basic problem in deconvolution is: given $\overline{\mathbf{u}}+noise$ find $\mathbf{u}$ approximately. In other words

$$\text{given } \overline{\mathbf{u}} \text{ solve } G\mathbf{u} = \overline{\mathbf{u}} \text{ for } \mathbf{u}. \tag{2.4.1}$$

Many averaging operators $G$ are symmetric and positive semi-definite. If the averaging operator is smoothing, the deconvolution problem will be not stably invertible due to small divisor problems. With these constraints in mind we review a few examples of deconvolution operators and their properties. In this section we consider the filtering operation be given by (2.1.1).

1. **The van Cittert deconvolution operator**.

   The van Cittert method of approximate deconvolution, see [4], constructs a family $D_N$ of inverses to $G$ using $N$ steps of fixed point iterations.

   **Algorithm 2.4.1.** [*van Cittert Algorithm*]*: Choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

*For* $n = 0, 1, 2, ..., N-1$ *perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

*Set* $D_N\mathbf{u} := \mathbf{u}_N$.

A very detailed mathematical theory of the van Cittert deconvolution operator and resulting approximate deconvolution models are already known, see [3], [5], [47], [18] and [19]. We point out the following lemma, proved in [19], concerned with properties of the approximate deconvolution operator $D_N$.

**Lemma 2.4.1.** *The operator* $D_N : \mathrm{L}^2(Q) \to \mathrm{L}^2(Q)$ *is bounded, symmetric and positive definite.*

*Proof.* For the proof see [[19], Lemma 2.1]. □

2. **The Accelerated van Cittert deconvolution operator.**

   In our second example, approximations to $\mathbf{u}$ are obtained via the algorithm defined below.

**Algorithm 2.4.2.** [*Accelerated van Cittert Algorithm*]*: Given relaxation parameters* $\omega_n$, *choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

*For* $n = 0, 1, 2, ..., N-1$ *perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega_n\{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

*Set* $D_N^\omega\mathbf{u} := \mathbf{u}_N$.

**Proposition 2.4.1.** *Let the averaging operator be the differential filter* $G\boldsymbol{\varphi} := (-\delta^2\triangle + I)^{-1}\boldsymbol{\varphi}$. *If the relaxation parameters* $\omega_i$ *are positive, for* $i = 0, 1, ..., N$, *then the Accelerated van Cittert deconvolution operator* $D_N^\omega : \mathrm{L}^2(Q) \to \mathrm{L}^2(Q)$ *is symmetric positive definite.*

*Proof.* The proof follows from [[51], Lemma 3.2]. □

3. **Tichonov regularization deconvolution operator.**

Our third example is the Tichonov regularization deconvolution operator. More precisely, since $G := (-\delta^2\triangle + I)^{-1}$ is symmetric positive-definite, given $\overline{\mathbf{u}}$ and $\mu > 0$ small, an approximate solution to the deconvolution problem (2.4.1) can be calculated as the unique minimizer in $L^2(Q)$ of the functional

$$F_\mu(\mathbf{v}) = \frac{1}{2}(G\mathbf{v}, \mathbf{v}) - (\overline{\mathbf{u}}, \mathbf{v}) + \frac{\mu}{2}(\mathbf{v}, \mathbf{v}).$$

The resulting family of Tichonov regularization deconvolution operators is

$$D_\mu = (G + \mu I)^{-1} \tag{2.4.2}$$

and the approximate solution of (2.4.1) is $\mathbf{u}_\mu = (G + \mu I)^{-1}\overline{\mathbf{u}}$. The family of operators $D_\mu$ has the following properties, see [4]

1. for any $\mu > 0$, $D_\mu$ is a bounded linear operator,
2. $\lim_{\mu\to 0} D_\mu \overline{\boldsymbol{\varphi}} = \boldsymbol{\varphi}$ for all $\boldsymbol{\varphi} \in L^2(Q)$.

**Proposition 2.4.2.** *Let the averaging operator be the differential filter $G\boldsymbol{\varphi} := (-\delta^2\triangle + I)^{-1}\boldsymbol{\varphi}$. Let $\mu > 0$ be fixed. The operator $D_\mu : \mathrm{L}^2(Q) \to \mathrm{L}^2(Q)$ is one-to-one and onto, bounded, self-adjoint and positive definite.*

*Proof.* We remark that $G$ is a linear, self-adjoint positive definite operator, with spectrum contained in $[0, 1]$. Thus the spectrum of $G + \mu\, I$ is contained in $[\mu, 1 + \mu]$, and consequently, the spectrum of $D_\mu = (G+\mu\, I)^{-1}$ is a subset of the interval $[(1+\mu)^{-1}, \mu^{-1}]$. Consequently, we have that $D_\mu$ is one-to-one and onto, bounded, self-adjoint and positive definite. $\square$

4. **Geurts' approximate filter inverse.**

One of the first studies of deconvolution as a basis for LES models was done by Geurts in [30]. Let $\overline{\boldsymbol{\phi}}$ be the top hat filter, $\overline{\boldsymbol{\phi}}(\mathbf{x}) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} \boldsymbol{\phi}(x')dx'$. Briefly, he developed approximate inverse of the top-hat convolution filter,

$$D\boldsymbol{\phi} := \int_{x-\delta}^{x+\delta} d\,(x - x')\,\boldsymbol{\phi}(x')dx',$$

based on exactness of polynomials of degree $\leq \mu$.

If $\Pi_\mu = \{p(x) : p(x) = a_0 + a_1 x + ... + a_\mu x^\mu\}$ the criteria that determined the deconvolution kernel $d$ was

$$DG\phi = \phi, \text{ for all } \phi \in \Pi_\mu. \tag{2.4.3}$$

In $1D$ (in multiple dimension extension is by tensor product), the top-hat filter, $g(\cdot)$ has the special property that

$$g \star \{\text{ polynomial of degree } \leq \mu\} = \{\text{ polynomial of degree } \leq \mu\}.$$

Using this property, for $L \geq 0$, let

$$d(x) = \begin{cases} d_0 + d_1 x + ... + d_{2L} x^{2L}, & \text{if } |x| \leq 2\pi \\ 0, & \text{if } |x| > 2\pi. \end{cases}$$

On $[-\pi, \pi]$ the coefficients $d_0, d_1, ..., d_{2L}$ are uniquely determined by exactness of polynomials of degree $\leq 2L + 1$.

$$d \star g \star x^l = x^l, \text{ l} = 0, 1, ..., 2\text{L+1}. \tag{2.4.4}$$

The deconvolution operator $D\phi = d \star \phi$ is self adjoint, commutes with $G$. The theoretical development in [30] and associated test suggests that $D$ satisfies:

$$\begin{aligned} ||DG\phi - \phi|| &\leq C(\phi)\delta^{2L+1} \text{ for smooth } \phi \\ ||D||_{\mathcal{L}(L^2 \to L^2)} &\leq C(\delta, L) < \infty. \end{aligned}$$

The deconvolution operator $D_L$ are tabulated for $L = 0, 1, 2, 3$ in Geurts [[30], Table 1]. The top-hat filter, and thus the associated deconvolution operator, is important in many applications, but it is not clear if or how the theory developed herein could be extended to it. This is because $\widehat{g}(k) = \sin(k\delta/2)/(k\delta/2)$, both changes sign and has zeros. At this point it appears to be an interesting and important open question to extend Geurts construction to the other filters, such as the Gaussian. (Extension to dynamic inverse models has been done in [42].)

5. **A variation of the Geurts' Approximate Filter Inverse**, [30].

   The construction of Geurts [30], can be modified so as to fit in the theory herein. Indeed, first we shall interpret the construction in wave number space. With the differential filter $\widehat{g}(\delta k) = \frac{1}{\delta^2|k|^2+1}$ we have $|\widehat{g}(k)| \to 0$ as $|k| \to \infty$, since the filtering is smoothing. High order accuracy on the large scales means exactness on the high degree polynomials on $x$, (i.e. Guerts condition (3.2.22)) and, equivalently, high order contact of $\widehat{d}(k)$ to $\frac{1}{\widehat{g}(k)}$ at $k = 0$. Since the convolution should be a bounded operator and $\widehat{d}(0) = 1$ we can pose the problem seeking a rational deconvolution kernel

$$\widehat{d}(k) = \frac{1 + n_1 k + ... + n_l k^l}{1 + b_1 k + ... + b_l k^l}, \; b_l \neq 0.$$

   Then the accuracy conditions are

$$\begin{aligned} &\widehat{d}(0) = 1 \text{ satisfied by the choice of the } 0^{th} \text{ coefficients } n_0 \text{ and } b_0 \\ &\frac{d^m}{dk^m}\widehat{d}(k)\bigg|_{k=0} = \frac{d^m}{dk^m}\frac{1}{\widehat{a}(k)}\bigg|_{k=0}, \; m = 1, 2, ..., \mu. \end{aligned}$$

   If, additionally,

$$\begin{aligned} 1 + b_1 k + ... + b_l k^l \neq 0, \text{ for all } k \in \mathbb{R} \\ \widehat{d}(k) > 0, \text{ for all } k \in \mathbb{R}, \end{aligned}$$

   then the deconvolution operator satisfies the conditions of the theory.

6. Not all deconvolution operators used in image processing satisfy **P**1, **P**2, and **P**3 above. **Direct deconvolution**, $D = -\delta^2\triangle + I$ is not bounded and thus not satisfying **P**1. **The Accelerated van Cittert with negative relaxation parameters** is not positive definite, violating **P**2. The resulting deconvolution operator is not positive definite. Furthermore, there are very many iterative methods, such as *steepest descent* and *the conjugate gradient method*, that can be truncated to give deconvolution operators. However these often produce approximations to $\mathbf{u}$ which depend nonlinearly on $\overline{\mathbf{u}}$. Thus, the resulting approximate deconvolution operators are nonlinear, violating **P**1 .

## 2.5 EXTENSION TO OTHER FILTERS

When developing a mathematical foundation of an LES model, the first analytical problem that arises is existence of solutions of the model. We developed a general theory about existence of solutions of deconvolution models. The averaging operator chosen was a specific differential filter. More generally, if the filter $G$ satisfies $\widehat{g}(k) \neq 0$ for all $k$, then the exact filter inverse $A$ can be defined as an unbounded operator with dense domain and closed range. If additionally, $|\widehat{g}(k)| \to 0$ as $k \to \infty$ with $O(\frac{1}{|k|^2})$ (or faster) then the existence theory developed herein can be extended to the filter $G$. This includes the Gaussian filter, for example, but excludes the top filter and sharp spectral cutoff.

One main result of this work is finding near minimal conditions on the deconvolution operator that guarantee existence and uniqueness of the strong solution of a deconvolution model. We also proved that under $\mathbf{P}1, \mathbf{P}2$, and $\mathbf{P}3$ the models satisfy an energy equality, which describes the evolution of the kinetic energy in a fluid's flow. It is important to remark that there are many possible deconvolution operators that don't satisfy the conditions $\mathbf{P}1, \mathbf{P}2$, and $\mathbf{P}3$.

# 3.0 CHEBYCHEV OPTIMIZED APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

Most turbulent flows are heterogeneous mixes of laminar flows, transitional flows, large coherent structures, fully developed turbulent flows and boundary layer flows. Ultimately, turbulence models are used in simulations and one central issue is to use the available degrees of freedom in the simulation effectively. This suggests that the "building block" flows which have universal features should be modelled with high precision in a continuum turbulence model so that available degrees of freedom in the simulation are retained for other flow aspects. Building block flows with universal features include turbulent boundary layers and homogeneous, isotropic turbulence, considered herein.

Various turbulence models are used for simulations seeking to predict flow statistics or averages. In LES the evolution of local, spatial averages is sought. The accuracy of a model measured in a chosen norm, $||\cdot||$, i.e.

$$||averaged\ NSE\ solution - LES\ solution||,$$

can be assessed in several experimental and analytical ways. One important analytical approach is to optimize the model's *consistency error/residual stress* as a function of the averaging radius $\delta$ and the Reynolds number $Re$ and, most importantly, model parameters. One approach is to optimize model parameters for special flows, such as boundary layers or homogeneous, isotropic turbulence (considered herein). The complement (also considered herein) is to optimize over general velocity fields with finite kinetic energy. We analyze the residual stress in the model and give an analytic and numerical comparison of the deconvolution error of two different optimization strategies: for special vs. general velocities. With a *very efficient* LES solver, optimization can also be studied computationally by wrapping the solver in an optimization routine. This has been done in Hickel, Adams and Domaradski, [37] and Adams, Hickel and Franz, [1] and produces the *exactly* optimal parameters for a particular flow as guidance to automatic parameter selection. Model optimization should ultimately be done self-adaptively inside a simulation. However, until each approach is developed individually, comparison is not possible and it will not be easy to find the correct way to automate model optimization.

Numerical simulation of complex flows presents many challenges. Often, simulations are based on various regularizations of the NSE rather than the NSE themselves, [28], [39], [59]. The oldest example was proposed by Leray in 1934, [54]:

$$\mathbf{v}_t + \overline{\mathbf{v}} \cdot \nabla \mathbf{v} - \nu \triangle \mathbf{v} + \nabla p = \mathbf{f}, \text{ and } \nabla \cdot \mathbf{v} = 0, \tag{3.0.1}$$

where $\overline{\mathbf{v}} = G\mathbf{v}$ is a smoothed/averaged velocity. Herein, we select the differential filter of $\mathbf{v}$, introduced by Germano, [27], and given by $G = (-\delta^2 \triangle + I)^{-1}$, i.e.

$$-\delta^2 \triangle \overline{\mathbf{v}} + \overline{\mathbf{v}} = \mathbf{v}. \tag{3.0.2}$$

This combination is sometimes called the Leray-alpha regularization, [10], [8], [29]. The Leray regularization's solution is smoother, more stable, and possesses (marginally) fewer scales than the NSE's solution. Still, the resulting error, even with a high accuracy numerical method, cannot be better than the error committed in the first step, replacing $\mathbf{v}$ by $\overline{\mathbf{v}}$ in (3.0.1). From (3.0.2) the error is $\mathbf{v} - \overline{\mathbf{v}} = O(\delta^2)$ at best. Experiments in [49] have shown that, due to its low accuracy, (3.0.1) with the filter (3.0.2) can have catastrophic error growth and not adequately conserve physically important integral invariants. The experiments in [49] also indicate that the increase in accuracy resulting from using deconvolution models (replacing (3.0.1) with (3.0.3)) decreases error growth and improves conservation properties.

Approximate deconvolution operators, $D : L^2(\Omega) \to L^2(\Omega)$, have the property that

$$D(\overline{\mathbf{v}}) = \text{higher order approximation of } \mathbf{v}.$$

The van Cittert deconvolution procedure gives a family ($D = D_N$, where N=0,1,2,...) of deconvolution operators with accuracy

$$\mathbf{e}(\mathbf{u}) := \mathbf{u} - D_N(\overline{\mathbf{u}}) = O(\delta^{2N+2}), \text{ for smooth } \mathbf{u}.$$

More accurate regularization of the NSE, which surpass (3.0.1) and related models for numerical simulations include:

1. The Leray deconvolution family [48], [49]:

$$\mathbf{v}_t + D(\overline{\mathbf{v}}) \cdot \nabla\mathbf{v} - \nu\triangle\mathbf{v} + \nabla p = \mathbf{f} \text{ and } \nabla\cdot\mathbf{v} = 0. \tag{3.0.3}$$

2. The time relaxation regularization of Stolz, Adam, and Kleiser [62], [61], [50]:

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\triangle\mathbf{v} + \nabla p + \chi(I - DG)^2\mathbf{v} = \mathbf{f} \text{ and } \nabla\cdot\mathbf{v} = 0. \tag{3.0.4}$$

3. The deconvolution $\alpha$-regularization [58] (enhancing NS-$\alpha$ accuracy, e.g [10], [8], [29]):

$$\mathbf{v}_t + (\nabla\times\mathbf{v})\times D(\overline{\mathbf{v}}) - \nu\triangle\mathbf{v} + \nabla P = \mathbf{f} \text{ and } \nabla\cdot\mathbf{v} = 0. \tag{3.0.5}$$

4. The Approximate Deconvolution LES Models [61], [5], [47], [19]:

$$\mathbf{v}_t + \overline{D(\mathbf{v})\cdot\nabla D(\mathbf{v})} - \nu\triangle\mathbf{v} + \nabla p + \chi(I - DG)\mathbf{v} = \mathbf{f} \text{ and } \nabla\cdot\mathbf{v} = 0. \tag{3.0.6}$$

5. The NS-omega deconvolution models [53]:

$$\mathbf{v}_t + \mathbf{v}\times\nabla D(\overline{\mathbf{v}}) - \nu\triangle\mathbf{v} + \nabla P = \mathbf{f} \text{ and } \nabla\cdot\mathbf{v} = 0. \tag{3.0.7}$$

For all these (and others as well) the modelling error is dominated by the *deconvolution error*

$$\mathbf{e}(\mathbf{u}) := \mathbf{u} - D(\overline{\mathbf{u}}).$$

This chapter considers minimizations of the deconvolution error for general (non-smooth) velocity fields $\mathbf{u}$. Since these (and other) models exist only to be used on a basis for numerical simulations of under-resolved flows, we minimize the deconvolution error over the resolved scales (i.e. over the scales that can be represented on a computational mesh). We begin by reviewing the van Cittert deconvolution operator, in Section 3.1, and give, for completeness, a convergent result as $\delta \to 0$ for fixed $N$ (standard). Section 3.1 also considers convergence as $N \to \infty$ for fixed $\delta$, a highly singular limit since van Cittert is an asymptotic rather than convergent approximation. We prove an ergodic theorem for the deconvolution iterates for a general filter: the large scales of the averages of iterates converge as $N \to \infty$. In Sections 3.2 and 3.3, we show how to optimize the van Cittert procedure to substantially increase its accuracy with no increase in computational cost. Section 3.2 reduces optimization to a Chebychev optimization problem. From this reduction we recover the optimal van Cittert parameters and show that the model's error is $O(\delta^{2/3}\mathrm{e}^{-1.24N})$, Section 3.3. Section 3.0.2 below considers, as an example, one of the above regularizations and gives the analysis of the model error in terms of the deconvolution error (addressed in Section 3.2). Finally, Section 3.4 closes with a few illustrations of the optimized method.

### 3.0.1 The Formulation

Underlying all regularizations (3.0.3) - (3.0.7) are the true Navier-Stokes equations, (1.0.1). We consider the case of $L$-periodic boundary conditions

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t), \;\; j = 1, ..., n.$$

The Navier-Stokes equations are supplemented by the initial condition, the usual normalization condition in the periodic case of zero mean velocity and pressure, and the assumption that all data are square integrable with zero mean

$$\int_Q |\mathbf{u}_0(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \;\; \int_Q |\mathbf{f}(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \;\text{ and }\; \int_Q \mathbf{f}(\mathbf{x}, t) d\mathbf{x} = 0, \;\text{ for }\; 0 \le t. \qquad (3.0.8)$$

### 3.0.2 The Connection Between Deconvolution Error and Model Error

Consider, as an example, the time relaxation regularization (3.0.4). The true NSE can be rewritten as $\nabla \cdot \mathbf{u} = 0$ and

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} + \nabla p + \chi (I - DG)^2 \mathbf{u} = \mathbf{f} + \chi (I - DG)^2 \mathbf{u}. \tag{3.0.9}$$

An equation for the model error is

$$\mathbf{e}_{model} = \mathbf{u}_{NSE} - \mathbf{v}_{model}, \tag{3.0.10}$$

The model's error is driven by the deconvolution error, $\chi (I - DG)^2 \mathbf{u}$ and is obtained by subtracting the model (3.0.4) from (3.0.9)

$$\begin{aligned} &\frac{\partial}{\partial t} \mathbf{e}_{model} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{v} \cdot \nabla \mathbf{v} - \nu \triangle \mathbf{e}_{model} \\ &\qquad + \nabla (p - p_{model}) + \chi (I - DG)^2 \mathbf{e}_{model} = \chi (I - DG)^2 \mathbf{u} \\ &\mathbf{e}_{model}(0) = 0. \end{aligned} \tag{3.0.11}$$

From (3.0.11) it is clear that zero deconvolution error trivially implies zero model error. It is thus reasonable to hope that *small* deconvolution error (i.e. small $\|\chi (I - DG)^2 \mathbf{u}\|$ on the RHS) translates to small model error. For strong solutions this is indeed the case.

**Proposition 3.0.1.** *Consider the Navier-Stokes equations with periodic boundary conditions. If* $\nabla \mathbf{u} \in L^4(0, T; L^2(\Omega))$, *then the error in the time relaxation regularization model (3.0.4) satisfies*

$$\begin{aligned} &\sup_{[0,T]} \|\mathbf{e}_{model}\|^2 + \int_0^T \left( \nu \|\nabla \mathbf{e}_{model}\|^2 + \chi \left\|(I - DG)^2 \mathbf{e}_{model}\right\|^2 \right) \, dt \\ &\qquad \leq e^{C(\mathbf{u}) \nu^{-3} T} \int_0^T \chi \left\|(I - DG)^2 \mathbf{u}\right\|^2 \, dt. \end{aligned}$$

*Proof.* Taking the inner product of (3.0.11) with $\mathbf{e}_{model}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{e}_{model}\|^2 + (\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{v}\cdot\nabla\mathbf{v}, \mathbf{e}_{model}) + \nu\|\nabla\mathbf{e}_{model}\|^2 + \chi\|\mathbf{e}_{model} - D(\overline{\mathbf{e}}_{model})\|^2 \\ = \chi(\mathbf{u} - D(\overline{\mathbf{u}}), \mathbf{e}_{model} - D(\overline{\mathbf{e}}_{model})). \qquad (3.0.12)\end{aligned}$$

The standard splitting

$$\begin{aligned}(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{v}\cdot\nabla\mathbf{v}, \mathbf{e}_{model}) &= (\mathbf{e}_{model}\cdot\nabla\mathbf{u}, \mathbf{e}_{model}) + (\mathbf{v}\cdot\nabla\mathbf{e}_{model}, \mathbf{e}_{model}) \\ &= (\mathbf{e}_{model}\cdot\nabla\mathbf{u}, \mathbf{e}_{model}) \qquad (3.0.13)\end{aligned}$$

and the Cauchy Schwarz inequality give

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{e}_{model}\|^2 + \nu\|\nabla\mathbf{e}_{model}\|^2 + \frac{\chi}{2}\|\mathbf{e}_{model} - D(\overline{\mathbf{e}}_{model})\|^2 \\ \leq -(\mathbf{e}_{model}\cdot\nabla\mathbf{u}, \mathbf{e}_{model}) + \frac{\chi}{2}\|\mathbf{u} - D(\overline{\mathbf{u}})\|^2. \qquad (3.0.14)\end{aligned}$$

We have

$$\begin{aligned}|(\mathbf{e}_{model}\cdot\nabla\mathbf{u}, \mathbf{e}_{model})| &\leq \|\mathbf{e}_{model}\|^{1/2}\|\nabla\mathbf{e}_{model}\|^{3/2}||\nabla\mathbf{u}|| \\ &\leq \frac{\nu}{2}\|\nabla\mathbf{e}_{model}\|^2 + C\nu^{-3}\|\nabla\mathbf{u}\|^4\|\mathbf{e}_{model}\|^2.\end{aligned}$$

Using this in the RHS of (3.0.14) and then applying Gronwall's inequality we deduce

$$\begin{aligned}\sup_{[0,T]}\|\mathbf{e}_{model}\|^2 + \int_0^T\left(\nu\|\nabla\mathbf{e}_{model}\|^2 + \chi\left\|(I-DG)^2\mathbf{e}_{model}\right\|^2\right)\,dt \\ \leq e^{C(\|\nabla\mathbf{u}\|)\nu^{-3}T}\int_0^T\chi\left\|(I-DG)^2\mathbf{u}\right\|^2\,dt.\end{aligned}$$

□

The model's error is bounded by the deconvolution error $\mathbf{e} = \mathbf{u} - D(\overline{\mathbf{u}})$ evaluated at the true solution of the NSE. Since analogous bounds can be proven for the regularizations and models (3.0.3) through (3.0.7) we consider, we turn to minimizing the deconvolution error.

## 3.1 APPROXIMATE DECONVOLUTION METHODS

The basic problem in deconvolution is to find $\mathbf{u}$ from $\overline{\mathbf{u}}$, in other words:

$$\text{given } \mathbf{u} \ ( + \mathit{noise}) \text{ solve } G\mathbf{u} = \overline{\mathbf{u}}, \text{ for } \mathbf{u}. \tag{3.1.1}$$

If the averaging operator is smoothing, the deconvolution problem will be not stably invertible due to small divisor problems.

This section considers the van Cittert approximate deconvolution algorithm, [4]. The approximation $D_N(\overline{\mathbf{u}})$, for the operator equation (3.1.1), is computed by $N$ steps of first order Richardson iteration. Each step of van Cittert requires only one filtering step.

**Algorithm 3.1.1.** [*The van Cittert Algorithm*]: *Choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

*For* $n = 0, 1, 2, ..., N-1$*, perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

*Set* $D_N(\overline{\mathbf{u}}) := \mathbf{u}_N$*.*

For example, for N=0, 1, and 2 the deconvolution operator $D_N$ is

$$\begin{aligned}
D_0(\overline{\mathbf{u}}) &= \overline{\mathbf{u}}, \qquad \mathbf{u} \simeq D_0(\overline{\mathbf{u}}) + O(\delta^2),\\
D_1(\overline{\mathbf{u}}) &= 2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}, \qquad \mathbf{u} \simeq D_1(\overline{\mathbf{u}}) + O(\delta^4),\\
D_2(\overline{\mathbf{u}}) &= 3\overline{\mathbf{u}} - 3\overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}}}, \qquad \mathbf{u} \simeq D_2(\overline{\mathbf{u}}) + O(\delta^6).
\end{aligned}$$

For the Cauchy problem, $\Omega = \mathbb{R}^n$, the transfer function of $D_N$ (for N = 0, 1, 2) is

$$\widehat{D}_0(k) = 1, \quad \widehat{D}_1(k) = 2 - \frac{1}{k^2+1} = \frac{2k^2+1}{k^2+1} \quad \text{and} \quad \widehat{D}_2(k) = 1 + \frac{1}{k^2+1} + \left(\frac{k^2}{k^2+1}\right)^2.$$

These three and the transfer function of exact deconvolution $(k^2+1)$ are plotted in Figure 1. The graphs of the transfer functions have high order contact near 0. Thus $D_N$ leads to a very accurate solution of the deconvolution problem.

There are two convergence issues that arise immediately:

1. Convergence as $\delta \to 0$ for fixed $N$ and general $\mathbf{u} \in L^2(\Omega)$ (see Theorem 3.1.2).

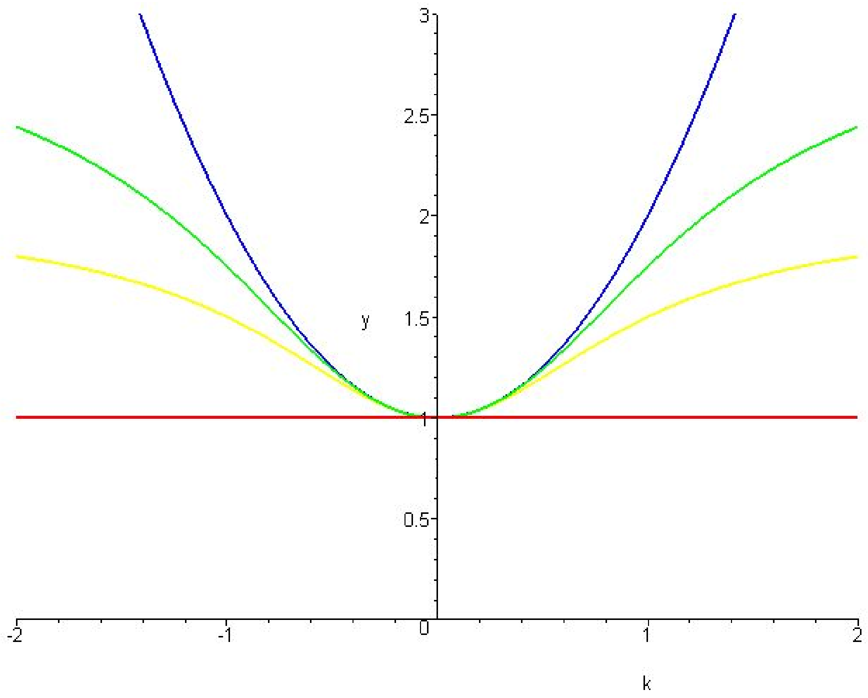


Figure 1: Exact and van Cittert Approximate Deconvolution Operators (N=0,1,2)

2. Convergence as $N \to \infty$ for $\delta$ fixed (possibly true for some specific filters, but likely not true in general, see Theorem 3.1.3 and [47]).

**Theorem 3.1.2.** [*Convergence as $\delta \to 0$ for general velocities*] *Suppose that $\overline{\mathbf{u}} \to \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$, for all $\mathbf{u} \in L^2(\Omega)$. Then, for $N$ fixed, we have $D_N(\overline{\mathbf{u}}) \to \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$.*

*Proof.* As $\delta \to 0$, we have $\mathbf{u}_0 = \overline{\mathbf{u}} \to \mathbf{u}$ and thus

$$\mathbf{u}_1 = \mathbf{u}_0 + \{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0\} \to \mathbf{u} + \{\mathbf{u} - \mathbf{u}\} = \mathbf{u}.$$

Similarly, each $\mathbf{u}_n \to \mathbf{u}$ and $D_N(\overline{\mathbf{u}}) \to \mathbf{u}$. $\square$

Since the deconvolution problem is ill-posed, convergence of $D_N(\overline{\mathbf{u}})$ to $\mathbf{u}$ as $N \to \infty$ is not expected. Nevertheless, it is possible to prove a type of ergodic theorem for averages predicted by the van Cittert algorithm for very general operators $G$. Sharper convergence theorems depend upon specific choices of the averaging operator $G$, see [48] for an example.

**Theorem 3.1.3.** [*Convergence as* $N \to \infty$] *Let* $X$ *be a Hilbert space and suppose that the averaging operator* $G : X \to X$ *is a bounded linear operator with* $\|I - G\|_{\mathcal{L}(X \to X)} \leq 1$. *For* $\mathbf{u} \in X$, *consider the van Cittert iteration*

$$\mathbf{u}_0 = G\mathbf{u}, \quad \mathbf{u}_{n+1} = \mathbf{u}_n + \{G\mathbf{u} - G\mathbf{u}_n\}.$$

*Let*

$$\mathbf{v}_N = \frac{\mathbf{u}_0 + \mathbf{u}_1 + \ldots + \mathbf{u}_N}{N+1}.$$

*Then* $G^2(\mathbf{u} - \mathbf{v}_N) \to 0$ *in* $X$ *as* $N \to \infty$, *specifically,*

$$\sup_{\mathbf{u} \in X} \frac{||G^2(\mathbf{u} - \mathbf{v}_N)||}{||G\mathbf{u}||} \quad \leq \quad \frac{2}{N+1}.$$

*Proof.* Let $B = I - G$. Then,

$$G\mathbf{u}_N = G\mathbf{u} - B^{N+1}G\mathbf{u} \;\text{ or }\; G\mathbf{e}_N = B^{N+1}G\mathbf{u},$$

where $\mathbf{e}_N = \mathbf{u} - \mathbf{u}_N = \mathbf{u} - D_N(\overline{\mathbf{u}})$. Consider $G(\mathbf{u} - \mathbf{v}_N)$ . A similar algebraic calculation gives

$$\begin{aligned} G^2(\mathbf{u} - \mathbf{v}_N) \quad &= \quad \frac{1}{N+1} G^2(\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N) = \\ &= \quad \frac{1}{N+1} G \sum_{n=0}^{N} B^{n+1} G\mathbf{u} = \\ &= \quad \frac{1}{N+1} \{BG\mathbf{u} - B^{N+2}G\mathbf{u}\} \to 0. \end{aligned}$$

as $N \to \infty$, since $\|B\| \leq 1$. Taking norms of both sides completes the proof. ☐

For LES, convergence of the van Cittert approximation $D_N(\overline{\mathbf{u}})$ to $\mathbf{u}$ as $N \to \infty$ is not as significant as convergence of $D_N(\overline{\mathbf{u}})$ to $\mathbf{u}$ as $\delta \to 0$ and the asymptotic order of accuracy as $\delta \to 0$ for fixed $N$. When the averaging is given by a differential filter, the accuracy of $D_N(\overline{\mathbf{u}})$ as an approximation to $\mathbf{u}$ for smooth functions was addressed by Stolz and Adams [61], Dunca [18], Dunca and Epshteyn [19].

**Lemma 3.1.1.** *Let the averaging operator be given by the differential filter* $G\mathbf{v} := (-\delta^2\triangle + I)^{-1}\mathbf{v}$. *For any* $\mathbf{v} \in L^2(\Omega)$,

$$\mathbf{v} - D_N(\overline{\mathbf{v}}) = (-1)^{N+1}\delta^{2N+2}\triangle^{N+1}G^{N+1}\mathbf{v}.$$

*Thus, if* $\triangle^{2N+2}\mathbf{v} \in L^2(\Omega)$ *we have*

$$\|\mathbf{v} - D_N(\overline{\mathbf{v}})\| \leq \delta^{2N+2}\left\|\triangle^{2N+2}\mathbf{v}\right\|.$$

*Proof.* For the proof, see for example [19]. □

The use of van Cittert as an asymptotic, rather than iterative, approximation of an ill-posed, rather than non-singular, linear problem as well as the associated convergence theory is very different than that of first order Richardson. However, the form of the iteration is the same. Exploiting this algorithmic similarity, relaxation parameters can be introduced at *no additional computational cost.* We shall optimize these parameters, for deconvolution of fluid velocities, in Section 3. In Algorithm 3.1.1 it is also clear that, at no extra cost, the parameters can be chosen to have different values in different regions. In fact, we expect different optimal values near walls (still an open problem), away from walls in free turbulence and for general velocities (considered herein).

**Algorithm 3.1.4.** [*Accelerated van Cittert Algorithm*]: *Given relaxation parameters* $\omega_n$, *choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

*For* $n = 0, 1, 2, ..., N-1$ *perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega_n\{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

*Set* $D_N^{\omega}(\overline{\mathbf{u}}) := \mathbf{u}_N$.

The operator $D_N^{\omega}$ is the *Accelerated van Cittert* deconvolution operator. The key to optimization is the following recursion formula for the operator $D_N^{\omega}$.

**Lemma 3.1.2.** *For N=0, 1, 2, ... , we have:*

$$D^{\omega}_{N+1} = D^{\omega}_{N} + \omega_N (I - D^{\omega}_{N} G). \tag{3.1.2}$$

*Proof.* First, we note that $D^{\omega}_0 = I$ on $L^2(\Omega)$ and $D^{\omega}_1$ is a linear combination of the identity and $G$. It follows that $D^{\omega}_1$ commutes with differentiation, since both $I$ and $G$ do. Using an induction argument we deduce that $D^{\omega}_N$ commutes with $G$ for N=0,1,2,... . Furthermore, for any positive integer N we have

$$\begin{aligned} D^{\omega}_{N+1}(\overline{\mathbf{u}}) = \mathbf{u}_{N+1} &= \mathbf{u}_N + \omega_N \{\overline{\mathbf{u}} - G\mathbf{u}_N\} \\ &= D^{\omega}_N(\overline{\mathbf{u}}) + \omega_N \ \{ \ \overline{\mathbf{u}} - G D^{\omega}_N(\overline{\mathbf{u}}) \ \} \\ &= ( \ D^{\omega}_N + \omega_N \ (I - D^{\omega}_N G) \ ) \, \overline{\mathbf{u}}. \end{aligned}$$

Thus, $D^{\omega}_{N+1} = D^{\omega}_N + \omega_N \ (I - D^{\omega}_N G)$ for every positive integer N. □

Next, we analyze in more detail properties of the Accelerated van Cittert deconvolution operator, $D^{\omega}_N$.

**Lemma 3.1.3.** *Let the averaging operator be the differential filter* $G\mathbf{v} := (-\delta^2 \triangle + I)^{-1}\mathbf{v}$. *If the relaxation parameters* $\omega_i$, $i = 0, 1, ..., N$, *are positive, then the Accelerated van Cittert deconvolution operator* $D^{\omega}_N : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ *is symmetric positive definite.*

*Proof.* The operator $G$ is bounded, compact and self adjoint. Indeed, multiplying (3.0.2) by $\overline{\mathbf{v}}$ and integrating over $\Omega$ we get

$$0 \leq \|G\mathbf{v}\|^2 \leq \|\mathbf{v}\|^2 .$$

This shows that $G$ is bounded and $\|G\| \leq 1$. To show $G$ is self-adjoint and positive definite, note that for every $\mathbf{v} \in \mathrm{L}^2(\Omega)$ we have

$$0 \leq \delta^2 \|\nabla \overline{\mathbf{v}}\|^2 + \|\overline{\mathbf{v}}\|^2 = (\mathbf{v}, \overline{\mathbf{v}}) = (\mathbf{v}, G\mathbf{v}).$$

Both $D_0^\omega$ and $D_1^\omega$ are symmetric, as linear combinations of $I$ and $G$. Proceeding by mathematical induction, assume $D_n^\omega$ is symmetric, for a positive integer $n$. From Lemma 3.1.2, we know

$$D_{n+1}^\omega = D_n^\omega + \omega_n(I - D_n^\omega G).$$

Thus $D_{n+1}^\omega$ is symmetric as linear combination of symmetric operators $I$, $G$, and $D_n^\omega$. To show $D_1^\omega$ is bounded, we apply the Spectral Mapping Theorem. We have

$$\|D_1^\omega\| = \lambda(D_1^\omega) = \lambda( \ \ D_0^\omega + \omega_0(I - D_0^\omega G) \ \ ) = 1 + \omega_0(1 - \lambda(G)).$$

Since $\|G\| \leq 1$, we deduce that

$$1 \leq \|D_1^\omega\| \leq 1 + \omega_0.$$

Proceeding by induction, it is easy to see that for every positive integer $n$ we have

$$1 \leq \|D_n^\omega\| \leq 1 + \sum_{i=0}^{n-1} \omega_i.$$

This concludes the proof. $\square$

## 3.2 CHEBYCHEV OPTIMIZED DECONVOLUTION

This section calculates relaxation parameters $\omega_i$ which minimize the deconvolution error

$$\mathbf{e}_N = \mathbf{u} - D_N^\omega(\overline{\mathbf{u}}),$$

for general (non-smooth) velocity fields $\mathbf{u}$. To begin, we give a recursion formula for the deconvolution error $\mathbf{e}_N$.

**Lemma 3.2.1.** *The deconvolution error $\mathbf{e}_N$, satisfies $\mathbf{e}_0 = \mathbf{u} - \overline{\mathbf{u}}$ and for all positive integers $N$ we have*

$$\mathbf{u} - D_N^\omega \overline{\mathbf{u}} = \prod_{i=0}^{N-1} (I - \omega_i G)\mathbf{e}_0. \tag{3.2.1}$$

*Proof.* We will use mathematical induction. Note that the conclusion holds true for $N = 1$:

$$\mathbf{e}_1 = (I - \omega_0 G)\mathbf{u} - (I - \omega_0 G)\overline{\mathbf{u}} = (I - \omega_0 G)\mathbf{e}_0,$$

since $\overline{\mathbf{u}} = G\mathbf{u}$. Assuming $\mathbf{e}_n = \prod_{i=0}^{n-1}(I - \omega_i G)\mathbf{e}_0$ for any integer $n \geq 1$, let us prove

$$\mathbf{e}_{n+1} = \prod_{i=0}^{n}(I - \omega_i G)\mathbf{e}_0.$$

Since $\mathbf{e}_{n+1}$ can be rewritten as $\mathbf{e}_{n+1} = (I - \omega_n G)\mathbf{u} - (I - \omega_n G)\mathbf{u}_n$, applying the induction hypothesis we obtain

$$\mathbf{e}_{n+1} = \prod_{i=0}^{n}(I - \omega_i G)\mathbf{e}_0, \text{ for all integers } n \geq 1. \tag{3.2.2}$$

Therefore (3.2.1) holds true. $\square$

Expand the velocity field $\mathbf{u}(\mathbf{x}, t)$ in Fourier series

$$\mathbf{u}(\mathbf{x}, t) = \sum_{k} \sum_{|\mathbf{k}|=k} \widehat{\mathbf{u}}(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ where } \widehat{\mathbf{u}}(\mathbf{k}, t) = \frac{1}{L^3}\int_{\Omega} \mathbf{u}(\mathbf{x}, t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \tag{3.2.3}$$

and $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ ($\mathbf{n} \in \mathbb{Z}^3$) is the wave number.

**Definition 3.2.1.** *Let $\delta$ be the filter's averaging radius. The resolved scales are span$\{e^{i\mathbf{k}\cdot\mathbf{x}} \mid |\mathbf{k}| \leq \pi/\delta\}$. If $\mathbf{u}$ is given by (3.2.3), its projection onto the resolved scales, $P_{RS}\mathbf{u}$, is*

$$P_{RS}\mathbf{u} = \sum_{k \leq \pi/\delta} \sum_{|\mathbf{k}|=k} \widehat{\mathbf{u}}(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.2.4}$$

We denote by $k_{\min}$, and $k_{\max}$ the smallest and the largest wave number of $P_{RS}\mathbf{u}$

$$0 < k_{\min} \leq k \leq k_{\max} = \pi/\delta < \infty.$$

The total kinetic energy at point $\mathbf{x}$ in space and at time $t$ is $E(\mathbf{u})(t)$. Using Parceval's equality, we deduce $\widehat{E}(k, t)$, the kinetic energy at wave number $\mathbf{k}$, and also

$$E(P_{RS}\mathbf{u})(t) = \frac{2\pi}{L}\sum_{k \leq \pi/\delta} \widehat{E}(k, t), \text{ where } \widehat{E}(k, t) = \frac{L}{2\pi}\sum_{|\mathbf{k}|=k} \frac{1}{2}|\widehat{\mathbf{u}}(\mathbf{k}, t)|^2. \tag{3.2.5}$$

**Lemma 3.2.2.** *Let* $\mathbf{u} \in L^2(\Omega)$. *For any positive integer* $N$, *the deconvolution error* $\mathbf{e}_N = \mathbf{u} - D_N^\omega \overline{\mathbf{u}}$ *satisfies*

$$\|P_{RS}(\mathbf{u} - D_N^\omega \overline{\mathbf{u}})\|^2 = \sum_{k\leq\pi/\delta} \sum_{|\mathbf{k}|=k} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1})^2 (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2. \quad (3.2.6)$$

*Proof.* From (3.0.2) and (3.2.3) we deduce

$$\widehat{\overline{\mathbf{u}}}(\mathbf{k}, t) = \frac{1}{1 + \delta^2 k^2} \widehat{\mathbf{u}}(\mathbf{k}, t). \quad (3.2.7)$$

With this, using Parceval's equality and Lemma 3.2.1, we have

$$\|P_{RS}(\mathbf{u} - D_1^\omega \overline{\mathbf{u}})\|^2 = \sum_{k\leq\pi/\delta} \sum_{|\mathbf{k}|=k} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1})^2 (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2 .$$

Using mathematical induction, we prove (3.2.6), for any positive integer $N$. □

With Lemma 3.2.2, the optimization problem reduces to minimizing the expression:

$$\sum_{k_{\min}\leq k\leq\pi/\delta} \sum_{|\mathbf{k}|=k} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1})^2 (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2. \quad (3.2.8)$$

Consider thus the function $F_N : \mathbb{R}_+^N \to \mathbb{R}_+$, where

$$F_N(\omega_0, ..., \omega_{N-1}) = \sum_{k_{\min}\leq k\leq\pi/\delta} \sum_{|\mathbf{k}|=k} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1})^2 (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2. \quad (3.2.9)$$

We are seeking for relaxation parameters $\omega_i$ to minimize the error in deconvolution. In other words, we want to find $\min_{\omega_i} F_N(\omega_0, ..., \omega_{N-1})$. We have

$$\min_{\omega_i} F_N(\omega_0, ..., \omega_{N-1}) \leq \min_{\omega_i} \max_{k_{\min}\leq k\leq k_{\max}} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1})^2$$
$$\sum_{k_{\min}\leq k\leq\pi/\delta} \sum_{|\mathbf{k}|=k} (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\widehat{\mathbf{u}}(\mathbf{k}, t)|^2.$$

Thus, minimizing the deconvolution error for a general velocity field leads to the problem of minimizing, with respect to $\omega_i$, the expression

$$\min_{\omega_i} \max_{k_{\min}\leq k\leq k_{\max}} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1}). \quad (3.2.10)$$

The change of variable, $x \leftarrow \frac{1}{\delta^2 k^2+1}$ gives that for $k_{\min} \leq k \leq k_{\max}$ we have

$$0 \;<\; a := \frac{1}{\delta^2 k_{\max}^2 + 1} \;=\; \frac{1}{\pi^2+1} \leq \; x \;\leq\; \frac{1}{\delta^2 k_{\min}^2 + 1} =: b \;<\; 1.$$

Then, (3.2.10) leads to

$$\min_{\omega_i} \max_{k_{\min} \leq k \leq k_{\max}} \prod_{i=0}^{N-1} (1 - \omega_i \frac{1}{\delta^2 k^2 + 1}) \;\leq\; \min_{\omega_i} \max_{a \leq x \leq b} \prod_{i=0}^{N-1} (1 - \omega_i x). \tag{3.2.11}$$

To proceed, we denote by $\Pi_N^1$, the set of all polynomial functions of degree less than or equal to $N$, which are 1 at the origin, i.e.

$$\Pi_N^1 = \{p(x) | p(0) = 1\}.$$

Note that $\prod_{i=0}^{N-1}(1-\omega_i x) \in \Pi_N^1$ and thus

$$\min_{\omega_i} \max_{a \leq x \leq b} | \prod_{i=0}^{N-1} (1 - \omega_i x)| \leq \min_{\Pi_N^1} \max_{x_1 \leq x \leq x_2} | \prod_{i=0}^{N-1} (1 - \omega_i x)|. \tag{3.2.12}$$

It is well known, see for example Axelsson, [2] (page 180), that the least maximum is achieved by the Chebychev polynomials, namely

$$\min_{\Pi_N^1} \max_{a \leq x \leq b} | \prod_{i=0}^{N-1} (1 - \omega_i x)| = \max_{a \leq x \leq b} \frac{T_N\left(\frac{b+a-2x}{b-a}\right)}{T_N\left(\frac{b+a}{b-a}\right)} = \frac{1}{T_N\left(\frac{b+a}{b-a}\right)}, \tag{3.2.13}$$

where $T_N(x) = \cosh\left(N \cosh^{-1}(x)\right)$ is the $N^{th}$ Chebychev polynomial, for all $x \geq 1$.

**Remark 3.2.1.** *Following [2], further calculations in (3.2.13) show that*

$$\min_{\Pi_N^1} \max_{a \leq x \leq b} | \prod_{i=0}^{N-1} (1 - \omega_i x)| = 2\frac{\sigma^N}{1+\sigma^{2N}}, \;\; \textit{where } \sigma = \frac{1 - \sqrt{a/\,b}}{1 + \sqrt{a/\,b}}. \tag{3.2.14}$$

**Corollary 3.2.1.** *For* $\mathbf{u} \in L^2(\Omega)$ *we have*

$$\|P_{RS}(\mathbf{u} - D_1^{\omega}\overline{\mathbf{u}})\|^2 \;\leq\; \frac{1}{T_N^2\left(\frac{b+a}{b-a}\right)} \sum_{k_{\min} \leq k \leq \pi/\delta} \sum_{|\mathbf{k}|=k} (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\hat{\mathbf{u}}(\mathbf{k}, t)|^2. \tag{3.2.15}$$

*Proof.* This follows easily from Lemma 3.2.2 and (3.2.13). □

**Proposition 3.2.1.** *The parameters $\omega_j$ solving the min-max (3.2.10) problem are given by*

$$\omega_j = \frac{1}{\frac{b-a}{2} cos\left(\frac{2j+1}{2N}\pi\right) + \frac{b+a}{2}}, \tag{3.2.16}$$

*for all positive integers $N$ and $j = 0, 1, ..., N-1$.*

*Proof.* From (3.2.13), the optimal parameters for the optimization problem are given by the inverses of the zeros of $T_N$. So, (3.2.16) holds true. □

Further *useful* progress depends on $a > 0$, i.e. on either $k_{\max} < \infty$ or on restricting to the error in the resolved scales (for which $k_{\max} = \pi/\delta$).

### 3.2.1 Expected accuracy increase for turbulent flows

The $\omega_i$ in (3.2.16) optimize deconvolution models over general velocities fields. It is useful to compare the resulting errors to the case when $\omega_i$ are optimized over special velocities with a $k^{-5/3}$ energy spectrum. If the comparison is done for velocities with a $k^{-5/3}$ energy spectrum, it can be done exactly analytically (and will be most favorable for the latter case).

We shall assume that the turbulent flow is persistent, statistically stationary, homogeneous and isotropic so that time averaging is an appropriate tool. The most important components of the K-41 theory are the time (or ensemble) averaged energy dissipation rate, $\varepsilon$ and the distribution of the flow's kinetic energy across wave numbers, $E(k)$. Let $< \cdot >$ denote time averaging

$$< \boldsymbol{\phi} > (\mathbf{x}) := \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \boldsymbol{\phi}(\mathbf{x}, t) dt. \tag{3.2.17}$$

Given the velocity field of a particular flow, $\mathbf{u}(\mathbf{x}, t)$, the (time averaged) energy dissipation rate of that flow is defined to be

$$\varepsilon := \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \frac{1}{L^3} \int_{\mathbb{R}^3} \nu |\nabla \mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} \, dt, \tag{3.2.18}$$

where $|\nabla \mathbf{u}(\mathbf{x}, t)|^2 = \frac{\partial u^i}{\partial x_j}(\mathbf{x}, t) \cdot \frac{\partial u^i}{\partial x_j}(\mathbf{x}, t)$.

Let $\mathbf{U}$ be a representative (large scale) velocity (herein we take $\mathbf{U} :=< L^{-3}||\mathbf{u}||^2_{L^2(\Omega)} >^{1/2}$). The K-41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < U\nu^{-1} \simeq k_{\min} \leq k \leq k_{\max} \simeq \varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}} < \infty, \tag{3.2.19}$$

known as the inertial range, beyond which the kinetic energy in $\mathbf{u}$ is negligible, and in this range

$$E(k) \doteq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{3.2.20}$$

where $\alpha$ (in the range 1.4 to 1.7) is the universal Kolmogorov constant, $k$ is the wave number, and $\varepsilon$ is the particular flow's energy dissipation rate. The energy dissipation rate $\varepsilon$ is the only parameter which differs from one flow to another. For wave numbers larger than the inertial range (i.e. the dissipative range) the kinetic energy in the small scales decays exponentially. Thus, $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ since $E(k) \simeq 0$ for $k \geq k_{\max}$ and $E(k) \leq E(k_{\min})$ for $k \leq k_{\min}$.

Consider Chebychev optimized deconvolution. Time averaging (3.2.6) and using Parceval's equality, we obtain

$$< \frac{1}{L^3}\|P_{RS}(\mathbf{u} - D_N^\omega\overline{\mathbf{u}})\|^2 >= 2\frac{2\pi}{L}\sum_{k_{\min}<k<\pi/\delta}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})^2(1-\frac{1}{\delta^2k^2+1})^2\widehat{E}(k).$$

For turbulent velocity fields, the problem becomes

$$< \frac{1}{L^3}\ \|P_{RS}(\mathbf{u} - D_N^\omega\overline{\mathbf{u}})\|^2\ >, \text{ subject to } \widehat{E}(k) \simeq \alpha\varepsilon^{2/3}k^{-5/3}. \tag{3.2.21}$$

With (3.2.15) we calculate the deconvolution error as

$$< \frac{1}{L^3}\ \|P_{RS}(\mathbf{u} - D_N^\omega\overline{\mathbf{u}})\|^2 >\leq \frac{4\pi}{L}\sum_{0<k\leq\pi/\delta}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})^2(\frac{\delta^2k^2}{\delta^2k^2+1})^2\widehat{E}(k). \tag{3.2.22}$$

Additionally, using (3.2.13), for Chebychev optimized deconvolution, we obtain

$$\begin{aligned}
< \frac{1}{L^3}\ \|P_{RS}(\mathbf{u} - D_N^\omega\overline{\mathbf{u}})\|^2\ > \ &\leq\ (\min_{\omega_i}\max_{0\leq\ k\leq\frac{\pi}{\delta}}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})\ )^2 \\
&\qquad \alpha\ \epsilon^{2/3}\ \frac{4\pi}{L}\sum_{0<k\leq\pi/\delta}(\frac{\delta^2k^2}{\delta^2k^2+1})^2k^{-5/3} \\
&\leq\ \frac{1}{\cosh^2(0.62\ N)}\ \alpha\ \epsilon^{2/3}\ \frac{4\pi}{L}\int_0^{\frac{\pi}{\delta}}(\frac{\delta^2k^2}{\delta^2k^2+1})^2k^{-5/3}dk \\
&=\ \frac{1}{\cosh^2(0.62\ N)}\ \alpha\ \epsilon^{2/3}\ \delta^{2/3}\ \frac{4\pi}{L}\ 0.54.
\end{aligned} \tag{3.2.23}$$

Since $\frac{1}{\cosh(x)} \leq e^{-x}$, for $x \geq 0$, we obtain the bound for the time average deconvolution error

$$< \frac{1}{L^3} \; \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 \; > \; \leq \; 2.16 \; \alpha \; \epsilon^{2/3} \; \delta^{2/3} \; \frac{\pi}{L} \; e^{-1.24N}. \tag{3.2.24}$$

**Remark 3.2.2.** *The unoptimized case of $\omega_i \equiv 1$ was studied in [47] with result*

$$< \frac{1}{L^3} \; ||\mathbf{u} - D_N\overline{\mathbf{u}}||^2 > \leq (\frac{3}{2} + \frac{1}{4N + \frac{10}{3}}) \; \alpha \; \; \epsilon^{2/3} \; \delta^{2/3}. \tag{3.2.25}$$

### 3.2.2 K-41 Direct Optimization

There is also a second approach to solving the minimization problem (3.2.10). In this section we give a direct calculation of the optimal parameters for velocities with the power/energy spectrum $E(k) \sim \alpha\varepsilon^{2/3}k^{-5/3}$ of homogeneous, isotropic turbulence. We minimize $F_N$ in $\mathbb{R}^N$ by solving the $N \times N$ system:

$$\left(\frac{\partial F_N}{\partial \omega_0}, ..., \frac{\partial F_N}{\partial \omega_{N-1}}\right) = 0. \tag{3.2.26}$$

We solved the above system for $N = 1, ..., 5$. The resulting K-41 optimized relaxation parameters are given in Table 1.

| $N$ | $\omega_0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|---|
| 1 | 2.10 | - | - | - | - |
| 2 | 2.02 | 2.02 | - | - | - |
| 3 | 1.44 | 4.91 | 1.44 | - | - |
| 4 | 1.49 | 1.49 | 5.83 | 1.49 | - |
| 5 | 1.53 | 1.53 | 6.52 | 1.53 | 1.53 |

Table 1: Direct optimized parameters

## 3.3 COMPARISON OF ATTAINED ACCURACY

There are three versions of van Cittert to be compared: unoptimized, Chebychev-optimized for a general flow field (herein), and K-41 direct optimized (for this last case, the optimality problem was formulated for special velocity fields with the exact energy spectrum $\widehat{E}(k) \sim k^{-5/3}$).

To compare the three, we consider the case of velocity fields with energy spectrum of $k^{-5/3}$. Using (3.2.16), we first compute the values of the Chebychev parameters.

| $N$ | $\omega_0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|---|
| 1 | 1.83 | - | - | - | - |
| 2 | 1.15 | 4.44 | - | - | - |
| 3 | 1.06 | 1.83 | 6.54 | - | - |
| 4 | 1.03 | 1.38 | 2.68 | 7.90 | - |
| 5 | 1.02 | 1.23 | 1.83 | 3.58 | 8.75 |

Table 2: Chebychev optimized parameters

With these values, we plot the transfer functions of the Accelerated van Cittert deconvolution operator for N=0,1,2,3. The high order contact of the graphs near 0 reveals the high order of accuracy of $D_N^\omega$.

Because of the form of the RHS of estimates (3.2.24) and (3.2.25), we normalize the errors calculated by $\alpha\ \epsilon^{2/3}\ \delta^{2/3}$. Thus, using Lemma 3.2.2, we give in Table 3

$$\frac{< \frac{1}{L^3} \|P_{RS}(\mathbf{u} - D_N^\omega \overline{\mathbf{u}})\|^2 \ >}{\alpha\ \epsilon^{2/3}\ \delta^{2/3}}, \text{ when } N = 0, 1, 2, 3, 4, 5$$

for the three cases $\omega_i \equiv 1$, $\omega_i$ from Table 1, and $\omega_i$ from Table 2. Table 3 shows that both optimizations reduce the error over standard van Cittert significantly. Figure 3 gives a plot of (normalized) deconvolution error vs. wave number, for $N = 2$ for all three cases of standard

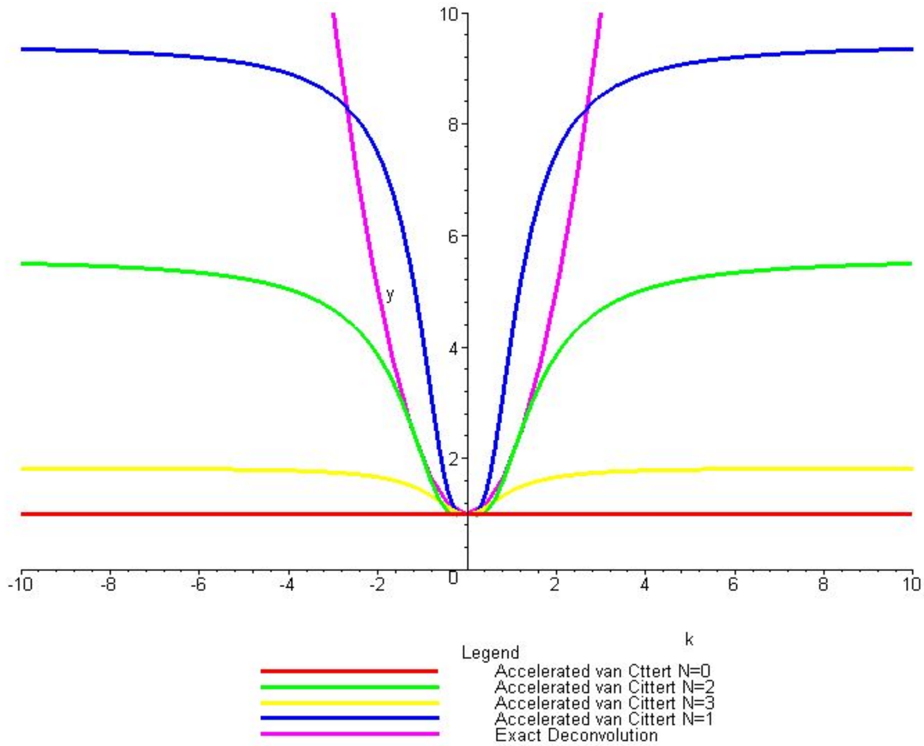


Figure 2: Exact and Accelerated van Cittert Approximate Deconvolution (N=0,1,2)

van Cittert, K-41 optimized and Chebychev optimized van Cittert. Figure 3 shows that both optimized van Citter improve the error in deconvolution for irregular velocities, while the unoptimized van Cittert is more accurate for very smooth velocity fields.

## 3.4 TWO ILLUSTRATIONS

To begin, we test the deconvolution error when both filtering and deconvolution are done discretely using a finite element approximation of the Laplace operator in (3.0.2). The computations were performed with the software FreeFem++, see [20]. We choose $\mathbf{u}$ =

| N | K-41 optimized $\omega_i$ | Chebychev optimized $\omega_i$ | $\omega_i = 1$ |
|---|---|---|---|
| 1 | 0.150 | 0.157 | 0.258 |
| 2 | 0.068 | 0.066 | 0.155 |
| 3 | 0.017 | 0.022 | 0.101 |
| 4 | 0.007 | 0.006 | 0.070 |
| 5 | 0.003 | 0.002 | 0.049 |

Table 3: Normalized deconvolution error

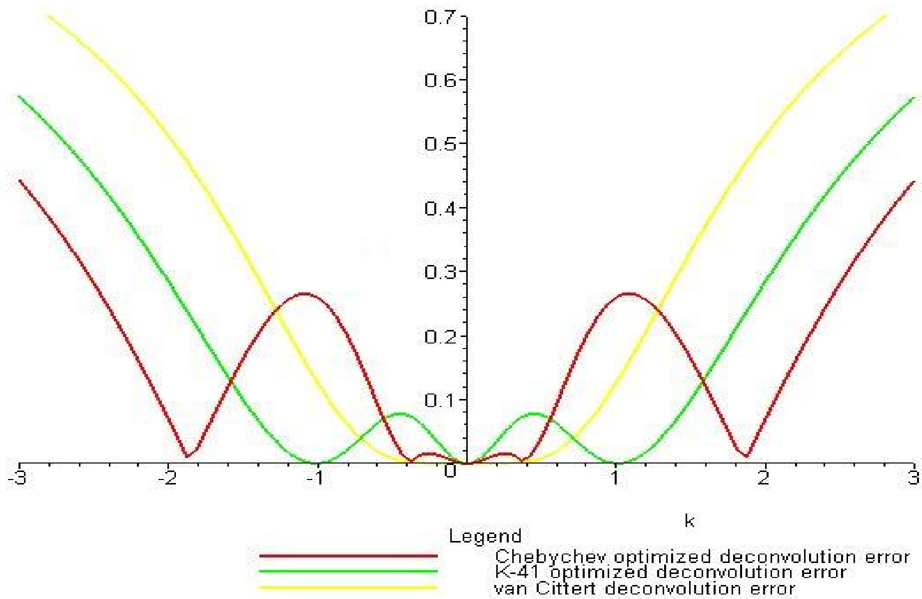


Figure 3: Deconvolution Error (N=2)

$(\sin(ky), \sin(kx))$ a known, divergence free velocity and calculate

$$\left[\frac{1}{|\Omega|}\int_{\Omega}|\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}}|^2 d\mathbf{x}\right]^{\frac{1}{2}},$$

where $\Omega = (0, 2\pi)^2$. P2 elements were used in the discretization, i.e. the velocity is approximated by continuous piecewise quadratics. For each value of N deconvolution involves the solution of N+1 discrete Poisson problems. We solve the resulting linear system with GMRES. For these calculations, we consider the meshwidth h = 1/10, 1/20, 1/30, 1/40 and N=1, 2, 3. We fix $\delta = 0.1$ and $k = 1$ and 8. The case $k = 1$ is very smooth and the theory predicts regular van Cittert to be more accurate. The case $k = 8$ oscillates faster and the theory predicts both Accelerated van Cittert to be more competitive.

Comparing tables 4, 5, and 6 we see that for a very smooth $\mathbf{u}$ (the case k=1) unoptimized van Cittert ($\omega_i = 1$) is indeed more accurate, as expected. In this case, Chebychev optimized is superior to K-41 optimized. This result is not expected since the Chebychev is for a general $L^2$ field while K-41 optimized is for fields that are slightly more regular.

| h | $\|(I - D_1G)\mathbf{u}\|$ | $\|(I - D_2G)\mathbf{u}\|$ | $\|(I - D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.000385707 | 0.000157297 | 0.000067544 |
| 1/20 | 0.000454801 | 0.000238021 | 0.000129238 |
| 1/30 | 0.000469269 | 0.000265614 | 0.000153671 |
| 1/40 | 0.000471633 | 0.000273757 | 0.000163838 |

Table 4: Unoptimized Deconvolution Error: k = 1, $\delta = 0.1$

| h | $\|(I - D_1^{\omega}G)\mathbf{u}\|$ | $\|(I - D_2^{\omega}G)\mathbf{u}\|$ | $\|(I - D_3^{\omega}G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.0102168 | 0.00862918 | 0.00590472 |
| 1/20 | 0.0102158 | 0.00862688 | 0.00590488 |
| 1/30 | 0.0102158 | 0.00862691 | 0.00590513 |
| 1/40 | 0.0102158 | 0.00862699 | 0.00590512 |

Table 5: K-41 Optimized Deconvolution Error: k = 1, $\delta = 0.1$

Next we consider the case a velocity field which is highly oscillatory with respect to the chosen filter radius, $k = 8$ and $\delta = 0.1$. We see in tables 7, 8, and 9 that, for rougher velocity fields, both optimized van Cittert are superior to unoptimized van Cittert, in accord with the predictions of the theory.

| h | $\|(I - D_1^\omega G)\mathbf{u}\|$ | $\|(I - D_2^\omega G)\mathbf{u}\|$ | $\|(I - D_3^\omega G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.00757131 | 0.00426287 | 0.00190707 |
| 1/20 | 0.00757049 | 0.0042709 | 0.00191 |
| 1/30 | 0.00757075 | 0.00427072 | 0.00191063 |
| 1/40 | 0.00757091 | 0.0042705 | 0.00191061 |

Table 6: Chebychev Optimized Deconvolution Error: k = 1, $\delta = 0.1$

| h | $\|(I - D_1 G)\mathbf{u}\|$ | $\|(I - D_2 G)\mathbf{u}\|$ | $\|(I - D_3 G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.166844 | 0.0797244 | 0.040896 |
| 1/20 | 0.154312 | 0.0200043 | 0.028794 |
| 1/30 | 0.152509 | 0.0191031 | 0.027441 |
| 1/40 | 0.15215 | 0.0186714 | 0.027323 |

Table 7: Unoptimized Deconvolution Error: k = 8, $\delta = 0.1$

| h | $\|(I - D_1^\omega G)\mathbf{u}\|$ | $\|(I - D_2^\omega G)\mathbf{u}\|$ | $\|(I - D_3^\omega G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.026368 | 0.027578 | 0.0372458 |
| 1/20 | 0.095817 | 0.028730 | 0.0200043 |
| 1/30 | 0.102346 | 0.027441 | 0.0191031 |
| 1/40 | 0.103715 | 0.027323 | 0.0186714 |

Table 8: K-41 Optimized Deconvolution Error: k = 8, $\delta = 0.1$

Note that, Table 9 shows that for the rougher velocity corresponding to k=8, $D_2^\omega$ leads to a worst deconvolution error than $D_1^\omega$, $D_3^\omega$. While this is a anomaly, Figure 4 shows that for the specific $k = 8$ and $\delta = 0.1$ ($\delta k = 0.8$), the case $N = 1$ is more accurate. Nevertheless, it is clear that in general the deconvolution error improves with the order of deconvolution.

| h | $\|(I - D_1^\omega G)\mathbf{u}\|$ | $\|(I - D_2^\omega G)\mathbf{u}\|$ | $\|(I - D_3^\omega G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.0359071 | 0.170887 | 0.0331716 |
| 1/20 | 0.0357505 | 0.195504 | 0.0383042 |
| 1/30 | 0.0398032 | 0.195824 | 0.0435825 |
| 1/40 | 0.0408305 | 0.195785 | 0.0446767 |

Table 9: Chebychev Optimized Deconvolution Error: k = 8, $\delta = 0.1$

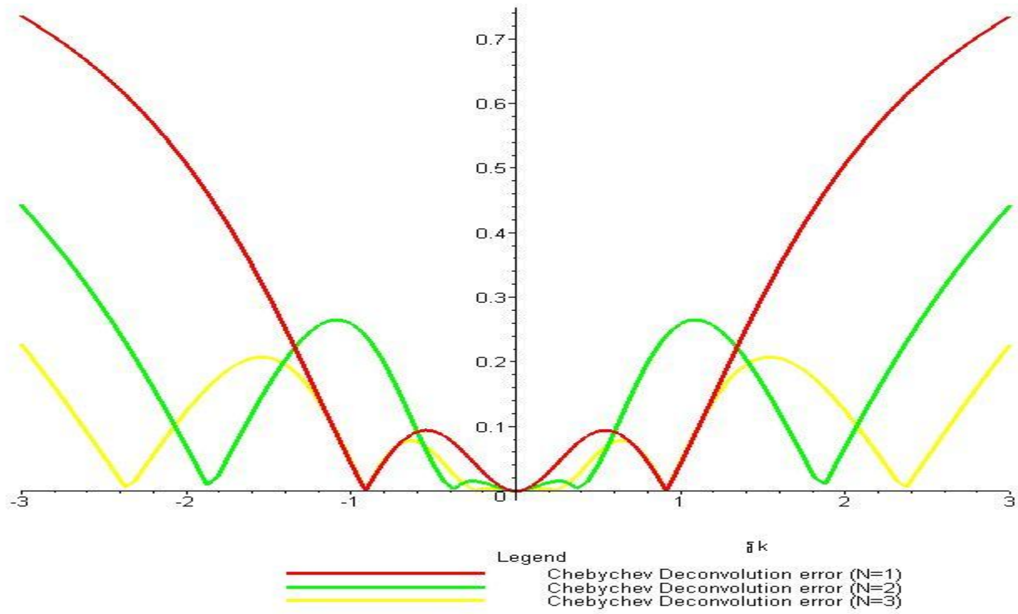


Figure 4: Deconvolution Error (N=1,2,3)

There is a secondary, but still interesting question, of whether a given model will over-damp large flow structures. The general ADM (3.0.6) is a *dispersive* regularization rather than a dissipative one, but the time relaxation regularization (usually added to it) is dissipative. We therefore consider:

$$\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} - \nu \Delta \mathbf{w} + \nabla q + \chi \left(I - D_N G\right)^2 \mathbf{w} = \overline{\mathbf{f}}$$
$$\nabla \cdot \mathbf{w} = 0. \tag{3.4.1}$$

The 2D forward-backward step for parameters values just above those for which eddies behind the step detach is a simple (not turbulent) flow, but one which can be difficult to simulate effectively on coarse meshes. Stabilization like (3.4.1) often cause the simulation to approach a non-physical equilibrium with a single attached eddy. Our illustration below, which is not a systematic test, indicates that the optimized $D_N$ in (3.4.1) behaves no worse than the usual van Cittert.

In (3.4.1), we study an under-resolved flow with recirculation, the flow across a step with $N = 1$. It is known that a particularity of this flow is a recirculating vortex behind the step, which detaches between $\nu^{-1} = 500$ and $\nu^{-1} = 700$. The parabolic inflow profile is given by $\mathbf{u} = (u_1, u_2)^T$, with $u_1 = y(10 - y)/25$ and $u_2 = 0$, no-slip boundary conditions are imposed on the top and bottom boundaries, and the "do nothing" boundary condition is used for the outflow.

The computations were also performed with the software FreeFem++, see [20]. The models were discretized in time with the implicit second order Crank-Nicolson scheme and in space with the Taylor Hood finite element method, i.e. the velocity was approximated by continuous piecewise quadratics and the pressure by continuous piecewise linears. Behind the step the flow simulation using both the optimal parameters (Figure 5 below) and the usual van Cittert ($\omega_i = 1$) (figures not included) correctly develop vortices separate from the step. Figure 5 shows the results at $T = 10, 20, 30, 40$ for $\nu^{-1} = 500$, $\chi = 0.001$, $dt = 0.005$, $\delta = 1.5$.

We conclude by remarking that the use of optimal parameters requires no extra computational effort. Two main results of this work are: the values of the optimal parameters (in Section 3.3) and the reduction in the model consistency error that results in their use is is at least 50%, (see Table 3). Interestingly, in all cases (N= 1,2,3,4,5) the Chebychev optimized parameters resulted in comparable or better errors to K-41 optimization. Since Chebychev optimization give parameter values good for all flow fields and the latter only for special ones, this suggests that Chebychev optimized deconvolution is to be strongly preferred.

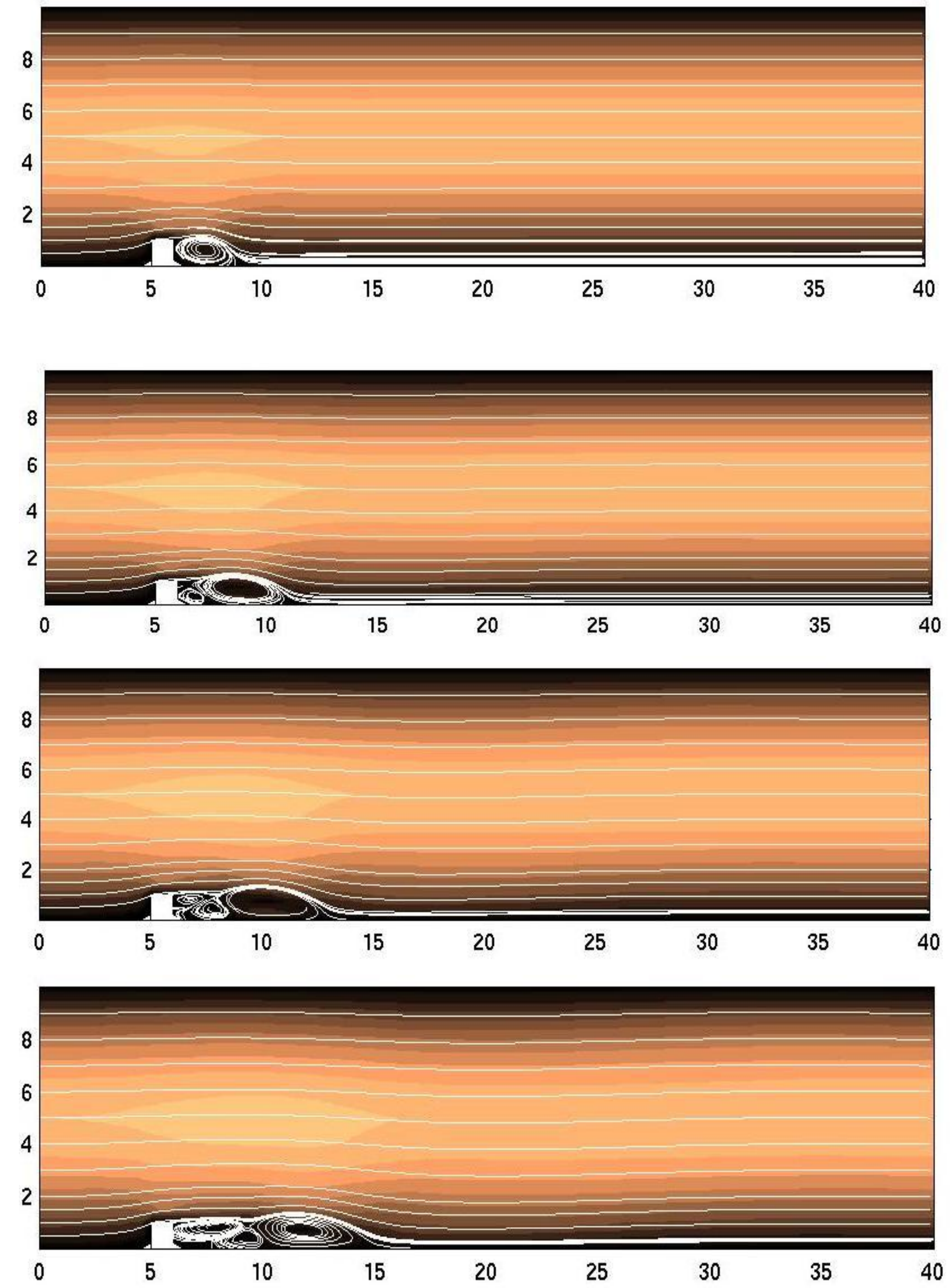


Figure 5: Accelerated van Cittert ADM: Flow field at T = 10, 20, 30, 40.

## 4.0 NUMERICAL ANALYSIS OF LERAY-TIKHONOV DECONVOLUTION MODELS OF FLUID MOTION

The Navier-Stokes equations, given by (1.0.1), are an exact model for the flow of a viscous, incompressible fluid, [24]. At higher Reynolds number, their solution contains so much information that they become impractical for many problems within typical time and resource limitations. Various models and tools have been developed seeking to give a reasonable treatment of this richness of information. The key is to capture all the relevant information with less (computational) work than that involved in solving the NSE. One of the ideas is to use regularizations of (1.0.1). Leray, [54], see also [7, 9, 38, 66], proposed the following:

$$\mathbf{u}_t + \overline{\mathbf{u}} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{u} = 0\,, \quad \text{in } \Omega \times (0,T), \tag{4.0.1}$$

where $\overline{\mathbf{u}} = G\mathbf{u}$ is a smooth/averaged velocity. He selected $G$ to be the Gaussian filter associated with a length scale $\delta$. He proved existence and uniqueness of strong solutions to (4.0.1) and showed that a subsequence $\mathbf{u}_{\delta_j}$ converges to a weak solution of the NSE as $\delta_j \to 0$. If that weak solution is a smooth, strong solution it is not difficult to prove additionally that $||\mathbf{u}_{NSE} - \mathbf{u}_{LerayModel}|| = O(\delta^2)$ using only $||\mathbf{u} - \overline{\mathbf{u}}|| = O(\delta^2)$.

Continuing his idea, new regularization models can be derived every time a suitable regularization operator is chosen. One modification is to replace the Gaussian filter by a differential filter, $\overline{\mathbf{u}} := (-\delta^2\Delta + 1)^{-1}\mathbf{u}$. Properties of the resulting Leray-$\alpha$ model (4.0.1) are derived by Geurts and Holm [32, 31] together with some tests in turbulent flow simulations and by Dunca [17] in a shape design problem.

The deconvolution problem is central in image processing, [4] and many algorithms can be adapted to give possible better regularizations of the NSE. The van Cittert deconvolution algorithm, see [17, 47, 62] is one such example. It is time to explore other operators, which can lead to possibly more accurate models. Based on the theory of inverse problems, we study a **new**, modified Tikhonov regularization operator, defined precisely in (4.1.21):

$$D_\mu(\overline{\mathbf{u}}) = \text{ approximation of } \mathbf{u}.$$

The operator $D_\mu$, $0 \leq \mu \leq 1$, is a modification of the Tikhonov-Lavrentiev regularization of the formal filter inverse *adapted to turbulence*, i.e. designed to accurately capture the large scales of a flow, while modeling the small (or under-resolved) scales (and truncating). The case $\mu = 0$, would result in $D_\mu(\overline{\mathbf{u}}) = \mathbf{u}$ (no regularization), whereas $\mu = 1$, leads to $D_\mu(\overline{\mathbf{u}}) = \overline{\mathbf{u}}$ (the Leray/Leray-$\alpha$ model (4.0.1)). For smooth velocity fields $\mathbf{u}$, we have $D_\mu(\overline{\mathbf{u}}) = \mathbf{u} + O(\mu\,\delta^2)$.

Replacing $\overline{\mathbf{u}}$ by $D_\mu(\overline{\mathbf{u}})$ in (4.0.1), we study the following Leray-Tikhonov model with time relaxation:

$$\begin{aligned} \mathbf{u}_t + D_\mu(\overline{\mathbf{u}}) \cdot \nabla\ \mathbf{u} - \nu \triangle \mathbf{u} + \nabla q + \chi(\mathbf{u} - D_\mu(\overline{\mathbf{u}})) &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0. \end{aligned} \tag{4.0.2}$$

The term $\chi(\mathbf{u} - D_\mu(\overline{\mathbf{u}}))$ is included to damp unresolved fluctuations over time, where $\chi \geq 0$ is the time relaxation parameter. It is a generalized fluctuation term, often included in Approximate Deconvolution Models of turbulence to damp marginally unresolved scales, see [62] and [50].

Our goal is to perform a convergence analysis of a discretization of (4.0.2), when $\mu, \delta, h \to 0$. The discretization consists of the finite element method in space, combined with the Crank-Nicolson algorithm in time. The notation and definitions necessary for the scheme and for the numerical analysis are in Section 4.1, where we also give a detailed mathematical theory of the Modified Tikhonov deconvolution operator. Section 4.2 develops the theory for the scheme, showing stability, existence of solutions, and analysis of convergence. Numerical experiments are presented in Section 4.3.

## 4.1 NOTATION AND PRELIMINARIES

Throughout this chapter we use standard notation for Lebesgue and Sobolev spaces and their norms. Let $\|\cdot\|$ and $(\cdot,\cdot)$ be the $L^2$ norm and inner product respectively. The $L^p(\Omega)$ norm and the Sobolev $W_p^k(\Omega)$ norm are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$. The semi-norm in $W_p^k(\Omega)$ is denoted by $|\cdot|_{W_p^k}$. The space $H^k$ represents the Sobolev space $W_2^k(\Omega)$ and $\|\cdot\|_k$ denotes the norm in $H^k$. For time dependent functions $\mathbf{v}(\mathbf{x},t)$, with $t \in (0,T)$, we define the norm

$$\|\mathbf{v}(\mathbf{x},t)\|_{m,k} = \begin{cases} \left(\int_0^T \|\mathbf{v}(\cdot,t)\|_k^m \, dt\right)^{1/m}, & \text{if } 1 \leq m < \infty \\ \operatorname{ess\,sup}_{0<t<T} \|\mathbf{v}(\cdot,t)\|_k, & \text{if } m = \infty. \end{cases}$$

The flow domain $\Omega$ is a regular, bounded, polyhedral domain in $\mathbb{R}^n$. The pressure and velocity spaces are

$$\begin{aligned} Q &= L_0^2(\Omega), \\ X &= H_0^1(\Omega). \end{aligned}$$

The dual space of $X$ is $X^\star$ and the corresponding norm is $\|\cdot\|_\star$. For the variational formulation we define the space of divergence free functions

$$V := \{\mathbf{v} \in X,\ (\nabla \cdot \mathbf{v}, q) = 0 \ \ \forall q \in Q\}.$$

The velocity-pressure finite element spaces $X_h \subset X$, $Q_h \subset Q$ are assumed to be conforming and satisfy the discrete inf-sup condition. Taylor-Hood elements are one common example of such a choice for $(X_h, Q_h)$ (see [30] and [34]). The discretely divergence free subspace of $X_h$ is defined as

$$V_h = \{\mathbf{v}_h \in X_h,\ (\nabla \cdot \mathbf{v}_h, q_h) = 0 \ \ \forall q_h \in Q_h\}.$$

For the convective term, we consider the following trilinear form.

**Definition 4.1.1.** [*The skew symmetric operator* $b^*$] *The skew-symmetric trilinear form* $b^* : X \times X \times X \to \mathbb{R}$ *is defined as*

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}). \tag{4.1.1}$$

**Lemma 4.1.1.** *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ such that $\mathbf{v} \in L^{\infty}(\Omega)$ and $\nabla \mathbf{v} \in L^{\infty}(\Omega)$, where indicated. The trilinear term $b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})$ can be bounded in the following ways*

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{1}{2} \left( \|\mathbf{u}\| \, \|\nabla \mathbf{v}\|_{\infty} \|\mathbf{w}\| + \|\mathbf{u}\| \, \|\mathbf{v}\|_{\infty} \|\nabla \mathbf{w}\| \right) \tag{4.1.2}$$

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_0(\Omega) \, \|\nabla \mathbf{u}\| \, \|\nabla \mathbf{v}\| \, \|\nabla \mathbf{w}\| \tag{4.1.3}$$

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_0(\Omega) \, \|\mathbf{u}\|^{1/2} \, \|\nabla \mathbf{u}\|^{1/2} \, \|\nabla \mathbf{v}\| \, \|\nabla \mathbf{w}\| \, . \tag{4.1.4}$$

*Proof.* For the proof see [44]. □

We also use the following approximation properties, see [6]:

$$\begin{aligned}
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\| \quad &\leq \quad Ch^{k+1} \|\mathbf{u}\|_{k+1}, \quad \mathbf{u} \in H^{k+1}(\Omega)^d, \\
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_1 \quad &\leq \quad Ch^{k} \|\mathbf{u}\|_{k+1}, \quad \mathbf{u} \in H^{k+1}(\Omega)^d, \\
\inf_{r \in Q_h} \|p - r\| \quad &\leq \quad Ch^{s+1} \|p\|_{s+1}, \quad p \in H^{s+1}(\Omega).
\end{aligned} \tag{4.1.5}$$

We often use some well known inequalities:

- Cauchy-Schwarz inequality: $|(f, g)| \leq \|f\| \, \|g\|$, for all $f$ and $g \in L^2(\Omega)$,
- Young's inequality: $ab \leq \frac{\epsilon}{p} \, a^p + \frac{\epsilon^{-q/p}}{q} \, b^q$, where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\epsilon > 0$ and $a, b \geq 0$,
- Poincaré-Friedrich's inequality: $\|\mathbf{v}\| \leq C_{PF} \, \|\nabla \mathbf{v}\|$, for all $\mathbf{v} \in X$.

### 4.1.1 Differential Filters

In our analysis we use differential filters, already introduced in Chapter 2. To begin, we recall the following definition.

**Definition 4.1.2.** [*Continuous differential filter*] *For $\boldsymbol{\phi} \in L^2(\Omega)$ and $\delta > 0$ fixed, denote the filtering operation on $\boldsymbol{\phi}$ by $\overline{\boldsymbol{\phi}}$, where $\overline{\boldsymbol{\phi}}$ is the unique solution (in $X$) of*

$$-\delta^2 \Delta \overline{\boldsymbol{\phi}} + \overline{\boldsymbol{\phi}} = \boldsymbol{\phi}. \tag{4.1.6}$$

*Set $A := -\delta^2 \Delta + I$, thus $\overline{\boldsymbol{\phi}} := A^{-1} \boldsymbol{\phi}$.*

Following Manica and Kaya Merdan [55], at this point, we define the discrete counterpart of the above differential filter.

**Definition 4.1.3.** [*Discrete differential filter*] *Given* $\mathbf{v} \in L^2(\Omega)$*, for a given filtering radius* $\delta > 0$*,* $\overline{\mathbf{v}}^h = A_h^{-1}\mathbf{v}$ *is the unique solution in* $X_h$ *of*

$$\delta^2(\nabla\overline{\mathbf{v}}^h, \nabla\boldsymbol{\chi}) + (\overline{\mathbf{v}}^h, \boldsymbol{\chi}) = (\mathbf{v}, \boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in X_h. \tag{4.1.7}$$

**Definition 4.1.4.** *The* $L^2$ *projection* $\Pi_h : L^2(\Omega) \to X_h$ *and discrete Laplacian operator* $\Delta_h : X \to X_h$ *are defined by*

$$(\Pi_h\mathbf{v} - \mathbf{v}, \boldsymbol{\chi}) = 0\,, \qquad (\Delta_h\mathbf{v}, \boldsymbol{\chi}) = -(\nabla\mathbf{v}, \nabla\boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in X_h. \tag{4.1.8}$$

**Remark 4.1.1.** *The extension from* $X_h$ *to* $X$ *in the definition of* $\Delta_h$ *is the extension by zero on the orthogonal complement of* $X_h$ *w.r.t.* $(\nabla\cdot, \nabla\cdot)$*. With* $\Delta_h$*, we can rewrite* $\overline{\mathbf{v}}^h = (-\delta^2\Delta_h + \Pi_h)^{-1}\mathbf{v}$ *and* $A_h = (-\delta^2\Delta_h + \Pi_h)$*.*

### 4.1.2 Why a modification of Tikhonov-Lavrentiev is needed?

The deconvolution problem is central in image processing, see [4]. The basic problem in deconvolution is: given $\overline{\mathbf{u}}$ solve for $\mathbf{u}$ the following equation

$$G\mathbf{u} = \overline{\mathbf{u}}. \tag{4.1.9}$$

Due to small divisor problems $G$ is not invertible and exact deconvolution is typically ill-posed.

Throughout the years, many approaches have been used to address and answer questions concerning ill-posed problems. Some were based on constrained least-square solutions, others determined the smoothest approximate solution compatible with the data within a given noise level. Tikhonov proposed a general approach, called *regularization*, which is a unification of the these two methods, [65]. The basic idea of regularization consists of constructing a family of approximate solutions depending on a positive *regularization parameter*, $\mu$. Using Tikhonov regularization, we compute a family of approximate solutions to the ill-posed deconvolution problem (4.1.9) as

$$\mathbf{u}_\mu = \arg\min_{\mathbf{u}} \left[ ||G\mathbf{u} - \overline{\mathbf{u}}||^2 + \mu||\mathbf{u}||^2 \right].$$

The main property of regularization is that, for a non-zero value of $\mu$, one can obtain an optimal approximation of the exact solution of the problem (4.1.9). Lavrentiev adapted Tikhonov's idea to symmetric positive definite (SPD) operators $G$. In this case the regularization process is called Tikhonov-Lavrentiev and leads to a family of approximate solutions given by

$$\mathbf{u}_\mu = \arg\min_{\mathbf{u}} \left[ \frac{1}{2}(G\mathbf{u},\mathbf{u}) - (\overline{\mathbf{u}},\mathbf{u}) + \frac{\mu}{2}(\mathbf{u},\mathbf{u}) \right].$$

**Remark 4.1.2.** *The Tikhonov-Lavrentiev regularization process gives an approximate solution to the deconvolution problem as follows: let $\mu > 0$ and let $G$ be SPD. Then,*

$$\mathbf{u}_\mu = (G + \mu I)^{-1}\overline{\mathbf{u}} \tag{4.1.10}$$

*solves (4.1.9) approximately as $\mu \to 0$.*

**Lemma 4.1.2.** [ *Error in Tikhonov-Lavrentiev approximation* ] *With the differential filter (4.1.6), we have*

$$\|\mathbf{u} - \mathbf{u}_\mu\| \le \mu(\delta^2 \|\triangle\mathbf{u}\| + \|\mathbf{u}\|). \tag{4.1.11}$$

*Proof.* Note that $\mathbf{u}, \triangle\mathbf{u} \in L^2(\Omega)$. Also, $G = (-\delta^2 \triangle + I)^{-1}$ and $A = -\delta^2 \triangle + I$. Now,

$$\begin{aligned}
\mathbf{u} - \mathbf{u}_\mu &= \mathbf{u} - (G + \mu I)^{-1} G\mathbf{u} = \mathbf{u} - (A^{-1} + \mu I)^{-1} A^{-1}\mathbf{u} \\
&= \mathbf{u} - (A(A^{-1} + \mu I))^{-1}\mathbf{u} = \mathbf{u} - (I + \mu A)^{-1}\mathbf{u} \\
&= (I + \mu A)^{-1} ((I + \mu A) - I)\mathbf{u} = \mu (I + \mu A)^{-1} A\mathbf{u} .
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_\mu\| &= \mu \|(I + \mu A)^{-1} A\mathbf{u}\| \\
&\le \mu \|(I + \mu A)^{-1}\| \, \|A\mathbf{u}\| .
\end{aligned}$$

All of our operators are self-adjoint. The spectrum of $I + \mu A$ is contained in the interval $[1 + \mu, \infty)$; and so the spectrum of $(I + \mu A)^{-1}$ is contained in the interval $(0, (1 + \mu)^{-1}]$. Hence,

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_\mu\| &\le \frac{\mu}{1+\mu} \|A\mathbf{u}\| \le \mu \|(-\delta^2 \triangle + I)\mathbf{u}\| \\
&\le \mu (\delta^2 \|\triangle\mathbf{u}\| + \|\mathbf{u}\|) .
\end{aligned}$$

□

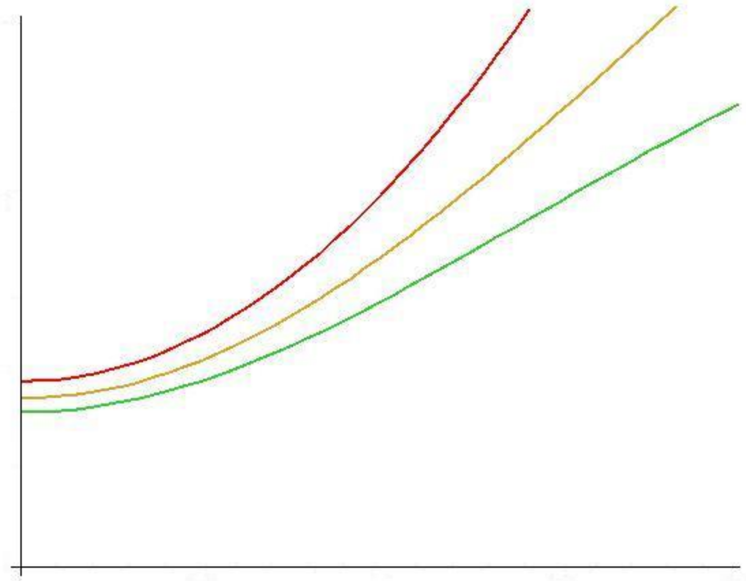

Figure 6: Tikhonov-Lavrentiev and Exact Deconvolution

With the differential filter (4.1.6), the transfer function of exact deconvolution is $\widehat{G}(k) = 1/(1+\delta^2 k^2)$, where $k$ is the wave number. Then, the transfer function of $(G+\mu I)^{-1}$ is

$$\widehat{(G+\mu I)}^{-1}(k) = \frac{1}{\mu + \widehat{G}(k)}. \tag{4.1.12}$$

This (for $\mu = 0.1$ and $0.01$) and exact deconvolution are plotted in Figure 6. It is clear that, for large scales (small wave number $k$), the operator $(G+\mu I)^{-1}$ is not a good approximation of the inverse of $G$. Thus, it is sensible to consider a modified Tikhonov-Lavrentiev operator, which is more consistent for large scales. With this, we obtain a new approximation of $\mathbf{u}$ and improve (4.1.11).

**Definition 4.1.5.** [ *Modified Tikhonov Approximate Deconvolution Operator*] *Let* $G$ *be a symmetric positive-definite operator and let* $0 < \mu < 1$*. Given* $\overline{\mathbf{u}}$*, an approximate solution to the deconvolution problem (4.1.9) is given by*

$$\mathbf{u}_\mu = ((1-\mu)G + \mu I)^{-1}\overline{\mathbf{u}}. \tag{4.1.13}$$

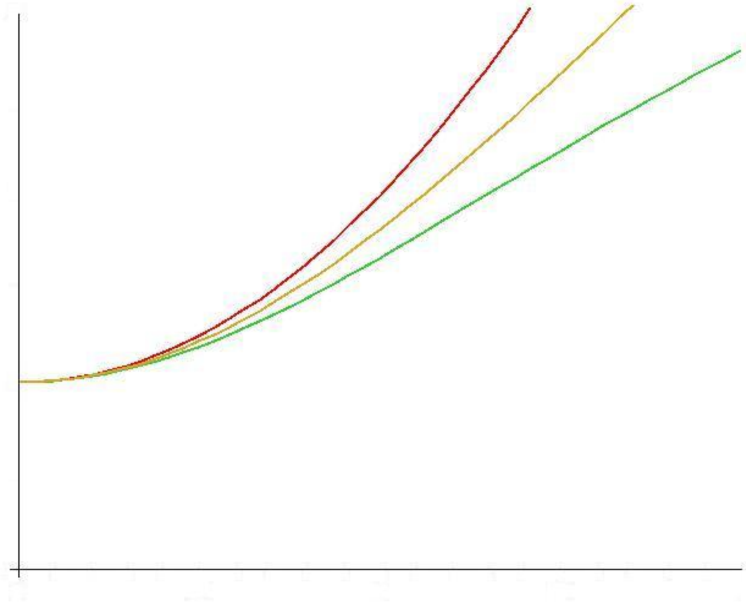

Figure 7: Modified Tikhonov and Exact Deconvolution

*Set* $D_\mu = ((1-\mu)G + \mu I)^{-1}$. *The operator* $D_\mu$ *is the modified Tikhonov deconvolution operator.*

**Remark 4.1.3.** *With the differential filter (4.1.6), the modified Tikhonov deconvolution operator can be defined variationally as: given* $\boldsymbol{\phi} \in X$*, for a given filtering radius* $\delta > 0$ *and* $0 < \mu < 1$, $\boldsymbol{\psi} = D_\mu \overline{\boldsymbol{\phi}}$ *is the unique solution in* $X$ *of the problem*

$$\mu\delta^2(\nabla\boldsymbol{\psi}, \nabla\boldsymbol{\chi}) + (\boldsymbol{\psi}, \boldsymbol{\chi}) = (\boldsymbol{\phi}, \boldsymbol{\chi}), \quad \forall \boldsymbol{\chi} \in X, \tag{4.1.14}$$

*where* $\boldsymbol{\phi}$ *and* $\overline{\boldsymbol{\phi}}$ *satisfy* $\overline{\boldsymbol{\phi}} = A^{-1}\boldsymbol{\phi}$ *in* $X$*, i.e.* $\delta^2(\nabla\overline{\boldsymbol{\phi}}, \nabla\boldsymbol{\chi}) + (\overline{\boldsymbol{\phi}}, \boldsymbol{\chi}) = (\boldsymbol{\phi}, \boldsymbol{\chi})$.

With the differential filter (4.1.6), the transfer function of $D_\mu$ is

$$\widehat{D}_\mu(k) = \frac{1}{\mu + (1-\mu)\widehat{G}(k)}. \tag{4.1.15}$$

Exact deconvolution and $\widehat{D}_\mu$ (for $\mu = 0.1$ and $0.01$) are plotted in Figure 7. The figure reflects a high order contact of the graphs for wave numbers near 0. Thus, $D_\mu$ leads to a very accurate solution of the deconvolution problem.

### 4.1.3 Continuous and Discrete Modified Tikhonov Deconvolution Operator

In this section we analyze in more detail properties of the modified Tikhonov deconvolution operator. In particular, we prove that it is a bounded, self adjoint, and positive definite operator. We also introduce and study a discrete version of the modified Tikhonov operator.

**Theorem 4.1.1.** *Let $G$ be SPD and $0 < \mu < 1$. Given $\overline{\mathbf{u}}$, the solution of (4.1.13) is given by the unique minimizer in $L^2(\Omega)$ of the functional*

$$F_\mu(\mathbf{u}) = \frac{1}{2}(G\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{u}) + \frac{\mu}{2}(\mathbf{u} - G\mathbf{u}, \mathbf{u}). \tag{4.1.16}$$

*Proof.* The unique minimizer in $L^2(\Omega)$ of the functional (4.1.16) is calculated as the minimum, when $t = 0$, of the function

$$F_\mu(\mathbf{u}+t\mathbf{w}) = \frac{1}{2}(G(\mathbf{u}+t\mathbf{w}), \mathbf{u}+t\mathbf{w}) - (\overline{\mathbf{u}}, \mathbf{u}+t\mathbf{w}) + \frac{\mu}{2}(\mathbf{u}+t\mathbf{w} - G(\mathbf{u}+t\mathbf{w}), \mathbf{u}+t\mathbf{w}), \ \ \forall \mathbf{w} \in L^2(\Omega). \tag{4.1.17}$$

Differentiating and setting $\frac{d}{dt}F_\mu(\mathbf{u}+t\mathbf{w})\big|_{t=0} = 0$ we obtain

$$((1-\mu)G\mathbf{u} + \mu\mathbf{u}, \mathbf{w}) = (\overline{\mathbf{u}}, \mathbf{w}), \quad \forall \mathbf{w} \in L^2(\Omega). \tag{4.1.18}$$

Thus, if $\mathbf{u}$ is a solution of (4.1.18), then $\mathbf{u}$ is also a solution of (4.1.13). □

**Proposition 4.1.1.** *Let the averaging operator be the differential filter $G\mathbf{u} := (-\delta^2\triangle + I)^{-1}\mathbf{u}$. Let $0 < \mu < 1$ be fixed. The operator $D_\mu : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is one-to-one and onto, bounded, self-adjoint and positive definite.*

*Proof.* We first recall that $G$ is a linear, self-adjoint positive definite operator, with spectrum contained in $[0, 1]$. Thus the spectrum of $(1-\mu)\,G + \mu\,I$ is contained in $[\mu, 1]$, and consequently, the spectrum of $D_\mu = ((1-\mu)\,G + \mu\,I)^{-1}$ is a subset of the interval $[1, \mu^{-1}]$. Consequently, we have that $D_\mu$ is one-to-one and onto, bounded, self-adjoint and positive definite. □

**Remark 4.1.4.** *Both $A^{-1} := (-\delta^2\triangle + I)^{-1}$, given by Definition 4.1.2, and $D_\mu$ are linear combinations of the Laplace operator. Thus, they commute with the Laplace operator and with each other.*

**Lemma 4.1.3.** *With the differential filter given by Definition 4.1.2 and for smooth* $\mathbf{u}$ *we have*

$$\|D_\mu \triangle \overline{\mathbf{u}}\| \leq \|\triangle \mathbf{u}\| \,. \tag{4.1.19}$$

*Proof.* We have that

$$\begin{aligned} D_\mu\, G &= ((1-\mu)\, A^{-1} + \mu\, I)^{-1}\, A^{-1} \\ &= (A\, ((1-\mu)\, A^{-1} + \mu\, I))^{-1} = ((1-\mu)\, I + \mu\, A))^{-1} \,. \end{aligned}$$

The spectrum of $A$ is contained in $[1, \infty)$, and thus the spectrum of $(1-\mu)\, I + \mu\, A$ is a subset of $[(1-\mu)+\mu, \infty) = [1, \infty)$, too. Hence, the spectrum of the self-adjoint operator $D_\mu\, G$ is contained in the interval $(0, 1]$; which implies that $\|D_\mu\, G\| \leq 1$.

Using Remark 4.1.4 and the Cauchy-Schwarz inequality, it follows that

$$\|D_\mu\, \triangle \overline{\mathbf{u}}\| = \|D_\mu\, G\, \triangle \mathbf{u}\| \leq \|D_\mu\, G\|\, \|\triangle \mathbf{u}\| \leq \|\triangle \mathbf{u}\|.$$

□

**Lemma 4.1.4.** [ *Error in Approximate deconvolution* ] *Consider the differential filter given by Definition 4.1.2. Then, for smooth* $\mathbf{u}$

$$\|\mathbf{u} - D_\mu \overline{\mathbf{u}}\| \leq \mu \delta^2\, \|\triangle \mathbf{u}\| \,. \tag{4.1.20}$$

*Proof.* Indeed, by algebraic manipulation, we have:

$$\begin{aligned} \mathbf{u} - D_\mu \overline{\mathbf{u}} &= (I - D_\mu A^{-1})\mathbf{u} \\ &= D_\mu (D_\mu^{-1} - A^{-1}\,)\mathbf{u} \\ &= \mu D_\mu (I - A^{-1})\mathbf{u}. \end{aligned}$$

But, $(I - A^{-1})\mathbf{u} = \mathbf{u} - \overline{\mathbf{u}} = -\delta^2 \triangle \overline{\mathbf{u}}$. Thus,

$$\mathbf{u} - D_\mu \overline{\mathbf{u}} = -\mu \delta^2 D_\mu \triangle \overline{\mathbf{u}}.$$

Using the proof of Lemma 4.1.3 and taking the $L^2$ norm of both sides we obtain the desired result. □

With the discrete filter (4.1.7), we define the discrete modified Tikhonov deconvolution operator.

**Definition 4.1.6.** [ *Discrete Modified Tikhonov Deconvolution Operator* ] *For a given filtering radius $\delta > 0$ and $0 < \mu < 1$, the discrete counterpart of $D_\mu$ is denoted by $D_\mu^h$, and is defined by*

$$D_\mu^h = \left((1-\mu)A_h^{-1} + \mu I\right)^{-1}.$$

*With the discrete differential filter 4.1.3 we define $D_\mu^h$ precisely. Given $\Psi \in X$, $\boldsymbol{\psi}^h = D_\mu^h \overline{\Psi}^h$ is the solution in $X_h$ of the problem*

$$\mu\delta^2(\nabla\boldsymbol{\psi}^h, \nabla\boldsymbol{\chi}) + (\boldsymbol{\psi}^h, \boldsymbol{\chi}) = \delta^2(\nabla\overline{\Psi}^h, \nabla\boldsymbol{\chi}) + (\overline{\Psi}^h, \boldsymbol{\chi}), \quad \forall\boldsymbol{\chi} \in X_h, \tag{4.1.21}$$

*where $\Psi$ and $\overline{\Psi}^h$ satisfy $\delta^2(\nabla\overline{\Psi}^h, \nabla\boldsymbol{\chi}) + (\overline{\Psi}^h, \boldsymbol{\chi}) = (\Psi, \boldsymbol{\chi})$.*

**Proposition 4.1.2.** *The operator $D_\mu^h$ is bounded self-adjoint and positive definite on $X_h$.*

*Proof.* The operator $D_\mu^h$ is the inverse of a convex combination of $A_h^{-1}$ and $I$. Both these operators are SPD on $X_h$. Thus, so is $D_\mu^h$. □

**Lemma 4.1.5.** *For $\mathbf{v} \in X$, we have the following bounds for the discretely filtered and approximately deconvolved $\mathbf{v}$*

$$\left\|\overline{\mathbf{v}}^h\right\| \leq \|\mathbf{v}\|, \tag{4.1.22}$$

$$\left\|\nabla\overline{\mathbf{v}}^h\right\| \leq \|\nabla\mathbf{v}\|, \tag{4.1.23}$$

$$\left\|D_\mu^h\overline{\mathbf{v}}^h\right\| \leq \|\mathbf{v}\|, \tag{4.1.24}$$

$$\left\|\nabla D_\mu^h\overline{\mathbf{v}}^h\right\| \leq \|\nabla\mathbf{v}\|. \tag{4.1.25}$$

*Proof.* Let $\boldsymbol{\chi} = \overline{\mathbf{v}}^h$ in (4.1.7) and apply the Cauchy-Schwarz inequality in the right hand side. We have

$$\delta^2 \left\|\nabla\overline{\mathbf{v}}^h\right\|^2 + \left\|\overline{\mathbf{v}}^h\right\|^2 \leq \left\|\overline{\mathbf{v}}^h\right\| \|\mathbf{v}\| .$$

Thus

$$\left\|\overline{\mathbf{v}}^h\right\|^2 \leq \left\|\overline{\mathbf{v}}^h\right\| \|\mathbf{v}\|$$

and (4.1.22) follows. For the proof of (4.1.23) we proceed in a similar way, we set $\boldsymbol{\chi} = \triangle_h\overline{\mathbf{v}}^h$ in (4.1.7) and obtain

$$\delta^2 \left\|\triangle_h\overline{\mathbf{v}}^h\right\|^2 + \left\|\nabla\overline{\mathbf{v}}^h\right\|^2 \leq \left\|\nabla\overline{\mathbf{v}}^h\right\| \|\nabla\mathbf{v}\| .$$

To prove (4.1.24), let $\Psi = \mathbf{v}$ and $\boldsymbol{\chi} = D^h_\mu\overline{\mathbf{v}}^h$ in (4.1.21). Definition 4.1.3 and the Cauchy-Schwarz inequality give

$$\mu\delta^2 \left\|\nabla D^h_\mu\overline{\mathbf{v}}^h\right\|^2 + \left\|D^h_\mu\overline{\mathbf{v}}^h\right\|^2 \leq \|\mathbf{v}\| \left\|D^h_\mu\overline{\mathbf{v}}^h\right\| , \tag{4.1.26}$$

proving (4.1.24). The proof of (4.1.25) is similar. Let $\Psi = \mathbf{v}$ and $\boldsymbol{\chi} = \triangle_h D^h_\mu\overline{\mathbf{v}}^h$ in (4.1.21). The definition of $\triangle_h$ and Cauchy-Schwarz inequality give

$$\mu\delta^2 \left\|\triangle_h D^h_\mu\overline{\mathbf{v}}^h\right\|^2 + \left\|\nabla D^h_\mu\overline{\mathbf{v}}^h\right\|^2 \leq \|\nabla\mathbf{v}\| \left\|\nabla D^h_\mu\overline{\mathbf{v}}^h\right\| . \tag{4.1.27}$$

Now the conclusion follows. □

**Theorem 4.1.2.** *Let $\mathbf{v}^*_h := \mathbf{v} - D^h_\mu\overline{\mathbf{v}}^h$, where $\mathbf{v} \in X$. Then, we have*

$$(\mathbf{v}^*_h, \mathbf{v}) > 0. \tag{4.1.28}$$

*Proof.* Let $\mathbf{v} \in X$. Poincaré's inequality together with (4.1.26) gives

$$\left\|D^h_\mu\overline{\mathbf{v}}^h\right\|^2 \leq \frac{1}{(C^2_{PF}\mu\delta^2 + 1)} \|\mathbf{v}\|^2 , \tag{4.1.29}$$

which leads to

$$\left(\mathbf{v} - D^h_\mu\overline{\mathbf{v}}^h, \mathbf{v}\right) \geq \frac{C^2_{PF}\mu\delta^2}{1 + C^2_{PF}\mu\delta^2} \|\mathbf{v}\|^2 . \tag{4.1.30}$$

and (4.1.28) follows for all $\mathbf{v} \in X$. □

**Lemma 4.1.6.** *Let* $\mathbf{v} \in X$*. Then*

$$(\mathbf{v}_h^*, \boldsymbol{\chi}_h) \leq \mu\delta^2 \|\nabla \mathbf{v}\| \|\nabla \boldsymbol{\chi}_h\|, \ \forall \boldsymbol{\chi}_h \in X_h. \tag{4.1.31}$$

*Proof.* Definition 4.1.6 leads to

$$(\mathbf{v}_h^*, \boldsymbol{\chi}_h) = \mu\delta^2(\nabla D_\mu^h \overline{\mathbf{v}}^h, \nabla \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in X_h.$$

Applying the Cauchy-Schwarz inequality and Lemma 4.1.5 in the RHS, we obtain (4.1.31). □

From Theorem 4.1.2 follows that $I - D_\mu^h A_h^{-1}$ is SPD. Fundamental in deriving energy estimates for the scheme outlined in the next section is the norm of $\mathbf{v}_h^*$ defined as

$$\|\mathbf{v}_h^*\|^2 := (\mathbf{v}_h^*, \mathbf{v}). \tag{4.1.32}$$

**Lemma 4.1.7.** *For all* $\mathbf{v} \in X$ *with* $\triangle \mathbf{v} \in L^2(\Omega)$*, we have*

$$\left\|\mathbf{v} - D_\mu^h \overline{\mathbf{v}}^h\right\| \leq \mu\delta^2 \|\mathbf{v}\|_2 + C\ (\delta h^k + h^{k+1}) \|\mathbf{v}\|_{k+1} + (\mu^{1/2}\delta h^k + h^{k+1}) \|D_\mu \overline{\mathbf{v}}\|_{k+1}. \tag{4.1.33}$$

*Proof.* Applying the triangle inequality, we obtain

$$\left\|\mathbf{v} - D_\mu^h \overline{\mathbf{v}}^h\right\| \leq \|\mathbf{v} - D_\mu \overline{\mathbf{v}}\| + \left\|D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}\right\| + \left\|D_\mu^h \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}^h\right\|. \tag{4.1.34}$$

We now look at each term in the right hand side separately. Lemma 4.1.4 gives

$$\|\mathbf{v} - D_\mu \overline{\mathbf{v}}\| \leq \mu\delta^2 \|\mathbf{v}\|_2. \tag{4.1.35}$$

For the second term, let $\boldsymbol{\chi} \in X_h$ in (4.1.14) and subtract it from (4.1.21). Let $\mathbf{e} := D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}} = D_\mu \overline{\mathbf{v}} - \mathbf{v}^h - D_\mu^h \overline{\mathbf{v}} + \mathbf{v}^h$ for all $\mathbf{v}^h \in X_h$. Using Galerkin orthogonality, we obtain

$$\mu\delta^2 \left\|\nabla D_\mu \overline{\mathbf{v}} - \nabla D_\mu^h \overline{\mathbf{v}}\right\|^2 + \left\|D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}\right\|^2 \leq \inf_{\mathbf{v}^h \in X_h} (\mu\delta^2 \left\|\nabla(D_\mu \overline{\mathbf{v}} - \mathbf{v}^h)\right\|^2 + \left\|D_\mu \overline{\mathbf{v}} - \mathbf{v}^h\right\|^2).$$

The approximation results (4.1.5) lead to

$$\left\|D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}\right\| \leq (\mu^{1/2}\delta h^k + h^{k+1}) \|D_\mu \overline{\mathbf{v}}\|_{k+1}. \tag{4.1.36}$$

To bound the last term we first apply Lemma 4.1.5

$$\left\| D_\mu^h \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}^h \right\| \leq \left\| \overline{\mathbf{v}} - \overline{\mathbf{v}}^h \right\| .$$

From Definitions 4.1.2 and 4.1.3, Galerkin orthogonality and then using the approximation results (4.1.5) we get

$$\left\| D_\mu \overline{\mathbf{v}} - D_\mu \overline{\mathbf{v}}^h \right\| \leq C \left( \delta h^k + h^{k+1} \right) \| \overline{\mathbf{v}} \|_{k+1} . \tag{4.1.37}$$

The final conclusion then follows from (4.1.35), (4.1.36), and (4.1.37). □

In the next section we study a Crank-Nicolson Finite Element Scheme for the Modified Tikhonov Approximate Deconvolution Model (1.3)."

## 4.2 CONVERGENCE OF THE DISCRETE MODEL

To begin, we define the scheme and show that its solutions are well defined, unconditionally stable, and optimally convergent to solutions of the NSE.

A strong solution of the Navier-Stokes equations satisfies $\mathbf{u} \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;X)$, $p \in L^2(0,T;Q)$ with $\mathbf{u}_t \in L^2(0,T;X^*)$ such that

$$(\mathbf{u}_t, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \, , \ \forall \mathbf{v} \in X \, , \tag{4.2.1}$$

$$(q, \nabla \cdot \mathbf{u}) = 0 \, , \ \forall q \in Q \, . \tag{4.2.2}$$

Throughout the analysis we use the following notation $\mathbf{v}(t_{n+1/2}) := \mathbf{v}((t_n + t_{n+1})/2)$ for the continuous variables and $\mathbf{v}_{n+1/2} := (\mathbf{v}_n + \mathbf{v}_{n+1})/2$ for both, continuous and discrete variables.

**Algorithm 4.2.1.** *[Crank-Nicolson Finite Element Scheme for Leray-Tikhonov deconvolution model] Let $\Delta t > 0$, $(\mathbf{w}_0, q_0) \in (X_h, Q_h)$, $\mathbf{f} \in X^*$ and $T := M\,\Delta t$ as $M$ is an integer. For $n = 0, 1, 2, \cdots, M-1$, find $(\mathbf{w}^h_{n+1}, q^h_{n+1}) \in (X_h, Q_h)$ satisfying*

$$\frac{1}{\Delta t}(\mathbf{w}^h_{n+1} - \mathbf{w}^h_n, \mathbf{v}^h) + b^*(D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{w}^h_{n+1/2}, \mathbf{v}^h) - (q^h_{n+1/2}, \nabla \cdot \mathbf{v}^h) + \nu(\nabla \mathbf{w}^h_{n+1/2}, \nabla \mathbf{v}^h)$$
$$+\chi(\mathbf{w}^h_{n+1/2} - D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h), \ \forall\ \mathbf{v}^h \in X_h \tag{4.2.3}$$

$$(\nabla \cdot \mathbf{w}^h_{n+1}, \boldsymbol{\phi}^h) = 0, \ \forall \boldsymbol{\phi}^h \in Q_h. \tag{4.2.4}$$

**Remark 4.2.1.** *Since $(X_h, Q_h)$ satisfies the discrete inf-sup condition, (4.2.3)-(4.2.4) is equivalent to*

$$\frac{1}{\Delta t}(\mathbf{w}^h_{n+1} - \mathbf{w}^h_n, \mathbf{v}^h) + b^*(D^h_N \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{w}^h_{n+1/2}, \mathbf{v}^h) + \nu(\nabla \mathbf{w}^h_{n+1/2}, \nabla \mathbf{v}^h)$$
$$+ \chi(\mathbf{w}^h_{n+1/2} - D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h), \ \forall\ \mathbf{v}^h \in V_h. \tag{4.2.5}$$

In the error analysis we use of the following lemmas and notation.

**Lemma 4.2.1.** *Assume $\mathbf{u} \in C^0(t_n, t_{n+1}; L^2(\Omega))$. If $\mathbf{u}$ is twice differentiable in time and $\mathbf{u}_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$ then*

$$\left\|\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2})\right\|^2 \ \leq\ \frac{1}{48}(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|^2 \, dt\,. \tag{4.2.6}$$

*If $\mathbf{u}_t \in C^0(t_n, t_{n+1}; L^2(\Omega))$ and $\mathbf{u}_{ttt} \in L^2((t_n, t_{n+1}) \times \Omega)$ then*

$$\left\|\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\right\|^2 \ \leq\ \frac{1}{1280}(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{ttt}\|^2 \, dt\,. \tag{4.2.7}$$

*If $\nabla \mathbf{u} \in C^0(t_n, t_{n+1}; L^2(\Omega))$ and $\nabla \mathbf{u}_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$ then*

$$\left\|\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}))\right\|^2 \ \leq\ \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^2 \, dt\,. \tag{4.2.8}$$

*Proof.* The proof is based on the Taylor expansion with remainder. $\square$

**Lemma 4.2.2.** [ *Discrete Gronwall Lemma* ] *Let* $\Delta t$*,* $H$*, and* $a_n, b_n, c_n, d_n$ *(for integers* $n \geq 0$*) be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \quad for\ l \geq 0. \tag{4.2.9}$$

*Suppose that* $\Delta t d_n < 1 \ \forall n$*. Then,*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq exp\left(\Delta t \sum_{n=0}^{l} \frac{d_n}{1 - \Delta t d_n}\right)\left(\Delta t \sum_{n=0}^{l} c_n + H\right) \quad for\ l \geq 0. \tag{4.2.10}$$

*Proof.* The proof follows from [36]. □

In the discrete case we use the analogous norms:

$$\begin{aligned}
|||\mathbf{v}|||_{\infty,k} &:= \max_{0 \leq n \leq M-1} \|\mathbf{v}_n\|_k\,, & |||\mathbf{v}_{1/2}|||_{\infty,k} &:= \max_{1 \leq n \leq M-1} \|\mathbf{v}_{n+1/2}\|_k\,, \\
|||\mathbf{v}|||_{m,k} &:= \left(\sum_{n=0}^{M-1} \|\mathbf{v}_n\|_k^m \triangle t\right)^{1/m}, & |||\mathbf{v}_{1/2}|||_{m,k} &:= \left(\sum_{n=1}^{M-1} \|\mathbf{v}_{n+1/2}\|_k^m \triangle t\right)^{1/m}.
\end{aligned}$$

**Lemma 4.2.3.** [ *Existence of Solutions and Stability of the Scheme* ] *At each time step, there exists a solution of the approximation scheme (4.2.5). Also, the scheme is unconditionally stable and satisfies the following á priori bound:*

$$\left\|\mathbf{w}_n^h\right\|^2 + \nu \Delta t \sum_{k=0}^{n-1} \left\|\nabla \mathbf{w}_{k+1/2}^h\right\|^2 + \chi \sum_{k=0}^{n-1} \left\|{\mathbf{w}_{k+1/2}^h}^*\right\|^2 \leq \left\|\mathbf{w}_0^h\right\|^2 + \frac{\Delta t}{\nu} \sum_{k=0}^{n-1} \left\|\mathbf{f}_{k+1/2}\right\|_*^2, \tag{4.2.11}$$

*for all integers* $1 \leq n \leq M$*.*

*Proof.* We begin by proving the à priori estimate (4.2.11). In (4.2.5), set $\mathbf{v}^h = \mathbf{w}^h_{k+1/2}$. Applying Young's inequality, we obtain

$$\frac{1}{\Delta t}(\left\|\mathbf{w}^h_{k+1}\right\|^2 - \left\|\mathbf{w}^h_k\right\|^2) + \nu\left\|\nabla \mathbf{w}^h_{k+1/2}\right\|^2 + \chi\left\|\mathbf{w}^h_{k+1/2}{}^{*}\right\|^2 \leq \frac{1}{\nu}\left\|\mathbf{f}_{k+1/2}\right\|^2_*, \quad \text{for every } k. \tag{4.2.12}$$

Summing from $k = 0$ to $n$, where $n$ is an integer, $1 \leq n \leq M$, we obtain the desired result.

The existence of a solution $\mathbf{w}^h_{k+1}$ to (4.2.5) follows from the Leray-Schauder Principle, [67]. We reformulate (4.2.5) as a fixed point problem, insert a parameter $\lambda$ and adapt the proof of the à priori bound to give a bound uniform in $\lambda$. To do this, we define the operator $T : X' \to V_h$, by $T(\mathbf{y}) := \mathbf{z}$, where

$$(\mathbf{y}, \mathbf{v}) := \triangle t\ \nu(\nabla\left(\frac{\mathbf{z} + \mathbf{w}^h_k}{2}\right), \nabla \mathbf{v}) + \triangle t\ \chi(\frac{\mathbf{z} + \mathbf{w}^h_k}{2} - D^h_\mu \overline{\frac{(\mathbf{z} + \mathbf{w}^h_k)}{2}}^h, \mathbf{v}), \quad \text{for all } \mathbf{v} \in V_h.$$

The bilinear form on the above right hand side is coercive. Then, by the Lax-Milgram theorem, the operator $T$ exists and is bounded. Note that $T$ is also linear. We also define the nonlinear operator $N : V_h \to X'$, via the Riesz Representation theorem,

$$(N(\mathbf{z}), \mathbf{v}) = -\triangle t\ b^*(D^h_\mu \overline{\frac{(\mathbf{z} + \mathbf{w}^h_k)}{2}}^h, \frac{\mathbf{z} + \mathbf{w}^h_k}{2}, \mathbf{v}) + (\mathbf{w}^h_k - \mathbf{z}, \mathbf{v}) + \triangle t(\mathbf{f}_{k+1/2}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V_h.$$

Since $V_h$ is finite dimensional, the operator $N$ is trivially bounded and continuous. Finally, we define $F : V_h \to V_h$, by $F(\mathbf{z}) = T(N(\mathbf{z}))$. Then, $\mathbf{z}$ is a solution of (4.2.5) if and only if it is a fixed point of $F$.

To show that $F$ has a fixed point in $V_h$, we apply the Leray-Schauder Principle. We first note that the operator $F$ is algebraic, hence continuous. Since $\dim V_h < \infty$, $F$ is also compact. By the Leray-Schauder Principle, we need to show that any solution $\mathbf{u}_\lambda$ of the fixed point problem $\mathbf{z} = \lambda F(\mathbf{z})$, where $0 \leq \lambda < 1$, satisfies $\|\mathbf{u}_\lambda\|_X \leq \gamma$, where $\gamma$ does not depend on $\lambda$. We have

$$\begin{aligned} &\triangle t\ \nu(\nabla\left(\frac{\mathbf{u}_\lambda + \mathbf{w}^h_k}{2}\right), \nabla \mathbf{v}) + \triangle t\ \chi(\frac{\mathbf{u}_\lambda + \mathbf{w}^h_k}{2} - D^h_\mu \overline{\frac{(\mathbf{u}_\lambda + \mathbf{w}^h_k)}{2}}^h, \mathbf{v}) \\ &\quad = -\lambda\triangle t\ b^*(D^h_\mu \overline{\frac{(\mathbf{u}_\lambda + \lambda\mathbf{w}^h_k)}{2}}^h, \frac{\mathbf{u}_\lambda + \mathbf{w}^h_k}{2}, \mathbf{v}) + \lambda(\mathbf{u}_\lambda - \mathbf{w}^h_k, \mathbf{v}) \\ &\quad\quad + \lambda\triangle t(\mathbf{f}_{k+1/2}, \mathbf{v}), \qquad \text{for all } \mathbf{v} \in V_h. \end{aligned} \tag{4.2.13}$$

Now, set $\mathbf{v} = \frac{\mathbf{u}_\lambda + \mathbf{w}_k^h}{2}$. Since $0 \le \lambda < 1$, proceeding as in the à priori estimate bounded we obtain the desired bound for $\|\nabla \mathbf{u}_\lambda\|$. It means that a solution of (4.2.5) exists at each time step. $\square$

**Remark 4.2.2.** *The same argument works in the infinite dimensional case, when (4.2.5) is posed in $X$ instead of $X_h$. The only modification is that compactness of $F$ (which holds) is verified separately using the Raleigh Lemma.*

### 4.2.1 Convergence Analysis

We now state and prove our main convergence estimate.

**Theorem 4.2.2.** *Let $(\mathbf{u}(t), p(t))$ be a sufficiently smooth, strong solution of the NSE satisfying periodic with zero-mean boundary conditions. Suppose $(\mathbf{w}_h(0), q_h(0))$ are approximations of $(\mathbf{u}(0), p(0))$ to the accuracy of (4.1.5), respectively. Then there is a constant $C = C(\mathbf{u}, p)$ such that*

$$|||\mathbf{u} - \mathbf{w}^h|||_{\infty,0} \le F(\triangle t, h, \delta) + Ch^{k+1}|||\mathbf{u}|||_{\infty,k+1}, \tag{4.2.14}$$

$$\left(\nu \triangle t \sum_{n=0}^{M-1} \|\nabla(\mathbf{u}_{n+1/2} - (\mathbf{w}_{n+1}^h + \mathbf{w}_n^h)/2)\|^2\right)^{1/2} \le F(\triangle t, h, \delta) + C\nu^{1/2}(\triangle t)^2\|\nabla \mathbf{u}_{tt}\|_{2,0} + Ch^k|||\mathbf{u}|||_{2,k+1}, \tag{4.2.15}$$

*where*

$$\begin{aligned}
F(\triangle t, h, \mu, \delta) \;=\;& C^* h^{k+1}\,(\nu^{-1/2} + \chi)(\,\|D_\mu \overline{\mathbf{u}}\|_{2,k+1} + \left\|\mathbf{u}_{n+1/2}\right\|_{k+1}\,) \\
& + \nu^{-1/2}\, h^{k+1/2}\left(|||\mathbf{u}|||_{4,k+1}^2 + |||\nabla \mathbf{u}|||_{4,0}^2\right) \\
& + \nu^{-1/2} h^k \left(|||\mathbf{u}|||_{4,k+1}^2 + \nu^{-1/2}(\|\mathbf{w}_0^h\| + \nu^{-1}|||\mathbf{f}|||_{2,\star})\right) \\
& + \delta h^k(\nu^{-1/2} + \chi)\left(\mu\,\|D_\mu \overline{\mathbf{u}}\|_{2,k+1} + \|\mathbf{u}\|_{2,k+1}\right) \\
& \nu^{-1/2} h^{s+1}|||p_{1/2}|||_{2,s+1} + (\nu^{-1/2} + \chi)\mu\delta^2\,\|\mathbf{u}\|_{2,2} \\
& + (\Delta t)^2\left(\|\mathbf{u}_{ttt}\|_{2,0} + \|\mathbf{f}_{tt}\|_{2,0} + \nu^{-1/2}\|p_{tt}\|_{2,0}\right. \\
& \left. + \nu^{-1/2}\|\nabla \mathbf{u}_{tt}\|_{4,0}^2 + \nu^{-1/2}|||\nabla \mathbf{u}|||_{4,0}^2 + \nu^{-1/2}|||\nabla \mathbf{u}_{1/2}|||_{4,0}^2\right)
\end{aligned}$$

*Proof.* First, we note that at time $t_{n+1/2}$, $\mathbf{u}$ given by (4.2.1)-(4.2.2) satisfies

$$
\begin{aligned}
(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}, \mathbf{v}^h) + b^*(D_\mu^h\overline{\mathbf{u}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \mathbf{v}^h) + \nu(\nabla\mathbf{u}_{n+1/2}, \nabla\mathbf{v}^h) - (p_{n+1/2}, \nabla\cdot\mathbf{v}^h) \\
+\chi(\mathbf{u}_{n+1/2} - D_\mu^h\overline{\mathbf{u}_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h) + Intp(\mathbf{u}^n, p^n; \mathbf{v}^h),
\end{aligned}
\tag{4.2.16}
$$

for all $\mathbf{v}^h \in X_h$, where $Intp(\mathbf{u}_n, p_n; \mathbf{v}^h)$, representing the interpolating error, denotes

$$
\begin{aligned}
Intp(\mathbf{u}_n, p_n; \mathbf{v}^h) \;=\; & \left(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}), \mathbf{v}^h\right) + \nu(\nabla\mathbf{u}_{n+1/2} - \nabla\mathbf{u}(t_{n+1/2}), \nabla\mathbf{v}^h) \\
& +b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v}^h) - b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}(t_{n+1/2}), \mathbf{v}^h) \\
& -b^*(\mathbf{u}_{n+1/2} - D_\mu^h\overline{\mathbf{u}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \mathbf{v}^h) \\
& +\chi(\mathbf{u}_{n+1/2} - D_\mu^h\overline{\mathbf{u}_{n+1/2}}^h, \mathbf{v}^h) - (p_{n+1/2} - p(t_{n+1/2}), \nabla\cdot\mathbf{v}^h) \\
& +(\mathbf{f}(t_{n+1/2}) - \mathbf{f}_{n+1/2}, \mathbf{v}^h)\,.
\end{aligned}
\tag{4.2.17}
$$

Subtracting (4.2.3) from (4.2.16) and letting $\mathbf{e}_n = \mathbf{u}_n - \mathbf{w}_n^h$ we have

$$
\begin{aligned}
\frac{1}{\Delta t}(\mathbf{e}_{n+1} - \mathbf{e}_n, \mathbf{v}^h) + b^*(D_\mu^h\overline{\mathbf{u}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \mathbf{v}^h) - b^*(D_\mu^h\overline{\mathbf{w}_{n+1/2}^h}^h, \mathbf{w}_{n+1/2}^h, \mathbf{v}^h) \\
+ \nu(\nabla\mathbf{e}_{n+1/2}, \nabla\mathbf{v}^h) + \chi(\mathbf{e}_{n+1/2} - D_\mu^h\overline{\mathbf{e}_{n+1/2}}^h, \mathbf{v}^h) \\
= (p_{n+1/2} - q, \nabla\cdot\mathbf{v}^h) + Intp(\mathbf{u}_n, p_n; \mathbf{v}^h), \qquad \forall\mathbf{v}^h \in X_h.
\end{aligned}
\tag{4.2.18}
$$

Let $\mathbf{e}_n = (\mathbf{u}_n - U_n) - (\mathbf{w}_n^h - U_n) := \boldsymbol{\eta}_n - \boldsymbol{\phi}_n^h$ where $\boldsymbol{\phi}_n^h \in X_h$ and $U$ represents the $L^2$ projection of $\mathbf{u}$ in $X_h$. Setting $\mathbf{v}^h = \boldsymbol{\phi}_{n+1/2}^h$ in (4.2.18) and using $(q, \nabla\cdot\boldsymbol{\phi}_{n+1/2}) = 0$ for all $q \in X_h$ we obtain

$$
\begin{aligned}
(\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n+1/2}^h) + \Delta t\nu\|\nabla\boldsymbol{\phi}_{n+1/2}^h\|^2 + \Delta t b^*(D_\mu^h\overline{\mathbf{w}_{n+1/2}^h}^h, \mathbf{e}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\
+ \Delta t b^*(D_\mu^h\overline{\mathbf{e}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) + \Delta t\chi\left\|{\boldsymbol{\phi}_{n+1/2}^h}^*\right\|^2 \\
= \Delta t(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n, \boldsymbol{\phi}_{n+1/2}^h) + \Delta t\nu(\nabla\boldsymbol{\eta}_{n+1/2}, \nabla\boldsymbol{\phi}_{n+1/2}^h) \\
+ \Delta t\chi(\boldsymbol{\eta}_{n+1/2} - D_\mu^h\overline{\boldsymbol{\eta}_{n+1/2}}^h, \boldsymbol{\phi}_{n+1/2}^h) \\
+ \Delta t(p_{n+1/2} - q, \nabla\cdot\boldsymbol{\phi}_{n+1/2}^h) - \Delta t\; Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}_{n+1/2}^h)\,.
\end{aligned}
\tag{4.2.19}
$$

Because of our choice of $U$, we have $(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n, \boldsymbol{\phi}^h_{n+1/2}) = 0$ and we can rewrite

$$
\begin{aligned}
\frac{1}{2}(\|\boldsymbol{\phi}^h_{n+1}\|^2 - \|\boldsymbol{\phi}^h_n\|^2) + \Delta t\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + \Delta t\chi \left\|{\boldsymbol{\phi}^h_{n+1/2}}^*\right\|^2 \\
= \Delta t\nu(\nabla\boldsymbol{\eta}_{n+1/2}, \nabla\boldsymbol{\phi}^h_{n+1/2}) - \Delta t b^*(D^h_\mu\overline{\boldsymbol{\eta}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
+ \Delta t b^*(D^h_\mu\overline{\boldsymbol{\phi}^h_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
- \Delta t b^*(D^h_\mu\overline{\mathbf{w}^h_{n+1/2}}^h, \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
+ \Delta t\chi(\boldsymbol{\eta}_{n+1/2} - D^h_\mu\overline{\boldsymbol{\eta}_{n+1/2}}^h, \boldsymbol{\phi}^h_{n+1/2}) \\
+ \Delta t(p_{n+1/2} - q, \nabla\cdot\boldsymbol{\phi}^h_{n+1/2}) + \Delta t\ Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}^h_{n+1/2})\,. \qquad (4.2.20)
\end{aligned}
$$

We now estimate the terms on the right hand side of (4.2.20) separately.

Using Cauchy-Schwarz and Young's inequalities we have

$$
\begin{aligned}
\nu\triangle t(\nabla\boldsymbol{\eta}_{n+1/2}, \nabla\boldsymbol{\phi}^h_{n+1/2}) &\leq \nu\triangle t\|\nabla\boldsymbol{\eta}_{n+1/2}\|\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\| \\
&\leq \frac{\nu\Delta t}{12}\left\|\nabla\boldsymbol{\phi}^h_{n+1/2}\right\|^2 + C\nu\Delta t\left\|\nabla\boldsymbol{\eta}_{n+1/2}\right\|^2. \qquad (4.2.21)
\end{aligned}
$$

$$
\begin{aligned}
\triangle t(p_{n+1/2} - q, \nabla\cdot\boldsymbol{\phi}^h_{n+1/2}) &\leq C\triangle t\|p_{n+1/2} - q\|\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\| \\
&\leq \frac{\nu\Delta t}{12}\left\|\nabla\boldsymbol{\phi}^h_{n+1/2}\right\|^2 + C\triangle t\nu^{-1}\|p_{n+1/2} - q\|^2. \qquad (4.2.22)
\end{aligned}
$$

Lemmas 4.1.1 and 4.1.5 and standard inequalities give

$$
\begin{aligned}
&\triangle t\, b^*(D^h_\mu\overline{\boldsymbol{\eta}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
\leq\ & C\Delta t\|D^h_\mu\overline{\boldsymbol{\eta}_{n+1/2}}^h\|^{1/2}\,\|\nabla D^h_\mu\overline{\boldsymbol{\eta}_{n+1/2}}^h\|^{1/2}\,\|\nabla\mathbf{u}_{n+1/2}\|\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\| \\
\leq\ & \frac{\nu\triangle t}{12}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\triangle t\,\nu^{-1}\|\boldsymbol{\eta}_{n+1/2}\|\,\|\nabla\boldsymbol{\eta}_{n+1/2}\|\|\nabla\mathbf{u}_{n+1/2}\|^2. \qquad (4.2.23)
\end{aligned}
$$

$$
\begin{aligned}
&\triangle t\, b^*(D^h_\mu\overline{\boldsymbol{\phi}^h_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
\leq\ & C\triangle t\|D^h_\mu\overline{\boldsymbol{\phi}^h_{n+1/2}}^h\|^{1/2}\,\|\nabla D^h_\mu\overline{\boldsymbol{\phi}^h_{n+1/2}}^h\|^{1/2}\,\|\nabla\mathbf{u}_{n+1/2}\|\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\| \\
\leq\ & C\triangle t\|\boldsymbol{\phi}^h_{n+1/2}\|^{1/2}\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^{3/2}\,\|\nabla\mathbf{u}_{n+1/2}\| \\
\leq\ & \frac{\nu\triangle t}{12}\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\triangle t\,\nu^{-3}\|\boldsymbol{\phi}^h_{n+1/2}\|^2\,\|\nabla\mathbf{u}_{n+1/2}\|^4. \qquad (4.2.24)
\end{aligned}
$$

$$
\begin{aligned}
& \triangle t\, b^*(D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h, \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\
\leq\ & C\| D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h \|^{1/2} \, \|\nabla D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h \|^{1/2} \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\
\leq\ & \frac{\nu \triangle t}{12} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + C \triangle t\, \nu^{-1} \|\mathbf{w}_{n+1/2}^h\| \, \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \,. \qquad (4.2.25)
\end{aligned}
$$

Lemma 4.1.6 and Young's inequality give

$$
\Delta t \chi(\boldsymbol{\eta}_{n+1/2} - D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h, \boldsymbol{\phi}_{n+1/2}^h) \ \leq \ \frac{\nu \triangle t}{12} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + C \triangle t\, \nu^{-1} \ \chi^2 \mu^2 \delta^4 \ \left\| \nabla \boldsymbol{\eta}_{n+1/2} \ \right\|^2 . \qquad (4.2.26)
$$

Substituting (4.2.21)-(4.2.25) into (4.2.20) and summing from $n = 0$ to $M - 1$ (assuming that $\|\boldsymbol{\phi}_0^h\| = 0$) we get

$$
\begin{aligned}
& \|\boldsymbol{\phi}_M^h\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + \chi \triangle t \sum_{n=0}^{M-1} \|{\boldsymbol{\phi}_{n+1/2}^h}^*\|^2 \\
\leq\ & \triangle t \sum_{n=0}^{M-1} C \nu^{-3} \|\nabla \mathbf{u}_{n+1/2}\|^4 \, \|\boldsymbol{\phi}_{n+1/2}^h\|^2 + \triangle t \sum_{n=0}^{M-1} C \nu \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} C \nu^{-1} \|\boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \|\nabla \mathbf{u}_{n+1/2}\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} C \nu^{-1} \|\mathbf{w}_{n+1/2}^h\| \, \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} C\, \nu^{-1} \ \chi^2 \mu^2 \delta^4 \ \left\| \nabla \boldsymbol{\eta}_{n+1/2} \ \right\|^2 + \triangle t \sum_{n=0}^{M-1} C \nu^{-1} \|p_{n+1/2} - q\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} |Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}_{n+1/2}^h)| \,. \qquad (4.2.27)
\end{aligned}
$$

We now bound each term in the right hand side of (4.2.27).

$$
\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C \nu \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \ & \leq \ \triangle t C \nu \sum_{n=0}^{M} \|\nabla \boldsymbol{\eta}_n\|^2 \leq \triangle t C \nu \sum_{n=0}^{M} h^{2k} |\mathbf{u}_n|_{k+1}^2 \\
& \leq \ C \nu h^{2k} |||\mathbf{u}|||_{2,k+1}^2 \,. \qquad (4.2.28)
\end{aligned}
$$

Next, we consider the term

$$
\begin{aligned}
& \triangle t \sum_{n=0}^{M-1} C\nu^{-1} \|\boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \mathbf{u}_{n+1/2}\|^2 \\
\leq \;& C\nu^{-1} \triangle t \sum_{n=0}^{M-1} \big( \|\boldsymbol{\eta}_{n+1}\| \, \|\nabla \boldsymbol{\eta}_{n+1}\| + \|\boldsymbol{\eta}_{n}\| \, \|\nabla \boldsymbol{\eta}_{n}\| \\
& \qquad + \|\boldsymbol{\eta}_{n}\| \, \|\nabla \boldsymbol{\eta}_{n+1}\| + \|\boldsymbol{\eta}_{n+1}\| \, \|\nabla \boldsymbol{\eta}_{n}\| \big) \|\nabla \mathbf{u}_{n+1/2}\|^2 \\
\leq \;& C \, \nu^{-1} \, h^{2k+1} \left( \triangle t \sum_{n=0}^{M-1} |\mathbf{u}_{n+1}|_{k+1}^2 \, \|\nabla \mathbf{u}_{n+1/2}\|^2 + \triangle t \sum_{n=0}^{M-1} |\mathbf{u}_{n+1}|_{k+1} |\mathbf{u}_{n}|_{k+1} \, \|\nabla \mathbf{u}_{n+1/2}\|^2 \right. \\
& \qquad \left. + \; \triangle t \sum_{n=0}^{M-1} |\mathbf{u}_{n}|_{k+1}^2 \, \|\nabla \mathbf{u}_{n+1/2}\|^2 \right) \\
\leq \;& C\nu^{-1} \, h^{2k+1} \left( \triangle t \sum_{n=0}^{M} |\mathbf{u}_{n}|_{k+1}^4 + \triangle t \sum_{n=0}^{M} \|\nabla \mathbf{u}_{n}\|^4 \right) \\
= \;& C\nu^{-1} \, h^{2k+1} \left( |||\mathbf{u}|||_{4,k+1}^4 + |||\nabla \mathbf{u}|||_{4,0}^4 \right) . \qquad (4.2.29)
\end{aligned}
$$

Using (4.2.11), we have

$$
\begin{aligned}
& \triangle t \sum_{n=0}^{M-1} C\nu^{-1} \left( \|\mathbf{w}_{n+1/2}^h\| \, \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \right) \\
\leq \;& C\nu^{-1} \triangle t \sum_{n=0}^{M-1} \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
\leq \;& C\nu^{-1} \triangle t \sum_{n=0}^{M-1} \left( \|\nabla \boldsymbol{\eta}_{n+1}\|^2 + \|\nabla \boldsymbol{\eta}_{n}\|^2 \right) \|\nabla \mathbf{w}_{n+1/2}^h\| \\
\leq \;& C\nu^{-1} h^{2k} \triangle t \sum_{n=0}^{M-1} \left( |\mathbf{u}_{n+1}|_{k+1}^2 + |\mathbf{u}_{n}|_{k+1}^2 \right) \|\nabla \mathbf{w}_{n+1/2}^h\| \\
\leq \;& C\nu^{-1} h^{2k} \left( \triangle t \sum_{n=0}^{M} |\mathbf{u}^n|_{k+1}^4 + \triangle t \sum_{n=0}^{M} \|\nabla \mathbf{w}_{n+1/2}^h\|^2 \right) \\
\leq \;& C\nu^{-1} h^{2k} \left( \; |||\mathbf{u}|||_{4,k+1}^4 + \nu^{-1} (\|\mathbf{w}_0^h\|^2 + \nu^{-1} |||\mathbf{f}|||_{2,\star}^2 ) \; \right) . \qquad (4.2.30)
\end{aligned}
$$

Lemma 4.2.1 and (4.1.5) give

$$\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C\nu^{-1}\|p_{n+1/2}-q\|^2 &\leq \triangle t C\nu^{-1}\sum_{n=0}^{M-1}( \; \|p(t_{n+1/2})-q\|^2 + \|p_{n+1/2}-p(t_{n+1/2})\|^2 \; ) \\
&\leq C\nu^{-1} \; (h^{2s+2}\triangle t \sum_{n=0}^{M-1} \|p(t_{n+1/2})\|^2_{s+1} \\
&\quad + \triangle t \sum_{n=0}^{M-1} \frac{1}{48}(\triangle t)^3 \int_{t_n}^{t_{n+1}} \|p_{tt}\|^2 \, dt \\
&\leq C\nu^{-1}(h^{2s+2}|||p_{1/2}|||^2_{2,s+1} + (\triangle t)^4 \|p_{tt}\|^2_{2,0}). \qquad (4.2.31)
\end{aligned}$$

Next, we bound the time relaxation term

$$\triangle t \sum_{n=0}^{M-1} \nu^{-1} \; \chi^2\mu^2\delta^4 \quad \left\|\nabla \boldsymbol{\eta}^h_{n+1/2} \; \right\|^2 \leq \nu^{-1} \; \chi^2\mu^2\delta^4 \; h^{2k}|||\mathbf{u}|||^2_{2,k+1} \, . \qquad (4.2.32)$$

We now bound the terms in $Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}^h_{n+1/2})$. Using Cauchy-Schwarz and Young's inequalities, Taylor's theorem, and Lemma 4.1.7,

$$\begin{aligned}
&\left( \frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2} \right) \\
\leq \; & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1/2}\|^2 \; + \; \frac{1}{2}\|\frac{\mathbf{u}^{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\|^2 \\
\leq \; & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1}\|^2 \; + \; \frac{1}{2}\|\boldsymbol{\phi}^h_{n}\|^2 \; + \; \frac{1}{2}\frac{(\Delta t)^3}{1280}\int_{t_n}^{t_{n+1}} \|\mathbf{u}_{ttt}\|^2 \, dt \, , \qquad (4.2.33)
\end{aligned}$$

$$\begin{aligned}
&(p_{n+1/2}-p(t_{n+1/2}), \nabla\cdot\boldsymbol{\phi}^h_{n+1/2}) \\
\leq \; & \varepsilon_1\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 \; + \; C\,\nu^{-1}\|p_{n+1/2}-p(t_{n+1/2})\|^2 \\
\leq \; & \varepsilon_1\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 \; + \; C\,\nu^{-1}\frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|p_{tt}\|^2 \, dt \, , \qquad (4.2.34)
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{f}(t_{n+1/2}) - \mathbf{f}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
\leq \; & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1/2}\|^2 \; + \; \frac{1}{2}\|\mathbf{f}(t_{n+1/2}) - \mathbf{f}_{n+1/2}\|^2 \\
\leq \; & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1}\|^2 \; + \; \frac{1}{2}\|\boldsymbol{\phi}^h_{n}\|^2 \; + \; \frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|\mathbf{f}_{tt}\|^2 \, dt \, , \qquad (4.2.35)
\end{aligned}$$

$$
\begin{aligned}
& (\nabla \mathbf{u}_{n+1/2} \,-\, \nabla \mathbf{u}(t_{n+1/2}), \nabla \boldsymbol{\phi}^h_{n+1/2}) \\
\leq\ & \varepsilon_2 \nu \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \,+\, C\,\nu \|\nabla \mathbf{u}_{n+1/2} \,-\, \nabla \mathbf{u}(t_{n+1/2})\|^2 \\
\leq\ & \varepsilon_2 \nu \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \,+\, C\,\nu \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^2 \, dt \,,
\end{aligned} \tag{4.2.36}
$$

$$
\begin{aligned}
& b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) - b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2}) \\
=\ & b^*(\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}), \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) + b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2}) \\
\leq\ & C \, \|\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}))\| \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \, \big(\|\nabla \mathbf{u}_{n+1/2}\| \,+\, \|\nabla \mathbf{u}(t_{n+1/2})\|\big) \\
\leq\ & C\,\nu^{-1} \big(\|\nabla \mathbf{u}_{n+1/2}\|^2 \,+\, \|\nabla \mathbf{u}(t_{n+1/2})\|^2\big) \, \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^2 \, dt \,+ \varepsilon_3 \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \\
\leq\ & C\,\nu^{-1} \frac{(\Delta t)^3}{48} \bigg( \int_{t_n}^{t_{n+1}} 2(\|\nabla \mathbf{u}_{n+1/2}\|^4 \,+\, \|\nabla \mathbf{u}(t_{n+1/2})\|^4) \, dt \\
& \qquad + \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^4 \, dt \bigg) + \varepsilon_3 \nu \|\nabla \phi^h_{n+1/2}\|^2 \\
\leq\ & C\,\nu^{-1} \, (\Delta t)^4 (\|\nabla \mathbf{u}_{n+1/2}\|^4 \,+\, \|\nabla \mathbf{u}(t_{n+1/2})\|^4) \\
& \qquad + C\,\nu^{-1} \, (\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^4 \, dt + \varepsilon_3 \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \,.
\end{aligned} \tag{4.2.37}
$$

$$
\begin{aligned}
& b^*(\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
\leq\ & \frac{1}{2} \big( \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \, \|\nabla \mathbf{u}_{n+1/2}\|_\infty \, \|\boldsymbol{\phi}^h_{n+1/2}\| \\
& \quad + \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \, \|\mathbf{u}_{n+1/2}\|_\infty \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \big) \\
\leq\ & C \, \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \\
\leq\ & \varepsilon_4 \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + C \nu^{-1} \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\|^2 \\
\leq\ & \varepsilon_4 \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + C \nu^{-1} \bigg( \, (\mu \delta^2 h^{2k} + h^{2k+2}) \, \big\|D_\mu \overline{\mathbf{u}_{n+1/2}}\big\|^2_{k+1} \\
& \quad + \, (\delta^2 h^{2k} + h^{2k+2}) \, \big\|\mathbf{u}_{n+1/2}\big\|^2_{k+1} + \mu^2 \delta^4 \, \big\|\mathbf{u}_{n+1/2}\big\|^2_2 \, \bigg) \,.
\end{aligned} \tag{4.2.38}
$$

$$
\begin{aligned}
& \chi(\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h, \boldsymbol{\phi}^h_{n+1/2}) \leq \chi \, \big\|\boldsymbol{\phi}^h_{n+1/2}\big\| \, \big\|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\big\| \\
& \leq \frac{1}{2} \|\boldsymbol{\phi}^h_{n+1}\|^2 \,+\, \frac{1}{2} \|\boldsymbol{\phi}^h_n\|^2 \,+ \chi^2 C \bigg( \, (\mu \delta^2 h^{2k} + h^{2k+2}) \, \big\|D_\mu \overline{\mathbf{u}_{n+1/2}}\big\|^2_{k+1} \\
& \quad + \, (\delta^2 h^{2k} + h^{2k+2}) \, \big\|\mathbf{u}_{n+1/2}\big\|^2_{k+1} + \mu^2 \delta^4 \, \big\|\mathbf{u}_{n+1/2}\big\|^2_2 \, \bigg) \,.
\end{aligned} \tag{4.2.39}
$$

Combine (4.2.33)-(4.2.39) to obtain

$$
\begin{aligned}
\sum_{n=0}^{M-1} \Delta t |Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}^h_{n+1/2})| \;\le\; & \sum_{n=0}^{M-1} [\; \Delta t\, C \|\boldsymbol{\phi}^h_{n+1}\|^2 \\
& +(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \Delta t\, \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \\
& + C\nu^{-1} \Big( \; (\mu\delta^2 h^{2k} + h^{2k+2}) \left\| D_\mu \overline{\mathbf{u}_{n+1/2}} \right\|^2_{k+1} \\
& \quad + \; (\delta^2 h^{2k} + h^{2k+2}) \left\| \mathbf{u}_{n+1/2} \right\|^2_{k+1} + \mu^2 \delta^4 \left\| \overline{\mathbf{u}_{n+1/2}} \right\|^2_2 \Big) \\
& + C\chi^2 \Big( \; (\mu\delta^2 h^{2k} + h^{2k+2}) \left\| D_\mu \overline{\mathbf{u}_{n+1/2}} \right\|^2_{k+1} \\
& \quad + \; (\delta^2 h^{2k} + h^{2k+2}) \left\| \mathbf{u}_{n+1/2} \right\|^2_{k+1} + \mu^2 \delta^4 \left\| \overline{\mathbf{u}_{n+1/2}} \right\|^2_2 \Big) \; ] \\
& + C(\Delta t)^4 \big( \|\mathbf{u}_{ttt}\|^2_{2,0} + \nu^{-1} \|p_{tt}\|^2_{2,0} + \|\mathbf{f}_{tt}\|^2_{2,0} \\
& \qquad + \nu \|\nabla \mathbf{u}_{tt}\|^2_{2,0} + \nu^{-1} \|\nabla \mathbf{u}_{tt}\|^4_{4,0} \\
& \qquad + \nu^{-1} |||\nabla \mathbf{u}|||^4_{4,0} + \nu^{-1} |||\nabla \mathbf{u}_{1/2}|||^4_{4,0} \big) \; . \qquad (4.2.40)
\end{aligned}
$$

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1/12$ and putting everything together, from (4.2.27) we obtain

$$
\begin{aligned}
& \|\boldsymbol{\phi}^h_M\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \chi \triangle t \sum_{n=0}^{M-1} \|{\boldsymbol{\phi}^h_{n+1/2}}^*\|^2 \\
\le \quad & \triangle t \sum_{n=0}^{M-1} C \; (\nu^{-3} \left\| \nabla \mathbf{u}_{n+1/2} \right\|^4 + 1) \; \left\| \boldsymbol{\phi}^h_{n+1/2} \right\|^2 \\
& + C \;\; \nu \; h^{2k} \; |||\mathbf{u}|||^2_{2,k+1} \; + \; C \; \nu^{-1} \, h^{2k+1} \; \left( \; |||\mathbf{u}|||^4_{4,k+1} + |||\nabla \mathbf{u}|||^4_{4,0} \; \right) \\
& + \; C \;\; \nu^{-1} h^{2k} \; \left( \; |||\mathbf{u}|||^4_{4,k+1} \; + \; \nu^{-1} (\|\mathbf{w}^h_0\|^2 \; + \; \nu^{-1} |||\mathbf{f}|||^2_{2,\star}) \; \right) \\
& + \; C \;\; \nu^{-1} \; \left( \; h^{2s+2} |||p_{1/2}|||^2_{2,s+1} \; + \; (\triangle t)^4 \|p_{tt}\|^2_{2,0} \; \right) \; + \; C \;\; \nu^{-1} \chi^2 \mu^2 \delta^4 \; h^{2k} |||\mathbf{u}|||^2_{2,k+1} \\
& + \; C \; (\nu^{-1} + \chi^2) \Big( \; (\mu\delta^2 h^{2k} + h^{2k+2}) \, \|D_\mu \overline{\mathbf{u}}\|^2_{2,k+1} \\
& \qquad + \; (\delta^2 h^{2k} + h^{2k+2}) \, \|\mathbf{u}\|^2_{2,k+1} \; + \; \mu^2 \delta^4 \, \|\mathbf{u}\|^2_{2,2} \Big) \\
& + C(\Delta t)^4 \big( \|\mathbf{u}_{ttt}\|^2_{2,0} + \|\mathbf{f}_{tt}\|^2_{2,0} + \nu \|\nabla \mathbf{u}_{tt}\|^2_{2,0} \\
& \qquad + \; \nu^{-1} \|\nabla \mathbf{u}_{tt}\|^4_{4,0} + \nu^{-1} |||\nabla \mathbf{u}|||^4_{4,0} + \nu^{-1} |||\nabla \mathbf{u}_{1/2}|||^4_{4,0} \big) \; . \qquad (4.2.41)
\end{aligned}
$$

Hence, with $\triangle t$ sufficiently small, i.e. $\triangle t < C(\nu^{-3}|||\nabla \mathbf{u}|||^4_{\infty,0} + 1)^{-1}$, from Gronwall's Lemma (see Lemma 4.2.2), we have

$$
\begin{aligned}
&\|\boldsymbol{\phi}^h_M\|^2 + \nu\triangle t \sum_{n=0}^{M-1} \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \chi\triangle t \sum_{n=0}^{M-1} \|{\boldsymbol{\phi}^h_{n+1/2}}^*\|^2 \\
&\le C^* \, \{ \, h^{2k+2} \, (\nu^{-1} + \chi^2)( \, \|D_\mu \overline{\mathbf{u}}\|^2_{k+1} + \|\mathbf{u}\|^2_{k+1} \, ) \\
&\quad + \nu^{-1} \, h^{2k+1} \left( |||\mathbf{u}|||^4_{4,k+1} + |||\nabla \mathbf{u}|||^4_{4,0} \right) + h^{2k}(\nu + \nu^{-1} \, \chi^2 \mu^2 \delta^4 \, )|||\mathbf{u}|||^2_{2,k+1} \\
&\quad + \nu^{-1} h^{2k} \left( |||\mathbf{u}|||^4_{4,k+1} + \nu^{-1}(\|\mathbf{w}^h_0\|^2 + \nu^{-1}|||\mathbf{f}|||^2_{2,\star}) \right) \\
&\quad + \delta^2 h^{2k} (\nu^{-1} + \chi^2) \left( \mu \, \|D_\mu \overline{\mathbf{u}}\|^2_{2,k+1} + \|\mathbf{u}\|^2_{2,k+1} \right) \\
&\quad \nu^{-1} h^{2s+2} |||p_{1/2}|||^2_{2,s+1} + (\nu^{-1} + \chi^2)\mu^2\delta^4 \, \|\mathbf{u}\|^2_{2,2} \\
&\quad + (\Delta t)^4 \left( \|\mathbf{u}_{ttt}\|^2_{2,0} + \nu^{-1} \|p_{tt}\|^2_{2,0} + \|\mathbf{f}_{tt}\|^2_{2,0} + \nu\|\nabla \mathbf{u}_{tt}\|^2_{2,0} \right. \\
&\quad \left. + \nu^{-1}\|\nabla \mathbf{u}_{tt}\|^4_{4,0} + \nu^{-1}|||\nabla \mathbf{u}|||^4_{4,0} + \nu^{-1}|||\nabla \mathbf{u}_{1/2}|||^4_{4,0} \right) \, \}
\end{aligned} \tag{4.2.42}
$$

where $C^* = C\exp(C\nu^{-3}T)$.

Estimate (4.2.14) then follows from the triangle inequality and (4.2.42).

To obtain (4.2.15), we use (4.2.42) and

$$
\begin{aligned}
& \|\nabla \left( \mathbf{u}(t_{n+1/2}) - (\mathbf{w}^h_{n+1} + \mathbf{w}^h_n)/2 \right) \|^2 \\
\le \;& \|\nabla(\mathbf{u}(t_{n+1/2}) - \mathbf{u}_{n+1/2})\|^2 \; + \; \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \; + \; \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \\
\le \;& \frac{(\triangle t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^2 \, dt \; + \; Ch^{2k}||\mathbf{u}_{n+1}||^2_{k+1} \; + \; Ch^{2k}||\mathbf{u}_n||^2_{k+1} \; + \; \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \, .
\end{aligned}
$$

□

## 4.3 NUMERICAL ILLUSTRATIONS

In this section, we present two numerical experiments. Our first test confirms the predicted rates of convergence. In our second experiment, we study a simple, under-resolved flow with recirculation: the flow across a step. The computations were performed with the software FreeFem++, see [20].

### 4.3.1 Convergence Rate Verification

To test the predicted convergence rates, we consider the Chorin vortex decay problem, [11, 40, 63]. The prescribed solution in $\Omega = (0,1) \times (0,1)$ is

$$
\begin{aligned}
u_1(x,y,t) &= -\cos(n\pi x)\sin(n\pi y)e^{-2n^2\pi^2 t/\tau} \\
u_2(x,y,t) &= \sin(n\pi x)\cos(n\pi y)e^{-2n^2\pi^2 t/\tau} \\
p &= -\frac{1}{4}(\cos(n\pi x) + \cos(n\pi y))e^{-2n^2\pi^2 t/\tau}.
\end{aligned}
$$

When the relaxation time $\tau = Re$, the pair $(\mathbf{u}, p)$ defined above is a solution of the NSE with $\mathbf{f} = 0$. This solution consists of an $n \times n$ array of oppositely signed vortices that decay as $t \to \infty$.

The model was discretized in time with the implicit second order Crank-Nicolson scheme and in space with the Taylor-Hood finite element method, i.e. the velocity was approximated by continuous piecewise quadratics and the pressure by continuous piecewise linears. Theorem 4.2.2 shows that under sufficient regularity of the solution of the variational problem, the error $||\nabla(\mathbf{u} - \mathbf{w}^h)||_{2,0}$ is of $O(h^2)$. This experiment suggests that it might be possible to establish an upper bound for the error $||\mathbf{u} - \mathbf{w}^h||_{2,0}$ as follows

$$
||\mathbf{u} - \mathbf{w}^h||_{2,0} \leq C\ h\ ||\nabla(\mathbf{u} - \mathbf{w}^h)||_{2,0},
$$

i.e., $||\mathbf{u} - \mathbf{w}^h||_{2,0} \simeq O(h^3)$. Generally, Nitsche's duality trick is employed to derive error estimates for $L^2$ norms. This is often referred to as the $L^2$ lift, [6]. In our test, we choose $n = 1$, $dt = 0.005$, $T = 0.5$, $\mu = 1/m$, $\chi = 0.1$, $\delta = \sqrt{1/m}$ and $h = 1/m$, where $m$ is the number of subdivisions of the interval $(0,1)$. We performed the same test for different Reynolds numbers. The results are in Tables 10 and 11 below. In both cases, $Re = 1$ and $Re = 10^4$, the convergence rate approaches the second order predicted for $||\nabla(\mathbf{u} - \mathbf{w}^h)||_{2,0}$. For $Re = 10^4$ we also see what appears to be an $L^2$ lift for $\left\|\mathbf{u} - \mathbf{w}^h\right\|_{2,0}$.

Note that the error plateau around $10^{-5}$ already on coarse mesh. This is likely related to the stopping criteria ($\mathbf{u}_{residual} < 10^{-5}$) or the $O(10^{-5})$ stabilization used in the (2,2) block of the linear Stokes system for the solver used.

| mesh | $\|\|\mathbf{u} - \mathbf{w}^h\|\|_{2,0}$ | ratio | $\|\|\nabla(\mathbf{u} - \mathbf{w}^h)\|\|_{2,0}$ | ratio |
|---|---|---|---|---|
| $10 \times 10$ | 0.000316786 | - | 0.00461599 | - |
| $20 \times 20$ | $6.66405 \cdot 10^{-5}$ | 2.25 | 0.00116309 | 1.98 |
| $30 \times 30$ | $5.31594 \cdot 10^{-5}$ | 0.6 | 0.000522003 | 1.97 |
| $40 \times 40$ | $5.16375 \cdot 10^{-5}$ | 0.1 | 0.000300615 | 1.91 |

Table 10: Errors and convergence rates for the Leray-Tikhonov model at $Re = 1$

| mesh | $\|\|\mathbf{u} - \mathbf{w}^h\|\|_{2,0}$ | ratio | $\|\|\nabla(\mathbf{u} - \mathbf{w}^h)\|\|_{2,0}$ | ratio |
|---|---|---|---|---|
| $10 \times 10$ | 0.0226085 | - | 1.35783 | - |
| $20 \times 20$ | 0.00428244 | 2.40 | 0.502447 | 1.43 |
| $30 \times 30$ | 0.00131237 | 2.91 | 0.23989 | 1.82 |
| $40 \times 40$ | 0.000531236 | 3.14 | 0.131774 | 2.08 |

Table 11: Errors and convergence rates for the Leray-Tikhonov model at $Re = 10^4$

### 4.3.2 Step Problem

In our second test, we consider a flow in transition via shedding of eddies behind the step. At a critical Reynolds number, for which the flow should be time dependent, some models are not able to capture the correct (non stationary) physical properties of the flow, e.g., [41]. Herein, we present results for a parabolic inflow profile, which is given by $\mathbf{u} = (u_1, u_2)^T$, with $u_1 = y(10 - y) = 25$ and $u_2 = 0$. No-slip boundary condition is prescribed on the top and bottom boundary as well as on the step. At the outflow we have "do nothing" boundary condition, an accepted outflow condition in CFD. The model was discretized in time with the Crank-Nicolson and in space with the Taylor Hood finite-element method. Figure 8 shows that expected behavior of the flow: behind the step the flow simulation correctly develops vortices separate from the step. Figure 8 shows the results for a course mesh (4585 degrees of freedom) at $T = 10, 20, 30, 40$, for $\nu^{-1} = 750$, $\chi = 0.01$, $\mu = 0.01$, $dt = 0.0025$, $\delta = 1.5$.

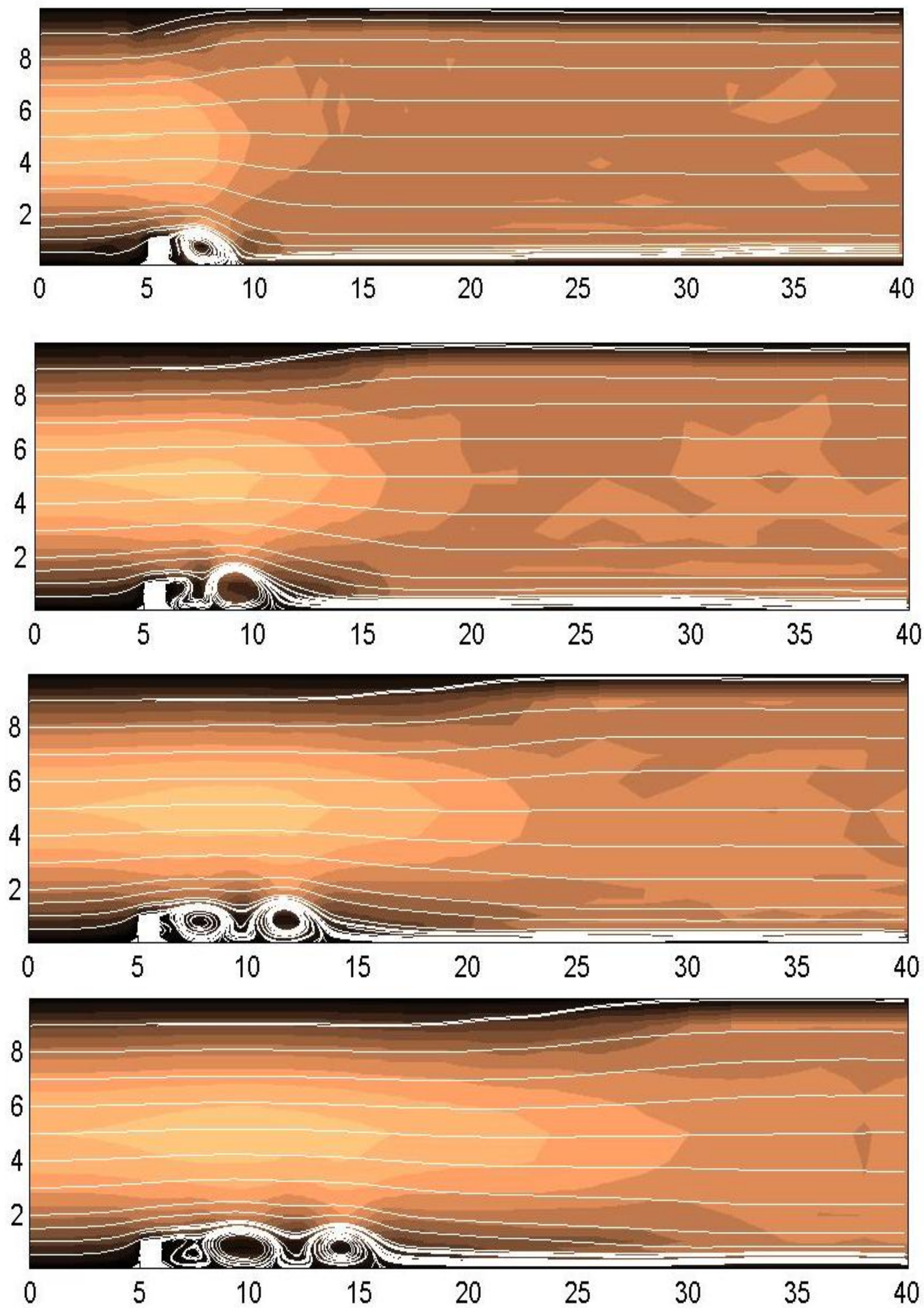


Figure 8: Leray-Tikhonov deconvolution model: Flow field at T = 10, 20, 30, 40.

# 5.0 CONCLUSIONS

Turbulence has been a long standing challenge for human mind. The Navier-Stokes equations probably contain all of turbulence. They describe every detail of the turbulent velocity field from the largest to the smallest length and time scales. Much progress was made throughout the years in studying the NSE, but the main problems, like existence and uniqueness of strong solutions, are still open. Thus various turbulence models are sought to better predict flow statistics and averages. Considering the diversity and complexity of fluid flows, every contribution in this field is important.

The main contribution of this thesis was the derivation of new such turbulence models and the analysis of their mathematical and numerical properties. The central idea was to adapt "off-the-shelf" methods for ill-posed problems to turbulence and use them to obtain new (or improve existing) models. In Chapters 2 and 3 we have focused on LES models, in particular, ADM, while in Chapter 4 we studied a new regularization model of the NSE.

In Chapter 2 we found near minimal conditions on the deconvolution operator that guarantee existence of weak solutions of a deconvolution model. Using techniques from functional analysis and the theory of weak solutions of PDEs, [23], we show that the weak solution is a unique strong solution and satisfies an energy equality. The averaging operator chosen was a specific differential filter. More generally, if the filter $G$ satisfies $\widehat{g}(k) \neq 0$ for all $k$, then the exact filter inverse $A$ can be defined as an unbounded operator with dense domain and closed range. If additionally, $|\widehat{g}(k)| \to 0$ as $k \to \infty$ with $O(\frac{1}{|k|^2})$ (or faster) then the existence theory developed in Chapter 2 can be extended to the filter $G$.

In Chapter 3, we derived deconvolution operators which minimize the deconvolution error for velocity fields with finite kinetic energy.

For a LES with deconvolution model to be feasible, the model's consistency error must be small for large radii $\delta$, which are large with respect to the problems inherent length scales (which correspond to computationally attainable meshwidths). Thus, selection of parameters to minimize the model's consistency error increases the problems for which LES is feasible, and increases the reduction in computational effort obtainable when using LES.

The use of optimal parameters requires no extra computational effort. Two main results of Chapter 3 are:

1. the values of the optimal parameters (in Section 3.3) and
2. the reduction in the model consistency error that results in their use is is at least 50%, (see Table 3).

We found two sets of parameters: Chebychev optimized (when the deconvolution error was optimized over a general, finite velocity field) and K-41 optimized (for special velocities, with a $k^{-5/3}$ energy spectrum).

To determine the resulting increase in accuracy, we considered the three versions of van Cittert, unoptimized, Chebychev optimized and K-41 optimized and compute the resulting deconvolution error in the case of velocity fields with energy spectrum of $k^{-5/3}$. Interestingly, in all cases (N= 1,2,3,4,5) the Chebychev optimized parameters resulted in comparable or better errors than K-41 optimization. Since Chebychev optimization gives parameter values good for all flow fields and the latter only for special ones, this suggests that Chebychev optimized deconvolution is to be strongly preferred. It is important to note that the relative increase in accuracy obtained using optimal parameters itself increases with the order of the model.

In Chapter 4, we developed and studied a new family of NSE-regularizations, based on a modification of Tikhonov-Lavrentiev regularization process for ill-posed problems.

Often, numerical simulations of complex flows are based on various regularizations of the NSE, rather than the NSE themselves. The resulting models often have positive effects on computation results, e.g. errors are observed to be much better over much larger time intervals. The main negative effects sometimes observed in regularizations are overdamping and delayed transitions. In the method presented in Chapter 4, the transition from one type of flow to another is not retarded. With this method, we obtained an approximation to the unfiltered solution by one filtering step. Using the differential filter (4.1.7), the error in approximation is $\mathbf{u} - D_\mu \overline{\mathbf{u}} = O(\mu\delta^2)$. We studied a fully discrete algorithm for the model, Crank-Nicolson in time and finite element in space. We have given a numerical analysis for the scheme and included proofs of unconditional stability and solvability. We have also given a convergence analysis which was also verified in our numerical computations.

Our analytical studies in Chapters 3 and 4 will be enhanced with more computational experiments. The work plan incudes:

- adapt other methods for solving ill-posed problems to turbulence and obtain new, more accurate turbulence models;
- optimize model parameters for special flows, such as boundary layers or homogeneous, isotropic turbulence;
- more numerical experiments with the discrete and exact Tikhonov Deconvolution Operator;
- extension of the theory in Chapter 2 to other filters;
- using the Accelerated van Cittert deconvolution operator, I would like to connect the work studying (2.0.3) as a continuum model with computational experiments by a numerical analysis of discretization of (2.0.3).

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