ESSAYS ON MACROECONOMIC DYNAMICS

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ABSTRACT

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This thesis deals with macroeconomic dynamics. In chapter 1, I study a one-sector growth model with endogenous discount rate of the sort proposed by Meng [2006]. I extend the model into a heterogeneous agents model with respect to initial wealth, and investigate whether the wealth distribution may converge to a degenerate distribution. I find that if an agent's decision only depends on his or her reference group and if consumption is more important in discounting than income around the steady state, then convergence to a degenerate distribution is a unique solution. Furthermore, if an agent's decision depends on average variables of overall society, I find that there exists a continuum of steady states.

In chapter 2, I introduce three mechanisms into otherwise standard Aiyagari [1994] model to generate a realistic wealth distribution. The three mechanisms include: i) a wealth-dependent shock: labor income shock is wealth-dependent; ii) misspecification: people do not take into account the dependence of the labor income process on wealth when they make consumption decisions; iii) status-seeking from some threshold: there is a direct utility gain from being wealthy. The main findings are as follows: i) Wealth-dependent labor income shock with misspecification helps to explain wealth concentration but cannot fully explain the share of the top 1% in wealth distribution. ii) In the full model with status-seeking, the share of top 1% becomes closer to the data.

In chapter 3, I build a simple model (two-dimensional discrete dynamical system) to study the interactive dynamics of short-term nominal interest rates of the U.S. and international risk appetite. Main implications from the research are the followings: First, strong interaction between short-term nominal interest rates of the U.S. and international risk appetite can induce bifurcations of the dynamical system: stable fixed point to limit cycle and then to chaos. Second, a numerical experiment suggests two possible explanations for rising variance ratios: the reduction of random shock and the bifurcation of a dynamical system. This finding hints the potential of complexity measures (such as Lyapunov exponent and permutation entropy) as early warning signals.

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PREFACE

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1.0 A CLOSER LOOK AT WEALTH DYNAMICS OF A ONE-SECTOR GROWTH MODEL WITH ENDOGENOUS DISCOUNT RATE

1.1 INTRODUCTION

Generally speaking, in simple macroeconomic models, an agent's utility depends only on his or her own consumption. Recently, social factors which have influence on consumption decisions have been extensively investigated. For example, people might be interested in the average consumption in the economy ("Keeping up with Jones"). Or, people might take into account the wealth level of the reference group ("status seeking"). In this paper, I investigate the consequence of these two motives in a model. Particularly, I am interested in the dynamics which emerges from these motives while focusing on the issue of wealth convergence. I think that the issue is particulary important in *economic integration*.[1]

Concretely speaking, in this paper, I study long-run wealth dynamics of a one-sector growth model with endogenous discounting suggested by Meng [2006]. Meng [2006] investigates the indeterminacy problem of equilibrium due to the opposing forces of average consumption and average income in discounting function in the framework of a representative agent model.[2] I extend the model to study the issue of wealth convergence by introducing

[1] On the other hand, the recent literature on "symmetry-breaking" such as Matsuyama [2004] pays more attention to endogenous inequality after international financial integration.

[2] Here, I briefly review the concept of indeterminacy. In a discrete time framework under the assumption of rational expectation, let A be the matrix governing dynamics of a model. Let $n_{||>1}$ be the number of eigenvalues of A outside the unit circle and m be the number of non-predetermined variables of the model. According to Blanchard and Kahn [1980], if $n_{||>1} = m$, the solution is unique. If $n_{||>1} < m$ there exist infinitely many solutions. In the literature, the latter case is also called 'indeterminate.' In a continuous time framework, $n_{||>1}$ is replaced by n_+, the number of eigenvalues with a positive real part. Nice discussions about indeterminacy problem are contained in Azariadis [1993, 28.5], Benhabib and Farmer [1999], and Mitra and Nishimura [2001], among others. Regarding the indeterminacy problem of rational expectation equilibrium (REE), related with interest rate rules, including the Taylor rule, I refer to Bullard and Mitra

heterogeneous agents with respect to initial wealth and examining the stability of the (symmetric) steady states in the model.

There are at least two ways of modeling social factors. One is to modify the utility function by adding social variables. [3] Another approach is to replace the standard constant discount rate with variable one which depends on some social variables. In this paper I consider the second approach.

This paper is related with the literature on an endogenous discounting rate starting with Uzawa [1968]. The general properties of endogenous discounting specification, including recursive property, are extensively analyzed by Epstein and Hynes [1983], Epstein [1987], and Obstfeld [1990] under the assumption that discounting depends on his or her own consumption profile. Drugeon [1998] studies a model including average consumption in discounting.

This paper is also related with the literature on long-run wealth dynamics. Wealth inequality has been one of the most controversial issues in our society. Since wealth is a stock variable determined by a flow variable saving, wealth dynamics is closely connected with saving decisions, in other words, consumption decisions. In an important paper, Stiglitz [1969] investigates economic factors which determine asymptotic convergence of wealth by using the Solow growth model. Given the saving function, he shows the convergence of wealth under the assumption of concavity of production function. But in that model, the consumer does not optimize her consumption. In optimal growth literature, Ramsey [1928] conjectures extreme wealth distribution in competitive equilibrium (where the agents with the lowest discounting rate hold the whole wealth in the economy) when agents have different discounting rates, which is confirmed by Becker [1980].[4] Recently, Bliss [2004] highlights these different results about convergence. Long and Shimomura [2004] shows that catching up by the poor may happen even under optimal consumption if status-seeking behavior is included in a model.

The main findings in this paper are as follows: In the heterogeneous agents model, I

[2007].

[3] For example, the utility function in Nakamoto [2009] includes private consumption, average consumption, and private capital.

[4] By dropping the assumption of constant discount rate, Epstein and Hynes [1983] and Lucas and Stokey [1984] show that the steady state where more patient agent holds a little more asset exists.

find that convergence of wealth can be a unique solution in optimization framework with the endogenous discounting which depends on social factors, if agents decision depends only on his or her reference group rather than on an economy-wide average and if consumption is more important in discounting around the steady state than income.[5]

In addition, an interesting question is whether the opposing forces, keeping up with Jones and status seeking, can induce the emergence of limit cycles. In Appendix C, I perturb a representative agent model in Meng [2006] and, by employing the Hopf bifurcation theorem and numerical investigation, I find that there exists an unstable limit cycle. I view the finding as an example of the so-called "corridor stability."

In section 1.2, I extend a model in Meng [2006] into a heterogeneous agents model with respect to initial wealth, and study the issue of wealth convergence. In particular, I consider different discounting functions between groups in subsection 1.2.2, and in subsection 1.2.3, I assume the same discounting function for each group.

1.2 HETEROGENEOUS AGENTS WITH ENDOGENOUS DISCOUNTING

I assume that there exist two types of agents in the economy. Agent of type i has initial endowment k_{i0} with i=1, 2. Labor supply is identical. I assume integrated capital and labor markets. Thus, the rate of return on capital and wage are identical to everybody. This set-up is interesting in analyzing the effect of intertemporal consumption choices on wealth distribution because other objective economic factors are the same to everybody except initial endowment. In this section, I consider several possible discounting functions. My primary concern in this section is the local stability of the symmetric steady state (s.s.s.), i.e., the situation where consumption and capitals are equalized in the long run.

[5]In other words, in that case, the model has saddle path stability proposed by Sargent and Wallace [1973]. This paper does not deal with the issue of coordination regarding saddle path stability. For the issue, I refer to Evans and Guesnerie [2005], for instance.

1.2.1 General set-up

The model is an extended version of Meng [2006] with heterogeneous agents. I consider one sector growth model with an endogenous discounting rate. I will assume that the instantaneous utility function is the CRRA (constant relative risk aversion).

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \tag{1.1}$$

For simplicity, I assume that the depreciation rate $\delta(t) = 0$ for all t$\geq$0. I assume the standard neoclassical aggregate production function,

$$y = f(k) = Ak^{\theta} \tag{1.2}$$

where k is the per capita capital:

$$k = \pi^1 \bar{k}^1 + \pi^2 \bar{k}^2, \ \pi^i \in (0,1), \pi^1 + \pi^2 = 1 \tag{1.3}$$

where π^i denotes the fraction of type i. Then in a competitive market, the rate of interest r and the wage rate w are determined by

$$r = f'(k) = A\theta k^{\theta-1} \tag{1.4}$$

$$w = f(k) - f'(k)k = A(1-\theta)k^{\theta} \tag{1.5}$$

I look at two different cases due to different discounting functions.

1.2.2 Case 1; different discounting

Agent of type i, i=1,2, solves the following utility maximization problem;

$$\max_{c^i(t)} \int_0^\infty u(c^i(t)) e^{[-\int_0^t \rho(\bar{c}^i(s), \bar{y}^i(s))ds]} dt \tag{1.6}$$

subject to

$$\dot{k}^i = rk^i + w - c^i, \;\; k(0) = k_{i0} \tag{1.7}$$

where $\bar{c}^i(s)$ and $\bar{y}^i(s)$ are average consumption and income of type i at time s, respectively. Note that the rate of time preference is $\rho(\bar{c}^i(s), \bar{y}^i(s))$, (Meng [2006, 2675]). The assumption that ρ is a function of average consumption and income of each type reflects the idea that an individual's consumption decision is affected by social factors. Later, in the Section 1.2.3, I also consider the case where ρ is a function of economy-wide average consumption and income. Let $\bar{k}^i(s)$ are average per capita capital of type i at time s. Then,

$$\bar{y}^i(s) = w + r\bar{k}^i(s) \tag{1.8}$$

Since there exists a continuum of agents, each agent takes $\bar{c}^i(s)$ and $\bar{y}^i(s)$ as exogenous variables. r is the rate of interest and w is the wage rate.

The current-value Hamiltonian for this problem is given by

$$H_i = u(c^i) + \lambda_i(rk^i + w - c^i), i = 1, 2 \tag{1.9}$$

where $\lambda_i(t) = \mu_i(t) e^{[\int_0^t \rho(\bar{c}^i(s), \bar{y}^i(s))ds]}$, $\mu_i(t)$ is the present-value lagrange multiplier. The necessary conditions for maximization are

$$\begin{aligned} u'(c^i) &= \lambda_i \\ \dot{\lambda}_i &= \rho(\bar{c}^i, \bar{y}^i)\lambda_i - r\lambda_i, i = 1, 2 \end{aligned} \tag{1.10}$$

with the transversality conditions

$$\lim_{t\to\infty} \lambda_i(t) e^{[-\int_0^t \rho(\bar{c}^i(s), \bar{y}^i(s))ds]} k_i(t) = 0, i = 1, 2 \tag{1.11}$$

From 1.10, the Euler equation is

$$\dot{c^i} = \frac{u'(c^i)}{u''(c^i)}[\rho(\bar{c}^i, \bar{y}^i) - r], i = 1, 2. \tag{1.12}$$

DEFINITION 1.2.1. A *Competitive equilibrium* is $\{r(t), w(t), c^i(t), k^i(t)\}_{t\geq 0, i=1,2}$ satisfying Eq. 1.3, 1.4, 1.5, 1.7, 1.11, and 1.12 with the equilibrium conditions ; $c^i(t) = \bar{c}^i(t)$, $k^i(t) = \bar{k}^i(t)$, (i=1,2).

$\rho(\cdot,\cdot)$ is affine to each variable as in Meng [2006, 2678].

$$\rho(\bar{c}^i(t), \bar{y}^i(t)) = \alpha\bar{c}^i(t) - \beta\bar{y}^i(t) + \gamma, \alpha, \beta, \gamma > 0 \tag{1.13}$$

Eq. 1.13 implies that if the average income of a reference group becomes higher, people discount the future less and that if the average consumption of a reference group becomes higher, people discount the future more. By using 1.4, 1.5, 1.8, and the equilibrium conditions that $c^i = \bar{c}^i, k^i = \bar{k}^i$, $(i = 1, 2)$, the model becomes as follows:

$$\dot{c^i} = -\frac{c^i}{\sigma}\big[\alpha c^i - \beta[A\theta k^{\theta-1}(k^i - k) + Ak^\theta] + \gamma - \theta A k^{\theta-1}\big], \tag{1.14}$$

$$\dot{k^i}(t) = A\theta k^{\theta-1}(k^i - k) + Ak^\theta - c^i, \tag{1.15}$$

where $k = \pi^1 k^1 + \pi^2 k^2$, $\pi^i \in (0, 1), \pi^1 + \pi^2 = 1$, $i = 1, 2$.

1.2.2.1 Basic results First, I show that in this model, the symmetric steady states are the only steady states.

Proposition 1.2.1. *If $\alpha \neq \beta$, then the symmetric steady states, $c^1 = c^2 = c^* = A(k^*)^\theta$ and $k^* = k^1 = k^2$, are the only steady states in the model.*

Proof. Note that the dynamics of capital per capita in an economy is given by

$$\dot{k} = \pi^1\dot{k}^1 + \pi^2\dot{k}^2, \tag{1.16}$$

With the steady state conditions, $\dot{k^i} = 0$, $i = 1, 2$ (hence $\dot{k} = 0$), we can see that from 1.15 and 1.16

$$Ak^\theta = c \tag{1.17}$$

should hold at a steady state where $c = \pi^1 c^1 + \pi^2 c^2$, per capita consumption in the economy. Similarly, the dynamics of per capita consumption in the economy is given by

$$\dot{c} = \pi^1\dot{c}^1 + \pi^2\dot{c}^2, \tag{1.18}$$

Since $\dot{c^i} = 0, i = 1, 2$ (hence $\dot{c} = 0$), should hold as well, we have from 1.14 and 1.18

$$\alpha c - \beta A k^{\theta} + \gamma - \theta A k^{\theta-1} = 0 \tag{1.19}$$

Then, (k^*, c^*) at steady states can be determined by 1.17 and 1.19. Note that I have the same steady state conditions for (k,c) as in Meng [2006] due to the linearity of the discounting function. *Given* (k^*, c^*), by imposing the conditions of $\dot{c^i} = 0$ and $\dot{k^i} = 0$ in Eqs. 1.14 and 1.15, Eqs. 1.14 and 1.15 become a system of linear equations with two unknowns, c^i and k^i. In other words, we have the following system of linear equations:

$$\begin{pmatrix} 1 & -A\theta(k^*)^{\theta-1} \\ -\alpha & \beta A\theta(k^*)^{\theta-1} \end{pmatrix} \begin{pmatrix} c^i \\ k^i \end{pmatrix} = \begin{pmatrix} (1-\theta)A(k^*)^{\theta} \\ -(1-\theta)\beta A(k^*)^{\theta} - \gamma + \theta A(k^*)^{\theta-1} \end{pmatrix} \tag{1.20}$$

If $\alpha \neq \beta$, then the matrix $\begin{pmatrix} 1 & -A\theta(k^*)^{\theta-1} \\ -\alpha & \beta A\theta(k^*)^{\theta-1} \end{pmatrix}$ is nonsingular and the solution of the system of linear equations is unique. Moreover, clearly $(c^i, k^i) = (k^*, c^*)$ where (k^*, c^*) satisfies Eqs. 1.17 and 1.19, is a solution for $\dot{c^i} = 0$ and $\dot{k^i} = 0$. □

Further, if $\alpha > \beta$, the symmetric steady state is uniquely determined.

Corollary. *If $\alpha > \beta$, the symmetric steady state is uniquely determined. If $\alpha < \beta$, multiple symmetric steady states are possible.*

Proof. From the Proposition 1.2.1, we know that symmetric steady states are the only steady states in the model. To determine the level of capital per capita at steady state, I obtain the following equation for k^* by combining 1.17 and 1.19

$$(\alpha - \beta)k + (\frac{\gamma}{A})k^{1-\theta} = \theta \tag{1.21}$$

which is the same condition for a representative model as in as Meng [2006, 2679] because of linear structure of discounting function. If $\alpha > \beta$ Eq. 1.21 uniquely pins down the level of capital per capita. If $\alpha > \beta$, then the left-hand side is strictly increasing and the right-hand side is constant, so there can be only one solution for k^*. If $\alpha < \beta$, the left-hand side is potentially nonmonotonic so there is the possibility of multiple solutions. Thus, the uniqueness of the symmetric steady state for $\alpha > \beta$ follows from Eq. 1.21. □

Before analyzing the local stability of the model, I briefly review the main results on local stability of nonlinear dynamical systems. Let's assume that the following dynamical system is given:

$$\dot{x} = f(x),\ x(0) = x_0 \in \mathbb{R}^n \tag{1.22}$$

Let $\phi(t, x_0)$ be the solution at time t with initial condition $x(0) = x_0$. Define

$$\phi_t(x) := \phi(t, x_0) \tag{1.23}$$

The set of mappings $\phi_t(x),\ x \in \mathbb{R}^n$, is called a flow (see Perko [2001, 96]). Let x^* be a steady state. Then, a local stable manifold $W^s_{loc}(x^*)$ of x^* is defined as follows:

$$W^s_{loc}(x^*) = \{x \in N \mid \phi_t(x) \underset{t\to\infty}{\rightarrow} x^*, \phi_t(x) \in N, \forall t \geq 0\} \tag{1.24}$$

where N is a neighborhood of x^*. A local unstable manifold of $W^u_{loc}(x^*)$ is defined similarly by reversing the direction of time (see Guckenheimer and Holmes [1983, 13]).

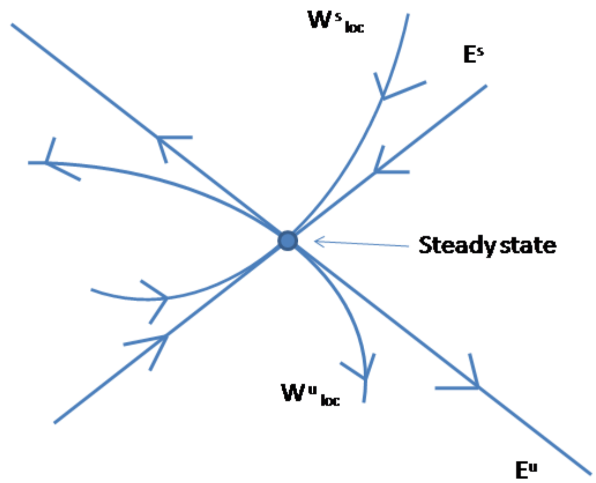
By linearizing Eq. 1.22 at x^*, we have the linearized system of f.

$$\dot{x} = J(x^*)(x - x^*),\ x(0) = x_0 \in \mathbb{R}^n \tag{1.25}$$

Let's assume that the Jacobian matrix $J(x^*)$ has no eigenvalues with zero real part. Then, the stable manifold theorem states that the dimension of a local stable manifold is equal to the dimension of the stable subspace of the linearized system. In turn, the dimension of stable subspace is equal to the number of eigenvalues with negative real parts. In short, if $J(x^*)$ has no eigenvalues with zero real part, we have the following relationship:

$$\begin{aligned} &\text{dimension of stable manifold} \\ &= \text{dimension of stable subspace} \\ &= \text{number of eigenvalues with negative real part} \end{aligned} \tag{1.26}$$

Figure 1: Local stable and unstable manifolds



Therefore, the stable manifold theorem justifies the convention of investigating the local stability of f by analyzing the linearized system of f around a steady state.[6] Figure 1 illustrates local stable and unstable manifolds in a two-dimensional dynamical system.[7] Note that the local stable manifold (W^s_{loc}) is a curve whereas the stable subspace (E^s) is a straight line.

If the dimension of the stable manifold is larger than the number of pieces of available information then there is indeterminacy. That is, for a given $k^1(0)$ and $k^2(0)$, there will be a continuum of choices of $(c^1(0), c^2(0))$ that give an equilibrium.

[6]For the stable manifold theorem, see Perko [2001, 107-108], for example.

[7]Figure 1 is based on Guckenheimer and Holmes [1983, Figure 1,3.1 (b)] and Perko [2001, p. 113, Figure 2].

Now, I check local dynamics around the symmetric steady state, $k^1 = k^2 = k^*$, $c^1 = c^2 = c^*$ where (k^*, c^*) is determined by Eqs. 1.17 and 1.19.

Proposition 1.2.2. *If $\alpha > \beta$, there exists a unique equilibrium which converges to the unique symmetric steady state. If $\alpha < \beta$, local indeterminacy can occur in a heterogeneous agents model with different discounting under some values parameters.*

Proof. Substituting the definition of $k = \pi^1 k^1 + \pi^2 k^2$ into 1.14 and 1.15, and linearizing 1.14 and 1.15, I obtain the following Jacobian matrix $J \mid_{s.s.s.}$ at symmetric steady state (s.s.s.):

$$J \mid_{s.s.s.} = \begin{pmatrix} -\frac{\alpha c^*}{\sigma} & 0 & \frac{c^*}{\sigma}[\beta A\theta(k^*)^{\theta-1} - \pi^1 A\theta(1-\theta)(k^*)^{\theta-2}] & -\frac{c^*}{\sigma}[\pi^2 A\theta(1-\theta)(k^*)^{\theta-2}] \\ 0 & -\frac{\alpha c^*}{\sigma} & -\frac{c^*}{\sigma}[\pi^1 A\theta(1-\theta)(k^*)^{\theta-2}] & \frac{c^*}{\sigma}[\beta A\theta(k^*)^{\theta-1} - \pi^2 A\theta(1-\theta)(k^*)^{\theta-2}] \\ -1 & 0 & A\theta(k^*)^{\theta-1} & 0 \\ 0 & -1 & 0 & A\theta(k^*)^{\theta-1} \end{pmatrix} \tag{1.27}$$

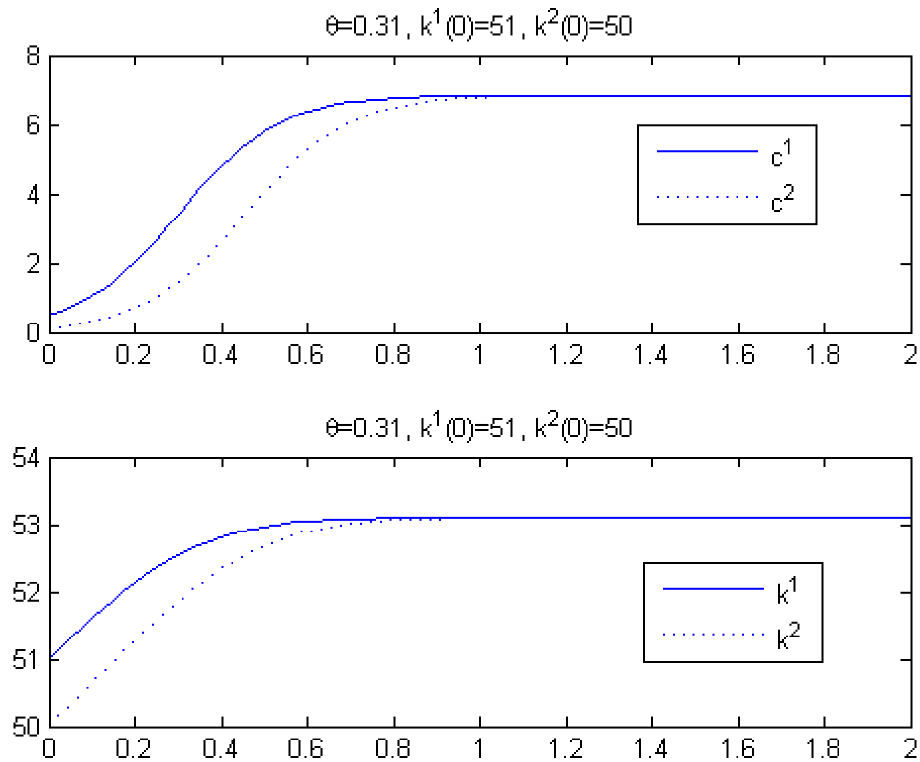
Then, the characteristic polynomial $P(\lambda)$ of $J \mid_{s.s.s.}$ is given by

$$\begin{aligned} P(\lambda) &= \Big[\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta)\Big]\Big[\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta) + ab_2)\Big] \\ &= \Big[\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta)\Big] P_1(\lambda) \end{aligned} \tag{1.28}$$

where $P_1(\lambda) = \Big[\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta) + ab_2)\Big]$, $a = -\frac{c^*}{\sigma}$, $b_1 = A\theta(k^*)^{\theta-1}$, and $b_2 = A\theta(1-\theta)(k^*)^{\theta-2}$. For derivation, see Appendix A.1. The stability at steady state is determined by 1.28. If $\alpha > \beta$ then, I have two negative and two positive eigenvalues. Then, by the stable manifold theorem, there exists a two dimensional stable manifold. Since there are two predetermined variables (the initial levels of capitals, k_{10} and k_{20}), the solution is uniquely determined. But if $\alpha < \beta$, then more than two negative eigenvalues can be obtained. □

if $\alpha > \beta$ (so, average consumption has more effect), then, I have a unique convergent solution for each initial conditions around the steady state. Notice that from the symmetric structure of differential equations, it is clear that if $k^1(0) = k^2(0)$ and $c^1(0) = c^2(0)$, then the paths of the two groups will be the same. This implies that the set, $M = \{k^1 = k^2$ and $c^1 = c^2\}$, is a two-dimensional invariant set in the system. On M, the system becomes two-dimensional, i.e., it reduces to a representative agent model. As we will see in Eq. C.8 in Appendix C, $P_1(\lambda)$ corresponds to the characteristic polynomial for a representative agent model. If $\alpha > \beta$, then from $P_1(\lambda)$ I always have one negative eigenvalue and one positive eigenvalue, which implies that if $\alpha > \beta$, the stable manifold in this system includes a one-dimensional subset of M, but the stable manifold is not a subset of M, because the stable manifold is two-dimensional. So we can conclude that by extending the model into heterogenous agents model, one more dimension for the stable manifold emerges. On the other hand, as shown in Meng [2006], if $\alpha < \beta$, then indeterminacy is possible. I illustrate the sample paths. In this case, since in this model the level of consumption directly affects

Figure 2: A convergent path (I): $\alpha > \beta$



the discounting function, so if the impact is sufficiently large enough relative to income, there exists a unique choice of initial level of consumption. Note that since the model has only a two dimensional stable manifold, given initial conditions, numerical approximation is not trivial.

I approximate a convergent path by using the *shooting* method. Here I closely follow an approach discussed in Stemp and Herbert [2006]. I use two MATLABMathworks [2010] programs; *ode45* as an initial value problem solver and *fminsearch* as a search method.[8] I

[8] *fminsearch* implements a Nelder-Meade direct simplex search.

set $\theta = 0.31, \pi^1 = 0.7, \pi^2 = 0.3$. I also set the other parameters as follows: $\alpha = 1, \beta = 0.8, \gamma = -1.3304, \sigma = 0.8$, and A=2.[9] Then, the values of k and c at steady state are given by $k^* \approx 53.1032, c^* \approx 6.8520$. By numerical computation, eigenvalues in that case approximately are -8.5325, -8.5330, 0.0075, and 0.0080. I pick $k^1(0) = 51,\ k^2(0) = 50$. Note that $k^2(0) < k^1(0) < k^*$. Then, I have $c^1(0) \simeq 0.4914,\ c^2(0) \simeq 0.1402$ and $(c^1, k^1, c^2, k^2)|_{t=2} =$ (6.8520, 53.1040, 6.8520, 53.1042). Approximation error in norm $(=\|(c^1, k^1, c^2, k^2)|_{t=2} - (c^*, k^*, c^*, k^*)\|)$ is 0.0013.[10] The path is shown in Figure 2.[11]

Now, I pick $k^1(0) = 54 > k^* > \ k^2(0) = 52$. Then, I have $c^1(0) \simeq 21.0043,\ c^2(0) \simeq 1.7219$ and $(c^1, k^1, c^2, k^2)|_{t=2}$ =(6.8520 ,53.1033, 6.8520, 53.1034). Approximation error in norm is 2.2422×10^{-4}. The path is illustrated in Figure 3.[12]

To get some intuitive understanding, I fix the initial level of $(k^1, c^2, k^2) = (54, 1.7219, 52)$ as before and compare three paths from different initial levels of c^1, including $c^1(0) = 21.0043$. Figure 28 in Appendix B exhibits three different paths. Given the choice of the other group, the figure shows that there is a unique level of $c^1(0)$ from which the path converges to steady state.

A necessary condition for indeterminacy is $\alpha < \beta$. To get some idea about dynamics, I approximate paths which start with different initial conditions. I set $\theta = 0.31, \pi^1 = 0.7, \pi^2 = 0.3$. And for other parameters I follow a numerical example in Meng [2006] by assuming that $\alpha = 0.012, \beta = 1, \gamma = 0.6, \sigma = 0.8, A = 0.2$. Then, with $\theta = 0.31$, the larger symmetric steady state is given by $k^* \approx 34.9538, c^* \approx 0.6019$. Approximately, eigenvalues in that case are -0.0018±0.0636i and -0.0018±0.0630i. Note that, in the case of indeterminacy,

[9] I pick $\alpha = 1, \beta = 0.8$ and A=2. Then I determine steady state values of k and parameter γ from the following two constraints;

$$A\theta k^{\theta-1} - [(\alpha - \beta)Ak^\theta + \gamma] = 0 \text{ from 1.17 and 1.19}$$
$$(\alpha - \beta)Ak^\theta + \gamma = 0.04$$

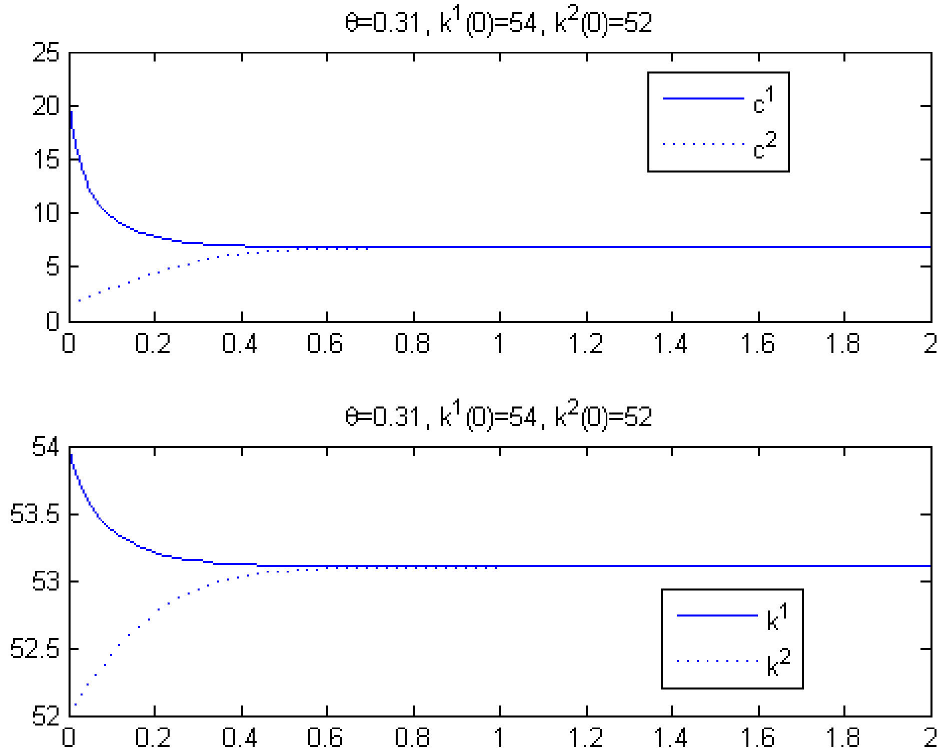
where I'm assuming that in standard constant rate of discounting $e^{-\rho t}$, $\rho = 0.04$. For instance, $\rho = 0.02$ in Barro and i Martin [1995, 78] and $\rho = 0.05$ in Carroll et al. [1997, 366].

[10] In this paper, approximation error means the distance between a trajectory and a steady state.

[11] Figure 26 in Appendix B displays an example of divergent paths where I pick initial levels for consumption randomly with the same initial levels for capital. One of the purposes of this exercise is to check whether the symmetric steady state has the saddle-path stability.

[12] If I pick arbitrary initial values of $c^1(0)$ and $c^2(0)$, then the result is shown in Figure 27 in Appendix B. We can see that the paths are diverging.

Figure 3: A convergent path (II): $\alpha > \beta$



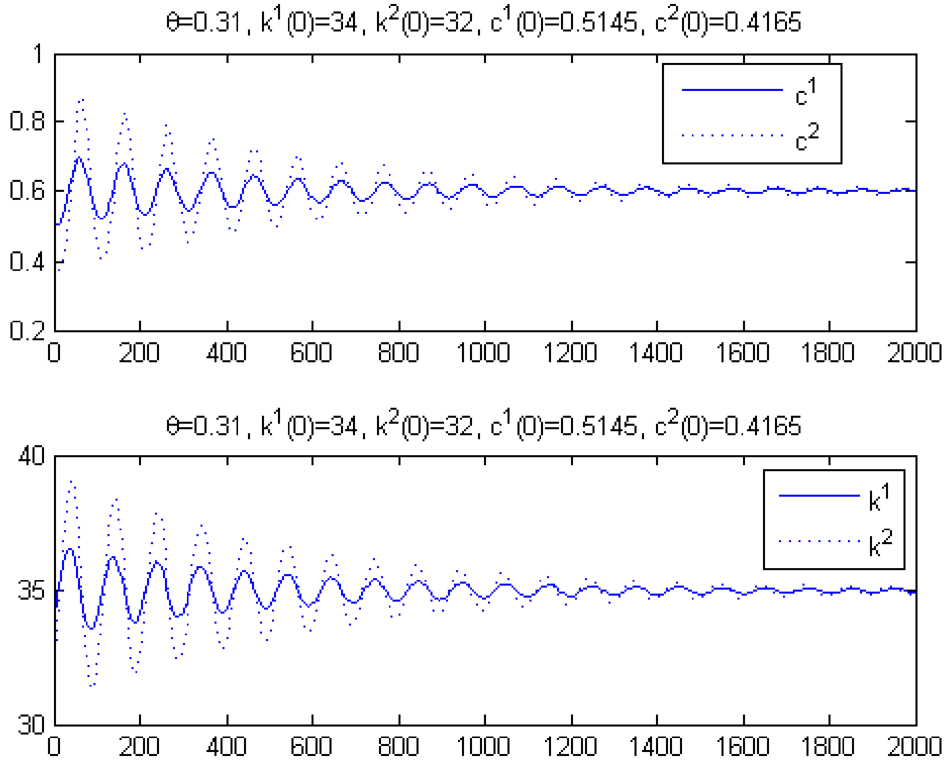
there are multiple solutions which converge to a steady state from the same initial condition of capitals. To verify the property, I pick arbitrary initial level of consumptions and compute the trajectory (recall that, if the solution is unique, this procedure generally yields a divergent trajectory).

I put $k^1(0) = 34,\ k^2(0) = 32, c^1(0) = 0.5145, c^2(0) = 0.4165$. With these initial values I obtain $(c^1, k^1, c^2, k^2)|_{t=2000}$ =(0.6025, 34.8847, 0.5964, 34.8323). Approximation error in norm is 0.1399. Figure 4 displays the path. Note that the path does converge.

Now, I compare two different initial conditions:

$$\begin{cases} IC1 := (c^1(0), k^1(0), c^2(0), k^2(0)) = (0.7870, 36, 0.4226, 32) \\ IC2 := (c^1(0), k^1(0), c^2(0), k^2(0)) = (0.7870, 36, 0.1, 32). \end{cases}$$

Figure 4: A convergent path: $\alpha < \beta$



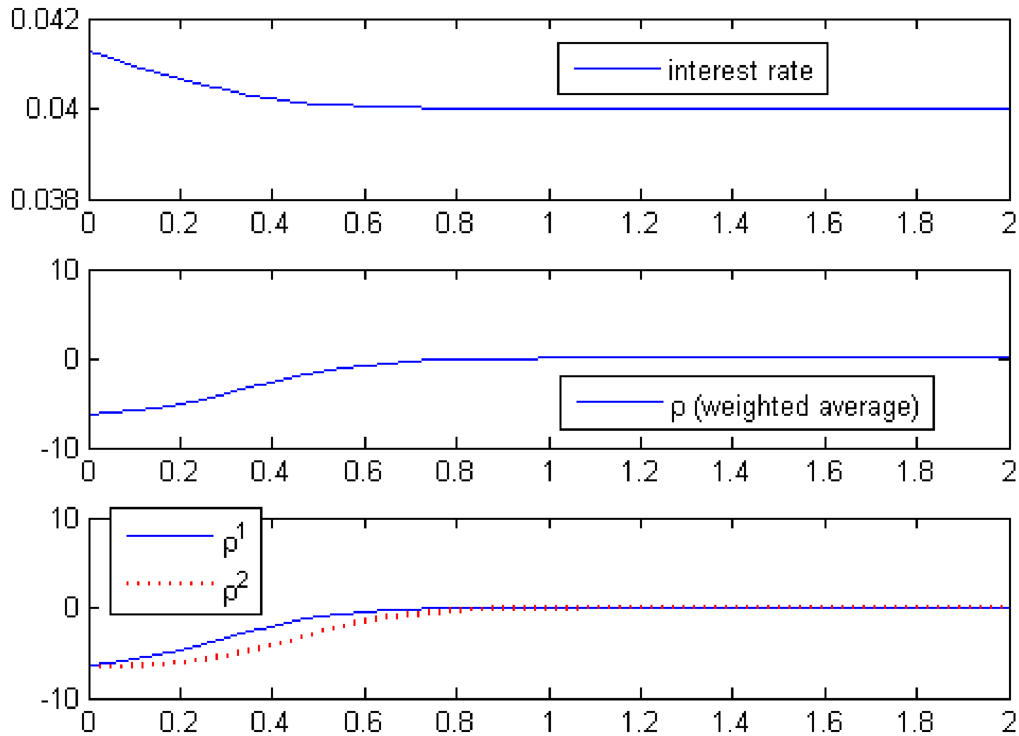
Note that IC1 and IC2 differ only in terms of $c^2(0)$. Figure 29 in Appendix B exhibits the paths. The two paths of type 1 look similar and converge to the same point. In contrast, the two paths of type 2 look different but still converge to the same point. This numerical experiment clearly shows that the steady state does not have saddle-path stability.

Note that, when a discount rate decreases because of higher capital, an interest rate decreases as well, which may offset the effect of decreasing discount rate. Is the movement of the interest rate necessary to get indeterminacy? To see the role of the interest rate in generating indeterminate solutions, in Section 1.2.2.2 I study a partial equilibrium where factor prices, wage and interest rate, are fixed.

Figure 5 displays the movements of the discount rates and the interest rate in the case

i) of Section 1.2.2.1. Note that the average discount rate (ρ) monotonically increases during the transition to the symmetric steady state.[13] The bottom panel of Figure 5 shows that the

Figure 5: Discount rates and the interest rate: $\alpha > \beta$

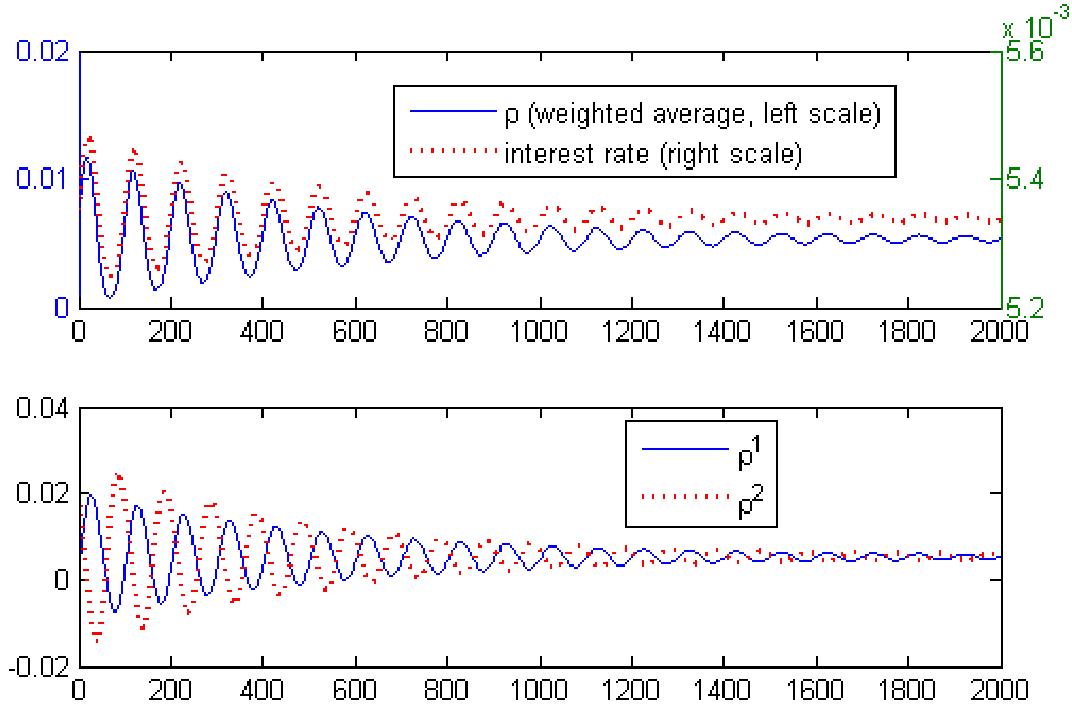


discount rate of the poor is indeed lower than that of the rich during the transition dynamics, which makes the catch-up possible.

Next, the *top panel* of Figure 6 exhibits the movements of the discount rates and the interest rate in the case ii) of Section 1.2.2.1. In contrast with the case of the unique solution ($\alpha > \beta$), the average discount rate and the interest rate oscillate in similar fashions. The bottom panel of Figure 6 exhibits that two discount rates of the two groups may move in different directions. But, the heterogenous discounting behaviors of the two groups eventually yield the convergence of wealth.

[13] The average discount rate ρ is defined by $\rho := \pi^1\rho^1 + \pi^2\rho^2$.

Figure 6: Discount rates and the interest rate: $\alpha < \beta$



1.2.2.2 Partial equilibrium analysis *The main question* in the partial equilibrium analysis is whether the oscillation of the interest rate is responsible for the indeterminacy.

Let's assume that there are two small villages in an economy.[14] Since they are small villages, their economic decisions do not affect economy-wide averages. Thus, we may assume that factor prices are given as constants to the two villages:

$$w = w_0 \tag{1.29}$$

$$r = r_0 \tag{1.30}$$

[14] In contrast, two groups in general equilibrium model may be viewed as two countries or races.

With Eqs. 1.29 and 1.30, the dynamical system becomes

$$\dot{c^i} = -\frac{c^i}{\sigma}\left[\alpha c^i - \beta[w_0 + r_0 a^i] + \gamma - r_0\right],\ i = 1, 2 \tag{1.31}$$

$$\dot{a^i} = r_0 a^i + w_0 - c^i, \quad i = 1, 2 \tag{1.32}$$

$$a^i(0) = a_0^i,\ i = 1, 2 \tag{1.33}$$

where a^i denotes wealth. Note that, by introducing the conditions of Eqs. 1.29 and 1.30, there is no interplay between the two villages. As a result, the system boils down to a two-dimensional continuous dynamical system. Because I am interested in the long-run behavior of a two-dimensional continuous dynamical system, the *Poincaré-Bendixson theorem* is a useful guidance (see Perko [2001, 245, Theorem 1], for instance). Suppose that a trajectory is bounded. The *Poincaré-Bendixson theorem* implies the the trajectory is attracted to a point or a limit cycle (= periodic orbit). The theorem makes the analysis of two-dimensional continuous dynamical systems easier than the analysis of higher dynamical systems. For example, according to the theorem, chaotic dynamics is impossible in a *two-dimensional continuous* dynamical system.

Let J be the Jacobian matrix at a steady state. Then, the eigenvalues of J can be obtained by the formula:

$$\lambda_{\pm} = \frac{1}{2}[tr(J) \pm \sqrt{(tr(J))^2 - 4det(J)}] \tag{1.34}$$

In short, the information about the trace and the determinant of the matrix J is sufficient to determine the local stability of a steady state.[15]

Proposition 1.2.3. *There exists a unique steady state if $\alpha \neq \beta$. And the unique steady state is given by $(c^*, a^*) = (\frac{r_0-\gamma}{\alpha-\beta}, \frac{c^*-w_0}{r_0})$.*

[15] For a detailed discussion about two-dimensional continuous dynamical systems , I refer to Hirsh et al. [2004, ch.4].

Proof. By imposing the steady state conditions, $\dot{c}^i = \dot{a}^i = 0$ in Eqs. 1.31 and 1.32, we have the following system of linear equations:

$$\begin{pmatrix} \alpha & -\beta r_0 \\ -1 & r_0 \end{pmatrix} \begin{pmatrix} c^i \\ a^i \end{pmatrix} = \begin{pmatrix} \beta w_0 + r_0 - \gamma \\ -w_0 \end{pmatrix} \tag{1.35}$$

If $\alpha \neq \beta$, the matrix $\begin{pmatrix} \alpha & -\beta r_0 \\ -1 & r_0 \end{pmatrix}$ is nonsingular. Thus, we have a unique solution, if $\alpha \neq \beta$. Then, the unique steady state, $(c^*, a^*) = (\frac{r_0-\gamma}{\alpha-\beta}, \frac{c^*-w_0}{r_0})$, is obtained from Eq. 1.35. □

Note that Proposition 1.2.3 rules out the possibility of a continuum of steady states. Thanks to the *Poincaré-Bendixson theorem*, I conclude that any bounded trajectory in the dynamical system converges to the unique steady state or to a limit cycle, if any. In the following, since my interest is in the convergence of wealth, I focus on the local stability of the unique steady state.

Regarding the convergence of wealth, Proposition 1.2.3 implies that if the levels of wealth of the two villages converge, then they converge to the same level. Note that there is no interaction between economic decisions of the two villages since factor prices are constants. *The levels of wealth of two villages converge to the same level because they independently confront the same long-run constraints, in other words, the same steady state conditions.*

Proposition 1.2.4. *The unique steady state has saddle-path stability if $\alpha > \beta$. If $\alpha < \beta$ and $-\frac{\alpha c^*}{\sigma} + r_0 < 0$ hold, indeterminacy occurs.*

Proof. The Jacobian matrix J at the steady state is given by

$$J = \begin{pmatrix} -\frac{\alpha c^*}{\sigma} & \frac{\beta r_0 c^*}{\sigma} \\ -1 & r_0 \end{pmatrix} \tag{1.36}$$

So, we have

$$tr(J) = -\frac{\alpha c^*}{\sigma} + r_0 \tag{1.37}$$

$$det(J) = (\beta - \alpha)\frac{c^* r_0}{\sigma} \tag{1.38}$$

If $\alpha > \beta$, then $det(J) < 0$ from Eq. 1.38. In turn, from Eq. 1.34, the matrix J has one negative and one positive eigenvalue. Hence, the unique steady state is a saddle point.

If $\alpha < \beta$, then $det(J) > 0$ from Eq. 1.38. Furthermore, if $\frac{\alpha c^*}{\sigma} > r_0$, then $tr(J) < 0$ from Eq. 1.37. With the two conditions, both eigenvalues of J have negative real part as we can see from Eq. 1.34. In other words, the unique steady state is a sink in that case. Therefore, the indeterminacy of rational expectation equilibrium occurs because the two-dimensional continuous dynamical system has the only one predetermined variable of wealth, as we can see from Eq. 1.33. □

Proposition 1.2.4 reveals that the movement of an interest rate is *not necessary* in order to obtain indeterminacy because even with a fixed interest rate, the model generates an indeterminate solution under some parameter values.

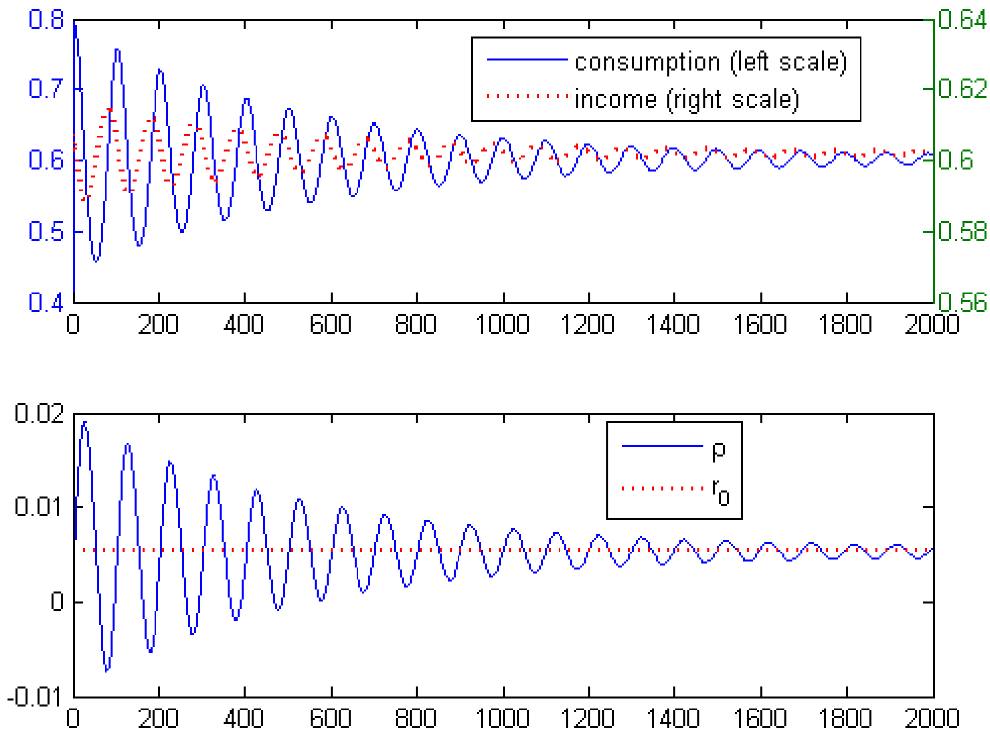
Assume that $\alpha = 0$ which implies that there is no centripetal force (consumption) in the discount function. Then, we have $tr(J) = r_0 > 0$ from Eq. 1.37 and $det(J) = \beta\frac{c^* r_0}{\sigma} > 0$ from Eq. 1.38. The two conditions imply that both eigenvalues of J have positive real part from Eq. 1.34. In other words, the unique steady state is a source. The same conclusion holds for the very small α.

Only when α gets large enough, the unique steady state becomes a sink and indeterminacy occurs. Which mechanism makes indeterminacy happen? From Eq. 1.36, we know that

$$\frac{\partial \dot{c}^i}{\partial a^i} = \frac{\beta r_0 c^*}{\sigma} > 0 \tag{1.39}$$

Suppose that the level of wealth becomes higher than the level at steady state. Then, discount rate decreases for a while due to higher wealth. But as people become wealthier, their consumption eventually grows as Eq. 1.39 implies. Indeterminacy conditions imply that if α is sufficiently large, rising consumption will eventually reverse the decreasing trend of the discount rate. Figure 7 reveals this mechanism. In the simulation of the model that generates Figure 7, the parameter values specified in case ii) of Section 1.2.2.1 are used. The initial condition for the simulation is (c(0), a(0))=(0.7870, 36). Note that in Figure 7, income goes through a trough before consumption goes through the trough. In that

Figure 7: Indeterminacy in partial equilibrium



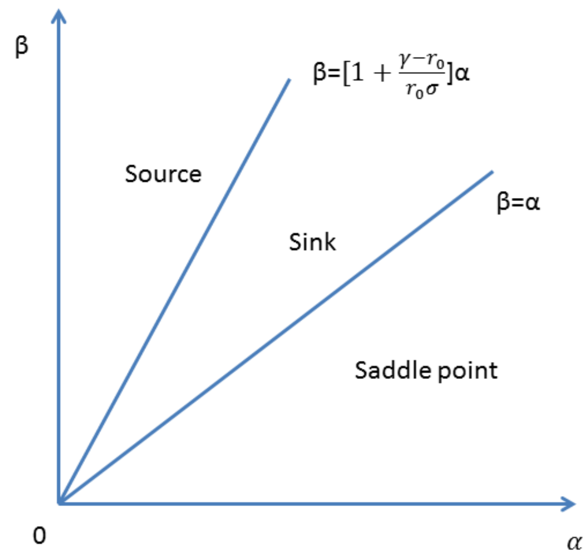
way, consumption follows the income trend with a time lag. The bottom panel shows that the interaction of the two oscillation of the two variables, consumption and income, makes discount rate oscillate around the fixed interest rate.

Now, suppose that α becomes even larger. Recall that if $\alpha > \beta$, the unique steady state is a saddle point. Thus, if α becomes too large relative to β, the convergence of a path gets sensitive on initial conditions. Therefore, I conclude that the existence of two conflicting forces and the balance of two forces as specified in indeterminacy condition are essential in generating indeterminacy.[16]

Figure 8 shows how the local stability of the dynamical system depends on the two parameters of α and β.

[16] The interpretation in this paper regarding indeterminacy problem is in line with Meng [2006, 2678].

Figure 8: Bifurcation diagram



1.2.2.3 Discussion: Comparison with earlier literature One of the factors causing the convergence of wealth in Stiglitz [1969] is the assumption that the saving function is an affine function of income. Note that, as a saving decision depends only on income, the growth rate of wealth may be a decreasing function of wealth under some conditions. The intuition is clarified by the assumption of the Kaldorian saving function in Stiglitz [1969, 391]. In the case of Kaldorian saving function, the growth rate of wealth of the poor is higher than that of the rich if the saving from labor income is positive. Bliss [2004] shows that under recursive preference framework, by using the Negishi approach in computing competitive equilibrium, strict wealth convergence cannot be optimal based on the fact that in the optimal equilibrium, agents with higher initial wealth have more weights in social utility function. Therefore convergence result in Stiglitz [1969] cannot be competitive equilibrium in recursive preference framework.[17] Long and Shimomura [2004] adds status seeking behavior into the model with the following felicity function under constant discounting;

$$u(c_i(t)) + v(\frac{k_i(t)}{\tilde{k}(t)}) \tag{1.40}$$

And they assume that u and v are strictly concave, which implies that the poor get more utility from a marginal increase in relative wealth. With this assumption, they showed that under some conditions, the symmetric steady state is the only a steady state in the model, and the symmetric steady state is saddle-path stable.

The model in this paper with the different reference levels (so, different discounting) do not require status-seeking in Long and Shimomura [2004]. Instead, the discount rate $\rho(\tilde{c}^i(s), \tilde{y}^i(s))$, is different across agents of different types. As Proposition 1.2.1 shows, the *heterogeneity* in discounting makes the symmetric steady states only possible steady states.[18] In other words, the heterogeneity in discounting rules out the solutions which converge to a state different from a symmetric steady state. Next, the convergence case with a unique solution in this paper is based on the assumption that the consumption has more importance in a discounting function (in the sense that $|\rho_c| > |\rho_y|$) around the steady state. So, I can say that the assumption is the mirror image of the "status seeking" in Long and Shimomura [2004] in that two different assumptions yield the similar result: stronger saving incentive for the poor, as the bottom panel of Figure 5 shows.

The indeterminacy example where income is more important in discounting function, shows that, in that case, the convergence of wealth still may happen even though the solution is not unique. Note that the interaction of a centrifugal force (income) in discounting function and a centripetal force (consumption) generates dynamic stability in the case of indeterminacy and the symmetric steady state becomes a sink of the dynamical system.[19] The pattern of the convergence of wealth is also different from the case of $\alpha > \beta$. The

[17] If I consider multi-agent balanced growth model under recursive preference, then it turned out that the restriction on the discount factor is quite tight. Farmer and Lahiri [2006] demonstrates that with these assumptions, only one of two things is possible; no balanced growth path or the same discount factor. To rescue the motivation of recurve preference framework, they suggest an exogenous time-dependent factor in preference.

[18] For the literature on the relation between heterogeneity and stability of equilibria, I refer to Grandmont [1992] and Herrendorf et al. [2000], for instance.

[19] Artige et al. [2004] provides an example with a limit cycle in a two region model.

bottom panel of Figure 6 shows that each of two groups *becomes more patient alternatively*, while the levels of wealth of the two groups eventually converge to the same level.

1.2.3 Case 2; same discounting

1.2.3.1 Basic results Now I assume that discounting depends on the overall average of consumption and income to explicitly consider interactions among agents due to interdependent intertemporal preferences via discounting functions. In this case, agent of type i, i=1,2, solves the following utility maximization problem;

$$\max_{c^i(t)} \int_0^\infty u(c^i(t)) e^{[-\int_0^t \rho(c(s),y(s))ds]} dt \tag{1.41}$$

subject to

$$\dot{k^i} = rk^i + w - c^i, \ k(^i0) = k_{i0} \tag{1.42}$$

where $c(s)$ and $y(s)$ are average consumption and income at time s, respectively. Then the system of differential equations for competitive equilibrium solution in this case becomes as follows:

$$\dot{c^i} = -\frac{c^i}{\sigma}\left[\alpha c - \beta(Ak^\theta) + \gamma - \theta Ak^{\theta-1}\right], \ i = 1, 2 \tag{1.43}$$

$$\dot{k^i} = A\theta k^{\theta-1}(k^i - k) + Ak^\theta - c^i, \ i = 1, 2 \tag{1.44}$$

$$k = \pi^1 k^1 + \pi^2 k^2, \tag{1.45}$$

$$c = \pi^1 c^1 + \pi^2 c^2, \ \pi^i \in (0, 1), \pi^1 + \pi^2 = 1 \tag{1.46}$$

In this case, by substituting 1.45 and 1.46 into 1.43 and 1.44, and linearizing 1.43 and 1.44 at symmetric steady state (s.s.s.), I get the following Jacobian matrix:

$$J\ |_{s.s.s.} = \begin{pmatrix} -\frac{\alpha c^*}{\sigma}\pi^1 & -\frac{\alpha c^*}{\sigma}\pi^2 & \frac{c^*}{\sigma}\pi^1[\beta A\theta(k^*)^{\theta-1} - A\theta(1-\theta)(k^*)^{\theta-2}] & \frac{c^*}{\sigma}\pi^2[\beta A\theta(k^*)^{\theta-1} - A\theta(1-\theta)(k^*)^{\theta-2}] \\ -\frac{\alpha c^*}{\sigma}\pi^1 & -\frac{c^*}{\sigma}\pi^2 & \frac{c^*}{\sigma}\pi^1[\beta A\theta(k^*)^{\theta-1} - A\theta(1-\theta)(k^*)^{\theta-2}] & \frac{c^*}{\sigma}\pi^2[\beta A\theta(k^*)^{\theta-1} - A\theta(1-\theta)(k^*)^{\theta-2}] \\ -1 & 0 & A\theta(k^*)^{\theta-1} & 0 \\ 0 & -1 & 0 & A\theta(k^*)^{\theta-1} \end{pmatrix}$$

$$= \begin{pmatrix} a\alpha\pi^1 & a\alpha\pi^2 & -a\pi^1[\beta b_1 - b_2] & -a\pi^2[\beta b_1 - b_2] \\ a\alpha\pi^1 & a\alpha\pi^2 & -a\pi^1[\beta b_1 - b_2] & -a\pi^2[\beta b_1 - b_2] \\ -1 & 0 & b_1 & 0 \\ 0 & -1 & 0 & b_1 \end{pmatrix}$$

Then, the characteristic polynomial $P(\lambda)$ of $J\mid_{s.s.s.}$ is given by

$$\begin{aligned} P(\lambda) &= \lambda(\lambda-b_1)\Big[\lambda^2-(a\alpha+b_1)\lambda+ab_1(\alpha-\beta)+ab_2\Big] \\ &= \lambda(\lambda-b_1)P_1(\lambda) \end{aligned} \tag{1.47}$$

Hence, the matrix $J\mid_{s.s.s.}$ always has a zero eigenvalue and one positive eigenvalue ($b_1>0$). So, by the Center Manifold theorem (see, for example, Perko [2001, 116]), there exists one dimensional center manifold.[20] The definition of a center manifold is as follows:

DEFINITION 1.2.2. *The definition of a center manifold* (Carr [1981, 3]).

$$\begin{aligned} \dot{x} &= Ax+f(x,y) \\ \dot{y} &= By+g(x,y) \end{aligned} \tag{1.48}$$

where $x\in\mathbb{R}^n,\ y\in\mathbb{R}^m$ and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. f and g are C^2 with f(0,0)=0, $f'(0,0)=0,\ g(0,0)=0,\ g'(0,0)=0$. In general, if y=h(x) is an invariant manifold for 1.48 and h is smooth, then it is called a *center manifold* if h(0)=0, $h'(0)=0$.

Thus, I will have to look at a center manifold to fully determine the local stability of the dynamic system.

Proposition 1.2.5. *There exists a continuum of steady states S,*
where $s\in S\ if\ s=\begin{pmatrix} c^1 \\ c^2 \\ k^1 \\ k^2 \end{pmatrix}=\begin{pmatrix} \frac{\pi^2 b_1 k^*}{\pi^1}+c^* \\ -b_1k^*+c^* \\ \frac{k^*}{\pi^1} \\ 0 \end{pmatrix}+j\begin{pmatrix} -(\frac{\pi^2}{\pi^1})b_1 \\ b_1 \\ -\frac{\pi^2}{\pi^1} \\ 1 \end{pmatrix}$
with $j\in(0,\infty)$. *Moreover,* $\begin{pmatrix} -(\frac{\pi^2}{\pi^1})b_1 \\ b_1 \\ -\frac{\pi^2}{\pi^1} \\ 1 \end{pmatrix}$ *is an eigenvector of zero eigenvalue of Jacobian matrix at each point of the continuum of steady state.*

[20]For a more detailed discussion about a center manifold in macroeconomic models, I refer to Gomis-Porqueras and Haro [2009].

Proof. By using the steady state conditions $\dot{k^1} = \dot{k^2} = 0$, I have from 1.44 that $c = Ak^\theta$. So, with the result, steady state values of averaged capital and consumption are determined from 1.43. Given k^* and c^*, from 1.44 - 1.46, I have the desired expression for steady states. To see why $v \equiv [-\frac{\pi^2}{\pi^1}b_1, b_1, -\frac{\pi^2}{\pi^1}, 1]$ is the eigenvector of zero eigenvalue, I write the system after substituting 1.45 and 1.46 as follows;

$$\dot{x} = f(x), x = [c_1, c_2, k_1, k_2]' \in \mathbb{R}^4 \tag{1.49}$$

Let $\hat{x} \equiv [\frac{\pi^2 b_1 k^*}{\pi^1} + c^*, -b_1 k^* + c^*, \frac{k^*}{\pi^1}, 0]'$. Then we have

$$f(\hat{x} + jv) \equiv 0,\ \ j \in (0, \infty) \tag{1.50}$$

From 1.50, by using chain rule,

$$\frac{d}{dj} f(\hat{x} + jv) = Df(\hat{x} + jv)v = 0 \tag{1.51}$$

So, we can conclude that since the directional derivative of f at $\hat{x} + jv$ in the direction v should be zero, v is the eigenvector of zero eigenvalue at $\hat{x} + jv$.

□

Proposition 1.2.5 shows that center subspace= $Span([-\frac{\pi^2}{\pi^1}b_1, b_1, -\frac{\pi^2}{\pi^1}, 1])$.

This different result about steady state, compared with the previous case, comes from the fact that discounting functions between different groups have the same arguments c and y. Note that with the condition of $\dot{c}^i = 0$, there is no c^i in Eq. 1.43 whereas c^i still appears in Eq. 1.14. Examples in macroeconomics which have a continuum of equilibria are Becker [1980] with the common discount factor in discrete time and Lucas [1988] in continuous time.[21]

Proposition 1.2.6. *The continuum of steady states S is a center manifold of symmetric steady state (s.s.s.).*

[21] Benhabib and Rustichini [1994] is a nice explanation about the existence of zero eigenvalue in balanced growth model.

Proof. Let's call center subspace E^c and stable subspace E^s. Assume that the origin is an equilibrium. Then, h(x) in the definition is a map such that

$$h : E^c \to E^s \textit{ with } h(0) = 0 \textit{ and } h'(0) = 0.$$

To apply the definition, I make the origin an equilibrium by changing coordinates;

$$s' = s - (c^*, c^*, k^*, k^*)$$

If $s \in S$, then

$$s' = \begin{pmatrix} \frac{\pi^2 b_1 k^*}{\pi^1} + c^* \\ -b_1 k^* + c^* \\ \frac{k^*}{\pi^1} \\ 0 \end{pmatrix} + j \begin{pmatrix} -(\frac{\pi^2}{\pi^1})b_1 \\ b_1 \\ -\frac{\pi^2}{\pi^1} \\ 1 \end{pmatrix} - \begin{pmatrix} c^* \\ c^* \\ k^* \\ k^* \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\pi^2 b_1 k^*}{\pi^1} \\ -b_1 k^* \\ \frac{\pi^2}{\pi^1} k^* \\ -k^* \end{pmatrix} + j \begin{pmatrix} -(\frac{\pi^2}{\pi^1})b_1 \\ b_1 \\ -\frac{\pi^2}{\pi^1} \\ 1 \end{pmatrix}$$

$$= (j - k^*) \begin{pmatrix} -(\frac{\pi^2}{\pi^1})b_1 \\ b_1 \\ -\frac{\pi^2}{\pi^1} \\ 1 \end{pmatrix} = (x) \begin{pmatrix} -(\frac{\pi^2}{\pi^1})b_1 \\ b_1 \\ -\frac{\pi^2}{\pi^1} \\ 1 \end{pmatrix} \in E^c \textit{ where } x \equiv j - k^*$$

Hence I have S-$(c^*, c^*, k^*, k^*) \subset E^c$. So, with new coordinates, if we think of S-$(c^*, c^*, k^*,$ $k^*)$ as a graph, we have $(x, h(x)) = (x, 0)\ \forall x$, i.e., h(x)$\equiv$ 0. Clearly, S-(c^*, c^*, k^*, k^*) is invariant, and h(0)=0, $h'(0) = 0$, trivially. It follows that S is a center manifold in original coordinates. □

Proposition 1.2.7. *There is no solution which converges to symmetric steady state in the heterogeneous agents model with the same reference level except the trivial cases, i.e. the paths which start with the same initial conditions.*

Proof. Since the center manifold is a continuum of equilibria, it does not increase the dimension of the stable region. From the symmetric structure of differential equations, it is clear that the set, $M = \{k^1 = k^2 \text{ and } c^1 = c^2\}$, is a two-dimensional invariant set in the system. On M, the system becomes two-dimensional, i.e., it reduces to a representative agent model. Since negative eigenvalues can come only from $P_1(\lambda)$ corresponding to the characteristic polynomial for a representative agent model, I can conclude that the stable manifold of symmetric steady state is always included in M. So, if $k^1(0) \neq k^2(0)$, the path cannot converge to the symmetric steady state. [22] □

To get the economic interpretation for this result, first note that 1.43 implies the same growth rate of consumption between different groups. If I consider convergent paths to the symmetric steady state, from this observation, it follows that initial consumption levels should be the same. Furthermore, from 1.44 I have

$$\begin{aligned}\frac{\dot{k^i}}{k^i} &= A\theta k^{\theta-1}(1-\frac{k}{k^i}) + \frac{Ak^\theta}{k^i} - \frac{c^i}{k^i} \\ &= A\theta k^{\theta-1} - \frac{A\theta k^\theta}{k^i} + \frac{Ak^\theta}{k^i} - \frac{c^i}{k^i} \\ &= A\theta k^{\theta-1} + \frac{(1-\theta)Ak^\theta}{k^i} - \frac{c^i}{k^i},\ i = 1, 2\end{aligned}$$

So, the difference in the growth rate of capital in the two groups is given by

$$\begin{aligned}\frac{\dot{k^1}}{k^1} - \frac{\dot{k^2}}{k^2} &= (1-\theta)Ak^\theta\Big\{\frac{1}{k^1} - \frac{1}{k^2}\Big\} - \Big\{\frac{c^1}{k^1} - \frac{c^2}{k^2}\Big\} \\ &= \Big\{(1-\theta)Ak^\theta - c\Big\}\Big\{\frac{1}{k^1} - \frac{1}{k^2}\Big\} \qquad (1.52)\\ &\text{since } c^1 = c^2 = c \text{ along convergent paths}\end{aligned}$$

Since all agents in this case consume the same amount at each time, and since I assume $k^1(0) > k^2(0)$, it follows that $k^1(t) > k^2(t)\ \forall t$. Hence the following holds

$$\frac{\dot{k^1}}{k^1} - \frac{\dot{k^2}}{k^2} > 0 \Leftrightarrow \Big\{(1-\theta)Ak^\theta - c\Big\} < 0 \qquad (1.53)$$

This condition is intuitive; since $(1-\theta)Ak^\theta$ is labor income share, the condition just says that $\frac{\dot{k^1}}{k^1} - \frac{\dot{k^2}}{k^2} > 0$ holds if agents consume more than the labor income. If agents consume

[22] So, this case violates the regularity condition in Epstein [1987].

exactly the amount of labor income, then the growth rates of capital will be the same: the rate of return on capital r. But, if they consume more than the labor income, because agent of type 2 has smaller capital, obviously agent of type 2 consumes *more portion* of capital income because everybody consumes *the same amount*. So, the growth rate of capital of agent of type 2 will be lower. Now, we can see that 1.53 implies the impossibility of the convergence to the symmetric steady state in this model since the converging path will satisfy 1.53 eventually, a contradiction.

1.2.3.2 Several special cases

Case 1. $\alpha = \beta = 0$. The model becomes a constant discount rate case like Becker [1980]. Then, the characteristic polynomial $P(\lambda)$ of $J\mid_{s.s.s.}(\theta)$ is given by

$$P(\lambda) = \lambda(\lambda - b_1)\left[\lambda^2 - b_1\lambda + ab_2\right] \tag{1.54}$$

Hence, $\lambda = b_1, 0, \frac{b_1 \pm \sqrt{b_1^2 - 4ab_2}}{2}$. Since $a < 0$, there is only one negative eigenvalue. Because the stable manifold is one-dimensional, the convergence to symmetric steady state can happen only with the same initial values. But, since there is a continuum of steady states, depending on initial values, the equilibrium solution may converge to a steady state (not necessarily a symmetric one). This case was extensively analyzed by Kemp and Shimomura [1992]

Case 2. $\beta = 0$. Then, the characteristic polynomial $P(\lambda)$ of $J\mid_{s.s.s.}(\theta)$ is given by

$$P(\lambda) = \lambda(\lambda - b_1)\left[\lambda^2 - (a\alpha + b_1)\lambda + ab_1\alpha + ab_2\right] \tag{1.55}$$

In this case, $\lambda = b_1, 0, \frac{a\alpha + b_1 \pm \sqrt{(a\alpha + b_1)^2 - 4(ab_1\alpha + ab_2)}}{2}$. So, the stability is the same as case 1. For all values of parameters, there exists only the one-dimensional stable manifold.

Case 3. $\alpha = 0$.The characteristic polynomial $P(\lambda)$ of $J\mid_{s.s.s.}(\theta)$ is given by

$$P(\lambda) = \lambda(\lambda - b_1)\left[\lambda^2 - b_1\lambda - ab_1\beta + ab_2\right] \tag{1.56}$$

$\lambda = b_1, 0, \frac{b_1 \pm \sqrt{b_1^2 - 4a(-b_1\beta + b_2)}}{2}$. Again, there exists at most one negative eigenvalue for all values of parameters.

Case 4. Dynan and Ravina [2007] suggest that relative concern may be more important for an above-average income group. To see this effect I assume one group has a constant discounting rate, i.e. $\alpha = \beta = 0$ for the second group. Then,

$$J\mid_{s.s.s.} = \begin{pmatrix} a\alpha\pi^1 & a\alpha\pi^2 & -a\pi^1[\beta b_1 - b_2] & -a\pi^2[\beta b_1 - b_2] \\ 0 & 0 & a\pi^1 b_2 & a\pi^2 b_2 \\ -1 & 0 & b_1 & 0 \\ 0 & -1 & 0 & b_1 \end{pmatrix}$$

$$P(\lambda) = \lambda(\lambda - b_1)\Big[\lambda^2 + (a\pi^1(\beta b_1 - \alpha) + ab_2 - b_1)\lambda + a\alpha\pi^1 b_1\Big] \tag{1.57}$$

$\lambda = b_1, 0, \frac{(a\pi^1(\beta b_1-\alpha)+ab_2-b_1)\pm\sqrt{(a\pi^1(\beta b_1-\alpha)+ab_2-b_1)^2-4a\alpha\pi^1 b_1}}{2}$. So, there exists only the one-dimensional stable manifold.

1.2.3.3 Transition dynamics to a continuum of steady states

Proposition 1.2.8. *Points in the continuum of steady states S share the same eigenvalues, therefore they show the same type of stability.*

Proof. The proposition follows from the fact that the characteristic polynomial at each point of the continuum of steady states is identical with the characteristic polynomial at the symmetric steady state. For the proof, see Appendix A.2. □

So, If the symmetric steady state has two negative eigenvalues, so does each point of the continuum of steady states. Since the model has two state variables, it implies that an indeterminacy of equilibria is possible if there exist two negative eigenvalues.

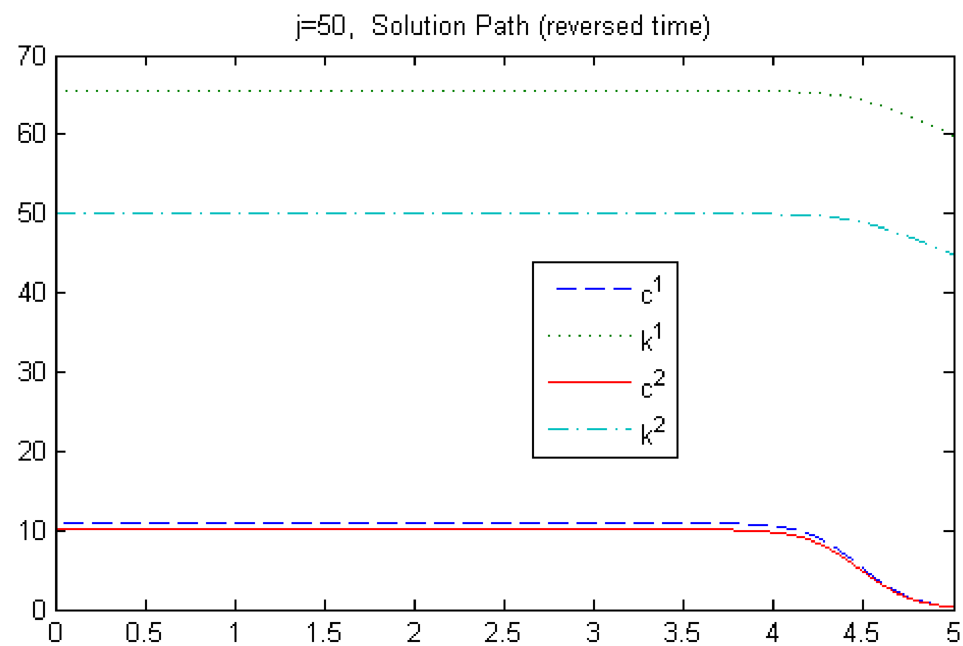
Here, I focus on the case of $\alpha > \beta$, which implies that each point at the continuum of steady states has one dimensional stable manifold. Hence, the dynamics now depends on initial conditions in the sense that solutions from different initial values converge to different points in the continuum of steady states. I study the following numerical example: $\alpha = 0.5, \beta = 0.495, A = 3, \gamma = 0.001, \sigma = 0.8, \theta = 0.31, \pi^2 = 0.3$. Then, I have $k^* = 60.8647, c^* = 10.7220$. I report two cases;

i) $j = 50, (c^1, k^1, c^2, k^2)\mid_{c.s.s.} = (10.9762, 65.5210, 10.1286, 50.0000)$

ii) $j = 70, (c^1, k^1, c^2, k^2) \mid_{c.s.s.} = (10.5082, 56.9495, 11.2208, 70.0000)$.

So, for the first case, the wealthier people have more weight. To approximate the solution, following [Judd, 1998, 335], I use the idea of reverse shooting. I change the direction of time by transforming t to (-t). Then, by using the standard *Matlab* program (*ode45*), I approximate the orbit. Figure 9 displays the path. And Figure 30 in Appendix B exhibits relative ratios of capital and consumption in two groups. Since the direction of time is

Figure 9: A convergent path (I) in reverse time



reversed, we have to look at the graph from the right to the left. So, along the transition path, capital and consumption are growing, and the ratio of k^1/k^2 is decreasing. And since the discounting functions are the same, the ratio of consumption is constant.

For the second case, again, we can see the same pattern. Figure 31 in Appendix B shows the path. And Figure 32 in Appendix B displays relative ratios of capital and consumption in two groups. Thus, the weight is not important for the result.

Finally, I pick one point on this path, (c^1, k^1, c^2, k^2)=(0.1111, 49.8289, 0.1187, 62.3715) and approximate the solution in the original time direction. Figure 33 in Appendix B shows the path.

1.2.3.4 Discussion In contrast with the heterogeneous discounting, if people's economic decisions depend only on *economy-wide averages*, there exists a continuum of steady states. In that case, if economy-wide averages satisfy certain conditions for steady states, there is no reason for people to change their decisions because economy-wide averages are only relevant variables for their decisions. As there is no more constraint for people's decisions, there exist infinitely many possible distributions of wealth.

But, in the case of heterogeneous discounting, two groups pay attention to their own economic variables, which is an element of *heterogeneity*. The element of heterogeneity excludes non-symmetric steady states by adding one more constraint at steady states. Further, note that the functional form of discounting function or the discounting rule is the same between the two groups, which is an element of *homogeneity*. The element of homogeneity together with the element of heterogeneity also reveals that there is no reason to believe the existence of non-symmetric steady states, considering the *same decision rule*, the *same conditions for steady states*, and *different reference groups*.

I think the analysis in this section can be applied to economic integration among nations. My analysis suggests the speed of convergence in preference does matter for wealth convergence. In other words, after economic integration, if discounting quickly converges to the new average level, the positive effect of economic integration on wealth convergence will be smaller. For German reunification, Alesina and Fuchs-Schundeln [2007] expect 1-2 generations for preference convergence.

Regarding the transition dynamics to the continuum of steady state, Kemp and Shimomura [1992] shows that with the same constant discounting, the wealth is more evenly distributed if the initial total wealth is larger than the steady state level and vice versa. My result with social effects seems to be the opposite. But, as my results depend only on one numerical method, it might be better to employ additional numerical procedures in order to draw firm conclusions about transition dynamics. I will leave the task to be a future research topic.[23]

[23]For example, recently, Trimborn et al. [2008] proposes "relaxation algorithm" for computing transition dynamics.

1.3 CONCLUSION

In this paper, I investigated long-run wealth dynamics of a one-sector growth model with endogenous discounting by extending a representative agent model in Meng [2006] into a heterogeneous agents model. In the heterogeneous agents model, if agent's decision depends only on his or her reference group, then the symmetric steady state is the unique steady state of the model. Moreover, I found that if consumption is more important in discounting around the steady state than income, the convergence of wealth distribution to the symmetric steady state is the unique solution in optimization framework with the endogenous discounting which depends on social factors. To my knowledge, this result is new. But, if discounting function depends only on economy-wide average, then the inclusion of social factors does not result in converging wealth dynamics except the special case of the same initial values. In general, because there exists the continuum of steady states, if consumption is more important in discounting around the steady state than income, then the resulting dynamics depends on initial conditions.

Regarding the policy implication of the models in this paper, this paper highlights the necessity of gradual economic integration in order to generate more even distribution of wealth after economic integration.

Finally, an interesting question is whether the opposing forces of *keeping up with Jones* and *status seeking* can lead to the emergence of limit cycles. In Appendix C, I explored this possibility in a representative agent model in [Meng, 2006, example 2.3]. I found an unstable limit cycle. I viewed the finding as an example of the so-called "corridor stability."

2.0 WEALTH DISTRIBUTION WITH WEALTH-DEPENDENT LABOR INCOME SHOCK, MISSPECIFICATION, AND STATUS-SEEKING

2.1 INTRODUCTION

It is a well-known regularity that the distribution of wealth is much more concentrated than the distribution of income. It has also been well recognized that replicating the regularity in a standard incomplete market model is a very hard task. This paper aims at contributing to the literature on the distribution of wealth by building an incomplete market model that can generate the highly concentrated distribution of wealth.

After seminal works of Huggett [1993] and Aiyagari [1994], there have been lots of efforts in the explanation of highly skewed wealth distribution. Several newly introduced mechanisms have made progress in replicating the highly concentrated distribution of wealth.[1] But, whereas our understanding about amplifying mechanisms related with uninsurable idiosyncratic income risk has been better, from my personal viewpoint, there has not been enough discussion about possible feedback from wealth to the income process and its recursive effect on wealth distribution. Recently, Campanale [2007] filles this gap in capital income by introducing the assumption of the increasing return function with respect to the level of wealth. Similarly but instead of capital income, I will show that the labor income process is also wealth-dependent via several channels and try to quantify the size of the effect on the wealth distribution (particularly on shares of the top 1%/5%/20% of population in wealth distribution).

On the other hand, recently, Nirei and Souma [2007] provides a model for a Pareto

[1] I will only briefly review the most closely related papers. For a up-to-date survey, see Cagetti and Nardi [2008].

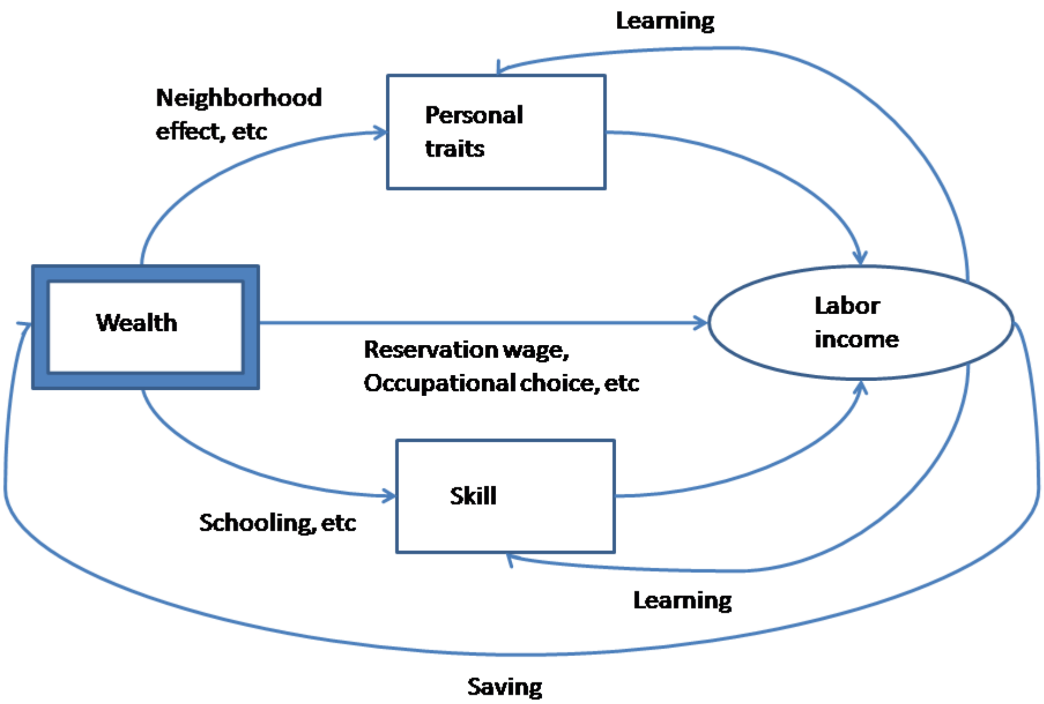
distribution of income and conjectures similar results would follow from incomplete market model. Instead of an income distribution, I will try to replicate a Pareto distribution of wealth in the right-hand tail in the model. Therefore, the main targets of this paper are i) wealth shares of the top 1%/5%/20% of the population and ii) Pareto distribution of the right-tail in wealth distribution.[2]

A model in this paper can be evaluated by looking at whether the model is able to replicate the main targets. Moreover, since there exist several competing models, we also need to compare testable predictions of the models. Regarding testable predictions, for example, Brückner et al. [2010] provides evidences for the positive impact of wealth inequality on real interest rates by time series analysis for Sweden, the UK, and the US. Aiyagari [1993, 30] shows that more volatile or persistent earning process results in lower real interest rates. In that case, the impact of the wealth inequality on real interest rates would be negative. As we see later, a model in this paper implies the positive impact of the wealth inequality on real interest rates.

Broadly speaking, in this paper, the following three mechanisms are employed for generating a wealth distribution similar to the data, especially with respect to the shares of the top 1%/5%/20% of the population in the distribution of wealth. First, the labor income shock is wealth-dependent. In other words, since the level of individual wealth is an endogenous state variable in the model, labor income shock is viewed as endogenous or state-dependent. Shocks in economic models are normally defined as an purely exogenous processes. But more generally, we can assume that the realization of shock depends on the state of the individual (therefore, my current decision affects the realization of shock in the future). In this paper, I will study the effect of this modification by focusing on the issue of wealth inequality. Figure 10 illustrates the mutual feedback between wealth and labor income.[3] In Figure 10, wealth, personal traits, and skill are *stock variables*, whereas labor income is a *flow variable*. In Figure 10, we can see three different ways by which wealth may be related with the realization of labor income shock. First, wealth directly affects labor income by influencing reservation wage or occupational choice. Second, wealth indirectly affects labor

[2] Besides, I will look at other several aspects of distribution.
[3] A literature review on the feedback from wealth to labor income is given in Section 2.2.1.

Figure 10: Mutual feedback between wealth and labor income



income by influencing the factors which are determinants of labor income such as skill and personal traits. Personal traits include work ethic and attitude. Finally, labor income also affects wealth by saving. Labor income has influence on determinants of labor income via learning at work as well. In this respect, we may expect that favorable labor income shock induces both higher wealth and better skill and personal traits. In this context, wealth may be a proxy variable for determinants of labor income, skill and personal traits. Note that, in standard incomplete market models, exogenous labor income shock depends only on the level of the current labor income. This means that if the poorest and the richest in a society happen to earn the same labor income in the current year, the expected labor income in the next year is the same. But, Figure 10 illustrates why we may expect *unequal opportunity* even conditional on the current labor income. In short, wealth-dependent labor income shock implies that labor income shock will be more favorable to the richest than to the poorest, on average.

Second, I will consider the case where people do not take into account the dependence of the labor income process on wealth when they make consumption decisions.[4] These two mechanisms generate skewed distributions. In other words, I will investigate the effect of the underparameterization of the stochastic process which is related with the policy function of consumers.[5] This mechanism is naturally related with the concept of *bounded rationality* as in Simon [1955] and Kahneman [2003], for instance. There are two rationales for the mechanism. First, the feedback from wealth to labor income is complex, and therefore hard to exactly measure the feedback effect. Second, learning optimal consumption seems to be more difficult if wealth is involved.[6] Note that knowing the distribution of wealth of coworkers is normally much harder than knowing the distribution of their labor income. In other words, wealth is much more private information than labor income. But, as we can expect, it turns out these two mechanisms are not sufficient to generate a high concentration at the right tail of distribution in data, since these two mechanisms cannot differentiate the very rich from the rich.

Third, to generate a realistic concentration of wealth in the right tail, I borrow the idea of the spirit of capitalism that wealth may affect utility directly (wealth in utility model). However, I apply the idea only after some threshold (or at the highest level of asset space).[7] I call this mechanism status seeking. I will analyze the effect of this modification.

Since wealth accumulation can be viewed as a combination of an income process and a saving decision at an individual level, I will briefly review the literature along the two dimensions. First, heterogeneity in saving behavior is introduced into a model either as an exogenous stochastic process like Krusell and Smith [1998] or as an endogenous process notably due to heterogeneous budget constraints (particulary, financial constraint) among different types of agents via occupation choices such as Quadrini [2000], Meh [2005], Boháĉek

[4]Another way of achieving similar effect would be introducing mortality into a model. I leave this approach an open research question.

[5]I do not investigate the case where people do consider the dependence of the labor income process on wealth. I leave the topic an open research question.

[6]For the difficulty related with individual learning about consumption, I refer to Allen and Carroll [2001].

[7]For example, the lowest level in “the Forbes 400” - the 400 richest Americans - is a candidate for the threshold.

[2006], Cagetti and Nardi [2006], Kitao [2008], and Flodén [2008]. Second, regarding the income process, a parsimonious but more skewed income process is considered in Castaneda et al. [2003]. Whereas Castaneda et al. [2003] look at the channel from income to wealth, Campanale [2007] looks at positive feedback from wealth to the capital income process. From this point of view, this paper is investigating a wealth-dependent labor income process and wealth-dependent saving behavior because of status-seeking as determinants of wealth distribution.

The fact that the saving behavior of the rich is different from the poor is discussed in Carroll [2000]. The functional form suggested by Carroll [2000] is extensively investigated by Francis [2009] and Luo and Young [2009]. I borrow the idea from the literature that wealth may be related with social status. But, I will model the idea in a simple way: there is a direct utility gain by wealth only after some threshold of asset space. It turns out that this simple modification helps to match the high share of the top 1% in wealth distribution *given wealth dependent labor income shock with misspecification*.

So, the main contribution of this paper is to pay attention to the labor markets regarding wealth distribution whereas other papers normally focus on capital markets. Rather than denying the importance of capital market in explaining a fat-tailed distribution of wealth, the point of this paper is that if we take into account frictions from labor markets, it seems to be easier to generate realistic distributions of wealth by incomplete market models. Moreover, since the feedback from wealth to labor income generally hinges on *historical*, *institutional*, and *social* factors, the model highlights these factors as determinants of the distribution of wealth. In this respect, this paper is in line with Mulder et al. [2009] where they provide evidences for the argument that both the technology and the institutions and norms are important in understanding the differences in "intergenerational wealth transmission" of different economic systems. I also expect that this approach is helpful in explaining *temporal* or *cross-country variations* of wealth distribution. Besides, from a policy perspective, controlling the distribution of wealth within some tolerable range is might be desirable since too high wealth inequality may jeopardize the social stability. In this context, this paper shed some light on the question of what would happen in general equilibrium framework if policies such as enhancing public education, aiming at reducing the feedback from wealth to labor income, are employed.

Another contribution of this paper is to discuss the Pareto distribution of wealth in the framework of incomplete market models. A Pareto distribution, or more generally a power law distribution has been often discussed as a characterization of the right-tail in wealth distribution.[8] Interestingly, several examples of fat-tailed wealth distributions before modern societies were found. For examples, Abul-Magd [2002] provides evidence in ancient Egypt with respect to house area and Hegyi et al. [2007] reports evidence in medieval Hungary with respect to the number of owned serf families. Those findings were one of the motivations of this paper in considering status-seeking behavior because they imply that we need general principles which apply to both ancient and modern societies. Several models have been proposed to explain fat-tailed distributions of income and wealth.[9] Recent contributions on the Pareto distribution of wealth are Levy [2003] and Benhabib and Bisin [2009].

The composition of this paper is the following: in Section 2.2, I provide empirical and theoretical evidence for the wealth-dependent labor income process. In Section 2.3, I introduce wealth-dependent labor income shock into Aiyagari [1994] and in Section 2.4 I analyze the effect by numerically solving the model (value function iteration). Section 2.5 is the conclusion.

2.2 EMPIRICAL INVESTIGATION

2.2.1 Review of mechanisms for the feedback from wealth to labor income

I list four mechanisms for the feedback from wealth to labor income. First, there is the intergenerational channel. Endowment is an important factor in lifetime earning. According to Keane and Wolpin [1997, 515], unobserved endowment heterogeneity at age 16 is responsible for 90 percent of the variation in lifetime utility. Bowles and Gintis [2002] provides a very

[8] Mitzenmacher [2004] and Newman [2005] contain nice discussion about power law distributions. Clauset et al. [2009] provides empirical analysis regarding power law distributions.

[9] Gabarix [2009] provides an extensive literature review on empirical and theoretical works regarding power laws in economics and finance. Yakovenko and Rosser [2009] reviews "econophysics" approach on distribution of income, money, and wealth.

interesting viewpoint related to the channel.

> It thus seems likely that the intergenerational persistence of wealth reflects, at least in part, parent-offspring similarities in traits influencing wealth accumulation, such as orientation toward the future, sense of personal efficacy, work ethic, schooling attainment and risk taking. Some of these traits covary with the level of wealth: for example, less well-off people may be more likely to be risk averse, to discount the future and have a low sense of efficacy (Bowles and Gintis [2002, 18]).

They speak about the traits related with wealth accumulation. But, similarly, *the traits related with the performance in labor markets such as work ethic, may co-move with wealth.* According to Bowles and Gintis [2002, 19], other variables which affect economic success include group membership. Note that group membership may depend on the level of wealth as well. Another particular mechanism works through the health of children as Currie [2009] shows. Condliffe and Link [2008] provides empirical evidence for the impact of economic status on child health for the U.S. Other mechanisms include education and crime. For example, Akee et al. [2010] investigates the long-run effect of exogenous increase in household income on children's outcome. According to their study, in the case of the poorest households, an additional increase in household income by $4,000 per each year resulted in two things: longer education by one year at age 21 and the reduction of the probability of committing a minor crime by 22 percent for 16 and 17 year olds.

Second channel works through social networks in labor market. For example, Weinberg et al. [2004] reports that the improvement of one standard deviation in the qualities of a neighborhood leads to increase in yearly working-hours by 6.1%. Weinberg et al. [2004] also finds that the effect is nonlinear in the sense that the effect is stronger for the poor. Magruder [2010] investigates intergeneration networks in South Africa. He documents that the fathers' network connections may account for a one-third increase in the employment rates of their sons.

Third, occupational choice like being entrepreneurs may depend on initial wealth. In general, more wealth enlarges the size of a choice set for individual utility optimization particularly when financial markets are not perfect (and in an actual economy, for example, asymmetric information and moral hazard make financial markets imperfect). Because the choice cannot be worse with a bigger choice set, we can expect some positive correlation between outcome and wealth. But, empirical evidence seems to be mixed. For example, Mondragón-Vélez [2009] documents evidence for a hump-shaped transition probability, whereas Hurst and Lusardi [2004] suggests that wealth matters only after the 95th percentile in wealth.

Finally, the reservation wage depends on the level of wealth. Bloemen and Stancanelli [2001] reports that financial wealth has a significantly positive impact on reservation wage whereas financial wealth has a negative impact on the probability of employment.

I do not explicitly model the above-mentioned channels. Rather, I model an implication of the channels on the labor income shock process. In other words, based upon the literature, I assume that the transition probability of labor income shock depends on *not only the current labor income but also the current wealth.* Besides, the literature implies the *nonlinear* effect of wealth on labor income. That is, the effect of wealth on labor income seems to be more important to the poor, which is a rationale for the nonlinear specification of wealth effect in this paper. I expect that these modifications of the stochastic endowment process will help to generate the highly concentrated distribution of wealth.

2.2.2 Review of distributions of labor income and wealth from the Panel Study of Income Dynamics (PSID)

In this Subsection, I briefly review the main characteristics of distributions of labor income and wealth from PSID. Particularly, I am interested in a Pareto distribution of wealth. According to Mitzenmacher [2004], a Pareto distribution is defined as follows:

$$Pr[X \geq x] = (\frac{x}{constant})^{-\alpha},\ constant, \alpha > 0, X : random\ variable \tag{2.1}$$

which implies that

$$\log(1 - F(x)) = constant - \alpha \log(x) \tag{2.2}$$

where F denotes the cumulative distribution function (=c.d.f.). The coefficient α is called *Pareto exponent* in this paper. Note that rare events or extreme values cannot be ignored if a random variable follows the Pareto distribution since the Pareto distribution has a fatter right-tail than other distributions such as a normal distribution.[10]

2.2.2.1 Labor income Tables 17 and 18 in Appendix D summarize several descriptive statistics for head's wage and labor income. It is interesting to notice from Tables 17 and 18 that the coefficients of variation ($:=\frac{standard\ deviation}{mean}$) of head's yearly wage and labor income are not very different from 1 except 2002 and 2004, given the fact that the coefficient of variation of the exponential distribution is 1.[11]

The higher coefficient of variation in recent years reflects rising inequality documented in Kennickell [2009].[12] Particularly, we can see the rising tendencies of share of the top 1% and the Gini coefficient of the top quartile from Tables 17 and 18. Autor et al. [2006, 189] characterizes the past trend of the US labor market as a "polarization".

Another interesting point regarding labor income is that labor income becomes more important in the composition of income. Kennickell [2009] reports the share of interest, dividends and capital gains in income composition declined from 1989 to 2007. Similarly, Piketty and Saez [2007, Figure 2] documents that the rising share of wage income in income composition of top 0.1% of tax units was important in the surge of the share of the top 0.1% from the 1970s.

2.2.2.2 Wealth From Table 19 in Appendix D, we can see that the Gini coefficients for wealth is much higher than the Gini coefficients for income.[13] In contrast with labor income, there is no clear rising tendency in the Gini coefficients for wealth.[14] To see the tail distribution of wealth (= V17389, total wealth in 1989 from PSID), I use a log-log plot

[10]For example, it is well-known that stock market returns follow power law distributions (Gabarix [2009, 276-9]).

[11]Using different data, Drăgulescu and Yakovenko [2001] provides some evidence for exponential distribution of individual income in the U.S.

[12]According to Kennickell [2009, Table 3] with the Survey of Consumer Finances (SCF), Gini coefficient of income increases from 0.5399 (1989) to 0.5643 (2001), 0.5406 (2004), and 0.5745 (2007). The long-run effect of rising income inequality is interesting. For example, Iacoviello [2007] shows that rising income inequality was responsible for rising household debt.

[13]To compute a Gini coefficient, I use trapezoidal rule for numerical integration.

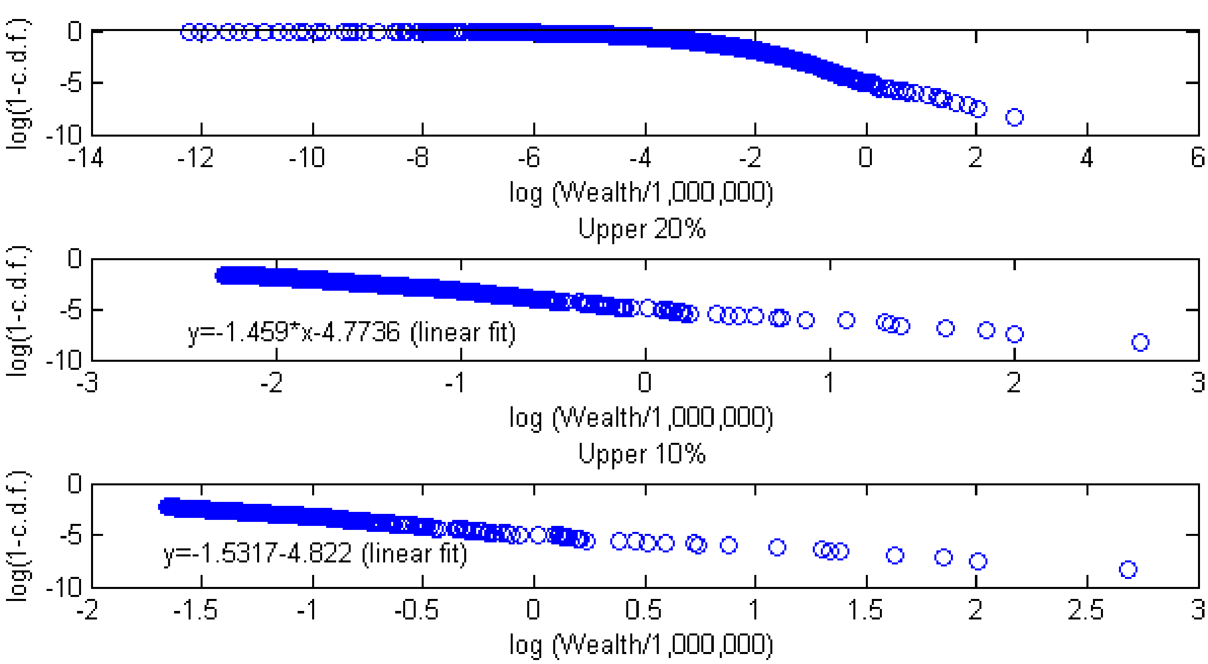
[14]But, according to Kennickell [2009, Table 3], Gini coefficient of net worth becomes higher from 0.7863 (1989) to 0.8030 (2001), 0.8047 (2004), and 0.8120 (2007). And according to Heathcote et al. [2010, 16], the Gini coefficient for net worth in SCF increases by 5 points between 1983 and 2007.

in Eq. 2.2. As we can see from Figure 11 the Pareto exponent of wealth is around 1.5, as mentioned in Gabarix [2009]. To see how stable estimated Pareto exponents of wealth are, I look at nine data sets from PSID. Table 19 in Appendix D shows the Pareto exponents stay within the interval of [1.3, 1.7].

By using Forbes data of 400 richest people in the U.S., Klass et al. [2006, 291] shows that in the period of 1988 - 2003, Pareto exponents belong roughly to the interval of [1.1, 1.6] and that there was a decreasing trend in the 1990s. But as Clauset et al. [2009] shows, it is not clear whether wealth distribution follows Pareto distribution in a strict sense.

Instead of going into the issue, I will focus on the fact that the log-log plot in the right tail of a distribution of wealth is a roughly straight line with the slope around 1.5. After I calibrate the model to match the shares of the top 1%/5%/20% of population in the wealth distribution from data, I will look at whether the model generates a similar slope in a log-log plot.

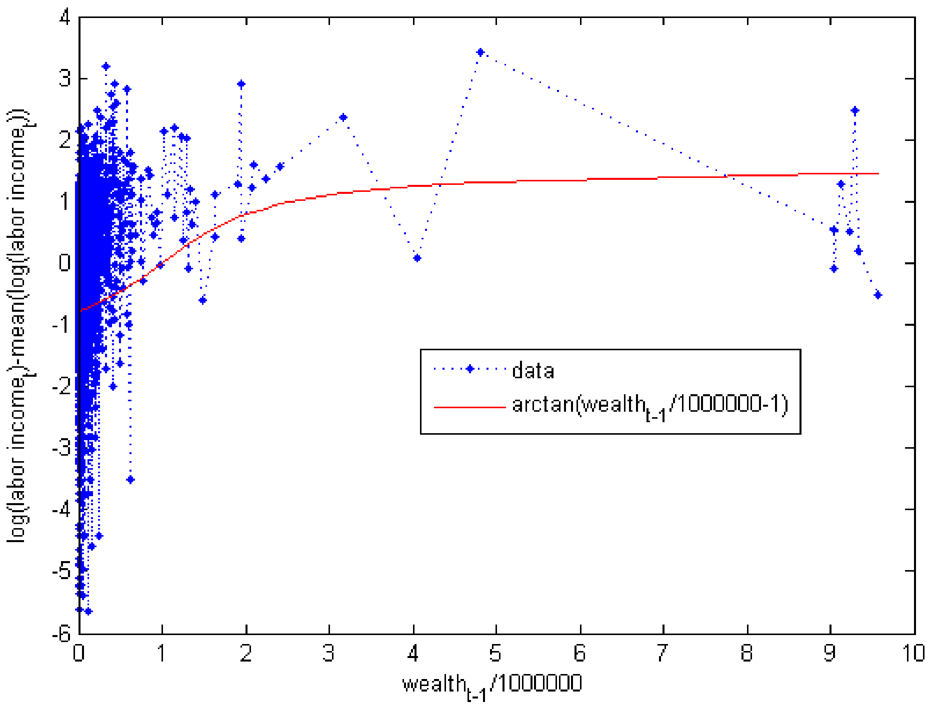
Figure 11: Complementary cumulative distribution function of wealth (data)



2.2.2.3 The relation between labor income and wealth

Figure 12 displays the relation between wealth in 1984 (=S117 in PSID) and labor income in 1985 (=V13624 in PSID).[15] As we can see from Figure 12, the relation between labor income and wealth is not

Figure 12: Data and inverse tangent



linear.

As we see later, possible values of labor income shocks in this paper are exogenously given and the same for the poor and the rich. Different specifications regarding the feedback from wealth to labor income affect only chances of good shocks and bad shocks, more formally, transition probabilities of labor income shock. In this paper I will employ an inverse tangent function to capture the feedback from wealth to labor because of the following three reasons: First, the specification is simple in the sense I need only one parameter (ξ, as we see later) to represent the strength of wealth effect. The feature makes comparative study easier. Second, since the function is nonlinear, the feature helps to capture the nonlinear effect of wealth. Third, since the function is bounded, the function is consistent with the idea that

[15] 1984 is the first year wealth data in PSID is available.

the feedback from wealth to labor income has only a limited impact. Since the function is monotonically increasing and bounded, the specification implies that the marginal effect of wealth on transition probabilities of the labor income shock process is negligible after some threshold. Other possible alternatives include a logistic function and a hyperbolic function as in Campanale [2007]. Note that all of the three functions are monotonically increasing. I leave the task of checking the robustness of the results in this paper by employing a logistic function or a hyperbolic function as a future research topic.

To quantify the effect of wealth on labor income for the model in this paper, I use a regression analysis, while controlling for other important variables such as age, schooling, previous labor income.[16] The results of the regression will guide the choices of values of parameters. First, I normalize wealth(=a) by the following transformation:

$$h(a) := arctan(10a/(a_{\max}) - \tilde{a}^*), \ \tilde{a}^*: \text{ reference level} \tag{2.3}$$

So, transformed data has the domain of $[-\tilde{a}^*, 10 - \tilde{a}^*]$ under the assumption of nonnegative wealth. For computational purposes, I choose the ratio of the max/mean of wealth around 45 in the main models in this paper. In the data (=S117 in PSID) if I drop the wealthiest 9 observations, the ratio of the max/mean of wealth is around 46.2076. For regression analysis, the following five variables from PSID are used: Age of 1984 head (V10419), education 1984 head (V11042), wealth 1984 (S117),[17] total head labor Y 84 (V12372),[18] total head labor Y 85 (V13624). In the model in this paper, I assume that wealth is non-negative because of borrowing constraint. And unemployment is not included in the model. Taking into account the restrictions, I extract data from raw data based on the following criteria:

$$Wealth \geq 0, \ Labor\ income > 0 \tag{2.4}$$

Then labor income is normalized as follows:

$$labor_{i,t} = \log(Total\ Head\ Labor\ Y)_{it} - mean(\log(Total\ Head\ Labor\ Y)_{it}) \tag{2.5}$$

[16]For a recent study on the effect of wealth on earning mobility, I refer to Morillas [2007].

[17]For detailed description of the variable, I refer to Hurst et al. [1998, appendix A]

[18]According to the PSID data custom codebook, labor income includes the followings: labor part of farm income, labor part of business income, head's wages income, head's bonuses, overtime, commissions, head's income from professional practice or trade, labor part of market gardening income, and labor part of roomers and boarders income.

Finally, the regression equations is as follows:

$$
\begin{aligned}
labor_{i,t} = &\beta_0 + \beta_1 age_{i,t-1} + \beta_2 education_{i,t-1} \\
&+ \beta_3 arctan(10(wealth_{i,t-1}/(\max\{wealth_{i,t-1}\})) - \tilde{a}^*) + \beta_4 labor_{i,t-1} + u_{i,t}
\end{aligned}
\tag{2.6}
$$

Regression results in Table 1 show that there is positive impact of wealth on labor income.[19] For a reference level $\tilde{a}^*$, I compare the alternatives, $\tilde{a}^* = 1$ and $\tilde{a}^* = 2$. Table 1 displays that the goodness-fit (adjusted R-square) is marginally better in the case of $\tilde{a}^* = 1$. Drawing upon the result, I will set $\tilde{a}^* = 1$. Based on the regression, I will set ξ $(= \beta_3) = 0.1$, and θ $(= \beta_4) = 0.8$ that belong to the confident intervals with sample size =4,131. ξ $(= \beta_3) = 0.1$ is chosen since it seems that ξ $(= \beta_3) = 0.158702$ is too high in a simulation of model. Instead of 0.780981, θ $(= \beta_4) = 0.8$ is chosen to emphasize the persistence of labor income shock.

Table 1: OLS of 2.6

	β_0	β_1	β_2	β_3	β_4
1984 -1985					
full sample (4,140 obs.), $\tilde{a}^* = 1$	0.113596	-0.004142	0.043190	0.223012	0.783486
C.I.	[-0.0198 0.2470]	[-0.0056 -0.0027]	[0.0320 0.0543]	[0.1059 0.3401]	[0.7632 0.8037]
	(0.136935)	(0.000037)	(0.000000)	(0.000277)	(0.000000)
	adj. R^2	DW	$\hat{\sigma}$		
	0.6511	1.9839	0.5733		
sub-sample (4,131 obs.), $\tilde{a}^* = 1$	0.094253	-0.004859	0.040282	0.158702	0.780981
C.I.	[-0.0173 0.2058]	[-0.0064 -0.0033]	[0.0290 0.0515]	[0.0884 0.2290]	[0.7604 0.8015]
	(0.160559)	(0.000003)	(0.000000)	(0.000024)	(0.000000)
	adj. R^2	DW	$\hat{\sigma}$		
	0.6509	1.9850	0.5727		
full sample (4,140 obs.), $\tilde{a}^* = 2$	0.137604	-0.003831	0.044323	0.190321	0.786217
C.I.	[-0.0354 0.3106]	[-0.0053 -0.0024]	[0.0332 0.0554]	[0.0608 0.3198]	[0.7661 0.8064]
	(0.149740)	(0.000113)	(0.000000)	(0.005404)	(0.000000)
	adj. R^2	DW	$\hat{\sigma}$		
	0.6507	1.9833	0.5738		
sub-sample (4,131 obs.), $\tilde{a}^* = 2$	0.1090	-0.0043	0.0425	0.1454	0.7847
C.I.	[-0.0248 0.2428]	[-0.0058 -0.0028]	[0.0314 0.0537]	[0.0619 0.2288]	[0.7643 0.8051]
	(0.152855)	(0.000021)	(0.000000)	(0.000575)	(0.000000)
	adj. R^2	DW	$\hat{\sigma}$		
	0.6502	1.9842	0.5732		

C.I.: confidence interval
(): White Heteroscedastic Consistent p-value

I also try a quadratic specification for wealth as in Eq. 2.7 as a robustness check. Table 2 shows that the linear and quadratic terms are statistically significant. In particular, thanks

[19] I use Matlab codes in "Econometrics Toolbox" written by James P. LeSage (LeSage [1999]) for regression .

to the quadratic term, the results confirm the nonlinear effect of wealth as well.

$$\begin{aligned} labor_{i,t} =& \beta_0 + \beta_1 age_{i,t-1} + \beta_2 education_{i,t-1} \\ &+ \beta_3 (10(wealth_{i,t-1}/(\max\{wealth_{i,t-1}\})) + \beta_4 (10(wealth_{i,t-1}/(\max\{wealth_{i,t-1}\}))^2 \\ &+ \beta_5 labor_{i,t-1} + u_{i,t} \end{aligned} \tag{2.7}$$

Table 2: OLS of 2.7

	β_0	β_1	β_2	β_3	β_4	β_5
1984 -1985						
full sample (4,140 obs.)	-0.050999	-0.004473	0.042068	0.253841	-0.022517	0.780308
C.I.	[-0.1354 0.0334]	[-0.0060 -0.0030]	[0.0309 0.0533]	[0.1259 0.3817]	[-0.0367 -0.0083]	[0.7598 0.8008]
	(0.353637)	(0.000014)	(0.000000)	(0.000174)	(0.005208)	(0.000000)
	adj. R^2	DW	$\hat{\sigma}$			
	0.6514	1.9841	0.5731			
sub-sample (4,131 obs.)	-0.020174	-0.005231	0.039117	0.168928	-0.017730	0.778380
C.I.	[-0.1061 0.0657]	[-0.0068 -0.0037]	[0.0278 0.0504]	[0.0941 0.2438]	[-0.0307 -0.0047]	[0.7577 0.7990]
	(0.715606)	(0.000001)	(0.000000)	(0.000041)	(0.007645)	(0.000000)
	adj. R^2	DW	$\hat{\sigma}$			
	0.6512	1.9860	0.5724			

C.I.: confidence interval
(): White Heteroscedastic Consistent p-value

2.3 MODEL

The model is identical to Aiyagari [1994] except for the labor income shock process. I differentiate the true labor income shock process (=Q) from the perceived labor income shock process (=$\tilde{Q}$).[20]

2.3.1 Household's problem

There exists a continuum of households of measure one. Each household solves the utility maximization problem with known wage (=w), real interest rate (=r), and perceived labor income shock process (=$\tilde{Q}$). Following Aiyagari [1994, footnote 11], I assume the labor supply in the physical term (=labor hour) is fixed. So labor income fluctuates only due to the change of labor in the effective term. l_t represents effective labor at time t. Let a_t and

[20] Appendix E provides general framework of the model including formal definitions of Q and $\tilde{Q}$

c_t be asset and consumption at time t respectively. Then, household is characterized by the state $s_t = (a_t, l_t)$ and a probability measure (λ) is defined on the state.[21] Preference is given by utility function $u(\cdot)$. The maximization problem of individual household is given as follows:

$$\begin{aligned}
&\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&s.t. c_t + a_{t+1} = a_t(1+r) + wl_t \\
&c \geq 0 \\
&a_t \geq -\phi \ a.s.
\end{aligned} \tag{2.8}$$

where β is discount factor, ϕ is borrowing constraint, and E_0 is conditional expectation operator at time 0.

Value function is

$$\begin{aligned}
V(a,l) = \max_{a'} \{u(a(1+r) + wl - a') + \beta \int_Z V(a', l') \tilde{Q}(l, dl')\} \\
s.t.\ a_t(1+r) + wl_t - a_{t+1} \geq 0 \\
a_t \geq -\phi \ a.s.
\end{aligned} \tag{2.9}$$

Note that in Eq. 2.9, the value function is integrated with respect to $\tilde{Q}$ instead of Q. In this sense, the representative agent solves a misspecified problem.[22] In other words, the representative agent does not take into account the fact that different levels of wealth (=a) will induce different labor income processes. Note that since labor income shock depends on the level of wealth as well, the total labor of the economy will depend on the wealth distribution in general, whereas the total labor of the economy is fixed with the exogenous labor shock process. To make the problem simple, I will only consider the stationary equilibrium of the economy in this paper. From Eq. 2.9, policy function $g_{\tilde{Q}}(s; w, r)$ is obtained.

[21] For more detail, I refer to Ljungqvist and Sargent [2004, 570].

[22] Actually, Eq. 2.9 is the set-up in Aiyagari [1994]. It is called a misspecified model because I assume true process also depends on wealth.

2.3.2 Wealth-dependent labor income shock: finite approximation of Q and $\tilde{Q}$

To approximate Q, I follow Tauchen [1986] with a straightforward modification. For the *true* labor income shock process (Q), I assume the following stochastic law of motion:

$$l_t = \theta l_{t-1} + \xi h(a_{t-1}) + u_t, \ \theta, \xi, \ and \ h' > 0, u_t \sim Normal(0, \sigma_u^2) \tag{2.10}$$

where $l_t : log(y_t'), y_t' := \frac{y_t}{\bar{y}}, \bar{y} = \exp^{E(log(y_t))}, y_t$: labor income, a_{t-1} : wealth. The function h in this paper is specified by Eq. 2.3.[23] Labor income shock ($=\tilde{l}_t$) takes discrete values as follows:

$$\tilde{l}_t \in Z := \{\bar{l}^1, \cdots, \bar{l}^N\} \ where \ \bar{l}^1 \leq \cdots \leq \bar{l}^N, N = 7 \tag{2.11}$$

As in Tauchen [1986], $\tilde{l}^i$ is determined by the following procedure: first, $\bar{l}^N$ is chosen as a multiple m of σ_u where σ_u is the standard deviation of random shock in the labor income process. Second, $\bar{l}^1 = -\bar{l}^N$, Finally, the remaining is equally distributed over the interval $[\bar{l}^1, \bar{l}^N]$. Let $w := \bar{l}^k - \bar{l}^{k-1}$. Then, the transition probability from $\bar{l}^j$ to $\bar{l}^k$ given $(\bar{l}^j, a)$ is the following:

$$\begin{aligned} p_{jk}(a) :&= Pr[l_t = \bar{l}^k \mid l_{t-1} = \bar{l}^j, a_{t-1} = a] \\ &= Pr[\bar{l}^k - w/2 \leq \theta\bar{l}^j + \xi h(a) + u_t \leq \bar{l}^k + w/2 \mid l_{t-1} = \bar{l}^j, a_{t-1} = a] \\ &= F(\frac{\bar{l}^k + w/2 - (\theta\bar{l}^j + \xi h(a))}{\sigma_u}) - F(\frac{\bar{l}^k - w/2 - (\theta\bar{l}^j + \xi h(a))}{\sigma_u}) \end{aligned} \tag{2.12}$$

where F is a cumulative distribution function of $Normal(0, \sigma_u^2)$. I will also use a finite number of states for asset space to employ finite state approximation for dynamic programming. Let $X := \{a_1, a_2, \cdots, a_K\}$ where K is the number of grid points for wealth. Let P(Z) be the power set of Z. With state space $S = X \times Z$, $p_{jk}(a)$ defines a true labor income shock process Q on $(S, P(Z))$ in the following way.

$$Q((a_i, \bar{l}^j), A) := \sum_{\bar{l}^k \in A} p_{jk}(a_i), A \subset Z \tag{2.13}$$

A perceived labor income shock process $\tilde{Q}$ in this paper is defined as a special case of Q with $\xi = 0$ in Eq. 2.12.

$$\tilde{Q}(\bar{l}^j, A) := Q((a_i, \bar{l}^j), A)_{|\xi=0} \tag{2.14}$$

[23] So, the choices of values of θ and ξ depends on the estimation in this paper

Table 20 in Appendix F displays a part of Q. A perceived labor income shock process $\tilde{Q}$ corresponds to a transition matrix with a_{t-1}=30. Now, if the person becomes the poorest (a_{t-1}=0) and if the labor income in this year is $\bar{l}^1$, then the probability of the same labor income in the next year is 50.811% . If the wealth in this year is a_{t-1}=30, the probability is 43.382%. If the person is the richest, the probability is 30.351%.

Summing up, given (a, l) of the state variables in the current period, household determines the level of wealth in the next period ($= a'$) by policy function $g_{\tilde{Q}}(s; w, r)$. Then by the nature (=Q), taking into account (a, l), the level of labor income ($= l'$) in the next period is determined. Note that this feature is the only deviation from Aiyagari [1994] where the nature determines (l') based on (l) alone. Then, in the next period, given (a', l'), the process will continue. Note that this procedure defines a stochastic process on the state space of (wealth×labor). Proposition E.0.1 in Appendix E formally shows that this procedure defines a transition function on the state space.[24] And since the state space is finite in numerical computation, the transition function can be expressed by a finite Markov chain which always has a stationary distribution. The uniqueness of a stationary distribution is guaranteed if the Markov chain is irreducible (Durret [1999, 55]).

2.3.3 Stationary Competitive Market Equilibrium

2.3.3.1 Definition of Stationary Competitive Market Equilibrium There exists a firm with the following aggregate production function:

$$F(K, L) = AK^{\alpha}L^{1-\alpha} \tag{2.15}$$

Following Ljungqvist and Sargent [2004, 573], I define stationary competitive market equilibrium $(g_{\tilde{Q}}(s), \lambda, K, L, r, w)$ as the following:

1. $g_{\tilde{Q}}$(s) solves the household's problem.
2. The probability measure λ is stationary with respect to adjoint operator $T^*_{\tilde{Q},Q}$.[25]

$$T^*_{\tilde{Q},Q}\lambda = \lambda \tag{2.16}$$

[24] Note that the Proposition E.0.1 is a minor modification from Theorem 9.13 in Stokey and Lucas [1989, 284] due to endogenous shock.

[25] The definition of adjoint operator $T^*_{\tilde{Q},Q}$ is given in Appendix E

3. Markets clear

$$K = \int_S g(s)\lambda(ds) \tag{2.17}$$

$$L = \int_S proj_Z(s)\lambda(ds) \tag{2.18}$$

4. Each factor price is competitively determined.

$$w = F_L \tag{2.19}$$

$$r = F_K - \delta \tag{2.20}$$

2.3.3.2 Computation of an equilibrium

Assumption 2.3.1. Labor income shock has bounded support.

$$l \in [\underline{l}, \bar{l}], \underline{l} > 0 \tag{2.21}$$

Assumption 2.3.2. I assume that there is a physical or institutional upper bound for asset accumulation. And I impose no borrowing restriction ($\phi = 0$). Then, the asset space is bounded:

$$a \in [0, a_{max}] \tag{2.22}$$

Assumptions 2.3.1 and 2.3.2 guarantee that the state space is compact. The argument for the existence of equilibrium is similar to Aiyagari [1994] with a minor modification due to endogenous determination of an aggregate amount of effective labor. I briefly review the argument together with algorithm for computation.[26]

step 1 Given r, w(r) is obtained from Eq. 2.19 and 2.20.

$$w(r) = (1-\alpha)(A(\alpha/(r+\delta))^{\alpha})^{1/(1-\alpha)} \tag{2.23}$$

step 2 Given r and w(r), policy function $g_{\tilde{Q}}(s; r, w(r))$ is given as a solution of Eq. 2.9. Particularly, with a finite state space, a policy function can be written as a matrix, $G_{(K\times N)}$.

[26] I use Matlab codes in Ljungqvist and Sargent [2004, 584, footnote 9] with a modification due to endogenous shock.

step 3 Policy function $g_{\tilde{Q}}(s; r, w(r))$ and the true labor income shock process Q define a transition function $P_{\tilde{Q},Q}$. More concretely, let kth state be $s_k = (a_i, \bar{l}^j)$ where $k = (i-1)N + j$. Similarly, let k'th state be $s_{k'} = (a_{i'}, \bar{l}^{j'})$ where $k' = (i'-1)N + j'$. Then, similar to Ljungqvist and Sargent [2004, 569],

$$P_{\tilde{Q},Q}(k,l) = Prob(s' = s_l \mid s = s_k) = \begin{cases} Q((a_i, \bar{l}^j), \{\bar{l}^{j'}\}) & if\ g_{\tilde{Q}}(s_k; r, w(r)) = a_{i'} \\ 0 & otherwise \end{cases} \tag{2.24}$$

where Q comes from Eq. 2.13. Note that if $\tilde{Q}$ replaces Q in Eq. 2.24, then the model becomes identical to Aiyagari [1994]. The transition function $P_{\tilde{Q},Q}(k,l)$ can be expressed as a matrix $P_{(K\times N)\times(K\times N)}$ if state space is finite. Then stationary probability measure $\lambda(r)$ satisfying Eq. 2.16 can be computed from the Markov chain $P_{(K\times N)\times(K\times N)}$. $\lambda(r)$ can be written as $\Lambda_{(K\times N)}$ in a matrix form.

step 4 Given stationary probability measure $\lambda(r)$, the aggregate supply of effective labor $L^s(r)$ is determined from the right-hand side of Eq. 2.18. For a finite state space S, the state space can be written as $S_{(K\times N)}$ in a matrix form. Let's define a projection map $proj_Z$ from S to Z by

$$proj_Z((a_i, \bar{l}^j)) = \bar{l}^j, (a_i, \bar{l}^j) \in S \tag{2.25}$$

Applying the projection map to $S_{(K\times N)}$ componentwise, we have the following matrix:

$$L^s_{(K\times N)} = proj_Z(S_{(K\times N)}) \tag{2.26}$$

Then, labor supply (=L^s) as finite version of right-hand side of L^s, is given by

$$L^s = < L^s_{(K\times N)}, \Lambda_{(K\times N)} > \tag{2.27}$$

where $< \cdot, \cdot >$ denotes an inner product in $\mathbb{R}^{(K\times N)}$. Under the condition that the labor market is clear in Eq. 2.18, I put $L(r) = L^s(r)$ where L depends on r via stationary probability measure $\lambda(r)$.

step 5 Given r, w(r), $g_{\tilde{Q}}(s; r, w(r))$, $\lambda(r)$, and L(r), we can obtain $K^d(r, L(r))$ from Eq. 2.20

$$K^d(r, L(r)) = L(r)(\frac{A\alpha}{r+\delta})^{\frac{1}{1-\alpha}} \tag{2.28}$$

step 6 Then, we can compute the distance: $|ED(r) := (K^d(r, L(r)) - \int_S g(s; r)d\lambda)|$ from 2.17. A finite version of $K^s = \int_S g(s; r)d\lambda)$ is given by

$$K^s = < G_{(K\times N)}, \Lambda_{(K\times N)} > \tag{2.29}$$

step 7 If some convergence criteria are not satisfied, r is updated by some numerical methods for root finding and the computation restarts with step 1.

In practice, I use the bisection method for an update rule as in Aiyagari [1994] and the following two distances as convergence criteria:

$$|\frac{r_t - r_{t-1}}{r_{t-1}}| < \varepsilon_1 \; and \; |ED(r)| < \varepsilon_2 \tag{2.30}$$

To show the existence of equilibrium with $r > -\delta$, it suffices to show that ED(r)=0 has a solution in $r > -\delta$. The main difference from Aiyagari [1994] is in Eq. 2.28. In Aiyagari [1994], L is a constant. But in this model, L depends on r due to endogenous shock. And L(r) seems to be an increasing function of r from numerical computation. So, now $K^d(r, L(r))$ may not be a strictly decreasing function of r in some range of r. But, this does not preclude the existence of equilibrium since we still have

$$\lim_{r\downarrow -\delta} K^d(r, L(r)) = \infty, \lim_{r\to\infty} K^d(r, L(r)) = 0 \tag{2.31}$$

due to boundedness of L. K^s is bounded as well, because asset space is also bounded ($0 \leq \int_S g(s; r)d\lambda \leq a_{max}$). Therefore, the existence of equilibrium is guaranteed with $r > -\delta$. But, the uniqueness of equilibrium is not guaranteed. And since now $K^d(r, L(r))$ may not be a strictly decreasing function of r in some range of r, it is more plausible to obtain multiple equilibria than in the case of the exogenous shock model.

2.4 RESULTS

2.4.1 The effect of wealth-dependent shock

Wealth-dependent shock in this Section always means wealth-dependent shock with *misspecification* since I assume that people do not consider the dependence.

2.4.1.1 Calibration The utility function is a constant relative risk aversion (=CRRA) function.

$$u(c) = \frac{c^{1-\mu} - 1}{1 - \mu} \tag{2.32}$$

A relative risk aversion coefficient ($= \mu$) is 1.5 as in Cagetti and Nardi [2006, 849]. θ in Eq. 2.10 is 0.8 roughly as estimated in this paper (Table 1). Then, I calibrate standard deviation ($=\sigma_u$) in Eq. 2.10 to match the coefficient of variation of labor income in data ($\simeq 1$) with a benchmark model where no endogenous shock and no status-seeking are considered. I set $\sigma_u = 0.42$. Table 3 summarizes values of all parameters in this paper. Other model-specific parameters will be explained later.

Table 3: Parameter values

parameter		
(Technology)		
Capital share α	0.36	Aiyagari [1994]
Depreciation rate of capital δ	0.08	Aiyagari [1994]
(Preference)		
Discount factor β	0.95	0.96 in Aiyagari [1994]
Relative risk aversion coefficient μ	1.5	Cagetti and Nardi [2006, 849]
(Spirit of capitalism)		
Reference level γ	$\{50, 150, 250, 500\}$	See the text
s	1.385	See the text
(Status seeking)		
s_s	0.00015	calibrated in this paper
(Labor endowment shock)		
Standard deviation σ_u	0.42	calibrated in this paper
coefficient of l_{t-1}, θ	0.8	calibrated, based on the estimation in this paper
m	3	Aiyagari [1993, footnote 33]
(Wealth-dependent shock)		
ξ	0.1	calibrated, based on the estimation in this paper
Reference level $\tilde{a}^*$	1 (=one-tenth of a_{max})	See the text
Upper bound for asset a_{max}	300	See the text
(Borrowing constraint)		
ϕ	0	No borrowing constraint

Now I try to quantify the effect of wealth dependent shock. Because I do not take into account frictions from capital market directly the goal of this exercise is assessing the relative importance of mechanisms introduced in this paper within plausible values of parameters. Choices of $\tilde{a}^*$ is based on the regression analysis. And θ and ξ are calibrated, based on the regression analysis as explained in the Section 2.2.2.3. I set $a_{max} = 300$ to match the ratio of the max/mean of wealth around 45 (see Table 4 and 5).[27] Table 20 in Appendix F illustrates finite approximation of Q with this specification.

2.4.1.2 The effect of wealth-dependent shock Table 4 reports the results for a benchmark model and a wealth-dependent labor income shock model. Grid points are uniformly distributed in logarithmic scale as in Francis [2009]. The statistics from data (SCF) come from Rodríguez et al. [2002, Table 1]. Note that statistics from SCF are a little higher than from PSID. We can see improvements by wealth-dependent shock in several dimensions compared with the benchmark model. Particularly, first, the Gini coefficient of wealth increases from 0.5157 to 0.6774. Second, the share of the top 1% increases from 5.86% to 13.47%. Finally, the Gini coefficients in quartiles of wealth become closer to the data. Recently, Sierminska et al. [2006, Table V] reports the distribution of net worth for five countries (Canada, Finland, Italy, Sweden, and US). Among them, the result with endogenous shock in Table 4 is closest to the distribution of net worth in Finland where shares of the top 1% / 5% are 13% / 31% respectively and the Gini coefficient of net worth is 0.68 according to Sierminska et al. [2006, Table V]). The result is also comparable with Campanale [2007] where the model with the assumption of increasing returns to saving matches the data except the top 1 percentile of wealth. So, it seems that the wealth-dependent (labor or capital) income process helps to understand wealth concentration except the very rich. But, it is also clear that the wealth-dependent labor income shock alone cannot match the share of the top 1%. So, to explain the share of the top 1%, I will try to modify the assumption about preference based on the observation that models with wealth-dependent (labor or capital) income process alone are not so successful in matching the share of the top 1% of the U.S. In Appendix G, I study wealth-in-utility (or the spirit of capitalism) models as in Carroll [2000], Francis [2009], and Luo and Young [2009] for comparison. The study points out that we need a utility function which generates stronger saving incentive *only when people become*

[27] But, I also report the results with different upper bounds in Table 23 in Appendix F.

very rich in some sense. In the Section 2.4.2, I consider alternative functional forms from this perspective.

Regarding the relation between wealth inequality and interest rate, Table 4 clearly displays the *positive* impact of wealth inequality on interest rate as the interest rate increases from 2.35% to 3.14%.

Table 4: Benchmark model and endogenous shock

	Data	benchmark model	endogenous shock ($\xi_{perceived}, \xi_{actual}$)=(0,0.1)
Number of grid points (wealth)		500	500
$[a_{min}, a_{max}]$		[0,300]	[0,300]
interest rate		0.0235	0.0314
a_{max}/a_{mean}		31.7535	48.1248
Gini coeff. (wealth/ income)	0.803 / 0.553	0.5157 / 0.3840	0.6774 / 0.4184
Gini coeff.(quartile in wealth)		0.4233/0.1654/0.1279/0.2252	0.6072/0.2458/0.1902/0.3854
Top. 1%/5%/20%	34.7 / 57.8 / 81.7	0.0586 / 0.2135 / 0.5425	0.1347 / 0.3768 / 0.7176
Coeff. of variation (wealth)	6.53	1.0482	1.8721
Coeff. of variation (labor income)		0.8735	0.9385
Corr (wealth, labor income)		0.3282	0.3807

2.4.2 Wealth-dependent shock and status seeking

Let's think about Robinson Crusoe living alone on an island. He has 100 apples. It might be plausible that the marginal utility from eating one more apple after he eats 90 apples would be lower than the marginal utility from eating one more apple after he eats 89 apples. Now, let's imagine that Robinson Crusoe goes to a high school in New York. The class size is 100. Suppose that he is ranked 10th in the second math exam. If he is ranked 9th in the third exam, his marginal utility due to his rising ranking from 10th to 9th would not be lower than the marginal utility due to his rising ranking from 11th in the first exam to 10th. In other words, the marginal utility from better status may not be decreasing as higher status gets obtained. And, since being a top student may entail extra utility, the relative distance between the current status and the highest status is an important factor affecting a person's utility.

Drawing upon the above discussion and the study in Appendix G, I look at a straightforward modification of the standard utility function. In other words, I assume wealth induces extra utility after some threshold, and the utility is linearly increasing in wealth. The threshold would be affected by diverse institutional and social factors, including tax systems. In this respect, it is interesting to observe that Hurst and Lusardi [2004] documents the positive relation between wealth and being an entrepreneur holds only after 95% of wealth distribution.

The assumption of status-seeking generates the following utility function:

$$u_n(c_t, a_{t+1}) = \begin{cases} \frac{c_t^{1-\mu}-1}{1-\mu}, & a_{t+1} < (a_{max} - \epsilon_n) \\ \frac{c_t^{1-\mu}-1}{1-\mu} + \frac{s_s a_{max}}{\epsilon_n}(a_{t+1} - (a_{max} - \epsilon_n)), & a_{t+1} \geq (a_{max} - \epsilon_n) \end{cases} \tag{2.33}$$

where the value of s_s is chosen to roughly match the share of the top 1%, if the utility function takes the final functional form in this paper. I call the utility function u_n in Eq.2.33 *the utility function with status-seeking*. So, the utility function with status-seeking is different from the standard utility function only after some threshold $(= (a_{max} - \epsilon_n))$ of asset space. And if wealth reaches the maximum level, the utility gain is $s_s(a_{max})$. Intuitively speaking, this modification does not affect the poor's consumption decisions because they are far away from the threshold.[28] But as a person becomes richer, he or she may pay more attention to the possibility of reaching the threshold, which may induce different consumption decisions from the poor. So, I view this specification as a straightforward modeling of the insight from Carroll [2000] that *the rich are different because they are rich*. Table 21 in Appendix F shows the results of simulation as ϵ_n goes to zero(0). Note that as the threshold becomes higher, the share of the top 1% increases.

$$\begin{aligned} &14.27\%(\epsilon_n = 100) \to 15.94\%(\epsilon_n = 75) \to 19.95\%(\epsilon_n = 50) \\ &\to 20.36\%(\epsilon_n = 40) \to 24.59\%(\epsilon_n = 30) \to 27.73\%(\epsilon_n = 3) \end{aligned} \tag{2.34}$$

Finally, motivated from this observation, I define the utility function with the *strongest* status-seeking in this paper as follows:

$$u(c_t, a_{t+1}) =: \lim_{n \to \infty} u_n(c_t, a_{t+1}) = \frac{c_t^{1-\mu} - 1}{1 - \mu} + s_s(a_{max}) \mathbf{1}_{\{a_{max}\}}(a_{t+1}) \tag{2.35}$$

[28]The threshold level considered in this paper is higher than wealth level of 99% of population

where **1** denotes the indicator function. In other words, u is a pointwise limit of the sequence $\{u_n\}_{n\geq 0}$ where $\epsilon_n \downarrow 0$. Note that u is discontinuous at $a = a_{max}$ whereas u_n is clearly continuous function. But, discontinuity of u at boundary is not a critical issue because close approximation to u by u_n is feasible as we will see later. To highlight the intuition regarding status-seeking, I use the specification in Eq. 2.35 in this paper.

Now, I report the result from the specification in Eq. 2.35 and compare it with the model of wealth-dependent shock in Subsection 2.4.1. Table 5 shows that the strongest status-seeking behavior specified in this paper indeed has sizable effect on the right tail of the wealth distribution. With 500 grid points, first, the Gini coefficient of wealth increases from 0.6774 (wealth dependent shock) to 0.7206. Second, the shares (= 28.38 % / 49.17 % / 77.37%) of the top 1% / 5% / 20% become closer to the data Third, the correlation (= 0.2729) between capital and labor income is lower than the correlation (= 0.3282) between capital and labor income in the benchmark model without wealth-dependent shock. It is interesting to notice that change in the utility function only at the top of asset space induces remarkable impact on wealth distribution. Finally, in order to evaluate the accuracy of the numerical approximation of the value function (=V_{true}), I report the distance of $\|V_{n+1} - V_n\|_\infty$ at the last iteration in the numerical approximation, based on the fact that $\frac{1}{1-\beta}\|V_{n+1} - V_n\|_\infty$ is an upper bound of $\|V_{true} - V_n\|_\infty$ as shown in Judd [1998, 413]. As expected, the distance of $\|V_{n+1} - V_n\|_\infty$ decreases as the number of grid points increases. I also perform an accuracy test developed in Den Haan and Marcet [1994]. The results are reported in Appendix H.

Table 22 in Appendix F displays the results with different values of parameters. As we can expect, higher ξ or s_s induces a higher Gini coefficient. We can see also that an increase in ξ (=change in labor income process) induces a higher share of top 20% compared to an increase in s_s, whereas an increase in s_s (=change in preference at top level of wealth) results in a higher share of the top 1% compared to an increase in ξ, which seems to be consistent with our intuition. Finally, note that the results in Table 5 are not so different from the results in Table 21 in Appendix F with $\epsilon = 3$. So, it seems that the close approximation is feasible.[29]

[29] But, there is one unusual feature of this model: a concentration of measure at the upper bound of asset space. The portion of people in a population at the upper bound is 0.46%. But this is not an inevitable consequence of utility function with status-seeking in general. Table 23 in Appendix F displays that with

Table 5: Wealth-dependent shock and the strongest status-seeking

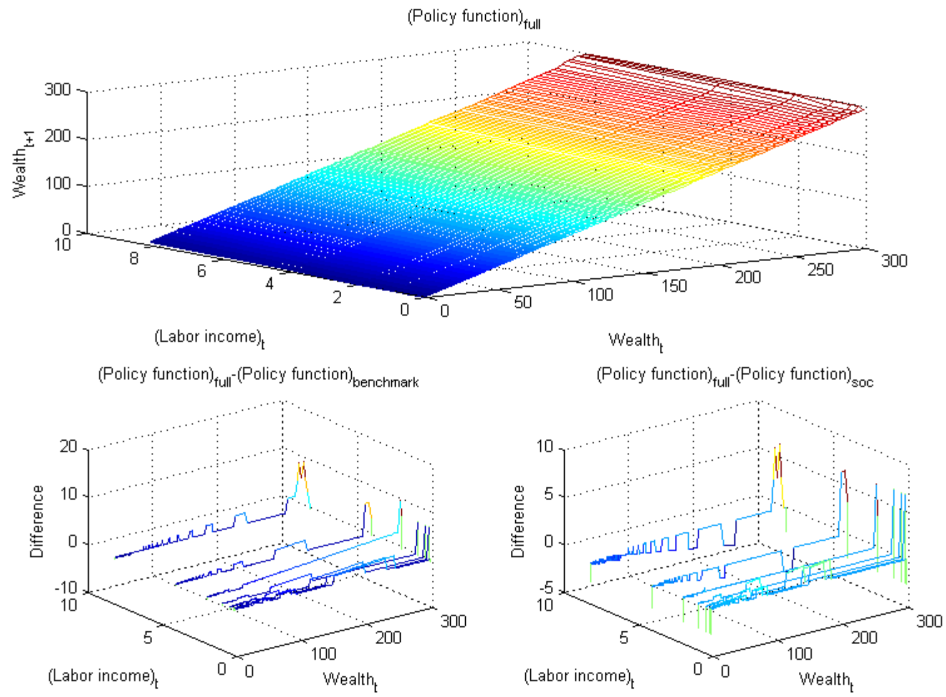
	endogenous shock (ES)	ES+Status-seeking	ES+Status-seeking	ES+Status-seeking
Number of grid points (wealth)	500	500	750	1,000
$[a_{min}, a_{max}]$	[0,300]	[0,300]	[0,300]	[0,300]
Interest rate	0.0314	0.0279	0.0281	0.0280
a_{max}/a_{mean}	48.1248	46.2061	46.3358	46.2347
Gini coeff. (wealth/ income)	0.6774 / 0.4184	0.7206 / 0.4270	0.7176 / 0.4271	0.7183 / 0.4262
Top. 1%/5%/20%	0.1347/0.3768/0.7176	0.2838/0.4917/0.7737	0.2794/0.4846/0.7696	0.2820/0.4861/0.7695
Prob(a_{max})		0.46%	0.46%	0.46%
Coeff. of variation (wealth)	1.8721	3.4123	3.4006	3.4170
Coeff. of variation (labor income)	0.9385	0.9390	0.9383	0.9385
Corr(wealth,labor income)	0.3807	0.2729	0.2709	0.2708
$\|V_{n+1} - V_n\|_\infty$		0.0085	0.0078	0.0034

To see why status-seeking has sizable effect on wealth distribution, I look at policy functions in Figure 13. The top panel of Figure 13 shows policy function of the model with the strongest status seeking. In bottom panels, to find differences among policy functions I look at the difference between policy functions. We can see from the panels that the full model is clearly different from the other specifications especially around the top level of asset.

Since the status-seeking parameter is calibrated to match the right tail of the wealth distribution, it is important to see whether the size of the effect of status-seeking is reasonable. To quantify the implied effect, I calculate the required compensation for the wealth effect. At the top level of wealth, minimum consumption is 8.6895 and maximum consumption is 18.6702. Given that the extra-utility of being at the top level of wealth is $0.00015 \times 300 = 0.045$, required compensation for not being at the top of wealth is the increase in consumption by 14.71% (minimum) or 22.70% (maximum). According to an estimated model in Clark et al. [2009] increase in relative income rank in the small neighborhood from 51% to the 100% corresponds to 0.55 point increase in satisfaction on an 1-6 scale. I also look at whether the slope of a log-log plot regarding Pareto distribution is similar to the data. Figure 14 displays slopes from simulated data. In the second plot, for the top 20% with the exclusion of the last grid, the slope is 1.1745. And if I drop the flatter

larger asset space, the model can generate a similar distribution to Table 5 without concentration of measure on the upper bound. For example, in the case of $a_{max} = 500$, $\epsilon = 250, \xi = 0.12$ and $s_s = 0.00015 \times (3/5)$, Gini coefficient is 0.7282 and the shares of the top 1% / 5% / 20% are 29.31% / 50.15% / 78.07%. But, the portion of people at the upper bound is 0.0087%.

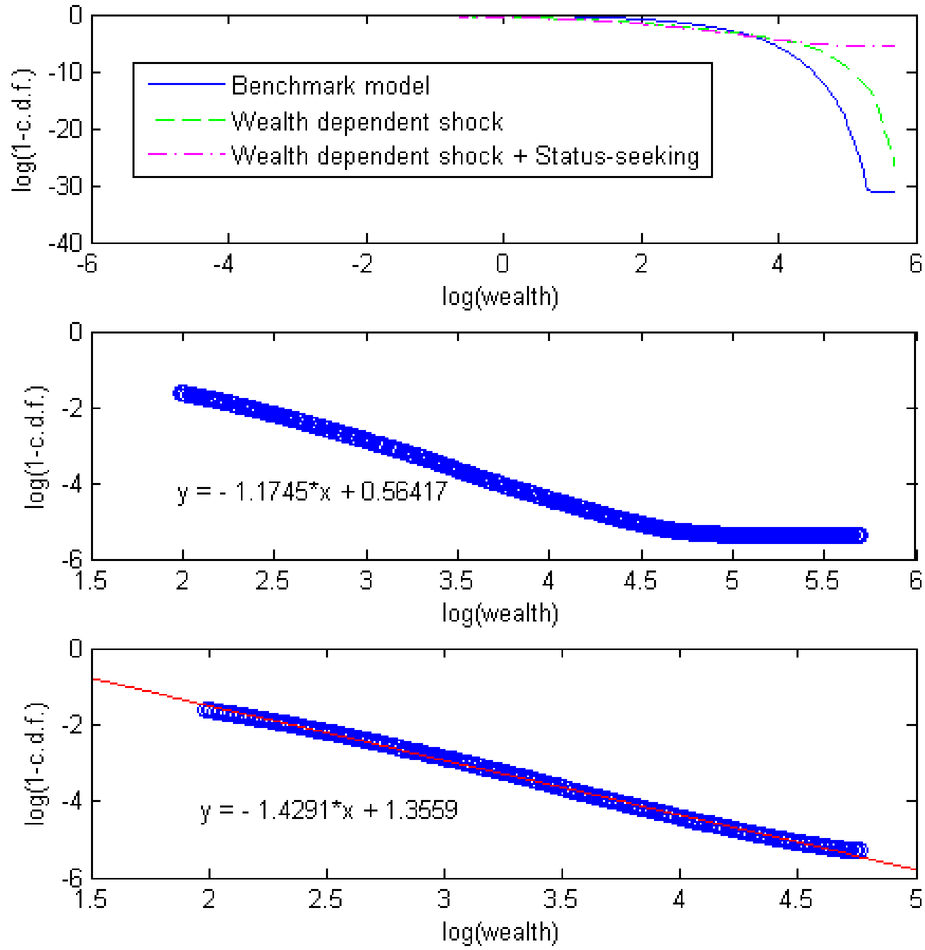
Figure 13: Policy function



part, the slope is 1.4291 for the region of (80th percentile, 99.5th percentile) in the lowest plot. The slope in the data is around 1.5. Figure 37 in Appendix F shows that as ϵ in Eq. 2.33 becomes smaller, the slopes tend to be lower but still higher than 1.4291. For the benchmark model, the slope is 2.8516 and for the wealth-dependent labor income shock the slope is 1.6147. So, we can see that the wealth-dependent labor income shock also plays a role in generating the Pareto exponent around 1.5. Another example is given by Figure 38 in Appendix F with $a_{max} = 500$. In this case, the model can generate roughly a straight line in the range of 80th percentile-99.9th percentile with slope = 1.28.

To see the implication of the full model regarding mobility, I calculate a mobility matrix regarding the wealth quintile based on a transition matrix obtained in the model. Since one-period in the model corresponds to one-year, I calculate five iterations of a one-year mobility matrix for the comparison with the data provided by Rodríguez et al. [2002, Table 15] where five year (1989 $\rightarrow$ 1994) mobility matrix is available. The result is reported in Table 24

Figure 14: Complementary cumulative distribution function of wealth (models)



in Appendix F. In terms of the Frobenius norm, the difference between a mobility matrix generated by full model and data is 0.2141 whereas the difference between the mobility matrix generated by the benchmark model and the mobility matrix from the data is 0.2566. Therefore, we can see the improvement of model performance in this dimension as well.

Note that in Table 5, the interest rate ($\simeq 2.79\%$) is lower than that in the case of wealth-dependent shock alone, which implies that status-seeking may induce the *negative* impact of wealth inequality on the interest rate. Table 22 in Appendix F shows that, given the same size of status seeking ($s = 0.00015$), a stronger effect of wealth-dependent shock ($\xi = 0.11$) results in the higher interest rate ($\simeq 2.82\%$).

To see whether the wealth dependent shock is essential for matching the data, I look at the effect of the strongest status-seeking alone.

Table 6: The strongest status seeking without endogenous shock

	Benchmark	**Status-seeking (SS)**	Endogenous shock (ES)	ES + SS
$[a_{min}, a_{max}]$	[0,300]	[0,300]	[0,300]	[0,300]
interest rate	0.0235	0.0234	0.0314	0.0279
a_{max}/a_{mean}	31.7535	31.7535	48.1248	46.2061
Gini coeff. (wealth/ income)	0.5157/0.3840	0.5163/0.3840	0.6774/0.4184	0.7206/0.4270
Top. 1%/5%/20%	0.0586/0.2135/0.5425	0.0586/0.2136/0.5426	0.1347/0.3768/0.7176	0.2838/0.4917/0.7737
Coeff. of variation (wealth)	1.0482	1.0489	1.8721	3.4123
Coeff. of variation (labor income)	0.8735	0.8735	0.9385	0.9390
Corr(wealth,labor income)	0.3282	0.3283	0.3807	0.2729

In Table 6, the numbers for the other three cases ('Benchmark,' Status-seeking (SS),' and 'ES+SS') come from Tables 4 and 5. Table 6 reports that status seeking without the wealth-dependent shock does not generate a fat-tailed distribution under the same parameter values. This is not surprising given that status-seeking is mostly relevant for a very high level of wealth, because without the wealth-dependent shock the chance of entering a high level of wealth is small. So, we can say that the wealth-dependent shock is essential in explaining the division between the bottom 80% and the top 20% and given the wealth-dependent shock, status-seeking speaks to the share of the top 1%. Summing up, two mechanisms play different roles in matching wealth distribution, and so two mechanisms are essential for the result.

Table 7: Models on wealth distribution

	Top. 1%/5%/20%	Gini (wealth)	Main mechanisms
Krusell and Smith [1998, 884, Table 1]	24 / 55 /88	0.82	stochastic discount factor
Quadrini [2000, 32, Table 11]	24.9 / 45.8 / 73.2	0.74	entrepreneurship
Castaneda et al. [2003, 845, Table 7]	29.85 /48.06 / 81.97	0.79	earning process
Cagetti and Nardi [2009, 97, Table 3]	30 / 60 / 85	0.82	entrepreneurship
This paper	28.38 / 49.17 / 77.37	0.72	wealth-dependent shock with misspecifiation & status-seeking

As we can see from Table 7, there are several works on wealth distribution which can match the data very well. My paper is different from others at least in two aspects. First, the main mechanisms in this paper (wealth dependent shock with misspecification + status seeking) are different from others mainly because this model pays more attention to frictions in labor markets than other models. Moreover, two main mechanisms play different roles as we saw before. So, this paper highlights the interaction between two sources of wealth inequality. Second, as we saw, the Pareto exponent ($\simeq 1.4291$) for the region of 80th percentile-99.5th percentile generated by the model in this paper roughly matches the Pareto exponent of wealth ($\simeq 1.5$).

2.5 CONCLUSION

In this paper, I introduce three mechanisms (wealth-dependent labor income shock, misspecification, and status-seeking) to generate wealth distribution similar to data. The main findings are as follows. First, wealth-dependent labor income shock with misspecification helps to explain wealth concentration. Second, the model with spirit of capitalism (or wealth in utility) does not generate wealth concentration like the data with the specification in this paper. But the model suggests the direction of modification of the utility function. Third, in the full model (wealth-dependent labor income shock with misspecification + the strongest status-seeking), a Gini coefficient of wealth (= 0.7206) and shares of the top 1% / 5% / 20% (= 28.38 % / 49.17% / 77.37 %) become closer to the data.

The full model also speaks to other aspects of the data. First, the Pareto exponent ($\simeq 1.4291$) in the region of 80th percentile-99.5th percentile generated by the model in this paper roughly matches the Pareto exponent of wealth ($\simeq$1.5), which is new in the literature to my knowledge. Second, regarding 5-year mobility, the difference between a mobility matrix generated by the full model and data is 0.2141 in terms of Frobenius norm, whereas for the benchmark model the difference is 0.2566.

Finally, I find wealth-dependent labor income shock with misspecification is essential for the result in the sense that the strongest status seeking without the wealth-dependent shock does not generate a fat-tailed distribution under the same parameter values. So, the conclusion is that two mechanisms play different roles in determining wealth distribution. Wealth-dependent labor income shock with misspecification is important in determining the share of the upper 20%. But status-seeking is essential for a Pareto-like distribution and for share of the top 1%.

Findings in this paper have important policy implication. If policy-makers are concerned about the share of the lower 80%, frictions regarding labor market are important.

Regarding a testable prediction, stronger wealth-dependent shock in this paper results in higher wealth inequality and a higher interest rate. In this respect, a long-run relationship between wealth inequality and interest rates is an interesting research topic. Besides, Piketty [2005] documents the remarkable drop of the top 1% income share in 1914-1945 and no recovery until the 1970s. Piketty [2005, 386] points out changes in tax systems as one of possible explanations. It will be also interesting to investigate whether the model in this paper can explain the temporal movement of the top 1% income share by integrating tax systems into the model.

3.0 FROM THE "GREAT MODERATION" TO THE FINANCIAL CRISIS OF 2007-9: ON THE INTERACTIVE DYNAMICS OF SHORT-TERM NOMINAL INTEREST RATES OF THE U.S. AND INTERNATIONAL RISK APPETITE WITH BIFURCATION ANALYSIS

> "The first theorem of financial instability hypothesis is that economy has financial regimes under which it is stable, and financial regimes in which it is unstable. The second theorem of financial instability hypothesis is that over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system (Minsky [1992])."

3.1 INTRODUCTION

The financial crisis of 2007-9 highlights the huge cost of a severe financial crisis. Moreover, Cerra and Saxena [2008] demonstrates that there is no automatic compensating recovery of the output growth after financial crises. Motivated by these observations, in this paper, I study the financial crisis of 2007-9 in order to learn lessons for preventing similar financial crises, or more realistically, for reducing the social cost of the next financial crisis. The financial crisis of 2007-9 was a surprising event to most people. One of the reasons for this surprise is the so-called "Great Moderation" (low volatility of economic variables such as output growth and inflation rate) before the financial crisis of 2007-9.[1] In this respect, the recent financial crisis can be viewed as an example of sudden shift of system dynamics. The ultimate goal of this research is to develop a framework that can coherently explain the very different states of an economy. As we see later, this paper counts on the concept of

[1] Blanchard et al. [2010] contains a nice discussion about the impact of the "Great Moderation" on ways of thinking of macroeconomists and policy makers.

bifurcation to reach the goal.

There are at least four important topics regarding the financial crisis of 2007-9. In chronological order, the first is about the causes of the housing bubble before the financial crisis of 2007-9. The second is understanding the boom and burst process of the housing bubble. The third is about the freeze of interbank markets in August 2007. The fourth is about the spill-over (or contagion) and real effect of the financial crisis after the failure of Lehman Brothers in September 2008. This paper focuses on the first topic and does not speak about the housing bubble.[2] Incorporating housing markets into a model in this paper would be an interesting research topic.

Several authors point out that low interest rates are one of the important sources for the recent housing bubble in the U.S. economy (see, for examples, Brunnermeier [2009, 77] and Allen et al. [2009]). In general, the movement of an asset bubble is closely related with the supply of credit. For example, regarding the "cycle of manias and panics" related with financial crises, Kindleberger and Aliber [2005, 10] points out procyclical movement of the supply of credit as one of sources of the cycle. Recent works such as Reinhart and Rogoff [2008] and Schularick and Taylor [2009] also highlight credit booms as one of the precursors of financial crises. In turn, the supply of credit is naturally related with short-term interest rates. Borio and Zhu [2008] presents a "risk-taking channel" of monetary policy with highlighting the possibility that risk appetite in financial markets are affected by policy interest rates. Similarly, Adrian and Shin [forthcoming] emphasizes the effect of short-term interest rates on "risk-taking capacity" of a banking system. The literature reveals that short-term nominal interest rates can affect the movement of an asset bubble indirectly by changing the supply of credit. Recently, Taylor [2009] documents the deviations of the short-term nominal interest rates of EU and U.S. from the Taylor rule, particularly during the period of low interest rates (2003-2004).

Drawing upon the literature, in this paper, I study the transition from the so-called the "Great Moderation" to the financial crisis of 2007-9 while focusing on short-term nominal interest rate dynamics before the financial crisis of 2007-9.[3]

[2] Allen et al. [2009] contains a nice survey about the literature on financial crises.

[3] The unusual movements in the money markets started on August 9, 2007, as Cecchetti [2009] and Taylor and Williams [2009] point out. Thus, I consider the date as the starting date of the financial crisis of 2007-9.

The co-movement of short-term nominal interest rates across countries is pointed out, among others, by Dokko et al. [2009] and Taylor [2009]. It seems that the data indeed suggests a close co-movement of short-term nominal interest rates. The top panel of Figure 15 shows the time series of a 3-month Euro Interbank Offered Rate (:=EURIBOR , henceforth) of EU and a 3-month Treasury Bill Rate (:= TB, henceforth) of US for 01/04/1999 - 12/29/2006.[4] The bottom panel displays the same data as a trajectory in the space of these two interest-rate variables. We can see that the two short-term nominal interest rates came back together around the initial position. The bottom panel suggests a nonlinear co-movement of the two rates.

The financial crisis of 2007-9 also highlights the close interconnectedness of the world economy, for example, by the unexpected remarkable contagion effects of the U.S. financial crisis. Motivated from the observations, I investigate the short-term nominal interest rates of several countries including the U.S. with special attention to EU-US.

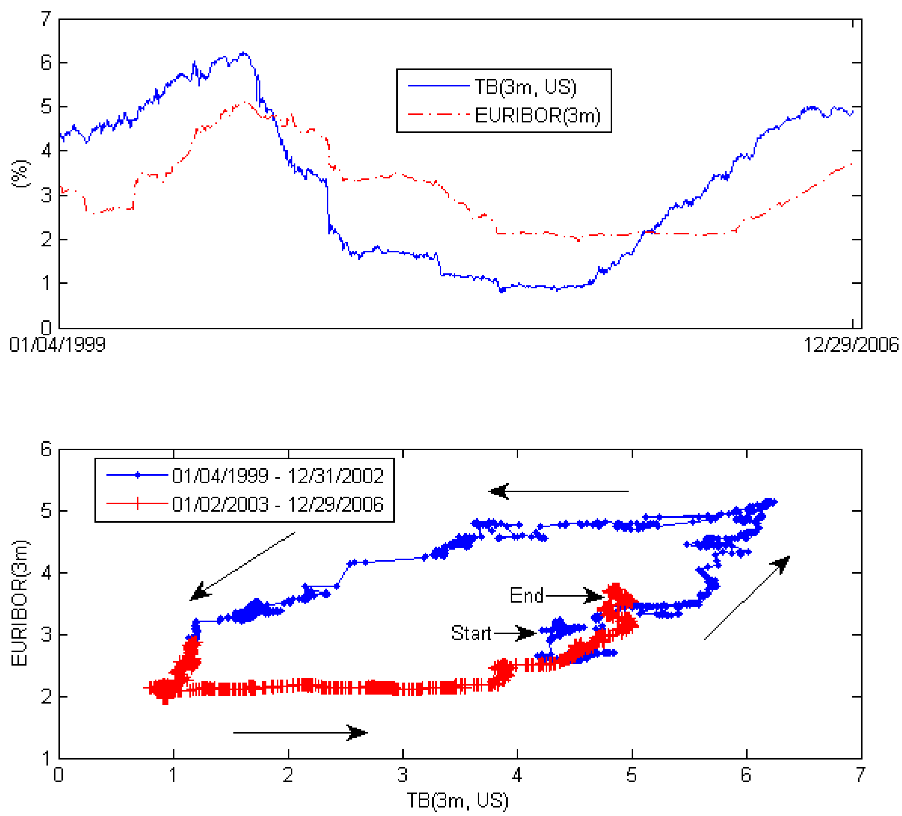
Instead of treating each country's interest rate separately, *this paper aims at contributing to understanding international co-movements and interactions of short-term nominal interest rates by building a simple model.*

The two key mechanisms of the model are as follows: One is the interaction between the short-term interest rates of the U.S. as a key currency country and the risk appetite of the world economy. The other is nonlinear dynamics with heterogeneous agents (domestic vs. international investors).

Regarding the interaction, on the one hand, there is the effect of the risk appetite of the world economy on the interest rates of the U.S. First, the concept of "flight to quality" indicates a higher demand for safe assets such as the U.S. Treasuries in bad times, like financial crises of emerging markets. The literature on global imbalances such as Bernanke [2005] points out Asian countries' demand for safe assets after the Asian crisis of 1997 as one

[4] Sahuc and Smets [2007] and Christiano et al. [2008] study the sources of apparent difference in amplitude in Figure 15 by employing New Keynesian Dynamic Stochastic General Equilibrium (DSGE) models in closed economy setups.

Figure 15: Three-month EURIBOR and Three-month U.S. TB



of reasons of low U.S. interest rates.[5] Second, an interesting question was what would happen if a financial crisis happened in the U.S. As we know now, the answer is that international capital surprisingly inflows to the U.S. in the middle of the financial crisis of 2007-9. These two observations lead to the assumption of the *asymmetric* role of the key currency country as a safe haven in the world economy. The assumption implies that lower risk appetite of the world economy makes U.S. interest rates lower via capital inflows to the U.S. in times of economic distress.

[5]Brunnermeier [2009, 77] points out two sources of lower U.S. interest rates: capital inflow from Asian countries after Asian crisis and benign monetary policy after bursting of IT bubble. Coulibaly and Millar [2008] documents that the share of aggregate investment in GDP in non-China Emerging Asia decreased roughly from 33 percent to 25 percent on average. And they find that this change is related with decline in corporate spending on fixed investment after Asian crisis.

On the other hand, there is literature on the leading role of the U.S. monetary policy in the world economy. In addition, the recent literature on "risk-taking channel" of monetary policy emphasizes the effect of level of short-term interest rates on investors' risk appetite. Following the literature, I assume that the risk appetite of the world economy is also affected mainly by U.S. interest rates.[6]

The second mechanism is nonlinear dynamics with heterogeneous agents. Note that the bottom panel of the Figure 15 looks like a closed curve, which suggests nonlinear movements of interest rates.[7] The unexpectedly huge repercussion of the financial crisis of 2007-9 demonstrated the importance of nonlinear effects in financial markets. Based on these considerations, I add a nonlinear mechanism due to heterogeneous agents (domestic vs. international investors) and international capital flow into a model. The reason a model with heterogeneous agents may have nonlinearity can be illustrated as follows: Let's assume two types of players and call their strategies g(x) and h(x), respectively, where x is a state variable. If the composition $(\alpha, 1-\alpha)$ of two types also depends on state variable x, then, the average of two strategies is given by $\alpha(x)g(x) + (1-\alpha(x))h(x)$, which must be nonlinear as long as α, g, and h are not constants. The distinction between domestic and international investors implies a difference in the portfolios of investors. I assume that international investors are more sensitive on the risks such as an exchange risk. As we see later, domestic (international) investors act as a centripetal (centrifugal) force in the model and international capital flows constantly change the relative strength of two forces. It turns out that this simple set-up can lead to complex dynamics such as chaos. In that respect, this paper highlights the uncertainty endogenously generated by chaotic dynamics. In economics, there is a well-known distinction between risk and Knightian uncertainty. In the case of risk, the distribution of an event is known, whereas in the case of Knightian uncertainty, it is not. Recently, Caballero and Krishnamurthy [2008] highlights Knightian uncertainty as a cause of "flight to quality." Considering the long-run unpredictability of chaos, the similarity between the uncertainty generated by chaotic dynamics and Knightian uncertainty would be an interesting research

[6] A "two-way causality" is an important mechanism in the literate on "symmetry-breaking" as nicely exposited in Matsuyama [2008].

[7] For a recent discussion about the nonlinearity of short-term interest rates such as the Federal funds rate and the 3-month Treasury Bill rate, I refer to Kyrtsou and Vorlow [2009].

topic.

In the literature, the possibility of chaotic dynamics with the Taylor rule in a closed economy is investigated by Benhabib et al. [2002]. Airaudo and Zanna [2005] studies the issue in an open economy. There is extensive literature on nonlinear dynamics with heterogeneous agents in financial markets.[8] Day and Huang [1990] is an early work in this field. This paper is also close to Brock and Hommes [1997, 1998] in this respect.

With the two main mechanisms in the model, the main analytical tool for the stability of dynamics is a bifurcation analysis as a nonlinear version of comparative statics.[9] A sudden and noticeable change of a system like the transition from the "Great Moderation" to the financial crisis of 2007-9 is observed in other several areas such as climate. Scheffer et al. [2009] discusses early warning signals for such "critical transitions" in complex dynamical systems and introduces two types of bifurcations as relevant bifurcations for the "critical transitions": One is a "catastrophic bifurcation." In a "catastrophic bifurcations," a dynamical system has multiple equilibria in some range of parameters and the system undergoes a dramatic change as the system passes thresholds (also called "cusp points" as in Strogatz [2000, 70]). The other is the bifurcations related with the change of attractor from a stable equilibrium point to a cyclic or chaotic attractor. One of the differences in the two types of bifurcations is that in "catastrophic bifurcation" both equilibria (an equilibrium before the bifurcation and another equilibrium after the bifurcation) are locally stable, whereas in the other type, an attractor such as a chaos after a bifurcation is not locally stable. In this paper, I study the second type of bifurcation mainly because it seems that unstable dynamics after bifurcation is more consistent with the data.[10]

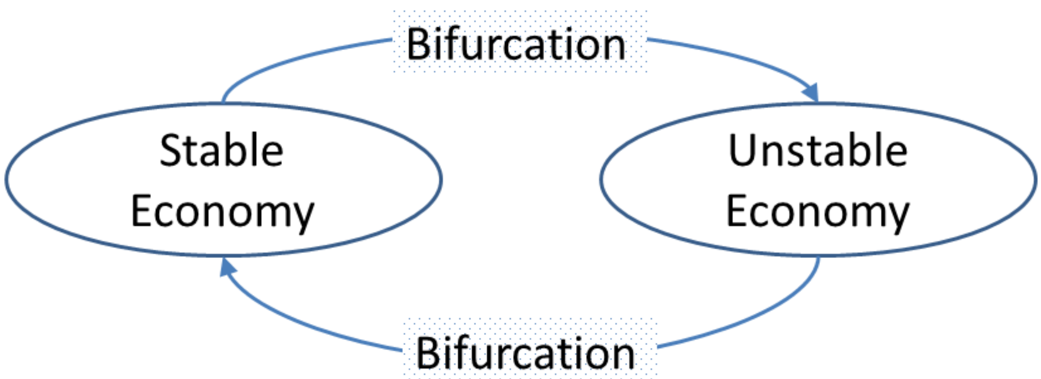
In a nutshell, in this paper, a bifurcation analysis is employed as a unifying device in analyzing two different states of an economy, stable and unstable as illustrated in Figure 16. The main contribution of this paper is two-fold. First, this paper provides a simple model

[8] Hommes [2006] is a nice survey on this subject.

[9] For the formal definition of bifurcation in dynamical systems, I refer to Kuznetsov [1998, 57]. Grandmont [2008] provides a nice overview on the subject.

[10] For more detailed discussion about applications of catastrophe theory in economics, I refer to Rosser [2007].

Figure 16: Regime shift as a bifurcation



for the interaction between the interest rates of the U.S. and international risk appetite. The model may be useful as a thought experiment for policy-making regarding necessary institutional reforms in international financial markets.[11] The bifurcation analysis in this paper sheds some light on the relation between the stability of interest rate dynamics and the institutional environment in the sense that the values of bifurcation parameters naturally depend on institutional environment. The absence of policy variables is one limitation of the model. Adding policy rates into the model may be an interesting extension of the model. Second, the empirical investigation in this paper also contributes to the literature on early warning indicators in financial markets by highlighting the potential of complexity measures

[11] For example, the extension of currency swaps related with the U.S. dollar is in line with the implications of the simple model.

as early warning indicators.

The composition of this paper is as follows: Section 3.2 is a literature review on the interaction between interest rates of the U.S. and international risk appetite. Section 3.3 is an empirical investigation of short-term nominal interest rates of selected countries including the U.S. In Section 3.4.1, I briefly review the other components of the model in this paper. In Section 3.4.2, I present a simple model for short-term nominal interest rates dynamics. Section 3.5 is the conclusion of this paper.

3.2 INTERACTION BETWEEN INTEREST RATES OF THE U.S. AND INTERNATIONAL RISK APPETITE

3.2.1 The U.S. as a key currency country

Because I am interested in the co-movement of international nominal interest rates, it seems that a version of interest rate parity condition, i.e. UIRP(:= uncovered interest rate parity) is a natural starting point.[12] Whereas UIRP treats each country in a symmetric way, one of the lessons from the financial crisis of 2007-9 is the *asymmetric* role of a key currency country like the U.S. as a provider for safe assets. Appreciation of the U.S. dollar after the advent of the financial crisis was not usual, particularly considering that 18 banking crises out of total 23 cases in 1980 - 1995 identified by Kaminsky and Reinhart [1999] were followed by currency crises (see Table 1 in Kaminsky and Reinhart [1999, 477]). Fratzscher [2009, Figure 1] displays that the U.S. dollar appreciated in the middle of the financial crisis of 2007-9 approximately from July 2008.[13]

In addition to the foreign exchange markets, several papers document that the U.S. is a leading country in bond markets as well (see Chinn and Frankel [2005], for example).

[12] One of possible resolution of the difficulty in empirically validating UIRP (for nice survey, see e.g. Engel [1996]) is time varying risk premium. There are several candidates for determinants of time varying risk premium: external habits (Verdelhan [2010]), rare disasters (Gabaix [2008] and Farhi and Gabaix [2008]), heterogeneity with segmented markets (Alvarez et al. [2009]).

[13] Fratzscher [2009] shows that three factors were important in determining the size of appreciation of the U.S. dollar against other currencies after summer of 2008: weight of the U.S. investors in domestic financial markets, size of foreign exchange (FX) reserves, and current accounts.

From these observations, in this paper, I consider the *asymmetric* role of a key currency country as a source of deviation from UIRP, following Blanchard et al. [2005] and Canzoneri et al. [2008].

3.2.2 International risk appetite

According to González-Hermosillo [2008], risk appetite is related with "the willingness of investors to bear risk." In a similar vein, I use the term 'international risk appetite' to indicate international investors' willingness to bear risk.[14] We can observe the co-movement of risk appetites measured by interest rate spreads in markets of the EU and the U.S. as shown in Figure 17. Bekaert et al. [2009] documents the co-movement of risk aversion in Germany and the U.S. So, there may be a common factor which is related with risk aversions of both regions. Indeed, according to Manganelli and Wolswijk [2009, 194] in the literature on bond markets of Euro area, a common factor has been recognized and called an "international risk aversion."[15]

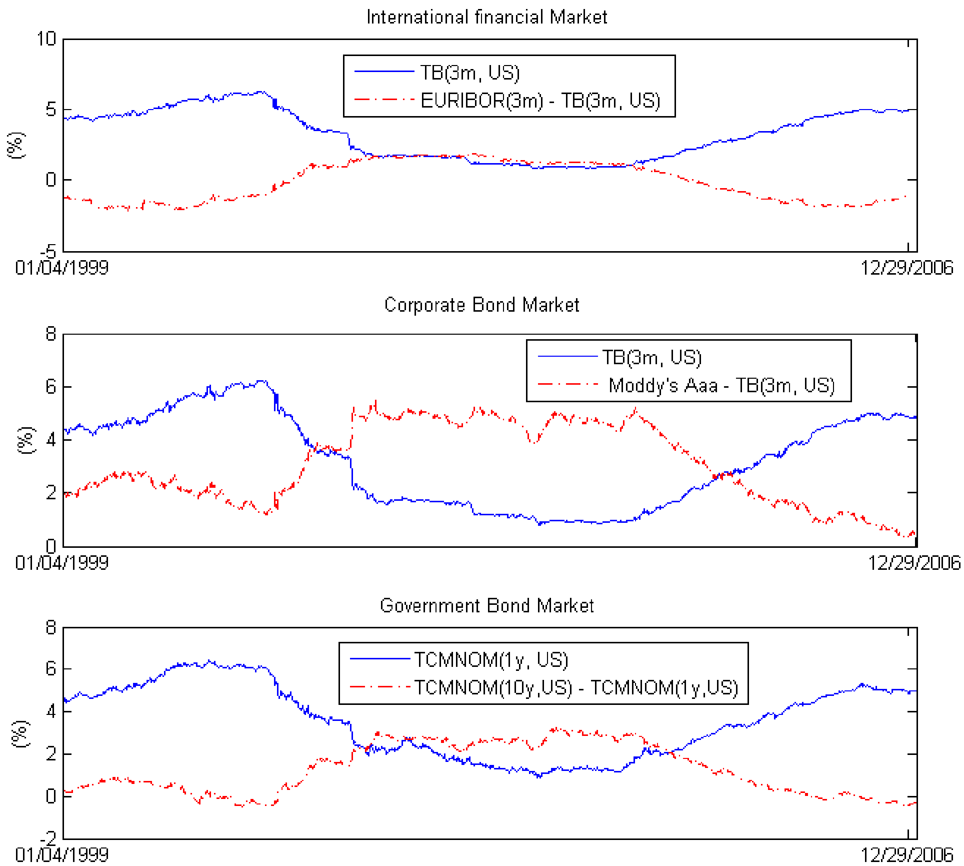
Manganelli and Wolswijk [2009, 194] also points out that the common factor may be affected by the level of short-term interest rates "in normal times." As we see below, there is growing evidence for the interaction between interest rate and risk appetite (or animal spirit in broad sense). For example, Felices et al. [2009] provides the evidence of reciprocal interaction ("flight to quality" and "search for yield") between emerging market sovereign debt spreads and the U.S. interest rates by employing structural VAR. Whereas this paper shares the viewpoints in Felices et al. [2009], special attention in this paper is given to possible *nonlinear effects* in mutual interaction between short-term nominal interest rates of the U.S. and international risk appetite as we see in Section 3.4.2.

Drawing upon the literature, in the model of this paper, I will assume the interaction between interest rates of the U.S. and international risk appetite as illustrated in Figure 18

[14] The term "risk appetite" is also used in Adrian et al. [2009], for example. In this paper, the two terms, risk appetite and risk aversion, are interchangeable. In other words, risk aversion does not mean a structural parameter related with preference.

[15] For Latin America, for example, see García-Herrero and Ortíz [2006].

Figure 17: Interest rates and interest spreads in three financial markets



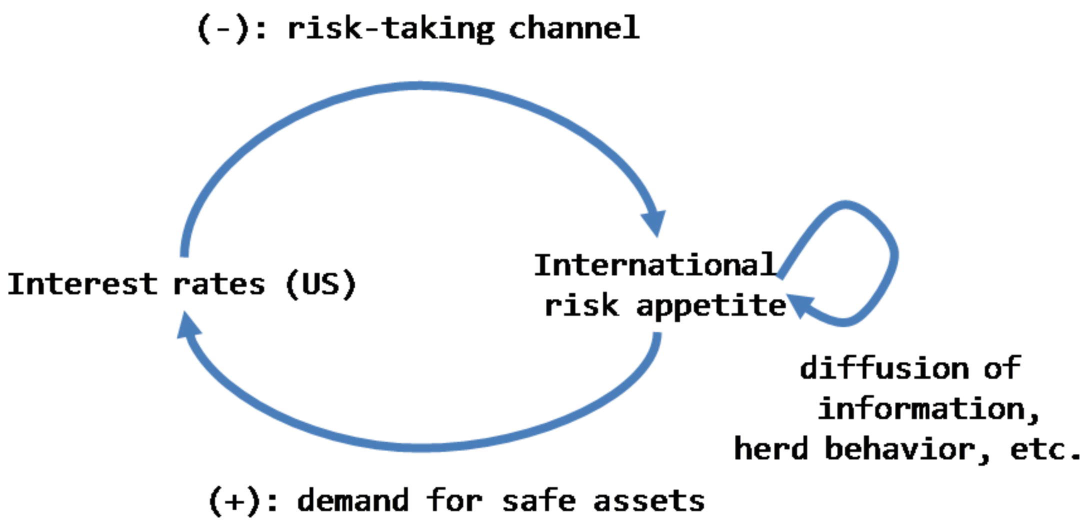
that is similar to the interaction in a well-known predator-prey model.[16]

3.2.3 The interaction between interest rates of the U.S. and international risk appetite

In the following, I review the literature on the interaction between interest rates of the U.S. and international risk appetite.

[16] We can think of interest rates of the U.S. as a predator and the risk appetite of the world as a prey. If I replace international risk appetite with population, the figure illustrates famous Malthusian Trap. Investment demand in Harrod [1939] is another example of self-reinforcing factors. Regarding credit cycle, Kiyotaki and Moore [1997, 215] also uses the analogy of predator (= debt)-prey (= landholding).

Figure 18: Interaction between interest rates (U.S.) and international risk appetite



One direction of the interaction is related with the increase in the demand for safe assets due to lower international risk appetite. And higher demand for safe assets normally induces stronger demand for the U.S. financial assets partially because the U.S. dollar is a key currency, which in turn may lower interest rates in the U.S. (Caballero et al. [2008]).[17]

$$\text{international risk appetite } \downarrow \rightarrow \text{ demand for safe assets} \uparrow \rightarrow \text{ U.S. interest rates} \downarrow \qquad (3.1)$$

[17] The another possible channel is the monetary policy which reacts on unusual behaviors of credit spreads as discussed in Paul and Toloui [2008] and in Taylor [2008]. Cúrdia and Woodford [2009] provides extensive analysis about the issue.

On the other hand, nominal interest rates of the U.S. may affect risk appetite of an economy (see Rajan [2006], Borio and Zhu [2008], and Manganelli and Wolswijk [2009], for instances). The effect of monetary policy on risk appetites of people is called the "risk-taking channel" of monetary policy by Borio and Zhu [2008]. Similar to Manganelli and Wolswijk [2009, 194], I call two main mechanisms of "risk-taking channel" of monetary policy direct and indirect mechanisms, respectively.

First, the direct mechanism also can be called the "search for yield" mechanism as in Rajan [2006, 501]. A lower interest rate means lower return for fund-managers. Hence, the "search for yield" by fund managers can occur given a fixed target rate of return with lower interest rates. And "career risk" will enhance this tendency.[18] The following remark made by Chuck Prince, the former CEO of Citigroup, in an interview with Financial Times succinctly illustrates the logic of the career risk.

> "When the music stops, in terms of liquidity, things will be complicated. But as long as the music is playing, you've got to get up and dance. We're still dancing" (Nakamoto and Wighton [2007]).

In short, the direct mechanism highlights the higher willingness of economic agents to take riskier projects due to low interest rates even if the risk structures of financial assets are the same as before.

$$\text{U.S. interest Rate} \downarrow, \overline{\text{target rate of return}} \overset{\text{(career risk)}}{\rightarrow} \text{international risk appetite} \uparrow \tag{3.2}$$

Second, the indirect mechanism works through the change of *objective* risk structure due to a change of an interest rate. This channel is also closely related with a business cycle. A lower interest rate means a lower borrowing cost for borrowers. Hence, the downside risk of the economy becomes smaller with respect to business cycle when an interest rate becomes lower.

$$\text{U.S. interest Rate} \downarrow \rightarrow \text{downward risk of economy} \downarrow \rightarrow \text{international risk appetite} \uparrow \tag{3.3}$$

[18] Guerrieri and Kondor [2009] is a recent contribution regarding career risk with general equilibrium framework.

Finally, the model in this paper will take into account the self-reinforcing mechanism of international risk appetite as shown in Figure 18 due to the diffusion of information and herd behavior. All in all, the *main contribution* of this paper is to analyze the stability of the dynamical system implied in Figure 18 rather than to add a new channel into the literature.

3.3 EMPIRICAL INVESTIGATION ON SHORT-TERM INTEREST RATES AND INTEREST SPREADS DURING THE "GREAT MODERATION"

In this Section, I empirically investigate short-term nominal interest rates of 10 selected countries (Australia, Denmark, France, Germany, Japan, New Zealand, Norway, Sweden, Switzerland, and UK) and their spreads against TB(3m, U.S.) to see whether there is some evidence for co-movement of international short-term nominal interest rates with TB(3m, U.S.). Figures 40 and 41 in Appendix J display the short-term nominal interest rates of 10 countries against TB(3m, U.S.) for 1990-1998 and for 1999-08/08/2007, respectively. Note that, compared to the trajectories of several countries in Figure 40, those in Figure 41 look more like closed curves. With regard to the finding, I also pay attention to the possibility of the structural change of interest rate dynamics. In particular, I divide the period of 1984 - 08/08/2007 into two roughly equal periods; 1984 - 1995 and 1996 - 08/08/2007. Note that the second period includes several important episodes in international financial markets before financial crisis of 2007-9; Asian crisis (1997-8), the introduction of Euro (1999), the zero interest rate policy in Japan (1999), and the adoption of the Financial Services Modernization Act in the U.S. (1999).

By employing simple correlation analysis, first, I provide empirical evidence for the *co-movement* of international short-term nominal interest rates. The correlation analysis reveals that the co-movement was particularly stronger during the second period (1996 - 08/08/2007). Second, I run a variance ratio test for each sub-sample.[19] In addition to

[19] Charles and Darné [2009] provides a recent review of variance ratio tests. Applying variance ratio test to subsamples to check the possibility of structural break is not new. For example, Liu and He [1991], revisiting random walk hypothesis of nominal exchange rate, divides full sample (08/07/1974 - 03/29/1989) into two subsampels; 08/07/1974 -10/10/1979 and 10/17/1979 - 03/29/1989. Another example is Yilmaz [2003] that uses moving subsample windows to take into account possible structural breaks.

the variance ratio test, I also look at the Durbin-Watson statistic (DW) from the Dickey-Fuller (DF) test for checking robustness, based on the observation that DW of DF test and variance ratio of the variance ratio test tend to move in opposite directions.[20] For the empirical investigation in this Section, I look at daily observations of interest rates of the U.S. and at those of short-term nominal interest rates of other selected countries. The data description is given in Appendix I.

3.3.1 Interest rates and spreads in the U.S.

Table 8 displays that correlations between TB (3m, U.S.) and interest rate spreads, on average, become stronger in the second period (1996-08/08/2007) with respect to the levels and the first differences of variables, respectively.[21] Peculiarly, the sign of the correlation between TB(3m, U.S.) and the credit spread of Baa-Aaa with respect to level changes from positive to negative.

Table 8: Daily correlations between domestic interest spreads and TB(3m,U.S.)

	Full sample	1984 -1995	1996-08/08/2007
FF(overnight) - TB(3m)	0.5873 / -0.1330	0.4646 / -0.0617	0.5114 / -0.1534
CD (3m) - TB (3m)	0.5203 / -0.5621	0.5674 / -0.5369	0.7040 / -0.7978
CP (financial, 3m) - TB (3m)	-	-	0.6902 / -0.7482
ED (London, 3m) - TB (3m)	0.5429 / -0.4072	0.6475 / -0.5195	0.7191 / -0.7339
Baa-Aaa	-0.3372 / -0.0340	0.4000 / -0.0379	-0.5996 / -0.0537
Aaa-TCMNOM (30y)	-0.3852 / -0.2520	-0.0866 / -0.3010	-0.1287 / -0.2723
TCMNOM (10y)-TB (3m)	-0.4073 / -0.7462	-0.3717 / -0.4431	-0.8585/ -0.4130
TCMNOM (30y)-TB (3m)	-0.5498 / -0.7952	-0.5224 / -0.5384	-0.7227 / -0.8620

Corr(x,TB(3m, U.S.)) / Corr(Δ x, Δ TB(3m, U.S.))

Now, regarding random walk tests, to investigate quarterly movements of interest rates and interest spreads, I use time lag = 65 days. The starting date is always the first date of

[20] I use Matlab built-in codes: adftest.m (DF test) and vraiotest.m (variance ratio test). Default options of the codes are maintained for all computations.

[21] Generally speaking, it is well-known that in the U.S. there have been several monetary policy regimes. For this point, I refer to Duffy and Engle-Warnick [2006].

the investigation period.[22] Table 9 shows the result of random walk tests.

First, in the case of spreads, estimated variance ratios of short-term interest spreads are below unity in both periods.[23] For each of the short-term interest rates, the variance ratio test rejects the null hypothesis of random walk at 1% significance level in the second period because the estimated variance ratio is much higher than unity. Moreover, in the second period, estimated variance ratios of short-term interest rates increase for all variables and DWs of DF test for the same variables decrease.

By contrast, for long-term nominal interest rate spreads, the null of random walk is accepted with an exception. And the variance ratio tests do not reject the null for all long-term nominal interest rates in both periods either. Moreover estimated variance ratios of long-term nominal interest rates are below unity in the second period, which is at odds with the result in the case of short-term nominal interest rates.[24]

All in all, the result of the variance ratio test indicates a difference between short-term and long-term interest rates movements. It is an interesting open problem whether the difference is related with the so-called "interest rate conundrum" in 2004-5.[25]

3.3.2 International comparison

Now, for international comparison, I examine short-term nominal interest rates of 10 countries. And since several events that were important in the world economy occurred in 1999, I also look at the period of 1999-08/08/2007. Table 10 reports the result of correlation analysis. In regard to the correlations between TB(3m, U.S.) and interest rates of other countries, there are six countries (Denmark, France, Germany, Japan, Switzerland, and UK) where the correlation becomes stronger in the second period (or in 1999-08/08/2007). Similarly, Table 10 displays stronger negative correlations between international interest rate spreads and TB(3m, U.S.). For the short-term interest rate spreads of 10 countries, the tendency of stronger co-movements with TB(3m, U.S.) in terms of correlation holds without an exception.

[22]For example, in the case of Aaa in Table 9, the staring date for the period of 1984 - 1995 is 01/03/1984. With time lag=65 days, the ending date is 10/27/1995. And the number of observations is 47.

[23]Short-term variables are CD,CP,ED,FF and TB. Aaa, Baa, TCMNOM(10y and 30y) belong to long-term variables.

[24] Table 27 in Appendix K exhibits the results of variance ratio tests with time lag = 30 days.

[25]For the recent contribution on "interest rate conundrum," see, for example, Craine and Martin [2009].

Table 9: Random walk tests of quarterly nominal interest rates and quarterly interest spreads in U.S. (time lag = 65 days)

	1984 - 1995			1996 - 08/08/2007		
Interest spreads						
Variance ratio	obs.	test stastistic (p-value)	ratio	obs.	test stastistic (p-value)	ratio
FF (overnight) - TB(3m)	47	-3.3666 (0.0008)***	0.6061	45	-2.0661 (0.0388)**	0.5463
CD (3m)-TB(3m)	47	-1.7464 (0.0807)*	0.4697	45	-1.8085 (0.0705)*	0.3648
CP (financial, 3m)	-	-	-	41	-1.7548 (0.0793)*	0.5408
ED (3m) - TB(3m)	47	-1.5290 (0.1263)	0.5675	45	-1.9177 (0.0552)*	0.4020
Aaa - TCMNOM (30y)	47	-1.5759 (0.1151)	0.8227	30	0.2720 (0.7856)	1.0751
Baa - Aaa	39	-2.1011 (0.0356)**	0.6830	45	-1.0935 (0.2742)	0.8030
TCMNOM (10y) - TB(3m)	47	-0.1766 (0.8598)	0.9713	45	1.5845 (0.1131)	1.2164
TCMNOM (30y) - TB(3m)	47	0.6420 (0.5209)	1.1018	30	0.0631(0.9497)	1.0126
DF		test statistic (p-value)	coefficient (DW)		test statistic (p-value)	coefficient (DW)
FF(overnight) - TB(3m)		-1.6786 (0.0876)	0.8547 (2.2953)		-3.8102 (0.0010)	0.5722 (2.1787)
CD (3m)-TB(3m)		-2.0280 (0.0418)	0.8341 (2.8485)		-1.6663 (0.0897)	0.8685 (2.9931)
CP (financial, 3m)-TB(3m)	-	-	-		-1.5032 (0.1227)	0.8771 (2.6048)
ED (3m)-TB(3m)		-1.8467 (0.0621)	0.8648 (2.7170)		-1.5482 (0.1133)	0.8797 (2.8539)
Aaa-TCMNOM (30y)		-1.0176 (0.2746)	0.9521 (2.1272)		-0.5786 (0.4318)	0.9796 (1.9843)
Baa-Aaa		-1.6083 (0.1003)	0.9705 (2.6968)		-0.2686 (0.5481)	0.9933 (2.4455)
TCMNOM (10y) - TB(3m)		-1.1600 (0.2226)	0.9620 (2.1084)		-1.0119 (0.2765)	0.9545 (1.4784)
TCMNOM (30y) - TB(3m)		-1.0149 (0.2756)	0.9689 (1.7759)		-1.7237 (0.0797)	0.8160 (1.8910)
Interest rates						
Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
FF (overnight)	47	1.3131 (0.1892)	1.1986	45	3.0921 (0.0020)***	1.5721
CD (3m)	47	1.8060 (0.0709)*	1.2573	45	2.6591 (0.0078)***	1.6293
CP (financial, 3m)	-	-	-	41	2.8779 (0.0040)***	1.6160
ED (3m)	48	0.6009 (0.5479)	1.0873	46	3.0485 (0.0023)***	1.6616
TB (3m)	47	2.6976(0.0070)***	1.3395	45	2.7795(0.0054)***	1.7302
Aaa	47	0.6501 (0.5156)	1.0992	45	-0.5807 (0.5614)	0.9121
Baa	39	-0.0505 (0.9597)	0.9908	45	-0.9509 (0.3416)	0.8520
10 y	47	1.0410 (0.2979)	1.1510	45	-0.3376 (0.7357)	0.9492
30 y	47	1.1274 (0.2596)	1.1666	30	-1.0194 (0.3080)	0.7965
DF		test statistic (p-value)	coefficient (DW)		test statistic (p-value)	coefficient (DW)
FF (overnight)		-1.3014 (0.1754)	0.9796 (1.6493)		0.6816 (0.3972)	0.9884 (0.8796)
CD (3m)		-1.1876 (0.2125)	0.9811 (1.5308)		-0.3990 (0.5005)	0.9936 (0.8846)
CP (financial, 3m)	-	-	-		-0.3978 (0.5005)	0.9928 (0.9116)
ED(3m)		-1.1672(0.2201)	0.9800(1.8686)		-0.4196(0.4931)	0.9935(0.8163)
TB (3m)		-1.3704(0.1561)	0.9804(1.3481)		-0.4684(0.4752)	0.9925(0.6713)
Aaa		-1.9132 (0.0538)	0.9856 (1.8749)		-0.6517 (0.4082)	0.9949 (2.0140)
Baa		-1.3393 (0.1641)	0.9891 (1.8988)		-0.5004 (0.4635)	0.9963 (2.1400)
TCMNOM (10 y)		-1.7570 (0.0748)	0.9815 (1.6938)		-0.4892 (0.4676)	0.9940 (1.9884)
TCMNOM (30 y)		-1.7878 (0.0702)	0.9830 (1.6585)		-0.5688 (0.4354)	0.9936 (2.1488)

Table 11 reports the results of random walk tests for quarterly movements of international short-term nominal interest rates of 10 countries. First, I inspect the results of random walk tests of interest spreads. The null of random walk in the variance test is rejected at 5% significance level more often in the second period (3 countries (the first period) $\rightarrow$ 7 countries (the second period)). And estimated variance ratios are higher than unity without an exception in the second period, which is the main difference from the U.S. domestic interest spreads with the same maturity as shown in Table 9.

Second, I examine the results of random walk tests of interest rates of 10 countries. Once

Table 10: Daily correlation between international interest spreads and TB(3m, U.S.)

Interest rates	Full sample	**1984 -1995**	**1996-08/08/2007**	1999-08/08/2007
Australia	0.7068 / 0.0139	0.7262 / -0.0160	0.5190 / 0.1080	0.6426 / 0.1207
Denmark	0.4622 / -0.0014	0.0242 / -0.0166	0.6443 / 0.0597	0.6334 / 0.0346
France	0.5151 / 0.0341	0.2008 / 0.0124	0.6014 / 0.1094	0.5671 / 0.1225
Germany	0.5984 / 0.0878	-0.1737 / 0.1171	0.6107 / 0.1509	0.6283 / 0.1959
Japan	0.7513 / 0.0208	0.5471 / -0.0062	0.7093 / 0.0173	0.6824 / 0.0601
New Zealand	0.6597 / -0.0196	0.6370 / -0.0393	0.5113 / 0.0839	0.4200 / 0.1422
Norway	0.5392 / 0.0013	0.3699 / 0.0034	0.2272 / 0.0065	0.2969 / -0.0005
Sweden	0.6657 / 0.0205	0.4029 / 0.0242	0.3676 / 0.0517	0.1742 / 0.0794
Switzerland	0.5623 / 0.0725	0.3528 / 0.0716	0.6935 / 0.1100	0.7292 / 0.0723
UK	0.8128 / 0.0822	0.6390 / 0.0892	0.7903 / 0.0940	0.8748 / 0.1155
Spreads	Full sample	**1984 -1995**	**1996-08/08/2007**	1999-08/08/2007
Australia	-0.0421 / -0.6007	0.3865 / -0.4310	-0.8835 / -0.7415	-0.9398/ -0.8080
Denmark	-0.2062 / -0.2184	-0.6002 / -0.1326	-0.8142 / -0.5998	-0.7889 / -0.6700
France	-0.1707 / -0.5501	-0.5977 / -0.3481	-0.8657 / -0.8250	-0.8504 / -0.8288
Germany	-0.4977 / -0.8243	-0.7668 / -0.8239	-0.8800 / -0.8839	-0.8537 / -0.8661
Japan	-0.3363 / -0.4050	-0.3941 / -0.4126	-0.9917 / -0.4212	-0.9952 / -0.7181
New Zealand	0.3349 / -0.2919	0.4240 / -0.2323	-0.6037 / -0.4443	-0.8132 / -0.7576
Norway	0.0265 / -0.1996	-0.1174 / -0.1318	-0.5388 / -0.4497	-0.4809/ -0.7036
Sweden	0.1440 / -0.2195	-0.3811 / -0.1439	-0.7367 / -0.8198	-0.8766 / -0.8415
Switzerland	-0.1417 / -0.6071	-0.4654 / -0.4610	-0.8376 / -0.7033	-0.8070 / -0.7708
UK	-0.0754 / -0.6855	-0.0176 / -0.3794	-0.7416 / -0.8114	-0.9378 / -0.8148

Corr(x,TB(3m, U.S.)) / Corr(Δ x, Δ TB(3m, U.S.))

again, the null of random walk in the variance ratio test is rejected at 5% significance level more often in the second period (2 countries (the first period) → 6 countries (the second period)). In addition, estimated variance ratios are higher than unity with an exception, Japan, in the second period. In particular, Table 9 and Table 11 show that for the five countries (Denmark, France, Germany, UK, and U.S.), the variance ratio tests reject the null hypothesis of random walk with 1% significance level during the second period and that 3-month Treasury Bill of the U.S. has the highest estimated variance ratio and the lowest DW among short-term nominal interest rates of 11 countries in the second period.[26] Table 29 in Appendix K shows the results of variance ratio tests for the period of 1999-08/08/2007 with time lag = 65 / 30 days. We can see that the estimated variance ratios are a little bit lower than in the period of 1997-08/08/2007 but still higher than unity without an exception with time lag = 65 days, for Euro area and the U.S., the estimated variance ratios are the

[26] Table 28 in Appendix K exhibits the results of variance ratio tests with time lag = 30 days.

highest among samples. And with time lag =30 days, the null of random walk is rejected at 5% significance level with the exception of Japan.

Table 11: Random walk tests: Quarterly nominal interest rates (time lag = 65 days)

		1984 -1995			1996-08/08/2007	
Interest spreads						
Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
Australia	46	-0.3753(0.7074)	0.9231	44	2.9841 (0.0028)***	1.3909
Denmark	27	1.1002(0.2713)	1.1988	44	1.2829 (0.1995)	1.2428
France	26	0.1329 (0.8942)	1.0180	44	2.2897(0.0220)**	1.4830
Germany	36	2.6311(0.0085)***	1.4223	35	2.7626(0.0057)***	1.5602
Japan	31	2.7288 (0.0064)***	1.5496	44	2.6365 (0.0084)***	1.4699
New Zealand	42	0.4596(0.6458)	1.1270	44	1.4643(0.1431)	1.2568
Norway	38	-0.4930(0.6220)	0.7835	44	2.8735(0.0041)***	1.5284
Sweden	45	-1.0243(0.3057)	0.5846	44	2.3509(0.0187)**	1.5313
Switzerland	27	2.8005(0.0051)***	1.5542	44	0.2979(0.7658)	1.0530
UK	46	-1.2998 (0.1937)	0.7926	44	3.4325(5.9796e-004)***	1.5055
DF		test statistic (p-value)	coefficient (DW)		test statistic (p-value)	coefficient (DW)
Australia		-0.8310 (0.3427)	0.9680(2.0260)		-0.7712(0.3644)	0.9788(1.0721)
Denmark		-0.8310 (0.3387)	0.9462(1.7232)		-1.1399(0.2297)	0.9390(1.4501)
France		-0.6971(0.3872)	0.9603(1.9786)	-	0.8425(0.3383)	0.9643(0.9182)
Germany		-1.1321(0.2312)	0.9422(1.2297)		-1.0633(0.2561)	0.9436(0.8327)
Japan		0.4850 (0.8135)	1.0272 (1.0657)		-0.5124 (0.4590)	0.9910 (1.1895)
New Zealand		-0.9862(0.2856)	0.9619(1.4972)		-0.7390(0.3761)	0.9738 (1.5540)
Norway		-1.6490(0.0926)*	0.8704(2.3547)		-1.0911(0.2475)	0.9449(1.0126)
Sweden		-1.8061(0.0676)*	0.8569(2.6719)		-2.0337(0.0414)**	0.8932(0.5470)
Switzerland		-0.8380(0.3361)	0.9420(0.9980)		-0.9274(0.3073)	0.9746(1.9080)
UK		-0.8613(0.3317)	0.9663(2.3875)		-0.8763(0.3260)	0.9692(1.0240)
Interest rates						
Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
Australia	47	0.3698(0.7115)	1.0573	46	2.2084(0.0272)**	1.3871
Denmark	28	-1.1542(0.2484)	0.5598	45	2.6015(0.0093)***	1.4164
France	27	1.1763(0.2395)	1.2711	45	3.3437(8.2654e-004)***	1.5757
Germany	47	3.9261(8.6336e-005)***	1.4846	45	3.2281(0.0012)***	1.4731
Japan	32	3.4670 (0.0005)***	1.7651	46	-0.9414 (0.3465)	0.8209
New Zealand	43	0.6742(0.5002)	1.1184	45	1.9324(0.0533)*	1.3798
Norway	39	0.3873(0.6986)	1.0902	45	2.1579(0.0309)**	1.4653
Sweden	46	-0.5000 (0.6170)	0.8970	45	1.6826(0.0925)*	1.3488
Switzerland	28	1.3000(0.1936)	1.2453	46	0.2038 (0.8385)	1.0339
UK	47	-0.2636(0.7921)	0.9589	46	2.7281(0.0064)***	1.4671
DF		test statistic (p-value)	coefficient (DW)		test statistic (p-value)	coefficient (DW)
Australia		-0.5390(0.4495)	0.9906(1.5389)		-0.6246(0.4182)	0.9939(1.3336)
Denmark		-0.7824(0.3567)	0.9643(2.8831)		-0.5414(0.4485)	0.9912(1.1667)
France		-0.7697(0.3611)	0.9850(1.5608)		-0.8110(0.3499)	0.9875(0.7483)
Germany		-0.7185(0.3840)	0.9918(1.1413)		-0.1415(0.5946)	0.9979 (1.0662)
Japan		-0.8934 (0.3175)	0.9826 (0.6605)		-1.1649 (0.2207)	0.9246 (2.2927)
New Zealand		-0.9030(0.3161)	0.9767(1.5699)		-0.5113(0.4595)	0.9915(1.3455)
Norway		-1.0668(0.2556)	0.9781(1.8846)		-0.5529(0.4443)	0.9891(1.1637)
Sweden		-0.6780(0.3986)	0.9875(2.2588)		-3.1351(0.0032)	0.9443(0.8458)
Switzerland		-0.6450(0.4069)	0.9850(1.4835)		-0.3996(0.5004)	0.9860(1.9695)
UK		-0.6314(0.4158)	0.9896(2.1505)		-0.4451(0.4838)	0.9951(1.1392)

In a nutshell, the correlation analysis of this Section reveals that correlation between international interest rate spreads and TB(3m, U.S.) becomes stronger before the financial crisis of 2007-9 for 10 countries without an exception whereas the correlation between international interest rates and TB(3m, U.S.) becomes stronger for 6 countries. Stronger correlation between international interest rate spreads and TB(3m, U.S.) before the financial crisis of 2007-9 is an encouraging evidence in light of the focus of this paper, interactive dynamics between short-term nominal interest rates of the U.S. and international risk appetite.

Regarding the variance ratio test, estimated variance ratios of interest rates are higher than unity in the second period with the exception of the Euro market rate of Japan yen. In particular, the estimated variance ratios of international short-term interest spreads are higher than unity in the second period without an exception, which is different from domestic interest spreads of the U.S. with the same maturity. Thus we can conjecture that an international factor might be responsible for the difference. And the variance ratio test is more often rejected in the second period. Note that these results from variance ratio tests may be useful characteristics of data in the sense that models for dynamics of international short-term nominal interest rates can be evaluated with regard to the characteristics. I will investigate whether the model in this paper is helpful to understand the results from the variance ratio tests. In other words, I will look at whether the model can generate some regularities of data such as a variance ratio higher than unity before the financial crisis of 2007-9.

3.4 A MODEL

3.4.1 The overview of the model

In addition to the interaction between the interest rates of U.S. and international risk appetite discussed in Section 3.2, in Section 3.4.1, I briefly discuss the remaining components of the model.

3.4.1.1 Two sources of nonlinearity of the model One of the sources of nonlinearity in the model is the fact that nominal interest rates are *bounded* below by zero, recently highlighted by Benhabib and Uribe [2001] where they show that a stable limit cycle exists around an active steady state with the Taylor rule under some conditions by using the Bogdanov-Takens bifurcation. The zero interest rate policy in Japan from 1999 is a typical example for the lower bound of nominal interest rates.

Another source of nonlinearity in dynamics of financial markets is related with adaptive behaviors of *heterogeneous agents* which are systematically studied under the concept of "adaptively rational equilibrium" in Brock and Hommes [1997] including a chaotic attractor via homoclinic bifurcation. Brock and Hommes [1998] also presents several bifurcation routes to complicated dynamics in financial markets including limit cycles and chaos. In this paper there exist two types of economic agents; domestic investors and international investors. And due to different portfolios, I assume international investors are more risk-sensitive. Moreover, I also assume that the weight of international investors in the U.S. financial markets is higher in times of low risk appetite because of higher demand for safe assets. In other words, the composition of two types in the U.S. financial markets depends on risk appetite via international flow of funds.

$$\begin{aligned}&\text{risk appetite} \downarrow \rightarrow \text{demand for safe assets} \uparrow \rightarrow \text{ fund inflows to the U.S.}\\&\rightarrow \text{the weight of international investors in the U.S. financial markets} \uparrow\end{aligned} \tag{3.4}$$

3.4.1.2 Bifurcation analysis The idea of bifurcation (including change of stability of equilibrium point) especially related with economic depression is not new in economics. Whereas the concept of the "natural price" of Adam Smith or the concept of the "natural rate of interest" of Knut Wicksell assumes the existence of stable equilibrium points of relevant dynamical systems, several destabilizing mechanisms of an economic system such as debt deflation of Irving Fisher and John Maynard Keynes' dynamic effect of lowering monetary wage have been proposed.[27] A seminal paper of Tobin [1975, 195] makes critical comments on Keynes's choice of equilibrium analysis and comparative statics.[28] The famous Minsky's financial instability hypothesis clearly shows its connection with the idea of bifurcation as I cited at the beginning of the paper.[29]

Since the model in this paper undergoes bifurcations eventually leading to chaotic motions, I briefly review relevant mechanisms for chaotic dynamics in this paper.[30] Assume

[27] Mendoza [forthcoming] is an interesting recent application of the idea of debt deflation in DSGE framework.

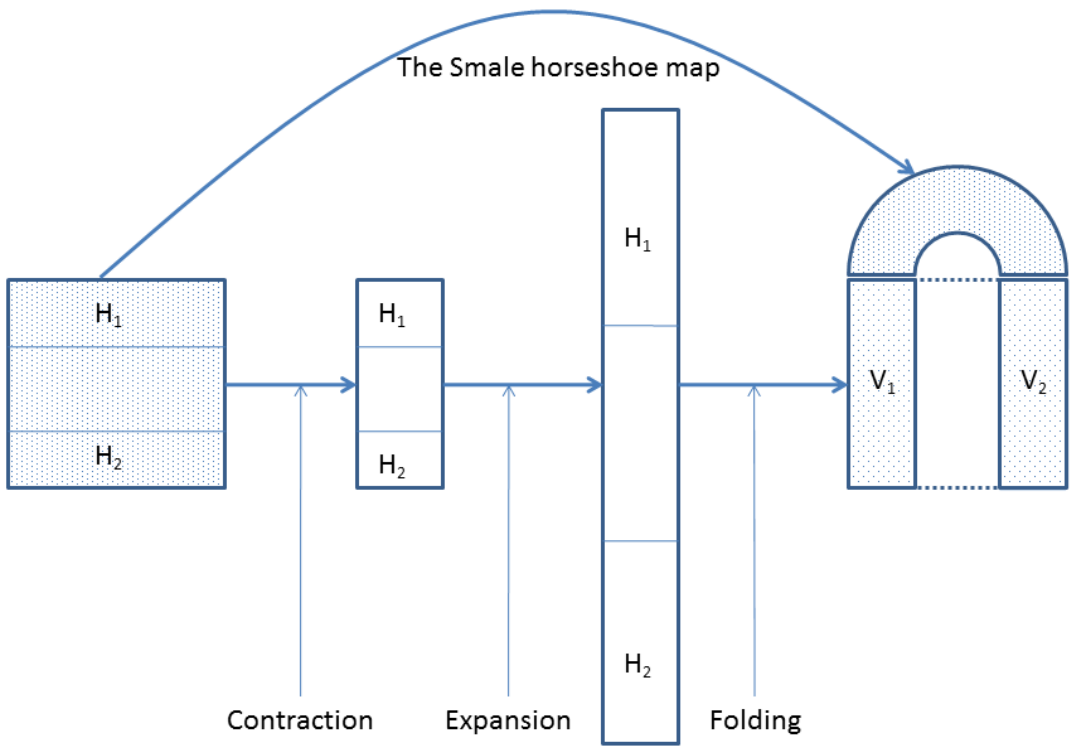
[28] Tobin [1975] presents a classical continuous time model for Keynesian depression where attracting basin of equilibrium point is bounded.

[29] See, for example, Minsky [1986, 10, footnote 9] where Minsky cited several early works on chaos in economic systems.

[30] Here I briefly review several bifurcation routes to nonlinear attractors such as limit cycles and chaos in economics. The list is not at all complete. For introduction to bifurcation in intertemporal equilibrium theory, I refer to Mitra and Nishimura [2001]. Boldrin and Montrucchio [1986] shows that in discrete optimal growth model even chaotic dynamics can be admissible under some values of parameters. In business cycle theory, a

that S is a square. Let F be a version of the well-known Smale horseshoe map: F maps S into $\mathbb{R}^2$ as Figure 19 shows.[31] Note that $V_1 = F(H_2)$ and $V_2 = F(H_1)$. We also can see that F(S) looks like a horseshoe. As Figure and Table 12 exhibit, the Smale horseshoe map has three elementary actions; contraction, expansion, and folding.

Figure 19: The Smale horseshoe map



Intuitively speaking, there are a centripetal force, a centrifugal force, and a change of relative strengths of two forces. Similarly, in Brock and Hommes [1997, 1998] there are three

bifurcation analysis is often related with the endogenous business cycle literature as explained in Grandmont [1985, 997]. In particular, homoclinic bifurcations in discrete time are also investigated in de Vildor [1996] and Agliari et al. [2007]. In development theory, for example, Matsuyama [1991] in continuous time with increasing returns presents a model with a homoclinic orbit. Artige et al. [2004] in discrete time with habit formation is an example of application of Hopf bifurcation to show a stable limit cycle in economic growth theory.

[31] $\Lambda = \bigcap_{j=-\infty}^{\infty} F^j(S)$ is an invariant cantor set of F. The interesting dynamics on Λ is normally analyzed by introducing symbolic dynamics. The explanation about the Smale horseshoe map in this paper is based on Perko [2001, 409-412]. For more details, I refer to Guckenheimer and Holmes [1983, 5.1]

corresponding components; rational expectation or fundamentalists, naive expectation or chartists, and switch of beliefs.[32] In line with those modeling strategies, the model in this paper contains three components; domestic investors, international investors, and capital flow between the U.S. and the rest of the world.

Table 12: Main actions of a model

Smale horseshoe map	Brock and Hommes [1997, 1998]	This paper
contraction	rational expectation or fundamentalists	domestic investors
expansion	naive expectation or chartists	foreign investors
folding	switch of beliefs	international movement of funds

There are two reasons I study a delayed-logistic map. One is that the bifurcation routes of the delayed-logistic map and the model in this paper turn out to be similar. The other is that it can be viewed as a simple nonlinear model for leader-follower relation because y (follower) just takes the previous value of x (leader).[33] Eq. 3.5 represents a delayed-logistic map.

$$\begin{aligned} x_{t+1} &= Ax_t(1 - y_t) \\ y_{t+1} &= x_t \end{aligned} \tag{3.5}$$

where $x_0 = 0.1, y_0 = 0.1$.

The map generates three different attractors depending on parameter A. A limit cycle emerges at A=2 through Neimark-Sacker bifurcation (Kuznetsov [1998, 142-4]). Chaos emerges around A=2.27. (Sprott [2003, 422]).[34]

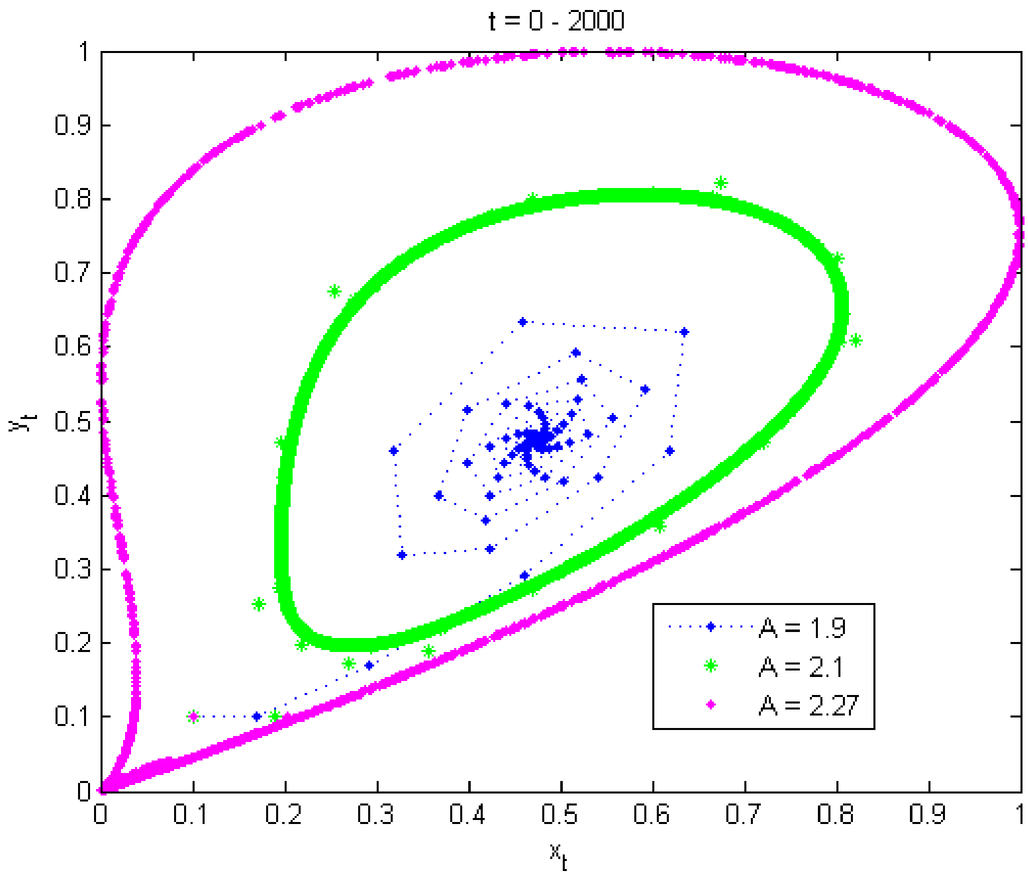
Figure 20 shows three different trajectories corresponding to three different values of A. Note that with A=1.9, the trajectory converges to the point attractor. But, with A=2.1, the trajectory does not converge to any point. Instead, the trajectory is attracted to an invariant closed curve (a limit cycle). Next, if A=2.7, the trajectory is attracted to a chaotic

[32] Kaizoji [2004] shows that the increase in the number of traders in a heterogeneous agents model may lead to an intermittent chaos.

[33] Belke and Cui [2010] investigates the monetary policy interdependence between the European Central Bank (ECB) and the Federal Reserve (Fed) from 1999 to 2006 (see Figure 15 for three-month interest rates of two regions during the period). By probing into the relation between the EONIA rate and the Fed Funds rate, they find an evidence for the leader (Fed) - follower (ECB) relation in one of two models (general VECM).

[34] For the detailed analysis including related homoclinic bifurcation, I refer to Aronson et al. [1982].

Figure 20: Delayed-logistic map



attractor. We can see from Figure 20 that *the diameter of a trajectory gets longer as the parameter of A becomes larger*. This is a useful observation in modeling the big swing of a state variable.

3.4.2 Model

Before I discuss the model, I make a few remarks in terms of methodological issues. The model in this paper is not strictly based upon the so-called micro foundations of macro models. With respect to the issue, I share the following viewpoint:

> To make the needed break from the past, macroeconomists must acknowledge that micro foundations are a choice variable of theorists. The appropriate choice cannot be determined

> a priori; it needs to be made in reference to empirical data and educated common sense in a way that will lead to useful macro models (Colander et al. [2008, 236]).

Regarding this point, we can find several cases where useful insights for policy-making are obtained without strict adherence to the micro foundations of macro models. Among them is the following observation made by a leading scholar of the regulation school:

> The stability of an equity-based regime depends on monetary policy which controls financial bubbles and thus the diffusion of finance may push the economy into a zone of structural instability. The next major financial crisis may originate in the USA whose economy approximates most closely to the model (Boyer [2000, 111]).

Given the fact that the financial crisis of 2007-9 was a surprising event to most people, the insight in Boyer [2000] is remarkable.

How can we evaluate different models that take competing methodological approaches? We can evaluate models by comparing explanatory power of each model. In other words, we can investigate whether each model replicates the essential properties of data. And I include several complexity measures as parts of essential properties of data (see Section 3.5 for more discussion). An advantage of complexity measures is that they are invariant at least to monotone transformation. Thus, it is not difficult to compare simulated data with real data with respect to complexity measures. Finally, as the model in this paper does not have solid micro foundations, I do not claim that it is a general model. Rather, I admit that the model is primarily specific to interest rates dynamics before the financial crisis of 2007-9.

3.4.2.1 A simple model for interactive dynamics of the U.S. interest rates and international risk appetite Here I present a simple two-country model for interest rate dynamics as a simple model to investigate the stability of interaction between interest rates of the U.S. and international risk appetite measured by interest rate spreads. The uncovered interest rate parity (=UIRP) holds if the following no-arbitrage condition holds:

$$(1 + i_{t,\$}) = (E_t S_{(t+1,€/\$)})^{-1}(1 + i_{t,€})S_{t,€/\$} \tag{3.6}$$

where $i_{t,\$}$ and $i_{t,€}$ denotes interest rates of two countries and $S_{t,€/\$}$ is a spot foreign exchange rate, and E_t is conditional expectation operator at time t, respectively. The model does not deal with the movement of a foreign exchange rate by making a simplifying assumption:

Assumption 3.4.1. Conditional expectation of foreign exchange rate is identical to current foreign exchange rate.

A sufficient condition for the assumption is that a foreign exchange rate follows a simple random walk. Note that with the above assumption, UIRP just means that

$$i_{t,\$} = i_{t,€}, \forall t. \tag{3.7}$$

But, due to several frictions including demand for safe assets, the financial assets of the U.S. are treated as special. So, modified UIRP becomes

$$i_{t,€} = i_{t,\$} + sp_t, \forall t. \tag{3.8}$$

where sp_t represents required spread (or premium) due to time-varying international risk appetite. I assume $i_{t,€}$ is fully determined by two variables $i_{t,\$}$ and sp_t.

sp_t is determined by autoregressive process with a little modification as follows:

$$sp_{t+1} = s_{sp} sp_t + s_i(i_{t,\$} - \bar{i_\$}), s_{sp} > 1, s_i > 0 \tag{3.9}$$

where $\bar{i_\$}$ is some reference level like natural rate of interest. I assume that $s_{sp} > 1$ and $s_i > 0$ because of the reasons reviewed in Section 3.2.3. We can imagine that $sp_t = 0$ represents market situation where investors accept no premium for holding other countries' financial assets given $i_{t,\$} = \bar{i_\$}$. So, the specification implies that with $sp_t = 0$, if $i_{t,\$} > \bar{i_\$}$, then since the probability of downturn of world economy becomes higher, investors want positive premium in the next period.

In bond market, there are two types of players, domestic and international investors.

Assumption 3.4.2. The supply of bond is fixed. Thus, the price of bond is demand-determined. The rate of return or interest rate is determined by a weighted average of the expectations of two types of players.

Even though a representative agent with rational expectation as a model-consistent expectation is popular in economics, there are several rationales for assuming adaptive expectation in financial markets in some cases, which are mainly related with the concept of *complexity*.[35] First, the rational expectation may not be obtainable just because the 'correct' model is not known due to the inherent complexity of the relevant system. Second, as in Brock and Hommes [1997], there may exist information cost in obtaining rational expectation. And if the benefit from using rational expectation instead of simpler expectation formation rules such as adaptive expectation is smaller than the cost, it would be better to use adaptive expectation with no information cost. And as a system becomes more complex, the information cost will be larger. I assume that the expectation of domestic investors is adaptive expectation.

$$i^d_{t+1,\$} = i_d \bar{i_\$} + (1 - i_d) i_{t,\$}, 0 < i_d < 1 \tag{3.10}$$

where $\bar{i_\$}$ is a reference level reflecting the natural rate of interest. Thus, domestic investors act like fundamentalists in a *fundamentalist-chartist* model.[36] If a price ($i_{t,\$}$) deviates from the fundamental value ($\bar{i_\$}$), domestic investors expect that the price will gradually come back to the fundamental value.[37] I assume that the expectation of international investors is also adaptive expectation with a little modification mainly due to a demand for safe asset.

$$i^w_{t+1,\$} = i_w(max\{\bar{i_\$} - i_{ws} sp_t, 0\}) + (1 - i_w) i_{t,\$}, i_{ws} > 0, 0 < i_w < 1 \tag{3.11}$$

Eq. 3.11 means that if $sp_t = 0$, then the reference level is $\bar{i_\$}$ but if international investor becomes pessimistic ($sp_t > 0$), then the demand for safe assets is stronger and his reference level becomes lower by $i_{ws} sp_t$. Caballero and Krishnamurthy [2009, 584] points out the demand for safe assets as the main cause of capital inflow to the U.S. Symmetrically, if $sp_t < 0$, international investors are optimistic, then the demand for safe assets is weaker and his reference level becomes higher by $i_{ws} sp_t$ mainly because he may find more investment

[35]For a recent overview on complexity in economics and finance, I refer to Anufriev and Branch [2009]. And also see the works at the *Santa Fe Institute*.

[36]For an overview on fundamentalist-chartist models, I refer to Hommes [2006]. Note that domestic investors (international investors) roughly correspond to α-investors (β-investors) in Day and Huang [1990].

[37]Another possible interpretation is that the strategy of domestic investors represents a monetary policy that employs a simple adaptive rule. Then, market interest rates would be determined by a weighted average of a monetary policy and strategy of international investors.

opportunities outside the U.S. The simple symmetric specification succinctly represents the idea that "the more fear is, the more greed is, and vice versa." Given sp_t, the extent of adjustment depends on the parameter of i_{ws}. I call i_{ws} a *parameter of risk-attitude*. Zero (0) is included in Eq. 3.11 because of the zero lower bound of nominal interest rates.
The weight of international investors in the market ($= \alpha_t$) depends on sp_t in the following way:

$$\alpha_t(sp_t) = \alpha_1(atan(-\alpha_2(sp_t - \hat{sp})^2)) + \alpha_0, \;\; \alpha_0, \alpha_1, \alpha_2, \; and \; \hat{sp} > 0 \tag{3.12}$$

Thus I assume that the weight of international investors in the U.S. bond markets is at the maximum ($=\alpha_0$) if $sp_t = \hat{sp} (> 0)$ due to the status of the U.S. as a key currency country. And since $sp_t > 0$ means a positive excess return in holding non-US assets, if $sp_t > \hat{sp}$, I assume that 'excess return effect' dominates the pessimism implied by $sp_t > 0$. Another rationale for the specification of Eq. 3.12 is related with foreign exchange rate movements. Note that rising α means the appreciation of the U.S. dollar because of capital inflow into the U.S. Therefore, unusually high $\alpha (> \alpha_0)$ would be related with the unusually appreciated U.S. dollar. In that case, investing in non-US assets would get gain from currency appreciation against the U.S. dollar. Thus, expected gain from foreign exchange markets provides another incentive for investing in risky non-US assets if $\alpha > \alpha_0$.[38]

Whereas Eq. 3.12 succinctly represents a folding mechanism which is necessary to generate complicated dynamics, the folding mechanism exogenously specified in Eq. 3.12 is obviously too simple. Thus, I suggest possible extensions of the model. First, considering heterogeneous predictors within international investors, including change of predictors as in Brock and Hommes [1997, 1998], would be interesting. Second, adding foreign exchange markets with heterogeneous predictors into the model would be promising as well.

Then, $i_{t+1,\$}$ is determined by the following equation:

$$i_{t+1,\$} = (1 - \alpha(sp_t)) i^d_{t+1,\$} + \alpha(sp_t) i^w_{t+1,\$} \tag{3.13}$$

This completes the description of the model.

[38] According to Farhi and Gabaix [2008], the failure of UIRP or the so-called "forward premium puzzle" is related with the empirical regularity that currencies with low interest rates experience depreciation. In the habit-based model of Verdelhan [2010], domestic investors are more risk averse and interest rates are low in bad times.

Plugging Eq. 3.10-3.12 into Eq. 3.13, we have the following:

$$\begin{aligned} i_{t+1,\$} &= (1 - (\alpha_1(atan(\alpha_2(sp_t - \hat{sp})^2)) + \alpha_0))(i_d\bar{i_\$} + (1 - i_d)i_{t,\$}) \\ &\quad + (\alpha_1(atan(\alpha_2(sp_t - \hat{sp})^2)) + \alpha_0)(i_w(max\{\bar{i_\$} - i_{ws}sp_t, 0\}) + (1 - i_w)i_{t,\$}) \end{aligned} \tag{3.14}$$

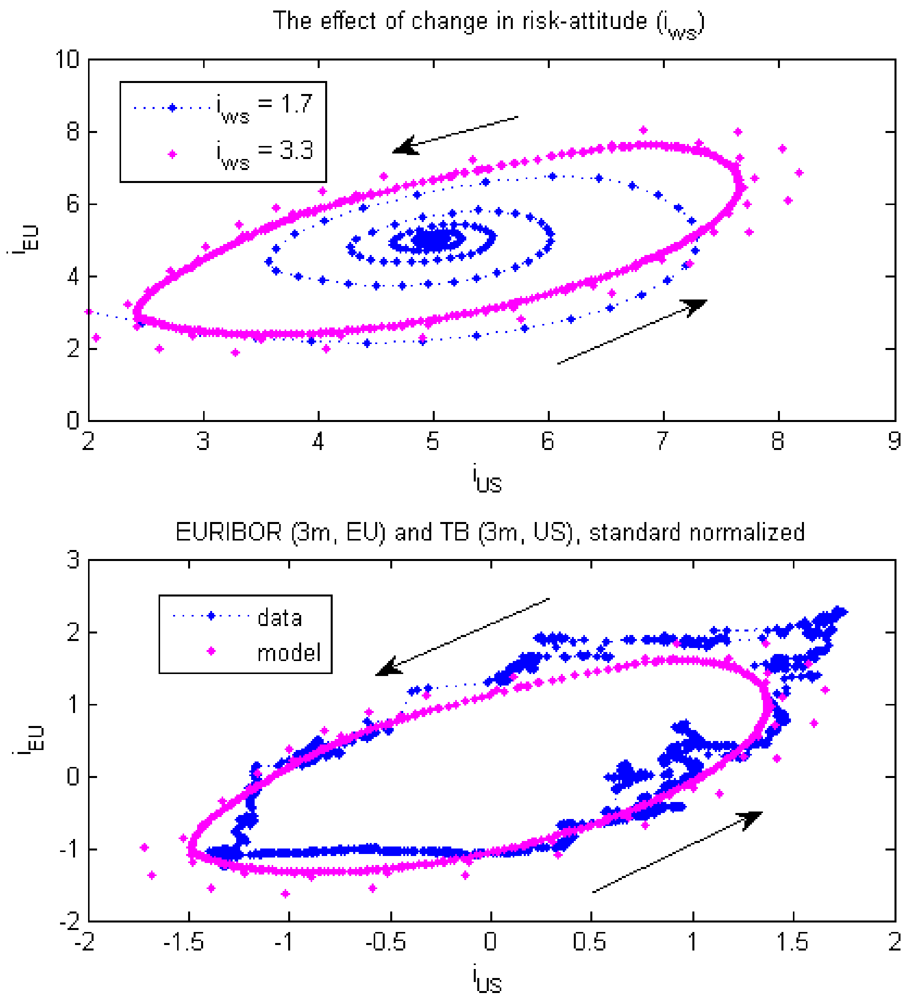
$\{sp_t, i_{t,\$}\}_{t\geq 0}$ is fully determined by Eq. 3.9 and Eq. 3.14 with initial conditions. Then, $i_{t,€}$ is passively determined by Eq. 3.8. Note that by construction, $(sp, i_\$) = (0, \bar{i_\$})$ is a steady state. It will be a formidable task to fully characterize the dynamics of this dynamical system. Instead of it, since I am interested in interactive dynamics between short-term nominal interest rates of the U.S. and international risk appetite, I probe into the effect of changes of i_{ws} and s_i on the stability of the dynamical system.

3.4.2.2 The stronger influence of international risk appetite on the U.S. interest rates ($i_{ws} \uparrow$) and its effect on dynamics of the model As Kindleberger and Aliber [2005, 55] points out, asset bubbles depend on the expansion of credit. Regarding the financial crisis of 2007-9, the global imbalance is identified as a source of supply of credit. In turn, the "saving-glut" of Asian countries is somewhat related with the Asian crisis of 1997 and the response of the IMF (see, Brunnermeier [2009, 77], Diamond and Rajan [2009, 606], and Allen et al. [2009], for instances). In other words, the historical events induce higher demand for safer assets, particularly for the U.S. financial assets. The related surge of international reserves in Asian countries especially after the Asian crisis of 1997 implies the change of attitude toward financial crises.[39] The trend can be modeled as a more sensitive risk-attitude or higher i_{ws} in the model. Thus, it is interesting to see the implication of change in risk-attitude in the simple model. Table 13 summarizes parameter values for the experiment about the effect of change in risk-attitude.

Parameter values are chosen to roughly match interest rate dynamics of EURIOBR (3m, EU) and TB (3m, U.S.) (the bottom panel of Figure 21) after bifurcation. Given the other parameters, I investigate the effect of increase in influence of international risk appetite on

[39] About the surge of international reserve in Asian countries after the Asian crisis of 1997, see Jeanne [2007] for example. Frankel and Saravelos [2010] reports that central bank reserves are one of the two most useful leading indicators for crisis incidence. Their finding supports the importance of international reserve as self-insurance.

Figure 21: Effect of change in risk-attitude: EU-US



interest rates of the U.S by looking at different trajectories due to different values of a parameter (i_{ws}). The upper panel of Figure 21 displays the result: As i_{ws} increases from 1.7 to 3.3, the steady state loses the stability and a new attractor (limit cycle) emerges. Regarding bifurcation analysis, a bifurcation diagram is a nice tool of looking at the effect of the change of a parameter on dynamics. The top panel of Figure 44 in Appendix shows a bifurcation diagram where i_{ws} is a bifurcation parameter.[40] The experiment exhibits that the more sensitive risk-attitude around the globe due to frequent financial crises before financial crisis of

[40] A bifurcation diagram for the delayed-logistic map is given in Sprott [2003, 170].

Table 13: Parameter values (I)

(sp)	s_{sp}	s_i	$\bar{i_\$}$				
	1.11	0.3	5				
$(i_\$)$	i_d	i_w	i_{ws}	α_0	α_1	α_2	$\hat{sp}$
	0.1	0.5	(**1.7** $\rightarrow$ **3.3**)	0.5	0.2	2.5	1
$(sp_0, i_{0,\$}) = (1, 2)$							

2007-9 may induce *not only low interest rates but also unstable interest rate dynamics.* Regarding the relation between ECB and Fed, the model suggests the following three elements are important: Asymmetric role of the U.S. as a key currency country, modified UIRP, and strong interaction between interest rates of the U.S. and international risk appetite.[41]

3.4.2.3 The stronger influence of the U.S. interest rates on international risk appetite ($s_i \uparrow$) and its effect on dynamics of the model As financial globalization has advanced, the influence of the the U.S. interest rates on the international financial markets has grown. For example, Ehrmann and Fratzscher [2005] provides evidences for stronger interdependence between Euro area and the U.S. after the introduction of Euro in 1999. In order to investigate the repercussion of this development on interactive dynamics, I look at the situation where the U.S. interest rates have a bigger impact on international risk appetite, which implies higher s_i in the model. Table 14 shows that the only difference from the previous experiment is higher s_i (0.3$\rightarrow$ 0.5).

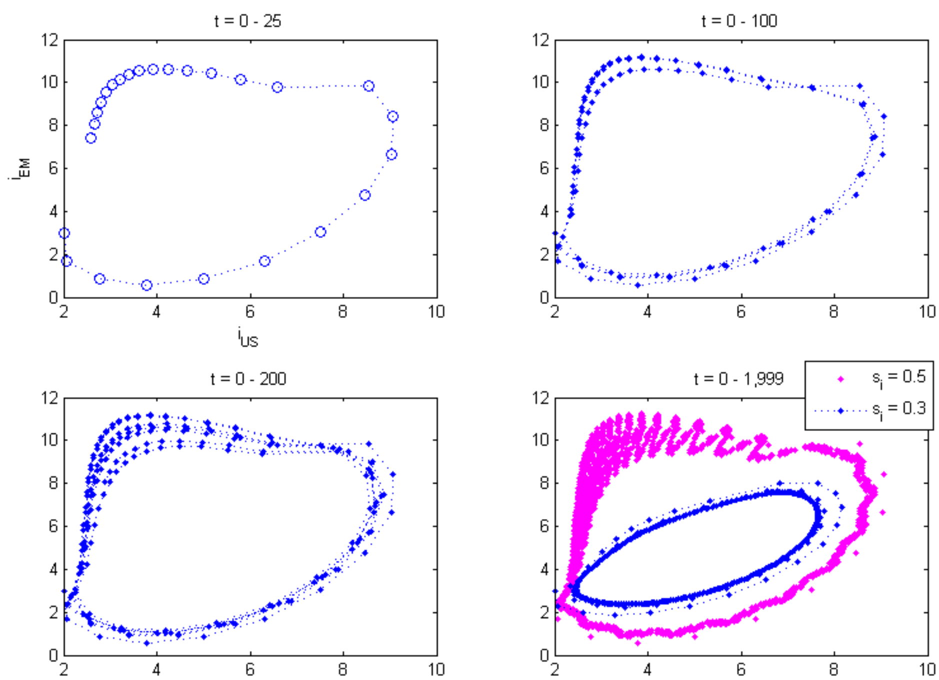
As Figure 22 illustrates, higher s_i has two impacts regarding levels of interest rates: Higher interest rate in the rest of the world when U.S. interest rate decreases and lower the lower-bound for the U.S. interest rate due to higher demand for safe assets. Several authors point out the Asian crisis of 1997 and related global imbalance lowered interest rates of the U.S. due to more sensitive risk attitude. Figure 22 represents the idea that, given mutual

[41] Interestingly, as Belke and Cui [2010, 778-779] points out, ECB did not follow Fed in early 2008 but made huge cuts in interest rates by the end of 2008.

Table 14: Parameter values (II)

(sp)	s_{sp}	s_i	$\bar{i_\$}$				
	1.11	$\mathbf{0.3 \rightarrow 0.5}$	2.5				
$(i_\$)$	i_d	i_w	i_{ws}	α_0	α_1	α_2	$\hat{sp}$
	0.1	0.5	**3.3**	0.5	0.2	2.5	1.5
$(sp_0, i_{0,\$}) = (1, 2)$							

Figure 22: Model with higher s_i



feedbacks between short-term nominal interest rate and international risk appetite, *stronger influence of interest rates on international risk appetite also may contribute to lowering the lower bounds of interest rates of the U.S.* The bifurcation diagram where s_i is a bifurcation parameter is given in the bottom panel of Figure 44.

Another implication of the model is about uncertainty, which comes from the fact that the dynamics under investigation is chaotic. Among several properties of a chaotic system, I focus on the condition of "sensitive dependence on initial conditions" mainly because the condition implies the unpredictability of paths due to computational errors involved in identifying initial conditions or in computing a path from an initial condition.[42] To highlight the uncertainty generated by chaotic dynamics, I look at the difference of two trajectories which start from very close initial conditions. I choose $(sp_{01}, i_{01,\$}) = (1, 2.00001)$ as a new initial condition. So the initial difference of two trajectories is 0.00001 with respect to the U.S. interest rate. Figure 42 in Appendix L reports the difference of two trajectories up to t=1,999 for three different attractors. From Figure 42, we can see that the difference of two trajectories in the case of chaos does not converge and displays irregular pattern (the so-called "butterfly effect").[43]

I also employ a Poincaré plot to appreciate the uncertainty generated by chaotic dynamics. Figure 23 shows more scattered points in Poincaré plots for chaotic dynamics that highlight endogenous uncertainty generated by endogenous dynamics rather than by exogenous shock.[44]

[42] A definition of chaotic map is given in Devaney [2003]:

> "Let V be a set. $f : V \to V$ is said to be chaotic on V if
> 1. f has sensitive dependence on initial conditions
> 2. f is topologically transitive
> 3. periodic points are dense in V (Devaney [2003, 50])."

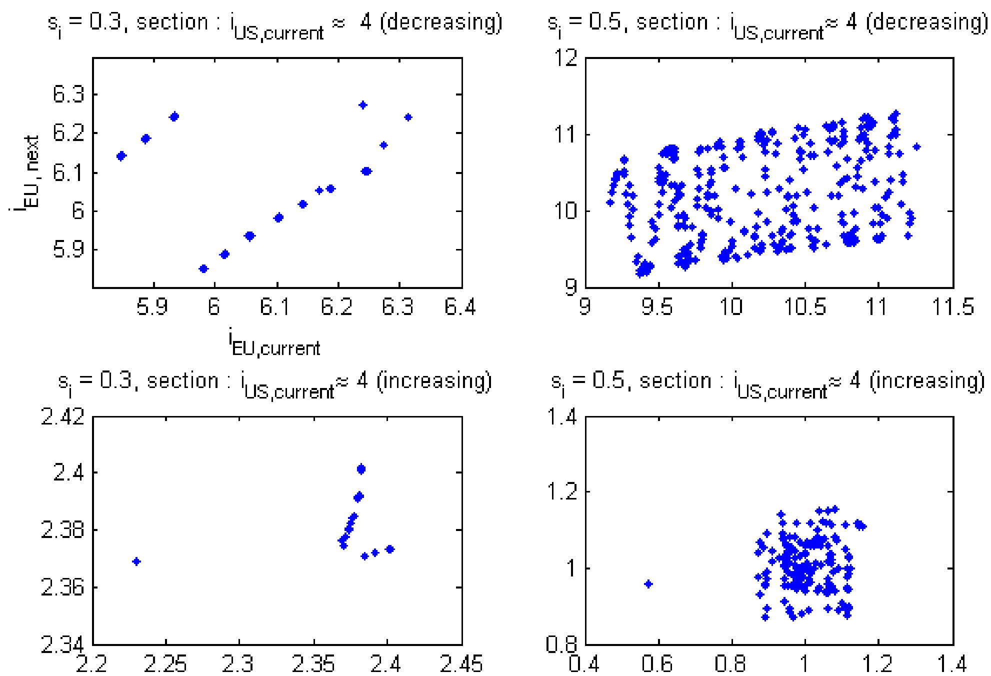
According to Hirsh et al. [2004, 321], sensitive dependence on initial conditions is "the hallmark of a chaotic system."

[43] The idea of connecting financial turbulence with chaotic movement is not new. See, for example, the following statement made by President of the European Central Bank.

> "The features of such exceptional circumstances are not only that they occur largely unexpectedly or that they make inference based on past observations inadequate, but also that the complex intertwining of economic relationships characterising this turbulent period can hardly be captured by our mostly linear models. These are times in which the 'flap of a butterfly's wings' causes tornadoes (chaos theory) (Trichet [2009])."

[44] For top panels, Poincaré plots are constructed as follows: Let $\{(x_t, y_t)\}_{1 \le t \le T}$ be a two-dimensional data. Let $A = \{y^j_{t(j)}\}_{1 \le j \le K}$ be the points in a Poincaré plot. $K \le T$ and $t(j)$ denotes the original index. Then $y_{t_0} \in A$ if the following condition hold: $4 \le x_{t_0} < 4.5$ and $x_{t_0+1} \le 4$.

Figure 23: Poincaré plots



Note that one possible mechanism for more sensitive risk attitude (higher i_{ws}) is rising uncertainty and precautionary demand for a key currency. Then, insofar as more uncertainty induces more sensitive risk attitude, the experiments in this paper implies that the following *vicious spiral of uncertainty* may occur under some circumstances where dynamics is chaotic:

$$Unceratainty \uparrow \rightarrow \text{More sensitive risk attitude } (i_{ws} \uparrow) \overset{chaotic\ motion}{\rightarrow} Uncertainty \uparrow \tag{3.15}$$

The two pillars for the *vicious spiral of uncertainty* are the endogenous uncertainty generated by a chaotic system and the uncertainty-related parameter of i_{ws} included in the chaotic system itself.[45]

The uncertainty discussed so far is general for any chaotic system because the uncertainty

[45]To fully model the *vicious spiral of uncertainty*, it would be better approach to treat risk attitude itself as a slowly changing endogenous variable. I leave this extension a future research topic.

depends only on the property of sensitive dependence on initial conditions. As we can see from Figure 22, one of the most interesting results from the experiment is that the impact of higher s_i is not symmetric: It has bigger impact when U.S. interest rate decreases in the sense that the paths become more diverse. In other words, It seems that there is more uncertainty for the movement of interest rates of the rest of the world when interest rates of the U.S. decreases. The asymmetric implication regarding uncertainty is indeed specific to the model rather than general.

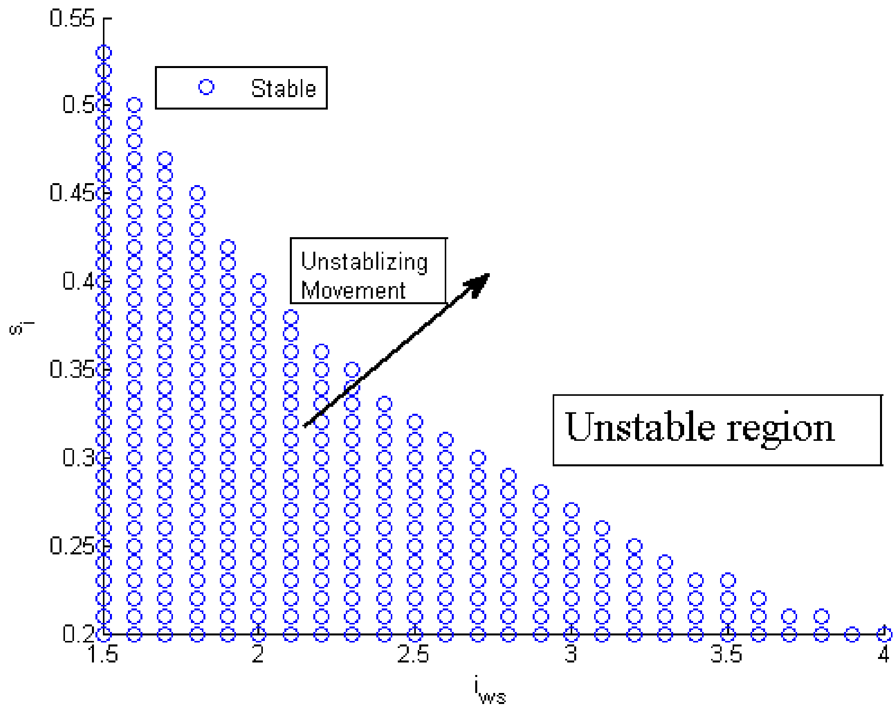
In a nutshell, strong interaction between interest rates of the U.S. and international risk appetite may generate chaotic dynamics. The resulting chaotic dynamics has two main implications for the financial crisis of 2007-9: First, it contributes to generating low interest rates of the U.S. Second, it highlights that the stronger interaction may generate endogenous uncertainty due to "butterfly effect" and in worse case the *vicious spiral of uncertainty*.

Figure 43 in Appendix M summarizes dynamics of main components of model. From the top panel of Figure 43, domestic investors ($=i^d_{t,\$}$) closely follow the realized interest rate ($=i_{t,\$}$) but international investors ($=i^w_{t,\$}$) deviate significantly from the realized interest rate and prevent the system from converging to a steady state. The bottom panel of Figure 43 displays that there is sometimes irregular increase in share of international investors ($=\alpha_t$).

3.4.2.4 The simultaneous effect of two parameters, i_{ws} and s_i Finally, Figure 24 shows the simultaneous effect of two parameters, i_{ws} and s_i on the stability of the dynamical system. The points denoted by circles represent the combinations of i_{ws} and s_i where the steady state is locally stable. Figure 24 clearly shows that, as the interaction between the short-term interest rates of the U.S. and international risk appetite becomes stronger, the dynamical system is more likely to lose local stability.

From the 1980's, the financial globalization made the U.S. interest rates more important in international financial markets ($s_i \uparrow$). And several financial crises in the 1990's may have resulted in a more sensitive risk attitude ($i_{ws} \uparrow$). These historical events are examples of the unstabilizing movement shown in Figure 24. In this way, Figure 24 displays a concrete

Figure 24: The simultaneous effect of two parameters, i_{ws} and s_i



mechanism for the bifurcation illustrated in Figure 16.

3.4.2.5 Random shock and variance ratios

Now, I investigate the effect of random shock with focusing on its impact on variance ratios.[46]

$$sp_{t+1} = s_{sp}sp_t + s_i(i_{t,\$} - \bar{i_\$}) + \epsilon_{sp,t+1},\ \epsilon_{sp,t+1} \sim i.i.d.\ Normal(0, \sigma^2_{sp}) \tag{3.16}$$

$$\begin{aligned} i_{t+1,\$} &= (1 - (\alpha_1(atan(\alpha_2(sp_t - \hat{sp})^2)) + \alpha_0))(i_d\bar{i_\$} + (1 - i_d)i_{t,\$}) \\ &+ (\alpha_1(atan(\alpha_2(sp_t - \hat{sp})^2)) + \alpha_0)(i_w(max\{\bar{i_\$} - i_{ws}sp_t, 0\}) \\ &+ (1 - i_w)i_{t,\$}) + \epsilon_{i_\$,t+1},\ \epsilon_{i_\$,t+1} \sim i.i.d.\ Normal(0, \sigma^2_{i_\$}) \end{aligned} \tag{3.17}$$

Table 15 shows the estimated variance ratios with different specifications.

[46] For a replicable simulation, I use the Matlab command, randn('state',0), in generating random shock. The initial condition is $(sp_0, i_{0,\$}) = (1, 2)$ as before.

Table 15: Random shock and variance ratio tests, t=0-49

	Equilibrium point $(i_{ws}, s_i) = (1.7, 0.3)$		Limit cycle $(i_{ws}, s_i) = (3.3, 0.3)$		Chaos $(i_{ws}, s_i) = (3.3, 0.5)$	
	coefficient (p-value)	ratio	coefficient (p-value)	ratio	coefficient (p-value)	ratio
$\sigma_{sp} = \sigma_{i_\$}$=0.5						
(sp)	2.8023 (0.0051)	1.4172	3.7155 (2.0280e-004)	1.5761	3.5238 (4.2534e-004)	1.6491
$(i_\$)$	2.6097 (0.0091)	1.4236	3.9275 (8.5834e-005)	1.6293	2.4554 (0.0141)	1.5626
$(i_€)$	1.6453 (0.0999)	1.1675	3.6923(2.2223e-004)	1.4812	2.8134(0.0049)	1.5639
$\sigma_{sp} = \sigma_{i_\$}$=0.4						
(sp)	2.9390(0.0033)	1.4820	4.0908 (4.2982e-005)	1.6922	3.8501(1.1808e-004)	1.7394
$(i_\$)$	2.8944(0.0038)	1.4895	4.2435(2.2004e-005)	1.7385	2.6977(0.0070)	1.6506
$(i_€)$	2.1634(0.0305)	1.2170	4.1342(3.5614e-005)	1.6543	3.1622(0.0016)	1.6563
$\sigma_{sp} = \sigma_{i_\$}$=0.3						
(sp)	3.0749(0.0021)	1.5772	4.3466(1.3828e-005)	1.7622	4.1747 (2.9834e-005)	1.8347
$(i_\$)$	3.5015(4.6259e-004)	1.6266	4.2076(2.5813e-005)	1.7966	3.9545(7.6696e-005)	1.8008
$(i_€)$	3.0983(0.0019)	1.3480	4.2997(1.7100e-005)	1.6886	3.7651(1.6650e-004)	1.7551
$\sigma_{sp} = \sigma_{i_\$}$=0.2						
(sp)	3.4226(6.2020e-004)	1.7166	4.8608(1.1689e-006)	1.8601	4.6714(2.9921e-006)	1.9025
$(i_\$)$	4.0414(5.3127e-005)	1.8069	4.6709(2.9995e-006)	1.8529	4.1408(3.4616e-005)	1.8252
$(i_€)$	3.9420(8.0806e-005)	1.5454	5.0068(5.5337e-007)	1.8244	4.4731(7.7087e-006)	1.8807
$\sigma_{sp} = \sigma_{i_\$}$=0.1						
(sp)	3.9457(7.9559e-005)	1.8697	5.2922(1.2085e-007)	1.9470	4.3285(1.5010e-005)	1.9679
$(i_\$)$	4.2580(2.0625e-005)	1.9561	5.0228(5.0934e-007)	1.9116	3.5641(3.6505e-004)	1.8739
$(i_€)$	4.4739(7.6803e-006)	1.8078	5.4577(4.8241e-008)	1.9144	3.7051(2.1127e-004)	1.9213
$\sigma_{sp} = \sigma_{i_\$}$=0						
(sp)	4.6466 (3.3750e-006)	1.9456	5.5332(3.1438e-008)	1.9985	5.1311(2.8801e-007)	1.9848
$(i_\$)$	4.4824(7.3803e-006)	2.0082	5.1794(2.2256e-007)	1.9584	4.8269(1.3866e-006)	1.9266
$(i_€)$	5.0788(3.7988e-007)	1.9997	5.7295(1.0073e-008)	1.9406	4.5588(5.1443e-006)	1.9272

Regardless of the size of random shock and the type of an attractor, estimated variance ratios are larger than unity for interest rates and spreads, respectively without an exception. In most cases, the null of random walk is rejected at 5% significance level. These results are consistent with the data we saw in Section 3.3. Figure 25 displays paths of $i_\$$ and $i_{t,€}$ with different sizes of random shock. We can see that circle-like motions in the cases of limit cycle and chaos are roughly preserved.[47]

Regarding higher variance ratios in the second period shown in Section 3.3, there are two important patterns in the above numerical experiment. First, given attractors (equilibrium point, limit cycle, and chaos), smaller random shock induces a higher estimated variance ratio without an exception in the Table 15. And the highest levels of variance ratios are obtained in the cases of deterministic dynamical systems. Since the variance ratios higher

[47] The basins of attraction in cases of limit cycle and chaos in the simple model are bounded, which implies that with random shock dynamic paths will eventually diverge. The boundedness of the basins of attraction is natural consequence of the existence of the zero-bound of a nominal interest rate, considering the stabilizing role of a nominal interest rate. Of course, in a real economy, there exist several unconventional monetary policies which may be conducted at the zero interest rate, including "quantitative easing." Regarding this point, I refer to Bernanke and Reinhart [2004] and Cecchetti [2009].

than unity imply positive autocorrelation of the first differences of data as in Charles and Darné [2009, 505-6], the result reveals that the models without random shock in this paper generate the highly positively autocorrelated first-differences. Adding random shock into dynamics turns out to reduce the positive autocorrelation. Second, given moderate random shock ($0.2 \leq \sigma_{sp} = \sigma_{i_\$} \leq 0.5$), change of attractors from equilibrium point to limit cycles or chaos results in a higher variance ratio.

Summing up, the numerical experiment shows that the model can easily generate variance ratios higher than unity. Regarding rising variance ratios, the numerical experiment suggests two possible explanations: One is the reduction of random shock and the other is the bifurcation of a dynamical system. Of course, two explanations may be intertwined. The emergence of limit cycles or chaos usually implies the existence of strong feedback mechanisms in a dynamical system. And these strong feedback mechanisms may reduce the effect of exogenous random shock. For example, suppose that an economy is dominated by self-reinforcing optimism. Then, in that case, unexpected negative shock like a sudden drop of durable goods orders would have smaller effect on the economy than in normal time. The following statement in Bloomberg.com succinctly states the point:

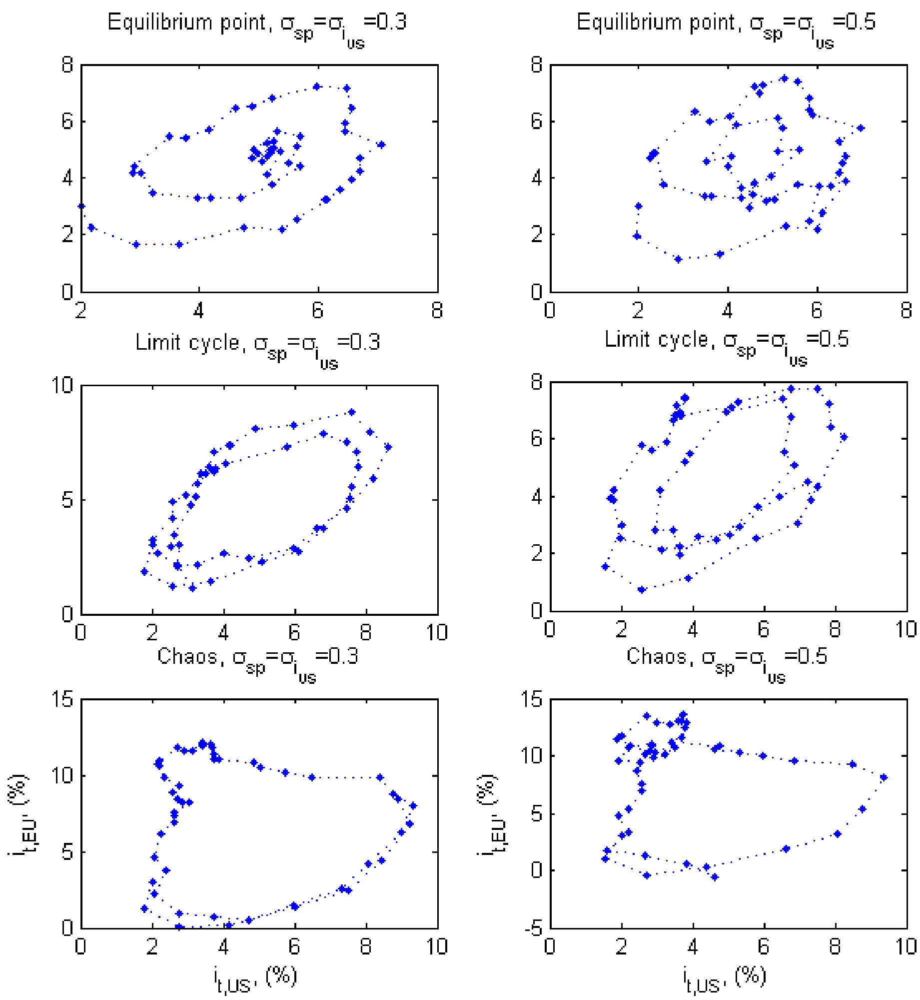
> "Against this background of heightened uncertainty, market participants focused on the deteriorating financial-market conditions while often ignoring positive macroeconomic news," the Basel, Switzerland-based BIS said in its quarterly report yesterday (Ross-Thomas [2010]).

3.5 CONCLUSION

In this paper, I studied the transition from the "Great Moderation" to the financial crisis of 2007-9 while focusing on the interactive dynamics of short-term nominal interest rates of the U.S. as a key currency country and international risk appetite as a main determinant of interest spread. The simple model includes a nonlinear dynamics due to heterogenous agents (= domestic investors and international investors) in financial markets and to international capital flow.

The main findings in this papers are two: First, strong interaction between short-term

Figure 25: Paths of $i_{t,\$}$ and $i_{t,€}$ with different sizes of random shock, t=0-49



nominal interest rates of the U.S. and international risk appetite can induce bifurcation of dynamical system. In particular, the simple model can undergo the following bifurcation route: stable fixed point → limit cycle → chaos. This finding provides a rationale for policy intervention in order to reverse the bifurcation route by affecting relevant bifurcation parameters. The simple model also suggests that the strong interaction between short-term nominal interest rates of the U.S. and international risk appetite is indispensable in understanding several economic phenomena such as the follower-leader relation between ECB and

Fed and low interest rates in the U.S. for 2003-4. The model also reveals the possibility of the *vicious spiral of uncertainty* insofar as a parameter of the model (risk attitude) depends on endogenous uncertainty generated by chaotic dynamics.
Second, regarding empirical investigation with the variance ratio test, the model can generate variance ratios higher than unity. Regarding rising variance ratios, the numerical experiment suggests two possible explanations: The reduction of random shock and the bifurcation of a dynamical system. How can we confirm these explanations? Note that there are several complexity measures which are sensitive on random shock.[48] For example, Hommes and Manzan [2006, Tabel 1] finds an evidence that Lyapunov exponent (LE) estimated by so-called "Jacobian method" is *negatively* correlated with the size of random shock. An experiment in Bandt and Pompe [2002, Figure 2, f)] displays the *positive* relation between the size of exogenous random shock and "permutation entropy" (PE). Moreover, taking into account the conjecture that more unstable endogenous dynamics might induce smaller effect of random shock, we may expect that if financial markets become unstable, the change may be detected by these measures (*higher* LEs and *lower* PEs). This reasoning suggests the potential of complexity measures as early warning signals for financial crises. In this regard, it is an interesting research topic to investigate the movements of LEs and PEs of financial variables before the financial crisis of 2007-9. That is where this research is going.

The global imbalance literature highlights Asian countries' strong demand for safe assets as one of factors responsible for the housing bubble in the U.S. via the supply of credit. This paper points out another possible effect of change of risk attitude of international investors; leading to unstable dynamics of interest rates in international markets and hence adding uncertainty in economies. Note that risk attitude of international investors is mainly affected by the environment of international financial markets. So, this paper supports the necessity of reforming current institutional setup in order to reduce the surge of demand for reserve currency in the period of international financial distress.

[48]The complexity measures include dimensions, entropies, and Lyapunov exponents (see Eckmann and Rulle [1985] for a classic discussion).

APPENDIX A

PROOFS

A.1 PROOF OF PROPOSITION 1.2.2

Note that

$$J\mid_{s.s.s.}=\begin{pmatrix} a\alpha & 0 & -a[\beta b_1-\pi^1 b_2] & a[\pi^2 b_2] \\ 0 & a\alpha & a[\pi^1 b_2] & -a[\beta b_1-\pi^2 b_2] \\ -1 & 0 & b_1 & 0 \\ 0 & -1 & 0 & b_1 \end{pmatrix}$$

Then, the characteristic polynomial $P(\lambda)$ of $J\mid_{s.s.s.}$ is as follows:

$$P(\lambda)=\begin{vmatrix} a\alpha-\lambda & 0 & -a[\beta b_1-\pi^1 b_2] & a[\pi^2 b_2] \\ 0 & a\alpha-\lambda & a[\pi^1 b_2] & -a[\beta b_1-\pi^2 b_2] \\ -1 & 0 & b_1-\lambda & 0 \\ 0 & -1 & 0 & b_1-\lambda \end{vmatrix}$$

$$=(a\alpha-\lambda)\begin{vmatrix} a\alpha-\lambda & a[\pi^1 b_2] & -a[\beta b_1-\pi^2 b_2] \\ 0 & b_1-\lambda & 0 \\ -1 & 0 & b_1-\lambda \end{vmatrix}$$

$$-\begin{vmatrix} 0 & -a[\beta b_1-\pi^1 b_2] & a[\pi^2 b_2] \\ a\alpha-\lambda & a[\pi^1 b_2] & -a[\beta b_1-\pi^2 b_2] \\ -1 & 0 & b_1-\lambda \end{vmatrix}$$

$$= (a\alpha - \lambda)(b_1 - \lambda) \begin{vmatrix} a\alpha - \lambda & -a[\beta b_1 - \pi^2 b_2] \\ -1 & b_1 - \lambda \end{vmatrix}$$

$$- \left[-(a\alpha - \lambda) \begin{vmatrix} -a[\beta b_1 - \pi^1 b_2] & a[\pi^2 b_2] \\ 0 & b_1 - \lambda \end{vmatrix} - \begin{vmatrix} -a[\beta b_1 - \pi^1 b_2] & a[\pi^2 b_2] \\ a[\pi^1 b_2] & -a[\beta b_1 - \pi^2 b_2] \end{vmatrix} \right]$$

$$= (a\alpha - \lambda)(b_1 - \lambda)\Big[(a\alpha - \lambda)(b_1 - \lambda) - a(\beta b_1 - \pi^2 b_2)\Big]$$

$$- (a\alpha - \lambda)(b_1 - \lambda)a(\beta b_1 - \pi^1 b_2) + a(\beta b_1 - \pi^1 b_2)a(\beta b_1 - \pi^2 b_2) - a(\pi^2 b_2)a(\pi^1 b_2)$$

$$= (a\alpha - \lambda)(b_1 - \lambda)\Big[(a\alpha - \lambda)(b_1 - \lambda) - a(2\beta b_1 - b_2)\Big]$$

$$+ a\beta b_1(a\beta b_1 - ab_2)$$

$$= \Big[(a\alpha - \lambda)(b_1 - \lambda) - a\beta b_1\Big]\Big[(a\alpha - \lambda)(b_1 - \lambda) - (a\beta b_1 - ab_2)\Big]$$

$$= \Big[\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta)\Big]\Big[\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta) + ab_2)\Big]$$

A.2 PROOF OF PROPOSITION 1.2.8

After substituting 1.45 and 1.46 into 1.43 and 1.44, I have

$$\begin{aligned} \dot{c^i} &= -\frac{c^i}{\sigma}\big\{\alpha[\pi^1 c^1 + \pi^2 c^2] - \beta(A[\pi^1 k^1 + \pi^2 k^2]^\theta) + \gamma - \theta A[\pi^1 k^1 + \pi^2 k^2]^{\theta - 1}\big\}, \ i = 1, 2 \\ \dot{k^i} &= A\theta[\pi^1 k^1 + \pi^2 k^2]^{\theta - 1}(k^i - [\pi^1 k^1 + \pi^2 k^2]) + A[\pi^1 k^1 + \pi^2 k^2]^\theta - c^i, \ i = 1, 2 \end{aligned} \tag{A.1}$$

At the continuum of steady states,

$$\begin{aligned}
\frac{\partial \dot{c^i}}{\partial c^i} &= -\frac{\alpha c^i \pi^i}{\sigma} \\
\frac{\partial \dot{c^i}}{\partial c^j} &= -\frac{\alpha c^i \pi^j}{\sigma} \\
\frac{\partial \dot{c^i}}{\partial k^i} &= -\frac{c^i}{\sigma}[-\beta\theta A(k^*)^{\theta-1}\pi^i - \theta(\theta-1)A(k^*)^{\theta-2}\pi^i] \\
\frac{\partial \dot{c^i}}{\partial k^j} &= -\frac{c^i}{\sigma}[-\beta\theta A(k^*)^{\theta-1}\pi^j - \theta(\theta-1)A(k^*)^{\theta-2}\pi^j] \\
\frac{\partial \dot{k^i}}{\partial c^i} &= -1 \\
\frac{\partial \dot{k^i}}{\partial c^j} &= 0 \\
\frac{\partial \dot{k^i}}{\partial k^i} &= A\theta(k^*)^{\theta-1} + A\theta(1-\theta)(k^*)^{\theta-1}[1-\frac{k^i}{k}]\pi^i \\
\frac{\partial \dot{k^i}}{\partial k^j} &= A\theta(1-\theta)(k^*)^{\theta-1}[1-\frac{k^i}{k}]\pi^j
\end{aligned} \tag{A.2}$$

Recall that $b_1 = A\theta(k^*)^{\theta-1}, b_2 = A\theta(1-\theta)(k^*)^{\theta-2}$. Let $b_{3i} = A\theta(1-\theta)(k^*)^{\theta-1}[1-\frac{k^i}{k^*}]$ (so, $b_{31}\pi^1 + b_{32}\pi^2 = 0$).

$$J\mid_{c.s.s.} = \begin{pmatrix} -\frac{\alpha c^1}{\sigma}\pi^1 & -\frac{\alpha c^1}{\sigma}\pi^2 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^1 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^2 \\ -\frac{\alpha c^2}{\sigma}\pi^1 & -\frac{c^2}{\sigma}\pi^2 & \frac{c^2}{\sigma}[\beta b_1 - b_2]\pi^1 & \frac{c^2}{\sigma}[\beta b_1 - b_2]\pi^2 \\ -1 & 0 & b_1 + b_{31}\pi^1 & b_{31}\pi^2 \\ 0 & -1 & b_{32}\pi^1 & b_1 + b_{32}\pi^2 \end{pmatrix}$$

Note that if $k^1 = k^2 = k^*, c^1 = c^2 = c^*$, then $b_{3i} = 0$, which is the case of the symmetric steady state.

$$
|J\ |_{c.s.s.} - \lambda I| = \begin{vmatrix} -\frac{\alpha c^1}{\sigma}\pi^1 - \lambda & -\frac{\alpha c^1}{\sigma}\pi^2 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^1 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^2 \\ -\frac{\alpha c^2}{\sigma}\pi^1 & -\frac{c^2}{\sigma}\pi^2 - \lambda & \frac{c^2}{\sigma}[\beta b_1 - b_2]\pi^1 & \frac{c^2}{\sigma}[\beta b_1 - b_2]\pi^2 \\ -1 & 0 & b_1 + b_{31}\pi^1 - \lambda & b_{31}\pi^2 \\ 0 & -1 & b_{32}\pi^1 & b_1 + b_{32}\pi^2 - \lambda \end{vmatrix}
$$

$$
= \begin{vmatrix} -\frac{\alpha c^1}{\sigma}\pi^1 - \lambda & -\frac{\alpha c^1}{\sigma}\pi^2 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^1 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^2 \\ \frac{c^2}{c^1}\lambda & -\lambda & 0 & 0 \\ -1 & 0 & b_1 + b_{31}\pi^1 - \lambda & b_{31}\pi^2 \\ 0 & -1 & b_{32}\pi^1 & b_1 + b_{32}\pi^2 - \lambda \end{vmatrix}
$$

$$
= \begin{vmatrix} -\frac{\alpha c^1}{\sigma}\pi^1 - \lambda & -\frac{\alpha c^1}{\sigma}\pi^2 & 0 & \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^2 \\ \frac{c^2}{c^1}\lambda & -\lambda & 0 & 0 \\ -1 & 0 & b_1 - \lambda & b_{31}\pi^2 \\ 0 & -1 & -b_1\frac{\pi^1}{\pi^2} + \frac{\pi^1}{\pi^2}\lambda & b_1 + b_{32}\pi^2 - \lambda \end{vmatrix}
$$

$$
= (b_1 - \lambda)\Big\{(-\frac{\alpha c^1}{\sigma}\pi^1 - \lambda)(-\lambda)(b_1 + b_{32}\pi^2 - \lambda)
$$

$$
- \frac{c^2}{c^1}\lambda[-\frac{\alpha c^1}{\sigma}\pi^2(b_1 + b_{32}\pi^2 - \lambda) + \frac{c^1}{\sigma}(\beta b_1 - b_2)\pi^2]\Big\}
$$

$$
+ (b_1\frac{\pi^1}{\pi^2} - \lambda\frac{\pi^1}{\pi^2})\Big\{-\lambda\frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^2 + b_{31}\pi^2[(-\frac{\alpha c^1}{\sigma}\pi^1 - \lambda)(-\lambda) + \frac{c^2}{c^1}\lambda\frac{\alpha c^1}{\sigma}\pi^2]\Big\}
$$

$$
= \lambda(b_1 - \lambda)\Big\{(\frac{\alpha c^1}{\sigma}\pi^1 + \lambda)(b_1 + b_{32}\pi^2 - \lambda)
$$

$$
+ [\frac{\alpha c^2}{\sigma}\pi^2(b_1 + b_{32}\pi^2 - \lambda) - \frac{c^2}{\sigma}(\beta b_1 - b_2)\pi^2]
$$

$$
- \frac{c^1}{\sigma}[\beta b_1 - b_2]\pi^1 + b_{31}\pi^1[(\frac{\alpha c^1}{\sigma}\pi^1 + \lambda) + \frac{\alpha c^2}{\sigma}\pi^2]\Big\}
$$

$$
= \lambda(b_1 - \lambda)\Big\{(\frac{\alpha c^*}{\sigma} + \lambda)(b_1 + b_{32}\pi^2 - \lambda) - \frac{c^*}{\sigma}(\beta b_1 - b_2) + b_{31}\pi^1[(\frac{\alpha c^*}{\sigma} + \lambda)\Big\}
$$

$$
= \lambda(b_1 - \lambda)\Big\{(\frac{\alpha c^*}{\sigma} + \lambda)(b_1 - \lambda) - \frac{c^*}{\sigma}(\beta b_1 - b_2)\Big\}, \text{ since } b_{31}\pi^1 + b_{32}\pi^2 = 0
$$

$$
= \lambda(\lambda - b_1)\Big\{\lambda^2 - (a\alpha + b_1)\lambda + a\alpha b_1 - a(\beta b_1 - b_2)\Big\}, a = -\frac{c^*}{\sigma}
$$

$$
= \lambda(\lambda - b_1)\Big\{\lambda^2 - (a\alpha + b_1)\lambda + ab_1(\alpha - \beta) + ab_2\Big\}
$$

APPENDIX B

SIMULATED PATHS

Figure 26: A divergent path (I): $\alpha > \beta$

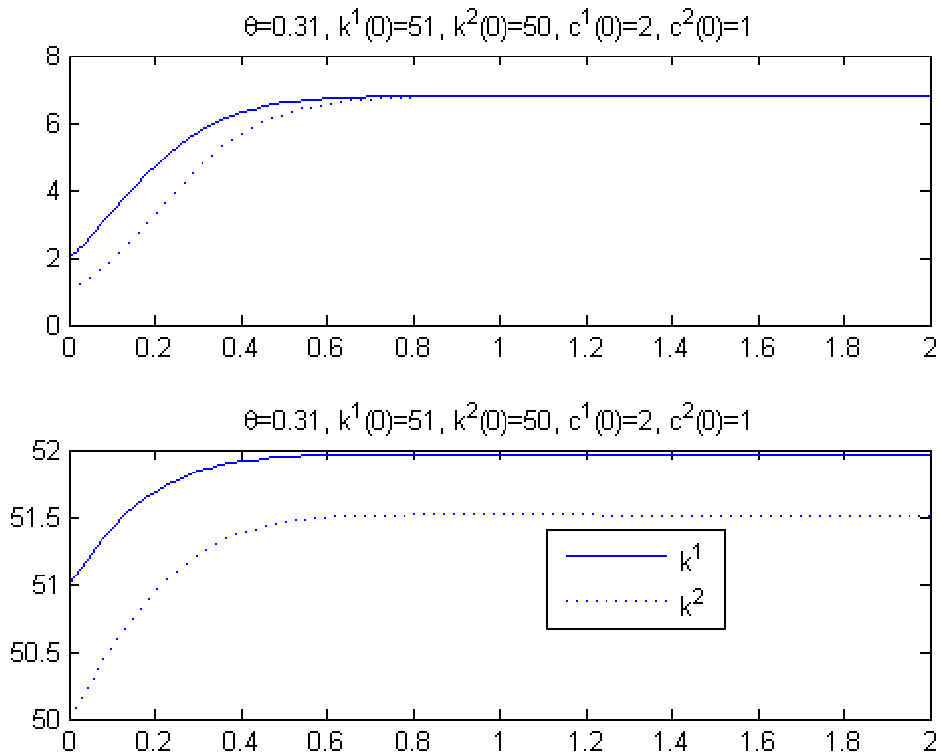


Figure 27: A divergent path (II): $\alpha > \beta$

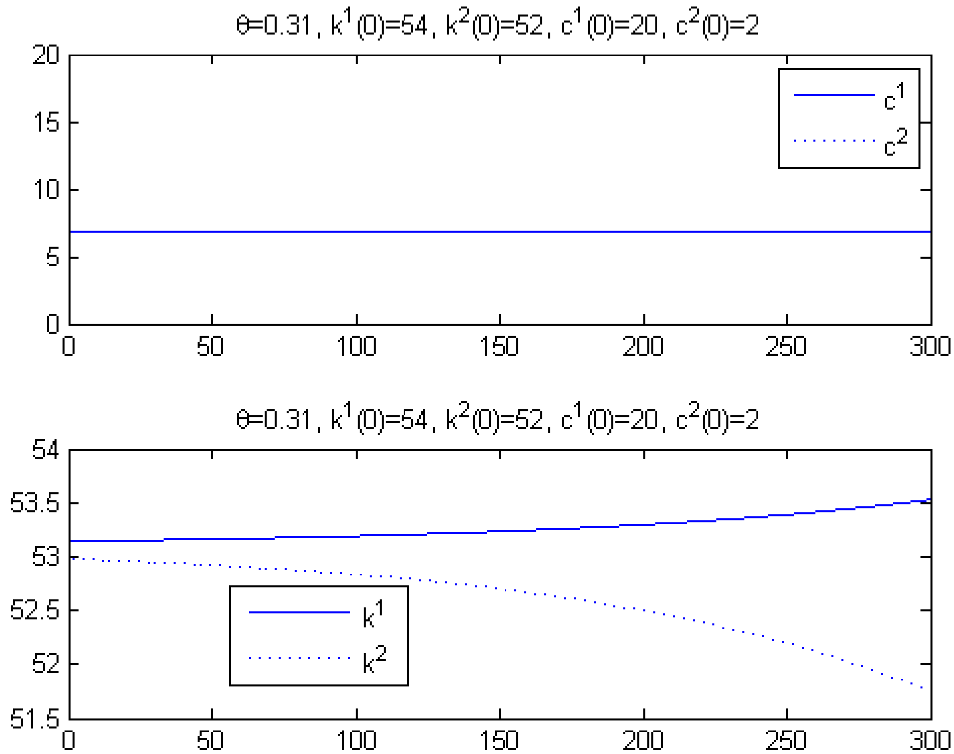


Figure 28: Three different paths: $\alpha > \beta$

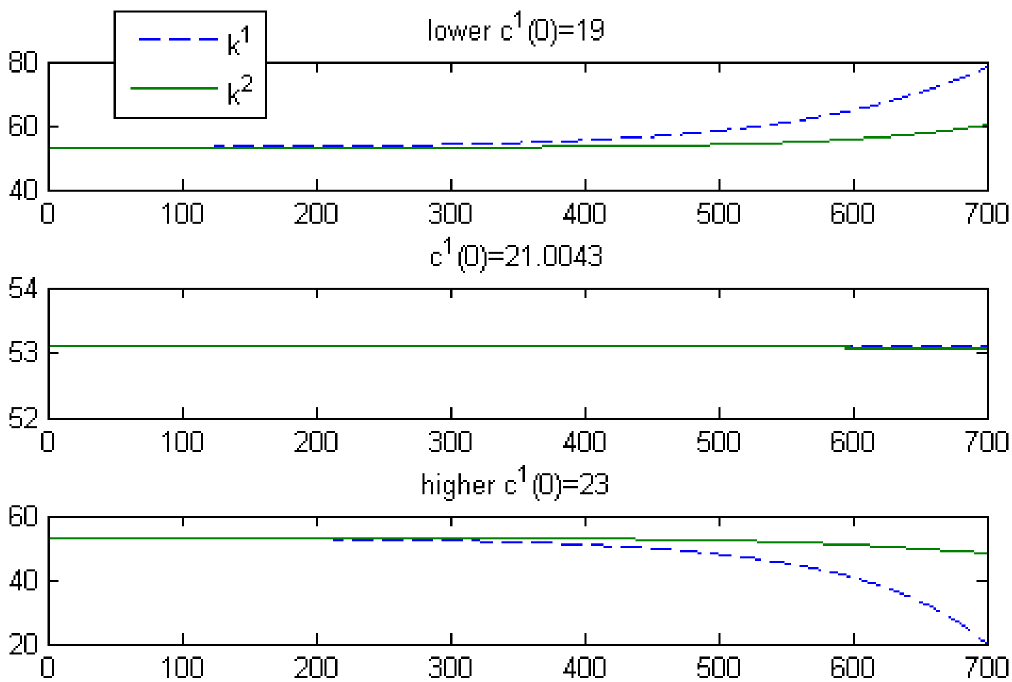


Figure 29: Comparison of two paths: $\alpha < \beta$

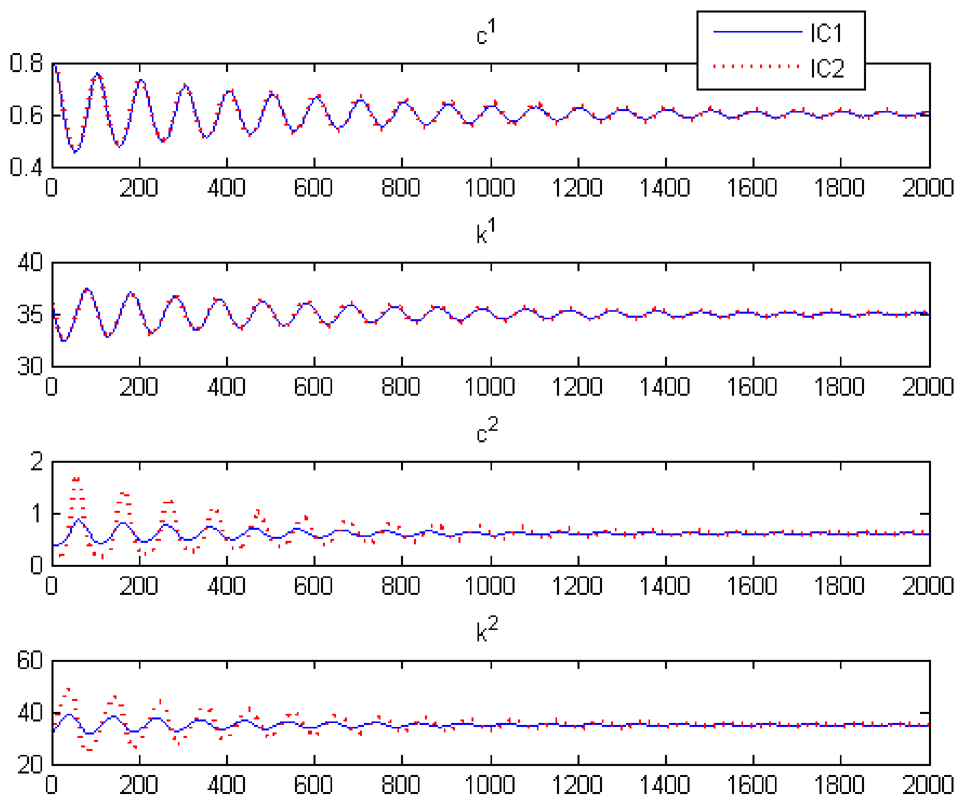


Figure 30: A convergent path (I) in reverse time: ratios

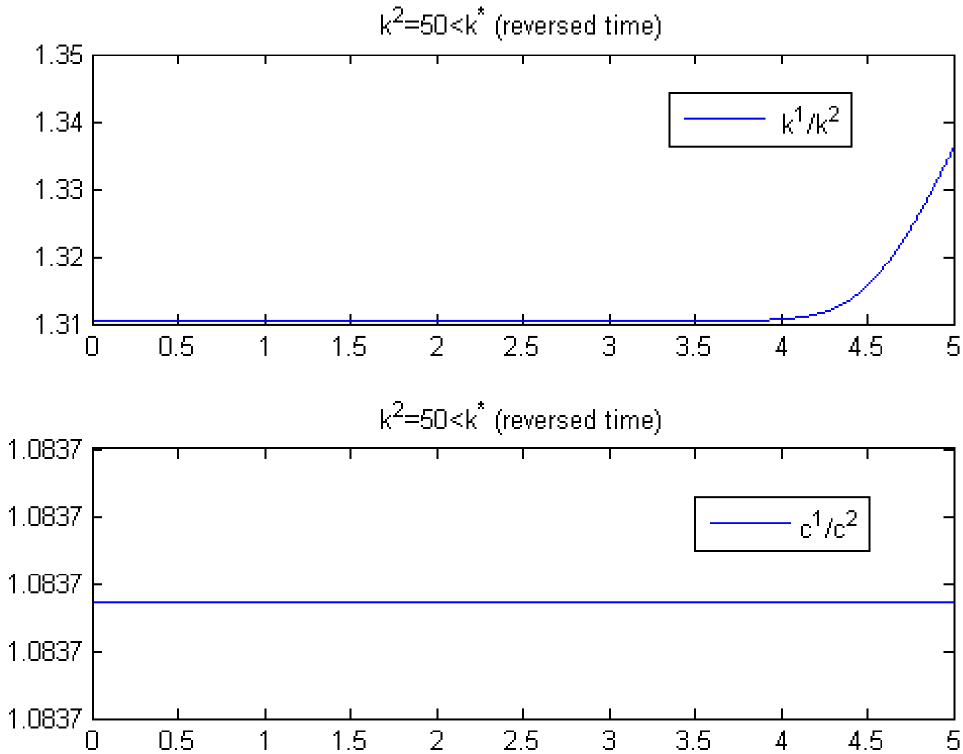


Figure 31: A convergent path (II) in reverse time

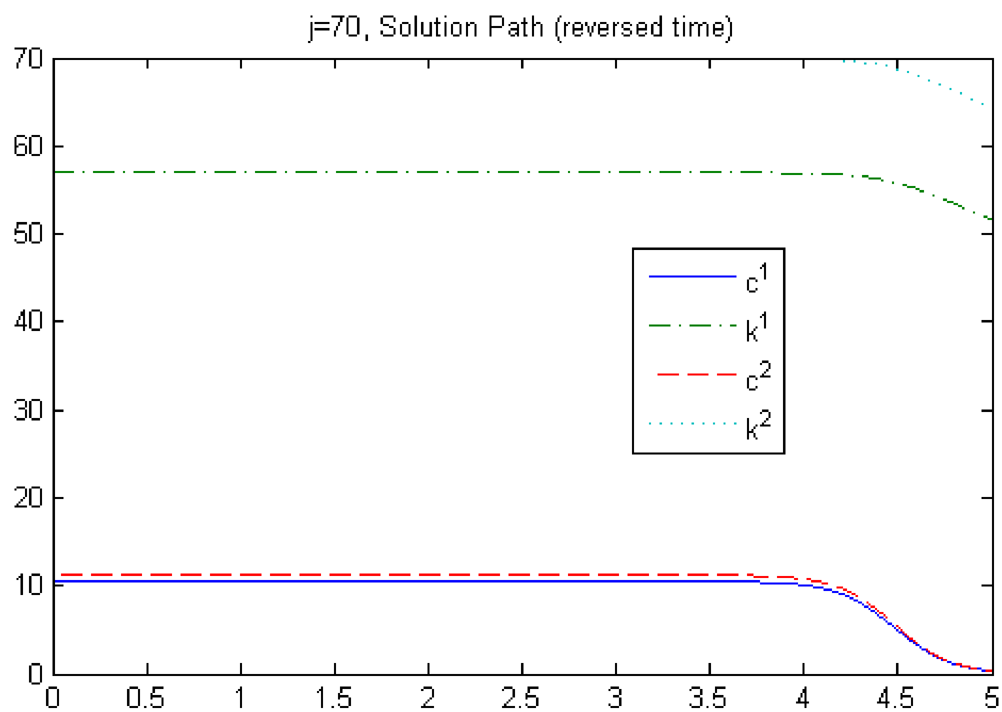


Figure 32: A convergent path (II) in reverse time: ratios

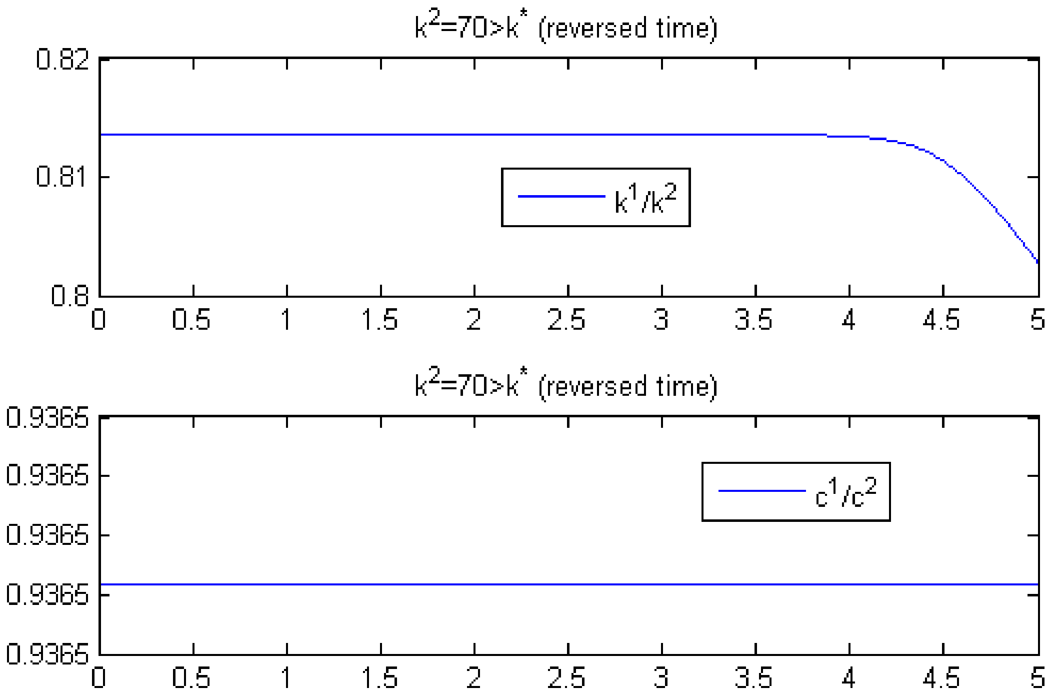
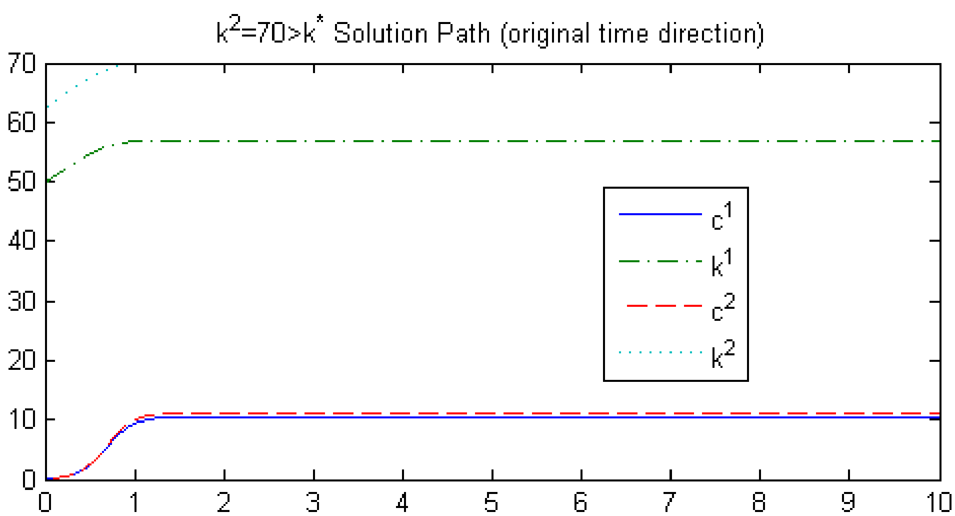


Figure 33: A convergent path (II) in original time



APPENDIX C

AN UNSTABLE LIMIT CYCLE AND THE NOTION OF "CORRIDOR STABILITY"

Here, I study an model provided by [Meng, 2006, example 2.3]. My interest in this section is whether I can find a limit cycle by employing bifurcation analysis. Ryder and Heal [1973] is an early example in an optimal growth model exhibiting a limit cycle. Benhabib and Nishimura [1979] is an early example of the application of the Hopf bifurcation theorem on optimal growth. More recently, several models investigating the influence of social factors on economic decision include limit cycles as equilibrium paths. For example, in continuous time framework, Shi [1999] shows a limit cycle may occur in some range of parameter values when fashion affects wealth accumulation decision. Drugeon [1998] finds an endogenous cycle in an endogenous discounting model where average consumption is included in the discounting function. Also, in discrete time framework, Artige et al. [2004] shows a limit cycle may occur in a two countries model with consumption habits. Recently, Ryoo [2010] provides a model of financial fragility where long waves and short cycles coexist.

The numerical investigation in this paper finds an *unstable* limit cycle.

C.1 A MODEL

In Meng [2006, example 2.3], the Euler equation and law of motion of k is given by :

$$\dot{c} = -\frac{c}{\sigma}[\alpha c - \beta A k^{\theta} + \gamma - \theta A k^{\theta-1}] \tag{C.1}$$

$$\dot{k} = A k^{\theta} - c \tag{C.2}$$

For numerical investigation, following Meng [2006, 2679], I assume that $\alpha_0 = 0.012, \beta_0 = 1, \gamma_0 = 0.6, \sigma_0 = 0.8, A_0 = 0.2$. Then, I can rewrite Eqs. C.1 and C.2 as follows:

$$\dot{c} = -\frac{c}{\sigma_0}[\alpha_0 c - \beta_0 A_0 k^{\theta} + \gamma_0 - \theta A_0 k^{\theta-1}] \equiv T_1(c, k; \theta) \tag{C.3}$$

$$\dot{k}(t) = A_0 k^{\theta} - c \equiv T_2(c, k; \theta) \tag{C.4}$$

So, the model becomes a two-dimensional dynamic system with one parameter θ. I assume $\theta \in [0.3, 0.37]$[1]. As Meng [2006, 2679] notes, there exist two steady state values $k_1^*(\theta) < k_2^*(\theta)$ within this range. Let $c_2^*(\theta)$ and $k_2^*(\theta)$ be the larger steady state. For notational convenience, from now on, I do not explicitly express the dependence of steady states on θ. By linearizing Eqs. C.3 and C.4 around the steady state, I have the following Jacobian matrix $J(\theta)$:

$$J(\theta) = \begin{pmatrix} -\frac{\alpha_0 c_2^*}{\sigma_0} & \frac{c_2^*}{\sigma_0}\left(\beta_0 \theta A_0 (k_2^*)^{\theta-1} + \theta(\theta-1) A_0 (k_2^*)^{\theta-2}\right) \\ -1 & \theta A_0 (k_2^*)^{\theta-1} \end{pmatrix} \tag{C.5}$$

Since $c_2^* = A_0 (k_2^*)^{\theta}$, I have as in [Meng, 2006, 2679]

$$tr(J;\theta) = A(k_2^*)^{\theta-1}(\theta - \frac{\alpha_0}{\sigma_0} k_2^*) \tag{C.6}$$

$$det(J;\theta) = \frac{\theta A_0 c_2^* (k_2^*)^{\theta-2}}{\sigma_0}[(\beta_0 - \alpha_0) k_2^* - (1-\theta)] \tag{C.7}$$

Let $a = -\frac{\tilde{c_2^*}}{\sigma_0},\ b_1 = A_0 \theta (k_2^*)^{\theta-1},\ b_2 = A_0 \theta (1-\theta)(k_2^*)^{\theta-2}$. Then, the characteristic polynomial $P(\lambda)$ is

$$P(\lambda) = \lambda^2 - (a\alpha_0 + b_1)\lambda + ab_1(\alpha_0 - \beta_0) + ab_2 \tag{C.8}$$

[1]In Meng [2006, 2679], $\theta = 0.3$. Here I change the value of θ. For example, According to Shao and Silos [2007], θ in the US was 0.36 on average in the post-war period. One reason of perturbing θ is that I think θ is less structural parameter than other parameters in the model. Why does capital share matter for stability? optimal growth model is about intertemporal decision of consumption. Today's consumption affects future consumption via capital accumulation. And the resulting impact also depends on capital share in the sense that capital share can be viewed as the elasticity of per capita output with respect to per capita capital. That is one reason capital share is related with stability issue.

Proposition C.1.1. *Given values of parameters, there exists unique* $\theta_0 \in [0.3, 0.37]$ *such that* $\lambda_{1,2}(\theta_0) = \omega_0 i$ *with* $\omega_0 > 0$

Proof. I show that given values of parameters, there exists $\theta_0 \in [0.3, 0.37]$ such that $tr(J;\theta_0) = 0$ and $det(J;\theta_0) > 0$, which is equivalent to the proposition.

step 1. $(k_2^*)'(\theta) < 0$.

By combining Eqs. C.3 and C.4 with the steady state condition $\dot{c} = \dot{k} = 0$, I obtain the following equation for k and θ as in [Meng, 2006, 2679].

$$S(k,\theta) = \theta + (\beta - \alpha)k - (\frac{\gamma}{A})k^{1-\theta} = 0 \tag{C.9}$$

Note that $S(\cdot,\theta)$ is continuous and concave. Moreover, given values of parameters, $S(1,\theta) < 0$, $\lim_{k\to 0} S(k,\theta) = \theta > 0$, and $\lim_{k\to\infty} S(k,\theta) = +\infty$. Hence, $k_1^* < 1$ and $k_2^* > 1$ for each $\theta \in [0.3, 0.37]$.

Then, I have

$$\begin{aligned}(k_2^*)'(\theta) &= -\frac{1 + \frac{\gamma}{A}\ln(\bar{k}_2)\bar{k}_2^{1-\theta}}{(\beta-\alpha) - \frac{\gamma}{A}(1-\theta)\bar{k}_2^{-\theta}} \quad \text{by the implicit function theorem} \\ &= -\frac{1 + \frac{\gamma}{A}\ln(\bar{k}_2)\bar{k}_2^{1-\theta}}{\theta[\frac{\gamma}{A}(k_2^*)^{-\theta} - (k_2^*)^{-1}]} < 0\end{aligned}$$

under given values of parameters since $k_2^* > 1$ where the second equality follows from Eq. C.9.

step 2. There exists unique $\theta_0 \in [0.3, 0.37]$ such that $tr(J;\theta_0) = 0$.

$tr(J;\theta_0) = 0$ holds iff θ_0 satisfies the following equation

$$\frac{\theta}{k_2^*(\theta)} = \frac{\alpha_0}{\sigma_0} = 0.015 \tag{C.10}$$

Let $g(\theta) = \frac{\theta}{k_2^*(\theta)}$. Then, $g'(\theta) = \frac{k_2^*(\theta) - \theta(k_2^*)'(\theta)}{(k_2^*(\theta))^2} > 0$ for $\theta \in [0.3, 0.37]$ and $g(0.3) \approx 0.00759 < \frac{\alpha_0}{\sigma_0}$, $g(0.37) \approx 0.01937 > \frac{\alpha_0}{\sigma_0}$. Since $g(\theta)$ is continuous and monotone, I can conclude that there exists unique $\theta_0 \in [0.3, 0.37]$ satisfying Eq. C.10.

step 3. $det(J; \theta_0) > 0$.

$$\begin{aligned} sign(det(J;\theta_0)) &= sign((\beta_0 - \alpha_0)k_2^* - (1-\theta_0)) \\ &= sign(\beta_0 k_2^* - 1 + \theta_0(1-\sigma_0)) \ (from\ Eq. C.10) \\ &> 0 \end{aligned}$$

□

To approximate θ_0, for given $\theta \in [0.3, 0.37]$ I solve Eq. C.9 numerically for k by using simple bisect method and then evaluate matrix $J(\theta)$ at the approximated values. Note that if $\theta = 0.3$, as in Meng [2006, 2679], k_1^* is a saddle and k_2^* is a stable focus. But as θ becomes larger, k_2^* becomes an unstable focus if $\theta > \theta_0$ (see the Table 1.).

Table 16: Local stability for different $\theta's$.

θ	k_1^*	eigenvalues	k_2^*	eigenvalues
0.3	0.0455	-0.6402,1.1608	39.5185	-0.0022 ± 0.0578i
0.31	0.0453	-0.6236,1.1467	34.9538	-0.0018 ± 0.0623i
0.32	0.0451	-0.6077,1.1333	31.1420	-0.0014 ± 0.0669i
0.345	0.0441	-0.5703,1.1020	23.9863	-0.0001 ± 0.0805i
0.348	0.0440	-0.5661, 1.0984	23.3013	-1.9514e-005±8.0089e-002i
0.3484	0.0440	-0.5655,1.0980	23.2123	2.7768e-006± 8.0279e-002i
0.35	0.0439	-0.5633,1.0961	22.8614	0.0001 ± 0.0810i
0.37	0.0429	-0.5361,1.0735	19.0936	0.0013 ± 0.0906i

My analytic proof for the existence of a limit cycle is based on the following normal form theorem for Hopf bifurcation.

Theorem C.1.2. *(Topological normal form for the Hopf bifurcation, in [Kuznetsov, 1998, Theorem 3.4], see also [Guckenheimer and Holmes, 1983, Theorem 3.4.2]) Any generic two-dimensional, one-parameter system*

$$\dot{x} = f(x, \eta)$$

having at $\eta = 0$ the equilibrium x=0 with eigenvalues $\lambda_{1,2}(0) = \pm i\omega_0, \omega_0 > 0$, is locally topologically equivalent near the origin to one of the following normal forms:

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} \kappa & -1 \\ 1 & \kappa \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The genericity conditions for theorem C.1.2 are the nondegeracy condition and the transversality condition [Kuznetsov, 1998, 99].

Proposition C.1.3. *There exists an unstable limit cycle.*

Proof. By proposition C.1.1 there exists a unique $\theta_0 \in [0.3, 0.37]$ which satisfies the condition for the theorem C.1.2. The transversality condition requires that $\frac{1}{2}\frac{\partial tr(J;\theta)}{\partial \theta} |_{\theta=\theta_0} \neq 0$ [Kuznetsov, 1998, 98].

Note that

$$\frac{\partial tr(J;\theta)}{\partial \theta} |_{\theta=\theta_0} = A(k_2^*(\theta_0))^{\theta_0 - 1}(1 - \frac{\alpha}{\sigma}(k_2^*)'(\theta_0)) > 0$$

The stability of a limit cycle is related with the nondegeracy condition (particularly, the sign of the first Lyapunov coefficient).[2] Since the computation normally requires information about the third derivative of the dynamic system, it is hard to interpret in economic terms. Instead, I numerically demonstrate that the limit cycle is unstable (See Section C.2). □

C.2 NUMERICAL ANALYSIS

To verify my argument, I numerically approximate the orbit of the unstable limit cycle. Since the limit cycle is unstable, that is not numerically observable. To overcome the problem, I use the idea of time reversal. I change the direction of time by transforming t to (-t). Then, by using a standard *Matlab* program (*ode45*), I approximate the orbit. In other words, I approximate the following system

[2] For the definition of the first Lyapunov coefficient, I refer to Kuznetsov [1998, 3.5].

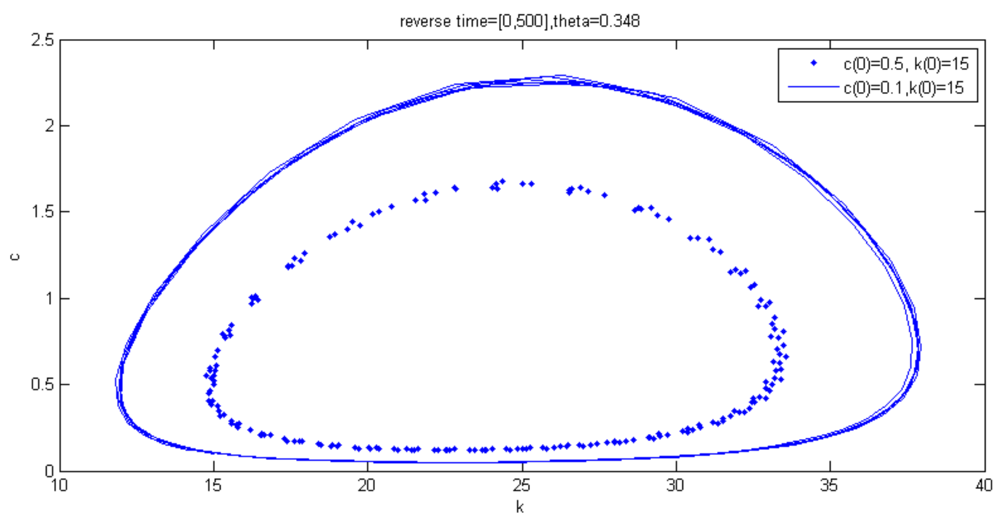
$$\dot{c}(t) = -T_1(c(t), k(t); \theta) \quad \text{(C.11)}$$

$$\dot{k}(t) = -T_2(c(t), k(t); \theta) \quad \text{(C.12)}$$

If a limit cycle is unstable in the original model, now the limit cycle becomes stable. Similarly, since the steady state is locally stable under the original system, now the steady state becomes unstable.

I compare two trajectories which start at (c(0), k(0)) = (0.1, 15) and at (c(0), k(0)) = (0.5, 15), respectively. The trajectory starting from (c(0), k(0)) = (0.1, 15) approaches to the steady state but the trajectory starting from (c(0), k(0)) = (0.5, 15) diverges from the steady state as time goes (see Figures 34-36).

Figure 34: Two trajectories I



By the uniqueness of solutions of ordinary differential equations, two trajectories cannot cross. So, I can locate a limit cycle between two trajectories.

Figure 35: Two trajectories II

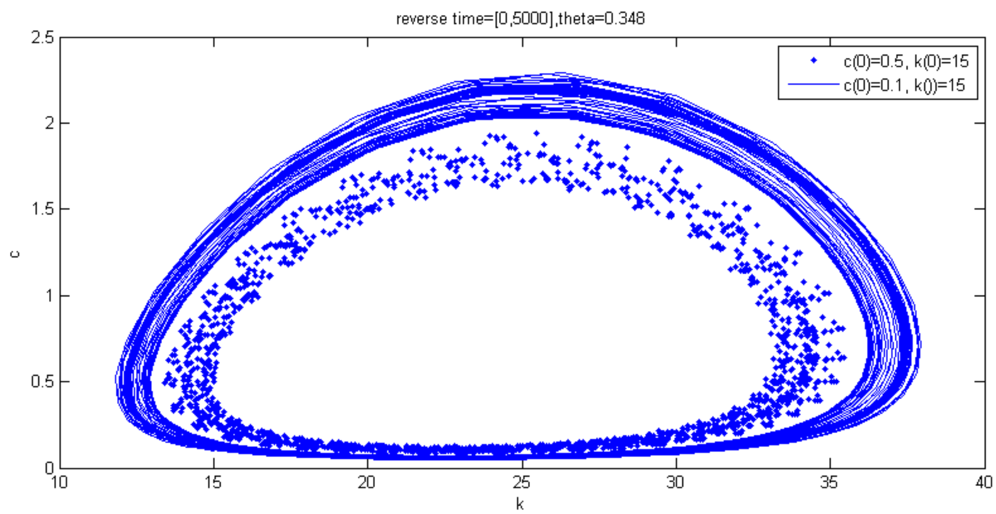
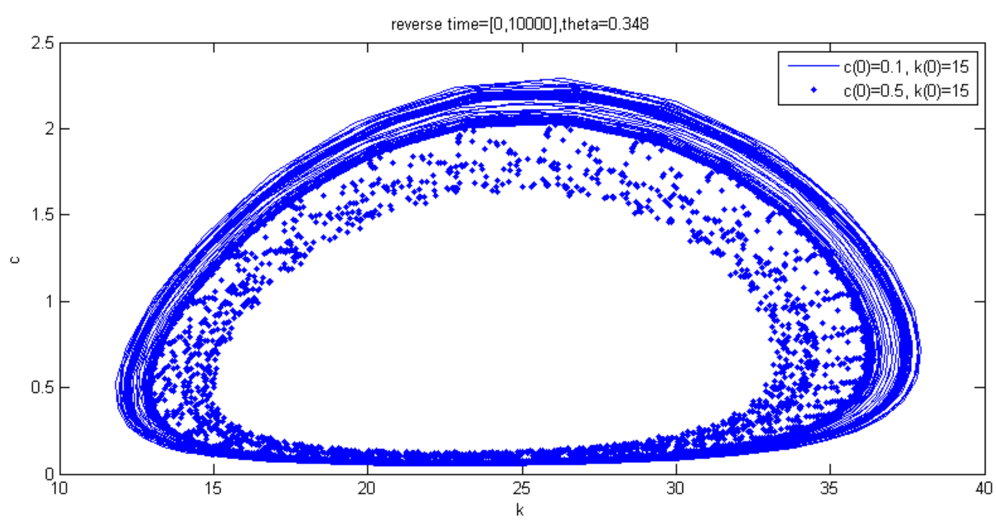


Figure 36: Two trajectories III



C.3 DISCUSSION

I found the model in [Meng, 2006, example 2.3] implies that the attracting region of the locally stable steady state is bounded by an unstable limit cycle within some range of the value of bifurcation parameter. According to Kind [1999, 147-148], an unstable limit cycle can be interpreted as "corridor stability" proposed by Leijonhufvud [1973] and Howitt [1978].

> The basic notion of the corridor is that although the economic system usually exhibits desirable stability properties there are limits to size of shock that it is capable of handling. Formally, the system is locally stable but globally unstable (Howitt [1978, 265]).

A famous model with corridor stability is Tobin [1975] regarding keynesian recession and depression. The notion of corridor stability is discussed in Leijonhufvud [2009] in the context of the financial crisis of 2007-9.[3] From the perspective of corridor stability, the finding in Appendix C implies that when consumption decisions are affected by social factors, the economy may be more vulnerable to exogenous shock.

[3]Another way of modeling a sudden dynamical change is to employ the catastrophe theory. In a recent paper, Rosser [2007] argues that abandoning catastrophe theory in economics was premature.

APPENDIX D

PSID

D.1 DATA I: HEAD WAGE

- V11397: Wages and Salaries of Head-1984
- V17829: Wages and Salaries of Head-1989
- ER6962: Wages and Salaries of Head-1994
- ER16493:Wages and Salaries of Head-1998
- ER20425: Wages and Salaries of Head-2000
- ER24117: Wages and Salaries of Head-2002
- ER27913: Wages and Salaries of Head-2004

Table 17: Descriptive statistics (Head Wage> 0)

	V11397	V17829	ER6962(<9,999,999)
obs.	5,155	6,715	6,287
Mean	$1.9684e+004	$2.3452e+004	$3.0101e+004
Maximum	600,000	800,000	775,029
coef. of variation	0.9736	1.0172	1.0357
Gini	0.3989	0.4016	0.4206
Gini (subsamples)	[0.3437 0.0969 0.0794 0.1917]	[0.3270 0.0971 0.0760 0.2062]	[0.3629 0.0945 0.0783 0.2301]
1/5/20	0.0638/ 0.1735 /0.4410	0.0691 /0.1809/ 0.4485	0.0732/ 0.1965/ 0.4687
pareto exponent (upper 20/upper 10)	3.0277/2.8947	2.787/2.659	2.6326/2.6118
	ER16493	ER20425	ER24117
obs.	5,423	5,802	6,046
Mean	$3.5597e+04	$3.9299e+004	$3.9880e+004
Maximum	850,000	885,467	3,500,000
coef. of variation	0.9370	1.1054	1.7121
Gini	0.4016	0.4104	0.4211
Gini (subsamples)	[0.2934 0.0886 0.0726 0.2204]	[0.2856 0.0851 0.0737 0.2617]	[0.2984 0.0864 0.0774 0.2731]
1/5/20	0.0654/ 0.1867/ 0.4579	0.0842 /0.2108 /0.4731	0.0978 /0.2199 /0.4823
pareto exponent (upper 20/upper 10)	2.8438/2.9502	2.3541/2.349	2.4092/2.3543
	ER27913		
obs.	6,313		
Mean	$4.1946e+004		
Maximum	2,710,000		
coef. of variation	1.5737		
Gini	0.4327		
Gini (subsamples)	[0.3120 0.0932 0.0809 0.2742]		
1/5/20	0.0985 /0.2236 /0.4895		
pareto exponent (upper 20/upper 10)	2.2822/2.2158		

D.2 DATA II: TOTAL HEAD LABOR INCOME

- V12372: Total Head Labor Y 84
- V18878: Total Head Labor Y 89
- ER6980: Labor Income of Head-1994
- ER16463: Labor Income-Head 1999
- ER20443 Labor Income of Head-2000
- ER24116: Head's 2002 Labor Income
- ER27931: Labor Income of Head-2004

Table 18: Descriptive statistics (Total Head labor income> 0)

	V12372	V18878	ER6980(<9,999,999)
obs.	5,504	7,151	6,352
Mean	$ 1.9943e+004	$2.3772e+004	$3.0668e+004
Maximum	600,000	800,000	900,033
coef. of variation	0.9904	1.0612	1.0357
Gini	0.4071	0.4091	0.4286
Gini (subsamples)	[0.3418 0.0982 0.0812 0.2037]	[0.3263 0.0982 0.0773 0.2182]	[0.3734 0.0959 0.0793 0.2394]
1/5/20	0.0670 /0.1814 /0.4501	0.0736 /0.1892 /0.4568	0.0797/ 0.2033/ 0.4757
pareto exponent (upper 20/upper 10/upper 5)	2.9156/2.854/2.717	2.6811/2.557/2.4021	2.5283/2.4739/2.3624
	ER16463	ER20443	ER24116
obs.	5,490	5,849	6,178
Mean	$3.6431e+004	$4.0669e+004	$4.0974e+004
Maximum	950,000	1,156,700	3,500,000
coef. of variation	1.0800	1.2330	1.7285
Gini	0.4076	0.4229	0.4334
Gini (subsamples)	[0.3204 0.0870 0.0730 0.2498]	[0.2916 0.0841 0.0744 0.2814]	[0.3002 0.0884 0.0783 0.2870]
1/5/20	0.0786/ 0.2029 /0.4683	0.0951/ 0.2245 /0.4851	0.1037 /0.2305 /0.4937
pareto exponent (upper 20 / upper 10 / upper 5)	2.4866/2.4817/2.4068	2.1828/2.1462/2.068	2.2817/2.2383/2.1276
	ER27931		
obs.	6,358		
Mean	$4.3129e+004		
Maximum	2,710,000		
coef. of variation	1.5897		
Gini	0.4394		
Gini (subsamples)	[0.3120 0.0929 0.0809 0.2852]		
1/5/20	0.1038 /0.2317 /0.4963		
pareto exponent (upper 20 / upper 10 / upper 5)	2.2034/2.1467/2.0087		

D.3 DATA III: WEALTH

- S117: Wealth2 84 (Main Home Equity Included)
- S217: Wealth2 89 (Main Home Equity Included)
- S317: Wealth2 94 (Main Home Equity Included)
- ER417: Wealth299 (Main Home Equity Included)
- S517: Wealth201 (Main Home Equity Included)
- S617: Wealth203 (Main Home Equity Included)
- S717: Wealth205 (Main Home Equity Included)
- V17609: 1984 Total Wealth
- V17389: 1989 Total Wealth

Table 19: Descriptive statistics (Wealth≥ 0)

	S117	S217	S317
obs.	6,296	6,436	7,797
Mean	$7.481e+04	$1.0120e+005	$28,984
Maximum	9560000	14609999	10584997
coef. of variation	4.7501	3.5746	3.0331
Gini	0.7379	0.7292	0.7219
Gini (subsamples)	[0.6413 0.2959 0.1897 0.5158]	[0.6447 0.2797 0.1913 0.4762]	[0.6705 0.2687 0.1843 0.4659]
1/5/20	0.3105/ 0.5144 /0.7893	0.2486 /0.4749 /0.7824	0.2258/ 0.4615 /0.7732
pareto exponent (upper 20/upper 10)	1.3699/1.4041	1.4505/1.5605	1.4683/1.5967

	ER417	S517	S617
obs.	6,268	6,586	6,925
Mean	$1.9108e+005	$2.0816e+005	$2.1473e+005
Maximum	27732000	43008000	36331000
coef. of variation	4.2037	4.2978	3.9110
Gini	0.7429	0.7372	0.7349
Gini (subsamples)	[0.5583 0.2539 0.1902 0.5222]	[0.5627 0.2518 0.2003 0.4999]	[0.5671 0.2522 0.1956 0.4945]
1/5/20	0.2961 /0.5205 /0.8016	0.2717/ 0.5010/ 0.7943	0.2650/ 0.4934 /0.7913
pareto exponent (upper 20/upper 10)	1.3258/1.4183	1.3954/1.4995	1.394/1.5112

	S717	V17609($<9,999,999$), 84	V17389($<99,999,999$), 89
obs.	7,001	5,883	5,751
Mean	$2.6453e+005	$8.1182e+004	$ 1.0869e+005
Maximum	$24,2047,000	9,560,000	$14,610,000
coef. of variation	3.7139	3.1865	3.1690
Gini	0.7378	0.6793	0.6954
Gini (subsamples)	[0.5677 0.2675 0.2030 0.4843]	[0.4415 0.2220 0.1680 0.4447]	[0.4235 0.2170 0.1760 0.4475]
1/5/20	0.2485 /0.4868 /0.7944	0.2184/ 0.4244 /0.7246	0.2189/ 0.4354/ 0.7473
pareto exponent (upper 20/upper 10)	1.4108/1.5677	1.5244/1.647	1.5247/1.6614

APPENDIX E

GENERAL FRAMEWORK

As Heathcote et al. [2009] points out, quantitative macroeconomics with *heterogeneous* agents has been developed in the last two decades with the development of computers. And because nice counting method is necessary to deal with rich heterogeneity, measure theory was chosen as one of mathematical backgrounds of this approach, as Ríos-Rull [1997] explained. General framework in this paper closely follows Stokey and Lucas [1989] with a minor modification due to endogenous shock.

- $(X, \mathcal{X})$, $a \in X$: endogenous state space
- $(Z, \mathcal{Z}), l \in Z$: shock
- $(S, \mathcal{S}) := (X \times Z, \mathcal{X} \times \mathcal{Z}), s := (a, l) \in S$: state space.

  DEFINITION E.0.1. (*(probability) kernel* from Kallenberg [2002, 20]).[1] Given two measurable spaces $(W, \mathcal{W})$ and $(Y, \mathcal{Y})$, a mapping K: $W \times \mathcal{Y} \to \bar{\mathbb{R}}_+$ is called a *(probability) kernel* from W to Y if:

  i) $K(\cdot, B) \in \mathcal{W}$, for all $B \in \mathcal{Y}$

  ii) $K(w, \cdot)$ is a (probability) measure on $(W, \mathcal{W})$.

- Q; *true probability kernel on* $(S, \mathcal{Z})$.

  Note that earning shock process depends on endogenous state variable as well. Normally, the shock process is characterized by transition function on $(Z, \mathcal{Z})$.[2]. Sometimes, the

[1]Probability kernel is the same as *stochastic kernel* in Stokey and Lucas [1989, 226].
[2]For example, please see Stokey and Lucas [1989, 241].

shock process is generalized to take into account the possibility that the realization of shock may depend on the state variables. If I assume that

$$Q((a_1,l),B) = Q((a_2,l),B), \forall a_1,a_2 \in X, l \in Z, B \in \mathcal{Z} \tag{E.1}$$

then, the model comes back to the standard specification.

- $\tilde{Q}$; *perceived probability kernel on* $(Z,\mathcal{Z})$. In this paper, $\tilde{Q}$ trivially satisfies the restriction of Eq. E.1. Note that in standard set-up, $\tilde{Q}$ is just true transition function of shock. It is viewed as misspecified in this paper just because now I assume shock depends on endogenous state variable as well.

- Policy function $g_{\tilde{Q}} : X \times Z \to X$ where $g_{\tilde{Q}}$ is $\mathcal{S}/\mathcal{X}$-measurable. I will use the notation of $g_{\tilde{Q}}$ whenever I need to emphasize the dependence of policy function only on $\tilde{Q}$(=perceived probability kernel).

Finally, given Q and $\tilde{Q}$, we can derive the transition function:

- Transition function $P_{Q,\tilde{Q}} : S \times \mathcal{S} \to [0,1]$ which is extended from[3]

$$P_{Q,\tilde{Q}}[(a,l), A \times B] := \begin{cases} Q((a,l),B) \ if \ g_{\tilde{Q}}(a,l) \in A, A \in \mathcal{X}, B \in \mathcal{Z} \\ 0 \ if \ g_{\tilde{Q}}(a,l) \notin A \end{cases} \tag{E.2}$$

The following proposition says this extension is well-defined.

Proposition E.0.1. *Let* $(X,\mathcal{X})$ *and* $(Z,\mathcal{Z})$ *be measurable spaces. Assume probability kernel* Q *from* $(X \times Z)$ *to* Z *and* $(\mathcal{X} \times \mathcal{Z})/\mathcal{X}$*-measurable function* g *are given. Define* $SA := \{A \times B : A \in \mathcal{X}, B \in \mathcal{Z}\}$*. Then, a mapping* $P_{SA} : (X \times Z) \times SA \to [0,1]$ *given by*

$$P_{SA}((a,l),(A \times B)) := \begin{cases} Q((a,l),B) \ if \ g(a,l) \in A \in \mathcal{X}, B \in \mathcal{Z} \\ 0 \ if \ g(a,l) \notin A \end{cases} \tag{E.3}$$

defines a transition function P *on* $(X \times Z, \mathcal{X} \times \mathcal{Z})$*.*

[3] Alternatively, we can have transition function from stochastic difference equations. Please see Stokey and Lucas [1989, 8.4], Medio [2004, 162-4], and Santos and Peralta-Alva [2005, 1941-4].

Proof. **i)** $P((a,l),\cdot)$ is a measure on $((X \times Z), \mathcal{X} \times \mathcal{Z})$.

Note that SA is semi-algebra (Durret [2005, 438,466]. Hence, by Durret [2005, theorem A.1.3,439] it suffices to show the followings:

- $P_{SA}((a,l),\emptyset) = 0$. It is trivial.
- If $A \times B = \underline{\bigcup}_{i \in I}(A_i \times B_i), I \subset \mathbb{N}, A \times B \in SA, (A_i \times B_i) \in SA, \forall i \in I$ where $\underline{\bigcup}$ denotes disjoint union, then,

$$P_{SA}((a,l), A \times B) = \sum_{i \in I} P_{SA}((a,l),(A_i \times B_i)) \tag{E.4}$$

To show Eq. E.4, if $g(a,1) \notin A$, Eq. E.4 holds clearly. Let's assume $g(a,1) \in A$ and $g(a,1) \in A_i, i \in F \subset I$. Then,

$$\underline{\bigcup}_{i \in F}(B_i) = B \tag{E.5}$$

should hold, which implies Eq. E.4.

Therefore, by Durret [2005, theorem A.1.3,439], there exists a unique extension of P_{SA} to P.

ii) $P(\cdot, A)$ with $A \in \mathcal{X} \times \mathcal{Z}$ is $\mathcal{X} \times \mathcal{Z}$-measurable.

Assume i). To show ii), let

$$\mathcal{D} = \{A \in \mathcal{X} \times \mathcal{Z} \mid P(\cdot, A) \in \mathcal{X} \times \mathcal{Z})\} \tag{E.6}$$

I will show $\mathcal{D} = \mathcal{X} \times \mathcal{Z}$ by using $\pi - \lambda$ theorem (Durret [2005, 444]).

a) Let $\mathcal{F} := \{C \times D : C \in \mathcal{X}, D \in \mathcal{Z}\}$. So, $\mathcal{X} \times \mathcal{Z} = \sigma(\mathcal{F})$ where $\sigma(\mathcal{F})$ denotes $\sigma-$algebra generated by $\mathcal{F}$. Note that $P((a,l), C \times D) = Q((a,l), D)1_C(g(a,l))$.

By basic properties of measurable functions, (Stokey and Lucas [1989, 284-5])

$$\mathcal{F} \subset \mathcal{D} \text{ (and clearly } \mathcal{F} \text{ is closed under intersection)} \tag{E.7}$$

b) I will show that $\mathcal{D}$ is a Dynkin system (or λ system). (i) Clearly, $X \times Y \in \mathcal{D}$, (ii) if $A, B \in \mathcal{D}$ and $A \subset B$, then, $B \setminus A \in \mathcal{D}$ because $P(\cdot, B \setminus A) = P(\cdot, B) - P(\cdot, A)$ from i). (iii) if $\{A_n\}_{n \geq 1} \subset \mathcal{D}, A_n \nearrow A$, then, $A \in \mathcal{X} \times \mathcal{Z}$ and $P((a,l), A_n) \nearrow P((a,l), A)$ for each $a \in \mathcal{X}, l \in \mathcal{Z}$ by the continuity of measure, which in turn implies that $P(\cdot, A) \in \mathcal{X} \times \mathcal{Z}$ because of limit function of measurable functions. By (i),(ii), and (iii),

$$\mathcal{D} \text{ is a Dynkin system (or } \lambda - \text{system).} \tag{E.8}$$

Hence, from a) and b), by $\pi - \lambda$ theorem, we have

$$\sigma(\mathcal{F}) \overset{E.7,E.8}{\subset} \mathcal{D} \overset{E.6}{\subset} \sigma(\mathcal{F}) \tag{E.9}$$

From Eq. E.9, we have the desired result.

□

- Transition operator T_P and adjoint T_P^* of T_P.

$$(T_P h)(s) := \int h(s') P(s, ds'), h \in B(S, \mathcal{S}) \tag{E.10}$$

where $B(S, \mathcal{S})$ is the set of all bounded, $\mathcal{S}$-measurable functions.

$$(T_P^* \lambda)(E) := \int P(s, E) \lambda(ds), \forall E \in \mathcal{S}, \lambda \in M(S, \mathcal{S}) \tag{E.11}$$

where $M(S, \mathcal{S})$ is the space of probability measures on $(S, \mathcal{S})$ as in Stokey and Lucas [1989, 215].

- Stationary measure λ_P^* *on* $(S, \mathcal{S})$: fixed point of adjoint operator T_P^*.

$$(T_P^* \lambda_P^*)(E) = \lambda_P^*(E), \forall E \in \mathcal{S} \tag{E.12}$$

APPENDIX F

RESULTS

Table 20: Transition Matrices with different wealth levels

($a_{\max} = 300$, $\xi = 0.1, \tilde{a}^* = 1$)	$\bar{l}^1$	$\bar{l}^2$	$\bar{l}^3$	$\bar{l}^4$	$\bar{l}^5$	$\bar{l}^6$	$\bar{l}^7$
$a_{t-1} = 0$							
$\bar{l}^1$	0.50811	0.44609	0.045403	0.00039848	2.5790×10^{-7}	1.1390×10^{-11}	0.00000
$\bar{l}^2$	0.094591	0.54361	0.34012	0.021561	0.00011341	4.3094×10^{-8}	1.1067×10^{-12}
$\bar{l}^3$	0.0040685	0.15956	0.59033	0.23675	0.0092656	2.9052×10^{-5}	6.4645×10^{-9}
$\bar{l}^4$	3.4506×10^{-5}	0.010327	0.24867	0.58718	0.15018	0.0035982	6.6940×10^{-6}
$\bar{l}^5$	5.3917×10^{-8}	0.00013295	0.023737	0.35327	0.53494	0.086659	0.0012625
$\bar{l}^6$	1.5024×10^{-11}	3.1845×10^{-7}	0.00046119	0.049386	0.45826	0.44609	0.045802
$\bar{l}^7$	7.3364×10^{-16}	1.3684×10^{-10}	1.6893×10^{-6}	0.0014411	0.093149	0.54361	0.36179
$a_{t-1} = 30$							
	0.43382	0.49938	0.066036	0.00077031	6.7129×10^{-7}	4.0160×10^{-11}	1.1102×10^{-16}
	0.066807	0.49938	0.40044	0.033144	0.00023251	1.1915×10^{-7}	4.1482×10^{-12}
	0.0023033	0.11937	0.56979	0.29341	0.015067	6.3190×10^{-5}	1.8990×10^{-8}
	1.5454×10^{-5}	0.0061942	0.19612	0.59534	0.19612	0.0061942	1.5454×10^{-5}
	1.8990×10^{-8}	6.3190×10^{-5}	0.015067	0.29341	0.56979	0.11937	0.0023033
	4.1482×10^{-12}	1.1915×10^{-7}	0.00023251	0.033144	0.40044	0.49938	0.066807
	1.5851×10^{-16}	4.0160×10^{-11}	6.7129×10^{-7}	0.00077031	0.066036	0.49938	0.43382
$a_{t-1} = 300$							
	0.30351	0.57190	0.12218	0.0024049	3.6337×10^{-6}	3.8172×10^{-10}	2.6645×10^{-15}
	0.032326	0.39586	0.50313	0.067872	0.00080909	7.2130×10^{-7}	4.4166×10^{-11}
	0.00073387	0.064239	0.49557	0.40500	0.034209	0.00024531	1.2862×10^{-7}
	3.1760×10^{-6}	0.0021990	0.11660	0.56760	0.29791	0.015618	6.6993×10^{-5}
	2.4928×10^{-9}	1.4510×10^{-5}	0.0059486	0.19235	0.59530	0.19992	0.0064652
	3.4605×10^{-13}	1.7506×10^{-8}	5.9610×10^{-5}	0.014533	0.28892	0.57190	0.12459
	8.3786×10^{-18}	3.7536×10^{-12}	1.1035×10^{-7}	0.00022033	0.032106	0.39586	0.57181

Table 21: Endogenous shock with status-seeking: different ϵ_ns

	$\epsilon_n = 100$	$\epsilon_n = 75$
$[a_{min}, a_{max}]$	[0,300]	[0,300]
interest rate	0.0309	0.0303
a_{max}/a_{mean}	47.7667	47.5139
Gini coeff. (wealth/ income)	0.6809 / 0.4196	0.6897 / 0.4213
Gini coeff.(quartile in wealth)	0.6177 / 0.2537 / 0.1903 / 0.3917	0.6136 / 0.2539 / 0.1894 / 0.4141
Top. 1%/5%/20%	0.1427 / 0.3826 / 0.7246	0.1594 / 0.4048 / 0.7340
Coeff. of variation (wealth)	1.9336	2.1143
Coeff. of variation (labor income)	0.9393	0.9406
Corr(wealth,labor income)	0.3758	0.3612
	$\epsilon_n = 50$	$\epsilon_n = 40$
$[a_{min}, a_{max}]$	[0,300]	[0,300]
interest rate	0.0297	0.0296
a_{max}/a_{mean}	46.9689	47.1146
Gini coeff. (wealth/ income)	0.7003 / 0.4234	0.6996 / 0.4223
Gini coeff.(quartile in wealth)	0.6137 / 0.2484 / 0.1845 / 0.4447	0.6094 / 0.2473 / 0.1853 / 0.4442
Top. 1%/5%/20%	0.1995 / 0.4339 / 0.7472	0.2036 / 0.4339 / 0.7470
Coeff. of variation (wealth)	2.6089	2.6502
Coeff. of variation (labor income)	0.9404	0.9399
Corr(wealth,labor income)	0.3146	0.3105
	$\epsilon_n = 30$	$\epsilon_n = 3$
$[a_{min}, a_{max}]$	[0,300]	[0,300]
interest rate	0.0288	0.0281
a_{max}/a_{mean}	46.5975	46.4533
Gini coeff. (wealth/ income)	0.7107 / 0.4251	0.7162 / 0.4255
Gini coeff.(quartile in wealth)	0.6055 / 0.2477 / 0.1858 / 0.4759	0.6140 / 0.2493 / 0.1864 / 0.4900
Top. 1%/5%/20%	0.2459 / 0.4635 / 0.7615	0.2773 / 0.4795 / 0.7675
Coeff. of variation (wealth)	3.0945	3.3865
Coeff. of variation (labor income)	0.9393	0.9380
Corr(wealth,labor income)	0.2845	0.2702

Table 22: Endogenous shock with the strongest status-seeking: different values of parameters

	$\xi = 0.11, s_s = 0,00015$	$\xi = 0.1, s_s = 0.0002$
$[a_{min}, a_{max}]$	[0,300]	[0,300]
interest rate	0.0282	0.0265
a_{max}/a_{mean}	47.7848	45.3330
Gini coeff. (wealth/ income)	0.7389 / 0.4339	0.7333 / 0.4299
Gini coeff.(quartile in wealth)	0.6535 / 0.2615 / 0.1943 / 0.5296	0.6286 / 0.2533 / 0.1889 / 0.5355
Top. 1%/5%/20%	0.3067 / 0.5253 / 0.7965	0.3368 / 0.5269 / 0.7881
Coeff. of variation (wealth)	3.6148	3.8071
Coeff. of variation (labor income)	0.9507	0.9386
Corr(wealth,labor income)	0.2957	0.2641

Table 23: Endogenous shock with status-seeking: different upper bounds

	$\xi = 0.11, s_s = 0.000125, \epsilon = 175$	$\xi = 0.12, s_s = 0.000125, \epsilon = 175$
$[a_{min}, a_{max}]$	[0,400]	[0,400]
interest rate	0.0325	0.0322
a_{max}/a_{mean}	69.2629	88.9648
Gini coeff. (wealth/ income)	0.7039 / 0.4204	0.7300 / 0.4280
Gini coeff.(quartile in wealth)	0.6268 / 0.2539 / 0.1899 / 0.4497	0.6584 / 0.2656 / 0.2016 / 0.4989
Top. 1%/5%/20%	0.2087 / 0.4417 / 0.7512	0.2512 / 0.4954 / 0.7835
Prob(a_{max})	4.0388e-005	8.9368e-005
Coeff. of variation (wealth)	2.6206	3.0604
Coeff. of variation (labor income)	0.9327	0.9447
Corr(wealth,labor income)	0.3108	0.3239
	$\xi = 0.12, s_s = 0.00009, \epsilon = 250$	$\xi = 0.12, s_s = 0.00009, \epsilon = 200$
$[a_{min}, a_{max}]$	[0,500]	[0,500]
interest rate	0.0332	0.0323
a_{max}/a_{mean}	92.1421	91.1411
Gini coeff. (wealth/ income)	0.7282 / 0.4242	0.7370 / 0.4260
Gini coeff.(quartile in wealth)	0.6589 / 0.2726 / 0.1875 / 0.5079	0.6712 / 0.2719 / 0.1942 / 0.5318
Top. 1%/5%/20%	0.2931 / 0.5015 / 0.7807	0.3338 / 0.5255 / 0.7913
Prob(a_{max})	8.7230e-005	6.2528e-004
Coeff. of variation (wealth)	3.7030	4.3221
Coeff. of variation (labor income)	0.9308	0.9307
Corr(wealth,labor income)	0.2696	0.2534

Figure 37: Log-log plots with different ϵ_ns

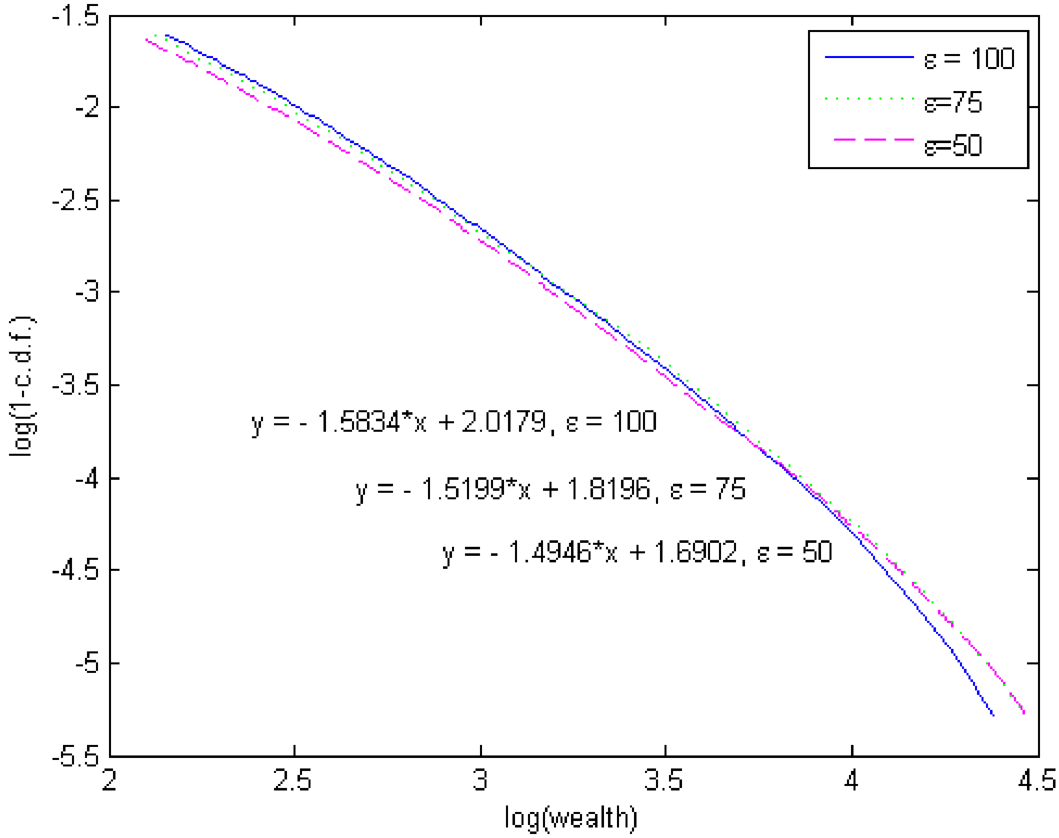


Figure 38: Log-log plot: $\xi = 0.12, s_s = 0.00009, \epsilon = 250$, and a_{max} =500

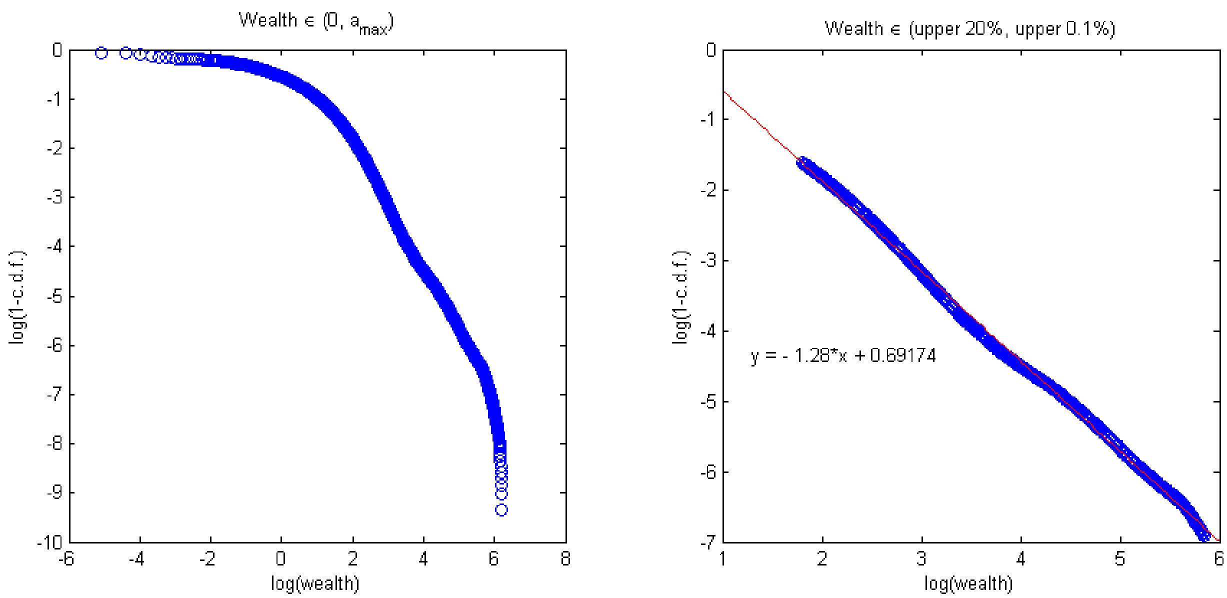


Table 24: Five iteration of one-year mobility matrix

Benchmark model	1st	2nd	3rd	4th	5th
1st	0.74020	0.21090	0.043750	0.0048998	0.00024258
2nd	0.21861	0.50228	0.23038	0.045333	0.0033836
3rd	0.042108	0.23967	0.47595	0.21288	0.029397
4th	0.0039688	0.047482	0.22413	0.55176	0.17266
5th	0.00011636	0.0029521	0.028780	0.17522	0.79293
Full model	**1st**	**2nd**	**3rd**	**4th**	**5th**
1st	0.59122	0.31624	0.081258	0.010809	0.00047399
2nd	0.31476	0.40451	0.22315	0.053506	0.0040717
3rd	0.078360	0.23234	0.43955	0.21952	0.030236
4th	0.0087161	0.053636	0.22884	0.53277	0.17604
5th	0.00027709	0.0036006	0.030877	0.17567	0.78957

APPENDIX G

WEALTH IN UTILITY: THE SPIRIT OF CAPITALISM

In this Section, I study the effect of modifying preferences by introducing wealth in utility on wealth distribution instead of changing the labor income process. The following specification of the utility function initially is proposed by Carroll [2000] and recently investigated by Francis [2009] and Luo and Young [2009].

Utility function with spirit of capitalism is specified as follows:

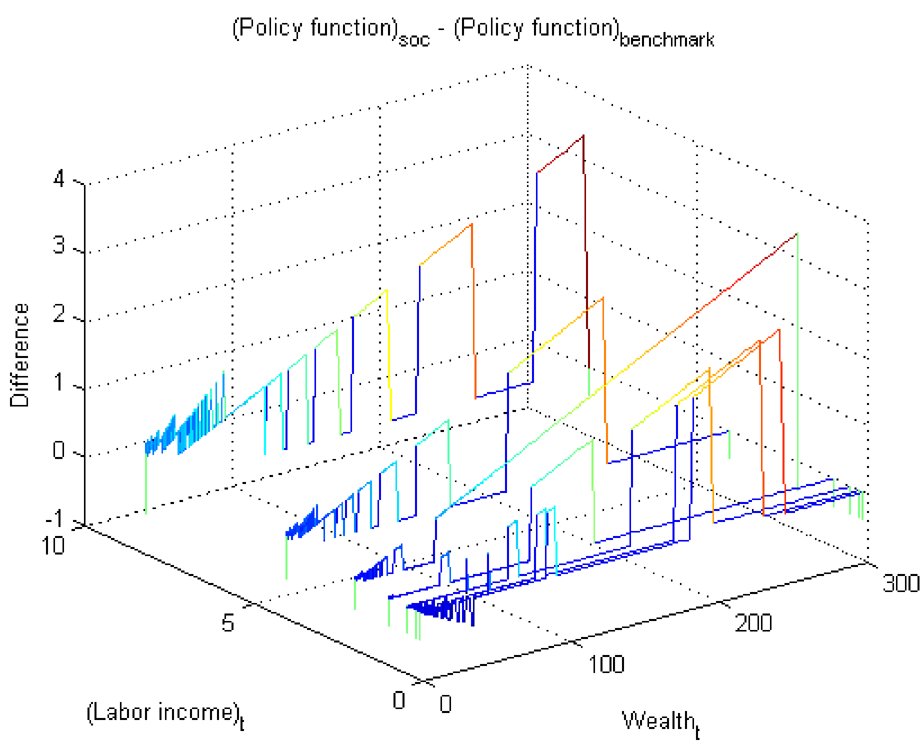
$$u(c_t, a_{t+1}) = \frac{c_t^{1-\mu} - 1}{1-\mu} + \frac{(a_{t+1} + \gamma)^{(1-s)}}{1-s} \tag{G.1}$$

γ determines the level of wealth where spirit of capitalism becomes an important factor. And s affects the speed by which importance of spirit of capitalism decreases since $\frac{\partial\ u}{\partial\ a_{t+1}} = \frac{1}{(a_{t+1}+\gamma)^s}$. In a finite horizon OLG model of Francis [2009, 402], s = 1.7 generates the closest approximation to the data with $\mu = 2$. Here I set s=1.385 ($= \frac{1.5+1.5*\frac{1.7}{2}}{2}$). And I use four different values of γ, $\{50, 150, 250, 500\}$. Table 25 shows the result. It turns out that the effect of the spirit of capitalism is not large enough to match the data within the specification of this paper as we can see from Table 25. So, the result is in line with Luo and Young [2009]. Luo and Young [2009] reports that the wealth concentration becomes less severe in an infinite horizon model with leisure choice and spirit of capitalism. The result in Table 25 implies that this specification generates too strong incentives for saving (note that interest rate is lower than the interest rates reported in Table 4). Figure 39 displays the difference between the policy function generated by the model with spirit of capitalism and the policy function generated by the benchmark model. The figure also shows that there is a surge of saving in the case of the model with spirit of capitalism. But Table 25 shows that it does not generate a fat-tailed distribution.

Table 25: Wealth in utility model

	γ=50	γ=150
Number of grid points (wealth)	500	500
$[a_{min}, a_{max}]$	[0,300]	[0,300]
interest rate	0.0157	0.0211
a_{max}/a_{mean}	28.0956	30.7276
Gini coeff. (wealth/ income)	0.5231 / 0.3868	0.5244 / 0.3847
Gini coeff.(quartile in wealth)	0.4256 / 0.1765 / 0.1447 / 0.2284	0.4295 /0.1728 / 0.1433 / 0.2289
Top. 1%/5%/20%	0.0607 / 0.2128 / 0.5457	0.0610 / 0.2160 / 0.5506
Coeff. of variation (wealth)	1.0695	1.0777
Coeff. of variation (labor income)	0.8735	0.8735
Corr(wealth,labor income)	0.3136	0.3186
	γ=250	γ=500
Number of grid points (wealth)	500	500
$[a_{min}, a_{max}]$	[0,300]	[0,300]
interest rate	0.0223	0.0229
a_{max}/a_{mean}	31.1595	31.5213
Gini coeff. (wealth/ income)	0.5206 / 0.3843	0.5176 / 0.3842
Gini coeff.(quartile in wealth)	0.4256 / 0.1684 / 0.1307 / 0.2296	0.4229 / 0.1616 / 0.1404 / 0.2259
Top. 1%/5%/20%	0.0607 / 0.2146 / 0.5458	0.0598 / 0.2149 / 0.5426
Coeff. of variation (wealth)	1.0667	1.0556
Coeff. of variation (labor income)	0.8735	0.8735
Corr(wealth,labor income)	0.3222	0.3260

Figure 39: Policy function



APPENDIX H

DHM TEST STATISTICS

To see how accurate the numerical method in this paper, I compute a DHM statistic developed in Den Haan and Marcet [1994]. The Euler equation for Eq. 2.9 with the utility function specified in Eq. 2.33 is as follows:

$$c_{t-1}^{-\mu} - \mathbf{1}_{\{a_t \geq (a_{max} - \epsilon_n)\}} \frac{sa_{max}}{\epsilon_n} = \beta(1+r)E_t[c_t^{-\mu}] \tag{H.1}$$

Multiplying both sides by c_{t-1}^{μ} and replacing the conditional expectation with a realized value, we can derive the following definition of expectation error:

$$error_{t-1} = 1 - \mathbf{1}_{\{a_t \geq (a_{max} - \epsilon_n)\}} \frac{sa_{max}}{\epsilon_n} c_{t-1}^{\mu} - \beta(1+r)(\frac{c_{t-1}}{c_t})^{\mu}, \ t \in \mathbb{N} \tag{H.2}$$

Then, DHM_T in this paper is defined as follows:

$$DHM_T = T \frac{(\frac{\sum_{t=1}^{T} error_{t-1}}{T})^2}{\frac{\sum_{t=1}^{T}(error_{t-1})^2}{T}} = \frac{(\sum_{t=1}^{T} error_{t-1})^2}{\sum_{t=1}^{T}(error_{t-1})^2} \tag{H.3}$$

According to Den Haan and Marcet [1994, 6], the DHM_T in this case follows a chi-square distribution with 1 degree of freedom:

$$DHM_T \underset{T\to\infty}{\overset{d}{\to}} \chi_1^2 \tag{H.4}$$

The detailed algorithm is similar to Den Haan and Marcet [1994, 9] and as follows:[1]

[1]The difference is that I choose the initial conditions randomly.

Step 1. Randomly choose initial conditions for wealth and labor. Then, generate data up to T=3,000 by simulating a Markov process. Compute DHM_T from Eq. H.3.

Step 2. Repeat the process 500 times.

Step 3. Compute the fractions of data which belong to the upper 5 percent or the lower 5 percent of the chi-square distribution with one degree of freedom.

I perform the test twice. The results are reported in Table 26. Since we can expect similar results for different ϵ_ns, I only report the case with $\epsilon_n = 100$.

Table 26: Accuracy of numerical method, $\epsilon_n = 100$, T=3,000

Number of grid points (wealth)	Lower 5%	Upper 5%
500	6.6% / 7.2%	2.0% / 1.4%
1,000	4.8% / 6.4%	3.2% / 3.2%
1,500	4.4% / 5.6%	3.4% / 3.0%
2,000	4.2% / 5.4%	3.4% / 3.2%

APPENDIX I

DATA

I.1 INTEREST RATES OF THE U.S.

Data about the interest rates of the U.S. all come from FRB's home page (H15. Selected Interest Rates).

- Aaa and Baa: Corporate bonds (Moody's seasoned)
- CD: Certificates of deposit
- CP (financial): Financial papers
- ED (London): Eurodollar deposits (London)
- FF: Federal funds (effective)
- TB: Treasury bills
- TCMNOM: Treasury constant maturities

I.2 SHORT-TERM NOMINAL INTEREST RATES OF 10 COUNTRIES

All data except Japan come from each country's central bank home page.

- Australia
  - Variable & Period: Bank accepted Bills (90 days), 04/07/1976 - 08/31/2009 (8,456 obs.)

- Source: http://www.rba.gov.au/Statistics/Bulletin/F01Dhist.xls

- Denmark
  - Variable & Period: Inter-bank interest rates (uncollateralized, 3 months maturity), 01/02/1989 - 07/16/2009 (5,155 obs.)
  - Source: http://nationalbanken.statistikbank.dk/statbank5a/SelectVarVal/Define.asp?MainTable=DNUDDAG&PLanguage=1&PXSId=0
- France
  - Variable & Period: 3-month Treasury Bill reference rate (bid rate), 01/03/1989 - 04/24/2009 (5,019 obs.)
  - Source: http://www.banque-france.fr/gb/poli_mone/taux/html/page4.htm
- Germany
  - Variable & Period: 3-month funds rate (ST0107, Money market rates reported by Frankfurt banks), 01/01/1970 - 04/26/2009 (9,855 obs.)
  - Source: http://www.bundesbank.de/statistik/statistik_zeitreihen.en.php?lang=en&open=&func=row&tr=ST0107
- Japan
  - Variable & Period: 3-month Euro-market interest rate, 11/02/1979 - 09/30/2009 (7,329 obs.)
  - Source: Bank of Sweden home page, http://www.riksbank.com/templates/stat.aspx?id=17206
- New Zealand
  - Variable & Period: Bank Bill Yields (90 days), 01/04/1985 - 07/24/2009 (6,183 obs.)
  - Source: http://www.rbnz.govt.nz/statistics/exandint/b2/hb2.xls
- Norway
  - Variable & Period: Norwegian Inter Bank Offered Rate (nominal, 3-month), 01/02/1986 - 07/15/2009 (5,928 obs.)
  - Source: http://www.norges-bank.no/webdav/stat/en/renter/renter_dag_e.csv
- Sweden
  - Variable & Period: Treasury Bill (3-month), 01/03/1983 - 04/24/2009 (6,565 obs.)
  - Source: http://www.riksbank.com/templates/stat.aspx?id=17187

- Switzerland
  - Variable & Period: Swiss franc Libor (3-month), 01/03/1989 - 10/31/2007 (4,759 obs.)
  - Source: http://www.snb.ch/en/iabout/stat/statpub/histz/id/statpub_histz_actual, Interest rates and yields - Historical time series 4, internet.xls
- UK
  - Variable & Period: Sterling interbank lending rate, mean (3-month), 01/03/1978 - 04/24/2009 (7,916 obs.)
  - Source: http://www.bankofengland.co.uk/mfsd/iadb/Index.asp?first=yes&SectionRequired=I&HideNums=-1&ExtraInfo=true&Travel=NIx

APPENDIX J

SCATTER PLOTS

Figure 40: Scatter plots of 10 countries against TB(3m, U.S.), 1990 - 1998

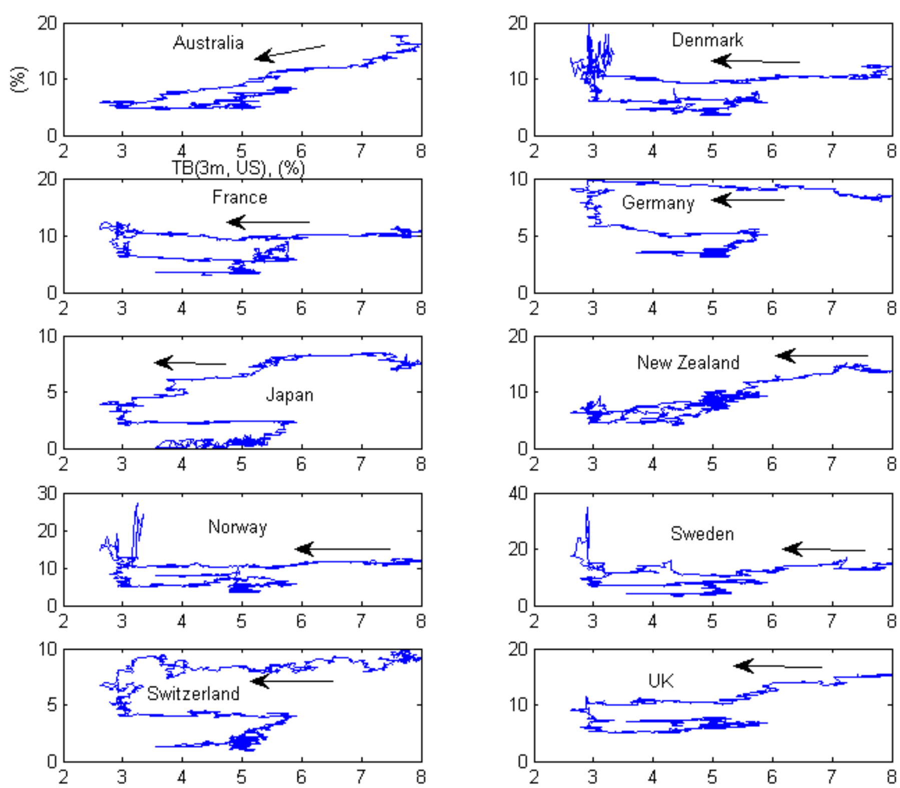
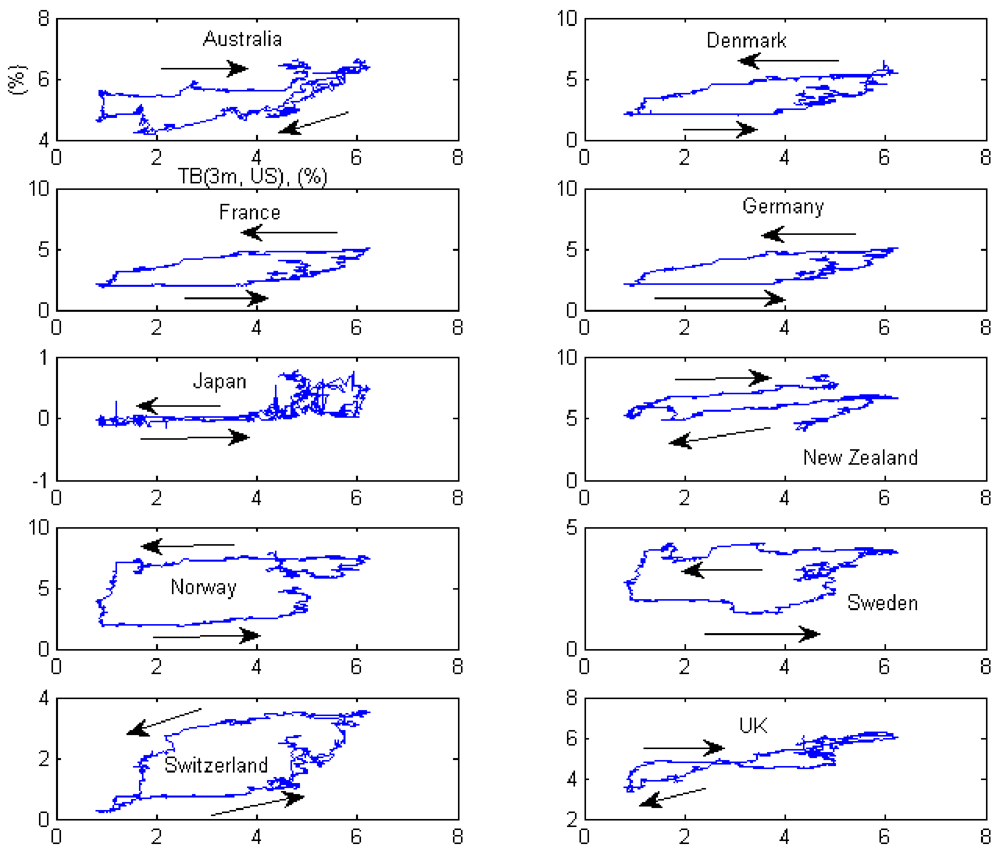


Figure 41: Scatter plots of 10 countries against TB(3m, U.S.), 1999 - 08/08/2007



APPENDIX K

VARIANCE RATIO TESTS

Table 27: Variance ratio tests of nominal interest rates in U.S. (time lag = 30 days)

	1984 - 1995			1996 - 08/08/2007		
Interest spreads						
Variance ratio	obs.	test stastistic (p-value)	ratio	obs.	test stastistic (p-value)	ratio
FF (overnight) - TB(3m)	100	-3.1498 (0.0016)***	0.4846	97	-1.8448 (0.0651)*	0.7630
CD (3m)-TB(3m)	100	-1.0276 (0.3041)	0.8649	97	-0.2788 (0.7804)	0.9619
CP (financial, 3m)	-	-	-	89	-1.9604 (0.0499)**	0.6631
ED (3m) - TB(3m)	100	-0.9118 (0.3619)	0.8839	97	-0.4529 (0.6506)	0.9415
Aaa - TCMNOM (30y)	100	-1.7228 (0.0849)*	0.8281	64	0.8201 (0.4121)	1.0814
Baa - Aaa	85	-1.1028 (0.2701)	0.8472	97	0.2942 (0.7686)	1.0298
TCMNOM (10y) - TB(3m)	100	0.4565 (0.6481)	1.0499	45	1.5845 (0.1131)	1.2164
TCMNOM (30y) - TB(3m)	100	1.4256 (0.1540)	1.1495	64	0.7622 (0.4460)	1.0654
Interest rates						
Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
FF (overnight)	101	0.5360 (0.5920)	1.0766	98	-0.6705 (0.5026)	0.8478
CD (3m)	100	1.8566 (0.0634)*	1.2340	97	3.3832 (0.0007)***	1.4715
CP (financial, 3m)	-	-	-	89	3.6530 (0.0003)***	1.5332
ED (3m)	104	1.2821 (0.1998)	1.1487	98	3.7953 (0.0001)***	1.5186
TB (3m)	100	1.4966 (0.1345)	1.2070	97	2.4324 (0.0150)**	1.3190
Aaa	102	0.3157 (0.7522)	1.0400	97	-1.0005 (0.3171)	0.8883
Baa	85	1.0016 (0.3166)	1.1107	97	-1.5152 (0.1297)	0.8259
10 y	100	0.3974 (0.6910)	1.0525	97	-0.4410 (0.6592)	0.9506
30 y	100	0.9041 (0.3659)	1.1172	64	-0.8069 (0.4197)	0.8874

Table 28: Variance ratio tests of nominal interest rates for 10 countries (time lag = 30 days)

	1984 -1995			1996-08/08/2007		
Interest spreads Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
Australia	99	0.6736 (0.5005)	1.0928	95	2.0469 (0.0407)**	1.2186
Denmark	57	-1.2165 (0.2238)	0.6769	95	0.0940 (0.9251)	1.0154
France	55	0.6380 (0.5235)	1.1136	94	2.3953 (0.0166)**	1.2270
Germany	78	2.4905 (0.0128)	1.2555	75	1.3638 (0.1726)	1.2443
Japan	66	2.5125(0.0120)***	1.3461	96	1.0705 (0.2844)	1.1388
New Zealand	90	1.2158 (0.2240)	1.1412	95	1.5895 (0.1120)	1.3036
Norway	82	0.0581 (0.9536)	1.0137	95	3.0168 (0.0026)***	1.3100
Sweden	97	0.7531 (0.4514)	1.0919	94	3.9525 (0.0001)***	1.6106
Switzerland	58	1.1005 (0.2711)	1.1269	95	0.2823 (0.7777)	1.0419
UK	99	0.9316 (0.3515)	1.1022	95	1.6185 (0.1056)	1.2415
Interest rates Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
Australia	102	2.8011 (0.0051)***	1.3124	98	2.7951 (0.0052)***	1.3430
Denmark	59	-1.3023 (0.1928)	0.5825	98	1.2835 (0.1993)	1.1932
EU	-	-	-	74	3.4299 (0.0006)***	1.3737
France	58	0.0337 (0.9731)	1.0065	97	4.0855 (4.3988e-005)***	1.4818
Germany	101	0.7871 (0.4312)	1.0728	98	2.7495 (0.0060)***	1.2494
Japan	68	2.9852 (0.0028)***	1.3452	99	0.6023 (0.5470)	1.0582
New Zealand	93	0.9230 (0.3560)	1.1294	98	2.7561 (0.0058)***	1.4588
Norway	85	-0.8373 (0.4025)	0.6325	98	2.8436 (0.0045)***	1.4204
Sweden	100	-0.7070 (0.4796)	0.6889	97	3.5080 (0.0005)***	1.5481
Switzerland	59	-1.1493 (0.2504)	0.8344	98	0.1997 (0.8417)	1.0253
UK	102	1.1148 (0.2649)	1.1506	98	3.2923 (0.0010)***	1.4610
U.S. (3m, TB)	100	1.4966 (0.1345)	1.2070	97	2.4324 (0.0150)**	1.3190
U.S. (3m, Euro-dollar)	104	1.2821 (0.1998)	1.1487	98	3.7953 (0.0001)***	1.5186

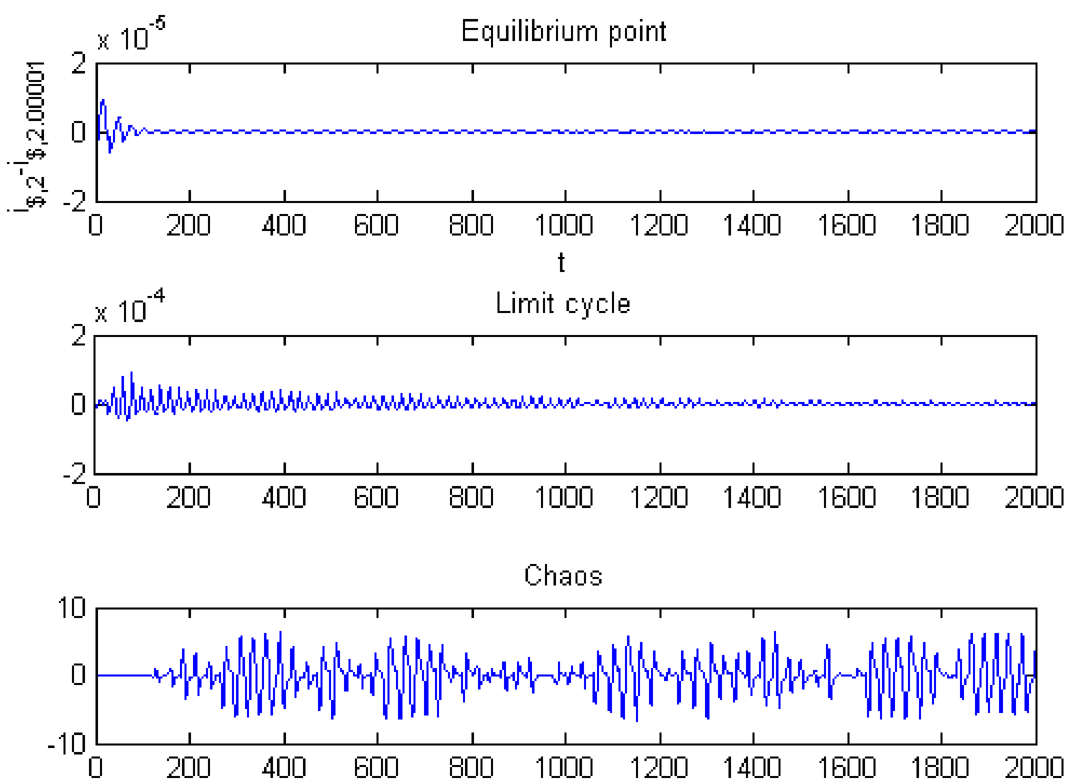
Table 29: Variance ratio tests (1999-08/2007)

		1999-08/08/2007 time lag = 65 days			1999-08/08/2007 time lag = 30 days	
Interest rates Variance ratio	obs.	test statistic (p-value)	ratio	obs.	test statistic (p-value)	ratio
Australia	34	1.3178 (0.1876)	1.2757	73	2.4363(0.0148)**	1.3844
Denmark	34	1.8684(0.0617)*	1.3577	73	2.1633(0.0305)**	1.3637
EU	34	3.0545(0.0023)***	1.5922	74	3.7423(1.8232e-004)***	1.4338
France	33	3.3018(9.6061e-004)***	1.6118	72	4.2086(2.5695e-005)***	1.5646
Germany	34	2.9495(0.0032)***	1.5744	73	3.2227(0.0013)***	1.3594
Japan	34	1.9406(0.0523)*	1.2391	72	1.5620(0.1183)	1.2554
New Zealand	34	2.6292(0.0086)***	1.3180	72	2.3195(0.0204)**	1.3150
Norway	34	2.3200(0.0203)**	1.3850	73	3.2445(0.0012)***	1.5137
Sweden	34	2.4904(0.0128)**	1.3929	72	3.6982(2.1710e-004)***	1.3921
Switzerland	34	1.4263(0.1538)	1.3139	73	2.4909(0.0127)**	1.2720
UK	34	1.9174(0.0552)*	1.1906	73	2.1419 (0.0322)**	1.2486
U.S.	34	3.0544(0.0023)***	1.6436	72	3.4935(4.7674e-004)***	1.5607

APPENDIX L

AN EXAMPLE OF THE "BUTTERFLY EFFECT"

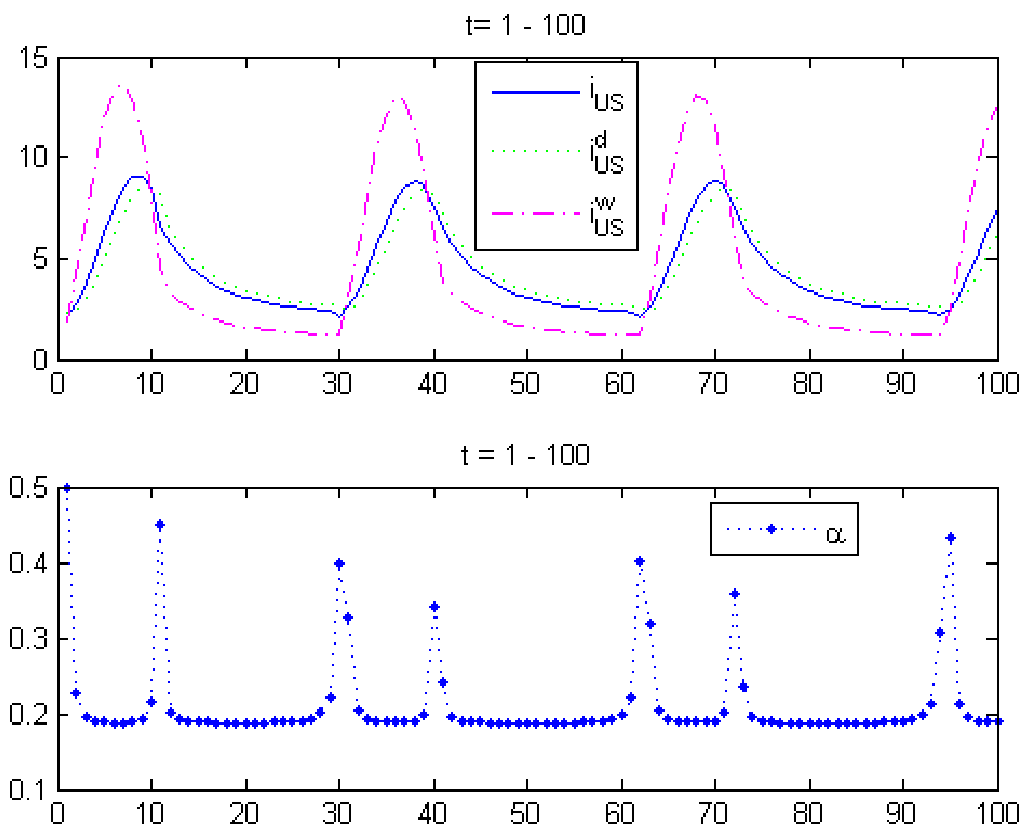
Figure 42: Example of sensitive dependence on initial conditions



APPENDIX M

PATHS OF COMPONENTS OF THE MODEL

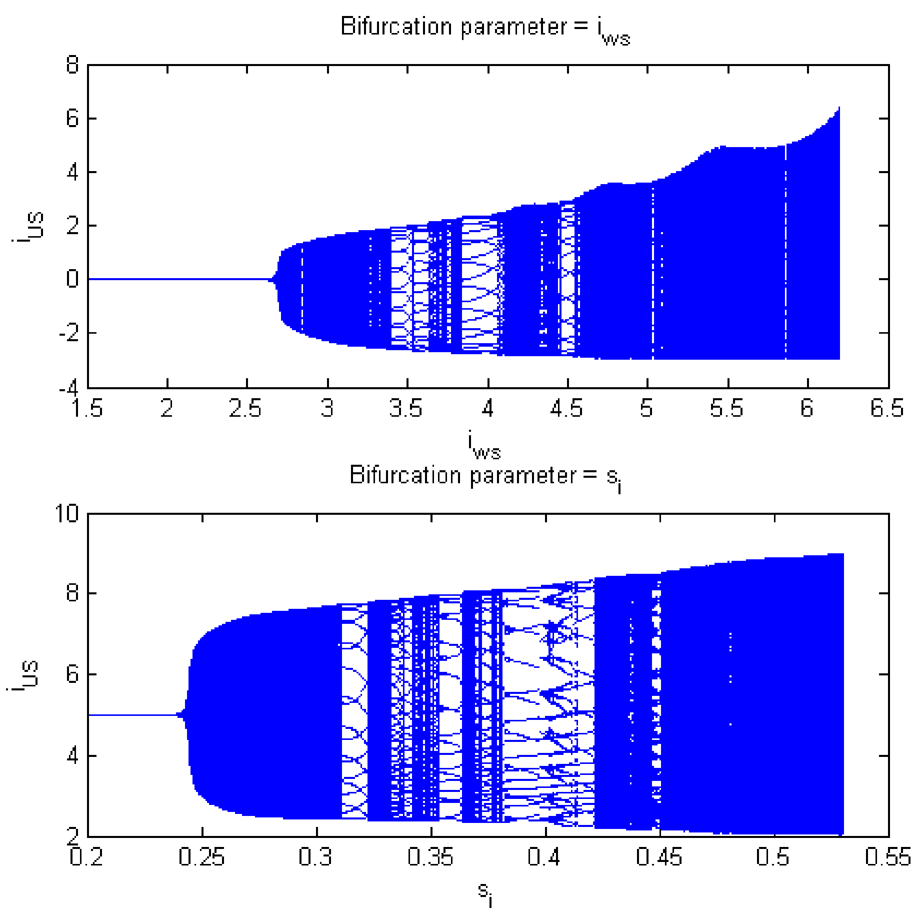
Figure 43: Paths of $i_{t,\$}, i^d_{t,\$}, i^w_{t,\$},$ and α_t



APPENDIX N

BIFURCATION DIAGRAMS

Figure 44: Bifurcation diagrams



BIBLIOGRAPHY

A. Y. Abul-Magd. Wealth distribution in an ancient Egyptian society. *Physical Review E*, 66(057104):1–3, 2002.

Tobias Adrian and Hyun Song Shin. Financial intermediaries and monetary economics. In *Handbook of Monetary Economics*. forthcoming.

Tobias Adrian, Erkko Etula, and Hyun Song Shin. Risk appetite and exchange rates. Staff Reports 361, Federal Reserve Bank of New York, 2009.

Anna Agliari, Roberto Dieci, and Laura Gardini. Homoclinic tangles in a kaldor-like business cycle model. *Journal of Economic Behavior & Organization*, 62:324–347, 2007.

Marco Airaudo and Luis-Felipe Zanna. Interest rate rules, endogenous cycles and chaotic dynamics in open economies. International Finance Discussion Papers 849, Board of Governors of the Federal Reserve System, 2005.

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. Working Paper 502, Federal Reserve Bank of Minneapolis, 1993.

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109(3):659–684, 1994.

Randall K. Q. Akee, William E. Copeland, Gordon Keeler, Adrian Angold, and E. Jane Costello. Parents incomes and childrens outcomes: A quasi-experiment using transfer payments from casino profits. *American Economic Journal: Applied Economics*, 2(1): 86–115, 2010.

Alberto Alesina and Nicola Fuchs-Schundeln. Good bye Lenin (or not?): The effect of communism on people's preferences. *American Economic Review*, 97(4):1507–1528, 2007.

Franklin Allen, Ana Babus, and Elena Carletti. Financial crises: Theory and evidence. *Annual Review of Financial Economics*, 1:97–116, 2009.

Todd W. Allen and Christopher D. Carroll. Individual learning about consumption. *Macroeconomic Dynamics*, 5:255–271, 2001.

Fernando Alvarez, Andrew Atkeson, and Patrick J. Kehoe. Time-varying risk, interest rates, and exchange rates in general equilibrium. *Review of Economic Studies*, 76:851–878, 2009.

Mikhail Anufriev and William A. Branch. Introduction to special issue on complexity in economics and finance. *Journal of Economic Dynamics and Control*, 33(5):1019–1022, 2009.

D. G. Aronson, M. A. Chory, G. R. Hall, and R. P. McGehee. Bifurcations from an invariant circle for two-parameter families of maps of the plane: A computer-assisted study. *Communications in Mathematical Physics*, 83:303–354, 1982.

Lionel Artige, Carmen Camacho, and David De La Croix. Wealth breeds decline: Reversals of leadership and consumer habits. *Journal of Economic Growth*, 9:423–449, 2004.

David H. Autor, Lawrence F. Katz, and Melissa S. Kearney. The polarization of the U.S. labor market. *American Economic Review*, 96(2):189–194, 2006.

Costas Azariadis. *Intertemporal Macroeconomics*. Blackwell, Cambrige, 1993.

Christoph Bandt and Bernd Pompe. Permutation entropy: A natural complexity measure for time series. *Physical Review Letters*, 88(17):174102–1–174102–4, 2002.

Robert J. Barro and Xavier Sala i Martin. *Economic Growth*. McGraw-Hill, New York, 1995.

Robert A. Becker. On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households. *Quarterly Journal of Economics*, 95(2):375–382, 1980.

Geert Bekaert, Marie Hoerova, and Martin Scheicher. What do asset prices have to say about risk appetite and uncertainty? Working Paper Series 1037, ECB, 2009.

Ansgar Belke and Yuhua Cui. US-Euro Area monetary policy interdependence: New evidence from Taylor rule-based VECMs. *World Economy*, 33(5):778–797, 2010.

Jess Benhabib and Alberto Bisin. The distribution of wealth and fiscal policy in economies with finitely lived agents. Working Paper 14730, NBER, 2009.

Jess Benhabib and Roger E. A. Farmer. Indeterminacy and sunspots in macroeconomics. In John B. Taylor and Michael Woodford, editors, *Handbook of Macroeconomics*, volume 1A, pages 387–448. Elsevier, New York, 1999.

Jess Benhabib and Kazuo Nishimura. The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth. *Journal of Economic Theory*, 21:421–444, 1979.

Jess Benhabib and Aldo Rustichini. Introduction to the symposium on growth, fluctuations, and sunspots: confronting the data. *Journal of Economic Theory*, 63:1–18, 1994.

Jess Benhabib and Martín Uribe. The perils of Taylor rules. *Journal of Economic Theory*, 96:40–69, 2001.

Jess Benhabib, Stephanie Schmitt-Grohé, and Martín Uribe. Chaotic interest-rate rules. *American Economic Review*, 92(2):72–78, 2002.

Ben S. Bernanke. The global saving glut and the U.S. current account deficit. *Sandridge Lecture, Virginia Association of Economics, Richmond, Virginia*, 2005.

Ben S. Bernanke and Vincent R. Reinhart. Conducting monetary policy at very low short-term interest rates. *American Economic Review*, 94(2):85–90, 2004.

Olivier Blanchard, Francesco Giavazzi, and Filipa Sa. International investors, the U.S. current account, and the dollar. *Brookings Papers on Economic Activity*, 2005(1):1–65, 2005.

Olivier Blanchard, Giovanni Dell'Ariccia, and Paolo Mauro. Rethinking macroeconomic policy. Staff Position Note 2010/03, IMF, 2010.

Olivier Jean Blanchard and Charles M. Kahn. The solution of linear difference models under rational expectations. *Econometrica*, 48(5):1305–1312, 1980.

Christopher Bliss. Koopmans recursive preferences and income convergence. *Journal of Economic Theory*, 117:124–139, 2004.

Hans G. Bloemen and Elena G. F. Stancanelli. Individual wealth, reservation wages, and transitions into employment. *Journal of Labor Economics*, 19(2):400–439, 2001.

Radim Boháĉek. Financial constraints and entrepreneurial investment. *Journal of Monetary Economics*, 53(8):2195–2212, 2006.

Michele Boldrin and Luigi Montrucchio. On the indeterminacy of capital accumulation paths. *Journal of Economic Theory*, 40:26–39, 1986.

Claudio Borio and Haibin Zhu. Capital regulation, risk-taking and monetary policy: a missing link in the transmission mechanism? Working Papers 268, BIS, 2008.

Samuel Bowles and Herbert Gintis. The inheritance of inequality. *Journal of Economic Perspectives*, 16(3):3–30, 2002.

Robert Boyer. Is a finance-led growth regime a viable alternative to fordism? A preliminary analysis. *Economy and Society*, 29(1):111–145, 2000.

William A Brock and Cars H. Hommes. A rational route to randomness. *Econometrica*, 65 (5):1059–1095, 1997.

William A. Brock and Cars H. Hommes. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, 22:1235–1274, 1998.

Markus Brückner, Kerstin Gerling, and Hans Peter Grüner. Wealth inequality and credit markets: Evidence from three industrialized countries. *Journal of Economic Growth*, 15: 155–176, 2010.

Markus K. Brunnermeier. Deciphering the liquidity and credit crunch 2007-08. *Journal of Economic Perspectives*, 23(1):77–100, 2009.

James Bullard and Kaushik Mitra. Determinacy, learnability, and monetary policy inertia. *Journal of Money, Credit and Banking*, 39(5):1177–1212, 2007.

Ricardo J. Caballero and Arvind Krishnamurthy. Collective risk management in a flight to quality episode. *Journal of Finance*, 63(5):2195–2230, 2008.

Ricardo J. Caballero and Arvind Krishnamurthy. Global imbalances and financial fragility. *American Economic Review*, 99(2):584–588, 2009.

Ricardo J. Caballero, Emmanuel Farhi, and Pierre-Olivier Gourinchas. An equilibrium model of "Global Imbalances" and low interest rates. *American Economic Review*, 98(1):358–393, 2008.

Marco Cagetti and Mariacristina De Nardi. Entrepreneurship, frictions, and wealth. *Journal of Political Economy*, 114(5):835–870, 2006.

Marco Cagetti and Mariacristina De Nardi. Wealth inequality: data and models. *Macroeconomic Dynamics*, 12(Supplement 2):285–313, 2008.

Marco Cagetti and Mariacristina De Nardi. Estate taxation, entrepreneurship, and wealth. *American Economic Review*, 99(1):85–111, 2009.

Claudio Campanale. Increasing returns to savings and wealth inequality. *Review of Economic Dynamics*, 10:646–675, 2007.

Matthew Canzoneri, Robert E. Cumby, Behzad Diba, and David Lopez-Salido. The macroeconomic implications of a key currency. Working Paper 14242, NBER, 2008.

Jack Carr. *Applications of Centre Manifold Theory*. Springer-Verlag, New York, 1981.

Christopher D. Carroll. Why do the rich save so much? In Joel B. Slemrod, editor, *Does Atlas Shrug? The Economic Consequences of Taxing the Rich*. Harvard University Press, Cambridge, 2000.

Christopher D. Carroll, Jody Overland, and David N. Weil. Comparison utility in a growth model. *Journal of Economic Growth*, 2:339–367, 1997.

Ana Castaneda, Javier Díaz-Giménez, and Jóse-Víctor Ríos-Rull. Accounting for the U.S. earnings and wealth inequality. *Journal of Political Economy*, 111(4):818–857, 2003.

Stephen G. Cecchetti. Crisis and responses: The Federal Reserve in the early stages of the financial crisis. *Journal of Economic Perspectives*, 23(1):51–75, 2009.

Valerie Cerra and Sweta Chaman Saxena. Growth dynamics: The myth of economic recovery. *American Economic Review*, 98(1):439–457, 2008.

Amélie Charles and Olivier Darné. Variance-ratio tests of random walk: An overview. *Journal of Economic Surveys*, 23(3):503–527, 2009.

Menzie Chinn and Jeffrey Frankel. The Euro area and world interest rates. *Working Paper*, 2005.

Lawrence Christiano, Roberto Motto, and Massimo Rostagno. Shocks, structures or monetary policies? the Euro area and US after 2001. *Journal of Economic Dynamics and Control*, 32(8):2476–2506, 2008.

Andrew E. Clark, Nicolai Kristensen, and Niels Westergård-Nielsen. Economic satisfaction and income rank in small neighbourhoods. *Journal of the European Economic Association*, 7(2-3):519–527, 2009.

Aaron Clauset, Cosma Rohilla Shalizi, and M. E. J. Newman. Power-law distributions in empirical data. *SIAM Review*, 51(4):661–703, 2009.

David Colander, Peter Howitt, Alan Kirman, Axel Leijonhufvud, and Perry Mehrling. Beyond DSGE models: Toward an empirically based macroeconomics. *American Economic Review*, 98(2):236–240, 2008.

Simon Condliffe and Charles R. Link. The relationship between economic status and child health: Evidence from the United States. *American Economic Review*, 98(4):1605–1618, 2008.

Brahima Coulibaly and Jonathan N. Millar. The Asian financial crisis, uphill flow of capital, and global imbalances: Evidence from a micro study. International Finance Discussion Papers 942, Board of Governors of the Federal Reserve System, 2008.

Roger Craine and Vance L. Martin. Interest rate conundrum. *The B.E. Journal of Macroeconomics*, 9(1):Article 8, 2009.

Vasco Cúrdia and Michael Woodford. Credit spreads and monetary policy. Working Paper 15289, NBER, 2009.

Janet Currie. Healthy, wealthy, and wise: Socioeconomic status, poor health in childhood, and human capital development. *Journal of Economic Literature*, 47(1):87–122, 2009.

Richard H. Day and Weihong Huang. Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, 14:299–329, 1990.

Robin de Vildor. Complicated endogenous business cycles under gross substitutability. *Journal of Economic Theory*, 71:416–442, 1996.

Wouter J. Den Haan and Albert Marcet. Accuracy in simulations. *Review of Economic Studies*, 61(1):3–17, 1994.

Robert L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Westview Press, Boulder, 2003.

Douglas W. Diamond and Raghuram G. Rajan. The credit crisis: Conjectures about causes and remedies. *American Economic Review: Papers & Proceedings*, 99(2):606–610, 2009.

Jane Dokko, Brian Doyle, Michael Kiley, Jinill Kim, Shane Sherlund, Jae Sim, and Skander Van den Heuvel. Monetary policy and the housing bubble. Finance and Economics Discussion Series 2009-49, FRB, 2009.

A. Drăgulescu and V.M. Yakovenko. Evidence for the exponential distribution of income in the USA. *European Physical Journal B*, 20:585–589, 2001.

Jean-Pierre Drugeon. A model with endogenously determined cycles, discounting and growth. *Economic Theory*, 12:349–369, 1998.

John Duffy and Jim Engle-Warnick. Multiple regimes in U.S. monetary policy? a nonparametric approach. *Journal of Money, Credit and Banking*, 38(5):1363–1377, 2006.

Rick Durret. *Essentials of Stochastic Processes*. Springer, New York, 1999.

Rick Durret. *Probability: Theory and Examples*. Thomson Brooks/Cole, Belmont, 2005.

Karen E. Dynan and Enrichetta Ravina. Increasing income inequality, external habits, and self-reported happiness. *American Economic Review*, 97(2):226–231, 2007.

J.-P. Eckmann and D. Rulle. Ergodic theory of chaos and strange attractor. *Review of Modern Physics*, 57(3):617–656, 1985.

Michael Ehrmann and Marcel Fratzscher. Equal size, equal role? interest rate interdependence between the Euro area and the United States. *Economic Journal*, 115:928–948, 2005.

Charles Engel. The forward discount anomaly and the risk premium: A survey of recent evidence. *Journal of Empirical Finance*, 3:123–192, 1996.

Larry G. Epstein. A simple dynamic general equilibrium model. *Journal of Economic Theory*, 41:68–95, 1987.

Larry G. Epstein and J. Allan Hynes. The rate of time preference and dynamic economic analysis. *Journal of Political Economy*, 91(4):611–635, 1983.

George W. Evans and Roger Guesnerie. Coordination on saddle-path solutions: the eductive viewpoint-linear multivariate models. *Journal of Economic Theory*, 124:202–229, 2005.

Emmanuel Farhi and Xavier Gabaix. Rare disasters and exchange rates. Working Paper 13805, NBER, 2008.

Roger E.A. Farmer and Amartya Lahiri. Recursive preference and balanced growth. *Journal of Economic Theory*, 125:61–77, 2006.

Guillermo Felices, Christian Grisse, and Jing Yang. International financial transmission: emerging and mature markets. Working Paper 373, Bank of England, 2009.

Martin Flodén. Aggregate savings when individual income varies. *Review of Economic Dynamics*, 11:70–82, 2008.

Johanna L. Francis. Wealth and the capitalist spirit. *Journal of Macroeconomics*, 31:394–408, 2009.

Jeffrey A. Frankel and George Saravelos. Are leading indicators of financial crises useful for assessing country vulnerability? Evidence from the 2008-09 global crisis. Working Paper 16047, NBER, 2010.

Marcel Fratzscher. What explains global exchange rate movements during the financial crisis? *Journal of International Money and Finance*, 28:1390–1407, 2009.

Xavier Gabaix. Variable rare disasters: A tractable theory of ten puzzles in macro-finance. *American Economic Review: Papers & Proceedings*, 98(2):64–67, 2008.

Xavier Gabarix. Power laws in economics and finance. *Annual Review of Economics*, 1: 255–293, 2009.

Alicia García-Herrero and Alvaro Ortíz. The role of global risk aversion in explaining sovereign spreads. *Economía*, 7(1):125–155, 2006.

Pere Gomis-Porqueras and Àlex Haro. A geometric description of a macroeconomic model with a center manifold. *Journal of Economic Dynamics and Control*, 33:1217–1235, 2009.

Brenda González-Hermosillo. Investors' risk appetite and global financial market conditions. Working Paper 08/85, IMF, 2008.

Jean-Michel Grandmont. On endogenous competitive business cycles. *Econometrica*, 53(5): 995–1046, 1985.

Jean-Michel Grandmont. Transformations of the commodity space, behavioral heterogeneity, and the aggregation problem. *Journal of Economic Theory*, 57:1–35, 1992.

Jean-Michel Grandmont. Nonlinear difference equations, bifurcations and chaos: An introduction. *Research in Economics*, 62:122–177, 2008.

John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, New York, 1983.

Veronica Guerrieri and Péter Kondor. Fund managers, career concerns, and asset price volatility. Working Paper 14898, NBER, 2009.

R. F. Harrod. An essay in dynamic theory. *Economic Journal*, 49(193):14–33, 1939.

Jonathan Heathcote, Kjetil Storesletten, and Giovanni L. Violante. Quantitative macroeconomics with heterogeneous households. *Annual Review of Economics*, 1:319–354, 2009.

Jonathan Heathcote, Fabrizio Perri, and Giovanni L. Violante. Unequal we stand: An empirical analysis of economic inequality in the United States, 1967-2006. *Review of Economic Dynamics*, 13:15–51, 2010.

Géza Hegyi, Zoltán Néda, and Maria Augusta Santos. Wealth distribution and Pareto's law in the Hungarian medieval society. *Physica A*, 380:271–277, 2007.

Berthold Herrendorf, Ákos Valentinyi, and Robert Waldmann. Ruling out multiplicity and indeterminacy: The role of heterogeneity. *Review of Economic Studies*, 67(2):295–307, 2000.

Morris W. Hirsh, Stephen Smale, and Robert L. Devaney. *Differential Equations, Dynamical Systems, and an Intorduction to Chaos*. Academic Press, San Diego, 2004.

Cars H. Hommes. Heterogeneous agent models in economics and finance. In Leigh Tesfatsion and Kenneth L. Judd, editors, *Handbook of Computational Economics*, volume 2. North-Holland, Amsterdam, 2006.

Cars H. Hommes and Sebastiano Manzan. Comments on testing for nonlinear structure and chaos in economic time series. *Journal of Macroeconomics*, 28:169–174, 2006.

Peter Howitt. The limits to stability of a full employment equilibrium. *Scandinavian Journl of Economics*, 80(3):265–282, 1978.

Mark Huggett. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17:953–969, 1993.

Erik Hurst and Annamaria Lusardi. Liquidity constraints, household wealth, and entrepreneurship. *Journal of Political Economy*, 112(2):319–347, 2004.

Erik Hurst, Ming Ching Luoh, Frank P. Stafford, and William G. Gale. The wealth dynamics of American families, 1984-94. *Brookings Papers on Economic Activity*, 1998(1):267–337, 1998.

Matteo Iacoviello. Household debt and income inequality, 1963-2003. *Journal of Money, Credit and Banking*, 40(5):929–965, 2007.

Olivier Jeanne. International reserves in emerging market countries: Too much of a good thing? *Brookings Papers on Economic Activity*, 2007(1):1–79, 2007.

Kenneth L. Judd. *Numerical Methods in Economics*. MIT Press, Cambridge, 1998.

Daniel Kahneman. Maps of bounded rationality: Psychology for behavioral economics. *American Economic Review*, 93(5):1449–1475, 2003.

Taisei Kaizoji. Intermittent chaos in a model of finanical markets with heterogenous agents. *Chaos, Solitions and Fractals*, 20:323–327, 2004.

Olav Kallenberg. *Foundations of Modern Probability*. Springer, New York, 2002.

Graciela L. Kaminsky and Carmen M. Reinhart. The twin crises: The causes of banking and balance-of-payments problems. *American Economic Review*, 89(3):473–500, 1999.

Michael P. Keane and Kenneth I. Wolpin. The career decisions of young men. *Journal of Political Economy*, 105(3):473–522, 1997.

Murray C. Kemp and Koji Shimomura. A dynamic model of the distribution of wealth among households and nations. *Annals of Operations Research*, 37:245–272, 1992.

Arthur B. Kennickell. Ponds and streams: Wealth and income in the U.S., 1989 to 2007. Finance and Economics Discussion Series 2009-13, FRB, 2009.

Christoph Kind. Remarks on the economic interpretation of Hopf bifurcations. *Economics Letters*, 62:147–154, 1999.

Charles P. Kindleberger and Robert Z. Aliber. *Manias, Panics and Crashes: A History of Financial Crises*. John Wiley & Sons, Hoboken, 2005.

Sagiri Kitao. Entrepreneurship, taxation and capital investment. *Review of Economic Dynamics*, 11:44–69, 2008.

Nobuhiro Kiyotaki and John Moore. Credit cycles. *Journal of Political Economy*, 105(2): 211–248, 1997.

Oren S. Klass, Ofer Biham, Moshe Levy, Ofer Malcai, and Sorin Solomon. The Forbes 400 and the Pareto wealth distribution. *Economics Letters*, 90:290–295, 2006.

Per Krusell and Anthony A. Smith. Income and welath heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, 1998.

Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer, New York, 1998.

Catherine Kyrtsou and Costas Vorlow. Modelling non-linear comovements between time series. *Journal of Macroeconomics*, 31:200–211, 2009.

Axel Leijonhufvud. Effective demand failures. *The Swedish Journal of Economics*, 75(1): 27–48, 1973.

Axel Leijonhufvud. Out of the corridor: Keynes and the crisis. *Cambridge Journal of Economics*, 33:741–757, 2009.

James P. LeSage. *Applied Econometrics using MATLAB*, 1999.

Moshe Levy. Are rich people smarter? *Journal of Economic Theory*, 110:42–64, 2003.

Christina Y. Liu and Jia He. A variance-ratio test of random walks in foreign exchange rates. *Journal of Finance*, 46(2):773–785, 1991.

Lars Ljungqvist and Thomas J. Sargent. *Recursive Macroeconomic Theory*. MIT Press, Cambridge, 2004.

Ngo Van Long and Koji Shimomura. Relative wealth, status-seeking, and catching-up. *Journal of Economic Behavior & Organization*, 53:529–542, 2004.

Robert E. Lucas. On the mechanics of economic development. *Journal of Monetary Economics*, 22:3–42, 1988.

Robert E. Lucas and Nancy L. Stokey. Optimal growth with many consumers. *Journal of Economic Theory*, 32:139–171, 1984.

Yolei Luo and Eric R. Young. The wealth distribution and the demand for status. *Macroeconomic Dynamics*, 13:1–30, 2009.

Jeremy R. Magruder. Intergenerational networks, unemployment, and persistent inequality in South Africa. *American Economic Journal: Applied Economics*, 2(1):62–85, 2010.

Simone Manganelli and Guido Wolswijk. What drives spreads in the euro area government bond market? *Economic Policy*, 24(58):191–240, 2009.

Mathworks, 2010. website: www.mathworks.com.

Kiminori Matsuyama. Increasing returns, industrialization, and indeterminacy of equilibrium. *Quarterly Journal of Economics*, 106(2):617–650, 1991.

Kiminori Matsuyama. Financial market globalization, symmetry-breaking and endogenous inequality of nations. *Econometrica*, 72(3):853–884, 2004.

Kiminori Matsuyama. Symmetry breaking. In Steven N. Durlauf and Lawrence E Blume, editors, *The New Palgrave Dictionary of Economics*. Palgrave Macmillan, New York, 2008.

Alfredo Medio. Invariant probability distributions in economic models: A general result. *Macroeconomic Dynamics*, 8:162–187, 2004.

Césaire A. Meh. Entrepreneurship, wealth inequality, and taxation. *Review of Economic Dynamics*, 8:688–719, 2005.

Enrique G. Mendoza. Sudden stops, financial crises and leverage. *American Economic Review*, forthcoming.

Qinglai Meng. Impatience and equilibrium indeterminacy. *Journal of Economic Dynamics and Control*, 30:2671–2692, 2006.

Hyman P. Minsky. *Stabilizing an unstable economy.* Yale University Press, New Haven, 1986.

Hyman P. Minsky. The financial instability hypothesis. Working Paper 74, The Jerome Levy Economics Institute of Berd College, 1992.

Tapan Mitra and Kazuo Nishimura. Introduction to intertemporal equilibrium theory: Indeterminacy, bifurcations, and stability. *Journal of Economic Theory*, 96:1–12, 2001.

Michael Mitzenmacher. A brief history of generative models for power law and lognormal distributions. *Internet Mathematics*, 1(2):226–251, 2004.

Camilo Mondragón-Vélez. The probability of transition to entrepreneurship revisited: Wealth, education and age. *Annals of Finance*, 5(3-4):421–441, 2009.

Juan Rafael Morillas. Assets, earnings mobility and the black/white gap. *Social Science Research*, 36:808–833, 2007.

Monique Borgerhoff Mulder, Samuel Bowles, Tom Hertz, Adrian Bell, Jan Beise, Greg Clark, Ila Fazzio, Michael Gurven, Kim Hill, Paul L. Hooper, William Irons, Hillard Kaplan, Donna Leonetti, Bobbi Low, Frank Marlowe, Richard McElreath, Suresh Naidu, David Nolin, Patrizio Piraino, Rob Quinlan, Eric Schniter, Rebecca Sear, Mary Shenk, Eric Alden Smith, Christopher von Rueden, and Polly Wiessner. Intergenerational wealth transmission and the dynamics of inequality in small-scale societies. *Science*, 326(5953):682–688, 2009.

Michiyo Nakamoto and David Wighton. Citigroup chief stays bullish on buy-outs. *Financial Times (FT.com)*, news on July 9, 2007.

Yasuhiro Nakamoto. Jealousy and underconsumption in a one-sector model with wealth preference. *Journal of Economic Dynamics and Control*, 33:2015–2029, 2009.

M.E.J. Newman. Power laws, Pareto distributions and Zipf's law. *Contemporary Physics*, 46(5):323–351, 2005.

Makoto Nirei and Wataru Souma. A two factor model of income distribution dynamics. *Review of Income and Wealth*, 53(3):440–459, 2007.

Maurice Obstfeld. Intertemporal dependence, impatience, and dynamics. *Journal of Monetary Economics*, 26:45–75, 1990.

McCulley Paul and Ramin Toloui. Chasing the neutral rate down: Financial conditions, monetary policy, and the taylor rule. *Global Central Bank Focus, PIMCO*, 2008.

Lawrence Perko. *Differential equations and dynamical systems*. Springer, New York, 2001.

Thomas Piketty. Top income shares in the long run: An overview. *Journal of the European Economic Association*, 3(2/3):382–392, 2005.

Thomas Piketty and Emmanuel Saez. How progressive is the U.S. federal tax system? A historical and international perspective. *Journal of Economic Perspectives*, 21(1):3–24, 2007.

Vincenzo Quadrini. Entrepreneurship, saving, and social mobility. *Review of Economic Dynamics*, 3:1–40, 2000.

Raghuram G. Rajan. Has finance made the world riskier? *European Financial Management*, 12(4):499–533, 2006.

F. P. Ramsey. A mathematical theory of saving. *Economic Journal*, 38(152):543–559, 1928.

Carmen M. Reinhart and Kenneth S. Rogoff. Banking crises: An equal opportunity menace. Working Paper 14587, NBER, 2008.

José-Víctor Ríos-Rull. Computation of equilibria in heterogeneous agent models. Research Department Staff Report 231, Federal Reserve Bank of Minneapolis, 1997.

Santiago Budria Rodríguez, Javier Díaz-Giménez, Vincenzo Quadrini, and Jóse-Víctor Ríos-Rull. Updated facts on the U.S. distributions of earnings, income, and wealth. *Federal Reserve Bank of Minneapolis Quarterly Review*, 26(3):2–35, 2002.

Emma Ross-Thomas. Investors ignoring positive news as crisis persists, BIS says. *Bloomberg.com*, news on June 13, 2010.

J. Barkley Rosser. The rise and fall of catastrophe theory applications in economics: Was the baby thrown out with the bathwater? *Journal of Economic Dynamics and Control*, 31:3255–3280, 2007.

Harl E. Ryder and Geoffrey M. Heal. Optimal growth with intertemporally dependent preferencs. *Review of Economic Studies*, 40(1):1–31, 1973.

Soon Ryoo. Long waves and short cycles in a model of endogenous financial fragility. *Journal of Economic Behavior & Organization*, 74:163–186, 2010.

Jean-Guillaume Sahuc and Frank Smets. Differences in interest rate policy at the ECB and the Fed: An investigation with a medium-scale DSGE model. *Journal of Money, Credit and Banking*, 40(2-3):505–521, 2007.

Manuel S. Santos and Andrian Peralta-Alva. Accurcy of simulatings for stochastic dynamics models. *Econometrica*, 73(6):1939–1976, 2005.

Thomas J. Sargent and Neil Wallace. The stability of models of money and growth with perfect foresight. *Econometrica*, 41(6):1043–1048, 1973.

Marten Scheffer, Jordi Bascompte, William A. Brock, Victor Brovkin, Stephen R. Carpenter, Vasilis Dakos, Hermann Held, Egbert H. van Nes, Max Rietkerk, and George Sugihara. Early-warning signals for critical transitions. *Nature*, 461(3):53–59, 2009.

Moritz Schularick and Alan M. Taylor. Credit booms gone bust: Monetary policy, leverage cycles and financial crises, 1870-2008. Working Paper 15512, NBER, 2009.

Enchuan Shao and Pedro Silos. Uninsurable individual risk and the cyclical behaivor of unemployment and vacancies. *Working Paper*, 2007.

Shouyong Shi. Fashion and wealth accumulation. *Economic Theory*, 14:439–461, 1999.

Eva Sierminska, Andrea Brandolini, and Timothy Smeeding. The Luxembourg wealth study: A cross-country comparable database for household wealth research. *Journal of Economic Inequality*, 4(3):375–383, 2006.

Herbert A. Simon. A behavioral model of rational choice. *Quarterly Journal of Economics*, 69(1):99–118, 1955.

Julien Clinton Sprott. *Chaos and Time-Series Analysis*. Oxford University Press, New York, 2003.

Peter J. Stemp and Ric D. Herbert. Solving non-linear models with saddl-path instabilities. *Computational Economics*, 28:211–231, 2006.

J. E. Stiglitz. Distribution of income and wealth among individuals. *Econometrica*, 37(3): 382–397, 1969.

Nancy L. Stokey and Robert E. Lucas. *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, 1989.

Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications To Physics, Biology, Chemistry, And Engineering*. Westview Press, Cambridge, 2000.

George Tauchen. Finite state markov-chain approximations to univariate and vector autoregressions. *Economics Letters*, 20:177–181, 1986.

John B. Taylor. Monetary policy and the state of the economy. *Testimony before the Committee on Financial Services U.S. House of Representatives*, 2008.

John B. Taylor. The financial crisis and the policy responses: An empirical analysis of what went wrong. *Critical Review*, 21(2&3):341–364, 2009.

John B. Taylor and John C. Williams. A black swan in the money market. *American Economic Journal: Macroeconomics*, 1(1):58–83, 2009.

James Tobin. Keynesian models of recession and depression. *American Economic Review*, 65(2):195–202, 1975.

Jean-Claude Trichet. Credible alertness revisited. *Intervention at the symposium on "Financial stability and macroeconomic policy," sponsored by the Federal Reserve Bank of Kansas City, Jackson Hole, Wyoming, 22 August*, 2009.

Timo Trimborn, Karl-Josef Koch, and Thomas M. Steger. Multidimensional transitional dynamics: A simple numerical procedure. *Macroeconomic Dynamics*, 12:301–319, 2008.

H. Uzawa. Time preference, the consumption function, and optimum asset holdings. In J.N. Wolfe, editor, *Value, Capital, and Growth: Papers in honour of Sir John Hicks*. Edinburgh University Press, Edinburgh, 1968.

Adrien Verdelhan. A habit-based explanation of the exchange rate risk premium. *Journal of Finance*, 65(1):123–146, 2010.

Bruce A. Weinberg, Patricia B. Reagan, and Jeffrey J. Yankow. Do neighborhoods affect hours worked? evidence from longitudinal data. *Journal of Labor Economics*, 22(4):891–924, 2004.

Victor M. Yakovenko and J. Barkley Rosser. Colloquium: Statistical mechanics of money, wealth, and income. *Reviews of Modern Physics*, 81:1703–1725, 2009.

Kamil Yilmaz. Martingale property of exchange rates and central bank interventions. *Journal of Business & Economic Statistics*, 21(3):383–395, 2003.