OPEN, ONLINE, CALCULUS HELP FORUMS: LEARNING ABOUT AND FROM A PUBLIC CONVERSATION

by

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Open, online, calculus help forums: Learning about and from a public conversation

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This study is an exploration of participation, community, and mathematical understanding in an open, online, calculus help forum. These forums, populated by members from around the world, are locations where students post queries from their coursework and receive assistance from volunteer tutors. The site under investigation has a spontaneous participation structure, meaning that any forum member can respond to a query and contribute to an ongoing discussion. From earlier work, we know that such forums foster mathematical dialogue, contain exchanges with sophisticated pedagogical moves, and exhibit a strong sense of community. In this study, we delve deeper into the functional aspects of activity (such as student positioning and pedagogical moves), the benefits that accrue from participation in tutoring as a communal activity, and the mathematical understanding that is evident in the way problems on limit and related rates are framed and solutions constructed.

Based on an observational methodology, we find that the forum provides tutoring for students and support for tutors that is unique from our expectations of other learning environments, such as one-on-one tutoring and computer-based tutoring systems. Students position themselves with authority in the exchanges by making assertions and proposals of action, questioning or challenging others' proposals, and indicating when resolution has been achieved. Tutors, who generally have more experience and expertise than students, provide mathematical guidance, and, in exemplary exchanges, draw the student into making a

mathematical discovery. The dedication of tutors to the forum community was evident in the presence of authentic, honest mathematical practices, in the generous provision of alternative perspectives on problems, and in the sincere correction of errors. Some student participants picked up on these aspects of community and expressed excitement and appreciation for this taste of mathematical discourse. The primary contribution of the tutors was their assistance in supporting students as they constructed productive framings for the exercises, and this was the help that students were most in need of. As a result of eavesdropping on this public conversation, we conclude that the forums are a public conversation that should be listened to by educational researchers, teachers, and designers of tutoring systems.

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A lim'rick of thanks I now send // to mathematician, advisor, and friend. // A main part of my life // he's seen joy, tears, and strife. // Frank Carroll: beginning and end.

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1.0 OPEN, ONLINE, CALCULUS FORUMS AS PUBLIC CONVERSATION

As students work on learning and understanding calculus, they assume more and more responsibility for their own understanding at a "distance" from the instructor. In doing so, they depend on a set of resources for self-guided learning: text books, goals set by the instructor, friends, and tutoring environments. Free, open, online help forums are a relatively recent addition to this set of resources and provide a means for students to voice their ideas, ask questions, and receive feedback on their understanding. Via the Internet, student access these forums and communicate with a network of others around the world, seeking help on specific questions regarding coursework wherever and whenever they arise. At the same time, these forums connect the "network of others," a group of people who have the time, experience, and willingness to work on mathematics with students and (in some cases) with one another.

Open, online mathematics help forums appear to be flourishing as a help-seeking resource for students in a variety of cultures *and* as a help-providing outlet and discussion arena for volunteer tutors of varying levels of knowledge and pedagogical expertise. For example, one such site based in the United States, FreeMathHelp.com, has attracted 10,494 members and received 85,173 total posts, contributing to 20,570 exchanges since June 2002.[1] Another site that was formed in 2005, MathHelpForum.com currently boasts 13,025 members and has received

[1] A sample of postings dated August 15, 2007 to December 31, 2007 (approximately the length of a semester) within the (sub)forum that is designated for calculus queries shows a median of 4 posts a day with participation from 195 different students and 31 tutors during this time.

132,354 total posts forming 30,462 threads. For the purposes of educational research, these forums provide a window into some very specific issues in the learning and teaching of calculus (such as the meaning of limit), function as a source of “real” questions that students do not usually vocalize in classroom (or other instructional) settings, and reveal nuances of instructor understandings and views on the role of particular practices (ranging from notational conventions to alternative solution paths) that are not otherwise broadcast. Understanding this emergent phenomenon requires systematic research into the nature of the participation, tutoring, and interaction that takes place in open, online help forums: For example, how do features of the site structure influence the nature of exchanges? What is the mathematical and pedagogical quality of the tutoring that takes place in these forums? In what sense are these forums a type of learning community? Addressing such questions positions us to understand better the role these forums play in student understanding and in the development and sustenance of a tutorial community, paving the way for the improvement of calculus learning and instruction.

This study is designed to contribute to our understanding of open, online, calculus help forums by framing tutoring as a collective activity and treating the exchanges as public conversation. Course-based calculus problems are the subject of this conversation that is taking place between students and volunteer tutors around the world. In this activity, participants are exposed to rich, intense, mathematical and pedagogical discourse that is not the norm in calculus learning and instruction. The very existence of open, online, calculus, help forms is evidence that many individuals desire to join in this conversation and extend the boundaries of their experiences in the calculus. Exploring the opportunities that this activity affords for both students engaged in learning the calculus and tutors who support these efforts is the subject of this work.

1.1 AVAILABILITY AND EXTENT

There are several open, online calculus help forums that are easily accessible to the general public through an Internet search on phrases such as “calculus help forum” or “free calculus help.” These sites exist in several countries, although participation in a given forum is by no means restricted to the nationality of the site. For example, FreeMathHelp (FMH) a North American site, has active participants from Canada, the United Kingdom, and New Zealand. An administrator of another North American site, MathHelpForum (MHF), has on two occasions published a map (created by Google Analytics) in the MHF Community sub-forum detailing the geographic location of people who accessed the forum over a recent time period. On these maps, circles mark locations of forum activity, and the size of a circle correlates with the number of unique visitors. Figure 1 shows the second of these maps, posted on February 6, 2006. On both occasions, this information generated discussion amongst forum members on this aspect of participation. For example, in the discussion accompanying the first published map (November 23, 2005), one member commented on the absence of an indication of her/his presence on the forum: “I always want to be incognito, but here, in your map, maybe it will be nice if you include or put a dot on Guam. I am in Guam. Thank you.” The administrator replied that s/he had seen Guam represented in the past and explained that the map only captured activity within a limited time frame, in this case visits just prior to November 23, 2005. Then, in the post that contained the publication of the second map (Figure 1), the administrator referred to this earlier exchange with the participant from Guam and commented: “I think someone will be happy to see Guam well represented.” On this map, the island of Guam is the perfectly circular island located in the Western Pacific Ocean since the size of the circle exceeds Guam’s land mass. To this, the member from Guam replied: “There! Very good. Graphically, I am now a member of this great forum. I am in Guam. In this map, Guam is that marker that is south of Japan, east of the Philippines, north of Indonesia and Australia, in the Pacific Ocean. (Guam is a Territory of the USA.)" For this participant from Guam, being 'on the map' was associated with claiming membership in the MHF forum, and s/he took obvious pride in being a contributor from this remote and out-of-the-way location.

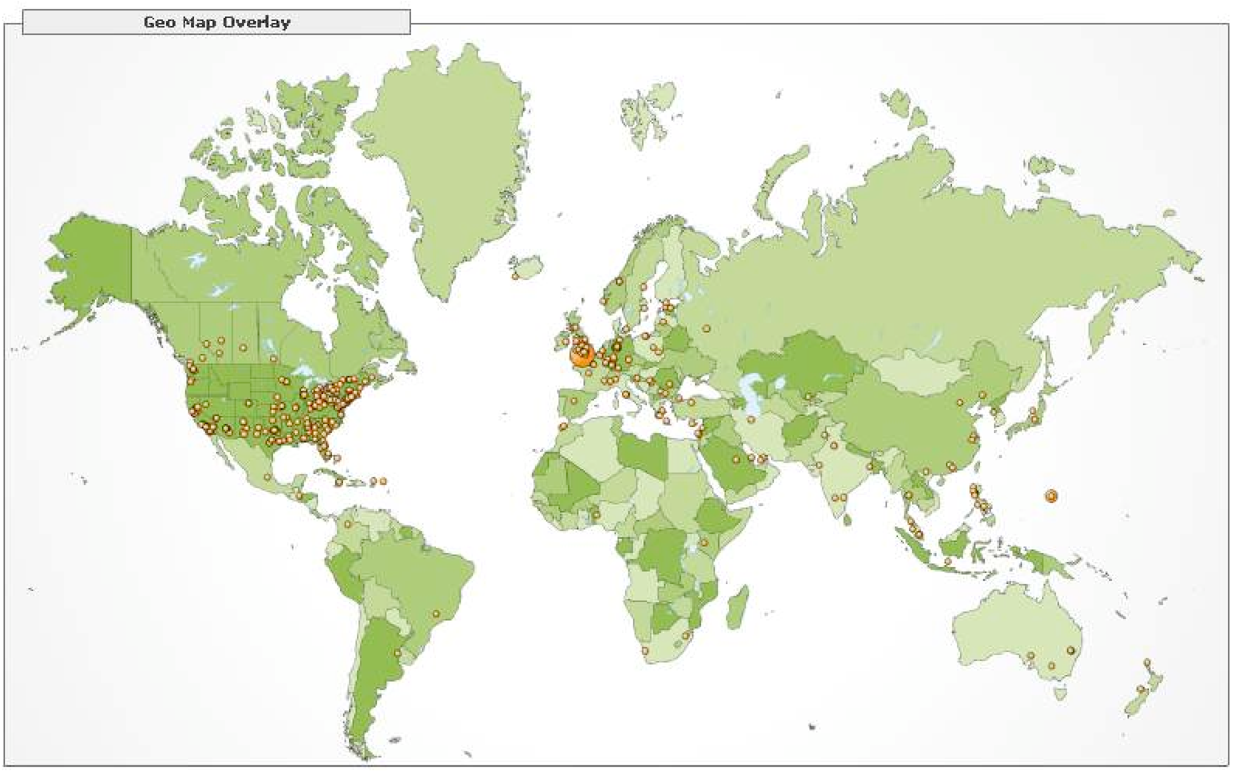

Figure 1. Locations of MHF visitors, February 6, 2006

In the excitement over pinpointing forum activity, the member from Guam was not alone; other MHF members, from the United States, South Africa, England, and Norway also joined in the discussion and pointed out their "blobs" on the map. As can be seen in Figure 1, the diversity of individuals interested in receiving or providing help jointly solving mathematics problems is certainly remarkable. People from virtually every continent are represented in this snapshot of activity on a single forum, demonstrating how these online forums are not merely *open* in name but rather the extent to which they are *open* in practice. Participation in open, online forums clearly spans geo-political boundaries and affords a unique opportunity for people around the world to hold discussions and exchange mathematical ideas.

1.2 SITE DESIGN AND STRUCTURE

Forums can be structured differently with respect to who may respond (tutor) and how monitoring is accomplished. In Spontaneous Online Help (SOH) sites, any forum member can respond to a query, whereas in Assigned Online Help (AOH) sites, only certain members (selected by the forum administration) can answer queries. In an AOH structure, there are various models for assignment of queries that are currently in use: incoming queries can be stored in a restricted-access database to be selected by (and thereafter assigned to) vetted tutors; or incoming queries can be assigned by the forum administration to participant tutors (based on pre-determined criteria such as subject area, tutor availability, etc.).

In addition to adhering to an SOH or AOH structure, a forum may also set moderation policies that address features of contributions such as quality, domain, and appropriateness. This moderation can take place prior to or following publication of postings on the forum. That is, incoming contributions may be subject to a screening process or, alternatively, may be published immediately (and possibly removed from the forum at a later time). Monitoring policy affects the speed of responses. If there are no intermediate monitoring actions, then the response rate can approximate that of a face-to-face exchange. If, on the other hand, a monitoring action requires both an assessment of content and an assessment for assignment, then there may be considerable delay – up to several days – between the time the query is posed and the time a response becomes available. The urgency of some of the queries (e.g. there is an assignment due or an examination pending) coupled with moderation policy presumably impacts forum activity, effectiveness, and longevity.

1.3 PURPOSE OF THIS STUDY

The purpose of this study is to extend our understanding of open, online, calculus help forums by exploring aspects of participation structure, by describing the benefits derived from communal tutoring, and by attending to students' mental models of difficult calculus concepts. It is part of a series of investigations of online tutoring in calculus (van de Sande, 2006; 2007a; 2007b; van de Sande & Greeno, 2008; van de Sande & Leinhardt, 2007a; 2007b; van de Sande & Leinhardt, 2008a; 2008b; 2008c; 2008d). Given the prevalence and widespread use of these forums as a help-seeking resource for students *and* a help-providing outlet for tutors, accounts of participation in these forums can give us a window into their intellectual efficacy, their organization as socially productive communities, and their potential for influencing future design activity. Students from around the world are accessing these forums routinely as they learn calculus and seek to fill in gaps in their understanding. In a complementary fashion, others with more mathematical experience (tutors) are voluntarily responding to this call by guiding and scaffolding students as they work through course-related problems. These forums, then, are an environment in which students and tutors interact with one another and with the procedures and concepts that constitute calculus as a subject-matter domain. In what ways are participants positioned with respect to one another and with respect to the calculus as they engage in this activity?

In this study, I focus on the participation structure of an SOH site since forums that allow spontaneous participation encourage extended, mathematically intense exchanges between multiple participants and exhibit a sense of community among tutors (van de Sande & Leinhardt, 2007b; 2008b). This context can provide a richer picture of how members engage with mathematics as a discipline and how they interact with one another, such as the distribution of agency, authority, accountability, leading and following, initiating, attending, accepting, and questioning or challenging (Greeno, 2006). I use the discussions surrounding the limit and related rates, two topics in calculus that reflect the diversity of problem types characteristic of the subject domain. The concept of limit is foundational to calculus and involves subtle mathematical ideas, whereas related rates problems are an application of calculus to real-world problems. Both of these topics are challenging to students and are the subject of many forum exchanges.

This intent of this study is to further develop an account of interaction in an open, online, calculus, help forum. In previous work, I have paved the way for this investigation by developing a methodology for describing some aspects of forum participation; establishing the mathematical and pedagogical quality of the exchanges; tracing implications of site structure and policy, and; examining the sense in which these forums resemble learning communities. The next step is to replicate the central findings of the previous body of work and extend our understanding of this emergent phenomenon that has radically re-defined and re-configured the practice of tutoring by performing detailed analyses of exemplary exchanges on complex mathematical topics.

2.0 CALCULUS, TUTORING, AND COMMUNITY

2.1 CALCULUS: THE LIMIT AND RELATED RATES

Calculus serves as the gatekeeper for the physical, biological, and many of the social sciences and is therefore a requisite part of the program of study for numerous students. At the same time, it is generally viewed as an extremely difficult subject and many students are unable to successfully complete an introductory course.[2] This state of affairs has fueled research on students' understandings and conceptions concerning important calculus ideas. The limit concept and related rates are two of these ideas for which there is considerable disparity between student and expert sense-making activities and problem solving abilities. The task for the learner is to gain an understanding of the concept of the limit in a way that supports performing mathematical activity and to grasp the activity of doing related rates problems in a manner that expands underlying mathematical concepts.

[2] In the late 1990's, the University of Massachusetts Dartmouth reported that the attrition rate in calculus courses was in the 40 percent range (Pendergrass & Dowd, 2005) and this range is consistent with findings in other institutions (Darken, Wynegar, & Kuhn, 2001).

2.1.1 The limit

The concept of limit sets the calculus apart from other branches of mathematics (such as algebra, trigonometry, or geometry) and is one of first topics covered in an introductory calculus course. It is foundational to the calculus in the sense that it is the basis for defining the core ideas of differentiation and integration. At the same time, the limit is a subtle and complex mathematical idea, as reflected by its definition, its controversial and prolonged historical evolution, and the challenge it continues to present to students and instructors.

The limit of a function at a point describes the behavior of the function as the independent variable approaches that point. Thus, the limit, L, of a function $f(x)$ at a point, $x = a$, can be informally defined as follows:

> If the values of $f(x)$ can be made as close as we like to L by making x sufficiently close to a (but not equal to a), then we write $\lim_{x \to a} f(x) = L$ which is read "the limit of $f(x)$ as x approaches a is L." (Anton, 1998, p. 115)

The values of the function are made *as close as we like* to the limit as the dependent variable *approaches* a certain value. This implies that the limit is a concept not based on a finite computation since getting *as close as we like* is a process that never ends (Cornu, 1981 as cited in Tall, 1993). Computing a limit, then, requires the contemplation of an infinite number of steps. Likewise, this definition brings into play notions of motion and travel, coupled with the premise of achieving an arbitrarily small (but nonzero) distance (Lakoff & Nunez, 2000). Since these notions conflict with everyday experience, the nature of the limit concept presents "epistemological obstacles" that make it difficult to grasp and reconcile with existing constructions (Bachelard, 1938 and Brousseau, 1983 as cited in Williams, 2001).

Historically, the absence of the limit concept presented an obstacle to the logical development of the calculus. Our current analytic definition of limit has its roots in criticisms leveled at the lack of rigor that was endemic to the early stages of its formulation. In a scathing essay directed at the mathematicians of the day, Bishop George Berkeley (1685-1753) pointedly ridiculed the mathematical structures (e.g. fluxions) on which calculus was based:

> And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities? (Dunham, 2005, p. 71)

Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716), recognized as the founders of the calculus, were performing computations and developing meaningful results based on mathematical constructions that were suspect. These constructions, termed infinitesimals, posited the existence of a least possible length, an infinitely small magnitude that could not be further subdivided. Of course, such constructions cannot exist over the real numbers since any length, no matter how small, can always be divided further, and this fact that did not escape 17th century mathematicians. Indeed, Leibniz himself recognized the problematic nature of infinitesimals, entities that could be both zero and nonzero, as the need dictated. However, Leibniz maintained, whether or not these constructions actually exist, they still serve as “fictions useful to abbreviate or speak universally” (as cited in Edwards, 1979, p. 265). In other words, Leibniz viewed the existence of infinitesimals as a philosophical issue that was independent of their operational validity, his primary concern. Others, most notably Berkeley, did not share this pragmatic perspective. “Error,” Berkeley wrote, “may bring forth truth, though it cannot bring forth science” (Dunham, 2005, p. 71). This concern – that the end does not justify the means –

continued to reverberate in the mathematical community and had repercussions that are evident in our current formulation of the calculus. In particular, the analytic definition of limit evolved in the aftermath of these objections. Although it took nearly a century, Berkeley's "ghosts" were eventually laid to rest when Augustin-Louis Cauchy (1789-1857) formulated the following definition of the limit that does not invoke infinitesimals:

> When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. (Dunham, 2005, p. 77)

Cauchy's definition is essentially the informal definition of the limit that is used today, as discussed above. Later, Karl Weierstrass (1815-1897) rigorously defined the notions "approach" and "as little as one wishes." The resulting formal definition of limit is expressed in terms of quantifiers, inequalities, and implications:

$$\lim_{x \to a} f(x) = L \text{ if and only if for all } \varepsilon > 0 \text{, there exists } \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

This formal definition of the limit is the culmination of the work of the mathematical community over a period of 200 years as ideas were introduced, challenged, wrestled with, and refined. Should there be any question that the underlying concept presents a challenge to students?

There is general agreement in the literature that students have trouble understanding the concept of limit (Davis & Vinner, 1986; Sierpinska, 1987; Tall, 1993; Tall & Vinner, 1981; Williams, 1991; 2001). In particular, the mental models that students construct and employ are not consistent with the mathematical concept of limit. These mental models include beliefs that the limit is a boundary that cannot be exceeded (Szydlik, 2000); that functions cannot attain their limits (Szydlik, 2000); and that the limit can be determined by sampling nearby values of the function or evaluating the function at the limit point (Bezuidenhout, 2001; Przenioslo, 2004). In addition, Oehrtman (2002) found that students use reasoning that mirrors that of Leibniz and Newton by thinking in terms of infinitesimals and positing the existence of a smallest possible number. Unfortunately, modern calculus instruction does not appear to move students toward a more mature, complete understanding of the limit, and neither do a number of proposed instructional innovations (Cottrill et al., 1996; Davis & Vinner, 1996; Monaghan, Sun, & Tall, 1994; Tall, 1992). Exploring tutoring exchanges on the limit concept will not only provide a window into students' mental models of limit in an authentic context but may also help identify techniques and explanations that lead to a more coherent understanding.

2.1.2 Related rates

Related rates problems have been referred to as "old chestnuts of calculus instruction" (Austin, Barry, & Berman, 2000). They are built around the use and coordination of the mathematical concepts of variable, function, and differentiation. In the solution process, changing quantities are represented with variables, functional relationships are constructed, and composite functions are differentiated (by invoking implicit differentiation and the chain rule). For example, consider the following related rates problem:

> A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder is slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladders is 6 ft from the wall? (Stewart, 2003, p. 257)

The changing quantities are the distance between the bottom of the ladder and the wall (increasing at a rate of 1 ft/s) and the distance from the top of the ladder to the ground (decreasing at an unknown rate); the ladder, wall, and ground form a right triangle so that the changing distances and the length of the ladder (10 ft) are related by a parametric equation based on the Pythagorean theorem, and; the distances are functions of time so that finding the rate at which they are changing involves implicit differentiation of the parametric equation.

Related rates problems frustrate many calculus students and are regarded by instructors as "contrived" and "too difficult for contemporary students" (Austin et al., 2000). In order to understand these reactions, it is useful to consider the class of related rates problems, their place in the curricular context (e.g. how these problems are typically presented), and the reasoning that supports the solution of such problems.

Consider the set of problems in Table 1. What do these scenarios of balloons, ladders cones, and shadows have in common? One answer is that they all describe a situation in which there is change of some sort (i.e. inflating, sliding, evaporating, and lengthening).

Table 1. Common related rates problems

Name	Problem
Balloon problem	Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm? (Stewart, 2003, p. 256)
Ladder problem	A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladders is 6 ft from the wall? (Stewart, 2003, p. 257)
Cone problem	A container has the shape of an open right circular cone. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating, so that its depth h is changing at the constant rate of -3/10 cm/hr. Find the rate of change of the volume of water in the container, with respect to time, when h = 5 cm. (Question 5, AP Calculus AB Exam, 2002)
Shadow problem	A man 6 ft tall is walking at the rate of 3 ft/s toward a streetlight 18 ft high. a) At what rate is his shadow length changing?; b) How fast is the tip of his shadow moving? (Anton, 1998, p. 275)

More specifically, "related rates" refers to a class of problems that involves the relationship(s) between two or more changing quantities, one of which is unknown and must be determined. For instance, in the Balloon problem in Table 1, the rate at which the volume is changing is known (100 cm^3/s) and is related to the rate at which the radius is changing. The task is to determine this rate of change at a certain time (when the diameter of the balloon is 50 cm). The Ladder problem requires a similar type of construction. The Cone problem and the Shadow problem also involve the relationship between changing quantities but require the construction of an auxiliary relationship. Thus, in the Cone problem, the fact that the ratio of the height and radius of the water in the cone is equal to the ratio of the height and radius of the cone itself (through similar triangles) is needed for the solution.

Most modern introductory calculus textbooks present related rates problems as physical applications of implicit differentiation and the chain rule. As can be seen from Table 1, exercises on related rates are generally framed as word or story problems that are meant to reflect authentic[3] situations. Students are expected to model the situations by appealing to geometric relations[4] (such as volume, area, and distance) and invoking the concepts of implicit differentiation and the chain rule. Figure 2 shows a modern textbook solution to the ladder problem in Table 1.[5]

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[3] The term authentic here refers to alignment with the outside world. This is one of four ways that the word "authentic" is used in the educational literature (Shaffer and Resnick, 1999).

[4] The standard solution model is not necessarily "faithful" to the situation. For example, in the classic falling ladder problem, Scholten and Simoson (1996) point out the fallacy of assuming that the tip of the ladder maintains contact with the wall until impact at ground level and describe the conditions under which the Pythagorean theorem model (used in a typical textbook solution) breaks down.

[5] The "Figure 1" and "Figure 2" that are referenced in the solution text were reproduced and can be found below the solution.

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time).

We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$, we have

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s.

Figure 1

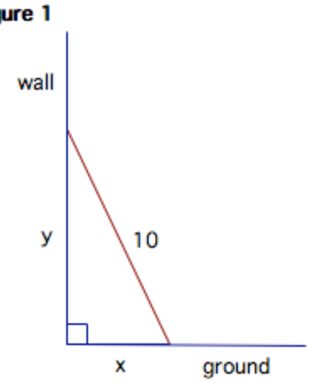

Figure 2

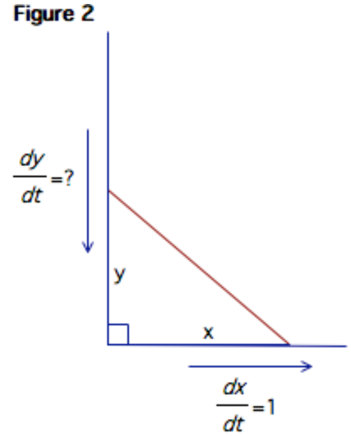

Figure 2. Solution to related rates ladder problem (Stewart, 2003, p. 257)

Related rates problems, then, provide students with a variety of scenarios in which they are to recognize the need for implicit differentiation and the chain rule, recall and perform these procedures, as well as make use of geometric relations, rules of differentiation, and algebraic operations. In effect, related rates problems are included in the calculus curriculum as an opportunity for students to *practice* principles of calculus and incorporate them with previously learned principles of geometry and algebra. The nature of the problems, however, also provides an *opportunity* for students to expand their conceptual understanding of ideas such as variable, function, and derivative and engage in sophisticated covariational reasoning[6].

Historically, related rates problems were introduced into the calculus curriculum as a means of *illustrating* and *exploring* the fundamental concept of derivative (Austin et al., 2000). Objecting to the presentation of calculus as "a science of symbols and mere algebraic formulae," Ritchie (1836) (as cited in Austin et al., 2000) sought to reform calculus instruction and, as part of this endeavor, developed the related rates problem as we know it today. Ritchie's text began with an intuitive introduction to the limit, followed by the presentation of an expanding square to illustrate the idea of a function *and* how uniform increases in the independent variable may cause the dependent variable to increase at an increasing rate. According to Ritchie, the object of the differential calculus was "to determine the ratio between the rate of variation of the independent variable and that of the function in which it enters." He used related rates problems such as the following in the service of this idea:

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[6] Covariational reasoning comes into play in making sense of dynamic, functional relationships and involves a coordination of the inputs and outputs over some interval (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

> "If the side of a square increases uniformly at the rate of three feet per second, at what rate is the area increasing when the side becomes 10 feet?" (p. 12)

Although several other reform-oriented textbooks followed suit and presented related rates as explanatory and motivating examples, the increasing emphasis at the time on formal rigor and abstraction turned the tide. By 1941, related rates problems were positioned in the curriculum as they are today – as an application, a set of exercises to be solved according to a series of prescribed steps "rather than an exciting and pedagogically important method by which to introduce calculus" (Austin et al., p. 10).

Many current calculus textbooks and instructional materials break down the solution process for related rates problems into multiple steps, emphasizing the procedural aspect of the activity. For example, Stewart (2003), author of a widely used calculus textbook, presents a seven-step solution strategy:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.

7. Substitute the given information into the resulting equation and solve for the unknown rate. (p. 258)

One interactive online tutorial on related rates (Waner & Costenoble, 2007) follows a similar scheme, prompting students to submit an equation that relates changing quantities, differentiate this equation with respect to time, rephrase the problems statement in terms of sought information, and substitute in the given information. Feedback is provided for each of these four solution stages. Although the number of steps in the solution scheme may differ depending on the curriculum, this basic strategy for solving related rates problems is the practice in most modern calculus instruction.

Research on student and expert understanding, however, demonstrates that solving related rates problems involves reasoning that is not addressed by conventional explanations and strategies and that contributes to the challenging nature of these problems (Engelke Infante, 2007; Martin, 1996, 2000; White & Mitchelmore, 1996). For example, students have particular difficulty solving geometric related rates problems that require the solution of an auxiliary problem (Martin, 1996; 2000), such as the Cone problem in Table 1 where variables representing dimensions of the cone are related by the principle of similar triangles[7]. This process apparently requires an understanding that differs from one that supports the solution of elementary problem situations and one that may not be adequately scaffolded by advice to "use the geometry of the situation to eliminate one of the variables by substitution" (as per Step 5 in Stewart, 2003). In

[7] This problem was taken from the 2002 Calculus AB Advanced Placement Examination. According to the College Entrance Examination Board (CEEB), students tried to solve the problem with two independent variables rather than developing the proper relationship between two variables (by solving an auxiliary problem) and this "led to mistakes when they failed to recognize the need for the product rule or failed to correctly use the chain rule to include both *dh/dt* and *dr/dt*." (CEEB, pp. 56-57)

addition, most approaches presume that students have a robust and flexible understanding of concepts that are embedded in the solution process (e.g. variable, function, and derivative) and are able to engage in covariational reasoning to make sense of dynamic functional relationships (Carlson et al., 2002). When confronted with related rates problems, however, students demonstrate an underdeveloped concept of variable that interferes with the construction of a solution (White and Mitchelmore, 1996). In particular, students fail to distinguish a general relationship from a specific value, resulting in confusion between entities in the problem statement that are a function of time and those that are values of a function at a particular time. This inability to make (and maintain) this distinction reflects a mental model that conflates the dynamic and static aspects of the problem situation. Finally, a "static view" of function and impoverished understanding of derivative also contribute to the difficulties students experience solving related rates problems and restrict their ability to invoke function composition and engage in covariational reasoning by considering how varying quantities change together (Engelke Infante, 2007). The solution of related rates problems, then, appears to rely on the construction of a mental model that captures *and* coordinates dynamic and static aspects of the problem situation. We would expect effective scaffolding and tutoring of related rates problems to promote this process (rather than adhering to the conventional solution strategy).

The limit and related rates are two topics in the calculus that require sophisticated reasoning and for which classroom instruction often proves inadequate. If students are motivated and have access to resources, they seek help in coming to grips with these topics. One of the most popular solutions is to approach someone more knowledgeable who can assist in filling in the gaps in understanding. Exploring the questions that are asked spontaneously can increase our understanding of what students find problematic under natural conditions. Exploring the help they receive can expand our understanding of tutoring, guide our design of course materials, and support teacher preparation.

2.2 TUTORING

Traditionally, (human) tutoring refers to an instructional activity that involves the delivery of an explanation on a pre-determined set of topics to a single student (or to a small group). Thus, a tutoring session may be designed to provide remediation or augment an instructional explanation for a specific topic (such as factorical design or fractions). Tutoring is a popular form of instruction (often funded and supported by schools) and has proven incomparably effective for academic performance (Cohen, Kulik & Kulik, 1982) and attitudes toward subject matter (Lepper, Aspinwall, & Mumme, 1990). There is no question that tutoring is an activity that offers benefits not generally provided by classroom instruction, mastery learning, computer-aided or programmed instruction and computer tutors (Chi, 1996).

One feature that distinguishes tutoring sessions from other means of instruction is the pattern of dialogue. The dialogue in tutoring sessions generally follows a 5-step dialogue frame (Graesser & Person, 1994; Graesser, Person, & Magliano, 1995):

Step 1: Tutor asks question (or presents problem).

Step 2: Learner answers question (or begins to solve problem).

Step 3: Tutor gives short immediate feedback on the quality of the answer (or solution).

Step 4: Tutor and learner collaboratively improve the quality of the answer.

Step 5: Tutor assesses learner’s understanding of the answer.

Although this dialogue frame differs in many ways from the frequent classroom discourse pattern of Initiation (teacher poses question), Response (student provides answer), and Evaluatison (teacher evaluates student contribution) (Cazden, 1986; Mehan, 1979), the instructor, in both cases, is positioned as leader of the interaction. The tutor selects the question(s) that begin the discussion, indicates the acceptability of answers or solutions, and assesses whether goals (in terms of learner understanding) have been met.

Alternatively, consider an instructional episode spawned by a particular problem that a student has encountered in coursework and has posed to a peer or more experienced other. In this case, the student selects the question(s) that begin the discussion, indicate the acceptability of the tutor's responses, and decides whether the goals of the interaction have been met (i.e. whether the exchange was helpful). Thus, a typical dialogue frame (for a single student-tutor pair) for this type of encounter might look like the following:

Step 1: Student asks question (or presents problem).

Step 2: Tutor answers question (or begins to provide scaffolding).

Step 3: Student gives feedback on the quality of the help.

Step 4: Student and tutor collaboratively work on solving the problem.

Step 5: Student assesses whether tutor's responses were helpful.

In order to distinguish this type of instructional episode from traditional tutoring sessions, we have referred to them as "tutorettes" (van de Sande, 2007a; van de Sande & Leinhardt, 2007a) and considered them as a form of student-initiated help-seeking. Although tutorettes share many features of tutoring sessions (such as personalized instruction and support), they are also different in terms of initiation (as discussed above), goals, and instructional objectives. My larger research agenda has been to explore interactions in environments where tutorettes occur:

university-sponsored help centers (face-to-face) and open, online, discussion forums (computer-mediated). The goal of this investigation is to look more deeply at the ways in which students and tutors are positioned in relation to other participants and in relation to the calculus (as the subject-matter domain) for tutoring exchanges within an open, online, help forum.

One key difference between online tutoring exchanges and tutoring sessions is the presence of an audience. Tutoring sessions are generally conducted between a single student-tutor pair in relative privacy, whereas the exchanges in open, online forums are public and can be witnessed by others. In addition, in SOH forums, any member can contribute alternative solutions, corrections, and commentary on mathematical issues as well as pedagogical approaches, and this often occurs as part of the tutors' practice (van de Sande & Leinhardt, 2007a; 2007b; 2008b; 2008d). The broader social dimension afforded by open, online help forums makes tutoring a collective activity in which the exchanges become a public conversation between individuals who share a common interest in doing mathematics and helping others. In this study, I will explore how participating in this conversation engages students and tutors in mathematical and pedagogical discourse in ways not characteristic of traditional tutoring and which might inform the designers of tutoring environments and calculus courses about the mathematical issues with which students grapple.

2.3 COMMUNITY

Recently, we have seen the decline of teacher-centered models of instruction coupled with the ascent of learner-centered and community-based models. In the process, however, the term "community" has been used indiscriminately to label many different types and degrees of interaction in many varied contexts. There are communities of learners, discourse communities, learning communities, knowledge-building communities, communities of practice, virtual communities, online communities, and so on. As Grossman, Wineburg, and Woolworth (2000) note "*community* has become an obligatory appendage to every educational innovation. Yet aside from linguistic kinship, it is not clear what features, if any, are shared across terms" (p.942). Before exploring a group of individuals jointly engaged in learning *or* designing and sustaining a public learning environment, then, one of the first tasks is to come to terms with community – that is, to anchor our understanding of what it means in practice for individuals to function as a community.

The concept of community has a rich socio-theoretical history evolving within a host of research communities, including anthropology, sociology, political science, and, education. Each of these traditions has contributed a different characterization of community – from people dwelling elbow-to-elbow in a village to the "imagined communities" that fuel nationalistic political movements. In addition, within each tradition, there are varied and evolving notions and characterizations of community. Thus, Barab and Duffy (2000) as educational researchers identified four necessary conditions of a learning community (common history; shared goals, practices, belief systems, and stories; sense of acting as collective whole; and enculturation of new members) and, later, Barab, MaKinster, and Scheckler (2004) added to this list (common practice and/or mutual enterprise; opportunities for participation; meaningful relationships; and respect for diverse perspectives and minority views). Other researchers have contributed different lists of requisite features of community. In short, there is no single definition of "community" but rather we see that the conception of community can itself be viewed as a communal artifact.

This diverse, multi-faceted, and complex character of “community” is reflected in the nature of the questions and issues that are addressed in the literature on learning communities. The driving goal behind the research is not to determine whether a particular group is (or is not) a community but rather how to describe and understand different *types* of community. This line of inquiry focuses attention on the relevance and importance of certain features of community, how these are manifest in groups of individuals working together in a particular context, and how to design supportive technical environments.

According to Riel and Polin (2004), there are three “distinct but overlapping” types of learning communities: task-based, practice-based, and knowledge-based. Task-based learning communities are groups whose defining purpose is produce a product during a specified amount of time[8]; practice-based learning communities are groups with shared goals that provide members with richly contextualized and supported arenas for learning; and, knowledge-based learning communities are similar to practice-based communities but are committed to the deliberate and formal production of external knowledge about the practice (rather than relying on ongoing participation for its transmission). In a similar vein, Bruckman (2006) identifies and cites three learning community prototypes: Papert’s (1980) “samba schools,”[9] Lave and Wenger’s (1991) “communities of practice”, and Scardamalia & Bereiter’s (2006) knowledge-building communities.

Of the above types described, open, online, help forums are most closely aligned with communities of practice, in which the practice is working with others to solve (course-related)

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[8] Task-based learning communities are different from individuals engaged in collaboration in that participants experience a strong sense of identification with their partners, the task, and the organization that supports them.
[9] Samba schools are a kind of Brazilian social club in which members work together to prepare a presentation for Carnival, an annual themed intergenerational festival of song and dance.

mathematics problems. To shape the exploration of these intellectual communities of practice and produce an account of what it means to “belong” to a forum, we need to consider four dimensions of community: membership, goals, participation structures, and mechanisms for further growth (Riel and Polin, 2004). *Membership* addresses the identity of individuals in the group, the life cycle of their activity, and issues of status. This is particularly important for informing analyses of positioning. For instance, does authority get attributed to senior forum members and, alternatively, are newcomers treated differently? *Goals* reflect the common purpose or intent of individuals engaged in an activity. In the online, help forums, what are the effects on participation when individual members adopt different goals than those espoused by the group (e.g. providing worked solutions)? *Participation structures* refer to the opportunities that the group affords for members to participate in various ways. Understanding how the adoption of a policy that restricts tutoring to a small set of members (AOH sites) versus allowing spontaneous participation by all (SOH sites) influences this dimension of community. Finally, *growth mechanisms* are the means by which a community perpetuates and sustains itself, building up a stable knowledge base that protects against the loss of key individuals while at the same time members are constructing experiential, dynamic knowledge in their practice. Thus, any explicit forum policies, as well as the current configuration of practices, should be part of the picture.

These elements of community can be thought of as a canvas on which to paint a portrait of open, online, calculus help forums. The goal of the endeavor is to contribute to the understanding of intellectual communities of practice by providing a baseline against which the design of online mathematical communities can be assessed. In the next section, I review the directions we have taken and our current knowledge about open, online help forums.

2.4 PREVIOUS WORK: OPEN, ONLINE, CALCULUS HELP FORUMS

Extensive efforts are underway to harness technology and create web-based environments to supplement, or even replace, traditional modes of learning and collaboration (for example, see the review of five online calculus courses in Appendix A).[10] Yet, our understanding about whether something that resembles a community of learners can be "made to order" or how to measure emergence lags far behind. A complement to the current approach of designing environments (e.g. based on usability strategies) and attempting to discern whether the design supports positive social interactions is to observe and analyze "organic," or naturally occurring, sites.

Open, online, help forums are perfectly suited for this purpose. In contrast to forums associated with a specific course, these open forums spring up and take root without being planted by design. Members are attracted to them by necessity and interest. Thus, students and tutors learn of the forums' existence, access the web sites, and then choose whether or not to join in the conversation. The record of this conversation is an artifact that is available to the public in general, and to educational researchers, in particular. By locating these tutoring conversations on several sites, developing a methodology for examining some aspects of the activity, and analyzing several corpora of exchanges that address complex, mathematical topics, we have

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[10] It was the analysis of stand-alone online calculus courses that first drew us to the presence of open, online help forums: "One remedy for discouragement is discussion, and three of the courses include an electronic forum that affords discussions of the material with others. For *Thinkwell Calculus*, the forum is included in the course management system and thus only links students enrolled in a course together. In contrast, *Karl's Calculus Tutor* and *Internet Calculus* provide forums that are open to any individual participant. Students can post questions and solutions in these forums but it is worth noting that these discussion forums do not appear to be thriving. Karl Hahn personally responds to most of the questions raised in Karl's Discussion Forum, and fellow students do not regularly respond to the postings in *Internet Calculus*."

begun to develop an account of this phenomenon. Table 2 lists and summarizes the main points of the 10 studies we have conducted, presented, or published on this subject. Six of these major points are discussed in Sections 2.4.1–2.4.5, and three of the preprints (van de Sande & Leinhardt, 2007a, 2007b, 2008c) and one unpublished manuscript (van de Sande, 2007b) can be found in Appendix B. In addition, one paper (under review) introduces a framework for analyzing stand-alone online calculus instruction as is in Appendix A.

Table 2. Studies of open, online, calculus help forums

Study	Authorship and Date	Main point(s)
Spontaneous online discussions in mathematics	van de Sande, 2006	• Many students are seeking and receiving calculus assistance on open, online help forums. • Participation codes indicate the presence of extended discussions between multiple participants and student engagement.
Help! Online calculus tutoring	van de Sande, 2007a	• Forums with spontaneous participation structure exhibit presence of student activity in joint construction of solutions.
Help! Tutorettes on the calculus concept of limit	van de Sande, 2007b	• Comparison of assigned (AOH) and spontaneous (SOH) participation structures with a university-sponsored help centers shows higher accuracy and sense of community within SOH site. • Proposal of model describing features of effective help in focused, student-initiated tutoring exchanges (tutorettes).
A framing of instructional explanations: Let us explain with you.	van de Sande & Greeno, 2008	• An instructional explanation can be viewed as an activity in which participant are constructing a new resource for framing. • Online tutoring exchanges can be viewed as “mini-explanations” and effective exchanges are those in which the tutor is sensitive to the student’s perspective and draws on resources that student has.
Help! Active student learning and error remediation in an online calculus e-help community	van de Sande & Leinhardt, 2007a	• Active student learning in an SOH site is influenced to some extent by tutor actions: providing solution sketches and asking questions encourages this, whereas complete or mostly complete worked solutions appears to have the opposite effect. • In an SOH site, students appear comfortable presenting incorrect or incomplete work and tutors are open and forthright in their commentaries and evaluations. Errors pertaining to calculus principles are accorded more recognition than arithmetic/algebraic “mistakes.”
Online tutoring in the Calculus: Beyond the limit of the limit	van de Sande & Leinhardt, 2007b	• AOH sites promote brief exchanges between single tutor-student pairs, whereas SOH sites (especially those with minimal publication delay) contain many extended and meaningful exchanges between multiple participants. • There appears to be a positive relationship between conversational complexity (based on number of contributions and participants in an exchange) and mathematical and pedagogical quality. • The SOH sites exhibit a strong sense of community (compared to AOH). In particular, tutor errors are caught and addressed in a wikipedia-like fashion.
Drawing conclusions about diagram use in	van de Sande & Leinhardt,	• Students do not appear to perceive diagrams as a natural part of the construction of a related rates solution, whereas forum tutors go to

an online help forum	2008a	considerable lengths to introduce diagrams into the discussion as part of mathematical practice. • The lack of a drawing tool in an open, online forum impacts exchange fluidity, depth, and quality.
Online tutoring: Complexity, Community, and calculus	van de Sande & Leinhardt, 2008b	• The operation of SOH sites as natural learning communities provides a baseline against which the design of online mathematical communities can be assessed.
The Good Samaritan effect: A lens for understanding patterns of participation	van de Sande & Leinhardt, 2008c	• Framing forum tutors as "Good Samaritans" provides a lens for understanding patterns of participation in an SOH site. • Being part of a community of helpful tutors can serve as encouragement, so that providing help becomes contagious.
How tutors benefit from participation in open, online homework forums with a spontaneous participation structure	van de Sande & Leinhardt, 2008d	• Tutors in an SOH site attend and respond to feedback from other forum members. • Tutors make mathematical advances and hone their understanding through forum participation as the "wikipedia effect" influences aspects of tutor practice.

2.4.1 Help-seeking and help-giving

Open, online help forums are a *resource* that is available to students as they seek to learn and understand calculus and this raises the issue of the nature of the problem-solving activity found in these environments. Recognizing the need for help, locating resources, and evaluating their effectiveness are legitimate learning activities and are beneficial for achievement, perceived self-efficacy, and meta-cognitive skill development (Nelson-Le Gall, 1985). Students using resources for building understanding and assuming responsibility for learning are engaging in instrumental or mastery-oriented help-seeking, whereas the use of resources as a means of obtaining information with minimal participation and effort signals executive or dependency-oriented help-seeking (Nelson-Le Gall; 1981). Exchanges in open, online calculus forums exhibit characteristics consistent with both of these types of help-seeking. That is, students may post questions and receive help without contributing to the problem-solving activity (either in the initial posting or subsequent thread); or students may be active participants throughout an exchange, proposing ideas, constructing mathematical solutions, and evaluating others' contributions (van de Sande, 2007a; van de Sande & Leinhardt, 2007a).

Many forums have policies that are aligned with instrumental help-seeking, with posting rules that emphasize the intent of the forum to provide help rather than answers and encourage students to show work. The degree to which forum practice conforms to these rules depends, at least in part, on how tutors respond to students who (consistently) fail to show their work and contribute to solving problems. It appears that if students are able to receive a full (or partial) worked solution or a detailed outline, then they are less likely to actively engage in solving problems, whereas if they are questioned directly or receive hints, then they are more likely to actively participate in the solution process (van de Sande, 2007a; van de Sande & Leinhardt, 2007a). Appendix B.2 contains a more complete discussion of these issues.

In addition to explicit forum policies, there are social mechanisms that may shape help-giving patterns of tutor participation in SOH sites. It is important to emphasize at this point in our discussion that the tutors who participate in open, online, help forums are *volunteers.* They have no face-to-face relationship with the students, receive no financial compensation, and are in no way answerable for the students' performance in calculus instruction. Yet, this group of individuals is willing to contribute their time and expertise to providing help. How can we describe this phenomenon (using purely observational methodology)? van de Sande and Leinhardt (2008c) have proposed that SOH forum tutors' actions are governed by the "Good Samaritan" effect, a variant of "the bystander effect" from social theory that is useful as a lens for understanding patterns of participation in online activity of students in a foreign language class (Hudson & Bruckman, 2001). Table 3 summarizes the four mechanisms that contribute to the Good Samaritan effect and contrasts these with the bystander effect. See Appendix B.4 for an explication of these ideas.

Table 3. The Good Samaritan effect as a variant of the bystander effect

Mechanism	**The bystander effect**	**The Good Samaritan effect**
Self-awareness	Individuals do not participate because they do not want to appear foolish in front of others.	Individuals participate because they want to appear helpful in front of others.
Social cues	Inactivity of others is taken as a cue and discourages participation.	Activity of others is taken as a cue and encourages participation.
Blocking/Inviting	Action of one bystander blocks others from taking action for fear of worsening situation.	Action of others encourages individual to take action in hopes of improving situation.
Responsibility	Each individual feels only limited responsibility for negative consequences of inaction.	Each individual feels substantial responsibility for positive consequences of action.[11]

2.4.2 Conversational complexity and quality

Although conversations have properties of both participation and topic, research in conversation often foregrounds aspects of participation and places the topic of discussion in the background. Here, the goal is to describe participation in mathematical discourse while addressing several complex issues in mathematics learning, and to address issues of framing information within the trajectory of an explanation. However, the concepts of complexity and quality are, in some ways, as nebulous as the concept of community, discussed previously. What are the features of a high quality tutoring exchange, and how can we identify exemplary

[11] In the 2008 ICLS Proceedings, this entry read: "Each individual feels substantial responsibility for negative consequences of inaction."

exchanges within a large corpus? The work described below is the beginning of an answer to these questions.

One way of describing participation in conversation is to attend to the number of participants, the sequencing of turns, and the number of contributions in an exchange. van de Sande and Leinhardt (2007a) introduced a coding scheme to mark these features of forum exchanges and used this as the basis for an index of conversational complexity (van de Sande & Leinhardt, 2007b; 2008b). Although this complexity index does not address many aspects of participant structure (such as positioning, agency, and accountability), it indicates general participation practices and provides a means for comparing participation over large numbers of exchanges. In particular, this index indicates that the design and structure of a site influence exchange participation: sites with an AOH structure favor brief exchanges between single student-tutor pairs (low complexity), whereas sites with an SOH structure (and minimal delay for publishing postings) contain extended exchanges between multiple participants (high complexity).

In addition to highlighting the effects of forum structure on participation, the complexity index appears to be positively related to the quality of exchanges, in terms of mathematical depth and pedagogical sophistication. Exchanges with a low complexity index are generally communications of sparse fragments of mathematical information (low quality), whereas exchanges with higher indices contain elaborated mathematical discussions with sophisticated pedagogical elements (high quality). In exemplary exchanges, mathematical principles are invoked (such as the meaning of indeterminate forms in discussions of the limit) and the problem-solving activity contains valued elements of instructional practice (such as Socratic dialogue).

2.4.3 Perspectives in explanations

Understanding spoken or written language involves the construction of an *interpretation* in which information is selected and organized (Rommetveit, 1974; MacWhinney, 2005). In formulating this notion, we use the metaphor of sight or vision and commonly refer to "seeing" things or situations in a certain way. The claim, then, is that the world appears to us, in our experience, as "seen" from some *perspective* (Linell, 2001). In order to refer to the informational contents of a problem or situation (the "stuff" of the perspective), theorists have proposed the use of the term *framing* (Hammer, Elby, Scherr, & Redish, in press). A framing, then, is a cognitive arrangement of entities and some properties organized in relation to one another.[12] An important characteristic of a framing is that some entities are foregrounded over others, which corresponds to their being in more central focus.

Given this stance on cognition and point of view, differences in understanding or "misunderstanding" can be treated as a misalignment between participants' perspectives (Greeno & van de Sande, 2007). That is, the individuals involved in the discussion or explanation do not share a common framing that is sufficient to support their shared activity. Open, online, help forums are a location where students publish their incomplete understandings of some course material (e.g. a problem they cannot solve, a piece of an explanation that they do not understand, or a solution that they are unsure of) and seek a "mini-explanation" from forum tutors.[13] These impasses can, in some cases, be attributed to an incorrect or unproductive framing of the problem

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[12] These arrangements have been referred to elsewhere in the literature as "perspectival understandings" (Greeno & van de Sande, 2007).

[13] There are many types of explanations (common, disciplinary, self, and instructional) that have defining features. I use the term "mini-explanation" to indicate a condensed and very focused explanation (centering on a single problem or issue) that shares many features with an instructional explanation.

situation (such as treating a limit problem as an algebraic substitution). In these mini-explanations, tutors work with students (and sometimes other tutors) toward the construction of a new resource for framing the information in the problem situation (van de Sande & Greeno, 2008). One hypothesis is that an effective and quality tutoring exchange is one in which the tutor is sensitive to the student's perspective, constructing a bridge between alternative framings by drawing on resources that the student presumably has and positioning the student as a co-explainer (rather than "as a sponge").

In addition to addressing unproductive framings, participants in an exchange can also extend and enhance the explanation through the introduction of a novel framing. Indeed, being able to "see" a problem from multiple perspectives is a mark of expertise and mathematical sophistication. The generation of multiple framings is then enhanced by showing that alternative framings lead to the same result (resolution) and establishing their isomorphism (reconciliation). It seems logical that people who are involved in a meaningfully mathematical discussion will deliberately introduce and discuss the problem from alternative perspectives and seek alignment, when called for.

2.4.4 Online versus face-to-face

Open, online help forums are instances of student-initiated help-seeking and, as such, are similar in many ways to university-sponsored assistance centers. These centers are designed to provide students at a particular institution with assistance on coursework and are generally organized around subject area (e.g. mathematics or physics). During hours of operations, tutors (advanced students) staff the center to answer students' questions from coursework in the subject area. However, there although assistance centers and open, online help forums share a defining purpose, there are key differences in the way that help is operationalized in these two contexts. In an assistance center, the tutors are familiar with the curricula from which the query stems, can interact with students face-to-face, and often have access to resources tailored to the situation (such as course solution manuals). In the online forums, the tutors are unfamiliar with where the query stands in relation to the mathematics program, must interact asynchronously using the computer as a medium, and do not have access to prepared solutions. Surprisingly, then, open, online help forums have been shown to outperform an assistance center model in terms of the mathematical accuracy of the help provided and the degree to which students are held accountable for participating in the construction of solutions (van de Sande, 2007b). Participants in online, help forums do not appear to be constrained by rules of politeness and universal conversational maxims in the same ways as participants in face-to-face encounters, and this may smooth the path for students to be positioned with authority (e.g. challenging ideas and critiquing explanations) and for tutors to hold students accountable for active participation.

One critical distinction between online and face-to-face delivery that may account for these results is the social breadth of the activity. In the assistance centers, tutoring (like traditional tutoring sessions) is a private rather than public activity. Tutors do not generally speak with or observe one another while engaged in practice, and students have access to only one tutor at any given time. In this arrangement, one would expect that alternative perspectives would be less likely to surface, tutors' mathematical errors would be less likely to be detected and corrected, and tutors' practices and mathematical understanding would be less likely to sharpen over time. Appendix B.1 contains a fuller discussion of these ideas.

2.4.5 Sense of community

Open, online, help forums are communal spaces where students and tutors interact with one another and engage in public mathematical conversation. Is there evidence that this broadens participants' sense of self beyond the "me" or "I" into the "we" and "us" (an issue raised by Putnam, 2001)? Responding to this question requires an examination of the activity and participation within a forum that is informed by a notion of what it means to function as a (successful) community.

As discussed earlier, there is no single definition of what constitutes a community of learners, much less in a virtual environment. However, there are many common themes that are found across theories of community, as well as feature shared norms and goals (Carter, 1998; Wertheimer, 1998): Participants share some common explicit and implicit goals; participants have an accessible meeting location; participants identify themselves as members of the community; participants assume responsibility for participation; the defining features of the community can be renegotiated and altered by the members, and; ideas can be questioned, elaborated, challenged, and revised safely (Grossman, Wineburg, & Woolworth, 2001; Lave, 1991; Palloff & Pratt, 1999; Pratt, 1996; Werry & Mowbray, 2001).

Participation in both AOH and SOH forums is consistent with this medley of community features. [14] Table 4 outlines the ways in which these characteristics of community were manifest in MathNerds.com (AOH) and FreeMathHelp.com (SOH) (van de Sande & Leinhardt, 2007b;

[14] Tutors form the core group of members that provide the sense of community since they are more regular participants (often multiple times daily over extended periods of time). Students participate on an "as needed" basis but also interact in ways consistent with community membership.

2008b). Appendix B.3 contains a fuller discussion of the ways in which a sense of community is evidenced in online forums.

Table 4. Community attributes across participation structure

Community Attribute	**Participation Structure**	
	AOH (MathNerds.com)	**SOH (FreeMathHelp.com)**
Shared explicit and implicit goals	Accurate, timely help	Accurate, timely help, *and* Encouraging participation in norms of mathematical discourse (e.g. use of parentheses)
Accessible meeting location	Access through registration	Access through registration *and* logging on concurrently
Identify as members	Sign postings using online handle	Sign postings using online handle *and* reference one another as resources
Assume responsibility for participation	Volunteer to field queries Contribute expertise	Volunteer to field queries, Contribute expertise, *and* Take on roles within exchange, Answer for one another within exchange, Distribute problems within exchange
Safe atmosphere to present, challenge, and revise ideas	Constructive criticism of student work	Constructive criticism of student work *and* "wikipedia-like" correction for errors made by tutors

Notice that the sense of community is stronger within the SOH site in which tutors work collaboratively while engaged in practice. In this forum, tutors helped students (sometimes acting in unison while other times contributing multiple voices) *and* attended to the contributions of other tutors so that they were jointly positioned as learners. The result was a self-correcting feature (the "wikipedia effect") that increased the accuracy of the information being discussed; tutor errors were either replaced or addressed by other tutors (and sometimes by the student) with the result that incorrect information did not often remain as the "last word" in an exchange.

2.5 QUESTIONS

The questions that emerge from our prior work and the literature that this research is designed to address through the investigation of tutoring as a communal activity and the treatment of this dialogue as a public conversation on which we can eavesdrop are the following:

- *Tutoring:* The big question concerning tutoring involves the mathematical and pedagogical quality of the tutoring that takes place in this officially unsupervised environment and how participants position themselves and are positioned by others in interaction.
  - *Complexity & Quality.* What are the predominant patterns of exchange participation? Are these related to the nature of the query topic? How do these results compare with previous work? What is the relationship between conversational complexity (based on number of contributions and participants in exchange) and exchange quality (mathematical and pedagogical)? Can we make a case that the conversational complexity index provides a good way of locating exemplary exchanges?
  - *Student positioning.* What are the functional aspects of forum participation in terms of student positioning? That is, in what ways do students position themselves in the forum, and, what is the extent of these activities? In particular, a) how do students contribute to the construction of mathematical solutions?; b) do we see students questioning or challenging the contributions of others?, and; c) in what ways do students indicate that an issue has been resolved?
  - *Pedagogical moves*. Next, looking at forum activity from a different perspective, what types of pedagogical moves are enacted in the forum, and, to what extent are these present? What is the context of the interaction associated with the various pedagogical moves?

- *Community*. What does it mean to be a member of a community of others who share an interest in mathematics and the desire to engage in the joint construction of solutions to "routine" problems?
    - What is the extent to which multiple forum tutors participate in any given exchange, and does this differ according to the topic of the query? What are the benefits that tutors derive from participation in an online help forum community? How can we discern the attention that (tutor) contributions within an exchange receive from the forum community?
- *Calculus*. How can we use the forum exchanges to discern what exercises are "bears" and learn about the way students "see" certain mathematical concepts as they work on course assignments?
    - What types of limit and related rates exercises are students posting on the forum? What aspects of these exercises are problematic for students? In what ways are students being helped to construct more coherent understandings of these concepts?

3.0 METHODS

3.1 CHOICE OF METHODOLOGY

This purpose of this study is to advance our understanding of tutoring exchanges in open, online, calculus help forums. The methodology that I chose is observational and non-intrusive. It allows us to “listen in” on – and learn from – a public conversation without interfering with existing forum practice or disrupting the learning environment. The careful analysis of activity in this organic tutoring environment can then guide experimental, empirical research and the design of virtual learning communities.

3.2 DESIGN OF THE STUDY

This study is designed to explore the functional aspects of activity in an open, online, calculus help forum with a spontaneous participation structure (e.g. SOH site). I have drawn a sample of 100 exchanges on the limit (from 4/29/08 to 1/09/07) and 100 exchanges on related rates (from 4/15/08 to 10/29/06) from the calculus help forum on FreeMathHelp.com. This corpus of 200 exchanges is then investigated with respect to participation sequences within exchanges, conversational complexity, mathematical and pedagogical quality, the positioning of individuals regarding the competence, authority, and accountability that are attributed to them by others and by themselves, and pedagogical moves. The intent is to frame this study of tutoring as an analysis of a public conversation that is neither student-centered nor tutor-centered but rather one that represents a polyphony of voices from a diverse group of individuals who share a common interest in mathematics.

3.2.1 Vocabulary

There is a vocabulary associated with asynchronous interaction in online environments that I have adopted for the discussion of online tutoring. An Internet *forum* is a web application for holding discussions and posting user-generated content.[15] The term "forum" is used to refer to the entire community as well as to any sub-forum dealing with a distinct topic. For instance, a calculus help forum may be one of many sub-forums of a larger (mathematics) help forum. A *post(ing)* is a contribution or message that is published on the site, either to initiate a discussion or in response to another's contribution. As in verbal discussions, participants generally take turns contributing to the conversation. The set of contributions pertaining to a single request for help constitute an *exchange* or *discussion*, sometimes referred to as a *topic* or *thread*. These threads are displayed on the entry webpage of the forum and designated by the *subject header* or title of the initial post.

Spontaneous Online Help (SOH) sites are forums that allow any member to participate in a thread and respond to a query. In contrast, *Assigned Online Help* (AOH) sites assign incoming queries to tutors who have been selected by the forum administration according to certain criteria. *Designed Online Help* (DOH) sites are computer-supported collaborative learning

[15] Internet forums are also commonly referred to as Web forums, newsgroups, message boards, discussion boards, (electronic) discussion groups, discussion forums, and bulletin boards.

(CSCL) environments that are created to serve a restricted community, such as members of a specific course.[16]

3.2.2 Site choice and description

The popularity and availability of open, online, calculus help forums means that there are several possible candidates for this study. I chose the calculus forum at FreeMathHelp.com because it has a history (extensive archives dating back to 2005); includes a search mechanism for locating exchanges by a keyword or phrase, and; is active in terms of daily postings and membership.[17] In addition, the forum policies (such as achievement of member status) are explicit, and member "reputation" is an implicit mechanism of the social arena rather than being quantified (e.g. by others' ratings of one's postings) and made an explicit part of forum identity.

FreeMathHelp.com is an advertisement-supported mathematics help portal established in 2002 by Ted Wilcox, an enterprising high school junior. In addition to the discussion forums, the site includes lessons, games, a graphing utility, and worksheet pages. There are nine homework help forums, organized by subject area (including algebra, differential equations, and calculus). Registration (which entails agreeing to abide by terms for permissible content and/or conduct, providing a username and e-mail address, and selecting a password) is the sole requirement for becoming a *forum member.* Forum members can initiate threads in a discussion forum (e.g. as students posting mathematics questions) and can respond to others' posts (e.g. as tutors

[16] DOH forums are the topic of mainline research activity (for example, Kortemeyer, 2005; Stahl, 2008) but are not represented in this project because they are closed to the general public.

[17] Whereas dysfunctional communities are characterized by low levels of activity and static membership, successful communities sustain themselves through multiple generations of members and do so without becoming "brittle." (Riel & Polin, 2004, p.18)

providing help)[18]. Forum members also have access to user profiles that include self-volunteered information on occupation, residence, contact information, as well as statistics on discussion board activity. Each member is characterized by total number of forum contributions to distinct threads: new (0-49), junior (50-249), full (250-999), senior (1000-2499), elite (more than 2500). There are several elite members who have contributed to more than 2500 threads, five of whom have contributed to more than 4000. Each forum has assigned moderators who may lock topics and move, delete, or edit postings. In addition, members can edit their own contributions after they have been posted: If this is done *after* the member has logged off of the forum, then a message is appended to the altered contribution: "Last edited by [member] on [date and time]; edited [number] times in total." If editing takes place *while* the member is still logged on to the forum, then there is no official evidence of the modification although the general practice is for the author to indicate that the contribution has been edited. The moderators are selected by the forum administrator, Ted Wilcox, who maintains the site, oversees forum activity, and establishes policies. Issues concerning the day-to-day running of the forum can be posted in an Administration Issues forum that is intended for discussions of policy, suggestions for improvement, and announcements of technical difficulties.

The prescribed etiquette for participation is located in a "sticky" that is the lead posting within each help forum. This covers administrative issues (e.g. posting to an appropriate category) and politeness (e.g. patience while waiting for response). In addition, there are three rules that specifically address the content and framing of posts: include problem context ("Post the complete text of the exercise"), show initial work ("Show all of your work [including

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[18] Any member of FreeMathHelp.com can post queries and provide assistance, but, in practice, there is very little overlap between student members (who initiate threads) and tutor members (who provide assistance).

intermediate steps that may contain errors]”), and attend to clarity (“Preview to edit your posts [to minimize errors]”).

The computer window for constructing posts contains traditional icons for highlighting text (e.g. italics, boldface, underlining, and font size and color), inserting material (e.g. external links and images)[19], and organizing text (e.g. forming lists). A large selection of graphic “emoticons” (faces) is available for expressing emotions and attitudes (such as  [Very Happy] and  [Confused]). In addition, there are format capabilities more specific to mathematical discussions since it is tedious and often impossible to create mathematical symbols and expressions using keyboard characters. Using LaTeX, a document preparation system designed to typeset mathematical text, participants can use command strings and code to produce mathematical symbols (such as ∞) and vertical expressions (such as $\frac{dA}{dt}$). In order to encourage the use of this software, FreeMathHelp includes a tutorial for LaTeX, as well as a link to a free equation editor that generates the LaTeX code, which, although powerful, can be difficult for the novice.

3.2.3 Sample characteristics

FreeMathHelp.com features participants’ profiles that include information on occupation, location, and interests. Whereas many student participants do not provide this information, the participating tutors in the calculus forum are self-reportedly students, educators, professionals,

[19] The forum does not have a whiteboard or analogous drawing tool, although members have expressed a desire for incorporating this feature (in one of the lengthiest threads in the Administration Issues forum). At present, forum tutors go to considerable lengths to introduce and encourage the use of diagrams in solutions of certain problem types by importing images created on other software and crafting “ascii art.” (van de Sande & Leinhardt, 2008a)

and retired mathematics professors. The most frequent tutor participants are from the U.S., although there are representatives from a variety of other countries as well. Most participants of open, online tutoring forums select names or "handles" (such as **ihatecalc** or **Skeeter**) that do not disclose personal information (location, knowledge level, etc.), and we refer to such participants using these self-designated handles. If a user name appears to correspond to a true surname, a pseudonym was assigned to protect the member's anonymity and privacy.

Although some tutors and students post more frequently, numerous tutors and students frequent MathHelpForum.com. The sample contained 100 related rates exchanges initiated by 65 different students, with responses from 18 different tutors and 100 limit exchanges initiated by 67 different students, with responses from 23 different tutors.[20] There was some overlap in participants (both students and tutors) across the two mathematical topics: 17% of these students posted queries on both limits and related rates, and 63% of the tutors provided assistance for both topics.

3.3 CODING AND ANALYSIS

Analyzing a sample of 100 limit exchanges and 100 related rates exchanges drawn from a single forum during the same time period positions us to make inferences about participation and activity in open, online, calculus, help forums. At the same time, it is important to designate the populations that will be addressed through the analyses. I infer that this corpus of limit/related

[20] The majority of these exchanges centered on a single problem statement, although 11 on limit and 7 on related rates contained more than one problem statement in the initial posting. If these threads develop into extended discussions, the practice of the moderators is to split the problems into separate threads. Also, 4 related rates exchanges contained an appended problem statement in the thread (following discussion of the problem posted in the initial posting), although this practice is discouraged.

rates exchanges is representative of any sample of the same size drawn from the same forum and supports claims about tutoring exchanges on these topics within this forum. The claims about community and functional aspects of activity extend to general participation between members of this forum, as well as other online forums that share key characteristics (such as participation structure and moderation policies). In order to assess the reliability of the coding used in the analyses, 10% of the sample was independently coded by a colleague with graduate level mathematical experience.

3.3.1 Tutoring

3.3.1.1 Conversational complexity

In order to locate exemplary exchanges, each exchange was assigned a *participation code* (as discussed in 2.4.2) that tracks the number of participants, the total number of contributions in the exchange and the sequence of participation. For example, a code of 1231 would characterize a discussion between 3 participants with 4 total contributions: a student [1] posted a problem and then two different tutors [2 and 3, respectively] responded, followed by a final contribution by the student [1]. These codes permit us to catalogue exchanges that involve multiple conversational turns, multiple participants, and multiple contributions by a single participant. In addition, although entries in the participation codes are agnostic with respect to the quality of the contribution (e.g. mathematical accuracy and depth, and pedagogical sensitivity), the codes do provide some indication of interaction within an exchange: for example, 1213121 is more likely to be an exchange in which two tutors are conversing with a student, whereas 1213232 is suggestive of dialogue between two tutors.

Based on these participation codes, each exchange received a *conversational complexity index* defined as the sum over code entries. While this index makes arbitrary use of the categorical indices – the numbers in the codes have no value beyond marking the sequence of participants – the index appears "well-behaved" in that lower sums correspond to exchanges that do not contain intense mathematical discussions and elements of pedagogical sophistication (van de Sande & Leinhardt, 2007b; 2008b). The relationship between the complexity index and the quality of exchanges was also explored in this study.

3.3.1.2 Quality

To assess the quality of the exchange, each exchange received a rating from 1 to 5 for its totality. A "1" was assigned to those exchanges that were both brief and contained little or no rich explanatory or mathematical material; a "5" was assigned to those exchanges that had a truly mathematical feel to them invoking principles, mathematical reasoning, and to some extent excitement. An important feature of these exemplary exchanges was that the student remained positioned as a focal participant in the exchange. Table 5 contains a description of some of the features that differentiated exchanges according to quality. This analysis differentiated exchanges containing elaborated, complete mathematical discussions from those that resembled sparse mathematical fragments. Inter-rater reliability for a sample of 20 exchanges was 85%, calculated as the number of agreements over the sum of the number of agreements and disagreements.[21] All differences involved assignment to neighboring rankings and there were no assignments that differed by more than a single ranking. Disagreements were resolved following discussion.

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[21] Cohen's kappa is not applicable.

Table 5. Select features distinguishing exchange

Rating	Features
1	Exchange contains little or no rich explanatory or mathematical material.
2	Exchange marked by sparse explanations or few connections to other mathematical material.
3	Exchange in which actions are prescribed but may not include reasons for application. Conditions of use largely absent when principles are invoked.
4	Exchange references principles but associated mathematical reasoning somewhat difficult to follow. Student may be peripheral participant.
5	Exchange in which principles of the calculus are invoked and perspicuous mathematical reasoning is evident. Student positioned as focal participant.

Figure 3 contains an exchange that received the lowest rating on quality. In this exchange, with a conversational complexity index of 3, the tutor, **pastel**, references some relevant formulas ("start…with the formula for the volume of a right circular cylinder, along with the formula for the area of a circle") without any explanation as to their connection to the problem situation or why they might be "a good place to start." There is no evidence that this interaction benefited the student, **Tascja**, who entered at a loss how to initiate a framing for this problem ("How would i go about solving this") and did not return to the exchange.

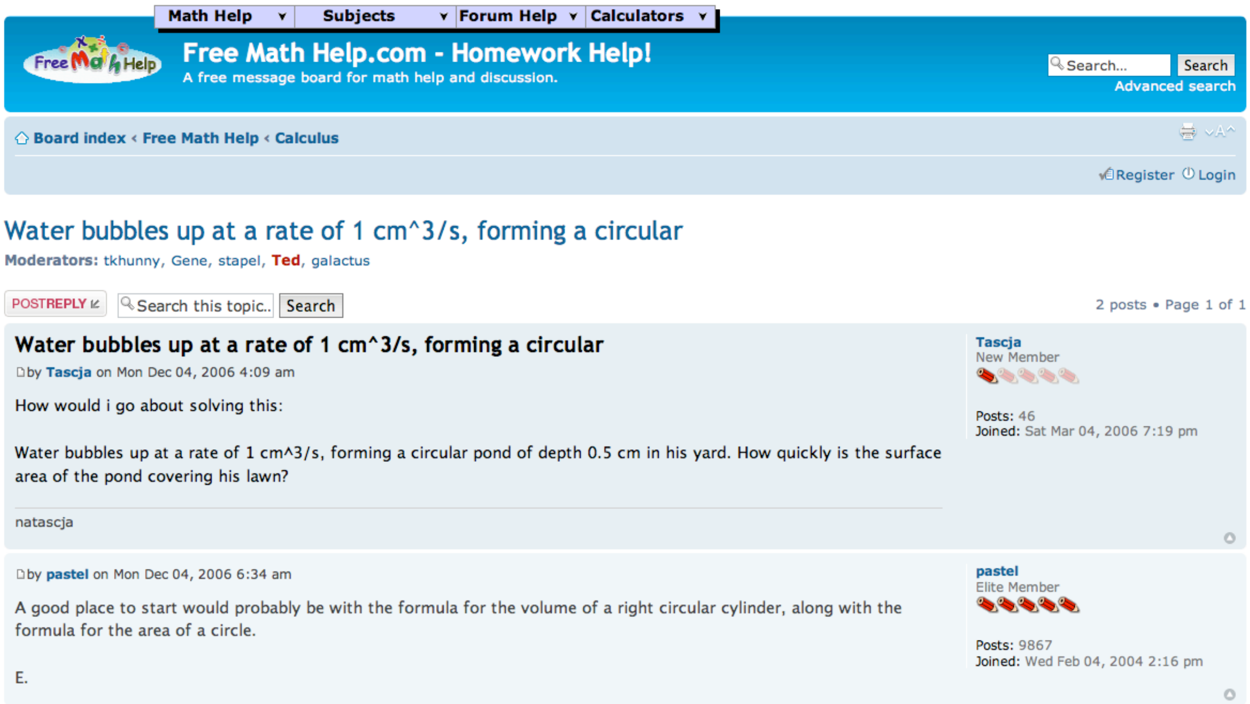

Figure 3. An example of a low quality exchange

In contrast, the exchange in Figure 12 with a complexity index of 26 is characteristic of exchanges of the highest quality, those that contain rich discussions of mathematical principles *and* revolve around providing support for the student's understanding and increased participation. In this exchange which is discussed in detail in Section 4.3.2, a forum tutor, **skeeter**, poses a sequence of leading questions to the student, **kimmy**, so that she discovers the cause of her misunderstanding and is able to establish a productive framing for this related rates problem. Finally, in order to get a feel for the multi-dimensional nature of "quality," consider the exchange in Figure 18 that received a quality rating of 4. Although this exchange contains a rich, intricate, and extended (complexity index of 25) discussion of mathematical principles and issues between forum member, it falls short of the highest quality ranking because the student, **toebo**, is not involved throughout the exchange that instead evolves into a discussion between forum tutors, **Hobostush**, **galactus**, and **pka**. The juxtaposition of these latter two examples shows that, in order to obtain the highest quality ranking, there must be evidence of mathematical depth and curiosity *as well as* student-focused positioning.

3.3.1.3 Positioning

In order to explore student positioning, each exchange was examined for the presence of three types of student activity: assertions and proposals for mathematical actions, questions and challenges of others' proposals, and degree of resolution. Cohen's κ, a conservative statistic for establishing reliability, indicated considerable inter-rater consistency on a sample of 20 exchanges: assertions and proposals for action (Cohen's $\kappa = 1$, standard error = 0); questions and challenges of others' proposals (Cohen's $\kappa = 0.77$, standard error = 0.22); degree of resolution (Cohen's $\kappa = 0.74$, standard error = 0.14); all differences in coding were resolved following discussion. Table 6 contains examples of exchange excerpts that were taken as instances within each of these categories for the two topics.

Table 6. Example excerpts showing ways in which students positioned themselves

Topic	Student makes assertions or proposes action	Student questions or challenges another's proposal	Student indicates issue has been resolved
Limit	THis is what i did. I don't know exactly how to solve for this but, my logic is that it'll go to zero eventually, because the bottom goes to infinity faster than the top so it'll to go zero. But i am not sure exactly if this is right can someone prove this or tell me if i'm intuatively correct. thanks	This is interesting, but I don't think it's what he [the instructor] wanted. As I said, he hinted us to make it into a limit that goes to 0, which would fit Galactus' solution. And to be honest, I don't think I've actually ever seen this inequality. Does it have a name? I'd have to know how to prove that first anyway, profs don't like us using theorems we can't prove.	I love you, people! I've tried L'Hôpital's rule but got stuck with the x^(n-1) and x^(m-1) parts. Now that I see the simplifications it seems so obvious! My sincerest gratitude for your help.

Related rates	1. made diagram the blank is 1, x is 2, and y is the distance that is increasing 2. I said dx/dt is 500mi/hr because that is the speed of the plane. 3. I said x^2 + 1^2 = y^2 4. I implicitly differentiated that to be 2x(dx/dt)=2y(dy/dt) 5. I plugged 500 in for (dx/dt), 2 in for x, and sqrt(5) in for y. 6. solved for dy/dt and got 1000/sqrt(5) wrong answer	for the derivative of law of cosines, i am confused how you got the last part : absin(O) (dO/dt) c^2 = a^2 + b^2 – 2abcosO to find the derivative of this last part, i get that deriv of cos is -sin, but where does the -2 go. did you use the product rule. if so, what did you make “f(x” and “g(x)” for product rule? when i tried, i made f(x) = 2a and g(x) = bcosO but i think I’m wrong.	i got it, forgot to factor, btw -14/sqrt(18) = -3.30, which is the correct answer

3.3.1.4 Pedagogy

The following scheme was used to address and investigate the pedagogical moves made by forum tutors: a) the tutor contributed mathematical information or advice to the construction of the solution or explanation; b) the tutor initiated a dialogue with the student inviting her/him to make inferences and draw conclusions leading to the problem solution; c) the tutor pressed the student to show her/his own work on the problem; or, d) the tutor referred the student to another resource (e.g. to a post involving a similar problem). Because a posting can contain multiple types of pedagogical moves, it was possible for a single contribution to be classified within more than one category. There was 100% agreement on the coding of a sample of 20 exchanges. Table 7 shows an example exchange excerpt that was taken as an instance of each type of pedagogical move.

Table 7. Example excerpts showing types of pedagogical moves

Topic	Contribution of information	Initiation of discovery	Pressing for work	Referral
Limit	sometimes you have to use the methods learned back in precalculus	**In order for a limit to exits** at all we **must have** $f(a-)=f(a+)=\lim_{x\to a}(x)$.	If you show what you tried – may be I can help to “uncomplicate”	Given the extent and fundamental nature of these questions, I think

	… $\frac{\sqrt{x}}{\sqrt{x+2}} = \sqrt{\frac{x}{x+2}} = \sqrt{\frac{1}{1+\frac{2}{x}}}$ now let $x \to \infty$	That is, **we must have a left-hand limit, a right-hand limit and those two must be equal.** Does that happen in this case?		that you should sit down with a live tutor. These are important concepts to grasp.
Related rates	Draw a picture, and look for right triangles!	your answer is correct, but I would like to know what **exactly** did you substitute in for dy/dt to determine dx/dt?	We'll be glad to look over your work and help you get going again, but you'll need to post it first.	Try this link: [link to posting in other online forum]

3.3.2 Community

Exchanges in which more than one tutor participated were surveyed to discern the nature of tutoring within a community, and three distinctive facets of tutoring in this environment emerged: mathematical discourse and debate between tutors, proposals from alternative perspectives, and the support of accuracy. Inter-tutor discussion was flagged by questions and comments directed at other forum tutors (e.g. "How would you find 'b' (=4) – without L'Hospital?(Or the logic that numerator must become 0 to able to get to 1)."); the introduction of an alternative perspective on the problem solution often just occurred as an extension of conversation but was sometimes marked by distinguishing it from those already in common ground (e.g. "When someone or something has a headstart,…I have my own approach…"), and; the support of accuracy for the mathematical information being disseminated became evident when contributions were challenged or advice from others was sought (e.g. "Soroban's answer is different than mine, which leads me to believe I may have an error. If anyone spots it let me know."). An exemplary exchange demonstrating each of these aspects was analyzed to demonstrate the benefits and affordances of engaging in tutoring as a communal activity; see

Figure 18, Figure 19, and Figure 21 – Figure 34. In addition, a scheme for representing the "weight" of a posting in an exchange (in terms of the amount and duration of attention it receives) was developed and used to illustrate the wikipedia-like nature of the forum in operation and the attention paid to posts within an exchange as described later.

3.3.3 Calculus

To discern the aspects of limit and related rates that are problematic for forum students, queries were examined and classified according to the type of exercise or task that was posed. For instance, one common type of exercise on limit involves the algebraic evaluation of the limit for a given expression. The exchanges were then analyzed according to the location and extent of student difficulty. Table 8 shows example exchange excerpts that demonstrate the different types of errors within each mathematical topic. A sample of 20 initating queries was coded for the attribution of primary student difficulty to framing (initiating vs. repairing), enacting procedures, or "other" sources: Cohen's κ = 0.76, standard error = 0.11.

Table 8. Example excerpts showing error types

Topic	Errors initiating framing	Errors interfering with framing	Errors enacting procedure	Other, e.g. "pre-calculus" errors
Limit	I do not understand some of my math problems. If someone could explain how to figure out the following. Once I see a problem explained I am able to do the rest. Here goes. It may be hard be I can't do all the symbols. consider f(x) = x-2 for x less than or equal 3	1) Find the limit as x -> infinity of: (sqrt(x^2+x)-x) I tried multiplying by the conjugate… lim x->infinity of: (x)/(sqrt(x^2+x)+x) But I can't tell if that's in the form of 0/0 or infinity/infinity, so I can't use "the hospital" rule yet.	Recently the class professor sent over l'Hospital's rule but didn't go into as much detail as I would have liked and was wondering if possible someone could inform me as to how you would do problems such as these. 1. Find lim goes to 0 (zero) ((e^11x)-1-11x)/x^2 (would that be	lim x^5-32/x-2; x>2 [limit as x approaches 2 of (x^5-32)/(x-2)] … I factored out x-2 in the denominator which left x^4-16 or (x^2-4)(x^2+4).

Topic	Errors initiating framing	Errors interfering with framing	Errors enacting procedure	Other, e.g. "pre-calculus" errors
	x-1 for x greater than 3 So ? Find lim f(x) x approaches 3+ from the right	Is there an obvious way to get this into a form I can use the rule on? Thanks	done by taking the derivative of the top and bottom using the derivative division rule something like: f=((e^11x)-1-11x) f'=((e^11x-11) g=(x^2) g'=2x then doing (g*f'-f*g')/g^2?	
Related rates	Please someone help me with this. I have done similar problems but I don't know what formulas to use in this problem. At least I hope some one will be able to tell me what formulas to use. I think I have to use C=2pi(r) as one but the other I don't know.	This is my diagram of what I think is going on. [diagram] I also get 100^2 + x^2=200^2 as the formula for the base of the triangle but I don't quite get how to find the rate of change. These are really weirding me out as each word problem is about something else and I don't know how to apply the concept to each one.	dA/dt is 40; dB/dt is 50; find dC/dt at what? Since 10 to 12 is 2 hours, A=80 B=100 and use law of cosines to find c c^2=a^2+b^2-2abcosC c= im having trouble finding the derivative of law of cosines, specifically the past 2abcosC is that part 2a(-sinc)db/dt + (bcosC(2a)(da/dt)	I drew a right angle triangle with the camera 1km from the tracks and the hyp as 2km x = 1km (distance from tracks to camera) y= 2km distance from train to camera (hyp) z = train tracks x^2 + z^2 = y^2 z = 2.24

3.4 QUESTIONS AND METHODS

Table 9 links the research questions on tutoring, community, and calculus that this project addresses to a summary of the source of the data and the methods used to answer each query. Both quantitative and qualitative methods are represented to provide converging support for the claims that will be made.

Table 9. Questions, data source, and methods

	Construct	Questions	Data Source	Methods
Tutoring	*Complexity & Quality*	What are the predominant patterns of exchange participation? Are these related to the query topic? How do these results compare with previous work?	Participation codes and complexity index applied to all exchanges in sample	Statistical comparison of distribution of complexity indices using Chi-square test
		What is the relationship between conversational complexity (based on number of contributions and participants in exchange) and exchange quality (mathematical and pedagogical)?	Complexity indices and application of 5 point quality rubric from van de Sande & Leinhardt, 2007b	Boxplot comparison of complexity index versus quality rating within each topic
	Student positioning	In what ways do students position themselves in the forum, and, what is the extent of these activities? In particular, a) how do students contribute to the construction of mathematical solutions?; b) do we see students questioning or challenging the contributions of others?, and; c) in what ways do students indicate that issue has been resolved?	Student contributions (initial and subsequent to tutor intervention)	a) Presence/absence of student assertions or proposals for action by location in thread; b) Presence/absence of questions or challenges directed at others' contributions; c) Indication of resolution according to "strength:" lack of resolution, weak resolution, and strong resolution Qualitative analyses of contrasting and exemplary exchanges to illustrate the manner in which these positioning moves play out in forum interaction
	Pedagogical moves	What types of pedagogical moves are enacted in the forum, and, to what extent are these present?	Tutor contributions	Categorized as one (or more) of following: a) provision of mathematical information; b) initiation of discovery; c) directive to show work, and; d) referral to other resource

	Construct	Questions	Data Source	Methods
		What is the context of the interaction associated with the various pedagogical moves?	Student and tutor contributions surrounding instantiation of certain pedagogical move	Comparison of number of exchanges exhibiting high (greater than 7) and low complexity indices for exchanges exhibiting particular pedagogical move Qualitative analyses of exemplary exchanges demonstrating pedagogical action to illustrate how these played out in interaction
Community	*Participation*	What is the extent to which multiple forum tutors participate in any given exchange, and does this differ according to the topic of the query?	Participation codes for exchanges in each topic	Distribution of number of tutors within an exchange
	Influence	What are the benefits that members derive from participation in an online help forum community?	Exchanges in which more than one tutor participated	Qualitative analysis of exemplary exchange containing: a) mathematical discourse and debate; b) introduction of alternative perspectives, and; c) support for maintaining high quality in terms of mathematical accuracy
		How can we discern the attention that (tutor) contributions within an exchange receive from the forum community?	Exemplary exchange containing tutor contributions that were inaccurate and were addressed by others	Construction of schematic representation displaying number and origin of direct references to each contribution in exchange Application of schematic representation to exemplary exchange and qualitative analysis of “fit” to demonstrate power of representation

	Construct	Questions	Data Source	Methods
Calculus	*Limit*	What types of limit exercises are students posting on the forum? What aspects of limit exercises are problematic for student?	Problem statements and queries on limit Student queries and proposals for action	Classification of exercises posted according to: a) evaluation; b) formal definition, and; c) "inverted" query Identification of primary source of difficulty as either involving re-organization or reframing of mathematical expression or enactment of procedure/technique Qualitative analysis of exchanges in each problem type to demonstrate how students are framing the exercises and the nature of the help received
	Related rates	What types of related rates exercises are students posting on the forum? What aspects of related rates exercises are problematic for students?	Problem statements and queries on related rates Student queries and proposals for action	Classification of exercises posted according to task complexity (simple or involved) Identification of primary source of difficulty as: a) initiating a frame; b) establishing/repairing a frame; c) enacting a procedure, or; d) other Qualitative analyses of exchanges within each category to demonstrate how framing issues are evident in exchange contributions

4.0 TUTORING

Research on tutoring has focused on two learning environments: tutoring sessions between single tutor-student pairs and computer-based tutoring systems. In the former case, the initiation of the encounter is off-stage; whether or not the student approached the tutor or was selected through assessment for remediation does not come into play. An account of this form of tutoring assesses the quality and effectiveness of the activity according to the presence of "ideal" pedagogical elements. For instance, Graesser, Person, and Magliano (1995) examined two large corpora of tutoring sessions for 8 indicators of quality instruction, such as explanatory reasoning and Socratic questioning. Within the computer-based tutoring tradition, the focus is on whether and how the tutoring program improves student performance on a specific type of problem and whether this learning transfers to other problems. Although the design of such tutoring systems may seek to emulate the presence and conduct of human tutors, there is no expectation that the student will interact with the technology in ways consistent with human interaction, for example by positioning themselves as inquirers. In both the human and computer-mediated tutoring traditions, the student is considered to be a receptive learner, so that the tutor intervenes when the student goes astray or produces hints to keep the student on track.

The presence of open, online, help forums offers a new scenario for tutoring in which students are interacting with human tutors in a computer-mediated setting. The "distance" and anonymity that characterizes these student-initiated interactions means that students can be more aggressive and can position themselves in ways that are not typical of other tutoring situations. In addition, because the forums are run by volunteers who are not obligated to produce answers and who pursue this activity as a hobby, there is an opportunity for tutors to make (relatively) sophisticated pedagogical moves, for instance by drawing the student into a discovery or requiring student demonstrations of understanding. The presence of these positioning and pedagogical opportunities raises questions concerning the activity and interaction that takes place in the forum. In this chapter, I describe the complexity of the exchanges in an open, online calculus help forum and the relationship between the conversational complexity and exchange quality. This provides the backdrop for analyzing the ways in which students position themselves and assume authority for their understanding, and the pedagogical moves that forum tutors employ as they seek to help students solve problems and "do mathematics."

4.1 CONVERSATIONAL COMPLEXITY AND QUALITY

Figure 4 shows the percentages of exchanges on limit and related rates according to the conversational complexity index (defined as the sum of the participation code entries). For the purpose of comparison, the indices for a different sample (n=100) on limit (van de Sande & Leinhardt, 2007b) from the same forum are also shown. There was no significant difference between the complexity indices for exchanges on related rates versus limits in the current sample, $\chi^2(4, N=200) = 1.34$, $p = .86$. In addition, the Chi-square test showed no significant difference between these two samples combined and the sample in van de Sande and Leinhardt (2007b), $\chi^2(5, N=300) = 4.77$, $p = .44$.

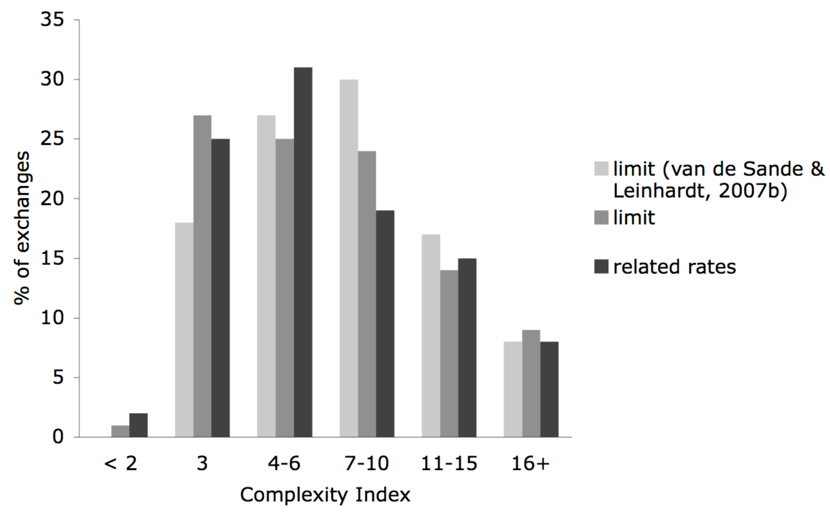

Figure 4. Conversational complexity index by topic[22]

Roughly 50% of the exchanges have a complexity index greater than 7, indicating the presence of extended exchanges and discussions involving multiple participants. In other words, the participants in the forum are engaging in conversation and public dialogue. The question that naturally arises, given this finding, is the nature of the relationship between conversational complexity and exchange "quality." One possibility is that exchanges might be brief (low complexity) but exhibit high quality, or they might be extended (high complexity) but trivial and superficial. Another possibility is that these two are positively related, so that high complexity serves as an indication of the presence of rich discussion and intricate pedagogical moves.

Figure 5 shows the relationship between the quality of the exchanges (ranked from trivial and superficial (1) to deep and mathematical (5)) and their conversational complexity.

[22] This sample contained three exchanges with complexity index less than 3. Both topics included one exchange in which the student who posted the query self-answered the question (complexity index of 2), and there was one unanswered posting on related rates. This unanswered posting involved a very common related rates problem situation and was identical to a query posted six days previously by a different student, so it is possible that the lack of response was intended as a reminder of the forum policy that students search the archives (especially recent entries) for identical or similar queries before posting.

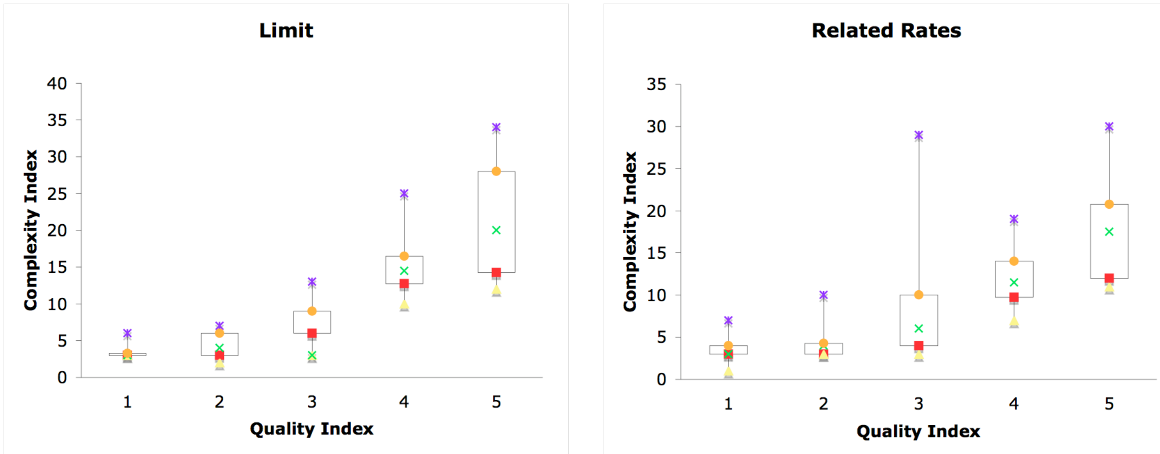

Figure 5. Relationship between complexity and quality

Within both mathematical topics, there is a positive relationship between complexity and quality. In particular, there are no exchanges of low complexity that were associated with exchanges invoking and discussing mathematical principles and containing intricate pedagogical moves. Exchanges that are low in complexity thus tend to be transmissions of mathematical information, similar in many ways to worked solutions or examples found in instructional materials. Of course, as can be seen in Figure 5, high complexity index does not, in and of itself, guarantee an exemplary exchange, and this can be attributed to the construction of the complexity index which is *not* a function of exchange content. Note for the vast majority of exchanges, a complexity index of 7 or above assures a quality index of 4 or 5. Focusing attention on the exchanges with a high complexity provides a means of locating the meaningful discussions, mathematical debates, and sophisticated pedagogical moves that are found in this tutoring environment.

4.2 POSITIONING

Although the forum participants acting as tutors are generally more experienced mathematically than the students acting as questioners, students can position themselves with authority in an exchange in three ways: by contributing to the construction of a solution, by questioning or challenging the contributions of others, and by indicating that an issue has been resolved. In this section, we look at each of these three indicators of student authority in turn to reveal how students are positioning themselves as they participate in the forum.[23]

4.2.1 Student makes assertion or proposes action

Students participate in the forum for different reasons: because they have reached an impasse while attempting a problem, because they wish to confirm the accuracy of a solution that they have constructed, or because they have questions regarding an explanation that they have encountered in their studies. When a student posts a query on the forum, s/he can assume either a passive or an active position in the construction of the solution or explanation. One mark of active participation involves making assertions or proposing mathematical actions (even if these are hedged or couched in uncertainty), and the exchanges were examined for this aspect of positioning.

Table 10 contains the number of exchanges in which the student made (or failed to make) an assertion or proposed a mathematical action within each topic by location in the thread (initial

[23] Of course, the way in which students *position themselves* in the interactions cannot be understood independently of the way in which students *are positioned* by others. Here, I focus on student initiative as part of a larger system of positioning that includes moves made by others and positioning relative to the discipline.

versus subsequent posting or both). The results in Table 10 indicate that students are generally positioning themselves as contributors to the discussions on solving the limit and related rates problems. The pattern of the location of these contributions in the thread is shared across topics, with students following the participation guideline to "show all of your work" in the initial posting 60% (limit) and 68% (related rates) of the time and contributing at some location in 69% (limit) and 79% (related rates) of the exchanges. Out of all 200 exchanges 74% involved students making some sort of assertion or action in the thread, and, of these, nearly three quarters involved exchanges with a complexity index greater than 6. That is to say, students proposed actions or made assertions in contexts that proved to have higher conversational complexity. The exchanges of higher complexity in which students did not participate often contained extended discussions among participating tutors that pursued alternative perspectives on the problem (such as whether or not a limit problem could be solved without resorting to l'Hôpital's Rule).

Table 10. Number of exchanges containing student proposals by location in thread

Topic	No assertion or proposal	Within initial posting only	Within initial and subsequent postings	Within subsequent posting only
Limit	31	45	15	9
Related Rates	21	54	13	12

Failing to contribute to the construction of a solution is an indication that the student is relying exclusively on others to make suggestions and provide information. That is, the student is acting as a passive recipient in the exchange rather than taking initiative and assuming responsibility for understanding. Figure 6 contains an exchange with a complexity index of 6 on related rates in which the student, **ihatecalc**, poses a query on how to find rates of change for a boat being pulled toward a dock and positioned her/himself in this manner.

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related rates: A dinghy is pulled toward a dock by a rope...

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

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related rates: A dinghy is pulled toward a dock by a rope...

by **ihatecalc** on Thu Dec 14, 2006 3:05 am

ihatecalc
New Member
Posts: 23
Joined: Mon Sep 11, 2006 9:10 pm

This question just makes NO sense to me:

A dinghy is pulled toward a dock by a rope from the bow through a ring of the dock 6 feet above the bow, as shown in the figure. Rhe rope is hauled in at the rate of 2 ft/sec.

a) How fast is the boat approaching the dock when 10 ft of rope are out?
b) St what rate is angle theta changing at that moment?

by **galactus** on Thu Dec 14, 2006 3:22 am

galactus
Elite Member
Posts: 4013
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Where's our picture as mentioned in the problem?.

Draw a triangle from the ring to the bow and then to a point 6' below the ring. You have a righ triangle.

Let x=the distance from the dock to the bow
Let y=the hypoteneuse of the triangle, that is, the length of the rope from the ring to the bow

For part a, you want dx/dt given that dy/dt=-2 ft/sec.

Use Pythagoras. $x^2 + 36 = y^2$

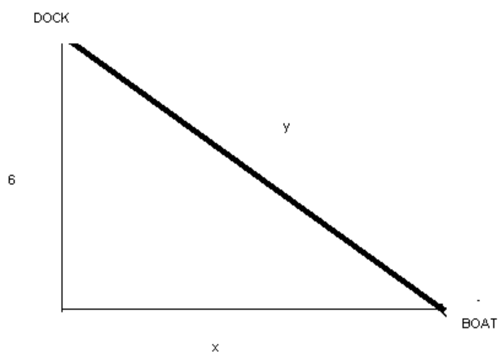
Differentiate: $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$\frac{y}{x} \cdot \frac{dy}{dt}$.....[1]

Using Pythagoras and y=10, we find x=8.

Using [1], $\frac{dx}{dt} = \frac{10}{8}(-2) = \frac{-5}{2} \quad ft/sec$

Now, you try an tackle the second part. OK?.



by **ihatecalc** on Thu Dec 14, 2006 4:24 am

is there a formula i'm supposed to use to find theta? :(i really think im too stupid to be taking AP calc.. lol

ihatecalc
New Member

Posts: 23
Joined: Mon Sep 11, 2006 9:10 pm

by **galactus** on Thu Dec 14, 2006 11:54 am

Yes, there is a formula, so to speak. Use your trig functions.

Try $sin(\theta)$

Look at the diagram. You know your trig functions, I hope.

You have $\frac{dy}{dt} = -2$

You want $\frac{d\theta}{dt}$ when y=10

$sin(\theta) = \frac{6}{y}$

Now, finish up?.

galactus
Elite Member

Posts: 4013
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Figure 6. Student adopts passive position in exchange

Analysis of exchange (passive)

Makes NO sense. In this exchange, the first indication of the position that **ihatecalc** is adopting [3:05 am][24] can be found in the way in which the query is posed ("This question just makes NO sense to me") and her/his failure to include an attempt at a solution to either part of the question: "a) How fast is the boat approaching the dock when 10 ft of rope are out?; and, b) St what rate is angle theta changing at that moment?"[25] Even though **ihatecalc** has access to a figure that was provided with the problem statement ("A dinghy is pulled toward a dock by a rope from the bow through a ring of the dock 6 feet above the bow, as shown in the figure."), s/he does not initiate a solution by defining variables corresponding to the problem situation or by suggesting relevant geometric relationships. A tutor, **galactus**, responds to the query [3:22 am] with a worked solution for the first part of the problem (constructing a diagram to take the place of the one that **ihatecalc** omitted: "Draw a triangle from the ring to the bow and then to a point 6' below the ring. You have a righ triangle. Let x=the distance from the dock to the bow. Let y=the hypoteneuse of the triangle, that is, the length of the rope from the ring to the bow") and then positions **ihatecalc** to take some initiative by suggesting that s/he should now attempt the second part of the problem, presumably using this solution as a model: "Now, you try an tackle the second part. OK?").

Fishing. **Ihatecalc**, however, does not take up this opportunity [4:24 am] and instead fishes for information ("is there a formula i'm supposed to use to find theta?"), accompanied by a self-depreciating remark that is consistent with this student's choice of a user name in the forum: "i really think im too stupid to be taking AP calc.. lol". **Galactus** responds [11:54 am] by providing the requested information ("Try $\sin(\theta)$") and setting up the problem by specifying the

[24] All times shown are Greenwich Mean Time, the forum display default.
[25] All quotations from exchanges are reproduced as written in the original forum context.

known and unknown rates of change ("You have $\frac{dy}{dt} = -2$. You want $\frac{d\theta}{dt}$ when y=10"), as well as the geometric relationship between the variables ("$\sin(\theta) = \frac{6}{y}$"). At this point, the only solution steps left for **ihatecalc** are implicit differentiation and substitution, and **ihatecalc** does not return to the exchange to propose a final answer following completion of these actions. In terms of positioning, then, this exchange can be characterized as one in which the student positioned her/himself as incompetent, and, although the tutor presented an opportunity for the student to take a more active part in the discussion, this effort was not successful so that all of the key elements of the problem solutions were contributed by the tutor.

In contrast, consider the following interaction between a student, **Kristy**, and tutors, **honey** and **galactus**, in an exchange with a complexity index of 15 on a similar problem involving the rate at which a boat is approaching a dock shown in Figure 7. Rather than positioning herself as a passive recipient of information, **Kristy** participates in the joint construction of the solution by making assertions and proposing mathematical actions throughout the discussion, making connections to previous learning, and evaluating the state of her understanding.

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Another moving ship problem I'm struggling with.

Moderators: tkhunny, Gene, stapel, Ted, galactus

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Another moving ship problem I'm struggling with.

by **Kristy** on Tue Jul 10, 2007 2:03 am

Kristy
Junior Member

Posts: 57
Joined: Wed Jun 27, 2007 12:24 am

A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/x, how fast is the boat approaching the dock when it is 8m from the dock?

(I'm trying to get my work to show up, I tried this way to draw it and it looked terrible so just a minute and you'll be seeing another post with more work.

Edit:

Okay, the vertical distance is 1 meter and is fixed.

The horizontal disstance between the dock and the boat is 8 m and is decreasing

Then I have the velocity for the hypotenuse as decreasing by 1 meter per second.

(I'm still trying to work on it.....)

Last edited by Kristy on Tue Jul 10, 2007 2:14 am, edited 1 time in total.

by **honey** on Tue Jul 10, 2007 2:12 am

honey
Elite Member

Posts: 5040
Joined: Wed Apr 13, 2005 2:56 am
Location: PA, USA

In your drawing, did you find a right triangle?

by **Kristy** on Tue Jul 10, 2007 2:15 am

Kristy
Junior Member

Posts: 57
Joined: Wed Jun 27, 2007 12:24 am

A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/x, how fast is the boat approaching the dock when it is 8m from the dock?

> **honey wrote:**
> In your drawing, did you find a right triangle?

Yes.

The base is 8m, and the height is 1 m.

Edit:

...................|
...................| 1 meter fixed
...................|
...................|
____
8m

This program is so frustrating. I had spaces because the line is supposed to be to the right, and it completely removed my spaces.
8 m and changing. Okay, ignore the ... those are just to fill in space.

So yes I do have a right triangle and I'm just trying to figure out how to use the stuff I used on the last problem to solve this.

Let x be the distance between the boat and the dock.
8-x at any particular time.
Okay here is where I'm getting confused applying the previous problem's knowledge. I think I should take 8-x and take the derivative, but I'm obviously missing something. I know the rate of change of the rope, and I know the vertical distance isn't changing. Somehow, hte horizontal distance must be related to the hypotenuse rope pulling in. I'm lost though. :(

Still struggling

by **Kristy** on Tue Jul 10, 2007 2:37 am

Kristy
Junior Member

Posts: 57
Joined: Wed Jun 27, 2007 12:24 am

$\frac{dz}{dt} = -1$

$\frac{dy}{dt} = 0$

$z^2 = (x)^2 + y^2$

$z^2 = (x)^2 + 1^2$

$z^2 = (x)^2 + 1$

Differentiate:

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 0$ this is eqn [1]

Okay, well I managed to make somethig look okay by copying and pasting from someonoe else using that "tex" notatoion. But I haven't gotten much further understanding this. I have an idea that I should solve for

$\frac{dx}{dt}$ =??? at x=8

by **galactus** on Tue Jul 10, 2007 3:05 am

This is a lot like the other problem.

You have another pythagoras deal.

$z^2 = x^2 + y^2$

Differentiate:

$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

$z = \sqrt{8^2 + 1^2} = \sqrt{65}$

$\sqrt{65}(1) = (8)\frac{dx}{dt} + (y)(0)$

Note dy/dt remains constant, so it is 0.

Solve the above equation for dx/dt.

galactus
Elite Member

Posts: 4016
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

by **Kristy** on Tue Jul 10, 2007 3:11 am

> **galactus wrote:**
> This is a lot like the other problem.
>
> You have another pythagoras deal.
>
> $z^2 = x^2 + y^2$
>
> Differentiate:
>
> $z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$
>
> So this equation z= is more to help me know what to put on the next line (like the line above and below are more related?
>
> $z = \sqrt{8^2 + 1^2} = \sqrt{65}$
>
> $\sqrt{65}(1) = (8)\frac{dx}{dt} + (y)(0)$
>
> Note dy/dt remains constant, so it is 0.
>
> Solve the above equation for dx/dt.

Kristy
Junior Member

Posts: 57
Joined: Wed Jun 27, 2007 12:24 am

by **Kristy** on Tue Jul 10, 2007 3:12 am

Kristy
Junior Member

Posts: 57
Joined: Wed Jun 27, 2007 12:24 am

> **galactus wrote:**
> This is a lot like the other problem.
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>
> $z = \sqrt{8^2 + 1^2} = \sqrt{65}$
>
> $\sqrt{65}(1) = (8)\frac{dx}{dt} + (y)(0)$
>
> Note dy/dt remains constant, so it is 0.
>
> Solve the above equation for dx/dt.

$\sqrt{65}(1) = (8)\frac{dx}{dt} + (y)(0)$

$\sqrt{65} = (8)\frac{dx}{dt}$

$\frac{sqrt65}{8} = \frac{dx}{dt}$

=8.062257748/8

=1.007782219

Does this seem like I'm doing the right thing at all????
It just seems like an approximately right answer (the height isn't very much), and so it would make sense that the boat is moving toward the dock at a pretty similar rate as the rope is pulling it in, but I just don't know if I've done everythin right.

by **Kristy** on Tue Jul 10, 2007 3:30 am

Kristy
Junior Member

Posts: 57
Joined: Wed Jun 27, 2007 12:24 am

Is this right? I feel fairly certain that I did do the right thing (after being confused on how to set it up intially.)

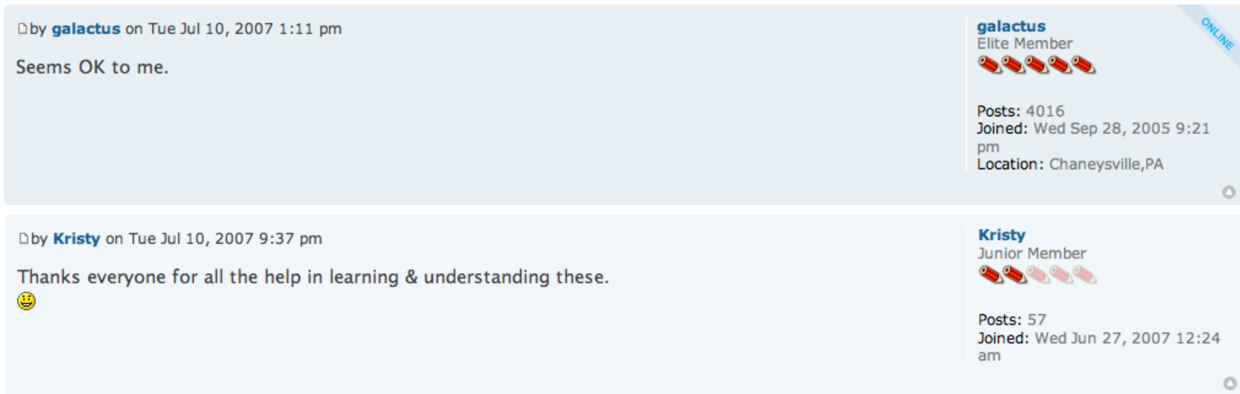


Figure 7. Student adopts active position in exchange

4.2.1.1 Analysis of exchange (active)

I'm trying. In this exchange, **Kristy** begins by posing the query and expressing her attempt to present a diagram of the problem situation: "I'm trying to get my work to show up, I tried this way to draw it and it looked terrible so just a minute and you'll be seeing another post with more work." [2:03 am]. A short time later (11 minutes), she edits this post with a verbal description of the diagram that she has constructed: "Okay, the vertical distance is 1 meter and is fixed. The horizontal disstance between the dock and the boat is 8 m and is decreasing. Then I have the velocity for the hypotenuse as decreasing by 1 meter per second." With these initial contributions, **Kristy** positions herself as someone who is working to solve the problem, despite the shortcomings of the user interface.[26]

What I've learned. Prior to the publication of **Kristy's** verbal description of her diagram, a tutor, **honey**, enters the discussion [2:12 am] and poses a hint in the form of a question: "In your drawing, did you find a right triangle?" By drawing attention to the presence of a right triangle, **honey**, is positioning **Kristy** to continue to contribute to the solution by noting that the

[26] The forum (like all others that we have encountered) does not contain a tool for constructing diagrams so that participants who wish to include figures must either create, and then upload, constructions from other software, or resort to "ascii art."

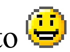
variables are related by the Pythagorean theorem. **Kristy** replies in the affirmative [2:15 am] and describes the right triangle: "Yes. The base is 8m, and the height is 1 m." In this exchange, she also expresses a desire to make use of help that she has received on another problem ("So yes I do have a right triangle and I'm just trying to figure out how to use the stuff I used on the last problem to solve this."); proposes an assignment of variables ("Let x be the distance between the boat and the dock. 8-x at any particular time."); and presents a meta-analysis of her understanding of the connection between the problems ("Okay here is where I'm confused applying the previous problem's knowledge. I think I should take 8-x and take the derivative, but I'm obviously missing something. I know the rate of change of the rope, and I know the vertical distance isn't changing. Somehow, hte horizontal distance must be related to the hypotenuse rope pulling in. I'm lost though."). With these actions, **Kristy** not only establishes herself as an active participant in the discussion of this particular problem but also positions herself more broadly as a calculus learner who is trying to build connections between problem situations: The "last problem" was posted by **Kristy** on the forum 47 minutes earlier and refers to a problem (also about ships, hence the current thread title "Another moving ship problem I'm struggling with") in which two ships are sailing away from one another (north- and eastbound) with certain velocities and the rate at which the distance between them is changing at a certain time is sought.

I'm making headway. A short time later in the thread on the boat approaching the dock, **Kristy**, posts another contribution [2:37 am] in which she specifies the rates of change of the length of the rope and the height of the dock ($\frac{dz}{dt} = -1$ and $\frac{dy}{dt} = 0$); establishes a relationship between the variables ($z^2 = (x)^2 + y^2$); differentiates the equation that relates the variables ($2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 0$); and, proposes a solution goal ("I have an idea that I should solve for

$\frac{dx}{dt}$ = ??? at x=8"). In this update on her understanding of the problem, **Kristy** provides a large portion of the solution concerning the dynamic aspect of the problem. There are correct values assigned to the rates of change, and the relationship that connects them is also described. The piece of the solution that is lacking concerns the static aspect of the problem, that is, the values of the variables at the time in question: The length of the rope (represented by z) needs to be specified at the time when the boat is 8 meters from the dock, which is 1 meter above the boat's bow. In response to this contribution, another tutor, **galactus**, enters the exchange [3:05 am] and refers to the similarity between this problem and one for which **Kristy** previously received help: "This is a lot like the other problem. You have another pythagoras deal." After specifying the relationship in terms of the same variables that **Kristy** introduced ("$z^2 = x^2 + y^2$"), **galactus** differentiates the equation ("$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$"); uses the relationship to determine the length of the rope at the moment in question ("$z = \sqrt{8^2+1^2} = \sqrt{65}$"); and, coordinates the dynamic and static information ("$\sqrt{65}(1) = (8)\frac{dx}{dt} + (y)(0)$"). With this presentation, **galactus**, explicitly provides the information that **Kristy** was lacking, namely the length of the rope when the boat is 8 meters from the dock, and shows how this information should be coordinated with the relationship between the variables and their rates of change.

Is this why? Six minutes later, **Kristy**, returns to the discussion [3:11 am] and (rhetorically) questions precisely this aspect of the solution, setting off her remark in red: "So this equation z= is more to help me know what to put on the next line (like the line above [$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$] and below [$z = \sqrt{8^2+1^2} = \sqrt{65}$] are more related?" That is, following **galactus's** intervention, **Kristy** appears to be cognizant of the way in which the static and dynamic aspects of the problem situation are coordinated and positions herself as someone seeking to understand the help being received in addition to arriving at the correct answer to the problem.

Sense-making. One minute later, **Kristy**, posts another contribution [3:12 am] containing computations that produce the sought-after value for $\frac{dx}{dt}$: "1.007782219." In addition, **Kristy** responds to her own question of whether she is "doing the right thing at all" by assessing the plausibility of this numerical value with respect to the physical situation: "It just seems like an approximately right answer (the height isn't much), and so it would make sense that the boat is moving toward the dock at a pretty similar rate as the rope is pulling it in, but I just don't know if I've done everythin right." This sense-making move in a patently contrived problem statement is remarkable since **Kristy** may hail from landlocked Arizona versus coastal Maine. Regardless of her past experience, however, **Kristy** is placing herself within the problem situation (either as an actor or spectator), playing out the scenario and recognizing constraints on the solution space.

Am I right? Eighteen minutes later, having received no reply, **Kristy** posts another message [3:30 am] seeking confirmation of the accuracy of the solution: "Is this right? I feel fairly certain that I did do the right thing (after being confused on how to set it up initially.)" Here, **Kristy** demonstrates an increase in confidence as she assesses the construction of the solution, which she takes ownership of ("I feel fairly certain that *I* did do the right thing…" [italics added]. Several hours later, **galactus** responds [1:11 pm] and confirms **Kristy's** conclusion: "Seems OK to me." The final posting in this exchange is made by **Kristy** [9:37 pm] as she expresses appreciation for the forum and the effective help that she has received: "Thanks everyone for all the help in learning & understanding these." In terms of positioning, this exchange is one in which the student adopts a position as an avid learner who makes assertions and proposes mathematical actions, works to understand the contributions of others, repeatedly assesses the state of her understanding, and engages in sense-making once a numerical result is produced. The interactions between the student and the forum tutors were instrumental in moving **Kristy** from being "lost" to an acknowledged better understanding of related rates problems, reflected in the progression of emoticons from  [Sad] to  [Smile] over the course of this exchange.

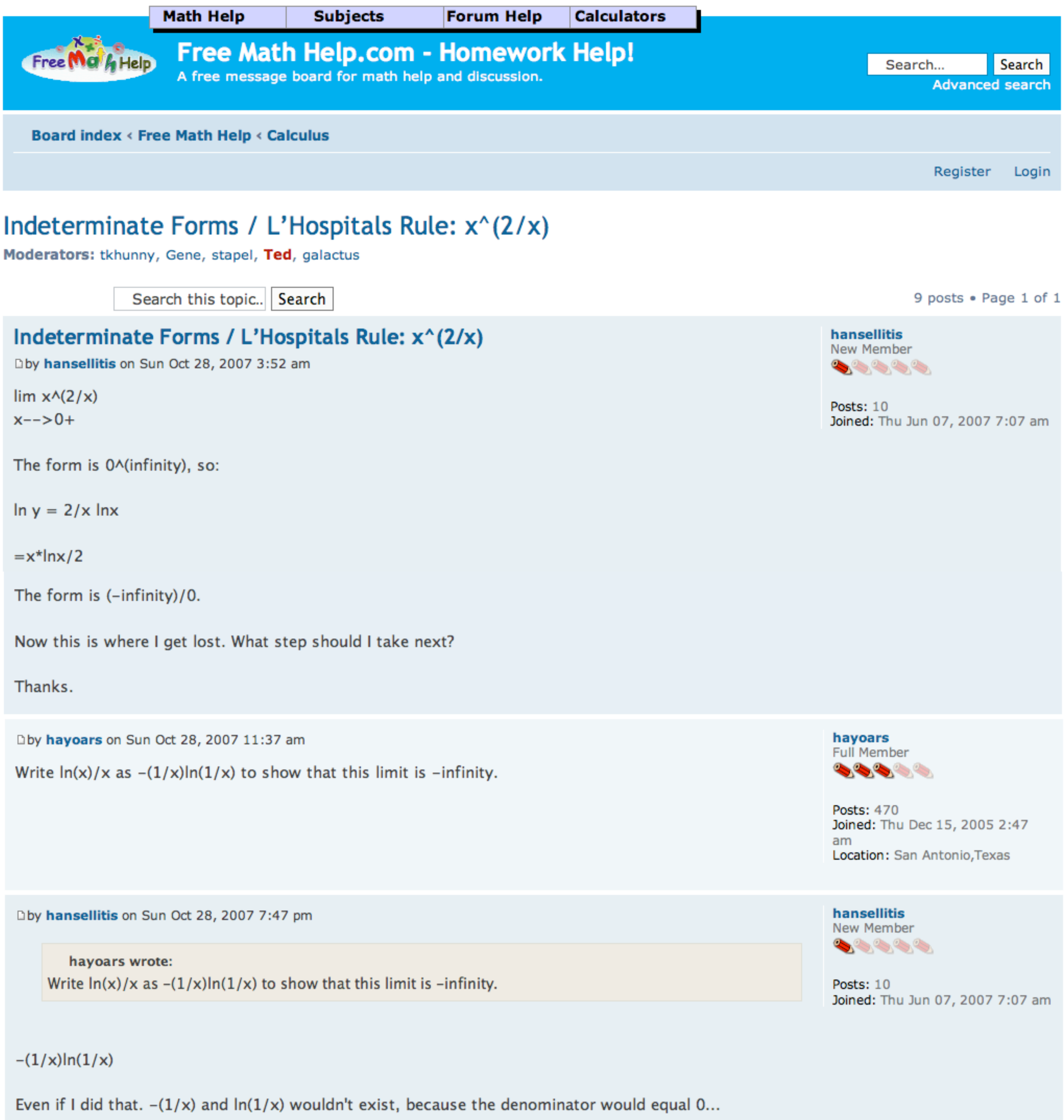
4.2.1.2 Summary of student assertions

These two exchanges on essentially the same related rates exercise exemplify the ways that students can position themselves with authority and a voice in tutoring encounters on the forum. In the first example, we see the student (**ihatecalc**) positioning her/himself as mathematically incompetent (*Makes NO sense*) and attempting to weasel information out of participating tutors (*Fishing*). The guiding intention of the student in this case appears to be to use the forum as a means of getting an answer to the problem. The second exchange exudes a remarkably different flavor in that the student (**Kristy**) acts as a partner in the exchange by proposing mathematical actions (*I'm trying*), making connections with other problems (*What I've learned*), persisting in her efforts (*I'm making headway*), engaging in self-explanation (*Is this why?*), analyzing the answer with regard to the physical situation (*Sense-making*), and repeatedly assessing her understanding of the solution (*Am I right?*). Here, the guiding intention of the student appears to be the construction of a mental model for understanding related rates. Taken together, the quantitative results on student assertions and proposals (Table 10) and the analysis of this latter exchange, demonstrate that the forum environment does not inhibit students acting with authority in tutoring exchanges. Indeed, an open, online homework forum may support this type of activity in ways that computer-tutoring systems cannot. Although these systems often provide feedback on the accuracy of student responses and sometimes produce hints based on student input, they do not generally allow students to make spontaneous assertions and proposals for actions in the way that is evident in the forum interactions.

4.2.2 Student questions or challenges assertion or proposal made by others

Contributions that question or challenge others' assertions or proposals are another indication of the way participants are positioned in the interaction. When a forum tutor offers advice, constructs part of a solution, or produces a hint, a student can either accept or question the information (just as contributions are either accepted or rejected in Clark's (1996) model of conversation). A student adopting a position as an active participant in a forum exchange may ask questions or challenge contributions as part of a self-regulatory learning strategy and in order to repair knowledge deficits.

As we examine lengthy exchanges, we can see students challenging ideas in distinctive ways. Nineteen percent of the exchanges on limit and 20% of those on related rates contained a contribution in which the student questioned or challenged the contribution of a forum tutor. Only one of these exchanges in each topic had a conversational complexity index less than 7, and both of these had a participation code of 1212 (so complexity index of 6). In other words, when a student introduced a question or challenge into a discussion, the outcome was an extension rather than a termination of the conversation. Figure 8 contains an exchange with a conversational complexity index of 18 in which the student, **hansellitis**, questions the contributions of three tutors, **hayoars**, **pastel**, and, **skeeter**. Each time that **hansellitis** questions the advice and information that s/he receives (in the third, fifth, and seventh posting of the exchange with participation code 121314141), another tutor responds. This interleaving of responses from multiple tutors creates, in effect, a single tutorial voice that steps **hansellitis** through the solution of this limit problem while touching on the nature of the limit concept, the domain of the logarithm function, and the meaning of indeterminate forms.



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Indeterminate Forms / L'Hospitals Rule: x^(2/x)

Moderators: tkhunny, Gene, stapel, Ted, galactus

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9 posts • Page 1 of 1

Indeterminate Forms / L'Hospitals Rule: x^(2/x)

by **hansellitis** on Sun Oct 28, 2007 3:52 am

lim x^(2/x)
x-->0+

The form is 0^(infinity), so:

ln y = 2/x lnx

=x*lnx/2

The form is (-infinity)/0.

Now this is where I get lost. What step should I take next?

Thanks.

hansellitis
New Member
Posts: 10
Joined: Thu Jun 07, 2007 7:07 am

by **hayoars** on Sun Oct 28, 2007 11:37 am

Write ln(x)/x as -(1/x)ln(1/x) to show that this limit is -infinity.

hayoars
Full Member
Posts: 470
Joined: Thu Dec 15, 2005 2:47 am
Location: San Antonio,Texas

by **hansellitis** on Sun Oct 28, 2007 7:47 pm

> **hayoars wrote:**
> Write ln(x)/x as -(1/x)ln(1/x) to show that this limit is -infinity.

-(1/x)ln(1/x)

Even if I did that. -(1/x) and ln(1/x) wouldn't exist, because the denominator would equal 0...

hansellitis
New Member
Posts: 10
Joined: Thu Jun 07, 2007 7:07 am

by **pastel** on Sun Oct 28, 2007 8:00 pm

> **hansellitis wrote:**
> Even if I did that. -(1/x) and ln(1/x) wouldn't exist, because the denominator would equal 0...

"Taking the limit" is not the same as "evaluating".

You are not being asked to evaluate the expression at x = 0; you are being asked to find the limit as x tends toward zero (gets very close to zero) from the right. So try taking that limit!

E.

pastel
Elite Member

Posts: 9682
Joined: Wed Feb 04, 2004 2:16 pm

by **hansellitis** on Sun Oct 28, 2007 10:29 pm

> **pastel wrote:**
>> **hansellitis wrote:**
>> Even if I did that. -(1/x) and ln(1/x) wouldn't exist, because the denominator would equal 0...
>
> "Taking the limit" is not the same as "evaluating".
>
> You are not being asked to evaluate the expression at x = 0; you are being asked to find the limit as x tends toward zero (gets very close to zero) from the right. So try taking that limit!
>
> E.

still though..

lim x^(2/x)
x-->0+

=(2*lnx)/x has the form -infinity/0, and the denominator approaches 0, so we need to differentiate again, right?

which gives us 2/x So we need to take the differentiate once more since it has the indeterminate type 0/0.

And we get 0/1 which is 0.

So

lim x^(2/x) = lim e^0 = 1
x-->0+ x-->0+

Is that right? Because in the graph, it appears x^(2/x) approaches 0.

What did I do wrong?

hansellitis
New Member

Posts: 10
Joined: Thu Jun 07, 2007 7:07 am

by **skeeter** on Sun Oct 28, 2007 10:48 pm

> **hansellitis wrote:**
> still though..
>
> lim x^(2/x)
> x-->0+
>
> =(2*lnx)/x has the form -infinity/0, and the denominator approaches 0, so we need to differentiate again, right?
>
> no ... if you get -inf/0, the limit is -inf ... you can't use L'Hopital again.
>
> However, there is an out ... note that lny -> -inf, so, what does y approach?

skeeter
Senior Member

Posts: 2138
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **hansellitis** on Sun Oct 28, 2007 10:53 pm

> **skeeter wrote:**
>> **hansellitis wrote:**
>> still though..
>>
>> lim x^(2/x)
>> x-->0+
>>
>> =(2*lnx)/x has the form -infinity/0, and the denominator approaches 0, so we need to differentiate again, right?
>>
>> no ... if you get -inf/0, the limit is -inf ... you can't use L'Hopital again.
>>
>> However, there is an out ... note that lny -> -inf, so, what does y approach?

e^-inf = 0

ahhhh thanks. But as the denominator approaches 0, doesn't x approach 0? Making it a non real answer?

hansellitis
New Member

Posts: 10
Joined: Thu Jun 07, 2007 7:07 am

by **skeeter** on Sun Oct 28, 2007 11:20 pm

x is approaching 0 from the right ... correct?

skeeter
Senior Member

Posts: 2138
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **hansellitis** on Sun Oct 28, 2007 11:25 pm

> **skeeter wrote:**
> x is approaching 0 from the right ... correct?

Oh my. haha, I'm so dumb.

So as x approaches 0 from the right, dividing by small values of x increases the quotient (2 lnx)/x.

Thanks for your help, my brain wasn't functioning correctly.

hansellitis
New Member

Posts: 10
Joined: Thu Jun 07, 2007 7:07 am

Figure 8. Student challenges others' contributions

4.2.2.1 Analysis of exchange (challenges)

Next step? The subject header of the thread flags this limit problem as one that pertains to indeterminate forms and that may appeal to l'Hôpitals Rule in the solution. After presenting the problem [3:52 am], **hansellitis** makes an assertion about the form that $\lim_{x \to 0^+} x^{2/x}$ takes and proposes a solution strategy for indeterminate forms of this type: "The form is 0^(infinity), so: ln y = 2/x lnx = x*lnx/2.[27] Having proposed that the limit of the logarithm of the function be examined, **hansellitis** next evaluates the form of this expression ("The form is (-infinity/0)") but is expresses uncertainty on how to proceed: "Now this is where I get lost. What step should I take next?" The query of this exchange, therefore, focuses on the interpretation or meaning of $-\infty/0$, a form that is *not* indeterminate but closely resembles forms such as $-\infty/-\infty$, ∞/∞, and $0/0$ that are.

Reorganize–why? The first response [11:37 am] comes from **hayoars** who provides a re-organization of the expression $\ln(x)/x$ as $-(1/x)\ln(1/x)$ based on the laws of logarithmic functions. This re-organization reframes the limit so that, instead of the form $-\infty/0$, it takes on the form $-\infty \cdot \infty$, a form which **hayoars** presumably views as one that **hansellitis** can more readily interpret as a limit that approaches $-\infty$. **Hansellitis**, however, questions this advice [7:47 pm]: "Even if I did that. –(1/x) and ln(1/x) wouldn't exist, because the denominator would equal 0…" Rather than accepting the proposal of the tutor (as an authority) and applying the supplied

[27] There is an error in the final expression of this equality. This is presumably a typographical mistake (reversal of 2 and x) since the form of the expression is correct in the next line ("-infinity/0"), and the expression (correctly) appears as (2*lnx)/x in a subsequent posting [10:29 pm].

information (that the limit approaches $-\infty$) to finish the problem solution, the student queries the usefulness of the mathematical move *and* presents a justification for the objection.

Reframe–New perspective. On the basis of this justification, another tutor, **pastel**, enters the exchange [8:00 pm], focusing on the way that **hansellitis** has framed the limit as substitution in her/his objection: “Taking the limit is not the same as “evaluating”. You are not being asked to evaluate the expression at x = 0; you are being asked to find the limit as x tends toward zero (gets very close to zero) from the right. So try taking the limit! ” Here, **hansellitis** is positioned as someone who has demonstrated a ‘shocking’ interpretation of the limit, but who, nevertheless, is capable of adopting a perspective that is consistent with the mathematical definition of limit and making use of this to complete the solution of the problem. In other words, the student is not stripped of the authority that s/he has assumed in the exchange, despite the weakness and invalidity of the proposed arguments.

Reframing–sort of. In the next contribution to the thread [10:29 pm], **hansellitis** demonstrates that s/he has adopted the framing of limit as values that approach (rather than substitution) *but* returns to her/his original framing of the problem as one of form $-\infty/0$: “still though.. lim $_{x -- >0+}$ x^(2/x) = (2*lnx)/x has the form - infinity/0, and the denominator *approaches* 0 (italics added), so we need to differentiate again, right?” In this posting, **hansellitis** again challenges the assertions of the forum tutors and positions her/himself with conceptual agency by seeking consistency across representations. S/he argues that, even if the limit is framed as a dynamic concept, a problem remains since the result of the calculus operations (applying l’Hôpital’s Rule repeatedly) is inconsistent with the graph of the function (which suggests a limit value of 0): “lim $_{x -- >0+}$ x^(2/x) = lim $_{x -- >0+}$ e^0 = 1 Is that right? Because in the graph, it appears x^(2/x) approaches 0. What did I do wrong?”

Reframing refined. A third tutor, **skeeter**, responds [10:48 pm] and immediately identifies the fallacy in **hansellitis's** argument: "no … if you get - inf/0, the limit is - inf … you can't use L'Hopital again." The form "- inf/0" is not indeterminate as **hansellitis** is assuming and therefore does not meet the conditions of use for l'Hôpital's Rule. **Skeeter** concludes the post with the proposal of an alternative action that presupposes the limit of the expression as negative infinity: "However, there is an out … note that lny -> - inf, so, what does y approach?" Once again, **hansellitis** is given authority to draw conclusions and positioned to contribute to a joint construction of the solution.

Answers and more questions. When **hansellitis** returns to the exchange [10:53 pm], s/he takes up **skeeter's** proposal and produces the solution to the problem: "e^-inf = 0". However, after expressing satisfaction that the representations now agree ("ahhhh thanks"), **hansellitis** voices another challenge: "But as the denominator approaches 0, doesn't x approach 0? Making it a non real answer?" Even though s/he has arrived at an answer to the limit problem *and* has converging evidence for its accuracy, **hansellitis** is not content to end the discussion and, using language consistent with framing the limit as approaching values, questions the construction of a solution that might produce "non real" results given the functions involved. **Skeeter** [11:20 pm] responds concisely with a single leading question that draws attention to the direction of approach since the domain of the natural logarithm is the set of positive real numbers: "x is approaching 0 from the right … correct?"

Ah, yes. The final post in the exchange [11:25 pm] indicates that this pedagogical move was helpful as **hansellitis** produces a correct justification of the conclusion: "So as x approaches 0 from the right, dividing by small values of x increases the quotient (2 lnx)/x." We can infer that the direction of approach to the limit had receded to the background in **hansellitis's** framing of the problem as s/he grappled with the form of the limit (whether it is indeterminate). In sum, we see that, as a consequence of this forum interaction in which **hansellitis** repeatedly questions the help provided by the tutors, s/he moves from a misclassification (and mistreatment) of –infinity/0 as an indeterminate form to an understanding of the behavior of limits that have this form, from a framing of limit as substitution to limit as approaching values, and from an uncoordinated treatment of the value of approach and the domain of the function to an analysis that brings together these aspects of limit.

4.2.2.2 Summary of student challenges

This exchange on a routine calculus exercise on limit illustrates how students can position themselves with authority in tutoring interactions on the forum by questioning and challenging the contributions of others. Approximately 20% of the exchanges contained a challenge that was initiated by the student, showing that the students are not shy about calling into question the information they receive in this context. In the example, the student, **hansellitis**, repeatedly presses the tutors on the mathematical information that they contribute during the joint construction of the solution (*Reorganize–why?*, *Answers and more questions*). The result is an extended conversation between multiple forum participants in which several unproductive framings of limit are addressed (*Reframe–New perspective, Reframing–sort of, Reframing–-refined*), so that, at the conclusion of the interaction, the student exhibits an improved and more coordinated understanding of the limit concept (*Ah, yes*). The patience and politeness that characterize this tutoring exchange are particularly noteworthy as an indication of how the forum functions as a tutoring environment in which students can safely challenge the contributions of more experienced others. Furthermore, the sense one gets from observing such exchanges is that this activity is supported rather than being inhibited, consistent with the finding that exchanges containing student questions and challenges have high conversational complexity indices (reflecting extended conversations between multiple participants).

4.2.3 Student indicates that the issue has been resolved

In addition to making assertions or proposals for action and to challenging and questioning others' contributions, students can adopt a position of authority in an exchange by indicating that the issue being discussed has been resolved to their satisfaction. In the classroom, it is generally the teacher who is positioned to evaluate the understanding of the student(s) and makes the decision whether to continue or terminate a discussion. Similarly, in tutoring sessions, the tutor is the participant who assesses the understanding of the student and decides whether to extend the discussion or move on to the next topic. In contrast, in an open, online forum, it is the student who initiates the exchange and who is ultimately responsible for deciding whether the goal of the interaction has been achieved to her/his satisfaction.

The perceived importance of indicating resolution within an exchange is underscored by the existence of automated "thank you" responses in some online help forums. For instance, MathHelpForum.com appends a "thanks" button to each post so that members can, with the click of a mouse, generate a response reading "The following users thank [name of contributor] for this useful post: [name of member] [date]." This feature was introduced in the forum to support and encourage public recognition of the usefulness of member contributions.[28] However, because

[28] An exploration of the ways in which this feature contributes to forum practices (e.g. whether it curtails the inclusion of stronger forms of resolution such as discussions of *how* a contribution was helpful) is part of our larger research agenda.

not all forums include this feature (including the forum chosen for the current study)[29], and because students may indicate that an issue is settled in other ways, it is worthwhile to consider a broader range of resolution markers.

There are several ways that a participant can indicate that an issue has (or has not) been resolved. First of all, participants can be silent and opt not to further contribute to an exchange. Silence in computer-mediated exchanges may indicate acceptance or rejection of another's contributions and does not offer evidence for (or against) the achievement of resolution. Thus, in the forum discussions, if a student does not return to the exchange beyond the initial posting or following tutor interventions, it is not clear whether the student feels that the issue has been settled or not. We refer to exchanges of this type as "*hangers*" since other forum participants are, in some sense, left hanging regarding the helpfulness of their contributions. On the other hand, when a student does acknowledge tutors' contributions, they can do so in either a *weak* or *strong* manner. For instance, an expression of appreciation, such as "Thank you," indicates a weak level of resolution on the part of the participant since this may simply be a residual of polite manners, that is, a customary response to receiving assistance. In contrast, the contribution of mathematical actions (e.g. the presentation of a solution to the problem) and assessments (e.g. reflections on differences in understanding) are stronger indications that the issue has been resolved to the satisfaction of the student. Finally, an exchange can evince a *lack of resolution*, as when a student receives no response to a query or receives a refusal from forum tutors to provide further assistance. Figure 9 shows the number of exchanges for each topic in which resolution

[29] On November 21, 2007, there was a thread in the Administration Issues forum in which a tutor proposed the incorporation of a thank you button tied to member credit points. However, other forum tutors objected, arguing that this would reward the provision of full worked solutions over tutoring. The forum administrator took members' opinions into account and chose not to introduce such a feature.

could not be determined (hangers), in which resolution was evident and the strength of the expression (weak versus strong), and in which there was no resolution.

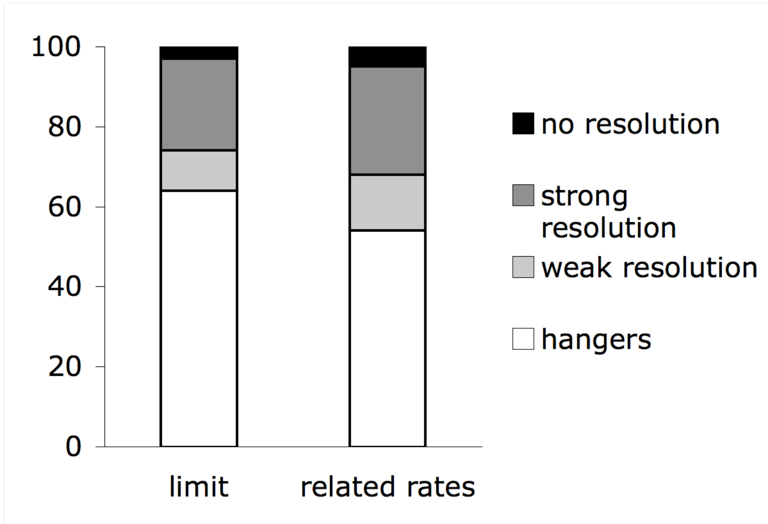


Figure 9. Student indications of resolution by topic

Although approximately 60% of the exchanges were "hangers" (with unspecified resolution), the majority of the remaining exchanges exhibited resolution from the student's perspective, in either a weak or strong manner. This means that nearly 80 of 200 exchanges show some level of resolution, surely a level that is higher than most classroom exchanges in which the teacher has little indication whether students "got it." In addition, the number of exchanges exhibiting characteristics of strong resolution outnumbered those in which only weak resolution was evident by a factor of two, a finding that is consistent with the amount of student activity in the forum. Over three quarters of the exchanges evincing weak resolution had low conversational complexity (index of 7 or less), so that exchanges in which a student received help and merely thanked the tutor(s) without demonstrating why the intervention was helpful were more likely to be brief transmissions of information rather than interactive discussions. Furthermore, there were very few exchanges (3 on limit and 5 on related rates) for which the issue was *not* resolved and in which the outcome of the exchange from the student's perspective could be characterized as inconclusive or unhelpful.

Figure 10 contains an exchange on limit in which the student, **johnk**, indicates the problem has been resolved to his satisfaction after contributions from two tutors, **pastel** and **skeeter**. Although the exchange is relatively brief, with a complexity index of 8, the interaction with the tutors is sufficient to help **johnk** resolve the problem within the initial framing that he had adopted (postings 1-3) as well as within an alternative framing that was introduced by the second tutor (postings 4-5).

[SPLIT] lim x->a (sinx - sina)/(x - a)

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

[SPLIT] lim x->a (sinx - sina)/(x - a)

by **johnk** on Sat Nov 10, 2007 2:29 pm

johnk
New Member

Posts: 23
Joined: Sat Jun 02, 2007 8:47 pm

We should do this without L'Hopital's rule.

The limit is:

$$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$$

Again the hardest part is probably figuring out the "right" substitution, anything I tried didn't seem to lead anywhere...

---

by **pastel** on Sat Nov 10, 2007 3:08 pm

pastel
Elite Member

Posts: 9748
Joined: Wed Feb 04, 2004 2:16 pm

Since the majority of the proofs that this limit is the cosine, and since they all use the "x + h" form, you might want to substitute "a + y" for "x", so you have:

. . . . . lim[y−>0] [(sin(a + y) − sin(a)] / [y]

Then use trig identities:

. . . . . sin(a + y) = sin(a)cos(y) + cos(a)sin(y) − sin(a)

. . . . . = sin(a)[cos(y) − 1] + cos(a)sin(y)

Split the limit into two pieces. As y −> 0, you have the sin(y)/y going to 1, and the [cos(y) − 1]/y going to zero. See if that helps. :)

E.

---

by **johnk** on Sat Nov 10, 2007 3:23 pm

johnk
New Member

Posts: 23
Joined: Sat Jun 02, 2007 8:47 pm

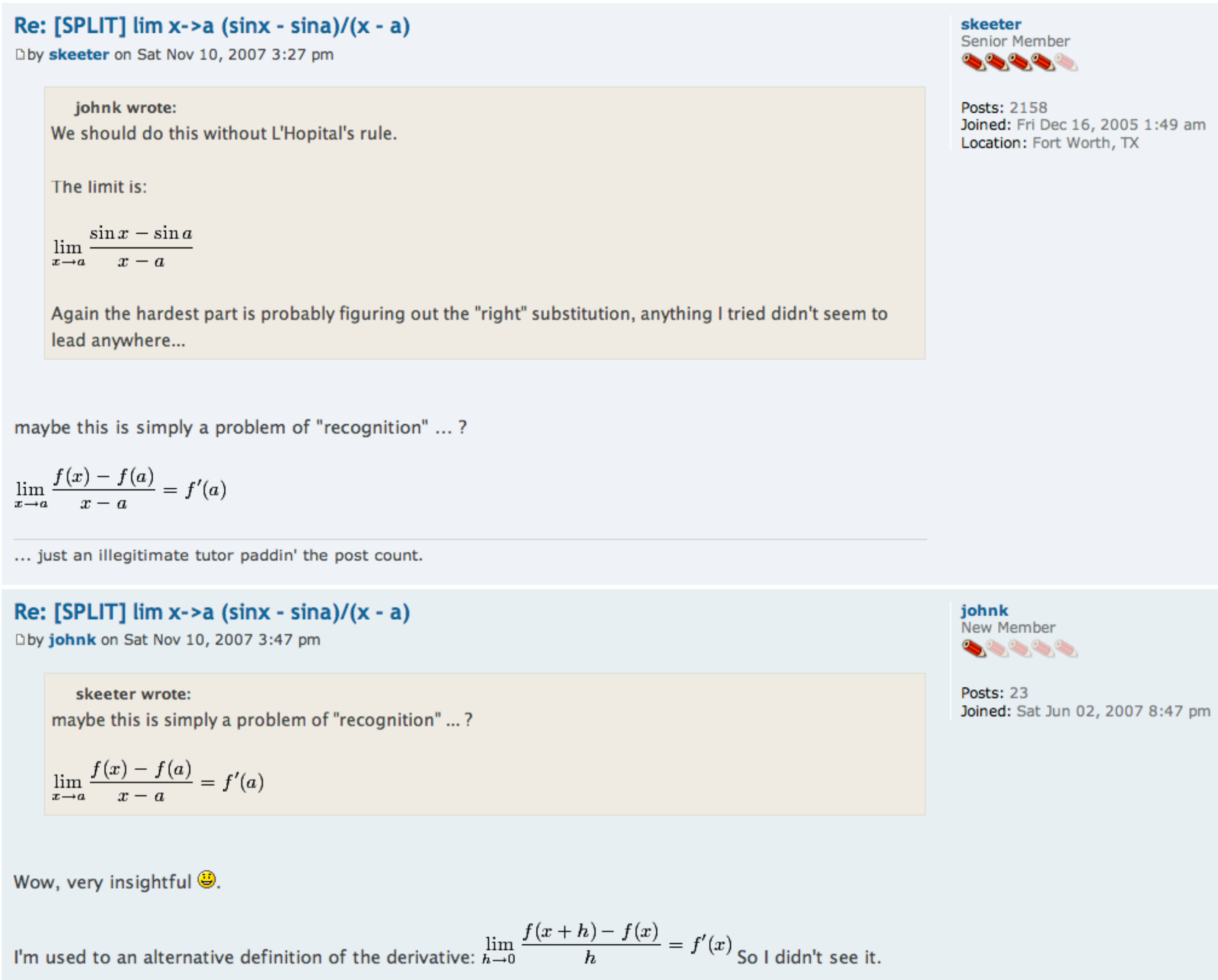
Thanks, E!

I actually tried that, but silly me didn't see that I could split it at the end.

So I get:

$$\lim_{y\to 0}\frac{\sin a\cdot(\cos y-1)}{y}+\lim_{y\to 0}\frac{\sin y\cdot\cos a}{y}=0+\lim_{y\to 0}\cos a=\cos a$$

Yay :D.



Re: [SPLIT] lim x->a (sinx - sina)/(x - a)
by **skeeter** on Sat Nov 10, 2007 3:27 pm

> **johnk wrote:**
> We should do this without L'Hopital's rule.
>
> The limit is:
>
> $\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$
>
> Again the hardest part is probably figuring out the "right" substitution, anything I tried didn't seem to lead anywhere...

maybe this is simply a problem of "recognition" ... ?

$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$

... just an illegitimate tutor paddin' the post count.

skeeter
Senior Member

Posts: 2158
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

Re: [SPLIT] lim x->a (sinx - sina)/(x - a)
by **johnk** on Sat Nov 10, 2007 3:47 pm

> **skeeter wrote:**
> maybe this is simply a problem of "recognition" ... ?
>
> $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$

Wow, very insightful.

I'm used to an alternative definition of the derivative: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$ So I didn't see it.

johnk
New Member

Posts: 23
Joined: Sat Jun 02, 2007 8:47 pm

Figure 10. Student indicates issue has been resolved

4.2.3.1 Analysis of exchange (resolved)

Framing–substitution. When **johnk** posts the query[30] [2:29 pm], he indicates that the desired solution to this limit, namely $\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$, should not appeal to l'Hôpital's Rule and proposes substitution as an alternative solution strategy: "Again the hardest part is probably figuring out the "right" substitution, anything I tried didn't seem to lead anywhere…" The first

[30] The original query contained two problems on limit that were "split" by the forum moderators into two separate threads.

tutor to respond, **pastel**, [3:08 pm] identifies the limit as an expression for cosine ("Since the majority of the proofs [use a form of this definition to show] that this limit is the cosine"); recommends a variable substitution ("and since they [the proofs] use the "x + h" form, you might want to substitute "a + y" for "x", so you have lim[y->0] [(sin(a + y)- sin(a)] / [y]"); provides a trigonometric identity for the expansion of the numerator ("sin(a + y) = sin(a)cos(y) + cos(a)sin(y) – sin(a)") [31]; and, finally, suggests reorganizing the expression ("Split the limit into two pieces.") to support the application of two well-known trigonometric limits ("As y->0, you have the sin(y)/y going to 1, and the [cos(y) – 1]/y going to zero.").

Yay. After receiving this detailed solution sketch (delivered with a friendly wink, ), **johnk** returns to the exchange [3:23 pm] and completes the remaining steps of the solution: "So I get: $\lim_{y\to 0}\frac{\sin a\cdot(\cos y-1)}{y}+\lim_{y\to 0}\frac{\sin y\cdot\cos a}{y}=0+\lim_{y\to 0}\cos a=\cos a$." In addition to this correct implementation of **pastel's** suggestions, **johnk** also indicates that the problem is resolved to his satisfaction by concluding the post with an expression of happiness or excitement: "Yay ." One would think that the exchange would end here; **johnk** has received help, jointly constructed a solution to the problem that he posed on the forum, indicated that he is pleased with the tutoring interaction, *and* acknowledged what piece of the tutoring was helpful: "I actually tried that, but silly me didn't see that I could split it at the end."

Framing–derivative. However, minutes later [3:27 pm] another forum tutor, **skeeter**, enters the exchange and proposes an alternative framing of the problem that satisfies the request for a solution method that does not rely on l'Hôpital's Rule: "maybe this is simply a problem of

[31] **Pastel** erroneously omits the second term of the numerator from the previous expression here. The equation should read "sin(a+y)-*sin(a)* = sin(a)cos(y)-cos(a)sin(y)-sin(a)."

"recognition" … ? $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$." Here, **skeeter** is framing the limit expression as the definition of the derivative of $\sin(x)$ at the value $x = a$, instead of framing the expression as a reorganization of trigonometric limits. Again, one might expect the exchange to end with this contribution, especially since the student, **johnk**, has demonstrated that the problem has been resolved to his satisfaction and therefore has no need to revisit or re-enter the exchange. This is not the case.

Wow. **Skeeter's** proposal draws a response from **johnk** [3:47 pm] that indicates an understanding of and appreciation for this alternative and novel perspective on the limit: "Wow, very insightful [Very Happy]. I'm used to an alternative definition of the derivative: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$ So I didn't see it." **Johnk** was apparently steeped in a particular form of the definition of the derivative, and this prevented him from adopting a framing of the problem as a derivative when it was first encountered. In terms of resolution, this final posting in the exchange is evidence that **johnk** has resolved the issue of this limit not once but *twice*: first with the help of **pastel** using a shared framing, and second with the help of **skeeter** by shifting to an alternative framing.

4.2.3.2 Summary of student resolution

This exchange exemplifies how students can position themselves with authority in the forum exchanges by taking initiative for demonstrating comprehension and providing indications of resolution. Although it is clearly the case that the forum tutors are leading the construction of the solution (in terms of introducing mathematical ideas and framings), the student is positioned to inform the community whether and how these contributions are helpful. That is, the student is the one who supplies information on the ways in which this exchange meets the goal of the forum to provide assistance on the joint construction of solutions to coursework problems. In this exchange, **johnk**, began without a single productive way of framing the limit and was able to construct two framings as a result of the interaction (*Framing–substitution, Framing–derivative*). Quantitatively, the low number of exchanges that demonstrate a definitive lack of resolution together with the large number of exchanges that exhibit some level of resolution is an indication that students experience a sense of closure after participating in a forum tutoring exchange.

4.3 PEDAGOGICAL MOVES

In addition to the ways that students position themselves in the forum exchanges (e.g. by making assertions and proposals for action, questioning and challenging others' contributions, and indicating that the issue has been settled), the tutors also contribute to the functional aspects of the activity through the selection and implementation of pedagogical moves. In this section, we describe four categories of pedagogical moves that were found in the forum exchanges: a) the tutor contributed mathematical information or advice to the construction of the solution or explanation; b) the tutor initiated a dialogue with the student inviting her/him to make inferences and draw conclusions leading to the problem solution; c) the tutor pressed the student to show her/his own work on the problem or clarify her/his meaning; or, d) the tutor referred the student to another resource (e.g. to a post involving a similar problem).

4.3.1 Tutor contributes mathematical information

As noted previously, the forum tutors generally have more mathematical experience and expertise than the students who pose the questions. Therefore, it is not surprising that the majority of the exchanges (94% limit and 93% related rates) contained at least one contribution from a tutor supplying mathematical information and advice pertinent to the construction of the solution posed by the student. However, the depth and extent of this information varied widely, from hints to solution sketches to partial or full worked solutions.

The exchanges in which the tutors contributed a full worked solution as the first intervention are particularly worth exploring since this practice could be antithetical to fostering student initiative and could relocate the forum as an answer service rather than as a tutoring environment. Table 11 shows the number of exchanges for each topic in which the student received a full worked solution from the first tutor who responded according to whether the complexity index was lower (less than or equal to 7) or higher (greater than 7).

Table 11. Provision of worked solution and complexity index

Topic	Complexity less than or equal to 7	Complexity greater than 7
Limit	16	7
Related rates	17	4

There are two things to note about the provision of worked solutions as a lead pedagogical move: first, less than 25% of the queries received a worked solution as the first response, and second, when this occurred, the discussions tended to be brief and end shortly after the introduction of the worked solution. Figure 11 contains an exchange on related rates with a complexity index of 4 that typifies this sort of exchange.

related rates: A weight is attached to one end of a rope....

Moderators: tkhunny, Gene, stapel, Ted, galactus

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3 posts • Page 1 of 1

related rates: A weight is attached to one end of a rope....

by **sickplaya** on Wed Feb 28, 2007 6:30 am

sickplaya
New Member

Posts: 21
Joined: Fri Jan 13, 2006 3:22 am

A weight is attached to one end of a rope, of length 8.5m, passing over a pulley 4.5m above a horizontal floor. The other end of the rope is attached to a point, 0.5m above the floor, on the rear of a moving tractor. If the tractor is moving at a constant speed of 1m/sec, how fast is the weight rising when the tractor is 3m from the point directly below pulley?

i got an answer of 0.555 m/sec

Re: related rates

by **soroban** on Wed Feb 28, 2007 12:35 pm

soroban
Elite Member

Posts: 3465
Joined: Fri Jan 28, 2005 10:13 pm
Location: Lexington, MA

Hello, sickplaya!

> A weight is attached to one end of a rope, of length 8.5m, passing over a pulley
> 4.5m above a horizontal floor. The other end of the rope is attached to a point,
> 0.5m above the floor, on the rear of a moving tractor. If the tractor is moving
> at a constant speed of 1m/sec, how fast is the weight rising when the tractor is 3m from the point directly below pulley?

CODE: SELECT ALL

```
                y+4 *          |4.5-y
                   *           |
                  *            |
               *               * W
            *                  |
       T *                     |
         |                     |y
    0.5|                       |
         |                     |
     - * - - - - - - - - - - + -
       A                       B
```

The weight is W, the pulley is P, the truck is T.

The height of the weight is: $y = WB.$
Then: $PW = 4.5 - y$ and: $TP = 8.5 - (4.5 - y) = y + 4$

At any time t, the diagram looks like this:

CODE: SELECT ALL

```
                          P
                          *
                      *   |
                  *       |
            y+4 *         |
              *           |4
          *               |
      *                   |
  *                       |
T * - - - - - - - - - - - * Q
              x
```

Let $x = TQ.$

Pythagorus tells us: $(y + 4)^2 = x^2 + 4^2$ [1]

Differentiate with respect to time: $2(y + 4) \cdot \frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$ [2]

When $x = 3$, from [1], we have: $(y + 4)^2 = 3^2 + 16 \Rightarrow y = 1$
We are told that: $\frac{dx}{dt} = 1$

Substitute into [2]: $2(5) \cdot \frac{dy}{dt} = (2)(3)(1)$

Therefore: $\frac{dy}{dt} = 0.6 \text{ m/sec}$

by **sickplaya** on Thu Mar 01, 2007 4:04 am

yeah i made a really stupid mistake and i get your answer of 0.6 now. thanks a lot =)

sickplaya
New Member

Posts: 21
Joined: Fri Jan 13, 2006 3:22 am

Figure 11. Worked solution as lead pedagogical move

4.3.1.1 Analysis of exchange (worked solution)

Here's how. In this exchange, the student, **sickplaya**, posed a related rates query [6:30 am] with a proposed numerical answer but without supporting work: "i got an answer of 0.555 m/sec." The first tutor who responded, **soroban**, provided a worked solution [12:35 pm] that would do any textbook justice, including diagrams and step-by-step descriptions of the solution process. The only thing absent from **soroban's** contribution is an inquiry into the reasoning behind **sickplaya's** answer; it is almost as if the query from the student did not contain the reference to this attempted solution or its outcome. The exchange ended when **sickplaya** returned the next day [4:04 am] to thank **soroban**. Although **sickplaya** alluded to the difference in the two solutions in this contribution ("yeah i made a really stupid mistake and i get your answer of 0.6 now"), the entire exchange has a shallow feel to it. There is no discussion of where or why the error occurred (or even whether this was "a really stupid mistake" instead of one that pointed to a deeper misunderstanding of related rates).

4.3.1.2 Summary of tutors providing information

The exchange in Figure 11 reflects the quality of the discussions that generally occurred when a full worked solution was the first pedagogical move – brief, surface-level, and restricted to the transmission (rather than co-construction) of mathematical information. By selecting this pedagogical move (*Here's how*), a tutor positions the student as someone who can profit from others' mathematical constructions and advice rather than as a participant in a dialogue who comes to the table with a perspective (even a flawed one) on the problem situation, e.g. someone the tutor is talking *to* rather than *with*. The result is a presentation of mathematical information that resembles a solution manual and eschews the affordances of the forum as a tutoring environment that connects people and supports dialogue in a symbiotic relationship.

4.3.2 Tutor initiates discovery

In contrast to the provision of full worked solutions (especially as the lead pedagogical move), the initiation of a dialogue aimed at discovery is a pedagogical move that takes full advantage of the affordances of an open, online forum. In such cases, the forum functions as a *resource* rather than as a *source*, i.e. it supports a dialogue between tutors with more mathematical expertise and students who wish to increase their mathematical understanding in the context of a specific problem. Table 12 shows the participation code and the corresponding conversational complexity index of exchanges in which a tutor invited a student to make inferences and draw conclusions about the construction of the solution to the problem.

Table 12. Initiation of dialogue: Participation codes and complexity

Topic	Participation code	Complexity Index
Limit	121	4
	1213	7
	12121212	12
	123124131	18
	123141415	22
Related rates	1212134	14
	1211311131	15
	12123432	18
	1213//13131311313131[32]	7//26

[32] The slashes here represent the start of an appended problem. The initiation of a dialogue by the tutor was the lead pedagogical move in the discussion of the second problem.

Arguably, the initiation of a dialogue was a relatively rare pedagogical move in the forum. However, its occurrence is worth noting since it generally resulted in very extended, back-and-forth conversations of high complexity in which the student was focally positioned as a co-constructor of the problem solution.[33] The participation codes in Table 12 reveal that, in all but one exchange (which developed into an inter-tutor discussion), the student remained an active participant throughout the exchange as tutors led the student through a mathematical discovery. Instead of acting as a source of information in these exchanges, the tutors appealed to resources available to the student and encouraged the student to ponder the mathematics involved in the construction of the solution. Figure 12 contains an exchange on related rates in which a tutor, **skeeter**, led a student, **kimmy**, to discover the relationship between the perspective taken on the problem situation and its mathematical representation and to adopt a productive framing for related rates problems.

[33] In the exchange with a complexity index of 4, the query itself was almost rhetorical and so the student figured out the answer after just one leading question. In the exchange with a complexity index of 7, a second tutor provided a worked solution and this marked the end of the exchange.

by **kimmy** on Sun Nov 12, 2006 4:48 pm

What about this question:

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 km north of the intersection, and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at 20km/h. If the cruiser is moving 60km/h at the instant of measurement, what is the speed of the car.

My work:

Let y be the police cruiser
Let x be the car being chased
Let r be the distance between them.
y= 0.6 km
dy/dt = 60 km/h
x=0.8 km
dx/dt = ?
r = ?
dr/dt = 20 km/r

Solve for r

r^2 = y^2 + x^2
r^2 = 0.6^2 + 0.8^2
r = 1
Differentiate

2r dr/dt = 2y dy/dt + 2x dx/dt
Sub everything in and get a value of 70 km/h for dx/dt

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **skeeter** on Sun Nov 12, 2006 4:56 pm

your answer is correct, but I would like to know what **exactly** did you substitute in for dy/dt to determine dx/dt?

skeeter
Senior Member

Posts: 2136
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **kimmy** on Sun Nov 12, 2006 4:58 pm

I substituted in 60 for dy/dt

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **skeeter** on Sun Nov 12, 2006 5:05 pm

> **kimmy wrote:**
> I substituted in 60 for dy/dt

then dx/dt wouldn't work out to be 70 km/hr, would it?

r dr/dt = y dy/dt + x dx/dt

(1)(20) = (.6)(60) + (.8)(dx/dt)

20 = 36 + .8(dx/dt)

16 = .8(dx/dt)

20 km/hr = dx/dt

where is the mistake?

skeeter
Senior Member

Posts: 2179
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **kimmy** on Sun Nov 12, 2006 5:08 pm

My mistake or a mistake in the problem?

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **skeeter** on Sun Nov 12, 2006 5:12 pm

your mistake.

skeeter
Senior Member

Posts: 2136
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **kimmy** on Sun Nov 12, 2006 5:12 pm

That wouldn't make sense, having a high speed car chase, and the one being chased driving at 20 km/h, now would it?

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **kimmy** on Sun Nov 12, 2006 5:14 pm

I have no idea what I did wrong.

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **skeeter** on Sun Nov 12, 2006 5:17 pm

why do you think I asked you about the value you used for dy/dt?

what is actually happening to the distance "y" during the chase?

skeeter
Senior Member

Posts: 2136
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **kimmy** on Sun Nov 12, 2006 5:23 pm

Distance y is decreasing, isn't it?

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **skeeter** on Sun Nov 12, 2006 5:29 pm

if a value is decreasing, what can you say about its rate of change (its derivative)?

skeeter
Senior Member

Posts: 2136
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **kimmy** on Sun Nov 12, 2006 5:31 pm

It is also decreasing! Ohh! That makes the 60 km/h a negative?

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

by **skeeter** on Sun Nov 12, 2006 5:37 pm

correct on the negative ... note that the derivative itself is not decreasing, it's constant.

Glad you figured that out yourself ... now you'll remember it.

skeeter
Senior Member

Posts: 2179
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **kimmy** on Sun Nov 12, 2006 5:46 pm

Thanks a lot!

kimmy
Junior Member

Posts: 73
Joined: Tue Sep 19, 2006 9:01 pm
Location: Canada

Figure 12. Initiation of dialogue leading to discovery and reframing

4.3.2.1 Analysis of exchange (discovery)

Greek pastry? This exchange involved a problem that was appended to another related rates query (not reproduced in Figure 12) in which **kimmy** sought verification for her solution.

For the first problem, **kimmy**'s supporting work was accurate and corresponded to the proposed numerical answer, although there was an error in her verbal statement of the results involving incorrect units. The first tutor to respond, **galactus**, failed to mention this error, instead affirming the numerical result ("Yes, your answer is correct. Good work.") and admonishing **kimmy** for using "pie" instead of "pi" ("Enough with the 'pie' already….DUH!.  It's a Greek letter, not your dessert. PI."). Following (or perhaps overlapping with) **kimmy's** expression of appreciation ("Excellent, thank you."), **skeeter** joined the exchange and noted the error of units: "what's 'off' are your units for dA/dt, the rate of change of area … dA/dt = 300pi m^2 / min," as well as drawing a distinction between the pastry and the symbol: "also … 'pie' is something you eat; 'pi' is the lower case Greek letter that represents the constant ratio of the circumference of a circle to its diameter."

Framing–inconsistent. Fourteen minutes later [4:48 pm], **kimmy** appends the question that begins the extended exchange in which **skeeter** initiates a dialogue that helps **kimmy** discover an important error that she has made and to reframe the problem (Figure 12). In this post, **kimmy** sets up the problem solution by defining variables, specifying the information provided, and noting the information that must be found ("Let y be the police cruiser; Let x be the car being chased; Let r be the distance between them; y=0.6 km; dy/dt = 60 km/h; x=0.8 km; dx/dt = ?; r = ?; dr/dt = 20 km/r"). This description indicates that **kimmy** has framed the problem as a set of variables (x, y, and r) and their corresponding derivatives (dx/dt, dy/dt, and dr/dt) and has perceived the assignment of a numerical value to each of these as the goal of the activity. There is no evidence that this framing includes a relationship between physical perspective and the rates of change in the problem situation; in other words, **kimmy's** solution does not address the relative positioning of the police cruiser and the car being chased (e.g. in a way that a diagram would). Using the numerical information that she has gleaned from the problem statement, **kimmy** calculates one of the two unknowns, namely the value of "r" for the moment at which the speed of the car is to be determined: "Solve for r; r^2 = y^2 + x^2; r^2 = 0.6^2 + 0.8^2; r = 1") and differentiates the equation relating the variables: "Differentiate; 2r dr/dt = 2y dy/dt + 2x dx/dt." The calculation of the sought rate of change, dx/dt, however, is not provided. Instead, **kimmy** glosses over the details and simply provides a numerical answer: "Sub everything in and get a value of 70 km/h for dx/dt." On the surface, it might appear as though kimmy has solved the problem since 70 km/h is accurate. However, this numerical answer does not correspond to the mathematical representation of the solution that **kimmy** provided, supported by her framing of the problem. Instead, it seems likely that **kimmy** found the numerical answer (e.g. in the back of the textbook or another source) and reproduced it in lieu of performing the calculations.

Framing–probed. One possible pedagogical move that a tutor could make in this situation would be to designate the location of the error and provide the correct information, for instance "your answer is correct, but dy/dt should be -60 instead of 60." However, **skeeter** chooses an alternative pedagogical path that addresses **kimmy's** framing of related rates. After evaluating the numerical answer, **skeeter** poses a question that positions **kimmy** to discover and diagnose the error herself [4:56 pm]: "your answer is correct, but I would like to know what exactly did you substitute in for dy/dt to determine dx/dt?" By framing her/his response in this manner, **skeeter** invites a response and initiates a dialogue in which **kimmy** is the one evaluating her solution. The next contribution to the exchange [4:58 pm] shows that **kimmy** is not aware of the inconsistency in her solution as she takes **skeeter's** question at face value: "I substituted in 60 for dy/dt." Although **skeeter's** question hinted at the error by emphasizing the adverb "exactly" when referring to the value of dy/dt, **kimmy** is not cognizant of her error. In her framing, 60 km/h was the value given in the problem statement for the speed of the police cruiser and this corresponds to the slot for dy/dt.

Framing–assessed. In the next contribution [5:05 pm], **skeeter** focuses attention on the discrepancy and demonstrates that a value of 60 for dy/dt would produce a different numerical value for dx/dt than the one that **kimmy** claimed: "then dx/dt wouldn't work out to be 70 km/hr, would it?" Although the tutoring has become more explicit with the calculation of the value for dx/dt that *would* result from assigning a value of 60 to dy/dt (namely, 20 km/hr), **skeeter** still positions **kimmy** to discover the error by ending the post with the question: "where is the mistake?" Once again, however, **kimmy** indicates that she is not aware of any problem with her conceptualization of the problem, although she requests information to help her locate the discrepancy [5:08 pm]: "My mistake or a mistake in the problem?" At this point in the exchange, one might expect that a tutor would resort to telling **kimmy** what was wrong with her solution, but this is not what occurs. The dialogue continues with **skeeter's** concise response that **kimmy** is responsible for the error [5:12 pm]: "your mistake." The back-to-back contributions from **kimmy** that follow next show clearly how she is endeavoring without success to diagnose the error according to the direction from **skeeter**, first trying to make sense of the physical situation [5:12 pm] ("That wouldn't make sense, having a high speed car chase, and the one being chased driving at 20 km/h, now would it?") and then, minutes later, admitting defeat [5:14 pm] ("I have no idea what I did wrong.").

Ohh!–framing revised. In response, **skeeter** directs **kimmy's** attention to the location and nature of the error with more pointed leading questions: "why do you think I asked you about the value you used for dy/dt? what is actually happening to the distance "y" during the chase?" This prompt is successful at focusing **kimmy's** attention on the relative positioning of entities in the problem situation so that "y" is now framed as the distance between the police cruiser and the intersection [5:23 pm]: "Distance y is decreasing, isn't it?" With this response, however, **kimmy** stops shy of making a connection between the adopted physical perspective (corresponding to the point of view of the police cruiser) and the mathematical representation of rates of change (negative), and **skeeter** poses another leading question [5:29 pm]: "If a value is decreasing, what can you say about its rate of change (its derivative)?" The exclamations in the response from **kimmy** demonstrate her excitement as realization dawns and she discovers her error through **skeeter's** coaching: "It is also decreasing! Ohh! That makes the 60 km/h a negative?" This contribution is evidence that **kimmy** has adopted a framing of the problem that includes relative positioning (or physical perspective), as she draws a connection between decreasing distance and negative rate of change. In the final pedagogical contribution to this exchange [5:31 pm], **skeeter** is careful to emphasize an important conceptual distinction between distance and rate of change (e.g. its derivative) which **kimmy** may have confused (based on the intended antecedent for "It is also decreasing!"): "correct on the negative … note that the derivative itself is not decreasing, It's constant." Adopting the perspective of the police cruiser, the *distance* between this vehicle and the intersection is *decreasing*, and it is the *rate of change* of the distance that is *negative* (e.g. the derivative represented by dy/dt).

You figured it out. **Skeeter** signs off by commenting on the positive and fruitful outcome of this dialogue in which **kimmy** was positioned as the one responsible for finding, diagnosing, and correcting the error: "Glad you figured that out yourself … now you'll remember it.  [Very Happy]" Clearly, the intent behind **skeeter's** adoption of this pedagogical maneuver was for **kimmy** to construct a deeper and more coherent understanding of related rates. The final contribution in the exchange is an enthusiastic expression of appreciation from **kimmy** [5:46 pm]: “Thanks a lot!”

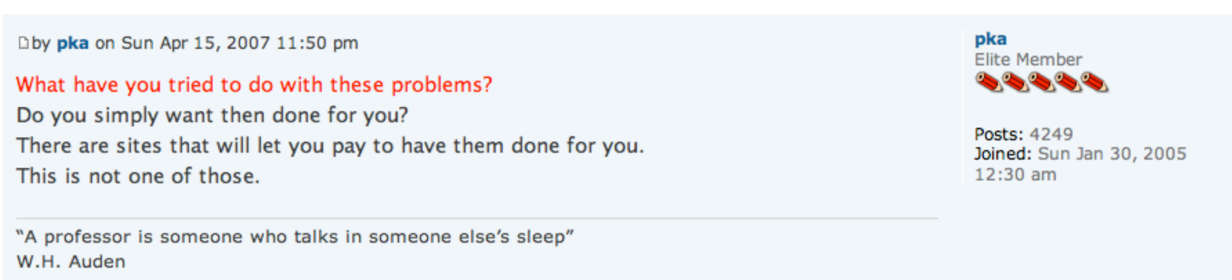
4.3.2.2 Summary of tutors initiating discovery

In approximately 5% of the exchanges, forum tutors drew the student into involved and elaborate discussion that paved the way for mathematical discovery. In the example, we saw how effective and exciting it can be when tutors in an online forum position students as authors of a solution by initiating a dialogue (*You figured it out*), posing leading questions, and focusing attention. The result was an extended, back-and-forth conversation that culminated in the student making a mathematical discovery that went well beyond an adjustment of a proposed solution. With the help of the tutor and a channel of leading questions (*Framing–probed, Framing–assessed*), the student was able to navigate a path from an unproductive (*Framing–inconsistent*) to a productive framing of related rates problems (*Ohh!–Framing revised*). This exchange exemplifies the quality of tutoring that is afforded by open, online forums and is not generally supported by computerized tutoring systems, in particular those that provide feedback on final numerical answers. Exploring such interactions gives us a window into the exciting potential of open, online forums as a tutoring environment and provides us with insight into how we can spark and sustain discovery learning in the context of routine problem solving.

4.3.3 “SHOW YOUR WORK!”

Another pedagogical move that was present in the forum involved pressing a student to cooperate by presenting or describing their mathematical ideas and reasoning. Such moves, represent the exercise of a tutor’s authority (by placing conditions on the provision of help), and position the student to participate in the discussion as a mathematical partner rather than as a recipient of information. There were 6 exchanges on limit and 11 on related rates that contained such a pedagogical move, and the contribution of this type of pedagogical move tended to occur in exchanges of higher complexity (index greater than 7). Of particular interest is the finding that, when this move occurred in the initial tutor intervention and *was* taken up by the student, there were no cases in which the exchange terminated or failed to grow into a discussion; in contrast, when this move occurred in the initial tutor intervention and *was not* taken up by the student, there were no cases in which the exchange grew into a discussion. It appears, then, that when students heed this directive to align their actions with the community expectations, the community responds favorably and the forum functions as intended.

There were three characteristics of student contributions associated with the directive to "Show your work" as a pedagogical move: a) perceived abuses of the forum; b) requests that lacked specificity; and, c) requests for help backed by insufficient mathematical argumentation or support. In cases in which a student appeared to be milking or gaming the system (e.g. by posing several queries without making contributions and showing effort), this move served as a public reminder to the student of the purpose of the forum to provide assistance rather than function as an answer service. As the exchange excerpt on related rates in Figure 13 shows, tutors were direct and did not tend to mince words when delivering these reprimands.


by pka on Sun Apr 15, 2007 11:50 pm

What have you tried to do with these problems?
Do you simply want then done for you?
There are sites that will let you pay to have them done for you.
This is not one of those.

"A professor is someone who talks in someone else's sleep"
W.H. Auden

pka
Elite Member

Posts: 4249
Joined: Sun Jan 30, 2005 12:30 am

Figure 13. Show your work directive prompted by perceived abuse of forum

In this thread, the student, **rofl**, a relatively new forum member (participation in 15 threads), posted 5 related rates queries at once without making any proposals for action or showing attempted solutions. After the student responded following **pka's** directive with a solution proposal for one of the problems, other forum tutors chimed in and provided support for assisting **rofl** with the remaining problems.

In addition to addressing slackness, this type of pedagogical move was also associated with requests that lacked specificity. Although students often used (a portion of) the problem statement as the topic for the thread, the request for help sometimes represented a general plea rather than a specific query. As the exchange excerpt in Figure 14 shows, this lack of information was perceived as being not conducive to forum activity. One of the main affordances of the forum (and one that distinguishes it from computerized help systems) is that responses can be tailored to the student's situation. However, this cannot happen if the student does not reveal their mathematical ideas and the "show your work" directive addresses this issue.

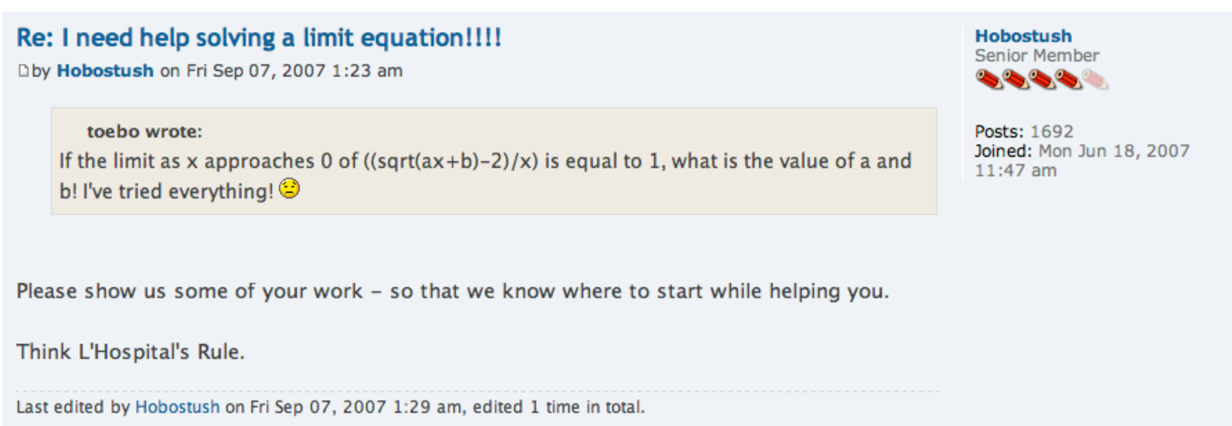


Figure 14. Show your work directive prompted by lack of specificity

In this case, the student, **toebo**, a new forum member (participation in 3 threads) alludes to her/his comprehensive mathematical labors ("I've tried everything! [Crying or Very sad]"), but does not provide any foothold for a specific tutoring intervention. There is no information on what mathematical techniques have been tried and why they were not successful. The tone of the response containing the directive to "show us some of your work" from senior member, **Hobostush**, is polite but direct, conveying the forum policy that students provide as much information as possible in order to facilitate tutoring in this environment.

The directive to show work was also prompted by student contributions that lacked sufficient mathematical information for the purpose of conversing. In such cases, the student provided limited mathematical information (such as only a numerical answer) with the expectation that the tutors could diagnose whatever error or misunderstanding produced this outcome. As the exchange excerpt in Figure 15 reveals, the forum tutors, who demonstrate impressive talent and mathematical prowess, do not aspire to clairvoyance and used this pedagogical move to impress upon students the difference between a *numerical answer* and a *solution*.

Related Rates: Water is leaking out of an inverted conical t

by **fish** on Sun Oct 15, 2006 12:11 am

Water is leaking out of an inverted conical tank at a rate of 500 cubic centimeters per min at the same time that water is being pumped into the tank at a constant rate. The tank has height 10 meters and the diameter at the top is 4.5 meters. If the water level is rising at a rate of 26 centimeters per minute when the height of the water is 1.0 meters, find the rate at which water is being pumped into the tank in cubic centimeters per minute.

I know the volume of the cone (with radius "r" and height "h") is given by the formula V = (1/3)pi r^2 h .

I get an answer of 413.5121, and it's not right. Neither is 4135.121 or 41351.2. I don't know what I'm doing wrong.

fish
New Member

Posts: 23
Joined: Sat Oct 14, 2006 6:19 am

by **pastel** on Sun Oct 15, 2006 12:20 am

> **fish wrote:**
> I don't know what I'm doing wrong.

Since you haven't shown your work, I'm afraid we can't tell what you're doing wrong, either. Sorry.

Please reply showing all of your work and reasoning. Thank you.

E.

pastel
Elite Member

Posts: 9673
Joined: Wed Feb 04, 2004 2:16 pm

Figure 15. Show your work directive prompted by lack of information

Although, on the surface, this type of contribution from a student could be attributed simply to negligence or laziness, it could also represent a more serious "missing link" phenomenon; the student does not recognize the important role that argument plays in mathematical practice as a chain that links query and answer. In this exchange, the student, **fish**, provides a spate of numerical results, none of which are supported by an argument, and **pastel's** response is to connect these results to the solution process by pressing **fish** to produce mathematical evidence. Thus, by pressing students to show their work, tutors are flagging the construction and production of an argument that can be shared with others as a vital part of mathematical practice and inviting students to participate in this practice.

4.3.3.1 Summary of SHOW YOUR WORK directive

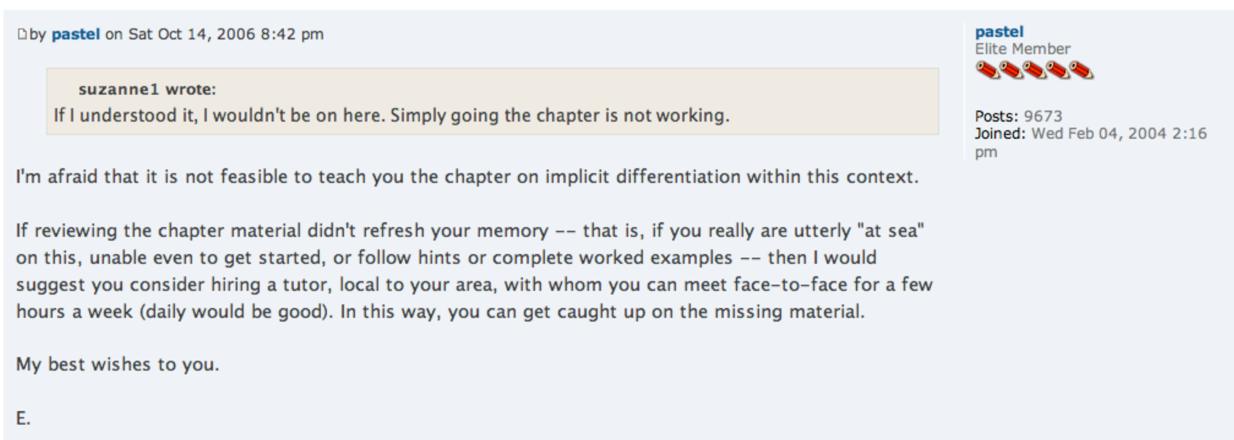
Although the pedagogical move directing a student to SHOW YOUR WORK was not used extremely often by forum tutors, it played a distinctive role in positioning students with respect to other forum members and mathematics. In particular, this move served as a reminder about forum expectations and as a message about mathematical practice and collaboration. One of the challenges associated with "distance tutoring" is that student queries do not have much context. That is, they do not reveal the background of the student, the curriculum, or the instructional practices in the student's course. In order for the forum to function efficiently and smoothly, the students need to reveal their mathematical understanding of the problem in question and indicate the point or location of confusion. The tone in which this directive was delivered ranged from polite and solicitous to blunt and pugnacious. The relationship between politeness and pedagogical effectiveness in the forum is something that warrants further investigation, since, in the context of face-to-face tutoring, these two have been shown to be at cross-purposes (Person, Kreuz, Zwaan, & Graesser, 1995).

4.3.4 Tutor refers student

In addition to providing mathematical information, initiating discovery, and pressing for work, forum tutors responded by directing students to other resources, although making referrals was not a common pedagogical move in this tutoring environment. Only 4 exchanges on limit and 6 on related rates included a contribution from a tutor sending the student to an alternative location for receiving help: either another online tutoring exchange or an alternative resource. Only 3 of these 10 exchanges had a complexity index greater than 7, and these three exchanges all contained a referral of the latter type. A referral to another exchange on a similar (or identical) problem, therefore, appears to be a pedagogical move that occurs early in an exchange and terminates discussion, whereas a referral to an alternative source of help is a move that tends to be embedded in more extended discussions.

Six of the 10 exchanges that contained a referral (2 on limit and 4 on related rates) pointed students to an online discussion of a similar or identical problem. Interestingly, the referent was not necessarily in the forum itself (e.g. FreeMathHelp.com), so that, in some cases, students were referred to exchanges in other open, online calculus help forums (such as MathHelpForum.com). The existence of such cross-referencing and inter-forum connections speaks to the extent of tutor interest and activity; not only do many tutors participate in several sub-forums on a single site (for different subject areas), but there are tutors who frequent and actively participate in more than one online forum community as well. Making referrals to other exchanges is consistent with the goal of the forum community to create an artifact (namely the archived exchanges) that can be useful to others. Thus, although private messaging is supported by the forum software (so that members can elect to allow this form of contact), it is a practice that appears to be generally discouraged for conducting mathematical discussions.

The second type of referral directed students to alternative resources and reflected the defining purpose of the forum community. This type of referral occurred when forum tutors inferred that a student required more extensive help and explanation than the forum is intended to provide, i.e. a tutorial on a calculus topic rather than the construction of a problem solution. As the exchange excerpt in Figure 16 illustrates, the message conveyed by this type of referral was that the forum is intended to *refine* and *support* student understanding rather than *replace* instructional explanations that can be found in textbooks and other instructional settings.


by **pastel** on Sat Oct 14, 2006 8:42 pm

> **suzanne1 wrote:**
> If I understood it, I wouldn't be on here. Simply going the chapter is not working.

I'm afraid that it is not feasible to teach you the chapter on implicit differentiation within this context.

If reviewing the chapter material didn't refresh your memory -- that is, if you really are utterly "at sea" on this, unable even to get started, or follow hints or complete worked examples -- then I would suggest you consider hiring a tutor, local to your area, with whom you can meet face-to-face for a few hours a week (daily would be good). In this way, you can get caught up on the missing material.

My best wishes to you.

E.

pastel
Elite Member

Posts: 9673
Joined: Wed Feb 04, 2004 2:16 pm

Figure 16. Referral to alternative resource

This contribution from forum tutor, **pastel**, occurred at the end of an exchange on related rates with a conversational complexity index of 12. In this exchange, the student, **suzanne1**, posted a query and then received help (in the form of a detailed solution sketch) from **galactus**. In response, **suzanne1** posted: “I [don’t] see where I can use Pythagoras to set up the problem. These type problems give me fits. I really am still pretty lost. Sorry.” Another tutor, **pka**, entered the exchange with the following advice for **suzanne1**: “These sorts of problems are ideal for implicit differentiation. You should review that topic in your textbook." According to **pka**, an understanding of implicit differentiation is a prerequisite for solving "these sorts of problems," and this understanding can (and should) be obtained from a textbook explanation. Although **pka's** contribution could be easily interpreted as a dismissal from engaging in further discussion, **suzanne1** returned to the exchange, pinpointing her difficulty understanding the help from **galactus** *and* responding to **pka's** advice: "I don't see how this solution addresses the idea that the ladder is sliding at a constant rate of 2.5 sec/ft along the floor. **pka**: If I understood it, I wouldn't be on here. Simply going the chapter is not working." This response demonstrates how students not only challenge the mathematical contributions of others (as discussed in the previous section) but also challenge forum tutors' pedagogical moves, a positioning that is hard to imagine occurring in other instructional settings. In response to these challenges, **pastel** enters the exchange with the contribution shown in Figure 16. In contrast to the tone of **pka's** contribution (authoritative but polite), the tone of this referral is decidedly derogatory and positions **suzanne1** as an incompetent participant; the forum has done its job and provided hints and worked examples (via **galactus**), and it is **suzanne1's** fault that she has not been helped and has remaining questions. The blame for the lack of resolution is thus placed firmly on the student's shoulders. Unlike **pka's** referral, this dismissal terminates the exchange, and **suzanne1**, notably a first-timer on the forum, neither returns to this exchange nor participates further in the forum.

4.3.4.1 Summary of tutor referrals

There were two types of referrals that occurred in the forum exchanges: students were referred to other tutoring exchanges or directed to more comprehensive instructional explanations. The practice of pointing students to view another online discussion of the same (or a similar) exercise is consistent with a vicarious learning strategy. In addition to direct instruction, students profit from watching others being tutored, e.g. a video of a face-to-face tutoring sesssion (Chi, Siler, Jeong, Yamauchi, & Hausmann, 2001). However, because students in the forum rarely returned to an exchange after being referred to another discussion, we do not know if this pedagogical move is similarly effective in an online environment. Vicarious learning in the context of online tutoring has not yet been systematically investigated. Directing students to another instructional explanation was a pedagogical move that tutors used when they detected that the student needed more extensive and broader instruction than can be afforded in the forum context. Although this move may be unpopular with students (and, in some cases, taken as a dismissal), it may better serve the student in the long run. Repairing the foundation of mathematical understanding is ultimately more beneficial than simply patching up holes.

4.4 CONCLUSIONS ON FORUM TUTORING

An open, online help forum with a spontaneous participation structure affords extended discussions between multiple forum members. Instead of brief transmissions of information from a single tutor, the tutoring in this environment often has the flavor of conversation and mathematical discussion. Furthermore, the mathematical and pedagogical quality of the tutoring was positively related to the conversational complexity index that was based on the number of contributions and participants in an exchange. Exchanges with a high complexity index often contained rich discussions of mathematical principles and sophisticated pedagogical moves. In particular, there were no exchanges of low complexity that demonstrated quality tutoring encounters. Thus, the conversational complexity index can be used to locate exemplary tutoring interactions for analysis.

The forum is a distinctive tutoring environment since participants are separated geographically and are not familiar with one another outside of these interactions. This context allows students to position themselves in nonstandard ways by making assertions and proposals, questioning and challenging others' proposals, and indicating resolution. Of particular interest is that finding that students in the forum freely engaged in self-reflection following tutor intervention. Student participation in such cases went beyond the construction of a solution to the problem that they had posed and which was ostensibly the purpose of participating in the forum. Self-reflection, or introspection, is a meta-cognitive skill characteristic of expertise and one that is much promoted in "reformed" curricula.

The forum tutors were more mathematically experienced than the students and brought this expertise to the forum. In some cases, the extent of their knowledge may have prompted them to divulge too much information (in the form of a full or partial worked solution), rather than making pedagogical moves that focused on the student. However, the existence of several exchanges in which forum tutors worked either individually or collectively to guide a student to a mathematical discovery is a mark of an effective tutoring environment. Finally, by pressing students to show their work, forum tutors not only promoted forum efficiency but, more importantly, encouraged students to be active learners and conveyed the message that mathematical learning entails cooperative participation and discussion.

5.0 COMMUNITY

The notion of community does not fit (and cannot be forced) into a pigeonhole. Defining community – delineating the necessary and sufficient conditions – has not proven to be a fruitful endeavor. Yet, as a paradigm for learning environments, community allows us to frame education as an activity in which members "grow" through participation and interaction with one another rather than through the transmission and accumulation of knowledge. Here, we are considering SOH forums as communities, with the tutors as the leading members, and examining what it means to participate in this activity. What benefits are derived from participating in a community of others who share an interest in mathematics and helping others?

In contrast to a collection of individuals separately engaged in tutoring, the participants in open, online forums, and especially the tutors, do exhibit a sense of community (van de Sande & Leinhardt, 2007b; 2008b). For example, forum tutors share explicit and implicit goals, identify themselves as members of the community, and assume shared responsibility for participation. This sense of community is particularly evident in forums with a spontaneous participation structure. Forums of this type afford opportunities for members to interact with one another and form relationships that are then manifest in distinctive patterns of participation (such as engaging in extended discussions of mathematical issues). In order to describe these patterns of participation, it is useful to view forum tutors as "Good Samaritans," who come to the aid of students in mathematical distress (van de Sande & Leinhardt, 2008c). For instance, there is a sense in which help becomes contagious as proposed solutions or perspectives on a problem prompt other forum tutors to contribute their ideas with the goal of improving and augmenting the discussion. In this chapter, these ideas are developed further by introducing three ways that tutoring within a community benefits participants: by bringing mathematical practices to life, by fostering alternative perspectives, and by supporting mathematical accuracy.

5.1 TUTOR PARTICIPATION

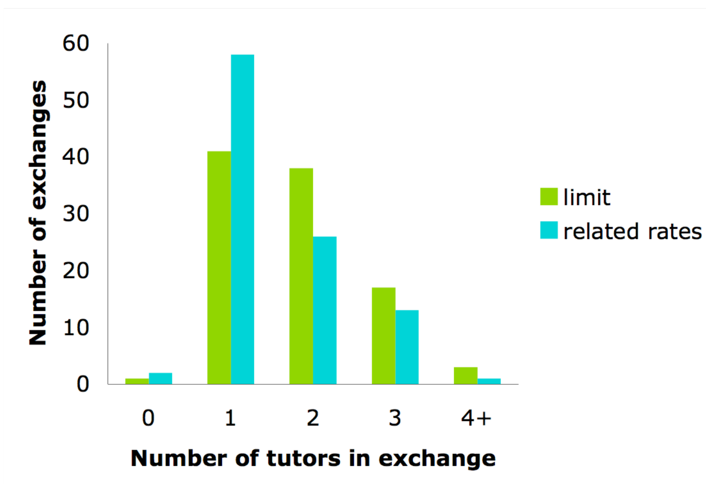
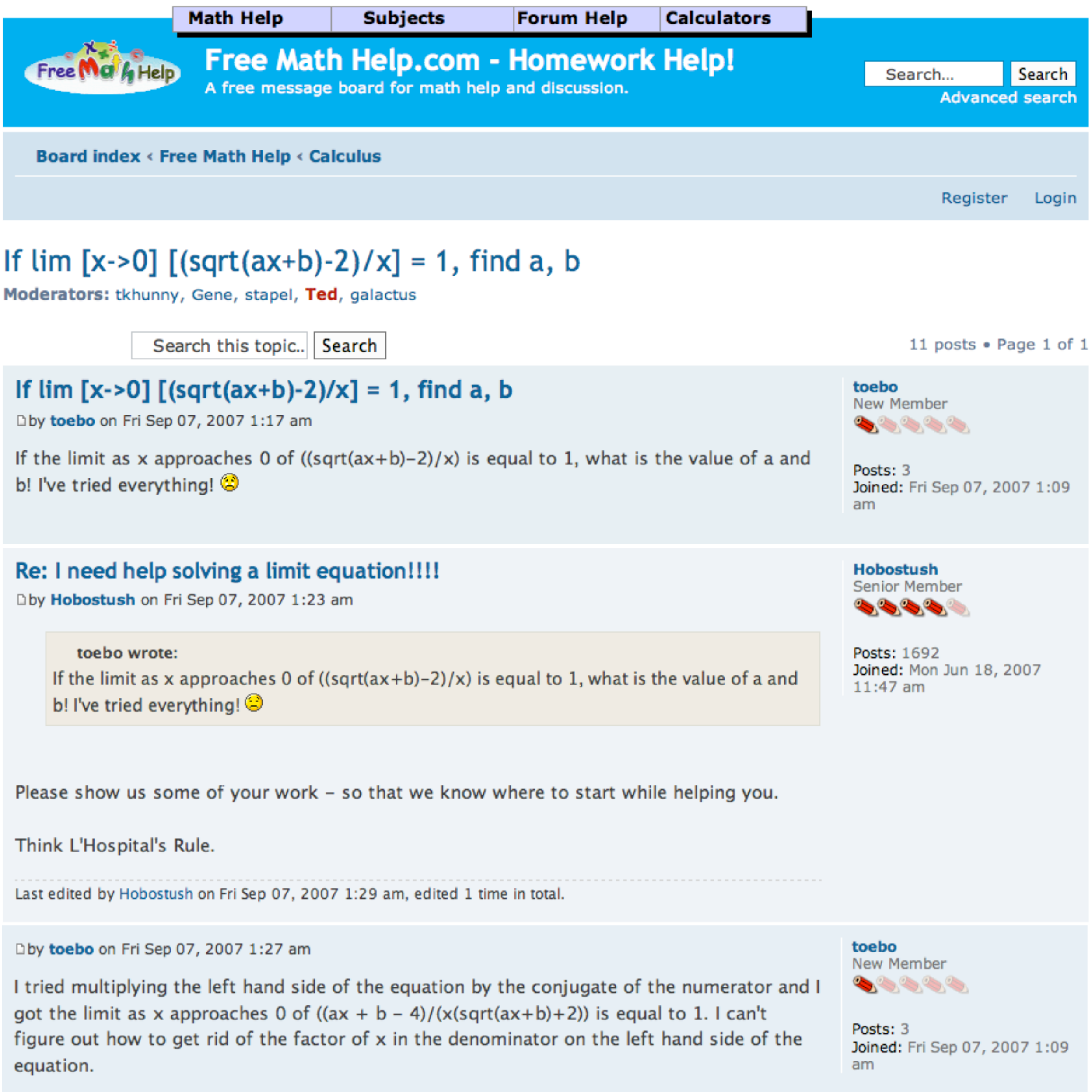


Figure 17. Number of tutors in exchanges

Figure 17 shows the number of exchanges on each mathematical topic according to the number of tutors that contributed. Notice that almost 60% of the exchanges on limit and 40% on related rates involved multiple tutors. The nature of the query, then, appears to influence the number of tutors who join in a given exchange. Exchanges on limit are more likely to attract multiple tutors to a given exchange, whereas related rates exchanges tend to involve a single tutor-student pair. A closer look at the contributions that tutors made in both cases suggests that it is the multiple framings of many limit queries that contributes to this difference. Tutors often joined into an exchange on limit to introduce an alternative framing (e.g. the definition of a derivative, see Figure 10) that had gone unnoticed or had not been recognized. Also, we see that it is rare to have exchanges involving more than three tutors, although there were a handful of instances in which this occurred. This finding indicates that three tutors is a natural boundary for participation and, as such, is a feature of interaction that the design of a tutoring system should take into account.

5.2 AUTHENTIC MATHEMATICAL DISCOURSE

One key aspect of a community is honesty in exchanges reflecting authentic practices and we take this as evidence that people care about and feel a sense of connection with the community. Engaging in authentic mathematical practices indicates that members value this activity as opposed to being counted as an answer service. However, students are usually not privileged to experience mathematicians in action as they construct arguments and debate possible solution paths with one another. Instead, students are presented with mathematics as a finished product rather than being granted access to the underlying process (that may include many false starts, questioning of assumptions, and reworking). A forum with a spontaneous participation structure affords the opportunity for students to witness mathematical discourse that is normally masked by the presentation of information in the classroom and in polished explanations. One benefit of conducting tutoring in a community, then, is that these discussions emerge naturally and expose students to authentic mathematical discourse on topics that are at their level of understanding. In addition, tutors themselves can benefit from these discussions as they have access through the community to a support system of knowledgeable others and can unpack the mathematics of "routine" problems. Figure 18 contains an exchange on limit involving three tutors (**Hobostush**, **galactus**, and **pka**) that illustrates how the mathematical practices associated with debate can be embedded in forum discussions.



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If lim [x->0] [(sqrt(ax+b)-2)/x] = 1, find a, b

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

Search this topic... Search

11 posts • Page 1 of 1

If lim [x->0] [(sqrt(ax+b)-2)/x] = 1, find a, b

by **toebo** on Fri Sep 07, 2007 1:17 am

If the limit as x approaches 0 of ((sqrt(ax+b)-2)/x) is equal to 1, what is the value of a and b! I've tried everything!

toebo
New Member
Posts: 3
Joined: Fri Sep 07, 2007 1:09 am

Re: I need help solving a limit equation!!!!

by **Hobostush** on Fri Sep 07, 2007 1:23 am

> **toebo wrote:**
> If the limit as x approaches 0 of ((sqrt(ax+b)-2)/x) is equal to 1, what is the value of a and b! I've tried everything!

Please show us some of your work - so that we know where to start while helping you.

Think L'Hospital's Rule.

Last edited by Hobostush on Fri Sep 07, 2007 1:29 am, edited 1 time in total.

Hobostush
Senior Member
Posts: 1692
Joined: Mon Jun 18, 2007 11:47 am

by **toebo** on Fri Sep 07, 2007 1:27 am

I tried multiplying the left hand side of the equation by the conjugate of the numerator and I got the limit as x approaches 0 of ((ax + b - 4)/(x(sqrt(ax+b)+2)) is equal to 1. I can't figure out how to get rid of the factor of x in the denominator on the left hand side of the equation.

toebo
New Member
Posts: 3
Joined: Fri Sep 07, 2007 1:09 am

by **Hobostush** on Fri Sep 07, 2007 1:29 am

Have you studied L'Hospital's rule?

Hobostush
Senior Member

Posts: 1692
Joined: Mon Jun 18, 2007 11:47 am

by **toebo** on Fri Sep 07, 2007 1:32 am

I haven't learned it yet.

toebo
New Member

Posts: 3
Joined: Fri Sep 07, 2007 1:09 am

by **galactus** on Fri Sep 07, 2007 2:15 pm

$$\lim_{x\to} \frac{\sqrt{ax+b}-2}{x}$$

I am not so sure we can use L'Hopital. It is not an indeterminate form.

We have $\frac{\sqrt{b}-2}{0}$, not 0/0.

This appears to be **undefined** to me.

Am I looking at it wrong?.

galactus
Elite Member

Posts: 4065
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

by **pka** on Fri Sep 07, 2007 3:12 pm

Try $a = 4 \;\; b = 4$.

"A professor is someone who talks in someone else's sleep"
W.H. Auden

pka
Elite Member

Posts: 4249
Joined: Sun Jan 30, 2005 12:30 am

by **Hobostush** on Fri Sep 07, 2007 3:25 pm

Hobostush
Senior Member

Posts: 1750
Joined: Mon Jun 18, 2007 11:47 am

> **galactus wrote:**
> $$L\lim_{x\to}\frac{\sqrt{ax+b}-2}{x}$$
>
> I am not so sure we can use L'Hopital. It is not an indeterminate form.
>
> We have $\frac{\sqrt{b}-2}{0}$, not 0/0.
>
> This appears to be **undefined** to me.
>
> Am I looking at it wrong?.

To change it to a definite value - 'b' must be such that the limit becomes 0/0.

For that to happen, b = 4

Then the function is:

$$\frac{\sqrt{ax+4}-2}{x}$$

Then applying L'Hospital we have

a/(2) = [sqrt](4)

a = 4

by **galactus** on Fri Sep 07, 2007 5:41 pm

galactus
Elite Member

Posts: 4065
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

OK. My bad. We had to find a and b that made the limit 1. Sorry

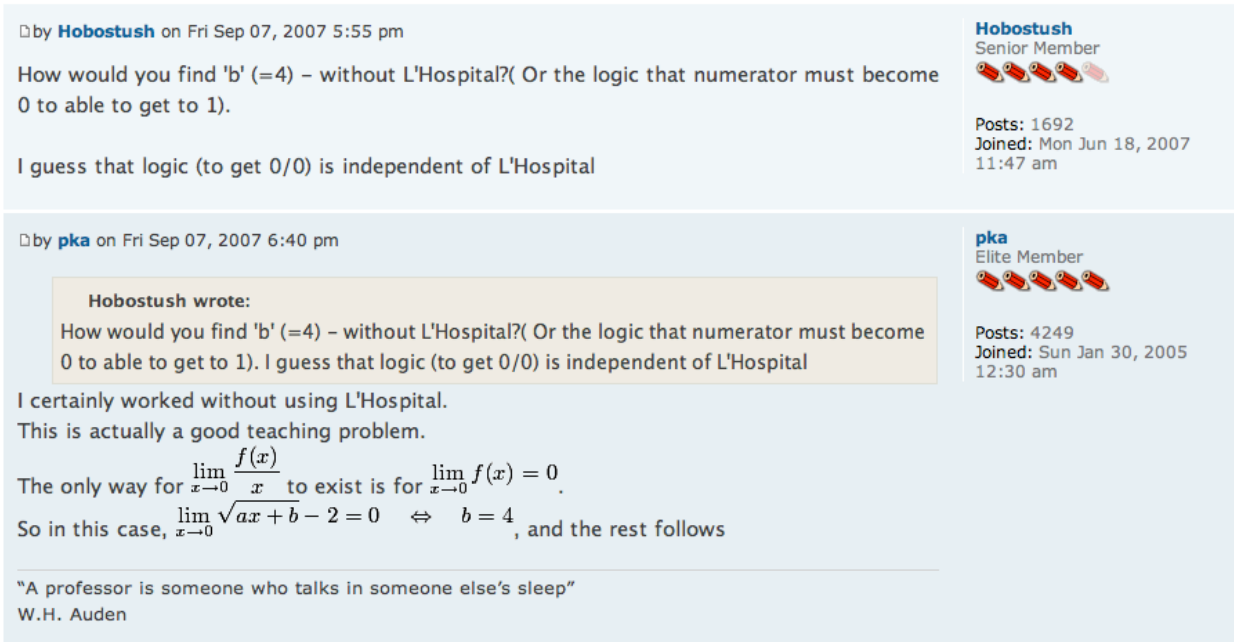
We can also use the conjugate instead of L'Hopital.

$$\frac{\sqrt{4x+4}-2}{x}\cdot\frac{\sqrt{4x+4}+2}{\sqrt{4x+4}+2}$$

Which gives:

$$\lim_{x\to}\frac{4x}{x(\sqrt{4x+4}+2)}$$

Now, it should fall into place nicely.



by **Hobostush** on Fri Sep 07, 2007 5:55 pm

How would you find 'b' (=4) – without L'Hospital?(Or the logic that numerator must become 0 to able to get to 1).

I guess that logic (to get 0/0) is independent of L'Hospital

Hobostush
Senior Member

Posts: 1692
Joined: Mon Jun 18, 2007 11:47 am

by **pka** on Fri Sep 07, 2007 6:40 pm

> **Hobostush wrote:**
> How would you find 'b' (=4) – without L'Hospital?(Or the logic that numerator must become 0 to able to get to 1). I guess that logic (to get 0/0) is independent of L'Hospital

I certainly worked without using L'Hospital.
This is actually a good teaching problem.
The only way for $\lim_{x \to 0} \frac{f(x)}{x}$ to exist is for $\lim_{x \to 0} f(x) = 0$.
So in this case, $\lim_{x \to 0} \sqrt{ax+b} - 2 = 0 \quad \Leftrightarrow \quad b = 4$, and the rest follows

"A professor is someone who talks in someone else's sleep"
W.H. Auden

pka
Elite Member

Posts: 4249
Joined: Sun Jan 30, 2005 12:30 am

Figure 18. Inter-tutor discourse

5.2.1 Analysis of exchange (math talk)

A challenge. The debate that occurs in this exchange centers on the relationship between a mathematical principle governing limit existence and l'Hôpital's Rule as a principle that applies to limits of certain form. After pressing the student, **toebo**, to show her/his work [1:23 am] and receiving a reply that did not include a reference to l'Hôpital's Rule[34], **Hobostush** makes a grounding move in the conversation and a proposal for action [1:29 am]: "Have you studied L'Hospital's rule?" and appends "Think L'Hospital's rule" to her/his earlier post. When **toebo** responds in the negative [1:32 am] "I haven't learned it yet. [Crying or Very sad]," **galactus** enters the exchange [2:15 pm] with a challenge for **Hobostush's** proposal ("I am not so

[34] English-speaking students often refer to "The Hospital Rule," but even in French, there are two accepted spellings of this mathematician's name: "Son nom s'écrit aussi L'Hospital." ("L'Hôpital – Wikipédia")

sure we can use L'Hopital. It is not an indeterminate form") supported by her/his reasoning ("We have $\frac{\sqrt{b}-2}{0}$, not 0/0. This appears to be **undefined** to me."). l'Hôpital's Rule only applies to indeterminate forms, and **galactus** questions whether the situation meets this condition of use, although s/he leaves the floor open for disagreement by ending the contribution with a hedge: "Am I looking at it wrong?"

The reply. At this point in the exchange, a third tutor, **pka**, enters [3:12 pm] with a numerical answer to the problem ("Try $a=4 \quad b=4$.") but without any accompanying explanation. The explanation of how these values are constructed comes in the next contribution from **Hobostush** [3:25 pm] that is a response to **galactus's** challenge: "To change it to a definite value - 'b' must be such that the limit becomes 0/0. For that to happen, b = 4." Here, **Hobostush** invokes the relevant mathematical principle as part of her/his argument: the existence of a limit for a quotient in which the denominator approaches zero requires that the numerator also approaches zero. The value for a is then constructed using l'Hôpital's Rule: "Then applying L'Hospital we have…a=4."

My bad. The fact that the student, **toebo**, has confirmed that s/he is unfamiliar with l'Hôpital's Rule shapes the reply to this contribution [5:41 pm] from **galactus**. We see the wikipedia-like nature of the forum as **galactus** acknowledges her/his mistake ("OK. My bad. We had to find a and b that made the limit 1. Sorry  [Embarassed]"), followed by an alternative proposal for constructing the value for a that does not require l'Hôpital's Rule: "We can also use the conjugate instead of L'Hopital."

A rebuttal. The response from **Hobostush** [5:55 pm] reveals that s/he assumed that l'Hôpital's Rule was the basis for invoking the guiding mathematical principle: "How would you find 'b' (=4) - without L'Hospital?(Or the logic that numerator must become 0 to able to get to

1). I guess that logic (to get 0/0) is independent of L'Hospital.” Notice how the forum is functioning as an arena for tutors to present, challenge, question, and revise mathematical proposals, where the problem posed by the student has stimulated a discussion between tutors on underlying mathematical principles and their application.

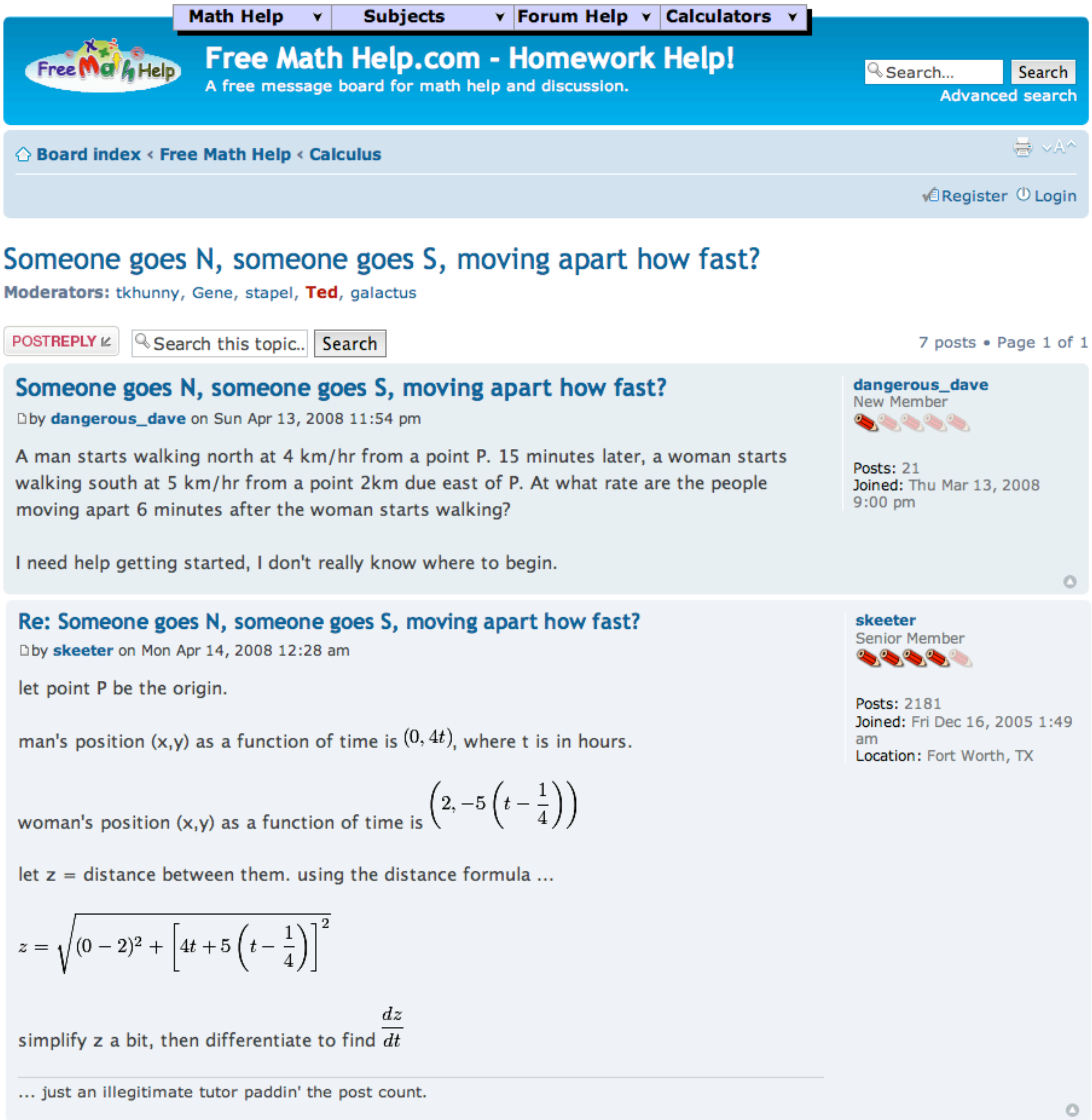
Resolved–A good problem. This question is answered in the final contribution in this exchange [6:40 pm] from the tutor, **pka**, who initially proposed the answer to the problem: “I certainly worked without using L'Hospital. This is actually a good teaching problem. The only way for $\lim_{x\to 0}\frac{f(x)}{x}$ to exist is for $\lim_{x\to 0} f(x)=0$.” In other words, the principle on which this problem rests involves the conditions under which a limit exists, whereas l’Hôpital’s Rule is a technique for determining the limit when certain conditions are met. The invocation of this principle marks an inverted problem type from standard limit exercises: typically the function is given and the problem is to determine the limit (using techniques such as conjugation and l’Hôpital’s Rule). In this case, though, the limit is provided and the goal is to find the function that produces this limit. The quality and depth of the problems that are posed in the forum does not escape the notice of the tutoring community; this characteristic of the problem is picked up on by **pka** and leads her/him to designate it as “a good teaching problem.”

5.2.2 Summary of tutor mathematical discourse

This exchange illustrates how tutoring in a community can stimulate dialogue that has benefits for both students and tutors. The forum environment exposes students to authentic mathematical discourse that occurs as others jointly work toward the construction of a solution that is part of the student’s curriculum. Students witness how mathematical proposals are made, taken up, questioned, and revised in practice. The importance of aligning principles with application is foregrounded as tutors debate the invocation of certain principles in the context of the problem. Although these practices can easily be described to students (for instance in textbook sidebars that advise students to select an appropriate technique or principle for application), they are embodied in authentic dialogue among members of a mathematical community. Furthermore, the tutors themselves are beneficiaries of engaging in these dialogues with other members of the community. In this exchange, we saw how **Hobostush** and **galactus** became aware of the nature of the underlying mathematical principle through their discussion and dialogue with one another (*A challenge, The reply, My bad, A rebuttal*) and with fellow tutor, **pka** (*Resolved–A good problem*). Because inter-tutor dialogue is not supported by tutoring systems that restrict participation to single tutor- or computer-student pairs, the community model of tutoring in the open, online forums is particularly worth exploring as a means of bringing mathematical practices to life.

5.3 ALTERNATIVE PERSPECTIVES

A robust community is one that, at the best, can make use of alternative perspectives and, at a minimum, can tolerate them. Here, we show that participating in tutoring within a forum community affords the introduction of alterative perspectives as members who view these exchanges contribute ways of viewing a problem that have not yet been addressed. According to the Good Samaritan hypothesis (van de Sande & Leinhardt, 2008c), the provision of assistance on the problem from one perspective can prompt fellow community members to contribute alternatives. The action of others stimulates actions aimed at providing further assistance, in this case through the contribution of a novel way of thinking about the problem at hand that may prove helpful. Figure 19 contains an exchange on related rates in which a tutor proposes an alternative perspective in response to the difficulties a student experiences working out the solution according to a framing of the problem that was introduced by another tutor.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Someone goes N, someone goes S, moving apart how fast?

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

POSTREPLY | Search this topic... | Search | 7 posts • Page 1 of 1

Someone goes N, someone goes S, moving apart how fast?

by **dangerous_dave** on Sun Apr 13, 2008 11:54 pm

A man starts walking north at 4 km/hr from a point P. 15 minutes later, a woman starts walking south at 5 km/hr from a point 2km due east of P. At what rate are the people moving apart 6 minutes after the woman starts walking?

I need help getting started, I don't really know where to begin.

dangerous_dave
New Member
Posts: 21
Joined: Thu Mar 13, 2008 9:00 pm

Re: Someone goes N, someone goes S, moving apart how fast?

by **skeeter** on Mon Apr 14, 2008 12:28 am

let point P be the origin.

man's position (x,y) as a function of time is $(0, 4t)$, where t is in hours.

woman's position (x,y) as a function of time is $\left(2, -5\left(t - \frac{1}{4}\right)\right)$

let z = distance between them. using the distance formula ...

$$z = \sqrt{(0-2)^2 + \left[4t + 5\left(t - \frac{1}{4}\right)\right]^2}$$

simplify z a bit, then differentiate to find $\frac{dz}{dt}$

... just an illegitimate tutor paddin' the post count.

skeeter
Senior Member
Posts: 2181
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **pastel** on Mon Apr 14, 2008 12:35 am

> **dangerous_dave wrote:**
> I need help getting started, I don't really know where to begin.

Draw a picture, and look for right triangles! :)

E.

pastel
Elite Member

Posts: 9803
Joined: Wed Feb 04, 2004 2:16 pm

Re: Someone goes N, someone goes S, moving apart how fast?

by **dangerous_dave** on Mon Apr 14, 2008 1:04 am

Thanks that helps heaps. Im having a bit of trouble with the simplifying. Does it follow the rules to simplify to this?

z = 2 + sqrt(4t) + 5(t-1/4)

Or does that break the rules?

@ pastel: I did, and puting that with skeeters help, I worked out how skeeter got to that point. Now just the differentiation...

dangerous_dave
New Member

Posts: 21
Joined: Thu Mar 13, 2008 9:00 pm

Re: Someone goes N, someone goes S, moving apart how fast?

by **skeeter** on Mon Apr 14, 2008 2:15 am

> **dangerous_dave wrote:**
> Thanks that helps heaps. Im having a bit of trouble with the simplifying. Does it follow the rules to simplify to this?
>
> z = 2 + sqrt(4t) + 5(t-1/4)
>
> Or does that break the rules? your algebra for simplifying z is incorrect ... follow the order of operations inside the radical first. you should get
>
> $$z = \sqrt{81t^2 - \frac{45}{2}t + \frac{89}{16}}$$

... just an illegitimate tutor paddin' the post count.

skeeter
Senior Member

Posts: 2181
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

Re: Someone goes N, someone goes S, moving apart how fast?

by **dangerous_dave** on Mon Apr 14, 2008 2:22 am

Oh of course :) thanks

dangerous_dave
New Member

Posts: 21
Joined: Thu Mar 13, 2008 9:00 pm

Re: Someone goes N, someone goes S, moving apart how fast?

by **galactus** on Mon Apr 14, 2008 12:18 pm

galactus
Elite Member

Posts: 4065
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Let's try it this way.

After the woman has walked 6 miutes, the man has walked 21 minutes. Therefore, he has walked (4)(21/60)=7/5 km

The woman has walked 5(6/60)=1/2 km.

From triangle ABC, The distance between them(AC) at 6 minutes after she starts walking is $\sqrt{(19/10)^2+(2)^2}=\frac{\sqrt{761}}{10}$ km.

So, we can use ol' Pythagoras:

$$D^2=x^2+y^2$$

$$D\frac{dD}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$$

They are moving apart vertically at dy/dt=4+5=9 km/hr. Since the distance between them stays at 2 km, then dx/dt=0.

$$D=\frac{\sqrt{761}}{10}, \quad x=2, \quad dx/dt=0, \quad y=19/10, \quad dy/dt=9$$

$$(\frac{\sqrt{761}}{10})\frac{dD}{dt}=2(0)+(\frac{19}{10})(9)$$

$$\frac{dD}{dt}=\frac{171}{\sqrt{761}}\approx 6.2 \quad km/hr$$

I think this is a little easier since we do not have to deal with radicals in our differentiation.

For the previous method, $\frac{dD}{dt}=\frac{1}{2}\cdot\frac{162t+18}{\sqrt{81t^2+18t+5}}$

Now, if you plug in t=1/10, you should get the same result.

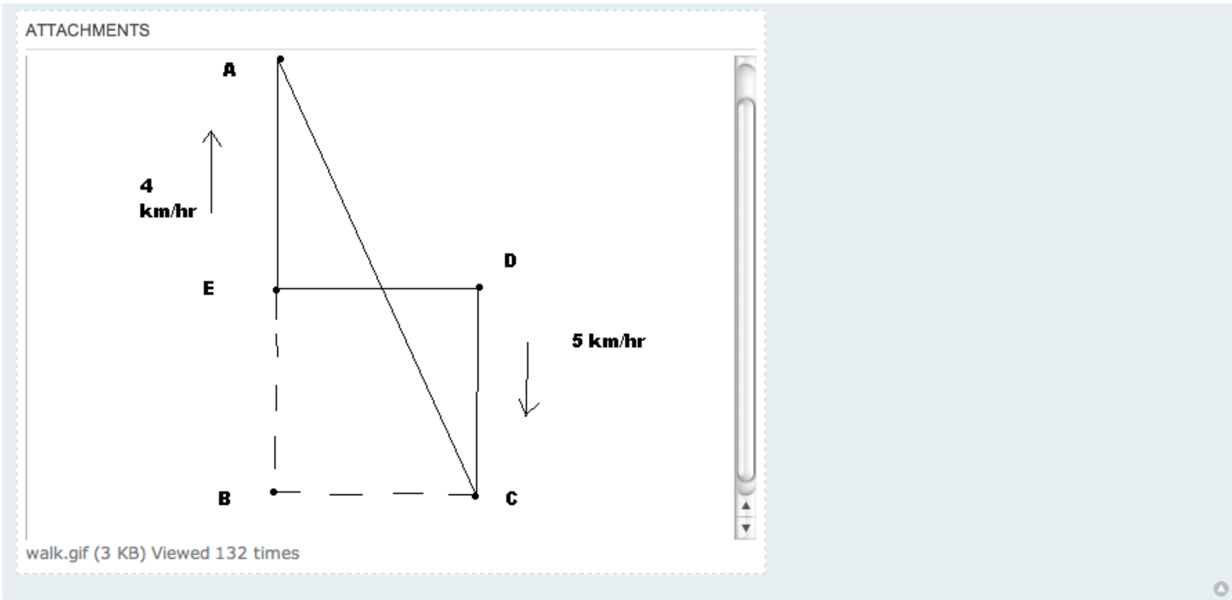


Figure 19. Alternative perspective

5.3.1 Analysis of exchange (framings)

Framing–needed and supplied. The student, **dangerous_dave**, initiates the thread [11:54 pm] by posing an involved related rates problem in which the timing of the 'objects' in motion is asynchronous, so that, the objects are traveling at different rates *and* begin moving apart at different times and from different starting locations. In response to **dangerous_dave's** request for help establishing a frame for this type of problem ("I need help getting started, I don't really know where to begin."), **skeeter** [12:28 am] introduces a framing for the problem based on a parameterization of the variables and specifies the relationship between the variables in this framing ($z = \sqrt{(0-2)^2 + [4t + 5(t - \frac{1}{4})]^2}$). The absence of a diagram (or reference to one) from either **dangerous_dave** or **skeeter** is notable, especially given the nature of the problem, and the advice from a second tutor, **pastel**, is for **dangerous_dave** to construct one and make use of its affordances: "Draw a picture, and look for right triangles! [Wink]"

Framing–hiccups. Putting these two responses together [1:04 am], **dangerous_dave** proposes the next step of the solution, namely the simplification of the relationship, $z = \sqrt{(0-2)^2 + [4t + 5(t - \frac{1}{4})]^2}$: “Thanks that helps heaps. Im having a bit of trouble with the simplifying. Does it follow the rules to simplify to this? z = 2 + sqrt(4t) + 5(t-1/4) Or does that break the rules?” As **skeeter** notes, however, in her/his red ink response [2:15 am], this simplification is incorrect (the square root of a sum of terms is not generally equal to the sum of the square roots of the terms) and should instead be $z = \sqrt{81t^2 - \frac{45}{2}t + \frac{89}{16}}$. At this point in the exchange, it is obvious that **dangerous_dave** is not fluent carrying out basic operations on algebraic expressions. In particular, he is struggling with the complicated mathematical expressions that result from the current framing of the problem, although his follow-up post [2:22 am] seems intended to reassure: “Oh of course thanks.”

Framing–Try this. In response to the difficulties that **dangerous_dave** is experiencing, **galactus** enters the exchange [12:18 pm] and provides an alternative framing of the problem (“Let’s try it this way.”) that results in much less complicated mathematical expressions (“I think this is a little easier since we do not have to deal with radicals in our differentiation.”) In addition, **galactus** initiates a resolution of the two framings (“For the previous method, $\frac{dD}{dt} = \frac{1}{2} \cdot \frac{162t + 18}{\sqrt{81t^2 + 18t + 5}}$. Now, if you plug in t=1/10, you should get the same result.”), clearly communicating to **dangerous_dave** that these two alternative framings are simply two different ways of viewing the problem that result in the same outcome.

5.3.2 Summary of alternative perspective emergence

In this exchange, we see how tutoring is enhanced through its enactment within a community. Here, one community member saw that a particular framing of the problem (*Framing–needed and supplied*) was problematic (*Framing–hiccups*) and proposed an alternative in order to alleviate the difficulties and provide further assistance (*Framing–try this*). As a consequence, a key mathematical practice (namely, resolution) became a part of the conversation. This sort of interaction does not generally take place in computer-based tutoring systems (in which a single framing predominates) or in single tutor-student pairs (although an expert tutor could perhaps be sensitive and responsive to alternative framings). It is the community aspect of the open, online forums with a spontaneous participation structure that seems to encourage the emergence of alternative perspectives. Clearly, the production and consideration of multiple alternative perspectives is mathematically stretching and enriching for the tutors involved. At the same time, the data suggest that students may also benefit from this practice. Recall, for instance, the excited reaction of **johnk** in Section 4.2.3 when the limit problem was re-framed as a derivative: "Wow, very insightful . I'm used to an alternative definition of the derivative: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$ So I didn't see it." Certainly, participation in this community has made some students aware of the role that alternative perspectives play in mathematical practice. As expressed by forum student, **kristy**, following an extended tutoring encounter with three forum tutors in which several framings emerged, "It seems like there are as many ways to solve it as there are people explaining it to me!" Such reactions from students engaging in homework activities on routine problems are remarkable and set a bar for the design of tutoring systems that should aspire to this level of student excitement in learning and doing mathematics.

5.4 SUPPORTING ACCURACY

The correction of errors can be a very sensitive issue, especially when the errors are made by someone positioned as an "expert" or teacher. Pursuit of this activity would be risky and distructive if it made people feel uncomfortable because they were zapped or "flamed,"[35] if wrong and would destroy the community. However, if participants care more about content and substance and getting ideas right than their ego, then there are ways of handling errors that are not disruptive but actually strengthen community. In a site with a spontaneous participation structure, the misinformation can be replaced with accurate information or directly addressed. This characteristic of community behavior has been referred to as the "wikipedia-like" nature of open, online help forums with a spontaneous participation structure (van de Sande & Leinhardt, 2007b). Here, we focus on the way participation in the community allows forum tutors to receive critiques from others and revise their contributions accordingly so that their mathematical understanding, and ultimately that of the whole community, is served.

Figure 20 contains a schematic that was designed to illustrate the way contributions receive attention from the community in an exchange. In the diagram, each shape represents a posting (or an edit of a previous posting) in the exchange, and each row contains the postings of a single participant (which are shade-coded). Here, the exchange involved one forum student

[35] Flaming is the hostile and insulting interaction between Internet users.

participant (**math**) and 3 tutors (**galactus**, **honey**, and **soroban**), who made 5, 7, 3, and 2 "postings", respectively.[36] Although the exact timing of each contribution is not indicated, sequential orientation is indicated by their placement along the time axis (on top). Arrows between shapes indicate a direct (causal) relationship between the postings, so that an arrow pointing from shape A to shape B signifies that posting A was directed at or made in response to posting B. For example, in this exchange, three forum tutors (**galactus**, **honey**, and **soroban)** responded to the query posted by forum student **math**. Dotted lines are used to depict relationships between posts that are less explicit but are plausible given the timing and contents of the contribution. The number of arrows pointing toward a given shape indicates its "weight" in the conversation, i.e. the number of postings in the exchange directly addressing this contribution. In addition, the length of the arrows provides some indication of the longevity of an exchange contribution; posts that contain ideas that have a large impact are those that have connections to postings that occur "far away," e.g. after an extended time and amount of interceding activity. In this example, notice that the first post from forum tutor **galactus** is the source of five arrows and these originate from posts that are both close by and farther away. In fact, this post, made in response to **math's** query on related rates, received a large amount of attention from the community *because* it contained several mathematical errors (calculus-based and algebraic). As the diagram illustrates, these errors were repeatedly addressed (by the student and other tutors) in a wikipedia-like fashion, with edits by **galactus** in response to contributions from **math**, **honey**, *and* **soroban**. The result of this process was that the incorrect information in the original post by **galactus** was revised and edited until it was transformed into a coherent and

[36] Notice that three of **galactus's** postings are labeled "edit". These are locations in the exchange in which **galactus** revised an earlier post. The archives indicate that **galactus** edited the original post a total of five times, but only three of these were directly referenced in the exchange and therefore included in the diagram.

accurate mathematical argument in an extended exchange (Figure 21–Figure 34) with a complexity index of 30. This exchange is presented here, but not analyzed in detail, in order to allow the reader to see how the schematic representation captures the attention paid to specific contributions, and, in particular, the way in which the incorrect contribution acted as a "magnet."

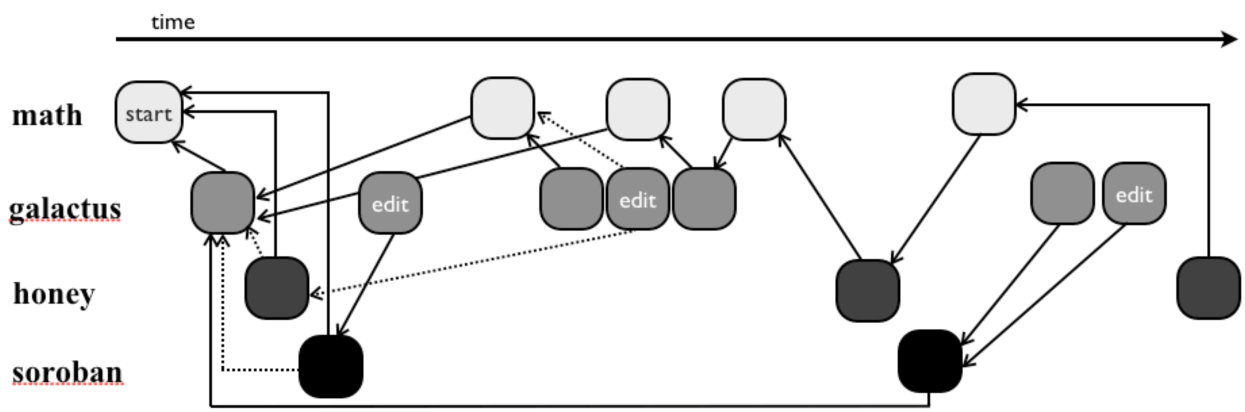
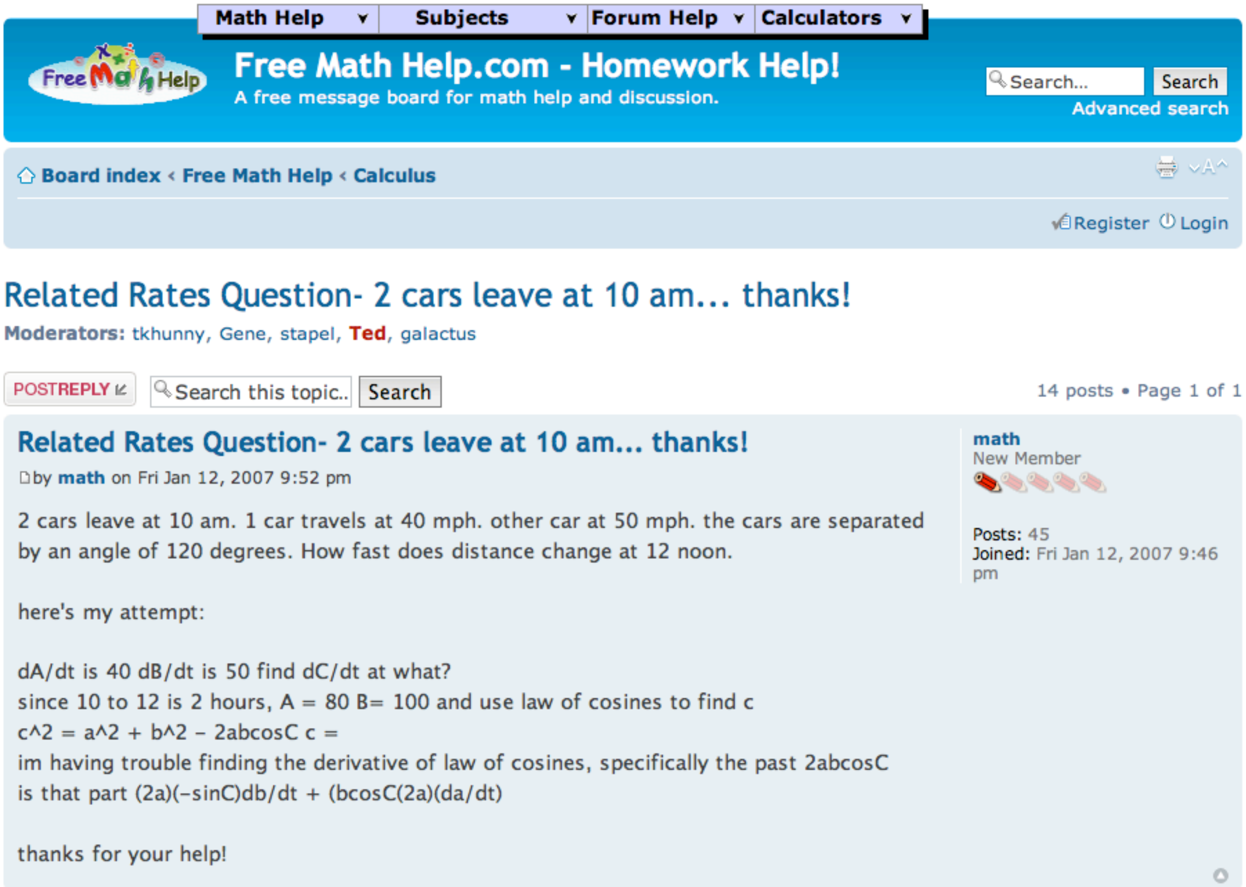


Figure 20. Schematic of exchange illustrating wikipedia-like nature of the forum



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Related Rates Question- 2 cars leave at 10 am... thanks!

Moderators: tkhunny, Gene, stapel, Ted, galactus

POSTREPLY Search this topic... Search

14 posts • Page 1 of 1

Related Rates Question- 2 cars leave at 10 am... thanks!

by **math** on Fri Jan 12, 2007 9:52 pm

2 cars leave at 10 am. 1 car travels at 40 mph. other car at 50 mph. the cars are separated by an angle of 120 degrees. How fast does distance change at 12 noon.

here's my attempt:

dA/dt is 40 dB/dt is 50 find dC/dt at what?
since 10 to 12 is 2 hours, A = 80 B= 100 and use law of cosines to find c
c^2 = a^2 + b^2 - 2abcosC c =
im having trouble finding the derivative of law of cosines, specifically the past 2abcosC
is that part (2a)(-sinC)db/dt + (bcosC(2a)(da/dt)

thanks for your help!

math
New Member

Posts: 45
Joined: Fri Jan 12, 2007 9:46 pm

Figure 21. (Post 1 of 5, row 1, Figure 20) Initiating post by forum student, math

by **galactus** on Fri Jan 12, 2007 10:22 pm

Let's give this a go. Check me out and make sure I didn't veer off the beaten path.

Law of Cosine is, indeed, the thing to use.

$c^2 = a^2 + b^2 - 2abCos(\theta)$

From 10 am to 12 noon the vehicles travel for 2 hours. That's 80 miles for A and 100 miles for B.

Differentiate:

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2ab(-1/2)$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}$$

$$2c\frac{dc}{dt} = (2a+b)\frac{da}{dt} + (a+2b)\frac{db}{dt}$$

We need dc/dt, knowing da/dt=40 and db/dt=50, a=80, b=100.

Use the law of cosines to find c when a=80 and b=100 and theta equals 120 degrees.

We find $c = 20\sqrt{61}$ when a=80 and b=100 and theta=120 deg.

$$2(20\sqrt{61})\frac{dc}{dt} = (2(80)+100)(40) + (80+2(100))(50)$$

$$\frac{dc}{dt} = \frac{24400}{40\sqrt{61}} \approx \boxed{78.1}$$

****EDIT**: Soroban's answer is different than mine, which leads me to believe I may have an error. If anyone spots it let me know.

Last edited by galactus on Sat Jan 13, 2007 6:01 pm, edited 5 times in total.

galactus
Elite Member

Posts: 4112
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Figure 22. (Post 1 of 4, row 2, Figure 20) Response from galactus that contained errors

by **honey** on Fri Jan 12, 2007 10:24 pm

Friendly Note:

Please try to make sense when you write something.

"im having trouble finding the derivative of law of cosines"

I dare you to tell me what that means.

Hint:

You may wish to observe that your angle is fixed.

cos(120º) = -1/2

That should simplify a few things.

honey
Elite Member

Posts: 4997
Joined: Wed Apr 13, 2005 2:56 am
Location: PA, USA

Figure 23. (Post 1 of 3, row 3, Figure 20) Contribution from forum tutor, honey

Re: Related Rates Question- 2 cars leave at 10 am... thanks!
by **soroban** on Fri Jan 12, 2007 11:02 pm

soroban
Elite Member

Posts: 3481
Joined: Fri Jan 28, 2005 10:13 pm
Location: Lexington, MA

Hello, math!

Here's my approach to the problem . . .

> "
> Two cars leave at 10 am.
> One car travels at 40 mph; the other car at 50 mph.
> The cars are separated by an angle of 120°.
> How fast does distance change at 12 noon?

CODE: SELECT ALL

```
  A *
     \  *
      \     *     x
   40t \        *
        \           *
         \ 120°         *
          * - - - - - - - - * B
         O         50t
```

The first car leaves point O at 40 mph.
In t hours, it has travelled $40t$ miles to point A.

The other car leaves point O at 50 mph.
In t hours, it has travelled $50t$ miles to point B.

Angle $AOB = 120^o$.Let $x = AB$.

Law of Cosines:
$x^2 = (40t)^2 + (50t)^2 - 2(40t)(50t)\cos 120^o = 6100t^2$

Differentiate with respect to time: $2x\left(\frac{dx}{dt}\right) = 12,200t$
and we have: $\frac{dx}{dt} = \frac{6100t}{x}$

When $t = 2:\ x^2 = 6100 \cdot 2^2 = 24,400 \Rightarrow x = 20\sqrt{61}$

Therefore: $\frac{dx}{dt} = \frac{6100(2)}{20\sqrt{61}} \approx \boxed{78.1\text{ mph}}$

Figure 24. (Post 1 of 2, row 4, Figure 20) "Here's my approach…" from tutor, soroban

by **math** on Sat Jan 13, 2007 3:01 am

galactus and soroban- thank you very much on your thorough replies.

galactus- O is theta

for the derivative of law of cosines, i am confused how you got the last part :
absin(O) (dO/dt)
and also, when you plugged in the values a few steps later, how did you get this part of the derivative equal to zero. how do we know the value of dO/dt?

i'm not trying to say there's a mistake, but just don't get the previous parts. thanks!

any final conclusions on the answer?

math
New Member

Posts: 45
Joined: Fri Jan 12, 2007 9:46 pm

Figure 25. (Post 2 of 5, row 1, Figure 20) Student, math, challenges galactus contribution

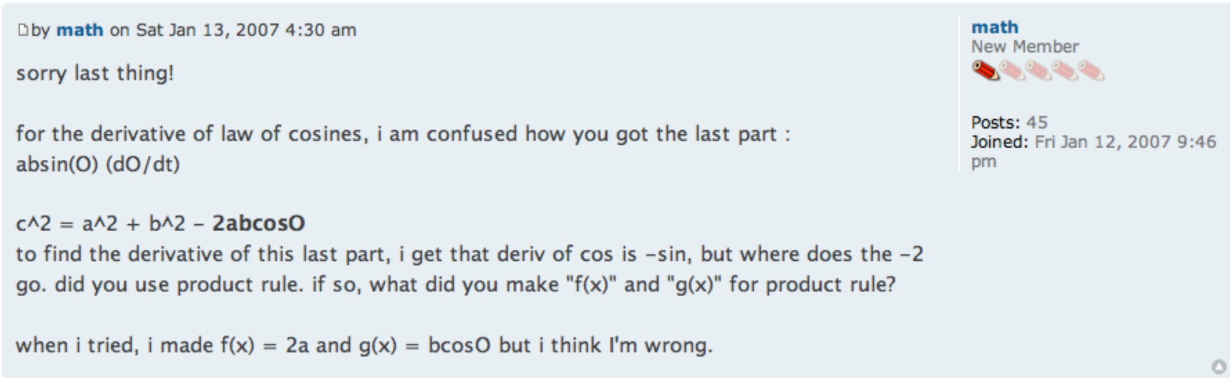
by **galactus** on Sat Jan 13, 2007 4:16 am

> **math wrote:**
> galactus and soroban- thank you very much on your thorough replies.
>
> galactus- O is theta
>
> > for the derivative of law of cosines, i am confused how you got the last part :
> > absin(O) (dO/dt)
>
> That's because the derivative of cos is -sin.
>
> > and also, when you plugged in the values a few steps later, how did you get this part of the derivative equal to zero. how do we know the value of dO/dt?
>
> Because the angle remains constant. There is no change, therefore $\frac{d\theta}{dt} = 0$
>
> i'm not trying to say there's a mistake, but just don't get the previous parts. thanks!
>
> any final conclusions on the answer?

galactus
Elite Member

Posts: 4112
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Figure 26. (Post 2 of 4, row 2, Figure 20) Tutor, galactus, responds to challenge



by **math** on Sat Jan 13, 2007 4:30 am

sorry last thing!

for the derivative of law of cosines, i am confused how you got the last part :
absin(O) (dO/dt)

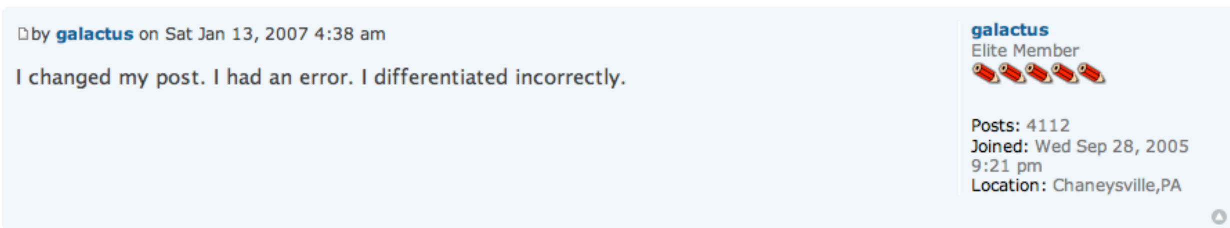
c^2 = a^2 + b^2 - **2abcosO**
to find the derivative of this last part, i get that deriv of cos is -sin, but where does the -2 go. did you use product rule. if so, what did you make "f(x)" and "g(x)" for product rule?

when i tried, i made f(x) = 2a and g(x) = bcosO but i think I'm wrong.

math
New Member

Posts: 45
Joined: Fri Jan 12, 2007 9:46 pm

Figure 27. (Post 3 of 5, row 1, Figure 20) Student, math, rechallenges galactus contribution



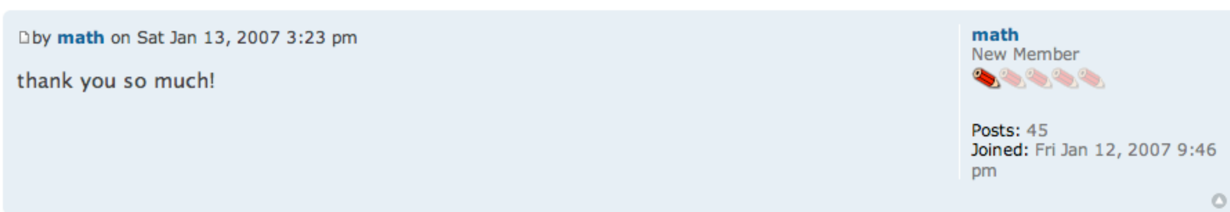
by **galactus** on Sat Jan 13, 2007 4:38 am

I changed my post. I had an error. I differentiated incorrectly.

galactus
Elite Member

Posts: 4112
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Figure 28. (Post 3 of 4, row 2, Figure 20) Tutor, galactus, responds to challenge with edit



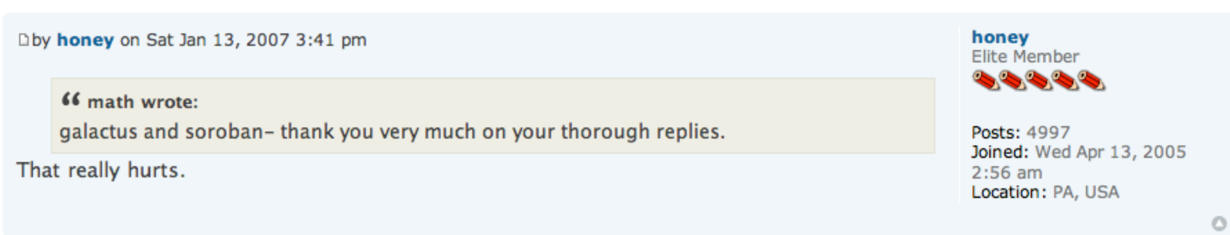
by **math** on Sat Jan 13, 2007 3:23 pm

thank you so much!

math
New Member

Posts: 45
Joined: Fri Jan 12, 2007 9:46 pm

Figure 29. (Post 4 of 5, row 1, Figure 20) Student, math, appreciates revision by galactus



by **honey** on Sat Jan 13, 2007 3:41 pm

> **math wrote:**
> galactus and soroban- thank you very much on your thorough replies.

That really hurts.

honey
Elite Member

Posts: 4997
Joined: Wed Apr 13, 2005 2:56 am
Location: PA, USA

Figure 30. (Post 2 of 3, row 3, Figure 20) Tutor, honey, on contribution not acknowledged

by **soroban** on Sat Jan 13, 2007 4:16 pm

soroban
Elite Member

Posts: 3481
Joined: Fri Jan 28, 2005 10:13 pm
Location: Lexington, MA

Hello, galactus!

I found your error . . . a silly one.
(Is there any other kind?)

> “
> Law of Cosine is, indeed, the thing to use.
>
> $$c^2 \;=\; a^2 + b^2 - 2ab\cos 120^o \;=\; a^2 + b^2 + ab$$
>
> From 10 am to 12 noon the vehicles travel for 2 hours.
> That's 80 miles for A and 100 miles for B.
>
> Differentiate:
>
> $$2c\frac{dc}{dt} \;=\; 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2ab\left(-\frac{1}{2}\right)$$
>
> $$2c\frac{dc}{dt} \;=\; 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}$$
>
> $$2c\frac{dc}{dt} \;=\; 3\left(a\frac{da}{dt} + b\frac{db}{dt}\right)$$ ← **here!**

We have:

$$2c\frac{dc}{dt} \;=\; \underbrace{2a\frac{da}{dt}} + \underbrace{2b\frac{db}{dt}} + \underbrace{a\frac{db}{dt}} + \underbrace{b\frac{da}{dt}}$$

$$2c\frac{dc}{dt} \;=\; (2a+b)\frac{da}{dt} \;+\; (a+2b)\frac{db}{dt}$$

Figure 31. (Post 2 of 2, row 4, Figure 20) Tutor, soroban publishes one of galactus's errors

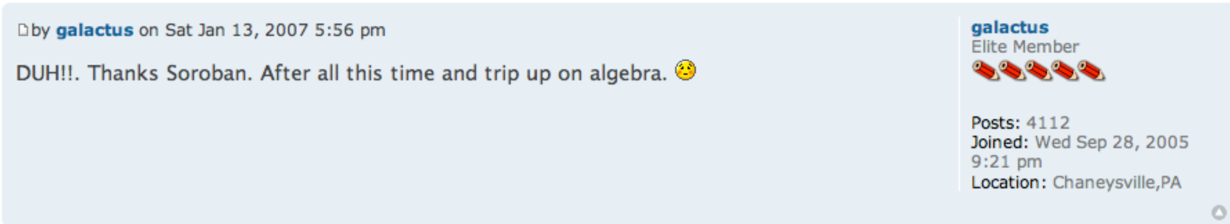
by **math** on Sat Jan 13, 2007 4:44 pm

math
New Member

Posts: 45
Joined: Fri Jan 12, 2007 9:46 pm

honey- i'm very sorry. thank you for pointing out that the angle is fixed.

Figure 32. (Post 5 of 5, row 1, Figure 20) Student, math, acknowledges honey’s contribution



by **galactus** on Sat Jan 13, 2007 5:56 pm

DUH!!. Thanks Soroban. After all this time and trip up on algebra.

galactus
Elite Member

Posts: 4112
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Figure 33. (Post 4 of 4, row 2, Figure 20) Tutor, galactus, thanks soroban for locating error

by **honey** on Sun Jan 14, 2007 1:01 am

> **math wrote:**
> honey- i'm very sorry. thank you for pointing out that the angle is fixed.

No worries. I'm just messing with your head.

honey
Elite Member

Posts: 4997
Joined: Wed Apr 13, 2005 2:56 am
Location: PA, USA

Figure 34. (Post 3 of 3, row 3, Figure 20) Tutor, honey, comments on belated thanks

Because **galactus** edited the post that contained the errors, the contents of the original version of this post must be inferred from subsequent contributions. However, it is clear that the post contained misinformation, and that the community functioned as a vigilant and supportive body in working through the mistakes. The error was first recognized because forum tutors came to different conclusions about the solution to the problem (the erred solution by **galactus**, a hint from **honey**, and an alternative solution from **soroban**); the student positioned her/himself as an inquirer and questioned the solution from **galactus** and this led **galactus** to further assess her/his mathematical work, and; third, fellow tutors in the community detected and diagnosed the misinformation. We can conclude that tutoring in a community supports mathematical quality and that interactions between forum members embody key mathematical practices (e.g. the location, diagnosis, and correction of errors) that can benefit all participants.

5.4.1 Summary of support for accuracy

One of the major concerns regarding unsupervised tutoring is the quality and mathematical accuracy of the instruction. This concern is legitimate, since, in an online help forum with a spontaneous participation structure, there is no guarantee that the tutors have formal qualifications, such as an advanced academic degree or professional experience. In fact, the tutors attracted to the forum often possess these qualifications, both self-reported and as evidenced in their contributions, but it is the case that the forum tutors still commit occasional mathematical errors. The community functions as a “review panel” that catches (and often directly addresses) these mistakes in a way that does not occur in other tutoring contexts (van de Sande, 2007b). Experiencing the process of review, remediation, revision, and resolution, allows students to see how experts “do mathematics” at an accessible level. At the same time, the tutors themselves profit from these conversations and the support that they receive from their peers.

5.5 CONCLUSIONS ON FORUM COMMUNITY

Despite the amorphous and multi-faceted nature of “community,” there is a need to understand more about what it means to participate and interact with others as a member of a community. For example, what benefits do individual members receive from this activity, and how do these, in turn, support the existence, life, and success of the community itself? In an SOH forum community, members who share a passion for mathematics and the desire to improve their understanding of this discipline interact with others on routine problems in a context that allows overlapping and responsive contributions. The first question is whether tutors making use of this affordance, and they do; approximately half of the exchanges included contributions from more than a single tutor. These exchanges provide a window into what it means to belong to a community of tutors since the forum tutors often addressed each other as well as the student. These inter-tutor conversations allow students to view and participate in authentic mathematical discourse at a mathematically accessible level. The subject of discussion is a “routine” problem that the student encountered in an introductory calculus course. At the same time, the forum community is functioning as an arena for tutors to engage in mathematical debate and supportive discussions. These discussions often center on alternative perspectives that are introduced to further support the student’s mathematical understanding. In this way, students are made aware that there are multiple ways of viewing and solving problems, and this experience of coming to a new mathematical realization proved exciting for students. The community also supports mathematical accuracy as tutors attend to the contributions of others and respond to feedback on their work. Incorrect contributions act as a “magnet” for attention, and evaluations are delivered with sensitivity and tact and received as an invitation to further participate, for instance through corrections and revisions. Participation within this community thus offers distinctive benefits – for the participants involved as well as the larger forum community – that are not afforded in traditional tutoring environments and can be considered as markers of success within a designed environment.

6.0 CALCULUS

The current research on calculus learning and understanding favors an empirical methodology, using tasks selected or designed to probe deeply into student understanding of a given concept or idea. In this tradition, students are given carefully crafted mathematical exercises to perform, and their responses are analyzed to assess the nature and extent of their abilities. For instance, Williams (1991) conducted a 2-phase, multiple session study on student understanding of limit in which participants were presented with opposing descriptions of limit and then asked to solve a series of problems that foregrounded the implications of the opposing viewpoints and produced anomalies with an informal conception of limit. The presence of open, online help forums creates an opportunity to conduct research using a complementary approach. Instead of *us giving students* exercises to solve, we can look at the exercises and solutions that *students bring to us*. Through eavesdropping on their calculus predicaments, woes, and foibles, we are supplied with a different and distinctive vantage on students' calculus experiences. The problems and difficulties that confront students in their daily (or perhaps nightly) performance of course assignments and in their preparation for examinations are available for observation and analysis.

Educational research on student understanding of the limit and related rates has revealed several "misunderstandings" regarding these topics. From the research on student understanding of limit, we know that students, even post instruction, often exhibit mental models of limit that are inconsistent with the mathematical concept. For instance, the limit is framed as the substitution of a single value, rather than as a dynamic process in which values of the function are approached. From the research on related rates, we know that students do not apply covariational reasoning, that is, the coordination of inputs and outputs to make sense of dynamic functional relationships. For instance, the variables in the problem statement that are functions of time are confused with values of the functions at a particular time. The online forums allow us to see how these issues surface as students are actively engaged in solving problems and, through this, may reveal what students find hard about the concept of limit and related rates.

6.1 STUDENT QUERIES ON LIMIT

There are at least two ways to approach the queries on limit that are found in the forum; we could take the limit problems and categorize them, or take the problems and issues that students post in response to the problems that they were assigned for analysis. The latter of these two options is what I have chosen as a means of revealing students' understanding of limit in a natural setting. There were three types of limit exercises that surfaced in the calculus forum: the evaluation of the limit for a given expression (85%), the use or meaning of the formal definition of limit (8%), and "inverted" problems in which the task was to constrain or construct a function to produce a certain limit (7%). Not surprisingly, this heavily skewed distribution of problem types is consistent with that found in many calculus textbooks in which much attention is paid to the mastery of procedures and less emphasis is placed on the use of definitions and the construction of proofs (Raman, 2004). In this section, we describe how each of these types of limit exercises was handled in the forum exchanges and draw conclusions about how students who participate in the forum think about "doing limits."

6.1.1 Evaluation exercises

Students posted exercises on evaluating limits that ranged from very elementary problems that surface toward the beginning of instruction on the limit (such as $\lim_{x\to\infty}(1/x+1)$) to more complex situations (such as $\lim_{x\to 0}[\sec(4x)]^{3/x}$) that require several operations and elaborate procedures. For these types of exercises, over 75% of the queries involved the transformation of the given expression into a form that could be interpreted, e.g. using "well-known" limits such as $\lim_{x\to 0}\frac{\sin x}{x}=1$ or procedures such as l'Hôpital's Rule). For example, the exchange in Figure 35 begins with forum student **MarkSA's** struggle to reorganize the expression $\sin x/(x+\tan x)$ in order to evaluate the limit at $x=0$.

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Find the limit of [sin(x)]/[x + tan(x)] as x -> 0

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

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Find the limit of [sin(x)]/[x + tan(x)] as x -> 0

by **MarkSA** on Tue Sep 25, 2007 4:58 pm

I'm having some trouble with this one:

Find the limit as x approaches 0 of:
[sin(x)]/[x + tan(x)]

First thing I did was to change tan(x) to sin(x)/cos(x). I then used the LCD of the denominator and ended up with:

[sin(x)cos(x)]/[(x * cos(x)) + sin(x)]

I'm not sure where to go from here. I can't see any further simplification using trig identities or otherwise. Can anyone help?

MarkSA
Junior Member

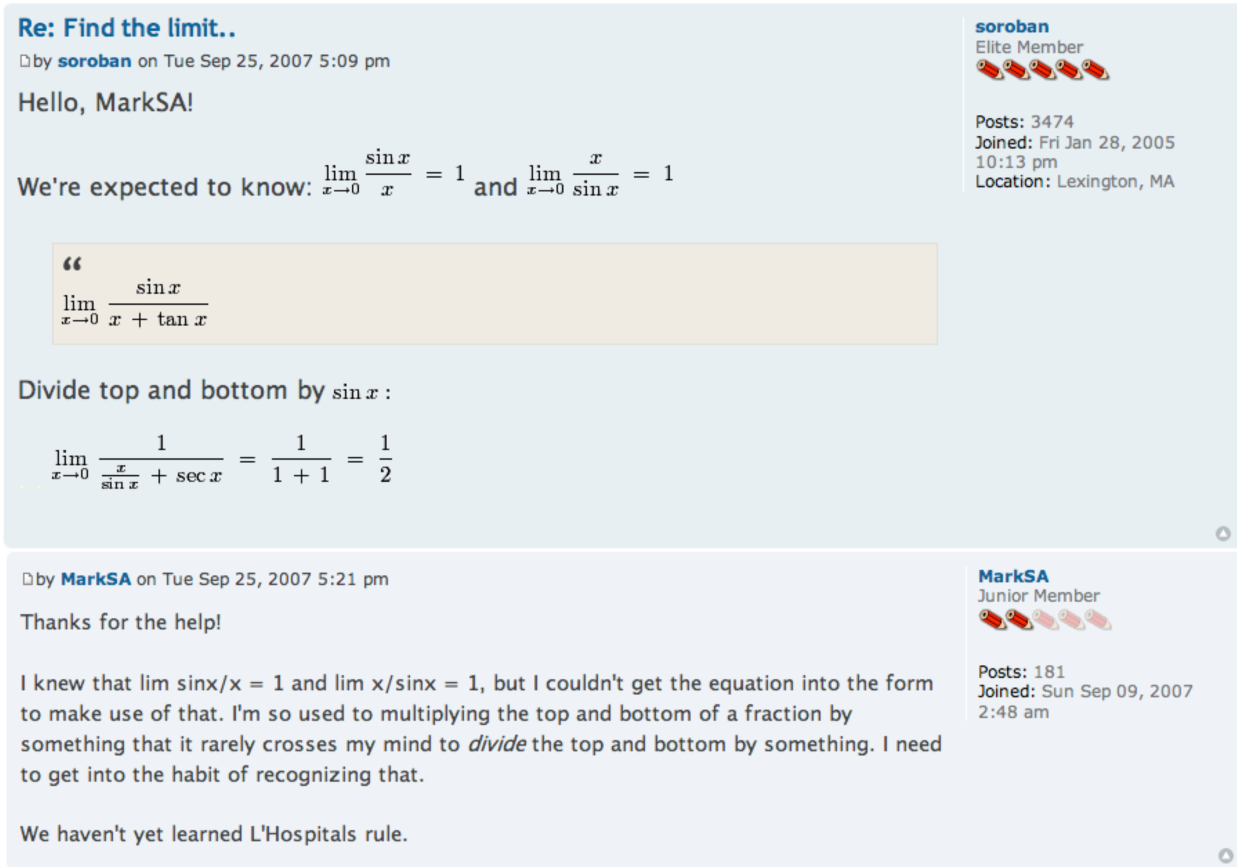
Posts: 181
Joined: Sun Sep 09, 2007 2:48 am

by **Hobostush** on Tue Sep 25, 2007 5:06 pm

Do you know L'Hospital's rule?

Hobostush
Senior Member

Posts: 1751
Joined: Mon Jun 18, 2007 11:47 am



Re: Find the limit..

by **soroban** on Tue Sep 25, 2007 5:09 pm

soroban
Elite Member
Posts: 3474
Joined: Fri Jan 28, 2005 10:13 pm
Location: Lexington, MA

Hello, MarkSA!

We're expected to know: $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{x}{\sin x} = 1$

> "
> $$\lim_{x\to 0}\frac{\sin x}{x + \tan x}$$

Divide top and bottom by $\sin x$:

$$\lim_{x\to 0}\frac{1}{\frac{x}{\sin x} + \sec x} = \frac{1}{1+1} = \frac{1}{2}$$

by **MarkSA** on Tue Sep 25, 2007 5:21 pm

MarkSA
Junior Member
Posts: 181
Joined: Sun Sep 09, 2007 2:48 am

Thanks for the help!

I knew that lim sinx/x = 1 and lim x/sinx = 1, but I couldn't get the equation into the form to make use of that. I'm so used to multiplying the top and bottom of a fraction by something that it rarely crosses my mind to *divide* the top and bottom by something. I need to get into the habit of recognizing that.

We haven't yet learned L'Hospitals rule.

Figure 35. Exchange on evaluating the limit of a given expression

6.1.1.1 Analysis of an exchange (evaluating limit)

Framing–needed. **MarkSA** is unable to complete the exercise because he is unable to start, that is, he is unable to "see" the expression in a certain way. Although he knows that a reorganization of the expression is necessary in order to evaluate the limit (which is an indeterminate form), he is unable to come up with a productive and enlightening transformation. His attempts [4:58 pm] have resulted in a dead end: "First thing I did was to change tan(x) to sin(x)/cos(x). I then used the LCD of the denominator and ended up with: [sin(x)cos(x)]/[(x * cos(x)) + sin(x)]. I'm not sure where to go from here. I can't see any further simplification using trig identities or otherwise. Can anyone help?" Indeed, roadblocks of this sort were the primary difficulty that students expressed in the forum when the exercise concerned evaluating a limit.

Framing–supplied. In this exchange, **soroban**, the second tutor to respond (who may have overlapped with the contribution of the first tutor just three minutes previously), provides a way of thinking about the expression [5:09 pm] that draws on the knowledge of a well-known limit: "We're expected to know: $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{x}{\sin x}=1$. **Soroban** is cognizant of this famous (and accessible) limit embedded in the given expression and therefore suggests the necessary transformation: "Divide top and bottom by $\sin x$." Seen in this light, the limit is obvious as a construction of limits of the individual terms. The last post in this exchange [5:21 pm] is a reflection from **MarkSA** on the novelty of thinking in these terms: "I knew that lim sinx/x = 1 and lim x/sinx = 1, but I couldn't get the equation into the form to make use of that. I'm so used to multiplying the top and bottom of a fraction by something that it rarely crosses my mind to *divide* the top and bottom by something. I need to get into the habit of recognizing that."

This exchange, from start to finish, provides a clear indication of the understanding that students are (and are not) bringing to this type of exercise (*Framing–needed*); they are relying on a set of "tricks" (such as trigonometric identities, algebraic operations, and unit multiplication) *without taking into account* the principles of calculus. This activity is, in some sense, akin to an attempt at constructing a bridge without a clear destination in mind; having a sense of the expression one desires (in this case, one that includes the well-known limit) guides the mathematical construction to get to it. The forum tutors, as more experienced mathematical learners (if not experts), are able to point students in the right direction. In this exchange, the tutor supplied the framing in its entirety (*Framing–supplied*). However, as we saw earlier in the exchange on related rates in which the tutor led the student to discover the flaw in her framing of the problem (Section 4.3.2), tutors can also position students to be active in the construction of the framing through the provision of hints and leading questions. Almost certainly, this latter approach is of more benefit in the long run, leading to a deeper understanding of the material and transfer to other problem situations.

The exchange in Figure 35 also demonstrates another (missing) characteristic of forum tutoring on exercises for evaluating limits; there is a notable absence of analytic discussion that touches on the behavior of the function in question and provides insight into the concept of limit. For instance, in this example, it can be easily inferred from the graphs of $y = \sin x$, $y = \tan x$, and $y = x$ that these functions behave very similarly in the neighborhood of $x = 0$. Therefore, the function $y = \frac{\sin x}{x + \tan x}$ behaves similarly to the function $y = \frac{x}{x + x} = \frac{x}{2x} = \frac{1}{2}$ for values of x that are close to zero, i.e. in the limit.[37] Although not rigorous, this analytic approach foregrounds the concept of limit (as the behavior of a function in a neighborhood) and sets the stage for an understanding of linear approximation and Taylor series, two topics that are routinely covered later in introductory calculus instruction. That is, it promotes a framing for limits that can then be called upon as a resource for constructing an understanding of other calculus concepts. However, this treatment of limit was extremely rare in the forum, even, as the exchange in Figure 36 shows, when introduced by the student.

[37] This same logic can be used to support the important limit, $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Limit of exponential fcn: [(2 ^ x) + 1] / [2^ (x + 1)]

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

Limit of exponential fcn: [(2 ^ x) + 1] / [2^ (x + 1)]

by **peblez** on Mon Nov 05, 2007 7:06 am

lim x--> infinity [(2 ^ x) + 1] / [2^ (x + 1)]

THis is what i did. I don't know exactly how to solve for this but , my logic is that it'll go to zero eventually, because the bottom goes to infinity faster than the top so it'll to go zero. But i am not sure exactly if this is right can someone prove this or tell me if i'm intuatively correct. thanks

peblez
New Member

Posts: 36
Joined: Tue Jan 30, 2007 2:44 am

Re: Limit of a exponential function

by **daon** on Mon Nov 05, 2007 8:15 am

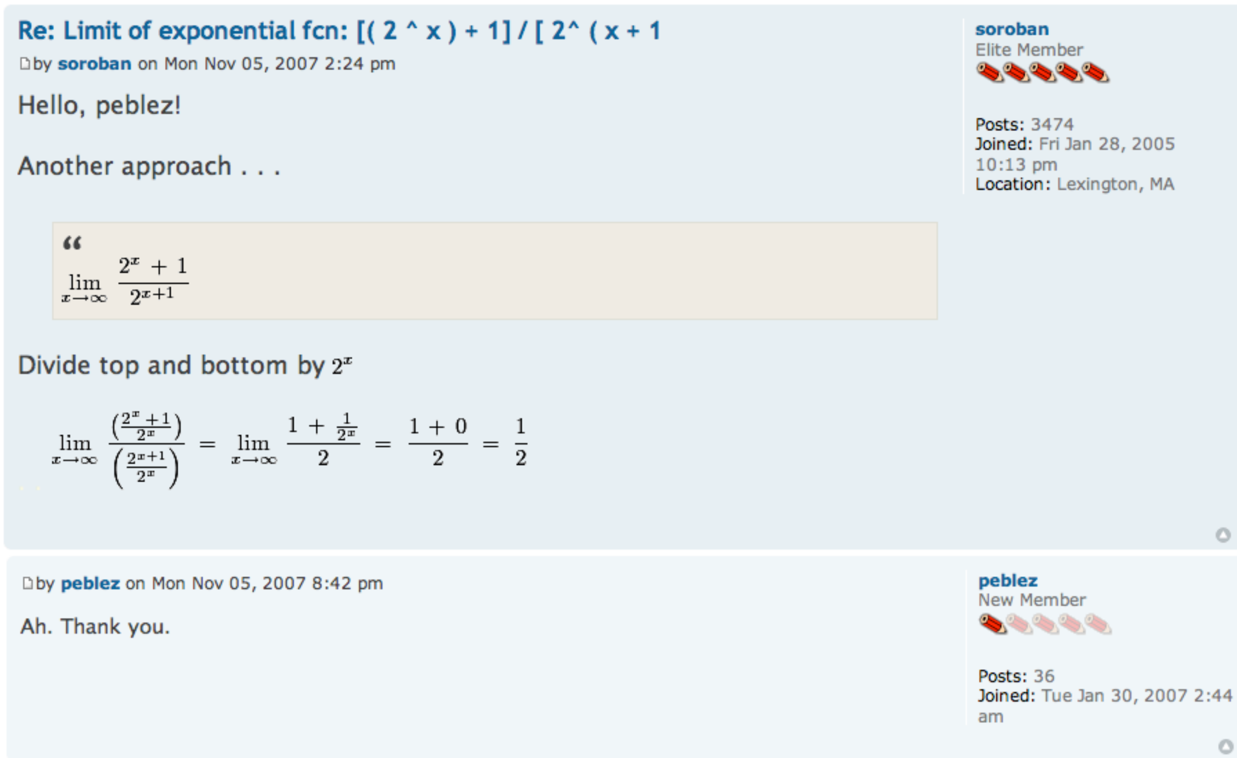
> **peblez wrote:**
> lim x--> infinity [(2 ^ x) + 1] / [2^ (x + 1)]
>
> THis is what i did. I don't know exactly how to solve for this but , my logic is that it'll go to zero eventually, because the bottom goes to infinity faster than the top so it'll to go zero. But i am not sure exactly if this is right can someone prove this or tell me if i'm intuatively correct. thanks

$$\frac{2^x+1}{2^{x+1}} = \frac{1}{2} + \frac{1}{2^{x+1}}$$

The second term approaches zero, but not the first. Since the limit of each term exists, use the fact that the limit of the sum is the sum of the limits. So it actually approaches 1/2, not zero.

daon
Full Member

Posts: 681
Joined: Fri Jan 27, 2006 7:51 am



Re: Limit of exponential fcn: [(2 ^ x) + 1] / [2^ (x + 1

by **soroban** on Mon Nov 05, 2007 2:24 pm

Hello, peblez!

Another approach . . .

> "
> $\lim_{x\to\infty} \frac{2^x + 1}{2^{x+1}}$

Divide top and bottom by 2^x

$$\lim_{x\to\infty} \frac{\left(\frac{2^x+1}{2^x}\right)}{\left(\frac{2^{x+1}}{2^x}\right)} = \lim_{x\to\infty} \frac{1 + \frac{1}{2^x}}{2} = \frac{1+0}{2} = \frac{1}{2}$$

soroban
Elite Member

Posts: 3474
Joined: Fri Jan 28, 2005 10:13 pm
Location: Lexington, MA

by **peblez** on Mon Nov 05, 2007 8:42 pm

Ah. Thank you.

peblez
New Member

Posts: 36
Joined: Tue Jan 30, 2007 2:44 am

Figure 36. Student introduction of analytic treatment not taken up

6.1.1.2 Analysis of exchange (analytic treatment)

In this exchange, forum student **peblez** proposes [7:06 am] an analytic treatment of $\lim_{x\to\infty}\frac{2^x+1}{2^{x+1}}$: "I don't know exactly how to solve for this but , my logic is that it'll go to zero eventually, because the bottom goes to infinity faster than the top so it'll to go zero." Although this analysis is incorrect (since the denominator and numerator are both growing at approximately the same rate), the approach itself is worth taking up, as discussed above. However, both tutors who respond propose alternative framings of limit that are based on reorganization: **Daon** [8:15 am] proposes splitting the expression into two terms, and **soroban** [2:24 pm] suggests a unit multiplication. These transformations are helpful (as acknowledged by **peblez** [8:42 pm]) but are not brought into resolution with the analytic framing that **peblez** introduced. (This could be achieved, for instance, by noting that, as x approaches infinity, the "+1" in the numerator does not contribute appreciatively, so that the expression can be thought of as $\lim_{x\to\infty}\frac{2^x}{2^{x+1}} = \lim_{x\to\infty}\frac{2^x}{2^x \cdot 2} = \frac{1}{2}$, producing a result that is consistent with the algebraic transformations.) Encouraging students to think of the evaluation of limits as an analysis of function behavior seems more conducive to achieving a mathematical mindset or "doing mathematics" than the approach that appears common to student experience in which sense-making within routine limit evaluations is not part of the picture.

6.1.1.3 Summary of limit evaluation exercises

Evaluation exercises were the most common type of limit query that students posted on the forum, accounting for 85% of the sample. These exercises represent routine problem solving and provide students with practice in applying the many techniques and principles of calculus. Interestingly, it was not the application of these techniques (some of which are very lengthy and quite involved) that caused students the most difficulty in limit evaluation exercises. Rather, the brunt of student difficulty was caused by an inability to reorganize a given expression into a form that can be evaluated in the limit. Students were unable to "see" a productive organization of the information and then construct a path to produce it. The tutors, who have more mathematical expertise and experience, pointed students in the right direction(s) by suggesting operations that could be performed and principles that could be invoked in the given situation. What was largely missing from the treatment of this type of problem, however, was an appeal to the behavior of a function in the neighborhood of interest. This analytical framing of limit is not only straightforward and intuitive but can be used as a future resource when students encounter other concepts in calculus and supports authentic mathematical practice.

6.1.2 Exercises on the definition of limit

In the current sample, there were only 8 limit queries that directly referred to the definition of limit, and only two of these had a complexity index greater than 7. This type of exercise, then, did not tend to spark extended discussion on the meaning or nature of the limit concept between multiple forum participants. Instead, the provision or explanation of a proof using the definition of limit typically ended the exchange. One notable exception (shown in Figure 37) involved a query from **Oneiromancy**, a relatively new forum student (participation in 14 threads), on using the formal definition of limit to construct a value of delta for a specific epsilon. In this exchange with a complexity index of 13, a tutor, **o_O**, helped **Oneiromancy** to interpret the formal definition in a meaningful fashion and establish a productive framing, rather than acting on it without connection to the limit concept. Thus, although rare, these discussions did occur and revealed how students are framing the definition in a way that divorces it from its object and need help adopting an interpretation that motivates the algebraic operations that shape the construction of a proof.

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A delta epsilon proof, using graph to find delta > 0

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

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A delta epsilon proof, using graph to find delta > 0

by **Oneiromancy** on Sun Jan 20, 2008 4:15 pm

I just need help on how to treat the absolute value sign.

I have this:

abs { (2 / sqrt (x + 1) + 2) * (x - 3) } < epsilon

What I really want is to move blah blah * (x - 3) to the other side so I can solve for delta (it tells me what epsilon is in the book). How do I do that? There might be another way, I'm new at this.

Oneiromancy
New Member

Posts: 14
Joined: Fri Sep 28, 2007 11:33 pm

Re: A delta epsilon proof.

by **o_O** on Sun Jan 20, 2008 6:05 pm

Are you trying to prove a limit? Can you post the problem in its entirety?

o_O
Full Member

Posts: 394
Joined: Sat Oct 20, 2007 8:08 pm
Location: Canada

Re: A delta epsilon proof.

by **Oneiromancy** on Sun Jan 20, 2008 6:11 pm

Sure:

Use the graphs to find a delta > 0 such that for all x, 0 < |x - a| < delta ==> |f(x) - L| < epsilon

10. (picture of graph showing delta and epsilon, but i still have to show work/proof other than pointing to graph)

f(x) = 2*sqrt (x+1)
a = 3
L = 4
epsilon = 0.2

The delta should eventually be 0.39 for the given epsilon, 0.2.

Oneiromancy
New Member

Posts: 14
Joined: Fri Sep 28, 2007 11:33 pm

Re: A delta epsilon proof.

by **o_O** on Sun Jan 20, 2008 6:18 pm

o_O
Full Member

Posts: 394
Joined: Sat Oct 20, 2007 8:08 pm
Location: Canada

So just using the definition of a limit:

Given e > 0, there exists d > 0 such that whenever 0 < |x - 3| < d, then it follows that |2sqrt{x + 1} - 4| < 0.2

Now, looking at the latter inequality, we want to get rid of the absolute value sign. So:

$$|2\sqrt{x+1} - 4 < 0.2$$

$$-0.2 < 2\sqrt{x+1} - 4 < 0.2$$

Can you carry on from here?

Last edited by o_O on Sun Jan 20, 2008 6:21 pm, edited 1 time in total.

Re: A delta epsilon proof.

by **Oneiromancy** on Sun Jan 20, 2008 6:21 pm

Oneiromancy
New Member

Posts: 14
Joined: Fri Sep 28, 2007 11:33 pm

Ohhh ya thanks.

So basically I'm going to add 4 to everything, square everything, etc. eventually I'll get x - 3 in which case I'll be able to convert it to absolute value (since both sides equal).

So what will that tell me once I do all that?

Re: A delta epsilon proof.

by **pka** on Sun Jan 20, 2008 6:24 pm

pka
Elite Member

Posts: 4254
Joined: Sun Jan 30, 2005 12:30 am

What you have written is hard for me to read.
If it is

$$abs\left(\frac{2}{\sqrt{x+1}+2}(x-3)\right) < \varepsilon$$

then you can write

$$\frac{4(x-3)^2}{\left(\sqrt{x+1}+2\right)^2} < \varepsilon^2$$

"A professor is someone who talks in someone else's sleep"
W.H. Auden

Re: A delta epsilon proof.
by **o_O** on Sun Jan 20, 2008 6:24 pm

Well eventually, you'll solve get:

|x – 3| < (some number) right?

Well, remember at the beginning we said that:

Given e = 0.2, there exists some **d** > **0** such that whenever **0** < |**x** – **3**| < **d** then it NECESSARILY FOLLOWS that |2sqrt{x + 1} – 4| < 0.2.

You basically worked backwards from 2sqrt{x+1} – 4 < 0.2 and got |x – 3| < (some number) which is what d should be.

o_O
Full Member

Posts: 394
Joined: Sat Oct 20, 2007 8:08 pm
Location: Canada

Re: A delta epsilon proof.
by **Oneiromancy** on Sun Jan 20, 2008 6:27 pm

That answered my question thanks.

Oneiromancy
New Member

Posts: 14
Joined: Fri Sep 28, 2007 11:33 pm

Figure 37. Exchange on the definition of limit exercise

6.1.2.1 Analysis of exchange (limit definition)

Framing–blah, blah. The goal of the exercise for **Oneiromancy** is apparent in her/his statement of the query [4:15 pm]: "I have this: abs { (2 / sqrt (x + 1) + 2) * (x - 3) } < epsilon. What I really want is to move blah blah * (x - 3) to the other side so I can solve for delta (it tells me what epsilon is in the book). How do I do that?" The focus here involves acting on the definition of limit without attending to its meaning so that the proposed rearrangement of the inequality lacks a conceptual motivation. The response [6:11 pm] to a query by forum tutor **o_O** on the nature of the exercise [6:05 pm] reveals that the curricular intent of the exercise is for the student to algebraically construct the constraint on the independent variable that satisfies the definition of limit for a specific value of epsilon and compare this result with the graphical representation (that is provided).

Framing–definition. This type of exercise on limit is fairly common in an introductory calculus treatment of the formal definition of limit since it encourages an unpacking of the string of inequalities and implications that comprise the definition. It is therefore not surprising that **o_O** begins her/his tutoring [6:18 pm] with the problem framed around the definition of limit: "So just using the definition of a limit: Given e > 0, there exists d > 0 such that whenever 0 < |x - 3| < d, then it follows that |2sqrt{x + 1} - 4| < 0.2." With the goal of satisfying this definition by constructing a constraint on the independent variable, **o_O** proposes an action ("Now, looking at the latter inequality, we want to get rid of the absolute value sign. So:... $-0.2 < 2\sqrt{x+1} - 4 < 0.2$") before positioning **Oneiromancy** to take over: "Can you carry on from here?"

Framing–questions. However, although **Oneiromancy** now sees [6:21 pm] the direction this line of argument is taking algebraically ("So basically I'm going to add 4 to everything, square everything, etc. eventually I'll get x - 3 in which case I'll be able to convert it to absolute value (since both sides equal)."), the concept of limit is still not part of her/his interpretation of the task: "So what will that tell me once I do all that?" The goal is in place but there is still no connection to the meaning of the definition that motivates these actions.

Framing–elaborated. The next two contributions in the exchange are posted at the same time [6:24 pm], one from **pka** addressing **Oneiromancy's** initial query at face value and the other from **o_O** focusing attention on the meaning of the definition and its relationship to the algebraic goal: "Well, remember at the beginning we said that: "Given e = 0.2, there exists some **d > 0** such that whenever **0 < |x - 3| < d** then it NECESSARILY FOLLOWS that |2sqrt{x + 1} - 4| < 0.2. You basically worked backwards from 2sqrt{x+1} - 4 < 0.2 and got |x - 3| < (some number) which is what d should be." The emphasis here is on the motivation for the algebraic operations, namely the identification of a neighborhood around the point of approach (x = 3) that ensures that the values of the function are within a certain distance of the limit (here, fixed at e = 0.2 but, in general, arbitrarily small). With this response, **o_O** is explicitly connecting the sequence of algebraic actions to the meaning of limit, an understanding that was clearly absent from **Oneiromancy's** approach to the exercise. The helpfulness of this dialogue is evident in the final contribution from the student [6:27 pm]: "That answered my question thanks." The fact that **o_O** did not complete the exercise for **Oneiromancy** and provide a worked solution lends credence to this expression of gratitude toward the construction of a deeper and more coherent understanding of the limit.

6.1.2.2 Summary of limit definition exercises

The de-emphasis of formal definitions in calculus instruction may have contributed to the low number of this type of limit exercises on the forum. The formal definition of limit is not a topic on the Calculus AP examinations so that even advanced or accelerated courses may not devote much time and attention to it in their treatment of the limit. In addition, the few queries on the forum that did center on the definition of limit did not typically result in exchanges of high complexity and mathematical depth. Rather than setting off discussions between multiple forum participants that invoked key mathematical principles and contained sophisticated or intricate pedagogical moves, such exercises were generally answered by brief, factual contributions. It appears as though drawing students and others into dialogue on this aspect of "doing calculus" is an open problem, although we get a taste of what this might look like in the example exchange; the tutor repeatedly focused attention on the meaning of the definition and the reversal reasoning in the construction of the solution, and the student positioned himself as an inquirer, supporting a move from a framing that focused on algebraic manipulations (*Framing–blah, blah*) to one that incorporated the meaning of the formal definition of limit (*Framing–definition, Framing–questions, Framing–elaborated*). Engaging in more such conversations might help students achieve a more coherent and consistent understanding of the limit concept and gain an appreciation for the beauty and elegance of the mathematical rigor and logical thought behind "the calculus."

6.1.3 "Inverted" exercises

The number of queries representing "inverted exercises," in which the task was to construct a function with a given limit or ascertain values for which the limit existed, was also low. Most introductory calculus textbooks do not contain a large number of these challenging and thought-provoking exercises. The solution of such exercises requires an analysis of function behavior and its relationship to the limit concept and cannot be carried out using a prescribed technique. For example, the solution to the query in Figure 18 rests on an understanding of the conditions under which the limit of the ratio of two functions exists. Of the exchanges concerning this type of exercise, approximately 50% grew into extended discussions between multiple forum participants. In such cases, the query was a springboard for inter-tutor inquiry and debate, as discussed in 5.2. Figure 38 contains an excerpt from an exchange with a complexity index of 19 in which two tutors engaged in an extended discussion on the query which was to find all values for which the limit of the function $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ x^2 \text{ if } x \text{ is irrational} \end{cases}$ exists and to justify the solution. The excerpt begins with a posting from **pka** after the student posed the query, received a response from **pka** ("Try to convince yourself that there two values: a=0 & a=1)", and replied with a proposed justification of this solution[38].

by **pka** on Tue Oct 16, 2007 4:04 pm

pka
Elite Member

Posts: 4277
Joined: Sun Jan 30, 2005 12:30 am

$x \approx 0 \quad \Rightarrow \quad x^2 \approx 0 \quad \& \quad x \approx 1 \quad \Rightarrow \quad x^2 \approx 1$

Those are the only two values that have that property.

"A professor is someone who talks in someone else's sleep"
W.H. Auden

by **Hobostush** on Tue Oct 16, 2007 4:17 pm

Hobostush
Senior Member

Posts: 1771
Joined: Mon Jun 18, 2007 11:47 am

pka,

Isn't it true for this function that the function is discontinuous everywhere - but limit exists everywhere?

The function has a definite value at every point. Thus I would think limit exists.

It is discontinuous everywhere because we could not define a interval for epsilon-delta test.

Am I thinking correctly about this problem?

by **pka** on Tue Oct 16, 2007 5:09 pm

pka
Elite Member

Posts: 4277
Joined: Sun Jan 30, 2005 12:30 am

> **Hobostush wrote:**
> It is discontinuous everywhere because we could not define a interval for epsilon-delta test.

Here is a proof that it is continuous at x=1.

If $\varepsilon > 0, \Rightarrow \quad \delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ and $abs\,(x-1) < \delta \Rightarrow \quad abs\,(x+1) < 3$.

Now we have $abs\,(f(x) - 1) = \left(\begin{array}{ll} abs\,(x-1) & x \in Q \\ abs\,(x^2-1) & x \notin Q \end{array} \right.$

In the first case we have $abs\,(x-1) < \delta = \varepsilon \Rightarrow \quad abs\,(f(x) - 1) < \varepsilon$.

In the second case we have
$abs\,(x-1) < \delta \le 1 \Rightarrow \quad abs\,\left(x^2-1\right) = abs\,(x+1)\,abs\,(x-1) < 3(\delta) < \varepsilon$.

QED

"A professor is someone who talks in someone else's sleep"
W.H. Auden

[38] This part of the exchange and the last two contributions are not included here because they are interactions between the student and tutors and not part of the inter-tutor discourse.

by **Hobostush** on Tue Oct 16, 2007 5:50 pm

Thanks, pka.

So the function is continuous –only – at the vicinity of x = 0 and x = 1.(I could not find that before).

However, the function has defined limit at every point though – correct?

Hobostush
Senior Member

Posts: 1771
Joined: Mon Jun 18, 2007 11:47 am

by **pka** on Tue Oct 16, 2007 7:12 pm

> **Hobostush wrote:**
> So the function is continuous only AT = 0 and x = 1.(I could not find that before).

Yes that is correct.

> **Hobostush wrote:**
> However, the function has defined limit at every point though – correct?

No that is incorrect.

Let's take an example, x=2. The sequence $\left(2+\frac{\sqrt{2}}{n}\right) \to 2$.

But every term is irrational so $f\left(2+\frac{\sqrt{2}}{n}\right) = \left(2+\frac{\sqrt{2}}{n}\right)^2 = \left(4+\frac{2\sqrt{2}}{n}+\frac{2}{n^2}\right) \to 4$.

On the other hand, the sequence $\left(2+\frac{1}{n}\right) \to 2$ and every term is rational.

Thus, $f\left(2+\frac{1}{n}\right) = \left(2+\frac{1}{n}\right) \to 2$.

In order for *f* to have a limit at 2, say L, for **any** number close to 2, but not 2, say t, then *f(t)* must be close to L. But that is false.

Because any real number is the limit of a sequence of rationals and also the limit of irrationals then the only time the limit will exist is when x=0 or x=1.
Because this requires $x = x^2$.

"A professor is someone who talks in someone else's sleep"
W.H. Auden

pka
Elite Member

Posts: 4277
Joined: Sun Jan 30, 2005 12:30 am

Figure 38. Excerpt from exchange on "inverted" limit exercise

This tutor dialogue was initiated by a two-part challenge [4:17 pm] made by forum tutor, **Hobostush**: "Isn't it true for this function that the function is discontinuous everywhere - but limit exists everywhere?" **Pka** had asserted that the limit only existed at two values, namely $x = 0$ and $x = 1$, and **Hobostush** questioned this claim and proposed, in addition, that the function is discontinuous everywhere. **Hobostush** has framed the limit as the value of the function ("The function has a definite value at every point. Thus I would think limit exists") so that the existence of the limit is contingent only on the function being defined at each point. In response to **pka's** rebuttal through a proof [5:09 pm] showing that the function *is* continuous at $x = 1$, **Hobostush** conceded this result [5:50pm] but repeated the challenge concerning the existence of the limit: ("So the function is continuous -only - at the vicinity of x = 0 and x = 1.(I could not find that before). However, the function has defined limit at every point though - correct?"). This challenge was met by **pka** [7:12 pm] with a proof by counterexample showing that the limit does not exist at $x = 2$: "In order for f to have a limit at 2, say L, for **any** number close to 2, but not 2, say t, then *f(t)* must be close to L. But that is false." **Pka** ends the contribution with an elaboration of her/his previous claim concerning the query: "Because any real number is the limit of a sequence of rationals and also the limit of irrationals then the only time the limit will exist is when x=0 or x=1. Because this requires $x = x^2$"

6.1.3.1 Summary of "inverted" limit exercises

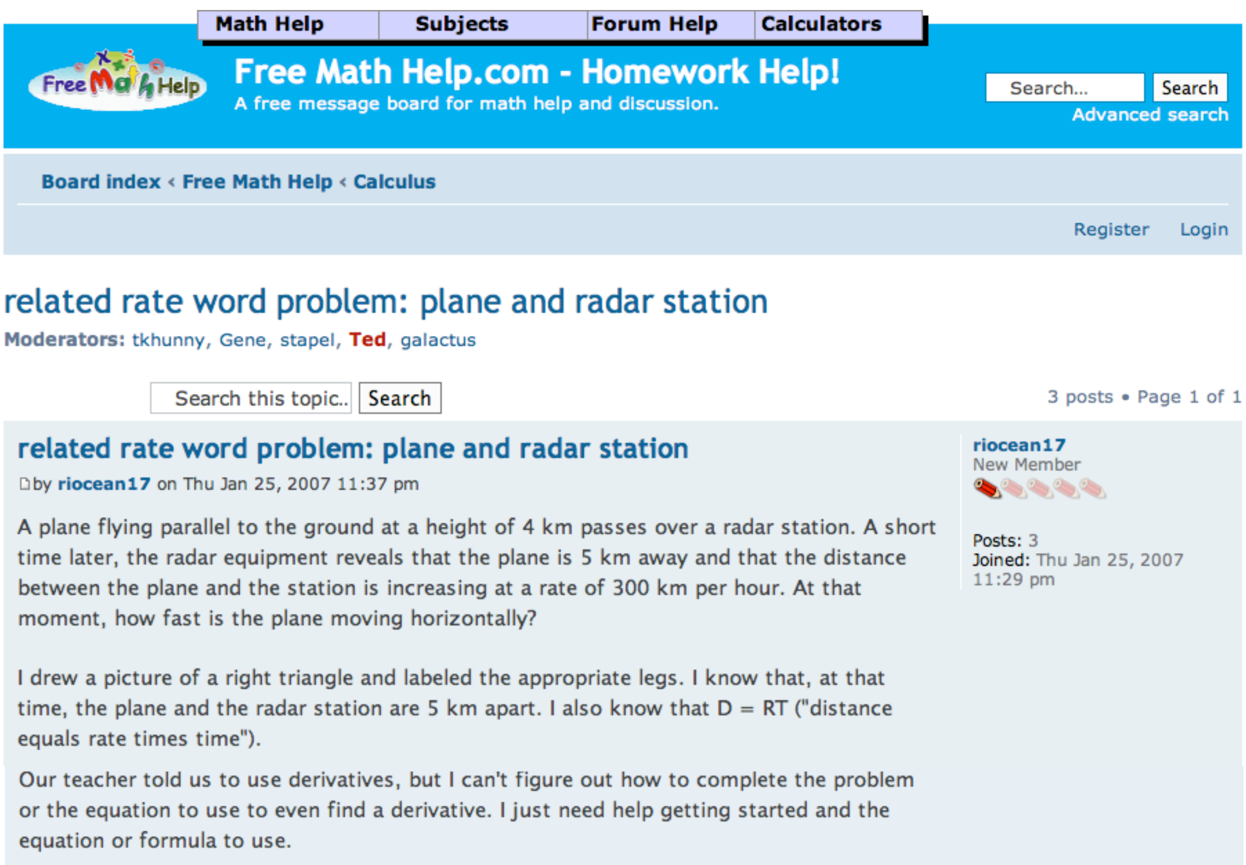
This exchange and the one in Figure 18 show how an unusual and involved query can stimulate lively and mathematically rich discussion amongst multiple forum tutors. In these discussions, we see authentic mathematical discourse as tutors engage with one another using arguments and justifications backed by mathematical principles, proofs, and counter-examples. These inverted exercises led to a refined understanding through interaction on what it means for a limit to exist, the very nature and essence of the concept. In this example, **Hobostush** began by applying a framing of limit based on the function having a defined value and a framing of continuity (a related concept) based on the requirement of a "connected" interval on which the function is defined. Another forum tutor, **pka**, responded with proofs that supported valid ways of framing this problem and conflicted with the framings that **Hobostush** had adopted. There is no question that wrapping one's mind around functions such as the one in this exercise is a difficult task. In particular, there is no way to construct a graph from which the limit and continuity of the function can be "observed" and used to support a productive framing. The discussion between the tutors filled this gap by invoking mathematical principles and techniques and resulted in the establishment of a shared and coherent understanding of the behavior of the function in terms of limits and continuity.

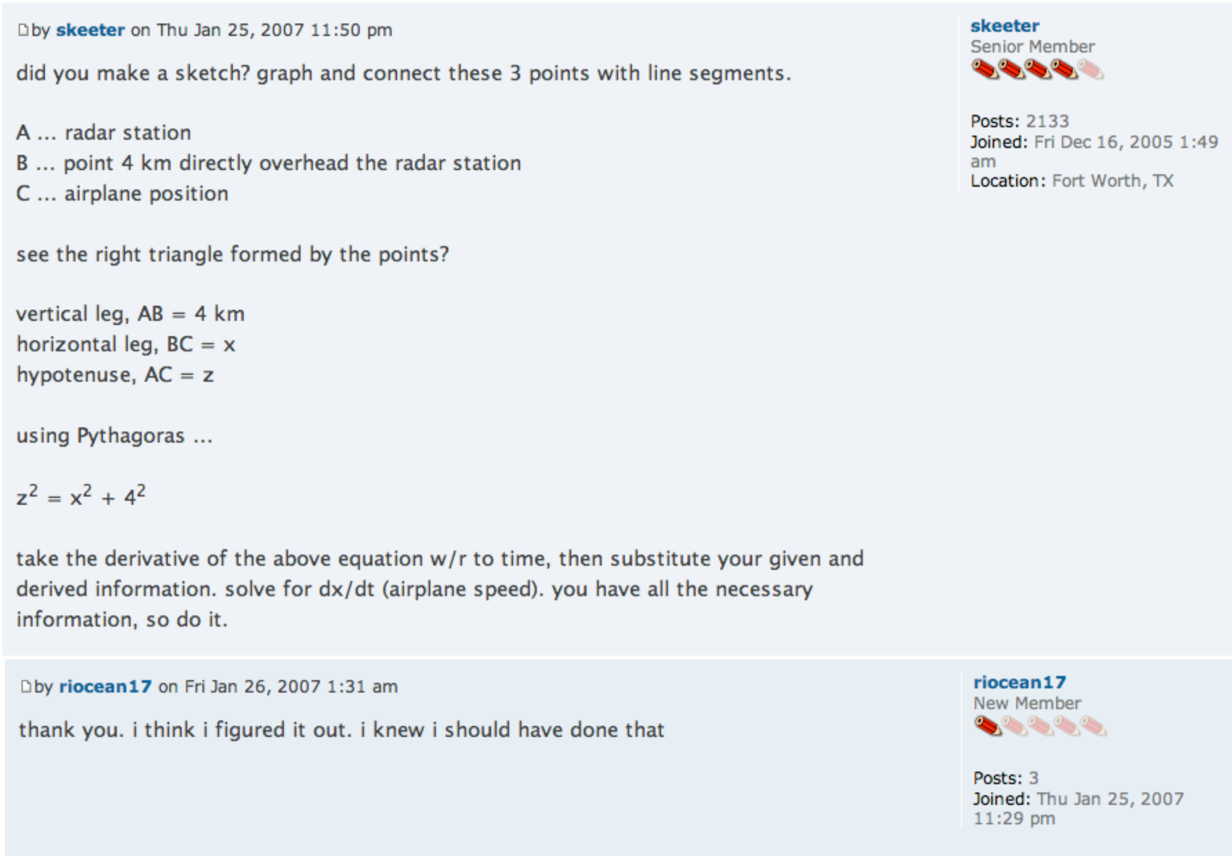
6.2 STUDENT QUERIES ON RELATED RATES

Unlike exercises on limit, related rates problems are cut from the same cloth and follow a single scheme, namely modeling the problem situation and finding an unknown rate of change (or other piece of "missing" information). The solution to such problems relies on a framing that includes the variables, their rates of change, the relationship that links them, and their meaning in the context of the problem. In addition, the solution requires the enactment of involved calculus procedures such as implicit differentiation and the chain rule that are based on the fundamental concept of the derivative. Each of these pieces of the construction of the solution, modeling and carrying out procedures, represents a location where errors can occur.

6.2.1 Framing

The solution of a mathematical problem requires the construction of a mental representation that organizes the elements in the problem situation and establishes the relationships between them. For instance, to solve a related rates problem, the quantities and their rates of change must first be mapped onto variables and these must be related to one another in a relevant fashion. In the forum, the most prevalent cause of student problems constructing solutions to related rates problems was either an inability to initiate a frame (42%) or conceptual difficulties encountered during the process of establishing a frame (30%). An inability to frame the problem was manifest in a complete absence of any mathematical constructions relevant to the solution of a related rates problem. For example, in the exchange in Figure 39, the student, **riocean17**, who posts the query, is unable to establish a mental model of the problem that incorporates the rates at which the objects are moving, the relationship between the changing quantities, and the configuration of objects at the time of interest.





by **skeeter** on Thu Jan 25, 2007 11:50 pm

did you make a sketch? graph and connect these 3 points with line segments.

A ... radar station
B ... point 4 km directly overhead the radar station
C ... airplane position

see the right triangle formed by the points?

vertical leg, AB = 4 km
horizontal leg, BC = x
hypotenuse, AC = z

using Pythagoras ...

$z^2 = x^2 + 4^2$

take the derivative of the above equation w/r to time, then substitute your given and derived information. solve for dx/dt (airplane speed). you have all the necessary information, so do it.

skeeter
Senior Member

Posts: 2133
Joined: Fri Dec 16, 2005 1:49 am
Location: Fort Worth, TX

by **riocean17** on Fri Jan 26, 2007 1:31 am

thank you. i think i figured it out. i knew i should have done that

riocean17
New Member

Posts: 3
Joined: Thu Jan 25, 2007 11:29 pm

Figure 39. Initial inability to establish framing for related rates problem

6.2.1.1 Analysis of an exchange (initiating frame)

Frame–use derivatives? Although **riocean17** knows [11:37 pm] that the objects are oriented in a triangle and that two of them are separated by a distance of 5 km ("I drew a picture of a right triangle and labeled the appropriate legs. I know that, at that time, the plane and the radar station are 5 km apart."), there is no evidence that the problem is being framed as a related rates problem that relies on covariational reasoning, a coordination of the quantities and their rates of change. Instead, the extent of **riocean17's** understanding is that the solution must somehow involve derivatives, and this realization is only due to "direction" from the teacher: "Our teacher told us to use derivatives, but I can't figure out how to complete the problem or the equation to use to even find a derivative. I just need help getting started and the equation or formula to use." When forum tutor, **skeeter**, responds [11:50 pm], s/he establishes a framing for the problem that includes the configuration of objects as it varies over time (labeling the distance between the radar station and the point overhead as fixed at 4 km, the horizontal distance between this point and the airplane as x [x(t)], and the distance between the radar station and the airplane as z [z(t)]) and the covariational relationship ("using Pythagoras ... $z^2 = x^2 + 4^2$"). In the sketch of the remainder of the solution, **skeeter**, emphasizes the covariation of the elements in the framing (an important move since the variables in the relationship are not explicitly notated as functions of time[39]) and designates which changing quantity is sought: "take the derivative of the above equation w/r to time, then substitute your given and derived information. solve for dx/dt (airplane speed). you have all the necessary information, so do it." The final posting in this exchange [1:31 am] is from **riocean17** expressing that the provision of the frame was helpful: "thank you. i think i figured it out. i knew i should have done that."

If we put faith in **riocean17's** reported construction of the solution (unfortunately not shared with the forum), then it was the establishment of a frame for this related rates problem – rather than procedures such as implicit differentiation – that posed a roadblock when the problem was encountered. As the exchange excerpt in Figure 40 shows, this sentiment was openly expressed on the forum by students, such as **Stealmylilhart**, as they lamented the challenging nature of related rates problems and candidly described why they felt these problems were so difficult.

[39] Engelke Infante (2007, p. 266) noted that, although mathematicians often use a single variable symbol (e.g. "*a*") to denote a function of time and then operate on it accordingly (e.g. treating it as "*a(t)*") in related rates solutions, students need to see "time" explicitly represented.

by **Stealmylilhart** on Fri Nov 03, 2006 1:38 am

I think that I have it fine...my main issue with these problems is setting them up and deciding on the write equation to use, because we do a lot of different area ones, etc. I can differentiate fine, i have no issues with that. Im just trying to practice on setting these up. Thanks for the help though. What you guys are writing is helping me to understand what I need to think about for setting them up.

Stealmylilhart
New Member

Posts: 11
Joined: Tue Oct 31, 2006 10:47 pm

Figure 40. Student view on related rates exercises

Admittedly, the cases in which students were unable to initiate a frame do not reveal much about the nature of their understanding of related rates (aside from the fact the establishing a frame is critical to success). However, the exchanges in which students experienced difficulties in the process of establishing a frame, provide insight into how students think about related rates problems and go about the process of solving them in an authentic setting. For example, the exchange in Figure 41 shows how divorcing the semantic and symbolic interpretation of variables can be disruptive to the establishment of a productive frame for solving related rates problems and how reuniting these alters the conceptualization of these exercises.

Related Rates: how fast is water height incr. inside tank?

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

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4 posts • Page 1 of 1

Related Rates: how fast is water height incr. inside tank?

by **Blkmage8** on Fri Feb 22, 2008 5:30 am

First of all, I'm not actually taking a calculus class right now but I'm sort of teaching myself. That said, I'm terribly confused on related rates. I was able to a few problems but I'm having a lot of trouble still.

*A cylindrical tank with a radius of 6 meters is filling with fluid at a rate of 108pi m^3/sec. How fast is the height increasing?

I know I need to find dh/dt and I need an equation that relates volume to height. I used:

V = (pi)(r^2)(h), so then

dV/dt = pi [(r^2 dh/dt) + (h * 2r dr/dt)]

I'm a little unsure of if I did that right but if I did, then regardless of substitution for the radius, I still can't solve for dh/dt without knowing the h or dr/dt. I can't figure out a way to find the height or eliminate it from the problem. I couldn't figure out a way to solve for dr/dt using a different equation(s) either. Any help would be appreciated.

Also, if any has any general tips for relating the functions that'd be great

Blkmage8
New Member

Posts: 6
Joined: Fri Feb 22, 2008 5:22 am

Re: Related Rates: Word Problems

by **galactus** on Fri Feb 22, 2008 11:25 am

The radius of a cylinder remains constant, therefore, dr/dt=0

galactus
Elite Member

Posts: 3985
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

Re: Related Rates: how fast is water height incr. inside tank?
by **Blkmage8** on Fri Feb 22, 2008 5:40 pm

Blkmage8
New Member

Posts: 6
Joined: Fri Feb 22, 2008 5:22 am

Oh I see now. I just got into the habit of monotonously just solving for the rates I wasn't really paying attention to what dr/dt really represented. So I got:

dV/dt = pi [(r^2 dh/dt} + (h * 2r dr/dt)]
108pi = pi [(36 dh/dt)]
3 m/sec ?

If you're willing to help with one more: (I'm really starting to get the hang of it now I think)

The sides of an equilateral triangle are increasing at the rate of 27 in/sec. How fast is the triangle's area increasing when the sides of the triangle are each 18 inches long?

I know I need to find dA/dt but I'm entirely sure what 27 in/sec is representing. I mean, I couldn't find an equation relating sides to area.

A = 1/2(bh)
dA/dt = 1/2 [(b dh/dt) + (h db/dt)]

Since it's equilateral then the base, b, is given by b=s=18 inches. But then that makes things more complicated
I'm just really bad at setting up these equations and area was all I could go by. Since it's asking how area is changing then I'm assuming that dh/dt and db/dt are probably changing as wlel so they can't be 0.

Re: Related Rates: how fast is water height incr. inside tank?
by **galactus** on Fri Feb 22, 2008 5:44 pm

galactus
Elite Member

Posts: 3985
Joined: Wed Sep 28, 2005 9:21 pm
Location: Chaneysville,PA

The area of an equilateral triangle is given by $A = \frac{\sqrt{3}}{4}s^2$. Where s is the length of a side

So, you have $\frac{ds}{dt} = 27$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}s\frac{ds}{st}$$

Finish it?.

Figure 41. Difficulty in process of establishing framing for related rates

6.2.1.2 Analysis of an exchange (framing repair)

Framing–disconnect. In the initial posting [5:30 am], there is evidence that the student, **Blkmage8**, has begun to construct a framing of the problem, establishing the relationship between the volume, radius, and height of the fluid in the cylindrical tank ("V = (pi)(r^2)(h)") and their rates of change ("dV/dt = pi [(r^2 dh/dt) + (h * 2r dr/dt)]"). However, this process is disrupted when **Blkmage8** is unable to proceed because of missing information: “I'm a little unsure of if I did that right but if I did, then regardless of substitution for the radius, I still can't solve for dh/dt without knowing the h or dr/dt. I can't figure out a way to find the height or eliminate it from the problem. I couldn't figure out a way to solve for dr/dt using a different equation(s) either. Any help would be appreciated.” This impasse has resulted because the meaning of the variables is disconnected from their symbols in the framing that **Blkmage8** is using, and it is precisely this relationship that forum tutor, **galactus**, foregrounds [11:25 am] in a succinct intervention: “The radius of a cylinder remains constant, therefore, dr/dt=0.”

Framing–repair. The “Aha” experience that often accompanies the introduction of a new or different way of looking at things and the realization that the framing of related rates problems must include a link between meaning and the symbolic representation are evident in **Blkmage8’s** follow-up [5:40 pm] contribution that also contained the correct completion of the problem solution: “Oh I see now. I just got into the habit of monotonously just solving for the rates I wasn't really paying attention to what dr/dt really represented.” It is worth noting here that this practice of self-reflection and diagnosis of what was wrong in a prior understanding was common in the forum; it occurred in 57% (limit) and 33% (related rates) of the exchanges that exhibited strong resolution. This type of interaction may set apart participation in this community versus other tutoring environments and may lead to increased retention of the help received in these relatively short and focused encounters.

Framing–impact. The impact that this altered framing has on **Blkmage8’s** understanding and treatment of related rates is evident in the solution attempt of a second problem that is appended to this exchange [5:40 pm]. Once again, the framing contains three variables and their rates of change (in this case, area, base, and height), but, this time, **Blkmage8** has incorporated their meaning when contemplating a reorganization: “Since it's asking how area is changing then I'm assuming that dh/dt and db/dt are probably changing as wlel so they can't be 0.” There is recognition that the area of the triangle cannot be increasing without some change in the base and height – in contrast to the lack of recognition that the radius of the fluid in a cylindrical tank does not change as the volume increases in the previous problem.

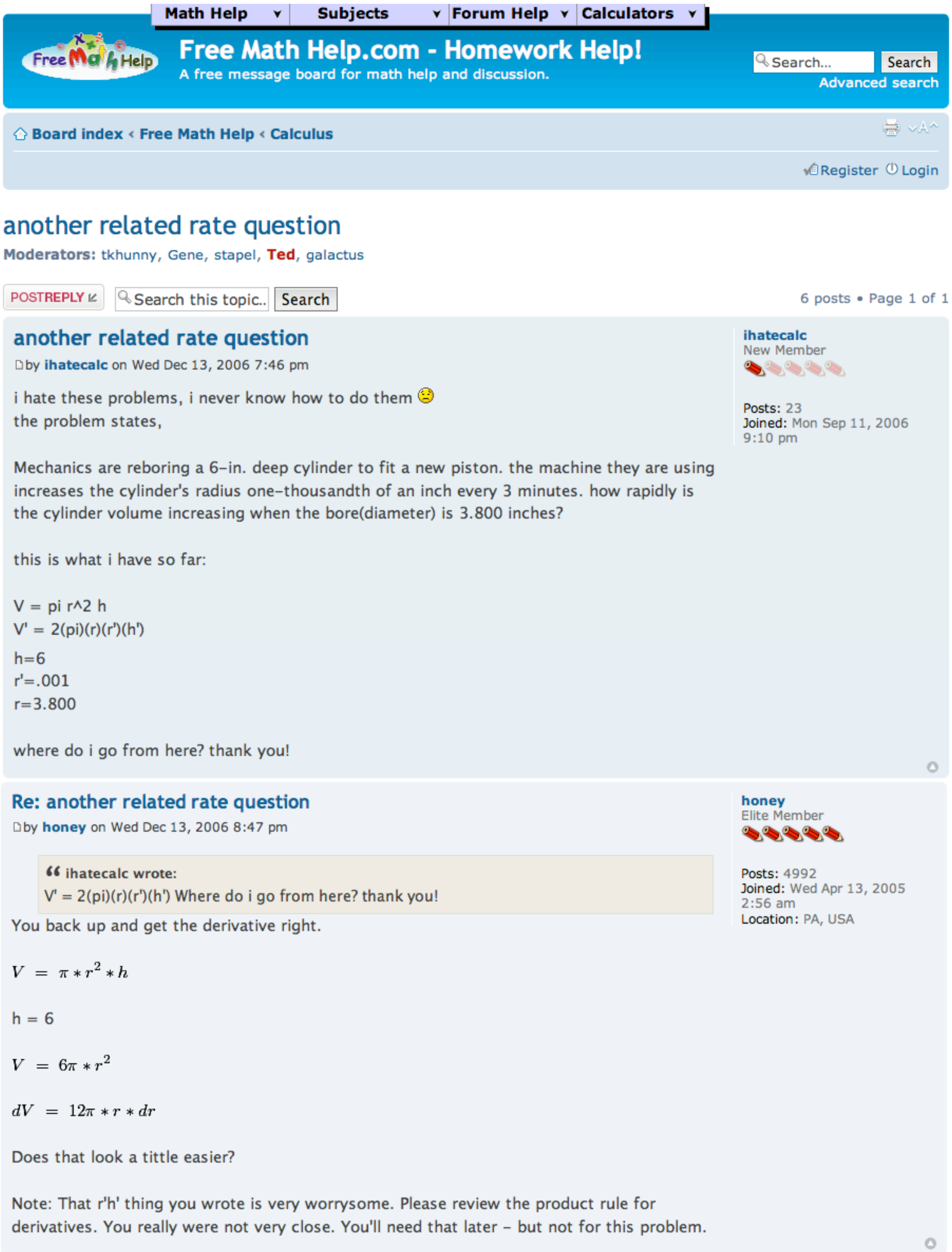
Framing–provided. Unfortunately, **Blkmage8** does not recognize that the base and height of an equilateral triangle are related and that this information can be used to reorganize the expression for area as a function of a single variable. **Galactus** [5:44 pm] elects to provide this information as a formula (“The area of an equilateral triangle is given by $A = \frac{\sqrt{3}}{4}s^2$. Where s is the length of a side”) and frames the problem accordingly without engaging in the construction of the relationship. Although **Blkmage8** does not return to the exchange, the message that s/he may have taken away was that the formula for the area of an equilateral triangle was pivotal in framing this problem, without realizing that her/his way of approaching the problem would have, with a bit of geometry, solidified a productive way of framing related rates problems of this type.

6.2.1.3 Summary of framing

These two exchanges demonstrate the ways in which forum students struggled with related rates problems; they are unable to initiate a framing from the outset (*Framing–use derivatives?*), or, they experience conceptual problems in establishing a productive framing (*Framing-disconnect*). In the former case, it is impossible to ascertain the nature of their understanding of related rates, but the large number of such instances does signify that current instructional treatments of this topic are not providing a strong foundation for developing a sense of covariational reasoning and the acquisition of good problem-solving skills. In the latter case, students are making headway at establishing a framing of the problem but become stymied in the process. For example, the student may be able to construct a framing that incorporates the variables, their rates of change, and the relationship between them, but will reach an impasse if these are not connected to the meaning of the variables in the problem situation. The intervention that addresses these issues is to assist the student in the construction of a productive framing (*Framing–repair*), ideally by appealing to resources that the student has rather than by explicitly providing the framing as a partially worked solution.

6.2.2 Enacting procedures

Related rates problems are generally presented as an application of implicit differentiation. That is, they are intended to serve as a real-life context in which students can practice implicit differentiation, a procedure that is usually taught just before the topic of related rates is introduced. Given that they were able to initiate and pursue a framing of the problem, the "procedural" aspect of the related rates problems did not appear to cause students who posted in the forum much difficulty; less than 20% of the errors in constructing a solution to the related rates problem could be directly linked to an error in the enactment of a procedure and approximately half of these were caused by computational mistakes, such as a solution of $z = 2.24$ for the equation $1^2 + z^2 = 2^2$. In this category of errors, however, one serious type of error emerged that reflected an impoverished understanding of (implicit) differentiation. For example, the exchange excerpt in Figure 42 shows how forum student, **ihatecalc**, constructed the derivative of a product as the product of the derivatives.



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another related rate question

Moderators: tkhunny, Gene, stapel, **Ted**, galactus

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another related rate question

by **ihatecalc** on Wed Dec 13, 2006 7:46 pm

i hate these problems, i never know how to do them
the problem states,

Mechanics are reboring a 6-in. deep cylinder to fit a new piston. the machine they are using increases the cylinder's radius one-thousandth of an inch every 3 minutes. how rapidly is the cylinder volume increasing when the bore(diameter) is 3.800 inches?

this is what i have so far:

V = pi r^2 h
V' = 2(pi)(r)(r')(h')

h=6
r'=.001
r=3.800

where do i go from here? thank you!

ihatecalc
New Member

Posts: 23
Joined: Mon Sep 11, 2006 9:10 pm

Re: another related rate question

by **honey** on Wed Dec 13, 2006 8:47 pm

> **ihatecalc wrote:**
> V' = 2(pi)(r)(r')(h') Where do i go from here? thank you!

You back up and get the derivative right.

$$V \ = \ \pi * r^2 * h$$

h = 6

$$V \ = \ 6\pi * r^2$$

$$dV \ = \ 12\pi * r * dr$$

Does that look a tittle easier?

Note: That r'h' thing you wrote is very worrysome. Please review the product rule for derivatives. You really were not very close. You'll need that later - but not for this problem.

honey
Elite Member

Posts: 4992
Joined: Wed Apr 13, 2005 2:56 am
Location: PA, USA

Figure 42. Difficulty enacting implicit differentiation

Instead of being differentiated as the product of two functions, the relationship $V = pi\, r\,^2\, h$ is treated as a string of functions and the derivative is sequentially applied to each part; thus, since the derivative of $pi\, r\,^2$ is $2(pi)(r)(r')$ and the derivative of h is h', **ihatecalc** constructs [7:46 pm] the derivative of the product as $V' = 2(pi)(r)(r')(h')$. This error is picked up on by forum tutor **honey**, who responds [8:47 pm] bluntly, "You back up and get the derivative right." After reframing the problem in a single variable (since the height of the cylinder is constant), **honey** further admonishes **ihatecalc**: "Note: That r'h' thing you wrote is very worrysome. Please review the product rule for derivatives. You really were not very close. You'll need that later - but not for this problem." Thus, this serious and consequential error in constructing the derivative is directly addressed and diagnosed through the tutoring interaction. Furthermore, there is evidence that the intervention was successful; later in the exchange, **ihatecalc**, who is still convinced that the problem must be framed using two variables, correctly applies the product rule to the relationship for volume: "$V' = (pi)(r\hat{}2)(dh/dt) + (h)(2pi(r))(dr/dt)$."

6.2.2.1 Summary of enacting procedures

Although the enactment of procedures did not emerge as a major source of difficulty in constructing related rates solutions for students in the forum, there were instances in which they reflected a deep, underlying misunderstanding of foundational calculus concepts, such as function and derivative. In the example above, **ihatecalc** did not conceptualize "r" and "h" as functions that should be operated on by the derivative and this created an impasse in reaching the solution to the related rates problem. However, the tone of interaction between members of the forum (in which bluntness is acceptable) allowed forum tutors to diagnosis and address errors that were perceived as being consequential and an impediment for the understanding of later topics and procedures in the calculus curriculum.

6.3 CONCLUSIONS ON CALCULUS UNDERSTANDING

The limit and related rates were chosen as topics of inquiry for this study because of their subtle and nuanced nature. The "limit" is the result of an infinite, never-ending process, and, therefore, in some sense, only exists in the imagination. Related rates problems involve modeling situations in which objects are moving at different rates and in relation to one another, sometimes even beginning their motion at different times or locations. Constructing a workable framing for problem situations of either type is nontrivial, especially if the resources to do so are not in place. The interaction on these topics in the forum revealed some aspects of the ways students are framing these problems and how tutors are assisting them in constructing more coherent understandings.

If the amount and nature of the activity on the forum is any indication of a "typical calculus experience," then exercises on the concept of limit and related rates are assuredly difficult for students. The queries that students brought to the forum ranged from introductory problems that are assigned for the purpose of practice and fluidity to challenging and thought-provoking exercises that are designed to push the limits of understanding. However, it was not the case that more complex or challenging exercises resulted in extended conversations whereas "trivial" exercises were limited to single line responses. The rich discussions occurred when a student positioned her/himself as an inquirer or when the tutors engaged in a debate concerning the application of a mathematical principle and introduced alternative perspectives on the problem. Regarding queries on the limit (predominantly evaluation exercises), the assistance that students required and that was provided by forum tutors was largely a nudge in the direction of a productive re-organization of the algebraic expression. The mathematical experience and expertise of the tutors enabled them to "see" the expression in a way that made it amenable to the principles of calculus. Regarding queries on related rates, tutors either initiated or helped students repair a framing that contained the variables, their rates of change, and the relationship between them. In both mathematical topics, the prevailing source of student difficulty did not appear to be "procedural" and students generally expressed (deserved) confidence in their ability to perform the relevant algorithms and techniques, some of which are quite involved. Effective calculus instruction, then, is that which prepares students to not just "see" the problem but rather to "see through" it and construct a solution path accordingly.

The analysis of exchanges using perspectival theory (framing) as a lens revealed aspects of solving limit and related rates exercises that were problematic for students and how, through computer-mediated interaction, the information could be reorganized productively. Perspectival theory was developed in face-to-face contexts using discourse analysis to account for reasoning, understanding, and learning in social interaction, and has focused on situations in which individuals were engaged in interactive problem solving. The forum exchanges are cases of computer-mediated interactive calculus problem solving in which participants are positioned in nonstandard ways. Analyzing the construction of solutions in this learning environment allows us to see how participants discern each other's perspectives and work to jointly construct productive framings in the absence of face-to-face conversational cues (such as gesture) and in situations in which students are encouraged to voice their misunderstandings. In the forum, individuals interact with one another using text and mathematical representations in stripped-down and extremely focused communications to one another. The student initiates an exchange by posing a calculus exercise *viewed from a certain perspective* and, in seeking help, communicates this framing of the problem to the forum. Forum tutors *interpret the student's framing*, and, if necessary, contribute information to assist in activating an existing schema or in making repairs, for example through foregrounding relevant entities or relationships. The ways in which individuals position themselves and are positioned by others contributes to the establishment of a shared framing that undergirds the construction of a solution to the exercise. Applying perspectival theory as a lens for calculus understanding in this novel environment suggests a definition for successful tutoring encounters: "seeing through a problem" means that the individuals have adopted and taken ownership of a productive framing for the exercise that is shared by others in the discussion.

7.0 CONCLUSIONS: FORUM TUTORING, COMMUNITY, AND CALCULUS

7.1 WHAT WE'VE LEARNED

The goal of this study was to explore the claim that open, online help forums are worthy of interest as a topic for educational research because they represent a learning environment that is popular with students and that connects a group of volunteer tutors (e.g. "Good Samaritans"). The argument is that these forums allow us to listen in on "calculus talk" that is normally not accessible to researchers and teachers, that is, talk concerning the solution of coursework exercises in progress. On these forums, students are upfront and candid about the mathematics that they find problematic and tutors provide assistance that can be (and is expected to be) viewed and responded to by others.

Despite the accessibility of these organic conversations that allow us to eavedrop on student and tutor calculus interactions, there are limitations attached to this particular study and to the study of online forums in general. The adoption of an experimental methodology poses the risk of disturbing and disrupting the phenomenon under study. On the other hand, the use of an oberservational methodology does not support any conclusions on learning other than that which is displayed in the exchanges. There is no way to discern whether the interactions on the forum transfer to other problem-solving situations in other instructional settings. Another limitation that applies to researching the forums, regardless of methodology, is the possibility of "behind the scenes" activity. Most forums allow members to use private messaging to communicate with one another and these interactions are not public record. Also, members may know and interact with one another outside of the forum. In this way, the forum archives represent an incomplete record of interaction and communication that contributes to the tutoring encounters, community, and participants' calculus understanding. Finally, the forums that are currently in operation tend to be somewhat traditional with regard to the type of exercises that are posed and the tutoring that takes place. It is not clear what the forum phenomenon might look like if fed by innovative, reformed calculus instruction. It is possible that a different picture of tutoring, community, and calculus understanding would emerge.

The picture that emerged from this study and the larger research program in which it is embedded is that the forums provide tutoring for students that is unique from what we would expect to see in other learning environments, such as one-on-one tutoring and computer-based tutoring systems. Here, the students set the initial topic of conversation and subsequent interaction is relatively brief and focused on the problem at hand. What does forum tutoring look like in terms of its quality, and, in particular, in what ways do students position themselves and become positioned by others? First, we find exchanges that contain deep discussions of mathematical principles and sophisticated pedagogical techniques; these are located in the more extended conversations between multiple forum participants. The power of the forum is not in the fast delivery of worked, annotated solutions from "experts." Instead, the power of the forum lies in the affordance of mathematical conversations that are spawned by a student query and often involve active student participation. Second, in this environment, students position themselves with authority in the interactions by making assertion and proposals for action, questioning or challenging the proposals of others, and indicating when they sense resolution.

Students appear to be comfortable expressing their thoughts and exposing their weaknesses in this context in which they are not being "watched" and judged by people they know; in terms of saving face and self-awareness, the only "face" at stake is a username, which is not part of their central or local mathematical identity. The community supports these actions, so that students are not ridiculed for committing what might be termed "major mathematical faux pas" and are encouraged to question the answers in the textbook and the information that is provided through the forum. Finally, tutors, as forum participants with more expertise and experience than students, provide mathematical guidance and evaluation of students' constructions. In the exemplary exchanges, this is done by initiating a dialogue with the student and drawing out a mathematical discovery. In such cases, the tutors called upon resources the student has (for example by posing leading questions) to assist in the construction of a productive and coherent framing of the problem situation. The excitement and eagerness with which students participate in these exchanges demonstrates how the forum not only helps students with the completion of their homework assignments and in their preparation for examinations but also introduces them to practices that are shared and valued by the larger mathematical community.

In terms of community, this study sought to explore how participating in tutoring as a communal activity was beneficial to the individual members, as well as promoting the vitality, continuity, and appeal of the forum itself. In a robust learning community, we expect to see the authentic, honest connection to mathematical practices, the generous provision of alternative perspectives, and the sincere correction of errors. All of these were present in the forum as tutors made use of affordances that allowed them to engage in dialogue and mathematical debate with one another. In these exchanges, the tables were, in some sense, turned as forum tutors became "students" and taught and collaborated with each other. The forum demonstrated a wikipedia-like effect when tutors reviewed others' contributions and either replaced the incorrect information or pointed out mathematical flaws and inconsistencies. The communal tutoring environment allowed students to see and experience mathematical practices, such as justification by proof, the use of counterexamples, and the role of alternative perspectives in understanding a concept. On some occasions, students expressed appreciation for this taste of "what mathematicians do," which is something they probably do not experience in other instructional settings. The effect of participation in this community on student attitudes towards mathematics remains to be seen but the outlook is promising.

Another question this work addresses is what can be learned about students' understanding of calculus concepts by listening in on these tutoring exchanges. The problems that students pose on the forum constitute a corpus of calculus exercises that reveal how students are framing these problems and the difficulties they encounter as they attempt to formulate answers and construct solutions. This method of inquiry is complementary to the more traditional mathematics education research that is conducted in a laboratory setting with designed tasks. To demonstrate the level of analysis that can be conducted using this observational methodology, perspectival theory was used as a lens for looking at the predominant causes of student problems as they grappled with exercises on the limit and related rates. Students had most of their difficulties in constructing a productive framing of the problems, rather than in carrying out the relevant procedures. For example, when constructing a framing of limit evaluation exercises, students were largely unaware of the mathematical principles that should be invoked in order to reorganize the given expression into something amenable to the techniques (such as l'Hôpital's rule) that they had been taught. When constructing a framing of related rates problems, students were prone to assigning names to the varying quantities and their rates of change (e.g. r and dr/dt) and letting the meaning of these variables in the context of the problem recede completely into the background. Forum tutors responded by sketching solutions that represented productive framings and helped repair students' framings by bringing relevant entities and relations into focus (e.g. by indicating that the radius of a cylinder is constant). The application of perspectival theory in a computer-mediated context showed how participants who initially did not understand the exercise in the same way achieved mutual understanding sufficient to support their shared activity through the foregrounding of relevant problem entitities and relationships.

7.2 WHAT'S LEFT TO LEARN

This research project focused on a particular context, namely an open, online help forum with a spontaneous participation structure (SOH site). However, this is only one context in which we find "tutorettes," or student-initiated problem-specific dialogues. They also occur in online forums with different participation structures and in university-sponsored help centers. We have looked at interaction and participation in some of these other contexts (van de Sande, 2007b; van de Sande & Leinhardt, 2007b; 2008b), but not nearly in the same level of detail or depth as we have for the SOH sites. We have not looked at all of the more formally designed forums (DOH) such as the Virtual Math Team project (Stahl, 2008) although we have begun conversations about the design of such research. The work so far indicates that the tutoring looks very different in these other contexts. We would like to know why and provide a better account for the interaction that occurs in different contexts as students seek and receive help on the completion of their assignments.

Another set of unanswered questions concerns the impact of forum participation over time. How does the forum influence students' attitudes, participation, and performance of mathematics? Does this exposure to authentic mathematical practices affect the way they think about doing mathematics and do they adopt some of the alternative perspectives that they've encountered? How do tutors become enculturated into such a community, and, in what ways does participation shape their tutoring style and mathematical contributions? In order to address these questions, we would need to conduct case studies and focus our attention on individuals as they participate over longer periods of time in a forum community. This work would enable us to construct trajectories of participation and learn more about the role open, online help forums play in establishing participants' identity as mathematical learners and instructors.

7.3 WHAT'S TO BE DONE

These analyses of open, online help forums suggest that this mode of tutoring represents an opportunity for teacher training programs and university help centers. Other research on online scaffolding has focused on elementary students' online solutions of *non-routine* challenge problems (Renninger, Farr, & Feldman-Riordan, 2000; Renninger, Ray, Luft, and Newton, 2006). Yet, there is no doubt that many students leave class with unanswered *routine* questions, either because the questions do not arise at the time of instruction *or* because the teacher is unprepared or unable to answer them. The asynchronous nature of online forums slows down the rapid pace of instruction and provides teachers time to produce and practice thoughtful responses (e.g. hinting and questioning strategies). In addition to preparing teachers to field questions in "classroom time," using these forums as a training site exposes pre-service teachers to many common student misunderstandings and prepares them to anticipate responses and adapt instruction by considering the underlying mental model of the student. Questions of perspective and how to efficiently work toward mutual understanding will naturally arise during this exposure to student work and other teachers' responses. Initial discussion of using the forums for instruction for teachers has begun with the faculty at Arizona State University.

Online help forums also hold great potential for enhancing university-sponsored help centers. Typically, such centers provide face-to-face tutoring help and interactions are limited to single student-tutor pairs. In this setting, there is substantial pressure for a tutor to respond quickly (in a matter of seconds versus minutes/hours for online forums). Finally, unlike online forums, tutoring exchanges in such help centers are not public and there is no public review of practice. Mathematical errors may go uncorrected and pedagogical skills are not reviewed: tutors rarely interact with fellow tutors in help centers and show reluctance to make use of other resources (such as solution manuals) (van de Sande, 2007b). Given these considerations, it appears that open, online forums would be a valuable addition to university-sponsored help centers and, at a minimum, could provide training for help center tutors. This inclusion would promote quality homework help for students and simultaneously refine the mathematical and pedagogical skills of teaching assistants as they participate in a virtual community. Initial discussions of doing just this have begun with the University of Pittsburgh help center director.

7.4 WHAT'S BEEN ADDED

Analyzing aspects of tutoring, community, and calculus understanding provided a detailed characterization of forum activity along these dimensions. In the process, the analyses of forum activity provide insight into these constructs as well. In terms of tutoring, the forum shows the power of having the student set the problem and communicate her/his intending meaning in a way that is consistent with participating in mathematical discourse. This sets up a different dialogue frame than traditional tutoring sessions (Graesser & Person, 1994; Graesser, Person, & Magliano, 1995). Although the more knowledgeable participant still leads the discussion, the tutee is entitled to initiate framings, question contributions, and indicate closure. From the perspective of the tutor, the activity of working with a student is expanded through participation in a community of others. Up to this point, tutoring has been treated as a uni-directional encounter in which the tutor is the resource and the beneficiary is the student. This study suggests that the prevailing notion of tutoring should be adapted to include tutors as beneficiaries of the activity and other tutors as resources and support.

The analysis of forum interaction also presses the notion of community and what it means to engage with others who share a common interest and desire to grow through interaction. Students participate in the forum because it is safe, efficient, and helpful. In this environment, they can ask questions that they otherwise might not and have access to help in the completion of their coursework activities. Although this study does not answer the question of why tutors come to the forum, the analyses of their actions suggest that their presence has to do with their identity and with their genuine affection for the domain. That is, they act as "Good Samaritans" because they want to continue to participate in mathematical discussions with others and because they want to assist students in their mathematical endeavors. In this sense, they are helping students with whom they are not familiar and for whom they are not responsible develop what might be termed "mathematical maturity," or the ability to participate in discussions that encompass accepted mathematical practices. Traditional considerations of community have emphasized shared goals and values, common locations, shared norms for activity, and an economy of exchange. These features are present in SOH communities, but new features, perhaps unique to online communities, have also emerged. In particular, there is a wikipedia-like effect that replaces formal monitoring and assessment and supports the forum as a community of practice. The popularity of the open, online, help forums for students willing to contribute in the service of receiving help and the voluntary participation of individuals who function as a resource for anonymous others thus embodies a novel “economy” for instructional communities of practice.

Finally, in terms of calculus understanding, this study points out a facet of instruction that has not been previously explored. In a mathematical exchange, it is the obligation of the more knowledgeable participant to ascertain the perspective taken by the less knowledgeable participant and to contribute information that assists in making repairs. This can be achieved in interaction through factual presentations of relevant mathematical entities and relationships or through appealing to resources that the student has available. Traditionally, in mathematical instruction and tutoring, the fact that individuals enter into a problem-solving experience with a perspective on the mathematical situation is not considered. Cognitive tutors, interactive computer-mediated help systems that are widely used in mathematical instruction (e.g. those marketed by Carnegie Learning™), assume a single framing of the problem and provide feedback and hints accordingly. The activity that occurs in the online, help forums suggests that it is possible to have an efficient, quality, interactive help system that provides many of the same tutoring features (such as feedback, hints, and advice) and also accommodates alternative framings. The nature of the forums – human-conducted and computer-mediated – affords discussions of mathematical ideas and scaffolding that is tailored to the individual learner and the perspective that s/he brings to the problem-solving situation. Listening in on the calculus exchanges on limit and related rates in an open, online help forum is a start in the development of instructional materials and approaches that draws more individuals to deeper participation and active engagement in mathematics.

APPENDIX A

ONLINE CALCULUS: A FRAMEWORK FOR ANALYSIS WITH APPLICATION TO FIVE COURSES[40]

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Abstract

As computers become a part of everyday life, universities, corporations, and hobbyists are producing online course materials made available for general public use. What differentiates these courses, and on what basis can an online course be selected? We address these questions by proposing and applying an analytic framework to five stand-alone online introductory calculus courses that are readily accessible to the general public. Our approach is informed by cognitive research and research in mathematics education on explanations, examples, exercises and explorations and includes a discussion of topic sequencing and a catalog of possible online resources. As a demonstration of the framework, an analysis is conducted on each course's instruction for the chain rule, a procedure that is mathematically interesting, common in introductory calculus instruction, and challenging to learn. The application of this framework provides a detailed picture of instruction in each course and captures the instructional practices resulting from different emphases and intentions on the part of the course authors.

[40] In February, 2007, a version of this paper entitled *A Framework for the Analysis of Online Calculus Courses* was presented at the 10th Conference on Research in Undergraduate Mathematics Education (CRUME), San Diego, CA.

Online Calculus: A Framework for Analysis with Application to Five Courses

The selection of a calculus course textbook, instructional materials, and pedagogical approach are core decisions that shape the way in which it is taught and learned. How should these critical decisions regarding the very essence or "stuff" of a course be made? In general, course text selections at the college level are made in an ad hoc fashion, yet, as increasingly sophisticated online courses become more available and more commonly integrated into college teaching, systematic approaches to selection become quite critical. In this paper, we argue that an analytic approach that specifies criteria provides a perspective on online courses that can greatly assist instructors (or students interested in learning a subject independently) in making informed decisions for course selection and can assist course designers in determining which course features to include and how to put these into practice. As our contribution to the establishment of a genre of analytic critiques of online courses, we propose a framework for analyzing stand-alone online calculus courses and show how this framework provides a comprehensive picture of five existing courses.

As a prelude to our discussion, we recount the experience of Noble Prize laureate, Richard Feynman, while serving on the California State Curriculum Commission (Feynman, 1997). The purpose of the commission was to select textbooks for the public schools, and Feynman was asked to contribute his professional expertise for the evaluation of several mathematics and science books. He accepted the assignment and chose to read the 300 pounds of books, declining an offer for someone to assist him with the reading task: "I couldn't figure out how you *do* that: you either read them or you don't read them." (p. 266) Other members of the committee, however, were not as circumspect, even granting favorable reviews to a textbook that contained only blank pages. In contrast to Feynman, who "would tell them, in detail, what was good and bad in all the books...[and] had a reason for very rating," (p. 268) other committee members were content with superficial examinations of the textbooks' contents. Although Feynman presented this anecdote in his usual humorous fashion, the reality of the situation is cause for concern and directs our attention to the way course materials might better be analyzed and selected for instruction. Feynman applied his expertise as a physicist to specific features of instruction such as accuracy and rigor of explanations (e.g., general principles of energy), appropriateness of examples (e.g., automobiles in the street for "sets"), and authenticity of exercises (e.g., adding stars' temperatures). We also feel the need for careful examination and close reading of course materials informed by expertise. In our case, the expertise that shapes our analytic framework for stand-alone online calculus courses is drawn from cognitive research on learning and instruction, research on students' mathematical understanding, and a developing genre of analytic frameworks for online courses.

Analytic frameworks

The unexplored potential and wide-spread use of computer-based instruction have inspired the development of analytic approaches for educational materials that are candidates for the systematic critique of online calculus courses. One of the most extensive efforts is based on cognitive load theory, a well-established description of human information processing (Sweller, 1989; Sweller, Van Merriënboer, & Paas, 1998). This work resulted in the formulation of nine design principles addressing features of a multimedia presentation, such as the temporal and spatial coordination of words and pictures (Mayer, 1999). Although the validity of the design principles has been replicated in numerous studies, the situations that support these principles are by no means authentic learning environments. In particular, the instructional messages in question are of extremely short duration (usually 30 seconds or less) and are generally viewed by human subjects in a laboratory setting. Thus, although cognitive load theory is relevant to the analysis of online experiences, it is neither formulated for nor tested in prolonged instructional sequences.

Nachmias, Mioduser, Oren, and Lahav (1999), science educators, have developed a taxonomy for evaluating and comparing educational websites along four dimensions: descriptive, pedagogical, knowledge, and communication. Each dimension is organized into categories that contain variables characterizing the instructional presentation. For example, the representational dimension contains two categories: representational structure (*e.g.,* linear, branching or web structure) and representational means (*e.g.,* frequency of text, image, sound, and animation). This taxonomy was subsequently revised and extended for specific application to scientifically oriented educational websites, largely through the addition of a fifth dimension entitled ``scientific content" that covers the representation of experiments and models, incorporation of mathematical descriptions, and reference to social issues (Nachmias & Tuvi, 2001). Although this taxonomy is largely generic, readily adaptable to specific subject domains, and easily applied, it has one significant shortcoming as a systematic tool: it does not necessarily allow the user to determine the quality of the instructional materials in question. Because the taxonomy relies heavily on frequency counts (*e.g.,* number of animations) and dichotomous judgments (*e.g.,* presence or absence of problem solving activity), it functions more as a descriptive tool rather than as an analytic one that can capture features of course design (such as exercise complexity) that promote effective learning.

An approach that does address learning effectiveness was developed for the analysis of workplace simulations by Ferrari, Taylor, and VanLehn (1999). In this framework, simulations are analyzed according to usability, content, learnability, and teacher support. Usability, for example, addresses the amount of requisite prior knowledge, the complexity of the user interface, and the potential for sustaining engagement. The application of this framework requires judgments on whether a given simulation is weak along any of the prescribed dimensions. This tool provides a very global assessment of instructional materials but does not differentiate at the fine-grained level of core content. The construction, revision, or selection of online course materials still calls for a more detailed analytic framework.

Towards this end, Larreamendy-Joerns, Leinhardt, and Corredor (2005) constructed a framework for analyzing stand-alone online statistics courses informed by cognitive research on explanations, examples, problem solving, and feedback. This framework includes an analysis of examples according to frequency, variability, and authenticity; an analysis of exercises according to frequency, authenticity, and cognitive complexity; and, an analysis of interactivity that covers the frequency and function of interactive learning objects. The instructional context, namely stand-alone instruction offered online, is accounted for by considering courses' implementation of online resources, such as search engines and electronic forums. The application of this analytic framework paints a clear picture of the nature of stand-alone online statistics courses and illustrates how closely these are aligned with well-established principles of learning and instruction. In addition, the generalizability and potential of this basic approach have been demonstrated through an application to stand-alone online chemistry courseware (Evans and Leinhardt, 2007).

Analyzing online mathematics courses

In a recent survey charting the scope of undergraduate mathematics information available on the Internet, Englebrecht and Harding (2005a; 2005b) discovered websites serving a wide variety of functions, from notice boards (containing only administrative information) to full courses (containing content and resources supporting personal interaction). In order to allow comparisons across such a range of materials, they constructed a graphical classification based on six features: dynamics and access; assessment; communication; content; richness; and, independence. The "independence" dimension designated the amount of instructor presence in the course, from fully lecture driven with minor website component to website conducted with no face-to-face contact. Our focus is on online introductory calculus courses in this latter category that can be easily accessed by the general public.

In this paper, we develop a framework for the analysis of stand-alone online calculus courses. Clearly, any such analysis should include a discussion of online resources (*e.g.,* animations, search engines, and external links) and general coverage. In addition, however, there should be a consideration of resources that are specific to mathematics instruction, such as graphing calculators and computer algebra systems (CAS). Although there is considerable debate as to the role of "technology" in calculus instruction, the general consensus is that students should be competent with a graphing utility. Thus, the College Board permits the use of a graphing calculator on portions of the Advanced Placement examination. Provided that resources are of high quality and well integrated with instruction, they can contribute to the sense of course "richness" (Englebrech & Harding, 2005).

A review of online mathematics courses, however, must go further than a consideration of resource implementation. In particular, there are two large issues that need to be resolved in order to build such a review. First, there must be a selection of instructional locations or elements for examination and a determination of the analytic criteria. Second, there must be a selection of one or more topics that will serve as the targeted comparison for the domain. Once a topic has been selected, it may be instructive to consider sequencing, or the location of the topic relative to others in the course. The sequencing of some topics is fixed by mathematical constraints whereas the location of other topics is a somewhat arbitrary decision. Courses may differ, then, along this dimension as well.

Selection of instructional locations. There are four major locations in instruction where learning opportunities occur and that form the skeleton of the analytic framework:

1. *Instructional explanations*: Explanations in instruction are used to introduce, describe, and demonstrate material, and it is at this location that learners are exposed to both the content and sense of the domain. Thus, definitions, theorems, proofs, and various representations often feature in an instructional explanation for a mathematical topic. Instructional explanations can be analyzed by identifying a system of interrelated goals (*e.g.,* framing a query) and their supporting actions (*e.g.,* introducing an impasse) (Leinhardt, 1987).
2. *Worked examples*: Examples, a component of instructional explanations, serve a myriad of functions (e.g., introducing, referencing, modeling, and bordering concepts) (Rissland, 1989) and therefore warrant analytical treatment in their own right. Examples can be analyzed by frequency (in order to support learning, more than a single example must be present) (Quillici & Mayer, 1996) and variability (in terms of the range of mathematical situations covered (Larreamendy-Joerns et al., 2005)) Within stand-alone learning environments, the presence of a variety of worked examples is particularly worthy of inspection.
3. *Exercises*: Exercises provide a location in instruction for learning through engagement. Exercises can be located both within or following an explanation and can support practice, leading to improvement in speed and accuracy of

performance (Anderson, 1993), as well as providing the learner with the opportunity to participate in authentic mathematical practices (Raman, 2004). In particular, expert mathematicians consider working numerous problems as a valuable means of understanding material (Szydlik, 2000). Like examples, exercises can be analyzed according to frequency and variability. In addition, an analysis of exercise complexity (e.g., cognitive task demands) provides a sense of the epistemological messages conveyed by a course.

4. *Explorations*: Explorations allow the learner to manipulate representations with ease and to experiment with a large variety of instances to infer principles. In these ways, they can function as an efficient means of prompting and answering "what if" questions. One of the assets of a computer-based learning environment is the natural way exploratory activities can support instruction, e.g., through simulations. An analysis of explorations can address the ways in which they contribute to explanations, the adequacy of instructional support, and comprehensibility (especially to naive learners).

Selection of topics. The subject matter location within a curriculum must also be identified, and this mathematical topic (or topics) fleshes out the analytical framework. In order to construct a framework that is general enough to permit application across several courses but detailed enough to provide a snapshot of a single course, the chosen topic that is the focus of analysis should be common to general instruction in the domain; important to the content (that is, without understanding the topic, other materials are likely to be hard to learn); challenging for students (that is, not trivial to learn), and representative pedagogically of the instructional style within a course.

METHODS

Selection of online calculus courses

Courses were identified through an Internet search of online calculus courseware, including materials advertised as online courses, course websites, and online textbooks. In order to insure that the review addressed courses that were both comprehensive and easily accessible to the general public, materials meeting the following five criteria were used to identify target courses: a) the courses needed to cover the scope of a college-level introductory calculus course, as defined by The College Board and tested in the Calculus AB Advanced Placement examination; b) the courses had to be stand-alone courses that did not necessarily require the intervention of an instructor or make reference to non-included materials (*e.g.,* texts or lecture discussions); c) the courses had to be available to distance learners, not only to enrolled campus students; d) the courses needed to run on multiple platforms; and e) the courses had to be free or low-cost (less than $100). In addition, an effort was made to include courses that represented a variety of philosophical stances regarding what it means to know calculus at the introductory level. As a result of the search, five courses (all previously unknown to us)[41] met the criteria for coverage, support, availability, and cost:

1. *Visual Calculus* (VC) (http://archives.math.utk.edu/visual.calculus/)
2. *Thinkwell Calculus I* (TC) (http://www.thinkwell.com/)
3. *Calculus on the Web* (COW) (http://www.math.temple.edu/ cow/)
4. *Karl's Calculus Tutor* (KCT) (http://www.karlscalculus.org/index.shtml)
5. *Internet Calculus* (IC) (http://hmco.tdlc.com/public/icalc/)

[41]Neither author has any personal or financial connection to any of the courses reviewed.

INSERT TABLE 1 ABOUT HERE

Table 1 contains an overview of the courses that were analyzed. With the exception of *Karl's Calculus Tutor*, all of the courses were authored by professional mathematicians associated with an academic institution. *Karl's Calculus Tutor* was written by Karl Hahn, who has degrees in electrical and software engineering, and was largely influenced by methods learned from his father, a professor of mathematics.

Each of the five courses was developed to teach calculus but with a different intent. *Visual Calculus* was originally designed to demonstrate ways in which technology (in particular, computers) could be used in calculus instruction and was then expanded to include other teaching materials (such as tutorials and problems). *Thinkwell Calculus* characterizes its online course materials as "next-generation" textbooks, with multimedia video lectures that are intended to take the place of the printed textbook. *Calculus on the Web* was developed to provide students with the opportunity to learn calculus and practice problems in a "friendly environment." *Karl's Calculus Tutor* was an endeavor intended to supplement rather than replace calculus instruction from other sources: "It is my hope that a student who is stuck on something might come to my web pages and find that an explanation of some topic, together with worked examples, might unstick him or her" (personal correspondence with Hahn, 2005). Finally, *Internet Calculus* was designed to provide calculus instruction "anytime, anywhere" and is an example of a participatory text (online educational interactive multimedia course materials with integrated communication features).

Each of the five courses also has a distinguishing technical feature. *Visual Calculus* provides animated explanations, allowing the user to control the rate at which text appears on the screen. *Thinkwell Calculus* is the only course reviewed that includes a lecture video component.

During the lecture, shown on one frame, notes highlighting or adding information (*e.g.,* definitions, graphs, examples) appear in a separate frame. *Calculus on the Web* produces immediate feedback to short answer questions and does not provide worked solutions. *Karl's Calculus Tutor* begins many of the tutorials with a story intended to motivate the topic. For example, the description of the foot race between Achilles and the tortoise precedes a discussion of limits and Wile E. Coyote's misadventures with a bouncing anvil set the scene for the determination of maxima and minima using derivatives. *Internet Calculus* also introduces each chapter with a scenario, but these are realistic and are accompanied by data that can be analyzed using a CAS. In addition, several of the exercises in *Internet Calculus* recommend the use of a CAS, either for producing or verifying solutions.

Three of the courses are available free of charge, whereas *Thinkwell Calculus* is available to students as a subscription that costs $25 per 6 months. All of the courses made an effort to use free plug-ins and run on standard browsers. Thus, *Visual Calculus* only requires the free (viewer) version of LiveMath, a computer algebra and graphing tool, and *Internet Calculus* supports but does not require the use of a CAS.

Selection of the chain rule as topic

From the long list of topics that are typical to instruction in introductory calculus, we chose the chain rule as the focus of our analysis. This theorem is common to instruction (indeed, it would be hard to imagine a course that did not cover the chain rule); important for gaining proficiency in other course topics, such as implicit differentiation and related rates; and, challenging for students to master and remember. These features make the chain rule well suited as a target for evaluating stand-alone calculus course instruction.

The chain rule in calculus.

The chain rule is a theorem that describes the differentiation of the composition of two functions, f and g:

> If g is differentiable at the point x and f is differentiable at the point $g(x)$, then the composition $f \circ g$ is differentiable at the point x and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. Using the notation of Leibniz and letting $y = f(g(x))$ and $u = g(x)$ so that $y = f(u)$, the chain rule can be expressed as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

In short, the chain rule states that the derivative of the composition of functions f and g at point x is the product of the derivative of function f evaluated at $g(x)$ and the derivative of function g evaluated at x. The chain rule can be extended to more than two functions and is one of the most useful results in differential calculus. In addition to being a requisite for differentiating a large number of functions encountered in an introductory calculus course, the chain rule continues to arise as the course develops (for example, in implicit and logarithmic differentiation, related rates, and integration by substitution). Students also encounter applications of the chain rule in subsequent courses, such as engineering, chemistry, and physics.

Although the chain rule appears to be a simple and straightforward rule of differentiation, the presentation and proof of this theorem touch on subtle issues that historically shaped the formulation of "the calculus." Using Leibniz notation, the chain rule is easily assembled because it appears to be the result of "canceling" numerators and denominators: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. However, when students are taught to construct the chain rule in this manner, they are sent a contradictory message: the familiar operation of simplifying products of fractions "works" in the chain rule *but they should not think of the entities* (e.g., $\frac{dy}{du}$ and $\frac{du}{dx}$) *as fractions*. Why, then, is this notation used? The answer is that the notation was considered by its founder as a "useful fiction" (Edwards, 1979) and not as a rigorous definition of the underlying concept, namely infinitesimals. When subsequent mathematicians effectively banished the notion of infinitesimals from the formulation of calculus, the fictive notation remained, largely due to its perspicuity.[42]

There are also noninfinitesimal issues concerning the proof of the chain rule. The most compact proof (4-5 lines in length) uses a technique that is a variation on the theme of adding and subtracting the same quantity to an expression in order to engineer a desired form. In the proof of the chain rule, the key difference is that the operations of multiplication and division replace those of addition and subtraction. This simple, elegant proof of the chain rule was once standard in many calculus textbooks (Gans, 1955) and still appears in textbooks today. However, an important – but subtle – consideration that renders this proof technique invalid is that the quantity in the multiplication and division process may take on the value of zero. Although rigorous proofs of the chain rule have subsequently been constructed, they are lengthier and not as accessible to most students enrolled in introductory calculus.

In short, the chain rule is a powerful result that has caught the attention of mathematicians who have worked to resolve nuances in its presentation and inconsistencies in its proof. We next turn our attention to how students develop an understanding of the chain rule and what this might imply.

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[42] Relatively recently, infinitesimals have been reintroduced in a rigorous fashion as "nonstandard analysis" (Robinson, 1996), an approach that has seen limited use in introductory calculus courses (Keisler, 2000).

Students' understanding of the chain rule. Educators have suggested many techniques for teaching the chain rule, such as verifying results using a CAS (Mathews, 1989), constructing arrow (or tree) diagrams (Thoo, 1995), and singing catchy lyrics (Gutman, 2006; NCSSM, 1994). One recent approach that encapsulates a theory of embodied cognition (cf. Pecher & Zwaan, 2005) introduces the chain rule as a hill-climbing experience so that students' intuitions of physical motion form the basis of their understanding (Lutzer, 2003). Despite differences in form and practice, all of these instructional strategies share a common goal – improving students' understanding and retention of the chain rule.

Why is the chain rule challenging to learn and difficult to remember? Clark et al. (1997) interviewed students on five tasks related to the chain rule and concluded that an understanding of the chain rule requires at least a process conception of function, function composition and decomposition. That is, students needed to be able to fluently compose and decompose functions, without explicitly performing all of the intermediary algorithmic steps.

Perspectival theory sheds some light on how function composition and decomposition contribute to an understanding of the chain rule. (See Greeno & van de Sande (2007) for a discussion of perspectives and framing in mathematics). In order to recognize conditions of use (that is, situations in which the chain rule is applicable), students must learn to look at familiar expressions that were differentiated using previously learned rules in new and different ways. For example, in order to apply the chain rule, the expression $\frac{1}{(x-3)^2}$ is viewed as the composition $f \circ g$ of the functions $f(x) = x^{-2}$ and $g(x) = x - 3$ rather than as a ratio that calls for the quotient rule. In addition to reconceptualizing expressions of functions prior to applying the chain rule, students must also shift perspectives during its application. Thus, when differentiating the function composition $(f \circ g)(x) = f(g(x))$ using the chain rule, the student must first focus on the “outer function” (and treat the “inner function” as a single variable in the differentiation process) before focusing attention on the “inner function” (and treating it as an expression in its own right that must be differentiated).

In addition to function composition and decomposition, understanding the chain rule also relies on conceptions of differentiation and rates of change. Chain rule instruction generally occurs at the tail end of the differentiation rules for operations on functions and is therefore separated from the presentation of the derivative concept (e.g., as a limit quotient). The focus of typical instruction is instead on the manipulation of symbols and application of appropriate formulae. The tendency of students, even after experiencing concept-based calculus instruction, to treat variables as symbols to be manipulated (White & Mitchelmore, 1996) is therefore relevant to any discussion of the chain rule. Even though students may possess intuitive knowledge and conceptual images of the relationship between functions and their derivatives (Nemirovsky & Rubin, 1992), these notions are unlikely to surface or become more coherent without purposeful intervention (cf. Tall, 1987). In addition, students’ weak foundations in graphs, tangents, ratio, and proportion contribute to incoherent understandings of rates of change (Orton, 1983). There is no question that the chain rule, as a complex and multi-layered rate of change, represents a challenging concept for students in introductory calculus.

Analytic criteria

Resources. There are numerous resources that online courses can include. Online courses can make use of representational resources (such as videos, animations, graphing utilities, and computer algebra systems), navigational resources (such as course maps, search engines and links to external sites), and resources for dialogue (such as question type, feedback, discussion forums, note-taking facilities, and course management systems). Decisions regarding both the inclusion and implementation of these resources affect not only the way students interact with the computer but also their understanding of what it means to know calculus.

Sequencing. The decision of where to locate a discussion of the chain rule is arbitrary to some extent; the chain rule could be located late in the discussion of differentiation or it could be introduced in the context of polynomial differentiation and then re-introduced in subsequent discussions of the differentiation of additional function types. There are advantages that could support either or these choices. Practicing a new rule in a varied context may help the student associate the rule with a large number of situations, whereas practicing a rule in a restricted setting (*e.g.,* on limited types of functions) and then returning to the rule later when other function types are introduced may give the student the opportunity to first master the rule and then revisit it as prior knowledge. For each course, we marked the location of instruction on the chain rule relative to "elementary" rules of differentiation (such as the power rule), product and quotient rules for polynomials, and the differentiation of trigonometric, exponential, and logarithmic functions and noted how sequencing was supported by instruction.

Explanation. Following Leinhardt (2005), we defined an instructional explanation as the discourse that communicates a portion of the subject matter to the student. The explanations of the chain rule were evaluated according to a revision of a theoretical model formalized as a planning net (Leinhardt & Greeno, 1986; VanLehn & Brown, 1980) that details action schemas, decision points, and goal structures relevant to mathematics instruction (Leinhardt, 1987).

The associated grammar designates possible configurations and relationships between elements; for example, higher order goal states may be partially achieved by actions embedded in other goal systems. In this formalism (Figure 1), goals (shown in hexagons) are achieved as a direct and indirect consequence of actions (shown in rectangles) and sub-goals. We examined the explanation of the chain rule in each course to see if it included a review of the subskills used in the explanation, demonstrated the nature of the problem that was resolved by the development and use of the chain rule, provided a verbal (informal) description of the procedure, presented an analogy, made connections to prior knowledge by supporting alternative perspectives, identified conditions and principles for use, established the legality of the procedure, and flagged common errors. Although a given explanation need not contain all of these elements, a good explanation will employ specific action schemas that support these goals to some degree.

INSERT FIGURE 1 ABOUT HERE

Examples. For coding purposes, we defined an example as a single problem or scenario used to model a target concept or procedure. We excluded instances that functioned solely as reminders of procedures previously covered (*e.g.,* function composition). Because the courses differed on the sequencing of topics, we located and counted examples that modeled the concept or procedure in later sections. However, we did not count examples from any section in which the procedure or concept in question was referenced but not explicitly demonstrated. We designed an example coverage index (with values ranging from 0 to 10) as an indicator of the variability of the examples present in each course. Points were awarded if the examples covered a range of situations that demonstrated the procedure in both simple and complex cases, showed how the new information related to prior knowledge while, at the same time, contributing to a new understanding or perspective, and flagged common errors. In particular, we looked for simple examples of the chain rule applied to compositions of standard functions (*e.g.,* polynomials, radicals, trigonometric, exponential, and logarithmic functions), examples demonstrating a repeated application of the chain rule, examples illustrating the chain rule in combination with the product or quotient rule, examples illustrating how the chain rule can be an alternative to previously learned rules of differentiation (*e.g.,* quotients with constant numerators), examples addressing conditions of use (*e.g.,* when applying the chain rule before or in lieu of another rule maximizes efficiency and when applying the chain rule is inefficient), and, finally, examples cautioning against common errors (*e.g.,* inserting the derivative of the inside function – instead of the function itself – into the derivative of the outer function).

Exercises. In each course, we identified exercises interspersed with the explanation of the chain rule, located at the end of a unit, or placed at a given location on the course website. We defined an exercise as a single problem or scenario that centered on the chain rule, and counted questions with multiple interdependent parts only once. We evaluated each exercise in terms of its cognitive complexity and coded a subset of the exercises to gauge the variability of practice that they afforded. To assess complexity, we applied a coding schema similar to that used by Larreamendy-Joerns et al. (2005) that differentiates between tasks that require decision making and interpretation, those that involve the execution of an algorithm, and those that can be accomplished by recalling information from memory. This schema, adapted from Stein and Lane (1996), is based on the idea that cognitive complexity is higher for tasks with greater degrees of freedom. Table 2 gives the definition and an example task for each category.

INSERT TABLE 2 ABOUT HERE

Focusing on the tasks with explicit instructions to differentiate a given expression, we performed an analysis similar to the analysis of examples and investigated the range of mathematical complexity and the opportunities for students to realize alternative perspectives on previously encountered functions and recognize conditions in which the chain rule is the “method of choice” (in terms of efficiency). We assigned each of these tasks to one of the following categories: single application, repeated application, rule combination, single application and perspectival shift, rule combination and perspectival shift, imposter, and reformulation. The first three categories refer to the complexity of the function being differentiated, whereas the latter categories reference the construction of an alternative perspective on the structure of the given function. The category definitions and an example function for each category are provided in Table 3.

INSERT TABLE 3 ABOUT HERE

Explorations. Tasks that function as explorations have less explicit guidelines than exercises. There are multiple goals of an instructional explanation that explorations could address: principles of use (what are the mathematics that allow us to decompose functions and then construct the derivative), connections to prior knowledge and conditions of use (why and in what situations is the chain rule more efficient than previously learned rules of differentiation), or the nature of errors (what happens if the procedure is incorrectly implemented). In the context of the chain rule, explorations may be particularly important because of perspectival issues. Students may not engage in constructing alternative perspectives if the information is presented in a passive manner (*e.g.,* through text or worked examples) or if it is presented implicitly (*e.g.,* through exercises that are merely suggestive).

RESULTS

Resources

Table 4 summarizes the online resources available in each course. As can be seen, very few of the resources are common to all of the courses, and note-taking facilities are not a feature of any course. In addition, courses vary widely in the presence and implementation of representational resources (such as videos, animations, and graphing utilities), navigational resources (such as course maps, search engines and links to external sites), and resources for dialog with the course user (such as question type, feedback, discussion forums, note-taking facilities, and course management systems).

INSERT TABLE 4 ABOUT HERE

Representational. The first four rows of the table address the differences in representational resources across courses. *Thinkwell Calculus* and *Internet Calculus* are the only two courses to make use of the multi-modal affordances of online presentations using video. In *Thinkwell Calculus*, an instructor lectures in one frame, while notes highlighting or adding information (*e.g.,* definitions, graphs, examples, pictures) appear in a separate frame. In *Internet Calculus*, videos supplement textually delivered explanations and are used as a tool for demonstrating concepts. For example, in the instruction on the chain rule, a video of a fire truck with siren blaring as it passes by a stationary observer illustrates the Doppler Effect. While this feature is "cool," it is not very closely related to the cognitive complexity of learning the chain rule.

Animations are used by three of the courses, but in very different ways. A distinctive feature of *Visual Calculus* is that most of the explanations and examples are dynamic in the sense that the user controls the rate at which sections of text (usually sentences or solution steps) appear ('fly in') on the screen. These animations also use color to focus attention by highlighting activities (*e.g.,* substituting) and co-referencing information. *Karl's Calculus Tutor* has occasional examples of procedures (*e.g.,* the product, quotient and chain rule) that are presented dynamically on a 'chalkboard' next to a cartoon figure of the stereotypical potbellied mathematics professor with receding hairline. In these presentations, the animation speed is not user controlled, but color is used to highlight operations and co-reference information. In contrast to *Visual Calculus* and *Karl's Calculus Tutor*, *Internet Calculus* does not animate text or algebraic expressions, but instead uses animations primarily to illustrate concepts. For example, the instruction on the chain rule includes the animation of a set of interconnected gears, illustrating that the number of times an axle must revolve in order to turn another axle is affected by their being ‘chained’ together.

The courses also differ in their inclusion of animated graphs. Both *Visual Calculus* and *Internet Calculus* include dynamic illustrations of concepts (such as secant lines approaching a tangent line) and allow the student to control presentation speed. Only two of the courses, *Karl’s Calculus Tutor* and *Internet Calculus*, include a graphing calculator. The graphing calculator in *Karl’s Calculus Tutor* functions largely as a supplementary feature of the site, whereas *Internet Calculus* uses the graphing calculator as a way for students to interact with graphs that often accompany examples. *Visual Calculus* does not include a graphing calculator but instead provides detailed programming instructions for the TI-85 and the TI-86 calculators.

Two of the courses, *Visual Calculus* and *Internet Calculus*, incorporate a CAS, although in very different ways. *Visual Calculus* uses LiveMath, a CAS designed to minimize programming activities on the part of the user, to produce (dynamic) graphs and demonstrate operations (*e.g.,* substitution, collection of terms, and expansion) on algebraic expressions. In these activities, the student can manipulate graphs (*e.g.,* by zooming in or out) and edit inputs through keystroke and mouse commands. Some tutorials also include detailed instructions for programs in Maple and MicroCalc. *Internet Calculus* includes notebooks for Derive, Maple, Mathematica, and Mathcad, as well as the TI-89 and TI-92 calculators. In this course, the CAS is presented as a tool to support data analysis and modeling, perform complex algebraic operations quickly, and confirm solutions. In neither of these courses, however, is the CAS central to instruction.

Navigational. The next five rows of Table 4 refer to the navigational resources of the online courses. Although a course map is featured in all of the courses, the presence of a glossary or search engine is not ubiquitous and only one course, *Thinkwell Calculus*, offers both. *Thinkwell Calculus* also contains select links to external sources, although *Karl's Calculus Tutor* is notable for the inclusion of numerous links, including access to online calculators (graphing and complex number), other calculus tutorials and texts, online programs that perform differentiation and integration, help sites (both computer- and human-based), collections of common mistakes and their explanations, and historical accounts of mathematics. It is worth mentioning that both *Thinkwell Calculus* and *Karl's Calculus Tutor* contain a link to *Visual Calculus*. No course offers note-taking facilities so that a student might keep a personal account of the material, although *Internet Calculus* offers a bookmark feature and *Thinkwell Calculus* keeps a checklist of materials viewed (*e.g.,* notes, animations, exercises, and videos) and most recent activity.

Dialogical. The final rows of Table 4 characterize the dialogical resources of each online course. These represent opportunities for students to display their knowledge of the material and receive assessments of their understanding or progress either from the computer or from others. With the exception of *Thinkwell Calculus*, all of the courses offer short answer questions. *Visual Calculus* and *Internet Calculus*, the two courses that employ both question types, use multiple-choice questions in quizzes and short answer questions in exercise settings.

The inclusion of feedback for exercises is a resource that is absent in many of the courses. The majority of the courses provide worked solutions independent of student performance, and none of the courses is built around a cognitive tutor that modifies feedback according to student input. *Calculus on the Web* is unique in providing judgments of accuracy and not offering worked solutions; in this course, students can attempt an exercise an unlimited number of times but are never given the correct solution. This course is also unique because it incorporates humor into feedback delivery. Both correct and incorrect feedback messages often contain puns and quips, seemingly tailored for the archetypical American college student (*i.e.,* a mug of beer is featured in one). If these function as intended, the result could be to bring the desired levity into the student's calculus predicament. However, there are also potential hazards that may accompany 'online' teasing. Quips could be taken seriously or misunderstood and thus have the potential for offending. Negative quips, in particular, may be misconstrued, especially by weaker students. If, for example, a student has honestly attempted a problem numerous times unsuccessfully, the feedback message consisting of a large picture of a fist next to the text "Get it right, pal, or else!" will, at best, not have any effect and may, at worst, actually discourage the student.

One remedy for discouragement is discussion, and three of the courses include an electronic forum that affords discussions of the material with others. For *Thinkwell Calculus*, the forum is included in the course management system and thus only links students enrolled in a course together. In contrast, *Karl's Calculus Tutor* and *Internet Calculus* provide forums that are open to any individual participant. Students can post questions and solutions in these forums but it is worth noting that these discussion forums do not appear to be thriving.[43] Karl Hahn personally responds to most of the questions raised in Karl's Discussion Forum, and fellow students do not regularly respond to the postings in *Internet Calculus*. It appears then that, when communication is not necessary for course participation (as is the case for stand-alone instruction), students are hesitant and reluctant to interact with others. (See Englebrecht & Harding (2005b) for a discussion of online communication, interaction, and collaboration in

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[43] This is in stark contrast with several free, open online homework help forums that are not affiliated with a particular course or institution (cf. van de Sande and Leinhardt, 2007).

online mathematics courses). Overall, *Internet Calculus* and *Thinkwell Calculus* stand out for the provision and implementation of online resources. *Visual Calculus* and *Calculus on the Web* contain the fewest.

Sequencing

Figure 2 shows the location of the discussion of the chain rule in each course relative to other rules of differentiation and the derivatives of exponential, logarithmic, and trigonometric functions. (Note: The length of each segment does not correspond to

the length or completeness of the explanations.)

INSERT FIGURE 2 ABOUT HERE

All of the courses introduce the "elementary" rules of differentiation (*e.g.,* the power rule) and rules for differentiating products and quotients of functions prior to the chain rule. However, the courses vary in the number of functional contexts in which the chain rule is discussed. Each course includes instruction on the chain rule in the differentiation of polynomial and trigonometric functions, but *Visual Calculus* and *Calculus on the Web* do not cover the differentiation of logarithmic functions, with or without the chain rule. The courses also differ in the location of the chain rule relative to the discussion of differentiation within these functional contexts. *Visual Calculus* is the only course that locates the chain rule after a discussion of derivatives of both trigonometric and exponential functions. *Karl's Calculus Tutor* and *Thinkwell Calculus* both introduce the chain rule directly following the product and quotient rules of differentiation applied to polynomials (or "fractonomials" such as $(x+3)^{\frac{1}{3}}(x-2)^{\frac{2}{3}}$). These two courses then introduce the differentiation of trigonometric, exponential, and logarithmic functions, incorporating instances within each topic that require the chain rule. *Internet Calculus* and *Calculus on the Web* introduce the differentiation of trigonometric functions prior to the chain rule so that students initially practice the chain rule in problems that involve compositions of trigonometric functions and polynomials (such as $\sin(2x)$). *Internet Calculus* then reintroduces the chain rule in the discussion of the differentiation of exponential and logarithmic functions. Although *Calculus on the Web* also introduces the differentiation of exponential functions following the chain rule, the chain rule is not reintroduced in this explanation. Instead, the explanation directs students to use the product and quotient rules to differentiate functions such as e^{2x} and e^{-x} and the chain rule is not mandated by any of the accompanying exercises, all of which involve integer powers ranging from -2 to 2.

Explanation

Table 5 summarizes each course's instructional explanation according to the major goal states depicted in Figure 1. *Thinkwell Calculus* and *Internet Calculus* stand out as addressing many of the goals.

INSERT TABLE 5 ABOUT HERE

Sub-skills available. The first row refers to the sub-skills needed to handle the components of a new procedure. In order to perform the chain rule, students must be able to discern whether a function is a composition of other functions, and, if so, how the function can be decomposed, as well as any of the differentiation rules that apply to the functions in question. All of the courses locate the chain rule directly after coverage of the differentiation rules for algebraic operations (*e.g.,* sums, products, and quotients), so that it may be assumed that this knowledge is still fresh. However, function composition is a topic that, if covered in a calculus course, occurs much earlier in the sequence, usually in an introductory section on functions. Three of the courses attempt to refresh this sub-skill by including either a review of function composition or a link to the material as part of the chain rule explanation, but none of these explanations includes a discussion of the non-uniqueness of function decomposition.

Nature of problem. The next row in Table 5 refers to the establishment of a query. This can be done by actions that extend previously encountered explanations or by actions that constrain solution paths so that students 'stumble onto' the problem. In the former case, the chain rule is presented as an extension of the discussion on how to differentiate functions combined in various ways. Functions can be combined by addition, subtraction, multiplication, and division, and a differentiation rule can be applied for each operation. Functions, unlike numbers, can also be composed with one another, and the query is how to differentiate these constructions. An alternative is to problematize the situation by presenting an example of a function that requires the chain rule for differentiation because previously learned methods are inadequate. With the exception of *Visual Calculus*, all of the courses include an introduction to the chain rule. However, *Thinkwell Calculus* is the only course that establishes a query by posing a problem that motivates a need for the chain rule; the function $(3x^2+1)^{200}$ bears a strong resemblance to functions such as $(3x^2+1)^2$ that students have previously differentiated but is obviously intractable using known rules (for products and sums). The other courses introduce the chain rule as an extension of the discussion on various rules of differentiation; the chain rule is presented as a method of differentiation that applies to an established way of combining functions, namely function composition.

Concrete demonstration. Analogies can facilitate learning by promoting an understanding of the abstract in terms of the concrete and form an important part of instructional explanations both in the classroom (Richland, Holyoak, & Stigler, 2004; Young & Leinhardt, 1998) and in texts (Iding, 1997). The next row in Table 5 refers to the presence of an analogy or illustration for the chain rule in the online courses. With the exception of *Visual Calculus* and *Calculus on the Web*, all of the courses use the analogy of an "inner" and "outer" function in explaining the chain rule. In the composition $(f \circ g)(x) = f(g(x))$, the function f is the outer function and g is the inner function. For functions such as $(x^3+1)^2$, the mapping is very compelling since (x^3+1) appears to be 'inside' the squaring operation (outer function), whereas, for exponential functions such as e^{2x}, the inner/outer relationship is less obvious. (Note: some authors denote exponentials as $\exp(x)$ instead of e^x, perhaps to help students "see" the inner and outer functions.) Using the inner/outer analogy draws attention to the nested property of function composition and, by labeling the functions, may facilitate a shift of perspective during implementation of the rule. In taking the derivative of the outer function, the inner function must first be placed in the background and then treated as an object. In *Thinkwell Calculus*, the lecturer emphasizes the foregrounding activity by physically covering over the inner function with putty and referring to it as "blop" while differentiating the outer function. *Thinkwell Calculus* also introduces an analogy of peeling fruits and vegetables; taking derivatives using the other differentiation rules is analogous to peeling bananas or oranges in which the fruit is immediately revealed, whereas applying the chain rule is analogous to peeling an onion in which the process must be repeated more than once. This nesting analogy is carried through in the accompanying worked examples and lecture notes that contain illustrations of outer and inner onion peelings next to the associated function components. *Internet Calculus* offers an illustration of the chain rule that, instead of emphasizing the nested characteristic, draws attention to the multiplicative nature of the chain rule. In an animated illustration of gears of given circumferences, the number of revolutions that one of the axles must make in order to turn another axle once (dy/dx) is shown to be the product of the number of revolutions related to an intermediate axle (dy/du and

du/dx). The potential of this analogy, however, may not be realized since the mapping between gears revolving and differentiating compositions is not explicitly unpacked in the instructional explanation.

Verbal description. The next row of Table 5 addresses the presence of a verbal demonstration of the procedure. If, in addition to the formal statement of the theorem, a course includes a verbal description, there is an increased likelihood that the student will be able to recall the rule and apply it in other situations. With the exception of *Visual Calculus*, all of the courses contain an informal restatement of the chain rule. Although revoicing is a powerful device, mathematical notation can also be suggestive and may provide an easy way to encode and remember information. As discussed earlier, the chain rule expressed using Leibniz notation is a prime example of this phenomenon; if the terms are conceptualized as quotients, then the chain rule appears to be the result of cancellation and can be easily constructed, especially for longer chains ($\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{du} \cdot \frac{du}{dx}$). *Karl's Calculus Tutor*, *Internet Calculus*, and *Thinkwell Calculus* include a statement of the chain rule using Leibniz notation. Only *Thinkwell Calculus* and *Internet Calculus*, however, mention the mathematical nuances that underlie the use of this notation as a mnemonic tool[44], although the explanation in *Internet Calculus* is in a later chapter.

Principles of use and establishment of legality. An instructional explanation may also identify the principles that permit a procedure to be used and prove its legality. For the chain rule, meeting these goals would require actions that address the relationship between differentiation and function composition. None of the courses identifies the principles underlying the chain rule

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[44] Tall (1985) provides an argument based on the original writings of Leibniz that permits the meaningful interpretation of $\frac{dy}{dx}$ as a quotient.

(although the gear animation in *Internet Calculus* does hint at the relationship between the derivative of a composition and its components). Three of the courses accomplish the related goal of establishing the legality of the chain rule. However, they differ in the number and construction of proofs they include. *Visual Calculus* contains a nongeneral proof followed by a note describing its shortcomings and mentions (but does not include) the existence of an ironclad alternative, whereas *Karl's Calculus Tutor* includes the nongeneral proof with no mention of the associated problems but also includes an alternative. *Internet Calculus* presents the nongeneral proof as part of the explanation, mentions its shortcomings, and includes a more general proof in an appendix.

Connections, conditions of use, and errors. The final rows of Table 5 refer to the goals of making connections to prior knowledge, illustrating conditions of use, and understanding the nature of errors. Because the supporting actions for these goals often center on worked examples, the discussion of these goal states is included in the analysis of the examples present in each course's explanation of the chain rule.

Examples

Figure 4 shows the number of worked examples and coverage index for the chain rule per course. *Thinkwell Calculus* and *Internet Calculus* stand out for having both a large number of examples and a high coverage index. *Calculus on the Web* also contains a comparable number of examples, but the coverage index is much lower.

INSERT FIGURE 4 ABOUT HERE

A closer look at the contributions to the coverage index reveals similarities and differences in the way the various courses use examples. The top half of Table 6 addresses the presence of worked examples modeling the chain rule in a variety of mathematical settings that the student is likely to encounter (*e.g.,* compositions of standard functions, functions that require a repeated application, and functions that require a combination of differential rules). These are the examples that the student might reference in order to successfully complete a similar exercise. The lower half of Table 6 refers to more conceptual issues surrounding the chain rule procedure (*e.g.,* approaching functions from alternative perspectives, recognizing conditions of use with respect to efficiency, and being alert for mistakes). These examples address questions of whether and when to use the chain rule and what to guard against. Notice that, with the exception of *Thinkwell Calculus*, all of the online courses concentrate on the former type of example although *Calculus on the Web* and *Karl's Calculus Tutor* do not provide simple examples for several instances.

INSERT TABLE 6 ABOUT HERE

Thinkwell Calculus is notable for containing examples that serve both purposes. In particular, *Thinkwell Calculus* is the only course that contains an example that demonstrates a condition in which the chain rule would *not* be the differentiation method of choice and addresses the tendency of students to overgeneralize a newly learned procedure. In the instruction of the chain rule, the function $f(x) = \ln(x^3)$ is an "imposter": it does not require the chain rule for differentiation but bears a strong physical resemblance to problems that do. Furthermore, using the chain rule in this situation, while mathematically correct, would be inefficient. It is important to note, however, that the decision of an online course to limit worked examples that highlight conceptual issues may be a reflection of a teaching philosophy that relies more on active participation to promote self-explanations (cf. Chi, Leeuw, Chiu, & LaVancher, 1994). *Internet Calculus* is a prime illustration of this decision; instead of worked examples, the course uses embedded exercises that direct students to think of previously encountered functions in new ways (*e.g.*, $\frac{-7}{(2x-3)^2} = -7 \cdot (2x-3)^{-2}$), compare the results of the chain rule with formerly learned rules of differentiation (in this case, the quotient rule), compare orders of application (*e.g.*, $\frac{(3x-1)^2}{(x^2+3)^2}$) differentiated using the quotient or product rule before the chain rule or vice-versa), and consider situations in which the chain rule is non-optimal (*e.g.*, $\ln(\sqrt{x+1})$). Provided students engage in these activities, this design encourages self-explanation more than a presentation in which students view (or listen to) worked examples. The remaining online courses contain exercises that illustrate some of these conceptual issues but, unlike *Internet Calculus*, do not explicitly address them.

Exercises

Although all of the courses provide some opportunity for the student to test his/her understanding of the chain rule, the number of exercises ranges considerably from 30 (*Karl's Calculus Tutor*) to 251 (*Internet Calculus*). The location of the exercises relative to worked examples also differs across the courses. *Karl's Calculus Tutor* and *Internet Calculus* are the two courses that minimize distance by placing exercises on the same "page" as worked examples that might serve as problem solving models. *Internet Calculus* draws attention to this affordance by using the labels "Try It" and "Exploration" for these embedded problem solving activities.

An analysis of the types of exercises in each course reveals a predominance of activities that are low in cognitive complexity. Although none of the courses contains exercises that can be accomplished by direct recall, three of the courses (*Visual Calculus*, *Thinkwell Calculus*, and *Calculus on the Web*) only provide exercises that qualify as simple applications of procedures. That is, in these environments, students' experience with the chain rule is limited to applying an algorithm to a set of given expressions. This emphasis on the application of a procedure without meaning attached may promote a narrow view of the very nature of mathematics (Hiebert & Carpenter, 1992). *Karl's Calculus Tutor* and *Internet Calculus* include exercises requiring cognitive effort beyond application of the algorithm, but there is a marked difference in the number and variability of these tasks within each course. The complex tasks in *Karl's Calculus Tutor* are limited to a few modeling situations in other domains (*e.g.,* chemistry), whereas *Internet Calculus* stands out for including a large number of tasks that promote mathematical practices such as conjecturing, deliberating, and justifying. *Internet Calculus* is also the only course that includes activities that instruct the student to use a CAS to compute the derivative of given functions and that incorporates exercises without performance standards which encourage students to freely explore the result of the chain rule in a variety of situations (although *Visual Calculus* contains a CAS editable example of the application of the chain rule to a given function).

Given that the focus of the chain rule instruction in each course is to teach students to apply the chain rule to given expressions, it is informative to inspect the types of situations that are represented by these exercises. Table 7 shows the number and type of exercises devoted to algorithmic practice according to mathematical complexity (single application, repeated application, and rule combination) and structure (single application with perspectival shift, rule combination with perspectival shift, imposter, and reformulation). As can be seen, all five courses devote the majority of these exercises to applications of the chain rule that do not require the construction of an alternative perspective. In these situations, the student must construct a decomposition of the given function, but the structure of the function does not need to be reconceptualized. For example, differentiating the function $\sin^2(2x)$ requires the recognition of three nested functions before applying the chain rule, whereas differentiating the function

$\frac{1}{(x-2)^3}$ affords a perspectival shift (the expression, given as a quotient, can be reconceptualized as $(x-2)^{-3}$).

The ability to shift perspectives is generally a useful skill that allows the problem solver to view a given situation in a way that leads to the solution satisfying pre-determined criteria (such as efficiency or simplicity). However, not all perspectival shifts are constructive, and some of the courses encourage the adoption of perspectives with respect to the chain rule that are inefficient. For example, *Calculus on the Web* requires the differentiation of $5(8v+1)$ using the chain rule, and *Thinkwell Calculus* directs students to use the quotient and chain rules to differentiate the quotient $\frac{x^3}{(3x^2)^2}$. Although applying the chain rule is not wrong in either of these two cases, it is clearly not as efficient as conceptualizing the functions without composition, and this alternative perspective (one that corresponds to ways of looking at functions prior to the introduction of the chain rule) is not mentioned or explored. *Internet Calculus* provides students with the most opportunities for productive perspectival constructions and encourages students in the habit of approaching problem solving using perspectives by explicitly calling attention to this activity (in tasks embedded in the explanation) and by the variability of the functions and their representation within larger practice settings.

INSERT TABLE 7 ABOUT HERE

Visual Calculus and *Thinkwell Calculus* are the two courses that give hints. In *Visual Calculus*, the hints are a decomposition of the given function, but *Thinkwell Calculus* often provides hint sequences for a given problem that include definitions, strategies, and solution steps. These hints, however, are independent of student performance and are sometimes incongruous with an efficient solution. For instance, when the imposter problem “Find $\ln'(e^4)$” is embedded in a set of problems for the differentiation of logarithmic functions, the first hint provides information about the derivative of the logarithmic function ($\ln'(x) = \frac{1}{x}$) and does not draw attention to the mathematical content of the given situation.

Explorations

Explorations are not a key component of the chain rule instruction for any of the online courses. An example of one of the four explorations included in *Internet Calculus* is to graph the function $y = \sin(bx + c)$ and its derivative for several values of b and c and then explain the effect that these parameters have on the derivative of the function. In this instance, the exploration could both connect to (prior) knowledge of the relationship between the graph of a function and its derivative as well as address a potential error, namely the failure to recognize the need for the chain rule in this example. Graphing the functions alongside their derivatives and noting the effect of the parameters could serve to increase the saliency of such an error. Whether or not this activity as it stands (that is, without feedback and guidance) is sufficient for helping students construct a better understanding of the chain rule is a question for empirical research.

Visual Calculus, as mentioned previously, includes a CAS editable worked example of the chain rule applied to two functions. However, in contrast to *Internet Calculus*, there are no explicit directions that accompany the activity; the accompanying text simply notes that it is possible for the user to change the functions in the example. One could imagine an activity that encouraged students to investigate the effects of using the chain rule on two different decompositions of the same function. For example, the function $f(g(x)) = \sqrt{x^2 + 1}$ can be decomposed as $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$ *or* $f(x) = \sqrt{x+1}$ and $g(x) = x^2$. The lesson to be learned is that, in either case, the chain rule produces the same result. In such an activity, the computations could be generated using a CAS so that students' focus could be on the result of various decompositions rather than on the mechanics of the procedure. Once again, empirical work is needed to ascertain whether such an activity would be productive and to determine effective design principles.

DISCUSSION

Analytic frameworks

The purpose of this analysis was to construct a framework for critiquing stand-alone online calculus instruction. Our primary interest was to ascertain how existing courses were aligned with established principles of instruction and learning and provide a picture of each course that captured features of interest for instructors and students. Accordingly, we selected analytic criteria based on cognitive research and tested the viability of our framework through its application to five courses that represented a variety of learning goals and educational philosophies.

Another research paradigm that often comes to mind when courses are positioned neck-to-neck is the "horse race." In this paradigm, two (or more) courses are selected, assessments are designed, student participation is elicited, and, following the experiment, one of the courses is declared as "the winner." Using this paradigm, one can answer questions, such as: Which course causes the greatest learning gains? Which course was more enjoyable for students? Which course was most appropriate for a given level of mathematical understanding? Clearly, gathering this type of information is helpful for choosing between a selection of specific courses that are under consideration, or "in the running." However, the horse race paradigm is not as efficient if the goal is to choose a course that meets an instructor's (or student's) unique needs. In this case, answers to a different set of questions are needed, for example: Which course provides more comprehensive instructional explanations? How can one maximize opportunities for practice?

What provisions are there for help on exercises? These are the types of questions addressed by our framework.

In contrast to descriptive characterizations of computer-based instruction, the framework proposed here has a greater analytic focus. Central to this analytic framework are dimensions that contribute to learning effectiveness according to cognitive research, namely explanations, examples, exercises, and explorations. In addition, a catalog of possible online resources was included to characterize the implementation of representational, navigational, and dialogical resources. This framework for stand-alone online calculus instruction is a member of a family of analytic frameworks that address online statistics (Larreamendy-Joerns et al., 2005) and online chemistry (Evans & Leinhardt, 2007).

Each of the analytic frameworks in this family is based around instruction covering an important and challenging topic in the discipline. Thus, stoichiometry was the topic chosen for the examination of online chemistry instruction, and the online statistics review analyzed units on statistical tools. Similarly, our analysis of stand-alone calculus courses was based on instruction for the chain rule. The disciplinary importance of the chain rule, together with the compactness of its instruction, made it well suited as an analytic focus. A reasonable question one could pose to this choice is the degree to which topic treatment is representative within a course. An overview of the courses showed that the instruction on the chain rule was representative of instruction on other topics within the course. Just as textbooks (Mesa, 2004, in press; Raman, 2004) and teachers (Leinhardt & Greeno, 1986; Gresalfi, 2004) are generally consistent in their instructional approach, the online courses we examined appeared to be consistent in their treatment of the many topics that comprise introductory calculus.[45]

Although based on chain rule instruction, the framework that we constructed is sufficiently general to be applied to another topic in calculus with modification. For instance, analyzing an instructional explanation requires that one determine the sub-skills that are relevant for the given topic. This set of skills will necessarily be different for different topics. However, of more interest than this discussion of generality and choice of topic is the question of viability. Is our proposed framework that was informed by cognitive research and based on instruction for the chain rule viable? The answer can be found in its application. When applied to five stand-alone online calculus courses, the framework not only provided a snapshot of instructional practices but also captured the differences between the courses. As a result of different emphases (*e.g.,* performing operations fluently versus forming and testing conjectures) and guiding intent (*e.g.,* supplementary help versus opportunity for practice), the courses differed along several of the dimensions (*e.g.,* the implementation of online resources and especially graphing calculators and CAS's, the depth of the explanations, and types of exercises). In this way, the proposed framework is part of a viable plan for a choosing a course that meets a potential calculus learner's unique needs.

Courses

Each course included in the analysis took on the nontrivial and laud worthy task of providing students with an understanding of calculus within an (potentially isolated) online environment. However, the courses vary widely in the implementation of resources and

[45] Our criteria for including only courses that covered the scope of college-level introductory calculus (versus a single or small set of topics) enabled us to compare the treatment of the chain rule with other topics.

instructional presentation of the chain rule. In terms of online affordances, the absence of responsive feedback was perhaps the most conspicuous shortcoming of the courses analyzed. Students, especially beginning learners, often overestimate the accuracy of their knowledge (Rozenblit & Keil, 2002) and may suffer from the lack of explicit judgments on the accuracy and quality of their solutions (whether or not the final answer is "correct."). Why then is this not a standard feature of online courses, especially those accessible to isolated learners? The implementation of adaptive feedback requires sophisticated programming and authoring technologies, as well as detailed cognitive analyses of task environments and student learning processes (Aleven & Koedinger, 2002; Koedinger, Aleven, & Heffernan, 2003), creating a tension between the provision of this resource and cost-effectiveness. *Calculus on the Web* has taken an important first step by providing evaluations of accuracy for algebraic responses, often for multiple problem-solving steps. This effort, along with the success of other mathematics tutoring programs (e.g., The Algebra Tutor and The Geometry Tutor), should spur on the investigation and further development of adaptive feedback for online instruction (Anderson, Corbett, Koedinger, & Pelletier, 1995).

In addition, the chain rule was sequenced with other topics in a variety of ways across courses. This raises the question of establishing the topic arrangements that are optimal for student learning. Although the order of presentation varies widely in practice, there is no precedent for selecting any particular sequence of topics, other than preserving the order of topics that build on others.

Thinkwell Calculus and *Internet Calculus* are the two courses that provide the most complete instructional explanations for the chain rule. It is worth noting that none of the courses highlight the mathematical principles underlying this procedure (addressing the questions of why the derivative of a composition is multiplicative and why the differentiation involves multiple variables), although several courses include one or more proofs. Another general weakness of the explanations is the discussion of conditions of use. This omission may be particularly important for instruction on the chain rule since recognizing situations that call for its application can be problematic for students.

Although all of the courses provide worked examples to support the explanation of the chain rule, the analysis revealed differences in both the number of examples and the ways they supported learning. The primary use of worked examples in all of the courses was to model applications of the chain rule in a variety of mathematical settings that the student is likely to encounter. *Thinkwell Calculus* is notable for also including worked examples that highlight alternative perspectives, demonstrate conditions of use, and warn against potential mistakes. The lower example coverage index of *Internet Calculus* reflects a pedagogical strategy that favors active participation over the use of worked examples to promote self-explanation.

Each course also provides opportunities for the student to practice applying the chain rule as an algorithm for a variety of functional compositions. *Internet Calculus* and *Thinkwell Calculus* are notable for including practice exercises that require reconceptualization and, therefore, afford perspectival shifts. However, *Internet Calculus* is the only course that encourages participation in other mathematical practices, such as conjecturing, deliberating, and justifying.

Explorations are, in principle, well suited for course instruction in an online environment. Surprisingly, the courses included in this analysis do not make much use of this resource. One could envision an activity, for example, that embodies mathematical principles by simulating a hill-climbing activity (Lutzer, 2003) or, as discussed earlier, one that explores the (non)effect of alternative function decompositions on the outcome of the chain rule. However, including exploration activities in the instructional materials for the sake of their presence is inadvisable; an analytic framework addressing this particular instructional component and experimental research are needed to discern the characteristics of explorations that contribute to their effectiveness.

Implications

Online courses allow students to pace themselves and respond to questions as the material is encountered, *e.g.,* through embedded problems or activities (Larreamendy-Joerns & Leinhardt, 2006). Such courses can stand alone, or, as is more common, be used in a melded setting where the lead instructor becomes a coach rather than a lecturer. Designing courses that allow students to master calculus in a computer-based environment requires numerous decisions on the implementation of online resources, the construction of explanations, the presentation and content of examples, the provision of exercises and feedback, and the inclusion and design of explorations. The development of such a course for introductory calculus can profit from the framework we have presented. In particular, the analysis of existing courses highlights the need for incorporating adaptive feedback, productive explorations, and an active social component for students who may otherwise be isolated learners.

Online courses are becoming a more common educational practice, and universal methods for analyzing these courses are needed. The framework developed here, in the context of calculus, contributes to a developing genre of online course analysis. From the languages to the sciences, online courses have the potential of providing students across the world with the opportunity to experience new domains in unique ways. Realizing this potential calls for the joint effort of course designers, domain experts, and educational researchers.

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Table 1

Online calculus course included in the analysis

Course	VC	TC	COW	KCT	IC
Author(s)	L. S. Husch	Edward Burger	G. Mendoza & D. Reich	Karl Hahn	R.Larson, B. Hostetler, & B. Edwards
Author Institution	U. Tennessee	Williams College	Temple University	None	Penn State University & U. Florida
Intent	Computer use in instruction	"Next generation" textbook	"Friendly environment"	Supplement	Participatory text
Technical Feature	Animated explanation	Video tutorials	Short-answer feedback	Stories motivating topics	CAS usage
Cost	Free	$68.95	Free	Free	$25 per 6 months
Plug Ins	LiveMath, Flash, Tech-Explorer, Acrobat Reader	Included	None	None	Acrobat Reader, Quicktime
Suggested Broswer	Safari 1.2.3	Explorer 4.0+	None	Explorer 5.x+ or Firefox 1.0+	Navigator 4.x or Explorer 5.x

Table 2

Definitions and examples of cognitive complexity

Category	Definition	Example
Complex use of procedures	Task requires students to make meaning of a situation by applying a given set of concepts or procedures, in order to make conjectures, provide justifications, construct models, or draw conclusions	Let f be a differentiable function of period p. (a) Is the function $f(x)$ periodic?; (b) Consider the function $g(x) = f(2x)$. Is the function $g(x)$ periodic?
Simple use of procedures	Task requires students to apply a well-rehearsed algorithm, without attending to the meaning of the outcome in the context of the problem situation.	Differentiate $f(x) = \sin(x)$.
Direct recall	Task requires students to reproduce previously learned facts, rules, or definitions.	State the chain rule for two functions.

Table 3

Definitions and examples of practice types by course

Category	Practice type	Example
Simple	Single application	$(x^3+1)^{1/2}$
Repeated	Multiple applications to single function	$\sin^2(x^3)$
Rule combination	Application with the product or quotient rule	$(x^3+2x)^{1/2}\cdot(2x^2+1)^3$
Simple with perspectival shift	Single application to function that can be differentiated using other rules	$(x^2+1)^2$
Rule combination with perspectival shift	Application with other rules to function that can be differentiated using other rules	$(x^4)(2x+1)^2$
Imposter	Application to expression with surface features that suggest chain rule but differentiation without is more efficient	$e^{\ln x}$ or $\ln(e^x)$
Reformulation	Application requires restructuring as composition	2^x or $-1/\sin^2(x)$

Table 4

Available online resources by course

Resource	VC	TC	COW	KCT	IC
Videos		✓			✓
Animations	✓			✓	✓
Graphing calculator				✓	✓
CAS	✓				✓
Course map	✓	✓	✓	✓	✓
Glossary		✓			✓
Search engine		✓		✓	
Links to external sources		✓		✓	
Note-taking facilities					
Multiple choice questions	✓	✓			✓
Short answer questions	✓		✓	✓	✓
Feedback		✓	✓		
Electronic forum		✓		✓	✓
Course management system		✓	✓		✓

Table 5

Instructional explanation goal states addressed by course

Goal state	VC	TC	COW	KCT	IC
Sub-skills available			✓	✓	✓
Nature of problem		✓	✓	✓	✓
Concrete demonstration		✓			✓
Verbal description		✓	✓	✓	✓
Principles of use					
Legality established	✓			✓	✓
Connections to prior knowledge		✓		✓	✓
Conditions of use		✓			
Nature of errors understood		✓	✓		

Table 6

Example coverage by course

Coverage	VC	TC	COW	KCT	IC
Polynomial	●	●	●	●	●
Radical (symbol)	●	○	●	●	●
Trigonometric	●	●	●	○	●
Exponential	●	●	○	○	●
Logarithmic	○	●	○	○	●
Repeated application	●	●	●	●	●
Within quotient or product rule	○	●	●	○	○
New perspective *and* demonstrate equivalence	○	○ ◓	○	●	○ ◓
Conditions of use (imposters)	○	●	○	○	○
Common errors	○	●	●	○	○

Table 7

Number of practice type by course

Practice	VC	TC	COW	KCT	IC
Single application	30	15	65	20	74
Repeated application	7	7	13	2	16
Rule combination	5	10	53	4	27
Single application with perspective shift	2	4	12	2	9
Repeated application with perspective shift	0	3	1	0	4
Imposter	5	3	0	0	29
Reformulation	9	4	6	0	3

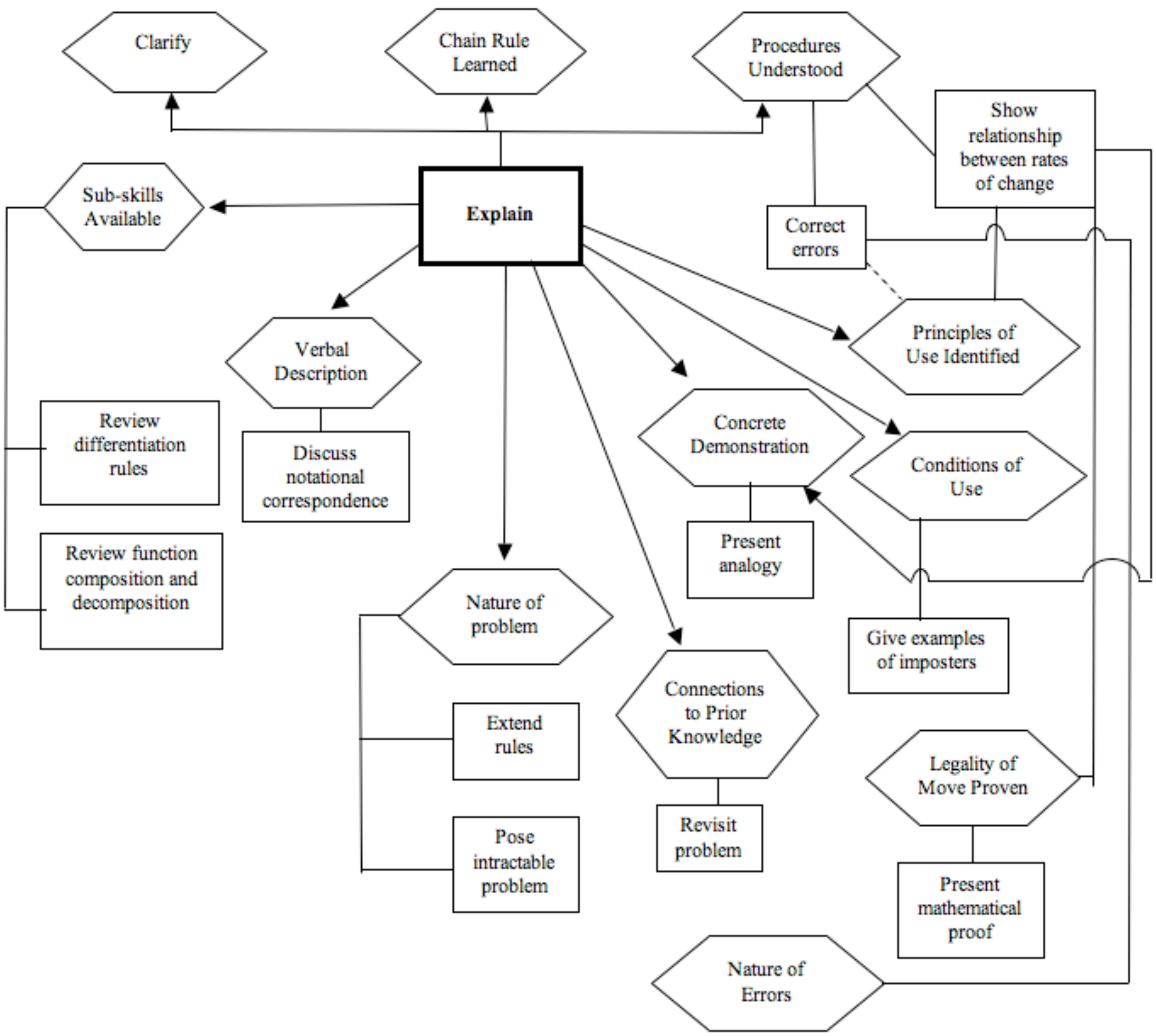
Figure Captions

Figure 1. Model of an instructional explanation adapted from Leinhardt (1987).

Figure 2. Sequencing of differentiation rules and function types by course.

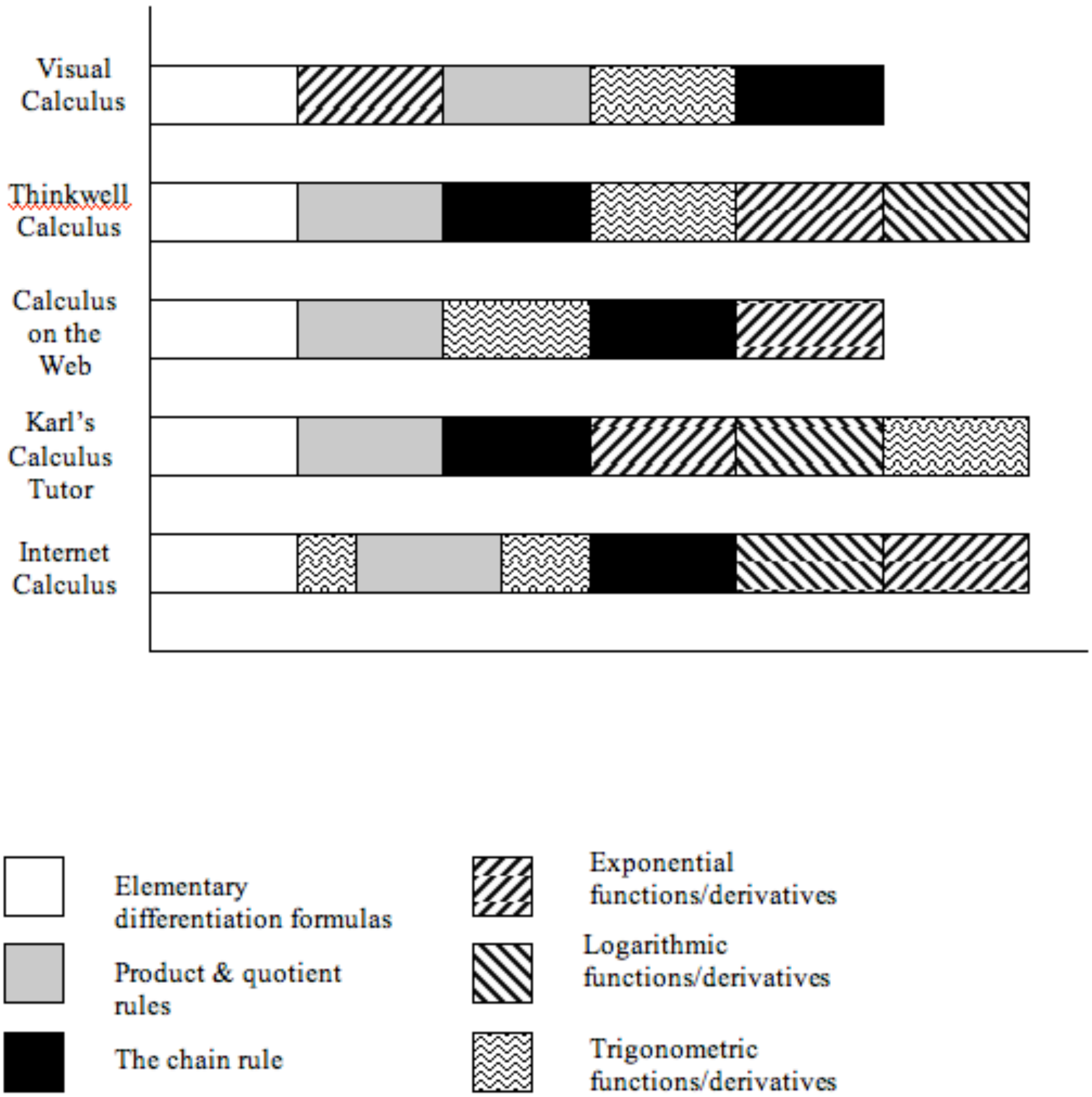
Figure 3. Number of examples and coverage index by course.

Figure 1



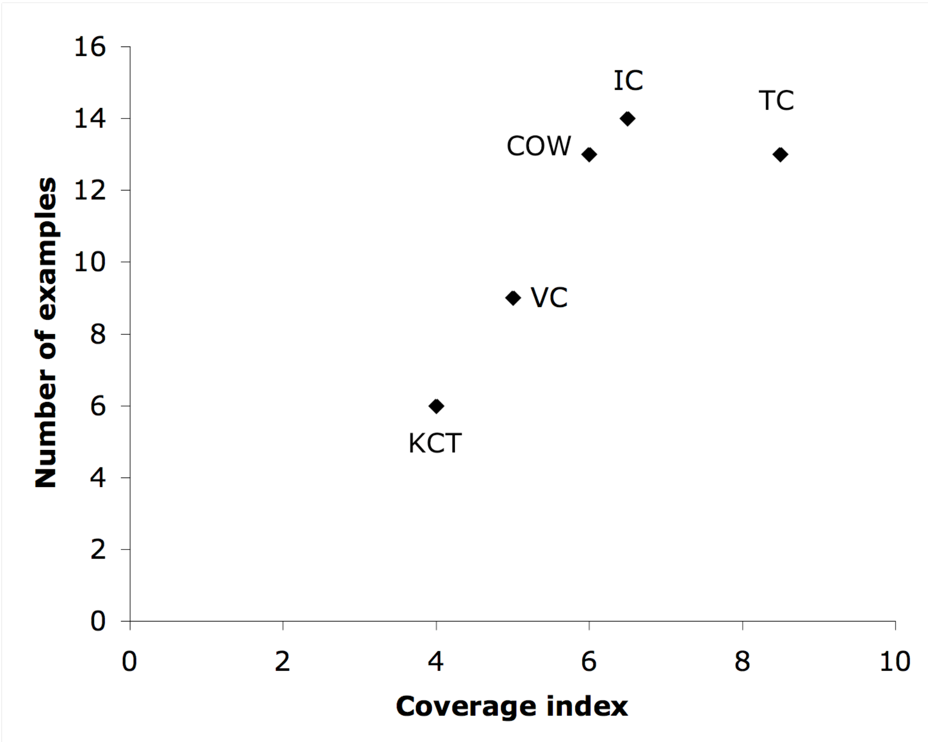
Clarify
Chain Rule Learned
Procedures Understood
Show relationship between rates of change
Sub-skills Available
Explain
Correct errors
Verbal Description
Principles of Use Identified
Review differentiation rules
Discuss notational correspondence
Concrete Demonstration
Conditions of Use
Present analogy
Review function composition and decomposition
Nature of problem
Give examples of imposters
Connections to Prior Knowledge
Extend rules
Legality of Move Proven
Pose intractable problem
Revisit problem
Present mathematical proof
Nature of Errors

Figure 2



Visual Calculus
Thinkwell Calculus
Calculus on the Web
Karl's Calculus Tutor
Internet Calculus
Elementary differentiation formulas
Product & quotient rules
The chain rule
Exponential functions/derivatives
Logarithmic functions/derivatives
Trigonometric functions/derivatives

Figure 3



16
14
12
10
8
6
4
2
0
Number of examples
IC
TC
COW
VC
KCT
0
2
4
6
8
10
Coverage index

APPENDIX B

PRE-PRINTS OF WORK ON OPEN, ONLINE HELP FORUMS

B.1 HELP! TUTORETTES ON THE CALCULUS CONCEPT OF LIMIT

Carla van de Sande

University of Pittsburgh

> Help! I need somebody.
> Help! Not just anybody.
> Help! You know I need someone —
> Help.
>
> (The Beatles, 1965)

The lyrics from this popular song of the 1960s still resonate with today's young people, particularly students as they grapple with homework assignments, struggle with difficult and often novel material, or prepare for examinations. Consequently, many students are turning to tutors for assistance with their schoolwork. Fueled by large and often impersonal classroom situations (VanderMeij, 1988), social barriers to asking for help in public (Graesser, McMahen, & Johnson, 1994), and complex instructional explanations, the demand for tutors is on the rise. Indeed, help-seeking is recognized as an effective means for students to cope with academic difficulties (Nelson-Le Gall, 1985). Student-initiated tutoring, in which a student seeks out a tutor for help (usually for a specific problem), is a common manifestation of this phenomenon.

Student-initiated tutoring has traditionally occurred in face-to-face settings, for example in university-sponsored help centers where students experiencing difficulties with their course work can receive help from more advanced students. These tutoring centers are generally organized according to subject area (e.g. mathematics) and have designated hours of operation. However, since computer networking has become ubiquitous, a new context for tutoring has emerged in which students and tutors are not necessarily linked by physical ortemporal proximity. Today's "net generation" (Oblinger & Oblinger, 2005) and "digital natives" (Prensky, 2001) are using the Internet to voice their cries for help. A large number of students are participating in asynchronous web-based forums that are subject-specific but are not affiliated with a particular institution. As participants in these forums, students can "meet" with tutors anonymously to receive free help on schoolwork. There are two major categories of gratuitous online mathematics help, depending on whether the postings can be spontaneously answered by any registered viewer ("Spontaneous Online Help" or SOH) or whether the postings are assigned to a specific volunteer tutor ("Assigned Online Help" or AOH). The exchanges in AOH sites are typically between a student and a single tutor and, in this way, resemble the configurations found in face-to-face help centers; in contrast, the exchanges in SOH sites often contain responses from more than one tutor.

Student-initiated tutoring, either face-to-face or web-based, may be the only recourse some students have for receiving help on preparing homework assignments or studying for examinations. In particular, this opportunity is critical for some students taking introductory calculus, a course that is required for numerous college majors but is renowned for its high attrition rate. However, there has been no systematic study of student-initiated tutoring, either in face-to-face settings or in web-based forums. The purpose of this research is to address this deficit by investigating face-to-face and web-based student-initiated tutoring for introductory calculus with a focus on a challenging mathematical topic, namely the limit. The concept of limit is foundational to calculus and is a recurring theme in any introductory course, from the definition of the derivative to that of the integral. However, the concept of limit is fraught with subtle nuances that took over a century to resolve. In particular, students often experience confusion about whether a function can reach its limit, whether a limit is actually a bound, whether limits are dynamic processes or static objects, and whether limits are inherently tied to motion concepts (Williams, 1991). Resolving these quandaries is at the heart of gaining an understanding of calculus that will support future learning. A study of student-initiated tutoring in calculus will provide insight into the practices that help students resolve authentic questions and suggest ways in which instruction might be better designed.

We have coined the term "tutorettes" for referring to focused, student-initiated tutoring exchanges (conducted either face-to-face or online) in order to distinguish them from tutoring sessions in which the tutor is assigned the task of covering a pre-determined topic or set of topics. In contrast to tutoring sessions, the topic of a tutorette is a single problem (or small set or problems), usually from homework, and the primary goal of a tutorette is to solve the given problem(s).

Although educational research on "tutoring" has focused on tutoring sessions (generally faee-to-face), there has been some research on online problem-specific dialogues. Kortemeyer (2006) analyzed dialogues between students in an introductory physics class according to type of contribution (e.g. emotional, surface, procedural, or conceptual). Each dialogue was associated with a specific homework problem.[46] However, these dialogues are different than most tutorettes since the participants shared a common course experience and the tutors were all fellow students. Renninger, Ray, Luft, and Newton (2006) investigated pre-service teachers' responses to elementary students' online solutions of non-routine challenge problems (the Math Forum's Problems of the Week) and constructed a coding scheme for "mathematical mentoring." Successful scaffolding reflected an accurate assessment of the learner's current level and the provision of feedback that enabled the learner to reach a new mathematical understanding. These dialogues technically qualify as tutorettes, although students seeking help on solving challenge problems may have very different goals than those seeking help for (graded) homework problems.

Because tutorettes are instances of tutoring, we begin with a review of the research on traditional tutoring. Second, we introduce a theoretical model of help. Third, we review the history of the limit concept and students' understanding of the limit. Finally, we analyze a corpus of tutorettes from three contexts according to some of the dimensions specified in the model.

Tutoring Research

Traditionally, a tutoring session involves a tutor and a student sitting together (e.g. "face-to-face") and working with instructional materials to cover a pre-determined topic or set of topics. This instructional approach has proven incomparably effective for academic performance and attitudes toward subject matter, spawning a wealth of educational research (See Cohen, Cohen, Kulik, and Kulik (1982) for a meta-analysis of 65 tutoring studies). In what has become a

[46] Each student saw a slightly different version of the problem (e.g. different numbers, coordinate system, etc.) in order to discourage students from sharing numerical answers.

classic piece of literature in the field, Bloom (1984) documented that students learning from tutors perform two standard deviations above students learning in a classroom situation. One direction taken by research is to identify characteristics of participants and exchanges that contribute to the tutoring advantage. Surprisingly, the tutoring advantage cannot be attributed to the tutors level of expertise (Graesser & Person, 1994) or the familiarity of the participating parties (McArthur, Stasz, & Zmuidzinas, 1990; Siler & VanLehn, 2005). Instead, the advantage of tutoring may be attributed to the opportunity it presents for students to ask questions (Graesser & Person, 1994), the intensity of the interaction (McArthur et al., 1990), and the cues from tutors that maximize the motivation to learn (Lepper, Aspinwall, & Mumme, 1990). While detailing a wide variety of characteristics that contribute to the effectiveness of face-to-face tutoring, all of these analyses reveal that face-to-face tutoring is a highly interactive process in which participants relate to one another in ways that are markedly different from those found in other instructional settings.

Another direction taken by research asks how tutoring embodies elements of idealized instruction. Graesser, Person, & Magliano (1995) constructed a set of 8 such elements that were emphasized in contemporary pedagogical theories and intelligent tutoring systems and examined the extent to which these were present in a large corpus of tutoring protocols. They found evidence for three of these elements (collaborative problem solving, question answering, and explanatory reasoning in the context of specific examples) but not for the remaining elements (active student learning, sophisticated pedagogical strategies, deep explanatory reasoning, convergence toward shared meaning, feedback, and motivation).

The positioning and goals of the participants in tutorettes, however, are not identical to those in tutoring sessions or classroom instruction. For instance, the purpose of a tutorette is to construct the solution of a particular problem (example) versus elucidate a given concept (with examples). Therefore, the elements of an effective tutorette will be somewhat different than those of tutoring. Tutorettes are, however, instances of instruction and take on some of the characteristics of instructional explanations. For example, the underlying principles in the subject domain (e.g. theorems and definitions) may be used to support the construction of the solution to a problem.

A model of help

Figure 1 shows a theoretical model of help that revises the best practices of teaching according to Graesser, Person, and Magliano (1995) and incorporates these with the features of an effective instructional explanation, as defined by Leinhardt (1987; 2005). The formalization of the model is a planning net of goals and supporting actions. There are three main goals of helping: *clarifying*, *solving* the problem, and *understanding* the procedure or concept. The action of *helping*, shown in the dominant rectangle immediately below these top goal states, has as a consequence the (partial) achievement of one or more of the top-level goals. First of all, a prerequisite for help is that the tutor *discern the query*. Although the problem statement is often identical to the student's query, there are cases in which the query is only indirectly voiced and discerning this is key to effective help. For example, if a student asks, "Does $\lim_{x \to 0} (x/x) = \lim_{x \to 0} 1 = 1$ mean that $0/0 = 1$?," the query may involve justifications for the progression of steps in the chain of expressions (e.g. the limit operation) rather than the definition of 0/0.

Another requisite of help is that the components of the explanation are at the *appropriate level*. If the tutor uses methods or procedures that are unknown to the student, then the help is unlikely to benefit the student. This is especially an issue when participants are unfamiliar with one another, as in the case in most tutorettes. In such cases, tutors and students must devote some effort to establishing common ground.

The presence of *alternative perspectives* on the problem may also contribute to the effectiveness of help. If the pursuit of one perspective is unfruitful, the introduction of another way of looking at the problem may help the student construct a solution. The availability of a large repertoire of perspectives contributes to the flexibility of the help.

Tutorettes are instances of help-seeking and, as such, should promote instrumental (mastery-oriented) versus executive (dependency-oriented) behavior of the students (Nelson-Le Gall, 1985). Thus, one goal of the tutor should be to promote *student activity*. In particular, students should contribute to the construction of the solution and not be positioned as mere recipients of information.

The actions that support the goal of promoting student activity (e.g. hinting) can also be seen as supporting the goal of understanding the *nature of errors*. When students have unsuccessfully attempted a problem, productive hints or corrections from a tutor can contribute not only to the correct solution of this particular problem but, more generally, to an understanding of the procedure or concept involved.

Help should also manifest good problem solving *techniques*. In particular, the written record of the tutorette should be perspicuous and error-free. The tutor can contribute to this construction by actions such as modeling and critiquing. The solution artifact can then serve as a resource for the student, for example when reviewing the work at a later time.

Finally, in the process of helping solve a particular problem, a tutor should identify *general principles* in the domain that underlie the solution. The problem can then be incorporated into the students' mental model as an instance of the general principle. Sometimes it may help to *contextualize* the problem, especially if the intent of the problem is to prepare the student for a future concept. This is particularly relevant to mathematical instruction where concepts may be assembled over several different courses (e.g. algebra and calculus).

The Limit

The concept of limit and associated ideas of the infinitely small and the infinitely large are foundational to calculus. Indeed, a consideration of infinitesimal quantities and infinity is a distinguishing feature of calculus compared to other branches of mathematics (such as algebra, trigonometry, geometry, etc.). Thus, it is typically within the context of an introductory calculus course that students become acquainted with the mathematical definition of limit together with various representations and procedures for computing limits.

The formal definition. The mathematical definition of limit used in calculus instruction today, with its string of quantifiers, absolute values, and implications, has a formidable appearance:

$\lim_{x \to a} f(x) = L$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

In order to appreciate the need for such a construct and the meaning of the definition, it is worthwhile to take a few moments to review the events surrounding its development. Historically, our current analytic definition of limit has its roots in criticisms leveled at the lack of rigor that was endemic to the early stages of the formulation of calculus. In a scathing essay directed at the mathematicians of the day, Bishop George Berkeley pointedly ridiculed the mathematical structures on which calculus was based:

> And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities? (Dunham, 2005)

Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716), recognized as the founders of calculus, were performing computations and developing meaningful results based on mathematical constructions that were suspect. These constructions, termed infinitesimals, posited the existence of a least possible length, an infinitely small magnitude that could not be further subdivided. Of course, such constructions cannot exist over the real numbers since any length, no matter how small, can always be divided further, and this fact that did not escape 17th century mathematicians. Indeed, Leibniz himself recognized the problematic nature of infinitesimals, entities that could be both zero and nonzero, as the need dictated. However, Leibniz maintained, whether or not these constructions actually exist, they still serve as "fictions useful to abbreviate or speak universally"(Edwards, 1979). In other words, Leibniz viewed the existence of infinitesimals as a philosophical issue that was independent of their operational validity, his primary concern. Others, most notably Berkeley, did not share this pragmatic perspective. "Error," Berkeley wrote, "may bring forth truth, though it cannot bring forth science"(Dunham, 2005). This concern – that the end does not justify the means – continued to reverberate in the mathematical community and had repercussions that are evident in our current formulation of calculus. In particular, the analytic definition of limit evolved in the aftermath of these objections. Although it took nearly a century, Berkeley's "ghosts" were eventually laid to rest.

We attribute the foundational status of the limit concept to Augustin-Louis Cauchy (1789-1857) who formulated the following definition:

When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. (Dunham, 2005)

Karl Weierstrass (1815-1897) refined this definition and provided us with the current analytic form, the formal definition of limit. Disturbingly vague notions of "approach" and "differing...as little as one wishes" that were present in the initial definition were formalized in a statement of inequalities that can be used as the foundation for proving not only the validity of a proposed limit in a given situation (e.g. $\lim_{x \to 3} x^2 = 9$) but for constructing proofs of general limit theorems (e.g. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$).

Although there is disagreement about the role that formal definitions should play in mathematical instruction (Raman, 2004), it is not unusual for students in an introductory calculus course to be required to apply the formal definition of limit definition to select cases. For example, a typical exercise might ask students to use the epsilon-delta definition of limit to prove that $\lim_{x \to 2}(5x - 3) = 7$. The solution involves forming a series of implications showing how to construct delta (usually a function of epsilon) such that, if the function is evaluated at any value (other than 2 itself) that is less than the distance of delta from 2, the result will be within epsilon of 7, where epsilon is some positive value that can be arbitrarily close to zero. That is, given $\varepsilon > 0$, $|(5x - 3) - 7| < \varepsilon \Leftrightarrow |5x - 10| < \varepsilon \Leftrightarrow 5|(x - 2)| < \varepsilon \Leftrightarrow |x - 2| < \varepsilon/5$.

The conclusion is that, given epsilon, delta can be assigned the value of $\varepsilon/5$ and this will guarantee that, for all values of x within delta of 2, the corresponding values of the function $(5x - 3)$ will be within epsilon of 7. Notice that the construction of the solution is the reversal of how the formal definition of limit frames the requirements; that is, constructing the solution requires working backwards from the desired result (an inequality involving values of the function and the limit) to establish certain conditions (on an inequality involving the independent variable and the point of approach) that would support it as a conclusion. It seems reasonable to hypothesize that the need to reason backwards through multiple implications in this manner contributes to the difficulty students experience understanding the formal definition and connecting this definition to the limit concept. Other difficulties involve the multitude of quantifiers and the algebraic complexity of the construction (Tall, 1992).

The situation can be even more complicated if the assignment for delta does not simply result from performing a sequence of algebraic manipulations as above. In addition to linear functions, students may also encounter exercises in applying the formal definition of limit in more complex situations, such as quadratic functions. These solutions involve restricting the interval around the point of approach (the potential searching ground for delta) and then constructing delta so that it satisfies all requirements. For example, in proving that $\lim_{x \to 2} x^2 = 4$, it would be usual to select an initial bound for delta, where the choice of this bound is arbitrary although a value of 1 is often used in practice. Given this restriction, the values of x under consideration (those within a distance of delta) are contained within the interval (1, 3) and the following holds $|x^2 - 4| = |x+2| \cdot |x-2| < 5 \cdot |x-2|$, since $|x+2| < 5$ for $x \in (1,3)$.

In addition, delta must be constructed so that $5 \cdot |x-2| < \varepsilon$ is less than epsilon in order to satisfy the definition of limit. If delta is the minimum of 1 and $\varepsilon/5$, then all of the requirements will be met. (If $1 < \varepsilon/5$, then, letting $\delta = 1$, we have $5 \cdot |x-2| < 5 \cdot 1 < 5 \cdot \varepsilon/5 = \varepsilon$ and if $\varepsilon/5 < 1$, then, letting $\delta = \varepsilon/5$, we have $5 \cdot |x-2| < 5 \cdot \varepsilon/5 = \varepsilon$.) Thus, as demonstrated, constructing delta in such cases is a two-stage process and requires that multiple scenarios be (at least hypothetically) considered. This type of construction employs several subtle mathematical tactics (such as the initial arbitrary restriction of delta and the ultimate assignment of a minimum of two values) that may be only cursorily treated in an exposition, possibly because these considerations are thought to constitute mere details rather than the substance of the argument. However, these mere details are a prime candidate as a source of confusion for students who have only recently been introduced to a formalization of the limit concept

Determining limits. Estimating and computing limits is the most common form of exercise that students encounter. Limits can be estimated from the graph or numerical values of the function in question, and this tack is often used to introduce the limit concept and emphasize its relation to the behavior of the function in a given neighborhood before proceeding to algebraic computations that are supported by theorems. Determining limits of rational functions (ratios of polynomials) is one common entrance to instruction on algebraically computing limits. Rational functions with removable discontinuities have the property of being undefined at the point of interest (denominator is zero) but can be algebraically transformed to functions that are. The limit of the discontinuous function is the same as the value of the transformed function at that point. For example, the limit of $(x^2 - 9)/(x - 3)$ at $x = 3$ is the same as the limit of $x + 3$ at $x = 3$, which is equivalent to the evaluation of $x + 3$ at $x = 3$, namely 6. Note that the two functions are not the same at $x = 3$; one is defined at this point whereas the other is not. Students, however, tend to equate the two, a phenomenon that Oehrtman (2002) referred to as "collapsing dimensions." Thus, in the example above, the student reasons that x *becomes* 3 for the computation of the limit.

Indeterminate forms, expressions that evaluate to 0/0 or ∞/∞, require special treatment and new ideas for which the students may not have adequate background (e.g. poor understanding of infinity and division by zero). Indeed, the nature of 0/0 figures among the most frequent queries on a popular mathematical forum (Ask Dr. Math) that provides explanations by experts. Students learn a set of procedures (such as multiplying by the conjugate) that are relevant for treating indeterminate forms but often cannot explain why these procedures work (Szydlik, 2000).

The definition of the derivative. The derivative of a function at a point is defined as a limit:

$$f'(x) = \lim_{h->0} \frac{f(x+h) - f(x)}{h}.$$

One common introductory calculus exercise is to compute the derivative of a given function directly from this definition. Algebraically, this may be very involved, and the student can become focused on the procedure (Orton, 1983; Zandieh, 2000) The idea of limit is then effectively overshadowed and the connection between the concept of limit and differentiation is obscured. One consequence is that when an expression that *could* be viewed as a derivative is encountered, it is not recognized as such. For example, the expression $\lim_{h->0}(\cos(\pi/2+h))/h$ could be conceptualized as the derivative of $\cos(x)$ at $x = \pi/2$ versus being viewed as an indeterminate form. Thus, problems of this type afford the construction of alternative perspectives.

METHODS

The locations

Tutorettes on the concept of limit were collected from three locations: a university-sponsored mathematics assistance center (MAC), MathGoodies.com (an SOH website), and MathNerds.com (an AOH website).

The MAC. The MAC, a mathematics help center at a large northwestern university, employs undergraduate and graduate tutors who are available for set hours at a designated location to help students with mathematics coursework. The graduate tutors are mathematics students who work at the MAC to earn a portion of their student stipend; some of these students also teach undergraduate courses and hold office hours at the MAC. The undergraduate tutors are students (mostly mathematics majors) that have applied for this position and been selected based on their major and having met the requirement of a minimum grade of B+ in all previous mathematics courses at the university. While on-duty, each tutor designates the courses (subjects) that he/she will tutor and sits at a table for 6. Students select a tutor/table as they enter and generally remain seated at the same table while receiving help. The room is equipped with white boards and plenteous scrap paper. In addition, the solution manuals for the university course textbooks and a large selection of mathematics textbooks for the course textbooks are provided for tutors. The MAC administrator, who is also a university mathematics instructor, has an office that is located within the MAC and is therefore present during many of the hours of operation.

SOH site. MathGoodies.com was chosen as a representative SOH website. Math Goodies is an advertisement-supported math help portal established in 1988 by Gisele Glosser, a former secondary teacher of mathematics and computer science. The site includes lessons, puzzles, worksheet pages and discussion forums for new visitors, educators, and parents. There are 7 homework help forums organized by subject area: basic math and pre-algebra, algebra, geometry and trigonometry, pre-calculus and calculus, probability and statistics, standardized test preparation help, and miscellaneous math topics. The postings for these forums (from the previous 4 months to present date) are available to the general public, but membership through registration is required in order to contribute or respond. Forum members also have access to user profiles that included volunteered information on identity, place of residence, contact information, as well as total number of posts, per day average, and recent topics. Each member is characterized by total number of topic contributions, either as a topic initiator or responder: starting member (0), new member (1-2), junior member (3-6), average member (7-19), senior member (20-49) or advanced member (more than 50). There are several advanced members with more than 1000 topic contributions, and the two most 'advanced' members have contributed to more than 5000 postings each. Each forum also has assigned moderators who have the right to lock topics and remove postings.

Within each subject area, topics are indexed by title, author (initiator), number of replies, number of views, and last post time. The topic titles, chosen by the posting initiator, vary from expressions of general need and desperation (e.g. "!!!!!!!!!!!!!!!!!!!!HELP ME!!!!!!!!!!!!!!!!!") to descriptions of problem type (e.g. "Limits involving Infinity"). Each posting contains the contributor's username, type/level of membership, country, and number of topic contributions to date. Included in the posting header, adjacent to the posting date and time, are icons for accessing the contributor's profile, sending an e-mail message to the participant, and replying to the posting with its contents quoted. Each posting also contains an icon for reporting the posting to a forum moderator. The website contains a search mechanism for locating postings containing key words or phrases. Members can elect to subscribe to topics and receive e-mail notification when a posting is contributed to that topic.

The rules for participation are listed in a "sticky" that is the lead posting within each help forum (see appendix). The rules cover administrative issues (e.g. managing subscriptions), a call for honor (e.g. the site is not to be used for exam questions), and general help-seeking advice (e.g. search forums for answer to question before posting and remain patient). In addition, there are 3 rules that specifically address the content and framing of posts: post a specific question ("Type the entire question including the instructions"), show all work ("Post all work you've done so far (even if you think it's wrong) so we can see what you are having difficulty with"), and communicate clearly ("Use the buttons to the left of the message form to insert mathematical symbols.") Figure 2 shows the computer window for constructing posts. This window contains traditional icons for highlighting text (e.g. italics, boldface, underlining, and font size and color), justifying text, inserting material (e.g. external links and images), and organizing text (e.g. forming lists). In addition, there are format capabilities that are more specific to tutorettes: a limited set of mathematical symbols, striking through text, and "smilies" to indicate emotion.

AOH site. Mathnerds.com was chosen as a representative AOH site. MathNerds is a non-profit corporation founded in 1999 by Valerio De Angelis and W. Ted Mahavier as an extension of The Math Doctor. The primary purpose of MathNerds is to provide "free, discovery-based, mathematical guidance via an international, volunteer network of mathematicians." In particular, MathNerds promotes help via guidance, references, and hints versus worked solutions. The site is available in both English and Spanish and includes links to other "useful" websites (intended for reference and supplementary materials).

MathNerds has 325 tutors ("volunteers"), the majority of whom have PhD degrees in mathematics, although this is not a requirement. Applicant tutors are given 5-10 practice questions in categories pertaining to interest with 7 days to respond and are selected based on pedagogical approach and clarity. Reflecting the site's adherence to an inquiry-based teaching philosophy, applicants who respond with complete worked solutions are not accepted. Accepted tutors then specify the number of questions within various categories that they will address each week.[47] The categories are arranged by (United States) grade level (e.g. K-5 Elementary) or by subject area (e.g. Calculus I).

MathNerds has a systematic way of matching volunteers with client questions based on volunteer availability and interest. Figures 3 and 4 are flowcharts that reflect client and volunteer activity, respectively. Upon visiting the site, a client first chooses the category that matches the content of the question. If there is a volunteer who has selected that subject area and has not met her/his weekly cap, then the client is presented with the terms and conditions of participation (e.g. expectations and obligations for response time and legal disclaimers). The client then posts the question in a screen (Figure 5) that includes areas for the subject (5 -10 words), the statement of the question, and any work already done by the client. Instructions for accurately typing mathematical information (e.g. notation) are provided directly above the area for posting the question via a link to Karl's Calculus Tutor, an online calculus course. Encouragement is given to show all attempts at solving the problem(s) (including incorrect ones) along with general help-seeking advice (e.g. searching for solutions to similar problems). After the client submits the question, two automated e-mails are sent: a confirmation of receipt to the client and the question (with a link to the online response form) to the volunteer. The volunteer can reject the question and move it to a general queue (where another volunteer may respond) or elect to respond within 2-7 days.[48] If the volunteer does not respond within 2 days (and has not indicated that a response is forthcoming), the question is automatically routed to the general queue, where it remains for 2 weeks. If the volunteer does respond within a week, the solution is archived and forwarded to the

[47]The greatest present need is for volunteers in the 6-8 middle school and 9-10 high school areas, followed by algebra, differential equations, and the "other" category.

[48]The published average response time is 22 hours. However, this information is not available in the archives and is thus unknown for the sample used in this study.

client along with a link for future assistance on the question. The client can then engage in further dialogue (exclusively) with the same volunteer that initially responded.

Data collection and analysis

During a week-long time period that corresponded roughly to the initial coverage of limits by university calculus courses, the researcher visited the MAC and videotaped exchanges between participant tutors and students who posed questions pertaining to limit. This resulted in 17 tutorettes from 7 student-tutor pairs. Each tutorette only involved a single tutor-student pair. The conversations and accompanying mathematical work were transcribed. For the SOH data, all tutorettes from the Pre-calculus and Calculus forum at mathgoodies.com were collected over approximately a 9-month period. Those pertaining to the limit concept were then selected, resulting in 39 tutorettes involving 17 tutors and 24 students. There was one participant who acted in both roles, crossing over once from his more frequent role of student to that of tutor. For the AOH data, tutorettes pertaining to the limit concept were identified by using the search mechanism provided by MathNerds for postings containing "lim." 49 of the most recent hits were selected, resulting in tutorettes involving at least 14 volunteers[49] and 32 students.

Each tutorette was assigned a participant code that characterized participant contributions and turn-taking behavior. Online turns corresponded to postings within a tutorette, face-to-face turns corresponded to utterances other than conversational indications of acceptance ("Mmm-hmm") or repetition. For instance, a participant code of 1231 would be assigned to a tutorette

[49]MathNerds volunteers are not required to sign their responses. 27 of the 49 tutorettes involved an unidentified tutor.

initiated by the student (1) with subsequent contributions from two tutors (2 and 3, respectively) and concluded by the student.

Problem-solving activity was defined as participation in the construction of a solution or explanation. Although there are clearly many different levels of participation as well as individual or joint constructions, students' contributions were judged dichotomously in order to detect problem-solving activity. Responses to low-level prompts from the tutor (e.g. "c over c is …") were not counted.

Tutor accuracy was assessed by counting the number of tutorettes in which a tutor produed a mathematical error. Each tutorette was classified as either error-free, error 'fixed', or with error. Error-free tutorettes contained no mathematical mistakes on the part of the tutor. Error 'fixed' tutorettes contained errors that were either resolved (by the tutor or by another party) or were 'replaced' by the introduction of a correct solution. In these cases, the student was party to (at least) one correct solution by the end of the tutorette. The final category consisted of tutorettes that contained inaccurate information from a tutor that was not 'fixed.'

RESULTS

Student activity

Participation codes. One gross measure of activity is length of participation code with longer participation codes presumably corresponding to more extensive discussions. As Figure 6 indicates, the different tutorette contexts produced very different participation patterns in terms of "length" of discussion. The briefest tutorette would be an unanswered query (participation code: '1'). Although it is almost impossible to imagine a face-to-face tutorette of this type, the asynchronous nature of the online context makes tutorettes of length one a reasonable configuration. Although it was not possible to obtain the data for the AOH site[50], none of the SOH queries were unanswered. The next most brief participation configuration has length two (participation code: "12") and involves a student query followed by a tutor response. This was the most common type of configuration in the AOH context and occurred infrequently in the other two. Conversations of length three (participation code: "121", "123", or "112") constituted about 25% of the online and none of the face-to-face tutorettes. The majority of the "121" conversations ended with an expression of thanks from the student, the "123" conversations occurred exclusively in the SOH context and often reflected the introduction of an alternative perspective by a second tutor, and the "112" conversations generally consisted of students' adding subsequent information (e.g. forgotten parts of a problem) to the posting. Finally, "extended" conversations (those of length greater than 3) were the most prevalent type of interaction in the face-to-face tutorettes but were much less common in the AOH context. Furthermore, less than half of the extended AOH conversations contained subsequent problem solving activity on the part of the student; instead, many were devoted to clarifying the problem statement in response to a tutor query. The extended conversations in the SOH site constituted 62% of all exchanges, with 92% of these involving more than a single student-tutor pair. Multiple participants was a general characteristic of the SOH tutorettes, with 67% consisting of more than one student-tutor pair. Although we discounted the conversational indications of acceptance, the conversations in the MAC still contained many turns, indicating that students were participating in this context.

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[50] The queuing policy of MathNerds makes it impossible to access any unanswered queries as these are not published in the archives. However, they boast a 98% response rate, and clients can re-submit questions that have not been answered within a week. There would be no reason to expect any unanswered questions in a face-to-face context.

Initial and subsequent activity. The presence of student contributions to the construction of the solution is one of the marks of effective help. Students can contribute through initial problem solving attempts as well as during the tutorette. Figure 7 shows the percentage of tutorettes in each context that exhibited a given configuration of initial and subsequent activity: no initial and no subsequent ([0 0]), initial but no subsequent ([1 0]), no initial but subsequent ([0 1]), initial and subsequent ([1 1]). By our measure, students in the MAC tutorettes showed noticeable higher subsequent activity, but this is due in part to the fact that we were capturing all student actions in this context (using videotape) whereas, in the case of the online tutorettes, we have no access to student actions outside the postings themselves. In contrast, the AOH context was notable for absence of student activity during tutorettes but very high initial activity. The designated area for "Work Done" in the posting screen (Figure 5) seems to be successful, i.e. encourages initial problem solving attempts. The absence of student activity during the AOH tutorettes corresponds to the shortness of the exchanges in this context (see Figure 6). The pattern of activity in the SOH context was very similar that in the AOH context, except that that there was more subsequent problem solving activity following no initial attempt. In some cases this was occasioned by an explicit prompt from a participating tutor, e.g. "Please show us your work – indicating exactly where you are stuck. That way we can assist you better."

Technique. One goal of help is to create an artifact that the student might turn in for a grade or use as a future reference. In order to attain this goal, a tutor's contributions should be accurate and perspicuous. On this dimension, one measure of effectiveness is the mathematical correctness of the work that the tutor produces.[51] Figure 8 shows the mathematical accuracy of tutor contributions in the tutorettes across the three contexts. The AOH context showed the most accuracy; indeed, there was only one tutor response contained a mathematical error and that was a “typo.” This was a sign error in an intermediate step of a solution. Likewise, the SOH context showed a high degree of accuracy. However, there were errors present, but, due to the wikipedia-like nature of the context,[52] these errors were either fixed or ‘replaced’ by alternative correct information. For example, in a tutorette concerning the proof of $\lim_{x->1} 2x^2 = 2$ using the formal definition of limit, the first tutor provided an incorrect response:

> I will use D for small delta and ∈ for epsilon.
> Prove
> $\lim_{x->1} 2x^2 = 2$
> |x^2-1|< ∈ if 0<|x-1|<D
> |x^2-1|=|(x+1)(x-1)|=|x+1||x-1|
> |x+1||x-1|<∈ if 0<|x-1|<D
> |x+1|<k
> |x+1||x-1|<k|x-1|, if x,(1)
> This implies:
> |x+1||x-1|<∈ if k|x-1|<∈
> Rewrite as:
> k|x-1|<∈ if 0<|x-1|<D
> |x-1|<∈\k if 0<|x-1|<D
> D=∈/k
> Can you go further?.

Aside from the lack of coherent explanation, this solution is inaccurate. In particular, the value of k, and therefore the interval for which the inequalities hold, is never specified. These are not mere details but constitute the main concept behind the formal definition of limit as discussed in the introduction. This error was ‘fixed’ by another tutor in a subsequent post:

> This type of limit question is usually done in two steps.
> The first step is to do some rough work to figure out
> how you should choose your delta (this does not need

[51] Of course, a mathematically accurate tutor contribution does not ensure the accuracy of the final written construction (produced by the student).

[52] Gaea Leinhardt first recognized this characteristic of SOH sites and brought it to my attention.

to show up in your solution).
The second step is to show that your choice of delta works.

(I will use e for epsilon and d for delta)
Step 1: Rough work:
You want |2x^2-2| < e when 0 <|x-1| < d.

2|x^2-1| < e
2|x-1||x+1| < e
The |x-1| term is easily handled if |x-1| < d, but we
need to do something about the |x+1| term.
Lets suppose d <= 1. There is nothing special about 1 here,
we are just saying what if x is in the interval (0,2).
Then |x+1| <= |x| + |1| < 2 + 1 = 3.

If |x-1| < d <= 1, then |x+1| < 3 and we can say
2|x-1||x+1| < 2|x-1|(3) = 6|x-1|.
If we make 6|x-1| less than epsilon, then
2|x-1||x+1| will also be less than epsilon.

We want 6|x-1| < e,
or |x-1| < e/6.

So, things should work out provided |x-1| < e/6 AND |x-1| < 1.
Let d = min (e/6, 1).

Step 2: Now you rewrite things in the standard fashion (this is what you submit).

Given any e > 0, let d = min(e/6, 1). Then if 0 < |x-1| < d <= 1, |x| < 2 and so |x+1| <= |x| + 1 < 3. If 0 < |x-1| < d, then |2x^2-2| = 2|x+1||x-1| < 2(3)|x-1| < 6 (e/6) = e.

Not only does the second tutor 'fix' the mistakes of the first tutor, he/she also provides a coherent explanation. While this tutor does provide a full worked solution (not prompting any problem solving on the part of the student), the nature of the task (namely constructing a rather complex proof) may necessitate this pedagogical approach. The intent is presumably that the student will use this explanation as a worked example for similar problems: "This type of limit question is usually done in two steps.")

The 'fixed' error in the SOH context can also correspond to a correct solution from a different perspective. The open response policy of the SOH site increases the opportunity for multiple perspectives to emerge as the following example on computing $\lim_{h \to 0}[\cos(\pi/2 + h)]/h$ illustrates:

> S: lim h->0 of [cos(π/2 + h)] / h Can someone help me with this? Is the answer DNE?
>
> T1: hint ... f'(a) = lim{h->0} [f(a+h) - f(a)]/h

After S responds to this hint with a numerical answer but no work, is provided with the correct numerical answer with no explanation by T1, and then requests clarification ("Could someone show the steps?"), another tutor enters the dialogue. T2 explicitly refers to the hint given by T1 and fleshes it out with problem-specific details:

> T2: T1 said: f'(a) = lim{h->0} [f(a+h) - f(a)]/h
>
> Another hint [associated with that] ...What is the derivative of cos(x)? How would you evaluate that at x = π /2 ?

S responds to this hint, attempting to map the hint contents (mathematical definition) onto the problem expression.

> S: I see how it is -1 by your explanation of deriv. of cos(x) = -sinx = -sin(π /2) = -1.
>
> How can you tell the answer is -1 by looking at the equation and the definition that T1 gave. I don't see the -f(a) in my problem. Am i being stupid? Sorry
>
> edit] is the -f(a) in my problem supposed to be -cos(π /2) ?

Note that the correspondence between the mathematical definition and the problem expression is somewhat difficult to "see" since one of the terms in the definition (namely f(a)) is zero in this case and therefore not present in the problem expression. This lack of correspondence contributes to the challenge of viewing the problem from this perspective that requires recognizing that this limit can be conceptualized as a derivative. However, the student successfully constructs this perspective, answering his own question ("I don't see the –f(a) in my problem.") in an edited addition to the post: "is the –f(a) in my problem supposed to be –

cos(π/2)?" T2 responds affirmatively and then, minutes later, posts a worked solution from this perspective:

> T2: EDIT: Note that here will be some difference in the way you have written the initial problem and the solution given so far. I will outline the problem now, as I see it:
>
> -sin(x) = d/dx(cos(x)) = lim h->0[cos(x + h) - cos(h)] / h
> So lim h->0[cos(π/2 + h) - cos(h)] / h = -sin(π /2) = -1
> L.S. = lim h->0[cos(π /2 + h)]/h - lim h->0[cos(h)]/h
>
> The last term will tend to infinity, so this does not bode well for the indicated solution as you have written the problem. If you mean it as written, then you have...
>
> lim h->0[cos(π /2 + h)]/h - oo= -1
> From which lim h->0[cos(π/2 + h)]/h = -1 + oo = oo

Ironically, this solution is incorrect since T2 provided the additional hints that enable the student to view the problem from this perspective. T2 uses an incorrect definition of the derivative, replacing the term cos(x) with cos(h). T2 produces a worked solution based on this faulty definition, with a final answer that is different than the one provided by T1. Hours later, two other tutors (T3 and T4) post correct solutions for the limit problem from alternative perspectives:

> T3: lim h->0 of [cos(π/2 + h)] / h = lim h->0 of [cos'(π/2 + h)] / h', used l'Hopital's rule = -1
>
> T4: I assume you know:
> lim(h->0) cos(h) = 1
> and
> lim(h->0) [sin(h)/h] = 1
> and cos(A+B) = cos(A)*cos(B) - sin(A)*sin(B)
> then
> cos(π /2 + h)
> = cos(π /2) * cos(h) - sin(π/2) * sin(h)
> = - sin(h)
> then
> lim(h->0) [{cos(π/2 + h)}/h]
> = lim(h->0)[-sin(h)/h] =
> -1
> BTW, you should plot {cos (π /2 + x)}/x and check your answer.

Thus, at least one correct solution was provided. However, it is unfortunate that the perspective that the student had adopted was left with an unresolved error.

In contrast to the online contexts, the MAC tutorettes exhibited errors that remained unresolved in the sense that the student left without a correct solution. The majority of these errors involved ideas critical to an understanding of the limit concept. For example, one tutor appealed to the mathematical principle, "the limit of a constant is a constant," that was not relevant for the problem. In this case, the student and tutor had already performed the limit operation, albeit unwittingly. Thus, the student's confusion remained: "Six? Don't I have to take the limit or anything?") There was also an unresolved error that involved faulty mathematical principles. In applying the definition of the derivative, the tutor used the premise that the square of a sum is equal to the sum of the squares (e.g. $(a+b)^2 = a^2 + b^2$). When the student expressed skepticism and remarked that it "looked different than the other ones" (perhaps referring to previous problems), the tutor assured him that they had achieved the "main goal [of canceling the h's]"; "it's just different algebra to get there." The errors in the MAC that were corrected were almost arithmetic and were caught by the students as often as by the tutor.

The final solution that the student produces is also an indication of the effectiveness of help, although this is usually not available in the online contexts. The student work in the MAC was generally written directly on homework papers and was often fraught with errors and lacked coherence. In most cases, although the final numerical answer was correct, the supporting work was bunk. Figure 9 (problem 8) illustrates the resulting student work for computing the derivative of $y = 2x^3 - 5x$ at $(-1,3)$. The limit symbol is dropped for the intermediate steps, inequivalent expressions are equated, and terms appear spontaneously (due to the fact that error corrections were incompletely executed).

A different story emerges from one of the few examples of a student's final solution from the AOH context. This example also involved finding the derivative of an expression at a given point using the definition. The student posted some initial work that was critiqued by the tutor based on clarity:

> S: Find the slope of the tangent linje to the curve y = x^3 at the point (-1,-1) using m = lim h>0 (f(a+h) - f(a))/h
>
> Work Done : Lim h>0 ((x+h)^3 - f(x))/h lim h>0 (x^3 + 3h^2x + 3hx^2 + h^3 +1)/h
>
> This seems to give 0 and I know the answer is m = 3 from other methods and checking on calculator.
>
> T: You're supposed to be taking the limit at a = 1, so I'm not sure where 'x' is coming from...? Also, if you're using 'f(a)', then you need to use 'f(a + h)'; otherwise, plug in the actual expressions.
>
> How did you get that the limit evaluates to zero? There seem to be some steps missing. Please reply with the steps between 'lim h>0 (x^3 + 3h^2x + 3hx^2 + h^3 +1)/h' and '=0', including where you plugged in the given value for 'a' (or 'x'??).
>
> Thank you.

The student replies, producing a solution that includes an explanation for the "missing" steps. Although the student claims to "have it now" when producing this solution (that gives the correct numerical answer), the written work remains incoherent as variables are not properly defined:

> S: I have it now. Just can't add. Glad I am math major and not arithmetic major. LOL
> lim h>0 (x^3 + 3h^2x + 3hx^2 + h^3 + 1)/h = 3
> 3hx^2 term gets h cancelling leaving 3x^2, 3h^2x and h^3 terms go to 0. Then x^3 + 3x^2 +1 = 3
> Thanks

The tutor is insistent that this solution, as it stands, is not correct and follows his/her critique with an exemplary solution:

> T: You still need, at some point, to use the formula they gave you (with the 'a' instead of 'x'), and at some point use the value that they gave you for 'a', in order to

obtain the numerical value that they seek. In my experience, to be counted as being correct, you need to follow the instructions and use legitimate methods.

(Since you are not given any value for 'x', I'm not sure on what basis you are saying that x^3 + 3x^2 + 1 = 3...? And at no stage do you show that you're taking any limit. Since the instructions require that, the entire exercise could be counted as incorrect.)

f(x) = x^3
a = -1
f(a) = f(-1) = (-1)^3 = -1
f(a + h) = f(-1 + h)
= (-1 + h)^3
= -1 + 3h - 3h^2 + h^3
f(a + h) - f(a)
= [-1 + 3h - 3h^2 + h^3] - [-1]
= -1 + 3h - 3h^2 + h^3 + 1
= h^3 - 3h^2 + 3h
[f(a + h) - f(a)] / h
= [h^3 - 3h^2 + 3h] / h
= h^2 - 3h + 3
Now take the limit as h goes to zero.

After the provision of this worked solution, the student recognizes and acknowledges the error of their ways:

S: ok thanks. I was just using a = -1 = x. Your point that one cannot arbitrarily substitute x for a is technically not correct is noted. Thanks for your help!

DISCUSSION & CONCLUSIONS

We investigated three communities for evidence of effective help based on student activity and tutor accuracy. Although these communities share a common goal, they have different explicit and implicit rules for participation that are then reflected in the extent of conversations, the relative contributions of participants, and the accuracy of the help. For online help, the short response time that characterizes the SOH community appears to facilitate more back-and-forth dialogues. In contrast, the longer latency characteristic of the AOH site favored short exchanges. This property may make the AOH context less suitable for homework help, given the pressure of due dates. Also, because many homework assignments involve sets of similar problems, it is likely that, if a student cannot do one problem, then he/she will be unable to do others and cannot afford to wait long for help.

All of the communities exhibited instrumental help-seeking in that a majority of the students actively engaged in constructing solutions. The difference across communities was accountability. The online communities had explicit policies regarding student participation that were sometimes enforced. In contrast, the MAC did not have any such policy and so initial student work was generally not referenced or built upon.

The accuracy and clarity of the explanations also varied across the communities. The online communication facilitated greater precision although formatting was often an issue. Many mathematical expressions, when written horizontally rather than vertically, require the addition of parentheses (e.g. $\frac{1}{x+1}$ is written as $1/(x+1)$). This led to misunderstandings and negotiations of the problem statement and delayed help.

This study has revealed a series of trade-offs that are features of tutorettes: latency versus accuracy, multiple perspectival constructions versus focus on single perspective, and active student participation versus frustration. The face-to-face tutorettes had the shortest latency but a substantially higher error rate, whereas the AOH produced very accurate help but the latency may have been unviable for the student. Has the SOH community achieved the best balance between these two extremes? The SOH community was also the only context where multiple perspectives were discussed for a given problem. The discussions that this inspired were clearly beneficial to the tutors involved, but there is less evidence that the students profited. It remains to be seen whether an in-depth understanding from a single point of view would be more helpful for the student. Finally, there is a trade-off between requiring active problem-solving student participation and causing undue frustration. Adhering to the usual rules of politeness and conversational maxims does not necessarily correspond to pedagogical objectives (Person, Kreuz, Zwaan, & Graesser, 1995). Although we have not yet performed any formal analysis, it appears that it was more acceptable for tutors to violate rules of politeness in the online communities.

There is no question that students are seeking help both in online and face-to-face communities. It is therefore important that we as educational researchers and instructors learn how to effectively help students in these communities. Even though the Beatles are no longer an item, students are still crying out for help.

Author Note

I would like to thank Gaea Leinhardt for her comments on this paper and for her support and encouragement throughout the project.

Figure Captions

Figure 1. Theoretical model of help.

Figure 2. Posting screen for SOH site mathgoodies.com.

Figure 3. Client flow chart for AOH site, mathnerds.com.

Figure 4. Volunteer flow chart for AOH site, mathnerds.com.

Figure 5. Posting screen for AOH site, mathnerds.com.

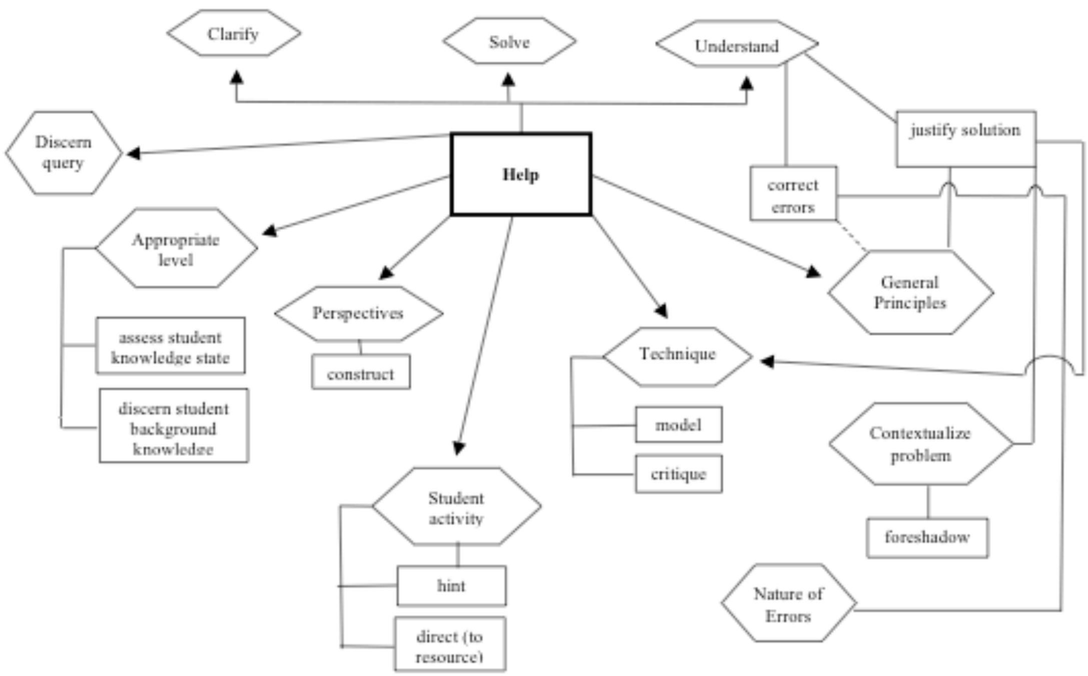
Figure 6. Percentage of tutorettes with participation codes of various lengths across contexts.

Figure 7. Percentage of tutorettes with initial and subsequent student activity configurations.

Figure 8. Percentage of tutorettes with errors, errors that are ‘fixed,’ and no errors by context.

Figure 9. Example of student work following tutorette in MAC.

Figure 1



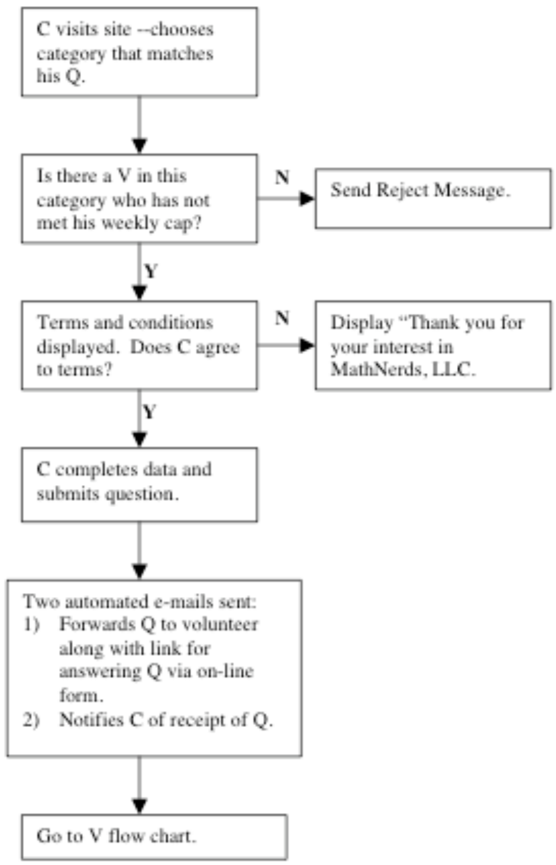
Clarify
Solve
Understand
Discern query
Help
justify solution
correct errors
Appropriate level
assess student knowledge state
discern student background knowledge
Perspectives
construct
General Principles
Technique
model
critique
Contextualize problem
foreshadow
Student activity
hint
direct (to resource)
Nature of Errors

Figure 2


Math Forums @ Math Goodies
http://mathgoodies.com/forums/post.asp?method=Topic&FORUM_ID=1
your teacher
Click Now to Try Online
Pre-Algebra · Algebra 1 · Geometry · Algebra 2
Get math help today!
Mrs. Glosser's Math Goodies™
Forums For Students, Teachers & Parents
Math Forums @ Math Goodies
Home | Profile | Register | Active Topics | Members | Search | FAQ
You are logged on as carlacvds
LOGOUT
All Forums
New Visitor Forum
Testing Forum
Screensize: 800 x 600
Format Mode: Prompt
Format:
Font
Size
Color
Subject:
Message:
* HTML is OFF
* Forum Code is ON
Math Symbols
Math Symbols created with MathType by Design Science, Inc.
Check here to subscribe to this topic.
Post New Topic
Preview
Math Forums @ Math Goodies
© 2000-2004 Snitz Communications
This page was generated in 0.28 seconds.
Snitz Forums 2000
1 of 1
4/17/07 2:40 PM

Figure 3

Client Flow Chart



Key:

C = client
GQ = general queue
TQ = trash queue
V = volunteer

Figure 4

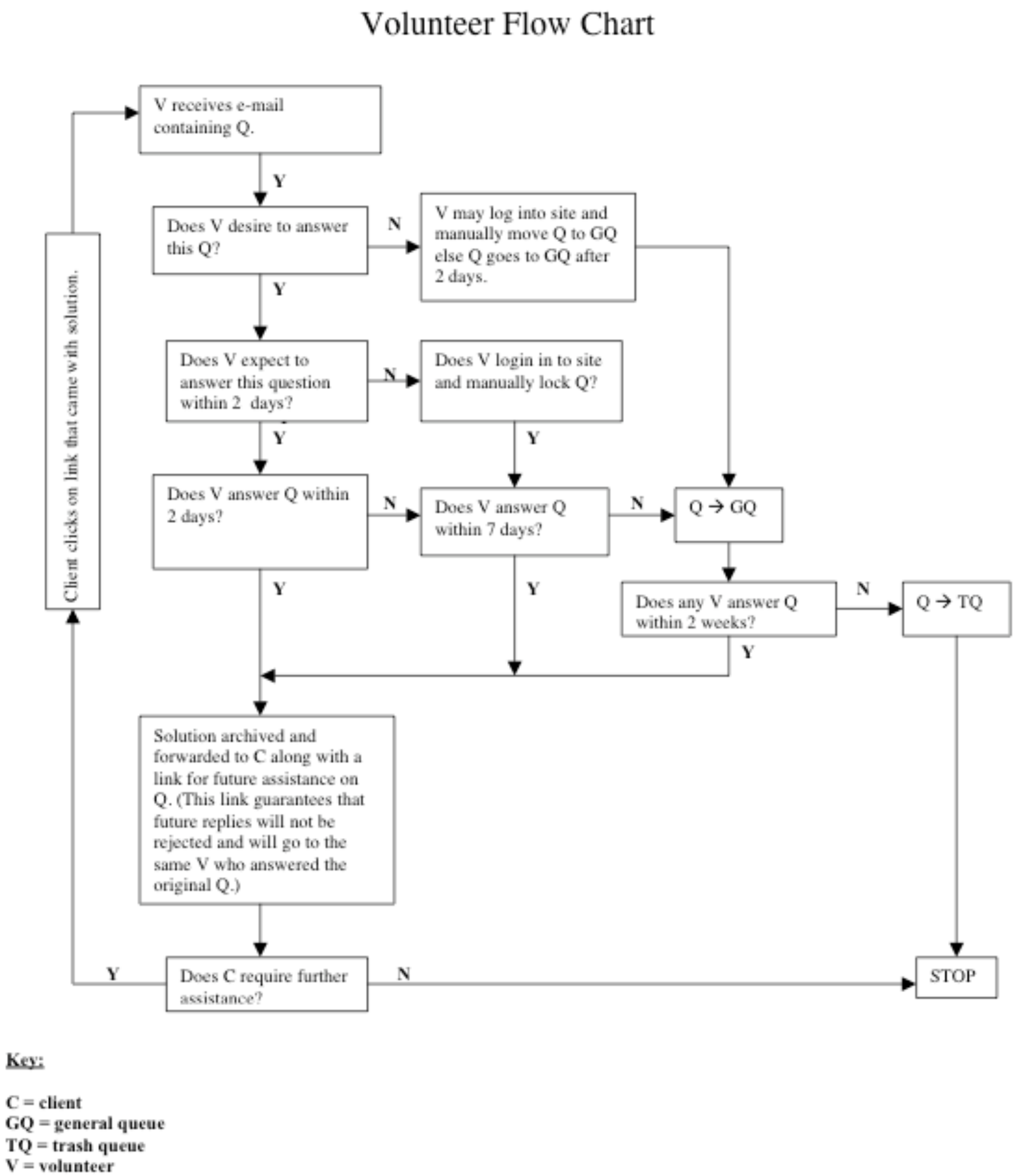

Volunteer Flow Chart
V receives e-mail containing Q.
Y
Does V desire to answer this Q?
N
V may log into site and manually move Q to GQ else Q goes to GQ after 2 days.
Y
Does V expect to answer this question within 2 days?
N
Does V login in to site and manually lock Q?
Y
Y
Does V answer Q within 2 days?
N
Does V answer Q within 7 days?
N
Q → GQ
Y
Y
Does any V answer Q within 2 weeks?
N
Q → TQ
Y
Solution archived and forwarded to C along with a link for future assistance on Q. (This link guarantees that future replies will not be rejected and will go to the same V who answered the original Q.)
Y
Does C require further assistance?
N
STOP
Client clicks on link that came with solution.
Key:
C = client
GQ = general queue
TQ = trash queue
V = volunteer

Figure 5

MATHNERDS

Home Best Texans Volunteer Logout

Archive
Ask a Question!
Client Login
Contact Us
FAQ
Guestbook
Home
Legal
Links
Sponsors
Team Members
Volunteer

Calculus I

☐ I have read the **terms and conditions**.

Subject (five to ten words) :

What is your question ? (**How to type math**)

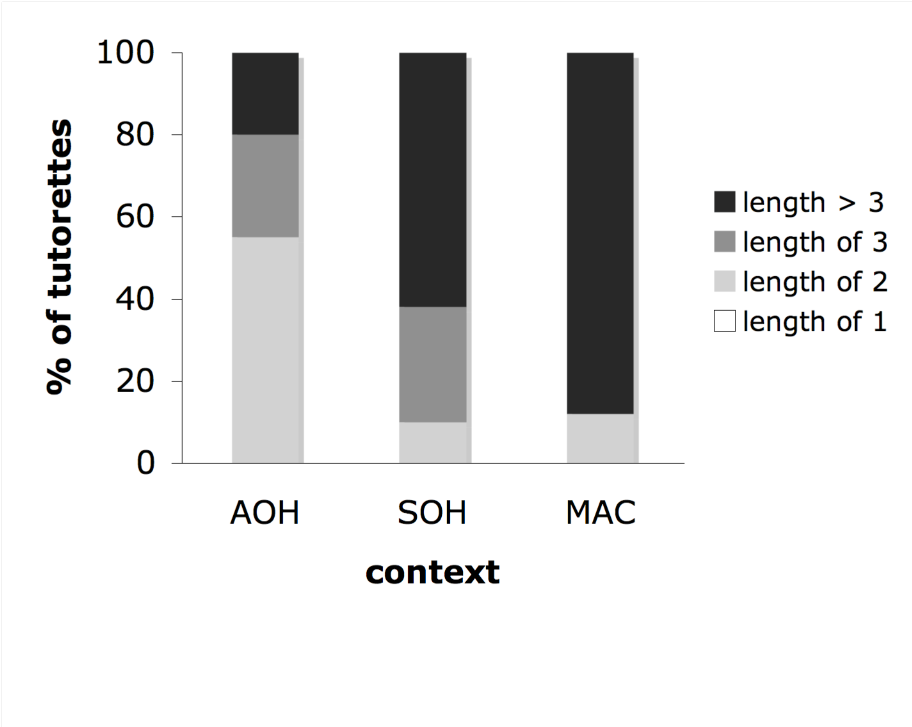
Show us what you've done so far. (see note below)

Show us your work! Did you work part of the problem and then get stuck? Do you not understand what the question is asking? Tell us about it, even if you're sure you have it all wrong. The more we know, the better we can help. Tips: Search Google for help. Try the methods in your textbook. Try our links page.

Want to make sure your question is properly formatted for nice viewing? Click on the **Preview Question** button to preview the question before submitting it.

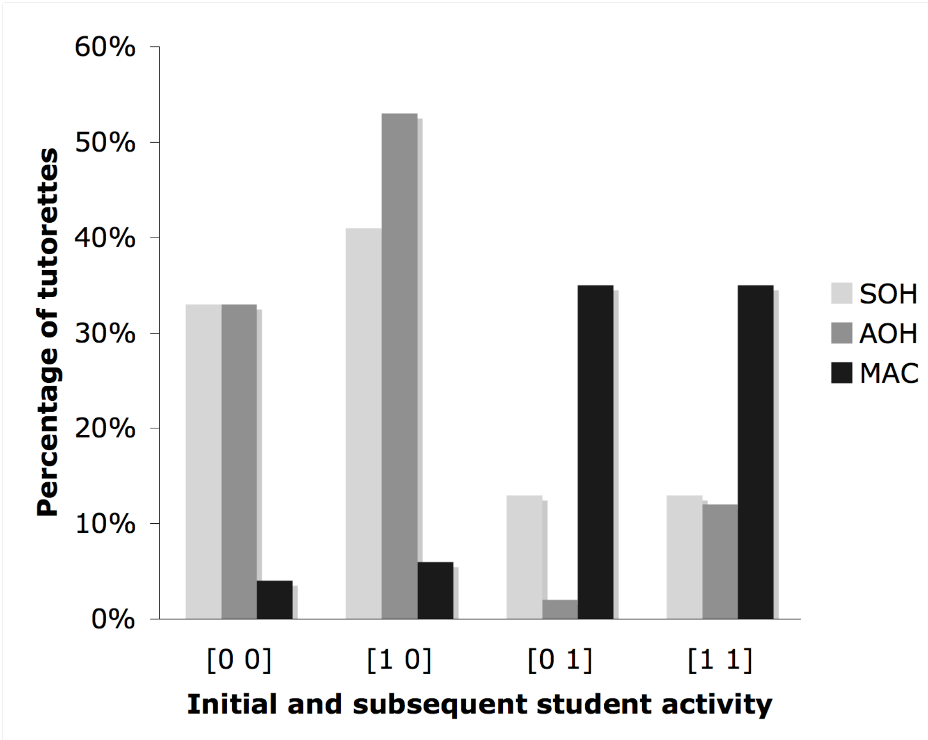
Preview Question Submit Question

Figure 6



100
80
60
40
20
0
% of tutorettes
AOH
SOH
MAC
context
length > 3
length of 3
length of 2
length of 1

Figure 7



60%
50%
40%
30%
20%
10%
0%
Percentage of tutorettes
[0 0]
[1 0]
[0 1]
[1 1]
Initial and subsequent student activity
SOH
AOH
MAC

Figure 8

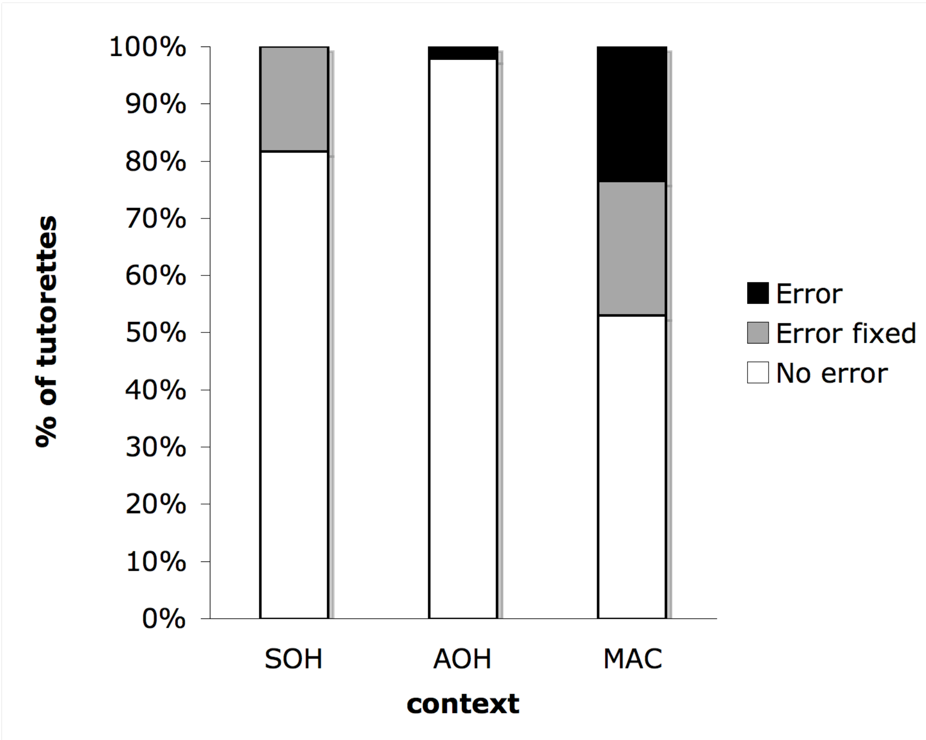
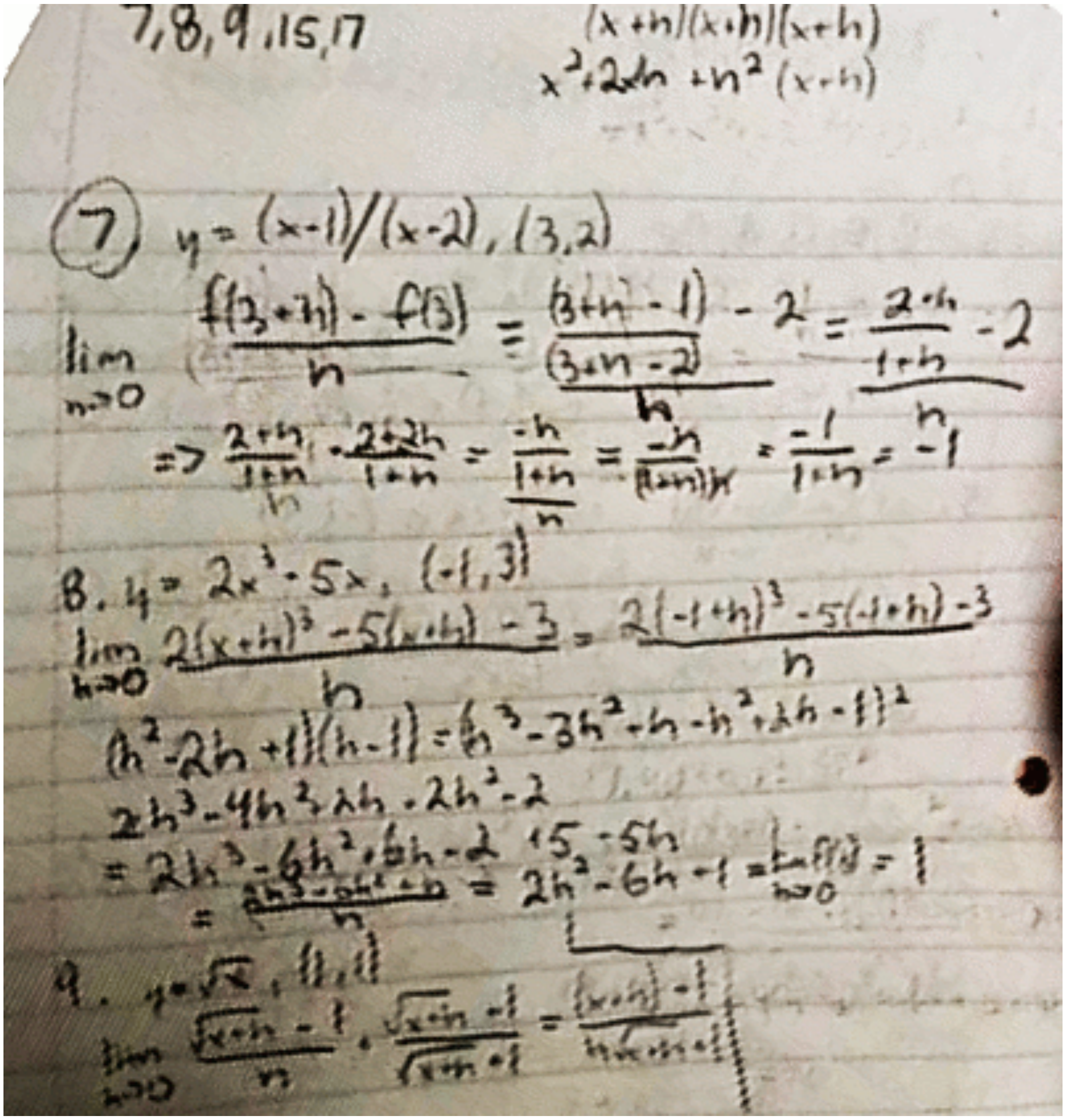

100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
% of tutorettes
SOH
AOH
MAC
context
Error
Error fixed
No error

Figure 9



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B.2 HELP! ACTIVE STUDENT LEARNING AND ERROR REMEDIATION IN A MATHEMATICS E-HELP COMMUNITY[53]

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Abstract: Free, open, online homework help sites appear to be extremely popular and exist for many school subjects. Students can anonymously post problems at their convenience and receive responses from forum members. This mode of tutoring may be especially critical for school subjects such as calculus that are intrinsically challenging and have high attrition rates. However, educational research has focused on tutoring sessions that instruct students on a pre-determined set of material or topics, and there has been no systematic research on these dynamic, free, open, online tutoring communities. In order to distinguish the student-initiated e-help episodes from traditional tutoring sessions, we refer to them as "tutorettes."

Each tutorette was assigned a participation code that contained information on the number of contributions by each participant, the sequence of contributions, and the number of different participants. Student problem solving activity, defined by mathematical contributions and efforts, was measured for initial postings and for subsequent contributions. Finally, each tutorette was examined for evidence of mathematical errors and these were classified according to type: pre-calculus, operational, and conceptual. A tutorette on the limit concept is provided to demonstrate how mathematical queries are resolved in an SOH e-help community.

Participation and problem solving attempts provided evidence of active student learning. Instead of simply using the tutors to do their homework, many students made initial attempts at solutions, queried tutor responses, and applied the help they received to make progress on solving problems. This behaviour appeared to be influenced by the actions of the tutor: Providing solution sketches accompanied by asking direct questions encouraged dialogue, whereas providing quasi-complete worked solutions seemed to have the opposite effect. In contrast

[53] A version of this paper entitled *Help! Online calculus tutoring* can be found in D. Remenyi (Ed.), *Proceedings of the 2nd International Conference on e-Learning* (pp. 497-506). Reading, UK: Academic Conferences Limited. A later version was published in the *Electronic Journal of e-Learning, 5*(3), 227-238, available online at www.ejel.org

> to classroom instruction, students in this e-help community appeared comfortable in presenting incorrect work and tutors were open and forthright in their commentaries, evaluations, and explanations. In addition, tutors modulated their responses according to the type of error. Pre-calculus errors and operational (calculus) errors were not accorded the same depth of explanation as conceptual misunderstandings.

Help-seeking is recognized as an effective means for students to cope with academic challenges (Nelson-Le Gall, 1985), and student-initiated tutoring, in which a student seeks a tutor for help (usually for a specific problem), is a common manifestation of this phenomenon. This type of tutoring traditionally occurs in face-to-face settings, such as in university-sponsored help centers. However, since computer networking has become ubiquitous, a new context for tutoring has emerged in which students and tutors are not necessarily linked by physical proximity. The "net generation" (Oblinger and Oblinger, 2005) and "digital natives" (Prensky, 2001) are using the Internet to voice their requests for help. Students are participating in topic-specific, free, open, asynchronous Web-based forums for help with homework problems and answers to questions. These e-help communities may be the only recourse some students have for receiving help outside of the classroom on homework assignments or on studying for examinations. In particular, this opportunity is critical for some students taking introductory calculus, a course that is renowned for its challenging nature and high attrition rate. A casual survey of the Internet posting subjects for mathematical discussion forums that include subjects such as geometry, trigonometry, and standardized test preparation reveals that calculus is one of the most frequented subject areas, with multiple postings daily. However, despite the existence and apparent functionality of several mathematical help forums for today's students, there has been no systematic study of student-initiated tutoring in Web-based forums.

In order to distinguish the tutoring exchanges that we study from the "tutoring" that has been the subject of previous educational research, we use the term "tutorettes" for student-initiated episodes that are typically brief and involve a specific problem situation. In contrast, the "tutoring'' that has been the subject of previous educational research resembles scaled-down classroom instruction in that the tutor is assigned the task of covering a pre-determined topic or set of topics (see Cohen, Kulik and Kulik, 1982 for a meta-analysis of 65 studies). In opening up this new area of research, many questions can be posed and answered: What are the effects of different participation structures (van de Sande and Leinhardt, 2007)? What are recurring patterns of questions around a given topic? What are the similarities and differences between face-to-face help center encounters and online tutoring in the same content (van de Sande in preparation)? The research reported here examines the extent to which Web-based tutorettes reflect active student learning and error remediation, two elements of effective instruction according to educational research. A tutorette on the mathematical concept of the limit is presented as an example of how a challenging topic is discussed and resolved in a tutoring e-help community.

Active student learning

In order to learn, an individual must become actively engaged with the material, ideas, and uses of concepts and procedures to be learned. However, there are a variety of ways in which a learner can be active. Most reform-based educators urge a particular kind of active learning, for instance, that instruction should position students as active participants in the construction of knowledge rather than as passive recipients of information (Greeno, 2003; Lave and Wenger, 1991; Rogoff, 1990). This position calls for both a definition of "active" participation and methods of assessment. According to Scardamalia and Bereiter, (1991), active student learners are those that select problems, ask questions, and self-monitor their understanding. Although there is ample evidence to suggest that productive student engagement and participation fosters learning, active student learning is not a feature of typical classroom instruction. One explanation for its absence involves pragmatics: Coordinating a large number of students learning simultaneously might interfere with active student participation on an individual basis. However, active student engagement, as measured by the initiation of exchanges and questioning behavior, was also not supported in face-to-face, one-on-one tutoring sessions (Graesser, Person and Magliano, 1995). Although student questions were more frequent in these tutoring contexts than in classroom settings (Graesser and Person, 1994), the majority of questions were asked by tutors and students rarely initiated exchanges.

Active student learning, then, does not appear to be a phenomenon that naturally occurs in face-to-face instructional settings. Are other instructional settings more amenable to active student learning? By definition, online homework help forums are likely locations for active student participation since it is the role of students exclusively in these forums to initiate questions and seek resolution. In addition, the asynchronous and anonymous nature of such exchanges would seem to encourage student participation. Students are not constrained by the pace of instruction, can pose questions as they arise, and are able to present ideas in an environment where face-saving is not an issue. However, there is the possibility that students participating in web-based tutorettes are "executive'' (or dependency-oriented) versus "instrumental" (or mastery-oriented) help-seekers (Nelson-Le Gall, 1981). That is, these students may appear to be active learners but may simply be seeking worked solutions to homework problems rather than seeking help on understanding the relevant procedure or underlying concept. In our corpus of online calculus exchanges, we distinguish between these types of help-seeking by looking for evidence of active student learning in terms of student participation within tutorettes and in terms of student problem-solving contributions.

Errors and error remediation

Errors are mistakes: Some errors are trivial and some represent a quite profound misunderstanding of the situation. When students produce errors in the process of engaging with mathematics, it can be a moment of learning if the error sets up an occasion of serious exchange and consideration of the ideas involved in making the mistake. Therefore, it is precisely in these situations that the response of a tutor is critical. Following an error, a tutor may provide information about the existence, the nature, or the consequences of the error, and may do so in an explicit manner or less directly by hinting. Analyzing the timing and informational content of feedback (McKendree, 1990), the manner in which it is presented (Lepper et al, 1990), and the underlying situational features that prompt different tutor responses (Hume et al, 1996), has been instrumental in understanding the effectiveness of (human) tutoring and in shaping the design of computer-based tutoring systems (Merrill et al, 1992). One key finding is that tutors appear to modulate their responses based on the perceived criticality of an error: Errors that are judged to be less consequential for learning are treated in a different manner than errors that are considered to involve focal goals or objects in the domain (Littman, Pinto and Solway, 1990; Merrill, Reiser and Landes, 1992).

Do tutors modulate their responses to student errors in online e-help communities? Although they are instructional by nature, tutorettes are quite different from the traditional tutor sessions that have been used for the evaluation of feedback. For instance, the tutors are not constructing "tutoring plans" that will support extended instruction with the student. Instead, their goal is to quickly and efficiently answer a given student query before moving on to the next. At the same time, the tutors in e-help communities are in a position to make some assessment of a student's knowledge state by the work that is posted, the way a question is framed, and the response to their actions. As a first step for investigating errors and error remediation in calculus tutorettes, we have constructed a system for classifying the mathematical errors in our corpus and have explored the corresponding patterns of remediation.

METHODS

The corpus

As part of our ongoing research, we have collected and analyzed tutorettes from free, open, online help sites that represent different participation structures, span international borders, and pertain to various mathematical topics. We have identified two basic participation structures: Spontaneous Online Help (SOH) sites permit any forum member to respond to postings, whereas Assigned Online Help (AOH) sites assign postings to vetted volunteer tutors. The corpus used in this study (Cohort 1) contained 100 sequential introductory calculus SOH tutorettes that were collected from www.mathgoodies.com. MathGoodies.com is representative of other math homework SOH sites and includes an active pre-calculus and calculus homework help forum. The Advanced Placement Calculus course description (College Entrance Examination Board, 2003) was used to delineate "introductory" (versus pre-calculus or advanced) calculus tutorettes that were included in the analysis. The Math Goodies homework help forums are part of an online resource founded in 1998 and maintained by former secondary mathematics and computer science instructor, Gisele Glosser. Although this is an SOH site, there are assigned moderators for each individual forum who can edit, delete, or prune posts. Other participants are categorized as New Members, Average Members, Senior Members, or Advanced Members depending on their number of postings, either giving or seeking help. The explicit rules for participation in the forum include a mandate not to request help on take-home exam questions, a request to search the forum prior to posting a question, and admonitions to specify the entire question (including instructions), to show one's work on the problem, and to use the provided mathematical symbol keys to facilitate communication.

Coding

Problem-solving activity was measured by student participation within a given tutorette and by contributions to the problem-solving activity. Each posting was assigned a "participation code" that differentiated the participants of that posting and characterized the sequence of activity. A "1" was used for the initiator of the posting, "2" was used for the next participant, and so on. For example, a participation code of 1231 would be a posting by three forum members in which a student (1) requested help, participants (2) and (3) responded, followed by a final contribution from the student (1). The length of a code, then, signifies the number of exchanges in the tutorette, the ordering of numbers within a code tracks the sequence of participation, and the largest number in the code reflects the number of different participants in the exchange. In addition, each tutorette was examined to see if the student demonstrated problem-solving activity in the initial posting and whether there was subsequent activity as the tutorette was enacted. There were four possible classifications, representing all possible initial/subsequent problem-solving activity configurations. In order to distinguish between executive and instrumental help-seeking, the classification was conservative; thus, to qualify as a problem-solving attempt, the effort had to extend beyond listing possible strategies or questioning a tutor to include an explicit proposal of solution steps. Two coders independently classified problem-solving activity with high inter-rater reliability ($\kappa = .93$, n=20), and all disagreement was resolved through discussion.

In order to investigate errors and error remediation, the content of each of tutorette was coded for mathematical accuracy. Errors were defined as statements that were logically inconsistent or demonstrated a misunderstanding regarding some aspect of mathematics as opposed to those that indicated a lack of knowledge. The errors were then classified according to type: pre-calculus, operational, and conceptual. Pre-calculus errors involved arithmetic miscalculations (such as incorrect summands) or violations of algebraic principles (such as the distributive law). Operational errors involved the incorrect implementation of an algorithm or procedure of calculus (such as the chain rule). Conceptual errors, as the name suggests, involved the misunderstanding of a calculus concept (such as the limit). A posting could contain errors of more than one type, and each error was classified separately.

RESULTS

Active student learning

In order to detect active student learning, we first examined the participant codes (Figure 1) for the presence of student participation beyond initiating the posting. One broad indication of student activity is the likelihood of a student re-entering a discussion; active learners would be more likely to make contributions and to extend exchanges. Although for many of the tutorettes there is no record of whether or not the student profited from the help or engaged in any further activity on the particular problem, the student who initiated the dialogue made at least one further contribution in 46 instances and made two or more additional contributions in 17 instances. These numbers indicate that many students are engaging in discussions in this e-community. However, this analysis, on its own, does not reveal the nature of their contributions and whether the participation is indicative of executive or instrumental help-seeking.

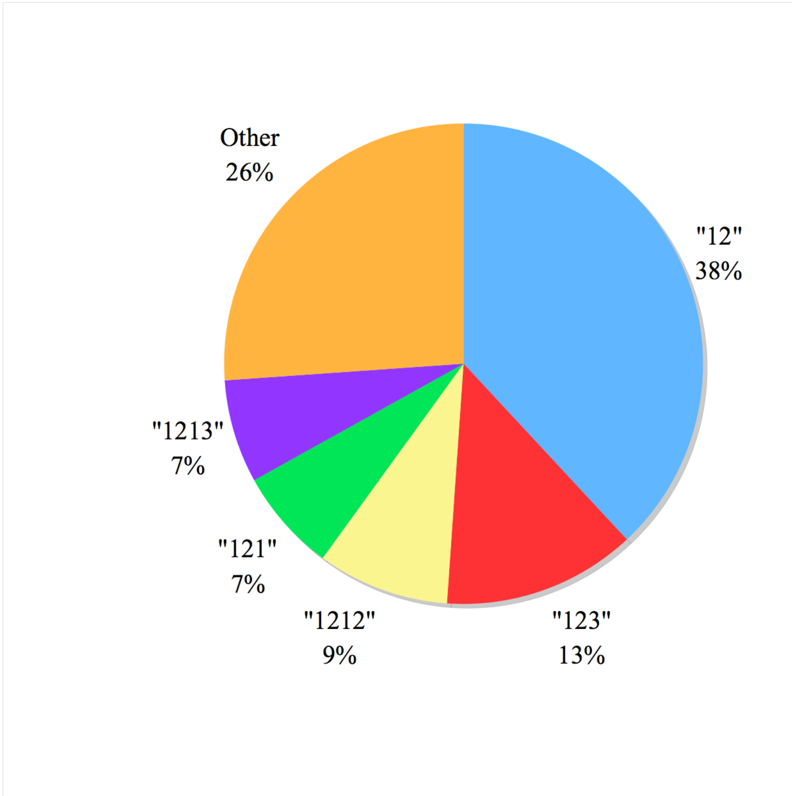

Figure 1: Percentages of participant codes. “Other” category includes two unanswered postings and other less common participant codes.

A closer examination of initial and subsequent problem-solving efforts by the student-initiators provides a more refined measure of student activity and helps discern between executive and instrumental help-seeking. Figure 2 shows the percentage of tutorettes that display initial and subsequent attempts by the student-initiator at solving the problem.

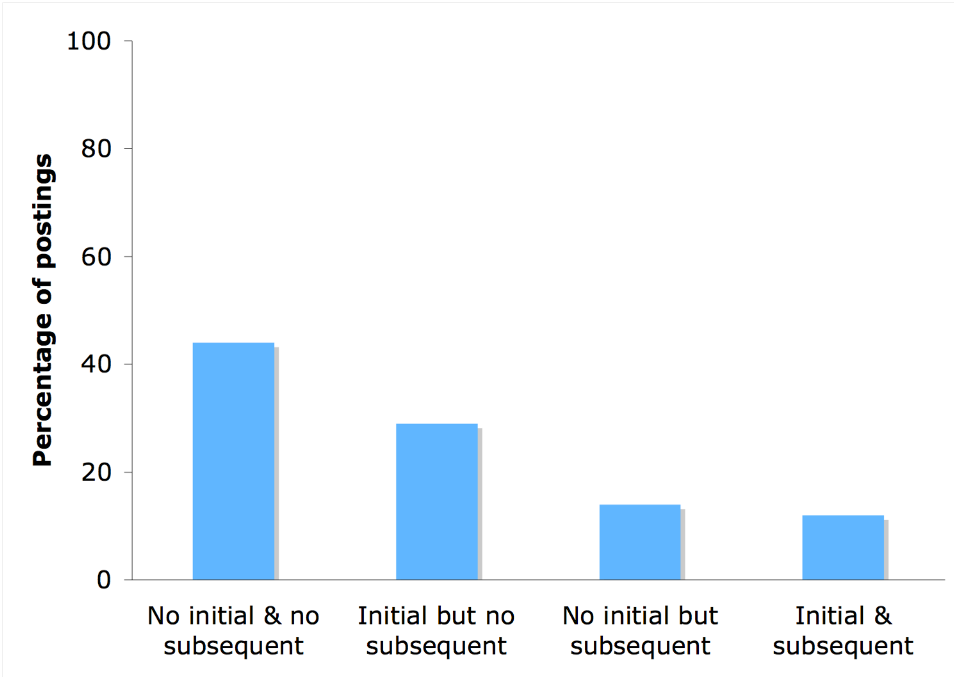

Figure 2: Student-initiator problem-solving activity.[54]

Twelve percent of the tutorettes reflect problem-solving activity both initially and subsequently, 14% reflect an increase in problem-solving activity, 29% reflect problem-solving activity by the student in the initial posting only, and 44% reflect no problem-solving activity by the student-initiator. These latter categories are potential indicators of web-based SOH sites enabling executive help-seeking and were examined more closely to determine tutor actions that may have contributed to the lack of student problem-solving activity. If, for example, tutors provide complete worked solutions, then there is less incentive for students to engage in problem solving.

The tutorettes in which the student-initiator did not participate in any problem-solving activity beyond perhaps an initial attempt revealed several characteristics of tutor activity that

[54] One posting was a request for a reference and was therefore excluded from this analysis.

may contribute to a low level of student problem-solving activity. In the majority of these tutorettes, the student received a complete worked solution (*n*=23 for no initial student attempt, *n*=10 for initial student attempt) or a partial solution or solution outline (*n*=13 for no initial student attempt, *n*=10 for initial student attempt). The level of detail in the solution sketches varied greatly, but there were several instances in which the challenging part of the problem was provided for the student with only a few remaining algebraic steps left for the student to complete. In some cases, one tutor responded to a student with a solution sketch, and, without any further contributions from the student, another tutor volunteered a full worked solution, potentially deterring the student from attempting to apply the sketched solution steps. Thus, the "spontaneous" characteristic of this web-based forum[55], although potentially a mechanism for catching mistakes and introducing multiple perspectives, is sometimes redolent of "too many cooks in the kitchen."

Alternatively, the examination of tutor actions in tutorettes that resulted in an *increase* in student activity may reveal ways of supporting and encouraging student problem-solving attempts. Although the number of such tutorettes was small in this corpus, some tutorial moves did appear to support instrumental help-seeking. For instance, the inclusion of partial solutions or solution sketches followed by a direct question, such as ``What do you need to integrate to find the arc-length?'', provided limited information and directly prompted students to work further on the problem. Hinting, in this fashion, is a common tactic used in traditional tutoring sessions that functions as a prompt for students to access information already known and to carry out the next solution step (Hume et al, 1993).

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[55] The "spontaneous" characteristic of this help site is reflected by the number of tutorettes (39) in the corpus in which more than one tutor participated, either addressing the student or another tutor.

Errors and error remediation

Because previous research has shown that tutors, as well as students, make mathematical errors in Web-based and face-to-face tutorettes (van de Sande and Leinhardt, 2007; van de Sande, in preparation), we examined the contributions of participants in both roles. The error rate for tutor contributions was impressively low for this corpus. Only three tutorettes contained mathematical errors made by tutors. Two of these were sign errors (one involving the computation of a derivative and the other the factorization of a quadratic) and the third concerned a trigonometric identity. In contrast to other SOH corpora that we have analyzed, none of these errors was discovered or addressed by another forum participant. In general, SOH communities are wikipedia-like and members share responsibility with one another by catching mistakes and publishing corrections. However, the fact that two of these tutor errors were relatively minor may have contributed to their slipping by unnoticed by others. One sign error result was a value to be squared so that its sign was, in some sense, irrelevant; the other sign error occurred in an explanation of a removable discontinuity and did not affect the ultimate conclusion. The remaining error occurred when a tutor utilized a trigonometric identity that is not generally well known and, therefore, may not have been detected by fellow participants.

In contrast, the error rate for student contributions was relatively high. Of the 55 tutorettes in which the student displayed problem-solving activity, 34 contained errors. This finding attests to the function of open, online, help forums as safe environments for students to present their work and tutors to critique this work, as well as to the social norms of this particular e-help community. Students did not appear to be concerned with "saving face" and tutors did not appear to be constrained by universal conversational maxims and politeness principles that, in face-to-face encounters, may conflict with pedagogical goals (Person et al, 1995). Also, the "rules for participation" for the `mathgoodies.com` community specified that students were responsible for showing all work, and students who routinely did not show work were sometimes chastised and denied help. This practice encourages students to publish their misunderstandings and incorrect results, thereby contributing to the magnitude of the error rate.

Table 1 Type and number of student errors with example of each.

Type	Number	Example
Pre-calculus	8	-sin t – (sint t + t cost) = -2sin t + tcos t
Operational	15	Use of the harmonic series, 1/n, to investigate the convergence of the series $n/(n+1)2^{(n-1)}$
Conceptual	14	f(x) = x^6 – 3e^x cos(x) + e^3.5 f'(x) = 6x^5 -3xe^(x-1)cosx + 3e^xsinx + 3.5e^2.5

Table 1 shows the number of student errors in this corpus according to type and an example of each. The majority of the errors that students made were calculus-based, either involving the implementation of an algorithm (15) or the misunderstanding of a concept (14). Only 8 errors pertained to arithmetic or algebraic operations. Of these, 2 were arithmetic errors, 4 were errors concerning the distributive property, and 2 resulted from incorrect calculations of a function value. The typical response of the tutors to this type of error was to draw attention to the mistake (directly or using hints), with minimal explanation of the violated principle. For instance, when a student was performing differentiation and, in the process, neglected to distribute the negative sign to both terms of an expression, a tutor responded by pointing to the line that contained the error and identifying the incorrect term in the expression using red-colored font: "–sint – (sint + tcost) = -2sin t + tcos t **< -- here is your mistake … check the sign of tcost**."

Despite the difference in the mathematical domain (calculus vs. pre-calculus), the errors that students made in operationalizing a concept met with much the same response: The error was corrected with little explanation of the underlying principle. For example, when a student was investigating the convergence of an infinite series using the limit comparison test and chose an unproductive comparison series, a tutor responded by providing an appropriate comparison series and solution but with no explanation regarding the student's failed attempt. This mode of response to operational errors in tutorettes is in keeping with research on traditional tutoring in that tutors generally do not perform detailed evaluations of students' knowledge (Putnam, 1987) or make inferences about specific student "bugs" (McArthur, Stasz and Zmuidzinas, 1990). Presumably, the errors that we classified as operational matched with those that the tutors judged to be less consequential for learning and were therefore treated somewhat cursorily.

The third type of error centered on conceptual misunderstandings or interpretations. Tutors responded to this type of error in a very different way from the way they responded to the other two error types and they generally provided explanations that invoked mathematical definitions and principles. For example, when a student overgeneralized the rule for differentiating powers to exponential functions, the tutoring included responses such as: "The derivative of an exponential term is the exponential terms times the derivative of the exponent" and "Remember, the derivative of a constant is 0, therefore, $e^{3.5}$dx is 0." Responses of this nature prompted students to reformulate their understanding of the concept. In this example, the student queried, "Oh, the derivative of -3e^x is itself?" and a tutor replied with a proof of this fact using the product rule on the terms -3 and e^x.

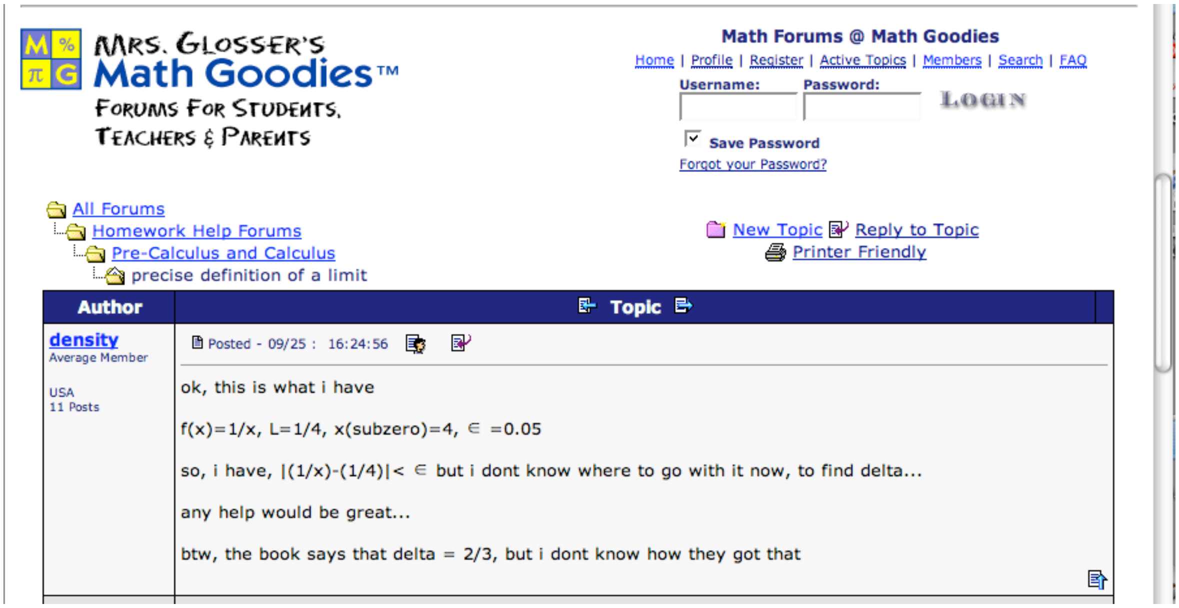
We hypothesize that many errors of this latter type may result from an unproductive perspective on the problem situation, specifically one that does not afford reasonable opportunities for solution (see Greeno and van de Sande, 2007 for a discussion of perspectival theory). Adopting a visual analogy, problem solvers reach an impasse when there are some (mathematical) objects that are not placed in the foreground or background in a helpful manner. In some cases, an unproductive perspective may be the result of trying to operate according to a schema that is too specific and therefore not applicable. This difficulty results from the tendency of students to latch onto simple examples that a mathematics instructor uses to introduce a new topic. Students then construct a schema for solving problems that is limited to these example types: "[It is] almost impossible to give students simple experiences without giving them correspondingly simple long-term conceptions of the concepts being introduced" (Davis and Vinner, 1986). Constructing an alternative perspectival understanding is an effortful process, but one that has implications for conceptual growth (Greeno and van de Sande, 2007). In the next section, we present an example of a tutorette that illustrates how an alternative perspective was introduced and adopted by a student.

A TUTORETTE ON THE LIMIT

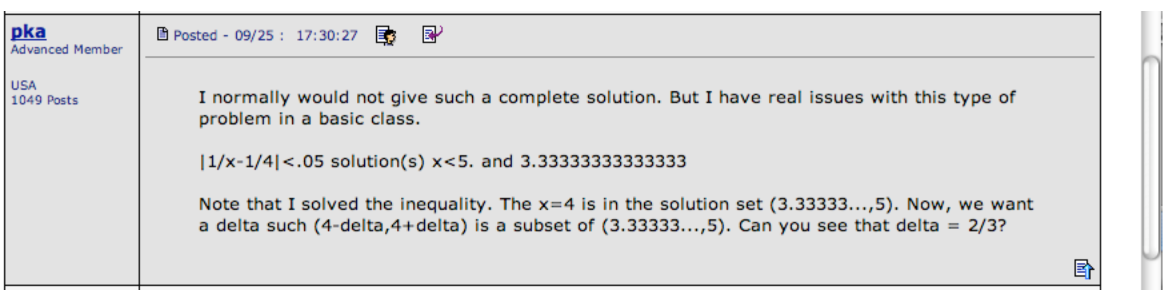
The following example illustrates how an SOH site can function as a collaborative tutoring effort to effectively help students understand a challenging calculus concept, namely the limit. The exchange is an example of instrumental help-seeking in which two tutors (**pka** and **tkhunny**) responded with alternative perspectives. The student (**density**) questioned the first tutor's solution. Because the first tutor was unsuccessful at framing an explanation for the student, s/he requested an additional tutoring help. Another tutor entered the dialogue and provides the sketch of a solution from an alternative perspective. (The participant code for this dialogue between the student and the two tutors is 1212131.) This second perspective was successfully understood by the student, and the situation was resolved.

The formal mathematical definition of the limit, in particular, is often a source of extreme difficulty for students (Tall, 1993), although its presentation, at least to some extent, is not unusual in an introductory calculus course. Typical problems include the application of the formal definition of the limit to a given function. Instruction usually begins with linear functions and then progresses to more complex cases, such as reciprocals (e.g., $f(x) = 1/x$). The application of the formal definition to a linear function can be performed using a sequence of algebraic manipulations (factoring followed by division) but this 'procedure' does not extend to more complex functions without significant modifications. This situation presents a difficulty for students who have acquired a schema for applying the formal definition but have not grasped the underlying limit concept.

In the initial posting, **density** posed the problem and an attempt (albeit weak) at starting a solution. As is often the case with homework assignment from a textbook, **density** knew the final answer but could not construct the accompanying solution steps:



A short time later, a tutor (**pka**) responded with a partial solution, preceded by a comment on the nature of such problems for introductory calculus:



Despite the characterization ("I normally would not give such a complete solution."), **pka** did not provide a complete worked solution but rather provided select solution steps and ended the posting with a question, "Can you see that delta = 2/3?" This move encouraged active learning since **density** was prompted to use this additional information to work through the problem. **Density** responded by questioning how **pka**'s solution supported the answer and presented his/her work on the problem. This work corresponded to the enactment of a schema for applying the formal definition of the limit to a linear function and resulted in an acknowledged impasse. **Density** was trying to manipulate the absolute value expression to resemble the desired algebraic form which would have |x-4| on one side of the inequality and a constant multiple of ∈ on the other:

density
Average Member

USA
11 Posts

Posted - 09/25 : 20:15:34

how did you get delta = 2/3?

this is what I did,

|(4/4)(1/x) - (x/x)(1/4)| < ∈
|(4/4x) - (x/4x)| < ∈
|(4-x)/(4x)| < ∈
|(1/4x)**x**(-1)(x-4)| < ∈
and then im left with |1/4x|**x**|x-4|< ∈
but now how do i isolate that |x-4| so that i can find epsilon ?

Pka, however, was apparently unable to explain the solution and called for help from other forum participants. This is evidence that, just as students seem comfortable voicing questions and producing imperfect work in this e-help community, tutors also appear comfortable publicly acknowledging difficulties:

pka
Advanced Member

USA
1049 Posts

Posted - 09/25 : 20:53:16

Can someone else help?
It seems so clear to me!
I do not know what else to say.

Density responded apologetically and clarified his/her state of understanding. **Density** understood how **pka** could arrive at the final answer if an earlier claim was accepted (**pka**: "Note that I solved the inequality") but did not understand the justification for this claim, especially in light of the impasse that **density** had reached:

density
Average Member

USA
11 Posts

Posted - 09/25 : 20:59:07

Sorry, I guess im just blind on this one...
I don't see how you came up with the interval 3.3333 < x <5
but I do understand that if the interval is (3.33333,5) than delta is 2/3, or .6666666666667 , that's easy to see

It is at this point that another tutor, **tkhunny**, entered the dialogue and presented an alternative perspective that focused on the *dynamic* nature of limits; the value of the limit of a function at a point (if it exists) is the value that the function is arbitrarily close to as the independent variable *approaches* that point. Thus, **tkhunny** suggested considering the behavior of the function for values LESS than 4 and values GREATER than 4. The absolute value – the source of **density**'s impasse – was then equivalent to a simple inequality for each case:

tkhunny
Advanced Member

USA
501 Posts

Posted - 09/25 : 21:40:36

Let's just hang around x = 4, close enough to avoid zero (0).

If we are LESS than x = 4, approaching from the low side, so that 1/x > 1/4
(or [(1/x)-(1/4)] is positive), we have:
|(1/x)-(1/4)| = [(1/x)-(1/4)] < 0.05

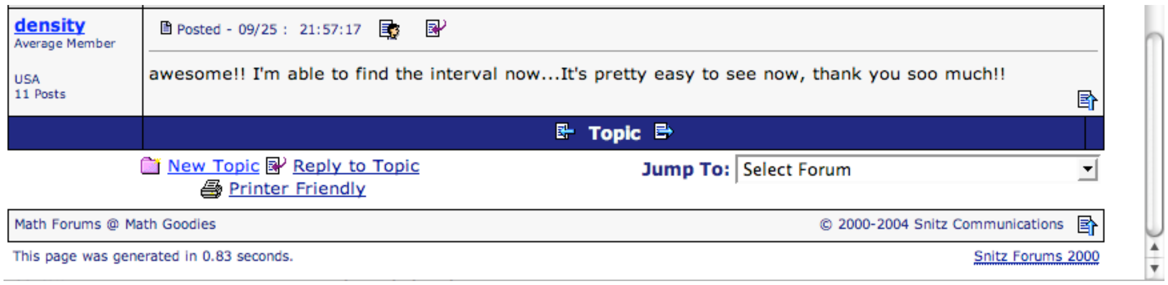
That should be easy enough to solve, recalling that it is valid ONLY for x < 4.

If we are GREATER than x = 4, approaching from the high side, so that 1/x < 1/4
(or [(1/x)-(1/4)] is negative), we have:
|(1/x)-(1/4)| = [(1/4)-(1/x)] < 0.05

That should be easy enough to solve, recalling that it is valid ONLY for x > 4.

Edited by - David on 08/26/2007 12:58:31

Although **tkhunny** did not provide a complete worked solution (leaving the solution of the inequalities to **density**), this sketch was sufficient for **density** to adopt the alternative perspective and thereby to understand the derivation of the interval in question. **Density** replied with gratitude and enthusiasm, demonstrating clearly the effectiveness of the help received:



Despite the apparent success of this tutorette, however, its outcome is not without concern. Although **density** was able to solve the problem by adopting the perspective of **tkhunny**, there is no indication that **density** made progress toward reconciling the original schema-based approach and this alternative perspective. In other words, it is most probable that, following the exchange, **density** retained two disconnected perspectives on the formal definition of the limit: a schema-based approach for linear functions and a dynamic approach for more complex functions, such as reciprocals. The relationship between these two was not constructed in the tutorette. This example calls attention to the importance of carefully examining the ways in which tutors specifically address and build upon student activity, especially in light of constructing an understanding of the student's perspective. Instruction as a collaborative activity requires that tutors take student perspectives into account rather than simply presenting alternatives.

CONCLUSIONS

Students are turning to discussion forums in order to receive help on mathematics homework assignments and studying for examinations. These sites are a resource that allows students to complete homework assignments and learn outside of classroom instruction and may be critical for the success of some students, especially in introductory calculus courses. Because participants are anonymous, these communities provide a relatively safe environment for asking questions, presenting solutions, and critiquing work. In addition, several of these homework help forums have the added benefit of being free of cost. While some of these forums provide tutoring from assigned volunteer tutors (usually mathematicians or upper-level mathematics students) who meet certain criteria (AOH sites), there are also several forums that provide spontaneous help by other members of the e-community (SOH sites). The research reported here investigated a corpus of 100 sequential tutorettes on introductory calculus topics from one such SOH site that is taken as representative of other web-based homework help forums of this type. The analyses focused on active student learning and error remediation, two elements of effective instruction.

Active student learning involves problem selection, questioning, and self-regulation and is a desirable element of instruction that is not often achieved in traditional classroom situations or in traditional face-to-face tutoring (Graesser, Person and Magliano, 1995). However, there was evidence of active student learning in the SOH tutorettes. If students in this community were solely "executive" or dependency-oriented help-seekers, then the participation codes would have been limited to instances of "12", that is, postings in which a tutor responds to a student query. This was not the case. Instead of simply using the tutors to do their homework for them, many students took part in these dialogues as "instrumental" or mastery-oriented help-seekers; students made initial attempts at solutions, queried tutor responses, and applied the help they received from tutors to make progress on solving problems. Furthermore, this behavior was influenced, at least to some extent, by the actions of the tutor. Some tutor actions seemed to encourage active student problem-solving, whereas others may have discouraged it. In particular, providing solution sketches (versus complete worked solutions) accompanied by asking direct questions encouraged dialogue; providing complete (or close to complete) worked solutions seemed to have the opposite effect.

Related to the issue of active student learning in instruction is the issue of how errors are handled. One mark of a learning community is that ideas can be questioned, elaborated, challenged, and revised safely. In practice, this has proven problematic for face-to-face instructional settings, where students tend to refrain from asking questions and presenting work that displays knowledge deficiencies and tutors are sometimes reluctant to criticize student contributions. In the SOH e-help community, however, students appeared comfortable in presenting incorrect work and tutors were open and forthright in their commentaries, evaluations, and explanations and vice-versa (van de Sande, in preparation). Saving face was clearly not the central concern, although members still adhered to a standard of politeness: Criticism was directed at the incorrect mathematical information rather than at contributors. In addition, e-help tutors in the SOH community modulated their responses according to the type of error. Pre-calculus errors and operational (calculus) errors were not accorded the same depth of explanation as conceptual misunderstandings.

The e-help community that we chose for this project was characterized as *spontaneous* online help. That is, any forum member could take on the role of tutor, regardless of mathematical expertise or instructional experience. This participation structure fostered collaboration between individuals with different abilities, specialties, and interests. In this corpus, the collaborative potential of an SOH site was evident in the participation by multiple tutors per posting; as many as 4 different tutors took part in a single tutorette. The spontaneous (SOH) feature of the discussion forum also encouraged and supported the contribution of alternative perspectives on problems. We concluded that this “party-line” characteristic of SOH sites has the potential of helping both students and tutors understand problems in a multitude of ways (many of which may be novel). However, as the tutorette on the formal definition of the limit demonstrates, the benefits may be curtailed if tutors do not connect their responses to a student's perspective and help reconcile alternatives.

The larger aim of this project aims to define and evaluate effective learning in the context of Web-based tutorettes. As a starting point, we have begun investigating features of ideal instruction that stem from cognitive research and that have been applied to traditional face-to-face tutoring corpora. Clearly, this is not an ideal fit since the goals, setting, and composition of the instruction are vastly different. On the other hand, at the core, tutorettes are instances of instruction and learning, and, as such, share many of the same ideals. Understanding how these ideals (and potentially others) are realized in e-help communities is important for a number of reasons. First of all, these communities are flourishing as instructional support for today's students. Given that these communities may become the new norm for seeking help on homework, it is important to understand how they function and how they impact students' understanding. Do tutorettes help students beyond the construction of a solution for the problem that is posted? A second reason for pursuing this research involves the variety of forum types available – gratis versus subscription and AOH versus SOH. Do these communities manifest different elements of ideal instruction, and, if so, which ones and why? For instance, it may be the case that SOH sites are more likely to introduce students to multiple perspectives on a given problem, whereas AOH sites tend to encourage more in-depth explanations. Knowing how the different e-based communities function could inform the formation or endorsement of such a community. Finally, this research has implications for the design of intelligent tutoring systems, particularly those that contain a dialogue component such as an automated pedagogical agent.

These systems reside in computer environments and, as such, have much in common with Web-based homework help sites. Identifying the ways tutors communicate with students in e-communities can inform the construction of more realistic and effective computerized pedagogical agents. In general, the message is clear: Students of today are voicing their appeals for help in web-based homework help forums. As educational researchers, what, then, is our response?

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B.3 ONLINE TUTORING IN THE CALCULUS: BEYOND THE LIMIT OF THE LIMIT[56]

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Abstract:

Students in many countries are participating in free, open, online tutoring forums for homework help, yet, to date, there has been no systematic research on the learning that takes place on these sites. Forums can be structured differently with respect to who may respond (tutor) and how monitoring is accomplished (when questions and answers get posted). In Spontaneous Online Help (SOH) sites, any participant can respond to a query, whereas in Assigned Online Help (AOH) sites, only select participants can answer queries. In this paper, we look at three calculus help sites (two U.S. and one French) that reflect these differences. We collected and analyzed 100 tutoring exchanges from each site on the challenging mathematical concept of the limit. Our objectives were to investigate patterns of participation, mathematical and pedagogical quality, and to provide a sense of how these forums resemble communities. To this end, we developed and applied measures of exchange complexity and quality and noted several ways in which these sites manifest characteristics of an online learning community. We found that AOH sites promote brief exchanges between single student-tutor pairs (low complexity), whereas SOH sites (particularly those that post queries and responses with minimal delay) encourage extended exchanges between multiple participants (high complexity). There appears to be a positive relationship between exchange complexity and quality. Exchanges that involve few participants and contributions (low complexity) are often devoid of mathematical and pedagogical sophistication (low quality). The SOH sites exhibit a stronger sense of community as members refer to one another by name, collaborate mathematically, and critique or correct each other's errors or mistakes. Of particular interest was our finding that, in the SOH sites, tutors were positioned jointly as learners.

This research is part of an ongoing effort to understand the impact that free, open, online tutoring has on students and tutors and to explore its potential for instruction and learning.

When a new technology appears in a culture the first stage of its presence is often felt as a simple replication of activity, but by the use of new means. Consider, for example, the movement of water supplies from hauling buckets from a near by river to having a town well to having running piped water in every home; or the support of writing from scribe to typewriter to computer. In each case the central activity remained the same for a while and then was

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transformed by the unique affordances and demands of the new technology. This transformation folds back on itself so that the culture that supports the activity reconsiders its fundamental purpose. We are at such a place with respect to student learning and teaching in mathematics. The current timing, sequencing, and resources available to the mathematics student seem to be more or less the following: the student learns in a classroom with a teacher; engages in individual or group assignments that appear in the text book or are given out; and checks the work by consulting with other students, other texts, or occasionally additional adults. If a student struggles consistently with the content and procedures of mathematics, perhaps there are arrangements for formal tutoring support either for additional payment outside of school or in a special support session inside the school.

The presence of free, open, online tutoring resources has altered this scenario. Students can simply log on to a website and immediately pose a concern, problem, or issue of confusion to a group of highly knowledgeable and willing responders. Students can ask their questions when they wish and from the convenience of their home or school. The contact is asynchronous in many senses – the student may ask the question at a time different from when the answerer is available, at a time different from when the ideas are discussed in school, and out of sequence from assigned work. The tutorial response is highly specific and goes directly to the question asked but is agnostic with respect to the particular slant that the teacher or textbook might have taken on the matter. Thus, the online conversation is both stripped down – it does not contain many entrance negotiation moves, it does not make internal references to other parts of the lessons or texts, but it is also more detailed and elaborate. Because the tutors have no idea what exactly has and has not been discussed or what the history of understanding is on the part of the particular enquirer, more specificity and detail must also be included in the question. In a

Gricean (1989) sense the conversational implicatures require simultaneously more and less information.

In order to start to understand this new technology and its impact, we have begun to gather a substantial corpus of online tutorial exchanges in calculus help forums. We have focused our efforts on three topics within the calculus: the limit, the chain rule, and related rates (conceptual, procedural, and integrated). We are inspecting this corpus through a number of lenses (cognitive, situative, perspectival) and posing a number of questions, a few of which are the following: Does the form of the arrangement of the help site influence the kind of exchanges that take place? Do the exchanges show evidence of explanatory completion? How do these online exchanges compare qualitatively to face-to-face tutoring sessions? In what sense are these online help sites communities? How do community norms for exchanges evolve over time? What can we learn about the way in which instruction might be better designed from examining the nature and depth of the questions posed? In this paper we examine the ideas of participation, quality instruction, and an emerging sense of community as students and tutors engage in a series of questions and responses. At a deeper level we want to examine how this new form of support for student learning may alter the very nature of instruction or what we take to be instruction.

Community

The notion of a community of learners is a central construct in analyzing and understanding instructional practices (Bruner, 1986; Brown, 1997). Classrooms are considered communities of practice and the participation, positioning, and growth of individuals within this community contribute to an understanding of the instruction and learning that is taking place. As the Internet is becoming a ubiquitous means of communication and instruction, the question of defining community in this new context arises. Some have posited the idea of a virtual community, while others have suggested that the idea of a virtual community is an oxymoron. While not wanting to take on the fundamental issue of what is and is not a community, we do feel that there are features in an online environment that make it community like. That is, the activity and participation of members in online help sites reflects the common themes found across theories of community as well as the feature shared norms and goals (Carter, 1998; Wertheimer, 1998). Appealing to the notion of what constitutes an online learning community, we consider the presence of the following attributes as indications of an open online tutoring community (Grossman, Wineburg, & Woolworth, 2001; Lave, 1991; Palloff & Pratt, 1999; Pratt, 1996; Werry & Mowbray, 2001):

1. Participants share some common explicit and implicit goals.
2. Participants have an accessible physical or virtual location in which they meet.
3. Participants identify themselves as members of the community.
4. Participants assume responsibility for participation.
5. The defining features of the community can be renegotiated and altered by the members.
6. Ideas can be questioned, elaborated, challenged, and revised safely.

The participants in open online help forums are positioned as students (those who request help) and tutors (those who provide assistance)[57]. In general, the tutors are more regular participants; students use the forum as the need dictates but tutors consistently participate (often on a daily basis). Therefore, the tutors are the core group of participants that provide the sense of community. In this paper, we identify several ways in which open online help forums manifest

[57] Generally, these two roles are independent although we have seen some cases in which a student took on the role of a tutor and vice-versa.

the features of community listed above, with special attention paid to the attributes concerning responsibility for participation, the establishment of principles, and the exchange of ideas.

Tutoring

Tutoring has often been considered a face-to-face, single tutor-student pair activity that has the goal of instructing the student on a pre-determined set of concepts or procedures. This form of instruction has proven effective for academic performance and attitudes toward subject matter. In what has become a classic piece of literature in the field, Bloom (1984) documented that students learning from tutors in this way perform two standard deviations above students learning in a classroom situation.

In order to account for the tutoring advantage, the characteristics of participants and exchanges have been examined. Somewhat surprisingly, the tutoring advantage does not appear to be attributable to the tutors level of expertise (Graesser & Person, 1994) or the familiarity of the participating parties (McArthur, Stasz, & Zmuidzinas, 1990; Siler & VanLehn, 2005). Instead, the advantage of tutoring may be attributed to the opportunity it presents for students to ask questions (Graesser & Person, 1994), the intensity of the interaction (McArthur, Stasz, & Zmuidzinas, 1990), and the cues from tutors that maximize the motivation to learn (Lepper, Aspinwall, & Mumme, 1990). Of key importance is the finding that tutoring sessions do not generally embody a large set of the elements of idealized instruction. Graesser, Person, & Magliano (1995) found evidence for only three elements (collaborative problem solving, question answering, and explanatory reasoning in the context of specific examples) in tutoring sessions.

One can conclude that tutoring is a highly interactive process in which support is provided in ways that are markedly different than other instructional settings. However, in addition to treating tutoring as one-on-one, face-to-face instruction covering a pre-determined set of material, the majority of the research has also been conducted in the laboratory. A more naturalistic approach to tutoring is worth pursuing. Students often seek help from others on particular problems they encounter while completing homework assignments or preparing for examinations. This form of tutoring or student-initiated help-seeking has not been as systematically studied.

Open online help communities are a relatively unexplored instantiation of tutoring, despite the fact that these communities are developing and flourishing across the world. Investigating these communities locates tutoring in a natural setting and is important because these sites may be the only recourse that some students have for gaining instructional support outside of the classroom. In this paper we address the issues of complexity and quality in the context of online help forums. Within a student-initiated help-seeking discussion, the number of participants and the duration of the discussion contribute to its complexity and the depth of explanation and pedagogical sophistication mark its quality.

The calculus

We have chosen to use online tutoring help sites on the calculus as a location of study. The calculus functions universally as a gate-keeper for the physical, biological, and many of the social sciences. It is viewed as extremely challenging by many students and introductory courses often have high attrition rates. Teachers often view success or failure in the calculus as an indication of underlying capability and, in the United States, success in the course is often a prerequisite for admission to programs that in fact use very little calculus in the content of their own domain. One of the first challenges facing a student in an introductory calculus course is coming to grips with the concept of the limit.

The concept of limit is foundational to calculus and is a recurring theme in any introductory course. However, the concept contains nuances that took mathematicians over a century to resolve (Dunham, 2005) and pose numerous problems for introductory calculus students (Szydlik, 2000). Students often experience confusion regarding the relationship between the value of a function at a point (or nearby points) and the limit, the meaning of indeterminate forms, and the notion of boundedness. In addition, there is a large set of procedures (such as factoring, multiplying by the conjugate, and rearranging terms) associated with the computation of limits. Deciding which technique to apply in a given situation can be a daunting task, and resolving these quandaries is at the heart of gaining an understanding of calculus that will support future learning (Tall, 1992).

As a part of our on-going effort to understand the ways in which the Internet has altered instruction and learning (writ large) and has become a support for topics such as the calculus more specifically, we address the following questions in this paper: How might we set up an appropriate methodology for studying these environments? What is the effect on participatory engagement of different kinds of online forums? What is the range of quality that we see in these environments? What is the nature of the “community” of participants?

METHODS

Vocabulary

There is a vocabulary associated with interaction in online environments that we have appropriated for our discussion of online tutoring. A *post(ing)* is a contribution that is published on the site, either to initiate a discussion or in response to another's contribution. As in verbal discussions, participants generally take turns contributing to the conversation. The set of contributions pertaining to a single request for help constitute an *exchange* or *discussion*, sometimes referred to as a *topic*.

Site*s*

There are a large number of free, open, online tutoring websites. These sites exist in many countries; and among those countries that share a language (English or French, for example) students and tutors can and seem to traverse geo-political boundaries. Although similar in many ways, online help sites can be structured differently with respect to who may respond (tutor) and how monitoring is accomplished (when questions and answers get posted). We selected three calculus help sites to reflect these differences. FreeMathHelp.com (U.S.) allows any registered participant to respond immediately and has select participants who subsequently moderate the discussions. Cyberpapy.com (French)[58] also allows any registered participant to respond but has moderators screen[59] the replies before they are made public. MathNerds.com (U.S.) only permits select tutors (based on mathematical qualifications and tutoring performance) to respond and assigns each query to a particular tutor. Based on the rules for whom may participate as tutors and how responses occur, we refer to FreeMathHelp.com and Cyberpapy.com as Spontaneous Online Help (**SOH**) and MathNerds.com as Assigned Online Help (**AOH**). One effect of the monitoring feature is on the speed of responses. If there are no intermediate monitoring actions, then the response can be as quick as a real time face-to-face question and answer; on the other hand, if a monitoring action requires both an assessment of the

[58] Although neither author is fluent in French, both have sufficient familiarity to read and work through the mathematical online postings.
[59] We were unable to ascertain the nature of the screening criteria from the site administrators.

question and an assessment of the availability of a specific tutor to deal with the question, then there may be a considerable delay – up to several days – between the time the question is asked and the answer is presented. Thus, there is a trade-off between the certainty of a response and its timing; in unmonitored sites it is possible that no one chooses to pick up on a question.

FreeMathHelp.com. FreeMathHelp is an advertisement-supported mathematics help portal established in 2002 by Ted Wilcox, an enterprising high school junior. In addition to the discussion forum, the site includes lessons, games, a graphing utility, and worksheet pages. There are 9 homework help forums, organized by subject area (such as algebra, differential equations, calculus). Forum members can contribute or respond to these postings and have access to user profiles that include volunteered information on occupation, residence, contact information, as well as amount of discussion board activity. Each member is characterized by total number of contributions to distinct postings: new (0-49), junior (50-249), full (250-999), senior (1000-2499), elite (more than 2500). There are several elite members with more than 2500 contributions, four of whom have contributed to more than 4000 postings each. Each forum has assigned moderators who have the right to lock topics and move or delete postings; but who do not affect the pace of responding.

The prescribed etiquette for participation is located in a "sticky" that is the lead posting within each help forum. This covers administrative issues (e.g. posting to an appropriate category) and politeness (e.g. patience while waiting for response). In addition, there are 3 rules that specifically address the content and framing of posts: include problem context ("Post the complete text of the exercise"), show initial work ("Show all of your work [including intermediate steps that may contain errors]”), and attend to clarity (“Preview to edit your posts [to minimize errors]”).

The computer window for constructing posts contains traditional icons for highlighting text (e.g. italics, boldface, underlining, and font size and color), inserting material (e.g. external links and images), and organizing text (e.g. forming lists). A large selection of graphic “emoticons” (faces) is available for expressing emotions and attitudes (such as gladness or perplexity). In addition, there are format capabilities more specific to mathematical discussions since it is tedious and often impossible to create mathematical symbols and expressions using keyboard characters. Using LaTeX, a document preparation system designed to typeset mathematical text, participants can use command strings and code to produce mathematical symbols (such as ∞) and vertical expressions (such as $\lim_{x \to -1} \frac{x^2-1}{x+1}$). In order to encourage the use of this software, FreeMathHelp includes a tutorial for LaTeX, as well as a link to a free equation editor that generates the LaTeX code, which, although powerful, can be difficult for the novice[60].

Cyberpapy.com. Cyberpapy was founded in 1997 by the Boulanger Foundation to connect youth and seniors, with the premise that many seniors[61] have the time, expertise, and willingness to help young people with academics. The site includes discussion forums for 10 subject areas (including mathematics). Within each subject areas, postings are indexed by title, number of responses, initiator, date, and school level.

The prescribed Cyberpapy.com participation etiquette is similar to that of FreeMathHelp.com with respect to administrative issues and politeness (e.g. students are

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[60] For example, the LaTeX code to produce this limit expression is: \lim{x \to -1}\frac{x^2-1}{x+1}.

[61] Although the primary purpose is to encourage academic contact between seniors and youth, Cyberpapy.com is a true SOH site in the sense that anyone (not just seniors) can respond to postings.

reminded to be patient). However, unlike FreeMathHelp.com, students are not specifically instructed to show work on the problem in question. Tutors are encouraged to provide support in understanding versus providing complete worked solutions.

The computer window for constructing posts contains areas for identifying the author, a short header, the message, and the author's e-mail address. There is also an equation editor to enable the inclusion of mathematical formulas and symbols. Upon request, contributors receive notification via e-mail when others contribute to the exchange. All contributions pass through a system of moderation before being published on the site, a process that may take several hours.

Mathnerds.com. MathNerds is a non-profit corporation founded in 1999 by Valerio De Angelis and W. Ted Mahavier. The primary purpose of MathNerds is to provide "free, discovery-based, mathematical guidance via an international, volunteer network of mathematicians." In particular, MathNerds promotes help via guidance, references, and hints versus worked solutions. The site is available in both English and Spanish and includes links to other "useful" websites (intended for reference and supplementary materials).

MathNerds has 325 mathematics tutors, the majority of whom are Ph.D.'s in mathematics. Tutors are selected through an application process that is based on the pedagogical approach and clarity demonstrated in response to 5-10 practice questions. Accepted tutors then specify the number of questions within various categories that they will address each week. The categories are arranged by grade level (e.g. U.S. K-5 Elementary) or by subject area (e.g. Calculus I).

MathNerds has a systematic way of assigning tutors to incoming queries. Upon visiting the site, a student first chooses the category that matches the content of the question. If there is a tutor who has selected that subject area and has not met her/his weekly cap, then the student is presented with the terms and conditions of participation (e.g. expectations and obligations for response time and legal disclaimers). The student then posts the question in a screen that includes areas for the subject (5 -10 words), the statement of the question, and any work already done. Instructions for accurately typing mathematical information (e.g. notation) are provided directly above the area for posting the question via a link to an online calculus course (Karl's Calculus Tutor). Encouragement is given to show all attempts at solving the problem(s) (including incorrect ones) along with general help-seeking advice (e.g. searching the published web questions and answers first for solutions to similar problems). After the student submits the question, two automated e-mails are sent: a confirmation of receipt to the student and the question (with a link to the online response form) to the assigned tutor. The tutor can reject the question and move it to a general queue (where another tutor may respond) or elect to respond within 2-7 days. If the tutor does not respond within 2 days (and has not indicated that a response is forthcoming), the question is automatically routed to the general queue, where it remains for 2 weeks. If the tutor does respond within a week, the solution is archived and forwarded to the student along with a link for future assistance on the question. The student can then engage in further dialogue (exclusively) with the same tutor that initially responded.

The sites we chose to investigate represent various configurations of online tutoring site structure and differ with respect to nationality, the requirements to participate, and the system of moderation. FreeMathHelp (U.S.) allows any registered member to participate as a tutor and moderates exchanges after they have been published. Cyberpapy (French) also allows any member to participate as a tutor but moderates exchanges before they have been published.

MathNerds (U.S.) only permits select tutors to participate and moderation is performed after exchanges have been published.

These sites were also chosen because they provide exemplary asynchronous online mathematical tutoring. The following example from Cyberpapy.com gives the flavor of the kinds of exchanges:

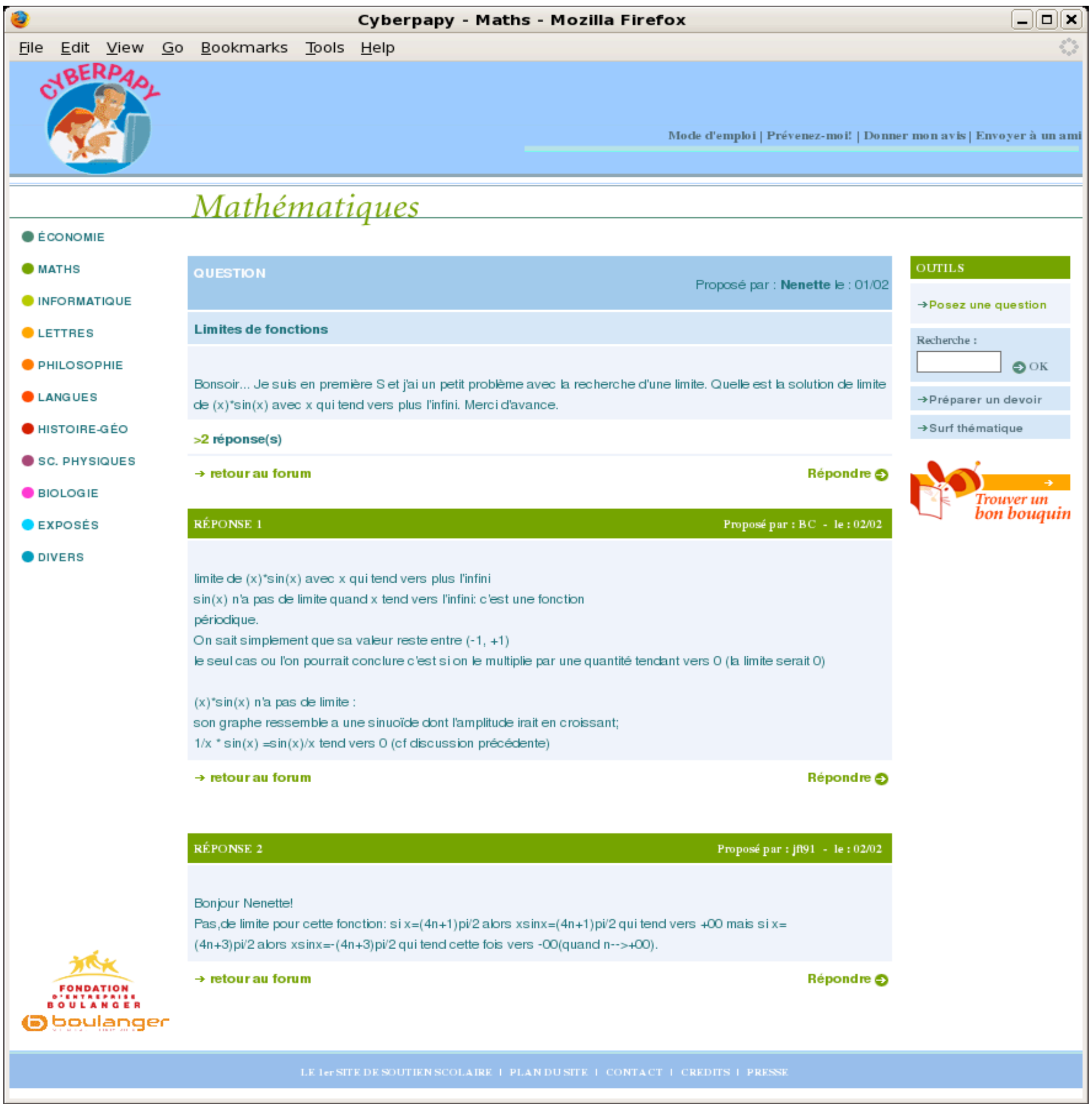


FIGURE 1 Online tutoring exchange from Cyberpapy.com.

In this exchange, two tutors provide correct and complementary responses to a student's query regarding the mathematical concept of limit. One tutor (**BC**) provides an explanation that builds on the general properties of the function $\sin(x)$, namely its periodicity and boundedness, and includes the conditions necessary for an alternative conclusion. **BC** also promotes the idea of coherence across multiple representations by describing the graph of the function as providing supporting evidence for the argument. The other tutor (**Jft91**) contributes a complementary explanation that is based on the consideration of specific sets of values, namely $(4n+1)\text{pi}/2$ and $(4n+3)\text{pi}/2$, that demonstrate conditions that support the conclusion. Together the two tutors give both an object and process sense of the functions involved.

Sample

After identifying the online tutoring sites, the next step in our investigation was the choice of a methodological approach. We deliberately chose a purely observational and non-intrusive approach for the investigation of these online help sites[62]. We have observed several online help sites for extended periods of time and collected a corpus of hundreds of calculus tutoring exchanges. We have catalogued those into sets by topic and by time of posting. For each investigation we draw a new sample (without replacement) in order to be careful not to over generalize our findings from one analysis to another.

A defining characteristic of free, open, online tutoring sites is the public availability of the discussions. However, some tutoring websites conserve resources by deleting exchanges following a set amount of time (usually, several months). The three sites we chose for this research have extensive archives (dating back several years) and a search mechanism for locating

[62] We sincerely hope that the education research community will respect this decision and not perform experimental studies in online help sites where others are performing observational research.

exchanges by a keyword or phrase. We selected 100 postings on the concept of limit from each of the three sites described.

There are numerous exercises that may accompany instruction on the limit concept. Traditional exercises involve estimating limits numerically or graphically, computing limits algebraically, and proving limits using the formal definition. Instruction may also address limits of sequences (versus functions). In recognition of the differing content and sequencing of mathematics instruction in France and U.S., we chose to select online tutoring exchanges that specifically pertained to the algebraic computation of the limit of a function[63]. The exchanges were selected from sequential postings in each site dating back from the same date. We included only those queries that received response, since unanswered queries are not publicly available for MathNerds.com, the AOH site. However, this decision should not strongly affect the data set since the three sites all report very high response percentages: Cyberpapy.com (90%), MathNerds.com (98%), FreeMathHelp.com[64] (94%).

Population

The availability of participants' profiles (tutors, in particular) is one of the features that free, open, online tutoring forums may include. Of the three sites, we chose, only FreeMathHelp.com has this feature. The participating tutors in this calculus forum are self-reportedly students, educators, professionals, and retired mathematics professors. Although individual profiles are not available for Cyberpapy.com and MathNerds.com, both of these sites elicit participation from tutors with Ph.D.'s in mathematics. MathNerds.com, in particular, is

[63] In order to focus solely on the limit concept, we did not include exchanges in which the computation of the limit was part of a larger problem, such as analyzing a function. However, these exchanges are good indicators of what problems and misunderstandings on the limit continue to crop up as students progress through an introductory calculus course.

[64] This site does not publish this information, so we based this estimate on the response rate over the time period of our study.

almost exclusively devoted to tutoring by mathematicians with Ph.D.'s. Students are not generally accepted as tutors and only a few high-school math teachers and undergraduate and graduate math majors are selected to participate.[65]

Most participants of open, online tutoring forums select names or "handles" (such as **Alice** or **Galactus**) that do not disclose personal information (location, knowledge level, etc.), and we refer to such participants using these self-designated handles. However, in order to respect privacy for data from this public forum, we refer to any participants whose handle appeared to reveal identifying information (such as surname) by pseudonyms that we constructed. We notice several individuals appear to frequent a variety of open, online help sites and preserve the same handle across the different sites.

The population of participants in each site is quite varied. Although some tutors and students post more frequently, each of the three sites has numerous tutor and student participants. Table 1 contains the number of unique tutors and students in our sample of 100 for each site.

TABLE 1

Percentage of Unique Tutors and Students in each Sample

	FreeMathHelp	Cyberpapy	MathNerds
Tutors	24	73	25-66
Students	67	84	81

FreeMathHelp.com has the smallest number of participants, indicating frequent repeated participation during the time period from which we were sampling (or possibly a higher density of exchanges since we 'stopped' when we had 100 exchanges). Cyberpapy.com is remarkable for the diversity of both student and tutor populations, although these numbers may be somewhat

[65] This is with the exception of two or three unusually knowledgeable high school students.

inflated due to the site registration policy. Because each participant enters a handle at each posting (instead of registering and accessing this information via "logging in"), there is a possibility that the same individual will be represented by different handles. MathNerds encourages but does not require tutors to sign their responses (by name or pseudonym), and, in our sample, only 24 tutors chose to do so. We can conclude that the number of different tutors in our sample ranges from 25 (if all of the unsigned responses were authored by a single tutor) to 66 (if each unsigned response was authored by a different tutor). The number of student participants for MathNerds was comparable to that of Cyberpapy.

Coding

In order to detect the impact of the different site structures on participation, we constructed codes that tracked the number of participants, the total number of contributions in the exchange and the sequence of participation. For example, a code of 1231 would characterize a discussion between 3 participants with 4 total contributions: a student [(1)] posted a problem and then two different tutors [(2) and (3), respectively] responded, followed by a final contribution by the student. As a gross measure of the complexity of each discussion, each discussion was assigned a *complexity index* defined as the sum total of the elements in the code. Thus, a discussion with participation code 1231 would be assigned an index of 7. Coding in this way blurs what may eventually turn out to be important distinctions in the exact pattern of exchange, but it helps collapse what becomes an increasingly differentiated sequence of possible configurations. For example, is a sequence of 1234141311 dramatically different from 1231414311, or is it simply a matter of timing? Coding exchanges in a way that retains sequencing, number of participants, and number of turns will allow us to address different questions at a future time.

To assess the quality of the exchange, we assigned a rating from 1 to 5 for the totality of each exchange. A 1 was assigned to those postings that were both brief and contained little or no rich explanatory or mathematical material; a 5 was assigned to those exchanges that had a truly mathematical feel to them invoking principles, mathematical reasoning, and to some extent excitement. An important feature of these exemplary exchanges was that the student remained positioned as a focal participant in the exchange. Table 2 contains a description of some of the features that differentiated exchanges. This analysis permitted us to describe both sites and specific topics as containing elaborated complete mathematical discussions or sparse mathematical fragments. Inter-rater agreement was 90% and all differences were resolved following discussion.

TABLE 2

Select Features Distinguishing Exchange Quality

Rating	Features
1	Brief exchange with little or no rich explanatory or mathematical material.
2	Generally brief exchange with sparse explanations or few connections to other mathematical material.
3	Exchange in which actions are prescribed but may not include reasons for application. Conditions of use are largely absent when principles are invoked.
4	Generally longer exchange invoking principles in which the mathematical reasoning is somewhat difficult to follow. Student may be peripheral participant.
5	Extended exchange in which principles of the calculus are invoked and perspicuous mathematical reasoning is evident. Student positioned as focal participant.

To discern whether participants in open online help sites act as members of a community, we surveyed the exchanges for attributes corresponding to the common themes and features found across theories of community, as discussed above. For example, referring to others by name is an indication of participants identifying themselves as members of the community. For some of the attributes, we identified a variety of indicators. For instance, taking turns within an exchange, adopting roles, and sharing the load are all indicators of an assumption of responsibility for participation.

RESULTS

Participation patterns

Table 3 contains the percentage of discussions in each homework help site according to the complexity index (defined as the summed participant code entries).

TABLE 3

Percentage of Exchanges in Complexity Range

Complexity Index	FreeMathHelp (SOH)	Cyberpapy (SOH)	MathNerds (AOH)
3	18	32	54
4-6	27	32	35
7-10	30	21	11
11-15	17	8	0
16+	8	7	0

Low indices reflect discussions that are brief (contain few turns) and involve a small number of participants. For example, the following exchange (Figure 2) from MathNerds.com, in which a student posts a problem and receives a single reply, has the lowest complexity possible (index = 3):

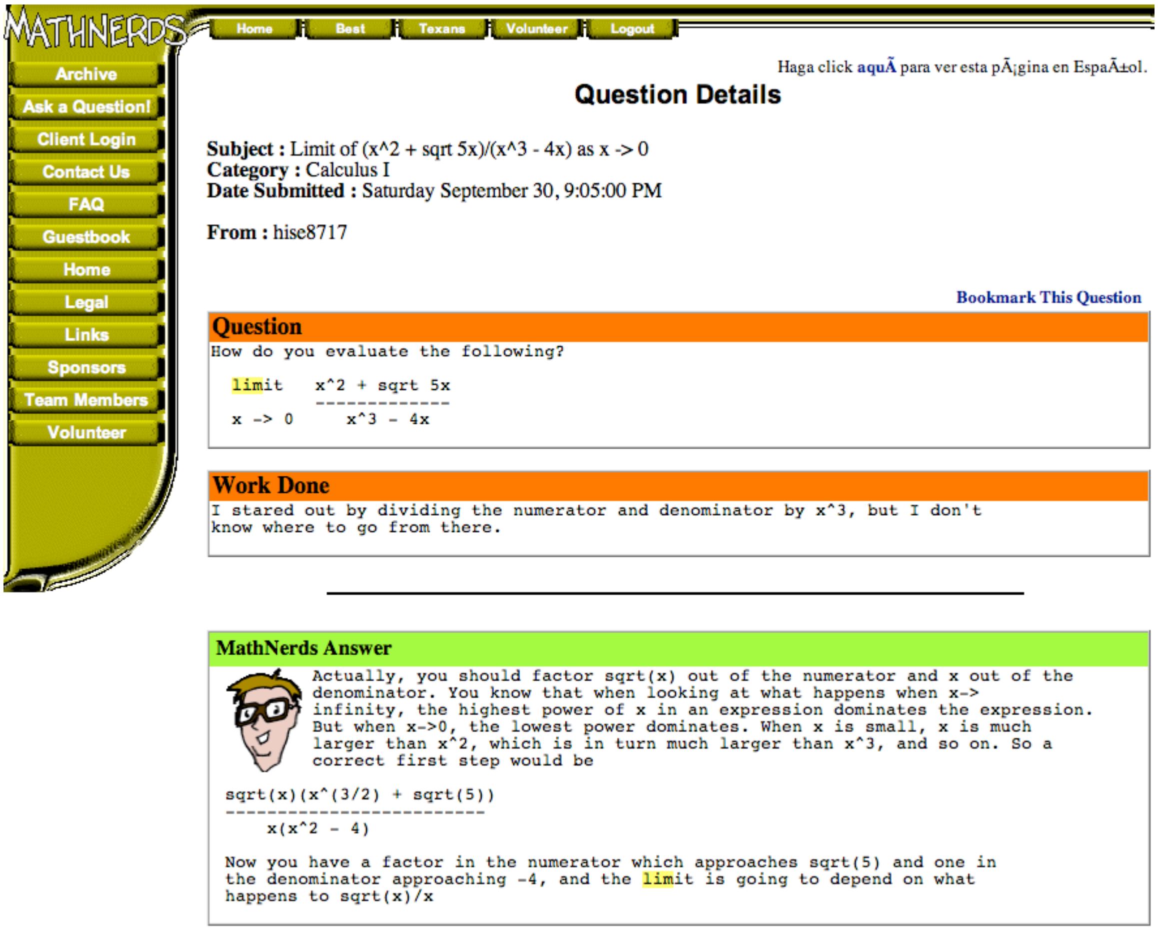


FIGURE 2 Online tutoring exchange of low complexity from MathNerds.com.

In contrast, higher indices are indicative of extended discussions (many contributions) between several participants. The following exchange (Figure 3) from Cyberpapy.com contains 7 contributions by six participants (index = 22):

Mathématiques

QUESTION — Proposé par : **-Alice-** le : 06/01

limite en un nombre a

bonjour,
je dois montrer que la limite en 2 de la fonction suivante:
f(x)=x²-x-2/x-2
tend vers 3, mais je n'arrive pas à écrire cette fonction différement pour parvenir à trouver cette limite, si vous pouviez me donner qqs pistes!
merci beaucoup d'avance
alice

≥6 réponse(s)

→ retour au forum — Répondre

RÉPONSE 1 — Proposé par : dd - le : 07/01

Il y a une erreur dans l'énonce.

→ retour au forum — Répondre

RÉPONSE 2 — Proposé par : Katy - le : 07/01

Attention à l'écriture !...

Si f(x) = (x²-x-2)/(x-2)

f n'est pas définie en 2 !

A toi.

Katy+++

→ retour au forum — Répondre

RÉPONSE 3 — Proposé par : mimi - le : 07/01

bonsoir Alice,

f(x)= (x-2)(x+1)/(x-2)= (x+1) pour x différent de 2

donc limf(x)=lim(x+1)=3 lorsque x tend vers 2.

→ retour au forum — Répondre

OUTILS
→Posez une question
Recherche :
OK
→Préparer un devoir
→Surf thématique

RÉPONSE 4 Proposé par : VI1378 - le : 07/01

Bonsoir,

Il me semble, mais je peux me tromper, que

f(x) = (x-2)(x+1)/x-2)

f(x) n'est pas d"finie pour x = 2.

Mais sur D = R - {2} f(x) = x+1
et sur D f(2) = 3

→ retour au forum Répondre

RÉPONSE 5 Proposé par : chamonix - le : 07/01

Bonsoir,
f(x)=(x+1)*(x-2)/(x-2)
Donc limite x->2, f(x) =limite x->2, (x+1) = 3.
Katy a raison, mais lorsqu'on parle de limite on ne se place pas en x=2, seulement très près.

→ retour au forum Répondre

RÉPONSE 6 Proposé par : -Alice- - le : 07/01

Merci beaucoup pour ces réponses!
Vous nous etes d'une grande aide!
Alice

→ retour au forum Répondre

FIGURE 3 Online tutoring exchange of high complexity from Cyberpapy.com.

As Table 3 shows, the most notable difference in patterns of participation occurs in row 1 and rows 4 and 5. The overall pattern suggests that the AOH site has more brief exchanges (54) and fewer extended (index > 11) complex exchanges (0) when compared to the SOH sites (18 and 25 respectively). All of the AOH discussions are relatively brief and among few participants. Indeed, none of the AOH discussions has a complexity index greater than 10 and there are relatively few with an index greater than 6. The most prevalent pattern of participation for this structure consists of a tutor replying to a student query (as in the example), perhaps followed by an expression of thanks.

The predominance of brief exchanges in the AOH structure may be attributable to the "assigned" characteristic that encourages discussions between a single student-tutor pair as well as to response latency. Upon receipt of a query, MathNerds assigns a tutor (based on interest and quotas) and informs her/him via e-mail. The tutor then has 2-7 days[66] to respond to the query. Due to this delay between requesting and receiving help, students may be disinclined to ask follow-up questions, particularly if the need for help is immediate (e.g. for a homework assignment).

The SOH sites, in contrast, contained quite a few extended discussions between multiple participants (as in the example from Cyberpapy.com), although FreeMathHelp.com supported even more discussions of this type. The following discussion (Figure 4) from FreeMathHelp.com (index = 13) is between a student and 3 tutors:

[66] The published average response time is 22 hours. However, this information is not available in the archives and is thus unknown for the corpus used in this study.

Forum Index • Profile • Private Messages • FAQ • Memberlist • Search • Usergroups

Log out [carlacvds]

Free Math Help.com - Homework Help! Forum Index » Calculus » Find the Limit

new topic | post reply

View posts since last visit
View your posts
View previous topic :: View next topic

Solve your tough algebra problems with Algebra Solved!

Find the Limit

Posted: Fri Mar 03, 2006 9:22 pm | quote

bittersweet
Junior Member

Joined: 12 Feb 2006
Posts: 105

what is the limit to this question? Im not sure how you find the limit for this!

lim (x is approaching 3) (7-x)square rooted - 2 / [1- (4-x)square rooted]

umm im not sure if I said that right.. I meant square rooted as the symbol that looks like a divided sign with a tic at the end.

profile | search | pm

Posted: Fri Mar 03, 2006 9:35 pm | quote

stapel
Elite Member

Joined: 04 Feb 2004
Posts: 7768

Do you mean the following?

$$\cdots\cdots \mathop{limit}_{x \to 3} \sqrt{7-x} - \frac{2}{1-\sqrt{4-x}}$$

If so, I would suggest rationalizing the denominator and then evaluating at x = 3.

If not, then please review the formatting articles in the "Forum Help" pull-down menu at the very top of the page, and reply with clarification.

Thank you.

Eliz.

profile | search | pm

Posted: Fri Mar 03, 2006 9:40 pm | quote

bittersweet
Junior Member

Joined: 12 Feb 2006
Posts: 105

sry I dont see the "forum help" but the -2 is part of the numerator and yes you have the denominator correctly written

profile | search | pm

Posted: Fri Mar 03, 2006 9:51 pm

soroban
Elite Member

Joined: 28 Jan 2005
Posts: 2997
Location: Lexington, MA

Hello, bittersweet!

Quote:

$$\lim_{x \to 3} \frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}$$

We must rationalize the numerator <u>and</u> the denominator . . .

$$\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}} \cdot \frac{\sqrt{7-x}+2}{\sqrt{7-x}+2} \cdot \frac{1+\sqrt{4-x}}{1+\sqrt{4-x}} \;=\; \frac{[(7-x)-4](1+\sqrt{4-x})}{[1-(4-x)](\sqrt{7-x}+2)}$$

$$=\; \frac{(3-x)(1+\sqrt{4-x})}{(x-3)(\sqrt{7-x}+2)} \;=\; \frac{-\cancel{(x-3)}(1+\sqrt{4-x})}{\cancel{(x-3)}(\sqrt{7-x}+2)} \;=\; -\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}$$

Therefore: $\displaystyle \lim_{x \to 3}\left[-\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}\right] \;=\; -\left(\frac{1+\sqrt{1}}{\sqrt{4}+2}\right) \;=\; -\frac{2}{4} \;=\; -\frac{1}{2}$

profile search pm

Posted: Fri Mar 03, 2006 10:03 pm

stapel
Elite Member

Joined: 04 Feb 2004
Posts: 7768

bittersweet wrote:

sry I dont see the "forum help"

Scroll all the way up to the very top of the page. "Forum Help" is above the "FreeMathHelp" logo, right next to "Forum Index" and "Math Subjects". These are also all visible when you have your browser open to the "Post a reply" box.

Eliz.

profile search pm

Posted: Fri Mar 03, 2006 11:05 pm

galactus
Elite Member

Joined: 28 Sep 2005
Posts: 2779
Location: Chaneysville,PA

Because it is of indeterminate form, $\frac{0}{0}$, we could

also use L'Hopital's rule:

Take the derivative of the numerator:

$$\frac{d}{dx}[\sqrt{7-x}-2] = \frac{-1}{2\sqrt{7-x}}$$

derivative of denominator:

$$\frac{d}{dx}[1-\sqrt{4-x}] = \frac{1}{2\sqrt{4-x}}$$

$$\lim_{x \to 3} \frac{\frac{-1}{2\sqrt{7-x}}}{\frac{1}{2\sqrt{4-x}}}$$

$$= \lim_{x \to 3} \frac{-\sqrt{4-x}}{\sqrt{7-x}}$$

$$= \frac{-1}{2}$$

FIGURE 4 Extended online tutoring exchange from FreeMathHelp.com.

Notice the conversational quality of this exchange that is reflected in the latency of responses. The initial response to the student's query is a request for clarification from **SE** that comes after only 13 minutes. In this request, **SE** points the student to the location on the site where information on formatting mathematical text is found. The student replies to this request for clarification after 5 minutes, indicating that he/she is unable to locate the information on formatting and verbally describes the mathematical expression. A second tutor, **Soroban**, provides help 11 minutes later that emphasizes the necessary mathematical action (rationalizing both the numerator *and* the denominator). Although the student did not indicate that he/she had attempted a solution, **Soroban** is anticipating one plausible initial source of difficulty: while limit exercises involving expressions that require rationalization of the numerator *or* the denominator are fairly common, this is not the case with expressions that require *both*. Shortly afterwards, **SE** follows up on the student's response to finding formatting information with a detailed description of its location on the webpage. Finally, less than 2 hours after the initial query was posted, a third tutor, **Galactus**, provides an alternative approach (l'Hôpital's rule) together with the mathematical conditions (indeterminate form of type 0/0) that permit its application.

The entire discussion that included posing the question, clarifying the expression in question, advice on finding information for formatting mathematical text, and the presentation of two alternative approaches took place in 1 hour and 43 minutes. (For our sample, the average time until the first response was 1 hour and 36 minutes.) This back-and-forth activity is encouraged by the moderation system of the FreeMathHelp site: Responses are immediately made available to the participants, with subsequent moderation only occurring if needed. In contrast, the other SOH site, Cyberpapy.com, introduces a delay in the latency of responses by subjecting them to moderation *before* they are made available for viewing. This publishing delay may account for the smaller percentage of extended back-and-forth exchanges on this site. In addition, the initial response latency for Cyberpapy.com was much larger than that of FreeMathHelp: 17% of the queries did not generate an initial response authored (much less published) on the same day.

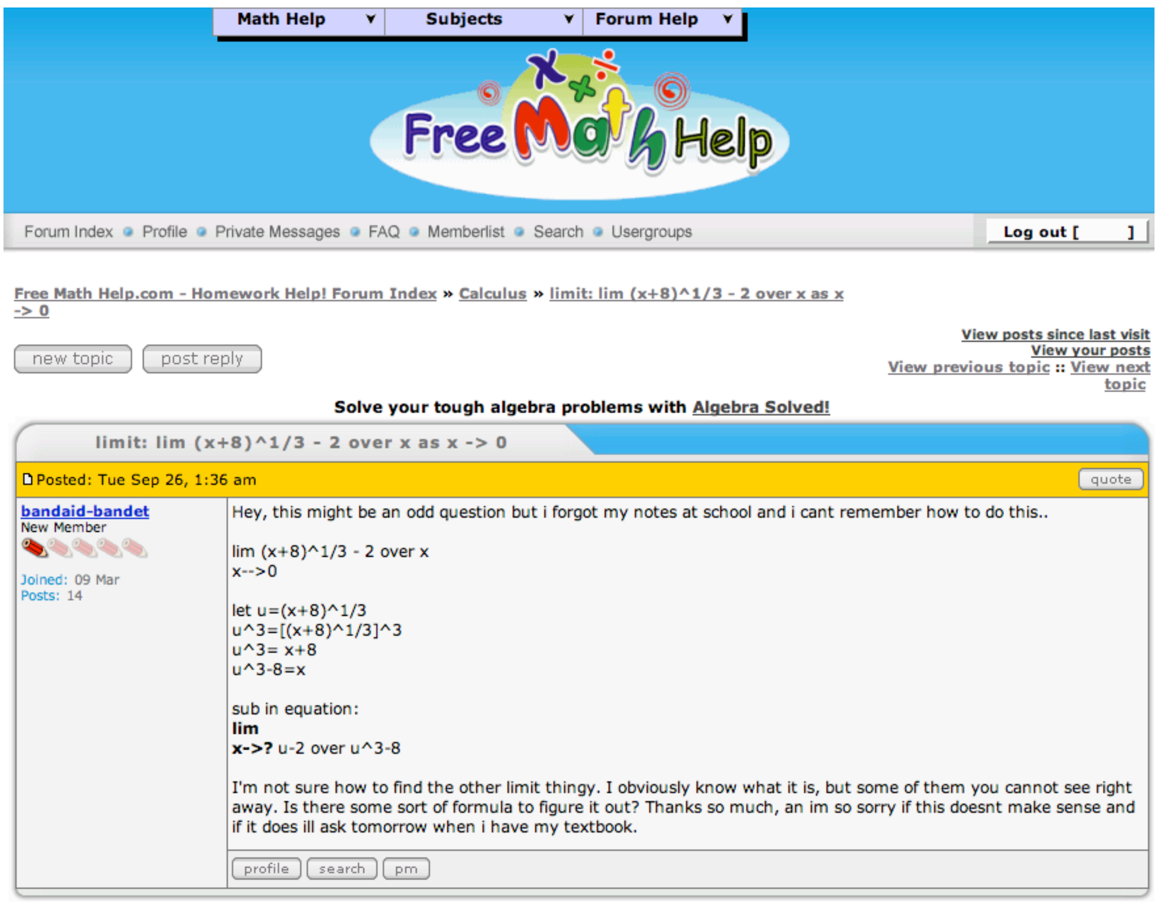
Not only does this delay impact the conversational quality of the communication, but it also fragments the exchanges between participants. If students do not have access to a response, then they may respond by re-posting the same query numerous times. 7% of the Cyberpapy queries in our sample (compared to 1% for FreeMathHelp) were of this type. When a different set of tutors responds to these identical postings, exclusive exchanges on the same query result. That is, the set of tutors responding to one posting may have no knowledge of the responses to the other posting, particularly if they are relying on e-mail confirmation for notification of contributions to a particular exchange.

These analyses of participation codes reveal that different structures encourage different participation patterns. In particular, an AOH structure promotes brief conversations between single student-tutor pairs, whereas a SOH structure promotes extended conversations between multiple participants. A delay imposed on the publication of responses in an SOH structure, however, dampens the effect.

Exchange quality

Obviously the pattern of participation is not the only important idea to investigate in the online help exchanges. Also important are discussions of mathematical and pedagogical quality. Responses might be brief but of high quality or extended (complex) but of trivial or superficial quality. Another possibility is that these two qualities correspond, so that extended exchanges tend to be more sophisticated with regard to the mathematical and pedagogical treatment of the query. To establish a complete mapping between complexity and quality is beyond the scope of this report, but, to give a flavor of the issue of quality, we explored one site.

Because the FreeMathHelp site contains the most interactive exchanges and is conducted in our native language, we chose to focus our analysis of the quality of exchanges on these discussions. The following exchange (Figure 6) is an example of low quality that is devoid of sophisticated mathematical and pedagogical moves. It also received the lowest possible complexity score of 3:



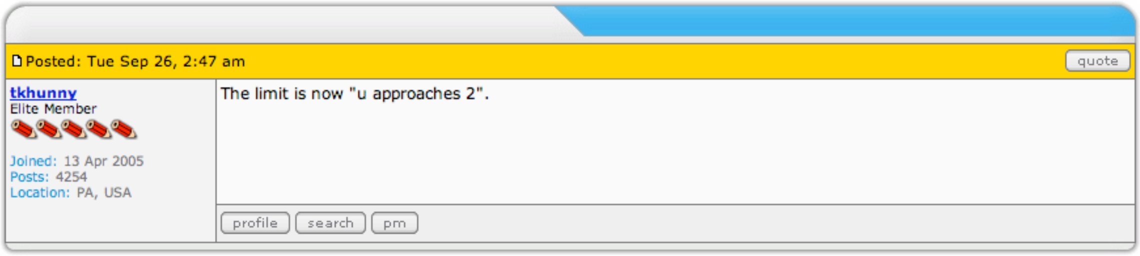


FIGURE 6 Low quality online tutoring exchange from FreeMathHelp.com.

The problem involves a change of variables and the student, **Bandaid-bandet**, has transformed the expression but expresses uncertainty about how to find the corresponding point of approach. A tutor, **Tkhunny**, responds with the correct numerical answer for the problem in question but does not explain how or why it is accomplished. In short, there is no explanation proffered although the student has explicitly requested one: "Is there some sort of formula to figure it out?" It is clear from the initial query that **Bandaid-bandet** does not understand the critical connection between transforming the point of approach and transforming the expression, and there is no evidence that this exchange has been instructional.

At the other extreme, there were exchanges such as the following (Figure 7) that were exemplary and reflected mathematical and pedagogical depth and sophistication. It also was scored as fairly[67] complex, with a 10:

[67] A complexity score of 10 is very difficult to realize with only two participants since it requires 7 total contributions (assuming participants alternate turns). The complexity score would have been higher had other tutors participated, but the exchange involved a line of reasoning that the tutor was pursuing with the student and was therefore well-suited to just two participants.

Math Help | Subjects | Forum Help

Free Math Help

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Free Math Help.com - Homework Help! Forum Index » Calculus » limit to infinity and cos

new topic | post reply

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limit to infinity and cos

Posted: Wed Feb 28, 1:31 am | quote

BW52
New Member

Joined: 20 May 2005
Posts: 30

I'm having trouble with one of my questions. I need to find the limit(or find that it does not exist). The question is:
lim(t-> infinity) cos(t+5t^(-2))

Plugging in infinity gets me to cos(infinity), which doesn't really help me much, so I've been trying to get the equation to a format that I can use with little success. I don't think I can use the squeeze theorem, either(but please correct me if I'm wrong).

so far:
f(t)= cos(t+5(1/(t^2)))
=cos(t+(5/(t^2)))
But I can't get anything useful beyond that, as if I try to combine the t and 5/(t^2) and divide by the greatest power of t in the denominator, it leads me full circle.

Any help would be appreciated.

profile | search | pm

Posted: Wed Feb 28, 3:48 am | quote

skeeter
Senior Member

Joined: 16 Dec 2005
Posts: 1648
Location: Fort Worth, TX

o.k. answer a more simple problem ...

does the following limit exist ?

$$\lim_{x \to \infty} \cos(x)$$

why or why not?

profile | search | pm

Posted: Wed Feb 28, 4:24 am | quote

BW52
New Member

Joined: 20 May 2005
Posts: 30

No, since cos graphs go up and down between -1 and 1 throughout the graph. It won't be closer to either value or to a value in between at infinity.

And replacing x with t+5t^-2 would get us the same answer to the previous question, correct? (although that would do strange things to the x->infinity , wouldn't it?)

profile | search | pm

Posted: Wed Feb 28, 2:05 pm | quote

skeeter
Senior Member

Joined: 16 Dec 2005
Posts: 1648
Location: Fort Worth, TX

bingo.

however, no strange behavior ... $\cos(t + 5/t^2)$ would behave pretty much the same as cos(x) as both x and t -> infinity.
why?

profile | search | pm

Posted: Thu Mar 01, 1:29 am quote

BW52
New Member

Joined: 20 May 2005
Posts: 30

I figured stuff out this morning. It doesn't matter what's inside the cos, because if there's nothing outside, then you know it will just keep going up and down forever between the same numbers.

I was just thinking that if you substituted t+5/t^2 for the x, then x->infinity would become t+5/t^2 ->infinity. Which, of course, would mess things up quite royally, except that that doesn't happen.

Thank you for the assistance!

Last edited by BW52 on Thu Mar 01, 1:58 am; edited 1 time in total

profile search pm

Posted: Thu Mar 01, 1:41 am quote

skeeter
Senior Member

Joined: 16 Dec 2005
Posts: 1648
Location: Fort Worth, TX

Quote:
> It doesn't matter what's inside the cos ...

careful ... what is the value of this limit?

$$\lim_{t \to \infty} \cos\left(\frac{\pi t}{2t+1}\right)$$

profile search pm

Posted: Thu Mar 01, 2:22 am quote

BW52
New Member

Joined: 20 May 2005
Posts: 30

Inside the cos can be simplified. It came out as cos pi/(2+(1/t))
Running that by the limit comes out as cos pi/2 = 0

It's because of dividing by the 2t+1, isn't it?

profile search pm

FIGURE 7 Exemplary online tutoring exchange from FreeMathHelp.com.

The student, **Bw52**, posts a request involving the computation of the limit approaching infinity of a composition of two functions, $\cos(t)$ and $t+5t^{-2}$. **Bw52** indicates that he/she has unsuccessfully attempted to transform the inner function, $t+5t^{-2}$, as an initial approach to solving the problem. A tutor, **Skeeter**, responds by posing a simpler problem for consideration: $\lim_{x\to\infty}\cos(x)$. The simplification of a problem is a key mathematical move as identified by Pólya

(1945) and also functions as a pedagogical move (Leinhardt & Schwarz, 1997) that switches the focus of the student's attention (from the behavior of the inner function to that of the outer function). **Bw52** reasons through this simpler problem and connects it back to the original expression: "And replacing x with t+5^-2 would get us the same answer to the previous question, correct?" However, the phrasing of this conclusion as a question reflects uncertainty, as does the accompanying parenthetical remark, "(although that would do strange things to the x-> infinity, wouldn't it?)" **Skeeter** responds by affirming **Bw52**'s conclusion ("bingo.") and addresses **Bw52**'s concern that the original expression may behave differently than the simpler one: "however, no strange behavior … cos(t + 5/t2) would behave pretty much the same as cos(x)." **Skeeter** concludes this remark by asking **Bw52** to explain why this statement is valid, a conversational move inviting the student to take another turn in the exchange and a pedagogical move supporting self-explanation. **Bw52** accepts this invitation and produces an explanation that references the bounded property of the cosine function: "It doesn't matter what's inside the cos, because if there's nothing outside, then you know it will just keep going up and down forever between the same numbers." This explanation, however, shows that **Bw52** has over-generalized the conclusion that can be drawn in this instance. The feature of the problem that was preserved in the reduction was that the inner function must approach infinity in both cases. Although **Bw52** indicates that he/she is now satisfied with the exchange and feels that the problem is resolved (ending this posting with an expression of appreciation for the assistance provided), **Skeeter** reopens the exchange with a warning ("careful…") that is supported by the framing of a counterexample ("what is the value of this limit? $\lim_{t\to\infty}\cos\left(\frac{\pi t}{2t+1}\right)$ "). The development of a counterexample is another key mathematical move (Rissland, 1989), and one that functions pedagogically as Socratic dialogue. Through answering this question, the student is confronted with a logical fallacy in his/her reasoning. The final posting in this exchange indicates that these mathematical and pedagogical moves were productive. **Bw52** reasons through the counterexample, produces a numerical answer that is not supported by the previous (overly general) claim (0 versus 'does not exist'), and pinpoints the difference between this case and the prior expression ("It's because of dividing by the 2t+1, isn't it?"). The evident productivity of the exchange may account for the fact that, although **Bw52** once again hedges, **Skeeter** does not reenter the exchange. Given the attention and response to incorrect conclusions in the discussion, the implicit message to **Bw52** is that he/she has now arrived at a correct conclusion.

These two examples, collected from the same site, illustrate the difference in quality that characterizes open online tutoring exchanges. Exchanges can be sparse fragments of mathematical information (as in the first example) or elaborated complete mathematical discussions in which sophisticated pedagogical elements are present (as in the second example). There also seems to be a positive relationship between our measures of complexity (based on number of participants and number of turns in the exchange) and quality (based on the extent of mathematical and pedagogical sophistication). Exchanges that involve few participants and contributions are often trivial communications of mathematical information that are devoid of complex pedagogical moves. On the other hand, exchanges that involve multiple participants and contributions tend to be imbued with mathematical issues and manifest intricate pedagogical moves.

Community

In addition to looking at the complexity and quality of online tutoring, we have also noted that the sites exhibit several features that are characteristic of community. That is, individuals with no connection or affiliation to one another outside of interacting in these forums have joined together for the purpose of receiving and providing mathematical tutoring support. As discussed previously, the group of tutors constitutes the core of the community although students participate in ways consistent with community membership. In order to provide a sense of how these online sites function as communities, we discuss the manifestation of each characteristic in turn:

1. Participants share common explicit and implicit goals.

All three of the online tutoring sites exist for the explicit purpose of providing students with accurate mathematical help. In addition, MathNerds.com (AOH) makes response timing an explicit goal by implementing a system in which there is a specified time period for the initial response to a query. Providing responses in a timely manner was also a goal of the SOH sites but as an implicit understanding amongst participants. This phenomenon was particularly evident in the FreeMathHelp.com site as tutors often exchanged light-hearted banter with one another about the speed of response. For example, when two tutors responded to a query virtually simultaneously (so that the postings were published just minutes apart), the second tutor edited his/her posting and appended the following: "You ornery **Soroban**, you beat me. Oh well, my approach is slightly different."

Another goal of the online help sites is to encourage students to communicate mathematics clearly. Tutors in all three sites frequently commented on the ambiguity and lack of clarity in the framing of the query. The neglect of parentheses, in particular, was a frequent culprit. Students often wrote mathematical expressions in a horizontal orientation and did not use parentheses to indicate the grouping of terms. Although it was evident that tutors could generally infer what the intended query was, they still chided students. In an exchange from Cyberpapy.com (Figure 3), the student, **Alice**, seeks help on showing that the limit as x approaches 2 of the expression, $x^2 - x - 2/x - 2$ is 3. As written, this expression would be interpreted as $x^2 - x - \frac{2}{x} - 2$. However, there are several clues that the intended expression is actually $(x^2 - x - 2)/(x - 2)$: the answer that **Alice** provided (namely, 3) corresponds to the limit of this expression, **Alice** expressed her inability to write the function in a different form (a move that would not otherwise be necessary), and, finally, this interpretation places the exercise in a traditional class of limit problems (functions with a removable discontinuity). The first two tutors that respond, **Dd** and **Katy**, both admonish **Alice** for the lack of clarity: "Il y a une erreur dans l'énonce" and "Attention à l'écriture !..." Three more tutors (**Mimi**, **VI1378**, and **Chamonix**) respond with advice based on a non-literal interpretation of the expression (that is, the intended versus the actual).

In face-to-face encounters, chiding or admonishing is considered a face-threatening act and, as such, a violation of universal rules of politeness (Brown & Levinson, 1987). However, these maxims do not necessarily extend to online interactions, where face-threatening acts are tempered by physical distance, written versus spoken communication, and anonymity. This is an unique property of online tutoring since adhering to conversational maxims and politeness strategies (such as refraining from critique) can impede pedagogical goals (such as perspicuity) (Person, Kreuz, Zwaan, & Graesser, 1995). The outcome of the above tutoring exchange illustrates how online interactions can succeed in mitigating this tension. The student, **Alice**, reposts the query the following day (perhaps after receiving only the first response due to the publishing delay) with an accurate formulation of the query that includes appropriate parentheses (Figure 8). Thus, the violation of politeness principles contributed to the productiveness of the exchange and the community goal promoting clear and accurate mathematical formulations:

FIGURE 8 Cyberpapy.com exchange following violation of politeness principles.

2. *Participants have an accessible physical or virtual location in which they meet.* Online help sites are virtual locations that are hosted on a server and participants have access to

these sites through a registration system. When members participate, their contributions are indexed by date and/or time, in some ways analogous to the societal practice of leaving a calling card. FreeMathHelp.com has an added feature that provides logged on participants a list of other participants who are currently visiting the site. This opportunity for effectively chatting with other members who are currently engaged in the same activity increases the sense of community in this site.

The costs of acquiring and maintaining an online help site are low, especially compared to face-to-face tutoring communities. Aside from the administrative costs (associated with performing upgrades, repairing technical glitches, and moderating), there are few expenses. The sites depend on benevolence (such as Foundation Boulanger) or advertising[68] to supply their financial needs and rely on search engines and reputation to broadcast their presence. The amount of participation in the online sites we investigated is evidence that these locations are very accessible to the student population ("FreeMathHelp.com served 1,891,472 pages to 616,839 visitors last year and another million pages to search engine robots and the like." [T. Wilcox, personal communication, June 9, 2007].

3. *Participants identify themselves as members of the community.*

In addition to sharing common goals and meeting together at a designated location, participants in the online help sites treat fellow members as colleagues rather than strangers or mere associates. For example, members reference one another by name as when **Chamonix** (Figure 3) elaborated on the posting of another tutor, **Katy**: "**Katy** a raison, mais lorsqu'on parle de limite on ne se place pas en x=2, seulement très près." The sense of

[68] We did not include sites where the "support" is offensive or intrusive.

community is also evident when tutors reference one another in ways that demonstrate an awareness of fellow members as a resource. For instance, when a FreeMathHelp tutor provided a solution that relied on l'Hôpital's rule, he qualified his/her posting with the following appeal to three fellow tutors by name, extended to other participating tutors: "I don't like to use L'Hopital[69] unless I have to. I think I'll use it on this one. Perhaps **pka**, **Soroban**, **Skeeter**, or someone will be along with a non-L'Hopital method. It's even precarious with L'Hopital."

As well as speaking *about* one another, participants in online help sites speak *to* one another, addressing each other by "name." In the following example (Figure 9) from FreeMathHelp.com, a tutor, **Galactus**, compliments another tutor, **Pka***,* on an alternative approach to the problem that does not rely on l'Hôpital's rule:

[69] In English, it is difficult to type accents and they are often neglected. We quote contributions as they appear in the exchange.

limit (e^x + e^(2x))^(2/x) as x tends to infinity

Posted: Fri Jun 09, 5:15 pm   quote

turkey
New Member
Joined: 09 Jun
Posts: 3

lim (e^x + e^(2x)) ^(2/x)
x->infinity

profile search pm

Posted: Fri Jun 09, 5:46 pm   quote

galactus
Elite Member
Joined: 28 Sep 2005
Posts: 2896
Location: Chaneysville,PA

$$\lim_{x \to \infty}(e^x+e^{2x})^{\frac{2}{x}}$$

Rewrite:

$$\lim_{x \to \infty} e^{\frac{2}{x}ln(e^x+e^{2x})}$$

$$e^{\lim_{x \to \infty}(\frac{2}{x}ln(e^x+e^{2x}))}$$

$$e^{2\lim_{x \to \infty}\frac{ln(e^x+e^{2x})}{x}}$$

Now, try L'Hopital's rule:

$$ln(e^x+e^{2x})dx = 2-\frac{1}{e^x+1}$$

$$e^{2\lim_{x \to \infty}\frac{2-\frac{1}{e^x+1}}{1}}$$

As can be seen, the limit of $2-\frac{1}{e^x+1} \to 2$

So, we have:

$$e^{2(2)} = e^4$$

profile search pm email

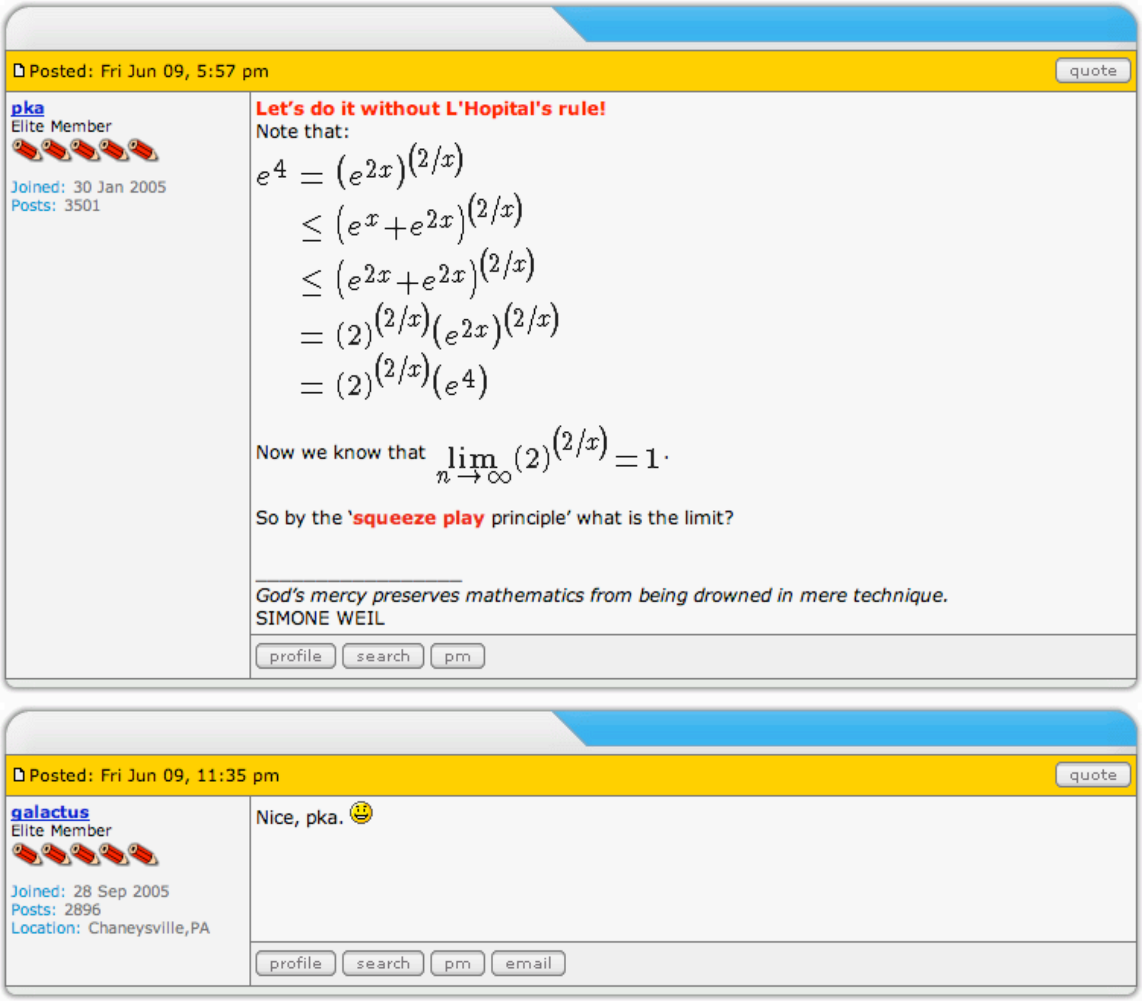

FIGURE 9 Tutors addressing one another by name in FreeMathHelp.com.

This exchange illustrates how tutors attend to the contributions of other tutors in open online help forums and demonstrates a "side-benefit" of this venue. In addition to supporting students (the intended population), online tutoring sites can also be instructive for tutors. Through **Pka**'s contribution, **Galactus** learns a novel way of approaching the problem mathematically. The "smiley face" emoticon that is part of **Galactus**'s compliment is indicative of the enthusiasm for novel approaches and perspectives in these online help sites.

What is more, this instructive potential for tutors extends beyond learning mathematics to becoming cognizant of pedagogical issues. When a FreeMathHelp tutor used the expression $\tan^{-1}(\infty)$ in reply to a student, a fellow tutor responded with: "PLEASE, PLEASE, PLEASE NEVER write $\tan^{-1}(\infty)=\frac{\pi}{2}$. ∞ is NOT A NUMBER. $\lim_{x\to-\infty}\arctan(x)=\frac{-\pi}{2}$." This emphatic plea for precision while tutoring mathematics concerns a common student misunderstanding, namely the treatment of the infinity concept as a number. Although it is not uncommon for mathematicians to refer casually to the limit of a function as "infinity" (as though infinity were a 'point' of approach), this is presumably accompanied by an implicit formal understanding of the underlying meaning. Students, on the other hand, are apt to over-generalize based on such linguistic expressions and, as a consequence, treat infinity as a number that is subject to the laws of arithmetic, concluding, for instance, that $\infty-\infty=0$ or $\infty/\infty=1$. The response to this plea was an apology from the offending tutor ("Sorry. ") and demonstrated the way pedagogical critique from fellow members in these sites is generally received.

Although there is much less evidence of student-student and student-tutor familiarity, there is unquestionably a sense in which students treat the online help sites as a community. They refer in plural to tutors when acknowledging them ("Thanks guys. You guys are truely helpful" and also address them by name ("Merci beaucoup a tous les deux…Malheureusement non, je n'ai pas vu cette méthode pour utiliser le théorème des gendarmes **kris**, ce qui me posait problème..Merci **diabolo** car je n'avais pas pensé a remplacer ln(1+X) par X tout court…Bonne fêtes de fin d'année !"). In general, students seem appreciative of these open online communities that provide them with free and timely quality mathematical help.

Finally, there is a sense in which each online site is itself a member of a larger community of help sites. As noted earlier, several tutors participate using the same "handle" in multiple online help sites. Although individuals may take on different roles in different communities (for example, **SE** acts as a moderator in one site but exclusively as a tutor in another), the cross-participation functions as a common thread through the larger community. The communities also refer to one another and inform students of the presence and location of other help sites. FreeMathHelp.com includes links to other help sites (including MathNerds.com).

4. *Participants assume responsibility for participation.*

We have discovered that participants, both tutors and students, in these online help sites assume responsibility for participation in numerous ways. Tutors collaborate with one another to provide quality and timely mathematical help, and students collaborate in the problem-solving activity. The collaboration between tutors takes on several forms that manifest themselves within a single tutoring exchange (sharing roles, taking turns and introducing alternative approaches) as well as across tutoring exchanges (sharing the load and distributing expertise). Students' collaborative efforts include the contribution of mathematical problem-solving steps and the questioning of others. Although by no means an exhaustive list, the following are examples of ways in which responsibility is assumed in the online help sites:

i. *Roles*. In addition to demonstrating how an SOH structure fosters extended discussions between multiple participants, the exchange in Figure 4 demonstrates how tutors may collaborate by taking on different roles in a single exchange. Here, each of the three tutors plays a different role in the tutoring activity: **SE** presses for clarity in the formulation of the query ("Do you mean the following? […] If not, then please review the formatting articles in the `Forum Help' pull-down menu at the very top of the page, and reply with clarification."), **Soroban** provides help in the form of a worked solution ("We must rationalize the numerator <u>and</u> the denominator . . .") , and **Galactus** provides an alternative approach for solving the problem ("Because it is of indeterminate form, $\frac{0}{0}$, we could also use L'Hopital's rule"). The result of this spontaneous and encompassing collaborative effort is that the student is tutored with respect to proper mathematical notation, mathematical procedures, as well as mathematical practices (specifically, mathematics as a domain in which multiple approaches lead to the same result).

ii. *Turns.* Another way in which tutors collaborate is by taking (conversational) turns in an exchange, for example by answering questions that are directed at another. In the following exchange (Figure 10) from Cyberpapy.com, a tutor, **La Flégère**, responds to a query that the student, **Flore/Gimoka**[70], poses to the first tutor, **Papi Gérard**:

[70] This is an example of a student participating in an online site using more than one handle.

CYBERPAPY

Mode d'emploi | Prévenez-moi! | Donner mon avis | Envoyer à un ami

Mathématiques

- ÉCONOMIE
- MATHS
- INFORMATIQUE
- LETTRES
- PHILOSOPHIE
- LANGUES
- HISTOIRE-GÉO
- SC. PHYSIQUES
- BIOLOGIE
- EXPOSÉS
- DIVERS

QUESTION

Proposé par : **flore** le : 15/02

fonction sln et exp

Bonjour,

on demande de justifier que:
lim x--->0 de (ln(1+x))/x =1

Cela fait 0/0 mais peut-on dire alors =1 ?

>8 réponse(s)

→ retour au forum | Répondre

RÉPONSE 1 — Proposé par : papi gérard - le : 15/02

Utiliser la dérivabilité de la fonction ln au point d'abscisse 1........

→ retour au forum | Répondre

RÉPONSE 2 — Proposé par : flore - le : 15/02

Bonjour,

Merci papy gérard mais je ne vois pas comment !

→ retour au forum | Répondre

RÉPONSE 3 — Proposé par : La Flégère - le : 15/02

bonjour flore,
oui.
Tu peux passer par la définition de la dérivée (ou mieux du nombre dérivé) ou par
l'approximation affine de ln(1+x) au voisinage de x=1 (cf cours).
Bon travail.

→ retour au forum | Répondre

OUTILS

→Posez une question

Recherche : OK

→Préparer un devoir

→Surf thématique

Trouver un bon bouquin

RÉPONSE 4 — **Proposé par : gimoka - le : 15/02**

bonjour,

mais papy gérard et la Flégère,
je vois bien qu'en posant y=1+x
on obtient la dérivée
(lny-ln1)/(y-1) pour y-->1
mais comment prouver que --->1

→ retour au forum — Répondre

RÉPONSE 5 — **Proposé par : flore - le : 16/02**

Bonjour,

Je pense avoir compris!
La dérivée de lny=1/y donc =1 pour y=1

Merci à tous

→ retour au forum — Répondre

RÉPONSE 6 — **Proposé par : La Flégère - le : 16/02**

Bonjour Flore et gimoka,
pour une fois je détaille:
a)Méthode du nombre dérivé.
On appelle nombre dérivé en x=a de ln(x) la limite quand x tend vers 0 de
(ln(a+x)-ln(a))/(x+a-a)
Si je fais a=1, j'ai bien
L=limite x->0 de ln(1+x)/x
Par ailleurs, chacun sait, après avoir appris le cours, que (ln(x)) ' =1/x et donc que
en x=1 ln(x)'=1/1=1=L
CQFD.
b)Méthode de l'approximation affine.
f(a+h)=f(a)+h*f'(a)+h*epsilon(h)
epsilon(h) est une fonction qui tend vers zéro avec h , petit écart de x autour de a.
Donc pour h très petit
f(a+h)~f(a)+h*f'(a) (1)
C'est l'approximation affine, puisqu'on assimile localement f(x) à un segment de droite.
Faisons f=ln, a=1 et h=x.
Quand x tend vers 0
ln(1+x)~x*1/1
Donc
ln(1+x)/x->1 CQFD
NB:
La première méthode n'est pas tj employable.
Je vous ai donc donné les deux.
Dans des cas cas retors, il faut développer plus "loin".
f(a+h)~f(a)+h*f'(a)+h^2/2*f"(a) (2)
On assimile f(x) à un arc de parabole.
Celà ne fait pas partie du cours. Certains profs le font découvrir en exercice, surtout pour
montrer l'évolution de cos(x) et sin(x) pour x petit.
Bon travail . Aimez les maths !

→ retour au forum — Répondre

RÉPONSE 7 — **Proposé par : flore - le : 16/02**

bonjour,

Merci, c'est plus net dans ma tete maintenant

@+

→ retour au forum — Répondre

RÉPONSE 8 — **Proposé par : Katy - le : 16/02**

La méthode la plus rapide, dans la mesure où on connaît le développement en série de la fonction
ln, est d'écrire:

ln(1+x) = x - (1/2) x² + (1/3)x^3 - 1/4 x^4+..

On divise par x.

La limite vaut donc 1.

Katy+++

→ retour au forum — Répondre

FIGURE 10 Example of turn-taking from Cyberpapy.com.

When **Flore/Gimoka** expresses uncertainty regarding the hint provided by **Papi Gérard** (“Merci papy gérard mais je ne vois pas comment !”), **La Flégère** responds with an affirmation (“oui.”) and a revoicing of the approach (“Tu peux passer par la definition de la derivée”), accompanied by an additional approach (“ou par l’approximation affine de ln(1+x) au voisinage de x=1”). In this exchange a sense of community is also evident from members referring to one another by name.

iii. *Load-sharing (with exchanges)*. Although the majority of the queries involve a single problem, it is not unusual for students to pose a set of problems simultaneously. In these instances, tutors may collaborate by dividing up the work, with different tutors taking on each problem. In the following exchange (Figure 11) from FreeMathHelp.com involving two problems, **Pka** answers the first question and **Soroban** answers the one remaining:

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Free Math Help

Forum Index • Profile • Private Messages • FAQ • Memberlist • Search • Usergroups   Log out [ ]

Free Math Help.com - Homework Help! Forum Index » Calculus » L'Hospital's Rule

new topic   post reply

View posts since last visit
View your posts
View previous topic :: View next topic

Solve your tough algebra problems with Algebra Solved!

L'Hospital's Rule

Posted: Sun May 07, 7:40 pm   quote

elcatracho
New Member

Joined: 09 Aug 2005
Posts: 25

Hello I had two questions about problems that use L'Hospital's **rule**. The instructions are find the indicated limit. Use L'Hospital's **rule** IF necesary. the first question is lim of x^x as x gets closer to 0+. So far I have done x times lnx. I dont know what to do from here. The next problem is lim of Sin(x)^(tan(x)) as x approaches pi/2-. Again, so far is did tan(x) times Sin(x) but i dont know what to do from there. When I did it on the calculator i got 1 for both answers but I'd really like to know how to do it by hand. Thank You:)

profile search pm

Posted: Sun May 07, 7:55 pm   quote

pka
Elite Member

Joined: 30 Jan 2005
Posts: 3502

$$y = x^x \;\Rightarrow\; \ln(y) = x\ln(x)$$

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(y)} = e^{\lim\limits_{x\to 0^+} \ln(y)} = 1$$

____
God's mercy preserves mathematics from being drowned in mere technique.
SIMONE WEIL

profile search pm

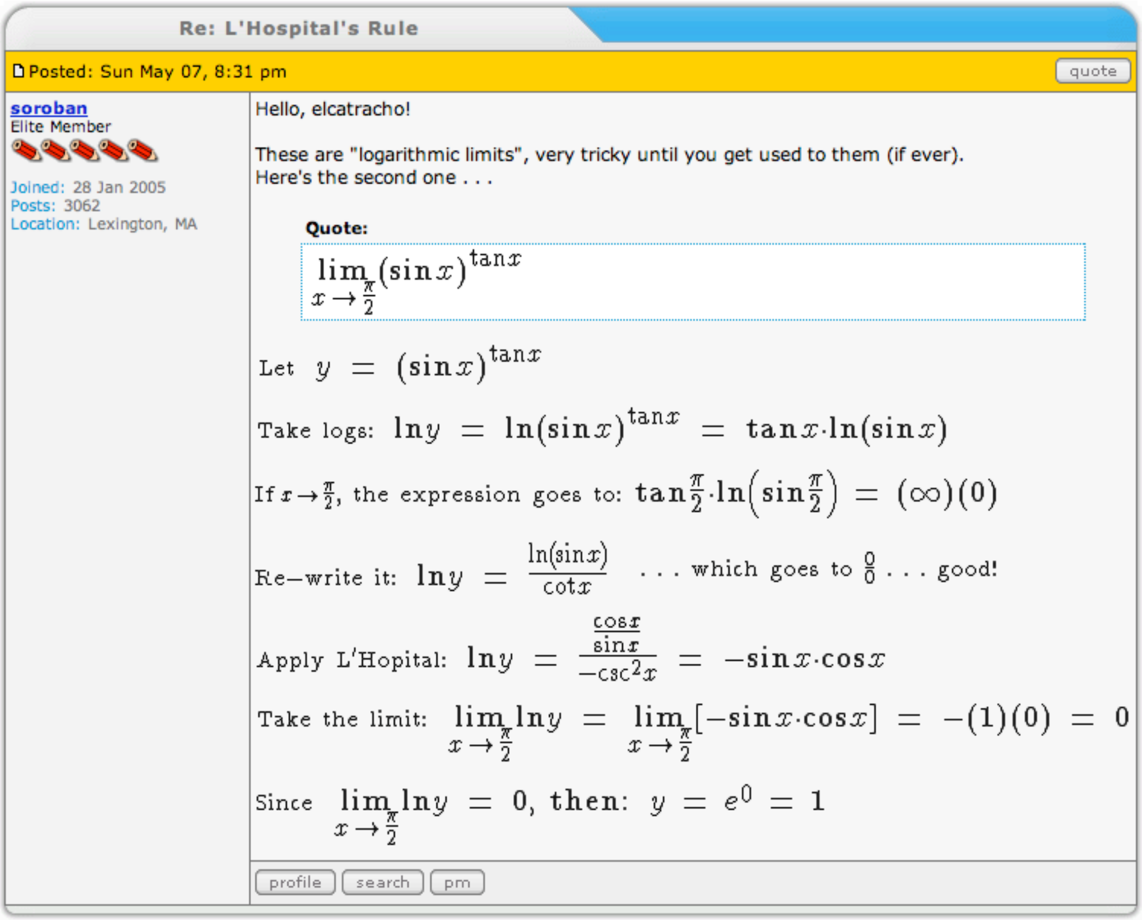

FIGURE 11 An example of load-sharing within an exchange from FreeMathHelp.com.

If the problems in the set are similar to one another (same solution method), then a collaborative effort to respond to all of the problems may constitute too much help; That is, it may discourage the student from using the solution to one as a model (worked example) for attempting the others. In the above exchange, the problems both involve indeterminate forms of exponential form and share a common solution method: taking the logarithm of the expression, writing it as an indeterminate quotient, applying l'Hôpital's Rule, and, finally translating the solution to correspond with the original expression. It is possible that

Soroban's contribution may have discouraged the student, **Elcatracho**, from attempting the second problem on his/her own. However, as **Soroban** notes, these logarithmic limit problems are "very tricky," it is clear from the student's initial posting that he/she does not understand a critical part of the procedure (expressing the form as an indeterminate *quotient*), and the solution of the first problem from **Pka** is sparse with regard to explanatory detail. Thus, in addition to the way tutors collaborate by distributing problems, this example illustrates how tutors attend to the contributions of others and use this information to assess whether the community has met the student's needs.

iv. *Load sharing (across exchanges).* These online help sites could not exist if tutors did not also work together to share the larger load - the constant stream of queries in need of quick response. All of the sites we investigated had a response percentage greater than 90%, a fact that is especially impressive since the tutors are volunteers. The AOH site relies on a large number of individuals who each commit to responding to a specified number of queries on a weekly basis. The SOH sites rely on a large number of individuals who voluntarily frequent the site and respond to queries on a regular basis. In both cases, the extensive amount of work required in order to provide a much-needed and valued service is distributed over individuals in a cooperative effort.

v. *Distributed expertise.* Individual tutors also contribute different types of expertise to the online help sites as a whole. The problems posed, even within a subject area (such as the calculus), range from elementary to advanced, from straightforward to subtle, and from procedural to conceptual. Individual tutors choose to respond to a set of queries based on interest and ability. If a tutor responds to a query but is not entirely satisfied with the result, then he/she may call on other members for support, as in the previous example from

FreeMathHelp.com: “I don’t like to use L’Hopital unless I have to…Perhaps **pka**, **Soroban**, **Skeeter**, or someone will be along with a non-L’Hopital method.” In this way, online help sites resemble a collaboration of experts, each contributing his/her subject knowledge.

vi. *Problem-solving activity.* It is important not to view the online help sites solely as a collaboration of tutors. The students are also a major part of the equation and contribute to the tutoring activity by submitting problems, attempting solutions and querying responses. As many of our examples have illustrated, student contributions reflect how online tutoring encourages students to reflect and engage in mathematical thinking. In an exchange from Cyberpapy.com (Figure 10), the student, **Flore/Gimoka**, poses a follow-up question to the tutors’ advice on one day (“mais **papy gérard** et **la Flégère**, je vois bien qu’en posant y=1+x on obtient la dérivée (lny-ln1)/(y-1) pour y-->1 mais comment prouver que --->1”) and then (without any intervening exchange) announces that she has figured it out the next (“Bonjour, Je pense avoir compris! La dérivée de lny=1/y donc =1 pour y=1 Merci à tous”). Evidently, the online exchange caused **Flore/Gimoka** to ruminate on the problem to the extent that eventual understanding was accompanied by excitement and a desire to share this understanding with others. In an investigation of problem solving activity for introductory calculus topics in another SOH site, 55% of the exchanges contained either initial or subsequent problem solving activity by the student (van de Sande, 2007).

5. *The defining features of the community can be renegotiated and altered by the members.*

All three online help sites provide a means for members to voice suggestions and comments. In MathNerds.com and Cyberpapy.com, members can contact the administrators using e-mail, although there is no indication whether individual or collective efforts have engendered change. FreeMathHelp.com, on the other hand, has a public forum devoted to “administration issues” where members post and respond to the structure, administration, and functioning of the site. On March 6, 2006, a member, **SE**, posted a suggestion to this forum containing 5 etiquette principles: Post to an appropriate category, preview or edit your posts for clarity, post the complete text of the exercise, show all of your work, and have patience. These principles were subsequently adopted and are now part of the “Read Before Posting” sticky that heads each forum and outlines the rules for site participation.

FreeMathHelp.com also allows members to author polls so that fellow members have the opportunity to vote on a given suggestion or issue. The administration can then act on these results or not. For example, one member questioned whether *we* (as a community) should enforce a policy regarding the names of threads and attached a poll for whether or not implementing such measures was a good or bad idea: “Just throwing an idea out here. Maybe we should implement some kind of policy that would require people to name their threads better. I'm tired of seeing “HELLPP!!!” or “calculus” or “Math suxxx!!11 one”. I don't know about anyone else, but they aren't very descriptive and remove from the apparent quality of the site itself. Consider something to the effect of first offense a warning to name their thread properly, second and concurrent offenses, continued deletions of topics until they get it.” Although only a small number of members responded to the poll, the majority was in favor of some such policy. The site administrator, however, vetoed the result, stating that warnings would not be effective for infrequent visitors, he was against a policy of deleting threads, that tutors might ignore posts with annoying names, and that moderators could edit the names to make them more descriptive.

6. Ideas can be questioned, elaborated, challenged, and revised safely.

One of the most valued and sought-after features of a learning community is that all members (learners and instructors) feel free and comfortable to exchange ideas. In particular, mistakes, misunderstandings, and uncertainty need to be resolved in a manner that is constructive and non-threatening. The anonymity of the participants and the public nature of the online exchanges make this venue particularly well suited for this activity. In all three online tutoring sites, there was ample evidence that students do not refrain from publishing mathematical attempts, voicing confusion, and questioning the tutors. In the following example (Figure 12) from MathNerds.com, the student, **Richard**, posts an initial attempt at solving a limit involving trigonometric functions followed by two responses to the tutor's reply:

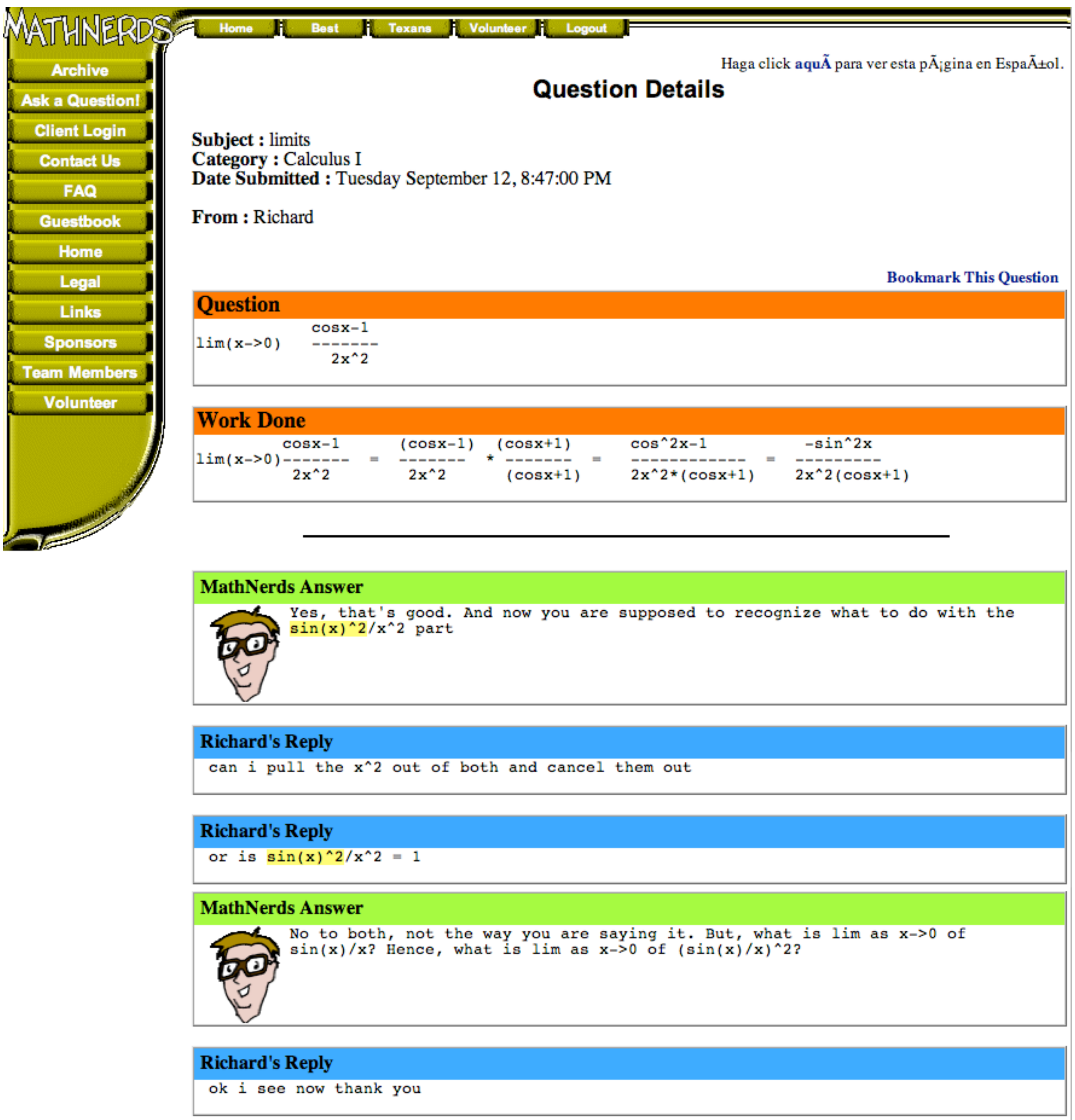

FIGURE 12 Exchange from MathNerds.com showing student comfort level.

Richard publishes two incorrect conclusions following the tutor's advice to focus on a particular part of the expression; One can neither "pull" out nor cancel the x^2 in $(\sin(x))^2$. The tutor gently refutes **Richard**'s conclusions ("No to both, not the way you are saying it.") and provides another hint that focuses attention more specifically on the relevant feature of the expression, namely that $\sin^2(x)/x^2 = (\sin(x)/x)^2$. This moves allows **Richard** to shift perspective and he responds with gratitude: "ok i see now thank you." In this exchange with its positive outcome, we see **Richard** presenting an incomplete solution and incorrect mathematical statements that reveal his poor understanding of the (pre-calculus) function concept, and the tutor responding in a constructive manner that is devoid of ridicule.

Tutors, as well as students, should feel comfortable exchanging ideas and addressing mistakes and misunderstandings. Although, ideally, tutors would not make errors, in reality it is not reasonable to assume that this will be the case, especially if the pace is rapid, the framing of the query is ambiguous, or participation is not restricted to experts. We have found evidence that the SOH sites, in particular, are wikipedia-like[71]. That is, the public nature of these sites generates a self-correcting feature. Mathematical errors are either replaced or addressed and do not generally remain long as the "last word" in a discussion. The following exchange (Figure 13) from FreeMathHelp.com demonstrates the wikipedia nature of the site in an exchange between a student and two tutors involving the limit of a quotient of trigonometric functions of indeterminate form:

[71] This feature was first noted and identified by Gaea Leinhardt. Wikipedia (`http://en.wikipedia.org`) is a multilingual, web-based, free content encyclopedia project in which entries are written and edited collaboratively by volunteer participants from across the world.

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Free Math Help.com - Homework Help! Forum Index » Calculus » Trig limit: lim [tan(3x)] / [3 tan(2x)] as x goes to zero

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new topic | post reply

Solve your tough algebra problems with Algebra Solved!

Trig limit: lim [tan(3x)] / [3 tan(2x)] as x goes to zero

Posted: Sat Feb 03, 7:49 am | quote

ku1005
New Member

Joined: 27 Oct 2006
Posts: 36

I have been working on the following limit for ages, it seems, and I just don't seem to be getting anywhere -- which ususally means I am missing something obvious. Any **pointers** would be greatly apprecited!

lim x-> 0 for tan(3x) / 3tan(2x)

I first tried the obvious route, substitution, getting a "value" of 0/0, which is no help. Then I converted the tangents into sines over cosines, and tried to work with their respective x-values. But I was unable to remove the sine from the denominator, meaning that I would always have something divided by zero. Hints?

Also, do you know of any sites with **pointers** for **solving trig** identities, like techniques?

With any luck, I'll have this figured out by the time somebody replies. In either case, thank you for looking at this, and thank you for any help you can offer.

profile | search | pm

Posted: Sat Feb 03, 11:20 am

OA
Full Member

Joined: 20 Feb 2005
Posts: 603

tan3x / 3 tan 2x lim x-->0 0/0 undefined by L'Hospitals rule take derivatives
1/3 [tan3x / tan 2x]
{1/3} { 1/cos^23x / 1/cos^22x}
{1/3} { cos2x/cos3x}^2 lim x-->0
1/3 answer

please check for errors
Arthur

Posted: Sat Feb 03, 12:50 pm

skeeter
Senior Member

Joined: 16 Dec 2005
Posts: 1647
Location: Fort Worth, TX

$$\lim_{x\to 0}\frac{\tan(3x)}{3\tan(2x)} =$$

$$\lim_{x\to 0}\frac{x\tan(3x)}{3x\tan(2x)} =$$

$$\lim_{x\to 0}\frac{\tan(3x)}{3x}\cdot\frac{x}{\tan(2x)} =$$

$$\lim_{x\to 0}\frac{\tan(3x)}{3x}\cdot\lim_{x\to 0}\frac{x}{\tan(2x)} =$$

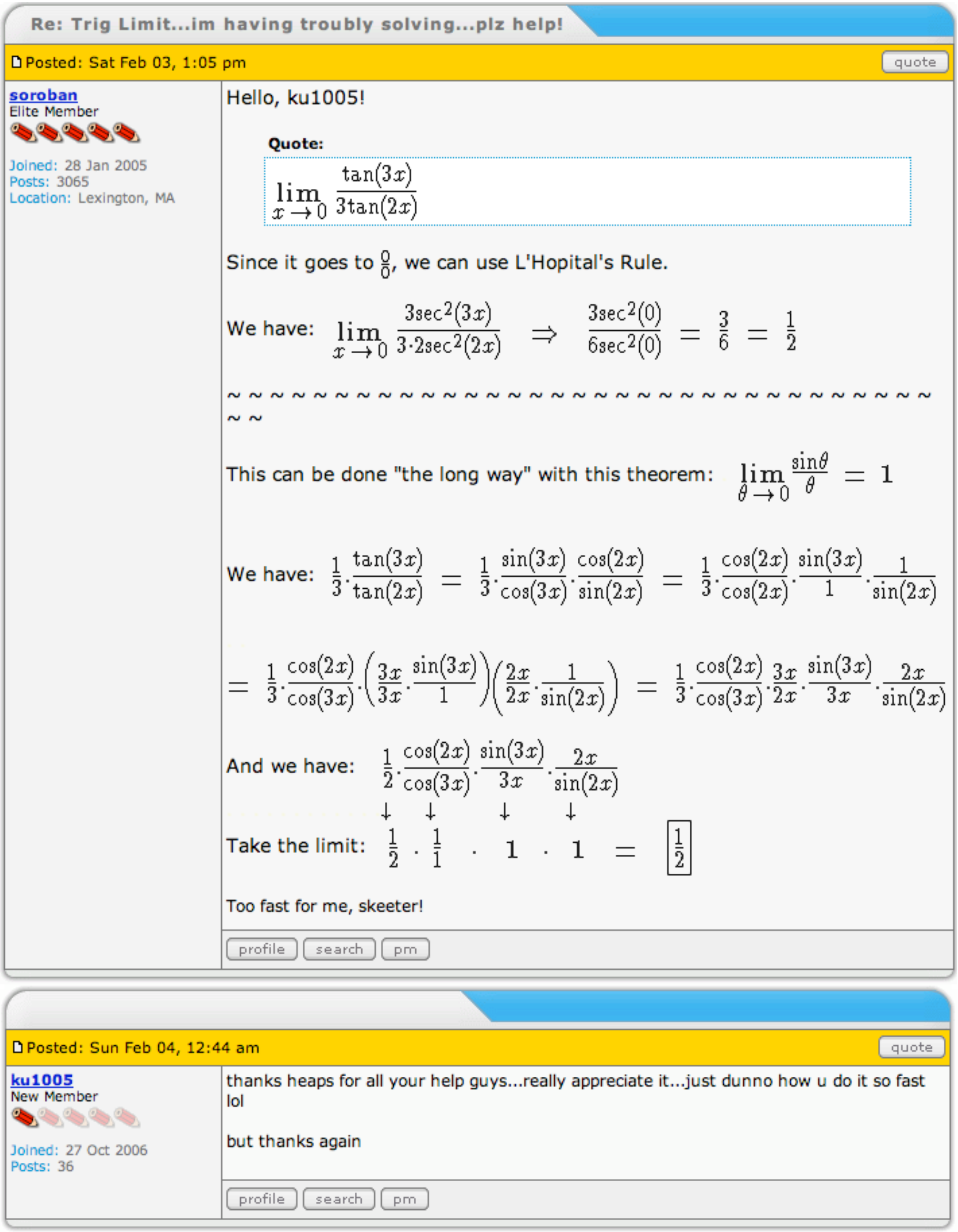
$$\lim_{x\to 0}\frac{\tan(3x)}{3x}\cdot\frac{1}{2}\lim_{x\to 0}\frac{2x}{\tan(2x)} =$$

$$\lim_{x\to 0}\frac{\sin(3x)}{3x\cos(3x)}\cdot\frac{1}{2}\lim_{x\to 0}\frac{2x\cos(2x)}{\sin(2x)} =$$

$$\lim_{x\to 0}\frac{\sin(3x)}{3x}\frac{1}{\cos(3x)}\cdot\frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin(2x)}\cos(2x) =$$

$$\lim_{3x\to 0}\frac{\sin(3x)}{3x}\frac{1}{\cos(3x)}\cdot\frac{1}{2}\lim_{2x\to 0}\frac{2x}{\sin(2x)}\cos(2x) =$$

$$1\cdot\frac{1}{2}\cdot 1 = \frac{1}{2}$$



Re: Trig Limit...im having troubly solving...plz help!

Posted: Sat Feb 03, 1:05 pm | quote

soroban
Elite Member
Joined: 28 Jan 2005
Posts: 3065
Location: Lexington, MA

Hello, ku1005!

Quote:

$$\lim_{x \to 0} \frac{\tan(3x)}{3\tan(2x)}$$

Since it goes to $\frac{0}{0}$, we can use L'Hopital's Rule.

We have: $\lim_{x \to 0} \frac{3\sec^2(3x)}{3 \cdot 2\sec^2(2x)} \;\Rightarrow\; \frac{3\sec^2(0)}{6\sec^2(0)} = \frac{3}{6} = \frac{1}{2}$

~ ~
~ ~

This can be done "the long way" with this theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

We have: $\frac{1}{3} \cdot \frac{\tan(3x)}{\tan(2x)} = \frac{1}{3} \cdot \frac{\sin(3x)}{\cos(3x)} \cdot \frac{\cos(2x)}{\sin(2x)} = \frac{1}{3} \cdot \frac{\cos(2x)}{\cos(2x)} \cdot \frac{\sin(3x)}{1} \cdot \frac{1}{\sin(2x)}$

$$= \frac{1}{3} \cdot \frac{\cos(2x)}{\cos(3x)} \cdot \left(\frac{3x}{3x} \cdot \frac{\sin(3x)}{1}\right)\left(\frac{2x}{2x} \cdot \frac{1}{\sin(2x)}\right) = \frac{1}{3} \cdot \frac{\cos(2x)}{\cos(3x)} \cdot \frac{3x}{2x} \cdot \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)}$$

And we have: $\frac{1}{2} \cdot \frac{\cos(2x)}{\cos(3x)} \cdot \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

Take the limit: $\frac{1}{2} \cdot \frac{1}{1} \cdot 1 \cdot 1 = \boxed{\frac{1}{2}}$

Too fast for me, skeeter!

profile | search | pm

Posted: Sun Feb 04, 12:44 am | quote

ku1005
New Member
Joined: 27 Oct 2006
Posts: 36

thanks heaps for all your help guys...really appreciate it...just dunno how u do it so fast lol

but thanks again

profile | search | pm

FIGURE 13 Exchange demonstrating wikipedia-like nature of an SOH site.

The first tutor to respond, **OA**, publishes an incorrect solution resulting from mistakes made in the implementation of l'Hôpital's Rule. **OA** neglects to apply the Chain Rule (necessary for the differentiation of function compositions) when differentiating the functions $\tan(3x)$ and $\tan(2x)$. This results in a final numeral answer that is off by a factor of 3/2. At the end of the posting, **OA** calls for other members to "please check for errors." The incorrect result is replaced 1½ hours later by a second tutor, **Skeeter**, using another approach. Instead of applying l'Hôpital's Rule, **Skeeter** presents a solution based on the `special' limit, $\lim_{x\to 0}\frac{\sin x}{x}=0$, that U.S. students typically learn prior to l'Hôpital's Rule. The wikipedia-like nature of the site is even more apparent through the actions of a third tutor, **Soroban**. **Soroban** replies almost concurrently with **Skeeter** ("Too fast for me, **skeeter**!") and posts the correct solution using two approaches: the one that **OA** incorrectly implemented and the alternative used by **Skeeter**. Mathematically, this move demonstrates that it was not the approach that was incorrect (as might be the case) but rather its implementation, and **Soroban** makes this important distinction explicit: "Since it [the limit expression] goes to $\frac{0}{0}$, we can use L'Hopital's Rule." Thus, the incorrect mathematical information published by **OA** is effectively erased through replacement with correct information that is promptly supplied by two fellow members of the community.

CONCLUSIONS

Today's youth belong to the "net generation" (Oblinger & Oblinger, 2005) and are accustomed to performing activities online that formerly required physical presence. These "digital natives" (Prensky, 2001) routinely use the Internet for a wide range of activities, from shopping, chatting, and playing to working, researching, and studying. Each of these activities is associated with an online venue where participants meet and transactions take place. For receiving help with homework problems, these meeting places have taken the form of free, open online help sites where students can access expertise at a click of the keyboard. Some of these help sites are staffed by volunteer tutors that have been vetted and are assigned to incoming requests (AOH), whereas others are staffed by volunteers who spontaneously visit and respond (SOH). Also, although the communications are asynchronous, some of the sites operate close to real time (publishing postings as they arrive), whereas other sites institute a delay by either the activity of matching tutor-student pairs or by screening responses.

Just as the structure of a classroom profoundly affects the activity of instruction and learning, so also the structure of an online site impacts the activity of tutoring. AOH sites promote brief exchanges (one or two turns) between single student-tutor pairs. A tutoring exchange in this context resembles a private consultation with an expert in the domain. A student poses a query and then has access to the ear of an expert, although the privacy normally associated with a consultation is lacking since the exchange is publicly available. SOH sites, on the other hand, encourage extended exchanges between multiple participants. A tutoring exchange in this context often resembles a collaboration of experts that may touch on both mathematical as well as pedagogical issues. A student poses a query and then has access to a community of experts, who jointly converse with the student and may also engage in discussions with one another.

In addition to transforming *when* tutoring is accomplished and *what* participation in this activity looks like, online help sites have also transformed *who* is involved and *how* those involved work together. Participants of online help sites are members of a community in the sense that they share common goals, meet at a specified location, identify with fellow members, assume responsibility for participation, negotiate features and practices, and are comfortable exchanging ideas. Although a given site does not need to possess a complete set of these features to qualify as a community, the presence of several of these strengthens the sense of a variety of individuals engaged in a collective effort. SOH sites, in particular, seem to exhibit a strong sense of a community. Members often refer to one another by name, collaborate mathematically, and critique or correct one another to address errors and mistakes. The result is that students, as part of the community, receive more than just mathematical help on the problem at hand (for instance, by witnessing mathematical discussions). Tutors, in turn, profit by being positioned jointly as learners (for instance, when other tutors introduce alternative approaches and ways of looking at a problem).

We hope that the research community can profit from this research that represents an alternative approach to the study of tutoring. In contrast to previous research efforts, we have applied an observational methodology to authentic occurrences of student-initiated online help-seeking. We have developed and applied measures for the complexity and quality of the exchanges and explored several ways that these online help sites resemble communities. This is only the beginning of the work needed to gain an understanding of this new and evolving form of tutoring, the impact that it has on students and tutors, and its potential for instruction and learning. Some of the exchanges on the calculus that we have observed show clearly how open online help sites are capable of taking students and tutors beyond the limit of the limit.

ACKNOWLEDGMENTS

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B.4 THE GOOD SAMARITAN EFFECT: A LENS FOR UNDERSTANDING PATTERNS OF PARTICIPATION[72]

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Abstract: We examine patterns of participation in an educational environment that exists solely for the purpose of providing help to those in distress. FreeMathHelp.com is a free, open, online homework help forum that is staffed by volunteers worldwide to help students to complete homework assignments in mathematics. We focus our attention on tutoring exchanges that concern related rates problems, a topic taught in introductory calculus that is often difficult for students. From social theory, the bystander effect has been used to explore online participation between members of a classroom. We propose a variant of the bystander effect in order to account for tutor participation patterns in online exchanges between anonymous participants. The Good Samaritan effect, named to capture the spirit of volunteers who come to the aid of strangers in distress, has four underlying mechanisms: self-awareness, social cues, blocking/inviting, and responsibility. The way each of these contributes to participation is discussed.

Heroism and helpfulness have long been a source of intrigue for sociologists, psychologists, and anthropologists. Indeed, Darwin's theory of evolution was not restricted to physical characteristics of individuals but extended to behavioral characteristics of a group. According to this view, human kindness could have evolved as a result of lethal warring between neighboring groups as unity and cohesiveness triumphed over division and discord. A century later, William Hamilton (1964) provided an explanation of how generosity can evolve and outlined its fostering conditions. More recently, as technology itself has evolved, evolutionary

[72] A version of this paper can be found in the *Proceedings of the Eighth International conference for the Learning Sciences* (Vol. 2, pp. 240-247). Utrecht, The Netherlands: International Society of the Learning Sciences, Inc.

biologists have begun to explore the possibility of a genetic underpinning for friendliness (Judson, 2007).

Without taking a stance on the "origin of kindness," we would like to draw attention to an environment in which kindness abounds. The purpose of this paper is to propose a lens for understanding patterns of participation in an environment that exists expressly for the purpose of providing help to those in distress – educational distress due to homework assignments. Students from across the world who face difficulties, reach impasses, or who simply wish to confirm their understanding of select problems now seek help in free, open, online homework forums. These forums are staffed by volunteer tutors who recognize that they are in a position to provide homework assistance to students and are willing to donate their time, energy, and knowledge to this cause. Access to such forums depends only on the presence of an Internet connection, which allows participation (both help-seeking and help-providing) from a large sector of the population. We wish to understand the mechanisms that contribute to the helpfulness proffered on these sites.

Because calculus, and mathematics in general, is universally known to cause high levels of student anxiety and success in such a course often depends on the completion of homework assignments, we chose to explore the issue of helpfulness in a free, open, online, calculus homework forum. We selected a challenging and intricate mathematical topic, namely related rates, as the context for investigating tutor participation patterns. Finally, in order to observe complex patterns of participation that might arise between multiple tutors, we chose to focus our attention on forums that allow any member to respond to queries (Spontaneous Online Help, or SOH), rather than forums that assign incoming queries to a select tutor (Assigned Online Help, or AOH).

The research reported here is part of a general research program to explore open, online homework help forums (van de Sande & Leinhardt, 2007a, 2007b). The questions investigated include: What are the effects of different site participation structures? Do these sites promote instrumental help-seeking? How do these sites constitute a learning community? How can we characterize the complexity and quality of the exchanges? In this paper, we propose a social psychological lens for examining participation patterns in open, online homework forums, namely the Good Samaritan effect. We draw on research that uses a well-established social psychological phenomenon, the bystander effect, as a way of characterizing student participation patterns in online discussions.

THE GOOD SAMARITAN EFFECT

The finding that individuals are less likely to offer assistance in an emergency when other witnesses are present has been termed 'the bystander effect' (cf. Latané & Darley, 1969; 1970). Although classroom events are generally not perceived as crises or emergencies, Hudson and Bruckman (2001) found that the bystander effect provides a useful lens for explaining differences in face-to-face and online patterns of student participation for discussions in foreign language instruction. The same four mechanisms that fuel the bystander effect in emergencies (self-awareness, social cues, blocking, and diffuse responsibility) contribute to the behavioral differences of students in the two instructional settings (see Table 1).

Our study of tutor participation patterns in open, online, homework forums has led us to posit a corresponding effect, that we term the Good Samaritan effect. The biblical account of the Good Samaritan, who overlooked cultural differences and provided aid to a complete stranger in desperate need, is legendary. While recognizing the obvious differences between the details of this account and the educational context we are exploring, we find the characterization of a volunteer helping those in distress appropriate. As Hudson and Bruckman (2001) emphasize with respect to the bystander effect, it is important not to reify the notion of the effect and attribute causality. Similarly, we do not argue that the helpful behavior we see in these online forums is *caused* by the Good Samaritan effect. We propose rather that the Good Samaritan effect may provide a useful way of making sense of tutor participation in a particular environment and has potential for informing the design of tutoring systems. Table 1 summarizes the four mechanisms that contribute to the Good Samaritan effect and contrasts these with the mechanisms that underlie the bystander effect.

Table 1. The Good Samaritan effect contrasted with the bystander effect.

	The bystander effect	The Good Samaritan effect
Self-awareness	Individuals do not participate because they do not want to appear foolish in front of others.	Individuals participate because they want to appear helpful in front of others.
Social cues	Inactivity of others is taken as a cue and discourages participation.	Activity of others is taken as a cue and encourages participation.
Blocking/Inviting	Action of one bystander blocks others from taking action for fear of worsening situation.	Action of others encourages individual to take action in hopes of improving situation.
Responsibility	Each individual feels only limited responsibility for negative consequences of inaction.	Each individual feels substantial responsibility for negative consequences of inaction.

METHODS

Vocabulary

There is a vocabulary associated with interaction in online environments that we have adopted for our discussion of online tutoring. A *post(ing)* is a contribution that is published on the site, either to initiate a discussion or in response to another's contribution. As in verbal discussions, participants generally take turns contributing to the conversation. The set of contributions pertaining to a single request for help constitute an *exchange* or *discussion*, sometimes referred to as a *topic* or *thread*.

Site

For our investigation of tutor participation patterns, we chose a representative SOH site. FreeMathHelp.com is an advertisement-supported mathematics help portal established in 2002 by Ted Wilcox, an enterprising high school junior. In addition to the discussion forum, the site includes lessons, games, a graphing utility, and worksheet pages. There are nine homework help forums, organized by subject area (such as algebra, differential equations, calculus). Forum members can contribute or respond to these postings and have access to user profiles that include self-volunteered information on occupation, residence, contact information, as well as amount of discussion board activity. Each member is characterized by total number of contributions to distinct threads: new (0-49), junior (50-249), full (250-999), senior (1000-2499), elite (more than 2500). There are several elite members who have contributed to more than 2500 threads, four of whom have contributed to more than 4000. Each forum has assigned moderators who may lock topics and move or delete postings. In addition, members can edit their own contributions after they have been posted: If this is done *after* the member has logged off of the forum, then a message is appended to the altered contribution: “Last edited by [member] on [date and time]; edited [number] times in total.” If editing takes place *while* the member is still logged on to the forum, then there is no official evidence of the modification although the general practice is to indicate that the contribution has been edited.

The prescribed etiquette for participation is located in a “sticky” that is the lead posting within each help forum. This covers administrative issues (*e.g.* posting to an appropriate category) and politeness (*e.g.* patience while waiting for response). In addition, there are three rules that specifically address the content and framing of posts: include problem context ("Post the complete text of the exercise"), show initial work ("Show all of your work [including intermediate steps that may contain errors]"), and attend to clarity ("Preview to edit your posts [to minimize errors]").

The computer window for constructing posts contains traditional icons for highlighting text (*e.g.* italics, boldface, underlining, and font size and color), inserting material (*e.g.* external links and images), and organizing text (*e.g.* forming lists). A large selection of graphic "emoticons" (faces) is available for expressing emotions and attitudes (such as gladness or perplexity). In addition, there are format capabilities more specific to mathematical discussions since it is tedious and often impossible to create mathematical symbols and expressions using keyboard characters. Using LaTeX, a document preparation system designed to typeset mathematical text, participants can use command strings and code to produce mathematical symbols (such as ∞) and vertical expressions (such as $\frac{dy}{dx}$). In order to encourage the use of this software, FreeMathHelp includes a tutorial for LaTeX, as well as a link to a free equation editor that generates the LaTeX code, which, although powerful, can be difficult for the novice. It is important to note (particularly with respect to our discussion of related rates problems) that there are no drawing tools available, so diagrams must be externally created and inserted as images or text-based (e.g. pieced together using ASCII characters).

Sample

In order not to interfere with existing participation patterns, we intentionally chose a purely observational, non-intrusive approach for the investigation. Because the tutoring exchanges in this site are open to public observation, we chose to cull existing information that reflects social interaction, rather than risk disrupting member participation.

As a context for investigating participation patterns, we selected the mathematical topic of related rates. Related rates problems represent a practical application in calculus instruction and are traditionally used to illustrate implicit differentiation and show the derivative as a rate of change. These problems are usually presented as word problems in which the rate of change for certain quantities is given and the rate of change of related quantities is sought. The following is an example of such a problem:

> Water is leaking out of an inverted conical tank at a rate of 500 cubic centimeters per minute at the same time that water is being pumped into the tank at a constant rate. The tank has height 10 meters and the diameter at the top is 4.5 meters. If the water level is rising at a rate of 26 centimeters per minute when the height of the water is 1.0 meters, find the rate at which water is being pumped into the tank in cubic centimeters per minute.

The problems provide a good context for studying participation patterns since they are challenging for students (and therefore the subject of frequent queries to the forum), require a multiple-step solution process (and are therefore more complex than the implementation of a simple algorithm), and can be solved using different perspectives (whether the general chain rule or the functional relationship is the focus of the model). In sum, these problems are sufficiently complex to generate discussion and present opportunities for tutor participation.

We made use of the search mechanism in FreeMathHelp.com to identify tutoring exchanges on related rates problems by searching for the keywords “related” or “rates” or “rate.” This search captured postings that involved related rates but did not contain the exact phrase “related rates,” as well as problems that involved rates (but not related rates per se). The 433 postings resulting from the search were then examined for relevancy to the related rates concept.

254 were classified as false hits (i.e. not pertaining to related rates) and were discarded. From the remaining exchanges, 176 were classified as discussions of solutions for related rates problems.

Population

FreeMathHelp.com features participants' profiles that include information on occupation, location, and interests. Whereas many student participants do not provide this information, the participating tutors in the calculus forum are self-reportedly students, educators, professionals, and retired mathematics professors. The most frequent tutor participants are from the U.S., although there are representatives from Canada and New Zealand as well.

Most participants of open, online tutoring forums select names or "handles" (such as **ihatecalc** or **Skeeter**) that do not disclose personal information (location, knowledge level, etc.), and we refer to such participants using these self-designated handles.

Although some tutors and students post more frequently, numerous tutors and students frequent MathHelpForum.com. Our sample contained exchanges between 116 different students, with responses from 31 different tutors.

RESULTS

Self-awareness

Self-awareness is an individual's conscious awareness of the judgments of others about that individual. That is, self-awareness is based on an individual's perception of the thoughts and attitudes of others. In the exchanges we observed, we saw evidence that tutors patterned their participation according to their perception of how helpful they might, in consequence, appear to others. This feature of the Good Samaritan effect was manifest when tutors vied to be "the first on the scene," offered reassurance to students, and responded to criticism.

Given the average latency of the first response to a query (only two hours and 16 minutes in this sample), the influence of self-awareness of tutors on participation is especially noteworthy. In order to be the first to respond, a tutor must be extremely attentive to incoming postings and efficient at preparing responses (even to lengthy, involved problems such as related rates). Yet being viewed as a quick tutor was clearly a determining factor in tutor participation. Thus, when a tutor discovered that another tutor had responded more quickly to a student's request for help, it was not unusual to find a quip, such as "You beat me, Soroban!," appended to the response. It is also a mark of self-awareness that, even in cases in which two responses were extremely similar, tutors chose to leave a record of their help and not remove their contribution.

Another indication that self-awareness shaped tutor participation came from tutors' reassurances to students. For example, when one student was involved in a lengthy back-and-forth exchange with a tutor, the student ended one posting apologizing: "Thanks for your help! Sorry to keep bothering you!" The tutor replied, "You're not bothering me. I wouldn't be here if I were bothered." Tutors made it clear that they were participating in order to be helpful and wanted students to be aware of this. Thus, tutors often encouraged students to continue a dialog by ending a posting with an invitation, such as "Write back if you need more help." One tutor, **Gene**, routinely included the message "I hope this helps. If you need more, come back with a post-reply" with his signature.

It is worth noting that tutors' sense of self-awareness extended beyond efforts aimed at receiving public recognitions of thanks for their helpfulness and involved the desire to be sincerely perceived as helpful. For example, when a tutor, **tkhunny**, and two other tutors responded to a student and only they were thanked by the student ("galactus and soroban- thank you very much on your thorough replies"), **tkhunnny** posted a reply to the exchange: "That really hurts." In response, the student apologized to **tkhunny** and made explicit mention of **tkhunny**'s contribution: "tkhunny- i'm very sorry. thank you for pointing out that the angle is fixed." The fact that this explicit expression of appreciation was clearly not **tkhunny**'s intent was evident in his/her reply to the student: "No worries. I'm just messing with your head." Thus, although tutors may wish to receive recognition for their helpfulness (and some discussion forums have instituted a packaged "thank you" message to encourage students to express appreciation), the sense of self-awareness stems from a larger desire to appear helpful.

Finally, there was evidence that tutors wanted other forum participants to view them as competent sources of help as providers of accurate mathematical information and clear pedagogical expositions. When an error made by a tutor was discovered (either by the author or another forum participant), the tutor who had erred often edited the posting to replace the incorrect information with a correct solution formulation. In many cases, the tutor not only edited their postings to make them mathematically correct but also publicly acknowledged their mistake: for example, after one student queried a tutor's solution, the tutor replied, "I changed my post. I had an error. I differentiated incorrectly." Thus, tutors appeared to want the helpfulness of their actions (providing accurate information as well as a willingness to amend contributions) to be part of the public record.

The following exchange shows how a tutor responded when criticized by a student for being unhelpful. Although such exchanges were not common, the interaction reveals how self-awareness influences the participation of both tutors and students in this environment:

Subject: please help Author: matt Posted: Mon Nov 14, 9:02 pm Status: Junior Member	the question is. A car traveling at 40 ft/s crosses a bridge over a canal 10s before a boat traveling at 20 ft/s passes under the bridge. the canal and the road are straight and at right angles to each other. At what rate are the car and boat seperating 10s after the boat passes under the bridge. THis is how I did it but it turned out worng.

	First I made a right triangle. Sinse the car was traveling at 40 ft/s and would travel for 30 sec I multiplied them to get 1200. I did the same for the boat and got 400. I then used the pathogream theorem to get the distance to be 1264.9. I then differentiated the theorem to get X dx/dt + Y dy/dt = Z dz/dt. I then plugged the numbers in and got dz/dt to equal 38.58 and it is the wrong answer.
Author: Eliz Posted: Mon Nov 14, 9:05 pm Status: Elite Member	matt wrote: "...Sinse the car was traveling at 40 ft/s and would travel for 30 sec...." Your general methodology looks good, but where are you getting the thirty seconds?
Author: matt Posted: Tue Nov 15, 12:08 am Status: Junior Member	because it had been traveling for 10 seconds longer but is that correct?
Author: Eliz Posted: Tue Nov 15, 12:16 am Status: Elite Member	matt wrote: "because it had been traveling for 10 seconds longer but is that correct?" Because *what* had been travelling longer than *which*? Is *what* correct? The thirty seconds? No.
Author: matt Posted: Tue Nov 15, 1:36 am Status: Junior Member	Im not trying to be rude but I do not appreciate the way that you answer the questions that I ask. Please, if you are not going to answer them dont make me sound stupid. I stated right in thr problem that the car had been traveling faster. You are always the one who responds to my questions but you never answer them. So please stop.
Author: Eliz Posted: Tue Nov 15, 1:56 am Status: Elite Member	I never argued with the stated rate of the car; I'm sorry I somehow gave you the impression that I thought you'd copied the exercise incorrectly, and would have no reason to think that you had. You had asked where you had gone wrong in working the exercise, I complimented your methodology and gave you a hint (how did you go from "10 + 10" to "30"?) to help you find the slight error. I apologize.
Author: matt Posted: Thu Nov 17, 1:06 am Status: Junior Member	no problem I found the answer to the problem, and the mistake. Sorry I became frustrated.

The tutor, **Eliz**, responded to **matt**'s proposed solution by complimenting his "general methodology" but, at the same time, hinting that there was an error: "…but where are you getting the thirty seconds?" After **matt** responded with a justification "because it had been traveling for 10 seconds longer", **Eliz** criticized its vagueness: according to the problem statement, the car had indeed been traveling for 10 seconds longer than the boat, but this information does not provide an account for **matt**'s use of 30 seconds in the solution. Instead of more closely examining the origin of the 30 seconds, **matt** responded with frustration directed at **Eliz**'s lack of helpfulness: "…I do not appreciate the way that you answer the questions that I ask." Following this public denouncement, **Eliz** continued helping **matt** discover his mistake with a more explicit hint ("how did you go from ‘10 + 10’ to ‘30’?”) and apologized, asserting that the intent of her contributions was “to help you find the slight error.” The end result of this exchange, following **matt**’s apology in turn (“Sorry I became frustrated”), was a picture of **Eliz**, trying earnestly to be viewed as helpful by other forum members (including **matt**) and **matt** trying not to appear to others as belligerent.

Social cues

In addition to the desire to appear helpful, tutors also responded to cues from others – tutors as well as students – that served as encouragements to participate. Often tutors cued other tutors “by name.” For example, a tutor, **galactus**, ended one of his contributions with “Soroban? Whatcha think?,” a direct invitation for a fellow tutor to participate in the exchange. Although students did not generally direct queries to a particular tutor (*e.g.* by name), they did provide cues that encouraged increased participation. When one tutor’s response was not productive, a student might request additional help. For example, when a tutor, **tkhunny**, provided a response that was not helpful for the student, the student replied: “still unclear. could you or someone else explain it in a different way.” In this case, two other forum tutors responded to this cue and provided the student with additional help. Participation, in both cases, was directly cued by requests from others.

There were also less direct cues that may have influenced tutor participation. For example, it was not uncommon for tutors to compliment fellow tutors (*e.g.* “Nice job!”), especially following the introduction of a solution that reflected an alternative perspective or showed particular insight. This positive feedback, together with the expressions of thankfulness from the students and the politeness with which errors and disagreements are handled, can be seen as cues that stimulate tutor participation.

One of the most distinctive features of this online forum as a learning environment was that disagreements and evaluations were invitations to further participate. As the following exchange illustrates, instead of treating expressions of doubt as a *negative* cue to withdraw participation, tutors responded to disagreement in a *positive* manner:

Subject: Another related rate Author: sigma Posted: Sat Feb 25, 1:45 am Status: Junior Member	Just had this one on an exam. The surface area of a cube is increasing at a rate 4 meters squared/sec. How much is the volume of the cube increasing when the length of the cube is 10? So a picture of a cube with a side labeld as 10. Then figuring out what rates I have and what I needed, there's what I did. Am I right? I don't think so. So a picture of a cube with a side labeld as 10. Then figuring out what rates I have and what I needed, there's what I did. Am I right? I don't think so. dsa/dt = 4 m^2/sec dv/dt = ??? da/dt = ??? ⇒ s a = 6 a ⇒ dsa/dt = 6 da/dt ⇒ 4 = 6 da/dt ⇒ da/dt = 2/3 ⇒ v = a^3 ⇒ dv/dt = 3 a^2 da/dt ⇒ when a = 10x10x10 = 1000, ⇒ dv/dt = 3(1000)^2 (2/3) ⇒ dv/dt = 1000^2 (2) ⇒ dv/dt = 20,000,000 !!! Dear god I messed up on this question! How do you figure out related rates involving cubes? One of the only types of questions we did not go over before the exam. The only real probelm is I couldn't remember what the surface area or volume of a cube was (the formulas). I semi guessed the surface area was equal to 6 times the total area because a cube has 6 sides and that the volume of a cube is equale to the area cubed but then again, I don't think thats right. Then trying to figure out how to relate surface area to volume I could not remember.
Author: galactus Posted: Sat Feb 25, 2:32 am Status: Elite Member	Let's give this a try sigma. [Original contents of this posting were deleted by galactus and replaced with the following text.] Sorry for the flub up. After I posted I thought

	something was amiss, but it was too late to log back on. [Last edited by galactus on Sat Feb 25 10:46 am; edited 1 time in total]
Author: sigma Posted: Sat Feb 25, 5:10 am Status: Junior Member	In other words, this has nothing to do with the area at all. I almost had the equations set up right for surface area and volume. Oh well, half to quarter marks I guess. Just for fun (or a redundant question) galactus, if you were marking this question and the question was worth 8 marks, how many marks would you give me for my work?
Author: daon Posted: Sat Feb 25, 6:39 am Status: Full Member	galactus, I'm curious how you've done this... Maybe if you explained your steps a bit more clearly? Especially the 3(S/6)dS/dt. Not sure where that came from... Thanks.
Author: Gene Posted: Sat Feb 26, 8:35 am Status: Elite Member	I'm not sure, LaTex looks so convincing! I would do it as $S=6x^2$ dS/dt = 12x dx/dt = 4 dx/dt = 1/(3x) V=x^3 dV/dx =3x^2 dV/dt = dV/dx*dx/dt = 3x^2*(1/3x) = x _________________ I hope this helps. If you need more, come back with a post-reply. Gene
Author: galactus Posted: Sat Feb 25, 2:51 pm Status: Elite Member	I'm back to fix my previous mistake. I made a bad substitution. Find dV/dt given dS/dt = 4; S = 6x^2; V = x^3; x=sqrt(S/6) V= (sqrt(S/6)^3 = (6S)^(2/3)/216 dV/dt = 6^(3/2)sqrt(S)/144 * dS/dt = sqrt(6S)/24 = x/4 * dS/dt x/4 * dS/dt = (10/4)(4) = 10 m^3/sec This is what I was getting at originally and messed up. I believe it agrees with your more efficient method, Gene.

In this exchange, **galactus** was the first tutor to respond to **sigma**'s query. However, an error in **galactus**'s response was subsequently caught by two other forum tutors, **daon** and **Gene**. It is notable that **daon**, who had a lower status in the forum (full member), appeared comfortable questioning a member of higher status, **galactus** (an elite member): "I'm curious how you've done this… Not sure where that came from…". Together, these two objections were a cue that led **galactus** to further action: s/he removed the incorrect solution, posted an apology in its place ("Sorry for the flub up."), and later returned to the exchange and posted a revised solution: "This is what I was getting at originally and messed up. I believe it agrees with your more efficient method, Gene."

Blocking

Instead of blocking action, the action of others in this online forum appears to stimulate actions aimed at providing further help using multiple voices as well as acting in concert. Following the contribution of one tutor in response to a student query, other forum tutors posted alternative methods or perspectives in order to help the student understand the problem. For example, after tutor, **pka**, had provided initial help to a student, **soroban** posted a reply that contained an alternative way of viewing the problem situation, prefacing his/her presentation with: "pka's suggestion is the best: the Law of Cosines. I would apply it differently." This activity was not only evident when it was obvious that previous help was ineffective (*e.g.* the student expressed continued confusion) but reflected the general tendency of participating to tutors to help students by producing alternative ways of viewing and solving problems.

Tutors also worked in concert, for example by following up on each other's actions. For example, after a student responded to help from **Eliz**, another tutor, **wjm11**, joined the exchange:

"Eliz has steered you quite close. Here's a bit more: …" The contributions of the first tutor in this case, stimulated a second tutor to participate and provide further assistance.

Diffuse responsibility

The response rate and initial response latency in the forum speak to the degree of responsibility felt by forum participants for the consequence of inactivity. In our sample, there was only one unanswered query, and the average latency for an initial response was 2 hours and 16 minutes. One gets the sense that participating tutors feel jointly responsible for providing students with quality help delivered in a timely fashion. This sense of duty is also evident from the apologies tutors offer for delays in response. In one memorable posting from a different sample, a tutor detailed an encounter with a deer that smashed up his car to explain why he was prevented from posting to the forum at an earlier time as he had wished.

DISCUSSION

Socially altruistic behavior is a common characteristic of virtual communities where individuals join together to meet the needs of the community. For instance, many wikipedia.org participants voluntarily commit considerable amounts of their time, effort, and expertise to produce a product (a reference) that is intended for the public good. Free, online, homework help forums are instances of virtual communities in which the defining goal or purpose is to respond to requests for help from unknown individuals (students). The "product" in this case is a service, and one that addresses a vital need in the educational system. Without this service, many students may not have the opportunity to engage in mathematical conversations outside of the classroom, much less to participate in discussions with "experts."

In this paper, we have drawn on research in online instructional participation between participants who are familiar with one another and extended this work to anonymous participation. We propose that the Good Samaritan effect – named to capture the spirit of individual volunteers who come to the aid of strangers in distress - provides a useful lens for examining the behavior of tutors in free, open, online, homework forums. We observed how mechanisms analogous to those that underlie the bystander effect (self-awareness, social cues, inviting, and a sense of responsibility) contribute to patterns of participation in such a forum. The tutoring communities that we have observed are rife with displays of altruistic behavior and exploring this behavior and its consequences is at the heart of our research efforts.

The Good Samaritan effect offers a new way of looking at volunteer efforts in educational settings and may also shed light on the design of tutoring programs. How can a tutoring environment be supported so that participants are enthusiastic and readily contribute, question, challenge, and revise mathematical ideas? Characteristically, university-sponsored academic help centers are arranged so that individual tutors "man" tables and incoming students select a table based on space availability. This arrangement encourages students to dialog with a single tutor during a visit and, at the same time, discourages tutors from interacting with one another. The Good Samaritan effect calls this design into question by suggesting that help can, in some conditions, be contagious. The research reported here represents part of our effort to understand complex social interaction in an educational setting. Clearly, much remains to be done.

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