COMPLEMENTARY GROUPS OF ANTOINE'S NECKLACES

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Submitted to the Graduate Faculty of

the Department of Mathematics in partial fulfillment

of the requirements for the degree of

Master of Science

University of Pittsburgh

2010

UNIVERSITY OF PITTSBURGH

DEPARTMENT OF MATHEMATICS

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August 16, 2010

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In 1921 Antoine constructed the first example of a *wild* Cantor set. A wild Cantor set is a subset of Euclidean 3–space, $\mathbb{R}^3$, which is homeomorphic to the Cantor set but which is not equivalent in $\mathbb{R}^3$ to the standard embedding of the Cantor set. Antoine's example is now called *Antoine's necklace*.

The purpose of this thesis is in to investigate the fundamental group of the complement of Antoine's necklace and other wild Cantor sets. First, a survey of known work on wild Cantor sets and their complementary groups is presented including: the Wirtinger presentation for knots and links, basic results on the complementary group of Antoine's necklace, Sher's theorem on canonical defining sequences for Antoine necklaces, and Skora's example of a wild Cantor set with trivial complementary group. Second, a complete presentation for the complementary group of Antoine's necklace (and some variants) is calculated. A plausible technique to construct a non–equivalent variant of Antoine's necklace with isomorphic complementary group is shown to fail. The thesis concludes with a survey of open problems on complementary groups and complements of Cantor sets embedded in Euclidean 3–space.

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1.0 PRELIMINARIES

1.1 INTRODUCTION

A central problem of topology is how a standard topological space can be embedded within another space. For example, knots (and more generally, links) — embeddings of a circle (respectively, of a disjoint sum of circles) into Euclidean 3–space, $\mathbb{R}^3$ — have been intensively studied since the birth of topology in the late 19th century. We have illustrated a simple knot and link in Figure 1.

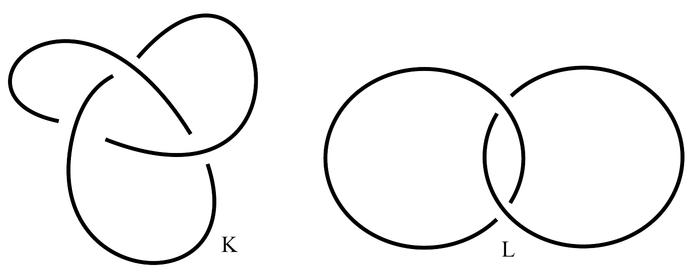

Figure 1: A Trefoil Knot K and a Link L

Knots and links are studied up to equivalence: two subsets A and B of a containing space X are said to be *equivalent* (in X) if there is a homeomorphism h of X with itself carrying A to B, $h(A) = B$. Note that if A and B are equivalent (say, via h), then $X - A$ and $X - B$ are homeomorphic (via $h \restriction (X - A)$).

In fact a well known theorem of Gordon and Luecke [6] states that knots are determined by their complements, i.e, that if K and K' are two knots embedded in $\mathbb{R}^3$, and $\mathbb{R}^3 - K$ is homeomorphic to $\mathbb{R}^3 - K'$, then K and K' are equivalent. However it was proven by Whitehead and illustrated in [7] that this is not in fact true for links.

In any case, it follows that topological invariants of $X - A$ are invariants, up to equivalence, of the pair (X, A). For knots the fundamental group of the complement, called the *knot group*, has been a widely studied invariant of the knot. These studies are aided by the fact that there is a simple algorithm called *Wirtinger's Presentation* for computing a presentation of the knot group of a given knot. For a general pair (X, A) let us call the fundamental group of $X - A$ the *complementary group* of A (in X).

The aim of this Thesis is to study the embeddings of Cantor sets in $\mathbb{R}^3$. More precisely we compute, apparently for the first time, the fundamental group of the complement of the Cantor subset of $\mathbb{R}^3$ known as *Antoine's necklace*. We have also tried to show that Antoine's necklace and a twisted variant have isomorphic complementary groups, with only limited success.

Recall that a *Cantor set* is anything homeomorphic to Cantor's classic 'middle thirds' subset of the real line. Identifying the real line with the x–axis in $\mathbb{R}^3$, we have a standard embedding of a Cantor set in $\mathbb{R}^3$. Call this the *standard Cantor set*, and any other Cantor set equivalent to it *trivial*. A non–trivial Cantor set in $\mathbb{R}^3$ is said to be *wild*. It is a remarkable result of Antoine [1] that there are wild Cantor sets. Antoine's Cantor set is called *Antoine's necklace*.

To construct an Antoine's necklace, first let V be a solid torus embedded in $\mathbb{R}^3$, and form a chain C_1 of $2n$ solid tori in V. We call this the 'top level' of our construction. Then in each component of C_1, construct a smaller chain of solid tori embedded in that component in the same manner as C_1 in V. Let C_2 denote the union of these smaller solid tori. This would be the 'second level' of the construction. This procedure yields Figure 2:

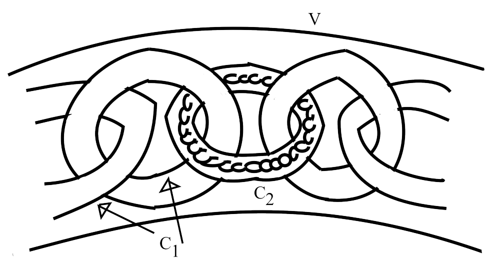

Figure 2: Construction of Antoine's Necklace

Now construct in each component of C_2 another chain of solid tori and call their union C_3. Continue this process a countable number of times to obtain:

$$C_1 \supset C_2 \supset C_3 \supset \ldots, \quad diam(C_{i,j}) \to 0, \quad i \to \infty$$

The intersection

$$A = \bigcap_{i=1}^{\infty} C_i$$

is called *Antoine's Necklace.* It is a non-empty compact subset of $\mathbb{R}^3$ homeomorphic to the Cantor set.

Sections 1.2 to 1.6 below lay out some preliminary results on the complementary groups of knots, and Cantor sets. This leads to the calculation that the complementary group of Antoine's necklace is non–trivial in Theorem 1.6.1. Since the standard

Cantor set in $\mathbb{R}^3$ has trivial complementary group, we see that Antoine's necklace is indeed wild (not equivalent to the standard Cantor set because their complements are non–homeomorphic, because they have distinct fundamental groups). Although this approach, through the complementary group, is now the standard way to show that Antoine's necklace is wild, Antoine himself gave a direct proof of wildness that we sketch.

We also present in Section 1.8 a Cantor set, W, constructed by Skora in [9] that, as we verify, is wild but has trivial complementary group ($\pi(\mathbb{R}^3 - W) = 1$). Clearly, as the standard Cantor set has trivial complementary group, Cantor sets are not determined by their complementary groups.

All of the results mentioned above are well known. But in Sections 2.1 to 2.5 we calculate a presentation for the complementary group of Antoine's necklace. We have not found any reference in the literature to such a calculation. From the work presented in the preliminaries we know that the complementary group of Antoine's necklace is the union of the complementary groups of the 'levels' going into the construction of the necklace. Using Wirtinger's Presentation (adapted to links) it is straightforward to calculate the complementary group of the top level (Section 2.1). We would then like to proceed by replacing one ring in the top level by a chain — and then repeating, first with the remaining rings on the top level, and then down into lower levels. To do so we have to adapt the proof of Wirtinger's presentation so as to calculate the complementary group of links *in a solid torus* (rather than in $\mathbb{R}^3$). This is the most technically difficult part of the work given here. From this the complete presentation for the complementary group of Antoine's necklace is obtained (Section 2.5).

We had hoped to show that Antoine's necklace and a twisted variant have the same complementary groups. We explain why this was plausible, and also where it appears to fall down in Sections 3.1 and 3.2.

The thesis concludes with a survey of open problems arising from this work.

1.2 LINKS AND CHAINS

1.2.1 LINKS:

Our first step in the development of a presentation for a general Antoine's necklace is to understand the Wirtinger presentation for knots and adapt it to links. Consider the following example:

Example: Let L be the link shown in Figure 3

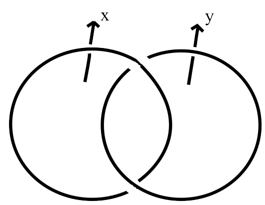

Figure 3: Link L

We add a generator onto each arc in the picture. In this case, we then have two generators, x and y. Each of these generators represents a loop in $\mathbb{R}^3 - L$ consisting of the oriented triangle from a fixed base point, to the tail of the generator, along the generator to the head, and back to the base point. It follows that at each crossing, there is some relation among the generators that must hold. Our crossings are illustrated in Figure 4.

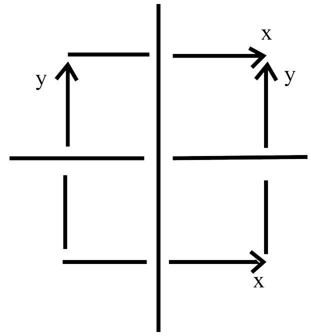

Figure 4: Adapting the Wirtinger Presentation to Links

Therefore our relation is $xy = yx$. The Wirtinger Presentation tells us that the group $\pi(\mathbb{R}^3 - L)$ is completely determined by these relations. Therefore $\pi(\mathbb{R}^3 - L) = (x, y; xy = yx) = \mathbb{Z} \times \mathbb{Z}$.

1.2.2 CHAINS:

Now that we have adapted the Wirtinger Presentation to a link, we can specialize this idea to a chain.

Consider the link C of $2n$ unknotted circles arranged in a chain running around the solid torus $V \cong D^2 \times S^1$ embedded in $\mathbb{R}^3$ given in Figure 5. By utilizing the same procedure as in the previous section, it is an easy exercise to show that the Wirtinger presentation for C is:

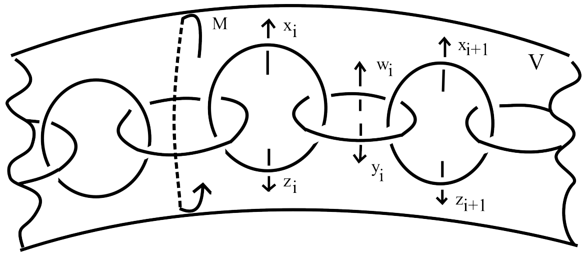

Figure 5: Link C of $2n$ circles in V

$$\pi(\mathbb{R}^3 - C) = \{x_i, y_i, z_i, w_i; y_i x_i = x_i w_i, y_i x_{i+1} = x_{i+1} w_i,$$
$$x_i y_i = y_i z_i, x_{i+1} y_i = y_i z_{i+1},\ \ 1 \leq i \leq n\}$$

These relations will play a very significant role in the presentation for the general Antoine's necklace.

Finally, we wish to use this result to prove an important proposition which will eventually be the key ingredient in showing that the complementary group of Antoine's necklace is non-trivial.

Proposition 1.2.1. *The meridian M of V is not homotopically trivial in $\mathbb{R}^3 - C$ or in $V - C$; in fact, M is of infinite order in $\pi(\mathbb{R}^3 - C)$.*

Proof. Consider the map of $\pi(\mathbb{R}^3 - C)$ onto the free group $F(x, y)$ given by the following:

$$x_i \to x \qquad y_i \to y \qquad z_i \to y^{-1}xy \qquad w_i \to x^{-1}yx$$

We note that this map is well defined, as the relations defined above become trivial in the free group. Using this map, the element $M = x_1^{-1} z_1$ is mapped to the commutator $x^{-1}y^{-1}xy$, which is of infinite order in a free group. Therefore M is of infinite order in $\pi(\mathbb{R}^3 - C)$ $\square$

Corollary 1.2.1. *The loop M is not contractible in the complement of the infinite chain pictured in Figure 6.*

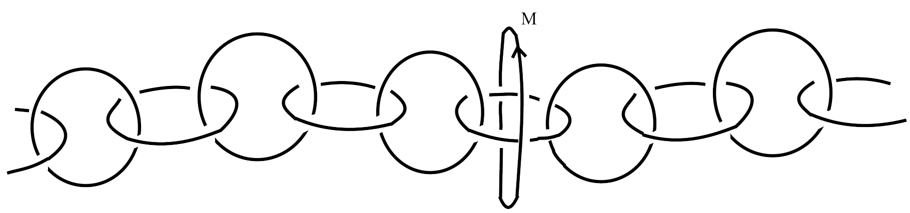

Figure 6: Infinite Chain with Loop M

Proof. Suppose there were a homotopy shrinking M to a point, missing the chain. Since the image of the homotopy is a compact set, each image h_i in the homotopy must also be contained in that compact set. Thus at some step in the process we could construct a finite chain which missed the homotopy, contradicting the previous proposition. $\square$

1.3 WHITEHEAD'S LINK

We now quickly illustrate one result of Whitehead.

If $L = L_1 \cup \cdots \cup L_n$ is a link with n components, we say that L_i is homotopically unlinked from the remaining components if there exists a homotopy h_t from L_i to the constant map such that the image of h_t and L_i are disjoint for all $t \in I$, $j \neq i$. We illustrate Whitehead's Link in Figure 7 and show that J is homotopically unlinked from K in Figure 8. It follows from the symmetric nature of $J \cup K$ that K is homotopically unlinked from J. When the link is embedded in $\mathbb{R}^3$, we see that this implies that each component of the link is contractible in the complement of the other.

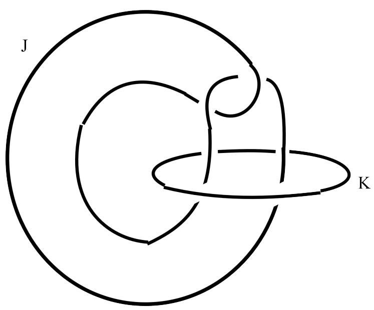

Figure 7: Whitehead's Link

Secondly, we enclose the link $J \cup K$ in a solid torus V (Figure 9). We then arrive at the following result:

Proposition 1.3.1. *K is not contractible in $V - J$.*

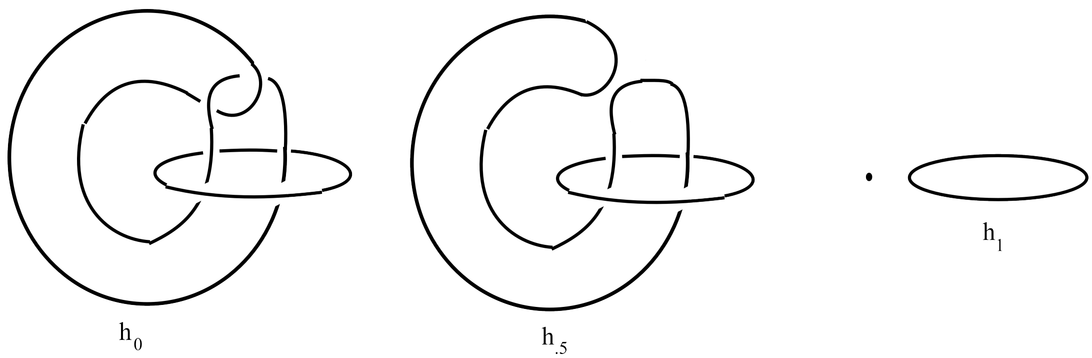

Figure 8: Diagram: J is homotopically unlinked from K

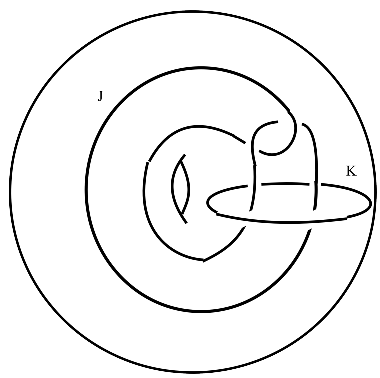

Figure 9: Whitehead Link in Torus V

Proof. Let $(\tilde{V}, p)$ be the universal cover of V, let $\tilde{J} = p^{-1}(J)$ and let $\bar{K}$ be one component of $p^{-1}(K)$. A homotopy which shrinks K in $V - J$ would lift to one which shrinks $\bar{K}$ to a point in $\tilde{V} - \tilde{J}$. But this contradicts the previous corollary, as $\bar{K}$ and $\tilde{J}$ are seen to be situated as M and the infinite chain. (See Figure 10)

□

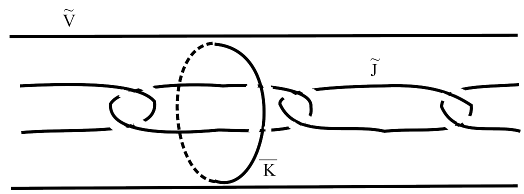

Figure 10: Diagram: K is not contractible in $V - J$

1.4 ANTOINE'S NECKLACE

We now proceed to show that Antoine's necklace is indeed a Cantor set. Recall that in Section 1.2 we constructed a chain C of $2n$ components in a solid torus V. To create Antoine's necklace, we now thicken each component of C slightly to form a chain C_1 of $2n$ solid tori in V. In each component of C_1, we construct a smaller chain of solid tori embedded in that component in the same manner as C_1 in V. Let C_2 denote the union of these smaller solid tori.

Now construct in each component of C_2 another chain of solid tori and call their union C_3. Continue this process a countable number of times to obtain:

$$C_1 \supset C_2 \supset C_3 \supset \ldots, \quad diam(C_{i,j}) \to 0, \quad i \to \infty$$

The intersection

$$A = \bigcap_{i=1}^{\infty} C_i$$

is called *Antoine's Necklace.*

Proposition 1.4.1. *Antoine's necklace is homeomorphic to a Cantor set.*

Proof. We use the following characterization:

C is a Cantor set if and only if C is totally disconnected, compact, and perfect.

Clearly, Antoine's necklace is totally disconnected, as there is some stage in the construction when any two points will lie in different tori. It is compact as the intersection of closed and bounded tori. Lastly, it is perfect, as every torus C_i contains at least two tori inside C_i in the following stage of the construction. Therefore we may choose, for any point $x \in A$, a sequence from $A - \{x\}$ converging to x. Thus every point in A is an accumulation point of A. □

1.5 SEIFERT VAN-KAMPEN THEOREM

We mentioned briefly in the introduction that we will compute the complete presentation for a general Antoine's necklace by first computing the group presentation for the top level of the construction, and then adding on the subsequent levels one torus at a time. To compute the presentation at these intermediate steps, we use the following well known theorem.

Theorem 1.5.1. *(Seifert Van-Kampen) Let $X = U_0 \cup U_1$, $U_2 = U_0 \cap U_1$, where U_i are open and path connected. Fix a base point $x_0 \in U_2$. Assume we know the following:*

$$\pi(U_0, x_0) = \{a_1, \ldots, a_m; r_1, \ldots r_n\}$$

$$\pi(U_1, x_0) = \{b_1, \ldots, b_p; s_1, \ldots s_q\}$$

Also let $c_1, \ldots c_t$ be generators for $\pi(U_2, x_0)$. Then

$$\pi(X, x_0) = \{a_1, \ldots, a_m, b_1, \ldots, b_p; r_1, \ldots r_n, s_1, \ldots s_q, u_1{v_1}^{-1}, \ldots u_t{v_t}^{-1}\}$$

where u_i (respectively v_i) is the ith generator, c_i, of $\pi(U_2, x_0)$, written in terms of $a_1 \ldots a_m$ (respectively $b_1 \ldots b_p$).

This theorem will be used extensively throughout this paper. We note that on occasion we will use non-open sets in our applications of the Seifert Van-Kampen Theorem. However, adding an ε-fringe on these sets will solve this problem without changing the complementary groups of the sets in question.

1.6 COMPLEMENTARY GROUPS OF CANTOR SETS

As we want to show that Antoine's necklace is a wild Cantor set, we must first show that the standard Cantor set has trivial complementary group. From Figure 11, it is clear that if K is the standard Cantor set embedded in $\mathbb{R}^3$, $\pi(\mathbb{R}^3 - K)$ is trivial:

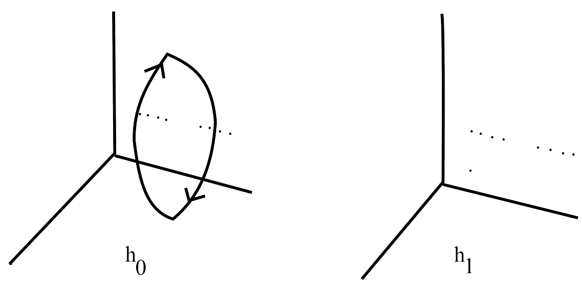

Figure 11: Standard Cantor Set has Trivial Complementary Group in $\mathbb{R}^3$

Indeed, it is obvious that we may simply slide any loop through the Cantor set and then simply contract the loop away.

On the contrary, we will now follow the work done in [7] to prove that the fundamental group of $\mathbb{R}^3 - A$ is not trivial, hence showing A is wild. Assume the same notation as in the previous description of Antoine's necklace. We prove the following:

Lemma 1.6.1. *The induced inclusion homomorphism $i_\star : \pi(\partial V) \to \pi(V - C_1)$ is injective.*

Proof. Since $\partial V \cong S^1 \times S^1$, we know that $\pi(\partial V) \cong \mathbb{Z} \times \mathbb{Z}$; call its generators λ and μ. (Representing longitude and meridian respectively.) Suppose the morphism is not injective, i.e., there exists an element of the form $\lambda^r \mu^s$ such that $i_\star(\lambda^r \mu^s) = 1$. In the *solid* torus V, the element $\lambda^r \mu^s$ is homotopic with λ^r. Therefore r must be zero. But by Proposition 1.2.1, we also know that μ is of infinite order in $\pi(V - C_1)$, so it follows that $s = 0$. Thus the kernel consists of exactly one element, so the morphism is injective. $\square$

Corollary 1.6.1. *For each component $C_{1,j}$ of C_1 $(j = 1, 2, \ldots, 2n)$ the inclusion $\partial C_{1,j} \subset C_{1,j} - int(C_2)$ induces injective fundamental group homomorphisms.*

Corollary 1.6.2. *The inclusion homomorphism $\pi(\partial C_{1,j}) \to \pi(V - int(C_1))$ is injective.*

Now we have the relationships shown in Figure 12.

By the Seifert Van-Kampen Theorem, we know that if the maps $\pi(\partial C_{1,1}) \to \pi(V - int(C_1))$ and $\pi(\partial C_{1,1}) \to (C_{1,1} - int(C_2))$, are injective, then so are the other two maps. Thus we can conclude that

$$\pi(V - int(C_1)) \to \pi((V - int(C_1) \cup (C_{1,1} - int(C_2))) \text{ is injective}$$

In other words, we have concluded that adding one component of $C_1 - int(C_2)$ to $V - C_1$ has simply enlarged the fundamental group.

$$
\begin{tikzcd}
& \Pi(V\text{-int}(C_1)) \arrow[dr] & \\
\Pi(\partial C_{1,1}) \arrow[ur] \arrow[rr] \arrow[dr] & & \Pi(V\text{-int}(C_1))U(C_{1,1}\text{-int}(C_2)) \\
& \Pi(C_{1,1}\text{-int}(C_2)) \arrow[ur] &
\end{tikzcd}
$$

Figure 12: Seifert Van-Kampen Theorem

We may add each of the components in the same fashion to conclude that the inclusion homomorphism $\pi(V - C_1) \to \pi(V - C_2)$ is injective. We can also perform a similar argument with $\mathbb{R}^3$ in place of V.

This argument may be applied repeatedly to show that the following inclusion homomorphisms are injective:

$$\pi(V - C_1) \to \pi(V - C_2) \to \pi(V - C_3) \to \dots$$

$$\pi(\mathbb{R}^3 - C_1) \to \pi(\mathbb{R}^3 - C_2) \to \pi(\mathbb{R}^3 - C_3) \to \dots$$

Proposition 1.6.1. *Each inclusion homomorphism $\pi(\mathbb{R}^3 - C_i) \to \pi(\mathbb{R}^3 - A)$ is injective.*

Proof. Assume to the contrary that there was a loop α in $\mathbb{R}^3 - C_i$ homotopic to a point in $\mathbb{R}^3 - A$. As in the proof of Corollary 1.2.1, we note that the image of the homotopy is a compact set. Therefore if α were contractible to a point, it would shrink to a point in $\mathbb{R}^3 - C_j$ for some $j > i$. But since $\pi(\mathbb{R}^3 - C_i) \to \pi(\mathbb{R}^3 - C_j)$ is injective, it follows that α must have already been trivial in $\mathbb{R}^3 - C_i$, i.e., the morphism is injective. □

Corollary 1.6.3. *The meridian M of V is not homotopically trivial in $\mathbb{R}^3 - A$*

Proof. Any loop in $\mathbb{R}^3 - A$, being compact, lies in $\mathbb{R}^3 - C_i$ for sufficiently large i. By Proposition 1.2.1, the meridian M is not homtopically trivial in $\mathbb{R}^3 - C_i$ for any i. □

Because of this fact, that any loop in $\mathbb{R}^3 - A$ lies in $\mathbb{R}^3 - C_i$ for large enough i, we may conclude the following:

Proposition 1.6.2. *The group $\pi(\mathbb{R}^3 - A)$ is the infinite union of the ascending chain of its subgroups $\pi(\mathbb{R}^3 - C_1) \subset \pi(\mathbb{R}^3 - C_2) \subset \dots$*

From this we may finally arrive at our results:

Corollary 1.6.4. *The group $\pi(\mathbb{R}^3 - A)$ is not finitely generated.*

Proof. If it were, by compactness all of the finitely many generators would lie in $\mathbb{R}^3 - C_i$ for sufficiently large i. We would then conclude that $\mathbb{R}^3 - C_i$ were the whole of $\pi(\mathbb{R}^3 - A)$, a contradiction. □

We have arrived at the following result:

Theorem 1.6.1. *There exists a Cantor Set A, Antoine's necklace, embedded in $\mathbb{R}^3$ so badly that the fundamental group of its complement (knot group) is nontrivial, and in particular, not finitely generated. This is in stark contrast to the standard Cantor Set, K, which when embedded in $\mathbb{R}^3$ has trivial knot group.*

1.7 ANTOINE'S ORIGINAL PROOF

It is known that the fundamental group of the complement of a tame Cantor set embedded in $\mathbb{R}^3$ is trivial, so the fact that A is wild follows immediately from our last result. However, below we give the proof of this fact originally given by Antoine [1].

Proposition 1.7.1. *Antoine's necklace is a wild Cantor set.*

Proof. It is known that any two tame Cantor sets in $\mathbb{R}^3$ are equivalent. It is clear from the above definitions that the standard Cantor set C is tame. Thus we must only show that Antoine's necklace, A, is not equivalent to C. It then would follow that A is not tame. Indeed, assume to the contrary, i.e., there is a homeomorphism h of $\mathbb{R}^3$ onto itself such that $h(C) = A$. Now let S be s sphere separating C, as in Figure 13:

Then $h(S)$ is also a sphere. We claim that $h(S)$ separates A. Indeed, let $C = C_I \cup C_E$, where C_I and C_E denote the interior and exterior of S respectively. Then $h(C) = h(C_I) \cup h(C_E)$ is a disjoint union of two open subsets of $h(C)$ such that $h(C_I)$ is interior to $h(S)$ and $h(C_E)$ is exterior. Thus $A = h(C)$ is separated in $\mathbb{R}^3$ by $h(S)$. But then this sphere misses all of the tori in some step of the construction of A. $\square$

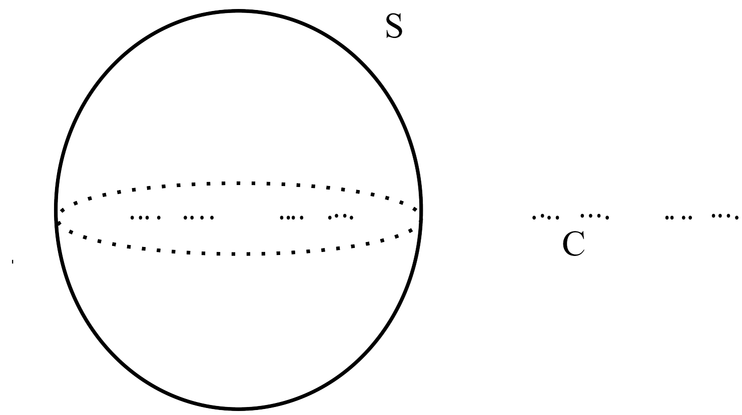

Figure 13: Sphere S Separating Cantor Set C

We now have proven that Antoine's necklace, A, is a wild Cantor set such that when embedded in $\mathbb{R}^3$, the fundamental group of its complement is not finitely generated. We will now discuss the following example of Skora's [9], in which a wild Cantor set $\mathbb{W}$ is constructed in $\mathbb{R}^3$ with simply connected complement. This fully illustrates that Cantor sets are not determined by their complements.

1.8 THE EXAMPLE W

The example $\mathbb{W}$ is drawn in Figure 14.

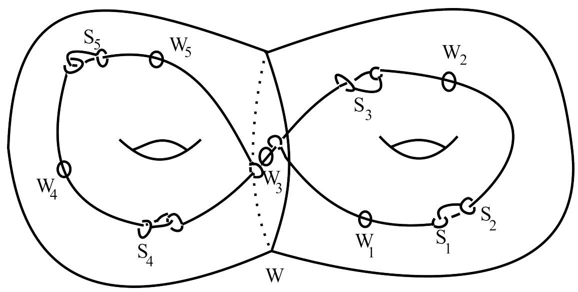

Figure 14: Skora's Cantor Set W

This figure shows a genus 2 handlebody S containing disjoint genus two handlebodies S_i in its interior. Also each S and S_i has a preferred loop W, W_i on its boundary, referred to as its waist.

Formally, we define $\mathbb{W}$ inductively as follows. Let $H_0 = S$, $H_1 = \cup S_i$. Then given H_N, define H_{N+1} as follows. For each component S' of H_N with waist W', there is a homeomorphism

$$h : (S, W) \to (S', W') \text{ such that } h(\cup S_i) = S' \cap H_{N+1}$$

Then the images of the W_i's will be the waists. We also choose the H_N's so that $diam(S_i) \to 0$ as $N \to \infty$. Then $\mathbb{W} = \bigcap H_\alpha$.

Note: We can describe Antoine's necklace in a similar fashion. If V is our solid torus, and C_1 is the union of all solid tori in the first step of the construction, then we may let $f : C_1 \to V$ be a homeomorphism when restricted to any component of C_1. Then let $C_2 = f^{-1}(C_1) = f^{-2}(V)$, and in general, $C_i = f^{-i}(V)$. Therefore the two constructions proceed in the same manner.

Corollary 1.8.1. $\mathbb{W}$ *is a Cantor set.*

Proof. Using the characterization that C is a Cantor set if and only if C is totally disconnected, compact, and perfect, the proof follows exactly in the same manner as the proof outlined for Antoine's necklace. $\square$

Theorem 1.8.1. $\mathbb{W}$ *is wild and* $\pi(\mathbb{R}^3 - \mathbb{W}) = 1$.

Proof. To show that $\mathbb{W}$ is wild, it suffices to show that $\pi(\partial S) \to \pi(S - \mathbb{W})$ is a monomorphism. Indeed, we first recall from our previous work that given a tame Cantor set C, $\pi(V - C) = \mathbb{Z}$, where V is the solid torus. Thus given a solid genus two handlebody, S, $\pi(S - C) = F(a, b)$, the free group on two generators. So by showing $\pi(\partial S) \to \pi(S - \mathbb{W})$ is a monomorphism, we will have proven the inductive step in the proof that $\pi(S - \mathbb{W}) \neq F(a, b)$, which in turn proves that $\mathbb{W}$ is not tame. The proof of this statement itself is similar to the proof given earlier for Lemma 1.6.1.

To see that $\pi(\mathbb{R}^3 - \mathbb{W}) = 1$, it suffices to show how one loop is shrunk. We first note that the linking of the S_i's shown in the diagram above does not affect our computation, as the links are still genus two handlebodies.

Then every meridian of a component S' of some H_N is homotopic in $\mathbb{R}^3 - \mathbb{W}$ to the waist of S'. And each waist of a component of H_N is homotopic in $\mathbb{R}^3 - \mathbb{W}$ to a waist of a component of H_{N-1}. The waist of H_0 is null homotopic in $\mathbb{R}^3 - \mathbb{W}$. By induction, it follows that $\pi(\mathbb{R}^3 - \mathbb{W}) = 1$. (See Figure 15.)

$\square$

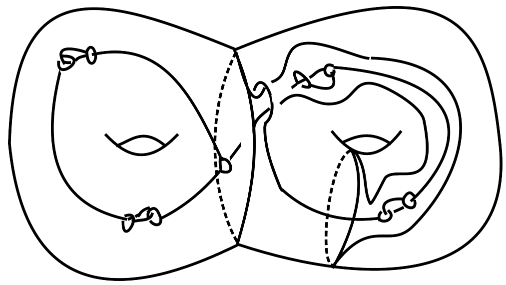

Figure 15: $\pi(\mathbb{R}^3 - \mathbb{W}) = 1$

1.9 SHER'S THEOREM

Lastly, we note the following theorem of Sher, stated in [8].

Theorem 1.9.1. *Two Antoine's necklaces,* $A = \bigcap_{i=1}^{\infty} T_i$ *and* $B = \bigcap_{i=1}^{\infty} R_i$*, are equivalently embedded in* $\mathbb{R}^3$ *if they always have the same number of refining tori, and they link in the same way. That is, there is a homeomorphism* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $h(T_i) = R_i$ *for all* i*.*

In the section of this thesis involving the twisted variant mentioned earlier, we will come back to this result.

2.0 GROUP PRESENTATIONS FOR ANTOINE'S NECKLACES

2.1 THE CASE N=4: THE TOP LEVEL

Our goal is to write a group presentation for a general Antoine's necklace with n tori in each refining step. To simplify the matter, we will first compute the presentation for an Antoine's necklace with $n = 4$ refining tori in each step, and generalize our results.

We will now change our notation slightly. We will use the notation $C_{\langle i \rangle}$, $i \in \{1, \dots 4\}$ to represent one of the four tori on the first level of the construction. When we move to the second level of the construction, the torus $D_{\langle i,j \rangle}$, $i, j \in \{1, \dots 4\}$ will represent the jth torus in the ith original torus. In other words, $D_{\langle 1,2 \rangle}$ represents the second torus inside of the torus $C_{\langle 1 \rangle}$. We will utilize a similar notation for generators. Thus on our top level of the construction, our generators will have the notation $x_{\langle i \rangle}$ and $y_{\langle i \rangle}$, $i \in \{1, \dots 4\}$. When we move to the second level of our construction, the generators $x_{\langle i,j \rangle}$ and $y_{\langle i,j \rangle}$ will correspond to the torus $D_{\langle i,j \rangle}$, etc.

So let $C_{\langle 1 \rangle}, C_{\langle 2 \rangle}, C_{\langle 3 \rangle}$ and $C_{\langle 4 \rangle}$ represent the first four tori in our necklace. Now we want to write the presentation for the top level of our construction, i.e., we want the presentation for:

$$\mathbb{R}^3 - \bigcup_i (intC_{\langle i\rangle})$$

From our preliminary work, we know that a chain of four tori should have 8 generators and relations. We illustrate this in Figure 16. (Note: The inside tori are drawn as circles for the sake of simplicity.)

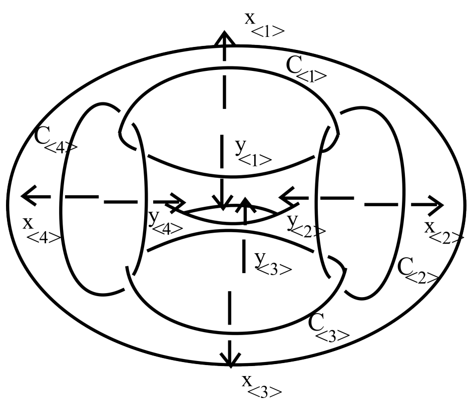

Figure 16: The Case $n = 4$

From Figure 16 we see that the 8 generators are $x_{\langle 1\rangle}, y_{\langle 1\rangle}, \ldots x_{\langle 4\rangle}, y_{\langle 4\rangle}$. Using the Wirtinger presentation defined in our preliminaries, we can determine that the 8 relations are the following:

$$y_{\langle 2\rangle}x_{\langle 1\rangle} = x_{\langle 1\rangle}x_{\langle 2\rangle}, x_{\langle 1\rangle}y_{\langle 2\rangle} = y_{\langle 2\rangle}y_{\langle 1\rangle}, x_{\langle 3\rangle}y_{\langle 2\rangle} = y_{\langle 2\rangle}y_{\langle 3\rangle}, y_{\langle 2\rangle}x_{\langle 3\rangle} = x_{\langle 3\rangle}x_{\langle 2\rangle}$$

$$y_{\langle 4\rangle}x_{\langle 3\rangle} = x_{\langle 3\rangle}x_{\langle 4\rangle}, x_{\langle 3\rangle}y_{\langle 4\rangle} = y_{\langle 4\rangle}y_{\langle 3\rangle}, x_{\langle 1\rangle}y_{\langle 4\rangle} = y_{\langle 4\rangle}y_{\langle 1\rangle}, y_{\langle 4\rangle}x_{\langle 1\rangle} = x_{\langle 1\rangle}x_{\langle 4\rangle}$$

Let us call these relations $w_{\langle 1\rangle}, w_{\langle 2\rangle}, \ldots w_{\langle 8\rangle}$. So we have:

$$\pi(\mathbb{R}^3 - \bigcup_i (intC_{\langle i\rangle})) \quad = \quad \{x_{\langle 1\rangle}, y_{\langle 1\rangle}, \ldots x_{\langle 4\rangle}, y_{\langle 4\rangle}; w_{\langle 1\rangle} \ldots w_{\langle 8\rangle}\}$$

2.2 ADAPTING THE WIRTINGER PRESENTATION FOR A TORUS

In order to continue computing the presentation of the Antoine's necklace with 4 refining tori in each step, we need to compute $\pi(C_{\langle 1\rangle} - \bigcup_j (intD_{\langle 1,j\rangle}))$ and then apply the Seifert Van-Kampen Theorem. To do this, we need to adapt the proof of the Wirtinger presentation, as we are now working inside of the solid torus. We will prove that the group $\pi(C_{\langle 1\rangle} - K)$, where K is a chain, can be presented using the Wirtinger presentation. We illustrate our situation in Figure 17.

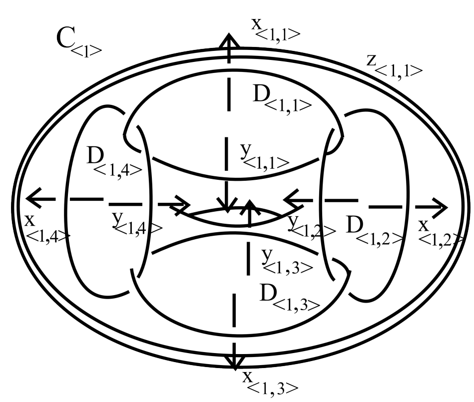

Figure 17: Generators in the Torus $C_{\langle 1\rangle}$

We note that the generator $z_{\langle 1\rangle}$ is trivial on the top level ($z_{\langle 1\rangle} = 1$) and is therefore not included in the presentation given in Section 2.1. However, inside the solid torus drawn in Figure 17, the longitude is no longer trivial, and thus we must add a new generator, $z_{\langle 1,1\rangle}$. Similarly, inside the torus $C_{\langle 2\rangle}$ we would have to add the generator $z_{\langle 1,2\rangle}$, while if we moved down a level, the torus $D_{\langle 1,2\rangle}$ would have a generator $z_{\langle 1,1,2\rangle}$, and so on.

Using Figure 17, we will adapt the proof given in [7] to prove Theorem 2.2.1.

Theorem 2.2.1. *The group $\pi(C_{\langle 1\rangle} - K)$ is generated by the $x_{\langle 1,i\rangle}, y_{\langle 1,i\rangle}, z_{\langle 1,1\rangle}$ and has presentation*

$$\pi(C_{\langle 1\rangle} - K) = \{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \dots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; w_{\langle 1,1\rangle}, \dots w_{\langle 1,8\rangle}\}$$

Note: The notation $w_{\langle 1,i\rangle}$, $i \in \{1, \dots 8\}$ refers to the 8 standard relations listed in Section 2.1, but down one level, inside of the torus $C_{\langle 1\rangle}$. Then the notation $w_{\langle 1,2,i\rangle}$, $i \in \{1, \dots 8\}$ would describe the 8 standard relations down two levels, inside of the torus $D_{\langle 1,2\rangle}$, etc.

Proof. Let $C_{\langle 1\rangle}$ be embedded in the plane such that our chain K lies in the plane $P = \{z = 0\}$, except where it dips down by an epsilon distance at each crossing. In order to apply the Seifert Van-Kampen Theorem, we want to break $C_{\langle 1\rangle} - K$ into 10 pieces, A, $B_1, \dots B_8$, and C.

Let $A = (\{z \geq -\epsilon\} \cap C_{\langle 1\rangle}) - K$. Then the lower boundary of A is the plane $P' = \{z = -\epsilon\}$ with 8 line segments, $\beta_1, \dots \beta_8$ removed. Let B_1 be a solid rectangular box whose top fits on P' and surrounds β_i. But we will remove β_i itself from B_i, and add an arc from the top to our base point, *, missing K. Finally, we let C be the closure of everything below $A \cup B_1 \cup \dots \cup B_8$.

First we note that $\pi(A)$ is the free group generated by $x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}$, as there is a deformation retract from A to a bouquet of circles. Thus

$$\pi(A) = \{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; \qquad \}$$

Now we adjoin B_1 to A. B_1 itself is simply connected, and $B_1 \cap A$ is a rectangle minus β_1, plus the arc to $*$, so $\pi(B_1 \cap A)$ is infinite cyclic, with generator $w_{\langle 1,1\rangle}$. We illustrate this in Figure 18:

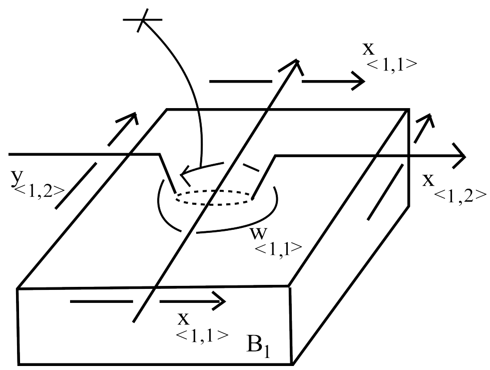

Figure 18: $B_1 \cap A$

We see that writing $w_{\langle 1,1\rangle}$ in terms of the generators of A gives us the relation

$$y_{\langle 1,2\rangle} x_{\langle 1,1\rangle} {x_{\langle 1,2\rangle}}^{-1} {x_{\langle 1,1\rangle}}^{-1} = 1$$

.

Therefore this relation is exactly $w_{\langle 1,1\rangle}$, which is what we claimed. Now applying the Seifert Van-Kampen Theorem, we have:

$$\pi(A \cup B_1) = \{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; w_{\langle 1,1\rangle}\}$$

By a similar argument, we may adjoin $B_2, B_3, \ldots B_8$ and get

$$\pi(A \cup B_1 \cup \cdots \cup B_8) = \{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; w_{\langle 1,1\rangle}, \ldots w_{\langle 1,8\rangle}\}$$

Lastly, we defined C as the closure of everything below $A \cup B_1 \cup \ldots B_8$, plus an arc to *. Thus it is clear that C has only one generator, the longitude $z_{\langle 1,1\rangle}$. Therefore $\pi(C) = \langle z_{\langle 1,1\rangle}\rangle$.

Furthermore, as $(A \cup B_1 \cup \cdots \cup B_8) \cap C$ retracts onto the circle, it also follows that

$$\pi((A \cup B_1 \cup \cdots \cup B_8) \cap C) = \langle z_{\langle 1,1\rangle}\rangle$$

Thus applying the Seifert Van-Kampen Theorem a final time allows us to arrive at the desired conclusion, i.e,

$$\pi(A \cup B_1 \cup \cdots \cup B_8 \cup C) = \pi(C_{\langle 1\rangle} - K) =$$

$$\{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; w_{\langle 1,1\rangle}, \ldots w_{\langle 1,8\rangle}\}$$

Thus we have determined that

$$\pi(C_{\langle 1\rangle} - \bigcup_j (intD_{\langle 1,j\rangle})) = \{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; w_{\langle 1,1\rangle}, \ldots w_{\langle 1,8\rangle}\}$$

$\square$

2.3 THE CASE N=4: THE SECOND LEVEL

Now that we have determined that we may use the Wirtinger presentation inside of the solid torus, we may begin applying the Seifert Van-Kampen Theorem in an attempt to compute the presentation for the Antoine's necklace with 4 refining tori in each step. Let $U_1 = \mathbb{R}^3 - \bigcup_i (intC_{\langle i \rangle})$ and let $V_1 = C_{\langle 1 \rangle} - \bigcup_j (intD_{\langle 1,j \rangle})$. Then we know that

$$\pi(U_1) = \pi(\mathbb{R}^3 - \bigcup_i (intC_{\langle i \rangle})) \quad = \quad \{x_{\langle 1 \rangle}, y_{\langle 1 \rangle}, \ldots x_{\langle 4 \rangle}, y_{\langle 4 \rangle}; w_{\langle 1 \rangle} \ldots w_{\langle 8 \rangle}\}$$

$$\pi(V_1) = \pi(C_{\langle 1 \rangle} - \bigcup_j (intD_{\langle 1,j \rangle})) = \{x_{\langle 1,1 \rangle}, y_{\langle 1,1 \rangle}, \ldots x_{\langle 1,4 \rangle}, y_{\langle 1,4 \rangle}, z_{\langle 1,1 \rangle}; w_{\langle 1,1 \rangle}, \ldots w_{\langle 1,8 \rangle}\}$$

It follows that $U_1 \cap V_1$ is the surface of the torus $C_{\langle 1 \rangle}$, i.e, the hollow torus $S^1 \times S^1$. We know that $U_1 \cap V_1$ has two generators, the meridian $v_{\langle 1,1 \rangle}$, and the longitude, which we will again call $z_{\langle 1,1 \rangle}$. We must represent each of these generators in terms of both the generators for U_1 and the generators for V_1. Figure 19 and Figure 20 represent these computations:

From these pictures, we determine that in U_1, $v_{\langle 1,1 \rangle} = {x_{\langle 1 \rangle}}^{-1}$, while in V_1, $v_{\langle 1,1 \rangle} = {x_{\langle 1,1 \rangle}}^{-1} y_{\langle 1,1 \rangle}$. On the other hand, when it comes to the longitude, in U_1, $z_{\langle 1,1 \rangle} = y_{\langle 2 \rangle} {y_{\langle 4 \rangle}}^{-1}$, and in V_1, $z_{\langle 1,1 \rangle} = z_{\langle 1,1 \rangle}$.

Therefore a final application of the Seifert Van-Kampen Theorem gives us that

$$\pi(U_1 \cup V_1) = \{x_{\langle 1 \rangle}, y_{\langle 1 \rangle}, \ldots x_{\langle 4 \rangle}, y_{\langle 4 \rangle}, x_{\langle 1,1 \rangle}, y_{\langle 1,1 \rangle}, \ldots x_{\langle 1,4 \rangle}, y_{\langle 1,4 \rangle}, z_{\langle 1,1 \rangle};$$
$$w_{\langle 1 \rangle} \ldots w_{\langle 8 \rangle}, w_{\langle 1,1 \rangle}, \ldots w_{\langle 1,8 \rangle}, {x_{\langle 1 \rangle}}^{-1} = {x_{\langle 1,1 \rangle}}^{-1} y_{\langle 1,1 \rangle}, z_{\langle 1,1 \rangle} = y_{\langle 2 \rangle} {y_{\langle 4 \rangle}}^{-1}\}$$

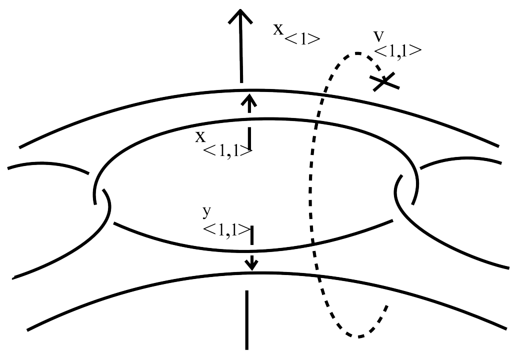

Figure 19: Representing Meridian in Generators for U_1 and V_1

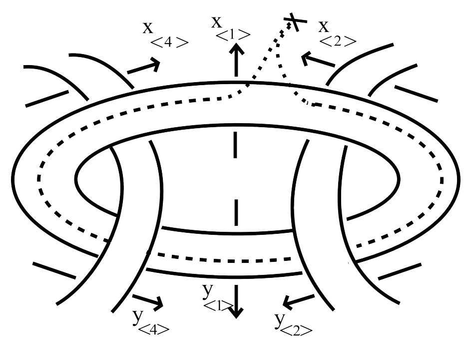

Figure 20: Representing Longitude in Generators for U_1 and V_1

Now we can use the same process as above to attach the other tori on the same level, $C_{\langle 2\rangle}$, $C_{\langle 3\rangle}$, and $C_{\langle 4\rangle}$.

So let $V_2 = C_{\langle 2\rangle} - \bigcup_j (intD_{\langle 2,j\rangle})$. We want to compute the fundamental group of $U_1 \cup V_1 \cup V_2$. To do so, we need to use Seifert Van-Kampen again. We know the presentations for $\pi(U_1 \cup V_1)$ and $\pi(V_2)$. Thus we simply need to compute the fundamental group of

$$(U_1 \cup V_1) \cap V_2.$$

We know that $(U_1 \cup V_1) \cap V_2 = (U_1 \cap V_2) \cup (V_1 \cap V_2)$. Then V_1 and V_2 don't intersect at all, so $V_1 \cap V_2 = \emptyset$ Further, computing the presentation for $U_1 \cap V_2$ differs only slightly from the computation we did previously for $U_1 \cap V_1$. Therefore we have the following:

$$\begin{aligned}\pi(U_1 \cup V_1 \cup V_2) = \{x_{\langle 1\rangle}, y_{\langle 1\rangle}, \ldots x_{\langle 4\rangle}, y_{\langle 4\rangle}, x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle},\\ x_{\langle 2,1\rangle}, y_{\langle 2,1\rangle}, \ldots x_{\langle 2,4\rangle}, y_{\langle 2,4\rangle}, z_{\langle 1,2\rangle};\\ w_{\langle 1\rangle} \ldots w_{\langle 8\rangle}, w_{\langle 1,1\rangle}, \ldots w_{\langle 1,8\rangle}, w_{\langle 2,1\rangle}, \ldots w_{\langle 2,8\rangle},\\ {x_{\langle 1\rangle}}^{-1} = {x_{\langle 1,1\rangle}}^{-1} y_{\langle 1,1\rangle}, z_{\langle 1,1\rangle} = y_{\langle 2\rangle} {y_{\langle 4\rangle}}^{-1},\\ {x_{\langle 2\rangle}}^{-1} = {x_{\langle 2,1\rangle}}^{-1} y_{\langle 2,1\rangle}, z_{\langle 1,2\rangle} = y_{\langle 3\rangle} {y_{\langle 1\rangle}}^{-1}\}\end{aligned}$$

Thus if we let $V_3 = C_{\langle 3\rangle} - \bigcup_j (intD_{\langle 3,j\rangle})$ and $V_4 = C_{\langle 4\rangle} - \bigcup_j (intD_{\langle 4,j\rangle})$, we can add the remaining tori on the first level of the construction. Following the argument outlined above, we have the following presentation:

$$\pi(U_1 \cup V_1 \cup V_2 \cup V_3 \cup V_4) = \{x_{\langle 1\rangle}, y_{\langle 1\rangle}, \dots x_{\langle 4\rangle}, y_{\langle 4\rangle}, x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \dots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle},$$

$$x_{\langle 2,1\rangle}, y_{\langle 2,1\rangle}, \dots x_{\langle 2,4\rangle}, y_{\langle 2,4\rangle}, z_{\langle 1,2\rangle},$$

$$x_{\langle 3,1\rangle}, y_{\langle 3,1\rangle}, \dots x_{\langle 3,4\rangle}, y_{\langle 3,4\rangle}, z_{\langle 1,3\rangle},$$

$$x_{\langle 4,1\rangle}, y_{\langle 4,1\rangle}, \dots x_{\langle 4,4\rangle}, y_{\langle 4,4\rangle}, z_{\langle 1,4\rangle};$$

$$w_{\langle 1\rangle} \dots w_{\langle 8\rangle}, w_{\langle 1,1\rangle}, \dots w_{\langle 1,8\rangle}, w_{\langle 2,1\rangle}, \dots w_{\langle 2,8\rangle},$$

$$w_{\langle 3,1\rangle} \dots w_{\langle 3,8\rangle}, w_{\langle 4,1\rangle}, \dots w_{\langle 4,8\rangle},$$

$$x_{\langle 1\rangle}{}^{-1} = x_{\langle 1,1\rangle}{}^{-1} y_{\langle 1,1\rangle}, z_{\langle 1,1\rangle} = y_{\langle 2\rangle} y_{\langle 4\rangle}{}^{-1},$$

$$x_{\langle 2\rangle}{}^{-1} = x_{\langle 2,1\rangle}{}^{-1} y_{\langle 2,1\rangle}, z_{\langle 1,2\rangle} = y_{\langle 3\rangle} y_{\langle 1\rangle}{}^{-1},$$

$$x_{\langle 3\rangle}{}^{-1} = x_{\langle 3,1\rangle}{}^{-1} y_{\langle 3,1\rangle}, z_{\langle 1,3\rangle} = y_{\langle 4\rangle} y_{\langle 2\rangle}{}^{-1},$$

$$x_{\langle 4\rangle}{}^{-1} = x_{\langle 4,1\rangle}{}^{-1} y_{\langle 4,1\rangle}, z_{\langle 1,4\rangle} = y_{\langle 1\rangle} y_{\langle 3\rangle}{}^{-1}\}$$

2.4 THE CASE N=4: THE REMAINING LEVELS

Now we want to proceed to add the tori in the remaining levels of the construction.

We know that there are four tori inside of the torus $C_{\langle 1\rangle}$, $D_{\langle 1,1\rangle}, D_{\langle 1,2\rangle}, D_{\langle 1,3\rangle}$ and $D_{\langle 1,4\rangle}$. We will use the methods outlined in the previous sections to illustrate how to add on

$$W_1 = D_{\langle 1,1\rangle} - \bigcup_k (intE_{\langle 1,1,k\rangle})$$

We want to compute the fundamental group of

$$U_1 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup W_1$$

Again we will use the Seifert Van-Kampen Theorem. We have already computed the presentation for $\pi(U_1 \cup V_1 \cup V_2 \cup V_3 \cup V_4)$. We also know the presentation for $\pi(W_1)$. Thus as before, we must compute the fundamental group of the intersection of the two sets, the set

$$(U_1 \cup V_1 \cup V_2 \cup V_3 \cup V_4) \cap W_1.$$

However, we know that this is equivalent to computing the fundamental group of:

$$(U_1 \cap W_1) \cup (V_1 \cap W_1) \cup (V_2 \cap W_1) \cup (V_3 \cap W_1) \cup (V_4 \cap W_1)$$

Luckily, these sets are all empty with the exception of $V_1 \cap W_1$. This intersection is exactly the surface of the solid torus $D_{\langle 1,1 \rangle}$.

We know from our previous work that the surface of the solid torus has two generators: the meridian and the longitude. If we draw the diagrams for the meridian and longitude down one level on our presentation, we can use the same logic as before to write the generators for $V_1 \cap W_1$ in terms of the generators of V_1 and W_1. We obtain the following:

$$x_{\langle 1,1 \rangle}^{-1} = x_{\langle 1,1,1 \rangle}^{-1} y_{\langle 1,1,1 \rangle}, \qquad z_{\langle 1,1,1 \rangle} = y_{\langle 1,2 \rangle} y_{\langle 1,4 \rangle}^{-1}$$

Therefore our new presentation becomes the following:

$$
\begin{aligned}
\pi(U_1 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup W_1) = \{&x_{\langle 1\rangle}, y_{\langle 1\rangle}, \dots x_{\langle 4\rangle}, y_{\langle 4\rangle}, \\
&x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \dots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}, \\
&x_{\langle 2,1\rangle}, y_{\langle 2,1\rangle}, \dots x_{\langle 2,4\rangle}, y_{\langle 2,4\rangle}, z_{\langle 1,2\rangle}, \\
&x_{\langle 3,1\rangle}, y_{\langle 3,1\rangle}, \dots x_{\langle 3,4\rangle} y_{\langle 3,4\rangle}, z_{\langle 1,3\rangle}, \\
&x_{\langle 4,1\rangle}, y_{\langle 4,1\rangle}, \dots x_{\langle 4,4\rangle}, y_{\langle 4,4\rangle}, z_{\langle 1,4\rangle}, \\
&x_{\langle 1,1,1\rangle}, y_{\langle 1,1,1\rangle}, \dots x_{\langle 1,1,4\rangle}, y_{\langle 1,1,4\rangle}, z_{\langle 1,1,1\rangle}; \\
&w_{\langle 1\rangle}, \dots w_{\langle 8\rangle}, w_{\langle 1,1\rangle}, \dots w_{\langle 1,8\rangle}, w_{\langle 2,1\rangle}, \dots w_{\langle 2,8\rangle}, \\
&w_{\langle 3,1\rangle}, \dots w_{\langle 3,8\rangle}, w_{\langle 4,1\rangle}, \dots w_{\langle 4,8\rangle}, w_{\langle 1,1,1\rangle}, \dots w_{\langle 1,1,8\rangle} \\
&{x_{\langle 1\rangle}}^{-1} = {x_{\langle 1,1\rangle}}^{-1} y_{\langle 1,1\rangle}, z_{\langle 1,1\rangle} = y_{\langle 2\rangle} {y_{\langle 4\rangle}}^{-1}, \\
&{x_{\langle 2\rangle}}^{-1} = {x_{\langle 2,1\rangle}}^{-1} y_{\langle 2,1\rangle}, z_{\langle 1,2\rangle} = y_{\langle 3\rangle} {y_{\langle 1\rangle}}^{-1}, \\
&{x_{\langle 3\rangle}}^{-1} = {x_{\langle 3,1\rangle}}^{-1} y_{\langle 3,1\rangle}, z_{\langle 1,3\rangle} = y_{\langle 4\rangle} {y_{\langle 2\rangle}}^{-1}, \\
&{x_{\langle 4\rangle}}^{-1} = {x_{\langle 4,1\rangle}}^{-1} y_{\langle 4,1\rangle}, z_{\langle 1,4\rangle} = y_{\langle 1\rangle} {y_{\langle 3\rangle}}^{-1}, \\
&x_{\langle 1,1\rangle}^{-1} = x_{\langle 1,1,1\rangle}^{-1} y_{\langle 1,1,1\rangle}, z_{\langle 1,1,1\rangle} = y_{\langle 1,2\rangle} y_{\langle 1,4\rangle}^{-1}\}
\end{aligned}
$$

By extending this same procedure, we can write the general presentation for an Antoine's necklace with $n = 4$ refining tori in each step. We have:

$$
\begin{aligned}
\pi(\mathbb{R}^3 - A) = \{ & x_{\langle 1\rangle}, y_{\langle 1\rangle}, \ldots x_{\langle 4\rangle}, y_{\langle 4\rangle}, x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}, \\
& x_{\langle 2,1\rangle}, y_{\langle 2,1\rangle}, \ldots x_{\langle 2,4\rangle}, y_{\langle 2,4\rangle}, z_{\langle 1,2\rangle}, \\
& x_{\langle 3,1\rangle}, y_{\langle 3,1\rangle}, \ldots x_{\langle 3,4\rangle} y_{\langle 3,4\rangle}, z_{\langle 1,3\rangle}, \\
& x_{\langle 4,1\rangle}, y_{\langle 4,1\rangle}, \ldots x_{\langle 4,4\rangle}, y_{\langle 4,4\rangle}, z_{\langle 1,4\rangle}, \\
& x_{\langle 1,1,1\rangle}, y_{\langle 1,1,1\rangle}, \ldots x_{\langle 1,1,4\rangle}, y_{\langle 1,1,4\rangle}, z_{\langle 1,1,1\rangle}, \\
& x_{\langle 1,2,1\rangle}, y_{\langle 1,2,1\rangle}, \ldots x_{\langle 1,2,4\rangle}, y_{\langle 1,2,4\rangle}, z_{\langle 1,1,2\rangle}, \ldots \\
& x_{\langle 2,1,1\rangle}, y_{\langle 2,1,1\rangle}, \ldots x_{\langle 2,1,4\rangle}, y_{\langle 2,1,4\rangle}, z_{\langle 1,2,1\rangle}, \ldots \\
& x_{\langle 4,1,1\rangle}, y_{\langle 4,1,1\rangle}, \ldots x_{\langle 4,1,4\rangle}, y_{\langle 4,1,4\rangle}, z_{\langle 1,4,1\rangle}, \ldots \\
& x_{\langle 4,4,1\rangle}, y_{\langle 4,4,1\rangle}, \ldots x_{\langle 4,4,4\rangle}, y_{\langle 4,4,4\rangle}, z_{\langle 1,4,4\rangle}, \\
& x_{\langle 1,1,1,1\rangle}, y_{\langle 1,1,1,1\rangle}, \ldots x_{\langle 1,1,1,4\rangle}, y_{\langle 1,1,1,4\rangle}, z_{\langle 1,1,1,1\rangle}, \ldots \\
& x_{\langle 4,4,4,1\rangle}, y_{\langle 4,4,4,1\rangle}, \ldots x_{\langle 4,4,4,4\rangle}, y_{\langle 4,4,4,4\rangle}, z_{\langle 1,4,4,4\rangle}, \ldots ;
\end{aligned}
$$

$$w_{\langle 1\rangle}, \dots w_{\langle 8\rangle}, w_{\langle 1,1\rangle}, \dots w_{\langle 1,8\rangle}, w_{\langle 2,1\rangle}, \dots w_{\langle 2,8\rangle},$$

$$w_{\langle 3,1\rangle}, \dots w_{\langle 3,8\rangle}, w_{\langle 4,1\rangle}, \dots w_{\langle 4,8\rangle}, w_{\langle 1,1,1\rangle}, \dots w_{\langle 1,1,8\rangle},$$

$$w_{\langle 1,2,1\rangle}, \dots w_{\langle 1,2,8\rangle}, \dots w_{\langle 2,1,1\rangle}, \dots w_{\langle 2,1,8\rangle}, \dots$$

$$w_{\langle 4,1,1\rangle}, \dots w_{\langle 4,1,8\rangle}, \dots w_{\langle 4,4,1\rangle}, \dots w_{\langle 4,4,8\rangle},$$

$$w_{\langle 1,1,1,1\rangle}, \dots w_{\langle 1,1,1,8\rangle}, \dots w_{\langle 4,4,4,1\rangle}, \dots w_{\langle 4,4,4,8\rangle}, \dots$$

$$x_{\langle 2\rangle}{}^{-1} = x_{\langle 2,1\rangle}{}^{-1} y_{\langle 2,1\rangle}, z_{\langle 1,2\rangle} = y_{\langle 3\rangle} y_{\langle 1\rangle}{}^{-1},$$

$$x_{\langle 3\rangle}{}^{-1} = x_{\langle 3,1\rangle}{}^{-1} y_{\langle 3,1\rangle}, z_{\langle 1,3\rangle} = y_{\langle 4\rangle} y_{\langle 2\rangle}{}^{-1},$$

$$x_{\langle 4\rangle}{}^{-1} = x_{\langle 4,1\rangle}{}^{-1} y_{\langle 4,1\rangle}, z_{\langle 1,4\rangle} = y_{\langle 1\rangle} y_{\langle 3\rangle}{}^{-1},$$

$$x^{-1}_{\langle 1,1\rangle} = x^{-1}_{\langle 1,1,1\rangle} y_{\langle 1,1,1\rangle}, z_{\langle 1,1,1\rangle} = y_{\langle 1,2\rangle} y^{-1}_{\langle 1,4\rangle},$$

$$x^{-1}_{\langle 1,2\rangle} = x^{-1}_{\langle 1,2,1\rangle} y_{\langle 1,2,1\rangle}, z_{\langle 1,1,2\rangle} = y_{\langle 1,3\rangle} y^{-1}_{\langle 1,1\rangle}, \dots$$

$$x^{-1}_{\langle 2,1\rangle} = x^{-1}_{\langle 2,1,1\rangle} y_{\langle 2,1,1\rangle}, z_{\langle 1,2,1\rangle} = y_{\langle 2,2\rangle} y^{-1}_{\langle 2,4\rangle} \dots$$

$$x^{-1}_{\langle 4,1\rangle} = x^{-1}_{\langle 4,1,1\rangle} y_{\langle 4,1,1\rangle}, z_{\langle 1,4,1\rangle} = y_{\langle 4,2\rangle} y^{-1}_{\langle 4,4\rangle}, \dots$$

$$, x^{-1}_{\langle 4,4\rangle} = x^{-1}_{\langle 4,4,1\rangle} y_{\langle 4,4,1\rangle}, z_{\langle 1,4,4\rangle} = y_{\langle 4,1\rangle} y^{-1}_{\langle 4,3\rangle},$$

$$x^{-1}_{\langle 1,1,1\rangle} = x^{-1}_{\langle 1,1,1,1\rangle} y_{\langle 1,1,1,1\rangle}, z_{\langle 1,1,1,1\rangle} = y_{\langle 1,1,2\rangle} y^{-1}_{\langle 1,1,4\rangle}, \dots$$

$$x^{-1}_{\langle 4,4,4\rangle} = x^{-1}_{\langle 4,4,4,1\rangle} y_{\langle 4,4,4,1\rangle}, z_{\langle 1,4,4,4\rangle} = y_{\langle 4,4,1\rangle} y^{-1}_{\langle 4,4,3\rangle}, \dots \}$$

Now our goal is to find a way to condense and simplify this presentation.

We recall from earlier that the 8 relations on the top level, $w_{\langle 1\rangle}, \dots w_{\langle 8\rangle}$, are given by the relations listed here:

$$y_{\langle 2\rangle} x_{\langle 1\rangle} = x_{\langle 1\rangle} x_{\langle 2\rangle}, x_{\langle 1\rangle} y_{\langle 2\rangle} = y_{\langle 2\rangle} y_{\langle 1\rangle}, x_{\langle 3\rangle} y_{\langle 2\rangle} = y_{\langle 2\rangle} y_{\langle 3\rangle}, y_{\langle 2\rangle} x_{\langle 3\rangle} = x_{\langle 3\rangle} x_{\langle 2\rangle}$$

$$y_{\langle 4\rangle} x_{\langle 3\rangle} = x_{\langle 3\rangle} x_{\langle 4\rangle}, x_{\langle 3\rangle} y_{\langle 4\rangle} = y_{\langle 4\rangle} y_{\langle 3\rangle}, x_{\langle 1\rangle} y_{\langle 4\rangle} = y_{\langle 4\rangle} y_{\langle 1\rangle}, y_{\langle 4\rangle} x_{\langle 1\rangle} = x_{\langle 1\rangle} x_{\langle 4\rangle}$$

We can now write a general form for these relations on any level of our presentation. We note that from here on we use the notation:

$$\alpha^n = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle, \;\; \alpha_i \in \{1, \ldots 4\}$$

We also use the convention that $\alpha^0 = \langle\rangle$. We have:

$$\begin{aligned}
w_{\langle \alpha^{n-1},1\rangle} &: y_{\langle \alpha^{n-1},2\rangle} x_{\langle \alpha^{n-1},1\rangle} = x_{\langle \alpha^{n-1},1\rangle} x_{\langle \alpha^{n-1},2\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},2\rangle} &: x_{\langle \alpha^{n-1},1\rangle} y_{\langle \alpha^{n-1},2\rangle} = y_{\langle \alpha^{n-1},2\rangle} y_{\langle \alpha^{n-1},1\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},3\rangle} &: x_{\langle \alpha^{n-1},3\rangle} y_{\langle \alpha^{n-1},2\rangle} = y_{\langle \alpha^{n-1},2\rangle} y_{\langle \alpha^{n-1},3\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},4\rangle} &: y_{\langle \alpha^{n-1},2\rangle} x_{\langle \alpha^{n-1},3\rangle} = x_{\langle \alpha^{n-1},3\rangle} x_{\langle \alpha^{n-1},2\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},5\rangle} &: y_{\langle \alpha^{n-1},4\rangle} x_{\langle \alpha^{n-1},3\rangle} = x_{\langle \alpha^{n-1},3\rangle} x_{\langle \alpha^{n-1},4\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},6\rangle} &: x_{\langle \alpha^{n-1},3\rangle} y_{\langle \alpha^{n-1},4\rangle} = y_{\langle \alpha^{n-1},4\rangle} y_{\langle \alpha^{n-1},3\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},7\rangle} &: x_{\langle \alpha^{n-1},1\rangle} y_{\langle \alpha^{n-1},4\rangle} = y_{\langle \alpha^{n-1},4\rangle} y_{\langle \alpha^{n-1},1\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N} \\
w_{\langle \alpha^{n-1},8\rangle} &: y_{\langle \alpha^{n-1},4\rangle} x_{\langle \alpha^{n-1},1\rangle} = x_{\langle \alpha^{n-1},1\rangle} x_{\langle \alpha^{n-1},4\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N}
\end{aligned}$$

With this notation in place, we will lastly consider the relations that connect the levels together. We can again write a general form for the relations that connect the levels together, which we will call c and d:

$$c_{\langle \alpha^{n-1}\rangle} : x^{-1}_{\langle \alpha^{n-1}\rangle} = x^{-1}_{\langle \alpha^{n-1},1\rangle} y_{\langle \alpha^{n-1},1\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N}$$

Lastly we define the relation d.

$$d_{\langle \alpha^{n-1}\rangle} : z_{\langle 1,\alpha^{n-1}\rangle} = y_{\langle \alpha^{n-2},\alpha_{n-1}+1(mod4)\rangle} y^{-1}_{\langle \alpha^{n-2},\alpha_{n-1}-1(mod4)\rangle}, \;\; \alpha_i \in \{1, \ldots 4\}, n \in \mathbb{N}$$

With this notation we can now give the presentation in a much more compact form. Therefore the presentation for the Antoine's necklace becomes:

$$\pi(\mathbb{R}^3 - A) = \{x_{\langle \alpha^n \rangle}, y_{\langle \alpha^n \rangle}, z_{\langle 1, \alpha^{n-1} \rangle} : \alpha_i \in \{1, \ldots, 4\},\ \ \forall n \in \mathbb{N}; z_{\langle 1 \rangle} = 1,$$
$$w_{\langle \alpha^{n-1}, \beta \rangle}, c_{\langle \alpha^{n-1} \rangle}, d_{\langle \alpha^{n-1} \rangle} : \alpha_i \in \{1, \ldots, 4\}, \beta \in \{1, \ldots, 8\}\ \ \forall n \in \mathbb{N}\}$$

2.5 THE GENERAL CASE

We know from our preliminary work that a chain of $2m$ tori should have $4m$ generators and $4m$ relations. Using this fact, we can generalize the presentation above to give a presentation for an Antoine's necklace with $2m$ refining tori on each level. We have:

$$\pi(\mathbb{R}^3 - A) = \{x_{\langle \alpha^n \rangle}, y_{\langle \alpha^n \rangle}, z_{\langle 1, \alpha^{n-1} \rangle} : \alpha_i \in \{1, \ldots, 2m\},\ \ \forall n \in \mathbb{N}; z_{\langle 1 \rangle} = 1,$$
$$w_{\langle \alpha^{n-1}, \beta \rangle}, c_{\langle \alpha^{n-1} \rangle}, d_{\langle \alpha^{n-1} \rangle} : \alpha_i \in \{1, \ldots, 2m\}, \beta \in \{1, \ldots, 4m\}\ \ \forall n \in \mathbb{N}\}$$

3.0 THE TWISTED VARIANT

3.1 CONSTRUCTION AND PROPERTIES

As mentioned in the introduction, our goal was to compare the presentation given for the Antoine's necklace listed above with a twisted variant. In this variant, we give one of the refining tori a double twist, as shown in Figure 21:

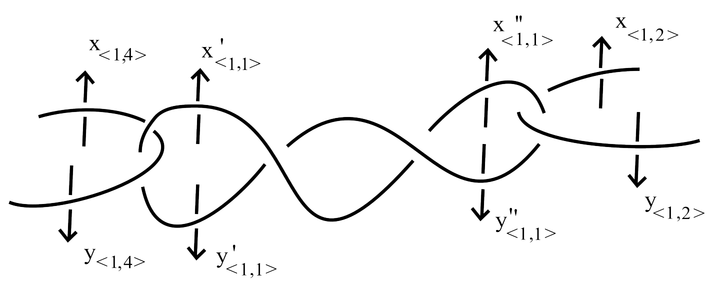

Figure 21: The Double Twist

Here we have taken the Antoine's necklace with 4 refining tori on each level that we worked with in the past, and added the double twist to the torus $D_{\langle 1,1\rangle}$ on the second level of the construction. Because of this extra twist, we have introduced another set of generators. Now the torus $D_{\langle 1,1\rangle}$ has the four generators:

$$x'_{\langle 1,1\rangle}, y'_{\langle 1,1\rangle}, x''_{\langle 1,1\rangle}, y''_{\langle 1,1\rangle}$$

Therefore our relations will change slightly as well. We note that the 8 relations on the second level inside the torus $C_{\langle 1\rangle}$ in the standard Antoine's necklace, $w_{\langle 1,1\rangle}, \ldots w_{\langle 1,8\rangle}$, are given by:

$$\begin{aligned}
y_{\langle 1,2\rangle}x_{\langle 1,1\rangle} &= x_{\langle 1,1\rangle}x_{\langle 1,2\rangle}, \quad & x_{\langle 1,1\rangle}y_{\langle 1,2\rangle} &= y_{\langle 1,2\rangle}y_{\langle 1,1\rangle},\\
x_{\langle 1,3\rangle}y_{\langle 1,2\rangle} &= y_{\langle 1,2\rangle}y_{\langle 1,3\rangle}, \quad & y_{\langle 1,2\rangle}x_{\langle 1,3\rangle} &= x_{\langle 1,3\rangle}x_{\langle 1,2\rangle}\\
y_{\langle 1,4\rangle}x_{\langle 1,3\rangle} &= x_{\langle 1,3\rangle}x_{\langle 1,4\rangle}, \quad & x_{\langle 1,3\rangle}y_{\langle 1,4\rangle} &= y_{\langle 1,4\rangle}y_{\langle 1,3\rangle},\\
x_{\langle 1,1\rangle}y_{\langle 1,4\rangle} &= y_{\langle 1,4\rangle}y_{\langle 1,1\rangle}, \quad & y_{\langle 1,4\rangle}x_{\langle 1,1\rangle} &= x_{\langle 1,1\rangle}x_{\langle 1,4\rangle}
\end{aligned}$$

Our new relations now become:

$$\begin{aligned}
y_{\langle 1,2\rangle}x''_{\langle 1,1\rangle} &= x''_{\langle 1,1\rangle}x_{\langle 1,2\rangle}, \quad & x''_{\langle 1,1\rangle}y_{\langle 1,2\rangle} &= y_{\langle 1,2\rangle}y''_{\langle 1,1\rangle},\\
x_{\langle 1,3\rangle}y_{\langle 1,2\rangle} &= y_{\langle 1,2\rangle}y_{\langle 1,3\rangle}, \quad & y_{\langle 1,2\rangle}x_{\langle 1,3\rangle} &= x_{\langle 1,3\rangle}x_{\langle 1,2\rangle}\\
y_{\langle 1,4\rangle}x_{\langle 1,3\rangle} &= x_{\langle 1,3\rangle}x_{\langle 1,4\rangle}, \quad & x_{\langle 1,3\rangle}y_{\langle 1,4\rangle} &= y_{\langle 1,4\rangle}y_{\langle 1,3\rangle},\\
x'_{\langle 1,1\rangle}y_{\langle 1,4\rangle} &= y_{\langle 1,4\rangle}y'_{\langle 1,1\rangle}, \quad & y_{\langle 1,4\rangle}x'_{\langle 1,1\rangle} &= x'_{\langle 1,1\rangle}x_{\langle 1,4\rangle}
\end{aligned}$$

Here we will denote the different relations by $w'_{\langle 1,i\rangle}$. We will also have a ninth and tenth relation occurring at the twists in the twisted torus. These relations are the following:

$$x'_{\langle 1,1\rangle}y''_{\langle 1,1\rangle} = y'_{\langle 1,1\rangle}x'_{\langle 1,1\rangle}$$

$$y''_{\langle 1,1\rangle}x'_{\langle 1,1\rangle} = x''_{\langle 1,1\rangle}y''_{\langle 1,1\rangle}$$

So we can compare this new variant to the original case. We recall that the presentation of the second level of the construction inside the torus $C_{\langle 1\rangle}$ was originally given by:

$$\pi(C_{\langle 1\rangle} - \bigcup_j (intD_{\langle 1,j\rangle})) = \{x_{\langle 1,1\rangle}, y_{\langle 1,1\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle}; w_{\langle 1,1\rangle}, \ldots w_{\langle 1,8\rangle}\}$$

Now after giving a double twist to the torus $D_{\langle 1,1\rangle}$, we have:

$$\begin{aligned}\pi(C_{\langle 1\rangle} - \bigcup_j (intD'_{\langle 1,j\rangle})) = \{&x'_{\langle 1,1\rangle}, y'_{\langle 1,1\rangle}, x''_{\langle 1,1\rangle}, y''_{\langle 1,1\rangle}, x_{\langle 1,2\rangle}, y_{\langle 1,2\rangle}, \ldots x_{\langle 1,4\rangle}, y_{\langle 1,4\rangle}, z_{\langle 1,1\rangle};\\ &w'_{\langle 1,1\rangle}, w'_{\langle 1,2\rangle}, w_{\langle 1,3\rangle} \ldots w_{\langle 1,6\rangle}, w'_{\langle 1,7\rangle}, w'_{\langle 1,8\rangle},\\ &x'_{\langle 1,1\rangle}y''_{\langle 1,1\rangle} = y'_{\langle 1,1\rangle}x'_{\langle 1,1\rangle}, y''_{\langle 1,1\rangle}x'_{\langle 1,1\rangle} = x''_{\langle 1,1\rangle}y''_{\langle 1,1\rangle}\}\end{aligned}$$

3.2 COMPARING PRESENTATIONS

Despite looking markedly different, we can actually show that these presentations are isomorphic. To do so, we fix a homeomorphism, h, from $C_{\langle 1\rangle}$ to itself. We illustrate the homeomorphism in Figure 22.

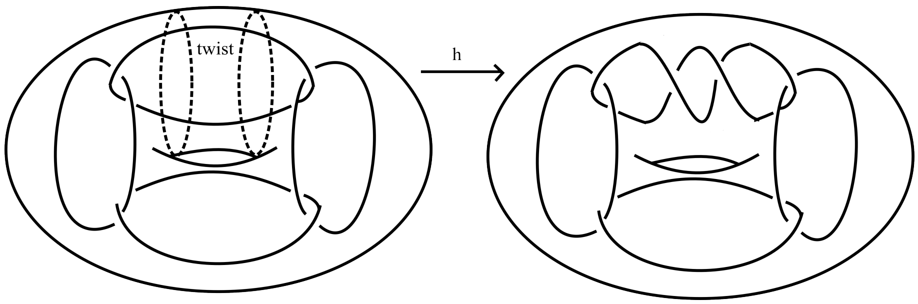

Figure 22: The Homeomorphism Between the Torus $C_{\langle 1\rangle}$ and Itself

Since the homeomorphism h carries $\bigcup_j (intD_{\langle 1,j\rangle})$ to the twisted variant, it follows that $C_{\langle 1\rangle} - \bigcup_j (intD_{\langle 1,j\rangle})$ and $C_{\langle 1\rangle} - \bigcup_j (intD'_{\langle 1,j\rangle})$ are homeomorphic via $\tilde{h} = h \mid (C_{\langle 1\rangle} - \bigcup_j (intD_{\langle 1,j\rangle}))$

In turn, this induces an isomorphism of fundamental groups.

$$\tilde{h}_* : \pi(C_{\langle 1\rangle} - \bigcup_j (intD_{\langle 1,j\rangle})) \to \pi(C_{\langle 1\rangle} - \bigcup_j (intD'_{\langle 1,j\rangle}))$$

given by $\tilde{h}_*([f]) = [h \circ f]$

We can actually explicitly say what the induced morphism $\tilde{h}_*$ does in terms of the known presentation. It is obvious from Figure 22 that

$$\tilde{h}_*(x_{\langle 1,i\rangle}) = x_{\langle 1,i\rangle}, \quad i \neq 1 \qquad \tilde{h}_*(y_{\langle 1,i\rangle}) = y_{\langle 1,i\rangle}, \quad i \neq 1$$

Therefore we have to check only a few cases. It is easy to check the following computations:

$$
\begin{aligned}
\tilde{h}_*(z_{\langle 1,1\rangle}) &= z_{\langle 1,1\rangle} {x'_{\langle 1,1\rangle}}^{-1} y'_{\langle 1,1\rangle} \\
\tilde{h}_*(x_{\langle 1,1\rangle}) &= {y''_{\langle 1,1\rangle}}^{-1} x''_{\langle 1,1\rangle} \\
\tilde{h}_*(y_{\langle 1,1\rangle}) &= x'_{\langle 1,1\rangle} y''_{\langle 1,1\rangle}
\end{aligned}
$$

Therefore, on one level of the construction, we can show that the original Antoine's necklace and the twisted variant actually have isometric complementary groups.

Unfortunately, when we proceed to try to use the Seifert Van-Kampen Theorem to add on additional levels in the construction, we run into some trouble. If U_1 represents the top level of the twisted Antoine necklace construction, and V_1 represents one of the sets of tori on the second tier, then to compute the presentation for $U_1 \cup V_1$, we must first determine the generators for $U_1 \cap V_1$ and write them in terms of the generators of both U_1 and V_1. As above, we determine that $U_1 \cap V_1$ has two generators: a meridian and a longitude. It is in the computation of the longitude that we encounter a problem.

From Figure 23 we can see that by writing the longitude in terms of the generators for both the top and second tier, we would have the relation

$$z_{\langle 1,1,1\rangle} = y_{\langle 1,2\rangle}(x'_{\langle 1,1\rangle})^{-1} y^{-1}_{\langle 1,4\rangle}(y''_{\langle 1,1\rangle})^{-1}$$

Clearly this relation differs from the relations obtained from adjoining levels in the construction of the standard Antoine's necklace. We therefore know that the two variations are actually inequivalent.

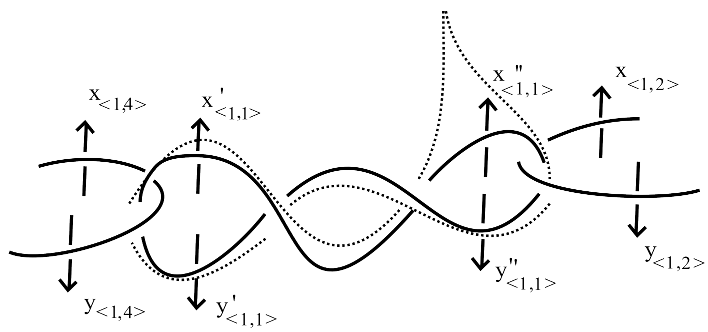

Figure 23: Tracing the Longitude in the Double Twist

4.0 OPEN PROBLEMS

Many open problems surround Cantor sets in $\mathbb{R}^3$, their complements and complementary groups. We consider a few of particular interest.

Complements Determine? Does the complement (up to homeomorphism) of a Cantor set in $\mathbb{R}^3$, determine the Cantor set (up to equivalence)?

The corresponding question for knots has a positive answer (Cameron & Luecke). But the proof techniques don't transfer to Cantor sets. For links the answer is negative. But again the examples of Whitehead and others don't help with Cantor sets. In particular, the complement of a link has 'large holes' around which things can be moved. This is not true for Cantor sets.

Complementary Groups Determine? Skora's example of a wild Cantor set with simply connected complement shows that the complementary group of a Cantor set *does not* determine the Cantor set. What if we restrict to certain classes of Cantor sets? For example, Antoine necklace type Cantor sets?

Are there two Antoine type necklaces with isomorphic complementary groups?

Complementary Groups Say What? In general, what can we determine about a Cantor set from its complementary group? More particularly, what about Antoine necklaces?

From the complementary group, is it possible to determine the number of rings in the top level? Lower levels? Whether any ring is twisted?

Other Invariants Perhaps the complementary group is not powerful enough to distinguish Antoine necklaces. What might work instead?

Can Antoine type Cantor sets be classified by knot/link invariants like the Jones polynomial? What about more geometric invariants?

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