Efficient PETSc Solvers for Discontinuous Galerkin Methods Applied to Elliptic Problems

by

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In this thesis we are creating several large scale relatively sparse linear systems generated by the Discontinuous Galerkin Method to numerically solve a two-point boundary value problem over the interval (0,1). These linear systems are then solved on a computer using three iterative Krylov methods all built into a Portable, Extensible Toolkit for Scientific Computation (PETSc). The methods that are used are Conjugate Gradient (CG), Bi-Conjugate Gradient-Stable (Bi-CGStab), and Generalized Minimal Residual (GMRES). The effectiveness and efficiency of these linear solvers are analyzed as two parameters of the system, namely EPS and penalty term SIG, are varied. Also the effects of several preconditioners are analyzed.

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Figure 31: BI-CGSTAB EPS = 0 SIG = 0.1 TOL = 1.0e-10 ... 121

Figure 32: BI-CGSTAB EPS = 0 SIG = 1 TOL = 1.0e-10 ... 121

Figure 33: BI-CGSTAB EPS = 0 SIG = 10 TOL = 1.0e-10 ... 122

Figure 34: BI-CGSTAB EPS = 0 SIG = 100 TOL = 1.0e-10 ... 122

Figure 35: BI-CGSTAB EPS = +1 SIG = 0 TOL = 1.0e-10 ... 123

Figure 36: BI-CGSTAB EPS = +1 SIG = 0.01 TOL = 1.0e-10 ... 123

Figure 37: BI-CGSTAB EPS = +1 SIG = 0.1 TOL = 1.0e-10 ... 124

Figure 38: BI-CGSTAB EPS = +1 SIG = 1 TOL = 1.0e-10 ... 124

Figure 39: BI-CGSTAB EPS = +1 SIG = 10 TOL = 1.0e-10 ... 125

Figure 40: BI-CGSTAB EPS = +1 SIG = 100 TOL = 1.0e-10 ... 125

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1.0 INTRODUCTION

The Discontinuous Galerkin Method (DG method) was initially introduced by Reed and Hill in 1973 as a way to solve neutron transport problems. For various reasons, the technique lay dormant for several years and has only recently become popular as a method for solving fluid dynamics and electromagnetic problems [5]. In this thesis we use the DG method to solve a two-point boundary value problem. When solving this problem, the DG method generates a sparse linear system $Ax = b$ that needs to be solved for the unknown x. We solve this system using three different iterative methods and analyze the efficiency when these methods are used alone and with preconditioning.

Iterative methods created to solve linear systems form a rich and lively area of research, and this has led to the publication of numerous books on the subject. One of the first books published was in 1962 by Varga [9]. This book contains much theory that is still relevant today. Methods absent from Varga's book are ones in the class of Krylov subspace methods, since at that time they lacked popularity. But, as circumstances would have it, these methods have grown in popularity over the years, and all three of the iterative methods we use are of the class of Krylov subspace methods.

Before presenting and analyzing our results from the numerical simulations, we first introduce the problem in greater detail and go a bit in depth describing and deriving the class of DG methods to be used. Next, the iterative methods are presented along with the algorithms that define them and relevant convergence theorems. Then the three preconditioners used to improve the convergence of the methods are presented. This will be followed by a study of the condition number of the global matrices generated by the DG method. Finally, the numerical simulation data along with conclusions are presented.

1.1 MODEL PROBLEM

Let us consider the following two-point boundary value problem:

$$\forall x \in (0,1), \quad -p''(x) = f(x), \qquad (1.1)$$

$$p(0) = 1, \qquad (1.2)$$

$$p(1) = 0, \qquad (1.3)$$

where $f \in C^0(0,1)$. A function p is said to be a solution to this BVP if $p \in C^2(0,1)$, and p satisfies equations (1.1)-(1.3) pointwise.

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition Γ_h of the interval (0,1) such that $x_i = ih$ and $h = \frac{1}{N}$. Denote the n$^{\text{th}}$ interval $I_n = (x_n, x_{n+1})$. Let $D_k(\Gamma_h)$ denote the space of piecewise discontinuous polynomials of degree k. That is

$$D_k(\Gamma_h) = \{v : v|_{I_n} \in \mathrm{P}_k(I_n), \quad \forall n = 0, \ldots, N-1\},$$

Where $\mathrm{P}_k(I_n)$ is the space of degree k polynomials on the interval I_n. Let $\varepsilon > 0$. If we denote $v(x_n^+) = \lim_{\varepsilon \to 0^+} v(x_n + \varepsilon)$ and $v(x_n^-) = \lim_{\varepsilon \to 0^+} v(x_n - \varepsilon)$, then we can define the jump and average of v at the endpoints of I_n by the following:

Jump at interior nodes: $[v(x_n)] = v(x_n^-) - v(x_n^+), \quad \forall n = 1, \ldots, N-1,$

Jump at endpoints: $[v(x_0)] = -v(x_0^+), \quad [v(x_N)] = v(x_N^-)$.

$$\text{Average at interior nodes: } \{v(x_n)\} = \frac{1}{2}(v(x_n^-) + v(x_n^+)), \quad \forall n = 1, \ldots, N-1$$

$$\text{Average at endpoints: } \{v(x_0)\} = v(x_0^+), \quad \{v(x_N)\} = v(x_N^-).$$

Finally we introduce the following penalty term:

$$J(v, w) = \sum_{n=0}^{N} \frac{SIG}{h}[v(x_n)][w(x_n)],$$

where $SIG \geq 0$.

1.2 DG METHOD

Let v be a function in $D_k(\Gamma_h)$. If we multiply (1.1) by v and use integration by parts on each interval I_n, we have the following:

$$\int_{x_n}^{x_{x+1}} p'(x)v'(x)dx - p'(x_{n+1}^-)v(x_{n+1}^-) + p'(x_n^+)v(x_n^+) = \int_{x_n}^{x_{n+1}} f(x)v(x), \qquad n = 0, \ldots, N-1.$$

If we add all N of the above equations, we have

$$\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} p'(x)v'(x)dx - \sum_{n=0}^{N} [p'(x_n)v(x_n)] = \int_0^1 f(x)v(x)dx. \qquad (1.4)$$

It is not hard to verify that for $1 \leq n \leq N-1$:

$$[p'(x_n)v(x_n)] = \{p'(x_n)\}[v(x_n)] + \{v(x_n)\}[p'(x_n)]. \qquad (1.5)$$

Also it is easy to verify that

$$[p'(x_n)v(x_n)] = \{p'(x_n)\}[v(x_n)], \text{ for } n = 0, N. \qquad (1.6)$$

Since the exact solution p to the BVP is in $C^2(0,1)$, it satisfies the equation $[p'(x_n)] = 0$ for all $1 \leq n \leq N-1$. Using this fact and applying both (1.5) and (1.6) to (1.4) we have

$$\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} p'(x)v'(x)dx - \sum_{n=0}^{N} \{p'(x_n)\}[v(x_n)] = \int_0^1 f(x)v(x)dx.$$

Again by continuity, the exact solution p also satisfies the equations $[p(x_n)]=0$ for all $1 \le n \le N$, and $[p(x_0)]=-1$. Thus, if p is the exact solution of (1.1)-(1.3), then p satisfies the following:

$$\sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} p'(x)v'(x)dx - \sum_{n=0}^{N}\{p'(x_n)\}[v(x_n)] + (EPS)\sum_{n=0}^{N}\{v'(x_n)\}[p(x_n)]$$

$$= \int_0^1 f(x)v(x)dx - (EPS)v'(x_0)p(x_0) + (EPS)v'(x_N)p(x_N)$$

$$= \int_0^1 f(x)v(x)dx - (EPS)v'(x_0).$$

Here EPS can be any real number, however we will restrict ourselves to the case when EPS is an element of the set {-1, 0, +1}.

Now define the DG bilinear form $a_\varepsilon : D_k(\Gamma_h)\times D_k(\Gamma_h) \to R$:

$$a_{EPS}(w,v) = \sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} w'(x)v'(x)dx - \sum_{n=0}^{N}\{w'(x_n)\}[v(x_n)] + (EPS)\sum_{n=0}^{N}\{v'(x_n)\}[w(x_n)] + J(w,v).$$

This bilinear form has the following properties:

If $EPS=-1$, the form is symmetric, that is to say

$$a_{-1}(v,w) = a_{-1}(w,v), \quad \forall v, w,$$

and we have

$$a_{-1}(v,v) = \sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} (v'(x))^2 dx - 2\sum_{n=0}^{N}\{v'(x_n)\}[v(x_n)] + J(v,v).$$

For $EPS \in \{0,+1\}$, the form is non-symmetric and we have

$$a_{+1}(v,v) = \sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} (v'(x))^2 dx + J(v,v) \ge 0.$$

$$a_0(v,v) = \sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} (v'(x))^2 dx - \sum_{n=0}^{N}\{v'(x_n)\}[v(x_n)] + J(v,v).$$

A class of DG methods for solving the BVP (1.1)-(1.3) is:

Find $P^{DG} \in D_k(\Gamma_h)$ such that

$$a_\varepsilon(P^{DG}, v) = L(v)\,, \quad \forall v \in D_k(\Gamma_h)\,, \tag{1.7}$$

Where $L : D_k(\Gamma_h) \to R$ is the linear form:

$$L(v) = \int_0^1 f(x)v(x)dx - (EPS)v'(x_0) + \frac{SIG}{h} v(x_0)\,.$$

Depending on the choices of the parameters EPS and SIG we obtain variations of the DG methods that have appeared in the literature at different times [1].

1.3 LINEAR SYSTEM

Now we derive the linear system obtained from the DG method in the case where discontinuous piecewise quadratic polynomials are used, i.e. $k = 2$, so we will be dealing with the space $D_2(\Gamma_h)$. For our local basis functions of $\mathrm{P}_2(I_n)$ we choose the monomial basis functions translated from the interval (-1, 1):

$$\mathrm{P}_2(I_n) = span\{\phi_0^n, \phi_1^n, \phi_2^n\}\,,$$

with

$$\phi_0^n(x) = 1, \quad \phi_1^n(x) = 2\frac{x - x_{n+1/2}}{x_{n+1} - x_n}, \quad \phi_2^n(x) = 4\frac{(x - x_{n+1/2})^2}{(x_{n+1} - x_n)^2}\,,$$

and where $x_{n+1/2} = \frac{1}{2}(x_n + x_{n+1})$ is the midpoint of the interval I_n. The local basis functions and their derivatives are reduced to:

$$\phi_0^n(x) = 1, \quad \phi_1^n(x) = \frac{2}{h}(x - (n + 1/2)h), \quad \phi_2^n(x) = \frac{4}{h^2}(x - (n + 1/2)h)^2,$$

$$\phi_0^n{}'(x) = 0, \quad \phi_1^n{}'(x) = \frac{2}{h}, \quad \phi_2^n{}'(x) = \frac{8}{h^2}(x - (n + 1/2)h).$$

We obtain the basis functions $\{\Phi_i^n\}$ for the space $D_2(\Gamma_h)$ by extending the local basis functions by zero:

$$\Phi_i^n = \begin{cases} \phi_i^n(x), & x \in I_n \\ 0, & x \notin I_n \end{cases}.$$

Now we can express the DG solution as:

$$P^{DG}(x) = \sum_{m=0}^{N-1} \sum_{j=0}^{2} \alpha_j^m \Phi_j^m(x), \quad \forall x \in (0,1),$$

where the coefficients α_j^m are unknown real numbers to be solved for. Putting this form of P^{DG} into (1.7), we have

$$\sum_{m=0}^{N-1} \sum_{j=0}^{2} \alpha_j^m a_{EPS}(\Phi_j^m, \Phi_i^n) = L(\Phi_i^n), \quad \forall 0 \le n \le N-1, \quad \forall 0 \le i \le 2,$$

where

$$L(\Phi_i^n) = \int_0^1 f(x)\Phi_i^n(x)dx - (EPS)\Phi_i^n{}'(x_0^+) + \frac{SIG}{h}\Phi_i^n(x_0^+).$$

Thus we obtain a linear system of the form $A\alpha = b$, where α is the vector with components α_j^m.

1.3.1 Global Matrix

Since the global basis functions have local support, the entries of the global matrix A can be obtained by computing and assembling local matrices. To assemble the local matrices, we will regroup the terms defining a_{EPS} into three groups: terms involving integrals over I_n, terms involving the interior nodes x_n, and terms involving the boundary nodes x_0 and x_N.

First consider the term corresponding to the integrals over I_n. Since on each interval I_n, the DG solution P^{DG} is a quadratic, we can write

$$P^{DG}(x) = \alpha_0^n \phi_0^n(x) + \alpha_1^n \phi_1^n(x) + \alpha_2^n \phi_2^n(x), \quad \forall x \in I_n.$$

Thus, the term $\int_{I_n} (P^{DG})'(x) v'(x) dx$ yields the vector $A_n \alpha_n$, where

$$\alpha^n = \begin{pmatrix} \alpha_0^n \\ \alpha_1^n \\ \alpha_2^n \end{pmatrix}, \quad (A_n)_{ij} = \int_{I_n} (\phi_i^n)'(x)(\phi_j^n)'(x) dx.$$

We can easily compute A_n as

$$A_n = \frac{1}{h} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16/3 \end{pmatrix}.$$

Next we consider the terms involving the interior nodes x_n. If we expand the average and jump terms, we can write:

$$-\{(P^{DG})'(x_n)\}[v(x_n)] + (EPS)\{v'(x_n)\}[P^{DG}(x_n)] + \frac{SIG}{h}[P^{DG}(x_n)][v(x_n)]$$

$$= b_n + c_n + d_n + e_n,$$

where the terms b_n, c_n, d_n, and e_n are defined as

$$b_n = \frac{1}{2}(P^{DG})'(x_n^+)v(x_n^+) - \frac{EPS}{2}P^{DG}(x_n^+)v'(x_n^+) + \frac{SIG}{h}P^{DG}(x_n^+)v(x_n^+),$$

$$c_n = -\frac{1}{2}(P^{DG})'(x_n^-)v(x_n^-) + \frac{EPS}{2}P^{DG}(x_n^-)v'(x_n^-) + \frac{SIG}{h}P^{DG}(x_n^-)v(x_n^-),$$

$$d_n = -\frac{1}{2}(P^{DG})'(x_n^+)v(x_n^-) - \frac{EPS}{2}P^{DG}(x_n^+)v'(x_n^-) - \frac{SIG}{h}P^{DG}(x_n^+)v(x_n^-),$$

$$e_n = \frac{1}{2}(P^{DG})'(x_n^-)v(x_n^+) + \frac{EPS}{2}P^{DG}(x_n^-)v'(x_n^+) - \frac{SIG}{h}P^{DG}(x_n^-)v(x_n^+).$$

These four terms will yield the following local matrices:

$$B_n = \frac{1}{h}\begin{pmatrix} SIG & 1-SIG & -2+SIG \\ -EPS-SIG & -1+EPS+SIG & 2-EPS-SIG \\ 2(EPS)+SIG & 1-2(EPS)-SIG & -2+2(EPS)+SIG \end{pmatrix},$$

$$C_n = \frac{1}{h}\begin{pmatrix} SIG & -1+SIG & -2+SIG \\ EPS+SIG & -1+EPS+SIG & -2+EPS+SIG \\ 2(EPS)+SIG & -1+2(EPS)+SIG & -2+2(EPS)+SIG \end{pmatrix},$$

$$D_n = \frac{1}{h}\begin{pmatrix} -SIG & -1+SIG & 2-SIG \\ -EPS-SIG & -1+EPS+SIG & 2-EPS-SIG \\ -2(EPS)-SIG & -1+2(EPS)+SIG & 2-2(EPS)-SIG \end{pmatrix},$$

$$E_n = \frac{1}{h}\begin{pmatrix} -SIG & 1-SIG & 2-SIG \\ EPS+SIG & -1+EPS+SIG & -2+EPS+SIG \\ -2(EPS)-SIG & 1-2(EPS)-SIG & 2-2(EPS)-SIG \end{pmatrix}.$$

Finally we consider the terms involving the boundary nodes x_0 and x_N.

$$f_0 = (P^{DG})'(x_0)v(x_0) - (EPS)v'(x_0)P^{DG}(x_0) + \frac{SIG}{h}P^{DG}(x_0)v(x_0),$$

$$f_N = -(P^{DG})'(x_N)v(x_N) + (EPS)v'(x_N)P^{DG}(x_N) + \frac{SIG}{h}P^{DG}(x_N)v(x_N).$$

These two terms will yield the following two matrices:

$$F_0 = \frac{1}{h}\begin{pmatrix} SIG & 2-SIG & -4+SIG \\ -2(EPS)-SIG & -2+2(EPS)+SIG & 4-2(EPS)-SIG \\ 4(EPS)+SIG & 2-4(EPS)-SIG & -4+4(EPS)+SIG \end{pmatrix}$$

$$F_N = \frac{1}{h}\begin{pmatrix} SIG & -2+SIG & -4+SIG \\ 2(EPS)+SIG & -2+2(EPS)+SIG & -4+2(EPS)+SIG \\ 4(EPS)+SIG & -2+4(EPS)+SIG & -4+4(EPS)+SIG \end{pmatrix}$$

Now that the local matrices have been computed, they can be assembled into the global matrix. The assembly of the global matrix depends on the order of the unknowns α_i^n. If we order the unknowns in the following way:

$$(\alpha_0^0, \alpha_1^0, \alpha_2^0, \alpha_0^1, \alpha_1^1, \alpha_2^1, \alpha_0^2, \alpha_1^2, \alpha_2^2, \ldots, \alpha_0^{N-1}, \alpha_1^{N-1}, \alpha_2^{N-1}),$$

we obtain the following global matrix that is block tri-diagonal:

$$\begin{pmatrix} M_0 & D_1 & & & & \\ E_1 & M & D_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & E_{N-2} & M & D_{N-1} \\ & & & & E_{N-1} & M_N \end{pmatrix},$$

where

$$M = A_n + B_n + C_{n+1}, \quad M_0 = A_0 + F_0 + C_1, \quad M_N = A_{N-1} + F_N + B_{N-1}.$$

2.0 LINEAR SOLVERS

Now we introduce the linear solvers that are used to solve the system developed in the previous chapter. Each solver is an iterative method. The idea behind iterative methods is to replace the given system by a different, yet related, system that can be more easily solved. So rather than solving the system $Ax = b$ for x, a simpler system $Kx_0 = b$ is solved for x_0 and this solution is taken as an approximation for x. These methods are iterative methods because of the way in which the approximations can be improved. The idea is that we would like to find a correction z such that

$$A(x_0 + z) = b.$$

From this equation we have

$$Az = b - Ax_0.$$

Now this system is solved by a different, yet somehow related, system. Let us assume that K is used again and we have

$$Kz_0 = b - Ax_0.$$

We can see that this leads to the new approximation $x_1 = x_0 + z_0$. Now we can repeat this procedure for x_1, and so on. This gives us an iterative method.

The way in which the system is altered in order to more easily solve for a good approximation to the original unknown x determines the method. Methods that attempt to generate better approximations from the Krylov subspace are referred to as Krylov subspace methods [2].

Definition: If A is an n-by-n matrix and r is an n-vector, then the p^{th} *dimensional Krylov subspace* generated by A and r is $\mathrm{K}^p(A;r)$, where

$$\mathrm{K}^p(A;r) = span\{r, Ar, A^2 r, \ldots, A^{p-1}r\}.$$

The Krylov subspace methods, for identifying suitable $x \in \mathrm{K}^p(A;r)$, can be categorized in four different classes (we assume that $x_0 = 0$):

(1) The *Ritz-Galerkin approach*: Construct the x_p for which the residual $b - Ax_p$ is orthogonal to the current subspace $\mathrm{K}^p(A;r_0)$.

(2) The *minimum norm residual approach*: Identify the x_p for which the Euclidean norm $\|b - Ax_p\|_2$ is minimal over $\mathrm{K}^p(A;r_0)$.

(3) The *Petrov-Galerkin approach*: Find an x_p so that the residual $b - Ax_p$ is orthogonal to some other suitable k-dimensional subspace.

(4) The *minimum norm error approach*: Determine x_p in $A^T \mathrm{K}^p(A^T;r_0)$ for which the Euclidean norm $\|x_p - x\|_2$ in minimal.

The Ritz-Galerkin approach leads to the Conjugate Gradient method, the minimum norm residual approach leads to the GMRES method, and the Bi-CGSTAB is a hybrid of these approaches [2].

2.1 CONJUGATE GRADIENT

When trying to solve the system $Ax = b$ with the Conjugate Gradient method, it is important to note that the matrix A must be symmetric. So for the systems we generate with the DG method, the only ones that we can solve with the CG method are the ones that have $EPS = -1$, since when $EPS \in \{0,+1\}$ the global matrix A will not be symmetric.

There are numerous books containing very detailed and well presented information about the CG method such as [2], [3], and [4]. Here we present this methods algorithm found on pages 311 and 312 of [3].

$k = 0$; $x_0 = 0$; $r_0 = b$; $p_1 = b$;

repeat

$k = k + 1$

$z = Ap_k$

$$v_k = \frac{r_{k-1}^T r_{k-1}}{p_k^T z}$$

$x_k = x_{k-1} + v_k p_k$

$r_k = r_{k-1} - v_k z$

$$\mu_{k+1} = \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$$

$p_{k+1} = r_k + \mu_{k+1} p_k$

until $\|r_k\|_2$ *is small enough*

The cost of the inner loop is one matrix-vector product $z = Ap_k$, two inner products ($r_{k-1}^T r_{k-1}$ and $r_k^T r_k$), three saxpys ($x_k = x_{k-1} + v_k p_k$, $r_k = r_{k-1} - v_k z$, and $p_{k+1} = r_k + \mu_{k+1} p_k$), and a couple of scalar operations. There are only four vectors that need to be stored and they are the current values of r, x, p, and $z = Ap$.

Theorem 1: Let x^* be the true solution to $Ax = b$, and let $q(\lambda)$ be a polynomial in λ (for example $q(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$). Then the iterates $\{x_k\}$ of the CG method satisfy

$$\|x^* - x_k\|_A = \min_{\deg(q)<k} \|x^* - q(A)b\|_A ,$$

where for any vector x the norm $\|x\|_A \equiv \sqrt{x^T Ax}$.

This theorem (page 566 of [4]) can give us a number of error results, varying with the properties of the matrix A. For example, let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of A repeated according to their multiplicity, and let $\psi_1, \ldots, \psi_n$ denote a corresponding orthonormal basis of eigenvectors. With this basis, we can write

$$x^* = \sum_{j=1}^{n} c_j \psi_j \quad \text{and} \quad b = Ax^* = \sum_{j=1}^{n} c_j \lambda_j \psi_j .$$

Then

$$q(A)b = \sum_{j=1}^{n} c_j \lambda_j q(\lambda_j) \psi_j \quad \text{and} \quad \|x^* - q(A)b\|_A = \left(\sum_{j=1}^{n} c_j^2 \lambda_j \left(1 - \lambda_j q(\lambda_j)\right)^2 \right)^{1/2} .$$

Now any choice of a polynomial $q(\lambda)$ with $\deg(q) < k$ will give a bound for $\|x^* - x_k\|_A$ [4].

2.2 GMRES

The method of GMRES, which stands for Generalized Minimum Residual, approximates the solution of $Ax = b$ by a vector in a Krylov subspace with minimal residual. To find this vector the Arnoldi iteration is used. The Arnoldi iteration is nothing more than an eigenvalues algorithm used to find the eigenvalues of a general (possibly non-Hermitian) matrix. There exist many variations for the method of GMRES. One variation used to avoid excessive storage requirements and computational costs for the orthogonalization is restarted after each m iteration steps. This algorithm is referred to as GMRES(m) [2]. Non-restarted versions are often called ‘full’ GMRES. The restarted version of GMRES is the one that the PETSc package uses, and the restart value we have chosen is the default of m=30.

The following algorithm (which can be found on page 76 of [2]) is for unpreconditioned GMRES(m) with modified Gram-Schmidt.

$r = b - Ax_0$, for a given initial guess x_0

$x = x_0$

for $j = 1,2,\ldots$

$\beta = \|r\|_2, \quad v_1 = \dfrac{r}{\beta}, \quad \hat{b} = \beta e_1$ (where e_1 is the canonical basis vector in R^n)

for $i = 1,2,\ldots,m$

$w = av_i$

for $k = 1,\ldots,i$

$h_{k,i} = v_k^T w, \quad w = w - k_{k,i} v_k$

$h_{i+1,i} = \|w\|_2, \quad v_{i+1} = \dfrac{w}{h_{i+1,i}}$

$r_{1,i} = h_{1,i}$

for $k = 2,\ldots,i$

$\gamma = c_{k-1} r_{k-1,i} + s_{k-1} h_{k,i}$

$r_{k,i} = -s_{k-1} r_{k-1,i} + c_{k-1} h_{k,i}$

$r_{k-1,i} = \gamma$

$\delta = \sqrt{r_{i,i}^2 + h_{i+1,i}^2}, \quad c_i = \dfrac{r_{i,i}}{\delta}, \quad s_i = \dfrac{h_{i+1,i}}{\delta}$

$r_{i,i} = c_i r_{i,i} + s_i h_{i+1,i}$

$\hat{b}_{i+1} = -s_i \hat{b}_i, \quad \hat{b}_i = c_i \hat{b}_i$

$\rho = \left|\hat{b}_{i+1}\right| \quad \left(= \left\|b - Ax_{(j-1)m+i}\right\|_2\right)$

if ρ is small enough **then**

($n_r = i$, goto SOL)

$$n_r = m\,, \quad y_{n_r} = \frac{\hat{b}_{n_r}}{r_{n_r,n_r}}$$

SOL: **for** $k = n_r - 1, \ldots, 1$

$$y_k = \frac{1}{r_{k,k}}\left(\hat{b}_k - \sum_{i=k+1}^{n_r} r_{k,i}\, y_i\right)$$

$x = x + \sum_{i=1}^{n_r} y_i v_i$, **if** ρ is small enough quit

$r = b - Ax$

According to van der Vorst [2], the convergence of GMRES cannot be adequately described in terms of the condition number of the matrix A. Also, in general, it cannot be described in terms of the eigenvalues. The following theorem (page 77 of [2]) gives the main result for the convergence of this method.

Theorem 2: Given a nonincreasing positive sequence $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1}$ and a set of nonzero complex number $\lambda_1, \lambda_2, \ldots, \lambda_n$, there exists a matrix A with eigenvalues λ_j and a right-hand side b with $\|b\|_2 = \alpha_0$ such that the residual vectors r_k of GMRES (for $Ax = b$, with $x_0 = 0$) satisfy $\|r_k\|_2 = \alpha_k$, $k = 0,1,\ldots,n-1$.

This theorem tells us that we need more information than the eigenvalues alone, we need information about the eigensystem. In the case where the eigensystem is orthogonal, such as for normal matrices, the eigenvalues are descriptive for the convergence. And for well-conditioned eigensystems the distribution of the eigenvalues can give insight into the actual convergence behavior of GMRES [2].

2.3 BI-CGSTAB

The method of Bi-CGStab, which stands for Bi-Conjugate Gradient-Stable, can be used to solve a larger class of linear systems than the Conjugate Gradient method, because unlike the Conjugate Gradient method, Bi-CGStab can be used to solve non-symmetric systems. The algorithm for Bi-CGStab is as follows:

x_0 is an initial guess; $r_0 = b - Ax_0$

choose $\tilde{r}$, for example, $\tilde{r} = r_0$

for $i = 1,2,\ldots$

$\rho_{i-1} = \tilde{r}^T r_{i-1}$

if $\rho_{i-1} = 0$ method fails

if $i = 1$

$p_i = r_{i-1}$

else

$$\beta_{i-1} = \left(\frac{\rho_{i-1}}{\rho_{i-2}}\right)\left(\frac{\alpha_{i-1}}{\varpi_{i-1}}\right)$$

$p_i = r_{i-1} + \beta_{i-1}(p_{i-1} - \omega_{i-1}v_{i-1})$

endif

$v_i = Ap_i$;

$$\alpha_i = \frac{\rho_{i-1}}{\tilde{r}^T v_i}$$

$s = r_{i-1} - \alpha_i v_i$

check $\|s\|_2$, if small enough: $x_i = x_{i-1} + \alpha_i p_i$ and stop

$t = As$, $\omega_i = \left(\frac{t^T s}{t^T t}\right)$

$x_i = x_{i-1} + \alpha_i p_i + \omega_i s$

$r_i = s - \omega_i t$

check convergence; continue if necessary

for continuation it is necessary that $\omega_i \neq 0$

end

According to van der Vorst [2], Bi-CGStab can be viewed as the product of GMRES(1) and another Krylov subspace method called Bi-Conjugate Gradient. Other product methods can be formulated. For instance, Gutknecht [11] proposes that the product of GMRES(2) and Bi-Conjugate Gradient leads to a method called BICGSTAB2. BICGSTAB is the method that is used in the PETSc package [7].

3.0 PRECONDITIONERS

When using Krylov subspace methods to solve a linear system, the number of iterations required to achieve an acceptable solution may be too large to be practical. Convergence of a method depends in a complicated way on the spectral properties of the matrix A, and this information may not be available in many cases. One way to get around this problem is to find an invertible matrix P such that $P^{-1}A$ has better spectral properties. The idea is that with an appropriately chosen P, a Krylov method applied to, for instance, $P^{-1}Ax = P^{-1}b$, would yield an acceptable solution to x in fewer iterations as compared to being applied to the original system $Ax = b$. Such a matrix P is called a *preconditioner* for the matrix A.

The problem of finding an efficient preconditioner P is a problem which involves finding a matrix P with the following properties:

(1) P is a good approximation to A in some sense.

(2) The cost of the construction of P is not prohibitive.

(3) The system involving P^{-1} is much easier to solve than the original system.

According to van der Vorst [2], research on preconditioning is a broad and active area of research with little structure, and there is no general theory on which to safely base an efficient selection. Thus the selection and construction of a good preconditioner for a given class of problems is at best an educated guess. However, there is a freedom in defining and constructing a preconditioner for Krylov subspace methods and this is one reason why these methods are as popular and successful as they are today.

Once a preconditioner is obtained, there are different ways of implementing it. Three different implementations are as follows:

(1) Left-preconditioning: Apply the iterative method to $P^{-1}Ax = P^{-1}b$.

(2) Right-preconditioning: Apply the iterative method to $AP^{-1}y = b$, with $x = P^{-1}y$

(3) Two-sided preconditioning: For a preconditioner P with $P = P_1P_2$, apply the iterative method to $P_1^{-1}AP_2^{-1}y = P_1^{-1}b$, with $x = P_2^{-1}y$.

For our numerical simulations we have chosen the methods of Jacobi, ILU, and Cholesky to create the preconditioning matrix P. Also, we have chosen to implement all preconditioners with left-preconditioning.

3.1 JACOBI

The Jacobi preconditioner is one of the simplest preconditioners to construct. It is a diagonal matrix that is constructed from the main diagonal of A. Since we want the preconditioner P to be non-singular, we must take care to insure that none of the components on the diagonal of P are zero. To construct the Jacobi preconditioner, we select the components of P in the following way

$$p_{i,j} = \begin{cases} a_{i,i} & if \quad i = j \quad and \quad a_{i,i} \neq 0 \\ 1 & if \quad i = j \quad and \quad a_{i,i} = 0 \\ 0 & otherwise \end{cases}.$$

Simply stated, the Jacobi preconditioning matrix is a diagonal matrix whose diagonal entries correspond to $a_{i,i}$ when $a_{i,i} \neq 0$ and 1 when $a_{i,i} = 0$.

3.2 ILU

When standard Gaussian elimination is used on a matrix A, it is equivalent to factoring the matrix as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix. When the actual computations are performed these factors are found explicitly. The problem that arises when dealing with a sparse matrix is that the factors are considerably less sparse than A, and this makes finding the solution expensive. The ILU preconditioner modifies Gaussian elimination to allow fill-ins at only a restricted set of positions in the lower factor L and the upper factor U. This restriction is done in the following way. Let S be the index set given by

$$S = \{(i, j) \mid a_{i,j} \neq 0\}.$$

Now let the allowable fill-in positions be given by the set S. That is to say

$$l_{i,j} = 0 \quad \text{if} \quad i < j \quad \text{or} \quad (i, j) \notin S$$

$$u_{i,j} = 0 \quad \text{if} \quad i > j \quad \text{or} \quad (i, j) \notin S.$$

So the only nonzero entries allowed in the LU factors are those for which the corresponding entries in A are nonzero.

Maintaining the definition of the set S as given above, we have the following ILU algorithm for a general n by n matrix A found on page 182 of [2]:

for $k = 1, 2, \ldots, n-1$

$d = \dfrac{1}{a_{k,k}}$

for $i = k+1, k+2, \ldots, n$

if $(i,k) \in S$

$e = da_{i,k}$; $a_{i,k} = e$

for $j = k+1, \ldots, n$

if $(i,j) \in S$ and $(k,j) \in S$

$a_{i,j} = a_{i,j} - ea_{k,j}$

end if

end j

end if

end i

end k

After the algorithm is completed, the incomplete factors $\tilde{L}$ and $\tilde{U}$ are stored in the corresponding lower and upper triangular parts of the array A. These factors then define the preconditioner $P = \tilde{L}\tilde{U}$.

3.3 CHOLESKY

The Cholesky preconditioner is created by the method of Cholesky matrix factorization. Given a symmetric positive definite matrix A, the Cholesky factorization produces a lower triangular matrix L having the property that $A = LL^T$. This matrix L is called the Cholesky triangle [10]. The preconditioning matrix P is then set to LL^T. Essentially, for symmetric positive definite matrices, the Cholesky factorization is the LU decomposition, except the algorithm for computing these factors is much more efficient when compared to the LU factorization.

When solving the system $Ax = b$, with symmetric positive definite A, PETSc intends the Cholesky preconditioner to be used as a direct solver. Thus in the case when EPS = -1, we must not consider Cholesky as a valid preconditioner. However, when EPS = 0 or +1 we can consider the effects of Cholesky preconditioning, since the global matrices will be non-symmetric. This will produce a preconditioner P such that $P^{-1} \neq A^{-1}$. The PETSc package computes the preconditioning matrix in an enigmatic way. With many sub-routines in the Cholesky preconditioning code it is not entirely clear how PETSc computes the preconditioner when A is non-symmetric.

4.0 NUMERICAL STUDY OF CONDITION NUMBER

The following tables give the condition number of the global matrix A generated by the DG method when solving the BVP (1.1)-(1.3) with $f(x) = 2e^{-x^2}(-3x + 1 - 2x^2 + 2x^3)$ and true solution $p(x) = (1 - x)e^{-x^2}$. We use EPS = -1, 0, and +1. When EPS = -1 we refer to the method as SIPG (Symmetric Interior Penalty Galerkin), when EPS = 0 we refer to the method as IIPG (Incomplete Interior Penalty Galerkin), and when EPS = +1 we refer to the method as NIPG (Non-symmetric Interior Penalty Galerkin). For every value of EPS we use SIG = 0.01, 0.1, 1, 10, and 100. In the case of NIPG we use the additional value SIG = 0.

Table 1: SIPG Condition Number SIG = 0.01

h	EPS	SIG	Cond. Number
$1/4$	-1	0.01	27.2828
$1/8$	-1	0.01	100.2164
$1/16$	-1	0.01	389.8348
$1/32$	-1	0.01	1.5474e+03
$1/64$	-1	0.01	6.1775e+03
$1/128$	-1	0.01	2.4698e+03

Table 2: SIPG Condition Number SIG = 0.1

h	EPS	SIG	Cond. Number
$1/4$	-1	0.1	26.1947
$1/8$	-1	0.1	96.0714
$1/16$	-1	0.1	373.2643
$1/32$	-1	0.1	1.4807e+03
$1/64$	-1	0.1	5.9098e+03
$1/128$	-1	0.1	2.3627e+04

Table 3: SIPG Condition Number SIG = 1

h	EPS	SIG	Cond. Number
$1/4$	-1	1	25.0438
$1/8$	-1	1	62.6562
$1/16$	-1	1	234.8822
$1/32$	-1	1	934.5871
$1/64$	-1	1	3.7330e+03
$1/128$	-1	1	1.4926e+04

Table 4: SIPG Condition Number SIG = 10

h	EPS	SIG	Cond. Number
$1/4$	-1	10	120.0838
$1/8$	-1	10	445.8831
$1/16$	-1	10	1.7511e+03
$1/32$	-1	10	6.9730e+03
$1/64$	-1	10	2.7861e+04
$1/128$	-1	10	1.1141e+05

Table 5: SIPG Condition Number SIG = 100

h	EPS	SIG	Cond. Number
$1/4$	-1	100	1.3982e+03
$1/8$	-1	100	5.2240e+03
$1/16$	-1	100	2.0526e+04
$1/32$	-1	100	8.1741e+04
$1/64$	-1	100	3.2661e+05
$1/128$	-1	100	1.3061e+06

Table 6: IIPG Condition Number SIG = 0.01

h	EPS	SIG	Cond. Number
$1/4$	0	0.01	1.0530e+03
$1/8$	0	0.01	1.5771e+03
$1/16$	0	0.01	2.6294e+03
$1/32$	0	0.01	4.7959e+03
$1/64$	0	0.01	9.5711e+03
$1/128$	0	0.01	2.2220e+04

Table 7: IIPG Condition Number SIG = 0.1

h	EPS	SIG	Cond. Number
$1/4$	0	0.1	95.8742
$1/8$	0	0.1	147.0048
$1/16$	0	0.1	303.0562
$1/32$	0	0.1	914.5993
$1/64$	0	0.1	3.3833e+03
$1/128$	0	0.1	1.3284e+04

Table 8: IIPG Condition Number SIG = 1

h	EPS	SIG	Cond. Number
$1/4$	0	1	10.4623
$1/8$	0	1	40.1277
$1/16$	0	1	159.5191
$1/32$	0	1	637.3554
$1/64$	0	1	2.5488e+03
$1/128$	0	1	1.0194e+04

Table 9: IIPG Condition Number SIG = 10

h	EPS	SIG	Cond. Number
$1/4$	0	10	129.7746
$1/8$	0	10	494.3801
$1/16$	0	10	1.9515e+03
$1/32$	0	10	7.7796e+03
$1/64$	0	10	3.1092e+04
$1/128$	0	10	1.2434e+05

Table 10: IIPG Condition Number SIG = 100

h	EPS	SIG	Cond. Number
$1/4$	0	100	1.4081e+03
$1/8$	0	100	5.2735e+03
$1/16$	0	100	2.0731e+04
$1/32$	0	100	8.2567e+04
$1/64$	0	100	3.2992e+05
$1/128$	0	100	1.3193e+06

Table 11: NIPG Condition Number SIG = 0

h	EPS	SIG	Cond. Number
$1/4$	1	0	18.2631
$1/8$	1	0	72.2014
$1/16$	1	0	288.1275
$1/32$	1	0	1.1518e+03
$1/64$	1	0	4.6066e+03
$1/128$	1	0	1.8426e+04

Table 12: NIPG Condition Number SIG = 0.01

h	EPS	SIG	Cond. Number
$1/4$	1	0.01	18.2724
$1/8$	1	0.01	72.2618
$1/16$	1	0.01	288.4312
$1/32$	1	0.01	1.1531e+03
$1/64$	1	0.01	4.6120e+03
$1/128$	1	0.01	1.8447e+04

Table 13: NIPG Condition Number SIG = 0.1

h	EPS	SIG	Cond. Number
$1/4$	1	0.1	18.4176
$1/8$	1	0.1	73.0466
$1/16$	1	0.1	292.0731
$1/32$	1	0.1	1.1685e+03
$1/64$	1	0.1	4.6743e+03
$1/128$	1	0.1	1.8698e+04

Table 14: NIPG Condition Number SIG = 1

h	EPS	SIG	Cond. Number
$1/4$	1	1	24.2615
$1/8$	1	1	97.8552
$1/16$	1	1	393.6743
$1/32$	1	1	1.5779e+03
$1/64$	1	1	6.3155e+03
$1/128$	1	1	2.5266e+04

Table 15: NIPG Condition Number SIG = 10

h	EPS	SIG	Cond. Number
$1/4$	1	10	142.1859
$1/8$	1	10	548.7608
$1/16$	1	10	2.1732e+03
$1/32$	1	10	8.6706e+03
$1/64$	1	10	3.4660e+04
$1/128$	1	10	1.3862e+05

Table 16: NIPG Condition Number SIG = 100

h	EPS	SIG	Cond. Number
$1/4$	1	100	1.4183e+03
$1/8$	1	100	5.3236e+03
$1/16$	1	100	2.0938e+04
$1/32$	1	100	8.3402e+04
$1/64$	1	100	3.3327e+05
$1/128$	1	100	1.3327e+06

The following plots compare the condition number of the global matrix generated by the NIPG, SIPG, and IIPG Galerkin methods for the given values of SIG.

Figure 1: Condition Numbers SIG = 1

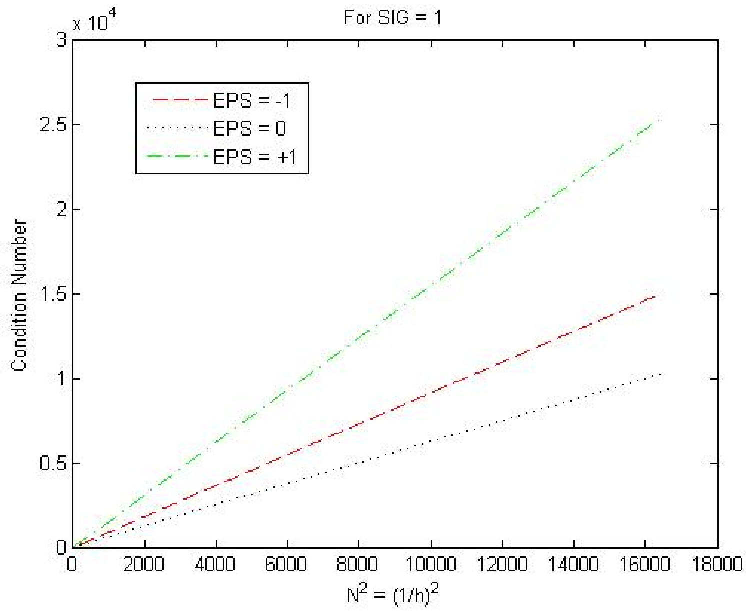


Figure 2: Condition Numbers SIG = 10

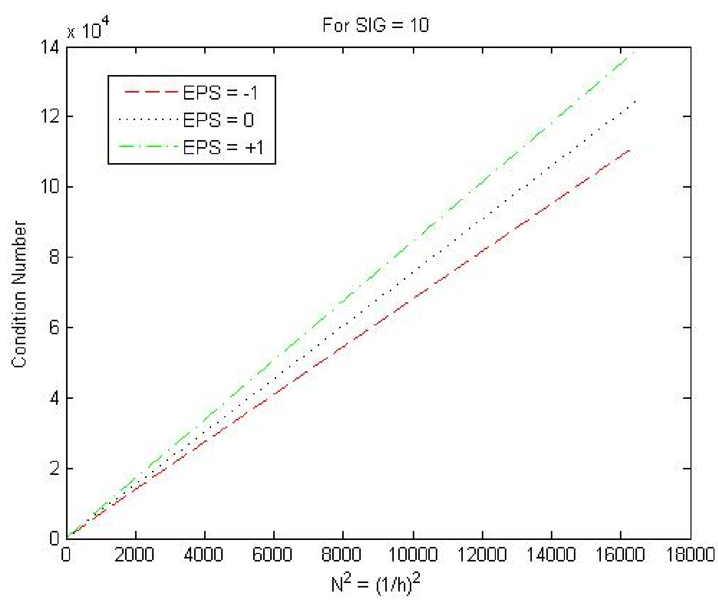
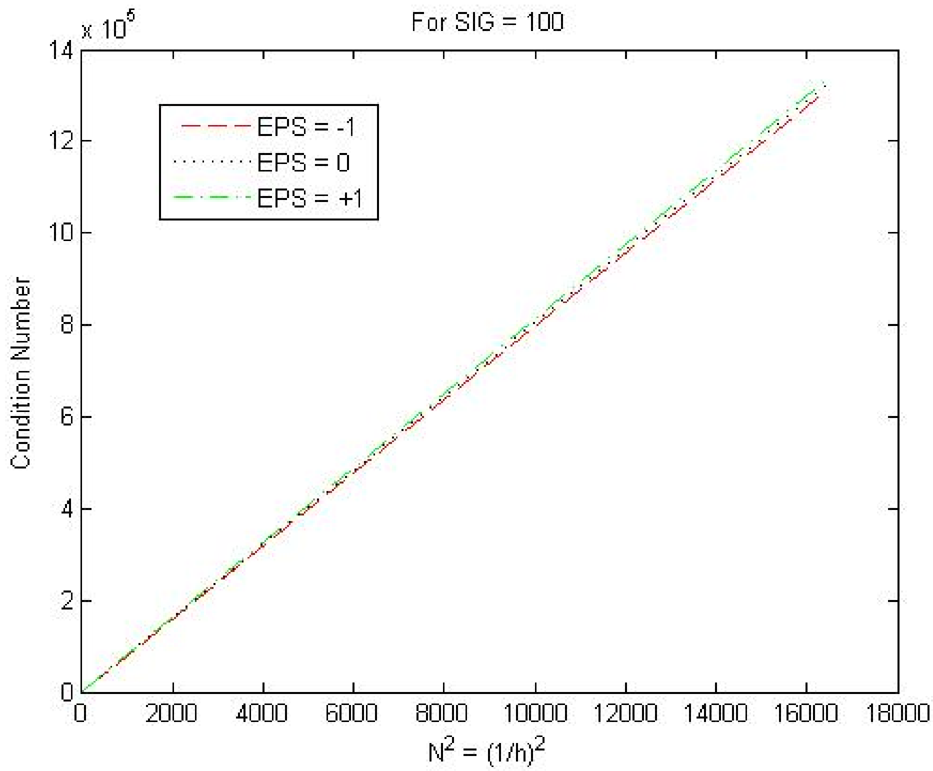


Figure 3: Condition Numbers SIG = 100



From the data we can see that, in general, as SIG is increased the condition number of the global matrix increases for all values of h. In all cases, when N increases, the condition number increases quadratically. This is why the plots of the condition number vs. N^2 appear linear. From these plots we can make the following observations:

(1) When SIG = 1, NIPG produces a global matrix with the largest condition number followed by SIPG followed by IIPG.

(2) When SIG = 10, NIPG produces a global matrix with the largest condition number followed by IIPG followed by SIPG.

(3) When SIG = 100, all three methods produce similar condition numbers that are larger in general than the ones obtained from smaller values of SIG.

5.0 NUMERICAL SIMULATION WITH PETSC

In Chapter 4.0 of this thesis we presented the condition number of the matrix A generated by the DG method to solve the BVP (1.1)-(1.3) with $f(x) = 2e^{-x^2}(-3x+1-2x^2+2x^3)$ and true solution $p(x) = (1-x)e^{-x^2}$ using various combinations of EPS = -1, 0, and +1; and SIG = 0, 0.01, 0.1, 1, 10, and 100. Now we present the following data generated when using PETSc [8] to solve the linear system $Ax = b$, with the same right hand side function $f(x)$, true solution $p(x)$, and the same combinations of EPS and SIG. The following tables provide the number of iterations required for convergence with the given relative tolerance, the residual norm, and the error in the approximate solution calculated using the L2 norm.

Let us make a remark about the convergence and divergence of the linear solver method. When running the simulations, the maximum number of iteration was set to the PETSc default of 10,000 iterations [6]. Anything exhibiting 10,000 iterations did not converge. Also if any row of a table has “nan” (meaning “not a number”) or “error” in it, the method did not converge in that case either. In the case of the Conjugate Gradient method with Preconditioning there is an additional left column in the tables titled “Conv or Div”. This column was added to eliminate confusion about convergence and divergence. Any row with -8 in this column did not converge because of an indefinite preconditioner, and any row with 2 in this column converged because the relative tolerance was reached. In all cases, if the method did not diverge for the reasons described above then it converged because the relative tolerance was reached.

In the chapters which give comparative plots (plots comparing the unpreconditioned method to the preconditioned methods), if a method did not converge because the maximum number of iterations was reached, the plots still display the value 10,000 iteration. But if the method did not converge for any of the other reasons described above, then the plot is just displayed with the number of iterations for which the method did converge. This is why some of the plots appear to end abruptly.

All of the following chapters are formatted the same. First the data for the Krylov subspace method without preconditioning is presented, then with preconditioning, then the plots are presented. Also the order of preconditioners remains consistent for CG, GMRES, and Bi-CGStab. First the methods are preconditioned with Jacobi, then ILU, and finally Cholesky.

5.1 CONJUGATE GRADIENT

Table 17: CG EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.45127984e-11	1.93196149e-03
$1/8$	27	2.74847179e-09	2.60547939e-04
$1/16$	61	3.46972201e-09	3.35060608e-05
$1/32$	124	1.13461222e-08	4.22322074e-06
$1/64$	256	2.58643257e-08	5.29087994e-07
$1/128$	528	4.29589806e-08	6.61748622e-08

Table 18: CG EPS = -1 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.45127984e-11	1.93196149e-03
$1/8$	23	1.08458389e-04	2.60568154e-04
$1/16$	50	4.13780249e-04	3.35060312e-05
$1/32$	110	8.53288137e-04	4.24959266e-06
$1/64$	239	1.10621131e-03	6.15800879e-07
$1/128$	506	3.35891170e-03	2.14266132e-07

Table 19: CG EPS = -1 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	9.14079297e-02	2.32585147e-03
$1/8$	21	1.01100226e-01	6.32504672e-04
$1/16$	43	2.04250578e-01	3.64761531e-04
$1/32$	73	1.42190244e+00	1.73275115e-01
$1/64$	111	2.69048587e+00	2.82666096e-01
$1/128$	163	5.61265318e+00	3.58108745e-01

Table 20: CG EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	6.10543055e-11	2.05471019e-03
$1/8$	30	2.28368606e-09	2.79605380e-04
$1/16$	61	6.55823937e-09	3.61402474e-05
$1/32$	122	2.62558102e-09	4.56487083e-06
$1/64$	258	7.13743585e-09	5.72374710e-07
$1/128$	553	5.54368973e-08	7.16148883e-08

Table 21: CG EPS = -1 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	6.10543055e-11	2.05471019e-03
$1/8$	23	2.39177264e-04	2.79623241e-04
$1/16$	51	4.67298773e-04	3.61425370e-05
$1/32$	109	8.48460721e-04	4.59461040e-06
$1/64$	239	2.75398812e-03	7.10184689e-07
$1/128$	514	4.14283033e-03	3.99377157e-07

Table 22: CG EPS = -1 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	8.77866966e-02	2.40786647e-03
$1/8$	21	9.79556626e-02	6.15202781e-04
$1/16$	44	2.50275459e-01	5.91397731e-04
$1/32$	74	1.36999523e+00	1.68725923e-01
$1/64$	117	2.66755067e+00	2.76122390e-01
$1/128$	172	5.27218246e+00	3.53224650e-01

Table 23: CG EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	3.60478101e-11	1.46470833e-02
$1/8$	21	1.04103229e-09	4.69518373e-03
$1/16$	53	3.85557949e-09	1.70023895e-03
$1/32$	143	7.35965531e-09	6.90388038e-04
$1/64$	456	1.99401518e-08	3.05814536e-04
$1/128$	1706	3.69912763e-08	1.43177234e-04

Table 24: CG EPS = -1 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	3.60478101e-11	1.46470833e-02
$1/8$	19	5.37562568e-07	4.69518372e-03
$1/16$	42	7.83949071e-05	1.70023852e-03
$1/32$	114	2.27906102e-04	6.90387490e-04
$1/64$	355	1.73514505e-03	3.05836693e-04
$1/128$	1242	3.12658764e-03	1.43245680e-04

Table 25: CG EPS = -1 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	3.60478101e-11	1.46470833e-02
$1/8$	19	5.37562568e-07	4.69518372e-03
$1/16$	35	3.23615298e-01	6.00542296e-02
$1/32$	80	7.39501286e-01	2.66364451e-01
$1/64$	227	1.44533807e+00	2.75981427e-01
$1/128$	542	4.14753877e+00	3.60373008e-01

Table 26: CG EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	14	3.69070383e-09	2.85353350e-04
$1/8$	31	4.87914771e-09	3.53050403e-05
$1/16$	62	1.15070243e-08	4.40399621e-06
$1/32$	117	2.37480493e-08	5.50443771e-07
$1/64$	223	8.15113003e-08	6.88200368e-08
$1/128$	431	6.79554982e-08	8.60400326e-09

Table 27: CG EPS = -1 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.34195609e-06	2.85353266e-04
$1/8$	25	4.68571428e-04	3.53697530e-05
$1/16$	49	1.44508535e-03	7.76127592e-06
$1/32$	99	2.88361097e-03	5.68404303e-06
$1/64$	196	5.52565205e-03	4.44718140e-06
$1/128$	390	1.51542304e-02	7.55701015e-06

Table 28: CG EPS = -1 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	8	3.95800886e-01	2.81034564e-02
$1/8$	14	1.03918552e+00	1.22194000e-01
$1/16$	21	1.91707412e+00	2.50576345e-01
$1/32$	27	3.63457223e+00	3.42881841e-01
$1/64$	27	7.36259476e+00	4.34241690e-01
$1/128$	27	1.45808414e+01	4.82435853e-01

Table 29: CG EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	1.13382445e-14	4.07582050e-04
$1/8$	39	4.84282907e-08	5.11562161e-05
$1/16$	87	1.27648547e-07	6.40858092e-06
$1/32$	174	4.01191268e-07	8.01934400e-07
$1/64$	348	5.24856052e-07	1.00295406e-07
$1/128$	687	1.03402706e-06	1.25421234e-08

Table 30: CG EPS = -1 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	3.81066415e-04	4.07644748e-04
$1/8$	30	7.17836928e-03	7.16406267e-05
$1/16$	68	9.15070851e-03	2.82342223e-05
$1/32$	139	5.24216985e-02	1.45367422e-04
$1/64$	295	9.95233974e-02	1.29150207e-04
$1/128$	615	1.47670539e-01	1.20263430e-04

Table 31: CG EPS = -1 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	4	5.26387511e+00	3.67088291e-01
$1/8$	4	1.07640674e+01	4.45425250e-01
$1/16$	4	2.12920223e+01	4.87460112e-01
$1/32$	4	4.24212088e+01	5.08933689e-01
$1/64$	4	8.47504315e+01	5.19741513e-01
$1/128$	4	1.69452406e+02	5.25157324e-01

5.1.1 Conjugate Gradient with Preconditioning

Table 32: CG \ Jacobi EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	8.46845361e+00	5.30578768e-01	-8
$1/8$	1	3.39861197e+00	5.30578768e-01	-8
$1/16$	1	1.75164746e+00	5.30578768e-01	-8
$1/32$	1	1.36037254e+00	5.30578768e-01	-8
$1/64$	1	1.28897650e+00	5.30578768e-01	-8
$1/128$	1	1.27618886e+00	5.30578768e-01	-8

Table 33: CG \ ILU EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.08175197e+00	5.30578768e-01	-8
$1/8$	1	1.50835373e+00	5.30578768e-01	-8
$1/16$	1	2.12505382e+00	5.30578768e-01	-8
$1/32$	1	3.00237820e+00	5.30578768e-01	-8
$1/64$	1	4.24497389e+00	5.30578768e-01	-8
$1/128$	1	7.88164737e-13	6.61750667e-08	2

Table 34: CG \ Cholesky EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.08175197e+00	5.30578768e-01	-8
$1/8$	1	1.50835373e+00	5.30578768e-01	-8
$1/16$	1	2.12505382e+00	5.30578768e-01	-8
$1/32$	1	3.00237820e+00	5.30578768e-01	-8
$1/64$	1	4.24497389e+00	5.30578768e-01	-8
$1/128$	1	1.28202742e-12	6.61750665e-08	2

Table 35: CG \ Jacobi EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.65287515e+00	5.30578768e-01	-8
$1/8$	1	1.39127603e+00	5.30578768e-01	-8
$1/16$	1	1.33169181e+00	5.30578768e-01	-8
$1/32$	1	1.31849854e+00	5.30578768e-01	-8
$1/64$	1	1.31546177e+00	5.30578768e-01	-8
$1/128$	1	1.31473762e+00	5.30578768e-01	-8

Table 36: CG \ ILU EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.08165609e+00	5.30578768e-01	-8
$1/8$	1	1.50834393e+00	5.30578768e-01	-8
$1/16$	1	1.63160626e-14	3.61402464e-05	2
$1/32$	1	4.64171753e-15	4.56487160e-06	2
$1/64$	1	1.21037012e-14	5.72374751e-07	2
$1/128$	1	2.69851024e-14	7.16149223e-08	2

Table 37: CG \ Cholesky EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.08165609e+00	5.30578768e-01	-8
$1/8$	1	1.50834393e+00	5.30578768e-01	-8
$1/16$	1	4.82848247e-15	3.61402464e-05	2
$1/32$	1	2.70468263e-14	4.56487160e-06	2
$1/64$	1	1.47793848e-14	5.72374751e-07	2
$1/128$	1	7.35446745e-14	7.16149222e-08	2

Table 38: CG \ Jacobi EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$\frac{1}{4}$	2	3.59985269e+00	5.33514213e-01	-8
$\frac{1}{8}$	2	6.97159851e+00	5.96940522e-01	-8
$\frac{1}{16}$	2	1.35433974e+01	6.92256638e-01	-8
$\frac{1}{32}$	2	2.66440548e+01	8.43035740e-01	-8
$\frac{1}{64}$	2	5.28394219e+01	1.07967846e+00	-8
$\frac{1}{128}$	2	1.05231897e+02	1.43925226e+00	-8

Table 39: CG \ ILU EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$\frac{1}{4}$	1	2.74262085e-16	1.46470833e-02	2
$\frac{1}{8}$	1	4.31941251e-16	4.69518373e-03	2
$\frac{1}{16}$	1	1.39752713e-15	1.70023895e-03	2
$\frac{1}{32}$	1	2.32162772e-15	6.90388037e-04	2
$\frac{1}{64}$	1	1.41407842e-14	3.05814536e-04	2
$\frac{1}{128}$	1	3.36371793e-14	1.43177242e-04	2

Table 40: CG \ Cholesky EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$\frac{1}{4}$	1	2.71782845e-16	1.46470833e-02	2
$\frac{1}{8}$	1	5.94193318e-16	4.69518373e-03	2
$\frac{1}{16}$	1	2.99342887e-15	1.70023895e-03	2
$\frac{1}{32}$	1	5.01959892e-15	6.90388037e-04	2
$\frac{1}{64}$	1	2.30536407e-14	3.05814536e-04	2
$\frac{1}{128}$	1	2.09347677e-14	1.43177242e-04	2

Table 41: CG \ Jacobi EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	14	1.51197065e-11	2.85353347e-04	2
$1/8$	31	1.41285918e-11	3.53050401e-05	2
$1/16$	65	7.96796229e-11	4.40399716e-06	2
$1/32$	127	6.81370410e-11	5.50443750e-07	2
$1/64$	243	7.01725593e-11	6.88199641e-08	2
$1/128$	467	6.63193832e-11	8.60404825e-09	2

Table 42: CG \ ILU EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.69509830e-15	2.85353348e-04	2
$1/8$	1	1.76176388e-15	3.53050402e-05	2
$1/16$	1	2.20235149e-14	4.40399589e-06	2
$1/32$	1	3.03693280e-14	5.50443622e-07	2
$1/64$	1	1.38843917e-13	6.88198044e-08	2
$1/128$	1	9.05605224e-13	8.60395814e-09	2

Table 43: CG \ Cholesky EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.90886643e-15	2.85353348e-04	2
$1/8$	1	4.41309198e-15	3.53050402e-05	2
$1/16$	1	5.20299854e-15	4.40399589e-06	2
$1/32$	1	1.81751531e-14	5.50443622e-07	2
$1/64$	1	4.24084551e-14	6.88198046e-08	2
$1/128$	1	1.14815290e-12	8.60395843e-09	2

Table 44: CG \ Jacobi EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	17	8.60015724e-17	4.07582050e-04	2
$1/8$	39	2.88225689e-11	5.11562143e-05	2
$1/16$	87	5.66248000e-11	6.40858104e-06	2
$1/32$	177	3.65032627e-11	8.01935888e-07	2
$1/64$	352	3.33257284e-11	1.00295069e-07	2
$1/128$	692	3.89802644e-11	1.25437854e-08	2

Table 45: CG \ ILU EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	2.76770172e-14	4.07582050e-04	2
$1/8$	1	4.45738773e-14	5.11562152e-05	2
$1/16$	1	6.37003714e-14	6.40858167e-06	2
$1/32$	1	1.16626295e-13	8.01935355e-07	2
$1/64$	1	4.02151687e-12	1.00294885e-07	2
$1/128$	1	1.24276261e-11	1.25401339e-08	2

Table 46: CG \ Cholesky EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error	Conv or Div
$1/4$	1	1.77389099e-14	4.07582050e-04	2
$1/8$	1	6.85403072e-14	5.11562152e-05	2
$1/16$	1	3.53899488e-14	6.40858167e-06	2
$1/32$	1	1.00222755e-13	8.01935356e-07	2
$1/64$	1	5.52240297e-13	1.00294888e-07	2
$1/128$	1	2.56438897e-11	1.25401333e-08	2

5.1.2 Conjugate Gradient (Numerical Comparative Plots)

Figure 4: CG EPS = -1 SIG = 0.01 TOL = 1.0e-10

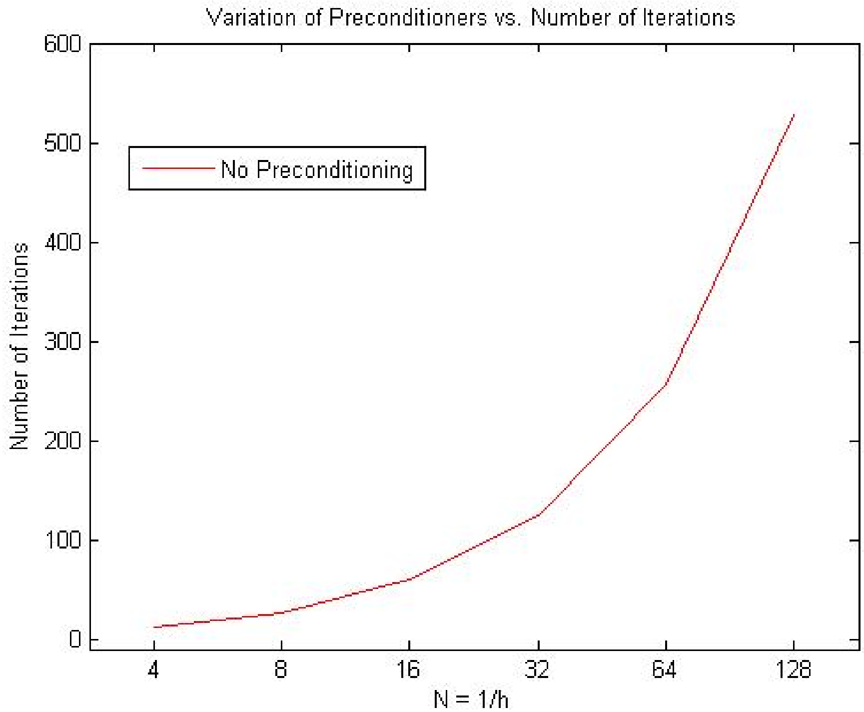


Figure 5: CG EPS = -1 SIG = 0.1 TOL = 1.0e-10

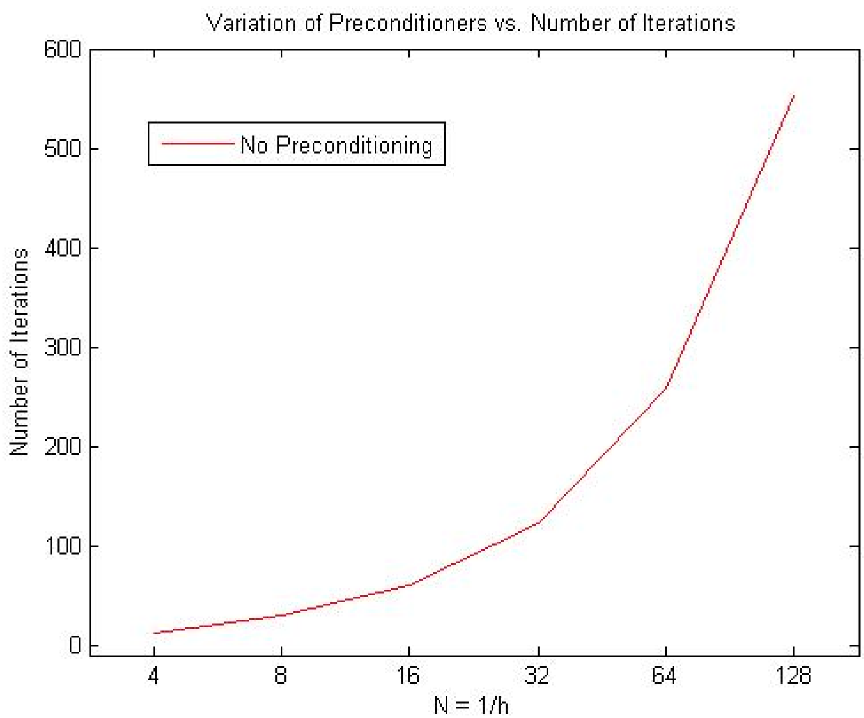


Figure 6: CG EPS = -1 SIG =1 TOL = 1.0e-10

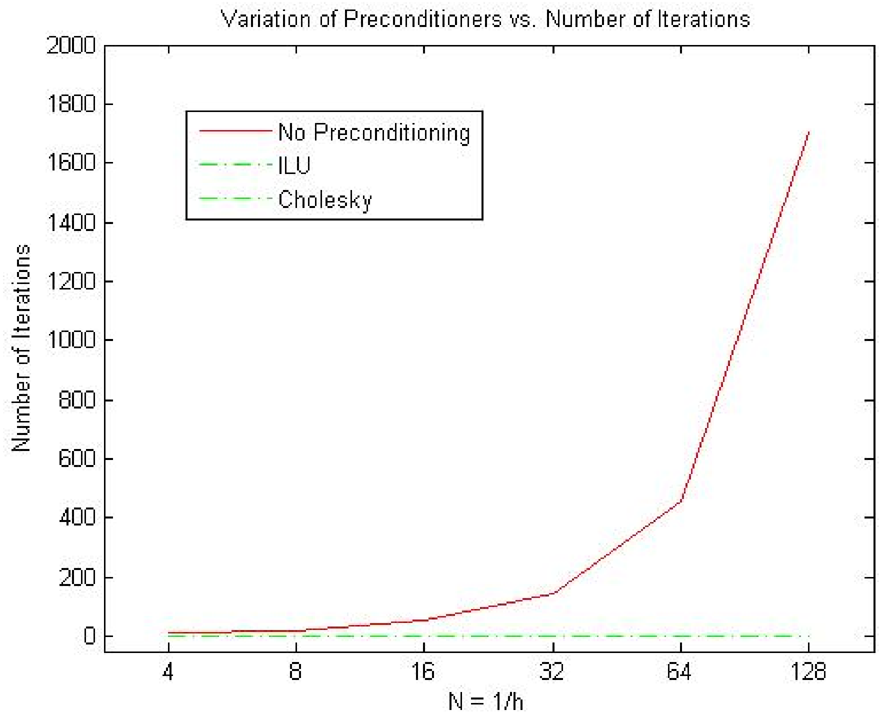


Figure 7: CG EPS = -1 SIG = 10 TOL = 1.0e-10

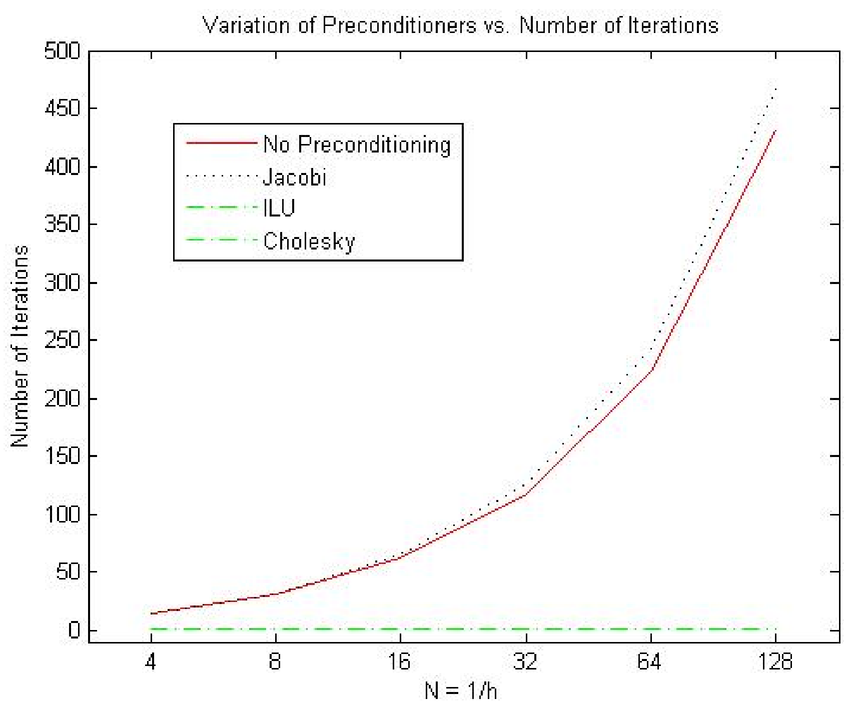
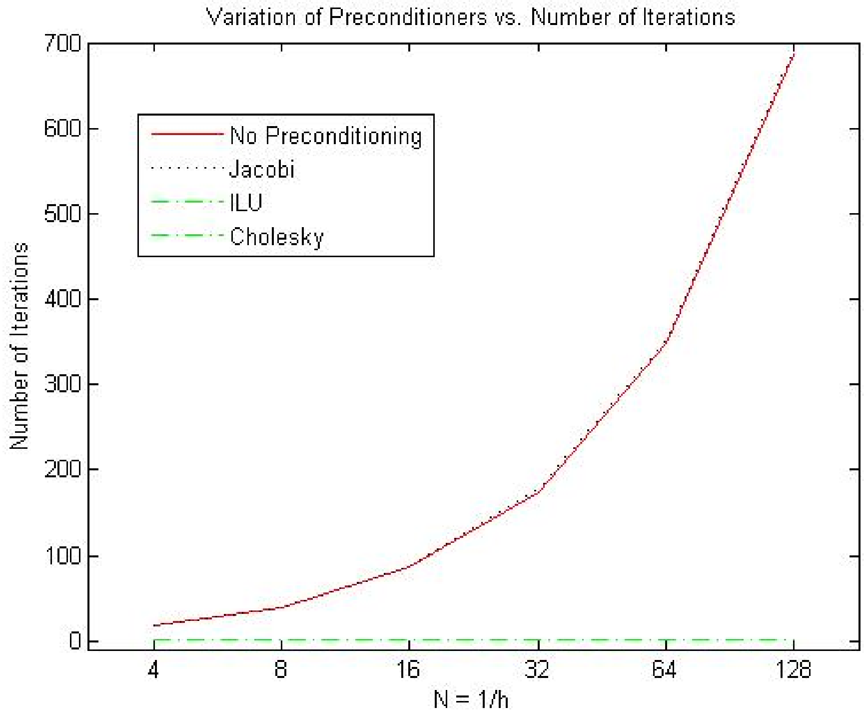


Figure 8: CG EPS = -1 SIG = 100 TOL = 1.0e-10



5.2 GMRES

Table 47: GMRES EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.1779788699e-15	1.93196149e-03
$1/8$	24	2.4107762791e-14	2.60547939e-04
$1/16$	617	7.1019867750e-09	3.35059760e-05
$1/32$	2871	1.4254061658e-08	4.22308340e-06
$1/64$	10000	4.5087340850e-08	5.29996156e-07
$1/128$	10000	5.5908454139e-03	6.39973978e-03

Table 48: GMRES EPS = -1 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.1779788699e-15	1.93196149e-03
$1/8$	23	1.0837339113e-04	2.60567157e-04
$1/16$	262	6.8341109847e-04	2.24478924e-04
$1/32$	1035	1.4259026849e-03	8.03737486e-04
$1/64$	3206	2.8529743946e-03	2.28590160e-03
$1/128$	9951	5.7041878689e-03	6.52910473e-03

Table 49: GMRES EPS = -1 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	9.0691785031e-02	2.97170969e-03
$1/8$	21	9.9089334667e-02	4.71691308e-03
$1/16$	40	6.9193402033e-01	2.31959796e-01
$1/32$	55	1.3649651272e+00	3.53706574e-01
$1/64$	55	2.7617627060e+00	4.40718028e-01
$1/128$	55	5.4744532181e+00	4.85657942e-01

Table 50: GMRES EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.1472267067e-14	2.05471019e-03
$1/8$	24	2.4417811052e-14	2.79605379e-04
$1/16$	591	6.7393535174e-09	3.61401560e-05
$1/32$	3024	1.3709525815e-08	4.56473815e-06
$1/64$	10000	1.0799894936e-07	5.78152099e-07
$1/128$	10000	6.9703277822e-03	7.97905367e-03

Table 51: GMRES EPS = -1 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.1472267067e-14	2.05471019e-03
$1/8$	23	8.9621393250e-05	2.79621783e-04
$1/16$	257	6.7020522653e-04	2.29332071e-04
$1/32$	1092	1.3745948971e-03	7.73818971e-04
$1/64$	3402	2.7693321027e-03	2.23442022e-03
$1/128$	10000	6.9703277822e-03	7.97905367e-03

Table 52: GMRES EPS = -1 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	8.7195355152e-02	2.89245641e-03
$1/8$	21	9.6260124978e-02	4.37517974e-03
$1/16$	43	6.6121841074e-01	2.24693792e-01
$1/32$	57	1.3333621220e+00	3.51589758e-01
$1/64$	57	2.7105525202e+00	4.39754261e-01
$1/128$	57	5.3725416973e+00	4.85172892e-01

Table 53: GMRES EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	2.5509529386e-15	1.46470833e-02
$1/8$	19	2.2290997521e-13	4.69518373e-03
$1/16$	6000	5.2187394330e-09	1.70023900e-03
$1/32$	10000	1.7156915365e-01	8.04509953e-02
$1/64$	10000	4.5668662018e-01	2.71457267e-01
$1/128$	10000	1.0137528910e+00	3.94049278e-01

Table 54: GMRES EPS = -1 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	2.5509529386e-15	1.46470833e-02
$1/8$	19	2.2290997521e-13	4.69518373e-03
$1/16$	3091	5.2508354371e-04	1.76680126e-03
$1/32$	10000	1.7156915365e-01	8.04509953e-02
$1/64$	10000	4.5668662018e-01	2.71457267e-01
$1/128$	10000	1.0137528910e+00	3.94049278e-01

Table 55: GMRES EPS = -1 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	2.5509529386e-15	1.46470833e-02
$1/8$	19	2.2290997521e-13	4.69518373e-03
$1/16$	106	5.3054199966e-01	1.92489998e-01
$1/32$	359	1.0606177652e+00	3.30731415e-01
$1/64$	395	2.1205165395e+00	4.27229019e-01
$1/128$	389	4.2402789269e+00	4.79180994e-01

Table 56: GMRES EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.1919977751e-14	2.85353348e-04
$1/8$	24	3.9989916921e-13	3.53050402e-05
$1/16$	171	2.1483516330e-08	4.40396118e-06
$1/32$	1258	4.4344736722e-08	5.50676126e-07
$1/64$	4429	9.0439424251e-08	9.97496825e-08
$1/128$	10000	4.2026535418e-05	4.80406862e-05

Table 57: GMRES EPS = -1 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.1919977751e-14	2.85353348e-04
$1/8$	22	2.1291925433e-04	3.53839167e-05
$1/16$	69	1.8877356425e-03	1.79489216e-04
$1/32$	403	4.4951221989e-03	2.46368469e-03
$1/64$	1168	9.0042231603e-03	7.05531877e-03
$1/128$	3279	1.8096197005e-02	2.06782150e-02

Table 58: GMRES EPS = -1 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	8	3.4841586246e-01	5.09006650e-02
$1/8$	11	1.0624705256e+00	2.44847194e-01
$1/16$	15	1.9196542707e+00	3.50122743e-01
$1/32$	15	3.8561664210e+00	4.38160852e-01
$1/64$	15	7.6349103131e+00	4.84282110e-01
$1/128$	14	1.8046556242e+01	5.10279320e-01

Table 59: GMRES EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.3925791667e-13	4.07582050e-04
$1/8$	24	1.1310897005e-12	5.11562152e-05
$1/16$	200	2.6193823386e-07	6.40738198e-06
$1/32$	2777	5.3991239566e-07	8.53716826e-07
$1/64$	10000	3.2152512956e-06	2.59672564e-06
$1/128$	10000	1.6451945975e-02	1.88400182e-02

Table 60: GMRES EPS = -1 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.3925791667e-13	4.07582050e-04
$1/8$	21	7.5240688664e-03	8.37241608e-05
$1/16$	62	2.5992907731e-02	3.13142501e-03
$1/32$	537	5.4295431071e-02	3.01810764e-02
$1/64$	1283	1.0838663865e-01	8.72314485e-02
$1/128$	1856	2.1710693395e-01	2.20920838e-01

Table 61: GMRES EPS = -1 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	4	4.6693205277e+00	3.67158239e-01
$1/8$	4	9.4957289673e+00	4.45496349e-01
$1/16$	4	1.8819886552e+01	4.87505238e-01
$1/32$	4	3.7521324887e+01	5.08958169e-01
$1/64$	4	7.4975726273e+01	5.19754143e-01
$1/128$	4	1.4991620009e+02	5.25163724e-01

Table 62: GMRES EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.0401691609e-12	7.74726005e-01
$1/8$	24	1.6809821376e-12	1.61723174e-01
$1/16$	10000	2.0397829007e-01	6.54366811e-01
$1/32$	10000	4.1099243000e-01	5.24430456e-01
$1/64$	10000	7.3248231947e-01	5.27938037e-01
$1/128$	10000	1.4434240274e+00	5.29734555e-01

Table 63: GMRES EPS = 0 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.0401691609e-12	7.74726005e-01
$1/8$	23	5.5074366185e-08	1.61723168e-01
$1/16$	10000	2.0397829007e-01	6.54366811e-01
$1/32$	10000	4.1099243000e-01	5.24430456e-01
$1/64$	10000	7.3248231947e-01	5.27938037e-01
$1/128$	10000	1.4434240274e+00	5.29734555e-01

Table 64: GMRES EPS = 0 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	4.8342360811e-03	7.44923378e-01
$1/8$	20	7.5445823701e-04	1.60025650e-01
$1/16$	10000	2.0397829007e-01	6.54366811e-01
$1/32$	10000	4.1099243000e-01	5.24430456e-01
$1/64$	10000	7.3248231947e-01	5.27938037e-01
$1/128$	10000	1.4434240274e+00	5.29734555e-01

Table 65: GMRES EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	8.2811924940e-14	7.74258964e-02
$1/8$	24	6.8764456325e-13	1.61691848e-02
$1/16$	3410	2.8780596301e-10	3.65158897e-03
$1/32$	8335	5.5912926338e-10	8.65574974e-04
$1/64$	10000	7.2380493602e-04	3.69799970e-04
$1/128$	10000	6.8243937032e-02	7.79138380e-02

Table 66: GMRES EPS = 0 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	8.2811924940e-14	7.74258964e-02
$1/8$	24	6.8764456325e-13	1.61691848e-02
$1/16$	1495	2.7911117705e-05	3.63708509e-03
$1/32$	3941	5.5893285004e-05	8.38640336e-04
$1/64$	10000	7.2380493602e-04	3.69799970e-04
$1/128$	10000	6.8243937032e-02	7.79138380e-02

Table 67: GMRES EPS = 0 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	8.2811924940e-14	7.74258964e-02
$1/8$	24	6.8764456325e-13	1.61691848e-02
$1/16$	471	2.7702390958e-02	9.23531645e-03
$1/32$	1265	5.5872227466e-02	2.93380033e-02
$1/64$	2513	1.1094531541e-01	8.81296815e-02
$1/128$	3832	2.2176811761e-01	2.23888456e-01

Table 68: GMRES EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	6.6439215399e-15	7.70621364e-03
$1/8$	19	2.7814972266e-14	1.61456464e-03
$1/16$	186	2.75973989634e-09	3.65005591e-04
$1/32$	595	5.4443490825e-09	8.65448608e-05
$1/64$	2098	1.1035525391e-08	2.10513447e-05
$1/128$	6201	2.2125127764e-08	5.16890125e-06

Table 69: GMRES EPS = 0 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	6.6439215399e-15	7.70621364e-03
$1/8$	19	2.7814972266e-14	1.61456464e-03
$1/16$	82	2.6968858406e-04	2.68795194e-04
$1/32$	249	5.4990503380e-04	2.05377812e-04
$1/64$	771	1.1000154104e-03	8.43923060e-04
$1/128$	2305	2.2149069699e-03	2.49479559e-03

Table 70: GMRES EPS = 0 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	3.2355624419e-03	7.72301645e-03
$1/8$	18	3.2028513597e-03	1.55551932e-03
$1/16$	40	2.7183110486e-01	9.54032521e-02
$1/32$	44	5.4963659520e-01	2.46882142e-01
$1/64$	47	1.1038633229e+00	3.63947840e-01
$1/128$	47	2.2120964533e+00	4.45891791e-01

Table 71: GMRES EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.4723747931e-13	8.36143053e-04
$1/8$	24	8.4579095017e-13	1.66830523e-04
$1/16$	179	2.3917828401e-08	3.68852857e-05
$1/32$	1128	5.5225869380e-08	8.65056299e-06
$1/64$	4075	1.1063809007e-07	2.01980644e-06
$1/128$	10000	6.5709803951e-05	7.45737832e-05

Table 72: GMRES EPS = 0 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.4723747931e-13	8.36143053e-04
$1/8$	22	7.8762879352e-04	1.66931509e-04
$1/16$	72	2.6447135006e-03	3.14218924e-04
$1/32$	398	5.5022523739e-03	3.03086427e-03
$1/64$	1200	1.1071550506e-02	8.73171322e-03
$1/128$	3268	2.2160325098e-02	2.51905858e-02

Table 73: GMRES EPS = 0 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	8	6.2138448211e-01	1.15414914e-01
$1/8$	12	1.1758636362e+00	2.51259580e-01
$1/16$	14	2.7197955088e+00	3.83648796e-01
$1/32$	14	5.4144047865e+00	4.56266620e-01
$1/64$	14	1.0750134493e+01	4.93433173e-01
$1/128$	14	2.1444457157e+01	5.12038151e-01

Table 74: GMRES EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.6361412856e-12	4.23076377e-04
$1/8$	24	7.3335939102e-12	5.45858837e-05
$1/16$	203	2.7466056988e-07	7.44821549e-06
$1/32$	2717	5.5253298660e-07	9.82897769e-07
$1/64$	9317	1.1076184107e-06	6.91438453e-07
$1/128$	10000	8.3141469624e-03	9.51828375e-03

Table 75: GMRES EPS = 0 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.6361412856e-12	4.23076377e-04
$1/8$	21	7.6848162432e-03	8.04681393e-05
$1/16$	67	2.7577218188e-02	3.28222705e-03
$1/32$	538	5.4910244822e-02	3.02095725e-02
$1/64$	1157	1.1046373233e-01	8.91827745e-02
$1/128$	1665	2.2154900394e-01	2.24091384e-01

Table 76: GMRES EPS = 0 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	4	4.8602739123e+00	3.68784971e-01
$1/8$	4	9.8581423461e+00	4.46294411e-01
$1/16$	4	1.9542832301e+01	4.87897611e-01
$1/32$	4	3.8967579673e+01	5.09152363e-01
$1/64$	4	7.7868741101e+01	5.19850703e-01
$1/128$	4	1.5570255709e+02	5.25211865e-01

Table 77: GMRES EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.23923576e-15	7.95816672e-03
$1/8$	24	3.78173286e-14	2.42102909e-03
$1/16$	435	6.76018220e-09	6.42109427e-04
$1/32$	1336	1.40872851e-08	1.63051932e-04
$1/64$	4242	2.85764213e-08	4.09046935e-05
$1/128$	10000	5.45948050e-05	5.21981035e-05

Table 78: GMRES EPS = +1 SIG = 0 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.23923576e-15	7.95816672e-03
$1/8$	22	6.47507655e-06	2.42103027e-03
$1/16$	178	6.81335547e-04	4.56869794e-04
$1/32$	501	1.42884680e-03	6.06549206e-04
$1/64$	1448	2.85153876e-03	2.24350950e-03
$1/128$	4612	5.71969150e-03	6.52288433e-03

Table 79: GMRES EPS = +1 SIG = 0 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	9.09170037e-02	5.78381079e-03
$1/8$	19	1.27688726e-01	9.67126913e-03
$1/16$	34	6.85402808e-01	2.37117849e-01
$1/32$	43	1.41064571e+00	3.54838801e-01
$1/64$	43	2.84052233e+00	4.40896303e-01
$1/128$	43	5.62968376e+00	4.85695802e-01

Table 80: GMRES EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.1836394655e-15	7.91131207e-03
$1/8$	24	3.5668317449e-14	2.40135548e-03
$1/16$	420	7.0144642719e-09	6.36227846e-04
$1/32$	1364	1.4309881199e-08	1.61489991e-04
$1/64$	4460	2.8704040042e-08	4.05053983e-05
$1/128$	10000	3.2369786576e-05	2.68983047e-05

Table 81: GMRES EPS = +1 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.183639465r5e-15	7.91131207e-03
$1/8$	22	9.8242261887e-06	2.40135643e-03
$1/16$	182	7.1668686408e-04	4.61332126e-04
$1/32$	497	1.4323887782e-03	6.18544480e-04
$1/64$	1478	2.8618297036e-03	2.23397783e-03
$1/128$	4571	5.7377679287e-03	6.54889125e-03

Table 82: GMRES EPS = +1 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	9.8409357221e-02	5.46555585e-03
$1/8$	19	1.3691799022e-01	1.08936975e-02
$1/16$	34	6.8306383453e-01	2.36765743e-01
$1/32$	43	1.3985278862e+00	3.53917764e-01
$1/64$	43	2.8174125394e+00	4.40394633e-01
$1/128$	43	5.5836796762e+00	4.85441618e-01

Table 83: GMRES EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	5.8710059237e-15	7.51472199e-03
$1/8$	24	4.5369590349e-14	2.23805472e-03
$1/16$	402	7.1892741238e-09	5.87829001e-04
$1/32$	1285	1.4730225029e-08	1.48681836e-04
$1/64$	4311	2.9472689677e-08	3.72357890e-05
$1/128$	10000	1.2061075127e-05	4.62945796e-06

Table 84: GMRES EPS = +1 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	5.8710059237e-15	7.51472199e-03
$1/8$	22	6.6638116174e-05	2.23798003e-03
$1/16$	170	6.9390059963e-04	3.76324808e-04
$1/32$	478	1.4706889457e-03	5.97119582e-04
$1/64$	1343	2.9343522011e-03	2.28684452e-03
$1/128$	4087	5.8953627496e-03	6.71913004e-03

Table 85: GMRES EPS = +1 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	1.6549104182e-01	5.24779610e-03
$1/8$	19	2.1493842141e-01	2.36685324e-02
$1/16$	32	7.2056945762e-01	2.43990090e-01
$1/32$	40	1.4461191286e+00	3.60318840e-01
$1/64$	40	2.9087255735e+00	4.43842791e-01
$1/128$	40	5.7655141843e+00	4.87180933e-01

Table 86: GMRES EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	6.2735891044e-15	5.05666010e-03
$1/8$	24	3.0251307348e-13	1.34192231e-03
$1/16$	177	7.8486103714e-09	3.35335470e-04
$1/32$	698	1.8498995949e-08	8.31134407e-05
$1/64$	2585	3.7802804723e-08	2.06198120e-05
$1/128$	8306	7.5596827976e-08	5.05845951e-06

Table 87: GMRES EPS = +1 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	6.2735891044e-15	5.05666010e-03
$1/8$	23	5.4857174968e-05	1.34190257e-03
$1/16$	79	9.2902308284e-04	1.38700225e-04
$1/32$	268	1.8621781674e-03	6.54956496e-04
$1/64$	810	3.7778849107e-03	2.86017164e-03
$1/128$	2363	7.5644858351e-03	8.59007588e-03

Table 88: GMRES EPS = +1 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	4.3929003617e-02	4.77885335e-03
$1/8$	18	4.5560742601e-01	8.30296405e-02
$1/16$	22	8.4954016899e-01	2.58233642e-01
$1/32$	23	1.8524062702e+00	3.82585099e-01
$1/64$	23	3.6910019640e+00	4.55874642e-01
$1/128$	22	7.5473179713e+00	4.94019443e-01

Table 89: GMRES EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.5352319536e-13	1.31276389e-03
$1/8$	24	4.5276600577e-12	2.83220204e-04
$1/16$	204	2.9402526040e-08	6.50268068e-05
$1/32$	1329	6.6980583263e-08	1.55284470e-05
$1/64$	4756	1.3399974012e-07	3.70007659e-06
$1/128$	10000	1.0499728968e-04	1.19127410e-04

Table 90: GMRES EPS = +1 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.5352319536e-13	1.31276389e-03
$1/8$	22	4.3492458727e-04	2.82992468e-04
$1/16$	82	3.3051066839e-03	7.76848845e-04
$1/32$	421	6.5836642680e-03	3.28129269e-03
$1/64$	1192	1.3377742385e-02	1.06881369e-02
$1/128$	3236	2.6841628695e-02	3.06109021e-02

Table 91: GMRES EPS = +1 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	8	7.2219083558e-01	1.35831952e-01
$1/8$	11	1.6634646420e+00	2.95579135e-01
$1/16$	14	2.9675923849e+00	3.90773778e-01
$1/32$	14	5.8973997635e+00	4.59992560e-01
$1/64$	13	1.3417915822e+01	4.98951212e-01
$1/128$	13	2.6780932387e+01	5.14790525e-01

Table 92: GMRES EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.7412556669e-12	4.50042142e-04
$1/8$	24	3.2788833760e-11	6.18795528e-05
$1/16$	204	2.2436929180e-07	9.75430833e-06
$1/32$	2693	5.5885155133e-07	1.62499480e-06
$1/64$	8968	1.1301191702e-06	5.07960262e-07
$1/128$	10000	1.3406382294e-03	1.53203524e-03

Table 93: GMRES EPS = +1 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	4.7412556669e-12	4.50042142e-04
$1/8$	21	7.8072657944e-03	8.01779720e-05
$1/16$	68	2.7339327753e-02	3.31906434e-03
$1/32$	519	5.6208453920e-02	3.15146393e-02
$1/64$	1154	1.1302546886e-01	9.13039290e-02
$1/128$	1470	2.2313512895e-01	2.22479867e-01

Table 94: GMRES EPS = +1 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	4	5.0488833246e+00	3.70356573e-01
$1/8$	4	1.0217198816e+01	4.47066684e-01
$1/16$	4	2.0259089892e+01	4.88277544e-01
$1/32$	4	4.0400360794e+01	5.09340450e-01
$1/64$	4	8.0734732407e+01	5.19944238e-01
$1/128$	4	1.6143482258e+02	5.25258501e-01

5.2.1 GMRES with Preconditioning

Table 95: GMRES \ Jacobi EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	7.55755187e-13	1.93196149e-03
$1/8$	24	1.42843959e-11	2.60547939e-04
$1/16$	10000	1.34690315e+00	5.00698582e-01
$1/32$	10000	1.22491918e+00	5.30096048e-01
$1/64$	10000	1.18118449e+00	5.30438897e-01
$1/128$	10000	1.17503539e+00	5.30521098e-01

Table 96: GMRES \ ILU EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	8.63629398e-16	1.93196149e-03
$1/8$	1	9.45600773e-16	2.60547939e-04
$1/16$	1	1.10841396e-15	3.35060612e-05
$1/32$	1	2.21158329e-15	4.22322352e-06
$1/64$	1	4.52569861e-15	5.29087793e-07
$1/128$	1	2.91497292e-15	6.61750670e-08

Table 97: GMRES \ Cholesky EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	6.92488898e-16	1.93196149e-03
$1/8$	1	1.20502901e-15	2.60547939e-04
$1/16$	1	1.52005463e-15	3.35060612e-05
$1/32$	1	3.37112724e-15	4.22322352e-06
$1/64$	1	3.93601831e-15	5.29087793e-07
$1/128$	1	3.66647775e-15	6.61750669e-08

Table 98: GMRES \ Jacobi EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.31058152e-14	2.05471019e-03
$1/8$	24	1.73270864e-14	2.79605379e-04
$1/16$	10000	3.65778515e-01	3.04072143e-01
$1/32$	10000	1.08990408e+00	5.18121663e-01
$1/64$	10000	1.11977788e+00	5.26123057e-01
$1/128$	10000	1.08787133e+00	5.27595326e-01

Table 99: GMRES \ ILU EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.03300821e-15	2.05471019e-03
$1/8$	1	3.94504975e-15	2.79605379e-04
$1/16$	1	4.20557265e-15	3.61402464e-05
$1/32$	1	3.31716494e-15	4.56487160e-06
$1/64$	1	6.67346995e-15	5.72374751e-07
$1/128$	1	7.12409422e-15	7.16149222e-08

Table 100: GMRES \ Cholesky EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.70749580e-15	2.05471019e-03
$1/8$	1	4.07793040e-15	2.79605379e-04
$1/16$	1	6.31451431e-15	3.61402464e-05
$1/32$	1	5.24338334e-15	4.56487160e-06
$1/64$	1	7.54703142e-15	5.72374751e-07
$1/128$	1	7.88942899e-15	7.16149222e-08

Table 101: GMRES \ Jacobi EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.25177879e-14	1.46470833e-02
$1/8$	24	1.62120169e-13	4.69518373e-03
$1/16$	10000	9.33025100e-04	1.03771629e-02
$1/32$	10000	1.09912918e+00	5.20054414e-01
$1/64$	10000	1.61109553e+00	5.24536660e-01
$1/128$	10000	1.60506060e+00	5.27541624e-01

Table 102: GMRES \ ILU EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	3.67928299e-16	1.46470833e-02
$1/8$	1	8.70262718e-16	4.69518373e-03
$1/16$	1	2.25151672e-15	1.70023895e-03
$1/32$	1	3.59913790e-15	6.90388037e-04
$1/64$	1	1.66216282e-14	3.05814536e-04
$1/128$	1	2.92115889e-14	1.43177242e-04

Table 103: GMRES \ Cholesky EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.45605452e-16	1.46470833e-02
$1/8$	1	1.01646693e-15	4.69518373e-03
$1/16$	1	1.45433717e-15	1.70023895e-03
$1/32$	1	6.39212133e-15	6.90388037e-04
$1/64$	1	8.20447101e-15	3.05814536e-04
$1/128$	1	2.37235882e-14	1.43177242e-04

Table 104: GMRES \ Jacobi EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	12	2.96692087e-15	2.85353348e-04
$\frac{1}{8}$	24	4.98068972e-15	3.53050402e-05
$\frac{1}{16}$	232	7.02988548e-11	4.40395650e-06
$\frac{1}{32}$	1434	8.04301384e-11	5.50878255e-07
$\frac{1}{64}$	4927	8.05973026e-11	1.07166102e-07
$\frac{1}{128}$	10000	3.36254696e-08	9.83759054e-05

Table 105: GMRES \ ILU EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	1	6.23886006e-16	2.85353348e-04
$\frac{1}{8}$	1	1.29753274e-15	3.53050402e-05
$\frac{1}{16}$	1	1.86122426e-14	4.40399589e-06
$\frac{1}{32}$	1	2.64300276e-14	5.50443622e-07
$\frac{1}{64}$	1	6.53744825e-14	6.88198044e-08
$\frac{1}{128}$	1	7.83867738e-13	8.60395795e-09

Table 106: GMRES \ Cholesky EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	1	7.83607217e-16	2.85353348e-04
$\frac{1}{8}$	1	2.69845702e-15	3.53050402e-05
$\frac{1}{16}$	1	7.70157896e-15	4.40399589e-06
$\frac{1}{32}$	1	2.90981448e-14	5.50443622e-07
$\frac{1}{64}$	1	4.47505577e-14	6.88198046e-08
$\frac{1}{128}$	1	1.19192019e-12	8.60395878e-09

Table 107: GMRES \ Jacobi EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	7.25224845e-16	4.07582050e-04
$1/8$	24	6.34733977e-15	5.11562151e-05
$1/16$	202	7.11522843e-11	6.40746995e-06
$1/32$	2673	8.61031812e-11	8.52610325e-07
$1/64$	8669	8.60888700e-11	8.85583902e-07
$1/128$	10000	2.97783898e-07	8.72769602e-03

Table 108: GMRES \ ILU EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.28660088e-14	4.07582050e-04
$1/8$	1	3.62898168e-14	5.11562152e-05
$1/16$	1	3.32213408e-14	6.40858167e-06
$1/32$	1	1.25384544e-13	8.01935355e-07
$1/64$	1	2.16756380e-12	1.00294882e-07
$1/128$	1	4.38791883e-12	1.25401319e-08

Table 109: GMRES \ Cholesky EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	8.17946574e-15	4.07582050e-04
$1/8$	1	1.64797850e-14	5.11562152e-05
$1/16$	1	9.47081773e-14	6.40858168e-06
$1/32$	1	1.39142474e-13	8.01935357e-07
$1/64$	1	1.15448517e-12	1.00294889e-07
$1/128$	1	7.46607969e-12	1.25401317e-08

Table 110: GMRES \ Jacobi EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	13	8.23906577e-10	7.74726030e-01
$1/8$	54	2.76121988e-11	1.61723175e-01
$1/16$	5027	1.23339861e-10	3.65179604e-02
$1/32$	10000	1.51477627e-03	3.56744831e-02
$1/64$	10000	9.71483778e-04	2.55020128e-02
$1/128$	10000	6.92041988e-04	3.93083017e-02

Table 111: GMRES \ ILU EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	3.32978220e-15	7.74726005e-01
$1/8$	1	1.64802799e-15	1.61723174e-01
$1/16$	1	1.77962358e-15	3.65179599e-02
$1/32$	1	1.76303419e-15	8.65588630e-03
$1/64$	1	1.94316800e-15	2.10604985e-03
$1/128$	1	5.36276685e-15	5.19361415e-04

Table 112: GMRES \ Cholesky EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.25110868e-12	7.74726005e-01
$1/8$	24	3.28583292e-13	1.61723174e-01
$1/16$	204	2.22941250e-12	3.65179598e-02
$1/32$	948	1.99210266e-12	8.65588622e-03
$1/64$	10000	9.48521941e-03	5.34355690e-01
$1/128$	10000	9.56288808e-03	5.32258947e-01

Table 113: GMRES \ Jacobi EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	8.20260408e-14	7.74258964e-02
$1/8$	54	3.51275692e-20	1.61691848e-02
$1/16$	908	5.86407051e-11	3.65158940e-03
$1/32$	1222	5.47279203e-11	8.65575219e-04
$1/64$	4094	5.48805134e-11	2.10603647e-04
$1/128$	10000	1.77903048e-05	4.60774112e-04

Table 114: GMRES \ ILU EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.06264555e-16	7.74258964e-02
$1/8$	1	4.85137871e-16	1.61691848e-02
$1/16$	1	9.16797136e-16	3.65158911e-03
$1/32$	1	5.98113615e-16	8.65575272e-04
$1/64$	1	5.27765053e-15	2.10604135e-04
$1/128$	1	5.00349043e-15	5.19360879e-05

Table 115: GMRES \ Cholesky EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	5.99437613e-13	7.74258964e-02
$1/8$	21	3.04112432e-12	1.61691848e-02
$1/16$	144	1.55303651e-10	3.65158693e-03
$1/32$	927	1.35732136e-10	8.65574481e-04
$1/64$	10000	5.51396529e-02	4.89924876e-01
$1/128$	10000	5.46489611e-02	5.01210098e-01

Table 116: GMRES \ Jacobi EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	5.46522514e-16	7.70621364e-03
$1/8$	19	1.14136139e-15	1.61456464e-03
$1/16$	119	8.47664815e-11	3.65006030e-04
$1/32$	457	9.59533707e-11	8.65448936e-05
$1/64$	1315	9.54961526e-11	2.10505098e-05
$1/128$	4413	9.57616083e-11	5.16680765e-06

Table 117: GMRES \ ILU EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.43172963e-16	7.70621364e-03
$1/8$	1	6.37646500e-16	1.61456464e-03
$1/16$	1	8.34933616e-16	3.65006036e-04
$1/32$	1	4.12972154e-15	8.65477308e-05
$1/64$	1	6.04478346e-15	2.10597926e-05
$1/128$	1	2.38795816e-14	5.19356970e-06

Table 118: GMRES \ Cholesky EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	1.37797855e-12	7.70621364e-03
$1/8$	20	7.97874891e-12	1.61456465e-03
$1/16$	206	1.20275053e-10	3.65007043e-04
$1/32$	10000	8.24011193e-03	1.72263754e-01
$1/64$	10000	1.09733386e-02	3.47740063e-01
$1/128$	10000	7.85346730e-02	4.98389235e-01

Table 119: GMRES \ Jacobi EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	3.03602698e-15	8.36143053e-04
$1/8$	24	4.01979014e-15	1.66830523e-04
$1/16$	178	7.43425647e-11	3.68851195e-05
$1/32$	1329	8.38036790e-11	8.65155533e-06
$1/64$	4543	8.61864099e-11	2.01952968e-06
$1/128$	10000	1.82672338e-08	5.29218782e-05

Table 120: GMRES \ ILU EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.85039896e-16	8.36143053e-04
$1/8$	1	1.28841696e-15	1.66830523e-04
$1/16$	1	5.17398943e-15	3.68862590e-05
$1/32$	1	3.17856654e-14	8.68056491e-06
$1/64$	1	2.84181042e-13	2.10764548e-06
$1/128$	1	1.37948101e-13	5.19462796e-07

Table 121: GMRES \ Cholesky EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	9.11145345e-16	8.36143053e-04
$1/8$	19	2.61416596e-10	1.66830613e-04
$1/16$	30	1.04225874e-10	3.68862340e-05
$1/32$	88	1.00371113e-09	8.65248335e-06
$1/64$	1287	6.77821845e-10	2.06336656e-06
$1/128$	10000	2.45443779e-07	3.98354768e-05

Table 122: GMRES \ Jacobi EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.76977643e-15	4.23076377e-04
$1/8$	24	1.19678266e-14	5.45858835e-05
$1/16$	203	8.28683290e-11	7.44924779e-06
$1/32$	2629	8.61965297e-11	9.83180493e-07
$1/64$	9045	8.64548745e-11	6.91348609e-07
$1/128$	10000	2.59385590e-07	7.60149442e-03

Table 123: GMRES \ ILU EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.71053816e-14	4.23076377e-04
$1/8$	1	1.00907363e-14	5.45858836e-05
$1/16$	1	6.60248606e-14	7.47667370e-06
$1/32$	1	1.37343362e-13	1.18950562e-06
$1/64$	1	6.76219117e-13	2.34008127e-07
$1/128$	1	5.55168273e-12	5.34784919e-08

Table 124: GMRES \ Cholesky EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	6.33655365e-11	4.23076372e-04
$1/8$	18	3.11722514e-11	5.45858914e-05
$1/16$	27	1.19364670e-10	7.47670318e-06
$1/32$	76	3.37315688e-10	1.18882555e-06
$1/64$	950	4.72534333e-10	1.93021169e-07
$1/128$	10000	4.41528199e-09	1.20083355e-06

Table 125: GMRES \ Jacobi EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	12	6.97959308e-15	7.95816672e-03
$\frac{1}{8}$	20	1.83499770e-11	2.42102909e-03
$\frac{1}{16}$	388	8.56244091e-11	6.42111646e-04
$\frac{1}{32}$	1058	9.22596797e-11	1.63059503e-04
$\frac{1}{64}$	2328	9.11887971e-11	4.09272556e-05
$\frac{1}{128}$	6254	9.08279736e-11	1.02419566e-05

Table 126: GMRES \ ILU EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	error	error	error
$\frac{1}{8}$	error	error	error
$\frac{1}{16}$	error	error	error
$\frac{1}{32}$	error	error	error
$\frac{1}{64}$	error	error	error
$\frac{1}{128}$	error	error	error

Table 127: GMRES \ Cholesky EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	error	error	error
$\frac{1}{8}$	error	error	error
$\frac{1}{16}$	error	error	error
$\frac{1}{32}$	error	error	error
$\frac{1}{64}$	error	error	error
$\frac{1}{128}$	error	error	error

Table 128: GMRES \ Jacobi EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	1.16554055e-11	7.91131208e-03
$1/8$	49	4.41085579e-11	2.40135547e-03
$1/16$	1210	1.57791561e-10	6.36229744e-04
$1/32$	1542	9.01946309e-11	1.61497758e-04
$1/64$	2396	9.73058619e-11	4.05281099e-05
$1/128$	7659	1.03321245e-10	1.01410840e-05

Table 129: GMRES \ ILU EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	4.80723696e-16	7.91131207e-03
$1/8$	1	5.71854402e-16	2.40135548e-03
$1/16$	1	1.16333060e-15	6.36229740e-04
$1/32$	1	1.94004442e-15	1.61497723e-04
$1/64$	1	3.92947128e-15	4.05281740e-05
$1/128$	1	1.06558068e-14	1.01413716e-05

Table 130: GMRES \ Cholesky EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	9.12361924e-17	7.91131207e-03
$1/8$	21	1.18779581e-15	2.40135548e-03
$1/16$	480	8.87611912e-11	6.36228293e-04
$1/32$	10000	2.70205012e-02	4.16838180e-01
$1/64$	10000	3.86246461e-02	4.91566488e-01
$1/128$	10000	3.51185118e-02	5.10363505e-01

Table 131: GMRES \ Jacobi EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.10901331e-12	7.51472198e-03
$1/8$	47	9.92037177e-12	2.23805472e-03
$1/16$	282	1.04260455e-10	5.87831554e-04
$1/32$	1036	1.02816067e-10	1.48689312e-04
$1/64$	2832	1.02272101e-10	3.72581209e-05
$1/128$	8000	1.01951256e-10	9.31432544e-06

Table 132: GMRES \ ILU EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.69098293e-16	7.51472199e-03
$1/8$	1	1.68201558e-16	2.23805472e-03
$1/16$	1	6.41253007e-16	5.87831623e-04
$1/32$	1	1.65077586e-15	1.48689621e-04
$1/64$	1	2.99933003e-15	3.72591266e-05
$1/128$	1	1.16931509e-14	9.31723573e-06

Table 133: GMRES \ Cholesky EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	4.62906754e-16	7.51472199e-03
$1/8$	21	1.26702713e-15	2.23805472e-03
$1/16$	10000	3.06138667e-03	8.37166474e-02
$1/32$	10000	2.42128886e-02	4.15319425e-01
$1/64$	10000	2.91037813e-02	4.73546761e-01
$1/128$	10000	2.73743245e-02	4.99500967e-01

Table 134: GMRES \ Jacobi EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	5.34719835e-16	5.05666010e-03
$1/8$	23	2.05181190e-11	1.34192230e-03
$1/16$	161	9.62643980e-11	3.35334538e-04
$1/32$	556	9.64148107e-11	8.31190279e-05
$1/64$	1778	9.76076483e-11	2.06396500e-05
$1/128$	6181	9.77592456e-11	5.11558193e-06

Table 135: GMRES \ ILU EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.12555427e-16	5.05666010e-03
$1/8$	1	3.12984518e-16	1.34192231e-03
$1/16$	1	6.82729394e-16	3.35335609e-04
$1/32$	1	3.82447486e-15	8.31219151e-05
$1/64$	1	6.04971805e-15	2.06492155e-05
$1/128$	1	6.27109661e-14	5.14334543e-06

Table 136: GMRES \ Cholesky EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	5.77189402e-15	5.05666010e-03
$1/8$	20	1.08450837e-10	1.34192227e-03
$1/16$	297	5.22471947e-10	3.35339405e-04
$1/32$	10000	8.42890116e-03	3.11143306e-01
$1/64$	10000	8.01997401e-03	3.84658002e-01
$1/128$	10000	8.06608848e-03	4.44897142e-01

Table 137: GMRES \ Jacobi EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.79974653e-16	1.31276389e-03
$1/8$	24	8.92750655e-15	2.83220205e-04
$1/16$	175	7.30452900e-11	6.50321387e-05
$1/32$	1140	8.96478501e-11	1.55354075e-05
$1/64$	3414	8.96655036e-11	3.71581914e-06
$1/128$	10000	1.01303686e-09	2.02038990e-06

Table 138: GMRES \ ILU EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.40993888e-15	1.31276389e-03
$1/8$	1	3.99378557e-15	2.83220204e-04
$1/16$	1	1.53640138e-14	6.50352470e-05
$1/32$	1	1.71730766e-14	1.55641767e-05
$1/64$	1	1.10134341e-13	3.80712424e-06
$1/128$	1	8.10429796e-13	9.41512356e-07

Table 139: GMRES \ Cholesky EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	1.10214183e-15	1.31276389e-03
$1/8$	21	1.44122273e-13	2.83220204e-04
$1/16$	60	1.42034033e-10	6.50352388e-05
$1/32$	800	2.10310753e-09	1.54305024e-05
$1/64$	7230	5.81704061e-09	3.15266026e-06
$1/128$	10000	4.16788515e-04	1.61449579e-01

Table 140: GMRES \ Jacobi EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	12	1.29155339e-15	4.50042141e-04
$\frac{1}{8}$	24	5.06494980e-15	6.18795536e-05
$\frac{1}{16}$	204	7.12237943e-11	9.75546779e-06
$\frac{1}{32}$	2454	8.60844212e-11	1.62941458e-06
$\frac{1}{64}$	8477	8.67273909e-11	4.92967468e-07
$\frac{1}{128}$	10000	2.86100770e-07	8.38474513e-03

Table 141: GMRES \ ILU EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	1	6.79235042e-15	4.50042141e-04
$\frac{1}{8}$	1	4.08110767e-14	6.18795536e-05
$\frac{1}{16}$	1	9.48060402e-14	9.79699480e-06
$\frac{1}{32}$	1	2.50432419e-13	1.90164241e-06
$\frac{1}{64}$	1	1.06487302e-12	4.29403737e-07
$\frac{1}{128}$	1	2.28867904e-12	1.03616263e-07

Table 142: GMRES \ Cholesky EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$\frac{1}{4}$	11	1.70665046e-15	4.50042141e-04
$\frac{1}{8}$	20	5.23125479e-11	6.18795297e-05
$\frac{1}{16}$	50	1.51528263e-10	9.79748793e-06
$\frac{1}{32}$	570	6.02442986e-10	1.87928103e-06
$\frac{1}{64}$	6805	8.20533728e-10	2.73447838e-07
$\frac{1}{128}$	10000	2.67527371e-04	1.17053834e-01

5.2.2 GMRES (Numerical Comparative Plots)

Figure 9: GMRES EPS = -1 SIG = 0.01 TOL = 1.0e-10

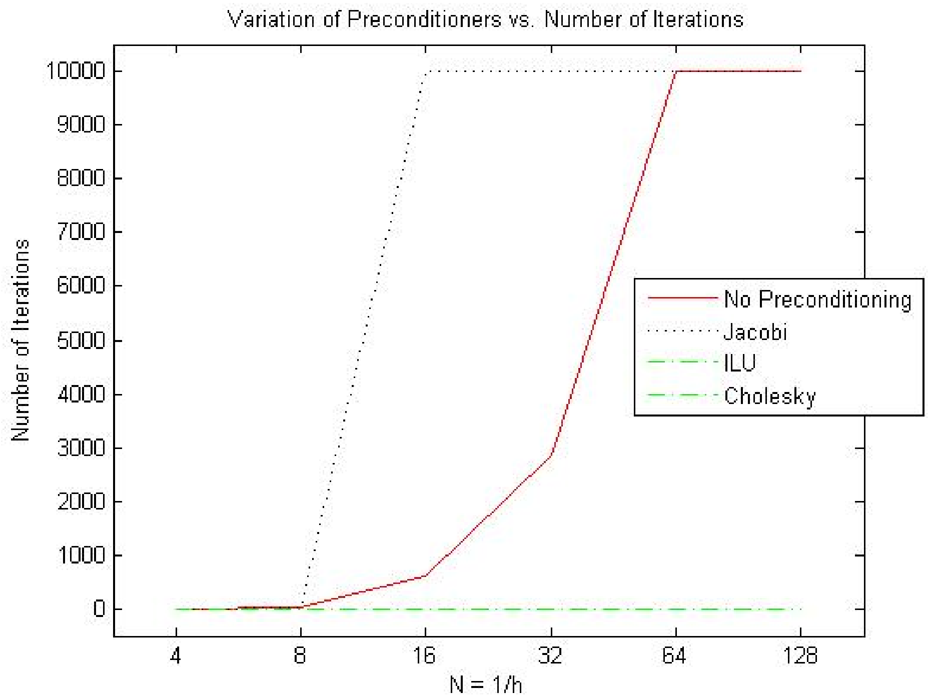


Figure 10: GMRES EPS = -1 SIG = 0.1 TOL = 1.0e-10

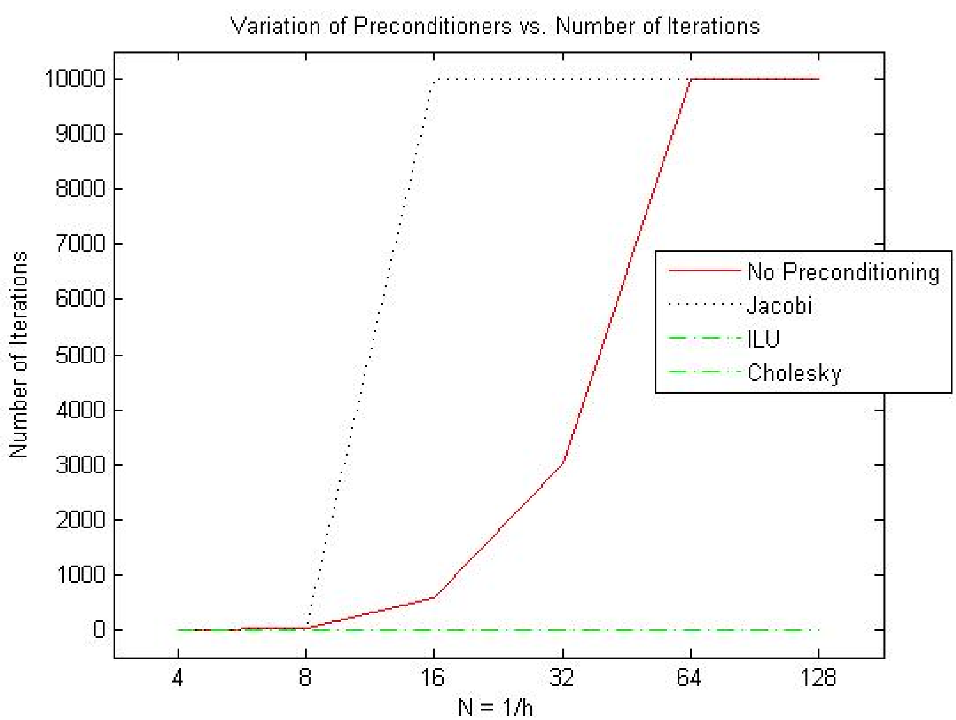


Figure 11: GMRES EPS = -1 SIG = 1 TOL = 1.0e-10

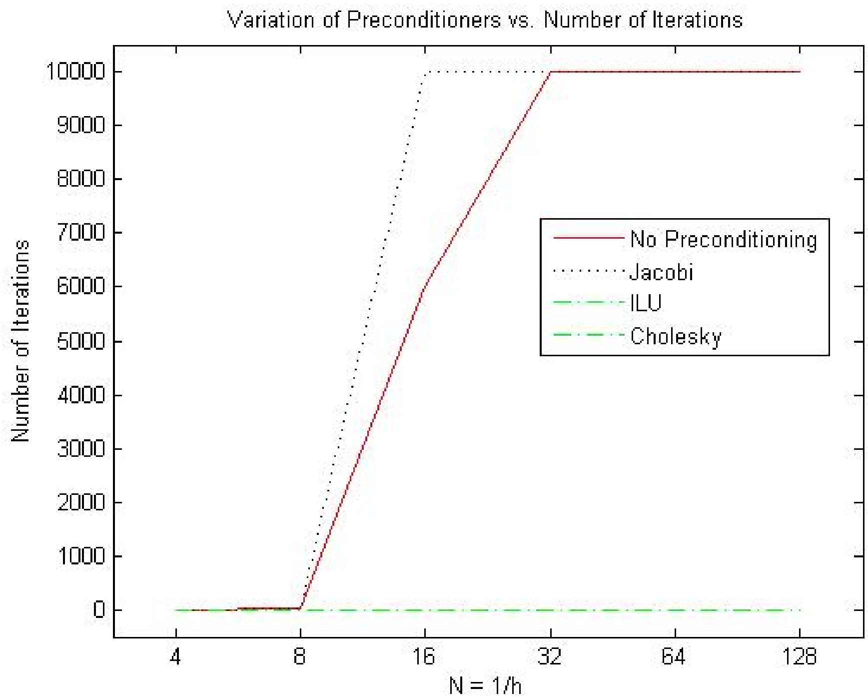


Figure 12: GMRES EPS = -1 SIG = 10 TOL = 1.0e-10

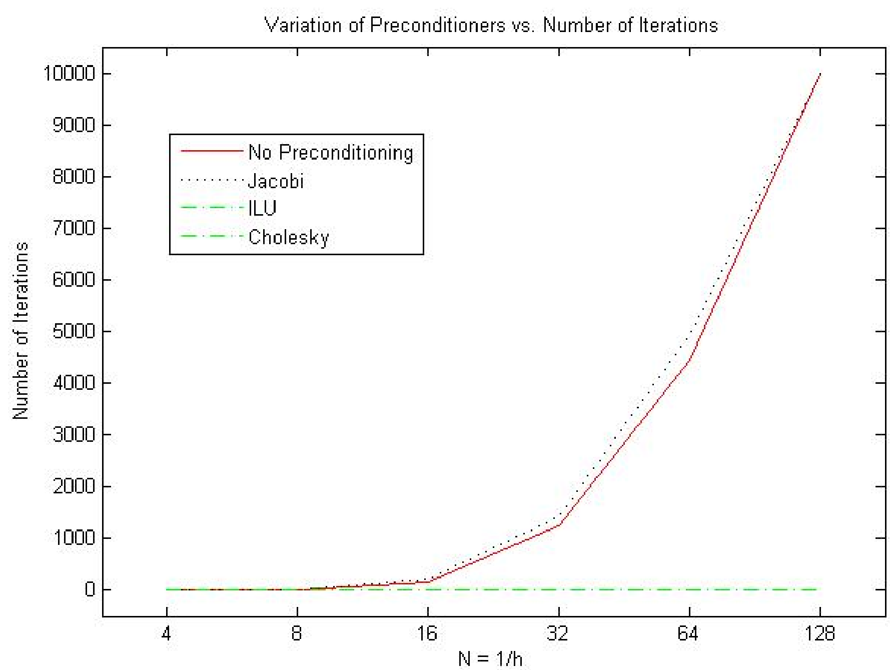


Figure 13: GMRES EPS = -1 SIG = 100 TOL = 1.0e-10

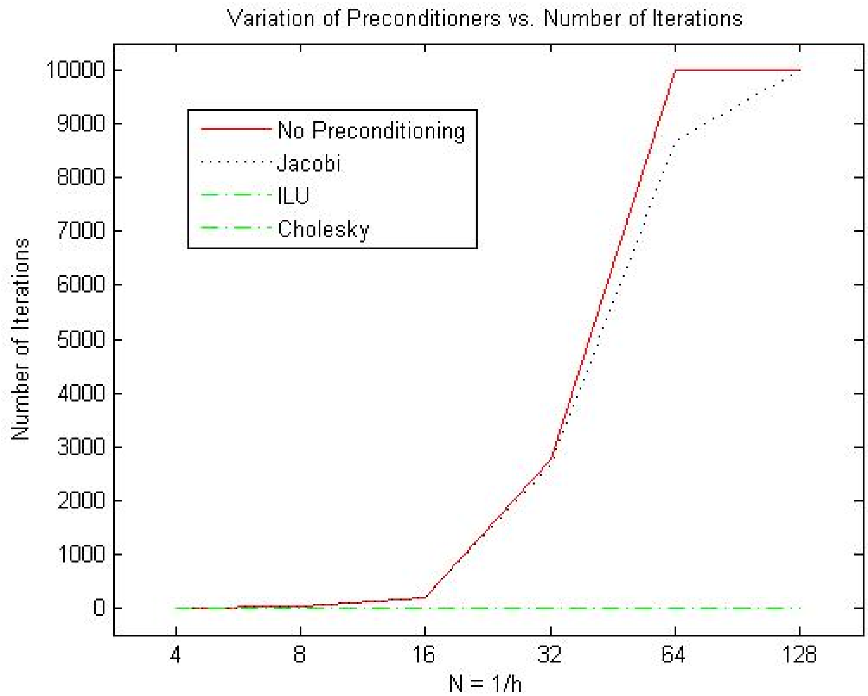


Figure 14: GMRES EPS = 0 SIG = 0.01 TOL = 1.0e-10

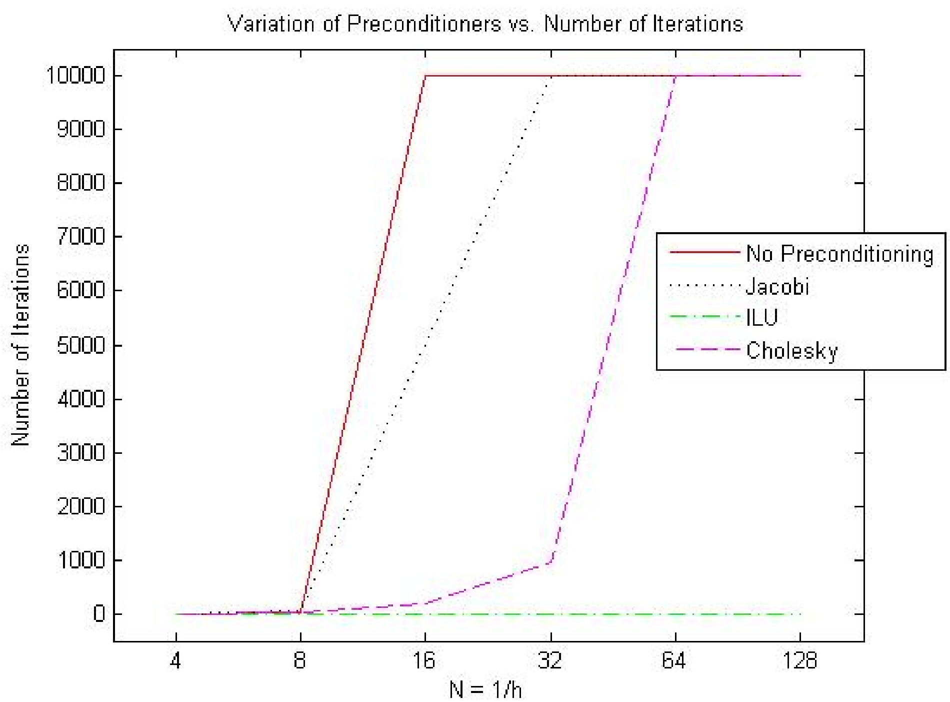


Figure 15: GMRES EPS = 0 SIG = 0.1 TOL = 0.1e-10

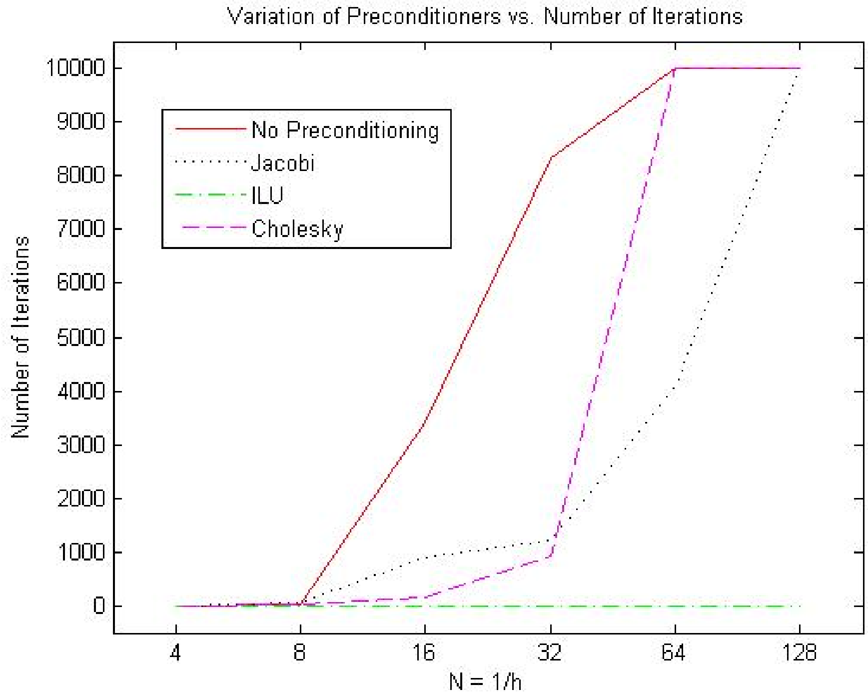


Figure 16: GMRES EPS = 0 SIG = 1 TOL = 1.0e-10

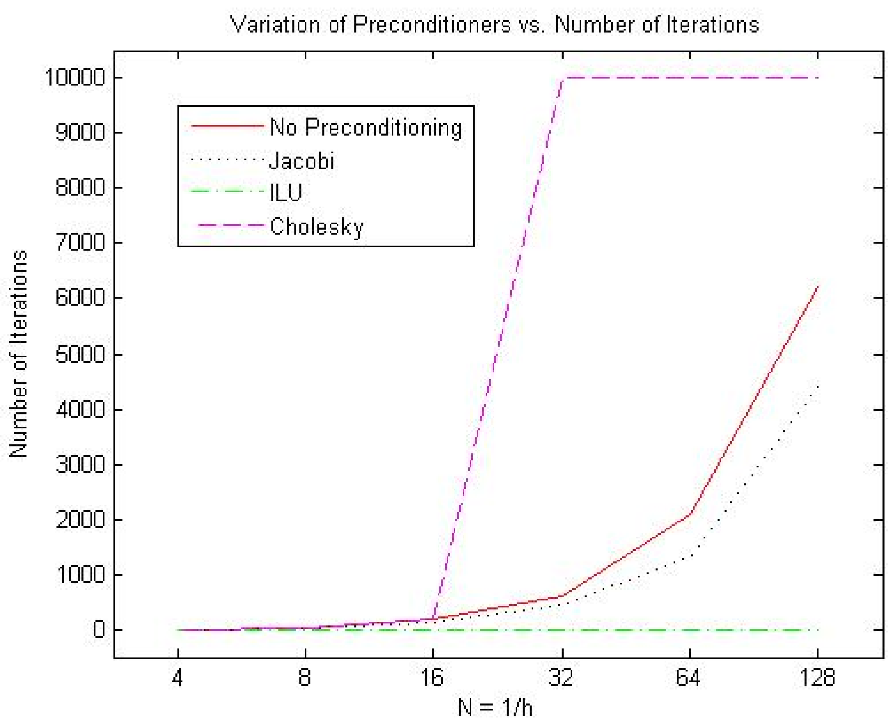


Figure 17: GMRES EPS = 0 SIG = 10 TOL = 1.0e-10

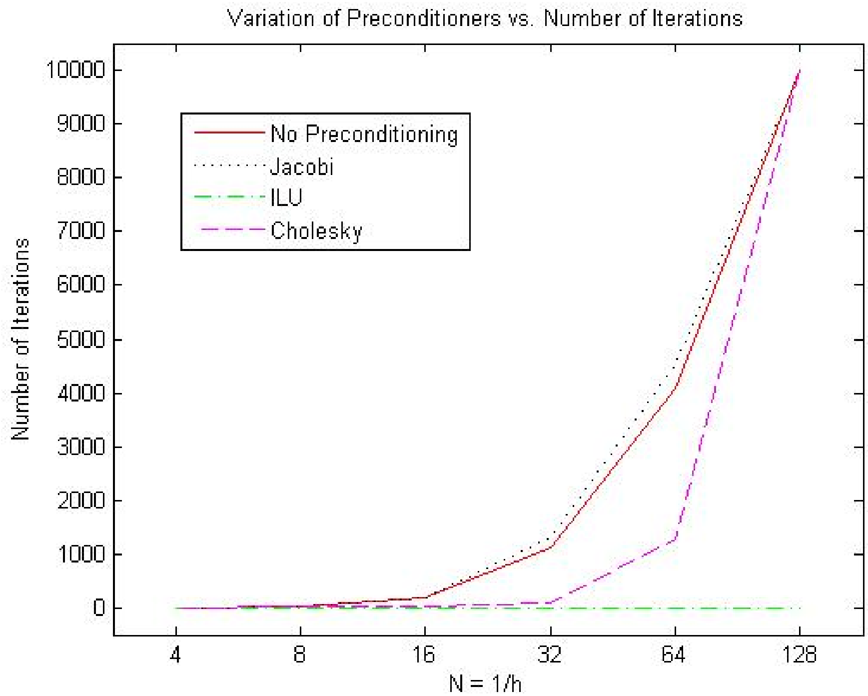


Figure 18: GMRES EPS = 0 SIG = 100 TOL = 1.0e-10

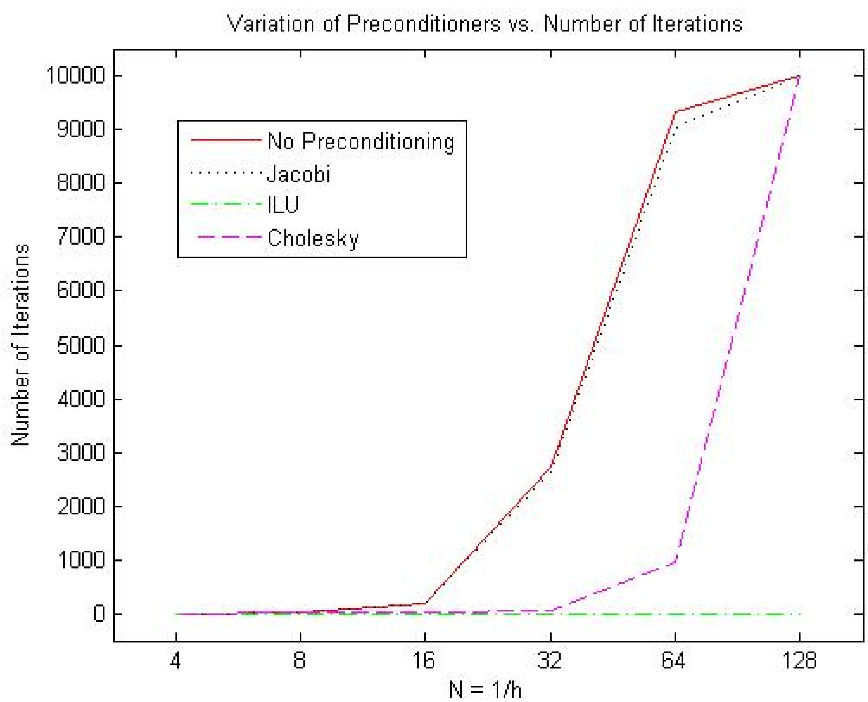


Figure 19: GMRES EPS = +1 SIG = 0 TOL = 1.0e-10

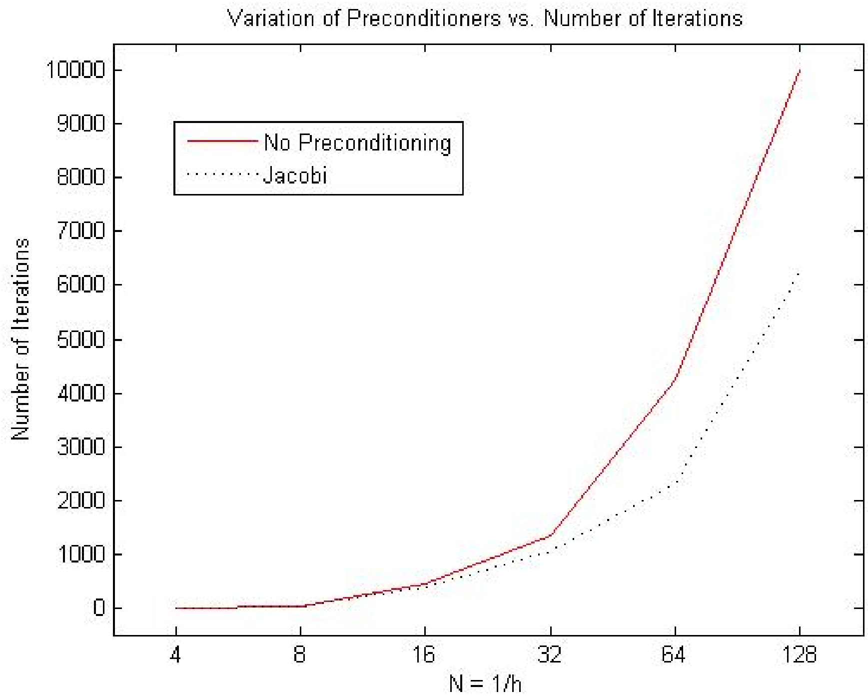


Figure 20: GMES EPS = +1 SIG = 0.01 TOL = 1.0e-10

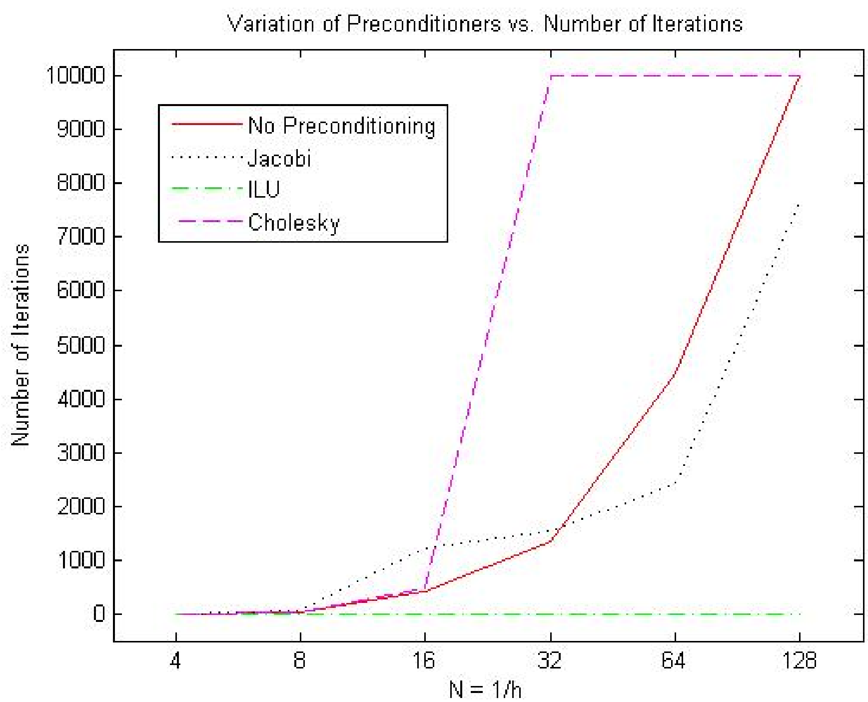


Figure 21: GMES EPS = +1 SIG = 0.1 TOL = 1.0e-10

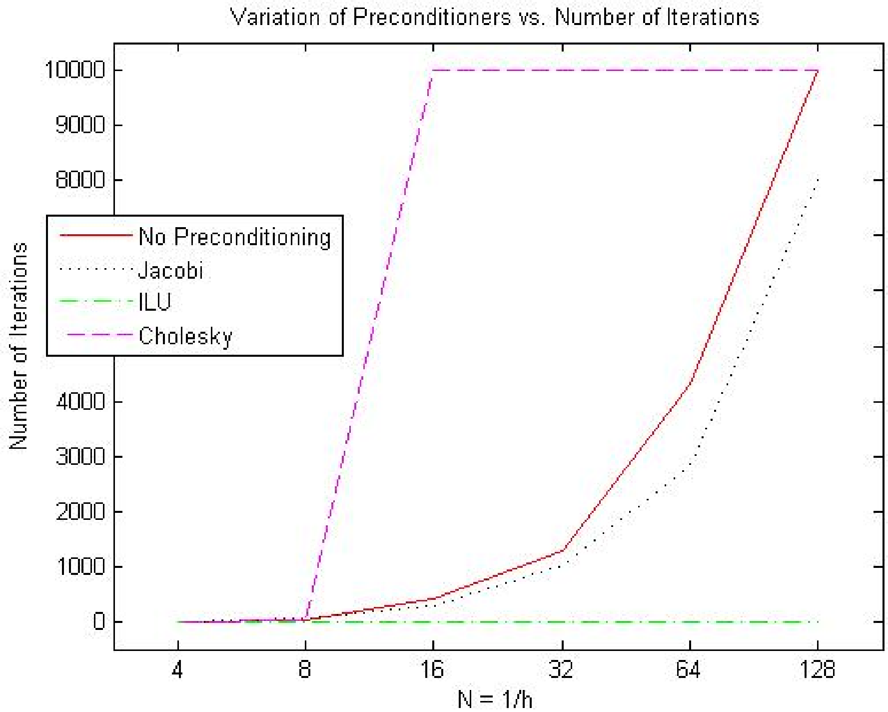


Figure 22: GMRES EPS = +1 SIG = 1 TOL = 1.0e-10

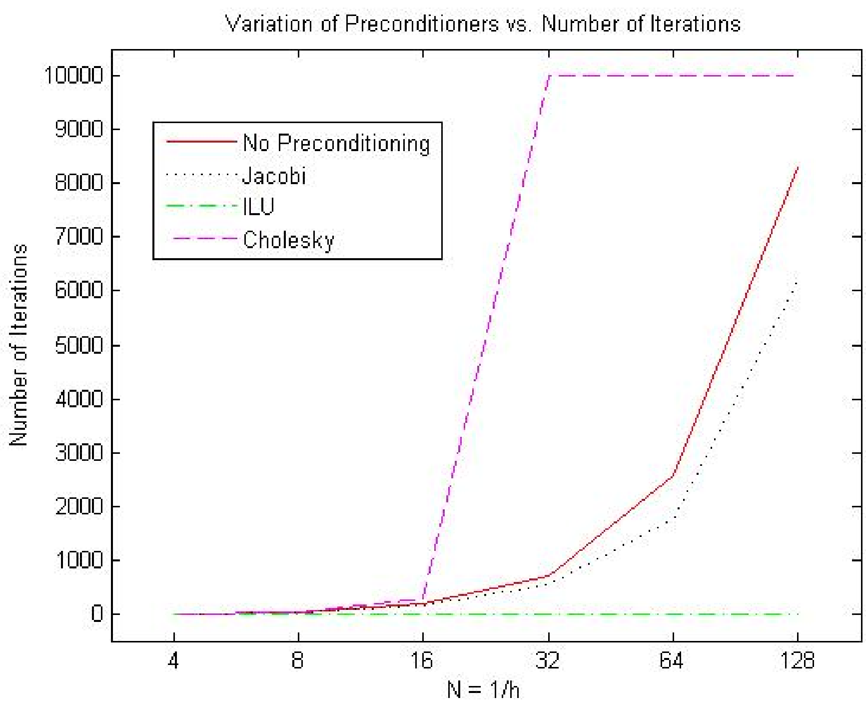


Figure 23: GMRES EPS = +1 SIG = 10 TOL = 1.0e-10

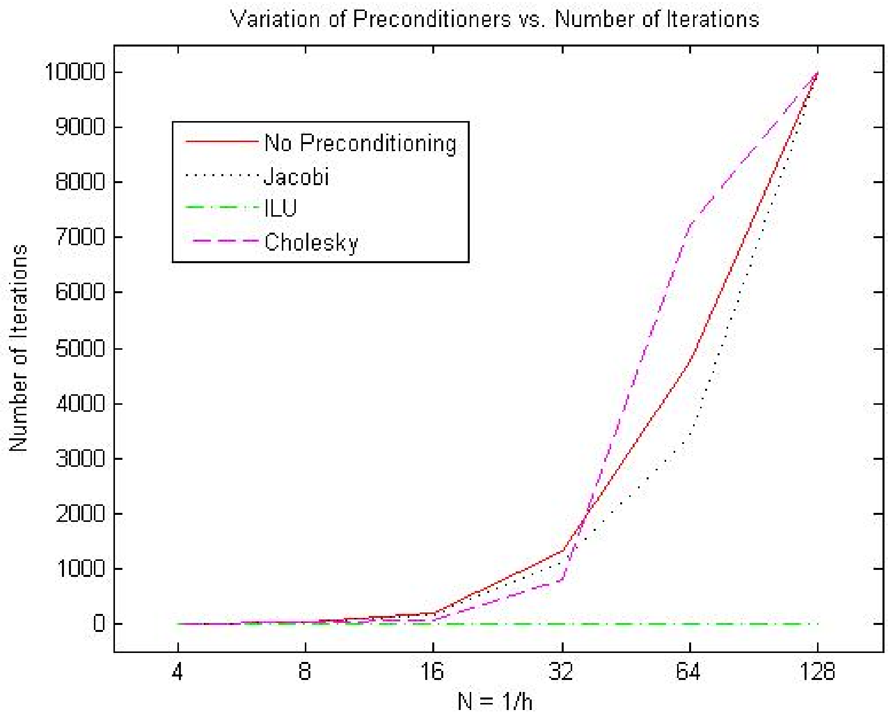
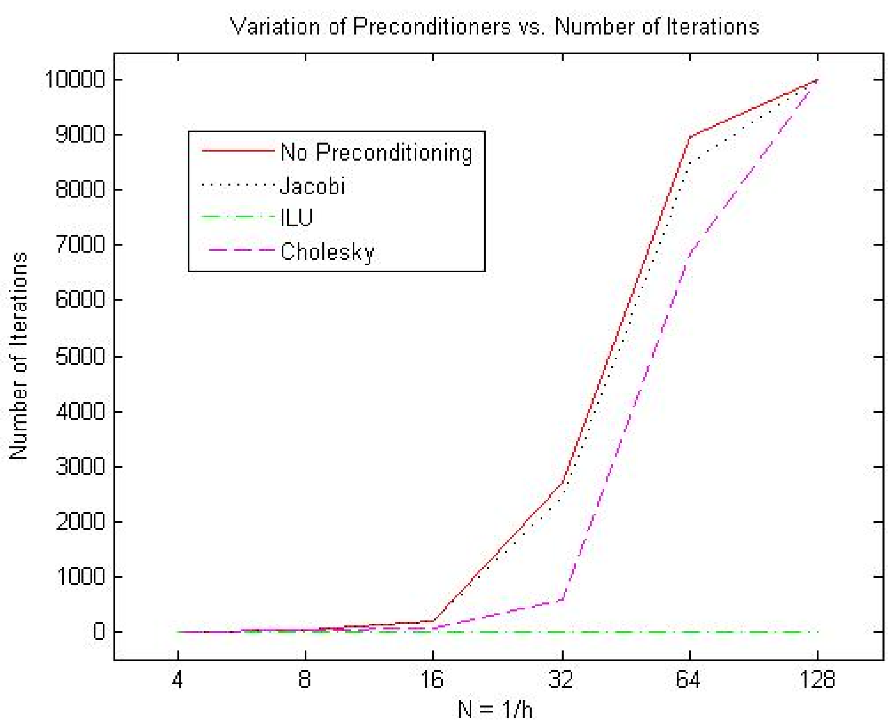


Figure 24: GMRES EPS = +1 SIG = 100 TOL = 1.0e-10



5.3 BI-CGSTAB

Table 143: BI-CGSTAB EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	1.3427583833e-13	1.93196149e-03
$1/8$	38	3.4061845638e-11	2.60547939e-04
$1/16$	88	4.4551688727e-10	3.35060672e-05
$1/32$	183	2.9957856815e-09	4.22323965e-06
$1/64$	463	2.0118161952e-08	5.29351921e-07
$1/128$	969	4.0202790217e-08	7.69636333e-08

Table 144: BI-CGSTAB EPS = -1 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	15	7.2806454888e-09	1.93196135e-03
$1/8$	31	1.9488659781e-04	2.61589124e-04
$1/16$	75	3.1043301346e-04	7.79059390e-05
$1/32$	173	7.3593356094e-04	3.45267695e-04
$1/64$	429	9.6115213088e-04	5.99752479e-04
$1/128$	873	3.5260543917e-03	3.31843633e-03

Table 145: BI-CGSTAB ESP=-1 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	1.5711830446e-01	2.37535389e-02
$1/8$	25	2.4890847029e-01	5.83621242e-02
$1/16$	37	7.0922864192e-01	2.39017941e-01
$1/32$	44	1.3885867241e	3.58028171e-01
$1/64$	46	2.7326970976e+00	4.41218137e-01
$1/128$	49	5.4820595921e+00	4.85351785e-01

Table 146: BI-CGSTAB EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	1.0110780352e-10	2.05471019e-03
$1/8$	36	1.6794750404e-09	2.79605346e-04
$1/16$	81	1.6400966672e-09	3.61402237e-05
$1/32$	191	1.3075535984e-08	4.56474133e-06
$1/64$	448	1.8321644790e-08	5.72365913e-07
$1/128$	1049	4.7770268974e-08	7.63981551e-08

Table 147: BI-CGSTAB EPS = -1 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	15	4.8334102184e-05	2.05599547e-03
$1/8$	31	2.9150605893e-04	2.85387869e-04
$1/16$	73	4.3037698421e-04	1.55547670e-04
$1/32$	174	3.6608760081e-04	1.06570309e-04
$1/64$	411	6.4815706587e-04	3.59765622e-04
$1/128$	963	4.3511989561e-03	3.53880475e-03

Table 148: BI-CGSTAB EPS = -1 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	1.5313255941e-01	2.67217052e-02
$1/8$	25	2.2323577818e-01	5.96487088e-02
$1/16$	45	6.4541880233e-01	2.28665216e-01
$1/32$	51	1.2325727808e+00	3.47391159e-01
$1/64$	46	2.7317751230e+00	4.35207557e-01
$1/128$	53	5.3726856784e+00	4.84336332e-01

Table 149: BI-CGSTAB EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	18	7.6719951370e-13	1.46470833e-02
$1/8$	36	7.7886308366e-10	4.69518378e-03
$1/16$	112	4.7292808520e-09	1.70023828e-03
$1/32$	346	5.4251251993e-09	6.90387045e-04
$1/64$	1510	2.0711719944e-08	3.05809705e-04
$1/128$	10000	1.9243375656e-04	2.09393071e-04

Table 150: BI-CGSTAB EPS = -1 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	1.1251598790e-08	1.46470815e-02
$1/8$	32	1.5412212276e-04	4.70383484e-03
$1/16$	93	1.2598875423e-04	1.68084911e-03
$1/32$	278	7.2658763303e-04	8.37594863e-04
$1/64$	1173	1.4143724202e-03	6.22592464e-04
$1/128$	8109	3.7220480625e-03	4.70719789e-04

Table 151: BI-CGSTAB EPS = -1 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	14	3.6031633500e-02	9.53277083e-03
$1/8$	28	2.1332455958e-01	5.56011549e-02
$1/16$	63	3.6984674388e-01	9.64678609e-02
$1/32$	149	9.3823750828e-01	2.71934193e-01
$1/64$	206	2.1142515685e+00	4.18503051e-01
$1/128$	276	4.2130439559e+00	4.72022173e-01

Table 152: BI-CGSTAB EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	15	1.9053160385e-09	2.85353355e-04
$1/8$	32	1.5724826862e-09	3.53050511e-05
$1/16$	66	5.9867623494e-09	4.40398760e-06
$1/32$	139	2.5437828756e-08	5.50464967e-07
$1/64$	270	2.4410538963e-08	6.88623071e-08
$1/128$	553	6.1721391905e-08	6.60223056e-08

Table 153: BI-CGSTAB EPS = -1 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	4.8066799747e-04	2.88332734e-04
$1/8$	22	6.6013942720e-04	1.51673155e-04
$1/16$	50	1.4480835221e-03	5.51657644e-04
$1/32$	115	3.7733723703e-03	1.81897941e-03
$1/64$	226	7.9752709207e-03	5.48964511e-03
$1/128$	479	1.6166265025e-02	1.78625499e-02

Table 154: BI-CGSTAB EPS = -1 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	7	2.7256003110e-01	5.03915231e-02
$1/8$	9	9.9937289535e-01	2.38344038e-01
$1/16$	10	2.1572510678e+00	3.65282056e-01
$1/32$	10	4.3193402864e+00	4.46322448e-01
$1/64$	10	8.5620715925e+00	4.88441855e-01
$1/128$	10	1.7065273678e+01	5.09552563e-01

Table 155: BI-CGSTAB EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	21	1.9804644284e-11	4.07582050e-04
$1/8$	54	1.0919344232e-07	5.11557975e-05
$1/16$	149	2.5948578692e-07	6.40624620e-06
$1/32$	399	2.4553608146e-07	8.12434020e-07
$1/64$	814	9.6149900981e-07	6.78456018e-07
$1/128$	1816	1.4575253754e-06	1.38287586e-06

Table 156: BI-CGSTAB TPS=-1 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	15	2.5630759913e-03	5.80279668e-04
$1/8$	38	9.2995757209e-03	2.21786392e-03
$1/16$	100	1.2304209357e-02	3.33461991e-03
$1/32$	300	5.2891323740e-02	2.46291554e-02
$1/64$	602	9.3666036984e-02	6.77764793e-02
$1/128$	786	2.1592707378e-01	2.19121940e-01

Table 157: BI-CGSTAB EPS = -1 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	2	5.8072313303e+00	3.67533518e-01
$1/8$	2	1.1710436713e+01	4.45728514e-01
$1/16$	2	2.3280386670e+01	4.87630898e-01
$1/32$	2	4.6464771955e+01	5.09022743e-01
$1/64$	2	9.2875507560e+01	5.19786762e-01
$1/128$	2	1.8572259200e+02	5.25180102e-01

Table 158: BI-CGSTAB EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	19	6.8574810362e-12	7.74726005e-01
$1/8$	60	3.4816587161e-11	1.61723174e-01
$1/16$	143	3.8082846727e-11	3.65179599e-02
$1/32$	869	4.8321138747e-11	8.65588630e-03
$1/64$	10000	5.2474041320e+01	2.35852652e+00
$1/128$	10000	8.57204402e+04	1.12609096e+03

Table 159: BI-CGSTAB EPS = 0 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	19	6.8574810362e-12	7.74726005e-01
$1/8$	49	6.0862652333e-08	1.61723158e-01
$1/16$	108	2.2687897578e-06	3.65188724e-02
$1/32$	624	5.0755358852e-06	8.65657851e-03
$1/64$	10000	5.2474041320e+01	2.35852652e+00
$1/128$	10000	8.57204402e+04	1.12609096e+03

Table 160: BI-CGSTAB EPS = 0 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	13	5.0381531161e-03	7.94919127e-01
$1/8$	31	4.3161507153e-03	1.63612210e-01
$1/16$	84	4.9810863980e-03	4.00331481e-02
$1/32$	449	2.8276128214e-03	9.02279164e-03
$1/64$	10000	5.2474041320e+01	2.35852652e+00
$1/128$	10000	8.57204402e+04	1.12609096e+03

Table 161: BI-CGSTAB EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
1/4	17	6.4171141089e-12	7.74258964e-02
1/8	87	4.7924350319e-11	1.61691848e-02
1/16	274	4.0058098052e+04	9.54202623e+02
1/32	10000	1.0577249643e+03	9.96113757e+00
1/64	10000	2.52051024e+07	8.36510791e+04
1/128	10000	3.13273362e+08	3.11399816e+05

Table 162: BI-CGSTAB EPS = 0 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
1/4	14	4.7835370861e-08	7.74259229e-02
1/8	61	1.6117779033e-05	1.61629927e-02
1/16	274	4.0058098052e+04	9.54202623e+02
1/32	10000	1.0577249643e+03	9.96113757e+00
1/64	10000	2.52051024e+07	8.36510791e+04
1/128	10000	3.13273362e+08	3.11399816e+05

Table 163: BI-CGSTAB EPS = 0 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
1/4	13	4.5351194487e-05	7.74011580e-02
1/8	52	1.2486641118e-02	2.12661636e-02
1/16	274	4.0058098052e+04	9.54202623e+02
1/32	10000	1.0577249643e+03	9.96113757e+00
1/64	10000	2.52051024e+07	8.36510791e+04
1/128	10000	3.13273362e+08	3.11399816e+05

Table 164: BI-CGSTAB EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	7.9846523142e-13	7.70621364e-03
$1/8$	21	1.3677005813e-09	1.61456497e-03
$1/16$	47	1.4875048592e-10	3.65006025e-04
$1/32$	111	3.1429311890e-09	8.65475778e-05
$1/64$	321	1.0732440915e-08	2.10598291e-05
$1/128$	975	1.1413312496e-08	5.19390098e-06

Table 165: BI-CGSTAB EPS = 0 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	1.7218272924e-06	7.70635965e-03
$1/8$	19	2.2880977432e-07	1.61451128e-03
$1/16$	41	3.2744602907e-05	3.62210611e-04
$1/32$	94	5.1254561650e-04	1.33965433e-04
$1/64$	245	9.6723441966e-04	6.24659014e-05
$1/128$	794	7.2256201648e-04	4.31754880e-05

Table 166: BI-CGSTAB EPS = 0 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	9	4.1142200656e-02	4.56067689e-03
$1/8$	19	2.2880977432e-07	1.61451128e-03
$1/16$	37	9.6111351068e-02	1.10765499e-02
$1/32$	64	3.6969053289e-01	2.92152086e-02
$1/64$	51	7.2051664898e-01	3.13900888e-01
$1/128$	81	2.1327035546e+00	3.36157320e-01

Table 167: BI-CGSTAB EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	1.1112937437e-10	8.36143075e-04
$1/8$	35	4.4924007643e-10	1.66830414e-04
$1/16$	71	2.6424962618e-08	3.68768368e-05
$1/32$	139	3.2699528618e-08	8.66909996e-06
$1/64$	281	7.6016097986e-08	2.06779410e-06
$1/128$	577	1.9987996203e-07	5.03690883e-07

Table 168: BI-CGSTAB EPS = 0 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.0826428541e-04	8.30760396e-04
$1/8$	27	7.2109706503e-04	5.28832927e-05
$1/16$	53	2.4938117967e-03	8.64841601e-04
$1/32$	106	4.4680782690e-03	2.05472420e-03
$1/64$	200	8.1813010562e-03	5.11782241e-03
$1/128$	494	1.1829121274e-02	2.10219454e-03

Table 169: BI-CGSTAB EPS = 0 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	7	5.0754639768e-01	9.06558137e-02
$1/8$	9	1.3219661717e+00	2.70696051e-01
$1/16$	11	2.1635563191e+00	3.68641066e-01
$1/32$	11	4.3221263370e+00	4.47900614e-01
$1/64$	11	8.5542968683e+00	4.89137154e-01
$1/128$	11	1.7040911787e+01	5.09880674e-01

Table 170: BI-CGSTAB EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	21	8.4589472543e-12	4.23076378e-04
$1/8$	51	5.4344232279e-08	5.45894333e-05
$1/16$	152	2.1219497395e-07	7.44255215e-06
$1/32$	373	2.9921171658e-07	1.27516277e-06
$1/64$	849	1.0399010344e-06	2.74559058e-07
$1/128$	2206	1.8679139065e-06	3.92846530e-07

Table 171: BI-CGSTAB EPS = 0 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	3.3324479749e-04	4.10287071e-04
$1/8$	36	3.9418297229e-03	9.13361655e-04
$1/16$	110	2.7532451670e-02	8.27222103e-03
$1/32$	268	5.5421666992e-02	2.24111661e-02
$1/64$	522	1.0728104952e-01	1.23754670e-02
$1/128$	1446	6.4075431917e-02	7.09132248e-03

Table 172: BI-CGSTAB EPS = 0 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	2	5.9745375654e+00	3.69147287e-01
$1/8$	2	1.2031928332e+01	4.46519743e-01
$1/16$	2	2.3920475026e+01	4.88019806e-01
$1/32$	2	4.7744366188e+01	5.09215203e-01
$1/64$	2	9.5434620959e+01	5.19882456e-01
$1/128$	2	1.9084083125e+02	5.25227811e-01

Table 173: BI-CGSTAB EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	1.03663765e-10	7.95816673e-03
$1/8$	33	3.08669808e-09	2.42102977e-03
$1/16$	95	4.63474489e-09	6.42111269e-04
$1/32$	292	4.21094951e-09	1.63059002e-04
$1/64$	1129	7.77914144e-09	4.09255186e-05
$1/128$	5894	3.02286229e-08	1.03012572e-05

Table 174: BI-CGSTAB EPS = +1 SIG = 0 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.60847948e-05	7.95625325e-03
$1/8$	27	3.40520076e-04	2.49554089e-03
$1/16$	84	1.63463417e-05	6.38147498e-04
$1/32$	247	1.08491609e-03	3.94517718e-05
$1/64$	890	1.42246437e-03	3.76277136e-04
$1/128$	4705	3.37203729e-03	1.36246457e-03

Table 175: BI-CGSTAB EPS = +1 SIG = 0 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	1.26517753e-02	6.60850108e-03
$1/8$	22	2.12364852e-01	4.23202726e-02
$1/16$	42	6.62788977e-01	1.95599849e-01
$1/32$	42	1.40975291e+00	3.45778578e-01
$1/64$	42	2.85722625e+00	4.36241513e-01
$1/128$	43	5.67823618e+00	4.83140739e-01

Table 176: BI-CGSTAB EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	1.3321286398e-10	7.91131209e-03
$1/8$	34	9.8655983094e-10	2.40135526e-03
$1/16$	99	6.5162636242e-09	6.36231573e-04
$1/32$	277	1.0299402228e-08	1.61500099e-04
$1/64$	1025	2.3077012519e-08	4.05214137e-05
$1/128$	5063	2.7873622270e-08	1.01510546e-05

Table 177: BI-CGSTAB EPS = +1 SIG = 0.01 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.6435026786e-06	7.91150034e-03
$1/8$	27	4.1320084699e-05	2.39267371e-03
$1/16$	83	3.5182448387e-04	7.24323800e-04
$1/32$	227	1.0241085742e-03	1.48944817e-04
$1/64$	773	2.8377241786e-03	4.21737821e-05
$1/128$	3308	4.3002835017e-03	1.56278749e-03

Table 178: BI-CGSTAB EPS = +1 SIG = 0.01 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	1.3951383560e-02	6.38659173e-03
$1/8$	21	3.1528164063e-01	6.60143084e-02
$1/16$	45	6.8864654172e-01	1.94858160e-01
$1/32$	44	1.3583794179e+00	3.45009078e-01
$1/64$	43	2.8485618453e+00	4.35886392e-01
$1/128$	44	5.5976778177e+00	4.83313850e-01

Table 179: BI-CGSTAB EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	1.0879100154e-10	7.51472197e-03
$1/8$	32	2.6215244701e-09	2.23805526e-03
$1/16$	86	5.5789971752e-09	5.87830099e-04
$1/32$	249	6.3771275347e-09	1.48690663e-04
$1/64$	892	1.4500157736e-08	3.72545121e-05
$1/128$	3082	1.4545385375e-08	9.31288521e-06

Table 180: BI-CGSTAB EPS = +1 SIG = 0.1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.6104506751e-06	7.51451238e-03
$1/8$	27	2.3913262165e-04	2.18492846e-03
$1/16$	73	6.0705952203e-04	4.22101626e-04
$1/32$	218	1.2262159019e-03	2.05385706e-04
$1/64$	647	2.6633711754e-03	8.78672998e-04
$1/128$	2218	5.5130148993e-03	9.88063163e-04

Table 181: BI-CGSTAB EPS = +1 SIG = 0.1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	2.5065377537e-02	4.48407805e-03
$1/8$	21	2.7351588996e-01	5.74276171e-02
$1/16$	35	6.9386237317e-01	2.08168207e-01
$1/32$	37	1.3695988282e+00	3.47745631e-01
$1/64$	37	2.8129224084e+00	4.36644211e-01
$1/128$	36	5.7900094680e+00	4.85141632e-01

Table 182: BI-CGSTAB EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	13	2.0336043026e-09	5.05666042e-03
$1/8$	27	4.0640426348e-09	1.34192137e-03
$1/16$	50	4.6586879008e-09	3.35334455e-04
$1/32$	112	1.1565561224e-08	8.31178548e-05
$1/64$	348	2.7726803252e-08	2.06407300e-05
$1/128$	1123	3.9202534890e-08	5.13101305e-06

Table 183: BI-CGSTAB EPS = +1 SIG = 1 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.1902188201e-04	5.03802266e-03
$1/8$	22	3.6087145246e-04	1.26023389e-03
$1/16$	43	4.0535082531e-04	2.37044435e-04
$1/32$	98	1.9610841818e-04	2.06527559e-05
$1/64$	280	3.5055231671e-03	9.72337625e-04
$1/128$	782	6.8569384252e-03	2.13938331e-03

Table 184: BI-CGSTAB EPS = +1 SIG = 1 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	9	1.5675410953e-02	3.99638781e-03
$1/8$	15	3.5902336985e-01	4.55333314e-02
$1/16$	15	9.4043130213e-01	2.72112252e-01
$1/32$	17	1.8138478596e+00	3.82688722e-01
$1/64$	17	3.6144227657e+00	4.55674218e-01
$1/128$	17	7.1640049680e+00	4.93122583e-01

Table 185: BI-CGSTAB EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	2.2372443102e-11	1.31276389e-03
$1/8$	32	1.6500110315e-09	2.83219763e-04
$1/16$	69	1.5545750553e-09	6.50352404e-05
$1/32$	146	4.4408865457e-08	1.55406288e-05
$1/64$	282	7.6054017583e-08	3.75550643e-06
$1/128$	593	1.8614782974e-07	1.09520388e-06

Table 186: BI-CGSTAB EPS = +1 SIG = 10 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	13	3.0793521064e-05	1.30734126e-03
$1/8$	25	7.7042847777e-04	4.81187292e-04
$1/16$	55	8.2899077185e-04	1.55335901e-04
$1/32$	118	6.2639205657e-03	3.36519638e-03
$1/64$	240	7.5905645847e-03	4.40179813e-03
$1/128$	496	2.5523733742e-02	2.14514653e-02

Table 187: BI-CGSTAB ESP=+1 SIG = 10 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	6	7.5173994799e-01	1.26069545e-01
$1/8$	8	1.6589107366e+00	2.93531649e-01
$1/16$	10	2.2819643314e+00	3.70333517e-01
$1/32$	10	4.5658772139e+00	4.49023150e-01
$1/64$	10	9.0563152405e+00	4.89759749e-01
$1/128$	10	1.8055165387e+01	5.10202237e-01

Table 188: BI-CGSTAB EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	22	4.2549376260e-12	4.50042142e-04
$1/8$	59	4.4337912966e-09	6.18795253e-05
$1/16$	153	4.6751685964e-08	9.78517692e-06
$1/32$	369	3.3934129802e-07	1.97548251e-06
$1/64$	871	4.4676631051e-08	4.10271746e-07
$1/128$	2000	1.5360415469e-06	1.04228603e-06

Table 189: BI-CGSTAB EPS = +1 SIG = 100 TOL = 1.0e-5

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	15	6.0473901912e-03	1.10922002e-03
$1/8$	38	1.0748423526e-02	2.87630071e-03
$1/16$	104	2.3922833183e-02	8.06938793e-03
$1/32$	273	3.8879629136e-02	1.72827720e-02
$1/64$	585	1.0062379949e-01	4.27698283e-02
$1/128$	793	2.2252177988e-01	2.00170892e-01

Table 190: BI-CGSTAB EPS = +1 SIG = 100 TOL = 1.0e-2

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	2	6.1411878290e+00	3.70706150e-01
$1/8$	2	1.2352734140e+01	4.47285285e-01
$1/16$	2	2.4559297782e+01	4.88396320e-01
$1/32$	2	4.9021430330e+01	5.09401578e-01
$1/64$	2	9.7988661567e+01	5.19975136e-01
$1/128$	2	1.9594891554e+02	5.25274020e-01

5.3.1 Bi-CGStab with Preconditioning

Table 191: BI-CGSTAB \ Jacobi EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	25	1.20096035e-12	1.93196149e-03
$1/8$	80	1.10354875e-10	2.60547942e-04
$1/16$	569	5.70290378e-11	3.35060606e-05
$1/32$	10000	2.20794832e+02	1.26333520e+00
$1/64$	10000	5.91901383e+02	2.20485463e+00
$1/128$	10000	6.28814620e+05	1.57794225e+03

Table 192: BI-CGSTAB \ ILU EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.29882675e-30	1.93196149e-03
$1/8$	1	1.31751508e-30	2.60547939e-04
$1/16$	1	9.01670889e-31	3.35060612e-05
$1/32$	1	1.68917785e-30	4.22322352e-06
$1/64$	1	7.13353482e-30	5.29087793e-07
$1/128$	1	9.12071142e-30	6.61750670e-08

Table 193: BI-CGSTAB \ Cholesky EPS = -1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	5.05783053e-30	1.93196149e-03
$1/8$	1	2.66149612e-30	2.60547939e-04
$1/16$	1	3.50525116e-30	3.35060612e-05
$1/32$	1	4.55583117e-30	4.22322352e-06
$1/64$	1	1.26906606e-29	5.29087793e-07
$1/128$	1	7.87271941e-29	6.61750670e-08

Table 194: BI-CGSTAB \ Jacobi EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	25	1.09602610e-10	2.05471019e-03
$1/8$	119	1.13364893e-10	2.79605381e-04
$1/16$	10000	3.89407683e+01	9.20879262e-01
$1/32$	10000	1.20334376e+04	1.73548781e+02
$1/64$	10000	1.12744181e+02	1.25220051e+00
$1/128$	10000	2.51191079e+02	1.90677193e+00

Table 195: BI-CGSTAB \ ILU EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.30250125e-29	2.05471019e-03
$1/8$	1	2.71045147e-29	2.79605379e-04
$1/16$	1	9.01827177e-29	3.61402464e-05
$1/32$	1	1.83499074e-28	4.56487160e-06
$1/64$	1	3.89824485e-30	5.72374751e-07
$1/128$	1	6.05648475e-29	7.16149222e-08

Table 196: BI-CGSTAB \ Cholesky EPS = -1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	9.86487499e-30	2.05471019e-03
$1/8$	1	1.55078932e-28	2.79605379e-04
$1/16$	1	6.09819662e-29	3.61402464e-05
$1/32$	1	6.28831734e-29	4.56487160e-06
$1/64$	1	2.62750449e-28	5.72374751e-07
$1/128$	1	5.55820093e-28	7.16149222e-08

Table 197: BI-CGSTAB \ Jacobi EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	21	1.69587378e-10	1.46470832e-02
$1/8$	57	4.95013983e-10	4.69518465e-03
$1/16$	268	1.23389994e-09	1.70023671e-03
$1/32$	10000	7.90790960e-02	3.82167751e-01
$1/64$	1501	nan	nan
$1/128$	1303	nan	nan

Table 198: BI-CGSTAB \ ILU EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	7.86428568e-32	1.46470833e-02
$1/8$	1	6.66307536e-31	4.69518373e-03
$1/16$	1	9.17828179e-31	1.70023895e-03
$1/32$	1	4.67513542e-30	6.90388037e-04
$1/64$	1	8.25386468e-29	3.05814536e-04
$1/128$	1	2.35402789e-28	1.43177242e-04

Table 199: BI-CGSTAB \ Cholesky EPS = -1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.89425809e-31	1.46470833e-02
$1/8$	1	2.62019884e-31	4.69518373e-03
$1/16$	1	3.13558307e-30	1.70023895e-03
$1/32$	1	1.13007566e-29	6.90388037e-04
$1/64$	1	6.29936430e-29	3.05814536e-04
$1/128$	1	5.61562741e-28	1.43177242e-04

Table 200: BI-CGSTAB \ Jacobi EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	1.98496299e-11	2.85353355e-04
$1/8$	37	3.07954988e-11	3.53050688e-05
$1/16$	71	1.47733259e-11	4.40402671e-06
$1/32$	141	3.77463884e-11	5.50721161e-07
$1/64$	277	7.22072954e-11	9.46639186e-08
$1/128$	568	2.23151726e-12	1.02952282e-08

Table 201: BI-CGSTAB \ ILU EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	5.10858576e-31	2.85353348e-04
$1/8$	1	3.75447307e-30	3.53050402e-05
$1/16$	1	8.45872443e-29	4.40399589e-06
$1/32$	1	2.74447574e-28	5.50443622e-07
$1/64$	1	2.72141714e-27	6.88198044e-08
$1/128$	1	8.80144598e-26	8.60395834e-09

Table 202: BI-CGSTAB \ Cholesky EPS = -1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	8.83011639e-31	2.85353348e-04
$1/8$	1	1.62833545e-29	3.53050402e-05
$1/16$	1	9.99616006e-29	4.40399589e-06
$1/32$	1	5.95757682e-28	5.50443622e-07
$1/64$	1	5.16995780e-27	6.88198046e-08
$1/128$	1	3.50124470e-26	8.60395824e-09

Table 203: BI-CGSTAB \ Jacobi EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	23	5.42608271e-13	4.07582061e-04
$1/8$	58	1.09037295e-12	5.11562268e-05
$1/16$	173	9.19303623e-12	6.40855330e-06
$1/32$	371	6.61112086e-11	8.23083054e-07
$1/64$	872	2.36951018e-11	1.42696373e-07
$1/128$	1837	4.12504275e-11	1.08563421e-06

Table 204: BI-CGSTAB \ ILU EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.62309409e-28	4.07582050e-04
$1/8$	1	4.03103455e-28	5.11562152e-05
$1/16$	1	2.96209025e-28	6.40858167e-06
$1/32$	1	7.87774008e-27	8.01935355e-07
$1/64$	1	1.36809828e-24	1.00294888e-07
$1/128$	1	5.94544068e-24	1.25401374e-08

Table 205: BI-CGSTAB \ Cholesky EPS = -1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.52833214e-28	4.07582050e-04
$1/8$	1	1.20797749e-27	5.11562152e-05
$1/16$	1	3.70602731e-27	6.40858167e-06
$1/32$	1	2.74482174e-26	8.01935356e-07
$1/64$	1	9.83711910e-25	1.00294888e-07
$1/128$	1	4.37421890e-23	1.25401414e-08

Table 206: BI-CGSTAB \ Jacobi EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	22	1.48417466e-11	7.74726005e-01
$1/8$	51	1.62418250e-10	1.61723174e-01
$1/16$	394	1.14943543e-10	3.65179613e-02
$1/32$	10000	2.42896589e+94	6.85873593e+93
$1/64$	8791	nan	nan
$1/128$	10000	5.85100057e+136	6.63672806e+135

Table 207: BI-CGSTAB \ ILU EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	8.32836035e-30	7.74726005e-01
$1/8$	1	1.35086250e-30	1.61723174e-01
$1/16$	1	7.26494627e-31	3.65179599e-02
$1/32$	1	1.40136265e-30	8.65588630e-03
$1/64$	1	1.64362441e-30	2.10604985e-03
$1/128$	1	2.43964654e-30	5.19361415e-04

Table 208: BI-CGSTAB \ Cholesky EPS = 0 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	19	8.14366336e-13	7.74726005e-01
$1/8$	49	1.00410226e-12	1.61723174e-01
$1/16$	116	2.58995585e-12	3.65179599e-02
$1/32$	325	1.70560702e-12	8.65588624e-03
$1/64$	977	1.39796603e-12	2.10604984e-03
$1/128$	4198	9.11855188e-13	5.19361415e-04

Table 209: BI-CGSTAB \ Jacobi EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	15	1.11436033e-10	7.74258966e-02
$1/8$	46	2.44754113e-11	1.61691848e-02
$1/16$	325	3.32337244e-11	3.65158915e-03
$1/32$	7924	nan	nan
$1/64$	9140	nan	nan
$1/128$	8405	nan	nan

Table 210: BI-CGSTAB \ ILU EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	4.12094319e-32	7.74258964e-02
$1/8$	1	1.16619858e-31	1.61691848e-02
$1/16$	1	1.36263435e-31	3.65158911e-03
$1/32$	1	1.91732249e-31	8.65575272e-04
$1/64$	1	1.92819378e-30	2.10604135e-04
$1/128$	1	8.22902608e-30	5.19360879e-05

Table 211: BI-CGSTAB \ Cholesky EPS = 0 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	1.07497186e-12	7.74258964e-02
$1/8$	35	4.77699322e-11	1.61691853e-02
$1/16$	76	4.75461431e-11	3.65158923e-03
$1/32$	201	1.25596693e-10	8.65574549e-04
$1/64$	506	1.54701083e-10	2.10604546e-04
$1/128$	2938	1.56499090e-10	5.19363306e-05

Table 212: BI-CGSTAB \ Jacobi EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.40173840e-12	7.70621364e-03
$1/8$	25	7.25870095e-13	1.61456465e-03
$1/16$	53	8.96018182e-11	3.65006043e-04
$1/32$	141	8.59240309e-11	8.65477307e-05
$1/64$	609	7.68607478e-11	2.10597713e-05
$1/128$	3010	nan	nan

Table 213: BI-CGSTAB \ ILU EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	2.01808236e-32	7.70621364e-03
$1/8$	1	5.62990454e-32	1.61456464e-03
$1/16$	1	3.15941852e-31	3.65006036e-04
$1/32$	1	3.80023695e-31	8.65477308e-05
$1/64$	1	2.16127098e-29	2.10597926e-05
$1/128$	1	1.59219944e-28	5.19356969e-06

Table 214: BI-CGSTAB \ Cholesky EPS = 0 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	14	1.61832095e-13	7.70621364e-03
$1/8$	40	6.38057893e-11	1.61456462e-03
$1/16$	117	1.08017891e-10	3.65005981e-04
$1/32$	580	1.63014948e-09	8.65456122e-05
$1/64$	4317	2.46466159e-10	2.10603756e-05
$1/128$	10000	2.85123173e-02	2.61433531e-01

Table 215: BI-CGSTAB \ Jacobi EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	2.02231024e-17	8.36143053e-04
$1/8$	37	1.04781943e-11	1.66830116e-04
$1/16$	70	3.45721117e-12	3.68858327e-05
$1/32$	142	8.48969871e-12	8.68289656e-06
$1/64$	280	3.61177466e-11	2.14097827e-06
$1/128$	577	8.11750905e-11	3.25589264e-07

Table 216: BI-CGSTAB \ ILU EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.61230707e-31	8.36143053e-04
$1/8$	1	9.98702968e-31	1.66830523e-04
$1/16$	1	1.98021152e-29	3.68862590e-05
$1/32$	1	8.90991545e-29	8.68056492e-06
$1/64$	1	1.11295183e-26	2.10764550e-06
$1/128$	1	3.20475566e-26	5.19462783e-07

Table 217: BI-CGSTAB \ Cholesky EPS = 0 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	3.39026769e-15	8.36143053e-04
$1/8$	18	1.57666033e-10	1.66830672e-04
$1/16$	31	1.37975044e-10	3.68852991e-05
$1/32$	70	6.66986283e-10	8.67870300e-06
$1/64$	178	5.34498020e-10	2.10201885e-06
$1/128$	637	7.57887223e-10	4.10708105e-07

Table 218: BI-CGSTAB \ Jacobi EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	21	8.36255569e-13	4.23076418e-04
$1/8$	58	8.21312091e-12	5.45855879e-05
$1/16$	145	6.46465744e-11	7.44450964e-06
$1/32$	384	8.62863326e-11	1.30839264e-06
$1/64$	777	6.26817559e-11	8.62057330e-07
$1/128$	1890	8.58768012e-11	2.32467278e-06

Table 219: BI-CGSTAB \ ILU EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	5.95579936e-30	4.23076377e-04
$1/8$	1	2.38574058e-28	5.45858836e-05
$1/16$	1	2.58633476e-27	7.47667369e-06
$1/32$	1	3.41827317e-26	1.18950557e-06
$1/64$	1	2.40631911e-26	2.34008251e-07
$1/128$	1	2.68557489e-24	5.34795052e-08

Table 220: BI-CGSTAB \ Cholesky EPS = 0 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	10	1.93326826e-14	4.23076377e-04
$1/8$	16	1.09973955e-10	5.45857883e-05
$1/16$	28	1.60963729e-10	7.47659261e-06
$1/32$	68	1.23595132e-10	1.18870219e-06
$1/64$	178	1.26948562e-10	2.25994990e-07
$1/128$	655	3.63680667e-10	4.79329519e-08

Table 221: BI-CGSTAB \ Jacobi EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	20	5.15972649e-12	7.95816672e-03
$1/8$	50	7.97013906e-13	2.42102909e-03
$1/16$	605	9.48146262e-11	6.42112815e-04
$1/32$	7156	nan	nan
$1/64$	4085	nan	nan
$1/128$	1685	nan	nan

Table 222: BI-CGSTAB \ ILU EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	error	error	error
$1/8$	error	error	error
$1/16$	error	error	error
$1/32$	error	error	error
$1/64$	error	error	error
$1/128$	error	error	error

Table 223: BI-CGSTAB \ Cholesky EPS = +1 SIG = 0 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	error	error	error
$1/8$	error	error	error
$1/16$	error	error	error
$1/32$	error	error	error
$1/64$	error	error	error
$1/128$	error	error	error

Table 224: BI-CGSTAB \ Jacobi EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	17	7.26903487e-10	7.91131212e-03
$1/8$	63	2.22173890e-10	2.40135551e-03
$1/16$	1146	4.05667516e-11	6.87992719e-04
$1/32$	2616	nan	nan
$1/64$	1897	nan	nan
$1/128$	1756	nan	nan

Table 225: BI-CGSTAB \ ILU EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	3.63045390e-30	7.91131207e-03
$1/8$	1	1.51595939e-30	2.40135548e-03
$1/16$	1	1.26885986e-30	6.36229740e-04
$1/32$	1	1.09024131e-30	1.61497723e-04
$1/64$	1	4.36890320e-30	4.05281740e-05
$1/128$	1	7.86120266e-30	1.01413716e-05

Table 226: BI-CGSTAB \ Cholesky EPS = +1 SIG = 0.01 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	14	2.90237928e-12	7.91131207e-03
$1/8$	29	7.96750932e-11	2.40135571e-03
$1/16$	82	6.44683074e-11	6.36229313e-04
$1/32$	398	6.49303224e-11	1.61498651e-04
$1/64$	1696	8.45024861e-11	4.05294062e-05
$1/128$	10000	1.50089463e-07	1.18648059e-05

Table 227: BI-CGSTAB \ Jacobi EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	19	2.15499303e-12	7.51472199e-03
$1/8$	46	9.20517422e-11	2.23805472e-03
$1/16$	275	9.34042381e-11	5.87831626e-04
$1/32$	8570	nan	nan
$1/64$	10000	4.61553726e+23	1.24007737e+22
$1/128$	10000	1.17399462e+75	2.46406077e+73

Table 228: BI-CGSTAB \ ILU EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	8.44647168e-32	7.51472199e-03
$1/8$	1	7.41687470e-32	2.23805472e-03
$1/16$	1	8.77891586e-32	5.87831623e-04
$1/32$	1	4.42195395e-31	1.48689621e-04
$1/64$	1	1.17455449e-30	3.72591266e-05
$1/128$	1	2.47011799e-29	9.31723573e-06

Table 229: BI-CGSTAB \ Cholesky EPS = +1 SIG = 0.1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.25114948e-11	7.51472201e-03
$1/8$	29	2.36809059e-12	2.23805472e-03
$1/16$	96	3.36836045e-11	5.87831886e-04
$1/32$	401	6.59188340e-11	1.48690903e-04
$1/64$	1843	2.75786314e-11	3.72586896e-05
$1/128$	10000	1.98436120e-03	5.42163210e-03

Table 230: BI-CGSTAB \ Jacobi EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	2.50438424e-11	5.05666009e-03
$1/8$	25	6.91813432e-11	1.34192209e-03
$1/16$	49	8.67555534e-11	3.35335354e-04
$1/32$	101	7.28979431e-11	8.31217057e-05
$1/64$	258	5.18991181e-11	2.06492476e-05
$1/128$	826	6.64502208e-11	5.14396006e-06

Table 231: BI-CGSTAB \ ILU EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	1.56648050e-32	5.05666010e-03
$1/8$	1	1.01055865e-31	1.34192231e-03
$1/16$	1	2.28607670e-30	3.35335609e-04
$1/32$	1	6.17661131e-30	8.31219151e-05
$1/64$	1	1.87455470e-29	2.06492155e-05
$1/128$	1	2.10875886e-28	5.14334542e-06

Table 232: BI-CGSTAB \ Cholesky EPS = +1 SIG = 1 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	14	4.90059247e-12	5.05666011e-03
$1/8$	37	1.15063149e-10	1.34192281e-03
$1/16$	96	3.01931915e-10	3.35334894e-04
$1/32$	369	8.23458131e-10	8.31077363e-05
$1/64$	2202	3.82725754e-09	2.06425287e-05
$1/128$	10000	5.67708124e-03	2.21759583e-01

Table 233: BI-CGSTAB \ Jacobi EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	16	7.17479562e-12	1.31276379e-03
$1/8$	31	4.35937854e-11	2.83221422e-04
$1/16$	65	2.76216506e-11	6.50341029e-05
$1/32$	135	8.82299938e-11	1.55531889e-05
$1/64$	279	5.85676831e-11	3.78665924e-06
$1/128$	580	3.31461099e-11	9.33816851e-07

Table 234: BI-CGSTAB \ ILU EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	7.82112189e-31	1.31276389e-03
$1/8$	1	3.13050274e-30	2.83220204e-04
$1/16$	1	1.15741077e-28	6.50352470e-05
$1/32$	1	8.20395177e-28	1.55641766e-05
$1/64$	1	3.94579909e-27	3.80712423e-06
$1/128$	1	1.88871670e-25	9.41512308e-07

Table 235: BI-CGSTAB \ Cholesky EPS = +1 SIG = 10 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	11	1.67218860e-12	1.31276389e-03
$1/8$	26	6.58824394e-10	2.83223425e-04
$1/16$	59	3.47568373e-10	6.50389373e-05
$1/32$	159	1.66714003e-09	1.55641178e-05
$1/64$	656	4.39388822e-09	3.90529466e-06
$1/128$	3081	7.42042248e-09	1.33855509e-06

Table 236: BI-CGSTAB \ Jacobi EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	20	3.48766725e-11	4.50044668e-04
$1/8$	55	3.20150769e-11	6.18772593e-05
$1/16$	157	8.43775758e-11	9.76716384e-06
$1/32$	433	7.75535999e-11	1.76623039e-06
$1/64$	872	5.20494287e-11	8.04744426e-07
$1/128$	1691	6.30948464e-11	2.22156430e-07

Table 237: BI-CGSTAB \ ILU EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	1	7.29262700e-29	4.50042141e-04
$1/8$	1	4.64095679e-29	6.18795536e-05
$1/16$	1	1.88379125e-27	9.79699482e-06
$1/32$	1	6.13283635e-26	1.90164237e-06
$1/64$	1	1.05670678e-25	4.29404016e-07
$1/128$	1	3.77319107e-24	1.03615584e-07

Table 238: BI-CGSTAB \ Cholesky EPS = +1 SIG = 100 TOL = 1.0e-10

h	Num. of Iterations	Residual Norm	L2 Error
$1/4$	12	1.21875364e-13	4.50042141e-04
$1/8$	26	2.55788432e-11	6.18795357e-05
$1/16$	47	1.74321562e-10	9.79665528e-06
$1/32$	132	6.77035408e-10	1.91138950e-06
$1/64$	471	4.46418206e-10	4.35149846e-07
$1/128$	2387	8.97086374e-10	1.66984265e-07

5.3.2 Bi-CGStab (Numerical Comparative Plots)

Figure 25: BI-CGSTAB EPS = -1 SIG = 0.01 TOL = 1.0e-10

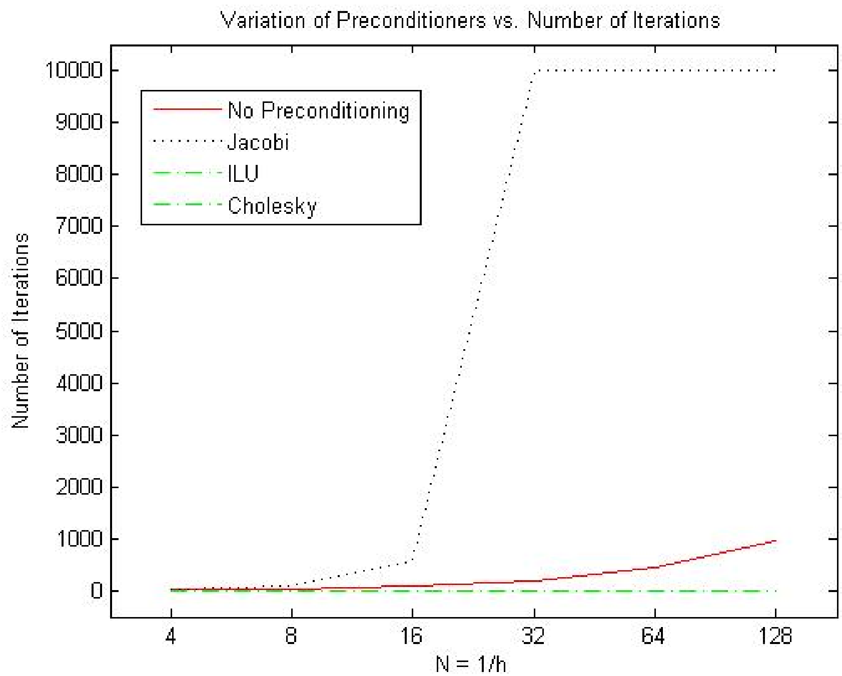


Figure 26: BI-CGSTAB EPS = -1 SIG = 0.1 TOL = 1.0e-10

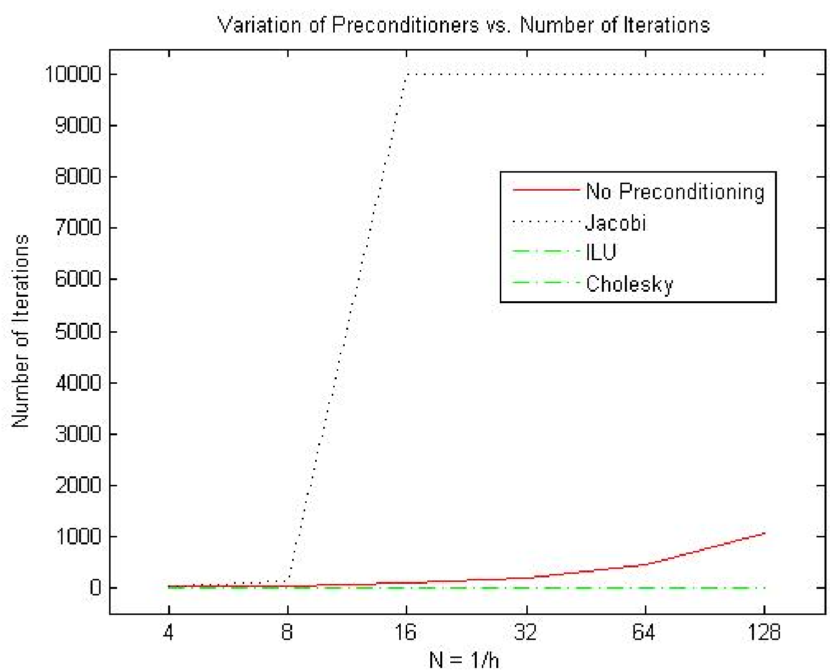


Figure 27: BI-CGSTAB EPS = -1 SIG = 1 TOL = 1.0e-10

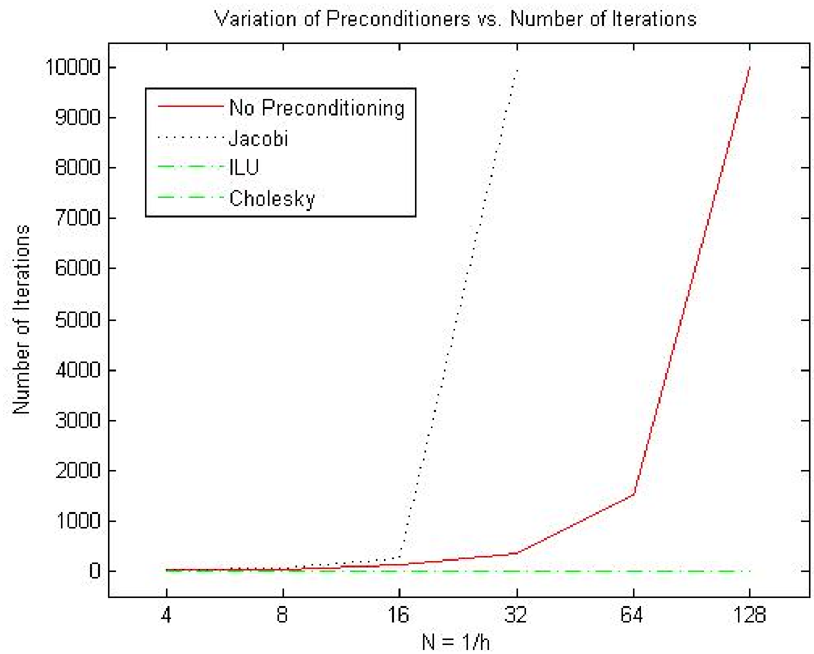


Figure 28: BI-CGSTAB EPS = -1 SIG = 10 TOL = 1.0e-10

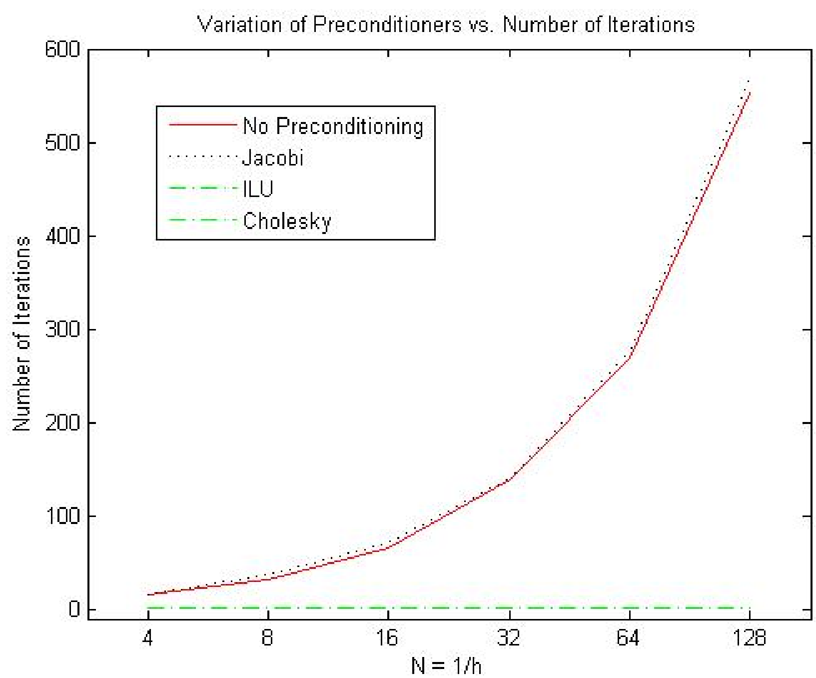


Figure 29: BI-CGSTAB EPS = -1 SIG = 100 TOL = 1.0e-10

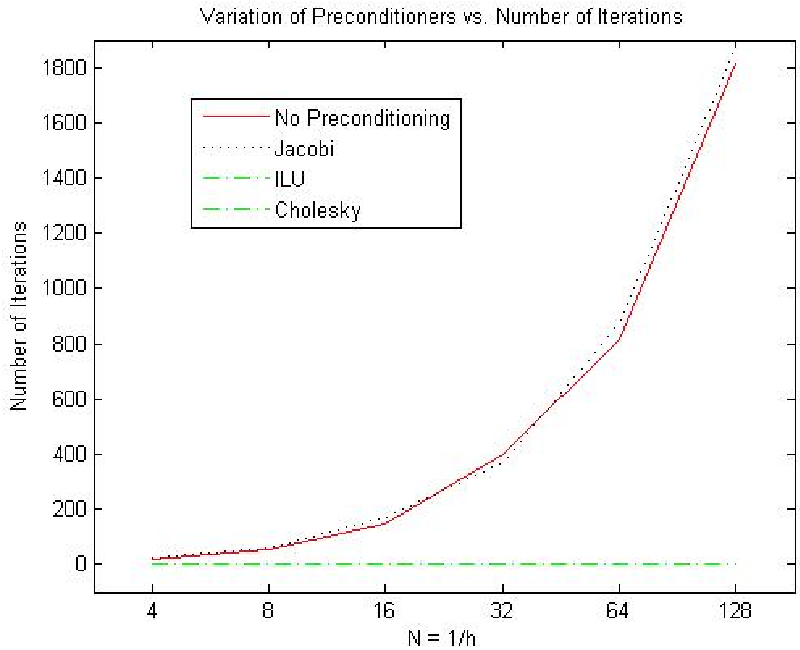


Figure 30: BI-CGSTAB ESP = 0 SIG = 0.01 TOL = 1.0e-10

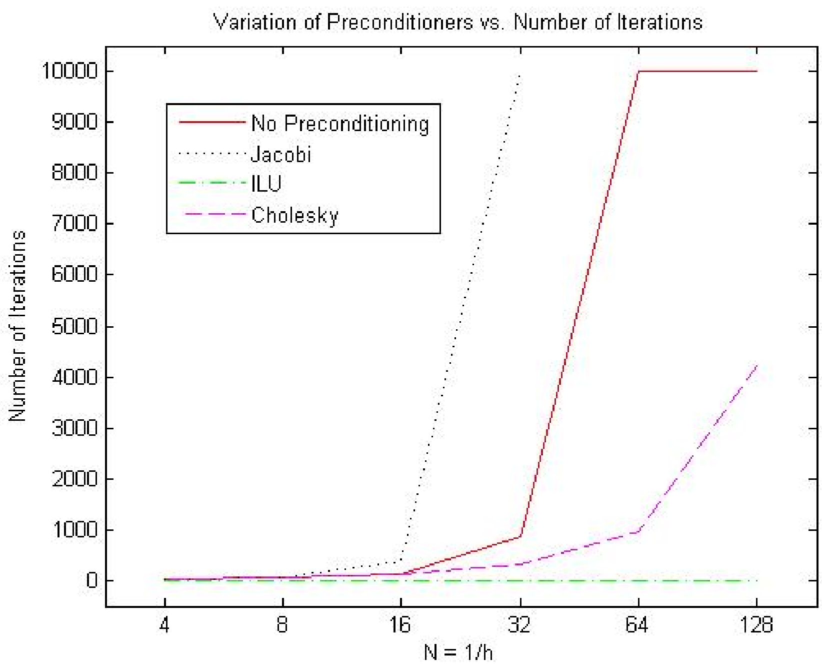


Figure 31: BI-CGSTAB EPS = 0 SIG = 0.1 TOL = 1.0e-10

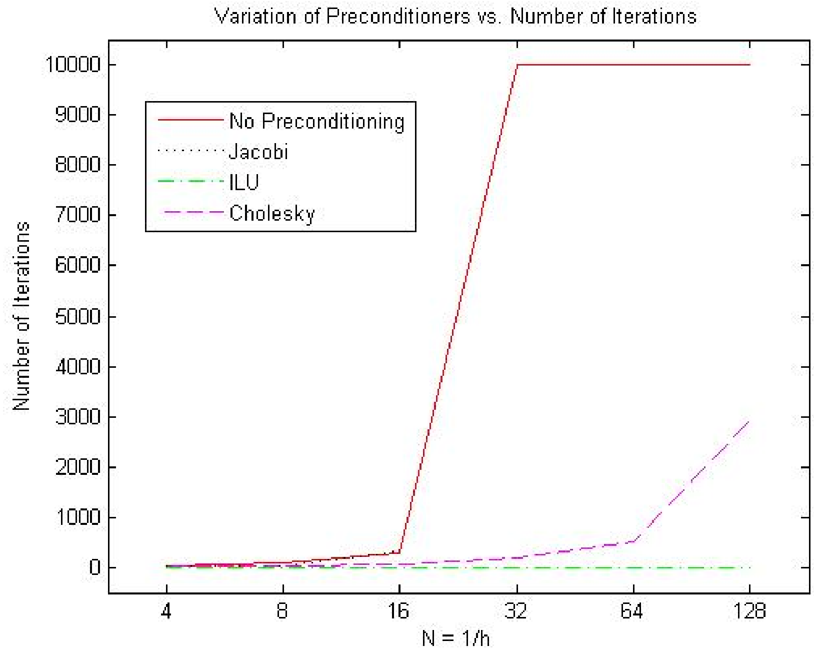


Figure 32: BI-CGSTAB EPS = 0 SIG = 1 TOL = 1.0e-10

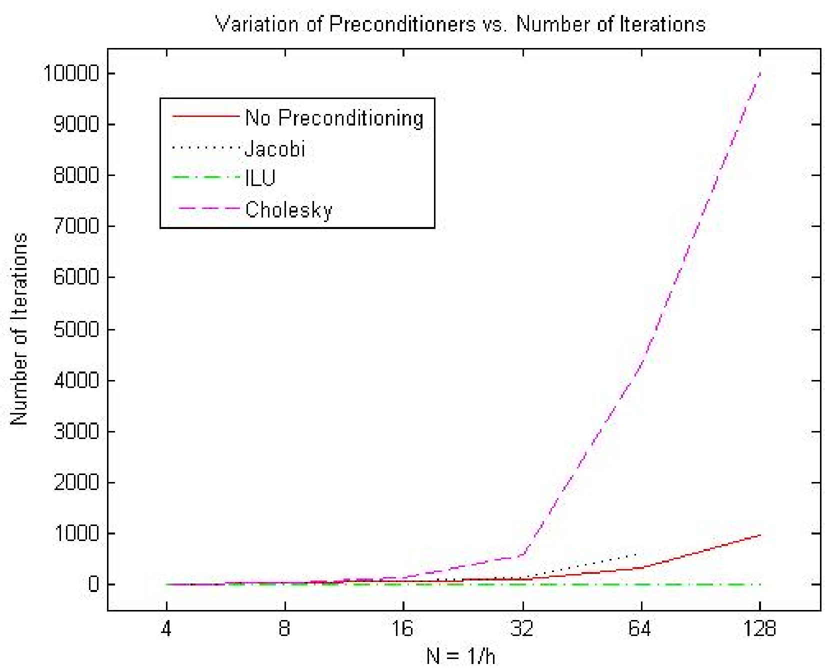


Figure 33: BI-CGSTAB EPS = 0 SIG = 10 TOL = 1.0e-10

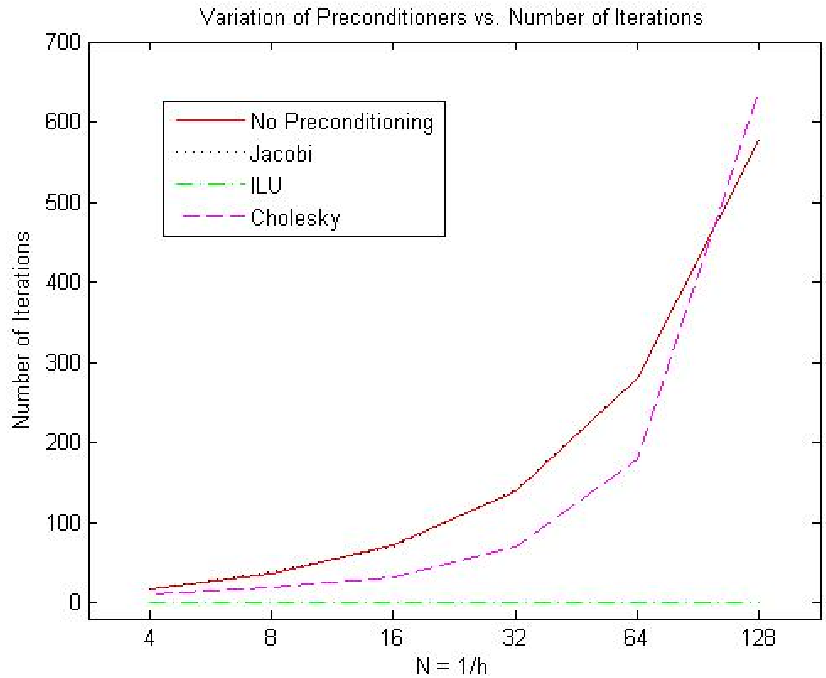


Figure 34: BI-CGSTAB EPS = 0 SIG = 100 TOL = 1.0e-10

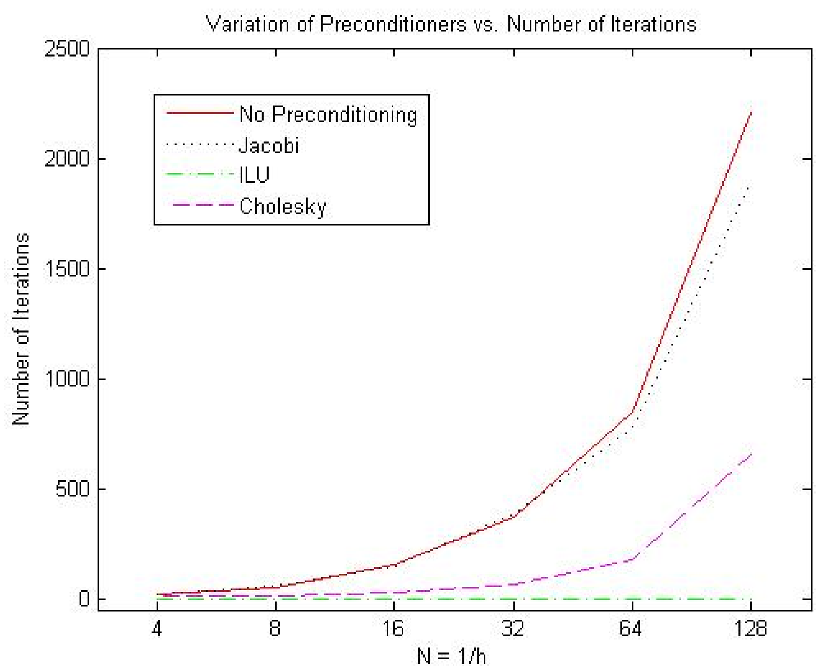


Figure 35: BI-CGSTAB EPS = +1 SIG = 0 TOL = 1.0e-10

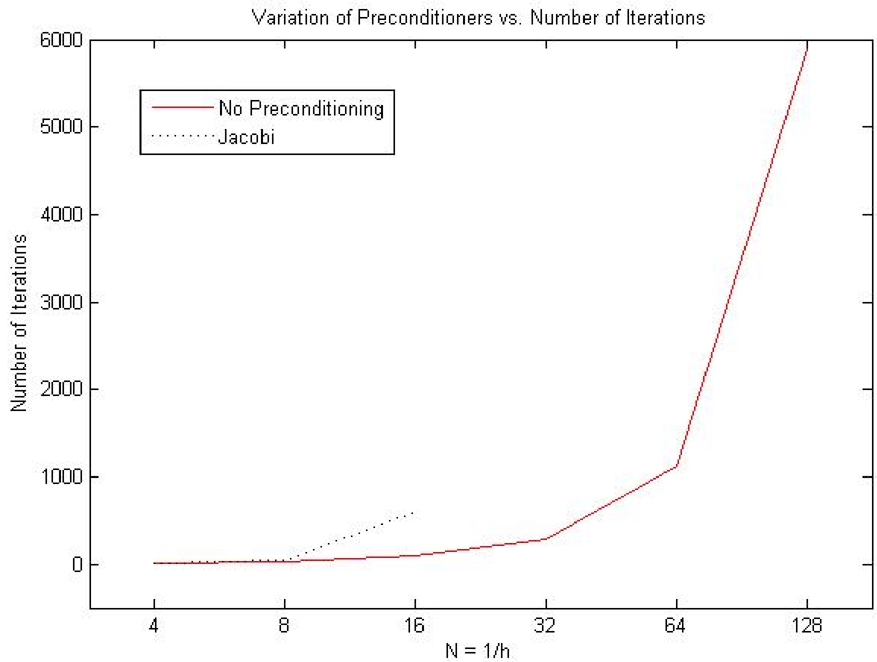


Figure 36: BI-CGSTAB EPS = +1 SIG = 0.01 TOL = 1.0e-10

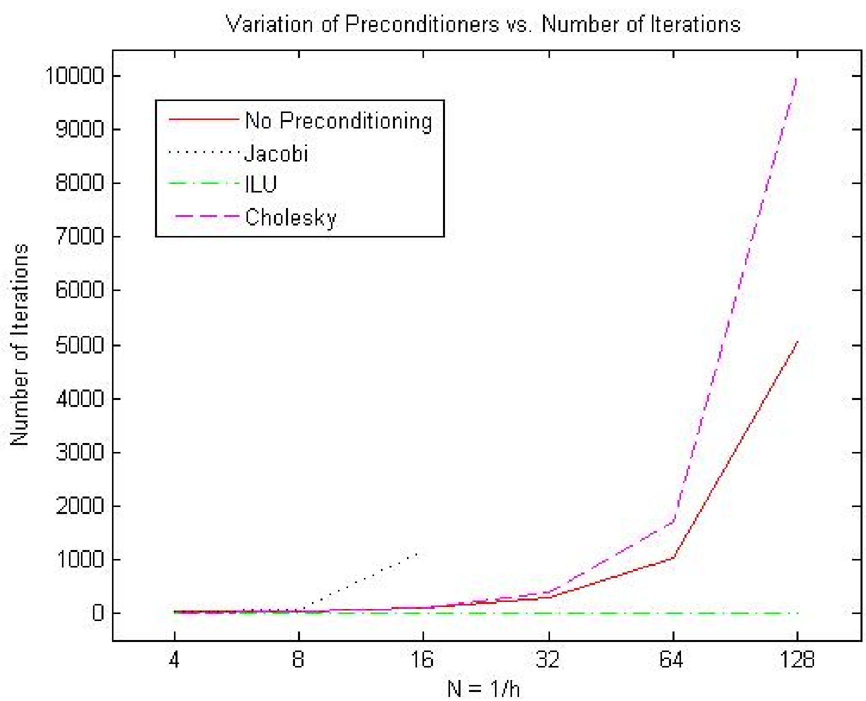


Figure 37: BI-CGSTAB EPS = +1 SIG = 0.1 TOL = 1.0e-10

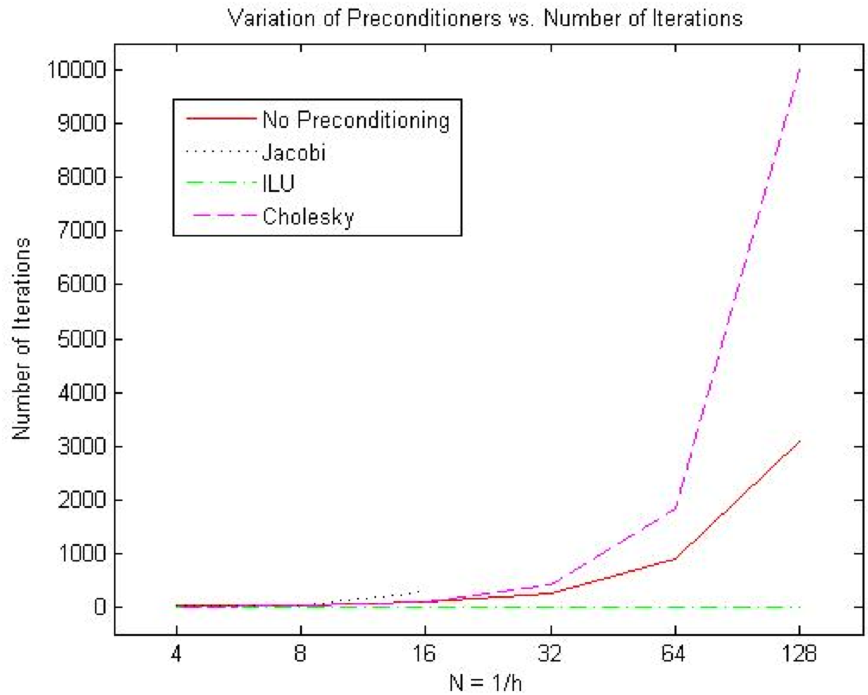


Figure 38: BI-CGSTAB EPS = +1 SIG = 1 TOL = 1.0e-10

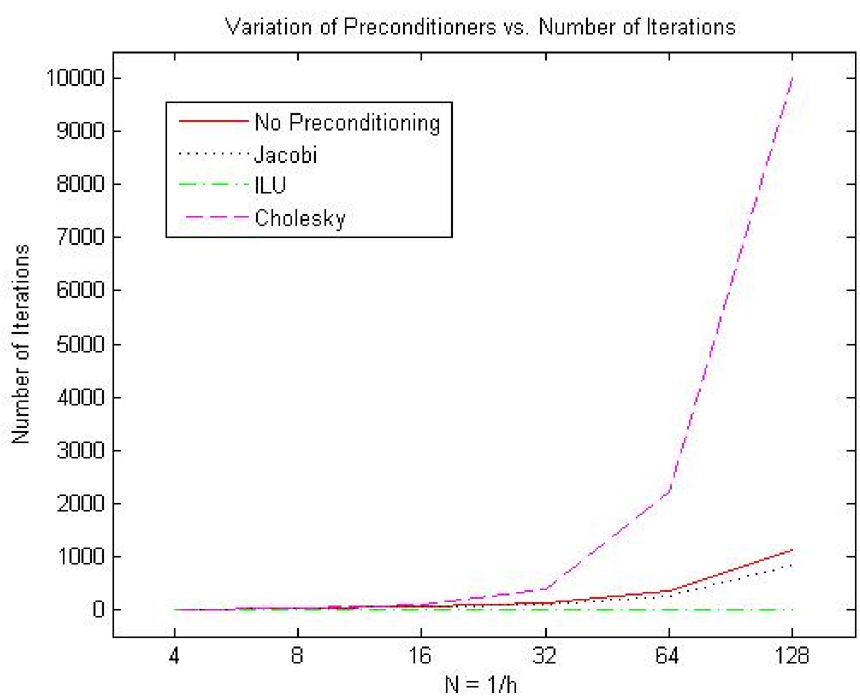


Figure 39: BI-CGSTAB EPS = +1 SIG = 10 TOL = 1.0e-10

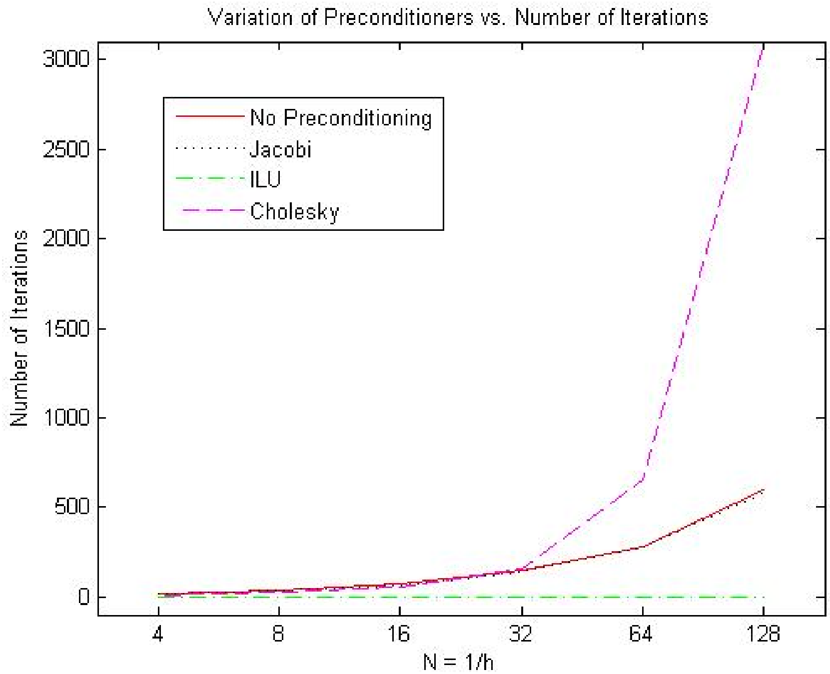
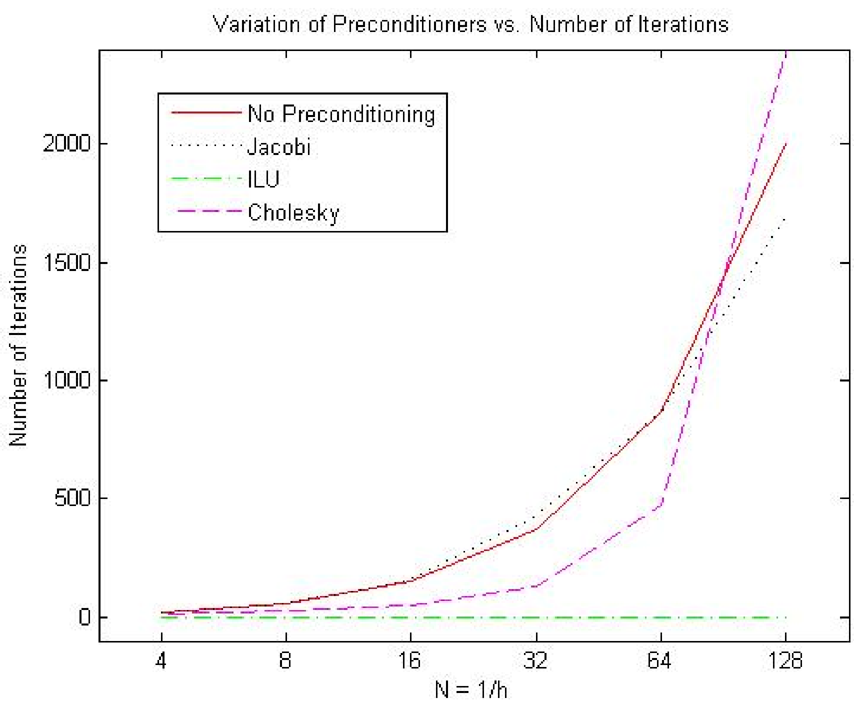


Figure 40: BI-CGSTAB EPS = +1 SIG = 100 TOL = 1.0e-10



6.0 CONCLUSIONS

The following are the main conclusions to the numerical simulations presented in the previous chapters. These conclusions were obtained from a detailed study of the data. The detailed study is presented in Chapters 6.1 through 6.3. One important thing to note is that the GMRES that is used for the numerical simulations has a restart value of 30 (as stated in Chapter 2.2). Much better convergence results can be obtained for GMRES if the restart value is increased. For instance (although the data is not presented in this thesis) when the restart value is increased to 200, GMRES converges very nicely in the cases in which it diverged in the presented data.

Conclusion for SIPG and no Preconditioning:

The CG method is by far the best method of the three. It converged in every case, and with the fewest number of iterations. The second best method was Bi-CGStab. The only case in which it did not converge was for SIG = 0 and h = 1/128. The worst method was GMRES. It did not converge on a fine mesh for every SIG.

Best Converging Method and Case for SIPG and no Preconditioning:

CG with SIG = 10 converged in 431 iterations and produced an L2 error of 8.6e-9 for h = 1/128.

Conclusion for IIPG and no Preconditioning:

Bi-CGStab is the best method to use. The only cases in which it sometimes did not converge was for SIG = 0.01 and 0.1. GMRES does not perform well for IIPG. It only converged for all h once, and it was when SIG = 1. When it did converge in this case (with SIG = 1 and h = 1/128) it took 6,201 iterations. This is more than six times the number of iterations Bi-CGStab used to converge.

Best Converging Method and Case for IIPG and no Preconditioning:

Bi-CGStab with SIG = 10 converged in 577 iterations and produced an L2 error of 5.0e-7 for h = 1/128.

Conclusion for NIPG and no Preconditioning:

Bi-CGStab is the best method to use. It converged in every case, and converged with the fewest number of iterations as compared to GMRES. GMRES does not perform well for NIPG. It only converged for all h once, and it was when SIG = 1. When it did converge in this case (with SIG = 1 and h = 1/128) it took 8,306 iterations. This is more than 10 times the number of iterations Bi-CGStab used to converge.

Best Converging Method and Case for NIPG and no Preconditioning:

Bi-CGStab with SIG = 10 converged in 593 iterations and produced an L2 error of 1.1e-6 for h = 1/128.

Conclusion for SIPG and Preconditioning:

Jacobi preconditioning is not effective for neither CG nor Bi-CGStab. It only aids convergence for GMRES in the case of SIG = 100. The other preconditioners must be disregarded because they compute the inverse matrix and solve the equation $x = A^{-1}b$ for x in one iteration. This does not qualify as an iterative method.

Conclusion for IIPG and Preconditioning:

Jacobi preconditioning only aids GMRES but not Bi-CGStab. Cholesky causes improvements to both GMRES and Bi-CGStab. ILU must be disregarded as a preconditioner.

Conclusion for NIPG and Preconditioning:

Jacobi causes improvement for GMRES but not Bi-CGStab. Cholesky aids GMRES but not Bi-CGStab. The best preconditioner for NIPG (for Bi-CGStab and GMRES) is Jacobi. ILU must be disregarded as a preconditioner.

6.1 CG WITHOUT PRECONDITIONING

SIPG

Converges for all SIG and for all h.

Comments: In all cases the number of iterations required for convergence stays consistent reaching a peak at about 600 to 700 iteration, except for when SIG = 1 and h = 1/128. In this case the number for required iterations is almost 2,000; much greater than any of the other cases.

Best Convergence Case: On a fine mesh, this method converges the best when SIG = 10. It converges in 431 iterations in this case when N = 128.

6.1.1 CG with Preconditioning

Preconditioner Jacobi:

SIPG

Does not converge for SIG = 0.01, 0.1 and 1 for all h.

Converges otherwise

Comments: When Jacobi-CG converges it takes about the same or more iterations to converge than CG without preconditioning. Thus Jacobi is not a good preconditioner for CG for DG method.

Preconditioner ILU:

SIPG

Does not converge for SIG = 0.01 and h = 1/4, … ,1/32, and 1/64.

Does not converge for SIG = 0.1 and h = 1/4 and 1/8.

Converges otherwise with one iteration.

Comments: The fact that with ILU preconditioning CG converges in one iteration (when the preconditioner is not indefinite) causes suspicion about whether ILU is computing the global matrix A as the preconditioning matrix. Indeed, after further numerical experiment (not presented in the data tables), we can conclude that ILU is computing the global matrix A as the preconditioning matrix. Thus, we must disregard ILU as a preconditioner for CG.

Preconditioner Cholesky:

SIPG

Does not converge for SIG = 0.01 and h = 1/4, … ,1/32, and 1/64.

Does not converge for SIG = 0.1 and h = 1/4 and 1/8.

Converges otherwise with one iteration.

Comments: The reason for convergence in one iteration (when the preconditioning matrix was not indefinite) is that Cholesky will produce precisely the global matrix A as the preconditioning matrix in the case of the SIPG since A is symmetric positive definite (refer to Chapter 3.3). Thus, we must disregard Cholesky as a preconditioner for the CG method.

6.2 GMRES WITHOUT PRECONDITIONING

SIPG

Does not converge for SIG = 0.01, 0.1, and 100 for h = 1/64 and 1/128.

Does not converge for SIG = 1 and h = 1/32, 1/64, and 1/128.

Does not converge for SIG = 10 and h = 1/128.

Converges otherwise.

IIPG

Does not converge for SIG = 0.01 and h = 1/16, 1/32, 1/64, and 1/128.

Does not converge for SIG = 0.1 and h = 1/64 and 1/128.

Does not converge for SIG = 10 and 100 and h = 1/128.

Converges otherwise, in particular for SIG =1 and all h.

NIPG

Does not converge for SIG = 0, 0.01, 0.1, 10, and 100 and h = 1/128.

Converges otherwise, in particular for SIG =1 and all h.

Comments: NIPG performs better overall on finer meshes. If all three methods converge, the number of iterations is about the same.

Best Convergence Case: On a fine mesh, this method converges the best for IIPG when SIG = 1. It converges in 6201 iterations in this case when N = 128.

6.2.1 GMRES with Preconditioning

Preconditioner Jacobi:

SIPG

Does not converge for SIG =0.01, 0.1, and 1 and h = 1/16, 1/32, 1/64, and 1/128.

Does not converge for SIG = 10 and 100 and h = 1/128.

Converges otherwise.

IIPG

Does not converge for SIG = 0.01 and h = 1/32, 1/64, and 1/128.

Does not converge for SIG = 0.1, 10, and 100 and h = 1/128.

Converges otherwise, in particular for SIG = 1 and all h.

NIPG

Does not converge for SIG = 10 and 100 and h = 1/128.

Converges otherwise, in particular for SIG = 0 and all h.

Comments: Jacobi-GMRES works better for NIPG, then for IIPG, then for SIPG. The number of iterations is smaller for NIPG. But if we compare GMRES alone with Jacobi-GMRES there is not a lot of gain. Both converge with the same number of iterations. Thus, we conclude that Jacobi is not a good preconditioner for GMRES for DG method.

Preconditioner ILU:

SIPG

Converges for all SIG and all h in one iteration.

IIPG

Converges for all SIG and all h in one iteration.

NIPG

Does not converge for SIG = 0 and all h.

Converges otherwise, in one iteration.

Comments: After further computer simulations we find that ILU is computing the global matrix as the preconditioner. Thus, we must disregard it as a preconditioner.

Preconditioner Cholesky:

SIPG

Converges in one iteration for all SIG and all h.

IIPG

Does not converge for SIG = 0.01 and 0.1 and h = 1/64 and 1/128.

Does not converge for SIG =1 and h = 1/32, 1/64, and 1/128.

Does not converge for SIG = 10 and 100 and h = 1/128.

Converges otherwise, and as SIG increases Cholesky-GMRES performs better.

NIPG

Does not converge for SIG = 0 and all h.

Does not converge for SIG = 0.01, 0.1, and 1 and h = 1/32, 1/64, and 1/128.

Converges otherwise, and as SIG increases Cholesky-GMRES performs better.

Comments: We must disregard Cholesky-GMRES for the SIPG case. Cholesky-GMRES works better for IIPG, then for NIPG. If we compare IIPG to NIPG with SIG = 10, the method converges 10 times faster for IIPG when h = 1/32; and 7 times faster when h = 1/64.

6.3 BI-CGSTAB WITHOUT PRECONDITIONING

SIPG

Does not converge for SIG = 1 and h = 1/128.

Converges otherwise.

IIPG

Does not converge for SIG = 0.01 and h = 1/64 and 1/128.

Does not converge for SIG = 0.1 and h = 1/32, 1/64, and 1/128.

Converges otherwise.

NIPG

Converges for all SIG and all h.

Comments: NIPG performs better overall on finer meshes. When all three methods converge, the number of iterations is about the same.

Best Convergence Case: On a fine mesh, this method converges the best for IIPG when SIG = 10. It converges in 577 iterations in this case when N = 128.

6.3.1 Bi-CGStab with Preconditioning

Preconditioner Jacobi:

SIPG

Does not converge for SIG = 0.01 and h = 1/32, 1/64, and 1/128.

Does not converge for SIG = 0.1 and h = 1/16, …, 1/128.

Does not converge for SIG = 1 and h = 1/32, …, 1/128.

Converges otherwise, in particular for SIG = 10 and 100 and all h.

IIPG

Does not converge for SIG = 0.01 and h = 1/32, …, 1/128.

Does not converge for SIG = 0.1 and h = 1/32, …, 1/128.

Does not converge for SIG = 1 and h = 1/128.

Converges otherwise, in particular for SIG = 10 and 100 and all h.

NIPG

Does not converge for SIG = 0 and h = 1/32, …, 1/128.

Does not converge for SIG = 0.01 and h = 1/32, …, 1/128.

Does not converge for SIG = 0.1 and h = 1/32, …, 1/128.

Converges otherwise, in particular for SIG = 1, 10, and 100 and for all h.

Comments: Jacobi-BICGSTAB does not perform as good as BICGSTAB alone for SIPG, and only slightly better for IIPG and NIPG when the mesh is fine. Thus Jacobi is not a good preconditioner for Bi-CGStab for the DG method.

Preconditioner ILU:

SIPG

Converges for all SIG and all h in one iteration.

IIPG

Converges for all SIG and all h in one iteration.

NIPG

Does not converge for SIG = 0 and all h.

Converges otherwise in one iteration.

Comments: ILU-BICGSTAB works exceptionally well in all cases except when SIG = 0.

Preconditioner Cholesky:

SIPG

Converges for all SIG and all h in one iteration.

IIPG

Does not converge for SIG = 1 and h = 1/128.

Converges otherwise.

NIPG

Does not converge for SIG = 0 and all h.

Does not converge for SIG = 0.01 and h = 1/128.

Does not converge for SIG = 0.1 and h = 1/128.

Does not converge for SIG = 1 and h = 1/128.

Converges otherwise, in particular for SIG = 10 and 100 for all h.

Comments: Cholesky-BICGSTAB works best for SIPG, then IIPG, then NIPG. Cholesky is a good choice for a preconditioner for SIPG, but not a good choice for NIPG or on a fine mesh with IIPG.

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