HARDY-TYPE SEQUENCE SPACES AND CESÀRO FRAMES

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ABSTRACT

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Cesàro averaging is used in conjunction with Hardy space and Hilbert space theory to realize certain types of convergence.

In Chapter 1, we study certain Hardy-type sequence spaces $\mathcal{H}^p$ and $\mathcal{H}^p_0$, $1 \leq p \leq \infty$, which are analogues of ℓ^∞ and c_0, respectively. We show that the Mazur product $\boxtimes : \mathcal{H}^p_0 \times \mathcal{H}^q \to c_0$ is not onto for every $p \in (1, \infty)$ with $q = p(p-1)^{-1}$, which provides a new solution of Mazur's Problem 8 in the Scottish Book. We present corollaries for spaces defined via weighted ℓ^p seminorms and for c_0.

In Chapter 2, we study the application of Cesàro operators on Bessel sequences to realize a weak version of frame reconstruction in Hilbert space. Conditions for reconstruction via Markuschevich bases that are certain linear combinations of orthonormal basis vectors are given.

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PREFACE

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In memory of my grandparents, Louis, Grace, Gilbert and Frances, and Dr. Thomas Metzger, who (along with my parents) all showed me the value of family and hard work.

1.0 THE MAZUR PRODUCT ON HARDY-TYPE SEQUENCE SPACES

1.1 INTRODUCTION

Define $\mathbb{N}_0$ to be the whole numbers, and S to be the space of all sequences with domain $\mathbb{N}_0$. Let c_{00} be the space of all sequences in S with finitely many nonzero entries; and denote by $c_{00}(n)$ the space of all sequences t in c_{00} with $t_0 = t_1 = ... = t_{n-1} = t_n = 0$. In this chapter, our focus will be on the action of an averaged Cauchy product mapping on the following spaces; the contents of this chapter are essentially the same as in Lennard and Radelet [12], augmented with extra details.

Definition 1. *Let $n \in \mathbb{N}_0$, $a \in S$, and $p \in [1, \infty)$ be given. We define the following:*

$$\zeta_n^p(a) := \frac{1}{(n+1)^{1/p}} \left(\inf_{t \in c_{00}(n)} \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^p(\Delta)} \right) , \tag{1.1}$$

$$\mathcal{H}^p := \left\{ a \in S \,\middle|\, \|a\|_{\mathcal{H}^p} := \sup_{n \in \mathbb{N}_0} \zeta_n^p(a) < \infty \right\} ,$$

$$\textit{and} \qquad \mathcal{H}_0^p := \left\{ a \in S \,\middle|\, \lim_{n \to \infty} \zeta_n^p(a) = 0 \right\} .$$

By $H^p(\Delta)$ we mean the usual Hardy spaces of analytic functions on the interior of the unit disc in $\mathbb{C}$:

$$H^p(\Delta) := \left\{ f : \Delta \to \mathbb{C} \,\middle|\, \|f\|_{H^p} := \lim_{r \to 1^-} \left(\int_{\theta=0}^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty \right\} .$$

Note that we can analogously define $\zeta_n^\infty(a)$ and $\mathcal{H}^\infty$. All spaces are complete (see the APPENDIX), and it can be shown that $\mathcal{H}_0^1$ is separable (see the end of section 1.3 for a proof of separability).

For $1 \le p \le 2$, note that $c_0 \subsetneq \mathcal{H}_0^p \subsetneq \mathcal{H}^p$, and we have $\ell^\infty \subsetneq \mathcal{H}^p$ (See section 1.5). In particular, since we have

$$\zeta_n^2(a) = \left(\frac{1}{n+1} \sum_{j=0}^{n} |a_j|^2 \right)^{1/2},$$

it is easy to see that $c_0 \subsetneq \mathcal{H}_0^2 \subsetneq \mathcal{H}^2$ and $\ell^\infty \subsetneq \mathcal{H}^2$. Also note that, for $p > 2$, $c_0 \not\subseteq \mathcal{H}_0^p$. While working with ideas related to Mazur's Problem 8 as found in the Scottish book [17], we considered the well-known space $\mathcal{H}^2$ as an analogue of the usual Hardy space $H^2(\Delta)$. This naturally led us to seek an analogue of $H^1(\Delta)$, $\mathcal{H}^1$, that would have the property that the Cauchy product continuously mapped $\mathcal{H}^2 \times \mathcal{H}^2$ into $\mathcal{H}^1$. With this motivation, we defined $\mathcal{H}^1$ and $\mathcal{H}_0^1$. By further analogy, we defined $\mathcal{H}^p$ for $1 < p < \infty$. The equivalent characterization of the $\mathcal{H}^1$-norm discussed in Section 1.3 arose naturally while proving our main result.

We thank the referee of [12] for informing us that for $1 < p < \infty$, the definition of $\mathcal{H}^p$ can be rewritten in a simpler way. Indeed, for $1 < p < \infty$, a sequence a belongs to $\mathcal{H}^p$ if and only if

$$\nu_p(a) := \sup_{n \in \mathbb{N}_0} \frac{1}{(n+1)^{1/p}} \left\| \sum_{j=0}^{n} a_j z^j \right\|_{H^p(\Delta)} < \infty.$$

In fact, ν_p is an equivalent norm on $\mathcal{H}^p$. To see this, let J be the directed set of all integers $\mathbb{Z}$, ordered in this way: $(0, 1, -1, 2, -2, \ldots)$. Note that for $1 < p < \infty$, $(e^{in\theta})_{n \in J}$ is a Schauder basis for $L^p(\mathbb{T})$, where for each $f \in L^p(\mathbb{T})$ the coefficient sequence is the sequence of Fourier coefficients of f (see, for example Wojtaszczyk [18], II.B.11). Thus, $(e^{in\theta})_{n \in \mathbb{N}_0}$, where $\mathbb{N}_0$ has its usual ordering, is a Schauder basis for the Hardy space $H^p(\mathbb{T})$, which is isometrically isomorphic to $H^p(\Delta)$. Therefore, $(z^n)_{n \in \mathbb{N}_0}$ is a Schauder basis for the Hardy space $H^p(\Delta)$. By Nikolskiǐ's criterion, there exists a constant $K_p \in [1, \infty)$ so that for all positive integers $M < N$, for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p(\Delta)$,

$$\left\| \sum_{n=0}^{M} a_n z^n \right\|_{H^p(\Delta)} \le K_p \left\| \sum_{n=0}^{N} a_n z^n \right\|_{H^p(\Delta)}.$$

It follows from the definitions, in a straightforward way, that for all $1 < p < \infty$, for every sequence $a \in \mathcal{H}^p$, $\|a\|_{\mathcal{H}^p} \le \nu_p(a) \le K_p \|a\|_{\mathcal{H}^p}$.

We will call the Cesàro-averaged Cauchy map the Mazur product map; it is defined here.

Definition 2. *We denote the Mazur product of sequences* $x = (x_n)_{n\in\mathbb{N}_0}$ *and* $y = (y_n)_{n\in\mathbb{N}_0}$ *by* $x \boxtimes y$. *We define*

$$(x \boxtimes y)_n := \frac{1}{n+1} \sum_{i=0}^{n} x_i y_{n-i}\,.$$

It is known that $\boxtimes : H^2 \times H^2 \to \ell^1$ is not onto via inequalities of Paley and Hardy, and we will generalize this idea to $\mathcal{H}^p$ and $\mathcal{H}_0^p$ using Paley's inequality and other techniques. Specifically, our main result is the following:

Theorem 3. *(a) The Mazur map* $\boxtimes : \mathcal{H}_0^p \times \mathcal{H}^q \longrightarrow c_0$ *is not onto* $\forall p \in (1, \infty)$, *where* $q = p(p-1)^{-1}$.
(b) Under the same conditions as in (a), $\boxtimes : \mathcal{H}^p \times \mathcal{H}_0^q \longrightarrow c_0$ *is not onto.*

In section 1.5, this result is extended to mixed-index weighted ℓ^p-like spaces, defined below.

Definition 4. *Let* $n \in \mathbb{N}_0$, $a \in S$, *and* $p, r \in (0, \infty)$ *be given. Define:*

$$\psi_n^{p,r}(a) := \frac{1}{(n+1)^{1/p}} \left(\sum_{j=0}^{n} |a_j|^r \right)^{1/r},$$

$$\lambda^{p,r} := \left\{ a \in S \,\middle|\, \sup_{n\in\mathbb{N}_0} \psi_n^{p,r}(a) < \infty \right\}$$

$$\textit{and} \qquad \lambda_0^{p,r} := \left\{ a \in S \,\middle|\, \lim_{n\to\infty} \psi_n^{p,r}(a) = 0 \right\}.$$

When $p = r$, these spaces are discrete analogues of spaces originally defined by Beurling [1] and considered, for example, in Lau and Lee [11].
This type of problem has been investigated previously by a number of authors, most of whom directly answer Problem 8 in [17]: do we have that $\boxtimes : c \times c \longrightarrow c$ is onto? (The converse statement is given below as Lemma 5.) This question was answered in the negative independently by Eggermont and Leung [7] as well as Kwapień and Pełczyński [10]. Problem 8 was also solved by Peller in [14], as described in [15]. Recently, Pełczyński and Sukochev [13] obtained results related to negative solutions of Problem 8, and Peller [15] was able to apply his own results [14] to answer open questions motivated by [13]. We generalize the question and the negative result to the sequence spaces $\mathcal{H}^p$ and $\mathcal{H}_0^p$. In particular, this leads to a new solution of Problem 8, which we discuss in Section 1.6.

Lemma 5. $c \boxtimes c \subseteq c$.

Proof. Let x and y be convergent sequences with $x_n \longrightarrow \lambda_x$ and $y_n \longrightarrow \lambda_y$, as $n \to \infty$. Fix $\varepsilon > 0$. There exists $M \in \mathbb{N}$ such that for every $j > M$,

$$|x_j - \lambda_x| < \varepsilon \qquad \text{and} \qquad |y_j - \lambda_y| < \varepsilon \qquad . \tag{1.2}$$

Also, we recall that convergent sequences are bounded, so we have the existence of constants A_x, A_y, B_x, B_y such that for all $j \in \mathbb{N}$,

$$|x_j| < B_x \qquad \text{and} \qquad |x_j - \lambda_x| < A_x, \tag{1.3}$$

$$|y_j| < B_y \qquad \text{and} \qquad |y_j - \lambda_y| < A_y. \tag{1.4}$$

Now, fix $n > 2M$. We have

$$\left| \frac{1}{n+1} \left(\sum_{j=0}^{n} x_j y_{n-j} \right) - \lambda_x \lambda_y \right| = \left| \frac{1}{n+1} \left(\sum_{j=0}^{n} x_j y_{n-j} - \lambda_x \lambda_y \right) \right|$$

$$\leq \frac{1}{n+1} \sum_{j=0}^{n} |x_j y_{n-j} - \lambda_x \lambda_y| = \frac{1}{n+1} \sum_{j=0}^{n} |x_j y_{n-j} + x y_{n-j} - x y_{n-j} - \lambda_x \lambda_y|$$

$$\leq \frac{1}{n+1} \left(\sum_{j=0}^{n} |(x_j - \lambda_x) y_{n-j}| + \sum_{j=0}^{n} |\lambda_x (y_{n-j} - \lambda_y)| \right).$$

If we split the sum into three parts, taking the first and last M terms as well as the terms in the middle, we have that the previous term is

$$= \frac{1}{n+1} \Bigg(\sum_{j=0}^{M} |(x_j - \lambda_x)||y_{n-j}| + \sum_{j=0}^{M} |\lambda_x||(y_{n-j} - \lambda_y)| + \sum_{j=M+1}^{n-M} |(x_j - \lambda_x)||y_{n-j}|$$

$$+ \sum_{j=M+1}^{n-M} |\lambda_x||(y_{n-j} - \lambda_y)| + \sum_{j=n-M+1}^{n} |(x_j - \lambda_x)||y_{n-j}| + \sum_{j=n-M+1}^{n} |\lambda_x||(y_{n-j} - \lambda_y)| \Bigg).$$

Using equations (1.2), (1.3), and (1.4), we get the preceding expression to be

$$\leq (A_x B_y + \varepsilon |\lambda_x| + \varepsilon B_y + A_y |\lambda_x|) \left(\frac{M+1}{n+1} \right) + (\varepsilon B_y + \varepsilon |\lambda_x|) \left(\frac{n-2M}{n+1} \right).$$

As n approaches infinity, the first term tends to zero and the second term tends to $K \cdot \varepsilon$, where $K := B_y + |\lambda_x|$. □

Notice that the Mazur map can be written as $\boxtimes = J \circ \boxdot : S \times S \to S$, where the J map is a multiplication operator and the $\boxdot$ map is the normal Cauchy product, defined as follows for $a, b \in S$:

$$J(a) := \left(\frac{a_n}{n+1}\right)_{n\in\mathbb{N}_0}, \quad (a \boxdot b)_n := \sum_{j=0}^{n} a_j b_{n-j}\,.$$

Recall that any analytic f on Δ can be represented by

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n\,, \qquad a_n \in \mathbb{C}\,, \qquad |z| < 1\,.$$

In this case, we will denote f by f_a. We have the following, via Hölder's inequality.

Lemma 6. *For every $f_a \in H^p$ and $f_b \in H^q$ with $1/p + 1/q = 1$, if we consider the product $f_\gamma = f_a f_b$, then we have $f_\gamma \in H^1$ and $\|f_\gamma\|_{H^1} \;\leq\; \|f_a\|_{H^p} \cdot \|f_b\|_{H^q}$.*

Theorem 7. *Let $p \in [1, \infty]$ with $1/p + 1/q = 1$. Then $\boxdot : \mathcal{H}^p \times \mathcal{H}^q \longrightarrow \mathcal{H}^1$ continuously.*

Proof. Fix $a \in \mathcal{H}^p$ and $b \in \mathcal{H}^q$, with $1/p + 1/q = 1$, and define $\gamma := a \boxdot b$. We have $\zeta_n^1(\gamma)$ as defined in (1.1). Fix $n \in \mathbb{N}_0$ and $\varepsilon > 0$. Then there exists $r, s \in c_{00}(n)$ such that

$$\zeta_n^p(a) \leq \frac{1}{(n+1)^{1/p}} \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{k} r_j z^j \right\|_{H^p} \leq \zeta_n^p(a) + \varepsilon\,,$$
$$\zeta_n^q(b) \leq \frac{1}{(n+1)^{1/q}} \left\| \sum_{j=0}^{n} b_j z^j + \sum_{j=n+1}^{l} s_j z^j \right\|_{H^q} \leq \zeta_n^q(b) + \varepsilon\,,$$

with k and l being the last nonzero coordinate of r and s, respectively. Without loss of generality, we may assume that $k > l$, so that all coordinates with index larger than k are zero for both sequences. We may also let $r_j := a_j$ and $s_j := b_j$ for $0 \leq j \leq n$, and define $y = (y_j) := ((r \boxdot s)_j)$ for $0 \leq j \leq k$. We may further define y for $k+1 \leq j \leq 2k$ as follows:

$$\begin{aligned}
y_{k+1} &:= r_k s_1 + r_{k-1} s_2 + \ldots + r_2 s_{k-1} + r_1 s_k \\
y_{k+2} &:= r_k s_2 + r_{k-1} s_3 + \ldots + r_3 s_{k-1} + r_2 s_k \\
&\;\vdots \\
y_{2k} &:= r_k s_k\,.
\end{aligned}$$

Using Lemma 6, we then have

$$\begin{aligned}
\zeta_n^1(\gamma) &= \frac{1}{n+1} \inf_{t\in c_{00}(n)} \left\| \sum_{j=0}^{n} \gamma_j z^j \,+\, \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^1(\Delta)} \\
&\le \frac{1}{n+1} \left\| \sum_{j=0}^{n} \gamma_j z^j \,+\, \sum_{j=n+1}^{2k} y_j z^j \right\|_{H^1} = \frac{1}{n+1} \left\| \left(\sum_{j=0}^{k} r_j z^j \right) \left(\sum_{j=0}^{k} s_j z^j \right) \right\|_{H^1} \\
&\le \frac{1}{(n+1)^{1/p}} \left\| \sum_{j=0}^{k} r_j z^j \right\|_{H^p} \frac{1}{(n+1)^{1/q}} \left\| \sum_{j=0}^{k} s_j z^j \right\|_{H^q} \\
&\le \left(\zeta_n^p(a) + \varepsilon\right) \left(\zeta_n^q(b) + \varepsilon\right) \, .
\end{aligned}$$

Since ε was arbitrary, we have shown that $\zeta_n^1(\gamma) \le \zeta_n^p(a) \cdot \zeta_n^q(b)$, for every n. If we take the supremum of this expression over all $n \in \mathbb{N}_0$, we see that $\gamma \in \mathcal{H}^1$, and
$||\gamma||_{\mathcal{H}^1} \le ||a||_{\mathcal{H}^p} ||b||_{\mathcal{H}^q}$. □

Corollary 8. *Let $p \in [1,\infty]$ with $1/p + 1/q = 1$. The Cauchy product is a continuous map of $\mathcal{H}_0^p \times \mathcal{H}^q$ and $\mathcal{H}^p \times \mathcal{H}_0^q$ into $\mathcal{H}_0^1$.*

Proposition 9. *$J : \mathcal{H}^1 \longrightarrow \ell^\infty$ continuously.*

Proof. Fix $x \in \mathcal{H}^1$, $n \in \mathbb{N}_0$, and $t \in c_{00}(n)$. Define $y := ((n+1)^{-1} x_n)_{n\in\mathbb{N}_0}$. Consider the power series

$$g(z) := \sum_{j=0}^{n} x_j z^j \,+\, \sum_{j=n+1}^{\infty} t_j z^j \qquad \forall z \in \Delta\, .$$

Note that $g \in H^1(\Delta)$. On the boundary $\mathbb{T}$ of the disk, we may change to Fourier series: for all $\theta \in [0, 2\pi)$ and $k \in \mathbb{N}_0$,

$$g(e^{i\theta}) := \sum_{j=0}^{n} x_j e^{ij\theta} \,+\, \sum_{j=n+1}^{\infty} t_j e^{ij\theta} \quad \text{and} \quad x_k \,=\, \int_0^{2\pi} g(e^{i\theta}) e^{-ik\theta} \, \frac{d\theta}{2\pi}\, .$$

Then we have

$$|x_n| \le \int_0^{2\pi} |g(e^{i\theta})| \frac{d\theta}{2\pi} = \|g\|_{L^1(\mathbb{T})}\, .$$

However, since $L^1(\mathbb{T}) \cong H^1(\mathbb{T}) \cong H^1(\Delta)$, we have $|x_n| \le \|g\|_{H^1(\Delta)}$. Now,

$$|y_n| \,=\, \frac{|x_n|}{n+1} \,\le\, \frac{1}{n+1} \cdot \|g\|_{H^1(\Delta)}.$$

So, taking the infimum over all $t \in c_{00}(n)$, we have that, for all n,

$$|y_n| \leq \zeta_n^1(x)\,. \tag{1.5}$$

Finally, taking the supremum over all $n \in \mathbb{N}_0$ gives $y \in \ell^\infty$ as claimed, with $\|y\|_\infty \leq \|x\|_{\mathcal{H}^1}$. This also shows that J is continuous. $\square$

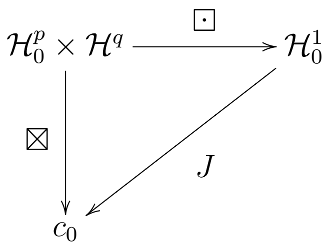
Corollary 10. *$J : \mathcal{H}_0^1 \longrightarrow c_0$ continuously.*

Corollary 11. *For every $x \in \mathcal{H}^1$, $f_x(z)$ is a holomorphic function on Δ.*

The proof of Corollary 11 follows from the proof of Proposition 9, as we have

$$\limsup_{n\in\mathbb{N}} \sqrt[n]{x_n} \leq \limsup_{n\in\mathbb{N}} (n+1)^{1/n} \cdot ||x||_{\mathcal{H}^1}^{1/n} \leq 1.$$

Now, because $\boxtimes = J \circ \boxdot$, we have established the following commutative diagram of continuous linear and bilinear mappings:



Now that we have identified the target space of the Mazur product map, we wish to investigate its surjectivity. Noting that J is also clearly injective, we can reason as follows: if J were not onto, then the Mazur map would not be onto. So, if the Mazur map is onto, we must have that J is also onto. In this case, applying the Open Mapping Theorem gives the existence of a continuous J^{-1} on all of c_0. For every $w \in c_0$, define $x = J^{-1}w$. Then there exists a constant $B < \infty$ such that $\|J^{-1}w\| \leq B\|w\|$. This would lead us to the following inequalities:

$$\|w\|_\infty \leq \|x\|_{\mathcal{H}^1} \leq B \cdot \|w\|_\infty\,. \tag{1.6}$$

The final inequality is the one we will contradict, as follows.

Theorem 12. *There exists a sequence of elements $(x^{(k)})$ in $\mathcal{H}_0^1$ with $\|x^{(k)}\|_{\mathcal{H}^1} \xrightarrow{k} \infty$, but $\|Jx^{(k)}\|_\infty \leq 1$ for every k.*

We will prove Theorem 12 in Section 1.4, contradicting the existence of any such constant B in equation (1.6). It follows that the Mazur map is not onto, establishing our main result, Theorem 3.

1.3 AN EQUIVALENT NORM

We have seen that the space $\mathcal{H}^1$ is the natural range of the Cauchy map $\boxdot$ independent of the choice of conjugate indices p and q, and thus is the natural focus of the Mazur mapping problem we are considering. However, the $\mathcal{H}^1$ norm we have is cumbersome for calculations, so in this section we seek to develop an equivalent (easier) norm $\|\cdot\|_{\mathcal{H}^1}^{\bigstar}$ on $\mathcal{H}^1$; i.e.,

$$A\|a\|_{\mathcal{H}^1}^{\bigstar} \leq \|a\|_{\mathcal{H}^1} \leq B\|a\|_{\mathcal{H}^1}^{\bigstar}\,. \tag{1.7}$$

Recall that the Fejér kernel

$$\Gamma_N(u) := \frac{1}{N+1}\sum_{n=0}^{N}\sum_{j=-n}^{n} e^{iju} = \frac{1}{N+1}\left(\frac{\sin\left[\left(\frac{N+1}{2}\right)u\right]}{\sin\left(\frac{u}{2}\right)}\right)^2$$

is non-negative and 2π-periodic, and

$$\int_{\theta=0}^{2\pi} |\Gamma_N(\theta)|\frac{d\theta}{2\pi} = 1\,.$$

For all $a \in S$, $N \in \mathbb{N}_0$, and $z \in \Delta$, define $f_{a,N}(z) := \sum_{j=0}^{N} a_j z^j$,

$$h_{a,N}(z) := \frac{1}{N+1}\sum_{n=0}^{N} f_{a,n}(z) \;=\; \frac{1}{N+1}\sum_{n=0}^{N}\sum_{j=0}^{n} a_j z^j\,,$$

$$\Omega_N(a) := \frac{1}{N+1}||h_{a,N}(z)||_{H^1(\Delta)}\,, \quad \text{and} \quad \|a\|_{\mathcal{H}^1}^{\bigstar} := \sup_{N\in\mathbb{N}_0} \Omega_N(a)\,.$$

Proposition 13. *For all $a \in \mathcal{H}^1$ and for all $N \in \mathbb{N}_0$,*

$$\sup_{N\in\mathbb{N}_0} \Omega_N(a) \;\leq\; \sup_{N\in\mathbb{N}_0} \zeta_N^1(a)\,.$$

Proof. Fix $N \in \mathbb{N}_0$ and $t \in c_{00}(N)$. Note that for every k in $\{-N, ..., N\}$, we have

$$a_k = \int_{\theta=0}^{2\pi} g_t(e^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi},$$

$$\text{where} \qquad g_t(z) = \sum_{j=0}^{N} a_j z^j + \sum_{j=N+1}^{\infty} t_j z^j = f_{a,N}(z) + \sum_{j=N+1}^{\infty} t_j z^j.$$

Now,

$$\begin{aligned}
\frac{1}{N+1} \|h_{a,N}(z)\|_{H^1(\Delta)} &= \frac{1}{N+1} \left\| h_{a,N}(e^{i\tau}) \right\|_{L^1(\mathbb{T})} \\
&= \frac{1}{N+1} \int_{\tau=0}^{2\pi} \left| \frac{1}{N+1} \sum_{n=0}^{N} \sum_{j=0}^{n} a_j e^{ij\tau} \right| \frac{d\tau}{2\pi} \\
&= \frac{1}{N+1} \int_{\tau=0}^{2\pi} \left| \frac{1}{N+1} \sum_{n=0}^{N} \sum_{j=-n}^{n} a_j e^{ij\tau} \right| \frac{d\tau}{2\pi}
\end{aligned}$$

because $a_j := 0$ for all $j < 0$. We can expand the last expression above into

$$\begin{aligned}
&\frac{1}{N+1} \int_{\tau=0}^{2\pi} \left| \frac{1}{N+1} \sum_{n=0}^{N} \sum_{j=-n}^{n} \left(\int_{\theta=0}^{2\pi} g_t(e^{i\theta}) e^{-ij\theta} \frac{d\theta}{2\pi} \right) e^{ij\tau} \right| \frac{d\tau}{2\pi} \\
&= \frac{1}{N+1} \int_{\tau=0}^{2\pi} \left| \int_{\theta=0}^{2\pi} g_t(e^{i\theta}) \left[\frac{1}{N+1} \sum_{n=0}^{N} \sum_{j=-n}^{n} e^{ij(\tau-\theta)} \right] \frac{d\theta}{2\pi} \right| \frac{d\tau}{2\pi} \\
&\le \frac{1}{N+1} \int_{\tau=0}^{2\pi} \int_{\theta=0}^{2\pi} \left| g_t(e^{i\theta}) \right| \left| \Gamma_N(\tau-\theta) \right| \frac{d\theta}{2\pi} \frac{d\tau}{2\pi},
\end{aligned}$$

and by Fubini's Theorem, we arrive at

$$\frac{1}{N+1} \|h_{a,N}(z)\|_{H^1(\Delta)} \le \frac{1}{N+1} \int_{\theta=0}^{2\pi} \int_{\tau=0}^{2\pi} \left| g_t(e^{i\theta}) \right| \left| \Gamma_N(\tau-\theta) \right| \frac{d\tau}{2\pi} \frac{d\theta}{2\pi} \quad .$$

Next, we use a substitution $u = \tau - \theta$ and the properties of the Fejér kernel:

$$\begin{aligned}
\frac{1}{N+1} \|h_{a,N}(z)\|_{H^1(\Delta)} &\le \frac{1}{N+1} \int_{\theta=0}^{2\pi} \int_{u=0}^{2\pi} \left| g_t(e^{i\theta}) \right| \left| \Gamma_N(u) \right| \frac{du}{2\pi} \frac{d\theta}{2\pi} \\
&= \frac{1}{N+1} \int_{\theta=0}^{2\pi} \left| g_t(e^{i\theta}) \right| \left[\int_{u=0}^{2\pi} \left| \Gamma_N(u) \right| \frac{du}{2\pi} \right] \frac{d\theta}{2\pi} \\
&= \frac{1}{N+1} \int_{\theta=0}^{2\pi} \left| g_t(e^{i\theta}) \right| \frac{d\theta}{2\pi} \\
&= \frac{1}{N+1} \left\| g_t(e^{i\theta}) \right\|_{L^1(\mathbb{T})} = \frac{1}{N+1} \| g_t(z) \|_{H^1} \\
&= \frac{1}{N+1} \left\| \sum_{j=0}^{N} a_j z^j + \sum_{j=N+1}^{\infty} t_j z^j \right\|_{H^1} .
\end{aligned}$$

Since the left-hand side of the inequality is independent of the choice of t, we have the desired result when we take the infimum over all $t \in c_{00}(N)$, and then the supremum over all $N \in \mathbb{N}_0$. $\square$

Proposition 14. *For all $a \in \mathcal{H}^1$ and for all $N \in \mathbb{N}_0$,*

$$\sup_{N\in\mathbb{N}_0} \zeta_N^1(a) \leq 5 \cdot \sup_{N\in\mathbb{N}_0} \Omega_N(a)\,.$$

Proof. Fix $a \in \mathcal{H}^1$ and $N \in \mathbb{N}_0$, and consider

$$\begin{aligned}
q_{a,N}(z) &:= \frac{1}{N+1}\sum_{n=N}^{2N} f_{a,n}(z) = \frac{1}{N+1}\sum_{n=N}^{2N}\sum_{j=0}^{n} a_j z^j \\
&= \frac{1}{N+1}\left[\sum_{j=0}^{N} a_j z^j + \sum_{n=N+1}^{2N}\sum_{j=0}^{n} a_j z^j\right] \\
&= \frac{1}{N+1} f_{a,N}(z) + \frac{1}{N+1}\sum_{n=N+1}^{2N}\left(\sum_{j=0}^{N} a_j z^j + \sum_{j=N+1}^{n} a_j z^j\right) \\
&= \frac{1}{N+1} f_{a,N}(z) + \frac{1}{N+1}\cdot N\sum_{j=0}^{N} a_j z^j + \frac{1}{N+1}\sum_{n=N+1}^{2N}\sum_{j=N+1}^{n} a_j z^j \\
&= f_{a,N}(z) + \frac{1}{N+1}\sum_{j=N+1}^{2N}\sum_{n=j}^{2N} a_j z^j \\
&= f_{a,N}(z) + \frac{1}{N+1}\sum_{j=N+1}^{2N}(2N-j+1)a_j z^j.
\end{aligned}$$

Now, using the definition of $\zeta_N^1(a)$, we have

$$\frac{1}{N+1}\left\|q_{a,N}(z)\right\|_{H^1} = \frac{1}{N+1}\left\|\sum_{j=0}^{N} a_j z^j + \sum_{j=N+1}^{2N} s_j z^j\right\|_{H^1} \geq \zeta_N^1(a),$$

where $s_j = (2N-j+1)a_j/(N+1)$. Reversing this and using the triangle inequality:

$$\begin{aligned}
\zeta_N^1(a) &\leq \frac{1}{N+1}\left\|\frac{1}{N+1}\sum_{n=N}^{2N} f_{a,n}(z)\right\|_{H^1(\Delta)} \\
&= \frac{1}{N+1}\left\|\frac{1}{N+1}\sum_{n=0}^{2N} f_{a,n}(z) - \frac{1}{N+1}\sum_{n=0}^{N-1} f_{a,n}(z)\right\|_{H^1(\Delta)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{N+1}\left\|\frac{1}{N+1}\sum_{n=0}^{2N} f_{a,n}(z)\right\|_{H^1(\Delta)} + \frac{1}{N+1}\left\|\frac{1}{N+1}\sum_{n=0}^{N-1} f_{a,n}(z)\right\|_{H^1(\Delta)} \\
&= \frac{(2N+1)^2}{(N+1)^2}\cdot\frac{1}{2N+1}\left\|\frac{1}{2N+1}\sum_{n=0}^{2N} f_{a,n}(z)\right\|_{H^1(\Delta)} \\
&\quad + \frac{N^2}{(N+1)^2}\cdot\frac{1}{N}\left\|\frac{1}{N}\sum_{n=0}^{N-1} f_{a,n}(z)\right\|_{H^1(\Delta)}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{2N+1}{N+1}\right)^2 \Omega_{2N}(a) + \left(\frac{N}{N+1}\right)^2 \Omega_{N-1}(a) \\
&\leq 4\cdot\Omega_{2N}(a) + \Omega_{N-1}(a).
\end{aligned}
$$

Now we take the supremum over all $N \in \mathbb{N}_0$ to see that

$$
\sup_{N\in\mathbb{N}_0} \zeta_N^1(a) \leq 5 \sup_{N\in\mathbb{N}_0} \Omega_N(a)\,. \qquad \square
$$

Taking $A = 1$ and $B = 5$ in (1.7), we have the following.

Corollary 15. $\|a\|^{\bigstar}_{\mathcal{H}^1} := \sup_{N\in\mathbb{N}_0} \Omega_N(a)$ *is an equivalent norm on* $\mathcal{H}^1$.

In particular, if we look at (1.6), we now have

$$
\|x\|^{\bigstar}_{\mathcal{H}^1} \leq B\cdot\|w\|_\infty\,, \tag{1.8}
$$

when we assume that the Mazur map is onto.

Claim 16. $\mathcal{H}^1_0$ *is a separable Banach space.*

Proof. Fix a in $\mathcal{H}^1_0$, and define

$$
U_N(a) := \frac{1}{N+1}\sum_{k=0}^{n} P_k(a) = (a_0, \frac{N}{N+1}a_1, \frac{N-1}{N+1}a_2, \dots, \frac{1}{N+1}a_N, 0, \dots);
$$

Note that the general coefficient of a_j is 0 if $j > N$, and $(N-j+1)/(N+1)$ if $j \leq N$. Further, define

$$
\begin{aligned}
V_N(a) := a - U_N(a) &= \frac{1}{N+1}\sum_{k=0}^{n}(a - P_k(a)) = \frac{1}{N+1}\sum_{k=0}^{n} Q_k(a) \\
&= (0, \frac{1}{N+1}a_1, \frac{2}{N+1}a_2, \dots, \frac{N}{N+1}a_N, a_{N+1}, \dots);
\end{aligned}
$$

Note that the general coefficient of a_j here is 1 if $j > N$, and $(j/(N+1))$ if $j \leq N$. We wish to calculate the norm of $V_N(a)$ in $\mathcal{H}_0^1$; we start with

$$\begin{aligned}
\|a\|_{\mathcal{H}^1} := \sup_{M\in\mathbb{N}_0} \Omega_M(a) &= \sup_{M\in\mathbb{N}_0} \frac{1}{M+1} \|h_{a,M}(z)\|_{H^1(\Delta)} \\
&= \sup_{M\in\mathbb{N}_0} \frac{1}{M+1} \left\| \frac{1}{M+1} \sum_{n=0}^{M} \sum_{j=0}^{n} a_j z^j \right\|_{H^1(\Delta)} \\
&= \sup_{M\in\mathbb{N}_0} \frac{1}{M+1} \left\| \sum_{k=0}^{M} \frac{M-k+1}{M+1} a_k z^k \right\|_{H^1(\Delta)} .
\end{aligned}$$

So, if we fix $N \in \mathbb{N}$,

$$\|V_N(a)\|_{\mathcal{H}^1} = \sup_{M\in\mathbb{N}_0} \Omega_M(V_N(a)) = \sup_{M\in\mathbb{N}_0} \frac{1}{M+1} \left\| \sum_{k=0}^{M} \frac{M-k+1}{M+1} (V_N(a))_k z^k \right\|_{H^1(\Delta)} .$$

Now, fix $M \in \mathbb{N}$, and first consider first the case where $M > N$. In this case, we can write

$$\begin{aligned}
&\Omega_M(V_N(a)) \\
&= \frac{1}{M+1} \left\| \sum_{k=0}^{N} \left(\frac{M-k+1}{M+1} \right) \cdot \left(\frac{k}{N+1} \right) a_k z^k + \sum_{k=N+1}^{M} \left(\frac{M-k+1}{M+1} \right) a_k z^k \right\|_{H^1(\Delta)} .
\end{aligned}$$

Now, if we fill out the last sum down to index zero, and compensate, we have

$$\begin{aligned}
\Omega_M(V_N(a)) = \frac{1}{M+1} \Bigg\| &\sum_{k=0}^{N} \left(\frac{M-k+1}{M+1} \right) \cdot \left(\frac{k}{N+1} \right) a_k z^k \\
&+ \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1} \right) a_k z^k - \sum_{k=0}^{N} \left(\frac{M-k+1}{M+1} \right) a_k z^k \Bigg\|_{H^1(\Delta)} .
\end{aligned}$$

Next, we use the triangle inequality in $H^1(\Delta)$ to see

$$\begin{aligned}
\Omega_M(V_N(a)) \leq &\frac{1}{M+1} \left\| \sum_{k=0}^{N} \left(\frac{M-k+1}{M+1} \right) \cdot \left(\frac{k}{N+1} - 1 \right) a_k z^k \right\|_{H^1(\Delta)} \\
&+ \frac{1}{M+1} \left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1} \right) a_k z^k \right\|_{H^1(\Delta)}
\end{aligned}$$

$$= \frac{1}{M+1}\left\| \sum_{k=0}^{N} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{k-N-1}{N+1}\right) a_k z^k \right\|_{H^1(\Delta)} + \Omega_M(a)$$

$$= \frac{1}{M+1}\left\| \sum_{k=0}^{N} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{N-k+1}{N+1}\right) a_k z^k \right\|_{H^1(\Delta)} + \Omega_M(a).$$

Note here that

$$\Omega_N(U_M(a)) = \frac{1}{N+1}\left\| \sum_{k=0}^{N} \left(\frac{N-k+1}{N+1}\right) \cdot \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)},$$

which is not what we have calculated above. However, we can refer to the proof of Claim 13 and argue as follows: for every j in $\{-M, \dots, 0, \dots, M\}$,

$$\left(\frac{M-k+1}{M+1}\right) a_j \;=\; \int_0^{2\pi} g_T(e^{i\theta}) e^{-ij\theta} \frac{d\theta}{2\pi},$$

where

$$g_T(z) := \sum_{k=0}^{M} a_k z^k \left(\frac{M-k+1}{M+1}\right).$$

We have previously established that

$$\frac{1}{N+1}\left\|h_{U_M(a),N}(z)\right\|_{H^1(\Delta)} \leq \frac{1}{N+1}\|g_T(z)\|_{H^1(\Delta)}$$

implies $\left\|h_{U_M(a),N}(z)\right\|_{H^1(\Delta)} \leq \|g_T(z)\|_{H^1(\Delta)}$; thus,

$$\begin{aligned}
\Omega_M(V_N(a)) &= \frac{1}{M+1}\left\| \sum_{k=0}^{N} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{N-k+1}{N+1}\right) a_k z^k \right\|_{H^1(\Delta)} + \Omega_M(a) \\
&= \frac{N+1}{M+1}\Omega_N(U_M(a)) \;+\; \Omega_M(a) \\
&= \frac{1}{M+1}\left\|h_{U_M(a),N}(z)\right\|_{H^1(\Delta)} \;+\; \Omega_M(a) \\
&\leq \frac{1}{M+1}\|g_T(z)\|_{H^1(\Delta)} \;+\; \Omega_M(a) \\
&= \frac{1}{M+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \;+\; \Omega_M(a) \\
&= 2\Omega_M(a).
\end{aligned}$$

Now, Claim 13 gives $\Omega_M(a) \leq \zeta_M(a)$, and we know that $a \in \mathcal{H}_0^1$ if and only if $\lim \zeta_M(a) = 0$; therefore, for every $M < N$, $\Omega_M(V_N(a)) \;\leq\; 2\Omega_M(a) \;\leq\; 2\zeta_M(a)$.

Now we examine the second case, where $M \le N$. Recall that

$$
\begin{aligned}
&\Omega_M(V_N(a)) \\
&= \frac{1}{M+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot (V_N(a))_k \, z^k \right\|_{H^1(\Delta)} \\
&= \frac{1}{M+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{k}{N+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&= \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{k}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&= \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{k}{M+1}\right) a_k z^k \right. \\
&\qquad \left. - \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k + \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&= \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{k-M-1}{M+1}\right) a_k z^k + \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&\le \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&\qquad + \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)}.
\end{aligned}
$$

Again, referring to previous work, we can see that

$$
\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \le \left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)},
$$

so for $M \le N$,

$$
\begin{aligned}
&\Omega_M(V_N(a)) \\
&\le \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) \cdot \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&\qquad + \frac{1}{N+1}\left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \\
&\le \frac{1}{N+1}\left[2 \cdot \left\| \sum_{k=0}^{M} \left(\frac{M-k+1}{M+1}\right) a_k z^k \right\|_{H^1(\Delta)} \right]
\end{aligned}
$$

$$= \frac{M+1}{N+1} \cdot 2 \cdot \Omega_M(a) \leq \begin{cases} 2\Omega_M(a) & \text{if } \sqrt{N} \leq M \leq N; \\ \frac{2}{\sqrt{N}} \cdot \|a\| & \text{if } M \leq \sqrt{N} - 1. \end{cases}$$

We now claim that there exists for fixed $\varepsilon > 0$ some $J(\varepsilon) \in \mathbb{N}$ such that for every N larger that J,

$$\|V_N(a)\|_{\mathcal{H}^1} = \sup_{M \in \mathbb{N}_0} \Omega_M(V_N(a)) < \varepsilon.$$

To prove this, we need to use the following 2 facts:

Remark 17. *For $a \in \mathcal{H}_0^1$, $\lim_{j\to\infty} \Omega_j(a) = 0$; so, for every $\eta > 0$, there exists some $K(\eta) \in \mathbb{N}$ such that $\forall j \geq K(\eta)$, $\Omega_j(a) < \eta$.*

Remark 18. *There exists some $\Gamma(\varepsilon) \in \mathbb{N}$ such that $\frac{2}{\sqrt{\Gamma(\varepsilon)}} \cdot \|a\|_{\mathcal{H}^1} < \frac{\varepsilon}{2}$; for every $N \geq \Gamma(\varepsilon)$, we also have $\frac{2}{\sqrt{N}} \cdot \|a\|_{\mathcal{H}^1} < \frac{\varepsilon}{2}$. Note that $\Gamma > (16\|a\|^2)/\varepsilon$.*

Now, fix $N \geq \Gamma(\varepsilon)$ such that $N \geq (K(\varepsilon/4))^2$; notice that for every $j > K(\varepsilon/4)$, $\Omega_j(a) < \varepsilon/4$. By Remark 18, if $M < \sqrt{N}$, we have

$$\Omega_M(V_N(a)) \leq \frac{2}{\sqrt{N}} \cdot \|a\|_{\mathcal{H}^1} < \frac{\varepsilon}{2}.$$

On the other hand, if $M \geq \sqrt{N}$, then we have $M \geq \sqrt{N} \geq K(\varepsilon/4)$, and so by Remark 17,

$$\Omega_M(V_N(a)) \leq 2\Omega_M(a) < 2\left(\frac{\varepsilon}{4}\right) = \frac{\varepsilon}{2}.$$

Defining $J(\varepsilon) := \max\{\Gamma(\varepsilon), (K(\varepsilon/4))^2\}$, we have that for every $N \geq J(\varepsilon)$,

$$\Omega_M(V_N(a)) < \frac{\varepsilon}{2} < \varepsilon$$

for every M. Consequently, for every $N \geq J(\varepsilon)$,

$$\|V_N(a)\|_{\mathcal{H}^1} = \sup_{M \in \mathbb{N}_0} \Omega_M(V_N(a)) < \varepsilon.$$

Therefore,

$$\lim_{N\to\infty} \|V_N(a)\|_{\mathcal{H}^1} = \lim_{N\to\infty} \left\| a - \frac{1}{N+1} \sum_{k=0}^{n} P_k(a) \right\| = 0,$$

proving that $\mathcal{H}_0^1$ is a separable Banach space with countable dense subset

$$\left\{ U_N(a) = \left(a_0, \frac{N}{N+1} a_1, \frac{N-1}{N+1} a_2, \ldots, \frac{1}{N+1} a_N, 0, \ldots \right) : \text{each } a_j \in \mathbb{Q} \text{ and } N \in \mathbb{N} \right\}.$$

□

1.4 PROOF OF MAIN THEOREM

We need a result on lacunary sequences from [18].

Lemma 19 (Paley's Inequality). *There exists a positive constant C such that for every $f = \sum \gamma_n z^n$ in $H^1(\Delta)$,*

$$\left(\sum_{k=0}^{\infty} |\gamma_{2^k}|^2 \right)^{\frac{1}{2}} \leq C \cdot \|f\|_{H^1(\Delta)} .$$

Adapting an argument from Zygmund [19] for Rademacher functions, we have the following argument. Fix τ in $[0, 2\pi)$. For every $k \in \mathbb{N}_0$, we can define

$$a^{(k,\tau)} := \left(0, 1 \cdot e^{i2\tau}, 2 \cdot e^{i2^2\tau}, \ldots, k \cdot e^{i2^k\tau}, 0, 0, \ldots\right) \quad \in \mathcal{H}_0^1 .$$

Fix $N \in \mathbb{N}_0$. Let's calculate the average value of $\Omega_N(a^{(N,\tau)})$:

$$\begin{aligned}
&\frac{1}{N+1} \int_{\tau=0}^{2\pi} \|h_{a^{(N,\tau)},N}(z)\|_{H^1(\Delta)} \frac{d\tau}{2\pi} \\
&= \frac{1}{N+1} \int_{\tau=0}^{2\pi} \left\| \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=0}^{n} k e^{i2^k\tau} e^{ik\theta} \right\|_{L^1(\mathbb{T})} \frac{d\tau}{2\pi} \\
&= \frac{1}{N+1} \int_{\tau=0}^{2\pi} \left\| \frac{1}{N+1} \sum_{k=0}^{N} \sum_{n=k}^{N} k e^{i2^k\tau} e^{ik\theta} \right\|_{L^1(\mathbb{T})} \frac{d\tau}{2\pi} \\
&= \frac{1}{N+1} \int_{\tau=0}^{2\pi} \left\| \frac{1}{N+1} \sum_{k=0}^{N} k(N-k+1) e^{i2^k\tau} e^{ik\theta} \right\|_{L^1(\mathbb{T})} \frac{d\tau}{2\pi} \\
&= \frac{1}{N+1} \int_{\theta=0}^{2\pi} \left(\int_{\tau=0}^{2\pi} \left| \frac{1}{N+1} \sum_{k=0}^{N} k(N-k+1) e^{i2^k\tau} e^{ik\theta} \right| \frac{d\tau}{2\pi} \right) \frac{d\theta}{2\pi}
\end{aligned}$$

using Fubini's Theorem again. Next, we use Paley's inequality to see that

$$\begin{aligned}
&\frac{1}{N+1} \int_{\tau=0}^{2\pi} \|h_{a^{(N,\tau)},N}(z)\|_{H^1(\Delta)} \frac{d\tau}{2\pi} \\
&\geq \frac{1}{N+1} \int_{\theta=0}^{2\pi} C \left(\sum_{k=0}^{N} \left| \frac{k(N-k+1)}{N+1} e^{ik\theta} \right|^2 \right)^{\frac{1}{2}} \frac{d\theta}{2\pi} \\
&= \frac{C}{(N+1)^2} \left[\sum_{k=0}^{N} k^2 (N-k+1)^2 \right]^{\frac{1}{2}} \\
&= \frac{C}{(N+1)^2} \left[(N+1)^2 \sum_{k=0}^{N} k^2 - 2(N+1) \sum_{k=0}^{N} k^3 + \sum_{k=0}^{N} k^4 \right]^{\frac{1}{2}} .
\end{aligned}$$

At this point, we evaluate these partial sums to get:

$$
\begin{aligned}
&= \frac{C}{(N+1)^2}\Bigg[(N+1)^2\left[\frac{N(N+1)(2N+1)}{6}\right] - 2(N+1)\left[\frac{N^2(N+1)^2}{4}\right] \\
&\qquad + \left[\frac{N(N+1)(6N^3+9N^2+N-1)}{30}\right]\Bigg]^{\frac{1}{2}} \\
&= \frac{C}{(N+1)^2}\left[\frac{N+1}{30}(N^4+4N^3+6N^2+4N+1-1)\right]^{\frac{1}{2}} \\
&= \frac{C}{(N+1)^2}\left[\frac{(N+1)^5-(N+1)}{30}\right]^{\frac{1}{2}} \geq \frac{C\cdot\sqrt{N}}{\sqrt{30}}\,.
\end{aligned}
$$

We can now see that for every $N \in \mathbb{N}_0$, there exists some $\tau_N \in [0, 2\pi)$ such that

$$
\Omega_N(a^{(N,\tau_N)}) \geq \frac{C\cdot\sqrt{N}}{\sqrt{30}}\,. \tag{1.9}
$$

On the other hand, assuming the Mazur map is onto, we have from (1.8) that

$$
\Omega_N(a^{(N,\tau_N)}) \leq \|a^{(N,\tau_N)}\|^{\bigstar}_{\mathcal{H}^1} \leq B\cdot\|Ja^{(N,\tau_N)}\|_\infty = \frac{B\cdot k}{k+1} < B\,. \tag{1.10}
$$

So, we have established from (1.9) and (1.10) that $C\cdot\sqrt{N}/\sqrt{30} < B$, for every $N \in \mathbb{N}_0$; which is clearly false. This establishes Theorem 12, thereby establishing the main result that the maps $\boxtimes : \mathcal{H}^p_0 \times \mathcal{H}^q \longrightarrow c_0$ and $\boxtimes : \mathcal{H}^p \times \mathcal{H}^q_0 \longrightarrow c_0$ are not onto.

1.5 FURTHER PROPERTIES OF $\mathcal{H}^P$ AND APPLICATIONS OF MAIN THEOREM

We noted the following in the Introduction.

Claim 20. *For $1 \leq p \leq 2$, $c_0 \subsetneq \mathcal{H}^p_0$ and $\ell^\infty \subsetneq \mathcal{H}^p$.*

Proof. To see why the first statement is true, first notice that $c_0 \subsetneq \mathcal{H}_0^2$, as assuming $a \in c_0$ implies that $(n+1)^{-1} \sum_{j=0}^{n} |a_n|^2 \to 0$ as $n \to \infty$, which in turn implies that $\zeta_n^2(a) \to 0$ as well. We can clearly see that the sequence $(x_j)_{\mathbb{N}_0}$ defined by
$\{x_j := \sqrt{n}$ for $j = 2^n$, $x_j := 0$ otherwise$\}$ is not in c_0. However,

$$(\zeta_{2^k}(x))^2 = \frac{1+2+\ldots+k}{2^k+1} \xrightarrow{k} 0,$$

and for $n \in [2^{k-1}, 2^k)$, $(\zeta_n^2(x))^2 \leq (\zeta_{2^{k-1}}^2(x))^2$; this shows that $\zeta_n^2(x) \to 0$, and therefore $x \in \mathcal{H}_0^2$. Finally, notice that for $1 \leq p \leq 2$,

$$\begin{aligned}\zeta_n^p(x) &\leq (n+1)^{-1/p} \left\| \sum_{j=0}^{n} x_j z^j \right\|_{H^p} \leq (n+1)^{-1/p} \left\| \sum_{j=0}^{n} x_j z^j \right\|_{H^2} \\ &= (n+1)^{-1/p} \left(\sum_{j=0}^{n} |x_j|^2 \right)^{1/2} \leq (n+1)^{-1/2} \left(\sum_{j=0}^{n} |x_j|^2 \right)^{1/2},\end{aligned}$$

which is the same as $\zeta_n^2(x)$. □

It is interesting that these containments are false for $p > 2$.

Proposition 21. *For $2 < p < \infty$, $c_0 \nsubseteq \mathcal{H}_0^p$ and $\ell^\infty \nsubseteq \mathcal{H}^p$.*

Proof. Fix $p \in (2, \infty)$ and $n \in \mathbb{N}_0$. We use the fact that $H^p \subset H^2$ to get the following estimate for $a \in S$:

$$\begin{aligned}\zeta_n^p(a) &\geq \frac{1}{(n+1)^{1/p}} \left(\inf_{t \in c_{00}(n)} \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^2(\Delta)} \right) \\ &= \frac{1}{(n+1)^{1/p}} \left(\sum_{j=0}^{n} |a_j|^2 \right)^{1/2}.\end{aligned}$$

Let $a \in c_0$ be defined by $a_j := (j+1)^{-\alpha}$ for some fixed $0 < \alpha < 1/2 - 1/p$. We see that $a \in c_0 \backslash \mathcal{H}^p$, and therefore $a \in c_0 \backslash \mathcal{H}_0^p$, because

$$\begin{aligned}\zeta_n^p(a) &\geq \frac{1}{(n+1)^{1/p}} \left(\sum_{j=0}^{n} \frac{1}{(j+1)^{2\alpha}} \right)^{1/2} \sim \frac{1}{(n+1)^{1/p}} \left(\int_{x=1}^{n+1} \frac{1}{x^{2\alpha}} dx \right)^{1/2} \\ &= \frac{1}{(n+1)^{1/p}} \cdot \frac{1}{(1-2\alpha)^{1/2}} \left((n+1)^{1-2\alpha} - 1 \right)^{1/2} \\ &\sim \frac{1}{(n+1)^{1/p}} \cdot \frac{1}{(1-2\alpha)^{1/2}} (n+1)^{1/2-\alpha} = \frac{(n+1)^{1/2-\alpha-1/p}}{(1-2\alpha)^{1/2}},\end{aligned}$$

which is unbounded as $n \to \infty$. (Here, for positive sequences, $\gamma_n \sim \beta_n$ means β_n/γ_n is bounded and bounded away from zero.) We also have $a \in \ell^\infty \setminus \mathcal{H}^p$. □

We wish to explore the case $p > 2$ further. Let q be the conjugate index of p; i.e., let $1/p + 1/q = 1$.

Proposition 22. *For $2 < p \leq \infty$, $\ell^q \subseteq \mathcal{H}^p$.*

Proof. Fix $a \in \ell^q$.

$$\zeta_n^p(a) \leq (n+1)^{-1/p} \left\| \sum_{j=0}^{n} a_j e^{ij\theta} \right\|_{H^p} = (n+1)^{-1/p} \sum_{j=0}^{n} |a_j| \, \|e^{ij\theta}\|_{H^p},$$

which, for any conjugate indices r and s, is

$$\leq (n+1)^{-1/p} \left(\sum_{j=0}^{n} |a_j|^r \right)^{1/r} \left(\sum_{j=0}^{n} 1^s \right)^{1/s} = (n+1)^{1/s - 1/p} \left(\sum_{j=0}^{n} |a_j|^r \right)^{1/r}.$$

If we specifically choose $r < q$, then it must be that $s > p$, so $\zeta_n^p(a)$ goes to zero as n goes to infinity. □

Recall the intermediary sequence spaces from Definition 4. We can show the following.

Proposition 23. *For all $1 \leq p \leq 2$, we have $\lambda^{p,p} \subseteq \mathcal{H}^p$ and $\lambda_0^{p,p} \subseteq \mathcal{H}_0^p$.*

Proof. Let $a \in S$ and $n \in \mathbb{N}_0$. Since we have $\zeta_n^2(a) = (n+1)^{-1/2} \left(\sum_{j=0}^{n} |a_j|^2 \right)^{1/2}$ and $\zeta_n^1(a) \leq (n+1)^{-1} \sum_{j=0}^{n} |a_j|$, we can use complex interpolation to conclude, for any $p \in [1,2]$, that

$$\zeta_n^p(a) \leq \frac{1}{(n+1)^{1/p}} \left(\sum_{j=0}^{n} |a_j|^p \right)^{1/p}.$$

□

Proposition 24. *(a) For $2 \leq p < \infty$, $\ell^q \subsetneq \lambda_0^{p,q}$.*

(b) For $2 < p \leq \infty$, $\lambda^{p,q} \subseteq \mathcal{H}^p$.

(c) For $2 < p < \infty$, $\lambda_0^{p,q} \subseteq \mathcal{H}_0^p$.

Proof. Part (a) is clear. Let $a \in S$ and $n \in \mathbb{N}_0$. Then $\zeta_n^\infty(a) \le \sum_{j=0}^{n} |a_j|$, and

$\zeta_n^2(a) = \left(\sum_{j=0}^{n} |a_j|^2 \cdot (n+1)^{-1} \right)^{1/2}$. By complex interpolation, for any $p \in (2, \infty)$,

$$\zeta_n^p(a) \le \left(\sum_{j=0}^{n} \left| \frac{a_j}{n+1} \right|^q (n+1) \right)^{1/q} = \frac{1}{(n+1)^{1/p}} \left(\sum_{j=0}^{n} |a_j|^q \right)^{1/q}.$$

□

The following theorem is an application of Theorem 3. The proof follows easily from the containments proven in Propositions 23 and 24.

Theorem 25. *For every $p \in [1,\ 2]$ and its conjugate index $q \in [2, \infty]$,*
(a)the Mazur map $\boxtimes : \lambda_0^{p,p} \times \lambda^{q,p} \longrightarrow c_0$ is not onto; and
(b)the Mazur map $\boxtimes : \lambda^{p,p} \times \lambda_0^{q,p} \longrightarrow c_0$ is not onto ($q < \infty$).

1.6 MAZUR'S THEOREM

We note that the main result, Theorem 3, with $p = q = 2$, provides an alternate solution of Problem 8 from [17]. Recall that the problems has been solved by others previously, as discussed in the Introduction.

Theorem 26. *The Mazur map $\boxtimes : c \times c \longrightarrow c$ is not onto.*

Proof. Assume that the map is onto. Fix $y \in c_0 \subset c$. Then, there exists $u, v \in c$ such that $y = u \boxtimes v = v \boxtimes u$. But $0 = \lambda_y = \lambda_u \cdot \lambda_v$, which implies $\lambda_u = 0$ or $\lambda_v = 0$. Thus, $\boxtimes : c \times c_0 \longrightarrow c_0$ is onto.
However, we know that $c \subseteq \mathcal{H}^2$ and $c_0 \subseteq \mathcal{H}_0^2$, and we have proven that $\boxtimes : \mathcal{H}^2 \times \mathcal{H}_0^2 \longrightarrow c_0$ is not onto. So, there must exist at least one $y_0 \in c_0$ such that

$$\forall u \in \mathcal{H}^2, \quad \forall v \in \mathcal{H}_0^2, \qquad u \boxtimes v \ne y_0\,;$$

which implies that $\forall u \in c$, $\forall v \in c_0$, $u \boxtimes v \ne y_0$. This is a contradiction. □

2.0 FRAMES AND N-CESÀRO BASES IN HILBERT SPACE

2.1 INTRODUCTION

The following discussion of frames is based on lectures and seminars given by Chris Lennard at the University of Pittsburgh. Also see, for example, the work of Ole Christensen in [5] or [4]. Most of the general introductory discussion can be found in the survey papers of Pete Casazza [2] or Heil and Walnut [9], or the original paper on abstract frames by Duffin and Schaeffer [6].

In this chapter, we are considering a separable, infinite-dimensional Hilbert space $(H, < \cdot, \cdot >)$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with orthonormal basis $(e_k)_{\mathbb{N}}$. We will abbreviate sequences $(x_i)_{i \in I}$ to $(x_i)_I$ when it is clear which index determines the sequence discussed.

Definition 27. *For an index set I, a family $(g_i)_I \subset H$ is a Hilbert frame for H if there are constants $0 < A \leq B < \infty$ such that for all $f \in H$,*

$$A||f||^2 \leq \sum_{i \in I} | < f, g_i > |^2 \leq B||f||^2.$$

If at least the upper frame bound B exists, then $(g_i)_I$ is called a Bessel sequence.

We will be considering the index set of natural numbers, $I := \mathbb{N}$. Let $(\xi_i)_{\mathbb{N}}$ be the usual orthonormal basis for ℓ^2.

If $(g_i)_{\mathbb{N}}$ is a frame for H, then $T : H \to \ell^2$ defined by

$$T(f) := (< f, g_j >)_{\mathbb{N}} = \sum_{j \in \mathbb{N}} < f, g_j > \xi_j$$

is a well-defined surjective linear mapping. T is also a Banach space isomorphism, as for all $f \in H$, $\sqrt{A}||f|| \leq ||T(f)||_{\ell^2} \leq \sqrt{B}||f||$. Since T is a continuous map, there exists a well-defined adjoint mapping $T^* : \ell^2 \to H$, where $T^*((x_j)_{\mathbb{N}}) = \sum_{j \in \mathbb{N}} x_j g_j$, for all $(x_j)_{\mathbb{N}} \in \ell^2$, and $||T^*|| = ||T|| \leq \sqrt{B}$. In this case, we can define the frame operator $S : H \to H$ by setting, for every vector $f \in H$,

$$S(f) := T^*T(f) = \sum_{j \in \mathbb{N}} < f, g_j > g_j.$$

S is self-adjoint, as $S^* = (T^*T)^* = T^*T^{**} = T^*T = S$; S is also positive definite, as $< Sf, f >= \sum_{j \in \mathbb{N}} | < f, g_j > |^2$ for all $f \in H$.

Claim 28. *S is an invertible operator, and $S^{-1} : H \to H$ is a bounded, linear, self-adjoint, positive definite operator.*

Proof. Based on the discussion about S above, there exists a unique bounded, linear, self-adjoint, positive definite operator R such that $R^2 = S$, or $R = S^{1/2}$. We have
$< Sf, f >=< S^{1/2}S^{1/2}f, f >= ||S^{1/2}f||^2$, so the fact that $A||f||^2 \leq < Sf, f > \leq B||f||^2$ implies that

$$\sqrt{A}||f|| \leq ||S^{1/2}f|| \leq \sqrt{B}||f||. \tag{2.1}$$

This means that $S^{1/2}$ is an invertible mapping of H onto H, and is in fact a Banach space isomorphism. So, for each $f \in H$, there must be some $g \in H$ with $f = (S^{1/2})^{-1}g$. From (2.1), we have

$$\sqrt{A}||(S^{1/2})^{-1}g|| \leq ||S^{1/2}(S^{1/2})^{-1}g|| = ||g||,$$

which implies that $||(S^{1/2})^{-1}|| \leq A^{-1/2}$. This means that $S = S^{1/2}S^{1/2}$ is invertible, and $S^{-1} = (S^{1/2})^{-1}(S^{1/2})^{-1}$. This in turn implies that

$$||S^{-1}|| \leq ||(S^{1/2})^{-1}|| \cdot ||(S^{1/2})^{-1}|| \leq \frac{1}{A}.$$

It is straightforward to check that the operator $S^{-1/2} := (S^{1/2})^{-1}$ is self-adjoint and positive definite; and so, S^{-1} also has these properties. □

For every $j \in \mathbb{N}$, define $h_j := S^{-1}g_j$. Notice the following two norm-convergent reconstruction properties hold for every $f \in H$; the first comes from writing $f = S^{-1}Sf$, and the second from writing $f = SS^{-1}f$.

$$f = \sum_{j \in \mathbb{N}} < f, g_j > h_j \tag{2.2}$$

$$f = \sum_{j \in \mathbb{N}} < f, h_j > g_j \tag{2.3}$$

In this case, $(h_j)_{\mathbb{N}}$ is also a Hilbert frame for H, called the dual frame, with lower frame bound $1/B$ and upper frame bound $1/A$.

For a general Bessel sequence $(g_j)_{\mathbb{N}}$, it is important to note that S is not generally onto H. In fact, if $S(H) = H$, the open mapping theorem gives the existence of S^{-1} as a bounded linear operator on all of H, and $(g_j)_{\mathbb{N}}$ would be a Hilbert frame for H. Note that, in this case, since S is onto, the closed linear span of $\{g_j : j \in \mathbb{N}\}$ equals H. Therefore, $\{g_j : j \in \mathbb{N}\}^{\perp} = \{0\}$, and consequently S is one-to-one. This explains why, in this situation, the mapping S^{-1} exists. In the case of a Bessel $(g_j)_{\mathbb{N}}$, we also note that T is not generally onto ℓ^2. If $T(H) = \ell^2$, then T would be both one-to-one and onto, and again the open mapping theorem would guarantee that T^{-1} existed on all of ℓ^2. Therefore, we would have

$$\sum_{j \in \mathbb{N}} | < f, g_j > |^2 = ||Tf||^2 \geq \frac{1}{||T^{-1}||^2} ||T^{-1}Tf||^2 = \frac{1}{||T^{-1}||^2} ||f||^2;$$

as before, $(g_j)_{\mathbb{N}}$ would be a Hilbert frame for H.

In this chapter, we will consider the vector sequences defined by

$$\left(g_j^{(n)} \right)_{j \in \mathbb{N}} := \left(\sum_{k=0}^{n} \binom{n}{k} e_{j+k} \right)_{j \in \mathbb{N}}$$

for any $n \in \mathbb{N}_0$. As an example, consider what we have when $n = 0$: $g_j^{(0)} = e_j$ for every $j \in \mathbb{N}$, and therefore $(g_j^{(0)})_{\mathbb{N}}$ is a tight frame for H with frame bounds $A = B = 1$. The dual frame for norm reconstruction of arbitrary $f \in H$ is then $(h_j^{(0)})_{\mathbb{N}} = (e_j)_{\mathbb{N}}$, i.e.

$$S_N^{(0)}(f) := \sum_{i=1}^{N} < f, e_i > e_i \xrightarrow{N} f.$$

The vector sequences $\left(g_j^{(n)} \right)_{j \in \mathbb{N}}$ have the following interesting properties.

Lemma 29. *For any fixed natural numbers n and j, $g_j^{(n)} = g_j^{(n-1)} + g_{j+1}^{(n-1)}$.*

Proof.

$$
\begin{aligned}
g_j^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} e_{j+k} = \binom{n}{0} e_j + \sum_{k=1}^{n-1} \binom{n}{k} e_{j+k} + \binom{n}{n} e_{j+n} \\
&= \binom{n-1}{0} e_j + \sum_{k=1}^{n-1} \binom{n-1}{k} e_{j+k} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} e_{j+k} + \binom{n-1}{n-1} e_{j+n} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} e_{j+k} + \sum_{k=1}^{n} \binom{n-1}{k-1} e_{j+k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} e_{j+k} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} e_{j+\ell+1} \\
&= g_j^{(n-1)} + g_{j+1}^{(n-1)}.
\end{aligned}
$$

□

Lemma 30. *For each $n \in \mathbb{N}$, $\left(g_j^{(n)}\right)_{j\in\mathbb{N}}$ is a Bessel sequence in H with constant $\sim \dfrac{4^n(n+1)}{\sqrt{\pi n}}$ for large n.*

Proof. Fix $n \in \mathbb{N}$ and $f \in H$.

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \left| < f, g_j^{(n)} > \right|^2 = \sum_{j=1}^{\infty} \left| < f, \sum_{k=0}^{n} \binom{n}{k} e_{j+k} > \right|^2 = \sum_{j=1}^{\infty} \left| \sum_{k=0}^{n} \binom{n}{k} < f, e_{j+k} > \right|^2 \\
&\leq \sum_{j=1}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}^2 \right) \left(\sum_{k=0}^{n} |< f, e_{j+k} >|^2 \right) = \left[\sum_{k=0}^{n} \binom{n}{k}^2 \right] \sum_{k=0}^{n} \sum_{j=1}^{\infty} |< f, e_{j+k} >|^2 \\
&\lesssim \frac{4^n(n+1)}{\sqrt{\pi n}} ||f||^2
\end{aligned}
$$

for large n, using Lemma 47. □

We will need the following definitions, which can be found in [16]:

Definition 31. *Let X be a Banach space and let X^* be its dual. A system $\{\{x_i\}, \{x_i^*\}\}$, $i \in I$, $x_i \in X$, $x_i^* \in X^*$, where I is some set of indices, is called a Markuschevich basis (an M-basis) of X if the closed linear span $[x_i,\, i \in I] = X$, $x_i^*(x_j) = \delta_{ij}$, and for every nonzero $x \in X$ there exists $i \in I$ with $x_i^*(x) \neq 0$.*

Definition 32. *The Cesàro averaging operator σ, applied to a vector or scalar sequence term, is defined by*

$$\sigma(x_N) := \sigma_N(x) := \frac{1}{N}\sum_{j=1}^{N} x_j.$$

We further recursively define $\sigma^n(x_N) := \sigma_N^n(x) := \frac{1}{N}\sum_{j=1}^{N} \sigma_j^{(n-1)}(x)$ for $n \in \mathbb{N}_0$, with the convention that $\sigma_N^0(x_k) := I_N(x_k) = x_N$.

Definition 33. *Let $\{\{x_n\}, \{x_n^*\}\}$ be an M-basis of a (separable) Banach space X. Let $S_n(x) := \sum_{i=1}^{n} x_i^*(x)x_i$ and $\sigma_n(x) := \sum_{i=1}^{n} S_i(x)/n$. If $||\sigma_n(x) - x|| \longrightarrow 0$ as $n \longrightarrow \infty$ for every $x \in X$, then the system $\{\{x_n\}, \{x_n^*\}\}$ is called a Cesàro basis of X.*

We consider a problem that was posed in Casazza, et al. [3], where a Banach frame for a Hilbert space was shown to not always be a Hilbert frame for the Hilbert space. Consider $(g_j^{(1)})_{j\in\mathbb{N}} := (e_j + e_{j+1})_{j\in\mathbb{N}} \subset H$. $(g_j^{(1)})_{\mathbb{N}}$ is complete and minimal, and for all $f \in H$,

$$\sum_{i\in\mathbb{N}} | < f, g_i^{(1)} > |^2 \leq 4||f||^2 \ ,$$

so that $(g_j^{(1)})_{\mathbb{N}}$ is a Bessel sequence for H. This implies the existence of a biorthogonal sequence in H [5],

$$h_j^{(1)} := \sum_{k=1}^{j} (-1)^{j+k} e_k \qquad \forall j \in \mathbb{N}.$$

Although $\left\{(h_j^{(1)}), (g_j^{(1)})\right\}_{\mathbb{N}}$ is an M-basis for H, $(g_i^{(1)})_N$ is not a frame for H. To see this, note that $||h_i^{(1)}||^2 = i$ for all i; this means that

$$\frac{1}{i}||h_i^{(1)}||^2 = 1 = \sum_{k=1}^{\infty} | < h_i^{(1)}, g_k^{(1)} > |^2,$$

so that there can be no nonzero lower frame bound A. Also, in this situation we have no reconstruction property for all vectors in H, as $e_1 \neq \sum c_k g_k^{(1)}$ for any choice $(c_k)_{\mathbb{N}} \in \ell^2$. Further, if we define

$$S_N^{(1)}(f) := \sum_{i=1}^{N} < f, g_i^{(1)} > h_i^{(1)} \ ,$$

we do not have $S_N^{(1)}(f) \xrightarrow{N} f$ for all $f \in H$. To see that this is the case, note that $S_N^{(1)}(f) = S_N^{(0)}(f) + < f, g_{N+1}^{(0)} > h_N^{(1)}$, and

$$|| < f, g_{N+1}^{(0)} > h_N^{(1)} || \geq \sqrt{N} \cdot | < f, g_{N+1}^{(0)} > |;$$

this expression will not converge to zero with N for certain vectors in H. For example, if we choose the sequence $(x_j) := (0, 1, 0, 0, 1/2, 0, 0, 0, 0, 1/3, \ldots) \in \ell^2$ as the coefficient sequence determining some $f_x \in H$, the limit of the sequence $(| < f_x, g_{N+1}^{(0)} > |)_{N \in \mathbb{N}}$ would not exist as N goes to infinity (there is a constantly 1 subsequence as well as a constantly 0 subsequence).

However, we can show that a weaker type of reconstruction is available here, as

$$\begin{aligned} ||\sigma_N(S^{(1)}(f)) - f|| &= ||\sigma_N(S^{(0)}(f)) - f + \sigma_N(< f, g_{k+1}^{(0)} > h_k^{(1)})|| \\ &\leq ||\sigma_N(S^{(0)}(f)) - f|| + ||\sigma_N(< f, g_{k+1}^{(0)} > h_k^{(1)})|| \\ &\longrightarrow 0 + 0. \end{aligned}$$

(See Theorem 54 for a proof.) By definition, this means that $\left\{(h_j^{(1)}), (g_j^{(1)})\right\}_{\mathbb{N}}$ is a Cesàro basis for H.

In general, for any f in H, if we define$(h_j^{(n)})_{\mathbb{N}}$ as in Theorem 49 and

$$S_k^{(n)}(f) := \sum_{j=1}^{k} < f, g_j^{(n)} > h_j^{(n)}$$

for every whole number n and natural number k, we have the following.

Theorem 34. *Consider $\sigma_N^m(S^{(n)}(f))$, for $0 \leq m \leq n$.*

(1) If $m < n$, there exists some f_0 in H such that $\sigma_N^m(S^{(n)}(f_0))$ is not norm convergent to f_0 as $N \to \infty$.

(2) $\sigma_N^n(S^{(n)}(f))$ is norm convergent to f as $N \to \infty$ for all f in H.

In the case given as part (2) of Theorem 34, we will say that $\left\{(h_j^{(n)}), (g_j^{(n)})\right\}_{\mathbb{N}}$ is a n-Cesàro basis for H. The proof of Theorem 34 is contained in section 2.5.

Notice that Theorem 34 addresses reconstruction via convergence variations on equation (2.2). Concerning norm reconstruction via (2.3), we make the following observation.

Claim 35. *For any Bessel sequence $(g_j)_{\mathbb{N}} \subset H$, assume that T is one to one and the kernel of T^* in ℓ^2 contains only the zero sequence. If we further assume that each g_j is in the range of S, we can norm reconstruct every vector in the range of S via (2.3).*

Proof. We can still define T, T^*, and S as above; because $(g_j)_{\mathbb{N}}$ is a Bessel sequence, $\sum_{j\in\mathbb{N}} c_j g_j$ converges unconditionally for all sequences $(c_j)_{\mathbb{N}}$ in ℓ^2, so $S(f)$ is well-defined. Assuming that T is one to one is the same as saying that f must be the zero vector whenever $(< f, g_j >)_{\mathbb{N}}$ is the zero sequence; this implies that the linear span of $(g_j)_{\mathbb{N}}$ is dense in H, and thus the closed linear span of $(g_j)_{\mathbb{N}}$ is exactly H. This in turn implies that S is one to one, and therefore has kernel consisting only of the zero vector. Therefore, the range of S is dense in H.

Similarly, assuming that the zero sequence is the only element of the kernel of T^* is equivalent to saying that the range of T is dense in ℓ^2. (Remember, if S and T were onto instead of just having dense range, $(g_j)_{\mathbb{N}}$ would automatically be a frame for H.)

Choose an arbitrary f in the range of S, and denote f_p as the unique element of H such that $Sf_p = f$. $S : H \to H$ is one-to-one, and so S^{-1} is well-defined on $S(H)$. Hence, for each $j \in \mathbb{N}$, the vector $h_j := S^{-1} g_j$ exists in H, so that for fixed $N \in \mathbb{N}$,

$$\left\| f - \sum_{j=1}^{N} < f,\, h_j > g_j \right\| = \left\| Sf_p - \sum_{j=1}^{N} < Sf_p,\, h_j > g_j \right\|$$
$$= \left\| Sf_p - \sum_{j=1}^{N} < f_p,\, S^* h_j > g_j \right\| = \left\| Sf_p - \sum_{j=1}^{N} < f_p,\, S h_j > g_j \right\|$$
$$= \left\| Sf_p - \sum_{j=1}^{N} < f_p,\, g_j > g_j \right\| \xrightarrow{N} 0.$$

□

Corollary 36. *For every $n \in \mathbb{N}$ and any f in the range of S, where $S(f) := T^*T(f) = \sum_{j\in\mathbb{N}} < f, g_j^{(n)} > g_j^{(n)}$, we have*

$$f = \sum_{j\in\mathbb{N}} < f, h_j^{(n)} > g_j^{(n)}$$

with $h_j^{(n)}$ given by Theorem 49 in section 2.3.

Proof. Fix $n \in \mathbb{N}_0$. We saw in Lemma 30 that $(g_j^{(n)})_{\mathbb{N}}$ is a Bessel sequence in H; verifying this Corollary will therefore depend on checking the hypotheses of Claim 35. We can use the biorthogonality developed in Corollary 51 to see that $S(h_j^{(n)}) = g_j^{(n)}$ for each j, so that each $g_j^{(n)}$ is in the range of S. To see that T is one to one, suppose $T(f) = T(g)$ for different vectors f and g in H. In this case, we must have $< f, g_j^{(n)} >=< g, g_j^{(n)} >$ for every j; this is equivalent to saying that for every j, $< f - g, g_j^{(n)} >= 0$. We wish to prove that $f - g$ must be the zero vector, and therefore $f = g$; this follows inductively from Lemma 29 and the fact that the sequence $(< f - g, e_j >)_{\mathbb{N}}$ must be in ℓ^2.

Finally, we need to show that the kernel of T^* is only the zero sequence in ℓ^2. Let $(x_j)_{\mathbb{N}}$ be in $\ker(T^*)$; this is the same as saying $\sum_{j\in\mathbb{N}} x_j g_j^{(n)} = 0$. However, if we define $x_j := 0$ for any $j \leq 0$, and make use of Lemma 29, we see that this is equivalent to saying

$$\sum_{j=1}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} x_{j-k} \right] e_j = 0.$$

It must be that the bracketed sequence is then the zero sequence; this means $x_1 = 0$, which implies $x_2 = 0$, and so on, to the desired result. □

2.2 FACTORIAL AND BINOMIAL PROPERTIES

Definition 37. *We denote the rising factorial, sometimes called the rising sequential product, by*

$$x^{<n>} := \prod_{r=0}^{n-1} (x + r) = \frac{(x + n - 1)!}{(x - 1)!},$$

and adopt the conventions $x^{<0>} := 1$ *as well as* $x^{<n>} := 0$ *whenever* $x \leq 0$ *or* $n < 0$.

This is also commonly denoted using the Pochhammer symbol $x^{(n)}$, or $(x)_n$, but unfortunately this may also represent the falling factorial, which replaces the plus sign in the product of our definition by a minus sign (especially in combinatorics). For these reasons, we'll stick with the notation in the definition above. We will require a few lemmas about the behavior of these rising factorials.

Lemma 38. *For any fixed positive integer k, we have for every whole number n*

$$\sum_{i=0}^{n} \frac{k^{<i>}}{i!} = \frac{(k+1)^{<n>}}{n!}.$$

Proof. If $n = 0$, we have

$$\frac{k^{<0>}}{0!} = 1 = \frac{(k+1)^{<0>}}{0!}.$$

If $n = 1$, we have

$$\frac{k^{<0>}}{0!} + \frac{k^{<1>}}{1!} = 1 + k = \frac{(k+1)^{<1>}}{1!}.$$

If $n = 2$, we have

$$\begin{aligned}\frac{k^{<0>}}{0!} + \frac{k^{<1>}}{1!} + \frac{k^{<2>}}{2!} &= 1 + k + \frac{k(k+1)}{2} = (k+1)\left(1 + \frac{k}{2}\right) \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)^{<2>}}{2!}.\end{aligned}$$

Now, assume that the lemma is true for every whole number n less than or equal to some positive integer $m-1$, with $m \geq 3$. Then, we have

$$\begin{aligned}\sum_{i=0}^{m} \frac{k^{<i>}}{i!} &= \sum_{i=0}^{m-1} \frac{k^{<i>}}{i!} + \frac{k^{<m>}}{m!} = \frac{(k+1)^{<m-1>}}{(m-1)!} + \frac{k^{<m>}}{m!} \\ &= \frac{m \cdot (k+1)^{<m-1>} + k^{<m>}}{m!} = \frac{m \cdot (k+1)^{<m-1>} + k \cdot (k+1)^{<m-1>}}{m!} \\ &= \frac{(m+k) \cdot (k+1)^{<m-1>}}{m!} = \frac{(k+1)^{<m>}}{m!}.\end{aligned}$$

□

Lemma 39. *For a fixed positive integer k, we have for every positive integer n*

$$\sum_{j=1}^{n} j^{<k-1>} = \frac{n^{<k>}}{k}.$$

Proof. Fix any positive integer k. If $n = 1$, we have

$$1^{<k-1>} = (k-1)! = \frac{k!}{k} = \frac{1^{<k>}}{k}.$$

If $n = 2$, we have

$$\begin{aligned} 1^{<k-1>} + 2^{<k-1>} &= \frac{1^{<k>}}{k} + 2^{<k-1>} = 2^{<k-1>}\left(\frac{1}{k}+1\right) \\ &= \frac{(k+1)!}{k} = \frac{2^{<k>}}{k}. \end{aligned}$$

If $n = 3$, we have

$$\begin{aligned} \sum_{j=1}^{3} j^{<k-1>} &= \sum_{j=1}^{2} j^{<k-1>} + 3^{<k-1>} = \frac{2^{<k>}}{k} + 3^{<k-1>} \\ &= \frac{2\cdot 3\cdots(k+1)}{k} + 3\cdot 4\cdots(k+1) = 3\cdot 4\cdots(k+1)\left(\frac{2}{k}+1\right) \\ &= \frac{3\cdot 4\cdots(k+1)\,(k+2)}{k} = \frac{3^{<k>}}{k}. \end{aligned}$$

Now, assume that the lemma is true for every whole number n less than or equal to some positive integer $m-1$, with $m \geq 4$. Then, we have

$$\begin{aligned} \sum_{j=1}^{m} j^{<k-1>} &= \sum_{j=1}^{m-1} j^{<k-1>} + m^{<k-1>} = \frac{(m-1)^{<k>}}{k} + m^{<k-1>} \\ &= \frac{(m-1)\cdot m\cdots(m+k-2)}{k} + m\cdot(m+1)\cdots(m+k-2) \\ &= m\cdot(m+1)\cdots(m+k-2)\left(\frac{m-1}{k}+1\right) \\ &= \frac{m\cdot(m+1)\cdots(m+k-2)\,(m+k-1)}{k} = \frac{m^{<k>}}{k}. \end{aligned}$$

□

Lemma 40. *For any positive integer k and any positive integer $n \geq 2$,*
$n^{<k>} - (n-1)^{<k>} = k \cdot n^{<k-1>}$.

Proof.

$$
\begin{aligned}
&n^{<k>} - (n-1)^{<k>} \\
&= \frac{(n+k-1)!}{(n-1)!} - \frac{(n-1+k-1)!}{(n-1-1)!} = \frac{(n+k-1)!}{(n-1)(n-2)!} - \frac{(n+k-2)!}{(n-2)!} \\
&= \frac{(n+k-1)! - (n-1)(n+k-2)!}{(n-1)(n-2)!} = \frac{(n+k-2)!\,[(n+k-1)-(n-1)]}{(n-1)!} \\
&= \frac{(n+k-2)! \cdot k}{(n-1)!} = k \cdot n^{<k-1>}.
\end{aligned}
$$

□

Corollary 41. *For any positive integers k and n,*

$$n^{<k>} + n^{<k-1>} = (n+1)^{<k>} \cdot \left(\frac{n}{n+k-1}\right).$$

Proof. If $n = k = 1$, we have $1^{<1>} + 1^{<0>} = 1 + 1 = 2^{<1>}$. For all other cases, we refer to Lemma 40, which tells us that $n^{<k>} - k \cdot n^{<k-1>} = (n-1)^{<k>}$, or

$$n^{<k>} + n^{<k-1>} - n^{<k-1>} - k \cdot n^{<k-1>} = (n-1)^{<k>}.$$

Rewriting this will isolate the left hand side of what we would like to show, and we proceed as follows:

$$
\begin{aligned}
&n^{<k>} + n^{<k-1>} = (n-1)^{<k>} + (k+1)n^{<k-1>} \\
&= (n-1)(n)(n+1)\cdots(n-1+k-1) + (k+1)(n)(n+1)\cdots(n+k-1-1) \\
&= n(n+1)\cdots(n+k-2)(n+k) \\
&= (n+1)^{<k>} \cdot \left(\frac{n}{n+k-1}\right).
\end{aligned}
$$

□

Before stating our next result for the rising factorials, we need a lemma pertaining to the alternating sum of the tail of a row of binomial coefficients in Pascal's triangle.

Lemma 42. *Fix a positive integer n. For any whole number j less than n,*

$$\binom{n-1}{j} = \sum_{k=j+1}^{n} (-1)^{j+k+1} \binom{n}{k}. \tag{2.4}$$

Proof. The lemma is easy to check for $n = 1, 2, 3$. For example, if $n = 3$, we have

$$\begin{aligned}
j = 2: \qquad & \binom{2}{2} = \binom{3}{3}; \\
j = 1: \qquad & \binom{2}{1} = \binom{3}{2} - \binom{2}{2} \\
& \qquad = \binom{3}{2} - \binom{3}{3}; \\
j = 0: \qquad & \binom{2}{0} = \binom{3}{1} - \binom{2}{1} \\
& \qquad = \binom{3}{1} - \binom{3}{2} + \binom{3}{3}.
\end{aligned}$$

In general, we have the following:

$$\begin{aligned}
j = n-1: \qquad & \binom{n-1}{n-1} = \binom{n}{n}; \\
j = n-2: \qquad & \binom{n-1}{n-2} = \binom{n}{n-1} - \binom{n-1}{n-1} \\
& \qquad = \binom{n}{n-1} - \binom{n}{n}; \\
j = n-3: \qquad & \binom{n-1}{n-3} = \binom{n}{n-2} - \binom{n-1}{n-2} \\
& \qquad = \binom{n}{n-2} - \binom{n}{n-1} + \binom{n}{n}.
\end{aligned}$$

Now, for any integer i with $4 \leq i < n$, assume that

$$\binom{n-1}{n-i} = \sum_{k=n-i+1}^{n} (-1)^{n-i+k+1} \binom{n}{k}.$$

Then, we can write

$$\begin{aligned}
\binom{n-1}{n-i-1} &= \binom{n}{n-i} - \binom{n-1}{n-i} \\
&= \binom{n}{n-i} - \sum_{k=n-i+1}^{n} (-1)^{n-i+k+1} \binom{n}{k} \\
&= (-1)^{n-i+n-i} \binom{n}{n-i} + \sum_{k=n-i+1}^{n} (-1)^{n-i+k} \binom{n}{k} \\
&= \sum_{k=n-i}^{n} (-1)^{n-i+k} \binom{n}{k}.
\end{aligned}$$

□

Lemma 43. *For any natural numbers k and n with $n > 1$,*

$$\sum_{i=1}^{n}(-1)^{1+i}\binom{n}{i}(k-i)^{<n-1>} = \binom{n-1}{0}(k-1)^{<n-1>} + (n-1)\sum_{i=1}^{n-1}(-1)^{1+i}\binom{n-1}{i}(k-i)^{<n-2>}.$$

Proof.

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^{1+i}\binom{n}{i}(k-i)^{<n-1>} \\
&= \sum_{i=1}^{n}(-1)^{1+i}\left[\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j} + \sum_{j=i+1}^{n}(-1)^{i+j+1}\binom{n}{j}\right](k-i)^{<n-1>} \\
&= \sum_{i=1}^{n}(-1)^{1+i}\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j}(k-i)^{<n-1>} \\
&\qquad + \sum_{\ell=2}^{n}(-1)^{\ell}\sum_{j=\ell}^{n}(-1)^{\ell+j}\binom{n}{j}(k-\ell+1)^{<n-1>} \\
&= \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}(k-1)^{<n-1>} \\
&\qquad + \sum_{i=2}^{n}(-1)^{i}\sum_{j=i}^{n}\binom{n}{j}\left[(-1)^{i+j}(k-i+1)^{<n-1>} + (-1)^{i+j+1}(k-i)^{<n-1>}\right] \\
&= \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}(k-1)^{<n-1>} \\
&\qquad + \sum_{i=2}^{n}(-1)^{i}\left[\sum_{j=i}^{n}\binom{n}{j}(-1)^{i+j}\right]\left[(k-i+1)^{<n-1>} - (k-i)^{<n-1>}\right]
\end{aligned}$$

We can now use Lemma 42 twice, as well as Lemma 40, to see that we can rewrite the previous expression as

$$\binom{n-1}{0}(k-1)^{<n-1>} + \sum_{i=2}^{n}(-1)^{i}\binom{n-1}{i-1}(n-1)(k-i+1)^{<n-2>}.$$

Finally, reindexing the sum using the substitution $\ell = i - 1$ gives us the desired expression

$$\binom{n-1}{0}(k-1)^{<n-1>} + (n-1)\sum_{\ell=1}^{n-1}(-1)^{1+\ell}\binom{n-1}{\ell}(k-\ell)^{<n-2>}.$$

□

Repeated applications of Lemma 43 lead us to the following result.

Corollary 44. *For any natural numbers k and n,*

$$\sum_{i=1}^{n}(-1)^{1+i}\binom{n}{i}(k-i)^{<n-1>} = k^{<n-1>}.$$

Proof.

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^{1+i}\binom{n}{i}(k-i)^{<n-1>} \\
&= \binom{n-1}{0}(k-1)^{<n-1>} + (n-1)\sum_{i=1}^{n-1}(-1)^{1+i}\binom{n-1}{i}(k-i)^{<n-2>} \\
&= \binom{n-1}{0}(k-1)^{<n-1>} \\
&\quad + (n-1)\left[\binom{n-2}{0}(k-1)^{<n-2>} + (n-2)\sum_{i=1}^{n-2}(-1)^{1+i}\binom{n-2}{i}(k-i)^{<n-3>}\right] \\
&= \frac{(n-1)!}{(n-1)!}\binom{n-1}{0}(k-1)^{<n-1>} + \frac{(n-1)!}{(n-2)!}\binom{n-2}{0}(k-1)^{<n-2>} \\
&\quad + \frac{(n-1)!}{(n-3)!}\sum_{i=1}^{n-2}(-1)^{1+i}\binom{n-2}{i}(k-i)^{<n-3>} \\
&= \frac{(n-1)!}{(n-1)!}\binom{n-1}{0}(k-1)^{<n-1>} + \frac{(n-1)!}{(n-2)!}\binom{n-2}{0}(k-1)^{<n-2>} \\
&\quad + \frac{(n-1)!}{(n-3)!}\left[\binom{n-3}{0}(k-1)^{<n-3>} + (n-3)\sum_{i=1}^{n-3}(-1)^{1+i}\binom{n-3}{i}(k-i)^{<n-4>}\right].
\end{aligned}$$

We continue to apply Lemma 43 ($n-4$ more times, in fact), and finally can write the initial expression as

$$\sum_{i=1}^{n}\frac{(n-1)!}{(n-i)!}\binom{n-i}{0}(k-1)^{<n-i>} = (n-1)!\sum_{j=0}^{n-1}\frac{(k-1)^{<j>}}{j!}.$$

An application of Lemma 38 then allows us to write this last expression as

$$(n-1)!\left[\frac{k^{<n-1>}}{(n-1)!}\right] = k^{<n-1>}.$$

□

It is interesting that the previous Lemma and its Corollary do not depend on the summation upper limit, as the following Lemma and Corollary demonstrate, with a specific choice for k.

Lemma 45. *Fix a positive integer n. For any positive integer M with $M < n$,*

$$\begin{aligned}
&\sum_{i=1}^{M}(-1)^{1+i}\binom{n}{i}(M+1-i)^{<n-1>} \\
&\qquad = \binom{n-1}{0}M^{<n-1>} + (n-1)\sum_{i=1}^{M}(-1)^{1+i}\binom{n-1}{i}(M+1-i)^{<n-2>}.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\sum_{i=1}^{M}(-1)^{1+i}\binom{n}{i}(M+1-i)^{<n-1>} \\
&= \sum_{i=1}^{M}(-1)^{1+i}\left[\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j} + \sum_{j=i+1}^{n}(-1)^{i+j+1}\binom{n}{j}\right](M+1-i)^{<n-1>} \\
&= \sum_{i=1}^{M}(-1)^{1+i}\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j}(M+1-i)^{<n-1>} \\
&\qquad + \sum_{\ell=2}^{M+1}(-1)^{\ell}\sum_{j=\ell}^{n}(-1)^{\ell+j}\binom{n}{j}(M+1-\ell+1)^{<n-1>}.
\end{aligned}$$

Replacing ℓ with i, and separating the final term in the sum, we see that the previous expression is equal to

$$\begin{aligned}
&\sum_{i=1}^{M}(-1)^{1+i}\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j}(M+1-i)^{<n-1>} \\
&\qquad + \sum_{i=2}^{M}(-1)^{i}\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j}(M+2-i)^{<n-1>} \\
&\qquad + (-1)^{M+1}\sum_{j=M+1}^{n}(-1)^{j+M+1}\binom{n}{j}(M+2-M-1)^{<n-1>}.
\end{aligned}$$

We now combine the common terms in the first two sums to get

$$\sum_{j=1}^{n}(-1)^{1+j}\binom{n}{j}M^{<n-1>}$$
$$+\sum_{i=2}^{M}(-1)^{i}\sum_{j=i}^{n}(-1)^{i+j}\binom{n}{j}\left[(M+2-i)^{<n-1>}-(M+1-i)^{<n-1>}\right]$$
$$+(-1)^{M+1}\sum_{j=M+1}^{n}(-1)^{j+M+1}\binom{n}{j}1^{<n-1>}.$$

Taking advantage of Lemmas 40 and 42 for each part of this expression, we can rewrite it as

$$\binom{n-1}{0}M^{<n-1>}+(n-1)\sum_{i=2}^{M}(-1)^{i}\binom{n-1}{i-1}(M+2-i)^{<n-2>}+(-1)^{M+1}\binom{n-1}{M}1^{<n-1>};$$

then, realizing that $1^{<n-1>}=(n-1)!=(n-1)(n-2)!=(n-1)\cdot 1^{<n-2>}$, and reindexing the sum, we arrive at

$$\sum_{i=1}^{M}(-1)^{1+i}\binom{n}{i}(M+1-i)^{<n-1>}$$
$$=\binom{n-1}{0}M^{<n-1>}+(n-1)\sum_{i=1}^{M}(-1)^{1+i}\binom{n-1}{i}(M+1-i)^{<n-2>}.$$

□

Applying Lemma 45 repeatedly achieves the following result.

Corollary 46. *Fix a positive integer n. For any positive integer M with $M<n$,*

$$\sum_{i=1}^{M}(-1)^{1+i}\binom{n}{i}(M+1-i)^{<n-1>}=(M+1)^{<n-1>}.$$

Proof.

$$\begin{aligned}
&\sum_{i=1}^{M}(-1)^{1+i}\binom{n}{i}(M+1-i)^{<n-1>}\\
&=\binom{n-1}{0}M^{<n-1>}+(n-1)\sum_{i=1}^{M}(-1)^{1+i}\binom{n-1}{i}(M+1-i)^{<n-2>}\\
&=\binom{n-1}{0}M^{<n-1>}\\
&\quad+(n-1)\left[\binom{n-2}{0}M^{<n-2>}+(n-2)\sum_{i=1}^{M}(-1)^{1+i}\binom{n-2}{i}(M+1-i)^{<n-3>}\right]\\
&=\frac{(n-1)!}{(n-1)!}\binom{n-1}{0}M^{<n-1>}+\frac{(n-1)!}{(n-2)!}\binom{n-2}{0}M^{<n-2>}\\
&\quad+\frac{(n-1)!}{(n-3)!}\sum_{i=1}^{M}(-1)^{1+i}\binom{n-2}{i}(M+1-i)^{<n-3>}\\
&=\frac{(n-1)!}{(n-1)!}\binom{n-1}{0}M^{<n-1>}+\frac{(n-1)!}{(n-2)!}\binom{n-2}{0}M^{<n-2>}\\
&\quad+\frac{(n-1)!}{(n-3)!}\left[\binom{n-3}{0}M^{<n-3>}+(n-3)\sum_{i=1}^{M}(-1)^{1+i}\binom{n-3}{i}(M+1-i)^{<n-4>}\right].
\end{aligned}$$

We continue to apply Lemma 45 ($n-M$ total times), and finally can write the initial expression as

$$\sum_{i=1}^{n-M}\frac{(n-1)!}{(n-i)!}\binom{n-i}{0}M^{<n-i>}+\frac{(n-1)!}{(M-1)!}\sum_{i=1}^{M}(-1)^{1+i}\binom{M}{i}(M+1-i)^{<M-1>}.$$

In the second sum, notice that the upper summation limit is now the same as the integer in the upper position of the binomial coefficient; this allows us to apply Corollary 44, and we see that the initial expression in this proof is the same as

$$\begin{aligned}
&(n-1)!\sum_{i=1}^{n-M}\frac{M^{<n-i>}}{(n-i)!}+\frac{(n-1)!}{(M-1)!}(M+1)^{<M-1>}\\
&=(n-1)!\sum_{j=M}^{n-1}\frac{M^{<j>}}{j!}+\frac{(n-1)!}{(M-1)!}(M+1)^{<M-1>}\\
&=(n-1)!\sum_{j=0}^{n-1}\frac{M^{<j>}}{j!}-(n-1)!\sum_{j=0}^{M-1}\frac{M^{<j>}}{j!}+\frac{(n-1)!}{(M-1)!}(M+1)^{<M-1>}.
\end{aligned}$$

Finally, we make use of Lemma 38 for each sum in this expression to arrive at the desired result:

$$\begin{aligned}
&\sum_{i=1}^{M}(-1)^{1+i}\binom{n}{i}(M+1-i)^{<n-1>} \\
&= (n-1)!\left[\frac{(M+1)^{<n-1>}}{(n-1)!} - \frac{(M+1)^{<M-1>}}{(M-1)!} + \frac{(M+1)^{<M-1>}}{(M-1)!}\right] \\
&= (M+1)^{<n-1>}.
\end{aligned}$$

□

The following result allows us to find a simple estimate for the Bessel constant of each Bessel sequence $(g_j^{(n)})_{j\in\mathbb{N}}$.

Lemma 47. *For large $n \in \mathbb{N}$,*

$$\left(\sum_{k=0}^{n}\binom{n}{k}^2\right)^{1/2} \sim \frac{2^n}{\sqrt[4]{\pi n}}.$$

Proof. Fix $n \in \mathbb{N}$. We have

$$q_n := \left(\sum_{k=0}^{n}\binom{n}{k}^2\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n}\binom{n}{k}e^{ik\theta}\right|^2 d\theta\right)^{1/2},$$

which is the same as saying

$$\begin{aligned}
2\pi q_n^2 &= \int_0^{2\pi}\left|\left(1+e^{i\theta}\right)^n\right|^2 d\theta = \int_0^{2\pi}\left(\left|1+e^{i\theta}\right|^2\right)^n d\theta \\
&= \int_0^{2\pi}\left[(1+\cos(\theta))^2+\sin^2\theta\right]^n d\theta \\
&= \int_0^{2\pi}\left[4\cos^2(\theta/2)\right]^n d\theta = 2\int_{u=0}^{\pi}4^n(\cos(u))^{2n}\,du.
\end{aligned}$$

Let's define

$$\frac{\pi q_n^2}{4^n} = \int_{u=0}^{\pi}(\cos(u))^{2n}\,du := I_n\,,$$

and see what we can find out about I_n. If $n = 1$, we have

$$I_1 = \int_{u=0}^{\pi}(\cos(u))^2\,du = \frac{\pi}{2};$$

If $n = 2$, we can use integration by parts to see

$$
\begin{aligned}
I_2 &= \int_{u=0}^{\pi} (\cos(u))^4 \, du = \int_{u=0}^{\pi} (\cos(u))^3 \cos(u) \, du \\
&= 3 \int_0^{\pi} \cos^2(u) \, du - 3 \int_0^{\pi} \cos^4(u) \, du.
\end{aligned}
$$

This is the same as saying $4I_2 = 3I_1$, so it must be that $I_2 = \pi \cdot (3 \cdot 1)/(4 \cdot 2)$. Assume now that we have $n \geq 2$, so that

$$
I_n = \int_0^{\pi} \cos^{2n-1}(u) \cos(u) \, du.
$$

We can use integration by parts again to see that

$$
I_n = (2n-1) \int_0^{\pi} \cos^{2n-2}(u)(1 - \cos^2(u)) \, du \,,
$$

which means that $I_n = (2n-1)(I_{n-1} - I_n)$, or $I_n = (2n-1)/(2n) I_{n-1}$. Using this recursively, we'll get

$$
I_3 = \frac{5 \cdot 3 \cdot 1 \cdot \pi}{6 \cdot 4 \cdot 2}\,, \qquad I_4 = \frac{7 \cdot 5 \cdot 3 \cdot 1 \cdot \pi}{8 \cdot 6 \cdot 4 \cdot 2}, \qquad \ldots
$$

and finally we get

$$
\begin{aligned}
\frac{\pi q_n^2}{4^n} &= \frac{(2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1 \cdot \pi}{(2n)(2n-2)\cdots 6 \cdot 4 \cdot 2} \\
&= \frac{(2n)(2n-1)(2n-2)(2n-3)\cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \pi}{2^n \cdot n! \cdot 2^n \cdot n!}.
\end{aligned}
$$

We can then use Stirling's formula to arrive at

$$
q_n^2 = \frac{(2n)!}{(n!)^2} = \binom{2n}{n} \sim \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi(2n)}}{\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right]^2} = \frac{4^n}{\sqrt{\pi n}}.
$$

□

2.3 BIORTHOGONAL SEQUENCES

For each sequence $(g_j^{(n)})_{\mathbb{N}}$, one could ask if there exists a biorthogonal sequence $(h_j^{(n)})_{\mathbb{N}}$ satisfying

$$f = \sum_{j\in\mathbb{N}} < f, g_j^{(n)} > h_j^{(n)}$$

for every vector f in H. This turns out to be impossible for $n \geq 1$. On the other hand, the equations

$$e_k = \sum_{j\in\mathbb{N}} < e_k, g_j^{(n)} > h_j^{(n)} \tag{2.5}$$

turn out to be true for every $k \in \mathbb{N}$, with $(h_j^{(n)})_{j\in\mathbb{N}}$ specified in Theorem 49. Therefore, we will derive our formula for each sequence $(h_j^{(n)})_{\mathbb{N}}$ assuming (2.5). Note that the converse is also true; i.e., the sequences $(h_j^{(n)})_{\mathbb{N}}$ we develop satisfy (2.5).

Lemma 48. *For any fixed $n \in \mathbb{N}_0$ and for any $k \in \mathbb{N}$, define $M(n,k) := min\{k-1, n\}$; then, assuming* (2.5),

$$e_k = \sum_{j=0}^{M(n,k)} \binom{n}{j} h_{k-j}^{(n)}. \tag{2.6}$$

Proof. For illustrative purposes, consider the case $n = 1$. We have from equation (2.5)

$$\begin{aligned}
e_1 &= \sum_{j=1}^{\infty} < e_1, g_j^{(1)} > h_j^{(1)} = \sum_{j=1}^{\infty} < e_1, e_j + e_{j+1} > h_j^{(1)} \\
&=< e_1, e_1 + e_2 > h_1^{(1)} = h_1^{(1)}; \\
e_2 &= \sum_{j=1}^{\infty} < e_2, g_j^{(1)} > h_j^{(1)} =< e_2, e_1 + e_2 > h_1^{(1)} + < e_2, e_2 + e_3 > h_2^{(1)} \\
&= h_1^{(1)} + h_2^{(1)};
\end{aligned}$$

$$\begin{aligned}
e_3 &= \sum_{j=1}^{\infty} < e_3, g_j^{(1)} > h_j^{(1)} =< e_3, e_2 + e_3 > h_2^{(1)} + < e_3, e_3 + e_4 > h_3^{(1)} \\
&= h_2^{(1)} + h_3^{(1)}; \\
&\vdots
\end{aligned}$$

$$e_k = \sum_{j=1}^{\infty} < e_k, g_j^{(1)} > h_j^{(1)} = \sum_{j=1}^{\infty} < e_k, e_j > h_j^{(1)} + \sum_{j=1}^{\infty} < e_k, e_{j+1} > h_j^{(1)}$$
$$= h_k^{(1)} + h_{k-1}^{(1)}.$$

In general, fix natural numbers k and n.

$$e_k = \sum_{j=1}^{\infty} < e_k, g_j^{(n)} > h_j^{(n)} = \sum_{j=1}^{\infty} \left\langle e_k, \sum_{i=0}^{n} \binom{n}{i} e_{j+i} \right\rangle h_j^{(n)}$$
$$= \sum_{j=1}^{\infty} \sum_{i=0}^{n} \binom{n}{i} \delta_{k,j+i} h_j^{(n)},$$

where $\delta_{k,j+i} = 1$ when $k = j + i$ and zero otherwise. So, setting $i = k - j$, we see that the resulting sum is only defined if $k - n \leq j \leq k$. Consider first the case $k > n$, where we have

$$e_k = \sum_{j=k-n}^{k} \binom{n}{k-j} h_j^{(n)} = \sum_{\ell=0}^{n} \binom{n}{\ell} h_{k-\ell}^{(n)}$$

as desired. On the other hand, if $k \leq n$,

$$e_k = \sum_{j=1}^{k} \binom{n}{k-j} h_j^{(n)} = \sum_{\ell=0}^{k-1} \binom{n}{\ell} h_{k-\ell}^{(n)}.$$

□

Notice that rewriting the conclusion of the previous Lemma gives us

$$h_k^{(n)} = e_k - \sum_{j=1}^{M(n,k)} \binom{n}{j} h_{k-j}^{(n)} \tag{2.7}$$

for any pair of natural numbers n and k with $k > 1$; if $k = 1$, we have $h_1^{(n)} = e_1$ for every $n \in \mathbb{N}_0$.

Theorem 49. *For each whole number n,*

$$\left(h_j^{(n)}\right)_{j\in\mathbb{N}} := \left(\frac{1}{(n-1)!} \sum_{k=1}^{j} (-1)^{j+k} (j-k+1)^{<n-1>} e_k\right)_{j\in\mathbb{N}}. \tag{2.8}$$

is the sequence satisfying equation (2.5).

Proof. The proof relies on equation (2.7) and the related comments following Lemma 48. If $n = 0$, $h_k^{(0)} = e_k$ for all $k \in \mathbb{N}$. If $n = 1$, it is easy to check that $h_2^{(1)} = e_2 - e_1$. If we assume that the Theorem is true for all j with $1 \le j \le L$, for some natural number $L > 2$, then equation (2.7) gives us

$$\begin{aligned} h_{L+1}^{(1)} &= e_{L+1} - h_L^{(1)} = e_{L+1} - \sum_{k=1}^{L} (-1)^{L+k} e_k \\ &= e_{L+1} + \sum_{k=1}^{L} (-1)^{(L+1)+k} e_k = \sum_{k=1}^{L+1} (-1)^{(L+1)+k} e_k. \end{aligned}$$

If $n = 2$, we can recursively use Lemma 48 and equation (2.7) to see $h_2^{(2)} = e_2 - 2e_1$ and $h_3^{(2)} = e_3 - 2e_2 + 3e_1$. If we assume that the Theorem is true for all j with $1 \le j \le L$, for some natural number $L > 2$, then equation (2.7) gives us

$$h_{L+1}^{(2)} = e_{L+1} - \sum_{k=1}^{2} \binom{2}{k} h_{L-k+1}^{(2)}.$$

Notice that the bounds on k give us $L - 1 \le L - k + 1 \le L$, so we may apply the induction hypothesis and see that

$$\begin{aligned} h_{L+1}^{(2)} &= e_{L+1} - \sum_{k=1}^{2} \binom{2}{k} \sum_{i=1}^{L-k+1} (-1)^{L-k+1+i} (L - k + 1 - i + 1) e_i \\ &= e_{L+1} - 2 \sum_{i=1}^{L} (-1)^{L+i} (L - i + 1) e_i - \sum_{i=1}^{L-1} (-1)^{L-1+i} (L - i) e_i \\ &= e_{L+1} - 2e_L - \sum_{i=1}^{L-1} (-1)^{L+i} \left[\sum_{\ell=1}^{2} (-1)^{1+\ell} \binom{2}{\ell} (L - i + 2 - \ell) \right] e_i. \end{aligned}$$

Corollary 44 allows us to then write

$$\begin{aligned} h_{L+1}^{(2)} &= e_{L+1} - 2e_L - \sum_{i=1}^{L-1} (-1)^{L+i} \left[(L - i + 2) \right] e_i \\ &= \sum_{i=1}^{L+1} (-1)^{L+1+i} (L + 1 - i + 1) e_i. \end{aligned}$$

If $n = 3$, we can recursively use Lemma 48 and equation (2.7) to see $h_2^{(3)} = e_2 - 3e_1$ and $h_3^{(3)} = e_3 - 3e_2 + 6e_1$. If we assume that the Theorem is true for all j with $1 \leq j \leq L$, for some natural number $L > 3$, then equation (2.7) gives us

$$h_{L+1}^{(3)} = e_{L+1} - \sum_{k=1}^{3} \binom{3}{k} h_{L-k+1}^{(3)}.$$

Notice that the bounds on k give us $L - 2 \leq L - k + 1 \leq L$, so we may apply the induction hypothesis and see that

$$\begin{aligned} h_{L+1}^{(3)} &= e_{L+1} - \sum_{k=1}^{3} \binom{3}{k} \left[\frac{1}{2!} \sum_{i=1}^{L-k+1} (-1)^{L-k+1+i} (L-k+2-i)^{<2>} e_i \right] \qquad (2.9) \\ &= e_{L+1} - \frac{1}{2!} \left[\sum_{i=1}^{L} (-1)^{L+i} \binom{3}{1} (L-i+1)^{<2>} e_i \right. \\ &\quad \left. + \sum_{i=1}^{L-1} (-1)^{L-1+i} \binom{3}{2} (L-i)^{<2>} e_i + \sum_{i=1}^{L-2} (-1)^{L+i-2} (L-i-1)^{<2>} e_i \right] \\ &= e_{L+1} - \frac{1}{2!} \left[\sum_{i=1}^{1} (-1)^{1+i} \binom{3}{i} (2-i)^{<2>} \right] e_L + \frac{1}{2!} \left[\sum_{i=1}^{2} (-1)^{1+i} \binom{3}{i} (3-i)^{<2>} \right] e_{L-1} \\ &\quad - \frac{1}{2!} \sum_{i=1}^{L-2} (-1)^{L+i} \left[\sum_{\ell=1}^{3} (-1)^{1+\ell} \binom{3}{\ell} (L-i+2-\ell)^{<2>} \right] e_i \end{aligned}$$

We can now use Corollary 44 on the final bracketed term, but we must use Corollary 46 to rewrite the first two bracketed terms:

$$\begin{aligned} h_{L+1}^{(3)} &= \frac{1}{2!} \left[1^{<2>} e_{L+1} - 2^{<2>} e_L + 3^{<2>} e_{L-1} + \sum_{i=1}^{L-2} (-1)^{L+1+i} (L-i+2)^{<2>} e_i \right] \\ &= \frac{1}{2!} \sum_{i=1}^{L+1} (-1)^{L+1+i} (L+1-i+1)^{<2>} e_i. \end{aligned}$$

However, it will be very useful to note an alternative method to that outlined in equation (2.9). We have

$$\begin{aligned} h_{L+1}^{<3>} &= e_{L+1} - \frac{1}{2!} \sum_{k=1}^{3} \binom{3}{k} \sum_{i=1}^{L-k+1} (-1)^{L-k+1+i} (L-k+2-i)^{<2>} e_i \\ &= e_{L+1} - \frac{1}{2!} \sum_{k=1}^{3} \binom{3}{k} \sum_{i=1}^{L-2} (-1)^{L-k+1+i} (L-k+2-i)^{<2>} e_i \\ &\quad - \frac{1}{2!} \sum_{k=1}^{2} \binom{3}{k} \sum_{i=L-1}^{L-k+1} (-1)^{L-k+1+i} (L-k+2-i)^{<2>} e_i. \end{aligned}$$

In the first double sum, we can exchange the summations; in the second double sum, we can reverse the order of summation to allow an exchange. If we further note that $(-1)^{A-k} = (-1)^{A+k}$ for any integer A, we have

$$h_{L+1}^{<3>} = e_{L+1} - \frac{1}{2!}\sum_{i=1}^{L-2}(-1)^{L+i}\left[\sum_{k=1}^{3}(-1)^{1+k}\binom{3}{k}(L-i+2-k)^{<2>}\right]e_i$$
$$-\frac{1}{2!}\sum_{i=L-1}^{L}(-1)^{L+i}\left[\sum_{k=1}^{L-i+1}(-1)^{1+k}\binom{3}{k}(L-i+2-k)^{<2>}\right]e_i,$$

which sets up nicely for the argument following equation (2.9). Now that we see the pattern, let's examine the general case. Fix a natural number n, and assume that the Theorem is true for all j with $1 \leq j \leq L$, for some natural number $L > n$. We again take advantage of equation (2.7), which gives us

$$h_{L+1}^{(n)} = e_{L+1} - \sum_{k=1}^{n}\binom{n}{k}h_{L-k+1}^{(n)}.$$

Notice that the bounds on k give us $L-n+1 \leq L-k+1 \leq L$, so we may apply the induction hypothesis and see that

$$\begin{aligned}
h_{L+1}^{<n>} &= e_{L+1} - \frac{1}{(n-1)!}\sum_{k=1}^{n}\binom{n}{k}\sum_{i=1}^{L-k+1}(-1)^{L-k+1+i}(L-k+2-i)^{<n-1>}e_i \\
&= e_{L+1} - \frac{1}{(n-1)!}\sum_{k=1}^{n}\binom{n}{k}\sum_{i=1}^{L-n+1}(-1)^{L-k+1+i}(L-k+2-i)^{<n-1>}e_i \\
&\quad - \frac{1}{(n-1)!}\sum_{k=1}^{n-1}\binom{n}{k}\sum_{i=L-n+2}^{L-k+1}(-1)^{L-k+1+i}(L-k+2-i)^{<n-1>}e_i \\
&= e_{L+1} - \frac{1}{(n-1)!}\sum_{i=1}^{L-n+1}(-1)^{L+i}\left[\sum_{k=1}^{n}(-1)^{1+k}\binom{n}{k}(L-i+2-k)^{<n-1>}\right]e_i \\
&\quad - \frac{1}{(n-1)!}\sum_{i=L-n+2}^{L}(-1)^{L+i}\left[\sum_{k=1}^{L-i+1}(-1)^{1+k}\binom{n}{k}(L-i+2-k)^{<n-1>}\right]e_i.
\end{aligned}$$

We can now apply Corollary 44 to the first sum and Corollary 46 to the second sum, and we have

$$\begin{aligned}
h_{L+1}^{<n>} &= e_{L+1} - \frac{1}{(n-1)!} \sum_{i=1}^{L-n+1} (-1)^{L+i} (L-i+2)^{<n-1>} e_i \\
&\quad - \frac{1}{(n-1)!} \sum_{i=L-n+2}^{L} (-1)^{L+i} (L-i+2)^{<n-1>} e_i \\
&= e_{L+1} + \frac{1}{(n-1)!} \sum_{i=1}^{L} (-1)^{L+1+i} (L+1-i+1)^{<n-1>} e_i \\
&= \frac{1}{(n-1)!} \sum_{i=1}^{L+1} (-1)^{L+1+i} (L+1-i+1)^{<n-1>} e_i .
\end{aligned}$$

In the case where $L \leq n$, $\min\{n, L+1-1\} = L$, and a similar argument works, starting from the equation

$$h_{L+1}^{(n)} = e_{L+1} - \sum_{k=1}^{L} \binom{n}{k} h_{L-k+1}^{(n)} .$$

□

Now that we are sure of the formula for the vector sequence $(h_j^{(n)})_{\mathbb{N}}$ for each $n \in \mathbb{N}_0$,we notice the following two properties.

Lemma 50. *Fix $n \in \mathbb{N}$. For any $j \in \mathbb{N}$, we have $h_{j-1}^{(n)} + h_j^{(n)} = h_j^{(n-1)}$.*

Proof. The result follows easily from Theorem 49, if we apply Lemma 40.

$$\begin{aligned}
&h_{j-1}^{(n)} + h_j^{(n)} \\
&= \frac{1}{(n-1)!} \sum_{k=1}^{j-1} (-1)^{j-1+k} \left[(j-k)^{<n-1>} - (j-k+1)^{<n-1>} \right] e_k + \frac{1^{<n-1>}}{(n-1)!} e_j \\
&= \frac{1}{(n-1)!} \sum_{k=1}^{j-1} (-1)^{j+k} \left[(j-k+1)^{<n-1>} - (j-k)^{<n-1>} \right] e_k + \frac{1^{<n-1>}}{(n-1)!} e_j \\
&= \frac{1}{(n-1)!} \sum_{k=1}^{j-1} (-1)^{j+k} \left[(n-1)(j-k+1)^{<n-2>} \right] e_k + e_j \\
&= \frac{1}{(n-2)!} \sum_{k=1}^{j-1} (-1)^{j+k} (j-k+1)^{<n-2>} e_k + e_j \\
&= \frac{1}{(n-2)!} \sum_{k=1}^{j} (-1)^{j+k} (j-k+1)^{<n-2>} e_k .
\end{aligned}$$

□

Note that, for all natural numbers j and n, repeated use of Lemma 50 gives us

$$h_j^{(n)} = \sum_{i=1}^{j} (-1)^{j+i} h_i^{(n-1)}. \tag{2.10}$$

Corollary 51.

$$\left\langle g_i^{(n)}, h_j^{(n)} \right\rangle = \delta_{i,j}.$$

Proof.

$$\begin{aligned}\left\langle g_i^{(n)}, h_j^{(n)} \right\rangle &= \left\langle g_i^{(n-1)} + g_{i+1}^{(n-1)}, \sum_{k=1}^{j} (-1)^{j+k} h_k^{(n-1)} \right\rangle \\ &= \sum_{k=1}^{j} (-1)^{j+k} \left[\left\langle g_i^{(n-1)}, h_k^{(n-1)} \right\rangle + \left\langle g_{i+1}^{(n-1)}, h_k^{(n-1)} \right\rangle \right] \\ &= \sum_{k=1}^{j} (-1)^{j+k} \left[\delta_{i,k} + \delta_{i+1,k} \right].\end{aligned}$$

If $i > j$, this sum is obviously zero. If $i < j$, the sum simplifies to $(-1)^{j+i}(1) + (-1)^{j+i+1}(1) = 0$. Finally, if $j = i$, the sum simplifies to $(-1)^{j+i}(1) = 1$. □

Corollary 52. $\left\{ (h_j^{(n)}), (g_j^{(n)}) \right\}_{\mathbb{N}}$ *is a M-basis for H.*

Proof. Fix $n \in \mathbb{N}_0$. Referring to Definition 31, we see that Corollary 51 establishes the necessary biorthogonality condition. Also, equation (2.6) gives us that each orthonormal basis vector e_k in H is in the linear span of the sequence $(h_j^{(n)})_{\mathbb{N}}$; therefore, closing this linear span gives us H. Now, fix an arbitrary nonzero $f \in H$, and assume $< f, g_j^{(n)} >= 0$ for every $j \in \mathbb{N}$. We wish to prove that f must be the zero vector in this case; but this follows inductively from Lemma 29 and the fact that the sequence $(< f, e_j >)_{\mathbb{N}}$ must be in ℓ^2. □

Notice, however, that $(g_j^{(n)})_{\mathbb{N}}$ can't be a frame for H for any $N \in \mathbb{N}$; if it were, then $S_k^{(n)} \to f$ in norm as $n \to \infty$ for every $f \in H$, which contradicts the main Theorem proven in section 2.5. Also notice that for each f in the linear span of the orthonormal basis $(e_k)_{\mathbb{N}}$, $f = \sum_{j \in \mathbb{N}} < f, g_j^{(n)} > h_j^{(n)}$, where the right hand side has only finitely many nonzero terms.

2.4 SIZE ESTIMATES AND LIMITS

It will be useful to develop estimates for the norm of each $h_j^{(n)}$ for fixed n. We know that $(e_\ell)_{\mathbb{N}}$ is an orthonormal basis for the Hilbert space H, so direct calculation gives us

$$\begin{aligned}
||h_j^{(n)}||^2 &= \sum_{\ell=1}^{\infty} \left| \left\langle \frac{1}{(n-1)!} \sum_{i=1}^{j} (-1)^{i+j} (j-i+1)^{<n-1>} e_i, e_\ell \right\rangle \right|^2 \qquad (2.11)\\
&= \sum_{\ell=1}^{j} \left| \frac{(j-\ell+1)^{<n-1>}}{(n-1)!} \right|^2 \\
&= \frac{1}{(n-1)!^2} \sum_{k=1}^{j} \left(k^{<n-1>}\right)^2,
\end{aligned}$$

where the substitution $k = j - \ell + 1$ reverses the order of the terms in the sum. We want to develop upper and lower estimates for $(k^{<n-1>})^2$.

$$\begin{aligned}
k^{<n-1>} &= k(k+1)(k+2)\cdots(k+n-2);\\
(k^{<n-1>})^2 &= kk(k+1)(k+1)(k+2)(k+2)\cdots(k+n-2)(k+n-2).
\end{aligned}$$

A half-step shift to the right in the duplicate terms gives us

$$\begin{aligned}
(k^{<n-1>})^2 &\leq k(k+\frac{1}{2})(k+1)(k+\frac{3}{2})(k+2)(k+\frac{5}{2})\cdots(k+n-2)(k+n-\frac{3}{2})\\
&= \left(\frac{2k}{2}\right)\left(\frac{2k+1}{2}\right)\left(\frac{2k+2}{2}\right)\cdots\left(\frac{2k+2n-4}{2}\right)\left(\frac{2k+2n-3}{2}\right)\\
&= \frac{(2k)^{<2n-2>}}{2^{2n-2}},
\end{aligned}$$

while a half-step shift to the left gives us

$$\begin{aligned}
(k^{<n-1>})^2 &\geq (k-\frac{1}{2})k(k+\frac{1}{2})(k+1)(k+\frac{3}{2})(k+2)\cdots(k+n-\frac{5}{2})(k+n-2)\\
&= \left(\frac{2k-1}{2}\right)\left(\frac{2k}{2}\right)\left(\frac{2k+1}{2}\right)\cdots\left(\frac{2k+2n-5}{2}\right)\left(\frac{2k+2n-4}{2}\right)\\
&= \frac{(2k-1)^{<2n-2>}}{2^{2n-2}};
\end{aligned}$$

in summary, we have the estimates

$$\frac{(2k-1)^{<2n-2>}}{2^{2n-2}} \leq (k^{<n-1>})^2 \leq \frac{(2k)^{<2n-2>}}{2^{2n-2}}. \qquad (2.12)$$

We can also see that for any whole number m,

$$(2k)^{<m>} \leq (2k)(2k+2)(2k+4)\cdots(2k+2(m-1)) = 2^m k^{<m>}. \tag{2.13}$$

Proposition 53. *For each whole number n,*

$$||h_j^{(n)}||^2 \overset{j}{\sim} j^{2n-1}.$$

Proof. Using our work from equation (2.11), and then applying the upper estimate from the inequalities (2.12), we see that we have

$$||h_j^n||^2 \leq \frac{1}{2^{2n-2}(n-1)!^2}\sum_{k=1}^{j}(2k)^{<2n-2>}.$$

Now, using the estimate developed in inequality (2.13), we see that we have

$$||h_j^{(n)}||^2 \leq \frac{1}{(n-1)!^2}\sum_{k=1}^{j}k^{<2n-2>}.$$

This puts us in a position to utilize Lemma 39, and results in

$$\begin{aligned}
||h_j^{(n)}||^2 &\leq \frac{1}{(n-1)!^2}\cdot\frac{j^{<2n-1>}}{2n-1} \\
&= \frac{1}{(n-1)!^2(2n-1)}\cdot j(j+1)(j+2)\cdots(j+2n-2) \\
&\overset{j}{\sim} \frac{j^{2n-1}}{(n-1)!^2(2n-1)}.
\end{aligned}$$

On the other hand, since it is clear that $(2k-1)^{<n>} \geq (k-1)^{<n>}$, we have

$$\begin{aligned}
||h_j^{(n)}||^2 &\geq \frac{1}{2^{2n-2}(n-1)!^2}\sum_{k=1}^{j}(2k-1)^{<2n-2>} \\
&\geq \frac{1}{2^{2n-2}(n-1)!^2}\sum_{k=1}^{j}(k-1)^{<2n-2>} = \frac{1}{2^{2n-2}(n-1)!^2}\sum_{k=1}^{j-1}k^{<2n-2>},
\end{aligned}$$

so we can again apply Lemma 39 to see that

$$\begin{aligned}
||h_j^{(n)}||^2 &\geq \frac{1}{2^{2n-2}(n-1)!^2}\cdot\frac{(j-1)^{<2n-1>}}{2n-1} \\
&\overset{j}{\sim} \frac{j^{2n-1}}{2^{2n-2}(n-1)!^2(2n-1)}.
\end{aligned}$$

□

Theorem 54. *For any natural number n, we have*

(1) $\lim_{N\to\infty} \left|\left|\frac{1}{N^{n-1}} < f_0, e_{N+n} > h_N^{(n)}\right|\right| \neq 0$ *for some f_0 in H;*

(2) $\lim_{N\to\infty} \left|\left|\sigma_N\left(\frac{1}{k^{n-1}} < f, e_{k+n} > h_k^{(n)}\right)\right|\right| = 0$ *for all f in H.*

Proof. Fix $n \in \mathbb{N}$ and an arbitrary f in H. To prove (1), we refer to the proof of Proposition 53, which gives us the existence of some constant $C_L = C_L(n)$ such that

$$\left|\left|\frac{1}{N^{n-1}} < f, e_{N+n} > h_N^{(n)}\right|\right| = \frac{1}{N^{n-1}} \,|< f, e_{N+n} >|\, ||h_N^{(n)}||$$
$$> \frac{C_L}{N^{n-1}}| < f, e_{N+n} > |N^{n-1/2} = C_L \,| < f, e_{N+n} > |\, \sqrt{N},$$

which does not go to zero as $N \to \infty$ for certain f_0 in H.

To prove (2), we fix an $\varepsilon > 0$, and make the following observations. Because the sequence $(< f, e_k >)_{\mathbb{N}}$ is in ℓ^2, there must exist a natural number M such that for every natural number N with $N \geq M$,

$$\left(\sum_{j=N+1}^{\infty} | < f, e_{j+n} > |^2\right)^{1/2} < \varepsilon. \tag{2.14}$$

With this choice of M, there must also exist a natural number N_0 (with $N_0 \geq M$) such that for every $N \geq N_0$,

$$\frac{M+1}{N}||f|| < \varepsilon. \tag{2.15}$$

Now, fix $N \in \mathbb{N}$ with $N \geq N_0$. We will again refer to the proof of Proposition 53 for the existence of a constant $C_u = C_u(n)$ in an upper estimate for the norm of $h_k^{(n)}$, and rely on Hölder's inequality:

$$\left|\left|\sigma_N\left(\frac{1}{k^{n-1}} < f, e_{k+n} > h_k^{(n)}\right)\right|\right| = \left|\left|\frac{1}{N}\sum_{k=1}^{N} \frac{1}{k^{n-1}} < f, e_{k+n} > h_k^{(n)}\right|\right|$$
$$\leq \frac{C_u}{N}\sum_{k=1}^{N} \frac{1}{k^{n-1}}| < f, e_{k+n} > |\, k^{n-1/2}$$
$$= \frac{C_u}{N}\left[\sum_{k=1}^{M} | < f, e_{k+n} > |\sqrt{k} + \sum_{k=M+1}^{N} | < f, e_{k+n} > |\sqrt{k}\right]$$

$$\leq \frac{C_u}{N}\left[\left(\sum_{k=1}^{M} |< f, e_{k+n} >|^2\right)^{1/2}\left(\sum_{k=1}^{M} k\right)^{1/2}\right.$$
$$\left. + \left(\sum_{k=M+1}^{N} |< f, e_{k+n} >|^2\right)^{1/2}\left(\sum_{k=M+1}^{N} k\right)^{1/2}\right]$$
$$\leq \frac{C_u}{N}\left[||f||\left(\frac{M(M+1)}{2}\right)^{1/2}\right.$$
$$\left. + \left(\sum_{k=M+1}^{\infty} |< f, e_{k+n} >|^2\right)^{1/2}\left(\frac{N(N+1)}{2}\right)^{1/2}\right]$$
$$\leq C_u\left[\frac{M+1}{N}||f|| + \frac{N+1}{N}\left(\sum_{k=M+1}^{\infty} |< f, e_{k+n} >|^2\right)^{1/2}\right].$$

We can now put our observations from equations (2.14) and (2.15) to use to achieve the desired result. □

Now we consider differential operators that are essential to the main result. Fix R in $(0, \infty)$, and define $\Delta_R := \{z \in \mathbb{C} : |z| < R\}$. Let $\mathbb{A}_R$ be the set of all analytic functions defined on Δ_R that are scalar valued, and consider an arbitrary function $\Phi(z) \in \mathbb{A}_R$. Because Φ is analytic, it must have a power series representation for any z in Δ_R; say, $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ for some sequence $(a_j)_{\mathbb{N}_0} \subset \mathbb{C}$. If we make the additional assumption that Φ is not identically zero, then there must exist some smallest whole number ν with $a_\nu \neq 0$; in this case, we can write

$$\Phi(z) = \sum_{j=\nu}^{\infty} a_j z^j \qquad \forall z \in \Delta_R.$$

We want to think about $F_k := \Phi(1/k)$ for natural numbers k with $k > 1/R$.

Definition 55. *Fix a positive real number R. For all natural numbers k with $k - 1 > 1/R$,*

$$D_k F := F_k - F_{k-1} = \Phi\left(\frac{1}{k}\right) - \Phi\left(\frac{1}{k-1}\right).$$

If we fix a k as in the definition above, and define $z := 1/k$ as well as $w := 1/(k-1)$, we have $1/w = k - 1 = 1/z - 1 = (1-z)/z$, so $w = z/(1-z)$. In this case, we can write

$$D_k F = \Phi(z) - \Phi(w) = \Phi(z) - \Phi\left(\frac{z}{1-z}\right).$$

Now, consider $z \in \Delta_R$ with $R < 1/2$; in particular, consider z with $|z| < R/2 < 1/2$. We have

$$|w| = \frac{|z|}{|1-z|} < \frac{|z|}{1-|z|} < 2|z| < R;$$

in other words, if $\Phi(z) \in \mathbb{A}_{R/2} \subset \mathbb{A}_R$, then $\Phi(w) \in \mathbb{A}_R$ also. This gives us that $\Phi(z) - \Phi(z/(1-z)) \in \mathbb{A}_{R/2}$.

Lemma 56. *Define* $\Psi(z) := \Phi(z) - \Phi(z/(1-z))$. *We have* $\Psi(z) = \sum_{\alpha=\nu+1}^{\infty} b_\alpha z^\alpha$ *for some complex scalar-valued sequence* $(b_\alpha)_{\mathbb{N}}$, *with* $b_{\nu+1} = -\nu a_\nu$.

Proof.

$$\begin{aligned}
\Psi(z) &= \Phi(z) - \Phi\left(\frac{z}{1-z}\right) \\
&= \sum_{j=\nu}^{\infty} a_j z^j - \sum_{j=\nu}^{\infty} a_j z^j (1-z)^{-j} \\
&= \sum_{j=\nu}^{\infty} a_j z^j \left(1 - (1-z)^{-j}\right) \\
&= \sum_{j=\nu}^{\infty} a_j z^j \left(1 - \sum_{\ell=0}^{\infty} \binom{-j}{\ell} (-z)^\ell\right) \\
&= \sum_{j=\nu}^{\infty} \sum_{\ell=1}^{\infty} a_j \binom{-j}{\ell} (-1)^{\ell+1} z^{j+\ell} \\
&= \sum_{\ell=\nu+1}^{\infty} z^\ell \left[\sum_{j=\nu+1}^{\ell} (-1)^{j+\ell} a_{j-1} \binom{-(j-1)}{-j+\ell+1}\right].
\end{aligned}$$

For each $\ell \geq \nu+1$, define b_ℓ as the bracketed term above, and we have the desired result. □

Note that $\Psi(z)$ is in $\mathbb{A}_{R^*}$, where R^* is between zero and $R/2$.

Corollary 57. *Fix a positive real number* R. *For any function* $\Phi \in \mathbb{A}_R$ *with the property that* $\lim_{z \to 0} \frac{\Phi(z)}{z^m} = \gamma$ *for some* $m \in \mathbb{N}_0$ *and* $\gamma \in \mathbb{C}$, *we have*

$$\lim_{z \to 0} \frac{\Psi(z)}{z^{m+1}} = -m\gamma.$$

Proof. Notice that we must necessarily have $m \le \nu$, where again ν denotes the index of the first nonzero term in the power series expansion of Φ. In fact, if $m = \nu$, then $\gamma = a_\nu$; if $m < \nu$, then $\gamma = 0$. According to Lemma 56,

$$\lim_{z\to 0} \frac{\Psi(z)}{z^{m+1}} = \lim_{z\to 0} \sum_{\alpha=\nu+1}^{\infty} b_\alpha z^{\alpha-m-1}.$$

If $m > \nu$, this limit does not exist; if $m < \nu$, this limit is $0 = m \cdot 0$. Again using Lemma 56 in the case where $m = \nu$, the limit is $b_{\nu+1} = -\nu a_\nu = -m a_\nu$. □

We state a special case, with $z = 1/k$, for later reference.

Corollary 58. *Fix $\Phi \in \mathbb{A}_R$. If there is some whole number m such that $\lim_{k\to\infty} k^m \Phi(1/k) = \gamma \in \mathbb{C}$, then $\lim_{k\to\infty} k^{m+1}\Psi(1/k) = -m\gamma$.*

2.5 PROOF OF MAIN THEOREM

We are ready to prove the main result of this Chapter. We'll need to consider the following.

Definition 59. *for every whole number m and every natural number n, define for all f in H*

$$\tau_k^{(n;m)}(f) := \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} S_k^{(\ell+m)}(f).$$

Lemma 60. *For any natural number m, $\tau_k^{(1;m)} = < f, g_{k+1}^{(m)} > h_k^{(1+m)}$.*

Proof.

$$\begin{aligned}
\tau_k^{(1;m)} &= S_k^{(1+m)}(f) - S_k^{(m)}(f) \\
&= \sum_{j=1}^{k} < f, g_j^{(1+m)} > h_j^{(1+m)} - \sum_{j=1}^{k} < f, g_j^{(m)} > h_j^{(m)} \\
&= \sum_{j=1}^{k} < f, g_j^{(m)} > h_j^{(1+m)} + \sum_{j=1}^{k} < f, g_{j+1}^{(1+m)} > h_j^{(1+m)} - \sum_{j=1}^{k} < f, g_j^{(m)} > h_j^{(m)} \\
&= \sum_{j=1}^{k} < f, g_j^{(m)} > (h_j^{(1+m)} - h_j^{(m)}) + \sum_{j=1}^{k} < f, g_{j+1}^{(m)} > h_j^{(1+m)}.
\end{aligned}$$

We can now apply Lemma 50 and use the convention $[h_0^{(n)} := 0$ for any $n \in \mathbb{N}_0]$ to get

$$\begin{aligned}\tau_k^{(1;m)} &= \sum_{j=2}^{k} < f, g_j^{(m)} > (-h_{j-1}^{(1+m)}) + \sum_{\ell=2}^{k+1} < f, g_\ell^{(m)} > h_{\ell-1}^{(1+m)} \\ &= \sum_{j=2}^{k} < f, g_j^{(m)} > (h_{j-1}^{(1+m)} - h_{j-1}^{(1+m)}) + < f, g_{k+1}^{(m)} > h_k^{(1+m)} \\ &= < f, g_{k+1}^{(m)} > h_k^{(1+m)}.\end{aligned}$$

□

Lemma 61. *For every whole number m and every natural number $n \geq 2$,*

$$\tau_k^{(n;m)}(f) = \tau_k^{(n-1;m+1)}(f) - \tau_k^{(n-1;m)}(f).$$

Proof. For any positive integer α, recall that $\binom{\alpha}{j} := 0$ if $j < 0$ or $j > \alpha$. With that idea in mind, we have

$$\begin{aligned}\tau_k^{(n;m)}(f) &= \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} S_k^{(\ell+m)}(f) \\ &= \sum_{\ell=1}^{n} (-1)^{n-\ell} \binom{n-1}{\ell-1} S_k^{(\ell+m)}(f) + \sum_{\ell=0}^{n-1} (-1)^{n-\ell} \binom{n-1}{\ell} S_k^{(\ell+m)}(f) \\ &= \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} S_k^{(j+m+1)}(f) - \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \binom{n-1}{\ell} S_k^{(\ell+m)}(f) \\ &= \tau_k^{(n-1;m+1)}(f) - \tau_k^{(n-1;m)}(f).\end{aligned}$$

□

Theorem 62. *Fix any natural number $n > 1$. For all $m \in \mathbb{N}_0$ and for all $N \in \mathbb{N}$,*

$$\sigma_N^{n-1}\left(\tau^{(n;m)}(f)\right) = \sum_{\alpha=1}^{n-1} \sigma_N^{\alpha-1}\left(G_k^{(n;\alpha)} \left\langle f, g_{k+n-(\alpha-1)}^{(m)} \right\rangle h_{k-(\alpha-1)}^{(n+m)}\right),$$

where each $G_k^{(n;\alpha)}$ is a real number. Specifically, $G_N^{(n;1)} := N^{1-n}$, and

$$G_k^{(n;n-1)} := (-1)^{n-2} D_k^{n-2}(1/\ell).$$

Further, there exists sequences of positive real numbers $R_n \leq 1/n$ and functions $\Gamma^{(n;\alpha)} \in \mathbb{A}_{R_n}$ such that for every natural number k with $1/k < R_n$, *(1)* $G_k^{(n;\alpha)} = \Gamma^{(n;\alpha)}\left(\frac{1}{k}\right)$;

(2) $\gamma_{n,\alpha} := \lim_{z\to 0} \frac{\Gamma^{(n;\alpha)}(z)}{z^{n-1}} \in \mathbb{R}$.

Proof. We begin by checking that the Theorem is true when $n = 2$. We apply Lemma 61 and then Lemma 60 to see

$$\begin{aligned}
\tau_k^{(2;m)}(f) &= \tau_k^{(1;m+1)}(f) - \tau_k^{(1;m)}(f) \\
&=< f, g_{k+1}^{(m+1)} > h_k^{(m+2)} - < f, g_{k+1}^{(m)} > h_k^{(m+1)} \\
&=< f, g_{k+1}^{(m)} > h_k^{(m+2)} + < f, g_{k+2}^{(m)} > h_k^{(m+2)} - < f, g_{k+1}^{(m)} > h_k^{(m+1)} \\
&=< f, g_{k+1}^{(m)} > (-h_{k-1}^{(m+2)}) + < f, g_{k+2}^{(m)} > h_k^{(m+2)} \\
&=< f, g_{k+2}^{(m)} > h_k^{(m+2)} - < f, g_{k+1}^{(m)} > h_{k-1}^{(m+2)}.
\end{aligned}$$

Averaging $2 - 1 = 1$ times, we have

$$\begin{aligned}
\sigma_N(\tau^{(2;m)}(f)) &= \frac{1}{N}\left[\sum_{j=1}^{N} < f, g_{j+2}^{(m)} > h_j^{(m+2)} - \sum_{j=1}^{N} < f, g_{j+1}^{(m)} > h_{j-1}^{(m+2)}\right] \\
&= \frac{1}{N}\left[\sum_{\ell=2}^{N+1} < f, g_{\ell+1}^{(m)} > h_\ell^{(m+2)} - \sum_{j=2}^{N} < f, g_{j+1}^{(m)} > h_{j-1}^{(m+2)}\right] \\
&= \frac{1}{N} < f, g_{N+2}^{(m)} > h_N^{(m+2)}.
\end{aligned}$$

Defining $G_N^{(2;1)} := 1/N$ and $\Gamma^{(2;1)}(z) := z$, we can choose $R_2 = 1/2$ and see $G_N^{(2;1)} = \Gamma^{(2;1)}(1/N)$; finally, we complete our check of the case $n = 2$ by noting that

$$\lim_{z\to 0} \frac{\Gamma^{(2;1)}(z)}{z^{2-1}} = 1 =: \gamma_{2,1} \in \mathbb{R}.$$

For the general case, we assume that the Theorem is true for some $n \geq 2$, and show that the Theorem is true with n replaced by $n + 1$. Fix $m \in \mathbb{N}_0$ and $N \in \mathbb{N}$ arbitrarily. We can use

Lemma 61 to see that $\sigma_N^n(\tau^{(n+1;m)}(f)) = \sigma_N^n(\tau^{(n;m+1)}(f)) - \sigma_N^n(\tau^{(n;m)}(f))$, and then apply our induction hypothesis to see that $\sigma_N^n(\tau^{(n+1;m)}(f))$ is the same as

$$\begin{aligned}\sigma_N^n(\tau^{(n+1;m)}(f)) &= \sum_{\alpha=1}^{n-1}\sigma_N^\alpha\left(G_k^{(n;\alpha)}\left\langle f, g_{k+n-(\alpha-1)}^{(m+1)}\right\rangle h_{k-(\alpha-1)}^{(n+m+1)}\right) \\ &\quad - \sum_{\alpha=1}^{n-1}\sigma_N^\alpha\left(G_k^{(n;\alpha)}\left\langle f, g_{k+n-(\alpha-1)}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+m)}\right) \\ &= \sum_{\alpha=1}^{n-1}\sigma_N^\alpha\left(G_k^{(n;\alpha)}\left[\left\langle f, g_{k+n-(\alpha-1)}^{(m)}\right\rangle\left(-h_{k-(\alpha-1)-1}^{(n+m+1)}\right) + \left\langle f, g_{k+n-(\alpha-1)+1}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+m+1)}\right]\right),\end{aligned} \tag{2.16}$$

using Lemma 29 followed by Lemma 50. We want to apply one of the α averaging operators but first make a small adjustment to ensure that the averaged terms will telescope. Of course, we also have to compensate for this adjustment, and so the final term in (2.16) can be rewritten as

$$\begin{aligned}&\sum_{\alpha=1}^{n-1}\sigma_N^\alpha\left[\left(G_k^{(n;\alpha)}\left\langle f, g_{k+n+1-(\alpha-1)}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+m+1)}\right) - \left(G_{k-1}^{(n;\alpha)}\left\langle f, g_{k+n-(\alpha-1)}^{(m)}\right\rangle h_{k-\alpha}^{(n+m+1)}\right)\right] \\ &\quad + \sum_{\alpha=1}^{n-1}\sigma_N^\alpha\left[\left(G_{k-1}^{(n;\alpha)} - G_k^{(n;\alpha)}\right)\left\langle f, g_{k+n-(\alpha-1)}^{(m)}\right\rangle h_{k-\alpha}^{(n+m+1)}\right].\end{aligned}$$

Notice that $G_{k-1}^{(n;\alpha)} - G_k^{(n;\alpha)} = (-1)D_k(G_\ell^{(n;\alpha)})$, and apply one averaging operator to the telescopable difference in the first set of brackets above; we then have

$$\begin{aligned}&\sigma_N^n(\tau^{(n+1;m)}(f)) \\ &= \sum_{\alpha=1}^{n-1}\sigma_N^{\alpha-1}\left(\frac{1}{k}G_k^{(n;\alpha)}\left\langle f, g_{k+n+1-(\alpha-1)}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+m+1)}\right) \\ &\qquad + \sum_{\alpha=1}^{n-1}\sigma_N^\alpha\left((-1)D_k(G_\ell^{(n;\alpha)})\left\langle f, g_{k+n-(\alpha-1)}^{(m)}\right\rangle h_{k-\alpha}^{(n+m+1)}\right) \\ &= \frac{1}{N}G_N^{(n;1)} < f, g_{N+n+1}^{(m)} > h_N^{(n+m+1)} \\ &\qquad + \sum_{\alpha=2}^{n-1}\sigma_N^{\alpha-1}\left(\frac{1}{k}G_k^{(n;\alpha)}\left\langle f, g_{k+n+1-(\alpha-1)}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+m+1)}\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\beta=2}^{n-1}\sigma_N^{\beta-1}\left((-1)D_k(G_\ell^{(n;\beta-1)})\left\langle f, g_{k+n+1-(\beta-1)}^{(m)}\right\rangle h_{k-(\beta-1)}^{(n+m+1)}\right)\\
&+\sigma_N^{n-1}\left((-1)D_k(G_\ell^{(n;n-1)})\left\langle f, g_{k+2}^{(m)}\right\rangle h_{k-n+1}^{(n+m+1)}\right)\\
=\ &\sigma_N^0\left(\frac{1}{k}G_k^{(n;1)} < f, g_{k+n+1}^{(m)} > h_k^{(n+m+1)}\right)\\
&+\sum_{\alpha=2}^{n-1}\sigma_N^{\alpha-1}\left[\left(\frac{1}{k}G_k^{(n;\alpha)}+(-1)D_k(G_\ell^{(n;n-1)})\right)\left\langle f, g_{k+n+1-(\alpha-1)}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+m+1)}\right]\\
&+\sigma_N^{n-1}\left((-1)D_k(G_\ell^{(n;n-1)})\left\langle f, g_{k+2}^{(m)}\right\rangle h_{k-n+1}^{(n+m+1)}\right).
\end{aligned}
$$

To see that we have what we want, define

(a) $G_k^{(n+1;1)} := \frac{1}{k}G_k^{(n;1)} = \frac{1}{k}\cdot\frac{1}{k^{n-1}} = \frac{1}{k^n}$

(b) $G_k^{(n+1;\alpha)} := \frac{1}{k}G_k^{(n;\alpha)} + (-1)D_k(G_\ell^{(n;\alpha-1)}) \qquad 2 \le \alpha \le n-1$

(c) $G_k^{(n+1;n)} := (-1)D_k(G_\ell^{(n;n-1)}) = (-1)^{n-1}D_k^{n-1}\left(\frac{1}{\ell}\right)$.

In summary, we have shown that

$$
\sigma_N^n(\tau^{(n+1;m)}(f)) = \sum_{\alpha=1}^{n}\sigma_N^{\alpha-1}\left(G_k^{(n+1;\alpha)}\left\langle f, g_{k+n+1-(\alpha-1)}^{(m)}\right\rangle h_{k-(\alpha-1)}^{(n+1+m)}\right).
$$

We need to check the other conditions of the Theorem; first, it is clear that each $G_k^{(n+1;\alpha)}$ is a real number. For $2 \le \alpha \le n-1$ and for all $k \in \mathbb{N}$ with $1/k < R_{n+1} < R_n$, we can define

$$
G_k^{(n+1;\alpha)} = \frac{1}{k}\Gamma^{(n;\alpha)}\left(\frac{1}{k}\right) + (-1)D_k\left(\Gamma^{(n;\alpha-1)}\left(\frac{1}{\ell}\right)\right) =: \Gamma^{(n+1;\alpha)}\left(\frac{1}{k}\right) \in \mathbb{A}_{R_{n+1}},
$$

and utilize Corollary 57 with $z := 1/k$ to see that

$$
\begin{aligned}
\lim_{z\to 0}\frac{\Gamma^{(n+1;n)}(z)}{z^{n+1-1}} &= \lim_{z\to 0}\frac{\Gamma^{(n;\alpha)}(z)}{z^{n-1}} - \lim_{z\to 0}\frac{\Gamma^{(n;\alpha-1)(z)} - \Gamma^{(n;\alpha-1)}\left(\frac{z}{1-z}\right)}{z^n}\\
&= \gamma_{n,\alpha} - -(n-1)\gamma_{n,\alpha-1} \in \mathbb{R}.
\end{aligned}
$$

For the case $\alpha = 1$, define $G_k^{(n+1;1)} = (1/k)\Gamma^{(n;1)}(1/k) =: \Gamma^{(n+1;1)}(1/k)$, and notice that $z^{-n}\Gamma^{(n+1;1)}(z) = z^{1-n}\Gamma^{(n;1)}(z) \to \gamma_{n,1} \in \mathbb{R}$ as $z \to 0$. Finally, for the case $\alpha = n$, define $G_k^{(n+1;n)} := \Gamma^{(n+1;n)}(1/k)$, and notice that we can again rely on Corollary 57 to get
$z^{-n}\Gamma^{(n+1;n)}(z) = -z^{-n}[\Gamma^{(n;n-1)}(z) - \Gamma^{(n;n-1)}(z/(1-z))] = (n-1)\gamma_{n,n-1} \in \mathbb{R}$ as $z \to 0$. □

Proof 63 (Proof of Theorem 34)**.**

Fix an arbitrary $f \in H$ and $n \geq 2$. If we let $m = 0$ in Theorem 62, we have for every natural number k that

$$\sigma_N^{n-1}\left(\tau^{(n;0)}(f)\right) = \sum_{\alpha=1}^{n-1} \sigma_N^{\alpha-1}\left(G_k^{(n;\alpha)}\left\langle f, e_{k+n-(\alpha-1)}\right\rangle h_{k-(\alpha-1)}^{(n)}\right),$$

where each $G_k^{(n;\alpha)} \in \mathbb{R}$. Further, there exists sequences of positive real numbers R_n such that $1/k < R_n$ and functions $\Gamma^{(n;\alpha)} \in \mathbb{A}_{R_n}$ such that

(1) $G_k^{(n;\alpha)} = \Gamma^{(n;\alpha)}\left(\frac{1}{k}\right)$;

(2) $\gamma_{n,\alpha} := \lim_{z\to 0} \frac{\Gamma^{(n;\alpha)}(z)}{z^{n-1}} = \gamma_{n,\alpha} \in \mathbb{R}$.

For $\alpha \geq 2$, we see that $G_k^{(n;\alpha)} \sim (1/k)^{n-1} \cdot \gamma_{n,\alpha}$, and this implies the existence of some positive constant C_n such that for k large enough,

$$\left|G_k^{(n;\alpha)}\right| \leq C_n \left(\frac{1}{k}\right)^{n-1}.$$

Recall also the proof of Proposition 53, which gives us

$$||h_{k-(\alpha-1)}^{(n)}|| \leq C_u(n) \cdot (k-(\alpha-1))^{n-1/2};$$

together, we see that there exists some positive constant $\tilde{C}$, depending only on n, such that

$$\left|\left|G_k^{(n;\alpha)} h_{k-(\alpha-1)}^{(n)}\right|\right| < \tilde{C} \cdot \sqrt{k}.$$

Therefore, since $\alpha - 1 \geq 1$, Theorem 54 tells us that

$$\lim_{N\to\infty} \left|\left|\sigma_N^{\alpha-1}\left(G_k^{(n;\alpha)}\left\langle f, e_{k+n-(\alpha-1)}\right\rangle h_{k-(\alpha-1)}^{(n)}\right)\right|\right| = 0.$$

We still have to consider the case $\alpha = 1$. The proof of Proposition 53 gives us a positive constant $C_L(n)$ such that

$$\begin{aligned}\left|\left|G_N^{(n;1)} < f, e_{N+n} > h_N^{(n)}\right|\right| &= \frac{1}{N^{n-1}} | < f, e_{N+n} > | \, ||h_N^{(n)}|| \\ &\geq C_L \cdot \frac{N^{n-1/2}}{N^{n-1}} = C_L \cdot \sqrt{N},\end{aligned}$$

which means that there exists some $f_0 \in H$ with

$$\lim_{N\to\infty} \left|\left|G_N^{(n;1)} < f_0, e_{N+n} > h_N^{(n)}\right|\right| \neq 0. \tag{2.17}$$

Now, we know that $\sigma_N^{\alpha}(S^{(\alpha)}(f)) \to f$ in norm as n goes to infinity, for $\alpha = 0, 1$. Assume that this result hold for $\alpha = 0, 1, \ldots, n-1$, and realize that (2.17) means that there is some $f_0 \in H$ such that $\sigma_N^{\alpha}(S^{(\alpha)}(f_0)) \nrightarrow f_0$ in norm. Let's apply the averaging operator one more time:

$$\begin{aligned}
\left|\left|\sigma_N^n\left(\tau^{(n;0)}(f)\right)\right|\right| &\leq \left|\left|\sigma_N\left(G_k^{(n;1)} < f, e_{k+n} > h_k^{(n)}\right)\right|\right| \\
&\quad + \left|\left|\sigma_N\left[\sum_{\alpha=2}^{n-1}\sigma_k^{\alpha-1}\left(G_\ell^{(n;\alpha)}\left\langle f, e_{\ell+n-(\alpha-1)}\right\rangle h_{\ell-(\alpha-1)}^{(n)}\right)\right]\right|\right| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} C_L \cdot \sqrt{k} + \left|\left|\sum_{\alpha=2}^{n-1}\sigma_N^{\alpha}\left(G_k^{(n;\alpha)}\left\langle f, e_{k+n-(\alpha-1)}\right\rangle h_{k-(\alpha-1)}^{(n)}\right)\right|\right|,
\end{aligned}$$

So we can apply Theorem 54 to see that $\left|\left|\sigma_N^n\left(\tau^{(n;0)}(f)\right)\right|\right| \to 0$ as $N \to \infty$. Finally, we can write

$$\begin{aligned}
0 = \lim_{N\to\infty}\sigma_N^n\left(\tau^{(n;0)}(f)\right) &= \lim_{N\to\infty}\sigma_N^n\left(\sum_{\ell=0}^{n}(-1)^{n-\ell}\binom{n}{\ell}S_k^{(\ell)}(f)\right) \\
&= \lim_{N\to\infty}\sigma_N^n\left(S_k^{(n)}(f)\right) + \lim_{N\to\infty}\sigma_N^n\left(\sum_{\ell=0}^{n-1}(-1)^{n-\ell}\binom{n}{\ell}S_k^{(\ell)}(f)\right) \\
&= \lim_{N\to\infty}\sigma_N^n\left(S_k^{(n)}(f)\right) + \left[\sum_{\ell=0}^{n-1}(-1)^{n-\ell}\binom{n}{\ell}\right] f \\
&= \lim_{N\to\infty}\sigma_N^n\left(S_k^{(n)}(f)\right) - f,
\end{aligned}$$

So, in norm, it must be that

$$\lim_{N\to\infty}\sigma_N^n\left(S_k^{(n)}(f)\right) = f.$$

APPENDIX

COMPLETENESS PROOFS

Theorem 64. $(\mathcal{H}^1, \|\cdot\|_{\mathcal{H}^1})$ *is a complete normed linear space.*

Proof. First, we'll prove that $||\cdot||_{\mathcal{H}_1}$ as given in definition (1.1) is a norm. We know that $||\cdot||_{H^1(\Delta)}$ is a norm, and therefore nonnegative; this directly leads to the conclusion that $||\cdot||_{\mathcal{H}^1}$ is nonnegative as well. Next, assume that we define $a \in S$ to be the zero sequence; we have

$$||0||_{\mathcal{H}^1} = \sup_{n\in\mathbb{N}_0} \left(\inf_{t\in c_{00}(n)} \left\| \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^1(\Delta)} \right).$$

Fix $n \in \mathbb{N}_0$ and an arbitrary positive ε; choosing t as the zero sequence gives us $||0||_{\mathcal{H}^1} < \varepsilon$. Conversely, if we assume $||a||_{\mathcal{H}^1} = 0$, we need to show that a is indeed the zero sequence. We have

$$\inf_{t\in c_{00}(n)} \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^1(\Delta)} = 0.$$

Define $f(z) := \sum c_j z^j$, where $c_j := a_j$ for $0 \le j \le n$ and $c_j := t_j$ otherwise. Note that $f \in H^1(\Delta)$, so we can apply Hardy's inequality as follows:

$$\begin{aligned}\sum_{j=0}^{n} \frac{|a_j|}{j+1} &\le \sum_{j=0}^{n} \frac{|a_j|}{j+1} + \sum_{j=n+1}^{\infty} \frac{|t_j|}{j+1} = \sum_{j=0}^{\infty} \frac{|c_j|}{j+1} \\ &\le \pi \|f\|_{H^1(\Delta)} = \pi \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^1(\Delta)}.\end{aligned}$$

Taking the infimum over all $t \in c_{00}(n)$ gives us $a_j = 0$ for every j.

We move on to show that the triangle inequality holds in $\mathcal{H}^1$. Fix $n \in \mathbb{N}_0$ and a positive ε again; it will suffice to show that $\zeta_n(a+b) \leq \zeta_n(a) + \zeta_n(b)$ for arbitrary sequences a and b in $\mathcal{H}^1$. There are sequences r and s in $c_{00}(n)$ such that the following are true:

$$\frac{1}{n+1}\left\|\sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} r_j z^j\right\|_{H^1(\Delta)} \leq \zeta_n(a) + \varepsilon/2,$$

$$\frac{1}{n+1}\left\|\sum_{j=0}^{n} b_j z^j + \sum_{j=n+1}^{\infty} s_j z^j\right\|_{H^1(\Delta)} \leq \zeta_n(b) + \varepsilon/2.$$

Defining $t_j := r_j + s_j \in c_{00}(n)$, we have

$$\begin{aligned}
&\zeta_n(a) + \zeta_n(b) + \varepsilon \\
&\geq \frac{1}{n+1}\left[\left\|\sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} r_j z^j\right\|_{H^1(\Delta)} + \left\|\sum_{j=0}^{n} b_j z^j + \sum_{j=n+1}^{\infty} s_j z^j\right\|_{H^1(\Delta)}\right] \\
&\geq \frac{1}{n+1}\left\|\sum_{j=0}^{n} (a_j + b_j) z^j + \sum_{j=n+1}^{\infty} (r_j + s_j) z^j\right\|_{H^1(\Delta)} \\
&= \frac{1}{n+1}\left\|\sum_{j=0}^{n} (a_j + b_j) z^j + \sum_{j=n+1}^{\infty} t_j z^j\right\|_{H^1(\Delta)} \\
&\geq \zeta_n(a+b).
\end{aligned}$$

Finally, we fix $\beta \in \mathbb{C}$ and an arbitrary $a \in \mathcal{H}^1$, and need to show that $\|\beta a\|_{\mathcal{H}^1} = |\beta| \cdot \|a\|_{\mathcal{H}^1}$. We can assume that $\beta \neq 0$; it will suffice to show that $\zeta_n(\beta a) \leq |\beta| \zeta_n(a)$. Fix an arbitrary positive ε; there exists some $r \in c_{00}(n)$ such that

$$\frac{1}{n+1}\left\|\sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} r_j z^j\right\|_{H^1(\Delta)} \leq \zeta_n(a) + \frac{\varepsilon}{|\beta|}.$$

If we define $t_j := \beta r_j$ for all $j > n$, we see that

$$\begin{aligned}
|\beta|\zeta_n(a) + \varepsilon &\geq \frac{1}{n+1}\left\|\sum_{j=0}^{n} \beta a_j z^j + \sum_{j=n+1}^{\infty} \beta r_j z^j\right\|_{H^1(\Delta)} \\
&= \frac{1}{n+1}\left\|\sum_{j=0}^{n} \beta a_j z^j + \sum_{j=n+1}^{\infty} t_j z^j\right\|_{H^1(\Delta)} \geq \zeta_n(\beta a).
\end{aligned}$$

Now, take the supremum over all $n \in \mathbb{N}_0$, and we see that $\|\beta a\|_{\mathcal{H}^1} \leq |\beta| \cdot \|a\|_{\mathcal{H}^1}$. We can then use this fact to see that $|\beta| \cdot \|a\|_{\mathcal{H}^1} = |\beta| \cdot \|\beta\beta^{-1}a\|_{\mathcal{H}^1}$; using the result of the first part of this section of the proof, we then have $|\beta| \cdot \|a\|_{\mathcal{H}^1} \leq \|\beta a\|_{\mathcal{H}^1}$, which allows us to conclude that $\|\beta a\|_{\mathcal{H}^1} = |\beta| \cdot \|a\|_{\mathcal{H}^1}$ as desired. This concludes the proof that $\|\cdot\|_{\mathcal{H}^1}$ is a norm on $\mathcal{H}^1$. For the rest of this proof, it will be understood that $\|\cdot\| := \|\cdot\|_{\mathcal{H}^1}$.

To show that the space is complete, we prove the equivalent condition that every absolutely summable series in $(\mathcal{H}^1, \|\cdot\|_{\mathcal{H}^1})$ is summable. Let $(a^{(k)})_{k\in\mathbb{N}_0}$ be an arbitrary sequence contained in $\mathcal{H}^1$ with the property

$$S := \sum_{k=0}^{\infty} \|a^{(k)}\| < \infty. \tag{.1}$$

If we define the partial sum $P_N := \sum_{k=0}^{N} a^{(k)}$, we want to show the existence of an element $b \in \mathcal{H}^1$ such that, as $N \to \infty$, $\|b - P_N\| \to 0$. Fix an arbitrary positive ε; because of equation (.1), there is some $N_0 = N_0(\varepsilon) \in \mathbb{N}_0$ such that $\sum_{k=N_0}^{\infty} \|a^{(k)}\| < \varepsilon$. Notice that for $N > M > N_0$,

$$\|P_N - P_M\| = \left\| \sum_{k=M+1}^{N} a^{(k)} \right\| \leq \sum_{k=M+1}^{N} \|a^{(k)}\| < \varepsilon;$$

this shows that $(P_N)_{N\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathcal{H}^1$.

In fact, if we fix $n \in \mathbb{N}_0$, and examine the fixed-coordinate partial sum sequence $(P_{N,n})_{N\in\mathbb{N}_0} = \sum_{k=0}^{N} a_n^{(k)}$, we can see that $(P_{N,n})_{N\in\mathbb{N}_0}$ is a Cauchy sequence as well. To accomplish this, let $\varepsilon_0 := \varepsilon/(n+1)$, and choose any N and M larger than the aforementioned $N_0(\varepsilon_0)$. We have

$$\begin{aligned}
&\zeta_n(P_N - P_M) = \zeta_n\left(\sum_{k=M+1}^{N} a^{(k)}\right) \\
&= \frac{1}{n+1} \inf_{t\in c_{00}(n)} \left\| \sum_{j=0}^{n} \left(\sum_{k=M+1}^{N} a^{(k)}\right)_j z^j + \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^1(\Delta)} \\
&\leq \sup_{n\in\mathbb{N}_0} \frac{1}{n+1} \inf_{t\in c_{00}(n)} \left\| \sum_{j=0}^{n} \left(\sum_{k=M+1}^{N} a^{(k)}\right)_j z^j + \sum_{j=n+1}^{\infty} t_j z^j \right\|_{H^1(\Delta)} \\
&= \|P_N - P_M\| < \varepsilon_0.
\end{aligned}$$

Now, Hardy's inequality tells us that for any $x \in \mathcal{H}^1$, $|x_n|/(n+1) < \zeta_n(x)$ for all $n \in \mathbb{N}_0$; we then have

$$\frac{|(P_N - P_M)_n|}{n+1} \leq \zeta_n(P_N - P_M) < \varepsilon_0 = \frac{\varepsilon}{n+1},$$

which implies that $|(P_N - P_M)_n| < \varepsilon$.

In summary, for fixed n, $(P_{N,n})_{N\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathbb{C}$, and therefore must converge to some $b_n \in \mathbb{C}$; it also must be that $b_n := \sum_{k=0}^{\infty} a_n^{(k)}$. As $N \to \infty$, we want to show that

$$\left\| b - \sum_{k=0}^{N} a^{(k)} \right\| = \left\| \sum_{k=N+1}^{\infty} a_n^{(k)} \right\| \longrightarrow 0;$$

it will suffice to show that, for all $n \in \mathbb{N}_0$,

$$\zeta_n\left(\sum_{k=N+1}^{\infty} a^{(k)}\right) \leq \sum_{k=N+1}^{\infty} \zeta_n\left(a_n^{(k)}\right) + \beta_N \tag{.2}$$

for some null sequence $(\beta_N)_{N\in\mathbb{N}_0}$. Use the same ε that determined $N_0(\varepsilon)$ as above, and choose any $n \in \mathbb{N}_0$. For each $k \in \mathbb{N}$, it is possible to use definition (1.1) to choose a complex-valued sequence $t^{(k)}$ in $c_{00}(n)$ such that

$$\zeta_n(a^{(k)}) \leq \frac{1}{n+1}\left\|\sum_{j=0}^{n} a_j^{(k)} z^j + \sum_{j=n+1}^{\infty} t_j^{(k)} z^j\right\|_{H^1(\Delta)} \leq \zeta_n(a^{(k)}) + \frac{\varepsilon}{2^k}. \tag{.3}$$

We want to examine $\zeta_n\left(\sum_{k=N+1}^{\infty} a^{(k)}\right)$, which is less than or equal to

$$\frac{1}{n+1}\left\|\sum_{j=0}^{n}\left(\sum_{k=N+1}^{\infty} a_j^{(k)}\right) z^j + \sum_{j=n+1}^{\infty}\left(\sum_{k=N+1}^{\infty} t_j^{(k)}\right) z^j\right\|_{H^1(\Delta)}. \tag{.4}$$

If we can establish that $\sum_{k=N+1}^{\infty} t_j^{(k)}$ exists in $\mathbb{C}$, we could apply the triangle inequality to see that

$$\begin{aligned}
\zeta_n\left(\sum_{k=N+1}^{\infty} a^{(k)}\right) &= \frac{1}{n+1}\left\|\sum_{k=N+1}^{\infty}\left(\sum_{j=0}^{n} a_j^{(k)} z^j + \sum_{j=n+1}^{\infty} t_j^{(k)} z^j\right)\right\|_{H^1(\Delta)} \\
&\leq \sum_{k=N+1}^{\infty} \frac{1}{n+1}\left\|\sum_{j=0}^{n} a_j^{(k)} z^j + \sum_{j=n+1}^{\infty} t_j^{(k)} z^j\right\|_{H^1(\Delta)} \\
&\leq \sum_{k=N+1}^{\infty}\left(\zeta_n(a^{(k)}) + \frac{\varepsilon}{2^k}\right)
\end{aligned}$$

by equation (.3); this is what we wanted to show, namely equation (.2). Once we have this result, we can easily write

$$\zeta_n\left(\sum_{k=N+1}^{\infty} a^{(k)}\right) \leq \sum_{k=N+1}^{\infty} \|a^{(k)}\| + \varepsilon \sum_{k+N+1}^{\infty} \frac{1}{2^k};$$

notice that the right hand side is independent of n, and take the supremum over all n to arrive at

$$\left\|\sum_{k=N+1}^{\infty} a^{(k)}\right\| \leq \sum_{k=N+1}^{\infty} \|a^{(k)}\| + \beta_N.$$

We can make this as small as we like for N large enough, proving that, in norm, $\lim_{N\to\infty} \sum_{k=0}^{N} a^{(k)} = b$, and the space $\mathcal{H}^1$ is shown to be complete.

Finally, we must deal with the unresolved issue following equation (.4); to this end, consider the sequence $(T_N)_{N\in\mathbb{N}} \subset \mathbb{C}$, where $T_N := \sum_{k=1}^{N-1} t_\ell^{(k)}$. Recall that, if a function $h(z) \in H^1(\mathbb{T})$ has a representation $h(z) = \sum_{j=0}^{\infty} u_j z^j$, then $|u_j| \leq \|h\|_{H^1(\Delta)}$ for any j. Fix natural numbers N and M with $N_0 \leq M < N$. Because $t^{(k)} \in c_{00}(n)$, the function

$$h(z) := \sum_{j=0}^{n}\left(\sum_{k=M}^{N-1} a_j^{(k)}\right) z^j + \sum_{j=n+1}^{\infty}\left(\sum_{k=M}^{N-1} t_j^{(k)}\right) z^j$$

is in $H^1(\Delta)$; for any $\ell \geq n+1$,

$$\begin{aligned}
\frac{1}{n+1}|T_N - T_M| &= \frac{1}{n+1}\left|\sum_{k=M}^{N-1} t_\ell^{(k)}\right| \\
&\leq \frac{1}{n+1}\left\|\sum_{j=0}^{n}\left(\sum_{k=M}^{N-1} a_j^{(k)}\right) z^j + \sum_{j=n+1}^{\infty}\left(\sum_{k=M}^{N-1} t_j^{(k)}\right) z^j\right\|_{H^1(\Delta)} \\
&\leq \sum_{k=M}^{N-1}\left(\zeta_n(a^{(k)}) + \frac{\varepsilon}{2^k}\right)
\end{aligned}$$

by equation (.3) again; arguing as before, we finally arrive at

$$\frac{1}{n+1}|T_N - T_M| \leq \sum_{k=M}^{\infty} \|a^{(k)}\| + \varepsilon \sum_{k=M}^{\infty} \frac{1}{2^k} < 2\varepsilon.$$

This proves that $(T_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and must converge to some complex number $T := \sum_{k=1}^{\infty} t_\ell^{(k)}$ for each $\ell > n+1$. Removing finitely many terms from a convergent series doesn't change its convergence; hence, $\sum_{k=N+1}^{\infty} t_j^{(k)}$ exists in $\mathbb{C}$ as desired. □

Corollary 65. *$\mathcal{H}_0^1$ is a Banach space.*

Proof. Because $\mathcal{H}_0^1$ is a subset of the complete space $\mathcal{H}^1$, we only need to show that $\mathcal{H}_0^1$ is closed. Take any Cauchy sequence $(x^{(k)})_{k\in\mathbb{N}}$ contained in $\mathcal{H}_0^1$; because $\mathcal{H}^1$ is complete, there must be some $x^{(0)} \in \mathcal{H}^1$ such that $\|x^{(k)} - x^{(0)}\| \to 0$ as $k \to \infty$. We need to show that $\lim_{n\to\infty} \zeta_n(x^{(0)}) = 0$; for any fixed $k \in \mathbb{N}$, we know that

$$\lim_{n\to\infty} \zeta_n(x^{(k)}) = 0. \tag{.5}$$

Fix any $n \in \mathbb{N}$ and arbitrary positive ε. We have

$$\zeta_n(x^{(0)}) = \zeta_n(x^{(0)} - x^{(k)} + x^{(k)}) \le \zeta_n(x^{(k)} - x^{(0)}) + \zeta_n(x^{(k)}). \tag{.6}$$

There must be some $K(\varepsilon)$ such that for any $k \ge K(\varepsilon)$ and any n,
$\zeta_n(x^{(k)} - x^{(0)}) < \varepsilon/2$; also, if we fix $k = K(\varepsilon)$, equation (.5) gives us for arbitrary positive η the existence of some $N(\eta)$ such that $\zeta_n(x^{(k)}) < \eta$ for all n larger than $N(\eta)$. If we specifically choose $\eta := \varepsilon/2$, equation (.6) gives

$$\zeta_n(x^{(0)}) < \varepsilon.$$

□

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