THREE ESSAYS ON SEQUENTIAL AUCTIONS

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This dissertation examines the reasons for which a seller may decide to conduct a multi-unit auction sequentially rather than simultaneously. It analyzes the manner in which the information generated during a sequential auction can affect bidding to the seller's benefit and demonstrates the requirement of intertemporal commitment to the auction rules.

When the seller cannot commit not to alter the reserve price over time, the bidders are reluctant to reveal their valuations. Therefore, with single-unit demands, a symmetric monotone equilibrium exists only in a sequential Dutch auction. In the earlier rounds of this auction, because of the anticipation of lower reserve prices in the future, some buyers prefer not to submit a bid, although their valuations exceed the requested reserve price. Furthermore, any buyer submitting a bid shades it sharply. Consequently, under imperfect commitment, the optimal sequential auction results in lower expected revenue than its simultaneous counterpart.

In the presence of allocative and informational externalities, in particular, in a sale of two oligopoly licenses, a sequential auction succeeds in eliminating some of the payoff uncertainty by allocating the licenses in an ordered manner, according to the bidders' strength. Therefore, the weaker oligopolist can acquire his license at a lower price than the one he would pay in a simultaneous auction. In addition, he can avoid overpaying. Conversely, the stronger oligopolist pays a higher price, so that, when the bidders' production costs are independent, the two auction schemes generate the same expected revenue. Therefore, without affecting the seller's revenue or efficiency objectives, the sequential auction results in a more equal distribution of the wealth generated by the oligopoly.

Finally, when the preparation or submission of bids is costly, so that a buyer will not enter the auction unless he expects a substantial gain from it, low prices in the earlier rounds of a sequential sale trigger stronger participation, and, consequently, higher prices in the later rounds. A sequential auction, therefore, may result in higher expected revenue than a simultaneous sale, especially when the number of potential bidders is large, the participation cost small, or the distribution of valuations convex.

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PREFACE

I wish to express my deepest feelings of gratitude to my supervisor, professor Andreas Blume. Throughout the course of my doctorate study, he offered me his advice in the most generous manner; and at the same time, he granted me plenty of freedom, so as to learn to work, especially to develop my ideas, in an independent manner. He is a role model for me, an example to follow in the future. Indeed, I consider myself most fortunate to be his student.

In addition, I would like to thank the members of my PhD committee, professors Oliver Board, Esther Gal-Or, Paul J. Healy and Jack Ochs. They patiently read my papers and listened to my presentations; and their insightful comments, in particular, their challenges of my assumptions and their inquiries upon the validity of my insights, forced me to sharpen my earlier arguments as well as to investigate new directions. Much of their advice has been incorporated to this dissertation. The rest of it will be the subject of future research.

Some of the material on my dissertation has received the benefit of being discussed in various conferences and university seminars. Chapter 2 was presented in the Fall 2005 Midwest Economic Theory Conference, at the University of Kansas. Chapter 3 was presented at the universities of Bielefeld, Hannover and St Andrews. I wish to thank the audiences of my presentations and the people who invited me.

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Finally, now that my formal education is about to be concluded, I cannot escape lapsing momentarily into nostalgia, remembering my earliest years at school and the people who either supported or simply accompanied me in my first efforts to learn. During these years, I have had the good fortune to be in an environment that encouraged idealism. No doubt, its influence has been a long lasting one.

1.0 INTRODUCTION

Several multi-unit sales are typically conducted by means of sequential auctions, carried out either in rapid succession or over long periods of time. For example, wine, art, condominium units, used cars, agricultural products and fish are often auctioned sequentially. On internet auction sites, sellers often auction in a sequence many units of the same consumer product. Furthermore, procurement contracts are also auctioned sequentially, as the need for each project arises. Finally, the recent radio spectrum auction in the United States, as well as similar spectrum auctions in numerous other countries, was conducted by means of a dynamic procedure.[1]

The present dissertation examines the reasons for which a seller may decide to organize a sequential rather than a simultaneous auction, or vice versa. In particular, it compares the performance of these two auction formats in three different environments:

a. When the seller cannot commit not to alter the auction rules over time.

b. In the presence of allocative and informational externalities.

c. When the bidders face costs of preparing or submitting their bids.

In each case, a symmetric monotone equilibrium for the sequential auction is constructed, its properties are analyzed and its outcome is compared to that of a similar equilibrium for the simultaneous auction.

[1]The examples mentioned here have been documented by numerous empirical studies. Sequential auctions of wine have been studied by Ashenfelter [2] and Ginsburgh [36]; art, by Pesando [83], Beggs and Graddy [8] and Ashenfelter and Graddy [4, 5, 6]; real estate, by Ashenfelter and Genesove [3]; cars, by Genesove [34] and Raviv [85]; Dutch roses, by van den Berg et al. [9]; dairy cattle, by Engelbrecht-Wiggans and Kahn [25]; and fish, by Pezanis-Christou [84]. Sequential auctions on the internet have been studied, among others, by Zeithammer [97]. The procurement of highway paving contracts in California has been analyzed by Jofre-Bonet and Pesendorfer [54]. Finally, the radio spectrum auctions in the United States have been surveyed by Cramton [18, 19]; for similar auctions in Europe, see Jehiel and Moldovanu [47] and Klemperer [61].

In all of the above settings, we assume that there is a fixed number of identical objects being put for sale as well as a fixed number of potential buyers. Each buyer has single-unit demand and no time preference over the sequence of auctions. Finally, the buyers' private information is distributed identically, in a statistically independent manner. These general assumptions, which are typical of much of the literature on sequential auctions, describe realistically some of the real-world problems that we wish to analyze. In other problems, they can be accepted only as a compromise between realism and tractability. When such a compromise is not necessary, relaxing these assumptions can be the subject of future research.

In the case of imperfect intertemporal commitment, studied in chapter 2, the buyers are concerned about the consequences of revealing, by means of their bids, information to the seller. Therefore, a symmetric monotone equilibrium can be constructed only in a sequential Dutch auction. In this equilibrium, the seller lowers the second-period reserve price, according to the outcome of the first auction. Anticipating the seller's second-round behavior, several buyers prefer not to bid for the first object, even though their valuation exceeds the requested reserve price. Because of this reduction in competition, any buyer willing to submit a first-period bid shades it sharply. Consequently, the optimal sequential auction results in lower expected seller revenue than its simultaneous counterpart (or the sequential auction with commitment to the optimal reserve-price schedule). In addition to justifying the use of a simultaneous sale, this conclusion may also explain the adoption by the seller of costly commitment measures, such as the signing of a formal contract or the employment of a neutral auctioneer.

Chapter 3 compares a sequential and a simultaneous auction of two oligopoly licenses. Since the value of each license depends on the identity of both winners as well as on the information revealed by the wining bids, these are auctions with allocative and informational externalities. The sequential procedure creates two informational effects, both of which are absent from the simultaneous sale. First, the revelation of the first-round winning bid allows the remaining firms to form better estimates of their valuations. Thus, the possibility of overpayment by the winner, that is, of acquiring a license at a price that exceeds its revealed value, is eliminated. Second, by allocating the licenses in an ordered manner, according to the winners' strength, the sequential auction appends the first license with the promise of a stronger market presence, a promise that is realized on the equilibrium path; and it sells the second license along with the knowledge of weaker market presence. As a result, relative to the simultaneous auction, the bidders compete more aggressively for the first license; and less aggressively for the second one. Therefore, in the sequential auction, the stronger of the two oligopolists acquires his license at a higher price than the one he would pay in the simultaneous auction; while the weaker oligopolist pays a lower price. Eventually, the two auctions result in the same expected seller revenue and market outcome. Hence, without affecting the sellers' revenue or efficiency objectives, the sequential auction succeeds in distributing more equally the wealth created by the oligopoly.

The information revealed by the first-round bids turns out to affect the seller's revenue when the bidders face costs of preparing or submitting their bids, the problem studied in chapter 4. In this environment, a buyer will not participate in the auction, unless he expects a substantial gain from it. In equilibrium, therefore, each buyer adopts a cut-off strategy, according to which he enters the auction if and only if his valuation for the object exceeds a certain threshold. In a sequential auction, low prices in the earlier rounds can result in stronger participation and, consequently, higher prices in the later rounds. On the other hand, because of the opportunity of delaying one's entry, the first-round participation rate is lower than that of a simultaneous auction. The first effect turns out to be more important, so that the sequential auction can generate a higher expected revenue for the seller, in many cases, especially when the number of potential bidders is large, the participation cost small or the distribution of the buyers' valuations convex.

Primarily, this dissertation shows how the information present in a sequential (but not in a simultaneous) auction can affect bidding to the seller's benefit. In particular, two informational effects are identified, a direct one, originating from the information revealed by the buyers' actions during the earlier rounds of the auction, and an indirect one, originating from the type of equilibrium describing the buyers' behavior in the entire auction. This information can affect the distribution of the prices that the winners of the auction pay as well as the seller's overall expected revenue. Finally, this dissertation also shows that the profitable use of a sequential auction requires from the seller the ability to commit intertemporally to the auction rules.

It should be mentioned that the three main chapters of this dissertation have been organized independently of one another. In particular, the study of each chapter does not require any knowledge of the material discussed in the remaining parts of the dissertation. Although this may occasionally create some duplication, especially in the references to the literature, at the end, the advantage of referring directly to each of the problems appears to be of greater importance.

2.0 OPTIMAL RESERVE PRICES IN SEQUENTIAL AUCTIONS WITH IMPERFECT COMMITMENT

2.1 INTRODUCTION

In the study of sequential auctions, a usual assumption is that of intertemporal commitment. In particular, the seller is able to commit in a fully credible manner to a specific auction mechanism, or to a sequence of auction mechanisms, through which all sales will be made. The bidders can therefore reveal, in the earlier rounds of the auction process, private information about their valuations, without fearing that the seller will change the rules, for example, alter the reserve price, at their expense.

In reality, however, perfect intertemporal commitment is often infeasible. In many cases, the seller lacks the credibility that this assumption requires. In addition, even when he commits to a certain selling scheme by means of a contract, he may still decide to redesign the auction at a cost, if it becomes profitable for him to do so. Finally, in several cases in which the seller announces the auction of only a single item, the bidders may try to conceal some of their private information, in anticipation of similar sales in the future.

In this chapter, we study the effects of imperfect commitment. In particular, we examine the sequential auction of two identical goods by a seller who can change the reserve price for the second object after observing the outcome of the first sale. The same group of potential buyers is assumed to be present in both auctions; and each buyer has single-unit demand and a valuation that is persistent over time. The seller is therefore able to take full advantage of any information that the buyers may reveal in the first auction.

The problem we have described typically occurs in sequential procurement. In this setting, a number of contracts for similar projects, for example, for highway paving, is procured sequentially. Capacity constraints force the competing firms to limit the supply of their services.[1] If the buyer's commitment not to alter the rules pertaining the future auctions cannot be fully credible, then the issues involved in the design of the optimal procurement process will be identical to the ones we study.

The seller is restricted to using standard single-unit auctions, with the actual auction format fixed exogenously. In particular, we will consider the cases of English, Dutch, and sealed-bid, first- or second-price auctions. For a specific auction format, the seller can choose reserve prices r_1 and r_2 so as to enhance his expected payoff.

We introduce imperfect commitment by restricting the manner in which the seller can set the second-period reserve price r_2. More specifically, the seller cannot credibly commit at the beginning of the game to any reserve price, or rule for determining the reserve price, for the second auction. Rather, he must choose the reserve price r_2 at the beginning of the second period, after the end of the first auction.[2]

The impossibility of intertemporal commitment has important consequences for the bidders' behavior in the first round of the auction. Since the auctioneer cannot restrict the manner in which he will use in the second round the information revealed in the first round, the bidders have strong incentives not to reveal their valuations. In particular, the non-winning bidders are best-off not submitting any bid.

This incentive to conceal one's private information turns out to be extremely strong. For several auction formats, in particular, for the sealed-bid and the English auctions that we consider, there is no symmetric equilibrium in monotone bidding strategies. The auctioneer's unrestricted use of the revealed information forces any bidders who might wish to participate in the first round not to shade their bids (beyond the degree characterizing the equilibrium of the corresponding single-unit auction). As a result, a deviation to non-participation (or to minimal bidding) during that round becomes strictly profitable.[3]

[1] The relevance of such constraints has been documented by Jofre-Bonet and Pesendorfer [54], in their study of highway paving contracts in California.

[2] In this setting, therefore, imperfect commitment assumes an extreme form, namely, that of non-commitment. In section 2.6, we show that all intermediate situations, in which the seller is allowed to change the reserve price with some positive probability, are qualitatively identical to non-commitment.

[3] The cause of this non-existence result can be thought off as a variation of the "ratchet effect" from the literature of dynamic contracting. The agents are reluctant to reveal any information that the principal can use at their expense in the future. For further details, see Freixas et al. [28] and Laffont and Tirole [63, 64].

For a symmetric monotone equilibrium, one shall consider an auction format that allows the non-winning bidders to avoid revealing information. Such a format is that of the Dutch auction. In this auction, there is only one bid, that of the winner. The remaining bidders do not take any action in the first period, other than that of their refusal to bid above the winner's price. Therefore, their private information remains protected, thus enabling the construction of an equilibrium.

In this equilibrium, the auctioneer always sets a first-period reserve price that allows the sale of the first object, if a bidder with a sufficiently high valuation exists. Subsequently, given the outcome of the first auction, he updates his beliefs (in particular, he obtains a sharper upper bound for the remaining bidders' valuations) and, accordingly, he lowers the reserve price for the second object. Because of the anticipation of a lower future reserve price, several bidder types do not submit a bid in the first auction, even though their valuations exceed the first-period reserve price.[4] Consequently, those bidders who participate in the first round shade their bids sharply, knowing that they face limited competition. Both the strategic non-participation and the excessive shading of the submitted bids would be absent, if the auctioneer could commit not to change, in particular, not to lower, the reserve price over time.

Overall, the seller suffers a revenue loss. Although he is able to design the second auction in a better informed manner and, therefore, to derive a higher revenue from the sale of the second object than in the case of commitment, he cannot prevent severe losses in the first auction. Thus, we reaffirm the intuition favoring commitment[5], in the setting of sequential auctions. In particular, the seller would be willing to adopt costly measures to enhance his credibility, for example, he would be willing to pay for the services of a trusted intermediary, like a well-established auction house.

The literature on sequential auctions has paid relatively little attention to the possibility of a strategic auctioneer. Some of this literature has tried to explain the declining-price

[4] This strategic non-participation decision first appeared in the literature of dynamic bargaining; for example, see Hart and Tirole [39]. For examples of its occurrence in sequential auctions, see McAfee and Vincent [75] and Caillaud and Mezzetti [14]. In our case, we shall remark that it differs from the ratchet effect, despite its resemblance to it. A buyer decides not to participate in the first auction even though the seller cannot use against him, in the future, the information revealed by his bid.

[5] This result was formally established by Stokey [91] and Bulow [11].

anomaly[6], a problem in which the auctioneer plays no strategic role. Learning in sequential auctions has been studied by Ortega Reichert [82] and Jeitschko [52]. They concentrate, however, on the manner in which the bidders, rather than the auctioneer, can use the information revealed during a sequential auction. A strategically active auctioneer is present in McAfee and Vincent [74], who study the optimal reserve-price path in a sequence of first- and second-price auctions.[7] In particular, the auctioneer puts the same object for sale repeatedly, until it is sold. At each round he chooses a reserve price according to his (increasingly pessimistic) beliefs about the buyers' valuations. Since the game ends as soon as the object is sold, that is, as soon as a bid is submitted, the buyers do not face the problem of hiding their valuations. Prior to the end of the game, information can be revealed only in a passive manner, by the buyers' refusal to bid for the object at a given reserve price.

The issue of concealing, during the auction, information from the auctioneer has appeared in Caillaud and Mezzetti [14]. In a sequence of two auctions in which the buyers have multi-unit demands and persistent valuations, the bidders face a problem similar to the one in our setting. However, because of the multi-unit demands, the need to conceal information eventually concerns only the bidder with the highest valuation, all other bidders realizing that they cannot win either of the two auctions.[8] Therefore, it is the format of the English auction that is employed in this setting, as it allows the winning bidder not to reveal his valuation. Our work complements that of Caillaud and Mezzetti [14] by looking at an environment in which the problem of concealing information is faced by the non-winning bidders.

In both settings, it turns out that relevant information can be revealed only indirectly, through the actions of the bidders that have no further interest in the game. In addition, in both cases, imperfect commitment is costly for the seller. In our setting, however, the seller's loss is enhanced by the bid shading that occurs in the first auction.[9]

[6]Although intertemporal non-arbitrage requires that the expected prices remain constant over time, in practice it has been observed that earlier sales tend to be concluded at higher prices than later ones. For details, see Ashenfelter [2], Jeitschko [53], MacAfee and Vincent [74] and Milgrom and Weber [79].

[7]Sobel and Takahashi [89] have studied a similar problem in the context of dynamic bargaining. Skreta [88] has generalized this problem, with respect to the mechanisms that the seller can use in every period.

[8] In fact, on the equilibrium path, if there is a winning bidder in the first auction, then the second auction becomes a bargaining problem between the seller and that bidder.

[9]In Caillaud and Mezzetti [14], the participating bidders are always willing to bid up to their valuation, so as to overbid their competitors. Therefore, any revenue loss for the seller comes directly from the decrease in participation.

Finally, a different type of strategically minded auctioneer appears in Thomas [94] and Tu [95]. This auctioneer tries to maximize his expected revenue by controlling the information that is made available to the bidders at the end of the first auction. He does not set any reserve prices nor make any decisions during the game. Therefore, the issues involved in his interaction with the bidders differ from the ones in our setting.

In the sections 2.2 and 2.3, we present the model describing our problem and we summarize the results regarding optimal reserve prices under commitment of the auctioneer. In section 2.4, we examine the sequential Dutch auction and we derive a symmetric sequential equilibrium in weakly increasing first-period strategies. In section 2.5, we show that such an equilibrium does not exist in sequential English or sealed bid first- and second-price auctions. In section 2.6, we extend this non-existence result to the case of imperfect commitment, that is, to a setting in which the auctioneer can change the second-period reserve price with a small probability only, which reflects his lack of credibility. We conclude in section 2.7. Lengthy proofs have been placed in section A of the appendix.

2.2 GENERAL MODEL

There is one auctioneer with 2 identical objects for sale. The auctioneer's valuation for the objects is normalized to zero, so that he can derive no benefit from any object that may remain unsold.

The auctioneer faces $N > 2$ bidders, indexed by $i = 1, \ldots, N$. Each bidder has single-unit demand and private valuation $v_i \, \epsilon \, [\underline{v}, \bar{v}]$, for $0 \leq \underline{v} < \bar{v}$, which remains constant throughout the game. The valuations are independently drawn, according to a common distribution function $F : [\underline{v}, \bar{v}] \rightarrow [0, 1]$. We assume that the distribution function F is differentiable and that its derivative, $f : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}^{+}$, has full support. The payoff of bidder i, in case he wins one unit, equals his valuation v_i minus the price that he pays for it; otherwise, if he does not win any unit, it equals to zero.

The two objects are allocated to the bidders by means of a sequence of two standard auctions, conducted in periods (or rounds) $t = 1, 2$. For simplicity, we assume that the two auctions are of the same format.[10] In particular, we will consider the cases of sequential Dutch auctions, English auctions, and sealed-bid first- and second-price auctions.[11]

Given a certain auction format, the auctioneer can select reserve prices r_1 and r_2 so as to maximize his expected revenue. We assume that the auctioneer cannot restrict at the beginning of the game the manner in which the reserve price r_2 will be determined (impossibility of intertemporal commitment). Instead, he must choose r_2 at the beginning of the second period, according to the information revealed by the bids submitted in the first auction. His strategy, therefore, consists of a reserve price for the first auction and a reserve price rule for the second auction:[12]

$$
\begin{aligned}
&r_1 \,\epsilon\, \mathbb{R}^+; \\
&r_2 \;:\; (\, b_1^1, \,\ldots, \, b_N^1, \, r_1 \,) \;\longmapsto\; r^2 \,\epsilon\, \mathbb{R}^+.
\end{aligned}
$$

In each period, every bidder either submits a bid or abstains from the auction. His first-period bid depends on his valuation and the reserve price. His second-period bid, if he does not win the first auction, depends on his valuation, the new reserve price, and the information that has been revealed in the first auction. In the sealed-bid auctions, we assume that the auctioneer reveals directly[13] only the winning bid. On the other hand, in the open-bid auctions, the auctioneer does not have anything to reveal, since the bidders can observe all first-period behavior.

Given an information revelation scheme $h(b^1)$, for first-period bids $b^1 = (b_1^1, \ldots, b_N^1)$, determined by either the auction format or the auctioneer's information revelation policy, the strategy of each bidder i consists of the bidding rules

[10] Since the outcome of the entire auction will depend only on the auction format used in the first period (in particular, on the type of information that the first-period auction reveals), our results easily extend to the case of different auction formats in the two periods.

[11] In the sequential Dutch auction, the auction clock is set high, above $\bar{v}$, at the beginning of each round. The second item can therefore be sold at a higher price than the first one. In the same manner, in the sequential English auction, the auction clock is set low, at the seller's reserve price, in each round.

[12] To simplify the notation, we have allowed b_i^1 to take the value of "no-bid" or "abstain".

[13] Indirectly, through his choice of the reserve price r_2, the auctioneer may reveal additional information. In our analysis, however, this issue will be of trivial importance. Our negative results regarding sequential sealed-bid auctions will remain valid independently of the auctioneer's information revelation policy.

$$\beta_i^1 \,:\, (\,v_i \,|\, r_1\,) \;\longmapsto\; \{a\} \cup [r_1, \infty);$$

$$\beta_i^2 \,:\, (\,v_i \,|\, b_i^1,\, h(b^1),\, r_1,\, r_2\,) \;\longmapsto\; \{a\} \cup [\,r_2, \infty),$$

where a denotes the action of abstaining from an auction.

The solution concept will be that of perfect Bayesian equilibrium. At each decision node, each player must behave optimally, given the other players' strategies and his beliefs. On the equilibrium path, these beliefs are determined by Bayes' rule. Off the equilibrium path, we strengthen the equilibrium concept by requiring that the players cannot infer from an observed action information that the acting player does not have.[14] For example, an off-equilibrium reserve price r_1 cannot alter the bidders' beliefs about their opponents' valuations. Finally, we restrict attention to symmetric[15], pure-strategy equilibria, in weakly increasing bidding strategies.

2.3 OPTIMAL SEQUENTIAL AUCTIONS UNDER COMMITMENT

If there is only one unit for sale, then, according to Myerson [81], the auctioneer will choose the optimal reserve price by considering the bidders' virtual valuation function

$$\psi(v) = v - \frac{1 - F(v)}{f(v)}.$$

We make the standard regularity assumption that the function $\psi(v)$ is increasing. For this assumption to hold, it is sufficient that the hazard rate $\frac{f(v)}{1-F(v)}$ is increasing.

Given the regularity assumption, the auctioneer can maximize his expected revenue by allocating the object by means of any standard auction with reserve price

[14] For a formal definition, consult Fudenberg and Tirole [30], definition 8.2.

[15] In fact, wherever it is applicable, we impose a stronger symmetry requirement, one that rules out the possibility of using the first-period bids as a labeling device. For example, we will not allow symmetric strategies that prescribe different second-period bidding behavior to the second and the third highest bidders of the first auction.

$$r_0 \;=\; \begin{cases} \psi^{-1}(0), & \text{if } \psi(\underline{v}) < 0; \\ \underline{v}, & \text{otherwise.} \end{cases} \tag{2.1}$$

That is, the auctioneer must exclude all bidders with valuations below $\psi^{-1}(0)$.

If the players receive prior information that the upper bound of the valuations is $\hat{v}$, then they update their beliefs, so that they consider the bidders' valuations as i.i.d., according to the distribution function $F(\cdot)/F(\hat{v})$ on $[\underline{v}, \hat{v}]$. In this case, the bidders' virtual valuations will be given by the function

$$\psi(v|\hat{v}) = v - \frac{F(\hat{v}) - F(v)}{f(v)}.$$

It is easy to check that the function $\psi(v|\hat{v})$ also satisfies the regularity assumption. Therefore, given such prior information, the auctioneer maximizes his expected revenue by setting a reserve price:

$$r_0(\hat{v}) \;=\; \begin{cases} \psi(\cdot\,|\,\hat{v})^{-1}(0), & \text{if } \psi(\underline{v}|\hat{v}) < 0; \\ \underline{v}, & \text{otherwise.} \end{cases} \tag{2.2}$$

Notice that the reserve price $r_0(\hat{v})$ is increasing in $\hat{v}$. Finally, for any optimal reserve price $r_0(\hat{v}) > \underline{v}$, the condition $\psi(r_0(\hat{v})|\hat{v}) = 0$ implies

$$F(\hat{v}) - F[r_0(\hat{v})] \;=\; f[r_0(\hat{v})]\; r_0(\hat{v}). \tag{2.3}$$

The optimal reserve price $r_0(\hat{v})$ equals to the inverse hazard rate at $r_0(\hat{v})$.

If there are two units for sale, then, because of the single-unit demand and the regularity assumption, the solution to the auctioneer's revenue optimization problem is similar to that of the single-unit case. By Maskin and Riley [70], the optimal selling mechanism takes the form of any standard multi-unit auction with reserve price $r_0 \in [\underline{v}, \bar{v}]$, defined as in the case of a single-unit auction.

Under full commitment, the restriction to sequential allocation plays no role, since the assumptions of risk neutrality and i.i.d. private valuations imply that all allocation equivalent equilibria of the sequential and the single-round auctions are revenue equivalent. Hence, any sequential auction that allocates the two units to the bidders with the two highest valuations, as long as these valuations exceed the reserve price r_0, is optimal for the auctioneer.[16] In particular, the auctioneer can maximize his expected revenue by conducting a sequential Dutch auction with the same reserve price $r_1 = r_2 = r_0$ in both rounds.[17]

2.4 SEQUENTIAL DUTCH AUCTIONS

In the sequential Dutch auction, there can be at most one bid in each period, that of the winner. Therefore, when the buyers follow monotone strategies, the information revealed by the first period outcome will take the form of an upper bound for the valuations of the remaining bidders.[18]

We show that the following strategies can be part of an equilibrium:

a. Given a first-period reserve price $r_1 < \bar{r}_1$, for a certain threshold $\bar{r}_1$, each bidder i follows a bidding strategy $\beta^1(\cdot \,|\, r_1)$, such that he participates in the auction only if his valuation is $v_i \in [\underline{v}(r_1), \bar{v}]$, for some value $\underline{v}(r_1) > r_1$. In addition, in the region of participation, $[\underline{v}(r_1), \bar{v}]$, the strategy $\beta^1(\cdot \,|\, r_1)$ is strictly increasing. Thus, the winning bidder fully reveals his valuation. If $r_1 \geq \bar{r}_1$, then, in equilibrium, no bidder participates in the first-period auction.

b. If the first-period object is sold at a price $\hat{b}^1 = \beta^1(\hat{v} \,|\, r_1)$, corresponding to a winning valuation $\hat{v} \,\epsilon\, [\underline{v}(r_1), \bar{v}]$, then the auctioneer and the bidders update their beliefs, so that the remaining bidders' valuations v_i are i.i.d., according to the distribution $F(\cdot)/F(\hat{v})$.

[16] Notice, in particular, that the optimality extends over sequential auctions in which the second-period reserve price is determined endogenously by the first-period bids, according to a reserve price schedule $r_2 = r_2(b_1^1, ..., b_N^1, r_1)$.

[17] A sequential English auction or a sequential first-price or second-price auction, with reserve prices $r_1 = r_2 = r_0$, is also optimal for the auctioneer. Any of the sequential auctions that we consider would also be optimal, if the auctioneer revealed all the bids submitted in the first round.

[18] We shall notice that there is a possibility of more than one bids in a Dutch auction, in case of a tie between two or more bidders. In this case, the valuation of some of the non-winning bidders will be fully revealed. Given our assumptions regarding $F(\cdot)$, this event has zero measure and can be ignored.

The auctioneer sets a new reserve price r_2, according to the updated virtual valuation function, and the bidders bid according to the standard first-price auction bidding strategies. Unless $r_1 \leq \underline{v}$, the new reserve price, r_2, is strictly lower than the first-period reserve price r_1. If the first-period object remains unsold at the reserve price r_1, then the same argument applies, with $\hat{v} = \underline{v}(r_1)$ being the revealed upper bound for the bidders' valuations.

For these strategies to support an equilibrium, a bidder with valuation $\underline{v}(r_1)$ must be indifferent between winning the first-period object at the minimal price $\beta^1(\underline{v}(r_1) \,|\, r_1) = r_1$ and waiting for the second auction, in which the reserve price will be lower. This requires that the bidders shade their second-period bids less than in the first period, so as a bidder with valuation $\underline{v}(r_1)$ will still bid r_1.

Notice that any bidder with valuation $v_i \in (r_1, \underline{v}(r_1))$, for a given reserve price r_1, does not participate in the first auction. He prefers to wait for the second auction, even if he can buy the first-period object at price r_1. This strategic non-participation decision, which also appears in McAfee and Vincent [75] and in Caillaud and Mezzetti [14], is entirely the consequence of the auctioneer's inability to commit not to lower the reserve price. It would not occur, if the auctioneer could commit to a second-period reserve price $r_2 \geq r_1$. In particular, it does not occur in the subgame following a first-period reserve price $r_1 = 0$. Therefore, it does not depend on the bidders' expectation of a smaller number of competing bidders in the future.[19]

We start our formal analysis by investigating the second-period auction. Since we are considering monotone first-period bidding strategies, we can abbreviate the notation for the second-period bidding strategy to $\beta^{2,M}(v_i \,|\, \hat{v}, r_2)$, where $M = N - 1$ or N is the number of participating bidders and $\hat{v} \in [\underline{v}, \bar{v}]$ is the upper bound for the participating bidders' valuations revealed in the first auction.[20]

[19] This expectation affects only how sharply the bidders shade their bids, not their decision to wait.

[20] Restricting attention to the game described in Lemma 2.1, notice that the possibility of $v_i > \hat{v}$ does not contradict the bidders' assumed beliefs and, therefore, does not violate the consistency requirement in the definition of equilibrium. We can simply assume that prior to the draw of the privately known valuation v_i, each bidder attaches zero probability to the event $v_i > \hat{v}$, for all i.

Lemma 2.1.

Consider a single Dutch auction with M bidders, whose valuations are i.i.d. according to the distribution function $F(\cdot)$ on $[\underline{v}, \bar{v}]$. Suppose that the auctioneer and the bidders believe that the unknown valuations are bounded above by the value $\hat{v} \in [\underline{v}, \bar{v}]$. Then, given a reserve price r_2, the following strategy constitutes a symmetric equilibrium:

$$\beta^{2,M}(v_i \,|\, \hat{v}, r_2) \;=\; \begin{cases} \mathbb{E}[\,\max\{v_1^{(M-1)}, r_2\} \,|\, v_1^{(M-1)} < v_i\,], & \text{if } r_2 \le v_i \le \hat{v}; \\[2ex] \mathbb{E}[\,\max\{v_1^{(M-1)}, r_2\} \,|\, v_1^{(M-1)} < \hat{v}\,], & \text{if } v_i > \hat{v}; \\[2ex] a, & \text{if } v_i < r_2. \end{cases}$$

In this auction, it is optimal for the auctioneer to set a reserve price:

$$r_2(\hat{v}) \;=\; \begin{cases} \psi(\cdot\,|\hat{v})^{-1}(0), & \text{if } \psi(\underline{v} \,|\, \hat{v}) < 0; \\[2ex] \underline{v}, & \text{otherwise.} \end{cases}$$

Proof:

The result follows directly from standard arguments regarding symmetric equilibria and optimal reserve prices in first-price auctions. In particular, for $r_2 \le v_i \le \hat{v}$, we derive the equilibrium bidding function

$$\beta^{2,M}(v_i \,|\, \hat{v}, r_2) \;=\; \frac{1}{F(v_i)^{M-1}} \left[\int_{r_2}^{v_i} v \; (M-1) F(v)^{M-2} f(v) \; dv \;+\; F(r_2)^{M-1}\, r_2 \right], \tag{2.4}$$

which corresponds to the upper branch of the first expression in the lemma.

□

Therefore, if the first-period auction reveals an upper bound $\hat{v}$ for the bidders' valuations, then, in the second period, the auctioneer's optimal reserve price and the bidders' strategies are described by Lemma 2.1. The possibility of a valuation $v_i > \hat{v}$ corresponds to an off-equilibrium path event, namely, to the case in which bidder i should have won the first-period unit but did not bid according to the prescribed strategy.

Moving backwards, suppose that the auctioneer has set a first-period reserve price r_1 and consider a bidder with valuation $v_i \geq \underline{v}(r_1)$, for some value $\underline{v}(r_1) > r_1$ that will be determined later. Suppose that all other bidders follow the strategies (β^1, β^2), where β^2 is as in Lemma 2.1. Furthermore, suppose that the auctioneer follows the strategy r^2, again described by Lemma 2.1.

Then by mimicking a type $\tilde{v}_i > v_i$, bidder i has an expected payoff

$$\begin{aligned}\Pi[\tilde{v}_i; v_i] &= F(\tilde{v}_i)^{N-1}\,[\,v_i - \beta^1(\,\tilde{v}_i \,|\, r_1\,)\,] \\ &+ (N-1)\,[1 - F(\tilde{v}_i)]\,F(v_i)^{N-2}\,\left[v_i - \int_{\tilde{v}_i}^{1} \beta^{2,N-1}(v_i \,|\, \hat{v}, r_2(\hat{v})\,)\,\frac{f(\hat{v})}{1 - F(\tilde{v}_i)}\,d\hat{v}\right].\end{aligned}$$

Similarly, by mimicking a type $\tilde{v}_i \in [\underline{v}(r_1), v_i)$, bidder i has an expected payoff

$$\begin{aligned}\Pi[\tilde{v}_i; v_i] &= F(\tilde{v}_i)^{N-1}\,[\,v_i - \beta^1(\,\tilde{v}_i \,|\, r_1\,)\,] \\ &+ (N-1)\,[1 - F(v_i)]\,F(v_i)^{N-2}\,\left[v_i - \int_{v_i}^{1} \beta^{2,N-1}(v_i \,|\, \hat{v}, r_2(\hat{v})\,)\,\frac{f(\hat{v})}{1 - F(v_i)}\,d\hat{v}\right] \\ &+ [F(v_i)^{N-1} - F(\tilde{v}_i)^{N-1}]\,\left[v_i - \int_{\tilde{v}_i}^{v_i} \beta^{2,N-1}(\hat{v} \,|\, \hat{v}, r_2(\hat{v})\,)\,\frac{(N-1)F(\hat{v})^{N-2}f(\hat{v})}{F(v_i)^{N-1} - F(\tilde{v}_i)^{N-1}}\,d\hat{v}\right].\end{aligned}$$

The third term corresponds to the possibility in which the first-period object is sold at a price $\hat{b}^1 \in (\beta^1(\tilde{v}_i|r_1), \beta^1(v_i|r_1))$. In this case, the winning bidder reveals the valuation $\hat{v} = \beta^1(\,\cdot\,|\,r_1)^{-1}(\hat{b}^1) \in (\tilde{v}_i, v_i)$. Therefore, in the second auction, with reserve price $r_2(\hat{v})$, bidder i bids $\beta^2[\hat{v}\,|\,\hat{v}, r_2(\hat{v})]$.

In either case, by solving the differential equation that results from the necessary first-order condition at the endpoint $\tilde{v}_i = v_i$ along with the boundary condition $\beta^1(\underline{v}(r_1)\,|\,r_1) = r_1$, we get the bidding function

$$\beta^1(v_i \,|\, r_1) = \frac{1}{G(v_i)}\,\left[\int_{\underline{v}(r_1)}^{v_i} \beta^{2,N-1}[v \,|\, v, r_2(v)]\;g(v)\;dv \;+\; G[\underline{v}(r_1)]\;r_1\right], \qquad (2.5)$$

where $G(v)$ and $g(v)$ denote respectively the distribution and the density of the highest of the competing $N-1$ bidders' valuations.

To determine the function $\underline{v}(r_1)$, consider a bidder with valuation $v_i = \underline{v}(r_1)$. Such a bidder shall be indifferent between winning the first-period object, at the minimal price $\beta^1(\underline{v}(r_1) \,|\, r_1) = r_1$, and abstaining from the first period so as to win the second-period object. This implies that the bidder shall pay the same price in either of the two auctions:

$$r_1 \;=\; \beta^{2,N}[\,\underline{v}(r_1) \,|\, \underline{v}(r_1)\,,\, r_2(\underline{v}(r_1))\,]. \tag{2.6}$$

Since the function $r_1(v) = \beta^{2,N}[\,v \,|\, v, r_2(v)\,]$ is increasing[21] in the valuation v, it follows that the threshold value function $\underline{v}(\cdot) : r_1 \mapsto \underline{v}(r_1)$ is increasing in the reserve price r_1. Let $\bar{r}_1$ be the minimal reserve price for which no bidder will participate in the first auction, that is, $\bar{r}_1 = \min\{r_1 : \underline{v}(r_1) = \bar{v}\}$. Then

$$\bar{r}_1 \;=\; \beta^{2,N}(\bar{v} \,|\, \bar{v}, r_2(\bar{v})). \tag{2.7}$$

Therefore, for all reserve prices $r_1 < \bar{r}_1$, we have $\underline{v}(r_1) \in [r_1, \bar{v})$; and, for all $r_1 \geq \bar{r}_1$, we have $\underline{v}(r_1) = \bar{v}$. Finally, notice that for all reserve prices $r_1 > \underline{v}$, we have $\underline{v}(r_1) < r_1$, implying non-participation for bidders with valuations $v_i \in [r_1, \underline{v}(r_1))$.

The above arguments lead to the following result, regarding the bidders' behavior in the game following a first-period reserve price r_1:

Proposition 2.2.
Consider a sequence of two Dutch auctions and suppose that the auctioneer has set a first-period reserve price $r_1 \in [\underline{v}, \bar{v}]$. Then there is an equilibrium for the continuation game, in which the strategies r_2, $\beta^{2,N}$ and $\beta^{2,N-1}$ are given by Lemma 2.1 and the strategy β^1 depends on the value of r_1:

- *If $r_1 < \bar{r}_1$, for the threshold value $\bar{r}_1 = \beta^{2,N}(\bar{v} \,|\, \bar{v}, r_2(\bar{v}))$, the strategy β^1 is given by*

[21] For a function $\Phi(x) = F[f_1(x), f_2(x)]$, we have $\Phi' = F_1\, f_1' + F_2\, f_2'$. Hence, if the derivatives F_1, f_1', F_2 and f_2' are all positive, then the function $\Phi(x)$ is increasing.

$$\beta^1(v_i \,|\, r_1) = \frac{1}{G(v_i)} \; [\int_{\underline{v}(r_1)}^{v_i} \beta^{2,N-1}[v \,|\, v, r_2(v)] \, g(v) \; dv \; + \; G[\underline{v}(r_1)] \; r_1 \,],$$

for all valuations $v_i \in [\underline{v}(r_1), \bar{v}]$*, where* $\underline{v}(r_1) \in (r_1, \bar{v})$ *is given by the equation*

$$r_1 \;\; = \;\; \beta^{2,N}[\, \underline{v}(r_1) \,|\, \underline{v}(r_1), \, r_2(\underline{v}(r_1)) \,].$$

Otherwise, for $v_i \in [\underline{v}, \underline{v}(r_1))$*, bidder* i *abstains from the first-period auction.*

- *If* $r_1 \geq \bar{r}_1$*, then all bidders abstain from the first-period auction.*

Having described the bidders' behavior, we can now consider the auctioneer's problem of determining the optimal first-period reserve price r_1^*. Since there is a bijective relation between a reserve price $r_1 \in [\underline{v}, \bar{r}_1]$ and the participation threshold $v(r_1) \in [\underline{v}, \bar{v}]$, namely,

$$r_1(v) = \beta^{2,N}[v \,|\, v, r_2(v)],$$

we can think of the auctioneer's problem as one of determining the revenue maximizing first-period participation threshold $v^* = \underline{v}(r_1^*)$.

Proposition 2.3.
The optimal first-period participation threshold v^* *in a sequence of two Dutch auctions solves the equation*

$$\frac{1 - F(v)}{f(v)} \;\; G[r_2(v)] \;\; \frac{dr_2}{dv}(v) \;\; = \;\; \int_{r_2(v)}^{v} \psi(u) \; g(u) \; du. \tag{2.8}$$

The auctioneer always induces participation by a positive measure of bidders' types, that is, $v^* < \bar{v}$*. In addition, for distributions* $F(\cdot)$ *such that* $r_0 > \underline{v}$*, we have* $r_1(v^*) < r_0 < v^*$*. Finally, if* $r_0 = \underline{v}$*, then* $r_1(v^*) = r_0 = v^* = \underline{v}$*.*

It is interesting to notice that the auctioneer's choice of second-period reserve price $r_2 = r_2(b^1, r_1)$ is better informed under non-commitment. Therefore, he expects a greater revenue in the second auction than in the case of commitment to a reserve price $r_1 = r_2 = r_0$. This gain, however, is dominated by the auctioneer's loss in the first period, so that, overall, it is more profitable for him to commit to r_0. The two revenues are equal only when $r_0 = \underline{v}$, which occurs when $\psi(\underline{v}) \geq 0$.

There are two reasons for the first-period loss. First, unless $r_0 = \underline{v}$, there is a smaller measure of bidder types participating in the auction. In addition, the participating types bid less aggressively than in the case of commitment. In this respect, our auctioneer's decision has a stronger effect on the bidders' behavior than that of the auctioneer in Caillaud and Mezzetti [14]. In their setting, because of the use of English auctions and the bidders' multi-unit demands and persistent valuations, each participating bidder always bids up to his valuation.

We underline the need of the auctioneer to commit to a reserve price schedule, if this can be possible, in the following result:

Corollary 2.4.
Suppose that the bidders' lowest virtual valuation is $\psi(\underline{v}) < 0$. Then, in a sequence of two Dutch auctions, the auctioneer's revenue is strictly greater under commitment to reserve prices $r_1 = r_2 = \psi^{-1}(0)$ than in any reserve-price scheme under non-commitment.

Proof:
The result follows directly from the characterization of the optimal selling mechanism under commitment, from Maskin and Riley [70], the revenue equivalence of sequential auctions and appropriately defined single-round, multi-unit auctions and Proposition 2.3.

□

We conclude by applying the above results to the case of uniformly distributed valuations.

Example:

Suppose that the bidders' valuations are uniformly distributed in [0,1]. In the second round, if the reserve price is r_2, the revealed upper bound for the bidders' valuations is $\hat{v}$ and there are M competing bidders, each bidder acts according to the bidding strategy described by Lemma 2.1:

$$\beta^{2,M}(v_i \,|\, \hat{v}, r_2) \;=\; \begin{cases} \frac{1}{M\, v_i^{M-1}} \left[r_2^M \;+\; (M-1)v_i^M\right], & \text{if } r_2 \le v_i \le \hat{v}; \\[2ex] \frac{1}{M\, \hat{v}^{M-1}} \left[r_2^M \;+\; (M-1)\hat{v}^M\right], & \text{if } v_i > \hat{v}; \\[2ex] a, & \text{if } v_i < r_2. \end{cases}$$

In this auction, it is optimal for the auctioneer to set a reserve price

$$r_2(\hat{v}) \;=\; \frac{1}{2}\,\hat{v}.$$

In the first auction, there will be a positive measure of participating bidder types, if and only if the reserve price is $r_1 < \bar{r}_1$, where $\bar{r}_1 \in [0,1]$ is given by the equation (2.7):

$$\bar{r}_1 \;=\; \beta^{2,N}(1 \,|\, 1, r_2(1)) \quad\Longrightarrow\quad \bar{r}_1 = \frac{N-1+(1/2)^N}{N}.$$

Otherwise, all bidders will wait for the second period.

When $r_1 < \bar{r}_1$, bidders with valuations $v_i \ge \underline{v}(r_1)$ bid according to the strategy

$$\beta^1(v_i \,|\, r_1) \;=\; \frac{1}{v_i^{N-1}} \int_{\underline{v}(r_1)}^{v_i} r_2(v)^{N-1} \;+\; (N-2)v^{N-1}\, dv \;+\; \frac{\underline{v}(r_1)^{N-1}}{v_i^{N-1}}\, r_1,$$

given by the equation (2.5), while bidders with valuations $v_i < \underline{v}(r_1)$ abstain from the auction. By imposing the indifference condition (2.8) defining the type $v_i = \underline{v}(r_1)$, we get

$$\bar{r}_1 \;=\; \beta^{2,N}(\bar{v} \,|\, \bar{v}, r_2(\bar{v})) \quad\Longrightarrow\quad \underline{v}(r_1) \;=\; \frac{N}{N-1+(1/2)^N}\, r_1 \;=\; \frac{r_1}{\bar{r}_1}.$$

Clearly $\underline{v}(r_1) > r_1$, because of the bidders' anticipation of a lower second-round reserve price. In addition, $\underline{v}(\bar{r}_1) = 1$, according to the definitions (2.6) and (2.7) of $\underline{v}(r_1)$ and $\bar{r}_1$.

By using the expressions for $r_2(\bar{v})$ and $\underline{v}(r_1)$ that we derived above, we can simplify the function describing the first-period strategy of a bidder with valuation $v_i \geq \underline{v}(r_1)$:

$$\beta^1(v_i \,|\, r_1) \;=\; \left(\bar{r}_1 - \frac{1}{N}\right) v_i \;+\; \frac{1}{N} \left(\frac{r_1}{\bar{r}_1}\right)^N \frac{1}{v_i^{N-1}}.$$

Finally, suppose that the auctioneer chooses a reserve price $r_1 \leq \bar{r}_1$ corresponding to a participation threshold $v = \underline{v}(r_1) = r_1/\bar{r}_1$. Then his expected payoff will be

$$\begin{aligned} R(v) \;&=\; \int_v^{\bar{v}} \left(\bar{r}_1 - \frac{1}{N}\right) v_1^N \;+\; \frac{1}{N} \left(\frac{r_1}{\bar{r}_1}\right)^N \, dv_1 \\ &+\; \int_{v/2}^{v} \left(\frac{v}{2}\right)^N \;+\; (N-1)\, v_1^N \; dv_1 \\ &+\; \int_{v/2}^{v} \int_{v_1/2}^{v_1} N \left[\left(\frac{v_1}{2}\right)^{N-1} \;+\; (N-2)\, v_2^{N-1} \right] \, dv_2 \, dv_1. \end{aligned}$$

The necessary condition (2.8) from Proposition 2.3 yields:

$$(1-v)\, v^{N-1} \;=\; 2^N \int_{v/2}^{v} (2u-1)\,(N-1)\, u^{N-2} \; du.$$

Therefore, the auctioneer maximizes his expected revenue by inducing a first-period participation threshold or, equivalently, by setting a first-period reserve price

$$v^* \;=\; \frac{N\,(2^N-1)}{2\,(N-1)\,(2^N-1)\;+N} \quad \Longleftrightarrow \quad r_1^* \;=\; \frac{N\,(2^N-1)}{2\,(N-1)\,(2^N-1)\;+N}\; \bar{r}_1.$$

Since $v^* < 1$, the auctioneer is always willing to sell the first-period object. In fact, the first-period reserve price will be $r_1^* < \frac{1}{2}$, the optimal reserve price under commitment. However, since $v^* > \frac{1}{2}$, there are strictly fewer bidder types participating in the first auction than in the optimal sequential auction under commitment.

$\square$

2.5 SEQUENTIAL SEALED-BID AND ENGLISH AUCTIONS

Under non-commitment, the auctioneer is able to make full use, in the second auction, of any information about the bidders' valuations that the first auction may reveal. For example, if a non-winning bidder in the first auction is revealed to have a valuation $v_i \geq v_L$, then the auctioneer will set a reserve price $r_2 \geq v_L$. The bidders, therefore, have a strong incentive to conceal, in the first auction, their valuations from the auctioneer. Non-surprisingly, this incentive leads to strong negative results.

Proposition 2.5.

In a sequential English or sealed-bid first- or second-price auction, there does not exist any symmetric perfect Bayesian equilibrium in weakly increasing first-period bidding strategies.

The non-existence of a symmetric perfect Bayesian equilibrium[22] in weakly increasing first-period strategies originates from the asymmetric effects of the deviations to $\beta^1(\hat{v}_i)$, for $\hat{v}_i > v_i$, and to $\beta^1(\check{v}_i)$, for $\check{v}_i < v_i$. In particular, a deviation to mimicking a type $\hat{v}_i > v_i$ does not decrease the expected second-period surplus of the bidder; this will still be zero. On the other hand, a deviation to mimicking a type $\check{v}_i < v_i$ does increase the expected second-period surplus of the bidder, as it leads to a lower second-period reserve price. Hence, to avoid deviations to $\check{v}_i < v_i$, the bidders shall shade their first-period bids so much so that the deviation to $\hat{v}_i > v_i$ becomes strictly profitable.

[22]Notice that if we adopted a weaker notion of perfect Bayesian equilibrium, one that would not impose any restriction on the players beliefs off the equilibrium path, then an equilibrium would exist in a rather generic manner. In particular, in the equilibrium path, the auctioneer would set $r_1 = \bar{v}$, so that effectively only the second-period auction, with $r_2 = r_0 = \psi^{-1}(0)$, would take place. Off the equilibrium path, for $r_1 < \bar{v}$, we could allow each bidder to believe, in an inconsistent manner, that he does not have the highest valuation; thus, all bidders would abstain from the first auction. Therefore, the auctioneer could not benefit from lowering the reserve price.

2.6 IMPERFECT COMMITMENT

We consider a variation of the sequential auctions that we analyzed in the previous two sections, one in which the auctioneer is able to commit in a non-fully credible manner not to change the second-period reserve price. In particular, at the beginning of the game, the auctioneer announces a reserve price $r_1 \in [\underline{v}, \bar{v}]$. After the end of the first auction, with a small probability $\rho > 0$, which reflects the auctioneer's lack of credibility (so, it is commonly known), the auctioneer can set a new reserve price $r_2 = r_2(b^1, r_1)$ for the second auction; otherwise, with probability $1-\rho$, we have $r_2 = r_1$, that is, the reserve price does not change.[23] One would hope that for $\rho \approx 0$ there can exist an equilibrium with participation in the first round. As the following result shows, this turns out not to be possible.

Proposition 2.6.
In a sequential English or sealed-bid first- or second-price auction under imperfect commitment, there does not exist any symmetric perfect Bayesian equilibrium in weakly increasing first-period bidding strategies.

On the other hand, in the case of sequential Dutch auctions, it is easy to show that there exists a sequential equilibrium demonstrating similar characteristics to the ones of the equilibrium described in the case of non-commitment. Therefore, the two cases, of imperfect commitment and of non-commitment, are qualitatively identical.

In conclusion, however small the auctioneer's lack of credibility may be, the bidders will be concerned about not revealing their valuations in the first round. Because of this concern a symmetric, monotone, pure-strategy equilibrium exists only in the case of sequential Dutch auctions. In the absence of sufficiently strict legal assurances or other means of establishing credibility[24], the use of this auction format, or of a single-round, multi-unit auction, is the only manner in which the auctioneer can induce a positive outcome.

[23] We assume that the auctioneer learns his type, whether he can change the reserve price or not, only after the end of the first auction. Hence, his choice of r_1 does not signal any information to the bidders.

[24] For example, the seller might try to trade though a well-established auction house. One can assume that reputation concerns may be more important for such an institution than for a single individual.

2.7 CONCLUSIONS

In a sequential auction in which the buyers have single-unit demands and the seller is unable to commit perfectly to the auction rules, we have shown that a perfect Bayesian equilibrium in symmetric monotone strategies exists only in the case of Dutch auctions. It is precisely this auction format that allows the non-winning bidders to hide their valuations. Our result complements Caillaud and Mezzetti [14], where the multi-unit demands, along with the persistent valuations, placed the problem of concealing one's valuation upon the winning bidder and, therefore, forced the use of the English auction. In both settings, information is revealed only indirectly, through the action of the bidders that do not have any further interest in the game.

It would be interesting to consider an environment in which all bidders try to hide their valuations, independently of whether they win the first object or not. This could be a sequential auction with multi-unit demands and non-persistent valuations or decreasing marginal returns. Alternatively, one could try to derive an equilibrium for the case of sealed-bid auctions. In all these cases, in equilibrium, the bidders may need to mix their strategies.

Finally, one could consider the sequential allocation problem in its full generality, by allowing the auctioneer to use a sequence of any single-object selling mechanisms he wishes. In this case, it would be interesting to learn whether the seller can profit from using a first-period mechanism that extracts information from the non-winning bidders.

3.0 SIMULTANEOUS AND SEQUENTIAL AUCTIONS OF OLIGOPOLY LICENSES

3.1 INTRODUCTION

Many important auctions aim at creating new markets or at expanding markets that already exist. For example, the radio spectrum auctions conducted by the Federal Communications Commission in the United States (and by similar government agencies elsewhere in the world) have allowed the wireless communication companies to expand their services. In the same manner, the auction of other state-owned resources, such as oil fields or timber tracts, have enabled the related companies to expand their operations. In addition, in several regulated markets, companies need to compete periodically for the acquisition or renewal of the license to supply their product. Finally, the development of new technologies often forces those companies that can benefit from their adoption to compete for the acquisition of the right to use them.[1]

Such auctions are typically characterized by the presence of allocative and informational externalities. In particular, the value of the resources or of the licenses that are sold, that is, the market profit that these assets may generate, depends on the entire outcome of the auction. First, it depends on the characteristics of the bidders against which a firm will compete in the market. In addition, it may depend on the identities of the winning and the losing bidders. Finally, when the bidders' characteristics are privately known, their bids

[1]For a survey of the FCC auctions, see Cramton [18, 19]; for a survey of similar auctions in Europe, see Jehiel and Moldovanu [47] and Klemperer [61]. For information regarding the auctions of oil sources or of timber tracts, see, respectively, Cramton [20] and Hendricks and Porter [40] as well as the references therein. For the relation between an auction and the market created by it, see Dana and Spier [22]. Regarding patent licensing schemes, see Kamien [55], Kamien et al. [56] and Kamien and Tauman [57]. Finally, for a general survey of auctions of public assets and the issues that typically arise in them, see Janssen [43].

convey information about them, in a manner that allows for the possibility of signaling. For example, an oligopoly license is more valuable for its recipient when the firms that gain the other licenses are weaker. Similarly, an oligopolist's overall benefit from acquiring the rights to use a new technology increases when his market competitors, which are prevented from accessing this technology, can derive a greater benefit from it. Finally, in both examples, a firm can profit from exaggerating or from understating its private information during the auction. Hence, because of these externalities, the value of the assets sold is determined endogenously, by the types of the bidders, the final allocation and the information that the auction reveals.[2]

In this chapter, we study a particular environment in which such externalities are present: an auction of two licenses to supply in a Cournot oligopoly or in a Bertrand oligopoly for differentiated products.[3] For each of the competing firms, the value of the auctioned licenses depends, primarily, on its own production costs. In addition, it depends on the production costs of the winner of the other license, that is, of the firm against which it will compete in the market. Finally, when the firms' production costs cannot be revealed otherwise, there is the possibility of signaling through bidding. In a Cournot oligopoly, a firm can increase its market profit by signaling, during the auction, a stronger type, so as to beguile its opponent into supplying a smaller quantity. Similarly, in a Bertrand oligopoly, a firm can increase its profit by signaling a weaker type, so as to lure its opponent into setting a higher price.

We compare two procedures for allocating the oligopoly licenses, a simultaneous "pay-your-bid" auction[4] and a sequential first-price auction. We assume that in each auction scheme, the seller reveals the same information, namely, the two winning bids. The two schemes differ, however, in the timing of revelation of that information. In the simultaneous format, all information is revealed at the end of the entire procedure, prior only to market

[2]This environment differs from that of an auction with symmetric interdependent valuations, such as the one studied in Milgrom and Weber [78]. There, the value of the assets sold depends on the bidders' private information only. It is independent of the outcome of the auction; in particular, it does not depend on who wins the auction or on the information that the auction may reveal.

[3]For simplicity, we have assumed that the sale of the licenses results in the creation of a new market. Our results, however, easily extend to an auction of two licenses to enter an already existing oligopoly or monopoly. In such an auction, the structure of the firms' payoffs and incentives is identical to that in our model. Similarly, one can modify our model so as to describe the auction of two licenses to use a process innovation. Again, the structure of the firms' incentives will not change.

[4]In Krishna [62] and elsewhere, this auction is referred to as a "discriminatory" auction.

competition. On the other hand, in the sequential format, some information, the bid that won the first license, is revealed during the bidding process, in between the two auction rounds. As a result, in the sequential auction, the firms can bid in a better-informed manner, as some of the payoff uncertainty present in this environment is eliminated.

When the winners' production costs are truthfully revealed after the auction[5], so that the information conveyed by their bids does not affect the ensuing market competition, then for each auction scheme there exists a symmetric equilibrium in strictly monotone bidding strategies. The two licenses are therefore allocated to the two strongest firms. The difference in the information structure, however, affects the degree by which the firms shade their bids and, therefore, the prices that they eventually pay.

In the first round of the sequential auction, the firms know that they will win the license only if they have the strongest type. Therefore, since they do not take into account the possibility of having to compete against a stronger market opponent, they shade their bids less than they would do in the simultaneous auction. Consequently, the stronger of the two market competitors, as revealed by the outcome of the auction, pays a higher price in the sequential auction than in the simultaneous one. Similarly, the firms participating in the second round know that, by winning the second license, they will necessarily have to face a stronger market competitor. Therefore, on average, they shade their bids more than they would do in the simultaneous auction. As a result, the weaker of the two market competitors pays a higher price in the simultaneous auction.[6]

Despite the differences in the bidders' behavior, the two auction schemes turn out to be revenue equivalent.[7] The excess of aggression that the bidders show in the first round of the

[5] The revelation of the two oligopolists' production costs can be the consequence of the actions that they need to take in the time period between the end of the auction and the beginning of the market. Truthful revelation can also be assumed when the effects of false signaling are negligible, for example, when the oligopolists can quickly adjust their market strategies. This assumption is present in much of the literature on auctions with externalities, for example, in Jehiel and Moldovanu [45].

[6] In addition, in the simultaneous auction, the weaker of the two oligopolists may gain his license at a price that exceeds its revealed value, thus, regretting his participation to the market. Such a negative outcome, however, cannot occur in the sequential auction.

[7] Since we assume that the firms' production costs are independent, the intuition of the Linkage Principle for auctions with interdependent valuations (cf. Milgrom and Weber [78]) does not apply to our setting. For this principle to impose a revenue ranking in favor of the auction scheme that reveals more information during the bidding procedure, in this case, in favor of the sequential auction, the firms' production costs must be affiliated.

sequential auction, which stems from the good news that a victory in this round will convey, is balanced, on average, by the excess of restraint that the bidders show in the second round, in reaction to the bad news they have received. Therefore, without incurring any cost to the seller, the sequential auction results in a more even distribution of the wealth generated by the introduction of the new market.[8]

When the oligopolists' production costs are not automatically revealed after the auction, but have to be inferred from their bids, the possibility of signaling is introduced. In this case, the firms adjust their valuations by incorporating the informational rents that they can extract. In each auction scheme, there are two incentive trade-offs, the signaling and the non-signaling one. Since the firms' market profit functions are separable in their real and signalled production costs, the two trade-offs can be separated and treated independently. Therefore, the firms' non-signaling incentives can always be balanced, as in the case in which their costs are truthfully revealed. On the other hand, the possibility of balancing the signaling incentives depends on the auction scheme as well as on the type of the market.

In the simultaneous auction, the firms can adjust their strategies so that any gains from false signaling can be offset by an increase in the expected payment, in the case of the Cournot market, or by a decrease in the probability of winning a license, in the case of the Bertrand market. In the sequential auction, a similar trade-off is possible only in the Cournot oligopoly. In the Bertrand oligopoly, a firm can always profit from mimicking a weaker type in the first round, as this would increase its expected market profit and decrease its expected payment, without changing its overall probability of winning one of the two licenses. Therefore, in this case, no symmetric monotone equilibrium exists.

Hence, in the Cournot oligopoly, the non-signaling results remain valid under signaling. The two auction schemes are revenue equivalent, even though the sequential auction favors the weaker (ex-post) of the two oligopolists. On the other hand, in the Bertrand oligopoly, a comparison between the two auction schemes is not possible, since a separating equilibrium exists only in the simultaneous auction.[9]

[8]The seller's concern for a more even distribution of the oligopolists' overall profit may stem from a desire to maintain balance, over time, among the competing firms. In this case, therefore, the choice of a sequential auction can be thought off as an indirect subsidy to the weaker market participant. For a discussion of this issue, see Maasland et al. [69].

[9]In light of the results regarding information sharing in oligopoly (cf. Gal-Or [32, 33]), our signaling

Overall, our results regarding the comparison of the two auction schemes are based on the ability of the sequential auction to generate implicitly, by the characterization of the equilibrium that describes the firms' bidding behavior, information about the two winners' relative strengths. Because of this information, the firms modify their interim valuations for each of the auctioned licenses. The first license becomes more valuable, relative to a license won in a simultaneous auction, since its acquisition promises a stronger presence in the market. On the other hand, the second license becomes less valuable, since its recipient knows that he will have a weaker market presence. As a result, the two licenses (as well as the information that accompanies their acquisition) are sold at prices that differ from the ones in the simultaneous auction.

The early study of auctions with externalities[10] assumed that the externalities depend only on the number of allocated objects (and not on the bidders' types and identities). In their study of the persistence of a monopoly, Gilbert and Newbery [35] show that a monopolist who faces a potential entrant may bid for a technological innovation for which he has no use, in a preemptive manner. Within the context of patent licensing and vertical contracting, Katz and Shapiro [59], Kamien and Tauman [57] and Kamien et al. [56] compare some typical licensing mechanisms, such as auctions, fixed fees and royalties, and show the superiority of auctions.

Optimal selling mechanisms in the presence of type-dependent externalities were first studied by Jehiel et al. [50, 51]. If the agents' private information is multi-dimensional, then it is optimal for the seller to employ identity-dependent "threats", which can exploit the negative allocative externalities in order to extract payments from bidders that do not acquire the license. For very strong negative externalities, the bidders may even pay the seller not to allocate the license at all. Clearly, this mechanism is not efficient. In fact, Jehiel and Moldovanu [46, 48] show that, with multi-dimensional signals, efficiency cannot be implemented. If the agents' private information is single-dimensional, however, efficiency is feasible. Figueroa and Skreta [27] show that sometimes the optimal mechanism is efficient; at other times, though, it allocates the auctioned objects in a random manner.

results are hardly surprising. In particular, when signaling is possible, a separating equilibrium exists only in the oligopoly in which the firms are willing to share information about their production costs.

[10]For an extensive survey of the literature on this subject, see Jehiel and Moldovanu [49].

Since the application of the optimal mechanism, in particular, the differential treatment of the bidders, may not be possible, Jehiel and Moldovanu [44, 45] examine relatively simpler selling schemes, such as auctions with fixed reserve prices or entry fees. They find that some bidders may prefer to abstain from the auction, if their participation can have an adverse effect on the other bidders' behavior and, eventually, on the winner's identity. Conversely, to encourage participation, the seller may set a reserve price below his own reservation value. In a multi-unit setting, in particular, in the sale of the rights to use a cost-reducing innovation in an oligopoly, Schmitz [87] and Bagchi [7] examine simultaneous auctions of a predetermined number of licenses. As the bidders' information is single-dimensional, the licenses are allocated efficiently. In addition, the seller can be better off auctioning multiple licenses rather than the exclusive rights to use the technology. Finally, in an auction of multiple licenses to enter an already existing oligopoly, Hoppe et al. [41] show that the resulting market can be less competitive if more licenses are made available.

Signaling in auctions with externalities was introduced in Goeree [37], who examined the auction of a single license to compete against a monopolist with known marginal cost. Das Varma [23], in a problem of bidding for the acquisition of a cost-reducing patent, identified conditions for the existence of equilibrium in the presence of negative informational externalities. Katzman and Rhodes-Kropf [60] extended the study of signaling to more general schemes of information revelation, showing the revenue equivalence of the auction schemes that result to the same allocation and reveal the same information. Finally, Molnár and Virág [80] determine the revenue maximizing allocation and information mechanism in environments with post-auction interaction.

The present work contributes to the existing literature on auctions with externalities by extending the analysis of the multi-unit case to sequential auctions. When there is no signaling, we show that both the simultaneous and the sequential scheme lead to an efficient allocation of the licenses[11] while they raise the same revenue for the seller. In addition, we demonstrate the effects of the better-informed bidding that is allowed by the sequential format, showing that it favors the weaker, ex-post, of the two oligopolists. Hence,

[11] Since the bidders' private information is single-dimensional, the efficiency result in our setting does not contradict Jehiel and Moldovanu [46, 48].

in the absence of informational externalities, we conclude that the sequential auction can be recommended as a policy device to an auctioneer who prefers a more even distribution of the wealth generated by the creation of the oligopoly. Finally, we explore the implications of signaling, showing that the incentive to understate one's strength, which is present in the case of the Bertrand oligopoly, eliminates the possibility of efficient allocation.

In sections 3.2 and 3.3, we present the model describing our problem and we analyze the firms' behavior in the oligopoly created by the auction of the two licenses. In section 3.4, we examine the two auction procedures when there is no signaling, deriving symmetric equilibria in strictly monotone bidding strategies and comparing them. In section 3.5, we analyze the case of positive signaling, which is present in the Cournot oligopoly, showing that the non-signaling results fully extend. In section 3.6, we study the case of negative signaling, present in the Bertrand oligopoly, showing that a symmetric equilibrium in strictly monotone strategies exists only for the simultaneous auction. We conclude in section 3.7. Lengthy proofs have been placed in section B of the appendix.

3.2 GENERAL MODEL

We study the auction of 2 licenses for participating in a newly formed oligopoly, which will take the form of either Cournot competition or Bertrand competition for differentiated products. The market profits of the two oligopolists depend, respectively, on the quantities they supply to the market or on the prices they set for their product. These decisions depend, in turn, on the oligopolists' production costs.

There are $N > 2$ firms competing for the acquisition of the oligopoly licenses. The firms have linear production technologies without fixed costs. Therefore, for each firm i, its technology is characterized by the privately known marginal cost c_i, which is drawn independently, at the beginning of the game, from a distribution $F : [\underline{c}, \bar{c}] \to [0, 1]$, where $0 < \underline{c} < \bar{c} < \frac{1+\underline{c}}{2}$.[12] We assume that F is twice differentiable, with a density function $f : [\underline{c}, \bar{c}] \to \mathbb{R}^+$ that has full support. Finally, we assume that the inverse hazard rate

[12]In particular, this last assumption implies that $\bar{c} < 1$.

$[1 - F(c)]/f(c)$ is decreasing.[13]

For any firm i, we denote by c^1_{-i} and c^2_{-i} the random variables describing respectively the lowest and the second-lowest marginal costs of firm i's competitors (and by c^1 and c^2 the values these random variables take). In addition, we denote by $G(c^1) = 1 - [1 - F(c^1)]^{N-1}$ the cumulative distribution function of c^1_{-i} and by

$$g(c^1) = (N-1)\,[1 - F(c^1)]^{N-2}\, f(c^1)$$

the corresponding density function. Finally, we denote by $G(c^1,\, c^2)$, for $c^1 \leq c^2$, the joint cumulative distribution function of c^1_{-i} and c^2_{-i} and by

$$g(c^1,\, c^2) = (N-1)(N-2)\,[1 - F(c^2)]^{N-3}\, f(c^2)\, f(c^1),$$

for $c^1 \leq c^2$, the joint density function.

We consider two auction formats:

a. A simultaneous pay-your-bid auction, with the two winning bids announced at the end of the auction.

b. A sequence of two first-price auctions, with the winning bid announced at the end of each auction.

In both formats, the winners' bids are publicly known by the end of the auction process, so that the information they reveal affects the ensuing market competition. Furthermore, in the sequential auction, the winning bid in the first round becomes known prior to the beginning of the second round, so that the information it conveys also affects the bidding for the second license. Since there are no reserve prices in any of the auctions, we can assume that the two licenses are always sold, even at a zero price.

[13] This assumption is satisfied by many well-known distributions, such as the uniform, exponential, normal, power (for $\alpha \geq 1$), Weibull (for $\alpha \geq 1$) and gamma (for $\alpha \geq 1$) distributions. It is consistent with the assumption of logconcave distribution of the firms' strength, made elsewhere in the literature, in particular, in Das Varma [23] and Goeree [37]. If the firms' strength $\theta \in [\underline{c}, \bar{c}]$, defined by $\theta(c) = (\underline{c} + \bar{c}) - c$, is distributed according to a logconcave density $\tilde{f}$, then the rate $\tilde{F}(\theta)/\tilde{f}(\theta)$ must be increasing. This, in turn, implies that the hazard rate $[1 - F(c)]/f(c)$ must be decreasing. For the definition and properties of logconcave probability density functions, consult An [1] and Caplin and Nalebuff [15]; for more on the assumption of increasing inverse hazard rate, consult Hoppe et al. [42].

We will restrict attention to equilibria in symmetric strategies, strictly monotone in the firm's own marginal cost. Therefore, in the simultaneous auction, each firm i bids $b_i = \beta(c_i)$, according to its marginal cost c_i and the strategy

$$\beta : [\underline{c}, \bar{c}] \longrightarrow \mathbb{R}^+.$$

Similarly, in the sequential auction, each firm i bids $b_i^1 = \beta^1(c_i)$ in the first round, according to its marginal cost c_i and the strategy

$$\beta^1 : [\underline{c}, \bar{c}] \longrightarrow \mathbb{R}^+.$$

If it fails to win the first round, then firm i bids $b_i^2 = \beta^2(c_i \,|\, b_i^1, b^1)$ in the second round, according to its marginal cost c_i, the first-round history

$$h_i^1 = (b_i^1, b^1) \;\in\; H^1_{\mathrm{sqc}} = \mathbb{R}^+ \times \mathbb{R}^+,$$

consisting of the privately known bid b_i^1 and the publicly known price b^1, and the strategy

$$\beta^2 : [\underline{c}, \bar{c}] \times H^1_{\mathrm{sqc}} \longrightarrow \mathbb{R}^+.$$

Following the completion of the auction, each of the winning firms enters the oligopoly. The information that firm i has at the end of the simultaneous auction,

$$h_i = (b_i, b^1, b^2) \;\in\; H_{\mathrm{sim}} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,$$

consists of its privately known bid b_i and the publicly known prices b^1, b^2.[14] Similarly, the information that firm i has at the end of the sequential auction is

$$h_i^2 = (b_i^1, b_i^2, b^1, b^2) \;\in\; H^2_{\mathrm{sqc}} = \mathbb{R}^+ \times \big(\mathbb{R}^+ \cup \{\emptyset\}\big) \times \mathbb{R}^+ \times \mathbb{R}^+,$$

allowing for the absence of a second-round bid, in the case of a first-round win.

[14]Without loss of generality, the two prices are in descending order.

Given this information and its marginal cost c_i, firm i will supply $q_i = q(c_i \,|\, b_i, b^1, b^2)$ or $q_i = q(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$ in the Cournot oligopoly, according to the strategy

$$q : [\underline{c}, \bar{c}] \times H \longrightarrow \mathbb{R}^+,$$

for $H \in \{H_{\text{sim}},\, H^2_{\text{sqc}}\}$, following respectively a simultaneous or a sequential auction.

Similarly, in the Bertrand oligopoly, firm i will set a price $p_i = p(c_i \,|\, b_i, b^1, b^2)$ or $p_i = p(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$, according to the strategy

$$p : [\underline{c}, \bar{c}] \times H \longrightarrow \mathbb{R}^+,$$

for $H \in \{H_{\text{sim}},\, H^2_{\text{sqc}}\}$, following respectively a simultaneous or a sequential auction.

Overall, we will impose a stronger symmetry requirement, one that rules out the possibility of using past histories as a labeling device for asymmetric continuation strategies. This assumption will rule out, in particular, asymmetric supply or price-setting strategies for the two oligopolists.[15]

In the sequel, we will use the strict monotonicity of the bidding strategies, with respect to the firm's own marginal cost, to simplify the notation in the following manner:

Notation:

In the sequential auction, we will denote[16] *firm i's second-period bid* b_i^2*, following a first-period bid* b_i^1 *and a price* $b^1 = \beta^1(c^1)$*, by*

$$\beta^2(c_i \,|\, b_i^1, \beta^1(c^1)) \;\equiv\; \beta^2(c_i \,|\, c^1).$$

In the Cournot oligopoly, following either a simultaneous or a sequential auction, we will denote firm i's supplied quantity, $q_i = q(c_i \,|\, b_i, b^1, b^2)$ *or* $q_i = q(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$*, by*

$$q_i \;\equiv\; q(c_i \,|\, \tilde{c}_i, \tilde{c}_j),$$

[15] For example, in the sequential setting, this assumption does not allow the possibility of prescribing different oligopoly strategies to the winners of the first and the second sequential auctions.

[16] This simplification is customary in sequential auctions; for example, see Krishna [62], chapter 15. It is based on the strict monotonicity of the equilibrium bidding strategy β^1 as well as on the independence of the equilibrium bidding strategy β^2 of the first-round bid b_i^1.

where $\tilde{c}_i$ and $\tilde{c}_j$ are the marginal costs corresponding to the bids, under the equilibrium bidding strategies, with which firms i and j won their oligopoly licenses.

Similarly, in the Bertrand oligopoly, following either of the two auction formats, we will denote firm i's requested price, $p_i = p(c_i \,|\, b_i, b^1, b^2)$ or $p_i = p(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$, by

$$p_i \;\equiv\; p(c_i \,|\, \tilde{c}_i, \tilde{c}_j).$$

The notational simplification of the oligopoly supply or price setting strategies is also based on the independence of these strategies of the firm's privately known bids. Indeed, as it will turn out, each oligopolist's behavior depends only on its own marginal cost, c_i, its opponent's inferred marginal cost, $\tilde{c}_j$, and its own marginal cost as perceived by its opponent, $\tilde{c}_i$. Since the costs $\tilde{c}_i$ and $\tilde{c}_j$ are inferred by the publicly known prices, the privately known bids provide no information.

The game payoff of firm i, in case it wins a license, equals its profit from the oligopoly minus the price that it paid for the license. Otherwise, if it does not win any license, it equals zero.

The solution concept is that of perfect Bayesian equilibrium. The players must therefore behave optimally at each decision point, given their knowledge of the other players' strategies and their beliefs. On the equilibrium path, these beliefs are formed by applying Bayes' rule while, off the equilibrium path, they are arbitrary.

3.3 MARKET COMPETITION

When the two winners' marginal costs, c_i and c_j, are truthfully revealed at the end of the auction, the firms cannot use their bids to manipulate their market opportunities, that is, no signaling is possible. In this case, each oligopolist supplies a quantity $q_i = q^{NS}(c_i, c_j)$ or sets a price $p_i = p^{NS}(c_i, c_j)$, which is independent of the bids submitted to or the prices reported in the auction.

When the oligopolists must infer their opponent's marginal cost by the reported prices, the opportunity of signaling arises. In this case, the two winning bids b_i^t and $b_j^{t'}$ perfectly reveal, through the inversion of the corresponding strategies $\beta^t(.)$ and $\beta^{t'}(.)$, the marginal costs $\tilde{c}_i$ and $\tilde{c}_j$ that the winners mimicked in the auction.[17] Therefore, each firm supplies $q_i = q(c_i \,|\, \tilde{c}_i, \tilde{c}_j)$ or sets a price $p_i = p(c_i \,|\, \tilde{c}_i, \tilde{c}_j)$. Since we consider only unilateral deviations, in examining the incentives of player i we will assume that $\tilde{c}_j = c_j$, so that $q_i = q(c_i \,|\, \tilde{c}_i, c_j)$ and $p_i = p(c_i \,|\, \tilde{c}_i, c_j)$.

Clearly, in the equilibrium path, the two firms reveal their marginal costs truthfully. As a result, for all $c_i, c_j \in [\underline{c}, \bar{c}]$, we have $q(c_i \,|\, c_i, c_j) = q^{NS}(c_i, c_j)$ and $p(c_i \,|\, c_i, c_j) = p^{NS}(c_i, c_j)$. In analyzing the firms' market behavior, therefore, we will consider only the case of signaling, treating the absence of signaling as one of its particular outcomes.

3.3.1 Cournot Oligopoly

We consider a Cournot oligopoly, in which the inverse demand function is given by $p = 1 - q$, where p is the market price and $q = q_i + q_j$ is the aggregate supply of oligopolists i and j.

If firm i reveals its marginal cost c_i truthfully, then, by supplying $q_i \in [0, 1 - q_j]$ in response to $q_j \in [0, 1]$, it will make a market profit

$$\pi(q_i, q_j) \;=\; q_i \;(1 - q_i - q_j - c_i).$$

Therefore, in equilibrium, firm i will supply[18]

$$q(c_i \,|\, c_i, c_j) \;\;=\;\; \tfrac{1}{3}\;(1 + c_j - 2c_i),$$

for a profit of

$$\pi(c_i \,|\, c_i, c_j) \;\;=\;\; (\tfrac{1}{3})^2\,(1 + c_j - 2c_i)^2.$$

[17] In the sequential auction, it is possible for a bidder to deviate into mimicking two different types, if he does not win in the first round. In this case, however, only the type mimicked in the second auction will be revealed and, therefore, be relevant in the analysis of the post-auction competition.

[18] The assumption of $\bar{c} < \frac{1+\underline{c}}{2}$ ensures that the market does not become a monopoly. In equilibrium, it is optimal for a firm to supply a positive quantity to the market, regardless of its own marginal cost and its beliefs about the marginal cost of the other firm.

Off the equilibrium path, if firm i mimics a type $\tilde{c}_i \neq c_i$ in the auction that it wins, firm j will supply $q(c_j \,|\, c_j, \tilde{c}_i) \;=\; \frac{1}{3}\,(1 + \tilde{c}_i - 2c_j)$. Therefore, firm i will maximize its profit by supplying

$$q(c_i \,|\, \tilde{c}_i, c_j) \;=\; \tfrac{1}{3}\,\left(1 + c_j - \tfrac{3}{2}c_i - \tfrac{1}{2}\tilde{c}_i\right).$$

In this case, its market profit will be

$$\pi(c_i \,|\, \tilde{c}_i, c_j) \;=\; \left(\tfrac{1}{3}\right)^2 \left(1 + c_j - \tfrac{3}{2}c_i - \tfrac{1}{2}\tilde{c}_i\right)^2.$$

Clearly, we have

$$\pi_2 \;=\; \frac{\partial \pi}{\partial \tilde{c}_i} \;<\; 0,$$

so, prior to market competition, during the auction process, each firm has an incentive to overstate its power by mimicking a lower marginal cost.

3.3.2 Bertrand Oligopoly

We consider a Bertrand oligopoly, in which each firm i faces a linear demand function $q_i \;=\; 1 - p_i + \gamma\, p_j$, where p_i and p_j are the prices set respectively by firm i and its rival, firm j, while $\gamma \in [0, 1)$ is a parameter reflecting the degree of product differentiation.[19] Since $\gamma < 1$, the demand faced by each firm is more responsive to changes in the price charged by this firm than to changes in the price charged by its rival.

If firm i reveals its marginal cost c_i truthfully, then, by setting a price $p_i \in [0, 1 + \gamma\, p_j]$ in response to a price $p_j \in [0, 1]$, it will make a market profit

$$\pi(p_i, p_j) \;=\; (1 - p_i + \gamma\, p_j)\;(p_i - c_i).$$

[19] Our results will not change, if we consider a more general linear demand function $q_i = \alpha - \beta\, q_i + \gamma\, p_j$, for $\beta \geq \gamma$, as the induced equilibrium market profit function will demonstrate the same properties.

Therefore, in equilibrium, firm i will set a price

$$p(c_i\,|\,c_i, c_j) \;=\; \frac{2+\gamma+\gamma\,c_j+2c_i}{4-\gamma^2}\,,$$

for a profit of

$$\pi(c_i\,|\,c_i, c_j) \;=\; \frac{[2+\gamma+\gamma\,c_j-(2-\gamma^2)\,c_i]^2}{(4-\gamma^2)^2}\,.$$

Off equilibrium, if firm i mimics a type $\tilde{c}_i \neq c_i$, firm j will set a price $p_j = p(c_j\,|\,c_j, \tilde{c}_i)$. Therefore, firm i will be best-off by setting a price

$$p(c_i\,|\,\tilde{c}_i, c_j) \;=\; \frac{2(2+\gamma)+2\gamma\,c_j+(4-\gamma^2)\,c_i+\gamma^2\,\tilde{c}_i}{2(4-\gamma^2)}\,,$$

for a market profit of

$$\pi(\tilde{c}_i\,|\,c_i, c_j) \;=\; \frac{[\,2(2+\gamma)+2\gamma\,c_j-(4-\gamma^2)\,c_i+\gamma^2\,\tilde{c}_i\,]^2}{4\,(4-\gamma^2)^2}\,.$$

In this market, we have

$$\pi_2 \;=\; \frac{\partial\pi}{\partial\tilde{c}_i} \;>\; 0,$$

so, during the auction process, each firm has an incentive to understate its power by mimicking a higher marginal cost.

3.3.3 General Remarks

For both oligopolies, the market profit function of each firm i is decreasing in its own marginal cost c_i and increasing in its rival's marginal cost c_j. That is,

$$\pi_1 \;=\; \frac{\partial\pi}{\partial c_i} < 0$$

and

$$\pi_3 \; = \; \frac{\partial \pi}{\partial c_j} > 0.$$

Furthermore, for all $c \in [\underline{c}, \bar{c}]$, we have

$$\frac{d}{dc}[\pi(c\,|\,c,c)] \; = \; \pi_1(c\,|\,c,c) + \pi_2(c\,|\,c,c) + \pi_3(c\,|\,c,c) \; < \; 0,$$

so that, with truthful revelation of the firms' marginal costs, a firm's market profit will be affected more by a change in its own marginal cost than by the same change in its rival's marginal cost.

The two oligopolies differ in the sign of the derivative

$$\pi_2 \; = \; \frac{\partial \pi}{\partial \tilde{c}_i},$$

that is, in the signaling incentives of the firms. In the Cournot oligopoly, there is positive signaling, that is, each firm has an incentive to overstate its power. On the other hand, in the Bertrand oligopoly, there is negative signaling, that is, each firm is better off understating its power. Other than that, in particular, when there is no signaling, the profit functions in the two oligopolies induce the same, qualitatively, incentives.[20]

3.4 NO SIGNALING

When signaling is not possible, the value of each oligopoly license,

$$\pi^{NS}(c_i, c_j) \; = \; \pi(c_i|c_i, c_j),$$

[20] The opposite signaling incentives correspond to the distinction between strategic substitutes and strategic complements, introduced by Bulow et al. [12]. In particular, in the Cournot oligopoly that we have described, quantities are strategic substitutes, since an increase in q_i causes a decrease in the q_j (as well as a decrease in the profit of firm j). On the other hand, in the Bertrand oligopoly, prices are strategic complements, since an increase in p_i causes an increase in p_j (and in the profit of firm j).

is fully determined by the actual marginal costs of the two firms that compete in the market. Mimicking a different type during the auction process cannot affect a firm's potential market profits. It only affects the firm's probability of winning the auction and its expected payment in it.

3.4.1 Simultaneous Auction

Suppose that all firms follow a strictly decreasing bidding strategy $b = \beta(c)$ and consider firm i with marginal cost c_i. Then, by mimicking a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$ during the auction, firm i will win a license if and only if $\tilde{c}_i \leq c^2_{-i}$. In this case, the actual value of this license will be equal to firm i's market profit, $\pi^{NS}(c_i, c^1_{-i})$, which depends on the marginal cost c^1_{-i} of the winner of the other license. Therefore, for a bid $\tilde{b}_i = \beta(\tilde{c}_i)$, the expected payoff of firm i is

$$\Pi(\tilde{c}_i \,|\, c_i) \;=\; \mathbb{P}[c^2_{-i} \geq \tilde{c}_i] \;\times\; [\,\mathbb{E}_{c^1_{-i}}[\pi^{NS}(c_i, c^1_{-i}) \,|\, c^2_{-i} \geq \tilde{c}_i\,] \;-\; \beta(\tilde{c}_i)\,]$$

or, by expanding the term for the firm's expected market profit,

$$\begin{aligned}
\Pi(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(c_i, c^1)\,(N-1)[1 - F(\tilde{c}_i)]^{N-2} f(c^1)\,dc^1 \\
&+\; \int_{\tilde{c}_i}^{\bar{c}} \pi^{NS}(c_i, c^1)\,(N-1)[1 - F(c^1)]^{N-2} f(c^1)\,dc^1 \\
&-\; \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\;\beta(\tilde{c}_i).
\end{aligned}$$

The first-order condition with respect to $\tilde{c}_i$ results in the equation

$$\frac{d}{d\tilde{c}_i}\{\mathbb{P}[c^2_{-i} \geq c_i]\,\beta(c_i)\} \;=\; -\int_{\underline{c}}^{c_i} \pi^{NS}(c_i, c^1)\,(N-1)(N-2)\,[1 - F(c_i)]^{N-3}\, f(c_i) f(c^1)\,dc^1,$$

which requires, in a manner that is standard for "pay-your-bid" auction schemes, that any increase in the firm's expected market profit from a deviation to $\tilde{c}_i \neq c_i$ must be offset by an increase in the firm's expected payment in the auction.

The differential equation derived from the first-order condition, along with the boundary condition expressing the equilibrium behavior of the weakest type $c_i = \bar{c}$,

$$\beta(\bar{c}) \;=\; \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c}, c)\, f(c)\, dc,$$

which guarantees the uniqueness of the solution to that differential equation, provides the equilibrium bidding strategy for this setting.

Proposition 3.1.
In the simultaneous pay-your-bid auction of two oligopoly licenses, in which the winners' marginal costs are revealed truthfully after the auction, the following strategy constitutes a symmetric separating equilibrium:

$$\beta(c_i) \;=\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi^{NS}(c^2, c^1)\, \frac{(N-1)(N-2)\,[1 - F(c^2)]^{N-3}\, f(c^2) f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2 \qquad (3.1)$$

The strategy $\beta(c_i)$ can also be expressed as

$$\beta(c_i) \;=\; \int_{c_i}^{\bar{c}} v^{NS}(c^2)\, \frac{(N-1)(N-2)\,[1 - F(c^2)]^{N-3}\, F(c^2) f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^2,$$

where

$$v^{NS}(c) \;=\; \int_{\underline{c}}^{c} \pi^{NS}(c, c^1)\, \frac{f(c^1)}{F(c)}\, dc^1,$$

for $c \in [\underline{c}, \bar{c}]$, is the expected market profit of a firm with marginal cost c, assuming that its market opponent is stronger. Therefore, in equilibrium, each firm submits a bid equal to the expected market profit of the strongest non-winning firm.

The value of the licences that the two winners of the auction gain is determined endogenously, as a function of the marginal costs of the winning bids. Since these costs are unknown prior to the end of the auction process, it is possible for a firm, when its market opponent turns out to be stronger than expected, to acquire a licence at a price above its ex-post value.[21]

Corollary 3.2.
In the simultaneous auction, the firm with the lowest marginal cost gains a license at a price below its ex-post value. The firm with the second-lowest marginal cost, however, may gain a license at a price above its ex-post value.

Hence, in equilibrium, the stronger of the two oligopolists will always make a positive profit. On the other hand, the weaker oligopolist may regret his participation to the market, because of the price of the license.

3.4.2 Sequential Auction

When the two licenses are allocated by means of a sequence of first-price auctions, then, assuming that the firms follow strictly monotone bidding strategies, the winning bid in the first auction reveals the marginal cost c^1 of the strongest oligopolist. This information affects the bidding for the second license in two distinct manners. First, it allows the remaining firms to learn, prior to the second auction, the actual value of the license for which they compete. In addition, the revealed marginal cost c^1 forms a lower bound for the marginal costs of the remaining firms. Therefore, after the end of the first auction, the firms update their beliefs, so that, for $c \in [c^1, \bar{c}]$,

$$c_i \ \sim \ \tilde{F}(c) \ = \ \frac{F(c) - F(c^1)}{1 - F(c^1)}.$$

[21]We emphasize the difference between this phenomenon and the winner's curse for an auction with interdependent valuations (as well as for our environment). The winner's curse refers to the bad news that a victory in such an auction conveys, namely, that the winner's estimate of the value of the auctioned object has been the most optimistic one. In the equilibrium path, the winner's curse is eliminated by means of an adjustment of the bidders' estimates. Still, it is possible that the losing bidders' private information will be very negative, so as to defy the winner's reasonable expectation and to result in a value that is below the price the winner must pay. It is this phenomenon to which we refer as the winner's regret.

Since the privately known first-period bids do not affect the firms' behavior in the second period, the second auction takes the form of a standard first-price auction with independent private values.

Lemma 3.3.
Suppose that $N-1$ firms, whose marginal costs are i.i.d. according to the distribution function $F(\cdot)$ on $[\underline{c}, \bar{c}]$, compete in a first-price auction for a license to participate in an oligopoly against a firm with known marginal cost $c^1 \in [\underline{c}, \bar{c}]$. In addition, suppose that the firms believe that the unknown marginal costs are bounded below by the value c^1. Then, assuming that the winner's marginal cost is revealed truthfully at the end of the auction, the following strategy constitutes a symmetric equilibrium:

For a marginal cost $c_i \geq c^1$, firm i bids

$$\beta^2(c_i\,|c^1) \;=\; \int_{c_i}^{\bar{c}} \pi^{NS}(c, c^1)\, \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc,$$

while for a marginal cost $c_i < c^1$, firm i bids $b^2 = \beta^2(c^1\,|c^1)$.

A marginal cost $c_i < c^1$ corresponds to an event off the equilibrium path, namely, to the case in which firm i should have won the first license but did not bid according to the strategy β^1 that was prescribed in the first auction.[22] Such a firm enters the second auction knowing that it has the highest valuation and that it will be best-off bidding as if it has marginal cost c^1.

For the analysis of the firms' behavior in the first auction, we will need the strategy $\beta^2(c_i|c_i)$ to be decreasing with respect to the marginal cost c_i. Without this condition, the strategy $\beta^1(c_i)$ that we derive may fail to be strictly decreasing, thus invalidating the argument leading to it. Notice, therefore, that the derivative of $\beta^2(c_i|c_i)$ equals to

[22] Restricting attention to the game described in Lemma 3.3, notice that the possibility of $c_i < c^1$ does not contradict the firms' beliefs and, therefore, does not violate the consistency requirement in the definition of Nash equilibrium. We can simply assume that prior to the draw of the privately known marginal costs c_i, each firm attaches zero probability to the event $c_i < c^1$, for all i.

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;&=\; -\pi^{NS}(c_i,c_i)\;\frac{(N-2)f(c_i)}{1-F(c_i)} \\
&+\;\int_{c_i}^{\bar{c}} \pi_2^{NS}(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&+\;\int_{c_i}^{\bar{c}} \pi^{NS}(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc\;\frac{(N-2)f(c_i)}{1-F(c_i)},
\end{aligned}
$$

or, after integrating the last term by parts, to

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;&=\; \int_{c_i}^{\bar{c}} \pi_2^{NS}(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&+\;\int_{c_i}^{\bar{c}} \pi_1^{NS}(c,c_i)\,\frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc\;\frac{(N-2)f(c_i)}{1-F(c_i)}.
\end{aligned}
\tag{3.2}
$$

Since the derivatives $\pi_1^{NS}(c_i\,|\,c^1)$ and $\pi_2^{NS}(c_i\,|\,c^1)$ are respectively decreasing and increasing in the valuation c_i, we have

$$
\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;\geq\; \pi_2^{NS}(\bar{c},c_i) \;+\; \pi_1^{NS}(c_i,c_i) \;\times\; [\,\mathbb{E}[c^1_{-i,j}\,|\,c^1_{-i,j}\geq c_i]-c_i\,]\;\frac{(N-2)f(c_i)}{1-F(c_i)},
$$

where $c^1_{-i,j}$ denotes the lowest value among N-2 realizations of the marginal cost c_i. Hence, since $\pi_1^{NS} < 0 < \pi_2^{NS}$, if the hazard ratio $f(c_i)/[1-F(c_i)]$ is too small, then the derivative $\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,]$ can be positive.

We avoid this possibility by imposing the following condition:

Assumption 3.4.

The distribution of the firms' marginal costs satisfies the inequality

$$
\int_{c_i}^{\bar{c}} \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc \;\geq\; \sup_{c\geq c_i}\left\{\frac{\pi_2^{NS}(c,c_i)}{-\pi_1^{NS}(c,c_i)}\right\}\;\frac{1-F(c_i)}{(N-2)\,f(c_i)},
$$

for all $c_i \in [\underline{c},\bar{c}]$.

In the case of uniformly distributed marginal costs, Assumption 3.4 is satisfied for the Cournot oligopoly that we have described. Indeed, for $c_i \sim U[\underline{c}, \bar{c}]$, it requires that

$$\frac{\bar{c} - c_i}{N-1} \;\geq\; \frac{\bar{c} - c_i}{2(N-2)},$$

which is true for all $N \geq 3$. On the other hand, for a Bertrand oligopoly with parameter $\gamma \in [0, 1)$, the assumption reduces to requiring that

$$\frac{N-2}{N-1} \;\geq\; \frac{\gamma}{2-\gamma^2},$$

which is satisfied only if the number of firms, N, is sufficiently large, relative to γ.

Assumption 3.4 requires that the inverse hazard rate does not decrease too rapidly. More precisely, as it is shown by the proof of the next Lemma, the inequality

$$\frac{1-F(c)}{f(c)} \;>\; \frac{\pi_2^{NS}(c, c_i)}{-\pi_1^{NS}(c, c_i)} \, \frac{1-F(c_i)}{f(c_i)}$$

remains valid for a sufficiently large interval of values $c \geq c_i$, so that the negative term in the equation (3.2) defining $\frac{d}{dc_i}[\beta^2(c_i|c_i)]$ dominates the positive one.

Lemma 3.5.

Under Assumption 3.4, the function $\beta^2(c_i|c_i)$ is decreasing in $c_i \in [\underline{c}, \bar{c}]$.

In the first auction, suppose that all firms follow a strictly monotone bidding strategy $b^1 = \beta^1(c)$ and consider firm i with marginal cost c_i. Then, by mimicking a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$ in this auction, firm i will win the first license if and only if $\tilde{c}_i \leq c^1_{-i}$. In this case, the actual value of this license will be equal to firm i's market profit, $\pi^{NS}(c_i, c^1_{-i})$, which depends on the marginal cost c^1_{-i} of the winner of the second license.

Therefore, by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \leq c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} [\,\pi^{NS}(c_i, c^1) - \beta^1(\tilde{c}_i)\,]\ (N-1)[1 - F(c^1)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{\underline{c}}^{\tilde{c}_i} [\,\pi^{NS}(c_i, c^1) - \beta^2(c_i \,|\, c^1)\,]\ (N-1)[1 - F(c_i)]^{N-2} f(c^1)\ dc^1,\end{aligned}$$

while by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \geq c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} [\,\pi^{NS}(c_i, c^1) - \beta^1(\tilde{c}_i)\,]\ (N-1)[1 - F(c^1)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{\underline{c}}^{c_i} [\,\pi^{NS}(c_i, c^1) - \beta^2(c_i \,|\, c^1)\,]\ (N-1)[1 - F(c_i)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{c_i}^{\tilde{c}_i} [\,\pi^{NS}(c^1, c^1) - \beta^2(c^1 \,|\, c^1)\,]\ (N-1)[1 - F(c^1)]^{N-2} f(c^1)\ dc^1.\end{aligned}$$

In the second case, the extra term results from the possibility of selling the first license to a firm with marginal cost $c^1 \in [c_i, \tilde{c}_i]$. In this case, firm i bids $b^2 = \beta^2(c^1 \,|\, c^1)$ in the second auction, knowing that it has the lowest marginal cost among the remaining firms.

In both cases, the necessary first-order condition at the endpoint $\tilde{c}_i = c_i$ results in

$$\frac{d}{d\tilde{c}_i}\{\,[1 - F(c_i)]^{N-1}\,\beta^1(c_i)\} \;=\; -\,\beta^2(c_i \,|\, c_i)\ (N-1)\,[1 - F(c_i)]^{N-2}\,f(c_i). \tag{3.3}$$

By solving this differential equation, along with the boundary condition

$$\beta^1(\bar{c}) = \pi^{NS}(\bar{c}, \bar{c})$$

that expresses the bidding behavior of the weakest possible type, we get the strategy

$$\beta^1(c_i) \;=\; \int_{c_i}^{\bar{c}} \beta^2(c^1 \,|\, c^1)\, \frac{(N-1)[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}}\ dc^1,$$

which is part of the equilibrium in our sequential auction.

Proposition 3.6.

In a sequential first-price auction of two oligopoly licenses, in which the winners' bids are announced at the end of each round and their marginal costs are truthfully revealed at the end of the entire auction, the following strategy profile constitutes a symmetric separating equilibrium:

– *In the first auction, each firm i bids*

$$\beta^1(c_i) \; = \; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi^{NS}(c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1. \qquad (3.4)$$

– *In the second auction, if firm i has a marginal cost $c_i \geq c^1 = (\beta^1)^{-1}(b^1)$, where b^1 is the price at which the first license was sold, then it bids*

$$\beta^2(c_i \,|c^1) \; = \; \int_{c_i}^{\bar{c}} \pi^{NS}(c^2, c^1) \, \frac{(N-2)[1-F(c^2)]^{N-3} f(c^2)}{[1-F(c_i)]^{N-2}} \; dc^2, \qquad (3.5)$$

while with a marginal cost $c_i < c^1$, it bids $b^2 = \beta^2(c^1 \,|c^1)$.

The equation (3.4) defining the bidding strategy $\beta^1(c_i)$ is a non-arbitrage condition for the winner of the first auction, which is, as it has turned out, the firm with the lowest marginal cost c_i. If this firm does not participate in the first auction, then it can win the second auction with a bid equal to $\beta^2(c^1|c^1)$, where $c^1 \geq c_i$ is the revealed lowest competing marginal cost. Therefore, to be indifferent, this firm must bid in the first auction an amount equal to its expected bid in the second round.

In the sequential first-price auction, the firms can infer from the first-round price, prior to submitting their second-round bids, the marginal cost of the strongest firm. Therefore, in the sequential auction, unlike the case of the simultaneous auction, it is possible for both winning firms to avoid paying for their licenses prices that exceed their ex-post values.

Corollary 3.7.

In the sequential first-price auction, both licenses are sold at prices below their ex-post values.

Proof:

The proof for the license sold in the first auction, to the firm with the lowest marginal cost, follows from a direct argument, identical to the one for the simultaneous auction.

In the second auction, the remaining bidders know the marginal cost c^1 of the first firm and, therefore, their value for the license, so that they never bid an amount above it.

$\square$

The revelation of the marginal cost of the strongest firm after the end of the first auction makes the second license less profitable for the remaining firms. As a result, these firms' bidding for the second license becomes less aggressive.

Corollary 3.8.

In the sequential first-price auction, the prices at which the two licenses are sold form a super-martingale:

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,c_i] \;\leq\; \beta^1(c_i).$$

Proof:

Suppose that the first license is sold at a price $b^1 = \beta^1(c_i)$, corresponding to a marginal cost c_i. Then, conditional on this information, the expected price for the second license will be

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,b^1 = \beta^1(c_i)] \;=\; \mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,c^1_{-i} \geq c_i]$$

and, since $\beta^2(c^1|c^1) \;>\; \beta^2(c^1|c_i)$, for all $c^1 \in [c_i, \bar{c}]$,

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,b^1 = \beta^1(c_i)] \;<\; \mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c^1_{-i})\,|\,c^1_{-i} \geq c_i] \;=\; \beta^1(c_i),$$

as required for the result.

$\square$

The super-martingale property implies that the (ex-ante) expected price of the second license is lower than the expected price of the first license:

$$
\begin{aligned}
\mathbb{E}_{c_i, c_{-i}^1}[\,\beta^2(c_{-i}^1|c_i)\,] &= \mathbb{E}_{c_i}[\,\mathbb{E}_{c_{-i}^1}[\beta^2(c_{-i}^1|c_i)\,|\,b^1 = \beta^1(c_i)]\,] \\
&< \mathbb{E}_{c_i}[\,\beta^1(c_i)\,].
\end{aligned}
$$

Hence, the information revealed in the process of the sequential auction makes the expected prices decrease.

3.4.3 Comparison of Auction Schemes

The two auction schemes that we have examined, the simultaneous pay-your-bid auction and the sequential first-price auction, have turned out to be allocation equivalent. The licenses are allocated to the two strongest firms, that is, to the firms with the lowest marginal costs. However, the manner in which the firms bid in each scheme is different.

In the simultaneous auction, the firms submit their bids without knowing the actual value of the licenses that they try to acquire. This value is determined endogenously, by the marginal costs of the firms that will compete in the market, and is revealed only at the end of the auction. In addition, the firms cannot know whether, in case they win one of the two licenses, they will face a stronger or a weaker market competitor. Therefore, while bidding, they need to take both possibilities into account.

On the other hand, in the sequential auction, the firms bidding for the second license know its actual value, since the marginal cost of the first oligopolist has been revealed by the winning bid in the first auction. Furthermore, in the auction for the first license, the firms know that if they win, then they will face a weaker market competitor. Therefore, they can bid more aggressively, since they are protected from the more negative of the two possibilities.

The following result shows that these informational differences do not affect the revenue generated by the auctioneer in the two schemes.

Proposition 3.9.

The simultaneous pay-your-bid auction and the sequential first-price auction of two Cournot oligopoly licenses result in the same expected revenue for the auctioneer.

Proof:

It is easy to show, by changing the order of integration in the definition of $\beta(c_i)$, that

$$\begin{aligned} \mathbb{P}[c^2_{-i} \geq c_i]\ \beta(c_i) &= [1-F(c_i)]^{N-1}\ \beta^1(c_i) \\ &+ (N-1)[1-F(c_i)]^{N-2} F(c_i) \int_{\underline{c}}^{c_i} \beta^2(c_i|c^1) \frac{f(c^1)}{F(c_i)}\, dc^1. \end{aligned} \tag{3.6}$$

This means that the expected payments of a firm with marginal cost c_i in the simultaneous auction, $R^D_{NS}(c_i)$, and in the sequential auction, $R^S_{NS}(c_i)$, are equal.

Therefore, since this is true for any $c_i \in [\underline{c}, \bar{c}]$, it follows that

$$N \int_{\underline{c}}^{\bar{c}} R^D_{NS}(c_i) f(c_i)\, dc_i \;=\; N \int_{\underline{c}}^{\bar{c}} R^S_{NS}(c_i) f(c_i)\, dc_i,$$

so that the two auction schemes raise the same expected revenue.

□

Hence, the auctioneer is indifferent, with respect to the revenue that he expects to raise, between the two auction schemes. Similarly, the bidders are indifferent, with respect to the payments that they expect to make, between the simultaneous and the sequential auction.

The two auction formats, however, allocate each of the two licenses at different prices.

Proposition 3.10.

The stronger of the two oligopolists pays a higher price for his license in the sequential first-price auction than in the simultaneous pay-your-bid auction; the weaker oligopolist pays a lower price for his license in the sequential auction than in the simultaneous auction:

$$\beta^1(c_i) \;>\; \beta(c_i) \;>\; \mathbb{E}_{c^1_{-i}}[\,\beta^2(c_i\,|\,c^1_{-i})\,|\,c^1_{-i} < c_i].$$

Therefore, in the first auction of the sequential format, the firms bid more aggressively than in the simultaneous auction, knowing that if they win, they will necessarily face a weaker market competitor. On the other hand, the firms participating in the second auction bid less aggressively, on average, since they know that they will have to face a stronger market competitor.

Corollary 3.11.

The stronger of the two oligopolists makes a higher total profit in the simultaneous pay-your-bid auction. The weaker oligopolist makes a higher total profit in the sequential first-price auction.

Proof:

Since both auction formats result to the same market supply and profits, any change in the firms' total profits will be the consequence of a change in the prices that the firms pay for their licenses. Therefore, the result follows directly from Proposition 3.10.

$\square$

Hence, an auctioneer aiming at a more equal distribution of the wealth generated in the oligopolistic market will prefer the sequential first-price auction to the simultaneous pay-your-bid auction.

3.5 POSITIVE SIGNALING: COURNOT COMPETITION

When signaling is possible, we need to revise the firms' valuations for the oligopoly licenses so as to incorporate to them the informational rents that the firms can extract. In the case of Cournot competition, in which the signaling incentives are positive, the firms' valuations shall be adjusted upwards.

To demonstrate the need for this adjustment, consider an auction of a single license to compete against a monopolist with known marginal cost c^1. When signaling is not possible, a firm i with marginal cost $c_i \in [\underline{c}, \bar{c}]$ would be willing to bid for the license an amount up to

$$\pi^{NS}(c_i, c^1) \;=\; \pi(c_i \,|\, c_i, c^1).$$

If signaling becomes possible, then, by mimicking a marginally stronger type $\tilde{c}_i < c_i$, firm i can increase its market profit, in case it wins the license, by approximately

$$-\pi_2(c_i \,|\, \tilde{c}_i, c^1)\, d\tilde{c}_i \;>\; 0.$$

Therefore, the maximal amount that the firm would be willing to bid exceeds $\pi^{NS}(c_i, c^1)$.

Since the effects of mimicking a different type depend on the auction format and on the equilibrium strategies that the bidders use, the manner in which the firms adjust their valuations will also depend on these elements. Therefore, the firms' valuations will be different in each auction environment that we consider.

Overall, under positive signaling, a firm's deviation to signaling a stronger type will have two effects. First, assuming that the bidding strategies are monotone, it will increase the probability of acquiring a license. Second, it will increase the profitability of the license that the firm may win. Hence, to offset both these effects, the firms must bid more aggressively than they would do if signaling were not possible.

3.5.1 Simultaneous Auction

Suppose that all firms follow a strictly decreasing bidding strategy $b = \beta(c)$ and consider firm i with marginal cost c_i. If firm i mimics a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$ during the auction, by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$, then its expected total payoff will be

$$\begin{aligned}
\Pi(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} \pi(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(c^1)\,dc^1 \\
&+\; \int_{\tilde{c}_i}^{\bar{c}} \pi(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\,dc^1 \\
&-\; \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\;\beta(\tilde{c}_i).
\end{aligned}$$

The first-order condition with respect to $\tilde{c}_i$ results in the equation

$$\begin{aligned}
\frac{d}{d\tilde{c}_i}\{\mathbb{P}[c^2_{-i} \geq c_i]\,\beta(c_i)\} \;&=\; \int_{\underline{c}}^{c_i} \pi_2(c_i \,|\, c_i, c^1)\;(N-1)\,[1-F(c_i)]^{N-2}\,f(c^1)\;dc^1 \\
&+\; \int_{c_i}^{\bar{c}} \pi_2(c_i \,|\, c_i, c^1)\;(N-1)\,[1-F(c^1)]^{N-2}\,f(c^1)\;dc^1 \qquad (3.7) \\
&-\; \int_{\underline{c}}^{c_i} \pi(c_i \,|\, c_i, c^1)\,(N-1)(N-2)\,[1-F(c_i)]^{N-3}\,f(c_i)f(c^1)\,dc^1,
\end{aligned}$$

which requires that any increase in the firm's expected market profit from a deviation to $\tilde{c}_i \neq c_i$, as this may be augmented by the expected gains from false signaling[23], must be offset by an increase in the firm's expected payment in the auction.

The differential equation derived from the first-order condition, along with the boundary condition expressing the behavior of the weakest type, $c_i = \bar{c}$,

$$\beta(\bar{c}) \;=\; \int_{\underline{c}}^{\bar{c}} \pi(\bar{c} \,|\, \bar{c}, c)\,f(c)\,dc,$$

[23]In particular, when $c_i < c^1_{-i} < c^2_{-i}$, the change in the firm's expected market profit is entirely the consequence of false signaling.

provides the equilibrium strategy for this setting.

Proposition 3.12.

In the simultaneous pay-your-bid auction of two Cournot oligopoly licenses, in which the winners' bids are revealed at the end of the auction, there is a symmetric separating equilibrium given by the strategy

$$\begin{aligned} \beta(c_i) &= \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, f(c^2)\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^1 \, dc^2 \\ &+ \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)\,[1-F(c^2)]^{N-2}\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^1 \, dc^2 \\ &+ \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)\,[1-F(c^1)]^{N-2}\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^1 \, dc^2. \end{aligned} \tag{3.8}$$

In this equilibrium, the firm with the lowest marginal cost gains its license at a price below its ex-post value. The firm with the second-lowest marginal cost, however, may gain its license at a price above its ex-post value.

Proof:

Notice that the strategy $\beta(c_i)$ can be expressed as

$$\beta(c_i) = \int_{c_i}^{\bar{c}} u(c^2) \, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, F(c^2)\, f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^2,$$

where

$$\begin{aligned} u(c) &= \int_{\underline{c}}^{c} \pi(c \,|\, c, c^1) \times \frac{f(c^1)}{F(c)} \, dc^1 \\ &+ \int_{\underline{c}}^{c} [\, -\pi_2(c \,|\, c, c^1) \, \frac{1-F(c)}{(N-2) f(c)} \,] \times \frac{f(c^1)}{F(c)} \, dc^1 \\ &+ \int_{c}^{\bar{c}} [\, -\pi_2(c \,|\, c, c^1) \, \frac{1-F(c)}{(N-2) f(c)} \,] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c)]^{N-2} F(c)} \, dc^1, \end{aligned}$$

for $c \in [\underline{c}, \bar{c}]$, is the valuation of a firm with marginal cost c, assuming that its market opponent is stronger and taking into account the informational rents from false signaling.

Therefore, the proof of this result parallels the one of Proposition 3.1, with $u(c)$ in place of $v^{NS}(c)$. Its details, in particular, the argument establishing that $u(c)$ is decreasing, can be found in the Appendix.

$\square$

The first term in the bidding strategy,

$$\beta_{NS}(c_i) \;=\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, f(c^1) f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2,$$

corresponds to the amount that a firm with marginal cost c_i would bid, if signaling were not possible.

The second and third terms,[24]

$$\begin{aligned}\beta_S(c_i) \;&=\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2 \\ &+\; \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2,\end{aligned}$$

correspond to the amount by which the firms should augment their bids so as to offset possible gains from false signaling by the other firms. Even though there can be no false signaling in equilibrium, without this amount, it would be possible for a firm to deviate into mimicking a stronger type and, therefore, to increase both the probability of winning a license and, through false signaling, the value of that license.

[24] The two double integrals do not allow, of course, for the possibility of $c^1_{-i} > c^2_{-i}$. Rather, in each case, the outer integral determines a value $c^2 \in [c_i, \bar{c}]$ such that $c^2 \leq c^2_{-i}$, while $c^1_{-i} \leq c^2_{-i}$. This creates two possibilities, namely, either $c^1_{-i} \leq c^2 \leq c^2_{-i}$, corresponding to the first signaling term, or $c^2 \leq c^1_{-i} \leq c^2_{-i}$, corresponding to the second signaling term.

3.5.2 Sequential Auction

In the sequential auction, the winning bid in the first round reveals the marginal cost c^1 of the strongest oligopolist. Therefore, similarly to the case in which signaling is not possible, in the second round, the firms know precisely the value of the license for which they bid. In addition, they know that the other firms' marginal costs are bounded below by c^1; thus, they update their beliefs, so that, for all $c \in [c^1, \bar{c}]$,

$$c_i \sim \tilde{F}(c) = \frac{F(c) - F(c^1)}{1 - F(c^1)}.$$

Since the privately known first-period bids do not affect the firms' incentives in the second auction, our second-period bidding environment belongs to the class of auctions studied by Das Varma [23], Goeree [37] and Katzman and Rhodes-Kropf [60]. In the following Lemma, we apply their analysis to our setting:

Lemma 3.13.
Suppose that $N-1$ firms, whose marginal costs are i.i.d. according to the distribution function $F(.)$ on $[\underline{c}, \bar{c}]$, compete in a first-price auction for a license to participate in a Cournot oligopoly against a firm with known marginal cost c^1. In addition, suppose that the firms believe that the unknown marginal costs are bounded below by the value $c^1 \in [\underline{c}, \bar{c}]$. Then, assuming that the winner's bid is revealed at the end of the auction, the following strategy constitutes a symmetric equilibrium:

For a marginal cost $c_i \geq c^1$, firm i bids

$$\beta^2(c_i\,|c^1) \;=\; \int_{c_i}^{\bar{c}} \left[\pi(c|c, c^1) - \pi_2(c|c, c^1)\,\frac{1 - F(c)}{(N-2)f(c)}\right] \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc,$$

while for a marginal cost $c_i < c^1$, firm i bids $b^2 = \beta^2(c^1\,|c^1)$.

In addition, for any $c^1 \in [\underline{c}, \bar{c}]$, the strategy $\beta^2(c_i|c^1)$ is strictly decreasing in $c_i \in (c^1, \bar{c}\,]$, so that, along the equilibrium path, an auction price $b^2 < \beta^2(c^1|c^1)$ fully reveals the marginal cost of the winning firm.

Every firm i submits a bid that is equal to the value that its strongest competitor is expected to have for the license, assuming that this competitor has marginal cost $c \geq c_i$,

$$\beta^2_{NS}(c_i \,|\, c^1) \;=\; \int_{c_i}^{\bar{c}} \pi(c \,|\, c, c^1)\, \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc,$$

augmented by the amount needed to offset possible gains from false signaling by its competitors, namely,

$$\beta^2_{S}(c_i \,|\, c^1) \;=\; \int_{c_i}^{\bar{c}} -\pi_2(c \,|\, c, c^1)\, \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\, dc.$$

Without this amount, it would be possible for a firm i to deviate into mimicking $\tilde{c}_i < c_i$ and to increase both the probability of winning a license and, through false signaling, the value of that license. For the second gain to be offset, each firm needs to bid above $\beta^2_{NS}(c_i \,|c^1)$, by an amount at least as large as $\beta^2_{S}(c_i \,|c^1)$.

To ensure that the strategy $\beta^2(c_i|c_i)$ is decreasing with respect to the marginal cost c_i, we will need to modify Assumption 3.4 in the following manner:

Assumption 3.14.
The distribution of the firms' marginal costs satisfies the inequality

$$\int_{c_i}^{\bar{c}} \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\, dc \;\geq\; \sup_{c \geq c_i} \left\{ \frac{v_2(c, c_i)}{-\tilde{v}_1(c, c_i)} \right\} \; \frac{1-F(c_i)}{(N-2)\, f(c_i)},$$

for all $c_i \in [\underline{c}, \bar{c}]$*, where*

$$\tilde{v}_1(c, c^1) \;=\; \frac{d}{dc}[\pi(c \,|\, c, c^1)] \;-\; \frac{d}{dc}[\pi_2(c \,|\, c, c^1)]\, \frac{1-F(c)}{(N-2)\, f(c)}.$$

For the Cournot duopoly that we have described, we have

$$\sup_{c \geq c_i} \left\{ \frac{v_2(c, c_i)}{-\tilde{v}_1(c, c_i)} \right\} \;=\; \sup_{c \geq c_i} \left\{ \frac{\pi_1(c \,|\, c, c_i)}{-\pi_2(c \,|\, c, c_i)} \right\},$$

so that Assumption 3.14 reduces to Assumption 3.4. In particular, for marginal costs $c_i \sim U[\underline{c}, \bar{c}]$, the assumption is always satisfied.

Lemma 3.15.

Under Assumption 3.14, the function $\beta^2(c_i|c_i)$ is decreasing in $c_i \in [\underline{c}, \bar{c}]$.

In the first auction, arguing in the same manner as in the non-signaling case, suppose that all firms follow a strictly monotone bidding strategy $\beta^1(c)$ and consider firm i with marginal cost c_i. By bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \le c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\tilde{c}_i}^{\bar{c}} [\, \pi(c_i \,|\, \tilde{c}_i, c^1) - \beta^1(\tilde{c}_i) \,]\; (N-1)[1 - F(c^1)]^{N-2} f(c^1)\; dc^1 \\ &+\; \int_{\underline{c}}^{\tilde{c}_i} [\, \pi(c_i \,|\, c_i, c^1) - \beta^2(c_i \,|\, c^1) \,]\; (N-1)[1 - F(c_i)]^{N-2} f(c^1)\; dc^1,\end{aligned}$$

while by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \ge c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\tilde{c}_i}^{\bar{c}} [\, \pi(c_i \,|\, \tilde{c}_i, c^1) - \beta^1(\tilde{c}_i) \,]\; (N-1)[1 - F(c^1)]^{N-2} f(c^1)\; dc^1 \\ &+\; \int_{\underline{c}}^{c_i} [\, \pi(c_i \,|\, c_i, c^1) - \beta^2(c_i \,|\, c^1) \,]\; (N-1)[1 - F(c_i)]^{N-2} f(c^1)\; dc^1 \\ &+\; \int_{c_i}^{\tilde{c}_i} [\, \pi(c_i \,|\, c^1, c^1) - \beta^2(c^1 \,|\, c^1) \,]\; (N-1)[1 - F(c^1)]^{N-2} f(c^1)\; dc^1.\end{aligned}$$

In both cases, the necessary first-order condition at the endpoint $\tilde{c}_i = c_i$ results in the differential equation

$$\begin{aligned}\frac{d}{d\tilde{c}_i}\{\, [1 - F(c_i)]^{N-1}\, \beta^1(c_i)\} \;&=\; \int_{c_i}^{\bar{c}} \pi_2(c_i \,|\, c_i, c^1)\; (N-1)\, [1 - F(c^1)]^{N-2}\, f(c^1)\; dc^1 \\ &-\; \beta^2(c_i \,|\, c_i)\; (N-1)\, [1 - F(c_i)]^{N-2}\, f(c_i).\end{aligned}$$

By solving this differential equation, along with the boundary condition

$$\beta^1(\bar{c}) = \pi(\bar{c} \,|\, \bar{c}, \bar{c}),$$

we get the strategy

$$\begin{aligned}
\beta^1(c_i) &= \int_{c_i}^{\bar{c}} \beta^2(c^1 \,|\, c^1) \, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^1 \\
&- \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} \pi_2(c^2 \,|\, c^2, c^1) \; \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^1 \, dc^2,
\end{aligned}$$

for the equilibrium of the sequential auction.

Proposition 3.16.

In a sequential first-price auction of two Cournot oligopoly licenses, in which the winners' bids are revealed at the end of each auction, the following strategy profile constitutes a symmetric separating equilibrium:

– *In the first auction, each firm i bids*

$$\begin{aligned}
\beta^1(c_i) &= \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1 \\
&+ \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \; \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1 \qquad (3.9) \\
&+ \int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1) \; \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1.
\end{aligned}$$

– *In the second auction, if firm i has a marginal cost $c_i \geq c^1 = (\beta^1)^{-1}(b^1)$, where b^1 is the price at which the first license was sold, then it bids*

$$\begin{aligned}\beta^2(c_i\,|\,c^1) &= \int_{c_i}^{\bar{c}} \pi(c^2\,|\,c^2,c^1)\,\frac{(N-2)[1-F(c^2)]^{N-3}f(c^2)}{[1-F(c_i)]^{N-2}}\;dc^2\\ &+ \int_{c_i}^{\bar{c}} -\pi_2(c^2\,|\,c^2,c^1)\,\frac{[1-F(c^2)]^{N-2}}{[1-F(c_i)]^{N-2}}\;dc^2, \end{aligned} \tag{3.10}$$

while with a marginal cost $c_i < c^1$, it bids $b^2 = \beta^2(c^1\,|c^1)$.

Proof:

Notice that the strategy $\beta^1(c_i)$ can be expressed as

$$\beta(c_i) = \int_{c_i}^{\bar{c}} v(c^1)\,\frac{(N-1)[1-F(c^1)]^{N-2}\,f(c^1)}{[1-F(c_i)]^{N-1}}\,dc^1,$$

where

$$\begin{aligned} v(c) &= \int_{c}^{\bar{c}} \pi(c\,|\,c,c^1)\times\frac{(N-2)\,[1-F(c^1)]^{N-3}f(c^1)}{[1-F(c)]^{N-2}}\;dc^1\\ &+ \int_{c}^{\bar{c}} [\,-\pi_2(c\,|\,c,c^1)\,\frac{1-F(c)}{(N-2)f(c)}\,]\times\frac{(N-2)\,[1-F(c^1)]^{N-3}f(c^1)}{[1-F(c)]^{N-2}}\;dc^1\\ &+ \int_{c}^{\bar{c}} [\,-\pi_2(c\,|\,c,c^1)\,\frac{1-F(c)}{(N-2)f(c)}\,]\times\frac{(N-1)[1-F(c^1)]^{N-2}f(c^1)}{[1-F(c)]^{N-1}}\;dc^1, \end{aligned}$$

for $c \in [\underline{c},\bar{c}]$, is the valuation of a firm with marginal cost c, assuming that its market opponent is weaker.

Therefore, the proof of this result parallels the one of Proposition 3.6, with $v(c)$ in place of $u(c)$. Its details can be found in the Appendix.

□

Similarly to the case of non-signaling, the equation (3.9) defining the bidding strategy $\beta^1(c_i)$ is a non-arbitrage condition for the firm with the lowest marginal cost, c_i. For this firm to be indifferent between winning the first or the second auction, its bid in the first round must exceed its expected bid in the second round by precisely its expected gain from signaling a stronger type.

The first term of the bidding strategy for the first auction,

$$\beta^1_{NS}(c_i) \;\; = \;\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^1) f(c^2)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1,$$

corresponds, again, to the amount that firm i would bid if signaling were not possible. The remaining two terms,

$$\begin{aligned} \beta^1_S(c_i) \;\; &= \;\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1 \\ &+ \;\; \int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1, \end{aligned}$$

correspond to the amounts that the firms must add to their bids in order to offset possible gains from false signaling by their competitors.

Because of the information revealed in the first round of the sequential auction, the weaker of the two oligopolists is able to avoid the possibility of winning his license at a price above its ex-post value. Thus, along the equilibrium path, both oligopolists make a positive profit.

In addition, the revelation of the marginal cost of the strongest firm makes the second license less profitable and, therefore, the firms' bidding for it less aggressive. As a result, the prices of the two licenses form a super-martingale,

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|\, c_i) \,|\, c_i] \;\leq\; \beta^1(c_i),$$

so that the expected price of the second license is lower than that of the first license.

3.5.3 Comparison of Auction Schemes

Our analysis of the simultaneous and the sequential auctions under positive signaling parallels the analysis of the same auctions without signaling. Non-surprisingly, so do the results regarding the comparison of the equilibria that we derived.

Proposition 3.17.
The simultaneous pay-your-bid auction and the sequential first-price auction of two Cournot oligopoly licenses result in the same expected revenue for the auctioneer.

Therefore, the informational differences between the two auction schemes do not affect the expected revenue of the auctioneer or the expected payment of the bidders.

Proposition 3.18.
The stronger of the two oligopolists pays a higher price for his license in the sequential first-price auction than in the simultaneous pay-your-bid auction; the weaker oligopolist pays a lower price for his license in the sequential auction than in the simultaneous auction:

$$\beta^1(c_i) \ > \ \beta(c_i) \ > \ \mathbb{E}_{c^1_{-i}}[\,\beta^2(c_i \,|\, c^1_{-i}) \,|\, c^1_{-i} < c_i].$$

Therefore, the stronger of the two oligopolists makes a higher total profit in the simultaneous pay-your-bid auction while the weaker oligopolist makes a higher total profit in the sequential first-price auction.

Hence, an auctioneer aiming at a more equal distribution of the wealth generated in the Cournot duopoly will still prefer the sequential auction to the simultaneous one, even when signaling is possible.

3.6 NEGATIVE SIGNALING: BERTRAND COMPETITION

In the case of Bertrand competition, the firms have an incentive to signal a weaker type. Therefore, opposite to the case of Cournot competition, the firms' valuations shall be adjusted downwards. Because of this adjustment, if the firms' signaling incentive is too strong, it is possible that a positive measure of bidder types will have valuations below zero. To avoid this problem, we need to assume the presence of a large number of firms competing in the auction.

Under this assumption, we can construct an equilibrium in strictly monotone bidding strategies for the simultaneous auction. On the other hand, in the sequential auction, since it is not possible to balance the bidders' signaling profits from deviating into waiting for the second round, such an equilibrium turns out not to exist.

3.6.1 Simultaneous Auction

By repeating the argument that we used for the Cournot oligopoly, that is, by assuming the use of a strictly decreasing bidding strategy $b = \beta(c)$ and considering the necessary first-order condition for the expected payoff function $\Pi(\tilde{c}_i|c_i)$ of some firm i at $\tilde{c}_i = c_i$, we can derive the equilibrium for this setting.

Proposition 3.19.

In the simultaneous pay-your-bid auction of two Bertrand oligopoly licenses, with the winners' bids revealed at the end of the auction, if there are sufficiently many bidders, then there is a symmetric separating equilibrium given by the strategy

$$\begin{aligned}
\beta(c_i) \;&=\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, f(c^2)\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2 \\
&+\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)\,[1-F(c^2)]^{N-2}\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2 \qquad (3.11)\\
&+\; \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)\,[1-F(c^1)]^{N-2}\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2.
\end{aligned}$$

In this equilibrium, the firm with the lowest marginal cost gains its license at a price below its ex-post value. The firm with the second-lowest marginal cost, however, may gain its license at a price above its ex-post value.

Notice that by understating its strength, a firm gains in terms of its expected market profit and of a lower payment in the auction, assuming that it wins an oligopoly license. On the other hand, it suffers the cost of a lower probability of winning the auction. This cost increases as the number of the bidders in the auction, N, becomes larger. Therefore, if N is sufficiently large, the cost is so severe that it can always counter-balance possible gains from false signaling.

3.6.2 Sequential Auction

In the sequential auction, the firms' incentive to signal a weaker type turns out to be too strong. Contrary to the case of the simultaneous auction, it is not possible to construct a symmetric separating equilibrium.

Proposition 3.20.
In a sequential first-price auction of two Bertrand oligopoly licenses, in which the winners' bids are revealed at the end of each auction, there is no symmetric equilibrium in monotone strategies.

In the presence of a sufficiently large number of bidders, as shown in Das Varma [23], the strategy $\beta^2(c_i|c^1)$, given in Lemma 3.13, forms the unique symmetric equilibrium for the continuation game that follows the allocation of the first license to a firm with marginal cost $c^1 \in [\underline{c}, \bar{c}]$. In addition, by adapting Assumption 3.14 to the Bertrand oligopoly setting, one can show that $\beta^2(c|c)$ is decreasing in c. Finally, by replicating the argument leading to Proposition 3.16, one can derive the bidding strategy $\beta^1(c_i)$, identical to the one used in the Cournot oligopoly, as the unique solution to the necessary first-order condition.

This strategy, however, cannot be part of an equilibrium. Although, the non-signaling component of $\beta^2(c_i|c^1)$ is sufficiently more aggressive than the non-signaling component of $\beta^1(c_i)$, so that to just eliminate the incentive to wait for the second round (if signaling were not possible), the signaling component of $\beta^2(c_i|c^1)$ cannot counter-balance the corresponding component of $\beta^1(c_i)$. As a result, each firm has a profitable deviation from β^1 into waiting, $b_i^1 = 0$, for the second round.

In particular, trying to diminish the potential gains from signaling by increasing the number of firms, as in the case of the simultaneous auction, cannot produce any result. The deviation into waiting for the second round does not cost any firm in terms of the probability of acquiring a license, so, changing the number of firms is ineffective.

3.7 CONCLUSIONS

We have examined two multi-unit auction schemes with allocative and, possibly, informational externalities, in particular, two auctions of oligopoly licenses.

When there is no signaling, we have provided a rationale for the use of a sequential procedure. The information generated during this procedure leads to more informative bidding. Even though this does not affect the seller's expected revenue, or the bidders' expected payments, the two winners are protected from the possibility of regret, that is, from buying a license at a price that exceeds its ex-post value. In addition, the strongest oligopolist has to pay a higher price for his license than he would pay in a simultaneous auction, whereas the weaker oligopolist pays a lower price. Therefore, the sequential auction results in a more even distribution of the wealth generated in the oligopoly.

When signaling is possible, these results remain valid only in the case of positive signaling incentives, as in the Cournot oligopoly. On the other hand, with negative signaling incentives, as in the Bertrand oligopoly, there is no symmetric monotone equilibrium for the sequential auction. Hence, in this environment, an efficient allocation is achieved only by means of a simultaneous auction.

The two auction formats will cease to be revenue equivalent, if we consider affiliated marginal costs. In this case, according to the intuition of the linkage principle, the sequential format will dominate, in terms of revenue, the simultaneous auction. The two auctions will also generate different expected seller revenues, if the firms face participation costs. In particular, if the winning bid in the first round of the sequential auction is sufficiently low, then some bidder types that would otherwise not participate may decide to bid in the second round. In this case, however, the auction scheme that is preferable for the seller may depend on the distribution of the firms' marginal costs.

A seller may also increase his expected revenue by adopting different information revelation rules and, therefore, allowing for different signaling possibilities. According to the intuition derived from the study of the auction of a single license, schemes that reveal more information about the winners will be revenue dominant in the case of positive signaling incentives, while schemes that disable signaling will be dominant in the case of negative signaling incentives.

Finally, it would be interesting to investigate experimentally the bidding behavior in a sequential auction with negative informational externalities. An experimental study may reveal patterns of behavior that can be of interest to sellers that would like to consider the use of a sequential procedure.

4.0 SEQUENTIAL AUCTIONS WITH PARTICIPATION COSTS

4.1 INTRODUCTION

In many auctions, the bidders can participate only at a cost, associated with either the preparation or the submission of their bids. For example, they may need to travel to the auction site or to hire an agent to bid on their behalf. It is also possible that they may enter the auction only at a fee, paid to the auctioneer (who can differ from the seller). Furthermore, in order to participate in the auction, the bidders may have to forego other profitable opportunities. Finally, the bidders may need to finance their bids by posting security bonds, aiming to protect the seller from spurious bidding. These costs, which are independent from those of learning one's valuation for the auctioned object, may affect a potential bidder's decision to participate in an auction as well as his bidding behavior in it.

When such bidding costs are present, the seller typically suffers from low levels of participation as well as from more restrained bidding by the buyers who enter the auction. To combat these effects, the seller may attempt to finance the bidders' entry by reimbursing part of their expenses. In addition, the seller may post a reserve price, in order to force the participating buyers to bid more intensively.

In the case of multi-unit sales, an additional option is often available to the seller, that of auctioning the goods sequentially, one unit at a time, rather than simultaneously. A sequential procedure has the advantage of generating information during the bidding process, in a manner that affects the buyers' subsequent participation and bidding behavior. In particular, unfavorable outcomes in the early rounds, characterized by relatively low sale prices, are likely to encourage stronger participation in the later rounds, thus amplifying the bidding competition and enhancing the seller's revenue.

The present chapter examines this possibility, by analyzing a sequential first-price auction for selling two identical units to buyers with single-unit demands. We assume that the buyers who participate in the first-round of the auction can also bid in the second round at no additional cost; and that the seller reveals the first-round price prior to the beginning of the second round. In this setting, we construct a symmetric equilibrium and, for particular cases of valuation distributions, we compare its expected seller revenue to that of the symmetric monotone equilibrium in a simultaneous "pay-your-bid" auction.

In both auction formats, the buyers adopt cut-off strategies, such that they enter the auction if and only if their valuations exceed a certain threshold. Because of the possibility of second-round entry, threshold for the first-round of the sequential auction is higher than that of the simultaneous auction. However, if the first unit is sold at a low price or not at all, the second-round threshold may decrease, possibly below that for the simultaneous auction.

The participating buyers bid in a strictly monotone manner, with the lowest type, therefore, bidding zero. In the sequential auction, as it turns out, the price paths do not behave in a uniform manner. Rather, there is a certain threshold price such that, for a first-round prices above it, the second-round price is expected to decrease while for first-round prices below it, because of stronger participation, the second-round price is expected to increase.

To compare the expected seller revenue from the sequential and simultaneous auctions, we need to use simulations, since analyticity is lost even in simple cases. For valuations that follow a uniform or a power distribution, we observe that the sequential auction generates a higher revenue when the number of potential bidders is large relative to the participation cost and to the degree of concavity. Given a fixed number of buyers (that is not too low), the sequential auction dominates when the participation cost is below a certain value. This value becomes larger, rather rapidly, for larger numbers of buyers or for more convex distributions.

Most of the literature on auctions with costly participation[1] has focussed on the case of single-unit sales. Green and Laffont [38] and Samuelson [86] calculate symmetric equilibria

[1] There are two other directions in the literature on auctions with costly participation. The first direction has studied auctions in which the bidders must incur a fixed or variable cost in order to learn their valuations, possibly in the form of pre-bid investment in R&D. For example, see Matthews [71], French and McCormic [29], McAfee and McMillan [72], Engelbrecht-Wiggans [24], Tan [92], Levin and Smith [66], Chakraborty and Kosmopoulou [17], Ye [96] and Lu [67]. The other direction has looked at sales in which it is costly for the seller to elicit price offers; for example, see McAfee and McMillan [73], Burguet [13] and Crémer et al. [21]. For a general survey of the subject of costly information acquisition, see Bergemann and Välimäki [10].

in cut-off strategies for the second- and the first-price auctions respectively. In these equilibria, the seller's expected revenue may decrease as the number of potential bidders increases. Stegeman [90] shows that the second-price auction can always implement the ex ante efficient outcome (that is, it has an equilibrium that maximizes the expected total surplus) whereas the first-price auction has an efficient equilibrium if and only if the efficient equilibrium of the second-price auction is symmetric. In addition, he constructs an equilibrium in asymmetric cut-off strategies for the first-price auction. Furthermore, Tan and Yilankaya [93] show that the second-price auction always has a symmetric equilibrium and identify conditions on the distribution of the buyers' valuations for this equilibrium to be unique (and therefore, efficient). In addition, they identify conditions under which the second-price auction has asymmetric cut-off equilibria, describe these equilibria and show that they are suboptimal for the seller. In an alternative set up, a second-price auction in which the buyers' valuations are commonly known but the entry costs are private information, Kaplan and Sela [58] construct an equilibrium in cut-off strategies in which a bidder's expected payoff may, somewhat surprisingly, decrease in his valuation. Finally, for a general class of games, including standard auctions, Landsberger [65] shows that even small participation costs can have a dramatic effect upon entry, especially when the number of players is large.

Regarding the issue of revenue maximization, Gal et al. [31] show that it is beneficial for the seller to commit to a partial reimbursement policy. Celik and Yilankaya [16] construct the revenue-maximizing auction mechanism. In this mechanism, first, the object is given to the participant with the highest valuation and, second, participation is determined in a non-stochastic manner by appropriately designed thresholds. Under certain conditions on the distribution of the buyers' valuations, these thresholds can be asymmetric. In this case, the optimal mechanism, which is ex post inefficient, can be implemented either by a second-price auction with individualized reserve prices and entry fees (or subsidies) or by an asymmetric equilibrium of an anonymous second-price auction. Finally, Lu [68] explores revenue maximization in an environment in which both the buyers' valuations and participation costs are private information. He first shows that in any implementable mechanism, a bidder participates if and only if his bidding cost is below a certain threshold, which is determined by his valuation. In addition, the probability of win in such a mechanism is independent of the bidder's privately known bidding cost. Finally, he characterizes the classes of symmetric ex post efficient and ex post revenue maximizing mechanisms and proposes a second-price auction with uniform reserve price and entry subsidy for their implementation.

In the case of multi-unit sales, Menezes [76] and Engelbrecht-Wiggans and Menezes [26] examine sequential second-price auctions of stochastically equivalent objects with delay or continuation costs. In these auctions, the bidders can participate freely in the first round but must incur a cost in order to remain in subsequent rounds. Similarly, Menezes and Monteiro [77] analyze a two-round sequential auction with participation costs, such that a bidder can enter the second round only if he has also bid in the first round, always at a cost. These studies try to understand why bidders may drop out from an auction as well as to provide a cost-based explanation for the "declining price anomaly".[2]

We depart from the earlier work on sequential auctions with participation costs by looking at the problem of the seller and trying to provide an explanation for his choice to conduct the sale sequentially rather than simultaneously. We construct an equilibrium in which the information revealed by the first-round price may encourage stronger participation in the second-round. Overall, we find that the sequential and the simultaneous auctions result in different outcomes. In particular, for valuations that follow a uniform or a power distribution, the seller may expect a higher revenue from a sequential auction, especially when the number of potential bidders is large, the participation cost small or the distribution convex.

We introduce our model in the next section. In sections 4.3 and 4.4, we describe the symmetric equilibrium for the simultaneous and the sequential auctions respectively. We compare the corresponding expected seller revenues, for the case of uniform valuations, in section 4.5. We conclude in section 4.6. All proofs, along with the source code for the revenue comparison, are in section C of the appendix.

[2] In many real-world sequential auctions, for example, auctions for wine, art, real estate or dairy cattle, it has been observed that the prices tend to drift downward over time, contrary to the theoretical prediction of a constant or an increasing price-path, respectively for the cases of independent or affiliated bidders' valuations. For more on the declining price anomaly, see Ashenfelter [2], McAfee and Vincent [74], Milgrom and Weber [79] and Krishna [62].

4.2 MODEL

There is one seller with two identical units for sale. His reservation value is normalized to zero. There are $N > 2$ potential buyers, each with single-unit demand. Each buyer i has a private valuation $v_i \in [\underline{v}, \bar{v}]$, for $0 \leq \underline{v} < \bar{v}$, drawn randomly at the beginning of the game. The valuations are distributed independently, according to a common distribution function $F : [\underline{v}, \bar{v}] \rightarrow [0, 1]$. We assume that the distribution F is differentiable and that the corresponding density $f : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}^+$ has full support.

The seller can allocate the two units either simultaneously, by means of a sealed-bid pay-your-bid auction, or sequentially, by means of two sealed-bid first-price auctions. In the sequential auction, we assume that the seller reveals the first-round winning bid $b^1 \in \mathbb{R}^+$ (or that the first unit has remained unsold) prior to the beginning of the second round.

In either selling scheme, each buyer can participate only at a cost $c > 0$, which is identical across all buyers. Obviously, in the simultaneous auction, this cost must be incurred prior to the submission of the bids. On the other hand, in the sequential auction, the buyers can decide whether to participate prior either to the first or to the second round. We assume that the buyers who participate in the first round can also participate in the second round at no additional cost. Finally, in both the simultaneous and the sequential auction, buyers cannot observe the number of the participating bidders.

We restrict attention to symmetric equilibria. Therefore, in the simultaneous auction, each buyer i follows a participation and bidding strategy[3]

$$\beta : [\underline{v}, \bar{v}] \longrightarrow \mathbb{R}^+ \cup \{a\},$$

where a denotes the action of abstaining from the auction. Similarly, in the first round of the sequential auction, buyer i follows a strategy

$$\beta^1 : [\underline{v}, \bar{v}] \longrightarrow \mathbb{R}^+ \cup \{a\}.$$

[3] Without loss of generality, the buyers' participation and bidding decisions are made simultaneously.

In the second round of the sequential auction, following a first round private history

$$h_i^1 = (b^1, b_i^1) \ \in \ H^1 = \left[\mathbb{R}^+ \times \left(\mathbb{R}^+ \cup \{a\}\right)\right] \cup \{(a,a)\}\,,$$

consisting of the publicly known first-round highest bid b^1 and the privately known[4] first-round decision b_i^1, each remaining buyer i follows a strategy

$$\beta^2 : [\underline{v}, \bar{v}] \times H^1 \longrightarrow \mathbb{R}^+ \cup \{a\}.$$

It will turn out that the bidders' second-round behavior does not depend on the particular bid that they may have submitted in the first round. It only depends on whether they participated or not as well as on the first-round outcome. In particular, every bidder who participates in the first round will also bid in the second round.

In case bidder i participates in the auction and wins a unit, his game payoff will be equal to his valuation v_i minus the price that he paid, minus the cost c. In case he participates in the auction but fails to win any of the two units, his game payoff will be $-c$. Finally, if buyer i does not participate to the auction, his payoff will be 0.

The solution concept is that of Bayesian equilibrium for the simultaneous auction and of perfect Bayesian equilibrium for the sequential auction. The players must therefore behave optimally at each decision point, given their knowledge of the other players' strategies and their beliefs. On the equilibrium path, the players' beliefs are formed by applying Bayes' rule while, off the equilibrium path, they are arbitrary.[5]

4.3 SIMULTANEOUS AUCTION

A buyer will be willing to participate to the simultaneous auction only if his expected gain from participating exceeds the cost c. When the buyers' participation and bidding strategy $\beta(v_i)$ is monotone, the participation decision is characterized by a cut-off valuation $v^* \in [\underline{v}, \bar{v}]$,

[4] Trivially, when $b^1 = a$, the bidders can infer their opponents' private first-round decisions.

[5] In fact, in the equilibrium that we construct for the sequential auction, the players' beliefs will also be Bayesian off the equilibrium path, for all first-round winning bids $b^1 \leq \beta^1(\bar{v})$.

defined by

$$\mathbb{P}[v_{-i}^2 \le v^*] \, v^* \;=\; c,$$

where v_{-1}^2 denotes the second order statistic among $N-1$ valuations. According to $\beta(\cdot)$, a buyer with valuation v_i will participate in the auction if and only if $v_i \ge v^*$. In particular, a buyer with valuation v^* is indifferent between participating and non-participating. When he enters the auction, he bids $\beta(v^*) = 0$, so that he wins a unit if and only if there is at most one other participating bidder. Finally, it is easy to check that the participation threshold $v^* = v^*(c)$ is unique and increasing in the cost c.

Proposition 4.1.
The following strategy constitutes a symmetric equilibrium for the simultaneous pay-your-bid auction of two identical units to buyers with single-unit demands and participation cost c:

A buyer with valuation $v_i \ge v^*$, where $v^* \in [\underline{v}, \bar{v}]$ is defined by

$$\{F(v^*)^{N-1} + (N-1)F(v^*)^{N-2}[1-F(v^*)]\} \, v^* \;=\; c, \tag{4.1}$$

participates in the auction and bids

$$\beta(v_i) = \int_{v^*}^{v_i} v \, \frac{(N-1)(N-2)\, F(v)^{N-3}[1-F(v)]\, f(v)}{F(v_i)^{N-1} + (N-1)F(v_i)^{N-2}[1-F(v_i)]} \, dv\,; \tag{4.2}$$

while for $v_i < v^*$, a buyer abstains from the auction.

Clearly, under participation cost c, each buyer's participation decision is identical to his decision to submit a bid in an auction without entry cost but with reserve price $r = v^*$. Therefore, since his bid is

$$\beta(v_i) \;=\; \mathbb{E}[\min\{v_{-i}^2, v^*\} \,|\, v_{-i}^2 \le v_i] \;-\; \frac{c}{\mathbb{P}[v_{-i}^2 \le v_i]},$$

this auction is equivalent with respect to allocation and seller revenue to an auction with reserve price $r = v^*$ and entry subsidy c.

4.4 SEQUENTIAL AUCTION

In the sequential auction, we will construct a symmetric equilibrium in strategies (β^1, β^2) such that a buyer participates in the auction if and only if his valuation exceeds a threshold value v_1^* or v_2^*, with the value v_2^* determined endogenously, as a function of the first-round outcome. Since a buyer who participates in the first round can also bid in the second round at no additional cost, it follows that $v_1^* \geq v_2^*$. Finally, in both rounds, the participating buyers bid in a strictly monotone manner, with the lowest participating type bidding zero.

In such a strategy profile, the information provided by the revelation of the first-round winning bid $b^1 \in \mathbb{R}^+$ (or $b^1 = a$, when the first unit remains unsold) takes the form of an upper bound for the remaining buyers' valuations, namely[6],

$$v^1 = \begin{cases} (\beta^1)^{-1}(b^1), & \text{for } b^1 \neq a; \\ \\ v_1^*, & \text{for } b^1 = a. \end{cases} \tag{4.3}$$

Thus, prior to the beginning of the second round, the buyers update their beliefs so that, for $v \in [\underline{v}, v^1]$,

$$v_i \sim F(v \,|\, v^1) = \frac{F(v)}{F(v^1)}.$$

In addition, by learning whether the first unit was sold, the buyers can infer the number, $N-1$ or N, of the players who remain interested in acquiring the second unit.

Given the above information update, some of the buyers who abstained from the first round may decide to enter the second round of the auction. They will do so if their expected gain with respect to their updated beliefs exceeds the cost c.

[6] Off the equilibrium path, for $b^1 > \beta^1(\bar{v})$, we can assume that $v^1 = \bar{v}$.

Notation:

We will denote[7] each buyer i's second-period action (bid or decision not to participate), following a first-period action $b_i^1 \in \mathbb{R}^+ \cup \{a\}$ and a price $b^1 = \beta^1(v^1) \in \mathbb{R}^+$, by

$$\beta^2(v_i \,|\, \beta^1(v^1), b_i^1) \;\equiv\; \beta^2(v_i \,|\, v^1).$$

Similarly, following no sale in the first period, so that $b^1 = a$, buyer i's second-period action will be $\beta^2(v_i \,|\, a, a) \equiv \beta^2(v_i \,|\, a)$.

If the first object remains unsold, so that $v^1 = v_1^*$ and N buyers remain in the auction, the bidders who will enter the second round have valuations $v_i \in [v_2^*, v_1^*)$, where $v_2^* = v_2^*(a)$ is defined by the condition

$$F(v_2^* \,|\, v_1^*)^{N-1}\; v_2^* \;=\; c. \tag{4.4}$$

These buyers will bid, as in Samuelson [86],

$$\beta^2(v_i \,|\, a) = \int_{v_2^*(a)}^{v_i} v \;\frac{(N-1)F(v)^{N-2}\, f(v)}{F(v_i)^{N-1}}\; dv\,. \tag{4.5}$$

On the other hand, buyers with valuations $v_i < v_2^*$ will abstain from the auction. Finally, off the equilibrium path, buyers with valuations $v_i \geq v_1^*$ will participate to the second auction and bid $b_i^2 = \beta^2(v_1^* \,|\, a)$.

If the first object is sold at a price $b^1 = \beta^1(v^1)$, for some valuation $v^1 \in [v_1^*, \bar{v}]$, then every buyer who participated in the first round will also bid in the second round. In addition, on the equilibrium path, a non-participating buyer i will enter the second round if and only if his valuation is $v_i \geq v_2^*$, where $v_2^* = v_2^*(v^1) \in [\underline{v}, v^1)$ is defined[8] by the condition

$$F(v_2^* \,|\, v^1)^{N-2}\; v_2^* \;=\; c. \tag{4.6}$$

[7] This simplification is customary in sequential auctions; for example, see Krishna [62], section 15. It is based on the strict monotonicity of the equilibrium strategy β^1, which allows us to identify any first-round price $b^1 \in \mathbb{R}^+$ with a winning valuation $v^1 \in [v_1^*, \bar{v}]$, as well as on the independence of the equilibrium strategy β^2 of the first-round bid b_i^1.

[8] In a manner that is consistent with our notation for the strategy β^2, following a first-round sale, we describe the threshold $v_2^*(\cdot)$ as a function of the winning valuation v^1 rather than of the price b^1.

In particular, there is a value $\hat{v}_1 = \hat{v}_1(v_1^*) \in (v_1^*, \bar{v})$, defined by the condition $v_2^*(\hat{v}_1) = v_1^*$, that is, by

$$F(v_1^* \,|\, \hat{v}_1)^{N-2} \; v_1^* \;=\; c, \tag{4.7}$$

such that for all $v^1 \geq \hat{v}_1$ no additional bidder will enter the second round of the auction; and, for $v^1 \in [v_1^*, \hat{v}_1)$, buyers with valuations $v_i \in [v_2^*(v^1), v_1^*)$ will enter the second round.

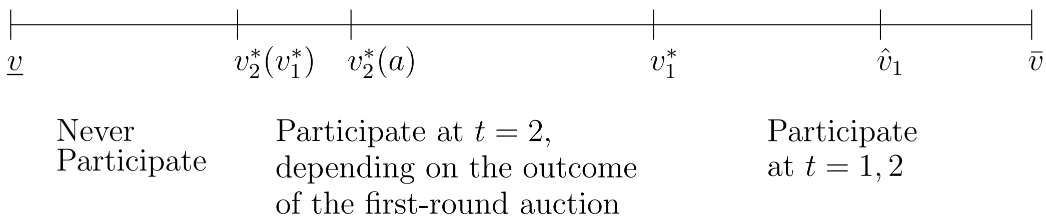

Figure 4.1: Participation Thresholds.

Clearly, following a first-round sale, the buyers' participation threshold is monotonically increasing in the price b^1. In comparison to the case of non-sale, however, because of the difference in the number of the bidders remaining in the auction, we have $v_2^*(v_1^*) < v_2^*(a)$, so that monotonicity does not hold overall.

Each participating buyer will bid

$$\beta^2(v_i \,|\, v^1) = \int_{v_{2s}^*(v^1)}^{v_i} v \; \frac{(N-2)F(v)^{N-3}\, f(v)}{F(v_i)^{N-2}} \; dv \,, \tag{4.8}$$

where $v_{2s}^*(v^1)$, defined by

$$v_{2s}^*(v^1) \;=\; \min\{v_1^*,\, v_2^*(v^1)\} \;=\; \begin{cases} v_2^*(v^1), & \text{if } v^1 \in [v_1^*, \hat{v}_1]; \\ v_1^*, & \text{if } v^1 \in [\hat{v}_1, \bar{v}], \end{cases}$$

denotes the lowest valuation possibly participating in the second round.

To describe the buyers' behavior off the equilibrium path, first, we define the upper bound $\bar{v}_1 \in [\hat{v}_1, \bar{v}]$ by

$$\begin{cases} \bar{v} - \beta^2(\bar{v}_1 \,|\, \bar{v}_1) \;=\; c, & \text{if a solution } \bar{v}_1 \in [\underline{v}, \bar{v}] \text{ exists;} \\ \\ \bar{v}_1 \;=\; \bar{v}, & \text{otherwise.} \end{cases}$$

This is the valuation corresponding to the largest first-round winning bid, $\bar{b}^1 = \beta^1(\bar{v}_1)$, for which a bidder with valuation $v_i = \bar{v}$ will have a second-round expected payoff that exceeds the participation cost c.[9]

In addition, for all $v^1 \in [v_1^*, \bar{v}_1]$, we define the participation threshold $\tilde{v}_2^* = \tilde{v}_2^*(v^1)$ by

$$F(\tilde{v}_2^* \,|\, v^1)^{N-2} \left[\, \tilde{v}_2^* - \beta^2(\, \min\{\tilde{v}_2^*, v^1\} \,|\, v^1 \,) \,\right] \;=\; c. \tag{4.9}$$

In particular, for $v^1 \in [v_1^*, \hat{v}_1]$, we have $\tilde{v}_2^*(v^1) = v_2^*(v^1)$ and $\beta^2(\, \min\{\tilde{v}_2^*, v^1\} \,|\, v^1 \,) = 0$.

A buyer i with valuation $v_i \in [v_1^*, \bar{v}]$ who did not participate in the first round will enter the second round, if and only if both $v^1 < \bar{v}_1$ and $\tilde{v}_2^*(v^1) \leq v_i$.[10] In this case, he will bid $b_i^2 = \beta^2(v_i \,|\, v^1)$, for $v_i \leq v^1$, and $b_i^2 = \beta^2(v^1 \,|\, v^1)$, for $v_i \geq v^1$. In addition, a buyer i with valuation $v_i \in [v^1, \bar{v}]$ who participated in the first round with a bid $b_i^1 < b^1$ will submit a bid $b_i^2 = \beta^2(v^1 \,|\, v^1)$ in the second round. Finally, a buyer i with valuation $v_i \in [\underline{v}, v_1^*]$ who participated in the first round with a bid $b_i^1 < b^1$ will submit a bid $b_i^2 = \beta^2(v_i \,|\, v^1)$, for $v_i \geq v_2^*(v^1)$, and $b_i^2 = 0$, for $v_i \leq v_2^*(v^1)$.

In the first round of the auction, a bidder i with valuation $v_i \in [\underline{v}, \bar{v}]$ will submit a bid, if his overall expected gain, prior to any information update, exceeds the participation cost c. Therefore, bidder i will participate in the first round if $v_i \geq v_1^*$, where v_1^* is defined by

$$(N-1)F(v_1^*)^{N-2}[1 - F(\hat{v}_1)]\, v_1^* \;+\; F(v_1^*)^{N-1}\, \beta^2(v_1^* \,|\, a) \;=\; [1 - F(\hat{v}_1)^{N-1}]\, c, \tag{4.10}$$

[9]The expected second-period payoff of a buyer with valuation $v_i = \bar{v}$ exceeds the participation cost c regardless of the first-round winning bid if and only if $c \leq \bar{v} - \beta^2(\bar{v} \,|\, \bar{v})$. Therefore, for sufficiently small participation cost c, in particular, for

$$c \;\leq\; \bar{v} - \mathbb{E}[v_1^{(N-2)}],$$

where $v_1^{(N-2)}$ denotes the highest among $N-2$ independent realizations of v_i, the $\bar{v}$-type will always be willing to enter the 2nd round.

[10]In particular, for all $v^1 \in [v_1^*, \hat{v}_1]$, buyer i will enter the second round.

expressing the v_1^*-type's indifference between entering the auction in the first round and waiting for the beginning of the second round in order to decide whether to participate.

Finally, each buyer participating in the first round bids according to the strategy

$$\beta^1(v_i) \;=\; \int_{v_1^*}^{v_i} \beta^2(v \,|\, v)\, \frac{(N-1)F(v)^{N-2}\, f(v)}{F(v_i)^{N-1}}\, dv,$$

that is, he bids an amount equal to the price that he would expect to pay in the second round, conditional on the possibility of a first-round win, $v_{-i}^1 \leq v_i$.

Substituting equation (4.8) for $\beta^2(v \,|\, v)$ results in

$$\beta^1(v_i) = \int_{v_1^*}^{v_i} \int_{v_{2s}^*(v)}^{v} w\, \frac{(N-1)(N-2)F(w)^{N-3}\, f(w) f(v)}{F(v_i)^{N-1}}\, dw\, dv\,, \tag{4.11}$$

Therefore, bidder i bids an amount equal to the expected valuation of the highest losing buyer, among those who may potentially participate in the auction, conditional on the realization of a first-round win, $v_{-i}^1 \leq v_i$.

Proposition 4.2.

The bidding strategies β^1 and β^2, along with the participation thresholds v_1^*, v_2^* and $\tilde{v}_2^*$, define a symmetric perfect Bayesian equilibrium for the sequential first-price auction.

The equilibrium price paths of the sequential auction do not demonstrate uniform behavior. Instead, there is a value $\tilde{v}_1 \in (v_1^*, \hat{v}_1)$, defined by

$$\int_{v_2^*(\tilde{v}_1)}^{v_1^*} \int_{v_2^*(\tilde{v}_1)}^{v} w\, F(w)^{N-3}\, f(w) f(v)\, dw dv \;=\; \int_{v_1^*}^{\tilde{v}_1} \int_{v_2^*(v)}^{v_2^*(\tilde{v}_1)} w\, F(w)^{N-3}\, f(w) f(v)\, dw dv,$$

such that the expected price in the second period is

$$\mathbb{E}[b^2 \,|\, b^1 = \beta^1(v^1)] \;\substack{\leq \\ >}\; b^1, \quad \text{for } v^1 \;\substack{\geq \\ <}\; \tilde{v}_1.$$

Therefore, a relatively low (high) price in the first round of the auction is likely to be followed by a higher (respectively, lower) price in the second round.

4.5 REVENUE COMPARISON

To compare the expected seller revenue in the two auction formats, we need to assume a particular form of the distribution of the bidders' valuations. First, we analyze the problem of uniformly distributed valuations, deriving insights about the cases in which the sequential auction may generate a higher seller revenue. Subsequently, we briefly examine whether these insights are valid for certain convex or concave distributions.

Let $v_i \sim U[0,1]$ and suppose that the buyers follow bidding strategies (β^1, β^2), with their participation in the auction determined by the thresholds v_1^*, v_2^* and $\tilde{v}_2^*$, as described in the previous section.

Since the strategy β^1 is strictly monotone in $[v_1^*, \bar{v}]$, the first-round outcome reveals an upper bound $v^1 \in [v_1^*, \bar{v}]$ for the remaining buyers' valuations. Therefore, in the second round, the buyers update their beliefs so that, for $v \in [\underline{v}, v^1]$,

$$v_i \;\sim\; F(v \,|\, v^1) \;=\; \frac{v}{v^1}.$$

In addition, the buyers learn whether there are $N-1$ or N players still in the auction.

If the first object remains unsold, the second-round threshold will be

$$v_2^*(a) \;=\; [(v_1^*)^{N-1}\, c\,]^{\frac{1}{N}}$$

and the buyers entering the second round will bid

$$\beta^2(v_i \,|\, a) \;=\; \frac{N-1}{N}\left[v_i - \left(\frac{v_1^*}{v_i}\right)^{N-1} c\right].$$

If the first object is sold at a price $b^1 = \beta^1(v^1)$, for some valuation $v^1 \in [v_1^*, \bar{v}]$, then every bidder that participated in the first round will bid again the second round. Furthermore, when $v^1 \in [v_1^*, \hat{v}_1]$, where

$$\hat{v}_1 \;=\; \left[(v_1^*)^{N-1}\, c^{-1}\,\right]^{\frac{1}{N-2}},$$

additional bidders may enter the auction. These bidders will have valuations $v_i \geq v_2^*(v^1)$, where

$$v_2^*(v^1) \;=\; [(v^1)^{N-2}\, c\,]^{\frac{1}{N-1}}.$$

In this case, each participating buyer i will bid

$$\beta^2(v_i \,|\, v^1) \;=\; \frac{N-2}{N-1}\,\left[v_i \;-\; \left(\frac{v^1}{v_i}\right)^{N-2} c \right].$$

On the other hand, when $v^1 \in [\hat{v}_1, \bar{v}]$, no new bidders will enter the second round. The bidders remaining from the first round will bid

$$\beta^2(v_i \,|\, v^1) \;=\; \frac{N-2}{N-1}\,\left[v_i \;-\; \frac{(v_1^*)^{N-1}}{(v_i)^{N-2}} \right].$$

In the first round, a buyer will enter the auction if and only if his valuation is $v_i \geq v_1^*$, where $v_1^* \in [\underline{v}, \bar{v}]$ is the solution to the equation

$$(N-1)(v_1^*)^{N-1}(1-\hat{v}_1) \;+\; \frac{N-1}{N}\,(v_1^*)^{N-1}\,(v_1^* - c) \;=\; [1 - (\hat{v}_1)^{N-1}]\, c.$$

For $v_i \in [v_1^*, \hat{v}]$, bidder i will bid

$$\beta^1(v_i) \;=\; \frac{1}{(v_i)^{N-1}}\,\left[\frac{N-2}{N}\,[(v_i)^N - (v_1^*)^N] \;-\; \frac{N-2}{N-1}\,[(v_i)^{N-1} - (v_1^*)^{N-1}]\, c \right],$$

while, for $v_i \in [\hat{v}, \bar{v}]$, he will bid

$$\begin{aligned}\beta^1(v_i) \;=\; \frac{1}{(v_i)^{N-1}}\,\Bigg[& \frac{N-2}{N}\,[(v_i)^N - (v_1^*)^N] \\ & -\; (N-2)\,(v_1^*)^{N-1}(v_i - \hat{v}_1) \;-\; \frac{N-2}{N-1}\,[(\hat{v}_i)^{N-1} - (v_1^*)^{N-1}]\, c \Bigg].\end{aligned}$$

The equilibrium price path of the sequential auction is increasing for all $v^1 \in [v_1^*, \tilde{v}_1]$, where $\tilde{v}_1 \in [v_1^*, \hat{v}_1]$ is the solution to the equation

$$\frac{1}{N}\left[\,(v_1^*)^N - v_2^*(\tilde{v}_1)^N\,\right] \;+\; v_2^*(\tilde{v}_1)^N \;\;=\;\; \tilde{v}_1\, v_2^*(\tilde{v}_1)^{N-1} \;\;-\;\; \frac{1}{N-1}\left[\,(\tilde{v}_1)^{N-1} - (v_1^*)^{N-1}\,\right] c,$$

while they are decreasing for $v^1 \in [\tilde{v}_1, \bar{v}]$.

In the simultaneous auction, a buyer will participate if and only if his valuation is $v_i \geq v^*$, where $v_1^* \in [\underline{v}, \bar{v}]$ is the solution to the equation

$$(v^*)^N \;+\; (N-1)(v^*)^{N-1}\,(1-v^*) \;\;=\;\; c.$$

In this auction, each participating bidder i will bid

$$\beta(v_i) \;\;=\;\; \frac{N-2}{N}\;\frac{N(v_i)^{N-1} - (N-1)\,(v_i)^N \;\;-\;\; N(v^*)^{N-1} + (N-1)(v^*)^N}{(v_i)^{N-1} \;+\; (N-1)(v_i)^{N-2}\,(1-v_i)}.$$

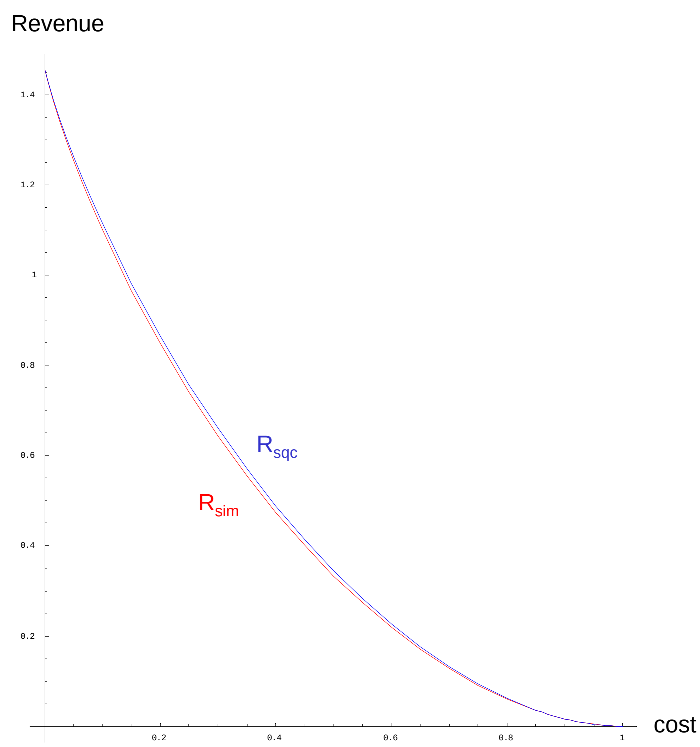

Figure 4.2: Expected Seller's Revenue for $N = 10$.

Comparing numerically the expected seller revenues from the two auction schemes, it appears that the sequential auction generates a higher seller revenue for larger numbers of bidders and for smaller participation costs. For example, when $N = 4$, the sequential auction generates a higher seller revenue only for $c \leq 0.1295$.Similarly, when $N = 5$, the sequential auction dominates for $c \leq 0.4441$. Finally, when $N = 10$, the case depicted in Figure 4.2, the sequential auction dominates for all $c \leq 0.89$. In Figure 4.3, we graph the difference in the expected seller's revenue from the sequential and the simultaneous auctions, as a function of $c \in [0, 1]$, for various values of N.[11]

The insights regarding the revenue comparison of the two auction schemes do not change when the buyers' valuations are distributed according to the power distribution, $F(v) = v^\alpha$, for $v \in [0, 1]$ and $\alpha > 0$. Again, for a fixed parameter $\alpha > 0$ and a fixed number of bidders, N, the sequential auction generates a higher seller revenue if and only if the cost c is below a certain threshold value; and this threshold value increases, as the number of bidders increases. In addition, as the 'degree of convexity", α, increases, the results from the revenue comparison become more pronounced, that is, the revenue gains from using a sequential auction increase while the region of dominance expands. Therefore, convexity appears to favor the use of a sequential procedure while concavity the use of a simultaneous one. In Figure 4.4, we demonstrate the effects of convexity for the case of $N = 8$ bidders.

[11] When $N = 3$, the simultaneous auction appears to dominate for all values of c. In general, for the sequential auction to dominate for some values of c, the number of buyers must not be too low relative to the concavity of the distribution.

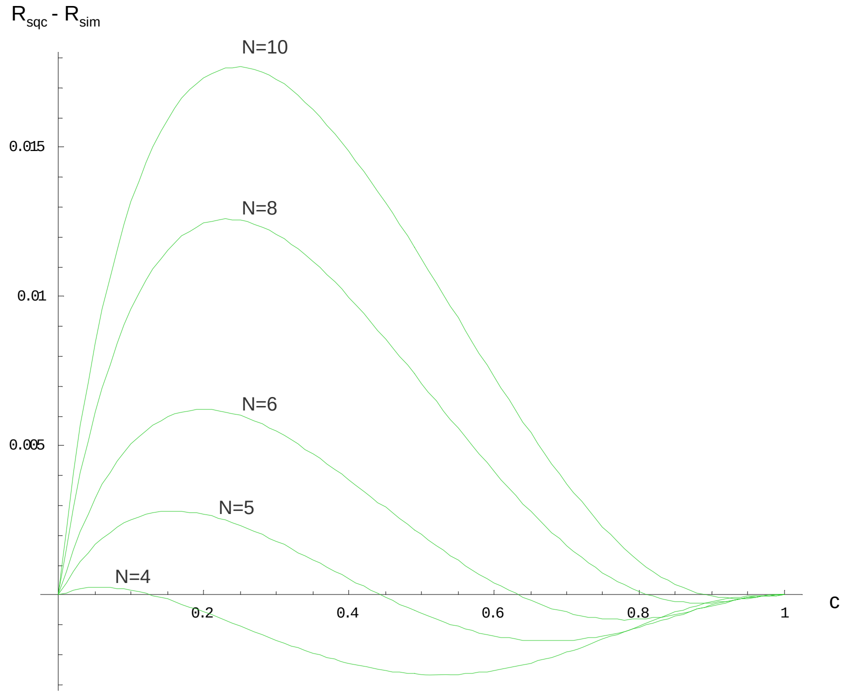

Figure 4.3: Difference in the Expected Seller's Revenue.

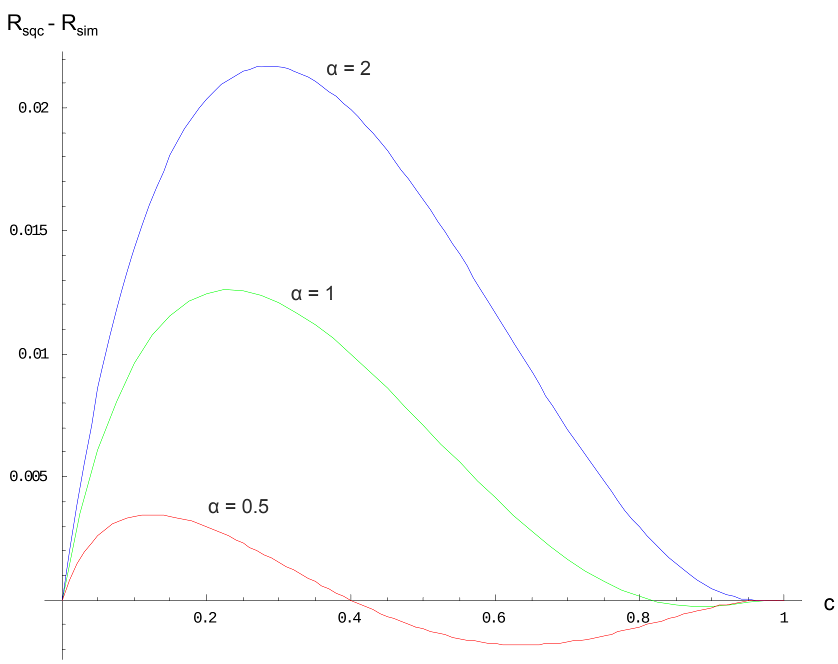

Figure 4.4: Revenue Effects of Convexity for $N = 8$.

4.6 CONCLUSIONS

For a sequential auction in which bidding is costly, we have constructed an equilibrium in cut-off strategies, such that the second-period participation threshold is determined endogenously, as a function of the information that the first-round price reveals. In this equilibrium, the price paths can be both decreasing, for a relatively high first-period price, and increasing, for a first-period price that is so low so as to encourage sufficiently stronger second-round participation. Compared to its simultaneous counterpart, the sequential auction generates a higher expected seller revenue, in cases in which the number of potential bidders is large, the participation cost small or the distribution of valuations convex.

This study can be extended in several directions. First, a more detailed comparison between the outcomes of the sequential and the simultaneous auctions, for example, a comparison of the prices paid by the stronger and the weaker of the two winners in each of the two auction schemes, will shed more light upon the reasons favoring the use of a particular auction format. In addition, one may introduce a positive marginal cost for bidding in the second round, for the bidders who participated in the first round. In this case, when the first-round price is high, some of the participating bidders may exit the auction in the second round. Finally, one can allow the seller to adopt different information revelation policies, such as announcing only whether the first object was sold, and to examine whether revealing the first-round price is optimal.

5.0 CONCLUSION

This dissertation has aimed at explaining the reasons for which a seller may decide to conduct a multi-unit auction sequentially rather than simultaneously. In particular, it has tried to analyze the manners in which the information generated in a sequential auction can affect the bidders' behavior to his benefit.

Overall, two informational effects have been identified:

a. A direct effect, generated by the bidders' actions in the first round, which leads to better informed bidding in the second round of the sequential auction (chapters 3 and 4). In the sale of the oligopoly licenses, it allows the winner of the second round to avoid overpaying for his license. In the presence of participation costs, it gives the opportunity to some of the buyers who would abstain from the simultaneous auction to enter the second round of the sequential auction.

b. An indirect effect, generated by the type of equilibrium that is possible in a sequential auction, which leads to better informed bidding in both rounds (chapter 3). In the sale of the oligopoly licenses, since the sequential auction allocates the two assets in an ordered manner, the winner of the first license expects a stronger presence in the market while the winner of the second license knows that he will have a weaker presence. As a result, the degree of competition in each of the rounds of the sequential auction is different from that in the simultaneous auction.

The second effect turns out to affect only the distribution of the auction prices, in a manner which favors the weaker of the two winners. On the other hand, the first effect may affect the seller's total revenue from the auction, as in the case of participation costs, possibly in a positive manner.

Finally, in order to profit from organizing a sequential auction, the seller must commit not to use, during the later rounds, the information that the bidders reveal in the earlier rounds (chapter 2). If such intertemporal commitment is not possible, the seller will be better off using a simultaneous procedure.

Several extensions have been proposed to the problems analyzed in this dissertation. A promising direction for future research appears to be the introduction of alternative information disclosure policies by the auctioneer. Such policies will alter the effect of the first-round outcome upon the second-round behavior and, in the problem of the auction of oligopoly licenses, upon the market competition. It would be interesting to identify the environments in which revealing the first-round winning bid is optimal as well as the ones in which the seller can enhance his revenue by adopting different policies.

Finally, another direction for future work is that of generalizing the sale mechanisms which the seller can use in each round of a sequential allocation. Although single-round direct-revelation mechanisms are optimal in general, it will be interesting to determine the kinds of mechanisms that are revenue optimizing when the seller's options are restricted to sequential procedures.

APPENDIX

A PROOFS FOR CHAPTER 2

Proof of Proposition 2.2:

Suppose that $r_1 < \bar{r}_1$. The argument that led to the derivation of the bidding function $\beta^1(v_i \,|\, r_1)$ has shown that no bidder with valuation $v_i \in [\underline{v}(r_1), \bar{v}]$ can profit by deviating unilaterally to bidding $\beta^1(\tilde{v}_i, r_1)$, for $\tilde{v}_i \geq \underline{v}(r_1)$ such that $\tilde{v}_i \neq v_i$.

It remains to show that no bidder with valuation $v_i \geq \underline{v}(r_1)$ can profit by abstaining from the first auction; and that no bidder with valuation $v_i < \underline{v}(r_1)$ can profit by participating in the first auction. This result follows from the indifference of the $\underline{v}(r_1)$-type between abstaining and participating in the first period.

Consider a bidder with valuation $v_i \geq \underline{v}(r_1)$. Since $\Pi[v_i, v_i] > \Pi[\underline{v}(r_1), v_i]$, it suffices to show that the bidder's payoff from abstaining does not exceed his payoff from mimicking the type $\underline{v}(r_1)$. Obviously, these two payoffs differ only if all other bidders' valuations are below $\underline{v}(r_1)$. In this case, in the second period, it is optimal for a bidder with valuation $v_i > \underline{v}(r_1)$ to bid

$$\beta^{2,N}[\, \underline{v}(r_1) \,|\, \underline{v}(r_1),\, r_2(\underline{v}(r_1)) \,] \;=\; r_1 \;=\; \beta^1[\underline{v}(r_1) \,|\, r_1].$$

Thus, the bidder's payoff from abstaining at reserve price $r_1 < \bar{r}_1$ is equal to his payoff from acquiring the object at price $\beta^1[\underline{v}(r_1), r_1]$. Hence, it is optimal for the bidder to participate in the first auction with a bid $\beta^1[v_i \,|\, r_1]$.

Conversely, consider a bidder with valuation $v_i < \underline{v}(r_1)$. Since $\Pi[\tilde{v}_i, v_i] < \Pi[\underline{v}(r_1), v_i]$ for all $\tilde{v}_i > \underline{v}(r_1)$, it suffices to compare the bidder's payoff from abstaining with that from mimicking the type $\underline{v}(r_1)$. Since the type $\underline{v}(r_1)$ is indifferent between participating in the first auction and abstaining from it, we can consider, equivalently, the bidder's second-period payoffs from bidding according to his valuation v_i and mimicking the type $\underline{v}(r_1)$. Hence, by Lemma 2.1, the bidder cannot profit from participating in the first auction.

For reserve prices $r_1 > \bar{r}_1$, no bidder is supposed to participate in the first auction. Therefore any bidder can claim the first-period object at price r_1. Since

$$r_1 \;>\; \bar{r}_1 \;=\; \beta^{2,N}(\bar{v} \,|\, \bar{v}, r_2(\bar{v})),$$

claiming the first-period object would be inferior to abstaining from the first auction and mimicking, in the second auction, the type $\bar{v}$. Therefore, by Lemma 2.1, participation in the first auction cannot be profitable.

□

Proof of Proposition 2.3:

By setting a first-period reserve price $r_1 \in [\underline{v}, \bar{r}_1]$, corresponding to a participation threshold $v = v(r_1) \in [\underline{v}, \bar{v}]$, the auctioneer expects a revenue

$$\begin{aligned} R(v) \;&=\; \int_v^{\bar{v}} \beta^1(v_1 \,|\, v)\; Nf(v_1)G(v_1)\, dv_1 \\ &+\; \int_{r_2(v)}^{v} \beta^{2,N}[v_1 \,|\, v, r_2(v)]\; Nf(v_1)G(v_1)\, dv_1 \\ &+\; \int_v^{\bar{v}} \int_{r_2(v_1)}^{v_1} \beta^{2,N-1}[v_2 \,|\, v_1, r_2(v_1)]\; Nf(v_1)g(v_2)\; dv_2\, dv_1. \end{aligned}$$

The first two integrals correspond to the payment of the buyer with the highest valuation, when he wins either the first or the second of the two auctions, while the third integral corresponds to the payment of the bidder with the second-highest valuation, when he can acquire the second-period object.

By differentiating with respect to the threshold valuation v and substituting the expressions for $\frac{d}{dv}\beta^1(v_1|v)$ and $\frac{d}{dv}\beta^{2,N}[v_1|\, v, r_2(v)]$, we get

$$\begin{aligned}
\frac{dR}{dv}(v) &= N\,[1-F(v)]\,\left[g(v)\,r_1(v)\,+\,G(v)\,\frac{dr_1}{dv}(v)\right] \\
&- N\,[1-F(v)]\;g(v)\;\beta^{2,N-1}[v\,|\,v, r_2(v)] \\
&- N\;G[r_2(v)]\;f[r_2(v)]\;r_2(v)\;\frac{dr_2}{dv}(v) \\
&+ N\;[F(v)-F(r_2(v))]\;G[r_2(v)]\;\frac{dr_2}{dv}(v) \\
&- N\;f(v)\;\int_{r_2(v)}^{v} \beta^{2,N-1}[v_2\,|\,v, r_2(v)]\;g(v_2)\;dv_2.
\end{aligned}$$

Since $\psi(r_2(v)\,|\,v)=0$ and, therefore, $F(v)-F[r_2(v)] = f[r_2(v)]\,r_2(v)$, as in equation (2.3), we can simplify this sum by eliminating its third and fourth terms. In addition, by substituting the expressions for $r_1(v)$, $\frac{dr_1}{dv}(v)$ and $g(u)\,\beta^{N-1}[u; v, r_2(v)]$, we get

$$\begin{aligned}
\frac{dR}{dv}(v) &= N\,[1-F(v)\,]\,\left[g(v)\,v\,+\,G(r_2(v))\,\frac{dr_2}{dv}(v)\right] \\
&- N\,(N-1)\,f(v)\;[1-F(r_2(v))\,]\,F(r_2(v))^{N-2}\;r_2(v) \\
&- N\,(N-1)\,f(v)\;[1-F(v)\,]\int_{r_2(v)}^{v} u\;(N-2)\,F(u)^{N-3}\;du \\
&- N\,(N-1)\,f(v)\;\int_{r_2(v)}^{v} f(v_2)\int_{r_2(v)}^{v_2} u\;(N-2)\,F(u)^{N-3}\;du\;dv_2.
\end{aligned}$$

Finally, by integrating the last integral by parts, canceling the opposite-sign terms and substituting the expression $\psi(v) = v - \frac{1-F(v)}{f(v)}$, the derivative becomes

$$\frac{dR}{dv}(v) \;=\; N\,[1-F(v)\,]\;G(r_2(v))\;\frac{dr_2}{dv}(v)\;-\;N\,f(v)\;\int_{r_2(v)}^{v}\psi(u)\;g(u)\;du.$$

First, suppose that $r_0 > \underline{v}$. If $v \in [\underline{v}, r_0]$, then, since the function $\psi(v)$ is increasing, we have $\psi(u) < 0$ for all $u \in [\underline{v}, v)$. Therefore, the derivative $\frac{dR}{dv}(v) > 0$, for all $v \in [\underline{v}, r_0]$, implying that $v^* > r_0$. By the Intermediate Value Theorem, since $\frac{dR}{dv}(r_0) > 0$ and $\frac{dR}{dv}(\bar{v}) < 0$, we conclude that there exists a value $v^* \in (r_0, \bar{v})$ for which the auctioneer's revenue function $R(v)$ attains its maximum.

Now, suppose that $r_0 = \underline{v}$. Then, for all $v \in [\underline{v}, \bar{v}]$, we have $r_2(v) = \underline{v}$, implying that $\frac{dR}{dv}(v) < 0$. Hence, in this case, $v^* = \underline{v}$, as asserted.

□

Proof of Proposition 2.5:

We consider only the case of a sequential sealed-bid first-price auction. The argument for the sequential English and sealed-bid second-price auctions is very similar.

Suppose that there exists a perfect Bayesian equilibrium $[(\beta_i^1, \beta_i^2)_{i=1}^N, (r_1, r_2)]$ such that the first-period bidding strategies are symmetric and increasing in the valuation v_i. To derive a contradiction, it suffices to consider the restriction of the equilibrium to the continuation game following a first-period reserve price r_1, such that a positive measure of bidders' types, $[\underline{v}(r_1), \bar{v}]$, participates in the first auction. In particular, in any equilibrium of the continuation game following $r_1 = \underline{v}$, all bidder types must participate to the first auction.

First, we rule out the existence of an equilibrium involving first-period bidding strategies $\beta^1(\cdot \,|\, r_1)$ that are strictly increasing in $[\underline{v}(r_1), \bar{v}]$. Under such a bidding strategy profile, we have perfect revelation of the participating bidders's valuations. Therefore, these bidders expect to make zero profit in the second auction. Hence, on the strategy-realization path, the participating bidders treat the problem as that of a single-period, single-unit auction. According to the symmetric equilibrium of the first-price auction, we must have

$$\beta^1(v_i \,|\, r_1) \;=\; \begin{cases} \mathbb{E}[\, \max\{v_1^{(N-1)}, r_1\} \,|\, v_1^{(N-1)} < v_i \,], & \text{if } v_i \geq \underline{v}(r_1); \\ a, & \text{if } v_i < \underline{v}(r_1). \end{cases}$$

By splitting cases, according to the realized type-profile, we claim that each bidder i has a profitable deviation to $\tilde{\beta}^1(v_i \,|\, r_1) \equiv a$. Notice that, along the equilibrium path, the second-period reserve price r_2 will be

$$r_2(b_1^1, ..., b_N^1, r_1) \begin{cases} = \ [\beta^1(\cdot \,|\, r_1)]^{-1}(b_2^{(N)}), & \text{if } b_2^{(N)} \neq a; \\ \leq \ r_1, & \text{if } b_2^{(N)} = a. \end{cases}$$

If bidder i, with valuation $v_i \in [\underline{v}(r_1), \bar{v}]$, turns out to have the highest valuation, then he will still win an object, in the second auction, at an expected price

$$p_2 \ = \ \mathbb{E}[\,\max\{v_2^{(N-1)}, r_1\} \,|\, v_1^{(N-1)} < v_i\,] \ + \ \varepsilon.$$

For sufficiently small $\varepsilon > 0$, the price p_2 is smaller than $\beta^1(v_i \,|\, r_1)$, the price bidder i will pay, if he wins the object in the first auction.

If bidder i turns out to have the second-highest valuation, then he will again win an object in the second auction, at an expected price

$$p_2 \ = \ \mathbb{E}[\,\max\{v_2^{(N-1)}, r_1\} \,|\, v_2^{(N-1)} < v_i < v_1^{(N-1)}\,] \ + \ \varepsilon.$$

Again, for sufficiently small $\varepsilon > 0$, the price p_2 is strictly smaller than v_i, the price bidder i would pay in the second auction, if he revealed his valuation

Finally, if bidder i turns out to have the third-highest, or lower, valuation, then he will win no object, as he would do after bidding $\beta^1(v_i \,|\, r_1)$.

We conclude the proof by ruling out the existence of intervals of non-increase in $[\underline{v}(r_1), \bar{v}]$. Suppose that $\beta^1(v_i \,|\, r_1) = b$, for all $v_i \,\epsilon\, [v_L, v_H] \subset [\underline{v}(r_1), \bar{v}]$. If $b > v_L$, then any bidder i with type $v_i \,\epsilon\, V_1 = [v_L, \min\{b, v_H\}]$ is better off bidding $\tilde{b}^1(v_i \,|\, r_1) = v_i$, a bid that avoids winning the object at a price above v_i. If $b < v_L$, then there is an $\varepsilon > 0$ sufficiently small such that the deviation to the strategy $\beta^1(v_i \,|\, r_1) = b + \varepsilon$, for all $v_i \,\epsilon\, [v_L, v_H]$ is profitable. Finally, if $b = v_L$, then we can simply apply the argument for $b < v_L$ to the interval $[\frac{1}{2}(v_L + v_H), v_H]$. Hence, there cannot exist an interval of non-increase of $\beta^1(\cdot \,|\, r_1)$.

□

Proof of Proposition 2.6:

We consider only the case of a sequential sealed-bid second-price auction, the argument for the other cases being essentially the same. It suffices to rule out the existence of an equilibrium in the subgame following a first-period reserve price $r_1 = \underline{v}$.

Suppose that in the second auction, each participating bidder bids his valuation. The reserve price r_2 depends on the outcome of the first period and on whether the auctioneer turns out to be credible. It is $r_2 = r_1 = 0$ with probability $1-\rho$; and $r_2 = \beta^1(\cdot|0)^{-1}(p_1)$, the second-highest valuation revealed in the first period, with probability ρ.

If each bidder follows a first-period bidding strategy $\beta^1(v\,|\,0)$, the expected payoff of a bidder i with valuation v_i who mimics a type $\tilde{v}_i > v_i$ will be

$$\begin{aligned}
\Pi[\tilde{v}_i; v_i] &= F(\tilde{v}_i)^{N-1}\, v_i - \int_0^{\tilde{v}_i} \beta^1(v\,|\,0)(N-1)F(v)^{N-2} f(v)\, dv \\
&+ (N-1)\,[1-F(\tilde{v}_i)]\,(1-\rho)\,F(v_i)^{N-2}\, v_i \\
&- (N-1)\,[1-F(\tilde{v}_i)]\,(1-\rho) \int_0^{v_i} v(N-2)F(v)^{N-3} f(v)\, dv.
\end{aligned}$$

The necessary first-order condition $\frac{\partial \Pi}{\partial \tilde{v}_i}[v_i; v_i] \leq 0$ yields the inequality

$$F(v_i)^{N-2}\,\beta^1(v_i\,|\,0) \;\geq\; (1-\rho) \int_0^{v_i} v(N-2)F(v)^{N-3} f(v)\, dv \;+\; \rho\, F(v_i)^{N-2}\, v_i.$$

Therefore, to avoid first-period deviations to mimicking a type $\tilde{v}_i > v_i$, the bidders must bid more aggressively than in the case of perfect commitment.

If bidder i mimics a type $\tilde{v}_i < v_i$, his expected payoff will be

$$\begin{aligned}
\Pi[\tilde{v}_i; v_i] &= F(\tilde{v}_i)^{N-1}\, v_i - \int_0^{\tilde{v}_i} \beta^1(v\,|\,0)(N-1)F(v)^{N-2} f(v)\, dv \\
&+ (N-1)\,[1-F(\tilde{v}_i)]\,F(v_i)^{N-2}\, v_i \\
&- (N-1)\,[1-F(\tilde{v}_i)]\,(1-\rho) \int_0^{v_i} v(N-2)F(v)^{N-3} f(v)\, dv \\
&- (N-1)\,[1-F(\tilde{v}_i)]\;\rho \int_0^{v_i} \max\{v, \tilde{v}_i\}\,(N-2)F(v)^{N-3} f(v)\, dv.
\end{aligned}$$

The necessary first-order condition $\frac{\partial \Pi}{\partial \tilde{v}_i}[v_i; v_i] \geq 0$ yields the inequality

$$\begin{aligned} F(v_i)^{N-2}\,\beta^1(v_i,0) \;\; &\leq \;\; (1-\rho)\int_0^{v_i} v(N-2)F(v)^{N-3} f(v)\,dv \,+\, \rho\,F(v_i)^{N-2}\,v_i \\ &- \;\; \rho\;F(v_i)^{N-2}\;\frac{1-F(v_i)}{f(v_i)}. \end{aligned}$$

Therefore, to avoid first-period deviations to mimicking a type $\tilde{v}_i > v_i$, the bidders must not bid too aggressively.

Clearly, for any $\rho > 0$, the two necessary conditions cannot hold simultaneously. Hence, as asserted, there cannot exist any symmetric perfect Bayesian equilibrium in increasing first-period bidding strategies.

$\square$

B PROOFS FOR CHAPTER 3

Proof of Proposition 3.1:

It is straightforward to verify that the function $\beta(c_i)$ is a solution to the differential equation that resulted from the necessary first-order condition. In addition, by using L'Hospital's rule, it is easy to check that

$$\lim_{c_i \to \bar{c}} \beta(c_i) \;=\; \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c} \,|\, \bar{c}, c)\, f(c)\, dc,$$

as required by the boundary condition.

Since the equation that produced the strategy $\beta(c_i)$ was only a necessary condition, we still need to establish that it is optimal for any bidder i with marginal cost c_i to bid $b_i = \beta(c_i)$, if all other bidders follow this bidding strategy.

Suppose that firm i bids $\tilde{b}_i = \beta(\tilde{c}_i)$, for $\tilde{c}_i \in [\underline{c}, \bar{c}]$ while having a marginal cost c_i. Then, by changing its bid marginally, that is, by mimicking a marginally different type, it can change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; -\frac{d}{d\tilde{c}_i}\{\, \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\, \beta(\tilde{c}_i)\,\} \\
&\quad -\int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(c_i, c^1)\; (N-1)(N-2)\,[1-F(\tilde{c}_i)]^{N-3}\, f(\tilde{c}_i) f(c^1)\; dc^1.
\end{aligned}$$

Substituting the expression (3.1) defining $\beta(c_i)$ results in

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(\tilde{c}_i, c^1)\; (N-1)(N-2)[1-F(\tilde{c}_i)]^{N-3} f(\tilde{c}_i) f(c^1)\, dc^1 \\
&\quad -\int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(c_i, c^1)\; (N-1)(N-2)[1-F(\tilde{c}_i)]^{N-3} f(\tilde{c}_i) f(c^1)\, dc^1.
\end{aligned}$$

Since the function $\pi^{NS}(c_i, c^1)$ is decreasing in the marginal cost c_i, the change in the firm's expected payoff is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

showing that the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ attains its maximum at $\tilde{c}_i = c_i$.

To show that the strategy $\beta(c_i)$ is decreasing, we can calculate its derivative to be

$$\frac{d\beta}{dc_i}(c_i) = \frac{-\frac{d}{dc_i}\mathbb{P}[c_{-i}^2 \geq c_i]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \times \left[-v(c_i) + \int_{c_i}^{\bar{c}} v(c^2) \frac{-\frac{d}{dc^2}\mathbb{P}[c_{-i}^2 \geq c^2]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \, dc^2 \right],$$

where

$$v^{NS}(c) = \int_{\underline{c}}^{c} \pi^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1,$$

for $c \in [c_i, \bar{c}]$, is the expected market profit of the strongest non-winning firm. Therefore, if the function $v(c)$ is decreasing, we can conclude that

$$\frac{d\beta}{dc_i}(c_i) < \frac{-\frac{d}{dc_i}\mathbb{P}[c_{-i}^2 \geq c_i]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \times [\, -v(c_i) + v(c_i)\,] = 0,$$

as required for the strategy $\beta(c_i)$ to be decreasing.

By differentiating the function $v^{NS}(c)$, we get

$$\begin{aligned} \frac{dv^{NS}}{dc}(c) &= \pi^{NS}(c, c) \, \frac{f(c)}{F(c)} \\ &+ \int_{\underline{c}}^{c} \pi_1^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1 \\ &- \int_{\underline{c}}^{c} \pi^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1 \, \frac{f(c)}{F(c)}, \end{aligned}$$

and, after integrating the last term by parts,

$$\frac{dv^{NS}}{dc}(c) \;=\; \int_{\underline{c}}^{c} \pi_1^{NS}(c,c^1)\,\frac{f(c^1)}{F(c)}\,dc^1 \;+\; \int_{\underline{c}}^{c} \pi_2^{NS}(c,c^1)\,\frac{F(c^1)}{F(c)}\,dc^1\;\frac{f(c)}{F(c)}.$$

Since the firms' marginal costs are distributed in a logconcave manner, the expression $f(c)/F(c)$ is decreasing. Therefore,

$$\frac{dv^{NS}}{dc}(c) \;\leq\; \int_{\underline{c}}^{c} [\,\pi_1^{NS}(c,c^1)+\pi_2^{NS}(c,c^1)\,]\,\frac{f(c^1)}{F(c)}\,dc^1,$$

and, since $\pi_1^{NS}+\pi_2^{NS}<0$, we conclude that

$$\frac{dv^{NS}}{dc}(c)<0,$$

completing the argument.

□

Proof of Corollary 3.2:

Suppose that firm i has the lowest marginal cost among all firms, namely, $c_i \in [\underline{c},\bar{c}]$. Then, in equilibrium, it will win one of the two licenses, at a price $\beta(c_i)$, for a market profit $\pi^{NS}(c_i,c^1)$, where $c^1 \geq c_i$ is the lowest competing marginal cost. Its overall payoff, therefore, will be $\pi^{NS}(c_i,c^1)-\beta(c_i)$. Since the function $\pi^{NS}(c_i,c^1)$ is increasing in c^1,

$$\pi^{NS}(c_i,c^1)-\beta(c_i) \;\geq\; \pi^{NS}(c_i,c_i)-\beta(c_i),$$

so, it suffices to show that

$$\mathbb{P}[c^2_{-i}\geq c_i]\;\{\pi^{NS}(c_i,c_i)-\beta(c_i)\}\;\geq\;0.$$

Notice that this last inequality is true for the boundary value $c_i = \bar{c}$. Furthermore,

$$\begin{aligned}
\frac{d}{dc_i}\{\,\mathbb{P}[c^2_{-i} \geq c_i]\,[\pi^{NS}(c_i, c_i) - \beta(c_i)]\,\} \;\; = & \\
& - (N-1)(N-2)[1 - F(c_i)]^{N-3} F(c_i) f(c_i)\; \pi^{NS}(c_i, c_i) \\
& + \{\,[1 - F(c_i)]^{N-1} + (N-1)[1 - F(c_i)]^{N-2} F(c_i)\,\}\; \frac{d}{dc_i}[\pi^{NS}(c_i, c_i)] \\
& + \int_{\underline{c}}^{c_i} \pi^{NS}(c_i, c^1)\; (N-1)(N-2)\,[1 - F(c_i)]^{N-3}\, f(c_i) f(c^1)\; dc^1.
\end{aligned}$$

By integrating the last term by parts, this derivative becomes

$$\begin{aligned}
\frac{d}{dc_i}\{\,\mathbb{P}[c^2_{-i} \geq c_i]\,[\pi^{NS}(c_i, c_i) - \beta(c_i)]\,\} \;\; = & \\
& \{\,[1 - F(c_i)]^{N-1} + (N-1)[1 - F(c_i)]^{N-2} F(c_i)\,\}\; \frac{d}{dc_i}[\pi^{NS}(c_i, c_i)] \\
& - \int_{\underline{c}}^{c_i} \pi^{NS}_2(c_i, c^1)\; (N-1)(N-2)\,[1 - F(c_i)]^{N-3}\, F(c_i)\, f(c^1)\; dc^1.
\end{aligned}$$

Since both terms are negative, it follows that

$$\frac{d}{dc_i}\{\,\mathbb{P}[c^2_{-i} \geq c_i]\,[\pi^{NS}(c_i, c_i) - \beta(c_i)]\,\} \;\leq\; 0,$$

proving firm i's realized payoff to be always positive.

To show that the firm with the second-lowest marginal cost may win a license at a price above its ex-post value, consider a firm with marginal cost $c_i = \bar{c}$. In equilibrium, such a firm will bid

$$\beta(\bar{c}) \;=\; \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c}, c)\, f(c)\, dc,$$

the expected value of the license, given that $N - 2$ firms have marginal cost equal to $\bar{c}$. If the firm wins the license, then, in the market, it may face an opponent with marginal cost

$\underline{c}$. In this case, it will make a market profit $\pi^{NS}(\bar{c}, \underline{c}) \leq \pi^{NS}(\bar{c}, c)$, for all $c \in [\underline{c}, \bar{c}]$. Therefore,

$$\pi^{NS}(\bar{c}, \underline{c}) < \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c}, c)\, f(c)\, dc,$$

for a negative overall profit.

$\square$

Proof of Lemma 3.5:

The derivative of the function $\beta^2(c_i|c_i)$ with respect to the variable c_i is equal to

$$\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;=\; \int_{c_i}^{\bar{c}} -\pi_1^{NS}(c, c_i)\; w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc,$$

where

$$w(c, c_i) \;=\; \frac{\pi_2^{NS}(c, c_i)}{-\pi_1^{NS}(c, c_i)} \;-\; \frac{f(c_i)/[1-F(c_i)]}{f(c)/[1-F(c)]}.$$

The expression $w(c, c_i)$ is negative for $c = c_i$, positive for $c = \bar{c}$, continuous and increasing with respect to $c \in [c_i, \bar{c}]$. Hence, there exists a value $c^* = c^*(c_i) \in (c_i, \bar{c})$ such that

$$w(c, c_i) \begin{cases} < 0, & \text{for } c \in [c_i, c^*); \\ = 0, & \text{for } c = c^*; \\ > 0, & \text{for } c \in (c^*, \bar{c}\,]. \end{cases}$$

It follows that

$$\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;<\; -\pi_1^{NS}(c^*, c_i) \int_{c_i}^{\bar{c}} w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc$$

and, since Assumption 3.4 implies that

$$\int_{c_i}^{\bar{c}} w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc \;\leq\; 0,$$

we conclude that $\beta^2(c_i|c_i)$ is decreasing.

$\square$

Proof of Proposition 3.6:

Suppose that all firms follow the bidding strategy (β^1, β^2) and consider firm i with marginal cost $c_i \in [\underline{c}, \bar{c}]$. The optimality of bidding $\beta^2(c_i \,|\, c^1)$ in the second auction, following the sale of the first license at a price $b^1 = \beta^1(c^1)$, has been established in Lemma 3.3.[1] Therefore, we only need to examine the optimality of bidding $\beta^1(c_i)$ in the first auction.

Obviously, firm i cannot gain from submitting a bid above $\beta^1(\underline{c})$ or below $\beta^1(\bar{c})$. So, suppose that it mimics a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$, that is, it bids $\beta^1(\tilde{c}_i)$. If $\tilde{c}_i \leq c_i$, then, by changing its bid marginally, firm i will change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \pi^{NS}(c_i, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \pi^{NS}(c_i, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \beta^2(c_i|\tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \frac{d}{d\tilde{c}_i}\{\,[1-F(\tilde{c}_i)]^{N-1}\,\beta^1(\tilde{c}_i)\,\}.
\end{aligned}$$

Substituting the first-order condition (3.3) in the last term gives

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \pi^{NS}(c_i, \tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \pi^{NS}(c_i, \tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \beta^2(c_i|\tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&+\; \beta^2(\tilde{c}_i|\tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i).
\end{aligned}$$

[1] If the first license is sold at a price $b^1 > \beta^1(\underline{c})$, an event outside the equilibrium path, we can assume that the remaining firms attribute a marginal cost $c^1 = \underline{c}$ to the winner of the license. Similarly, for $b^1 < \beta^1(\bar{c})$, we can assume that $c^1 = \bar{c}$.

The difference between the last two terms equals to

$$\begin{aligned}
&\beta^2(\tilde{c}_i|\tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2}f(\tilde{c}_i) \\
&-\beta^2(c_i|\tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2}f(\tilde{c}_i) \;\; = \\
&\qquad\qquad \int_{\tilde{c}_i}^{c_i} \pi^{NS}(c^2,\tilde{c}_i)\,(N-2)[1-F(c^2)]^{N-3}f(c^2)\,dc^2\;(N-1)f(\tilde{c}_i)
\end{aligned}$$

and, since $\pi^{NS}(c_i, c^1) < \pi^{NS}(c^2, c^1)$, for all $c^2 \in [\tilde{c}_i, c_i]$, to

$$\begin{aligned}
&\beta^2(\tilde{c}_i,\tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2}f(\tilde{c}_i) \\
&-\beta^2(c_i,\tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2}f(\tilde{c}_i) \;\; > \\
&\qquad\qquad -\,\pi^{NS}(c_i,\tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2}f(\tilde{c}_i) \\
&\qquad\qquad +\,\pi^{NS}(c_i,\tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2}f(\tilde{c}_i).
\end{aligned}$$

Hence, we can conclude that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i) \;\geq\; 0,$$

with equality only when $\tilde{c}_i = c_i$.

Similarly, if firm i mimics a marginal cost $\tilde{c}_i \geq c_i$, then, by changing its bid marginally, it will change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i) \;\; &= \;\; -\,\beta^2(\tilde{c}_i,\tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2}f(\tilde{c}_i) \\
&\quad\;\; -\,\frac{d}{d\tilde{c}_i}\{\,[1-F(\tilde{c}_i)]^{N-1}\,\beta^1(\tilde{c}_i)\,\}.
\end{aligned}$$

After substituting the second term, we get

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i) \;\leq\; 0,$$

with equality only when $\tilde{c}_i = c_i$.

We have therefore shown that the derivative of the firm's expected profit is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

as required for the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ to attain its maximum at $\tilde{c}_i = c_i$.

To show that the bidding strategy $\beta^1(c_i)$ is strictly decreasing, notice that

$$\frac{d\beta^1}{dc_i}(c_i) \;=\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times \left[\beta^2(c_i \,|\, c_i) + \int_{c_i}^{\bar{c}} \beta^2(c^2 \,|\, c^2) \frac{(N-1)[1-F(c^2)]^{N-2} f(c^2)}{[1-F(c_i)]^{N-1}} dc^2 \right].$$

Since Assumption 3.4 implies that $\beta^2(c_i \,|\, c_i)$ is decreasing in c_i, we can conclude that

$$\frac{d\beta}{dc_i}(c_i) \;<\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times [\, -\beta^2(c_i \,|\, c_i) + \beta^2(c_i \,|\, c_i) \,] \;=\; 0,$$

as required for the bidding strategy β^1 to be strictly decreasing.

□

Proof of Proposition 3.10:

First, notice that by rearranging the terms of the equation (3.6), in the proof of Proposition 3.9, we get

$$\begin{aligned} &[1-F(c_i)]^{N-1} \;\; [\,\beta^1(c_i) - \beta(c_i)\,] \;\; = \\ &\qquad (N-1)[1-F(c_i)]^{N-2} F(c_i) \;\times\; \left[\beta(c_i) - \int_{\underline{c}}^{c_i} \beta^2(c_i|c^1) \, \frac{f(c^1)}{F(c_i)} \; dc^1 \right]. \end{aligned}$$

Therefore, for the entire result, it suffices to show that $\beta^1(c_i) > \beta(c_i)$.

Using the equations (3.1) and (3.4), defining respectively the strategies $\beta(c_i)$ and $\beta^1(c_i)$, we can show, by means of a direct calculation, that $\beta^1(c_i) > \beta(c_i)$ if and only if

$$\int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi^{NS}(c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \quad >$$
$$\int_{\underline{c}}^{c_i} \int_{c_i}^{\bar{c}} \pi^{NS}(c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{(N-1)[1-F(c_i)]^{N-2} F(c_i)} \, dc^2 \, dc^1.$$

That is, we shall show that

$$\mathbb{E}_{c^1_{-i}, c^2_{-i}} [\, \pi^{NS}(c^2, c^1) \,|\, c^2_{-i} \geq c^1_{-i} \geq c_i \,] \; > \; \mathbb{E}_{c^1_{-i}, c^2_{-i}} [\, \pi^{NS}(c^2, c^1) \,|\, c^2_{-i} \geq c_i \geq c^1_{-i} \,].$$

Since the market profit function $\pi^{NS}(c^2, c^1)$ is increasing in c^1, we have

$$\begin{aligned}
&\mathbb{E}_{c^1_{-i}, c^2_{-i}} [\, \pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c^1_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^2_{-i}} [\, \mathbb{E}_{c^1_{-i}} [\pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^1_{-i} \in [c_i, c^2_{-i}]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad > \; \mathbb{E}_{c^2_{-i}} [\, \mathbb{E}_{c^1_{-i}} [\pi^{NS}(c^2_{-i}, c_i) \,|\, c^1_{-i} \in [c_i, c^2_{-i}]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^2_{-i}} [\, \pi^{NS}(c^2_{-i}, c_i) \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^2_{-i}} [\, \mathbb{E}_{c^1_{-i}} [\pi^{NS}(c^2_{-i}, c_i) \,|\, c^1_{-i} \in [\underline{c}, c_i]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad > \; \mathbb{E}_{c^2_{-i}} [\, \mathbb{E}_{c^1_{-i}} [\pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^1_{-i} \in [\underline{c}, c_i]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^1_{-i}, c^2_{-i}} [\, \pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c_i \geq c^1_{-i} \,],
\end{aligned}$$

as required for the result.

$\square$

Proof of Proposition 3.12:

It is straightforward to verify that the function $\beta(c_i)$ is a solution to the differential equation (3.7) that resulted from the necessary first-order condition. In addition, by using L'Hospital's rule, it is easy to check that

$$\lim_{c_i \to \bar{c}} \beta(c_i) = \int_{\underline{c}}^{\bar{c}} \pi(\bar{c} \,|\, \bar{c}, c)\, f(c)\, dc,$$

as required by the boundary condition.

Since the equation (3.7) was only a necessary condition, we still need to establish that it is optimal for any bidder i with marginal cost c_i to bid $b_i = \beta(c_i)$, if all other bidders follow this bidding strategy.

Suppose that firm i bids $\tilde{b}_i = \beta(\tilde{c}_i)$, for $\tilde{c}_i \in [\underline{c}, \bar{c}]$ while having a marginal cost c_i.[2] Then, by changing its bid marginally, that is, by mimicking a marginally different type, it can change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;=\;& -\frac{d}{d\tilde{c}_i}\{\, \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\, \beta(\tilde{c}_i) \,\} \\
& - \int_{\underline{c}}^{\tilde{c}_i} \pi(c_i \,|\, \tilde{c}_i, c^1)\ (N-1)(N-2)\, [1 - F(\tilde{c}_i)]^{N-3}\, f(\tilde{c}_i) f(c^1)\ dc^1 \\
& + \int_{\underline{c}}^{\tilde{c}_i} \pi_2(c_i \,|\, \tilde{c}_i, c^1)\ (N-1)\, [1 - F(\tilde{c}_i)]^{N-2}\, f(c^1)\ dc^1 \\
& + \int_{\tilde{c}_i}^{\bar{c}} \pi_2(c_i \,|\, \tilde{c}_i, c^1)\ (N-1)\, [1 - F(c^1)]^{N-2}\, f(c^1)\ dc^1.
\end{aligned}$$

Substituting the definition (3.8) of the strategy $\beta(c_i)$ and gathering the corresponding terms together result in

[2] If a license is sold at a price $b > \beta(\underline{c})$, an event outside the equilibrium path, we can assume that the remaining firms attribute a marginal cost $c = \underline{c}$ to the winner of that license. Similarly, for $b < \beta(\bar{c})$, we can assume that $c = \bar{c}$. In either case, no firm can profit from mimicking a type $\tilde{c}_i \notin [\underline{c}, \bar{c}]$.

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) &= \int_{\underline{c}}^{\tilde{c}_i} [\pi(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi(c_i \,|\, \tilde{c}_i, c^1)]\,(N-1)(N-2)[1-F(\tilde{c}_i)]^{N-3} f(\tilde{c}_i) f(c^1)\, dc^1 \\
&- \int_{\underline{c}}^{\tilde{c}_i} [\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi_2(c_i \,|\, \tilde{c}_i, c^1)]\,(N-1)\,[1-F(\tilde{c}_i)]^{N-2} f(c^1)\, dc^1 \\
&- \int_{\tilde{c}_i}^{\bar{c}} [\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi_2(c_i \,|\, \tilde{c}_i, c^1)]\,(N-1)\,[1-F(c^1)]^{N-2} f(c^1)\, dc^1.
\end{aligned}
$$

Since the functions $\pi(c_i \,|\, \tilde{c}_i, c^1)$ and $-\pi_2(c_i \,|\, \tilde{c}_i, c^1)$ are decreasing in c_i, the change in the firm's expected profit is

$$
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}
$$

showing that the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ attains its maximum at $\tilde{c}_i = c_i$.

To show that the strategy $\beta(c_i)$ is decreasing, we can calculate its derivative to be

$$
\frac{d\beta}{dc_i}(c_i) = \frac{-\frac{d}{dc_i}\mathbb{P}[c^2_{-i} \geq c_i]}{\mathbb{P}[c^2_{-i} \geq c_i]} \times \left[-v(c_i) + \int_{c_i}^{\bar{c}} v(c^2) \frac{-\frac{d}{dc^2}\mathbb{P}[c^2_{-i} \geq c^2]}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^2 \right],
$$

where

$$
\begin{aligned}
v(c_i) &= \int_{\underline{c}}^{c_i} \pi(c_i \,|\, c_i, c^1) \times \frac{f(c^1)}{F(c^1)}\, dc^1 \\
&+ \int_{\underline{c}}^{c_i} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{1-F(c_i)}{(N-2)f(c_i)} \right] \times \frac{f(c^1)}{F(c^1)}\, dc^1 \\
&+ \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{1-F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-2} F(c_i)}\, dc^1.
\end{aligned}
$$

Therefore, if the function $v(c_i)$ is decreasing, we can conclude that

$$
\frac{d\beta}{dc_i}(c_i) < \frac{-\frac{d}{dc_i}\mathbb{P}[c^2_{-i} \geq c_i]}{\mathbb{P}[c^2_{-i} \geq c_i]} \times [\, -v(c_i) + v(c_i)\,] = 0.
$$

For the monotonicity of the function $v(c_i)$, it suffices to show that each of the three terms in its definition is decreasing. We demonstrate the result for the third term only, since the argument for the first two integrals is similar.

By rewriting this term as

$$\int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{[1 - F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \right] \times \frac{[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}} \, dc^1,$$

we can calculate its derivative to be

$$\begin{aligned}
\frac{d}{dc_i} \left\{ \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1 - F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-2} F(c_i)} \, dc^1 \right\} \quad = \\
\pi_2(c_i \,|\, c_i, c_i) \, \frac{1 - F(c_i)}{(N-2)f(c_i)} \\
+ \int_{c_i}^{\bar{c}} \frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{[1 - F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \right] \times \frac{[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}} \, dc^1 \\
+ \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1 - F(c_i)}{(N-2)F(c_i)} \right] \times \frac{(N-1)[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}} \, dc^1.
\end{aligned}$$

By integrating the last term by parts, we get

$$\begin{aligned}
\frac{d}{dc_i} \left\{ \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1 - F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-2} F(c_i)} \, dc^1 \right\} \quad = \\
\int_{c_i}^{\bar{c}} \frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{[1 - F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \right] \times \frac{[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}} \, dc^1 \\
+ \int_{c_i}^{\bar{c}} \frac{d}{dc^1} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1 - F(c_i)}{(N-2)F(c_i)} \right] \times \frac{[1 - F(c^1)]^{N-1}}{[1 - F(c_i)]^{N-1}} \, dc^1,
\end{aligned}$$

and, since the assumption of the decreasing inverse hazard ratio $\frac{1-F(c_i)}{f(c_i)}$ implies that the term

$$\frac{[1 - F(c_i)]^2}{(N-1)(N-2)F(c_i)f(c_i)} \quad = \quad \frac{[1 - F(c_i)]}{(N-1)F(c_i)} \, \frac{[1 - F(c_i)]}{(N-2)f(c_i)}$$

is also decreasing, we can drop a negative term from the first integral, so as to get

$$\frac{d}{dc_i}\left\{\int_{c_i}^{\bar c}\left[-\pi_2(c_i\,|\,c_i,c^1)\,\frac{1-F(c_i)}{(N-2)f(c_i)}\right]\times\frac{[1-F(c^1)]^{N-2}f(c^1)}{[1-F(c_i)]^{N-2}F(c_i)}\,dc^1\right\}\ \leq$$
$$\int_{c_i}^{\bar c}\frac{d}{dc_i}\left[-\pi_2(c_i\,|\,c_i,c^1)\right]\frac{[1-F(c_i)]^2}{(N-2)F(c_i)f(c_i)}\times\frac{[1-F(c^1)]^{N-2}f(c^1)}{[1-F(c_i)]^{N-1}}\,dc^1$$
$$+\int_{c_i}^{\bar c}\frac{d}{dc^1}\left[-\pi_2(c_i\,|\,c_i,c^1)\right]\frac{1-F(c_i)}{(N-2)F(c_i)}\times\frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}}\,dc^1.$$

Using again the assumption of the decreasing inverse hazard rate,

$$\frac{d}{dc_i}\left\{\int_{c_i}^{\bar c}\left[-\pi_2(c_i\,|\,c_i,c^1)\,\frac{1-F(c_i)}{(N-2)f(c_i)}\right]\times\frac{[1-F(c^1)]^{N-2}f(c^1)}{[1-F(c_i)]^{N-2}F(c_i)}\,dc^1\right\}\ \leq$$
$$\int_{c_i}^{\bar c}\frac{d}{dc_i}\left[-\pi_2(c_i\,|\,c_i,c^1)\right]\frac{1-F(c_i)}{(N-2)F(c_i)}\times\frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}}\,dc^1$$
$$+\int_{c_i}^{\bar c}\frac{d}{dc^1}\left[-\pi_2(c_i\,|\,c_i,c^1)\right]\frac{1-F(c_i)}{(N-2)F(c_i)}\times\frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}}\,dc^1.$$

Finally, since

$$\frac{d}{dc_i}\left[-\pi_2(c_i\,|\,c_i,c^1)\right]\ +\ \frac{d}{dc^1}\left[-\pi_2(c_i\,|\,c_i,c^1)\right]\ =$$
$$-\left[\pi_{21}(c_i\,|\,c_i,c^1)+\pi_{22}(c_i\,|\,c_i,c^1)+\pi_{23}(c_i\,|\,c_i,c^1)\right]\ <\ 0,$$

for all marginal costs $c_i \in [\underline{c},\bar c]$ and all $c^1 \in [c_i,\bar c]$, we conclude that the derivative is negative.

The argument establishing that the stronger of the two oligopolists always makes a positive profit, unlike the weaker oligopolist who may regret his participation to the market, is identical to that in the proof of Corollary 3.2, with $\pi(c_i\,|\,c_i,c^1)$ in place of $\pi^{NS}(c_i,c^1)$. It is therefore omitted.

$\square$

Proof of Lemma 3.10:

For arbitrary $c^1 \in [\underline{c}, \bar{c}]$, suppose that all firms follow the bidding strategy $\beta^2(\,.\,|\,c^1)$ and consider firm i with marginal cost $c_i \in [c^1, \bar{c}]$. Obviously, firm i cannot profit by bidding above $\beta^2(c^1\,|\,c^1)$ or below $\beta^2(\bar{c}\,|\,c^1)$.[3] So, suppose that firm i mimics a type $\tilde{c}_i \in [c^1, \bar{c}]$, that is, it bids $\beta^2(\tilde{c}_i\,|\,c^1)$. Then its expected payoff will be

$$\Pi(\tilde{c}_i\,|\,c_i) = [1-\tilde{F}(\tilde{c}_i)]^{N-2}\,[\,\pi(c_i\,|\,\tilde{c}_i, c^1) - \beta^2(\tilde{c}_i|c^1)\,].$$

By changing its bid marginally, firm i will change its expected payoff by

$$\begin{aligned}\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i) \;&=\; -\,\pi(c_i\,|\,\tilde{c}_i, c^1)\;(N-2)[1-\tilde{F}(\tilde{c}_i)]^{N-3}\tilde{f}(\tilde{c}_i)\\ &\quad +\,\pi_2(c_i\,|\,\tilde{c}_i, c^1)\;[1-\tilde{F}(\tilde{c}_i)]^{N-2}\\ &\quad -\,\frac{\partial}{\partial \tilde{c}_i}\{\,[1-\tilde{F}(\tilde{c}_i)]^{N-2}\,\beta^2(\tilde{c}_i\,|\,c^1)\,\}.\end{aligned}$$

By calculating the derivative in the last term, we get

$$\begin{aligned}\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i) \;&=\; [\,\pi(\tilde{c}_i\,|\,\tilde{c}_i, c^1) - \pi(c_i\,|\,\tilde{c}_i, c^1)\,]\;(N-2)[1-\tilde{F}(\tilde{c}_i)]^{N-3}\tilde{f}(\tilde{c}_i)\\ &\quad +\;[\,\pi_2(c_i\,|\,\tilde{c}_i, c^1) - \pi_2(\tilde{c}_i\,|\,\tilde{c}_i, c^1)\,]\;[1-\tilde{F}(\tilde{c}_i)]^{N-2}.\end{aligned}$$

Since both the profit function $\pi(c_i\,|\,\tilde{c}_i, c^1)$ and the derivative $-\pi_2(c_i\,|\,\tilde{c}_i, c^1)$ are decreasing in c_i, it follows that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i)\;\begin{cases} >0, & \text{for } \tilde{c}_i < c_i;\\ =0, & \text{for } \tilde{c}_i = c_i;\\ <0, & \text{for } \tilde{c}_i > c_i,\end{cases}$$

as required for the optimality of bidding $\beta^2(c_i\,|\,c^1)$.

[3] If the license is sold at a price $b^2 > \beta^2(c^1|c^1)$, an event outside the equilibrium path, we can assume that the remaining firms, in particular, the competing oligopolist, will attribute a marginal cost $c^2 = c^1$ to the winner of the license. Similarly, for $b^2 < \beta^2(\bar{c}|c^1)$, we can assume that $c^2 = \bar{c}$.

In addition, when firm i has marginal cost $c_i \in [\underline{c}, c^1]$, the previous analysis shows that $\Pi_1(\tilde{c}_i \,|\, c_i) < 0$, for all $\tilde{c}_i \in [c^1, \bar{c}]$. Hence, firm i is best-off bidding $\beta^2(c^1 \,|\, c^1)$.

Finally, for the monotonicity of the strategy $\beta^2(c_i \,|\, c^1)$, notice that, since the inverse hazard rate $[1 - F(c)]/f(c)$ is decreasing, the expression

$$v(c \,|\, c, c^1) \;=\; \pi(c \,|\, c, c^1) \;-\; \pi_2(c \,|\, c, c^1)\, \frac{1 - F(c)}{(N-2)f(c)}$$

is decreasing in c. Therefore, the derivative

$$\begin{aligned} \frac{\partial \beta^2}{\partial c_i}(c_i \,|c^1) \;&=\; -\, v(c_i \,|\, c_i, c^1)\, \frac{(N-2)f(c_i)}{1 - F(c_i)} \\ &\quad + \int_{c_i}^{\bar{c}} v(c \,|\, c, c^1)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \,\times\, \frac{(N-2)f(c_i)}{1 - F(c_i)} \end{aligned}$$

is negative, showing that the strategy $\beta^2(c^1 \,|\, c^1)$ is strictly decreasing in $c_i \in (c^1, \bar{c}\,]$.

$\square$

Proof of Lemma 3.15:

The derivative of $\beta^2(c_i \,|\, c_i)$ equals to

$$\begin{aligned} \frac{d}{dc_i}[\,\beta^2(c_i \,|\, c_i)\,] \;&=\; -v(c_i, c_i)\, \frac{(N-2)f(c_i)}{1 - F(c_i)}, \\ &\quad + \int_{c_i}^{\bar{c}} v_2(c, c_i)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \\ &\quad + \int_{c_i}^{\bar{c}} v(c, c_i)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \,\times\, \frac{(N-2)f(c_i)}{1 - F(c_i)}, \end{aligned}$$

or, after integrating the last term by parts, to

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i\,|\,c_i)\,] &= \int_{c_i}^{\bar{c}} v_2(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&+ \int_{c_i}^{\bar{c}} v_1(c,c_i)\,\frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc \times \frac{(N-2)f(c_i)}{1-F(c_i)}.
\end{aligned}
$$

For all $c_i \in [\underline{c},\bar{c}]$, since the inverse hazard ratio $[1-F(c)]/f(c)$ is decreasing, we have

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i\,|\,c_i)\,] &< \int_{c_i}^{\bar{c}} v_2(c\,|\,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&+ \int_{c_i}^{\bar{c}} \tilde{v}_1(c\,|\,c_i)\,\frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc\,\frac{(N-2)f(c_i)}{1-F(c_i)},
\end{aligned}
$$

where

$$
\tilde{v}_1(c_i\,|\,c^1) = \frac{d\pi}{dc_i}(c_i\,|\,c_i,c^1) - \frac{d\pi_2}{dc_i}(c_i\,|\,c_i,c^1)\,\frac{1-F(c_i)}{(N-2)f(c_i)}.
$$

Since the expression

$$
\begin{aligned}
w(c,c_i) &= \frac{v_2(c\,|\,c_i)}{-\tilde{v}_1(c\,|\,c_i)} - \frac{f(c_i)/[1-F(c_i)]}{f(c)/[1-F(c)]} \\
&= \frac{1}{2} - \frac{f(c_i)/[1-F(c_i)]}{f(c)/[1-F(c)]}
\end{aligned}
$$

is negative for $c = c_i$, positive for $c = \bar{c}$, continuous and increasing with respect to $c \in [c_i,\bar{c}]$, there exists a value $c^* = c^*(c_i) \in (c_i,\bar{c})$ such that

$$
w(c,c_i)\;\begin{cases} <0, & \text{for } c \in [c_i,c^*); \\ =0, & \text{for } c = c^*; \\ >0, & \text{for } c \in (c^*,\bar{c}\,]. \end{cases}
$$

Therefore, since the function $-\tilde{v}_1(c, c^1)$ is decreasing in $c \in [c_i, \bar{c}]$ and positive, we have

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i\,|\,c_i)\,] \;&<\; \int_{c_i}^{c^*} -\tilde{v}_1(c\,|\,c_i)\; w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\, dc \\
&+\; \int_{c^*}^{\bar{c}} -\tilde{v}_1(c\,|\,c_i)\; w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\, dc \\
&<\; -\tilde{v}_1(c^*\,|\,c_i)\; \int_{c_i}^{\bar{c}} w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\, dc
\end{aligned}
$$

Finally, Assumption 3.14 implies that

$$
\int_{c_i}^{\bar{c}} w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\, dc \;\leq\; 0,
$$

which suffices for $\beta^2(c_i|c_i)$ to be decreasing.

□

Proof of Proposition 3.16:

Suppose that all firms follow the bidding strategy (β^1, β^2) and consider firm i with marginal cost $c_i \in [\underline{c}, \bar{c}]$. The optimality of bidding $\beta^2(c_i\,|\,c^1)$ in the second auction, following the sale of the first license at a price b^1 corresponding to a marginal cost $c^1 = (\beta^1)^{-1}(b^1)$, has been established in Lemma 3.13.[4] Therefore, we only need to examine the optimality of bidding $\beta^1(c_i)$ in the first auction.

Obviously, firm i cannot gain from submitting a bid above $\beta^1(\underline{c})$ or below $\beta^1(\bar{c})$. So, suppose that it mimics a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$, that is, it bids $\beta^1(\tilde{c}_i)$. If $\tilde{c}_i \leq c_i$, then, by changing its bid marginally, firm i will change its expected payoff by

[4] In case the first license is sold at a price $b^1 > \beta^1(\underline{c})$, an event outside the equilibrium path, we can assume that the remaining firms attribute a marginal cost $c^1 = \underline{c}$ to the winner of the license. Similarly, for $b^1 < \beta^1(\bar{c})$, we can assume that $c^1 = \bar{c}$.

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \pi(c_i \,|\, c_i, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \pi(c_i \,|\, \tilde{c}_i, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\,dc^1, \\
&-\; \beta^2(c_i \,|\, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \frac{d}{d\tilde{c}_i}\{\,[1-F(\tilde{c}_i)]^{N-1}\,\beta^1(\tilde{c}_i)\,\}.
\end{aligned}
$$

After substituting the appropriate expression for the last term, the change in the expected payoff of firm i becomes

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \pi(c_i \,|\, c_i, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \pi(c_i \,|\, \tilde{c}_i, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\,dc^1 \\
&+\; \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\,dc^1 \\
&-\; \beta^2(c_i \,|\, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&+\; \beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i).
\end{aligned}
$$

The difference between the last two terms equals to

$$
\begin{aligned}
&\beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i \,|\, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \;=\; \\
&\qquad \int_{\tilde{c}_i}^{c_i} \pi(c^2 \,|\, c^2, \tilde{c}_i)\,(N-2)[1-F(c^2)]^{N-3} f(c^2)\,dc^2\;\,(N-1) f(\tilde{c}_i) \\
&\quad +\; \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\,dc^2\;\,(N-1) f(\tilde{c}_i).
\end{aligned}
$$

Since $\pi(c_i \,|\, \tilde{c}_i, c^1)$ is decreasing in c_i, we have

$$\begin{aligned}
&\beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i \,|\, \tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \quad > \\
&\qquad\qquad \int_{\tilde{c}_i}^{c_i} \pi(c_i \,|\, c^2, \tilde{c}_i)\ (N-2)[1-F(c^2)]^{N-3} f(c^2)\, dc^2\ (N-1) f(\tilde{c}_i) \\
&\qquad + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2\ (N-1) f(\tilde{c}_i)
\end{aligned}$$

and, by integrating the first term by parts,

$$\begin{aligned}
&\beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i \,|\, \tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \quad > \\
&\qquad\qquad \pi(c_i \,|\, \tilde{c}_i, \tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&\qquad - \pi(c_i \,|\, c_i, \tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&\qquad - \int_{\tilde{c}_i}^{c_i} -\pi_2(c_i \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2\ (N-1) f(\tilde{c}_i) \\
&\qquad + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2\ (N-1) f(\tilde{c}_i).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \quad \geq \quad & - \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
& + \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
& - \int_{\tilde{c}_i}^{c_i} -\pi_2(c_i \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2\ (N-1) f(\tilde{c}_i) \\
& + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2\ (N-1) f(\tilde{c}_i)
\end{aligned}$$

and, since the derivative $-\pi_2(c_i \,|\, \tilde{c}_i, , c^1)$ is decreasing in c_i, we can conclude that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; \geq \;\; 0,$$

with equality only when $\tilde{c}_i = c_i$.

Similarly, if firm i mimics a marginal cost $\tilde{c}_i \geq c_i$, then, by changing its bid marginally, it will change its expected payoff by

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; &= \;\; - \int_{\tilde{c}_i}^{\bar{c}} - \pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1, \\ &\quad - \beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\ &\quad - \frac{d}{d\tilde{c}_i}\{\, [1-F(\tilde{c}_i)]^{N-1}\, \beta^1(\tilde{c}_i)\,\}. \end{aligned}$$

By substituting the appropriate expression for the last term, we get

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; &= \;\; - \int_{\tilde{c}_i}^{\bar{c}} - \pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\ &\quad + \int_{\tilde{c}_i}^{\bar{c}} - \pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1, \end{aligned}$$

which implies that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; \leq \;\; 0,$$

with equality only when $\tilde{c}_i = c_i$.

We have therefore shown that the derivative of the firm's expected profit is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

as required for the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ to attain its maximum at $\tilde{c}_i = c_i$.

Finally, to show that the bidding strategy $\beta^1(c_i)$ is strictly decreasing, notice that we can write its derivative as

$$\frac{d\beta^1}{dc_i}(c_i) \;=\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times \left[-v^1(c_i) + \int_{c_i}^{\bar{c}} v^1(c^2) \, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^2)}{[1-F(c_i)]^{N-1}} \, dc^2 \right],$$

where

$$\begin{aligned} v^1(c_i) \;&=\; \beta^2(c_i \,|\, c_i) \\ &+\; \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1-F(c_i)}{(N-1)f(c_i)} \right] \times \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^1. \end{aligned}$$

Because of Assumption 3.14, the term $\beta^2(c_i \,|\, c_i)$ is decreasing in c_i. In addition, by an argument similar to the one used for the corresponding term in $\frac{d}{dc_i}\beta(c_i)$, we can show that the second term is also decreasing in c_i. Therefore, the function v^1 is decreasing.

It follows that

$$\frac{d\beta}{dc_i}(c_i) \;<\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times \left[\, -v(c_i) + v(c_i) \,\right] \;=\; 0,$$

as required for the bidding strategy β^1 to be strictly decreasing.

$\square$

Proof of Proposition 3.18:

First, notice that by rearranging the terms of the equation (3.6), relating the bidding strategies $\beta(c_i)$, $\beta^1(c_i)$ and $\beta^2(c_i \,|\, c^1)$ in the proof of Proposition 3.9, we get

$$\begin{aligned} &[1-F(c_i)]^{N-1} \; [\,\beta^1(c_i) - \beta(c_i)\,] \;= \\ &\qquad (N-1)[1-F(c_i)]^{N-2} F(c_i) \; \left[\beta(c_i) - \int_{\underline{c}}^{c_i} \beta^2(c_i \,|\, c^1) \, \frac{f(c^1)}{F(c_i)} \, dc^1 \right]. \end{aligned}$$

Therefore, for the entire result, it suffices to show that $\beta^1(c_i) > \beta(c_i)$.

Using the equations (3.8) and (3.9) defining the strategies $\beta(c_i)$ and $\beta^1(c_i)$, we can show, by means of a direct calculation, that $\beta^1(c_i) > \beta(c_i)$ if and only if

$$\begin{aligned}
&\int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} v(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^2\, dc^1 \quad + \\
&\int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^2\, dc^1 \qquad > \\
&\qquad \int_{\underline{c}}^{c_i} \int_{c_i}^{\bar{c}} v(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{(N-1)[1-F(c_i)]^{N-2} F(c_i)}\, dc^2\, dc^1,
\end{aligned}$$

where

$$v(c^2 \,|\, c^2, c^1) \;=\; \pi(c^2 \,|\, c^2, c^1) \;-\; \pi_2(c^2 \,|\, c^2, c^1)\, \frac{1-F(c^2)}{(n-2) f(c^2)}.$$

Since the second double integral is positive, it suffices to show that

$$\begin{aligned}
&\int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} v(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^2\, dc^1 \quad > \\
&\int_{\underline{c}}^{c_i} \int_{c_i}^{\bar{c}} v(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{(N-1)[1-F(c_i)]^{N-2} F(c_i)}\, dc^2\, dc^1,
\end{aligned}$$

that is, to show that

$$\mathbb{E}_{c^1_{-i}, c^2_{-i}}[\, v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \ge c^1_{-i} \ge c_i \,] \;>\; \mathbb{E}_{c^1_{-i}, c^2_{-i}}[\, v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \ge c_i \ge c^1_{-i} \,].$$

Since the function $v(c^2 \,|\, c^2, c^1)$ is increasing in c^1, we have

$$
\begin{aligned}
&\mathbb{E}_{c^1_{-i},c^2_{-i}}[\, v(c^2_{-i}\,|\,c^2_{-i},c^1_{-i})\,|\,c^2_{-i}\geq c^1_{-i}\geq c_i\,] \\
&\qquad = \quad \mathbb{E}_{c^2_{-i}}[\,\mathbb{E}_{c^1_{-i}}[v(c^2_{-i}\,|\,c^2_{-i},c^1_{-i})\,|\,c^1_{-i}\in[c_i,c^2_{-i}]\,]\,|\,c^2_{-i}\geq c_i\,] \\
&\qquad > \quad \mathbb{E}_{c^2_{-i}}[\,\mathbb{E}_{c^1_{-i}}[v(c^2_{-i}\,|\,c^2_{-i},c_i)\,|\,c^1_{-i}\in[c_i,c^2_{-i}]\,]\,|\,c^2_{-i}\geq c_i\,] \\
&\qquad = \quad \mathbb{E}_{c^2_{-i}}[\,v(c^2_{-i}\,|\,c^2_{-i},c_i)\,|\,c^2_{-i}\geq c_i\,] \\
&\qquad = \quad \mathbb{E}_{c^2_{-i}}[\,\mathbb{E}_{c^1_{-i}}[v(c^2_{-i}\,|\,c^2_{-i},c_i)\,|\,c^1_{-i}\in[\underline{c},c_i]\,]\,|\,c^2_{-i}\geq c_i\,] \\
&\qquad > \quad \mathbb{E}_{c^2_{-i}}[\,\mathbb{E}_{c^1_{-i}}[v(c^2_{-i}\,|\,c^2_{-i},c^1_{-i})\,|\,c^1_{-i}\in[\underline{c},c_i]\,]\,|\,c^2_{-i}\geq c_i\,] \\
&\qquad = \quad \mathbb{E}_{c^1_{-i},c^2_{-i}}[\,v(c^2_{-i}\,|\,c^2_{-i},c^1_{-i})\,|\,c^2_{-i}\geq c_i\geq c^1_{-i}\,],
\end{aligned}
$$

as required for the result.

$\square$

Proof of Proposition 3.19:

Similarly to the proof of Proposition 3.12, notice that the strategy $\beta(c_i)$ can be expressed as

$$
\beta(c_i) \quad = \quad \int_{c_i}^{\bar{c}} u(c^2\,|\,c^2)\ \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\,F(c^2)\,f(c^2)}{\mathbb{P}[c^2_{-i}\geq c_i]}\ dc^2,
$$

where

$$
\begin{aligned}
u(\tilde{c}\,|\,c) \quad = \quad & \int_{\underline{c}}^{\tilde{c}} \pi(c\,|\,\tilde{c},c^1)\ \times\ \frac{f(c^1)}{F(\tilde{c})}\ dc^1 \\
- \quad & \int_{\underline{c}}^{\tilde{c}} \left[\pi_2(c\,|\,\tilde{c},c^1)\,\frac{1-F(\tilde{c})}{(N-2)f(\tilde{c})}\right]\ \times\ \frac{f(c^1)}{F(\tilde{c})}\ dc^1 \\
- \quad & \int_{c}^{\bar{c}} \left[\pi_2(c\,|\,\tilde{c},c^1)\,\frac{1-F(\tilde{c})}{(N-2)f(\tilde{c})}\right]\ \times\ \frac{[1-F(c^1)]^{N-2}f(c^1)}{[1-F(\tilde{c})]^{N-2}F(\tilde{c})}\ dc^1,
\end{aligned}
$$

for $\tilde{c}$, $c \in [\underline{c}, \bar{c}]$. In particular, $u(c) = u(c \,|\, c)$ is the valuation of a firm with marginal cost c, assuming that its market opponent is stronger.

Therefore, we have

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;=\; (N-1)(N-2)\,[1 - F(\tilde{c}_i)]^{N-3}\, F(\tilde{c}_i)\, f(\tilde{c}_i)\; [\, u(\tilde{c}_i \,|\, \tilde{c}_i) - u(\tilde{c}_i \,|\, c_i)\,]$$

and

$$\frac{\partial^2 \Pi}{\partial \tilde{c}_i \partial c_i}(\tilde{c}_i \,|\, c_i) \;=\; -\,(N-1)(N-2)\,[1 - F(\tilde{c}_i)]^{N-3}\, F(\tilde{c}_i)\, f(\tilde{c}_i)\; u_2(\tilde{c}_i | c_i).$$

Hence, if we can show that

$$u_2(\tilde{c}_i \,|\, c_i) \;>\; 0,$$

then, since $\frac{\partial \Pi}{\partial \tilde{c}_i}(c_i \,|\, c_i) = 0$, we can conclude that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

as it suffices for firm i's expected profit function $\Pi(\tilde{c}_i | c_i)$ to attain its maximum at $\tilde{c}_i = c_i$.

Notice that

$$\begin{aligned} u_2(\tilde{c}_i \,|\, c_i) \;=\; & \int_{\underline{c}}^{\tilde{c}_i} \pi_1(c_i \,|\, \tilde{c}_i, c^1) \times \frac{f(c^1)}{F(\tilde{c}_i)}\, dc^1 \\ - & \int_{\underline{c}}^{\tilde{c}_i} \left[\pi_{21}(c_i \,|\, \tilde{c}_i, c^1)\, \frac{1 - F(\tilde{c}_i)}{(N-2) f(\tilde{c}_i)} \right] \times \frac{f(c^1)}{F(\tilde{c}_i)}\, dc^1 \\ - & \int_{c_i}^{\bar{c}} \left[\pi_{21}(c_i \,|\, \tilde{c}_i, c^1)\, \frac{1 - F(\tilde{c}_i)}{(N-2) f(\tilde{c}_i)} \right] \times \frac{[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(\tilde{c}_i)]^{N-2} F(\tilde{c}_i)}\, dc^1. \end{aligned}$$

Therefore, if N is sufficiently large, then the positive term dominates the negative ones, so that $u_2(\tilde{c}_i \,|\, c_i) > 0$. Moreover, it is possible to find N^* such that for $N > N^*$ we have $u_2(\tilde{c}_i \,|\, c_i) > 0$, for all $\tilde{c}_i, c_i \in [\underline{c}, \bar{c}]$, uniformly.

The rest of the proof is identical to that of Proposition 3.12, so, it is omitted.

$\square$

Proof of Proposition 3.20:

Suppose, contrary to our assertion, that there exists a symmetric equilibrium in monotone bidding strategies (β^1, β^2). Since β^1 is strictly decreasing, the announcement of the first-round winning bid reveals the marginal cost c^1 of the strongest oligopolist. Therefore, in the second round, the firms update their beliefs, so that, for all $c \in [c^1, \bar{c}]$,

$$c_i \sim \tilde{F}(c) = \frac{F(c) - F(c^1)}{1 - F(c^1)}.$$

If the number of firms participating in the second auction, $N-1$, is sufficiently large, so as to satisfy the inequality

$$\frac{\pi_2(c,c)}{\pi_{12}(c,c)} \;>\; \frac{[1-F(c)]}{(N-2)\,f(c)}\,,$$

for all $c \in [\underline{c}, \bar{c}]$, then, as shown in Das Varma [23], the strategy

$$\begin{aligned} \beta^2(c_i \,|c^1) \;&=\; \int_{c_i}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1)\, \frac{(N-2)[1-F(c^2)]^{N-3} f(c^2)}{[1-F(c_i)]^{N-2}}\; dc^2 \\ &+\; \int_{c_i}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{[1-F(c^2)]^{N-2}}{[1-F(c_i)]^{N-2}}\; dc^2, \end{aligned}$$

for $c_i \geq c^1$, forms the unique equilibrium in the auction of the second license. In particular, for a marginal cost $c_i < c^1$, firm i bids $b^2 = \beta^2(c^1 \,|c^1)$.

By assuming, as we did in the case of the Cournot oligopoly, that, for all $c_i \in [\underline{c}, \bar{c}]$,

$$\int_{c_i}^{\bar{c}} \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\, dc \;\geq\; \sup_{c \geq c_i} \left\{ \frac{v_2(c, c_i)}{-\tilde{v}_1(c, c_i)} \right\}\; \frac{1-F(c_i)}{(N-2)\, f(c_i)},$$

where

$$\tilde{v}_1(c, c^1) \;=\; \frac{d}{dc}[\pi(c \,|\, c, c^1)] \;-\; \frac{d}{dc}[\pi_2(c \,|\, c, c^1)]\, \frac{1 - F(c)}{(N-2)\, f(c)}\,,$$

we can ensure that the strategy $\beta^2(c_i|c_i)$ is decreasing in the marginal cost c_i.

In the first auction, the optimization of the profit function $\Pi(\tilde{c}_i|c_i)$ of a firm i with marginal cost c_i results in the strategy

$$\begin{aligned}
\beta^1(c_i) \;&=\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1)\, \frac{(N-1)(N-2)[1 - F(c^2)]^{N-3} f(c^2) f(c^1)}{[1 - F(c_i)]^{N-1}}\; dc^2\, dc^1 \\
&+\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)[1 - F(c^2)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}}\; dc^2\, dc^1 \\
&+\; \int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1)\, \frac{(N-1)[1 - F(c^1)]^{N-2} f(c^1)}{[1 - F(c_i)]^{N-1}}\; dc^2\, dc^1\,,
\end{aligned}$$

as the unique solution of the differential equation derived from the necessary first-order condition $\Pi_1(c_i \,|\, c_i) = 0$.

To check sufficiency, we can calculate, for $\tilde{c}_i \geq c_i$,

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\tilde{c}_i}^{\bar{c}} \pi_2(c_i \,|\, \tilde{c}_i, c^1)\, (N-1)[1 - F(c^1)]^{N-2} f(c^1)\, dc^1 \\
&-\; \int_{\tilde{c}_i}^{\bar{c}} \pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\, (N-1)[1 - F(c^1)]^{N-2} f(c^1)\, dc^1.
\end{aligned}$$

Since $\pi_2(c_i \,|\, \tilde{c}_i, c^1) > \pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)$, for $\tilde{c}_i \geq c_i$, we conclude that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;>\; 0,$$

showing that the firm's deviation from $\beta^1(c_i)$ to $\beta^1(\tilde{c}_i)$, for $\tilde{c}_i > c_i$, is profitable. Hence, the strategy $\beta^1(c_i)$ fails to support an equilibrium.

Since the strategy β^1 was the unique solution to the necessary condition, we conclude that the sequential auction has no equilibrium in strictly monotone strategies.

$\square$

C PROOFS FOR CHAPTER 4

Proof of Proposition 4.1:

Suppose that all buyers follow the bidding strategy $\beta(\cdot)$, with their participation in the auction determined by the threshold value v^*. Consider buyer i with valuation $v_i \in [\underline{v}, \bar{v}]$.

First, for $v_i \geq v^*$, suppose that buyer i submits a bid $\tilde{b}_i = \beta(\tilde{v}_i)$, for some $\tilde{v}_i \in [v^*, \bar{v}]$ such that $\tilde{v}_i \neq v_i$. Then his expected payoff is

$$\Pi(\tilde{v}_i \,|\, v_i) \;=\; \mathbb{P}[v^2_{-i} \leq \tilde{v}_i]\;[\,v_i - \beta(\tilde{v}_i)\,] \;-\; c,$$

or, after substituting equation (4.2) defining the strategy $\beta(\cdot)$, integrating by parts the resulting expression, and applying equation (4.1) defining the threshold v^*,

$$\Pi(\tilde{v}_i \,|\, v_i) \;=\; \mathbb{P}[v^2_{-i} \leq \tilde{v}_i]\;(\,v_i - \tilde{v}_i\,) \;+\; \int_{v^*}^{\tilde{v}_i} \mathbb{P}[v^2_{-i} \leq v]\;dv.$$

Hence, buyer i's expected profit from this deviation is

$$\Pi(\tilde{v}_i \,|\, v_i) - \Pi(v_i \,|\, v_i) \;=\; \mathbb{P}[v^2_{-i} \leq \tilde{v}_i]\;(\,v_i - \tilde{v}_i\,) \;+\; \int_{v_i}^{\tilde{v}_i} \mathbb{P}[v^2_{-i} \leq v]\;dv.$$

For either $\tilde{v}_i > v_i$ or $\tilde{v}_i < v_i$, we have

$$\Pi(\tilde{v}_i \,|\, v_i) - \Pi(v_i \,|\, v_i) \;\leq\; \mathbb{P}[v^2_{-i} \leq \tilde{v}_i]\;(\,v_i - \tilde{v}_i\,) \;+\; \mathbb{P}[v^2_{-i} \leq \tilde{v}_i]\;(\,\tilde{v}_i - v_i\,) \;\leq\; 0,$$

showing that the deviation to bidding $\tilde{b}_i = \beta(\tilde{v}_i)$ is not profitable.

If buyer i abstains from the auction, his payoff will be

$$0 \;=\; \Pi(v^* \,|\, v^*) \;\leq\; \Pi(v^* \,|\, v_i) \;\leq\; \Pi(v_i \,|\, v_i),$$

the first inequality following from the monotonicity of $\Pi(\tilde{v}_i \,|\, v_i)$ in the buyer's type v_i, the second inequality from the argument used above. Therefore, buyer i is better-off participating in the auction and bidding $b_i = \beta(v_i)$.

Finally, for $v_i < v^*$, suppose that buyer i participates to the auction by submitting a bid $\tilde{b}_i = \beta(\tilde{v}_i)$, for some $\tilde{v}_i \in [v^*, \bar{v}]$. Then

$$\Pi(\tilde{v}_i \,|\, v_i) \;\leq\; \Pi(\tilde{v}_i \,|\, v^*) \;\leq\; \Pi(v^* \,|\, v^*) \;=\; 0,$$

so that buyer i is better-off abstaining from the auction.

□

Proof of Proposition 4.2:

Suppose that all buyers follow the bidding strategy (β^1, β^2), with their participation in the auction being determined by the threshold values v_1^*, v_2^* and $\tilde{v}_2^*$.

First, it is easy to check that the strategy β^1 is strictly increasing in $[v_1^*, \bar{v}]$. Therefore, the first-period price $b^1 \in [0, \beta^1(\bar{v})] \cup \{a\}$ reveals an upper bound v^1 for the valuations of the remaining buyers, as described in equation (4.3).

In the second period, an argument similar to the one in Samuelson [86] shows that the strategy profile $b_i^2 = \beta^2(v_i \,|\, v^1)$, for all i, along with the participation rule defined by the thresholds v_1^*, v_2^* and $\tilde{v}_2^*$, defines a symmetric equilibrium for the continuation game following any first-period outcome described by the price $b^1 \in \mathbb{R}^+ \cup \{a\}$.

In particular, for any $v^1 \in [v_1^*, \bar{v}]$, the function $y(v \,|\, v^1) = F(v \,|\, v^1)^M v$, for $v \in [\underline{v}, v^1]$, where $M = N-2$ or $M = N-1$, is strictly increasing, continuous, and takes values in $[0, v^1]$. Since $v^1 \geq v_1^* \geq c$, there exists a unique value $v_2^* = v_2^*(v^1) \in [\underline{v}, v^1]$ such that $y(v_2^* \,|\, v^1) = c$. Therefore, on the equilibrium path, the buyers' second-period participation decision, given by the equations (4.4) and (4.6), is well defined.

Similarly, the function $\tilde{y}(v \,|\, v^1) = F(v \,|\, v^1)^{N-2}\,[v - \beta^2(\min\{v, v^1\} \,|\, v^1)]$, for $v \in [\underline{v}, \bar{v}]$ and $v^1 \in [v_1^*, \bar{v}_1]$, is strictly increasing, continuous, and takes values in $[0, \bar{v} - \beta^2(v^1 \,|\, v^1)]$. Since $v^1 \leq \bar{v}_1$, it follows that $\bar{v} - \beta^2(v^1 \,|\, v^1) \geq c$. Hence, there exists a unique value $\tilde{v}_2^* \in [\underline{v}, \bar{v}]$ such that $\tilde{y}(\tilde{v}_2^* \,|\, v^1) = c$. Therefore, off the equilibrium path, the buyers' second-period participation decision, given by the equation (4.9), is well defined.

In the first period, the type v_1^* is indifferent between participating in the first round with a minimal bid $\beta^1(v_1^*) = 0$ and waiting for the second round in order to decide, after observing the outcome b^1, whether to participate in the auction. Equating the two expected payoffs results in

$$\begin{aligned}
& (N-1)\, F(v_1^*)^{N-2}\, [1 - F(\hat{v}_1)]\; v_1^* \\
- \;& [1 - F(\hat{v}_1)^{N-1}]\; c \\
- \;& (N-1)\, F(v_1^*)^{N-2} \int_{\hat{v}_1}^{\bar{v}} \beta^2(v_1^* \,|\, v^1)\, f(v^1)\, dv^1 \\
+ \;& F(v_1^*)^{N-1}\; \beta^2(v_1^* \,|\, a) \qquad\qquad = \; 0,
\end{aligned}$$

which, since $\beta^2(v_1^* \,|\, v^1) = 0$, for all $v^1 \in [\hat{v}_1, \bar{v}]$, reduces to equation (4.10).

To show existence and uniqueness of the value v_1^*, we proceed to defining the function $\phi : [\underline{v}, \bar{v}] \to \mathbb{R}$ as the difference in the v-type's payoff between entering in the first round and waiting for the second round to decide whether to participate to the auction, under the assumption that the first-round participation threshold is v.

First, for $v \in [\underline{v}, \bar{v}]$, we define the upper bound $\hat{v}_1(v) \in [\underline{v}, \bar{v}]$ by

$$\begin{cases} F[v \,|\, \hat{v}_1(v)]^{N-2}\, v \;=\; c, & \text{if } v \in [\underline{v}, \check{v}_1]; \\ \hat{v}_1(v) \;=\; \bar{v}, & \text{if } v \in [\check{v}_1, \bar{v}], \end{cases} \tag{C.1}$$

where $\check{v}_1$ is given by

$$F(\check{v}_1)^{N-2} \check{v}_1 \;=\; c.$$

For $v \in [\underline{v}, \check{v}_1]$, the value $\hat{v}_1(v)$ corresponds to the first-round winning bid $b^1 = \beta^1(\hat{v}_1(v))$ that would make the v-type indifferent between entering the auction in the second round (to compete against $N-2$ potential buyers) and abstaining from the entire procedure, under the assumption that the first-round participation threshold is v. For $v \in [\check{v}_1, \bar{v}]$, the value $\hat{v}_1(v)$ corresponds to the winning bids $b^1 = \beta^1(\hat{v}_1(v))$ for which the v-type prefers abstaining from the entire procedure to entering the auction in the second round, again under the assumption that the first-round participation threshold is v.

Given the upper bound $\hat{v}_1(v)$, we define the function $\phi : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}$ by

$$\phi(v) \;=\; (N-1)\, F(v)^{N-2}\, [1 - F(\hat{v}_1(v))]\, v \;-\; [1 - F(\hat{v}_1(v))^{N-1}]\; c \;+\; F(v)^{N-1}\; \beta^2(v\,|\,a),$$

or, after using equation (4.5), defining $\beta^2(v\,|\,a)$,

$$\begin{aligned} \phi(v) \;&=\; (N-1)\, F(v)^{N-2}\, [1 - F(\hat{v}_1(v))]\, v \\ &-\; [1 - F(\hat{v}_1(v))^{N-1}]\; c \\ &+\; \int_{v_2^*(a)}^{v} v\,(N-1) F(v)^{N-2} f(v)\, dv, \end{aligned}$$

with the threshold $v_2^*(a)$ being defined by equation (4.4) as a function of the value v. Clearly, for all $v \in [\check{v}_1, \bar{v}]$, the function reduces to $\phi(v) \;=\; F(v)^{N-1}\; \beta^2(v\,|\,a)$.

Since the function ϕ is continuous in $[\underline{v}, \bar{v}]$ and takes values $\phi(\underline{v}) < 0$ and $\phi(\check{v}_1) > 0$, by the Intermediate Value Theorem, there exists a value $v_1^* \in (\underline{v}, \check{v}_1)$ such that $\phi(v_1^*) = 0$, as required for equation (4.10) to have a solution.

In addition, for $v \in [\underline{v}, \check{v}_1]$, the derivative of $\phi(v)$ is

$$\begin{aligned} \frac{d\phi}{dv}(v) \;&=\; (N-1)(N-2)\, F(v)^{N-3}\, [1 - F(\hat{v}_1(v))]\, f(v)\, v \\ &+\; (N-1)\, F(v)^{N-2}\, [1 - F(\hat{v}_1(v))] \\ &+\; (N-1)\, F(v)^{N-2}\, f(v)\, v \\ &+\; (N-1)\, [\, F(v)^{N-2} v - F(\hat{v}_1(v))^{N-2} c\,]\, f(\hat{v}_1(v))\, \frac{d\hat{v}_1}{dv}(v)\, c \\ &-\; (N-1)\, F(v_2^*(a))^{N-2}\, f(v_2^*(a))\;\, v_2^*(a)\, \frac{dv_2^*(a)}{dv}(v). \end{aligned}$$

Equation (C.1) implies that

$$F(v)^{N-2}\, v \;=\; F(\hat{v}_1(v))^{N-2}\, c,$$

so that we can eliminate the fourth term.

In addition, differentiating both sides of the equation (4.4), which defines the threshold $v_2^*(a)$ as a function of v, results in

$$
\begin{aligned}
(N-1)\, F(v_2^*(a))^{N-2}\, f(v_2^*(a))\; v_2^*(a)\, \frac{dv_2^*(a)}{dv}(v) \quad &= \quad -\; F(v_2^*(a))^{N-1}\; \frac{dv_2^*(a)}{dv}(v) \\
&\quad + (N-1)\, F(v)^{N-2}\, f(v)\, c.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d\phi}{dv}(v) \quad &= \quad (N-1)(N-2)\, F(v)^{N-3}\, [1 - F(\hat{v}_1(v))]\, f(v)\, v \\
&+ \quad (N-1)\, F(v)^{N-2}\, [1 - F(\hat{v}_1(v))] \\
&+ \quad (N-1)\, F(v)^{N-2}\, f(v)\, (v - c) \\
&+ \quad F(v_2^*(a))^{N-1}\; \frac{dv_2^*(a)}{dv}(v).
\end{aligned}
$$

It follows that $\frac{d\phi}{dv}(v) > 0$, so that the function $\phi(v)$ is strictly increasing in $[\underline{v}, \check{v}_1]$. Therefore, the value $v_1^* \in [\underline{v}, \check{v}_1]$ satisfying equation (4.10) is unique. In addition, for all $v \in [\check{v}_1, \bar{v}]$, $\phi(v) > 0$. Hence, the value v_1^* is the unique root of $\phi(v)$, for all $v \in [\underline{v}, \bar{v}]$.

Having shown that the threshold v_1^* is well defined, we now examine the optimality of the bidding strategy β^1 for the bidders that participate in the first round.

If buyer i has a valuation $v_i \in [v_1^*, \bar{v}]$, then his expected payoff from bidding $\tilde{b}_i^1 = \beta^1(\tilde{v}_i)$ in the first round, for any $\tilde{v}_i \geq v_i$, and following the strategy β^2 truthfully in the second round will be

$$
\begin{aligned}
\Pi(\tilde{v}_i \,|\, v_i) \quad &= \quad \{F(\tilde{v}_i)^{N-1} \,+\, (N-1)\, F(v_i)^{N-2}\, [1 - F(\tilde{v}_i)]\}\; v_i \;-\; c \\
&- \quad F(\tilde{v}_i)^{N-1}\; \beta^1(\tilde{v}_i) \\
&- \quad \int_{\tilde{v}_i}^{\bar{v}} \beta^2(v_i \,|\, v^1)\; (N-1)\, F(v_i)^{N-2}\, f(v^1)\, dv^1.
\end{aligned}
$$

Similarly, his expected payoff from bidding $\tilde{b}_i^1 = \beta^1(\tilde{v}_i)$, for any $\tilde{v}_i \leq v_i$, will be

$$
\begin{aligned}
\Pi(\tilde{v}_i \,|\, v_i) \;&=\; \{F(\tilde{v}_i)^{N-1} \;+\; (N-1)\, F(v_i)^{N-2}\,[1 - F(\tilde{v}_i)]\}\; v_i \;-\; c \\
&-\; F(\tilde{v}_i)^{N-1}\; \beta^1(\tilde{v}_i) \\
&-\; \int_{v_i}^{\bar{v}} \beta^2(v_i \,|\, v^1)\; (N-1)\, F(v_i)^{N-2}\, f(v^1)\; dv^1 \\
&-\; \int_{\tilde{v}_i}^{v_i} \beta^2(v^1 \,|\, v^1)\; (N-1)\, F(v^1)^{N-2}\, f(v^1)\; dv^1.
\end{aligned}
$$

In either case, the necessary first-order condition, $\Pi_1(v_i \,|\, v_i) = 0$, results in the differential equation

$$
\frac{d}{dv_i}\left\{ F(v_i)^{N-1}\; \beta^1(v_i) \right\} \;=\; (N-1)\, F(v^1)^{N-2}\, f(v^1)\; \beta^2(v_i \,|\, v_i), \tag{C.2}
$$

which, solved along with the boundary condition $\beta^1(v_1^*) = 0$, gives the bidding strategy β^1 that has been described in equation (4.11).

For sufficiency, first consider the case of $\tilde{v}_i \in [v_i, \bar{v}]$. After differentiating with respect to the mimicked type $\tilde{v}_i$ and substituting, first, the necessary first-order condition (C.2) for the term $\frac{d}{d\tilde{v}_i}\{ F(\tilde{v}_i)^{N-1}\; \beta^1(\tilde{v}_i)\}$ and, second, the definition (4.8) for $\beta^2(v_i \,|\, v^1)$, we get

$$
\begin{aligned}
\Pi_1(\tilde{v}_i \,|\, v_i) \;&=\; (N-1)\,[F(\tilde{v}_i)^{N-2} - F(v_i)^{N-2}]\, f(\tilde{v}_i)\; v_i \\
&-\; (N-1)\, f(\tilde{v}_i)\; \int_{v_i}^{\tilde{v}_i} v\; (N-2)\, F(v)^{N-3}\, f(v)\; dv \\
&<\; (N-1)\,[F(\tilde{v}_i)^{N-2} - F(v_i)^{N-2}]\, f(\tilde{v}_i)\; v_i \\
&-\; (N-1)\, f(\tilde{v}_i)\; v_i\; \int_{v_i}^{\tilde{v}_i} (N-2)\, F(v)^{N-3}\, f(v)\; dv \\
&=\; 0,
\end{aligned}
$$

so that a first-round deviation to bidding $\tilde{b}_i^1 = \beta^1(\tilde{v}_i)$, for $\tilde{v}_i \geq v_i$, is non-profitable.

Now, consider the case of $\tilde{v}_i \in [v_1^*, v_i]$. After differentiating with respect to the mimicked type $\tilde{v}_i$ and substituting the necessary first-order condition (C.2), we get

$$\Pi_1(\tilde{v}_i \,|\, v_i) \;=\; 0,$$

so that buyer i cannot profit from submitting a lower bid in the first round.

To complete the argument, we need to show that it is optimal for any buyer i with type v_i to participate in the first round, for $v_i \in [v_1^*, \bar{v}]$, and to abstain from the first round, for $v_i \in [\underline{v}, v_1^*)$, that is,

$$\Pi(a \,|\, v_i) \begin{cases} \leq \;\Pi(v_i \,|\, v_i), & \text{for } v_i \in [v_1^*,\, \bar{v}]\,; \\ \\ \geq \;\Pi(\tilde{v}_i \,|\, v_i), & \text{for } v_i \in [\underline{v}, v_1^*),\;\; \tilde{v}_i \in [v_1^*,\, \bar{v}]. \end{cases}$$

We have shown that $\Pi(v_i \,|\, v_i) = \Pi(v_1^* \,|\, v_i)$, for $v_i \in [v_1^*,\, \bar{v}]$. In addition, by replicating the argument that was used to rule out a first-round deviation to overbidding, it is easy to show that $\Pi(\tilde{v}_i \,|\, v_i) \leq \Pi(v_1^* \,|\, v_i)$, for $v_i \in [\underline{v}, v_1^*)$ and $\tilde{v}_i \in [v_1^*,\, \bar{v}]$. Therefore, our task reduces to showing that

$$\Pi(a \,|\, v_i) \begin{cases} \leq \;\Pi(v_1^* \,|\, v_i), & \text{for } v_i \in [v_1^*,\, \bar{v}]\,; \\ \\ \geq \;\Pi(v_1^* \,|\, v_i), & \text{for } v_i \in [\underline{v}, v_1^*). \end{cases}$$

Since, by the definition of v_1^*,

$$\Pi(v_1^* \,|\, v_1^*) - \Pi(a \,|\, v_1^*) \;=\; 0,$$

it suffices to show that, for all $v_i \in [\underline{v}, \bar{v}]$,

$$\frac{d}{dv_i}\{\,\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} \;\geq\; 0.$$

First, consider the case of $v_i \in [v_1^*, \bar{v}]$. Then

$$\begin{aligned}
\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i) &= (N-1)\, F(v_i)^{N-2} [1 - F(\hat{v}_1(v_i))]\, v_i \\
&- \; [1 - F(\hat{v}_1(v_i))^{N-1}]\; c \\
&+ \; F(v_1^*)^{N-1}\; \beta^2(v_1^* \,|\, a) \\
&- \; \int_{\hat{v}_1(v_i)}^{\bar{v}} \beta^2(v_i \,|\, v^1)\; (N-1)\, F(v_i)^{N-2}\, f(v^1)\; dv^1.
\end{aligned}$$

Differentiating with respect to bidder i's type v_i results in, after canceling the opposite-sign terms,

$$\begin{aligned}
\frac{d}{dv_i}\{\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} &= (N-1)\, f(\hat{v}_1(v_i))\, \frac{d\hat{v}_1}{dv_i}(v_i)\, [F(\hat{v}_1(v_i))^{N-2} c \; - \; F(v_i)^{N-2} v_i] \\
&+ \; (N-1)\, F(v_i)^{N-2}\; [\, 1 \; - \; F(\hat{v}_1(v_i))\,] \\
&+ \; (N-1)\, F(v_i)^{N-2}\, f(\hat{v}_1(v_i))\; \beta^2(v_i \,|\, \hat{v}_1(v_i))\; \frac{d\hat{v}_1}{dv_i}(v_i)
\end{aligned}$$

By using the equation (C.1) defining the cut-off value $\hat{v}_1(v_i)$, we can eliminate the first term, so as to get

$$\begin{aligned}
\frac{d}{dv_i}\{\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} &= (N-1)\, F(v_i)^{N-2}\; [\, 1 \; - \; F(\hat{v}_1(v_i))\,] \\
&+ \; (N-1)\, F(v_i)^{N-2}\, f(\hat{v}_1(v_i))\; \beta^2(v_i \,|\, \hat{v}_1(v_i))\; \frac{d\hat{v}_1}{dv_i}(v_i)
\end{aligned}$$

which is positive, showing that buyer i's deviation to abstaining is non-profitable.

If buyer i has a valuation $v_i \in [\underline{v}, v_1^*)$, then, because of the different forms that the payoff functions $\Pi(v_1^* | \, v_i)$ and $\Pi(a| \, v_i)$ take, we need to consider three sub-cases, namely, $v_i \in [\underline{v}, v_2^*(v_1^*))$, $v_i \in [v_2^*(v_1^*), v_2^*(a))$ and $v_i \in [v_2^*(a), v_1^*)$.

When $v_i \in [v_2^*(a), v_1^*)$, bidder i's benefit from submitting a bid $\beta^1(v_1^*) = 0$ is

$$\begin{aligned}
\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i) \;&=\; \{\, F(v_1^*)^{N-1} \,-\, F(v_i)^{N-1} \,+\, (N-1)\, F(v_1^*)^{N-2}\, [1 - F(\hat{v}_1)] \,\}\; v_i \\
&-\; [1 - F(\hat{v}_1(v_i))^{N-1}]\, c \;+\; F(v_i)^{N-1}\, \beta^2(v_i \,|\, a) \\
&+\; \int_{\hat{v}_1(v_i)}^{\hat{v}_1} (N-1)\, F(v_2^*(v^1))^{N-2}\, f(v^1)\, dv^1 \;\, v_i.
\end{aligned}$$

After differentiating with respect to bidder i's type v_i, using the identity $v_2^*(\hat{v}_1(v_i)) = v_i$, eliminating the opposite-sign terms and applying the equation (C.1) defining $\hat{v}_1$, we get

$$\begin{aligned}
\frac{d}{dv_i}\{\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} \;&=\; F(v_1^*)^{N-1} \,-\, F(v_i)^{N-1} \\
&+\; (N-1)\, F(v_1^*)^{N-2}\, [1 - F(\hat{v}_1)] \\
&+\; \int_{\hat{v}_1(v_i)}^{\hat{v}_1} (N-1)\, F(v_2^*(v^1))^{N-2}\, f(v^1)\, dv^1.
\end{aligned}$$

Therefore,

$$\frac{d}{dv_i}\{\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} \;\geq\; 0,$$

showing that it is not profitable for a buyer with type $v_i \in [v_2^*(a), v_1^*)$ to deviate into participating to the first round of the auction.

When $v_i \in [v_2^*(v_1^*), v_2^*(a))$, bidder i's benefit from submitting a bid $\beta^1(v_1^*) = 0$ is

$$\begin{aligned}
\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i) \;&=\; \{\, F(v_1^*)^{N-1} \,+\, (N-1)\, F(v_1^*)^{N-2}\, [1 - F(v_1^*)] \,\}\; v_i \\
&-\; [1 - F(\hat{v}_1(v_i))^{N-1} + F(v_1^*)^{N-1}]\; c \\
&+\; \int_{\hat{v}_1(v_i)}^{\hat{v}_1} (N-1)\, F(v_2^*(v^1))^{N-2}\, f(v^1)\, dv^1 \;\, v_i.
\end{aligned}$$

Arguing exactly as in the case of $v_i \in [v_2^*(a), v_1^*)$, we can calculate

$$\begin{aligned} \frac{d}{dv_i}\{\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} &= F(v_1^*)^{N-1} + (N-1)\, F(v_1^*)^{N-2}\, [1 - F(v_1^*)] \\ &+ \int_{\hat{v}_1(v_i)}^{\hat{v}_1} (N-1)\, F(v_2^*(v^1))^{N-2}\, f(v^1)\, dv^1, \end{aligned}$$

which is positive, so that it is not profitable for buyer i, with type $v_i \in [v_2^*(v_1^*), v_2^*(a)]$ to participate to the auction.

Finally, when $v_i \in [\underline{v}, v_2^*(v_1^*))$, we have

$$\begin{aligned} \Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i) &= \{\, F(v_1^*)^{N-1} + (N-1)\, F(v_1^*)^{N-2}[1 - F(\hat{v}_1)]\,\}\; v_i \;-\; c \\ &+ \int_{v_1^*}^{\hat{v}_1} (N-1)\, F(v_2^*(v^1))^{N-2}\, f(v^1)\, dv^1 \; v_i, \end{aligned}$$

so that

$$\begin{aligned} \frac{d}{dv_i}\{\Pi(v_1^* \,|\, v_i) - \Pi(a \,|\, v_i)\} &= F(v_1^*)^{N-1} + (N-1)\, F(v_1^*)^{N-2}[1 - F(\hat{v}_1)] \\ &+ \int_{v_1^*}^{\hat{v}_1} (N-1)\, F(v_2^*(v^1))^{N-2}\, f(v^1)\, dv^1, \end{aligned}$$

which is positive, completing the argument.

□

Computer Code for Revenue Comparisons:

This is the source code used for the revenue comparison between the simultaneous and the sequential auctions. It runs on *Mathematica*, a quite common scientific calculation software; in particular, for this paper, it was ran on *Mathematica*, version 5.2.

```
Clear[n, s, c, v, w, w1, t];
Needs["Miscellaneous`RealOnly`"]

n = 4;
s = 0.01;

ψ[v_, c_] := v^n + (n - 1) v^(n-1) (1 - v) - c;
u1[v_, c_] := (v^(n-1)/c)^(1/(n-2));
ϕ[v_, c_] :=
  (n - 1) v^(n-1) (1 - Min[1, u1[v, c]]) + (n - 1)/n v^(n-1) (v - c) - (1 - Min[1, u1[v, c]^(n-1)]) c;

ThrSim[c_] := NSolve[ψ[w, c] == 0, w];
ThrSqc[c_] := NSolve[ϕ[w1, c] == 0, w1];

w[c_] := w /. ThrSim[c][[n - 1]];
w1[c_] := w1 /. ThrSqc[c][[1]];

Rsim[w_] := (2 (n - 2))/(n + 1) - n (n - 2) w^(n-1) + 2 (n - 1) (n - 2) w^n - (n (n - 1) (n - 2))/(n + 1) w^(n+1);

Rsqc[w1_, c_] := (2 (n - 2))/(n + 1) - (n - 1)^2/(n (n + 1)) w1^(n+1) - (n - 2) w1^n - (n - 1)/n w1^n c + n w1^(n-1) -
   1/(n - 1) w1^(n-1) c + (n - 1) w1 c - n c - (n (n - 2))/(n - 1) w1^((n-1)^2/(n-2)) c^(-1/(n-2)) + (n (n - 1))/(n + 1) w1^((n+1) (n-1)/n) c^((n+1)/n) -
   ((n - 1)^2 (n - 2))/(n (n - 2) + (n - 1)) w1^((n (n-2) + (n-1))/(n-1)) c^(n/(n-1)) + ((n - 1)^2 (n - 2))/(n (n - 2) + (n - 1)) w1^((n (n-2) + (n-1))/(n-2)) c^(-1/(n-2));


Rev[m_] := Table[{s*t, w[s*t], w1[s*t], Rsim[w[s*t]],
    Rsqc[w1[s*t], s*t], Rsqc[w1[s*t], s*t] - Rsim[w[s*t]] }, {t, 1, 1/s - 1}];
R[m_] := Join[{{0, 0, 0, N[(2 (m - 2))/(m + 1)], N[(2 (m - 2))/(m + 1)], 0}}, Rev[m], {{1, 1, 1, 0, 0, 0}}];
Rev = R[n]


RsimGraph = ListPlot[Rev[[All, {1, 4}]], AspectRatio → 1/1,
  PlotStyle → {RGBColor[1, 0, 0]}, PlotJoined → True, AxesLabel → {"c", "Rsim"}]

RsqcGraph = ListPlot[Rev[[All, {1, 5}]], AspectRatio → 1/1,
  PlotStyle → {RGBColor[0, 0, 1]}, PlotJoined → True, AxesLabel → {"c", "Rsqc"}]

Show[RsqcGraph, RsimGraph, AxesLabel → {"c", "Revenue"}]

RdiffGraph = ListPlot[Rev[[All, {1, 6}]], AspectRatio → 1/1,
  PlotStyle → {RGBColor[0, 1, 0]}, PlotJoined → True, AxesLabel → {"c", "Rsqc-Rsim"}]
```

Prior to running this code, one should specify the number of buyers, $\mathtt{n} \geq 3$, and the step size for the iteration, $\mathtt{s} \in \{\frac{1}{k} : k \in \mathbb{N}\}$. Subsequently, *Mathematica* will compute for all values $c =$ `s*t`, where `t` $= 0, 1, 2, ..., k$, the thresholds `w` and `w1`, respectively for the simultaneous and the first round of the sequential auction, the expected seller revenues[5] `Rsim` and `Rsqc` and their difference `Rdiff = Rsqc-Rsim`. In fact, the output takes the form of a $(k+1) \times 5$ matrix, consisting of row vectors (`c`, `w`, `w1`, `Rsim`, `Rsqc`, `Rdiff`). Finally, the graphs of the revenues `Rsim` and `Rsqc` and of their difference `Rdiff` will be plotted, as functions of the cost c. Clearly, smaller values for the step of iteration `s` correspond to smoother graphs.

The graph of the difference in the expected seller revenue, `Rsqc-Rsim`, in the case of a convex or a concave power distribution, shown in Figure 4.4, was produced in a similar manner. However, one must adjust the way in which the root `w` or `w1` is picked by the program. In the case of the convex distribution $F(v) = v^2$, this is done by replacing the commands

```
w[c_]:=w\.ThrSim[c][[2(n-1)]]
w1[c_]:=w1\.ThrSim[c][[1]]
```

In the case of the concave distribution $F(v) = v^{1/2}$, one should also alter the way in which the root `w1[c_]` is calculated. The new commands are

```
w[c_]:=w\.ThrSim[c][[2]]
ThrSqc[c_]:=FindRoot[\[Phi][w1,c]==0,{w1,0.5,0,1}];
w1[c_]:=w1\.ThrSim[c][[1]]
```

The rest of the source code is identical to the one used in the uniform case (apart for the obvious adjustments for the threshold and the payoff functions) and is therefore omitted.

[5] Notice that this computation presupposes knowledge of the expected seller revenue as a function of the number of buyers n, the cost c, and the appropriate threshold. In an alternative manner, one could compute the two expected prices by integrating numerically the bidding functions, over the region of participation, with respect to the distribution of the first- and second-order statistics.

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