SCALES OF FUNCTION AND MATRIX SPACES

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ABSTRACT

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The following work is divided into three chapters. In the first chapter, we extend the classical definition of L_p-spaces to include values of $p < 0$. If (Ω, Σ, μ) is a finite, non-atomic measure space, μ a positive measure, then we denote by $M(\mu)$ the space of equivalence classes of Σ-measurable functions. For all $p > 0$, $L_{-p}(\mu)$ is the set $M(\mu)$ together with a complete, translation invariant metric, d_{-p}, defined using the decreasing rearrangement of functions $f \in M(\mu)$. Defined as such, we can extend the inclusion $L_q(\mu) \subseteq L_p(\mu)$ to all $p, q \in \mathbb{R}$, $p < q$. Furthermore, $L_{-p}(\mu)$ can be equipped with an F-norm defined by $\|f\|_{-p} = d_{-p}(f, 0)$.

The second chapter deals with the theory of Hilbert frames. We prove several inequalities relating the Schatten norm of the frame operator, S, to the ℓ_p-norms of the frame elements, f_j. This is done first in finite dimensional Hilbert spaces, then extended to infinite dimensions using a truncated frame operator for finite subsets of the frame. In the final section of this chapter, we construct a frame for which the averaged ℓ_1-norm of the the associated Gram matrix exhibits an optimal growth rate.

In the paper *Generalized Roundness and Negative Type*, Lennard, Tonge, and Weston show that the geometric notion of generalized roundness in a metric space is equivalent to that of negative type. Using this equivalent characterization, along with classical embedding theorems, the authors prove that for $p > 2$, L_p fails to have generalized roundness q for any $q > 0$. It is noted, as a consequence, that the Schatten class $\mathcal{C}_p$, for $p > 2$, has maximal generalized roundness 0. In the third chapter, we prove that this result remains true for p in the interval $(0, 2)$.

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1.1 INTRODUCTION

The notion of L_p-spaces for $p < 0$ is a topic that my advisor, Dr. Christopher Lennard, and I have been discussing since 2003. One motivation for defining such spaces was the possible connection they may have to capacitance in series circuits and resistance and inductance in parallel circuits, in which case the totals are calculated by summing reciprocals. Another, purely theoretical motivation, stemmed from classical embedding theorems in L_p for $p > 0$. It is a well-known fact, going back to P. Levy, that the norms of elements of certain subspaces of L_p admit a special representation. In particular, an n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p for $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere S^{n-1} in $\mathbb{R}^n$ so that

$$\|x\|^p = \int_{S^{n-1}} |\langle x, \xi \rangle|^p \, d\mu(\xi) \tag{1.1}$$

for every $x \in \mathbb{R}^n$. Using Fourier analytic techniques to study the geometry of convex bodies, Alexander Koldobsky noticed a similarity between the above norm representation and those of *intersection bodies*. If K and L are are origin-symmetric star bodies in $\mathbb{R}^n$, then K is the intersection body of L if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction. Intersection bodies played an important role in the solution of the Busemann-Petty problem posed in 1956: Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ so that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller than the same for L. Is it true that the n-dimensional volume of K is smaller than that of L? If the dimension n is 3 or 4 the answer is affirmative. For $n \geq 5$, however, the answer was shown to be negative.

As it turns out, K is the intersection body of L if and only if

$$\|x\|_K^{-1} = \frac{1}{2} \lim_{\epsilon \to 0} \epsilon \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^{-1+\epsilon} \chi_L(\xi)\, d\xi \tag{1.2}$$

for every $x \in \mathbb{R}^n$, where $\|\cdot\|_K$ is the Minkowski functional of K and χ_L is the indicator function of L. Comparison of (1.1) and (1.2) provided Koldobsky with the motivation for introducing the concept of embedding in L_p for $p < 0$. For $p \leq -1$, convergence problems associated with the integral in (1.2) where handled using distributions, and the following definition was introduced.

Definition 1.1.1. Let D be an origin-symmetric star body in $\mathbb{R}^n$. Then the space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p}, where $0 < p < n$, if there exists a finite Borel measure μ on S^{n-1} so that for every test funtion ϕ

$$\int_{\mathbb{R}^n} \|x\|_D^{-p} \phi(x) dx = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1} \hat{\phi}(t\xi) dt \right) d\mu(\xi).$$

With this definition, Koldobsky shows that k-intersection bodies can be considered as the unit balls of subspaces of L_{-k}.

Theorem 1.1.2. *The following are equivalent:*
(i) An origin-symmetric star body D is a k-intersection body.
(ii) $\|\cdot\|_D^{-k}$ is a positive definite distribution
(iii) The space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k}.

Notice that no attempt is made to define L_{-p} as a function space of any kind. The goal of this first Chapter is to furnish such a definition, which at some stage we hope will give an explicit meaning to such embeddings. It should be noted that Emmanuele DiBenedetto does give a definition for L_{-p} in his book *Real Analysis*. He says that for $p < 0$ a measurable function $f : \Omega \to \mathbb{R}$ is in $L_p(\Omega)$ if

$$0 < \int_\Omega |f|^p \, d\mu < \infty.$$

In this case, $f \in L_p(\Omega)$ implies that $f \neq 0$ a.e. on Ω so that $L_p(\Omega)$ is not a linear space.

1.2 INCREASING REARRANGEMENTS

In all of what follows, let (Ω, Σ, μ) be a finite measure space with μ non-atomic. We denote by $M(\mu)$ the set of equivalence classes of μ-measurable functions $f : \Omega \to \mathbb{R}$ and, in analogy with the distribution function of f, we define $q_f : (0, \infty) \to [0, \mu(\Omega)]$ by

$$q_f(\lambda) = \mu \{x \in \Omega : |f(x)| < \lambda\}$$

for all $f \in M(\mu)$ and for every $\lambda \in (0, \infty)$. Furthermore, we define the *increasing rearrangement of* f, denoted $f^{\circledast}$, by

$$f^{\circledast}(t) = \inf \{\lambda \in (0, \infty) : q_f(\lambda) \geq t\}$$

for all $f \in M(\mu)$ and for every $t \in (0, \mu(\Omega))$. Observe that $|f(x)| < \infty$ for all $x \in \Omega$ implies that $f^{\circledast}(t) < \infty$ for all $t \in (0, \mu(\Omega))$. Also, since q_f is nondecreasing, it follows from the above definitions that

$$f^{\circledast}(t) = \sup \{\lambda \in (0, \infty) : q_f(\lambda) < t\} = m \{\lambda \in (0, \infty) : q_f(\lambda) < t\} = q_{q_f}(t),$$

where m is the Lebesgue measure on $(0, \infty)$. The following example illustrates the relationship between simple functions and their increasing rearrangements.

Let ϕ be a simple function on Ω defined by

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \chi_{A_i}(x)$$

where $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \infty$ and $A_i = \{x \in \Omega : \phi(x) = \alpha_i\}$ for all $1 \leq i \leq 4$. Then

$$q_\phi(\lambda) = \mu \{x \in \Omega : \phi(x) < \lambda\} = \sum_{i=1}^{4} \beta_i \chi_{(\alpha_{i-1}, \alpha_i]}(\lambda) + \mu(\Omega) \chi_{(\alpha_4, \infty)}(\lambda)$$

where $\beta_i = \mu\Big(\Omega \setminus \bigcup_{j=i}^{4} A_j\Big)$ and $\alpha_0 = 0$. The increasing rearrangement of ϕ is then given by

$$\phi^{\circledast}(t) = \inf \{\lambda \in (0, \infty) : q_\phi(\lambda) \geq t\} = \sum_{i=1}^{3} \alpha_i \chi_{(\beta_i, \beta_{i+1}]}(t) + \alpha_4 \chi_{(\beta_4, \mu(\Omega))}(t).$$

Notice that q_ϕ and $\phi^{\circledast}$ are also simple functions.

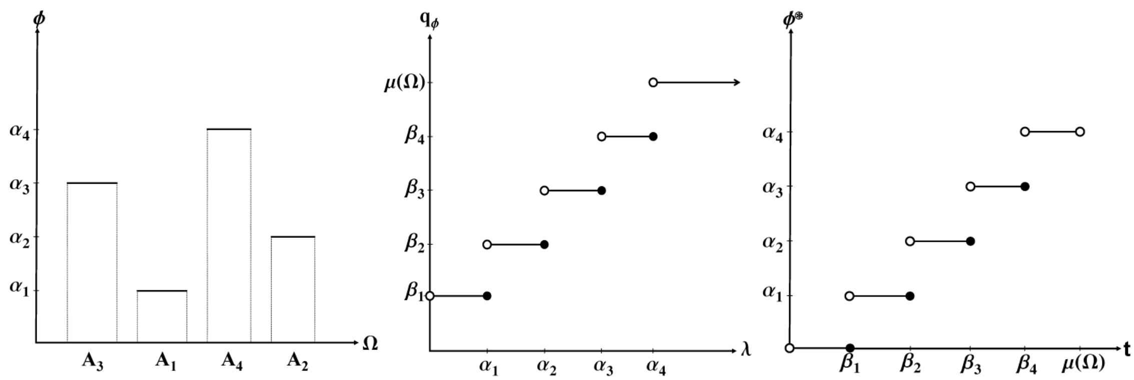

Figure 1: Simple Function

In the next section we use the increasing rearrangement (or more precisely, its relationship to the decreasing rearrangement) of a measurable function to define a new set of metrics for $M(\mu)$. Before doing so, we verify several elementary properties of the functions q_f and $f^{\circledast}$.

Lemma 1.2.1. *Let $f \in M(\mu)$. Then q_f is left continuous on $(0, \infty)$.*

Proof. Let $\lambda_0 \in (0, \infty)$ and define $A(\lambda_0) = \{x \in \Omega : |f(x)| < \lambda_0\} = \bigcup_{n \in \mathbb{N}} A\left(\lambda_0 - \frac{1}{n}\right)$. Since $\left(A(\lambda_0 - \frac{1}{n})\right)_{n \in \mathbb{N}}$ is an increasing sequence of sets, we have

$$q_f(\lambda_0) = \mu(A(\lambda_0)) = \lim_{n \to \infty} \mu\left(A\left(\lambda_0 - \tfrac{1}{n}\right)\right) = \lim_{n \to \infty} q_f\left(\lambda_0 - \tfrac{1}{n}\right). \qquad \square$$

Lemma 1.2.2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $M(\mu)$ such that $0 \leq f_{n+1} \leq f_n$ for all $n \in \mathbb{N}$ and suppose there exists $f \in M(\mu)$ such that $f_n \downarrow f$ μ-a.e. Then $(q_{f_n})_{n \in \mathbb{N}}$ is increasing and $q_{f_n} \uparrow q_f$ m-a.e.*

Proof. Without loss of generality, assume the convergence is everywhere on Ω. As in the proof of 1.2.1, define the sets $A(\lambda) = \{x \in \Omega : f(x) < \lambda\}$ and $A_n(\lambda) = \{x \in \Omega : f_n(x) < \lambda\}$ for all $\lambda > 0$ and for every $n \in \mathbb{N}$. Since $f_{n+1} \leq f_n$ we find that $A_n(\lambda) \subseteq A_{n+1}(\lambda)$ which means $q_{f_n}(\lambda) \leq q_{f_{n+1}}(\lambda)$ for all $\lambda > 0$ and for all n. Fix $\lambda > 0$ and $x \in A(\lambda)$. Since $f_n(x) \downarrow f(x)$ there exists $N \in \mathbb{N}$ such that $x \in \bigcap_{n=N}^{\infty} A_n(\lambda)$. Hence, for any fixed $\lambda > 0$, we can write

$$A(\lambda) = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n(\lambda).$$

Since $A_N(\lambda) = \bigcap_{n=N}^{\infty} A_n(\lambda) \subseteq \bigcap_{n=N+1}^{\infty} A_n(\lambda)$ we have

$$q_f(\lambda) = \mu(A(\lambda)) = \mu\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n(\lambda)\right) = \lim_{N\to\infty} \mu\left(\bigcap_{n=N}^{\infty} A_n(\lambda)\right) = \lim_{N\to\infty} q_{f_N}(\lambda). \qquad \square$$

Lemma 1.2.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $M(\mu)$ such that $0 \leq f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and suppose there exists $f \in M(\mu)$ such that $f_n \uparrow f$ μ-a.e. Then $(q_{f_n})_{n\in\mathbb{N}}$ is decreasing and $q_{f_n} \downarrow q_f$ m-a.e.*

Proof. Again, we assume the convergence is everywhere on Ω. For $\lambda \in (0,\infty)$ define $A(\lambda)$ and $A_n(\lambda)$ as in the proof of 1.2.2 and observe that $f_n \leq f_{n+1}$ implies $A_n(\lambda) \supseteq A_{n+1}(\lambda)$ so that $q_{f_n} \geq q_{f_{n+1}}$ for all $n \in \mathbb{N}$. Fix $\lambda \in (0,\infty)$ and let $x \in A(\lambda)$. Since $f_n(x) \leq f(x) < \lambda$ for all $n \in \mathbb{N}$, we find that $A(\lambda) \subseteq \bigcap_{n=1}^{\infty} A_n(\lambda)$. Hence

$$q_f(\lambda) = \mu(A(\lambda)) \leq \mu\left(\bigcap_{n=1}^{\infty} A_n(\lambda)\right) = \lim_{n\to\infty} \mu(A_n(\lambda)) = \lim_{n\to\infty} q_{f_n}(\lambda).$$

Furthermore, $x \in \bigcap_{n=1}^{\infty} A_n(\lambda)$ implies $f(x) \leq \lambda$ so that $\bigcap_{n=1}^{\infty} A_n(\lambda) \subseteq \bigcap_{k=1}^{\infty} A\left(\lambda + \frac{1}{k}\right)$. From above, we have

$$q_f(\lambda) \leq \lim_{n\to\infty} q_{f_n}(\lambda) \leq \mu\left(\bigcap_{k=1}^{\infty} A\left(\lambda + \tfrac{1}{k}\right)\right) = \lim_{k\to\infty} \mu\left(A(\lambda + \tfrac{1}{k})\right) = \lim_{k\to\infty} q_f\left(\lambda + \tfrac{1}{k}\right).$$

Since q_f is increasing, it can have at most a countable set of discontinuities. That is, $q_f\left(\lambda + \frac{1}{k}\right) \to q_f(\lambda)$ as $k \to \infty$ except possibly on a set of m-measure 0. The above can then be written as

$$q_f(\lambda) \leq \lim_{n\to\infty} q_{f_n}(\lambda) \leq q_f(\lambda), \quad m\text{-a.e. on } (0,\infty). \qquad \square$$

Lemma 1.2.4. *Let $(f_n)_{n\in\mathbb{N}}$ and f be as in 1.2.3. Then $f_n^{\circledast} \uparrow f^{\circledast}$.*

Proof. By 1.2.2, 1.2.3 and previous remarks we have

$$f_n \uparrow f \Rightarrow q_{f_n} \downarrow q_f \Rightarrow q_{q_{f_n}} \uparrow q_{q_f} \Leftrightarrow f_n^{\circledast} \uparrow f^{\circledast}. \qquad \square$$

Lemma 1.2.5. *Let $\phi \in M(\mu)$ be a nonnegative simple function. Then*

$$\int_{\Omega} \phi \, d\mu = \int_0^{\mu(\Omega)} \phi^{\circledast} \, dm \,.$$

Proof. For $x \in \Omega$, let

$$\phi(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x)$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \infty$ are the n distinct, nonzero values of ϕ, and $A_i = \{x \in \Omega : \phi(x) = \alpha_i\}$ for every $i \in \{1, \dots n\}$. Then $\phi^{\circledast}$ is a simple function of the form

$$\phi^{\circledast}(t) = \sum_{i=1}^{n-1} \alpha_i \chi_{(\beta_i, \beta_{i+1}]}(t) + \alpha_n \chi_{(\beta_n, \, \mu(\Omega))}(t)$$

for every $t \in (0, \mu(\Omega))$, where $\beta_i = \mu\Big(\Omega \backslash \bigcup_{j=i}^{n} A_j\Big)$ for $i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\int_0^{\mu(\Omega)} \phi^{\circledast} \, d\mu &= \alpha_1(\beta_2 - \beta_1) + \alpha_2(\beta_3 - \beta_2) + \cdots + \alpha_n(\mu(\Omega) - \beta_n) \\
&= \alpha_1 \Big(\mu(\Omega \backslash \bigcup_{i=2}^{n} A_i) - \mu(\Omega \backslash \bigcup_{i=1}^{n} A_i)\Big) + \alpha_2 \Big(\mu(\Omega \backslash \bigcup_{i=3}^{n} A_i) - \mu(\Omega \backslash \bigcup_{i=2}^{n} A_i)\Big) \\
&\qquad + \cdots + \alpha_n \Big(\mu(\Omega) - \mu(\Omega \backslash A_n)\Big) \\
&= \alpha_1 \mu(A_1) + \alpha_2 \mu(A_2) + \cdots + \alpha_n \mu(A_n) \\
&= \int_{\Omega} \phi \, d\mu \,. \qquad \square
\end{aligned}$$

The following is an immediate consequence of 1.2.4, 1.2.5 and the Monotone Convergence Theorem.

Proposition 1.2.6. *Let $f \in M(\mu)$. Then*

$$\int_0^{\mu(\Omega)} f^{\circledast}(t) \, dt = \int_{\Omega} |f(x)| \, d\mu(x).$$

Proposition 1.2.7. *Fix $0 < \epsilon < \mu(\Omega)$ and let $f \in M(\mu)$. Then*

$$\int_0^\epsilon f^{\circledast}(t)\,dt = \min_{\substack{E\in\Sigma \\ \mu(E)\geq\epsilon}} \int_E |f(x)|\; d\mu(x).$$

Proof. Let $E \in \Sigma$ be such that $\mu(E) \geq \epsilon$. Observe that for $t \in (0, \mu(E))$,

$$\begin{aligned}
(f\chi_E)^{\circledast}(t + \mu(\Omega\backslash E)) &= \inf\{\lambda : q_{f\chi_E}(\lambda) \geq t + \mu(\Omega\backslash E)\} \\
&= \inf\{\lambda : \mu\{x \in \Omega : |f(x)|\,\chi_E(x) < \lambda\} \geq t + \mu(\Omega\backslash E)\} \\
&= \inf\{\lambda : \mu(\Omega\backslash E) + \mu\{x \in E : |f(x)| < \lambda\} \geq t + \mu(\Omega\backslash E)\} \\
&= \inf\{\lambda : \mu\{x \in E : |f(x)| < \lambda\} \geq t\} \\
&\geq \inf\{s : \mu\{x \in \Omega : |f(x)| < s\} \geq t\} \\
&= f^{\circledast}(t)
\end{aligned}$$

That is, for every $t \in (0, \mu(E))$,

$$f^{\circledast}(t) \leq (f\chi_E)^{\circledast}(t + \mu(\Omega\backslash E)). \tag{1.3}$$

Since $(f\chi_E)^{\circledast} \equiv 0$ on $(0, \mu(\Omega\backslash E)]$, by (1.3) and 1.2.6

$$\begin{aligned}
\int_E |f(x)|\; d\mu(x) &= \int_\Omega |f(x)|\,\chi_E(x)\,d\mu(x) = \int_0^{\mu(\Omega)} (f\chi_E)^{\circledast}(t)\,dt = \int_{\mu(\Omega\backslash E)}^{\mu(\Omega)} (f\chi_E)^{\circledast}(t)\,dt \\
&= \int_0^{\mu(E)} (f\chi_E)^{\circledast}(t + \mu(\Omega\backslash E))\,dt \geq \int_0^{\mu(E)} f^{\circledast}(t)\,dt \geq \int_0^\epsilon f^{\circledast}(t)\,dt.
\end{aligned}$$

Hence,

$$\int_0^\epsilon f^{\circledast}(t)\,dt \leq \inf_{\substack{E\in\Sigma \\ \mu(E)\geq\epsilon}} \int_E |f(x)|\; d\mu(x).$$

It remains to show there exists an $E \in \Sigma$ with $\mu(E) = \epsilon$ such that

$$\int_0^\epsilon f^{\circledast}(t)\,dt = \int_E |f(x)|\,d\mu(x). \tag{1.4}$$

To prove (1.4) we will consider 2 cases.

Case 1: Suppose there exists $\lambda_0 \in (0, \infty)$ such that $q_f(\lambda_0) = \epsilon$.

Define $E = \{x \in \Omega : |f(x)| < \lambda_0\}$ so that $q_f(\lambda_0) = \mu(E) = \epsilon$. Then

$$\begin{aligned} q_{f\chi_E}(\lambda) &= \mu\{x \in \Omega : |f(x)|\,\chi_E(x) < \lambda\} \\ &= \mu(\Omega \backslash E) + \mu\{x \in E : |f(x)| < \lambda\} \\ &= \begin{cases} \mu(\Omega\backslash E) + q_f(\lambda), & 0 < \lambda \le \lambda_\circ \\ \mu(\Omega), & \lambda_\circ < \lambda < \infty. \end{cases} \end{aligned}$$

And so,

$$(f\chi_E)^{\circledast}(t) = \begin{cases} 0, & 0 < t \le \mu(\Omega\backslash E) \\ f^{\circledast}(t - \mu(\Omega\backslash E), & \mu(\Omega\backslash E) < t < \mu(\Omega). \end{cases}$$

Integration over E then yields

$$\begin{aligned} \int_E |f(x)|\, d\mu(x) = \int_\Omega |f(x)|\,\chi_E(x)\,d\mu(x) = \int_0^{\mu(\Omega)} (f\chi_E)^{\circledast}(t)\,dt = \int_{\mu(\Omega\backslash E)}^{\mu(\Omega)} f^{\circledast}(t - \mu(\Omega\backslash E))\,dt \\ = \int_0^{\mu(E)} f^{\circledast}(t)\,dt = \int_0^{\epsilon} f^{\circledast}(t)\,dt \end{aligned}$$

Equivalently, we can write

$$\int_0^{q_f(\lambda_\circ)} f^{\circledast}(t)\,dt = \int_{\{x:|f|<\lambda_\circ\}} |f(x)|\, d\mu(x)$$

Case 2: Suppose $q_f(\lambda) \neq \epsilon$ for all $\lambda \in (0, \infty)$.

First assume that $q_f(\lambda) < \epsilon$ for all $\lambda > 0$ so that $\mu\{x \in \Omega : |f(x)| < \infty\} = \mu(\Omega) \le \epsilon$, which is a contradiction. Next, suppose that $q_f(\lambda) > \epsilon$ for all $0 < \lambda < \infty$ so that

$$f^{\circledast}(t) = 0 \quad \forall\, t \in (0, \epsilon]. \tag{1.5}$$

Define the sets $A = \{x \in \Omega : f(x) = 0\}$ and $A_n = \left\{x \in \Omega : |f(x)| < \frac{1}{n}\right\}$ for all $n \in \mathbb{N}$. Then $A = \bigcap_{n\in\mathbb{N}} A_n$ and, by assumption, $\mu(A_n) = q_f(\frac{1}{n}) > \epsilon$ for all $n \in \mathbb{N}$. Since $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets, we have

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) \ge \epsilon.$$

Since μ is non-atomic, there exists a measurable set $E \subseteq A$ such that $\mu(E) = \epsilon$. Then, by definition of A and (1.5)

$$\int_0^\epsilon f^{\circledast}(t)\,dt = \int_E |f(x)|\,d\mu(x) = 0.$$

Finally, suppose there exists $\lambda_1,\ \lambda_2 \in (0,\infty)$ such that $q_f(\lambda_1) < \epsilon < q_f(\lambda_2)$. Set $c = f^{\circledast}(\epsilon) = \inf\{\lambda : q_f(\lambda) \geq \epsilon\} = \inf\{\lambda : q_f(\lambda) > \epsilon\} > 0$. Since q_f is left continuous it must have a jump-discontinuity at c with $q_f(c) < \epsilon$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence converging to c and let $L = \lim\limits_{n\to\infty} q_f(\lambda_n)$. Then $L \geq \epsilon$ and $f^{\circledast} \equiv c$ on the interval $[q_f(c), L]$.

Again, we define $A(\lambda) = \{x \in \Omega : |f(x)| < \lambda\}$ for all $\lambda > 0$ so that with $(\lambda_n)_{n\in\mathbb{N}}$ as above, $(A(\lambda_n))_{n\in\mathbb{N}}$ is a decreasing sequence of measurable sets with

$$\bigcap_{n=1}^{\infty} A(\lambda_n) = A(c) \bigcup B\,,$$

where $B := \{x \in \Omega : |f(x)| = c\}$. Since $A(c)$ and B are disjoint,

$$L = \lim_{n\to\infty} q_f(\lambda_n) = \lim_{n\to\infty} \mu(A(\lambda_n)) = \mu\left(\bigcap_{n=1}^{\infty} A(\lambda_n)\right) = q_f(c) + \mu(B)$$

so that $\mu(B) = L - q_f(c)$. Since μ is non-atomic, there exists a measurable set $G \subset B$ such that $\mu(G) = \epsilon - q_f(c)$. Using this set G we define $E \in \Sigma$ to be $E = \{x \in \Omega : |f(x)| < c\} \bigcup G$ so that $\mu(E) = q_f(c) + \epsilon - q_f(c) = \epsilon$. Integrating over E yields the desired result:

$$\begin{aligned}
\int_E |f(x)|\,d\mu(x) &= \int_{\{x:|f|<c\}} |f(x)|\,d\mu(x) + \int_G |f(x)|\,d\mu(x) = \int_0^{q_f(c)} f^{\circledast}(t)\,dt + c\,\mu(G) \\
&= \int_0^{q_f(c)} f^{\circledast}(t)\,dt + c\,(\epsilon - q_f(c)) = \int_0^{q_f(c)} f^{\circledast}(t)\,dt + \int_{q_f(c)}^{\epsilon} f^{\circledast}(t)\,dt \\
&= \int_0^{\epsilon} f^{\circledast}(t)\,dt.
\end{aligned}$$

□

Lemma 1.2.8. *Let $f \in M(\mu)$. Then $(f^{\circledast})^p = (|f|^p)^{\circledast}$ for every $p > 0$.*

Proof. Since $q_{|f|^p}(\lambda) = \mu\{x \in \Omega : |f(x)|^p < \lambda\} = \mu\{x \in \Omega : |f(x)| < \lambda^{1/p}\} = q_f(\lambda^{1/p})$ we have that

$$\begin{aligned}
(|f|^p)^{\circledast}(t) &= \inf\{\lambda : q_{|f|^p}(\lambda) \geq t\} = \inf\{\lambda : q_f(\lambda^{1/p}) \geq t\} = \inf\{\lambda^p : q_f(\lambda) \geq t\} \\
&= (f^{\circledast})^p(t).
\end{aligned}$$

□

Before proceeding to the next section, we recall that the decreasing rearrangement, f^*, of a measurable function f is defined by

$$f^*(t) = \inf \{\lambda : \mu_f(\lambda) \leq t\}$$

for all $t > 0$ where

$$\mu_f(\lambda) = \mu \{x \in \Omega : |f(x)| > \lambda\}$$

is the distribution function of f. Just as the decreasing rearrangement can be used to give an alternate description of the the norm of a function $f \in L_p$, $0 < p < \infty$, we shall use the reciprocal, $\frac{1}{f^*}$, to define a metric for L_{-p}. To this end, we note that $\frac{1}{f^*} = \left(\frac{1}{f}\right)^{\circledast}$, except possibly on a set of Lebesgue measure 0. Indeed,

$$\left(\tfrac{1}{f}\right)^{\circledast}(t) = \inf\left\{\lambda : q_{1/f}(\lambda) \geq t\right\} = \inf\left\{\lambda : \mu\left\{x \in \Omega : \left|\tfrac{1}{f(x)}\right| < \lambda\right\} \geq t\right\}$$

$$= \inf\left\{\lambda : \mu\left\{x \in \Omega : |f(x)| > \tfrac{1}{\lambda}\right\} \geq t\right\} = \inf\left\{\tfrac{1}{\lambda} : \mu\{x \in \Omega : |f(x)| > \lambda\} \geq t\right\}$$

$$= \inf\left\{\tfrac{1}{\lambda} : \mu_f(\lambda) \geq t\right\} = \frac{1}{\sup\{\lambda : \mu_f(\lambda) \geq t\}}.$$

Since μ_f is decreasing, $\sup\{\lambda : \mu_f(\lambda) \geq t\} = \inf\{\lambda : \mu_f(\lambda) \leq t\}$ $m-$a.e. In particular, if $f^* \equiv 0$ on the inverval $[a, \mu(\Omega))$, then $\left(\frac{1}{f}\right)^{\circledast} \equiv \infty$ on $[a, \mu(\Omega))$ except possibly at a.

1.3 $L_{-P}(\Omega, \Sigma, \mu)$, $0 < P < \infty$

To find a method to measure the distance between functions, so that the triangle inequality was satisfied, presented a major obstacle to overcome in defining $L_{-p}-$spaces. For instance, for $p = 1$, one possibility was to define L_{-1} as the set of equivalence classes of measurable functions $f \in M(\mu)$ for which $\gamma(f) < \infty$, where

$$\gamma(f) = \left(\int_{\Omega} \frac{1}{|f|} \, d\mu \right)^{-1}. \tag{1.6}$$

Then $0 \leq \gamma(f) < \infty$ for all $f \in M(\mu)$. However, if $f, g \in M(\mu)$ are nonnegative almost everywhere on Ω, then one can show that $\gamma(f+g) \geq \gamma(f) + \gamma(g)$. Modifying (1.6), we define

$$\mathcal{I}_f^p(\epsilon) = \left(\int_0^{\epsilon} \frac{1}{(f^*)^p} \, dm \right)^{-1/p}$$

for every $\epsilon > 0$ and $0 < p < \infty$. As is customary, we set $f^*(t) = 0$ for all $t > \mu(\Omega)$. With this convention, observe that $\mathcal{I}_f^p(\epsilon) = 0$ if $\epsilon > \mu(\Omega)$. The following theorem shows that $\mathcal{I}_f^p$ satisfies a Minkowski-type inequality.

Theorem 1.3.1. *Fix ϵ, $\eta > 0$ and let f, $g \in M(\mu)$. Then*

$$\mathcal{I}_{f+g}^p(\epsilon + \eta) \ \leq \ \mathcal{I}_f^p(\epsilon) + \mathcal{I}_g^p(\eta) \tag{1.7}$$

for every $p \in (0, \infty)$.

Proof. Fix $p \in (0, \infty)$ and let $\gamma = \int_0^{\epsilon} \frac{dm}{(f^*)^p}$ and $\delta = \int_0^{\eta} \frac{dm}{(g^*)^p}$. Since f and g are real-valued functions, we have that γ, $\delta \in (0, \infty]$. Then (1.7) can be written as

$$\frac{1}{\left(\int_0^{\epsilon+\eta} \frac{dm}{[(f+g)^*]^p} \right)^{1/p}} \ \leq \ \frac{1}{\gamma^{1/p}} + \frac{1}{\delta^{1/p}} \tag{1.8}$$

Here, $1/\infty = 0$. Note that $|f+g| \leq |f| + |g|$, and so $(f+g)^* \leq (|f| + |g|)^*$. Thus, to prove (1.8), we may without loss of generality assume that f and g are $[0, \infty)$-valued.

Observe that if $\epsilon + \eta > \mu(\Omega)$, the left side of (1.8) is 0 and we are done. So, without loss of generality, assume $\epsilon + \eta \leq \mu(\Omega)$.

We first consider the case where $\gamma = \delta = \infty$. Then there exists $\alpha \in (0,\epsilon)$ and $\beta \in (0,\eta)$ such that $f^* = 0$ on (α,∞) and $g^* = 0$ on (β,∞). Hence, the sets $U := \{x \in \Omega : f(x) > 0\}$ and $V := \{x \in \Omega : g(x) > 0\}$ satisfy $\mu(U) \leq \alpha < \epsilon$ and $\mu(V) \leq \beta < \eta$. But $U \cup V = W := \{x \in \Omega : f(x) + g(x) > 0\}$. Therefore, $\mu(W) \leq \mu(U) + \mu(V) < \epsilon + \eta$. Consequently, both the left and right sides of (1.8) are equal to 0.

Our second case is where precisely one of γ and δ is infinite. Without loss of generality, assume that $\gamma = \infty$ and $\delta \in (0,\infty)$. Also, without loss, we may assume that $(f+g)^* > 0$ on $[0, \epsilon+\eta)$, so that $W := \{x \in \Omega : f(x) + g(x) > 0\}$ satisfies $\mu(W) \geq \epsilon + \eta$. Applying 1.2.6 and 1.2.7, with Ω replaced by W and $|f|$ replaced by $1/(f+g)$, we see that there exists $E \subseteq W$ with $E \in \Sigma$ such that $\mu(E) = \epsilon + \eta$ and

$$\int_0^{\epsilon+\eta} \frac{dm}{[(f+g)^*]^p} = \int_0^{\epsilon+\eta} \left(\frac{1}{(f+g)^p}\right)^{\circledast} dm = \int_E \frac{1}{(f+g)^p}\, dm \ .$$

So, in this case, showing (1.8) is equivalent to proving that

$$\int_E \frac{1}{(f+g)^p}\, dm \geq \delta \ .$$

Let $C := \{x \in E : f(x) = 0\}$. Note that

$$\begin{aligned}
\mu(C) &= \mu(E) - \mu(\{x \in E : f(x) > 0\}) \geq \mu(E) - \mu(\{x \in \Omega : f(x) > 0\}) \\
&= \epsilon + \eta - \mu(\{x \in \Omega : f(x) > 0\}) > \epsilon + \eta - \epsilon \\
&= \eta \ .
\end{aligned}$$

Consequently, by the first part of the proof of 1.2.7 (which is still true for extended real-valued Σ-measurable functions on Ω),

$$\int_E \frac{1}{(f+g)^p}\, dm \geq \int_C \frac{1}{g^p}\, dm \geq \int_0^{\eta} \left(\frac{1}{g^p}\right)^{\circledast} dm = \int_0^{\eta} \frac{1}{(g^*)^p}\, dm = \delta \ .$$

The third and final case is where $\gamma \in (0,\infty)$ and $\delta \in (0,\infty)$. So, $f^* > 0$ on $[0,\epsilon)$ and $g^* > 0$ on $[0,\eta)$. Also, without loss, we may assume that $(f+g)^* > 0$ on $[0,\epsilon+\eta)$, so that $W := \{x \in \Omega : f(x) + g(x) > 0\}$ satisfies $\mu(W) \geq \epsilon + \eta$. Without loss we may further assume that $\Omega = W$.

Define the $(0,\infty]$-valued Σ-measurable functions F and G on Ω by $F := \frac{1}{f}$ and $G := \frac{1}{g}$. Note that $(F^{\circledast})^p = \frac{1}{(f^*)^p}$ and $(G^{\circledast})^p = \frac{1}{(g^*)^p}$.

Further, for all $x \in \Omega = W$, either $f(x) > 0$ or $g(x) > 0$. Therefore, for all $x \in \Omega$, either $F(x) < \infty$ or $G(x) < \infty$. If both $F(x) < \infty$ and $G(x) < \infty$, then

$$\frac{F(x)\,G(x)}{F(x)+G(x)} = \frac{1}{f(x)+g(x)} \ .$$

When $F(x) = \infty$, then $G(x) < \infty$, and we will define $\frac{F(x)}{F(x)+G(x)}$ to be 1 and $\frac{G(x)}{F(x)+G(x)}$ to be 0. Since $f(x) = 0$, we have that

$$\frac{F(x)\,G(x)}{F(x)+G(x)} := (1)\,G(x) = \frac{1}{g(x)} = \frac{1}{f(x)+g(x)} \ .$$

Similarly, when $G(x) = \infty$, then $F(x) < \infty$, and we will define $\frac{G(x)}{F(x)+G(x)}$ to be 1 and $\frac{F(x)}{F(x)+G(x)}$ to be 0 . Since $g(x) = 0$, we have that

$$\frac{F(x)\,G(x)}{F(x)+G(x)} := F(x)\,(1) = \frac{1}{f(x)} = \frac{1}{f(x)+g(x)} \ .$$

In summary, we have $\frac{F\,G}{F+G} = \frac{1}{f+g}$ on Ω. By 1.2.8,

$$\frac{1}{[(f+g)^*]^p} = \left[\left(\frac{FG}{F+G}\right)^{\circledast}\right]^p = \left[\left(\frac{FG}{F+G}\right)^p\right]^{\circledast},$$

and (1.8) becomes

$$\frac{1}{\int_0^{\epsilon+\eta} \left[\left(\frac{FG}{F+G}\right)^p\right]^{\circledast} dm} \leq \frac{(\gamma^{1/p}+\delta^{1/p})^p}{\gamma\,\delta} \qquad \Leftrightarrow$$

$$\int_0^{\epsilon+\eta} \left[\left(\frac{FG}{F+G}\right)^p\right]^{\circledast} dm \geq \frac{\gamma\,\delta}{(\gamma^{1/p}+\delta^{1/p})^p}.$$

By 1.2.7 we can find $E \in \Sigma$ such that $\mu(E) = \epsilon + \eta$ and

$$\int_0^{\epsilon+\eta} \left[\left(\frac{FG}{F+G}\right)^p\right]^{\circledast} dm = \int_E \left(\frac{FG}{F+G}\right)^p d\mu.$$

We partition E into the sets

$$A = \left\{ x \in E : \frac{F(x)}{\gamma^{1/p}} \geq \frac{G(x)}{\delta^{1/p}} \right\}$$

and

$$B = \left\{ x \in E : \frac{F(x)}{\gamma^{1/p}} < \frac{G(x)}{\delta^{1/p}} \right\}$$

so that $E = A \cup B$, $A \cap B = \emptyset$, and $\mu(A) + \mu(B) = \epsilon + \eta$. Then

$$x \in A \;\Leftrightarrow\; \frac{F^p(x)}{(F(x)+G(x))^p} \;\geq\; \frac{\gamma}{(\gamma^{1/p} + \delta^{1/p})^p}$$

and

$$x \in B \;\Leftrightarrow\; \frac{G^p(x)}{(F(x)+G(x))^p} \;>\; \frac{\delta}{(\gamma^{1/p} + \delta^{1/p})^p}\,.$$

Furthermore,

$$\int_0^{\epsilon+\eta} \left[\left(\frac{FG}{F+G} \right)^p \right]^{\circledast} dm = \int_E \left(\frac{FG}{F+G} \right)^p d\mu = \int_A \left(\frac{FG}{F+G} \right)^p d\mu + \int_B \left(\frac{FG}{F+G} \right)^p d\mu$$

$$\geq \frac{\gamma}{(\gamma^{1/p} + \delta^{1/p})^p} \int_A G^p \, d\mu + \frac{\delta}{(\gamma^{1/p} + \delta^{1/p})^p} \int_B F^p \, d\mu\,.$$

If $\mu(A) \geq \eta$, by 1.2.7 we find

$$\int_0^{\epsilon+\eta} \left[\left(\frac{FG}{F+G} \right)^p \right]^{\circledast} \geq \frac{\gamma}{(\gamma^{1/p} + \delta^{1/p})^p} \int_0^{\eta} (G^p)^{\circledast} \, dm = \frac{\gamma\,\delta}{(\gamma^{1/p} + \delta^{1/p})^p}\,.$$

In case $\mu(A) < \eta$ then $\mu(B) \geq \epsilon$ so that

$$\int_0^{\epsilon+\eta} \left[\left(\frac{FG}{F+G} \right)^p \right]^{\circledast} \geq \frac{\delta}{(\gamma^{1/p} + \delta^{1/p})^p} \int_0^{\epsilon} (F^p)^{\circledast} \, dm = \frac{\delta\,\gamma}{(\gamma^{1/p} + \delta^{1/p})^p}\,. \qquad \square$$

As a result of 1.3.1, we can now define a new set of metrics on the set of measurable functions.

Theorem 1.3.2. *For all $f, g \in M(\mu)$, for all $p \in (0, \infty)$, define*

$$d_{-p}(f,g) = \inf_{\epsilon>0}\left[\left(\int_0^{\epsilon} \frac{dt}{[(f-g)^*]^p(t)}\right)^{-1/p} + \frac{\epsilon}{\mu(\Omega)}\right].$$

Then d_{-p} is a metric on $M(\mu)$.

Proof. Fix $p \in (0, \infty)$.

(i) $d_{-p} : M(\mu) \times M(\mu) \longmapsto [0, \infty)$.

If $f, g \in M(\mu)$ are such that $f = g$ almost everywhere on Ω, then $f - g = 0$ almost everywhere on Ω and $(f-g)^*(t) = 0$ for every $t \in (0, \mu(\Omega))$. Hence, for any $\epsilon > 0$,

$$\int_0^{\epsilon} \frac{dt}{[(f-g)^*]^p\,(t)} = \infty$$

so that

$$\left(\int_0^{\epsilon} \frac{dt}{[(f-g)^*]^p\,(t)}\right)^{-1/p} = 0$$

and we find that $d_{-p}(f,g) = 0$. If $f \neq g$ on a set of positive measure then there is some positive constant c such that $(f-g)^*(t) > 0$ for $t \in [0, c)$. This means that for any $\epsilon > 0$

$$0 < \int_0^{\epsilon} \frac{dt}{[(f-g)^*]^p\,(t)} \leq \infty$$

so that

$$0 \leq \left(\int_0^{\epsilon} \frac{dt}{[(f-g)^*]^p\,(t)}\right)^{-1/p} < \infty$$

and we have $0 \leq d_{-p}(f,g) < \infty$ for all $f, g \in M(\mu)$.

(ii) $d_{-p}(f,g) = 0 \Leftrightarrow f = g \quad \mu-$a.e.

From (i) $f = g \quad \mu-$a.e. $\Rightarrow d_{-p}(f,g) = 0$. Conversely, suppose $d_{-p}(f,g) = 0$. Then there exists a nonincreasing sequence, $(\epsilon_n)_{n\in\mathbb{N}}$, of positive scalars such that $\epsilon_n \to 0$ and

$$\int_0^{\epsilon_n} \frac{dt}{[(f-g)^*]^p\,(t)} \to \infty$$

as $n \to \infty$. Since $\frac{1}{[(f-g)^*]^p}$ is nonnegative on $[0, \mu(\Omega)]$,

$$\int_0^{\epsilon_n} \frac{dt}{[(f-g)^*]^p\,(t)} \leq \int_0^{\epsilon_m} \frac{dt}{[(f-g)^*]^p\,(t)}$$

whenever $n > m$. That is, $\left(\int_0^{\epsilon_n} \frac{dt}{[(f-g)^*]^p(t)}\right)_{n\in\mathbb{N}}$ is a nonincreasing sequence converging to infinity, which implies that

$$\infty = \int_0^{\epsilon_n} \frac{dt}{[(f-g)^*]^p\,(t)} \leq \frac{\epsilon_n}{[(f-g)^*]^p\,(\epsilon_n)}$$

for all $n \in \mathbb{N}$. We therefore conclude that $(f-g)^*(\epsilon_n) = 0$ for every $n \in \mathbb{N}$. Since the decreasing rearrangement is right continuous, $(f-g)^*(0) = 0$ which implies $(f-g)^*(t) = 0$ for all $t \in [0, \mu(\Omega)]$. Hence, $f = g \quad \mu-$a.e.

(iii) $d_{-p}(f,g) = d_{-p}(g,f)$ for all $f, g \in M(\mu)$.

This follows from the fact that $(f-g)^* = (g-f)^*$ for all $f, g \in M(\mu)$.

(iv) $d_{-p}(f,g) \leq d_{-p}(f,h) + d_{-p}(g,h)$ for all $f, g, h \in M(\mu)$.

By 1.3.1.

$$\begin{aligned}
d_{-p}(f,h) + d_{-p}(h,g) &= \inf_{\epsilon>0}\left[\mathcal{I}^p_{f-h}(\epsilon) + \frac{\epsilon}{\mu(\Omega)}\right] + \inf_{\eta>0}\left[\mathcal{I}^p_{h-g}(\eta) + \frac{\eta}{\mu(\Omega)}\right] \\
&= \inf_{\epsilon,\eta>0}\left[\mathcal{I}^p_{f-h}(\epsilon) + \mathcal{I}^p_{h-g}(\eta) + \frac{\epsilon+\eta}{\mu(\Omega)}\right] \\
&\geq \inf_{\epsilon,\eta>0}\left[\mathcal{I}^p_{f-g}(\epsilon+\eta) + \frac{\epsilon+\eta}{\mu(\Omega)}\right] \\
&= d_{-p}(f,g).
\end{aligned}$$

□

From now on, we shall refer to $M(\mu)$ endowed with the metric d_{-p} as $L_{-p} = L_{-p}(\Omega, \Sigma, \mu)$ for all $p \in (0, \infty)$.

Theorem 1.3.3. *Let $p \in (0,\infty)$. Then $L_{-p}(\Omega, \Sigma, \mu)$ is a complete metric space.*

Proof. Fix $p \in (0,\infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be Cauchy in L_{-p} so that

$$d_{-p}(f_n, f_m) = \inf_{\epsilon>0}\left[\left(\int_0^{\epsilon} \frac{dt}{[(f_n - f_m)^*(t)]^p}\right)^{-1/p} + \frac{\epsilon}{\mu(\Omega)}\right] \to 0$$

as $n, m \to \infty$. Let $\epsilon_{n,m} > 0$ be such that

$$\left(\int_0^{\epsilon_{n,m}} \frac{dt}{[(f_n - f_m)^*(t)]^p}\right)^{-1/p} + \frac{\epsilon_{n,m}}{\mu(\Omega)} \leq d_{-p}(f_n, f_m) + \frac{1}{2^{nm}}$$

for each $n, m \in \mathbb{N}$. Then $\epsilon_{n,m} \to 0$ and

$$\int_0^{\epsilon_{n,m}} \frac{dt}{[(f_n - f_m)^*(t)]^p} \to \infty \tag{1.9}$$

as n, $m \to \infty$. We claim that (1.9) implies that for any $\alpha > 0$

$$m\{t \in [0, \mu(\Omega)) : [(f_n - f_m)^*]^p(t) > \alpha\} \overset{n,m\to\infty}{\longrightarrow} 0 \tag{1.10}$$

which is equivalent to

$$m\left\{t \in [0, \mu(\Omega)) : \frac{1}{[(f_n - f_m)^*]^p(t)} < M\right\} \overset{n,m\to\infty}{\longrightarrow} 0 \tag{1.11}$$

for any positive constant M. Suppose, to the contrary, the implication is not true. Then for some $M_0 > 0$ and $\eta > 0$ there is a sequence, $([(f_{n_k} - f_{m_k})^*]^p)_{k\in\mathbb{N}}$, such that $m(A_k) \geq \eta$ and m_k, $n_k \geq k$ for each $k \in \mathbb{N}$, where

$$A_k = \left\{t \in [0, \mu(\Omega)) : \frac{1}{[(f_{n_k} - f_{m_k})^*]^p(t)} < M_0\right\}.$$

Since $\frac{1}{[(f_{n_k} - f_{m_k})^*]^p}$ is a nondecreasing function for each k, there is some $a_k \in (0, \mu(\Omega))$ such that $A_k = [0, a_k)$ (or $A_k = [0, a_k]$) so that $m(A_k) = a_k$. Thus, for all k sufficiently large,

$$\begin{aligned}\int_0^{\epsilon_{n_k,m_k}} \frac{dt}{[(f_{n_k} - f_{m_k})^*]^p(t)} &\leq \int_0^{\eta} \frac{dt}{[(f_{n_k} - f_{m_k})^*]^p(t)} \leq \int_0^{a_k} \frac{dt}{[(f_{n_k} - f_{m_k})^*]^p(t)} \\ &\leq M_0\, a_k \leq M_0\, \mu(\Omega),\end{aligned}$$

which contradicts (1.9). Furthermore, since a measurable function and its decreasing rearrangement are equimeasurable, (1.10) shows that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure. Indeed, for fixed $\epsilon > 0$ define

$$\begin{aligned}E_{n,m} &= \{t \in [0, \mu(\Omega)) : [(f_n - f_m)^*]^p(t) > \epsilon\} \\ &= \left\{t \in [0, \mu(\Omega)) : (f_n - f_m)^*(t) > \epsilon^{1/p}\right\}.\end{aligned}$$

Let μ_f and m_{f^*} represent the distribution functions of f with respect to μ and f^* with respect to the Lebesgue measure, m. From above, we find

$$m(E_{n,m}) = m_{(f_n - f_m)^*}(\epsilon^{1/p}) = \mu_{f_n - f_m}(\epsilon^{1/p}) \to 0$$

as $n,\ m \to \infty$. Since $M(\mu)$ is complete with respect to convergence in measure, there exists $f \in M(\mu)$ such that $(f_n)_{n\in\mathbb{N}}$ converges to f in measure. It remains to show that $d_{-p}(f_n, f) \to 0$.

Fix $\epsilon > 0$ and let $\delta = \epsilon^p$ and $M = \frac{1}{\delta}$. As $n \to \infty$,

$$\begin{aligned}
&\mu\left\{x \in \Omega : |f_n(x) - f(x)| > \epsilon\right\} \to 0 && \Leftrightarrow \\
&m\left\{t \in [0, \mu(\Omega)) : (f_n - f)^*(t) > \epsilon\right\} \to 0 && \Leftrightarrow \\
&m\left\{t \in [0, \mu(\Omega)) : [(f_n - f)^*]^p(t) > \delta\right\} \to 0 && \Leftrightarrow \\
&m\left\{t \in [0, \mu(\Omega)) : \frac{1}{[(f_n - f)^*]^p(t)} < M\right\} \to 0.
\end{aligned}$$

For each $n \in \mathbb{N}$ define the set

$$A_n^M = \left\{t \in [0, \mu(\Omega)) : \frac{1}{[(f_n - f)^*]^p(t)} < M\right\}.$$

As above, since $\frac{1}{[(f_n-f)^*]^p}$ is nondecreasing for every n, we find that there exists $a_n^M \in (0, \mu(\Omega))$ such that $A_n^M = [0, a_n^M)$ (or $[0, a_n^M]$), in which case $m(A_n^M) = a_n^M \to 0$ as $n \to \infty$. And so, for any $\beta > 0$, we find that when n sufficently large

$$\begin{aligned}
\int_0^\beta \frac{dt}{[(f_n - f)^*]^p(t)} &= \int_0^{a_n^M} \frac{dt}{[(f_n - f)^*]^p(t)} + \int_{a_n^M}^\beta \frac{dt}{[(f_n - f)^*]^p(t)} \\
&\geq \int_{a_n^M}^\beta \frac{dt}{[(f_n - f)^*]^p(t)} \ \geq\ M(\beta - a_n^M),
\end{aligned}$$

or

$$\left(\int_0^\beta \frac{dt}{[(f_n - f)^*]^p(t)}\right)^{-1/p} \ \leq\ \left[M(\beta - a_n^M)\right]^{-1/p}.$$

Letting $n \to \infty$ we have

$$\limsup_{n\to\infty} \left(\int_0^\beta \frac{dt}{[(f_n - f)^*]^p(t)}\right)^{-1/p} \ \leq\ (M\,\beta)^{-1/p}.$$

Since $M > 0$ was arbitrary,

$$\lim_{n\to\infty} \left(\int_0^\beta \frac{dt}{[(f_n - f)^*]^p(t)}\right)^{-1/p} = 0.$$

Hence, for any $r > 0$, there exists some $N = N(\beta, r) \in \mathbb{N}$ such that

$$\left(\int_0^\beta \frac{dt}{[(f_n - f)^*]^p(t)} \right)^{-1/p} + \frac{\beta}{\mu(\Omega)} \leq r + \frac{\beta}{\mu(\Omega)}$$

for all $n \geq N$. By definition of the distance in L_{-p} we see that $d_{-p}(f_n, f) \leq r + \frac{\beta}{\mu(\Omega)}$. Since β and r are arbitrary, this completes the proof. $\square$

The above proof shows that convergence with respect to the metric d_{-p} is equivalent to convergence in measure. Furthermore, for all $p \in (0, \infty)$ and for all f, g, $h \in L_{-p}$,

$$d_{-p}(f, g) = d_{-p}(f + h, g + h).$$

That is, d_{-p} is a complete, translation invariant metric on the space of measurable functions. If we try to define a norm-like function for L_{-p} by the formula $\rho(f) = d_{-p}(f, 0)$, then $\rho(f) \geq 0$, $\rho(f) = 0$ if and only if $f = 0$, and, by the invariance of d_{-p}, $\rho(f + g) \leq \rho(f) + \rho(g)$ for all f,$g \in L_{-p}$. Unfortunately, ρ is not absolutely homogeneous and so d_{-p} does not generate a norm on L_{-p}.

We can, however, define a topology for L_{-p} in the usual way, via the basis

$$\mathcal{B} = \left\{ B(f, \epsilon) : f \in L_{-p}, \ \epsilon > 0 \right\},$$

where $B(f, \epsilon)$ is the open ball centered at f with radius ϵ. We shall denote this topology by τ_{-p}. Clearly, L_{-p} is a vector space in which every point is a closed set with respect to the topology, τ_{-p}. The next result shows that L_{-p} is an F-space.

Proposition 1.3.4. *For all $p \in (0, \infty)$, $(L_{-p}(\mu), \tau_{-p})$ is a topological vector space, and hence an F-space.*

Proof. Fix $p \in (0, \infty)$. By the remarks in the preceeding paragraph, we need only verify the vector space operations are continuous with respect to τ_{-p}.

Let $f, g \in L_{-p}$ and suppose that V is a neighborhood of $f + g$ in L_{-p}. Let $\epsilon > 0$ be such that $B(f + g, \epsilon) \subset V$. Define $B_1 = B(f, \frac{\epsilon}{2})$, $B_2 = B(g, \frac{\epsilon}{2})$, and let $h \in B_1$ and $k \in B_2$ so that $h + k \in B_1 + B_2$. Since d_{-p} is translation invariant,

$$\begin{aligned} d_{-p}(f+g, h+k) = d_{-p}(f-h, k-g) &\leq d_{-p}(f-h, 0) + d_{-p}(0, k-g) \\ = d_{-p}(f, h) + d_{-p}(g, k) &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

That is, $B_1 + B_2 \subset B(f + g, \epsilon) \subset V$ so that addition is continuous.

Next, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers converging to some $\alpha \in \mathbb{R}$. Also, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in L_{-p} converging to some f in L_{-p}. Then, as mentioned previously, $(f_n)_{n\in\mathbb{N}}$ also converges to f in measure. Since, for all $x \in \Omega$,

$$|\alpha_n f_n(x) - \alpha f(x)| \leq |\alpha_n(f_n(x) - f(x))| + |(\alpha_n - \alpha) f(x)| \ ,$$

with $\epsilon > 0$ fixed we find that

$$\begin{aligned} &\mu\{x \in \Omega : |\alpha_n f_n(x) - \alpha f(x)| > \epsilon\} \leq \\ &\mu\{x \in \Omega : |\alpha_n(f_n(x) - f(x))| + |(\alpha_n - \alpha) f(x)| > \epsilon\} \leq \\ &\mu\left\{x \in \Omega : |\alpha_n(f_n(x) - f(x))| > \frac{\epsilon}{2}\right\} + \mu\left\{x \in \Omega : |(\alpha_n - \alpha) f(x)| > \frac{\epsilon}{2}\right\}. \end{aligned}$$

Now, $(\alpha_n - \alpha) f \to 0$ pointwise and $\mu(\Omega) < \infty$ implies $(\alpha_n - \alpha) f \to 0$ in measure. Hence,

$$\mu\left\{x \in \Omega : |(\alpha_n - \alpha) f(x)| > \frac{\epsilon}{2}\right\} \to 0. \tag{1.12}$$

Without loss of generality, assume that $\alpha_n \neq 0$ for all $n \in \mathbb{N}$. Define $a = \inf_{n\in\mathbb{N}} \left(\frac{\epsilon}{2|\alpha_n|}\right)$ so that $0 < a < \infty$. Then,

$$\begin{aligned} \mu\left\{x \in \Omega : |\alpha_n(f_n(x) - f(x))| > \frac{\epsilon}{2}\right\} &= \mu\left\{x \in \Omega : |f_n(x) - f(x)| > \frac{\epsilon}{2|\alpha_n|}\right\} \\ &\leq \mu\{x \in \Omega : |f_n(x) - f(x)| > a\}. \end{aligned}$$

Since $f_n \to f$ in measure, the above inequality shows that

$$\mu\left\{x \in \Omega : |\alpha_n(f_n(x) - f(x))| \ > \ \frac{\epsilon}{2}\right\} \to 0. \tag{1.13}$$

From (1.12) and (1.13) we see that $\alpha_n f_n \to \alpha f$ in measure, which implies convergence in L_{-p}, and scalar multiplication is continuous. $\square$

2.0 NORM INEQUALITIES FOR HILBERT FRAMES

2.1 INTRODUCTION

Throughout this chapter, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ will denote a separable Hilbert space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle$. The concept of Hilbert frames dates back to the work of R.J. Duffin and A.C. Schaeffer in 1952 ([13]). They were working on problems involving nonharmonic Fourier analysis and, in particular, were trying to determine when the family of exponentials, $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$, forms a complete system for $L_2[a,b]$, where $\lambda_n \in \mathbb{K}$ for all n. This led them to define the following.

Definition 2.1.1. A sequence of elements $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{H}$ is called a *Hilbert frame* (briefly, a *frame*) if there are constants A,$B > 0$ such that for every $f \in \mathcal{H}$

$$A\left\|f\right\|^2 \leq \sum_{j\in\mathbb{N}} \left|\langle f, f_j\rangle\right|^2 \leq B\left\|f\right\|^2. \tag{2.1}$$

The numbers A and B are referred to as *frame bounds*. From this point on, A shall be taken as the supremum of all lower bounds, and B the infimum of all upper bounds. If $A = B$, then $(f_j)_{j\in\mathbb{N}}$ is called a *tight frame*. Frames include and strictly extend the notion of an orthonormal basis for a Hilbert space, $\mathcal{H}$. For instance, if $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then setting $f_j = \phi_j$ in (2.1) we see that $(\phi_j)_{j\in\mathbb{N}}$ is a tight frame with constant $A = 1$. Furthermore, if $\mathcal{H}$ is finite dimensional, then one can show that $(f_j)_{j=1}^m$ is a frame for $\mathcal{H}$ if and only if $\operatorname{span}[f_j] = \mathcal{H}$. Hence, in the finite dimensional case, every basis for $\mathcal{H}$ is a frame, and, adding any arbitrary set of vectors to an existing basis will also produce a frame. Although widely studied today, the theory of frames had not received much attention until

Daubechies, Grossman, and Meyer's work in 1986 ([8]). Since then Hilbert space frames have played a fundamental role in areas such as signal and image processing, data compression, sampling theory, and quantum theory.

One very useful aspect of frames is that they can viewed as linear operators. For each frame $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{H}$, one can define a Banach space isomorphism $\Phi : (\mathcal{H}, \|\cdot\|) \to (\ell_2, \|\cdot\|_2)$ by

$$\Phi(f) = (\langle f, f_j\rangle)_{j\in\mathbb{N}}$$

for all $f \in \mathcal{H}$, where $\ell_2 = \ell_2(\mathbb{N};\mathbb{K})$. Φ is called the *analysis operator*. The Hilbert adjoint, $\Phi^* : (\ell_2, \|\cdot\|_2) \to (\mathcal{H}, \|\cdot\|)$, is called the *pre-frame operator* or *synthesis operator* and is given by

$$\Phi^*(x) = \sum_{j\in\mathbb{N}} x_j f_j$$

for every $x = (x_j)_{j\in\mathbb{N}} \in \ell_2$. One can then show that the *frame operator*, $S : \mathcal{H} \to \mathcal{H}$, defined by

$$S = \Phi^*\Phi$$

is a bounded, self-adjoint, positive definite, invertible linear operator on $\mathcal{H}$. It is easy to verify that

$$S(f) = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle f_j , \tag{2.2}$$

$$\langle S(f), f\rangle = \sum_{j=1}^{\infty} |\langle f, f_j\rangle|^2 , \quad \text{and} \tag{2.3}$$

$$f = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle S^{-1}(f_j) \tag{2.4}$$

for every $f \in \mathcal{H}$. Furthermore, we have that $A \cdot I \le S \le B \cdot I$ in the quadratic form ordering. (2.4), referred to as the frame decomposition, may very well be one of the most important results in frame theory. It is interesting to note that the sequence $(S^{-1}f_j)_{j\in\mathbb{N}}$ is also a frame for $\mathcal{H}$, called the *dual frame* to $(f_j)_{j\in\mathbb{N}}$.

For a given frame $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{H}$, define the bounded linear operator $G : \ell_2 \to \ell_2$ by

$$G = \Phi\Phi^*$$

where Φ and Φ^* are as above. If $(e_k)_{k\in\mathbb{N}}$ is the canonical orthonormal basis for ℓ_2, then we have

$$G(e_k) = (\langle f_k, f_j \rangle)_{j=1}^{\infty}.$$

Identifying G with its matrix representation, we write

$$G = (\langle f_k, f_j \rangle)_{j,k=1}^{\infty}.$$

In other words, G is the *Gram matrix* associated with the frame $(f_j)_{j\in\mathbb{N}}$. In fact G defines a bounded operator on ℓ_2 when $(f_j)_{j\in\mathbb{N}}$ is simply a Bessel sequence. In the next section we shall use the relationship between S and G and complex interpolation to obtain norm inequalities relating the frame operator S to the frame elements f_j.

2.2 NORM INEQUALITIES

To begin this section, we present a general result pertaining to arbitrary matrices $A = (a_{j,k}) \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the space of n x n complex matrices. Before doing so, recall the following.

Definition 2.2.1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. An operator $U \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is called a *partial isometry* if the restriction of U to the orthogonal complement of its kernel is an isometry.

Theorem 2.2.2. *Polar Decomposition*
Let $A \in \mathcal{B}(\mathcal{H})$. Then there exists a unique partial isometry $U \in \mathcal{B}(\mathcal{H})$ such that $A = U[A]$ and $[A] = U^*A$. If A is invertible then $U \in \mathcal{B}(\mathcal{H})$ is unitary. If $\dim(\mathcal{H}) < \infty$, then there exists a unitary U with $A = U[A]$; although such a U is generally not unique.

In the above theorem, $[A] = (A^*A)^{1/2}$ is the absolute value of A. Also recall that for $p > 0$, the Schatten p-class, $\mathcal{C}_p = \mathcal{C}_p(\mathcal{H})$, are those operators $A \in \mathcal{B}(\mathcal{H})$ for which

$$\|A\|_{\mathcal{C}_p} = (tr[A]^p)^{1/p} \ < \ \infty. \tag{2.5}$$

Equivalently, we may also express (2.5) as

$$\|A\|_{\mathcal{C}_p} = \left(\sum_{j=1}^{\infty} \sigma_j(A)^p\right)^{1/p} = \left(\sum_{j=1}^{\infty} \langle [A]^p \phi_j, \phi_j\rangle\right)^{1/p}$$

where $(\sigma_j(A))_{j\in\mathbb{N}}$ are the singular values of A and $(\phi_j)_{j\in\mathbb{N}}$ is any orthonormal basis for $\mathcal{H}$. Of course, if $A \in \mathcal{C}_p$ is positive definite, then $\sigma_j(A) = \lambda_j(A)$, where $(\lambda_j(A))_{j\in\mathbb{N}}$ are the eigenvalues of A. If $p = \infty$, then $\mathcal{C}_\infty = \mathcal{C}_\infty(\mathcal{H})$ is the space of compact operators on $\mathcal{H}$ with operator norm.

For $A = (a_{j,k}) \in M_n(\mathbb{C})$ and for all $0 < p < \infty$, let

$$\|A\|_p = \left(\sum_{j,k=1}^{n} |a_{j,k}|^p\right)^{1/p}.$$

Clearly, $\left\|(a_{j,j})_{j=1}^n\right\|_p \leq \ \|A\|_p$ for all $p > 0$. Furthermore, in case $p = 2$, it is well known that $\|A\|_{\mathcal{C}_2} = \|A\|_2$.

Lemma 2.2.3. *Let $A \in M_n(\mathbb{C})$. Then* $\left\|(a_{j,j})_{j=1}^n\right\|_p \ \leq \ \|A\|_{\mathcal{C}_p} \ \leq \ \|A\|_p$ *for all* $1 \leq p \leq 2$.

Proof. Fix $A \in M_n(\mathbb{C})$. The case when $p = 2$ was mentioned above. We now verify the result for $p = 1$. Recall that for every $x,\, y \in \mathbb{C}^n$,

$$|\langle Ax, y\rangle|^2 \ \leq \ \langle [A]x, x\rangle \, \langle [A^*]y, y\rangle \, . \tag{2.6}$$

Let $(e_j)_{j=1}^n$ be the usual orthonormal basis for $\mathbb{C}^n$. Since $\|A\|_{\mathcal{C}_p} = \|A^*\|_{\mathcal{C}_p}$ for every p, by (2.6) and Hölder's Inequality

$$\begin{aligned}
\left\|(a_{j,j})_{j=1}^n\right\|_1 &= \sum_{j=1}^n |\langle Ae_j, e_j\rangle| \ \leq \ \sum_{j=1}^n \langle [A]e_j, e_j\rangle^{1/2} \langle [A^*]e_j, e_j\rangle^{1/2} \\
&\leq \left(\sum_{j=1}^n \langle [A]e_j, e_j\rangle\right)^{1/2} \left(\sum_{j=1}^n \langle [A^*]e_j, e_j\rangle\right)^{1/2} \\
&= \|A\|_{\mathcal{C}_1}^{1/2} \, \|A^*\|_{\mathcal{C}_1}^{1/2} = \|A\|_{\mathcal{C}_1} \, ,
\end{aligned}$$

which proves the first half of the inequality. For the other, let $A = U[A]$ be the polar decomposition of A. Since $A \in M_n(\mathbb{C})$, we may assume that $U = (u_{j,k})$ is unitary so that $|u_{j,k}| \leq 1$ for all j, k. Therefore,

$$\begin{aligned}
\|A\|_{\mathcal{C}_1} &= \sum_{j=1}^n \langle [A]e_j, e_j\rangle = \sum_{j=1}^n \langle U^*Ae_j, e_j\rangle = \sum_{j=1}^n \langle Ae_j, Ue_j\rangle \\
&= \sum_{j=1}^n \left(\sum_{k=1}^n a_{k,j}\bar{u}_{k,j}\right) \ \leq \ \sum_{j,k=1}^n |a_{k,j}||\bar{u}_{k,j}| \\
&\leq \sum_{j,k=1}^n |a_{k,j}| = \|A\|_1 \, ,
\end{aligned}$$

which proves the remainder of the inequality for $p = 1$. Since the inequality is valid for $p = 1$ and $p = 2$, by Riesz-Thorin complex interpolation the inequality holds for the full range $1 \leq p \leq 2$. $\square$

Theorem 2.2.4. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $(f_j)_{j=1}^m$ a frame for $\mathcal{H}$ with frame operator S. Then*

$$\left\|(\|f_j\|^2)_{j=1}^m\right\|_p \leq \|S\|_{\mathcal{C}_p} \leq \left\|(\langle f_k, f_j\rangle)_{j,k=1}^m\right\|_p \tag{2.7}$$

for all $1 \leq p \leq 2$. In case $p = 1$ we have equality on the left and if $p = 2$ we have equality on the right.

Proof. Let $G \in M_m(\mathbb{C})$ be the associated Gram matrix for the frame $(f_j)_{j=1}^m$. Since $S = \Phi^*\Phi$ and $G = \Phi\Phi^*$, where is Φ is the analysis operator as defined in Section 2.1, S and G are positive definite with the same (positive) eigenvalues having the same multiplicities. Hence, $\|S\|_{\mathcal{C}_p} = \|G\|_{\mathcal{C}_p}$ for every $p > 0$. Futhermore, since $(G)_{j,j} = \|f_j\|^2$ and $\|G\|_p = \left\|(\langle f_j, f_k\rangle)_{j,k=1}^m\right\|_p$, the result follows by applying 2.2.3 to G. In particular, if $p = 1$, $\left\|(\|f_j\|^2)_{j=1}^m\right\|_1 = tr(G) = \|S\|_{\mathcal{C}_1}$. □

As an application of 2.2.4, for fixed $n \in \mathbb{N}$, $n \geq 3$, and for each $j \in \{0, 1, \ldots, n-1\}$, define the vectors

$$f_j = (\cos(2j\pi/n), \sin(2j\pi/n)).$$

Then one can show that $(f_j)_{j=0}^{n-1}$ is a tight frame for $\mathbb{R}^2$, constant $A = \frac{n}{2}$, with corresponding frame operator $S = \frac{n}{2}I$, where I is the identity on $\mathbb{R}^2$. The Schatten norm of S is then easily seen to be $\|S\|_{\mathcal{C}_p}^p = 2\left(\frac{n}{2}\right)^p$ for every p. Furthermore, the Gram matrix G has as its entries

$$\begin{aligned}(G)_{j,k} = \langle f_k, f_j\rangle &= \cos(2k\pi/n)\cos(2j\pi/n) + \sin(2k\pi/n)\sin(2j\pi/n) \\ &= \cos(2\pi(k-j)/n).\end{aligned}$$

Since $\|f_j\| = 1$ for all $0 \leq j \leq n-1$, by 2.2.4 we have

$$n \;\leq\; 2\left(\frac{n}{2}\right)^p \;\leq\; \sum_{j,k=0}^{n-1} |\cos(2\pi(k-j)/n)|^p$$

for all $1 \leq p \leq 2$.

We now wish to extend 2.2.4 to include the case when $\mathcal{H}$ is an infinite dimensional, separable Hilbert space. The problem is that in the infinite dimensional case, the norms in (2.7) cannot be expected to be finite. To see this, let $(\phi_j)_{j\in\mathbb{N}}$ be any orthonormal basis for $\mathcal{H}$, and hence a frame for $\mathcal{H}$. Then $\left\|(\|\phi_j\|^2)_{j\in\mathbb{N}}\right\|_p = \infty$ for every p. Furthermore, the corresponding frame operator S is the identity on $\mathcal{H}$, and therefore not compact. One way around this problem is to, in some sense, average the norms in (2.7) on certain finite dimensional subspaces of $\mathcal{H}$, to which we may apply the results of 2.2.4.

Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space and $(f_j)_{j\in\mathbb{N}}$ a frame for $\mathcal{H}$ with frame operator S. For each $n \in \mathbb{N}$, define the finite dimensional subspace

$$\mathcal{H}_n = \operatorname{span}\{f_1, \dots, f_n\}$$

of $\mathcal{H}$. By previous remarks, $\{f_1, \dots, f_n\}$ is a frame for $\mathcal{H}_n$. Define $\Phi_n : \mathcal{H}_n \to \ell_2^n$ by

$$\Phi_n(f) = (\langle f, f_j\rangle)_{j=1}^n$$

for every $f \in \mathcal{H}_n$, so that $\Phi_n^* : \ell_2^n \to \mathcal{H}_n$ is given by

$$\Phi_n^*(x) = \sum_{j=1}^n x_j f_j$$

for every $x = (x_j)_{j=1}^n \in \ell_2^n$. If we define $S_n : \mathcal{H}_n \to \mathcal{H}_n$ by

$$S_n = \Phi_n^* \Phi_n,$$

then we find

$$S_n(f) = \sum_{j=1}^n \langle f, f_j\rangle f_j \tag{2.8}$$

for every $f \in \mathcal{H}_n$. That is, S_n is the frame operator on $\mathcal{H}_n$ corresponding to $\{f_1, \dots, f_n\}$. Similarly, if we define $G_n : \ell_2^n \to \ell_2^n$ by $G_n = \Phi_n \Phi_n^*$, then $G_n = (\langle f_k, f_j\rangle)_{j,k=1}^n$ is the associated $n \times n$ Gram matrix for the frame $(f_j)_{j=1}^n$. Again, we find that $\|S_n\|_{\mathcal{C}_p(\mathcal{H}_n)} = \|G_n\|_{\mathcal{C}_p(\mathcal{H}_n)}$ for all $n \in \mathbb{N}$ and for all $p > 0$, and by 2.2.4,

$$\left\|(\|f_j\|^2)_{j=1}^n\right\|_p \le \|S_n\|_{\mathcal{C}_p(\mathcal{H}_n)} \le \left\|(\langle f_j, f_k\rangle)_{j,k=1}^n\right\|_p \tag{2.9}$$

for $1 \le p \le 2$. If $p = 1$ we have equality on the left, and if $p = 2$ we have equality on the right.

Lemma 2.2.5. *Let $P_n : \mathcal{H} \to \mathcal{H}_n$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$ and let B be the upper frame bound for $(f_j)_{j\in\mathbb{N}}$. Then $S_n P_n \to S$ in the strong operator topology.*

Proof. Let $f \in \mathcal{H}$. Since $f_j \in \mathcal{H}_n$ for all $1 \leq j \leq n$

$$S_n P_n(f) = \sum_{j=1}^{n} \langle P_n f, f_j \rangle f_j = \sum_{j=1}^{n} \langle f, f_j \rangle f_j.$$

Then

$$\begin{aligned}
\|S(f) - S_n P_n(f)\| &= \left\| \sum_{j=1}^{\infty} \langle f, f_j \rangle f_j - \sum_{j=1}^{n} \langle f, f_j \rangle f_j \right\| \\
&= \left\| \sum_{j=n+1}^{\infty} \langle f, f_j \rangle f_j \right\| \\
&= \sup_{\|g\|=1} \left| \left\langle \sum_{j=n+1}^{\infty} \langle f, f_j \rangle f_j, g \right\rangle \right| \\
&= \sup_{\|g\|=1} \left| \sum_{j=n+1}^{\infty} \langle f, f_j \rangle \langle f_j, g \rangle \right| \\
&\leq \sup_{\|g\|=1} \sum_{j=n+1}^{\infty} |\langle f, f_j \rangle| \, |\langle f_j, g \rangle| \\
&\leq \left(\sum_{j=n+1}^{\infty} |\langle f, f_j \rangle|^2 \right)^{1/2} \sup_{\|g\|=1} \left(\sum_{j=n+1}^{\infty} |\langle f_j, g \rangle|^2 \right)^{1/2} \\
&\leq \sqrt{B} \left(\sum_{j=n+1}^{\infty} |\langle f, f_j \rangle|^2 \right)^{1/2} \to 0
\end{aligned}$$

as $n \to \infty$ since $(\langle f, f_j \rangle)_{j \in \mathbb{N}} \in \ell_2(\mathbb{N})$. □

In the above lemma pointwise convergence cannot be replaced by uniform convergence. Otherwise S would be compact, which, as we have observed, is not true in general. Our first infinite dimensional result, averaging for $p = 2$, is the following.

Lemma 2.2.6. *Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space and $(f_j)_{j \in \mathbb{N}}$ a frame for $\mathcal{H}$ with upper bound B and frame operator S. Let $(S_n)_{n \in \mathbb{N}}$ be the sequence of frame operators given in (2.8). Then*

$$\sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \left\| \left(\|f_j\|^2 \right)_{j=1}^{n} \right\|_2 \leq \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \|S_n\|_{\mathcal{C}_2(\mathcal{H}_n)} = \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \left\| \left(\langle f_j, f_k \rangle \right)_{j,k=1}^{n} \right\|_2 \leq B.$$

Proof. By definition, for every $f \in \mathcal{H}$

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B \|f\|^2 .$$

Setting $f = f_j$, where f_j is an arbirary frame element, we have

$$\|f_j\|^4 \leq \sum_{k=1}^{\infty} |\langle f_j, f_k\rangle|^2 \leq B \|f_j\|^2$$

so that $\|f_j\|^2 \leq B$ for each $j \in \mathbb{N}$. Hence, by (2.9)

$$\begin{aligned}\frac{1}{\sqrt{n}} \|S_n\|_{\mathcal{C}_2} = \frac{1}{\sqrt{n}} \left\|(\langle f_j, f_k\rangle)_{j,k=1}^n\right\|_2 &= \left(\frac{1}{n}\sum_{j,k=1}^{n} |\langle f_j, f_k\rangle|^2\right)^{1/2} = \left(\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle f_j, f_k\rangle|^2\right)^{1/2} \\ &\leq \left(\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} |\langle f_j, f_k\rangle|^2\right)^{1/2} \leq \left(\frac{1}{n}\sum_{j=1}^{n} B\|f_j\|^2\right)^{1/2} \leq B\end{aligned}$$

so that

$$\frac{1}{\sqrt{n}} \left\|\left(\|f_j\|^2\right)_{j=1}^n\right\|_2 \leq \frac{1}{\sqrt{n}} \|S_n\|_{\mathcal{C}_2(\mathcal{H}_n)} = \frac{1}{\sqrt{n}} \left\|(\langle f_j, f_k\rangle)_{j,k=1}^n\right\|_2 \leq B$$

for every $n \in \mathbb{N}$. The result follows by taking the supremum over n. □

At this point we would like to prove a similar result when $p = 1$. Since $\|f_j\|^2 \leq B$ for each frame element f_j, we have

$$\left\|\left(\|f_j\|^2\right)_{j=1}^n\right\|_1 = \|S_n\|_{\mathcal{C}_1(\mathcal{H}_n)} \leq nB$$

for every n, so that

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \left\|\left(\|f_j\|^2\right)_{j=1}^n\right\|_1 = \sup_{n\in\mathbb{N}} \frac{1}{n} \|S_n\|_{\mathcal{C}_1(\mathcal{H}_n)} \leq B.$$

In keeping with 2.2.6, we can write

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \left\|\left(\|f_j\|^2\right)_{j=1}^n\right\|_1 = \sup_{n\in\mathbb{N}} \frac{1}{n} \|S_n\|_{\mathcal{C}_1(\mathcal{H}_n)} \leq \sup_{n\in\mathbb{N}} \frac{1}{n} \left\|(\langle f_j, f_k\rangle)_{j,k=1}^n\right\|_1 .$$

The problem is that the supremum on the right is not always finite. Indeed, since

$$\left(\frac{1}{n}\sum_{j=1}^{n} |a_j|^r\right)^{1/r} \leq \left(\frac{1}{n}\sum_{j=1}^{n} |a_j|^q\right)^{1/q}$$

when $0 < r < q$,

$$\frac{1}{n^2}\sum_{j,k=1}^{n}|\langle f_j, f_k\rangle| \le \left(\frac{1}{n^2}\sum_{j,k=1}^{n}|\langle f_j, f_k\rangle|^2\right)^{1/2} = \frac{1}{\sqrt{n}}\left(\frac{1}{n}\sum_{j,k=1}^{n}|\langle f_j, f_k\rangle|^2\right)^{1/2} \le \frac{1}{\sqrt{n}}B,$$

which means that $\frac{1}{n}\sum_{j,k=1}^{n}|\langle f_j, f_k\rangle|$ may grow as fast as $\sqrt{n}$. In the next section we will construct a frame that exhibits this optimal growth rate. For now, to extend the result of 2.2.6 to include the full range $1 \le p \le 2$, we define

$$\mathscr{A}\ell_p = \left\{A = (a_{j,k})_{j,k\in\mathbb{N}} : \|A\|_{\mathscr{A}\ell_p} = \sup_{n\in\mathbb{N}} \tfrac{1}{n^{1/p}}\|A_n\|_p < \infty\right\}$$

and

$$\mathscr{A}\mathcal{C}_p = \left\{A = (a_{j,k})_{j,k\in\mathbb{N}} : \|A\|_{\mathscr{A}\mathcal{C}_p} = \sup_{n\in\mathbb{N}} \tfrac{1}{n^{1/p}}\|A_n\|_{\mathcal{C}_p} < \infty\right\}$$

where $(A_n)_{j,k} = a_{j,k}$ for $1 \le j, k \le n$ and 0 otherwise.

Lemma 2.2.7. *For $1 \le p < \infty$, $\mathscr{A}\ell_p$ and $\mathscr{A}\mathcal{C}_p$ are Banach spaces.*

We prove the result for $\mathscr{A}\ell_p$. The proof for $\mathscr{A}\mathcal{C}_p$ is exactly the same with $(\ell_p, \|\cdot\|_p)$ replaced by $(\mathcal{C}_p, \|\cdot\|_{\mathcal{C}_p})$.

Proof. Fix $1 \le p < \infty$ and let $(A^{(m)})_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{A}\ell_p$ so that

$$\sup_{n\in\mathbb{N}} \tfrac{1}{n^{1/p}}\left\|A_n^{(m)} - A_n^{(l)}\right\|_p \to 0$$

as $m, l \to \infty$. Hence, we find that $(A_n^{(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in ℓ_p for all n. So, for each n there exists some B_n such that $A_n^{(m)} \to B_n$ in ℓ_p as $m \to \infty$. Clearly $(B_n)_{j,k} = 0$ unless $1 \le j, k \le n$. Define

$$C = B_1 + \sum_{i=1}^{\infty}(B_{i+1} - B_i)$$

so that $C_n = B_n$ for every n. Fix $\epsilon > 0$ and let $(m_n)_{n\in\mathbb{N}}$ be some subsequence of the natural numbers such that $\left\|C_n - A_n^{(k)}\right\|_p < \frac{\epsilon}{3}$ for every $k \ge m_n$ and for all $n \in \mathbb{N}$. Furthermore, since $(A^{(m)})_{m\in\mathbb{N}}$ is Cauchy, we can find some $K > 0$ such that $\left\|A^{(m)}\right\|_{\mathscr{A}\ell_p} \le K$ for every m. Then, for every $n \in \mathbb{N}$ we have

$$\|C_n\|_p = \left\|C_n - A_n^{(m_n)} + A_n^{(m_n)}\right\|_p \le \left\|C_n - A_n^{(m_n)}\right\|_p + \left\|A_n^{(m_n)}\right\|_p < \frac{\epsilon}{3} + \left\|A_n^{(m_n)}\right\|_p.$$

Hence, for all $n \in \mathbb{N}$,

$$\tfrac{1}{n^{1/p}} \|C_n\|_p \leq \frac{\epsilon}{3} + \left\|A^{(m_n)}\right\|_{\mathscr{A}\ell_p} \leq \frac{\epsilon}{3} + K,$$

and so

$$\|C\|_{\mathscr{A}\ell_p} \leq \frac{\epsilon}{3} + K < \infty,$$

which shows that $C \in \mathscr{A}\ell_p$.

Again, with $\epsilon > 0$ fixed as above,

$$\left\|A^{(l)} - A^{(m)}\right\|_{\mathscr{A}\ell_p} < \frac{\epsilon}{3}$$

for all l and m greater than or equal to some $L(\epsilon) \in \mathbb{N}$. Fix $l \geq L(\epsilon)$. Next, let $n \in \mathbb{N}$. Choose $k_n \in \mathbb{N}$ with $k_n \geq L(\epsilon)$ and $k_n \geq m_n$. Thus,

$$\left\|A_n^{(l)} - C_n\right\|_p \leq \left\|A_n^{(l)} - A_n^{(k_n)}\right\|_p + \left\|A_n^{(k_n)} - C_n\right\|_p < \left\|A_n^{(l)} - A_n^{(k_n)}\right\|_p + \frac{\epsilon}{3}.$$

So, for all $n \in \mathbb{N}$,

$$\tfrac{1}{n^{1/p}} \left\|A_n^{(l)} - C_n\right\|_p \leq \left\|A^{(l)} - A^{(k_n)}\right\|_{\mathscr{A}\ell_p} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

Therefore, for all $l \geq L(\epsilon)$,

$$\left\|A^{(l)} - C\right\|_{\mathscr{A}\ell_p} \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.$$

Hence $(A^{(m)})_{m\in\mathbb{N}}$ converges to C and $\mathscr{A}\ell_p$ is complete. □

The following theorem is a consequence of (2.9), 2.2.4, 2.2.6, and 2.2.7.

Theorem 2.2.8. *Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space and $(f_j)_{j\in\mathbb{N}}$ a frame for $\mathcal{H}$ with frame operator S and upper frame bound B. Suppose that $(\langle f_j, f_k\rangle)_{j,k\in\mathbb{N}} \in \mathscr{A}\ell_p$ for $1 \leq p \leq 2$. Then,*

$$\left\|(\|f_j\|^2)_{j\in\mathbb{N}}\right\|_{\mathscr{A}\ell_p} \leq \|S\|_{\mathscr{A}\mathcal{C}_p} \leq \|(\langle f_j, f_k\rangle)_{j,k\in\mathbb{N}}\|_{\mathscr{A}\ell_p} \tag{2.10}$$

for all $1 \leq p \leq 2$. In case $p = 1$ we have equality on the left and if $p = 2$ we have equality on the right. Moreover, if $p = 2$ all the norms in (2.10) are bounded above by B.

As was shown in 2.2.6, if $(f_j)_{j\in\mathbb{N}}$ is frame for $\mathcal{H}$, the corresponding Gram matrix $G = (\langle f_k, f_j\rangle)_{j,k\in\mathbb{N}} \in \mathscr{A}\ell_2$. The question remained as to what conditions $(f_j)_{j\in\mathbb{N}}$ must satisfy in order that $G \in \mathscr{A}\ell_1$. A sufficient condition is supplied by the following lemma.

Lemma 2.2.9. *Let $(f_j)_{j\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with upper bound B. If $\langle f_j, f_k\rangle \geq 0$ for all $j,k \in \mathbb{N}$, then*

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{j,k=1}^{n} \langle f_j, f_k\rangle \leq B.$$

Proof. First recall that if Φ_n^* is the synthesis operator for $(f_j)_{j=1}^n$ on $\mathcal{H}_n = \operatorname{span}\{f_1, \ldots, f_n\}$, then $\|\Phi_n^*\| \leq \sqrt{B}$. Fix $n \in \mathbb{N}$ and let $s_j,\ t_k \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be such that $|s_j|,\ |t_k|\ \leq\ 1$ for every $1\ \leq\ j,k\ \leq\ n$. Then

$$\begin{aligned}\frac{1}{n}\left|\sum_{j,k=1}^{n} s_j t_k \langle f_j, f_k\rangle\right| &= \frac{1}{n}\left|\left\langle \sum_{j=1}^{n} s_j f_j, \sum_{k=1}^{n} \bar{t_k} f_k \right\rangle\right| \leq \frac{1}{n}\left\|\sum_{j=1}^{n} s_j f_j\right\| \left\|\sum_{k=1}^{n} \bar{t_k} f_k\right\| \\ &= \frac{1}{n}\left\|\Phi_n^*\left((s_j)\right)\right\|_{\mathcal{H}_n} \left\|\Phi_n^*\left((\bar{t_k})\right)\right\|_{\mathcal{H}_n} \leq \frac{1}{n}\left\|\Phi_n^*\right\|^2 \left\|(s_j)\right\|_{\ell_2^n} \left\|(\bar{t_k})\right\|_{\ell_2^n} \leq B.\end{aligned}$$

Since $\langle f_j, f_k\rangle \geq 0$,

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{j,k=1}^{n} \langle f_j, f_k\rangle \leq \sup_{n\in\mathbb{N}} \sup_{|s_j|,|t_k|\leq 1} \frac{1}{n}\left|\sum_{j,k=1}^{n} s_j t_k \langle f_j, f_k\rangle\right| \leq B. \qquad \square$$

For the next result, we need the following.

Theorem 2.2.10. *Grothendieck's Inequality*

There is a universal constant $K_G > 0$ such that for any $n \in \mathbb{N}$ and any matrix $(a_{j,k}) \in M_n(\mathbb{K})$ we have

$$\sup_{\|x_j\|_{\mathcal{H}},\|y_k\|_{\mathcal{H}}\leq 1}\left|\sum_{j,k=1}^{n} a_{j,k}\langle x_j, y_k\rangle\right| \leq K_G \sup_{|s_j|_{\mathbb{K}},|t_k|_{\mathbb{K}}\leq 1}\left|\sum_{j,k=1}^{n} a_{j,k} s_j t_k\right|$$

where $\mathcal{H}$ is any Hilbert space and $x_j,y_k \in \mathcal{H}$ for every j and k.

Proposition 2.2.11. *Let $(f_j)_{j\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with upper bound B. Assume that $f_j \neq 0$ for all $j \in \mathbb{N}$. Then*

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{j,k=1}^{n} \frac{|\langle f_j, f_k\rangle|^2}{\|f_j\|\,\|f_k\|} \leq K_G B$$

where K_G is the constant in Grothendieck's Inequality.

Proof. Fix $n \in \mathbb{N}$ and let $x_j, y_k \in \mathcal{H}$ be such that $\|x_j\|, \|y_k\| \leq 1$ for all $1 \leq j, k \leq n$. Then, by 2.2.10 and the proof of 2.2.9 we have

$$\frac{1}{n}\left|\sum_{j,k=1}^{n} \overline{\langle f_j, f_k\rangle} \langle x_j, y_k\rangle\right| \leq K_G \sup_{|s_j|,|t_k|\leq 1} \frac{1}{n}\left|\sum_{j,k=1}^{n} s_j t_k \langle f_j, f_k\rangle\right| \leq K_G B.$$

The result now follows by letting $x_j = \frac{f_j}{\|f_j\|}$, $y_k = \frac{f_k}{\|f_k\|}$ and taking the supremum over n. $\square$

2.3 A FRAME WITH OPTIMAL GROWTH IN $\mathscr{A}\ell_1$

In the last section, we showed that for the frame $(f_j)_{j\in\mathbb{N}}$, $(\langle f_j, f_k\rangle)_{j,k\in\mathbb{N}} \in \mathscr{A}\ell_1$ provided $\langle f_j, f_k\rangle \geq 0$ for all j and k. There are many such examples of frames with this characteristic. For instance, if $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then

$$\left\{\phi_1\,,\, \frac{1}{\sqrt{2}}\phi_2\,,\, \frac{1}{\sqrt{2}}\phi_2\,,\, \frac{1}{\sqrt{3}}\phi_3\,,\, \frac{1}{\sqrt{3}}\phi_3\,,\, \frac{1}{\sqrt{3}}\phi_3\,,\, \ldots\right\}$$

is a tight frame for $\mathcal{H}$ with constant one. If we let f_j represent the jth frame element, then clearly $\langle f_j, f_k\rangle \geq 0$ for all j and k and we find that $\|(\langle f_j, f_k\rangle)_{j,k\in\mathbb{N}}\|_{\mathscr{A}\ell_1} \leq 1$.

As another example, fix $0 < \epsilon < \frac{1}{2}$ and define

$$\phi_j = \chi_{[j-1/2-\epsilon\,,\, j+1/2+\epsilon]}$$

for $j \in \mathbb{Z}$, where χ is the characteristic function on the given interval. Then $(\phi_j)_{j\in\mathbb{Z}}$ is a Riesz basis, and hence a frame, for its closed linear span in $L_2(\mathbb{R})$. Furthermore, we have

$$\begin{aligned}
\langle \phi_j, \phi_j\rangle &= 1 + 2\epsilon && \forall j \in \mathbb{Z},\\
\langle \phi_j, \phi_k\rangle &= 0 && \text{when } |j-k| \geq 2,\\
\langle \phi_j, \phi_{j+1}\rangle &= \langle \phi_{j-1}, \phi_j\rangle = 2\epsilon && \forall j \in \mathbb{Z},
\end{aligned}$$

and again we find that $(\langle \phi_j, \phi_k\rangle)_{j,k\in\mathbb{Z}} \in \mathscr{A}\ell_1$.

We also noted in the previous section that, in general, as $n \to \infty$ we may expect $\displaystyle\frac{1}{n}\sum_{j,k=1}^{n} |\langle f_j, f_k\rangle|$ to grow as fast as $\sqrt{n}$. We now construct a frame that exhibits this optimal growth rate.

Theorem 2.3.1. *There exists a frame $(f_j)_{j\in\mathbb{N}}$ for a closed subspace $\mathcal{K}$ of ℓ_2 and constant $C > 0$ such that*

$$C\sqrt{\gamma_n} \;\leq\; \frac{1}{\gamma_n}\sum_{j,k=1}^{\gamma_n} |\langle f_k, f_j\rangle| \;\leq\; \sqrt{\gamma_n}$$

for n sufficiently large, where $\gamma_n = \displaystyle\sum_{\nu=1}^{n} 2^{\nu}$.

Proof. To begin our construction we define the matrices

$$Q^{(1)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q^{(2)} = \begin{pmatrix} Q^{(1)} & Q^{(1)} \\ Q^{(1)} & -Q^{(1)} \end{pmatrix}$$

and the $2^\nu \times 2^\nu$ block matrix by

$$Q^{(\nu)} = \begin{pmatrix} Q^{(\nu-1)} & Q^{(\nu-1)} \\ Q^{(\nu-1)} & -Q^{(\nu-1)} \end{pmatrix}$$

for all $\nu \in \mathbb{N}$, $\nu > 1$. Let 0_n and I_n denote the $n \times n$ zero and identity matrix respectively. Observe that $\left(Q^{(\nu)}\right)^* = Q^{(\nu)}$ for all $\nu \in \mathbb{N}$ and $\left(Q^{(1)}\right)^2 = 2I_2$. Then by induction we have

$$\left(Q^{(\nu)}\right)^2 = \begin{pmatrix} \left(Q^{(\nu-1)}\right)^2 + \left(Q^{(\nu-1)}\right)^2 & \left(Q^{(\nu-1)}\right)^2 - \left(Q^{(\nu-1)}\right)^2 \\ \left(Q^{(\nu-1)}\right)^2 - \left(Q^{(\nu-1)}\right)^2 & \left(Q^{(\nu-1)}\right)^2 + \left(Q^{(\nu-1)}\right)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\left(Q^{(\nu-1)}\right)^2 & 0_{2^{(\nu-1)}} \\ 0_{2^{(\nu-1)}} & 2\left(Q^{(\nu-1)}\right)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2^\nu I_{2^{(\nu-1)}} & 0_{2^{(\nu-1)}} \\ 0_{2^{(\nu-1)}} & 2^\nu I_{2^{(\nu-1)}} \end{pmatrix}$$

$$= 2^\nu I_{2^\nu}$$

for every $\nu \in \mathbb{N}$. Also for all $\nu \in \mathbb{N}$, we define

$$U^{(\nu)} = \frac{1}{\sqrt{2^\nu}} Q^{(\nu)}$$

and set

$$U = \begin{pmatrix} U^{(1)} & & & \\ & U^{(2)} & & \\ & & U^{(3)} & \\ & & & \ddots \end{pmatrix}$$

$$= \operatorname{diag}[U^{(1)}, U^{(2)}, U^{(3)}, \ldots].$$

Hence, $U^* = U$ and $U^*U = UU^*$ is the identity on ℓ_p for all p. In particular, U is a unitary operator on ℓ_2. Next we define

$$P = \frac{1}{2}(I + U)$$

so that $P^* = P$ and $P^2 = P$. That is, P is the orthogonal projection of ℓ_2 onto the closed subspace $\mathcal{K} := \operatorname{Range}(P)$ of ℓ_2. Let $(e_j)_{j\in\mathbb{N}}$ be the usual orthonormal basis for ℓ_2. Then $(f_j = P(e_j))_{j\in\mathbb{N}}$ is a tight frame for $\mathcal{K}$ with constant 1. Indeed, if $f \in \mathcal{K}$ then,

$$\sum_{j=1}^{\infty} |\langle f, f_j \rangle|^2 = \sum_{j=1}^{\infty} |\langle f, P(e_j) \rangle|^2 = \sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2 = \|f\|_{\ell_2}^2 .$$

Observe that $f_j = P(e_j)$ is the the jth row of P and $\langle f_j, f_k \rangle = \langle P(e_j), P(e_k) \rangle = P_{j,k}$ for every j and k. Now, if we set

$$P^{(\nu)} = \frac{1}{2}\left(I_{2^\nu} + U^{(\nu)}\right)$$

and let $\delta_{j,k}$ be the Kronecker delta, then for each $2^\nu \,\mathrm{x}\, 2^\nu$ block of P we have

$$\begin{aligned}\sum_{j,k=1}^{2^\nu} \left|\left\langle P^{(\nu)}(e_j), P^{(\nu)}(e_k)\right\rangle\right| &= \sum_{j,k=1}^{2^\nu} \left|P_{j,k}^{(\nu)}\right| = \frac{1}{2}\sum_{j,k=1}^{2^\nu} \left|U_{j,k}^{(\nu)} + \delta_{j,k}\right| \\ &\geq \frac{1}{2}\sum_{j,k=1}^{2^\nu}\left(\left|U_{j,k}^{(\nu)}\right| - \delta_{j,k}\right) = \frac{1}{2}\left(2^{3\nu/2} - 2^\nu\right)\end{aligned}$$

for every $\nu \in \mathbb{N}$. Let $\gamma_n = \sum_{\nu=1}^{n} 2^\nu = 2(2^n - 1)$. Then

$$\begin{aligned}\frac{1}{\gamma_n}\sum_{j,k=1}^{\gamma_n} |\langle f_j, f_k\rangle| = \frac{1}{\gamma_n}\sum_{j,k=1}^{\gamma_n} |P_{j,k}| &\geq \frac{\frac{1}{2}\sum_{\nu=1}^{n}\left(2^{3\nu/2} - 2^\nu\right)}{2\,(2^n - 1)} \backsim \frac{\sum_{\nu=1}^{n} 2^{3\nu/2}}{4\cdot 2^n} = \frac{(2^{3/2})^{n+1} - 2^{3/2}}{4\cdot 2^n(2^{3/2} - 1)} \\ &\backsim \frac{(2^{3/2})^{n+1}}{4\cdot 2^n(2^{3/2} - 1)} = \frac{2^{3/2}}{4(2^{3/2} - 1)}\sqrt{2^n} \approx .3867\sqrt{2^n} \geq \frac{.3867}{\sqrt{2}}\sqrt{\gamma_n}.\end{aligned}$$

Also, since $(f_j)_{j\in\mathbb{N}}$ is a tight frame with constant 1,

$$\frac{1}{\gamma_n}\sum_{j,k=1}^{\gamma_n}|P_{j,k}| \leq \sqrt{\gamma_n}\,,$$

and we have

$$C\sqrt{\gamma_n} \leq \frac{1}{\gamma_n}\sum_{j,k=1}^{\gamma_n}|P_{j,k}| \leq \sqrt{\gamma_n}$$

for large $n\in\mathbb{N}$, where $C=\frac{.3867}{\sqrt{2}}$. $\square$

3.0 GENERALIZED ROUNDNESS OF THE SCHATTEN CLASS, $\mathcal{C}_P$

3.1 INTRODUCTION

We begin this section with two definitions used by C. Lennard, A. Tonge, and A. Weston in [26]. These concepts were originally introduced by P. Enflo in [16], [17], and [18] to study the uniform structure of metric spaces.

Definition 3.1.1. A metric space (X, d) has *roundness* q, denoted $q \in r(X, d)$, if whenever a_1, a_2, b_1, b_2 are in X we have

$$d(a_1, a_2)^q + d(b_1, b_2)^q \leq \sum_{1 \leq i\,,\, j \leq 2} d(a_i, b_j)^q.$$

Definition 3.1.2. A metric space (X, d) has *generalized roundness* q, denoted $q \in gr(X, d)$, if for every $n \geq 2$ and every pair $a_1, \ldots, a_n, b_1, \ldots, b_n$ of n-tuples in X,

$$\sum_{1 \leq i < j \leq n} (d(a_i, a_j)^q + d(b_i, b_j)^q) \leq \sum_{1 \leq i\,,\, j \leq n} d(a_i, b_j)^q.$$

These concepts can also be defined for quasi-metric spaces. Recall that d is a quasi-metric on X if it satisfies $d(x, x) = 0$ and $d(x, y) = d(y, x) \geq 0$ for every x,$y \in X$. It is easy to verify that $1 \in r(X, d)$ and $q \in gr(X, d)$ implies $q \in r(X, d)$ for every metric space X. The definitions given above originally appeared in [16] and [17], where Enflo defines the roundness of a metric space, X, to be $\sup\{q : q \in r\,(X, d)\}$ and the generalized roundness to be $\sup\{q : q \in gr\,(X, d)\}$. In [16], Enflo proves that for $1 \leq p \leq 2$, the maximal roundness of $L_p(\mu)$ is p, and as a result, proves that an infinite dimensional $L_{p_1}(\mu_1)$ is not uniformly homeomorphic with $L_{p_2}(\mu_2)$ for $p_1 \neq p_2$, $1 \leq p_1, p_2 \leq 2$ (a result conjectured by Lindenstrauss in [28]). In lectures given at Kent State University, he also indicated that for $1 \leq p \leq 2$, $L_p(\mu)$ has maximal generalized roundness p. This was proved explicitly for L_2 in [17], and was used to constuct a countable metric space that is not uniformly homeomorphic to any subset of $L_2(0,1)$.

In [26], Lennard, Tonge, and Weston develop a rudimentary theory of generalized roundness. In particular, they show that $q \in gr(X,d)$ if and only if (X,d) has negative type q. As a consequence, $q_1 \in gr(X,d)$ implies $q_2 \in gr(X,d)$ for all $0 \leq q_2 < q_1$. Recall that a metric space (X,d) has negative type q if for all $n \in \mathbb{N}$ it satisifies

$$\sum_{j,k=1}^{n} d(x_j, x_k)^q \xi_j \xi_k \leq 0$$

for all $x_1, \ldots, x_n \in X$ and for all choices of real numbers $\xi_1, \ldots, \xi_n$ such that $\sum_{j=1}^{n} \xi_j = 0$. If X is a normed space, this is also equivalent to $\exp(-\|\cdot\|_X^q)$ being positive definite on X. The main result of [26] is the following.

Theorem 3.1.3. *For $2 < p \leq \infty$, if $L_p(\Omega, \Sigma, \mu)$ is at least three-dimensional then it fails to have generalized roundness q for any $q > 0$.*

The aim of the present chapter is to determine the maximal generalized roundness of the Schatten class $\mathcal{C}_p$ on an infinite dimensional Hilbert space for $0 < p < 2$. The case $p \geq 2$ was resolved in [26]. For $p > 2$, $\mathcal{C}_p$ contains a copy of ℓ_p, which has maximal generalized roundess 0 from 3.1.3. Hence, for $p > 2$, $\max\{q : q \in gr(\mathcal{C}_p)\} = 0$. Since $\mathcal{C}_2$ is a Hilbert space, its maximal generalized roundness is 2.

The results in the following section depend heavily on the use of Boolean algebras of projections. Before proceeding, we recall the following definition. In the definition, $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X, and I_X denotes the identity on X. Furthermore, $P \in \mathcal{L}(X)$ is a projection if it satisfies $P^2 = P$.

Definition 3.1.4. A *Boolean algebra of projections* (briefly, B.a.) in a (complex) Banach space X is a commuting family $\mathcal{E} \subseteq \mathcal{L}(X)$ of projections such that

$$PQ \in \mathcal{E}$$

and

$$I_X - P \in \mathcal{E}$$

whenever P, $Q \in \mathcal{E}$.

The lattice operations are defined by

$$P \wedge Q = PQ$$

and

$$P \vee Q = P + Q - PQ$$

with complementation defined by $P^c = I_X - P$. The B.a. is said to be *bounded* if

$$\|\mathcal{E}\| = \sup\{\|P\| : P \in \mathcal{E}\} < \infty.$$

An *atom* of a B.a. is a nonzero element $P \in \mathcal{E}$ that cannot be written in the form $P = Q_1 \vee Q_2$ with Q_1, $Q_2 \in \mathcal{E}$, Q_1, $Q_2 \neq P$. Equivalently, a nonzero element P is an atom of $\mathcal{E}$ if and only if $P \wedge Q = P$ or $P \wedge Q = 0$ for every $Q \in \mathcal{E}$.

Given two commuting Boolean algebras $\mathcal{E}$ and $\mathcal{F}$ of projections in X, there exists a smallest Boolean algebra in $\mathcal{L}(X)$ containing $\mathcal{E}$ and $\mathcal{F}$. It is called the B.a. generated by $\mathcal{E}$ and $\mathcal{F}$, denoted by $\mathcal{E} \vee \mathcal{F}$. Elements of $\mathcal{E} \vee \mathcal{F}$ have the form

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \eta_{ij} P_i Q_j$$

where $\sum_{i=1}^{n} P_i = \sum_{j=1}^{m} Q_j = I_X$ with $P_i \in \mathcal{E}$, $Q_j \in \mathcal{F}$, and $\eta_{ij} \in \{0,1\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

3.2 NON-EXISTENCE OF A BICONTINUOUS OPERATOR FROM $\mathcal{C}_P$ TO L_Q, $0 < P, Q < 2$

In the concluding remarks of [32], McCarthy shows that if $0 < p < \infty$, $p \neq 2$, there is no linear, bi-continuous map between $\mathcal{C}_p$ on an infinite dimensional Hilbert space $\mathcal{H}$ and any subspace of any L_p-space. His argument, however, works just as well to show that $\mathcal{C}_p$ is not linearly isomorphic to L_q for any $0 < p,\ q < \infty$, p,$q \neq 2$. This result is of such importance to determining the maximal generalized roundness of $\mathcal{C}_p$ that the we present the full details of his argument in this section, with some modifications (we consider the special case $0 < p\,, q < 2$). His argument relies on the fact that if $\mathcal{E}$ and $\mathcal{F}$ are bounded Boolean algebras of projections on L_q, then $\mathcal{E} \vee \mathcal{F}$ is also bounded on L_q. This was proved explicity in [31] for $1 \leq q < \infty$, but happens to be valid in the full range $0 < q < \infty$ under additional assumptions. A verification is provided in the Appendix.

Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. For each $n \in \mathbb{N}$, let P_n be the orthogonal projection of $\mathcal{H}$ onto span$[\phi_n]$ defined by $P_n = \langle \,\cdot\,, \phi_n \rangle \phi_n$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{K})$. If $x \in \mathcal{H}$, and $m, n \in \mathbb{N}$ with $m > n$ then

$$\left\| \sum_{j=n+1}^{m} a_j P_j x \right\|_{\mathcal{H}}^2 = \left\| \sum_{j=n+1}^{m} a_j \langle x, \phi_j \rangle \phi_j \right\|_{\mathcal{H}}^2 = \sum_{j=n+1}^{m} |a_j|^2 |\langle x, \phi_j \rangle|^2$$

$$\leq \|a\|_{\ell_\infty}^2 \sum_{j=n+1}^{m} |\langle x, \phi_j \rangle|^2 \to 0$$

as n,$m \to \infty$. Hence $\displaystyle\sum_{n=1}^{\infty} a_n P_n$ is a bounded linear operator on $\mathcal{H}$ with $\left\| \displaystyle\sum_{n=1}^{\infty} a_n P_n \right\|_{\mathcal{B}(\mathcal{H})} \leq \|a\|_{\ell_\infty}$. For $0 < p < \infty$ and $n \in \mathbb{N}$, define the operators E_n and F_n on $\mathcal{C}_p$ by

$$E_n(T) = P_n T$$

and

$$F_n(T) = T P_n$$

for all $T \in \mathcal{C}_p$. We now make use of the following analogue of Holder's inequality for $\mathcal{C}_p$ (see, for instance, [12] or [32]).

Theorem 3.2.1. *Let $T \in \mathcal{C}_p$ and $S \in \mathcal{C}_q$. Then $TS \in \mathcal{C}_r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for $0 < p, q, r \leq \infty$. Moreover, $\|TS\|_{\mathcal{C}_r} \leq \|T\|_{\mathcal{C}_p} \|S\|_{\mathcal{C}_q}$. Further, if $p = r$ and $q = \infty$, the result remains true if $\mathcal{C}_\infty$ is replaced everywhere by $\mathcal{B}(\mathcal{H})$.*

As a result of 3.2.1 we have

$$\left\| \sum_{n=1}^{\infty} a_n E_n(T) \right\|_{\mathcal{C}_p} = \left\| \sum_{n=1}^{\infty} a_n P_n T \right\|_{\mathcal{C}_p} \leq \left\| \sum_{n=1}^{\infty} a_n P_n \right\|_{\mathcal{B}(\mathcal{H})} \|T\|_{\mathcal{C}_p} = \|a\|_{\ell_\infty} \|T\|_{\mathcal{C}_p}$$

so that for $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{K})$ we find that $\sum_{n=1}^{\infty} a_n E_n$ is a bounded linear operator on $\mathcal{C}_p$ and $\left\| \sum_{n=1}^{\infty} a_n E_n \right\|_{op} \leq \|a\|_{\ell_\infty}$. We have a similar statement for the F_n's.

Define the sets

$$\mathcal{E} = \left\{ \sum_{n=1}^{\infty} a_n E_n : a_n \in \{0, 1\} \right\}$$

and

$$\mathcal{F} = \left\{ \sum_{n=1}^{\infty} a_n F_n : a_n \in \{0, 1\} \right\}.$$

Then $\mathcal{E}$ and $\mathcal{F}$ are Boolean algebras of projections of bound 1 on $\mathcal{C}_p$ for $0 < p < \infty$ with $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ as atoms. We verify for the set $\mathcal{E}$. Since $a_n \in \{0, 1\}$

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} a_n E_n \right)^2 (T) &= \sum_{n=1}^{\infty} a_n E_n \left(\sum_{m=1}^{\infty} a_m E_m(T) \right) = \sum_{n=1}^{\infty} a_n E_n \left(\sum_{m=1}^{\infty} a_m P_m T \right) \\
&= \sum_{n=1}^{\infty} a_n \left(\sum_{m=1}^{\infty} a_m E_n(P_m T) \right) = \sum_{n=1}^{\infty} a_n \left(\sum_{m=1}^{\infty} a_m P_n P_m T \right) \\
&= \sum_{n=1}^{\infty} a_n^2 P_n^2 T = \sum_{n=1}^{\infty} a_n P_n T = \sum_{n=1}^{\infty} a_n E_n(T).
\end{aligned}$$

Commutativity can verfied by observing that $E_n E_m = E_m E_n$ for all n,$m \in \mathbb{N}$. Since $E_n E_m = 0$ if $n \neq m$,

$$\left(\sum_{n=1}^{\infty} a_n E_n \right) \left(\sum_{m=1}^{\infty} b_m E_m \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n b_m E_n E_m = \sum_{n=1}^{\infty} a_n b_n E_n \in \mathcal{E}$$

since $a_nb_n \in \{0,1\}$ for all n. Futhermore,

$$\left(I_{\mathcal{C}_p} - \sum_{n=1}^{\infty} a_n E_n\right) T = T - \sum_{n=1}^{\infty} a_n P_n T = \left(I_{\mathcal{H}} - \sum_{n=1}^{\infty} a_n P_n\right) T$$
$$= \left(\sum_{n=1}^{\infty} P_n - \sum_{n=1}^{\infty} a_n P_n\right) T = \sum_{n=1}^{\infty} (1-a_n) P_n T$$
$$= \sum_{n=1}^{\infty} b_n E_n(T)$$

where $b_n = 1 - a_n \in \{0,1\}$. Hence $I_{\mathcal{C}_p} - \sum_{n=1}^{\infty} a_n E_n \in \mathcal{E}$. It is easy to verify that the lattice operations for $\mathcal{E}$ satisty the commutative, associative, distributive, identity and complementation laws, and that $\|\mathcal{E}\| = 1$. Finally, if $\sum_{m=1}^{\infty} a_m E_m \in \mathcal{E}$ then for any $n \in \mathbb{N}$

$$E_n \wedge \sum_{m=1}^{\infty} a_m E_m = E_n \left(\sum_{m=1}^{\infty} a_m E_m\right) = \sum_{m=1}^{\infty} a_m E_n E_m = a_n E_n$$

where $a_n \in \{0,1\}$. So, we find that E_n is an atom of $\mathcal{E}$ for every n.

Observe that if $T \in \mathcal{C}_p$, then $E_nF_m(T) = E_n(TP_m) = P_nTP_m = F_m(P_nT) = F_mE_n(T)$ for any n,$m \in \mathbb{N}$, so that $\mathcal{E}$ and $\mathcal{F}$ commute. Let $\mathcal{G} = \mathcal{E} \vee \mathcal{F}$ be the Boolean algebra of projections generated by $\mathcal{E}$ and $\mathcal{F}$. Then elements of $\mathcal{G}$ have the form

$$\sum_{n,m=1}^{\infty} a_{nm} E_n F_m$$

where $a_{nm} \in \{0,1\}$ for all n and m. Viewing operators $T \in \mathcal{C}_p$ as $T = (t_{j,k})_{j,k\in\mathbb{N}} = (\langle T\phi_k, \phi_j\rangle)_{j,k\in\mathbb{N}}$ we see that

$$\left\langle \sum_{n,m} a_{nm} E_n F_m T \phi_k, \phi_j \right\rangle = \left\langle \sum_{n,m} a_{nm} P_n T P_m \phi_k, \phi_j \right\rangle = \sum_n \left\langle \sum_m a_{nm} P_n T P_m \phi_k, \phi_j \right\rangle$$
$$= \sum_n a_{nk} \langle P_n T \phi_k, \phi_j\rangle = \sum_n a_{nk} \langle T\phi_k, P_n \phi_j\rangle$$
$$= a_{jk} \langle T\phi_k, \phi_j\rangle = a_{jk} t_{j,k}.$$

Thus, an operator on $\mathcal{C}_p$ of the form $\sum_{n,m=1}^{\infty} a_{nm} E_n F_m$ carries the operator $(t_{j,k})_{j,k\in\mathbb{N}}$ to the operator $(a_{j,k} t_{j,k})_{j,k\in\mathbb{N}}$.

For the moment, we restrict attention to finite dimensional subspaces of $\mathcal{H}$. Let $\mathcal{H}^{(n)} = \text{span}[\phi_1, \ldots, \phi_n]$ and let $P^{(n)}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^{(n)}$ for all $n \in \mathbb{N}$. For $0 < p < \infty$, define

$$\mathcal{C}_p^{(n)} = \left\{ T^{(n)} = P^{(n)} T P^{(n)} : T \in \mathcal{C}_p \right\}.$$

Representing $T^{(n)} \in \mathcal{C}_p^{(n)}$ by its matrix entries we have

$$T^{(n)} = (t_{j,k})_{j,k=1}^n = (\langle T\phi_k, \phi_j \rangle)_{j,k=1}^n .$$

Define

$$\mathcal{E}^{(n)} = \left\{ \sum_{k=1}^n a_k E_k : a_k \in \{0,1\} \right\}$$

and

$$\mathcal{F}^{(n)} = \left\{ \sum_{k=1}^n b_k F_k : b_k \in \{0,1\} \right\}.$$

Then $\mathcal{E}^{(n)} \subset \mathcal{E}$ and $\mathcal{F}^{(n)} \subset \mathcal{F}$ are commuting Boolean algebras of projections of bound 1 on $\mathcal{C}_p^{(n)}$. Let $\mathcal{G}^{(n)} = \mathcal{E}^{(n)} \vee \mathcal{F}^{(n)}$ be the Boolean algebra of projections generated by $\mathcal{E}^{(n)}$ and $\mathcal{F}^{(n)}$. Then $\mathcal{G}^{(n)} \subset \mathcal{G}$ so that $\left\| \mathcal{G}^{(n)} \right\| \leq \|\mathcal{G}\|$ for all $n \in \mathbb{N}$. As above, we find that elements of $\mathcal{G}^{(n)}$ have the form

$$\sum_{j,k=1}^n a_{jk} E_j F_k$$

where $a_{jk} \in \{0,1\}$. For $n \in \mathbb{N}$ define the operator $U^{(n)} = (u_{j,k})_{j,k=1}^n$ on $\mathcal{H}^{(n)}$ by $u_{j,k} = \frac{1}{\sqrt{n}} \omega_n^{jk}$ where $\omega_n = e^{\frac{2\pi i}{n}}$ is a primitive n^{th} root of unity. Then $U^{(n)}$ is unitary and we have $\left\| U^{(n)} \right\|_{\mathcal{C}_p^{(n)}} = n^{1/p}$. Let $T^{(n)} = (t_{j,k})_{j,k=1}^n \in \mathcal{C}_p^{(n)}$ be given by $t_{j,k} = 1/\sqrt{n}$ for $1 \leq j,k \leq n$. Then $T^{(n)} = \sqrt{n} Q^{(n)}$ where $Q^{(n)} = (q_{j,k})_{j,k=1}^n$ is a rank one self adjoint projection given by $q_{j,k} = 1/n$ for $1 \leq j,k \leq n$ so that $\left\| T^{(n)} \right\|_{\mathcal{C}_p^{(n)}} = \sqrt{n} \, \|Q\|_{\mathcal{C}_p^{(n)}} = \sqrt{n}$. Hence, the operator $\sum_{j=1}^n \sum_{k=1}^n \omega_n^{jk} E_j F_k$ on $\mathcal{C}_p^{(n)}$ carries the operator $T^{(n)}$ to the operator $U^{(n)}$ and

$$\left\| \sum_{j=1}^n \sum_{k=1}^n \omega_n^{jk} E_j F_k \right\|_{op} \geq \frac{\left\| \left(\sum_{j=1}^n \sum_{k=1}^n \omega_n^{jk} E_j F_k \right) T^{(n)} \right\|_{\mathcal{C}_p^{(n)}}}{\| T^{(n)} \|_{\mathcal{C}_p^{(n)}}} = n^{1/p - 1/2}. \tag{3.1}$$

Let G be an operator of the form $G = \sum_{j,k=1}^{n} \alpha_{jk} E_j F_k$, where $\alpha_{jk} \in \mathbb{C}$ for all $1 \leq j,k \leq n$. Let $A_1 = \{j,k : \mathrm{Re}(\alpha_{jk}) \geq 0\}$, $A_2 = \{j,k : \mathrm{Re}(\alpha_{jk}) < 0\}$, $A_3 = \{j,k : \mathrm{Im}(\alpha_{jk}) \geq 0\}$, and $A_4 = \{j,k : \mathrm{Im}(\alpha_{jk}) < 0\}$. Then

$$\begin{aligned}\|G\| &= \left\| \sum_{A_1} \mathrm{Re}(\alpha_{jk}) E_j F_k - \sum_{A_2} |\mathrm{Re}(\alpha_{jk})| E_j F_k + i \sum_{A_3} \mathrm{Im}(\alpha_{jk}) E_j F_k - i \sum_{A_4} |\mathrm{Im}(\alpha_{jk})| E_j F_k \right\| \\ &\leq \sum_{i=1}^{2} \left\| \sum_{A_i} |\mathrm{Re}(\alpha_{jk})| E_j F_k \right\| + \sum_{i=3}^{4} \left\| \sum_{A_i} |\mathrm{Im}(\alpha_{jk})| E_j F_k \right\| \\ &\leq \max_{j,k} |\alpha_{jk}| \sum_{i=1}^{4} \left\| \sum_{A_i} E_j F_k \right\| .\end{aligned}$$

Since $\sum_{A_i} E_j F_k \in \mathcal{G}^{(n)}$ for each $1 \leq i \leq 4$, we find

$$\|G\| \leq 4 \max_{j,k} |\alpha_{jk}| \left\| \mathcal{G}^{(n)} \right\| .$$

As a result, from (3.1) we find

$$\|\mathcal{G}\| \geq \left\| \mathcal{G}^{(n)} \right\| \geq \frac{1}{4} n^{1/p - 1/2}$$

for every $n \in \mathbb{N}$. Hence, for $0 < p < 2$, $\mathcal{G}$ is unbounded.

Suppose that A is a linear, one-to-one operator from $\mathcal{C}_p$ onto some subspace of some L_q-space, $0 < p,q < 2$. Then $A\mathcal{E}A^{-1}$ and $A\mathcal{F}A^{-1}$ are commuting Boolean algebras of projections on L_q with bound at most $\|A\|_{op} \|A^{-1}\|_{op}$. By [31] (and results in the Appendix in case $0 < q < 1$), $A\mathcal{G}A^{-1} = A\mathcal{E}A^{-1} \vee A\mathcal{F}A^{-1}$ is a bounded Boolean algebra of projections on L_q, with bound at most $K \|A\|_{op}^2 \|A^{-1}\|_{op}^2$, where $K > 0$ is some constant valid uniformly for $0 < q < 2$ (see [31] or Appendix for details). Hence,

$$\|\mathcal{G}\| = \left\| A^{-1} A \mathcal{G} A^{-1} A \right\| \leq \left\| A^{-1} \right\|_{op} \left\| A \mathcal{G} A^{-1} \right\| \|A\|_{op} \leq K \|A\|_{op}^3 \left\| A^{-1} \right\|_{op}^3 .$$

Since $\mathcal{G}$ is unbounded in case $0 < p < 2$ we obtain a contradiction, so that no such A exists.

3.3 GENERALIZED ROUNDNESS OF $\mathcal{C}_P$

We begin this section with a result due to Bretagnolle, Dacunha-Castelle, and Krivine ([2]). The statement and proof can also be found in [39], Corollary 5.8, page 22.

Corollary 3.3.1. *If $0 < p \le 2$ and $\|x-y\|^{p/2}$ is of negative type on the real linear space B, then B is linearly isometric to a subspace of an L_q space for $0 < q < p$.*

In the above Corollary, $\|x-y\|^{p/2}$ having negative type is equivalent to saying that the space $(B, \|\cdot\|)$ has negative type p. Futhermore, it is only required that $\|\cdot\|$ be p-homogeneous in the sense that for some $p > 0$, $\|\lambda x\| = |\lambda|^p \|x\|$ for $x \in B$ and λ in the scalar field; for instance, the F-norm on $\mathcal{C}_p$ (and L_p) for $0 < p < 1$, defined by

$$\|T\|_{\mathcal{C}_p}^p = \sum_{j=1}^{\infty} \sigma_j(T)^p.$$

For $0 < p < 1$, the F-norm on $\mathcal{C}_p$ is p-homogeneous and satisfies the triangle. The quasi-norm

$$\|T\|_{\mathcal{C}_p} = \left(\sum_{j=1}^{\infty} \sigma_j(T)^p \right)^{1/p},$$

on the other hand, is 1-homogeneous but only satisfies

$$\|T+S\|_{\mathcal{C}_p} \le K_p \left(\|T\|_{\mathcal{C}_p} + \|S\|_{\mathcal{C}_p} \right)$$

where $K_p = 2^{1/p-1} > 1$.

Theorem 3.3.2. *For $0 < p < \infty$, $p \neq 2$, the maximal generalized roundness of $\mathcal{C}_p$ on an infinite dimensional, separable Hilbert space $\mathcal{H}$ is 0.*

Proof. We need only verify the result for $0 < p < 2$. Also, for $0 < p < 1$, we will be using the F-norm on $\mathcal{C}_p$, which we will denote by $\|\cdot\|_p$. Let then $0 < p < 2$ and suppose $\mathcal{C}_p$ over $\mathbb{C}$ has generalized roundness q_1 for some $0 < q_1 < 2$. Since $\mathcal{C}_p$ is an F-normed space over $\mathbb{C}$, it is also an F-normed space over $\mathbb{R}$ with generalized roundness q_1. Hence, by 3.3.1, the corresponding real $\mathcal{C}_p$ space is linearly isometric to some subspace of some (real) $L_q^{\mathbb{R}}$ space for $0 < q < q_1$. Write

$$\mathcal{C}_p = \mathcal{S}_p \oplus i\mathcal{S}_p$$

where

$$\mathcal{S}_p = \{T \in \mathcal{C}_p : T = T^*\}$$

is a real subspace of $\mathcal{C}_p$. Similarly we define the corresponding complex L_q space by

$$L_q \;=\; L_q^{\mathbb{R}} \,\oplus\, iL_q^{\mathbb{R}}.$$

Denote by V the linear isometry given by 3.3.1. Then V also linearly and isometrically embeds $\mathcal{S}_p$ into $L_q^{\mathbb{R}}$. Define the real linear operator $W : i\mathcal{S}_p \to iL_q^{\mathbb{R}}$ by

$$W(iT) = iV(T)$$

for all $iT \in i\mathcal{S}_p$ so that W is also a linear isometry. Using V and W we can now define the operator $A : \mathcal{C}_p \to L_q$ by

$$A(T) = V(T_1) + W(iT_2) = V(T_1) + iV(T_2)$$

for every $T = T_1 + iT_2 \in \mathcal{C}_p$, where T_1, $T_2 \in \mathcal{S}_p$. We check that A is complex linear. If $a+ib \in \mathbb{C}$, a, $b \in \mathbb{R}$, and $T = T_1+iT_2 \in \mathbb{C}_p$ for T_1, $T_2 \in \mathcal{S}_p$, then $(a+ib)T = aT_1 - bT_2 + i(aT_2 + +bT_1)$. Hence,

$$\begin{aligned}
A((a+ib)T) &= V(aT_1 - bT_2) + iV(aT_2 + bT_1)\\
&= aV(T_1) - bV(T_2) + iaV(T_2) + ibV(T_1)\\
&= (a+ib)V(T_1) + i(a+ib)V(T_2)\\
&= (a+ib)A(T).
\end{aligned}$$

Define an F-norm on $\mathcal{C}_p$ by

$$|||T|||_{\mathcal{C}_p} = |||T_1 + iT_2|||_{\mathcal{C}_p} = \|T_1\|_{\mathcal{C}_p} + \|T_2\|_{\mathcal{C}_p}.$$

We claim that $\mathcal{C}_p$ is complete with respect to $|||\cdot|||_{\mathcal{C}_p}$. Let $(T^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to $|||\cdot|||_{\mathcal{C}_p}$. If $T^{(n)} = T_1^{(n)} + iT_2^{(n)}$ for all $n \in \mathbb{N}$, where $T_j^{(n)} \in \mathcal{S}_p$ for $j = 1$, 2, then

$$|||T^{(n)} - T^{(m)}|||_{\mathcal{C}_p} = \left\|T_1^{(n)} - T_1^{(m)}\right\|_{\mathcal{C}_p} + \left\|T_2^{(n)} - T_2^{(m)}\right\|_{\mathcal{C}_p} \to 0$$

as $n,m \to \infty$. Then there exists a $\tilde{T}_1$, $\tilde{T}_2 \in \mathcal{C}_p$ such that $T_j^{(n)} \to \tilde{T}_j$ for $j = 1, 2$. Also observe

$$\left\| (T_i^{(n)} - \tilde{T}_i)^* \right\|_{\mathcal{C}_p} = \left\| T_i^{(n)} - \tilde{T}_i^{\,*} \right\|_{\mathcal{C}_p} \to 0$$

so that $\tilde{T}_j^{\,*} = \tilde{T}_j$ and we find $\tilde{T}_j \in \mathcal{S}_p$. If we set $\tilde{T} = \tilde{T}_1 + i\tilde{T}_2$, then $|||T^{(n)} - \tilde{T}|||_{\mathcal{C}_p} \to 0$. Note that $\|T\|_{\mathcal{C}_p} \leq |||T|||_{\mathcal{C}_p}$ for all $T \in \mathcal{C}_p$. By the Open Mapping Theorem (see [21], page 10, Corollary 1.5 and page 8 in case $0 < p < 1$), $||| \cdot |||_{\mathcal{C}_p}$ is a Lipschitz equivalent F-norm on $\mathcal{C}_p$.

Similarly, we define an equivalent norm on L_q by

$$|||f|||_q = |||f_1 + if_2|||_q = \|f_1\|_q + \|f_2\|_q$$

where $f_1,\ f_2 \in L_q^{\mathbb{R}}$. Since V and W are isometries, A is a complex linear isometry of the space $\mathcal{C}_p$ with Lipschitz equivalent F-norm $||| \cdot |||_{\mathcal{C}_p}$ onto the subspace $V(\mathcal{C}_p) \oplus iV(\mathcal{C}_p)$ of L_q with Lipschitz equivalent F-norm $||| \cdot |||_q$.

Let $B_1,B_2,C_1,C_2 > 0$ be such that

$$B_1(\|T_1\|_{\mathcal{C}_p} + \|T_2\|_{\mathcal{C}_p}) \leq \|T\|_{\mathcal{C}_p} \leq B_2(\|T_1\|_{\mathcal{C}_p} + \|T_2\|_{\mathcal{C}_p})$$

and

$$C_1(\|f_1\|_q + \|f_2\|_q) \leq \|f\|_q \leq C_2(\|f_1\|_q + \|f_2\|_q)$$

for $T = T_1 + iT_2$ and $f = f_1 + if_2$. If $A(T) = f$ then we find

$$\frac{C_1}{B_2} \|T\|_{\mathcal{C}_p} \leq \|f\|_q \leq \frac{C_2}{B_1} \|T\|_{\mathcal{C}_p}.$$

Hence $(\mathcal{C}_p, \|\cdot\|_{\mathcal{C}_p})$ is linearly isomorphic to a closed subspace of $(L_q, \|\cdot\|_q)$, and we obtain a contradiction. $\square$

APPENDIX

PALEY-LITTLEWOOD SYSTEMS

A.1

As noted in Section 3.2, the proof that $\mathcal{C}_q$ has generalized roundness 0 for $0 < q < 2$ is dependent on the Boolean algebra $A\mathcal{G}A^{-1} = A\mathcal{E}A^{-1} \vee A\mathcal{F}A^{-1}$ being bounded on L_p. In [31], Littman, McCarthy, and Riviere prove that if $\mathcal{E}$ and $\mathcal{F}$ are commuting Boolean algebras of projections on L_p for $1 \leq p < \infty$, then $\mathcal{E} \vee \mathcal{F}$ is also bounded on L_p. Their proof makes use of the dual space $L_{p'}$ of L_p, which is unavailable for $0 < p < 1$. However, by adapting their arguments to the class of operators defined in Section 3.2, we can show this result can be extended to include values of p in the interval $(0, 1)$ as well. Much of the work is contained the following analogue of Khintchine's inequality, which is Lemma 6.1, p. 207 ([31]).

We denote by T the infinite dimensional torus $T = \times_{n=1}^{\infty} \{\theta_n : 0 \leq \theta_n < 2\pi\}$ with measure $d\theta = \displaystyle\prod_{n=1}^{\infty} \frac{d\theta_n}{2\pi}$. Fix $M \in \mathbb{N}$ and let $N = (n_1, \dots, n_M)$, where $n_i \in \mathbb{N}$ for $1 \leq i \leq M$. Let $\mathcal{H}$ be a Hilbert space and $a_N \in \mathcal{H}$ for each multi-index $N \in \mathbb{N}^M$. Let $T^M = T^{(1)} \times \cdots \times T^{(M)}$ where $T^{(i)} = T$ for all $1 \leq i \leq M$. Endow T^M with the measure $d\Theta = d\theta^{(1)} \cdots d\theta^{(M)}$ so that $\left(T^{(M)}, d\Theta\right)$ is a probability space for each M. Let $\Theta_N = \theta_{n_1}^{(1)} + \cdots + \theta_{n_M}^{(M)}$.

Lemma A.1.1. *Let $0 < p < \infty$. There exists a constant K_p depending only upon p such that*

$$(K_p^p)^{-M} \left(\sum_N \|a_N\|_{\mathcal{H}}^2 \right)^{p/2} \leq \int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^p d\Theta \leq (K_p^p)^M \left(\sum_N \|a_N\|_{\mathcal{H}}^2 \right)^{p/2}.$$

Proof. First observe that $K_2 = 1$, in which case we have equality. That is, for any $M \in \mathbb{N}$

$$\int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^2 d\Theta = \sum_N \|a_N\|_{\mathcal{H}}^2 .$$

Indeed,

$$\begin{aligned}
\int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^2 d\Theta &= \int_{T^M} \left\langle \sum_N e^{i\Theta_N} a_N, \sum_K e^{i\Theta_K} a_K \right\rangle d\Theta \\
&= \sum_{N,K} \langle a_N, a_K \rangle \int_{T^M} e^{i(\Theta_N - \Theta_K)}\, d\Theta \\
&= \sum_N \|a_N\|_{\mathcal{H}}^2 \int_{T^M} d\Theta + \sum_{N \neq K} \langle a_N, a_K \rangle \int_{T^M} e^{i(\Theta_N - \Theta_K)}\, d\Theta \\
&= \sum_N \|a_N\|_{\mathcal{H}}^2 + \sum_{N \neq K} \langle a_N, a_K \rangle\, I_{N,K}
\end{aligned}$$

where

$$I_{N,K} = \int_{T^M} e^{i(\Theta_N - \Theta_K)}\, d\Theta = \prod_{j=1}^{M} \int_{T^{(j)}} e^{i(\theta_{n_j}^{(j)} - \theta_{k_j}^{(j)})} d\theta^{(j)}.$$

Since $N \neq K$ there is some $1 \leq \alpha \leq M$ such that $n_\alpha \neq k_\alpha$. For this α we have

$$\int_0^{2\pi} \int_0^{2\pi} e^{i(\theta_{n_\alpha}^{(\alpha)} - \theta_{k_\alpha}^{(\alpha)})} \frac{d\theta_{n_\alpha}^{(\alpha)} d\theta_{k_\alpha}^{(\alpha)}}{(2\pi)^2} = 0$$

so that $I_{N,K} = 0$.

Suppose $0 < p < 2$ so that $1 < \frac{2}{p}$. Since $\left(T^{(M)}, d\Theta\right)$ is a probability space, by Hölder's inequality

$$\int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^p d\Theta \leq \left(\int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^2 d\Theta \right)^{p/2} = \left(\sum_N \|a_N\|_{\mathcal{H}}^2 \right)^{p/2} .$$

Next suppose $2 < p < \infty$ so that $1 < \frac{p}{2}$. Again, by Hölder's inequality

$$\left(\sum_N \|a_N\|_{\mathcal{H}}^2 \right)^{p/2} = \left(\int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^2 d\Theta \right)^{p/2} \leq \int_{T^M} \left\| \sum_N e^{i\Theta_N} a_N \right\|_{\mathcal{H}}^p d\Theta.$$

We now prove the inequality in the opposite sense for the case $p > 2$ using induction on M. First consider the case $M = 1$, $p = 2Q$ for $Q \in \mathbb{N}$. Then

$$\begin{aligned}
\int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^{2Q} d\theta &= \int_T \left\langle \sum_n e^{i\theta_n} a_n, \sum_m e^{i\theta_m} a_m \right\rangle^Q d\theta = \int_T \left(\sum_{n,m} \langle a_n, a_m \rangle\, e^{i(\theta_n - \theta_m)} \right)^Q d\theta \\
&= \int_T \prod_{j=1}^{Q} \left(\sum_{n_j, m_j} \left\langle a_{n_j}, a_{m_j} \right\rangle e^{i(\theta_{n_j} - \theta_{m_j})} \right) d\theta \\
&= \sum_{\substack{n_1,\ldots,n_Q \\ m_1,\ldots,m_Q}} \langle a_{n_1}, a_{m_1} \rangle \cdots \left\langle a_{n_Q}, a_{m_Q} \right\rangle \int_T e^{i(\theta_{n_1}+\cdots+\theta_{n_Q})} e^{-i(\theta_{m_1}+\cdots+\theta_{m_Q})} d\theta
\end{aligned}$$

where

$$\int_T e^{i(\theta_{n_1}+\cdots+\theta_{n_Q})} e^{-i(\theta_{m_1}+\cdots+\theta_{m_Q})} d\theta = \begin{cases} 0 & \text{if } \exists\ j \text{ such that } m_j \neq n_i\ \forall\ 1 \le i \le Q \\ \\ 1 & \text{if } \forall\ 1 \le i \le Q\ \exists\ 1 \le j \le Q \text{ s.t. } n_i = m_j \end{cases}$$

Let S_Q denote the set of permutations, σ, of the set $\{1, \ldots, Q\}$. Then

$$\begin{aligned}
\int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^{2Q} d\theta &= \sum_{\substack{n_1,\ldots,n_Q \\ \sigma \in S_Q}} \left\langle a_{n_1}, a_{n_{\sigma(1)}} \right\rangle \cdots \left\langle a_{n_Q}, a_{n_{\sigma(Q)}} \right\rangle \\
&\le \sum_{\substack{n_1,\ldots,n_Q \\ \sigma \in S_Q}} \left| \left\langle a_{n_1}, a_{n_{\sigma(1)}} \right\rangle \right| \cdots \left| \left\langle a_{n_Q}, a_{n_{\sigma(Q)}} \right\rangle \right| \\
&\le \sum_\sigma \sum_{n_1,\ldots,n_Q} \|a_{n_1}\|_{\mathcal{H}} \cdots \|a_{n_Q}\|_{\mathcal{H}} \left\|a_{n_{\sigma(1)}}\right\|_{\mathcal{H}} \cdots \left\|a_{n_{\sigma(Q)}}\right\|_{\mathcal{H}} \\
&= Q! \sum_{n_1,\ldots,n_Q} \|a_{n_1}\|_{\mathcal{H}}^2 \cdots \|a_{n_Q}\|_{\mathcal{H}}^2 \\
&= Q! \left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^Q \\
&= \left(\frac{p}{2}\right)! \left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^{p/2}.
\end{aligned}$$

So, for $M = 1$ and p even,

$$\left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^{p/2} \le \int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^p d\theta \le \left(\frac{p}{2}\right)! \left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^{p/2},$$

in which case we may choose $K_p = \sqrt[p]{\left(\frac{p}{2}\right)!}$. For other $p > 2$ set $p = 2(Q+\epsilon)$ for $0 < \epsilon < 1$. Using the previous result for p even and Hölder's inequality with index $\frac{Q+1}{Q+\epsilon} > 1$,

$$\begin{aligned}
\int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^{2(Q+\epsilon)} d\theta &\leq \left(\int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^{2(Q+1)} d\theta \right)^{\frac{Q+\epsilon}{Q+1}} \\
&\leq \left((Q+1)! \left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^{Q+1} \right)^{\frac{Q+\epsilon}{Q+1}} \\
&= [(Q+1)!]^{\frac{Q+\epsilon}{Q+1}} \left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^{p/2}
\end{aligned}$$

which means we can choose $K_p = [(Q+1)!]^{1/2(Q+1)}$ for $p = 2(Q+\epsilon)$. This completes the argument for $2 < p < \infty$, $M = 1$.

For the case $0 < p < 2$, $M = 1$, define the set

$$A = \left\{ \theta \in T : \sum_n \|a_n\|_{\mathcal{H}}^2 \leq 2 \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^2 \right\}$$

and let $\lambda = \int_A d\theta$. Then $0 \leq \lambda \leq 1$ and $\int_{A^c} d\theta = 1 - \lambda$. Since $K_4^4 = 2$ we have

$$\begin{aligned}
\sum_n \|a_n\|_{\mathcal{H}}^2 &= \int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^2 d\theta = \int_A \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^2 d\theta + \int_{A^c} \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^2 d\theta \\
&\leq \left(\int_A \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^4 d\theta \right)^{1/2} \left(\int_A d\theta \right)^{1/2} + \frac{1}{2} \int_{A^c} \sum_n \|a_n\|_{\mathcal{H}}^2 \, d\theta \\
&\leq \sqrt{\lambda} \left(\int_T \left\| \sum_n e^{i\theta_n} a_n \right\|_{\mathcal{H}}^4 d\theta \right)^{1/2} + \frac{1}{2}(1-\lambda) \sum_n \|a_n\|_{\mathcal{H}}^2 \\
&\leq \sqrt{\lambda} \left(2 \left(\sum_n \|a_n\|_{\mathcal{H}}^2 \right)^2 \right)^{1/2} + \frac{1}{2}(1-\lambda) \sum_n \|a_n\|_{\mathcal{H}}^2 \\
&= \sqrt{2\lambda} \sum_n \|a_n\|_{\mathcal{H}}^2 + \frac{1}{2}(1-\lambda) \sum_n \|a_n\|_{\mathcal{H}}^2 .
\end{aligned}$$

Hence, λ must satisfy $1 \leq \sqrt{2\lambda} + \frac{1}{2}(1-\lambda)$. Solving this inequality for λ we find $\lambda \in (3-2\sqrt{2}, 3+2\sqrt{2})$ so that $\lambda > \frac{1}{7}$. Then, for $0 < p < 2$,

$$\begin{aligned}\int_T \left\|\sum_n e^{i\theta_n} a_n\right\|_{\mathcal{H}}^p d\theta &= \int_T \left(\left\|\sum_n e^{i\theta_n} a_n\right\|_{\mathcal{H}}^2\right)^{p/2} d\theta \geq \int_A \left(\left\|\sum_n e^{i\theta_n} a_n\right\|_{\mathcal{H}}^2\right)^{p/2} d\theta \\ &\geq \int_A \left(\frac{1}{2}\sum_n \|a_n\|_{\mathcal{H}}^2\right)^{p/2} d\theta = \frac{\lambda}{2^{p/2}}\left(\sum_n \|a_n\|_{\mathcal{H}}^2\right)^{p/2} \\ &> \frac{1}{14}\left(\sum_n \|a_n\|_{\mathcal{H}}^2\right)^{p/2}.\end{aligned}$$

This establishes the lemma for the case $M = 1$. Observe that in case $0 < p < 2$ the bound is independent of p. To complete the induction argument, assume the lemma is valid for $M-1$. Let $T^{M-1} = T^{(2)} \times \cdots \times T^{(M)}$, $N' = (n_2, \ldots, n_m)$, $\psi_{N'} = \theta_{n_2}^{(2)} + \cdots + \theta_{n_M}^{(M)}$, and $d\psi = d\theta^{(2)} \cdots d\theta^{(M)}$. Define H to be the Hilbert space of all $\mathcal{H}$-valued square integrable functions on T^{M-1}. If $b_{N'} \in \mathcal{H}$ for all multi-indices N' and $\psi \in T^{M-1}$, then elements of the form

$$h(\psi) = \sum_{N'} b_{N'} e^{i\psi_{N'}}$$

belong to H. Furthermore,

$$\|h\|_H^2 = \int_{T^{M-1}} \left\|\sum_{N'} b_{N'} e^{i\psi_{N'}}\right\|_{\mathcal{H}}^2 d\psi = \sum_{N'} \|b_{N'}\|_{\mathcal{H}}^2.$$

Let $n = n_1$, $N = (n, N')$, and $a_N \in \mathcal{H}$ for every N. Define the sequence $(g_n(\psi))_{n\in\mathbb{N}}$ by setting

$$g_n(\psi) = \sum_{N'} a_N e^{i\psi_{N'}}$$

so that $g_n \in H$ for every n. Applying the case $M = 1$ to the elements g_n in the Hilbert space H we have that for all $p > 0$ there is a constant K_p such that

$$K_p^{-p}\left(\sum_n \|g_n\|_H^2\right)^{p/2} \leq \int_T \left\|\sum_n e^{i\theta_n} g_n\right\|_H^p d\theta \leq K_p^p \left(\sum_n \|g_n\|_H^2\right)^{p/2}.$$

Now,

$$\sum_n e^{i\theta_n} g_n = \sum_n e^{i\theta_n} \sum_{N'} a_N e^{i\psi_{N'}} = \sum_{N'} \sum_n e^{i\theta_n} a_N e^{i\psi_{N'}}.$$

Expressing the H-norm in terms of the $\mathcal{H}$-norms of the coefficients as above, we may write

$$\left\|\sum_n e^{i\theta_n} g_n\right\|_H^p = \left(\left\|\sum_n e^{i\theta_n} g_n\right\|_H^2\right)^{p/2} = \left(\sum_{N'}\left\|\sum_n e^{i\theta_n} a_N\right\|_{\mathcal{H}}^2\right)^{p/2}$$

and

$$\sum_n \|g_n\|_H^2 = \sum_n \sum_{N'} \|a_N\|_{\mathcal{H}}^2 = \sum_N \|a_N\|_{\mathcal{H}}^2$$

so that the above inequality is equivalent to

$$K_p^{-p}\left(\sum_N \|a_N\|_{\mathcal{H}}^2\right)^{p/2} \leq \int_T \left(\sum_{N'}\left\|\sum_n e^{i\theta_n} a_N\right\|_{\mathcal{H}}^2\right)^{p/2} d\theta \leq K_p^p\left(\sum_N \|a_N\|_{\mathcal{H}}^2\right)^{p/2}.$$

Fix $\theta \in T$. Using the induction hypothesis

$$\begin{aligned} K_p^{-p(M-1)}\left(\sum_{N'}\left\|\sum_n e^{i\theta_n} a_{n,N'}\right\|_{\mathcal{H}}^2\right)^{p/2} &\leq \int_{T^{M-1}}\left\|\sum_{N'} e^{i\psi_{N'}}\sum_n e^{i\theta_n} a_{n,N'}\right\|_{\mathcal{H}}^p d\psi \\ &\leq K_p^{p(M-1)}\left(\sum_{N'}\left\|\sum_n e^{i\theta_n} a_{n,N'}\right\|_{\mathcal{H}}^2\right)^{p/2}. \end{aligned}$$

The proof is completed by integrating the above inequality with respect to θ since

$$\int_T\left(\int_{T^{M-1}}\left\|\sum_{N'} e^{i\psi_{N'}}\sum_n e^{i\theta_n} a_{n,N'}\right\|_{\mathcal{H}}^p d\psi\right) d\theta = \int_{T^M}\left\|\sum_N e^{i\Theta_N} a_N\right\|_{\mathcal{H}}^p d\Theta$$

and

$$K_p^{-p}\left(\sum_N \|a_N\|_{\mathcal{H}}^2\right)^{p/2} \leq \int_T \left(\sum_{N'}\left\|\sum_n e^{i\theta_n} a_{n,N'}\right\|_{\mathcal{H}}^2\right)^{p/2} d\theta \leq K_p^p\left(\sum_N \|a_N\|_{\mathcal{H}}^2\right)^{p/2}. \quad \square$$

Definition A.1.2. Let $0 < p < \infty$. A family of operators, $(A_n)_{n\in\mathbb{N}}$, on $L_p(\Omega, \Sigma, \mu)$ is a *Paley-Littlewood system* if there exists a constant $A > 0$ such that

$$A^{-p}\int_{\Omega}\left(\sum_{n}|A_nf(x)|^2\right)^{p/2} d\mu(x)\leq \|f\|_p^p\leq A^p\int_{\Omega}\left(\sum_{n}|A_nf(x)|^2\right)^{p/2} d\mu(x)$$

for every $f\in L_p(\Omega,\Sigma,\mu)$.

The authors of [31] use A.1.1 to prove that if $(A_n^{(1)})_{n\in\mathbb{N}},\ldots,(A_n^{(M)})_{n\in\mathbb{N}}$ are M Paley-Littlewood systems of operators on L_p, $0<p<\infty$, then the family $(A_N)_{N\in\mathbb{N}^M}$ is also a Paley-Littlewood system on L_p, where $N=(n_1,\ldots,n_M)$ and $A_N=A_{n_1}^{(1)}\cdots A_{n_M}^{(M)}$ (Theorem 6.2, p. 210, [31]). The result in Chapter 3 only requires this theorem to be valid for the case $M=2$.

Theorem A.1.3. *Let $0<p<\infty$ and $(A_n)_{n\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$ be two Paley-Littlewood systems of operators on $L_p(\Omega,\Sigma,\mu)$ with constants A and B respectively. Then $(A_nB_m)_{n,m\in\mathbb{N}}$ is a Paley-Littlewood system with constant K_p^4AB where K_p is the constant in A.1.1.*

Proof. Fix $0<p<\infty$ and $f\in L_p$. Since $(B_m)_{m\in\mathbb{N}}$ a Paley-Littlewood system

$$B^{-p}\int_{\Omega}\left(\sum_{m}|B_mf(x)|^2\right)^{p/2} d\mu(x)\leq \|f\|_p^p\leq B^p\int_{\Omega}\left(\sum_{m}|B_mf(x)|^2\right)^{p/2} d\mu(x).$$

Let K_p be the constant from A.1.1. Then we can write

$$(K_pB)^{-p}\int_{\Omega}K_p^p\left(\sum_{m}|B_mf(x)|^2\right)^{p/2} d\mu\leq \|f\|_p^p\leq (K_pB)^p\int_{\Omega}K_p^{-p}\left(\sum_{m}|B_mf(x)|^2\right)^{p/2} d\mu.$$

Using A.1.1 with $M=1$ and Hilbert space elements $a_m=B_mf(x)$, for almost every $x\in\Omega$ we have

$$K_p^{-p}\left(\sum_{m}|B_mf(x)|^2\right)^{p/2}\ \leq\ \int_T\left|\sum_m e^{i\psi_m}B_mf(x)\right|^p d\psi\ \leq\ K_p^p\left(\sum_{m}|B_mf(x)|^2\right)^{p/2}.$$

From the two previous inequalities we have

$$(K_pB)^{-p}\int_{\Omega}\int_T\left|\sum_m e^{i\psi_m}B_mf(x)\right|^p d\psi\, d\mu\leq \|f\|_p^p\leq (K_pB)^p\int_{\Omega}\int_T\left|\sum_m e^{i\psi_m}B_mf(x)\right|^p d\psi\, d\mu.$$

Fix $\psi \in T$ and let $g = \sum_m e^{i\psi_m} B_m f \in L_p(\Omega, \Sigma, \mu)$. Interchanging the order of integration, we can express the above inequality as

$$(K_p B)^{-p} \int_T \|g\|_p^p \, d\psi \;\leq\; \|f\|_p^p \;\leq\; (K_p B)^p \int_T \|g\|_p^p \, d\psi. \tag{.1}$$

Since $(A_n)_{n\in\mathbb{N}}$ is also a Paley-Littlewood system on L_p

$$(K_p A)^{-p} \int_\Omega K_p^p \left(\sum_n |A_n g(x)|^2 \right)^{p/2} d\mu \leq \|g\|_p^p \leq (K_p A)^p \int_\Omega K_p^{-p} \left(\sum_n |A_n g(x)|^2 \right)^{p/2} d\mu$$

where g is as above. Again, using A.1.1 with $M = 1$ and $a_n = A_n g(x) = \sum_m e^{i\psi_m} A_n B_m f(x)$, for almost every $x \in \Omega$

$$K_p^{-p} \left(\sum_n |A_n g(x)|^2 \right)^{p/2} \;\leq\; \int_T \left| \sum_n e^{i\theta_n} A_n g(x) \right|^p d\theta \;\leq\; K_p^p \left(\sum_n |A_n g(x)|^2 \right)^{p/2}.$$

Hence,

$$(K_p A)^{-p} \int_\Omega \int_T \left| \sum_n e^{i\theta_n} A_n g(x) \right|^p d\theta \; d\mu \leq \|g\|_p^p \leq (K_p A)^p \int_\Omega \int_T \left| \sum_n e^{i\theta_n} A_n g(x) \right|^p d\theta \; d\mu$$

Using (.1) we have

$$(K_p^2 AB)^{-p} \int_T \int_\Omega \int_T \left| \sum_n e^{i\theta_n} A_n g(x) \right|^p d\theta \, d\mu(x) \, d\psi \;\leq\; \|f\|_p^p$$

and

$$\|f\|_p^p \;\leq\; (K_p^2 AB)^p \int_T \int_\Omega \int_T \left| \sum_n e^{i\theta_n} A_n g(x) \right|^p d\theta \, d\mu(x) \, d\psi.$$

Interchanging the order of integration and using the definition of g,

$$(K_p^2 AB)^{-p} \int_\Omega \int_{T^2} \left| \sum_{n,m} e^{i(\theta_n + \psi_m)} A_n B_m f(x) \right|^p d\theta \, d\psi \, d\mu(x) \;\leq\; \|f\|_p^p$$

and

$$\|f\|_p^p \;\leq\; (K_p^2 AB)^p \int_\Omega \int_{T^2} \left| \sum_{n,m} e^{i(\theta_n + \psi_m)} A_n B_m f(x) \right|^p d\theta \, d\psi \, d\mu(x).$$

Applying A.1.1 once again for the case $M = 2$ with $a_{n,m} = A_n B_m f(x)$

$$\begin{aligned} K_p^{-2p} \left(\sum_{n,m} |A_n B_m f(x)|^2 \right)^{p/2} &\leq \int_{T^2} \left| \sum_{n,m} e^{i(\theta_n + \psi_m)} A_n B_m f(x) \right|^p d\theta\, d\psi \\ &\leq K_p^{2p} \left(\sum_{n,m} |A_n B_m f(x)|^2 \right)^{p/2} \end{aligned}$$

for almost every $x \in \Omega$, so that

$$(K_p^4 AB)^{-p} \int_\Omega \left(\sum_{n,m} |A_n B_m f(x)|^2 \right)^{p/2} d\mu \leq \|f\|_p^p \leq (K_p^4 AB)^p \int_\Omega \left(\sum_{n,m} |A_n B_m f(x)|^2 \right)^{p/2} d\mu.$$

That is, $(A_n B_m)_{m,n\in\mathbb{N}}$ is a Paley-Littlewood system on L_p. □

We remark at this point that for $0 < p < 2$, the constant obtained in the proof of A.1.3 is independent of p. Furthermore, the proof does not require that the families $(A_n)_{n\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$ be defined on all of L_p. The theorem remains valid for families acting on subspaces of L_p, provided that the range of B_m is contained in the domain of A_n for all n and m.

As in Section 3.2, we make the assumption that A is a linear, one-to-one operator from $\mathcal{C}_q$, $0 < q < 2$, onto some subspace of an L_p-space. In order to apply A.1.3 to the families of operators defined in Chapter 3, we must verify that $(AE_nA^{-1})_{n\in\mathbb{N}}$ and $(AF_mA^{-1})_{m\in\mathbb{N}}$ are Paley-Littlewood systems on the subspace $\mathcal{M} = A(\mathcal{C}_q)$ of L_p. For the remainder of the discussion, we restrict attention to the case $0 < p < 2$ (general results for $p \geq 1$ can be found in [31]).

Recall that if $\sum_n AE_nA^{-1} \in A\mathcal{E}A^{-1}$ and $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{K})$, then for any $f \in \mathcal{M}$

$$\left\| \sum_n a_n A E_n A^{-1} f \right\|_p \leq \|a\|_\infty \|A\|_{op} \left\|A^{-1}\right\|_{op} \|f\|_p .$$

We now obtain an estimate for the lower bound of the norm of such an an element in $\mathcal{M}$. Let $T \in \mathcal{C}_q$ and x be an element of the Hilbert space $\mathcal{H}$. As in Section 3.2, let $(\phi_j)_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and $P_n = \langle \cdot, \phi_n \rangle \phi_n$ be the orthogonal projection of $\mathcal{H}$ onto span$[\phi_n]$. Then

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_n P_n T x\right\|_{\mathcal{H}}^{2q} &= \left\|\sum_{j=1}^{\infty}\left\langle \sum_{n=1}^{\infty} a_n P_n T x, \phi_j\right\rangle \phi_j\right\|_{\mathcal{H}}^{2q} = \left\|\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} a_n \langle P_n T x, \phi_j\rangle \phi_j\right\|_{\mathcal{H}}^{2q} \\ &= \left\|\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} a_n \langle T x, P_n\phi_j\rangle \phi_j\right\|_{\mathcal{H}}^{2q} = \left\|\sum_{j=1}^{\infty} a_j \langle T x, \phi_j\rangle \phi_j\right\|_{\mathcal{H}}^{2q} \\ &= \left(\sum_{j=1}^{\infty} |a_j|^2 |\langle T x, \phi_j\rangle|^2\right)^q \geq \left(\inf_{j\in\mathbb{N}} |a_j|^2 \|Tx\|_{\mathcal{H}}^2\right)^q\end{aligned}$$

so that

$$\left\|\sum_{n=1}^{\infty} a_n P_n T x\right\|_{\mathcal{H}}^{q} \geq \left(\inf_{j\in\mathbb{N}} |a_j| \, \|Tx\|_{\mathcal{H}}\right)^q \tag{.2}$$

for every $x \in \mathcal{H}$. Let $[T] = (T^*T)^{1/2}$ denote the absolute value of T. Since $0 < q < 2$ and $T^*T \geq 0$, using the spectral theorem and Jensen's inequality one can show that

$$\langle [T]^q \phi_j, \phi_j\rangle = \left\langle (T^*T)^{q/2}\phi_j, \phi_j\right\rangle \leq \langle T^*T\phi_j, \phi_j\rangle^{q/2} = \|T\phi_j\|^q$$

with equality in case $(\phi_j)_{j\in\mathbb{N}}$ is orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of T^*T. Summing over j we find

$$\|T\|_{\mathcal{C}_q}^q \leq \sum_{j=1}^{\infty} \|T\phi_j\|_{\mathcal{H}}^q$$

with equality in case $(\phi_j)_{j\in\mathbb{N}}$ is orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of T^*T. Hence, for $0 < q < 2$,

$$\|T\|_{\mathcal{C}_q}^q = \inf \sum_{j=1}^{\infty} \|T\phi_j\|_{\mathcal{H}}^q$$

where the infimum is taken over all orthonormal bases of $\mathcal{H}$. Using $x = \phi_j$ and summing over j in (.2) yields

$$\sum_{j=1}^{\infty}\left\|\sum_{n=1}^{\infty} a_n P_n T \phi_j\right\|_{\mathcal{H}}^{q} \geq \left(\inf_{n\in\mathbb{N}} |a_n|\right)^q \sum_{j=1}^{\infty} \|T\phi_j\|_{\mathcal{H}}^q.$$

Taking the infimum over all orthonormal bases of $\mathcal{H}$ we find

$$\left\|\sum_{n=1}^{\infty} a_n P_n T\right\|_{\mathcal{C}_q}^{q} \geq \left(\inf_{n\in\mathbb{N}} |a_n|\right)^q \|T\|_{\mathcal{C}_q}^q.$$

Since $E_nT = P_nT$ for all $T \in \mathcal{C}_q$ and for all $n \in \mathbb{N}$,

$$\left\| \sum_{n=1}^{\infty} a_n E_n T \right\|_{\mathcal{C}_q} \geq \inf_{n\in\mathbb{N}} |a_n| \, \|T\|_{\mathcal{C}_q}.$$

Now, let $f \in \mathcal{M}$ and $T \in \mathcal{C}_q$ be such that $AT = f$. Then

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} a_n A E_n A^{-1} f \right\|_p = \left\| A \sum_{n=1}^{\infty} a_n E_n T \right\|_p &\geq \left(\left\|A^{-1}\right\|\right)^{-1} \left\| \sum_{n=1}^{\infty} a_n E_n T \right\|_{\mathcal{C}_q} \\
&\geq \left(\left\|A^{-1}\right\|\right)^{-1} \inf_{n\in\mathbb{N}} |a_n| \, \|T\|_{\mathcal{C}_q} \\
&= \inf_{n\in\mathbb{N}} |a_n| \left(\left\|A^{-1}\right\|\right)^{-1} \left\|A^{-1} f\right\|_{\mathcal{C}_q} \\
&\geq \inf_{n\in\mathbb{N}} |a_n| \left(\|A\| \left\|A^{-1}\right\|\right)^{-1} \|f\|_p .
\end{aligned}$$

Combining this with the previous result gives the following.

Lemma A.1.4. *Let $f \in \mathcal{M}$ and $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{K})$. Then*

$$\inf_{n\in\mathbb{N}} |a_n| \left(\|A\| \left\|A^{-1}\right\|\right)^{-1} \|f\|_p \leq \left\| \sum_{n=1}^{\infty} a_n A E_n A^{-1} f \right\|_p \leq \|a\|_\infty \|A\| \left\|A^{-1}\right\| \|f\|_p .$$

A.1.4 remains valid if the E_n's are replaced by the F_n's. We are now in a position to prove the following.

Lemma A.1.5. *Let $0 < p < 2$. The families of operators $(AE_nA^{-1})_{n\in\mathbb{N}}$ and $(AF_mA^{-1})_{m\in\mathbb{N}}$ are Paley-Littlewood systems on the subspace $\mathcal{M} = A(\mathcal{C}_q)$ of $L_p(\Omega, \Sigma, \mu)$ with constant $K \|A\| \|A^{-1}\|$, where K is the constant from A.1.1.*

Proof. We prove the lemma for the family $(AE_nA^{-1})_{n\in\mathbb{N}}$. Let $\theta_n \in [0, 2\pi)$ and define $T_n = AE_nA^{-1}$ for all $n \in \mathbb{N}$. If $f \in \mathcal{M}$, then by A.1.4

$$\left(\|A\| \left\|A^{-1}\right\|\right)^{-p} \|f\|_p^p \leq \int_\Omega \left| \sum_{n=1}^{\infty} e^{i\theta_n} T_n f(x) \right|^p d\mu(x) \leq \left(\|A\| \left\|A^{-1}\right\|\right)^p \|f\|_p^p . \tag{.3}$$

By A.1.1 there is a constant K (independent $p \in (0, 2)$) such that

$$K^{-p} \left(\sum_{n=1}^{\infty} |T_n f(x)|^2 \right)^{p/2} \leq \int_T \left| \sum_{n=1}^{\infty} e^{i\theta_n} T_n f(x) \right|^p d\theta \leq K^p \left(\sum_{n=1}^{\infty} |T_n f(x)|^2 \right)^{p/2} .$$

Let $J = K \|A\| \|A^{-1}\|$. Integrating over Ω and using (.3) we have

$$J^{-p} \int_{\Omega} \left(\sum_{n=1}^{\infty} |T_n f(x)|^2 \right)^{p/2} d\mu(x) \leq \|f\|_p^p \leq J^p \int_{\Omega} \left(\sum_{n=1}^{\infty} |T_n f(x)|^2 \right)^{p/2} d\mu(x). \qquad \square$$

We need one final lemma to justify the results of Chapter 3.

Lemma A.1.6. *Let $0 < p < 2$. Then $A\mathcal{G}A^{-1} = A\mathcal{E}A^{-1} \vee A\mathcal{F}A^{-1}$ is a bounded Boolean algebra of projections of $\mathcal{M}$.*

Proof. We need only verify $A\mathcal{G}A^{-1}$ is bounded. By A.1.3 and A.1.5 the family $(AE_nF_mA^{-1})$ is a Paley-Littlewood system on $\mathcal{M}$ with constant $J = K^6 \|A\|^2 \|A^{-1}\|^2$. Define $T_{nm} = AE_nF_mA^{-1}$ for all n and m. Then for every $f \in \mathcal{M}$

$$J^{-p} \int_{\Omega} \left(\sum_{n,m=1}^{\infty} |T_{nm} f(x)|^2 \right)^{p/2} d\mu(x) \leq \|f\|_p^p \leq J^p \int_{\Omega} \left(\sum_{n,m=1}^{\infty} |T_{nm} f(x)|^2 \right)^{p/2} d\mu(x). \quad (.4)$$

Recall that elements of $A\mathcal{G}A^{-1}$ have the form $\sum_{n,m=1}^{\infty} a_{nm} T_{nm}$ where $a_{nm} \in \{0,1\}$ for all m and n. Furthermore, since the operators T_{nm} are disjoint idempotents and $\sum_{n,m} a_{nm} T_{nm} f \in \mathcal{M}$ we have

$$\sum_{j,k=1}^{\infty} T_{jk} \left(\sum_{n,m=1}^{\infty} a_{nm} T_{nm} f \right) = \sum_{j,k=1}^{\infty} a_{jk} T_{jk} f$$

so that

$$J^{-p} \int_{\Omega} \left(\sum_{j,k=1}^{\infty} |a_{jk} T_{jk} f(x)|^2 \right)^{p/2} d\mu \leq \left\| \sum_{n,m=1}^{\infty} a_{nm} T_{nm} f \right\|_p^p \leq J^p \int_{\Omega} \left(\sum_{j,k=1}^{\infty} |a_{jk} T_{jk} f(x)|^2 \right)^{p/2} d\mu.$$

Since $\sum_{jk} |a_{jk} T_{jk} f(x)|^2 \leq \sum_{jk} |T_{jk} f(x)|^2$ for all $f \in \mathcal{M}$, by (.4) we have

$$\begin{aligned} \left\| \sum_{n,m=1}^{\infty} a_{nm} T_{nm} f \right\|_p^p &\leq J^{2p} J^{-p} \int_{\Omega} \left(\sum_{j,k=1}^{\infty} |a_{jk} T_{jk} f(x)|^2 \right)^{p/2} d\mu(x) \\ &\leq J^{2p} J^{-p} \int_{\Omega} \left(\sum_{j,k=1}^{\infty} |T_{jk} f(x)|^2 \right)^{p/2} d\mu(x) \\ &\leq J^{2p} \|f\|_p^p. \end{aligned}$$

Hence, $\|A\mathcal{G}A^{-1}\| \leq J^2$. $\square$

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