ESSAYS ON AUCTION THEORY

by

Hadi Yektaş

B.S. in Mathematics, Koç University, 2000

B.A. in Economics, Koç University, 2000

M.A. in Economics, University of Pittsburgh, 2003

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This dissertation was presented

by

Hadi Yektaş

It was defended on

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and approved by

Andreas Blume, Professor, Department of Economics

Oliver Board, Assistant Professor, Department of Economics

Esther Gal-Or, Professor, Katz School of Business

Utku Ünver, Assistant Professor, Department of Economics

Dissertation Director: Andreas Blume, Professor, Department of Economics

ABSTRACT

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Hadi Yektaş, PhD

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This work is composed of three essays on auction theory. In the first essay, we analyze the optimal auction of multiple non-identical objects when buyers are risk averse. We show that the auction forms that yield the maximum revenue in the risk neutral case are no longer optimal. In particular, selling the goods independently does not maximize the seller's revenue. On the other hand, the optimal auction remains weakly efficient. The optimal auction has the following properties: The seller perfectly insures all buyers against the risk of losing the object(s) for which they have high valuation. While the buyers who have high valuation for both objects are compensated if they do not win either object, the buyers who have low valuation for both objects incur a positive payment in the same event.

In the second essay, we question whether, in the all-pay auction, the seller's commitment to the reserve price is beneficial if she has the chance of repeating the auction, possibly with a different reserve price, in case there is no sale in the first period. We show that, for any number of potential buyers, non-commitment is preferable only if the seller is relatively more patient than the buyers. Moreover, as the number of potential buyers increases, the seller's incentive to commit increases if she maximizes the average bid, whereas it decreases if she maximizes the highest bid. A possible explanation is that if the seller maximizes the average (highest) bid then screening high types (highest type) becomes costlier (less costly) as more buyers participate in the auction.

The third essay studies collusive behavior in the Ausubel auction in an environment with incomplete information. The Ausubel auction is vulnerable to collusion due to two

main reasons: First, the mechanism has a dynamic nature that allows the bidders to detect and punish those that deviate from the agreed collusive strategy. Second, in case a bidder strategically reduces his demand to signal his intention to collude, the mechanism allows the opponents to correctly interpret the signal.

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PREFACE

It was six years ago that I was attracted to the analysis of interdependent decision making problems. A year later, I was appealed in particular by auction theory. Since then, I have studied three separate questions related to auctions the results of which collectively formed this dissertation.

My understanding of game theoretic situations was greatly shaped by Prof. Andreas Blume and Prof. In-Uck Park. I am deeply grateful to both of them for arousing my curiosity about the topic and for passing on their knowledge and enthusiasm.

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The first chapter of this dissertation is a joint work with my colleague Çağrı S. Kumru. I would like to thank him not only for the time and the effort that he contributed to this chapter but also for his cordial friendship.

I also would like to thank my lifelong friends Burak Güneri, Fatih Çekiç, Furkan Nur, Ömer Tezel, Ümit Yusuf Oğraş, Yılmaz Ak and many others (too numerous to mention) for always being with me since the day I had the pleasure to meet them.

Needless to say, any remaining errors and shortcomings of this dissertation are my own responsibility.

I would like to dedicate this work to my beloved parents Kemal and Kiraz, my caring brothers Salih and Rüstem, my dear sister Mehtap and my lovely daughter Selin Nihal. Their unending support, encouragement, love and prayers helped me go through many obstacles during my graduate life. I can honestly say that I learned as much, if not more, from them as from my professors. They are the light of my life and they deserve the best of the thanks.

1.0 OPTIMAL MULTI-OBJECT AUCTION WITH RISK-AVERSE BUYERS (WITH ÇAĞRI S. KUMRU)

1.1 INTRODUCTION

Optimal selling mechanisms for multiple objects have been analyzed extensively due to their theoretical and practical importance (*e.g.*, the spectrum auctions, second hand car auctions).[1] One of the main assumptions in these studies is that the buyers are risk neutral. However, in many situations this assumption is violated and further analysis is needed.[2]

The optimal design problem in the presence of risk averse buyers can be described as follows: When the number of objects is limited, the buyers face the risk of not getting the object(s) they want. And in order to reduce this risk, the risk averse buyers bid more aggressively compared to those who are risk neutral.[3] Therefore, when the buyers are risk averse, the seller will be tempted to increase the magnitude of the risk. Yet, this comes with a trade-off, as the high type buyers (namely, the ones who value the good highly), when confronted with too much risk, may find it more profitable to mimic the low type buyers or may even be discouraged to participate.[4] Therefore, a revenue maximizing scheme should

[1] See for example, Harris and Raviv [21], Maskin and Riley [29], Levin [24], Figueroa and Skreta [16].

[2] In many auctions, the buyers are firms and they are generally assumed to be risk neutral. Yet, firms whose ownership are non-diversified (e.g. most family owned companies), those that are bound by liquidity constraints or under a financial distress, and those that are subject to a nonlinear tax system should all be assumed to be risk averse. (Asplund [4]) Even a firm which is owned by risk-neutral shareholders may behave in a risk-averse manner if the control of the firm is delegated to a risk-averse manager and his payment is linked to the firm's performance.(i.e. through stock options.)

Smith and Walker [40] show that the overbidding relative to Nash predictions (for the risk neutral environment) which has been observed in the first-price auction cannot be attributed to noisy-decision making, supporting the hypothesis that it must be due to the risk aversion of the buyers.

[3] See, for example, Maskin and Riley [28] and Matthews [30].

[4] As we know from the optimal auction literature, it may be desirable to exclude the low-type (and in some environments the high-type (Bertoletti [10])) buyers from the auction. (Exclusion Principle) Yet, if

impose "the right amount risk" on "the right type of buyers".

For single object, Maskin and Riley [28], Matthews [30], and Eső [15] describe how the above mentioned trade-off should be balanced. They observe that once the risk neutrality assumption is relaxed the models deliver quite different results. In his seminal work, Myerson [34] shows that if the buyers are risk neutral and their private valuations are independently distributed, then it is optimal to give the object to the buyer who has the highest *virtual valuation* (not the actual valuation) that exceeds the seller's outside option.[5] Thus, the standard auctions, including the "high bid" and "English" auctions, with appropriately chosen reserve price are all optimal. He further shows that any two auctions with the same allocation rule are revenue equivalent if the expected utility of each buyer in some benchmark case is the same, the celebrated *revenue equivalence theorem*. To the contrary, if the buyers are risk averse, the standard auctions with appropriate reserve price neither generate the same expected revenue nor are they optimal. (Maskin and Riley [28], Matthews [30]).[6] Another contrast is observed when the buyers' valuations are correlated: If the buyers are risk neutral, then the seller can fully extract the informational rents using an efficient auction (Crémer and McLean [13]), but she cannot do so if the buyers are risk averse, unless the correlation is sufficiently strong. (Eső [15]).[7]

In the light of these works, the current paper studies the optimal design problem for the case of multiple objects and seeks answers to the following two natural questions:

1. How does the optimal multi-object auction with *risk-averse* buyers compare with that with *risk-neutral* buyers?
2. Which features of the optimal *single-object* auction carry over to the optimal *multi-object* auction?

the seller imposes too much risk on all types then she will herself face the risk of no sale, hence ending up with no profit.

[5] Virtual valuations are the adjusted valuations that take into account buyers' informational rents and, more precisely, are defined as $\psi_i(v_i) = v_i - [1 - F_i(v_i)]/f_i(v_i)$, if buyer i's valuation v_i is distributed according to cumulative distribution function $F_i(.)$ with associated density function $f_i(.)$.

[6] In a second price auction, the buyers bid truthfully regardless of their risk preference. But in the first price auction, a risk-averse buyer shades his bid less than a risk-neutral buyer. As a result, the first price auction yields more revenue than the second price auction. Nevertheless, the first price auction is not optimal because it imposes too much risk on the high type buyers.

[7] Optimal auction should remove the risk from high type buyers, which requires providing insurance (and hence leaving some surplus) to them.

To answer the first question, we compare our results with those of Armstrong [1] who, in a binary model, characterizes the optimal multi-object auction for risk-neutral buyers.[8] This comparison provides a twofold answer: One, in either case, the optimal auction is weakly efficient.[9,10]. Two, *none of the auction forms that are shown to be optimal in Armstrong* [1] *maximize the seller's revenue when the buyers are risk averse. In particular, it is not optimal to sell the two goods independently.* This sharp contrast is due to the way in which the objects are allocated when all buyers have low valuation for both objects. (That is, when all buyers are of type LL.)

The optimal auctions for risk-neutral buyers can take the form of independent auction, bundling auction, or mixed auction, depending on how their valuations are correlated *across objects*.[11,12] These three formats allocate each object independently and randomly if all buyers are of type LL. However, doing so does not impose enough risk on type LL. Contrarily, when the buyers are risk averse both objects must be given to the same (LL type) buyer.[13,14]

[8] Armstrong [1] inherited his setting from Armstrong and Rochet [2], who study a principal-agent problem. Both of these papers and the current paper assume that buyers/agents have multidimensional private information and, in this regard, differ from the references mentioned in footnote 1.

Manelli and Vincent [26] and Manelli and Vincent [27] also assume multidimensional private information, but different from the current paper, they assume a single buyer.

[9] Weak efficiency requires that each object is sold to the buyer with the highest valuation whenever it is sold. Some of the objects can be kept by the seller eventhough there is a buyer who has valuation that exceeds that of the seller. For strong efficiency, on the other hand, the objects valued more highly by a buyer than the seller must always be sold. In this sense, the optimal auctions in Myerson [34] are weakly efficient.

[10] It must be noted, though, that the optimal multi-object auction is no longer weakly efficient when the model assumes a continuous type space.

[11] In all three forms, the buyers have the same *expected* probability of winning the object(s) for which they have high valuation. These forms differ only in the *expected* probability of winning the objects for which buyers have low valuation. In a mixed auction, a buyer who has low valuation, say, for object A but high valuation for object B, is assigned object A more often than a buyer who has low valuation for both objects. While independent auctions don't distinguish between these two types for object A, bundling auction perfectly discriminates against the type that has low valuations for both objects. It should be noted that the bundling auction allows the goods to end up in the hands of different buyers.

[12] Avery and Hendershott [7] also consider risk-neutral buyers. While Armstrong [1] assumes that all buyers have demand for both objects, in Avery and Hendershott [7], only one buyer demands multiple objects and the remaining buyers demand only one or the other. Not surprisingly, the optimal auction in the latter paper may not be weakly efficient due to the good deal of asymmetry among buyers. Yet, even in that case, the optimal auction bundles the objects *probabilistically* for the multi-demand buyer.

[13] It is riskier to lose both objects than to lose a single object.

[14] In Armstrong [1], bundling is optimal only when buyers' valuations are negatively correlated across objects, or in other words, when a buyer's high value for one object, say A, is likely to be accompanied by a relatively low value for the other object, say B. The goods are bundled only for the types HL or LH. In this case, their incentive conditions in all directions are binding.

In the current paper, we show that the seller utilizes bundling not only to make the desired incentive conditions binding but also to increase the risk as much as possible for type LL.

When the buyers are risk neutral, the seller assigns each buyer a single expected payment that depends only on his type. On the other hand, we show that, when the buyers are risk averse, it is optimal to make each buyer's payment (a function of his report) conditional *also* on the *type* and the *number* of the objects he wins. Moreover, it is not optimal to make these expected payments random.[15]

For the second question, we do a robustness check in order see to what extent our results, which we obtain in a binary model, are comparable with those of the current literature, which assumes continuous distribution of types. (Namely, Maskin and Riley [28] and Matthews [30])[16] We observe that the optimal single-object auction in the binary model replicates the behavior of that of the continuous model at the two extremes of the type space. This analogy helps us interpret our results regarding the multi-object auction: The seller perfectly insures all buyers against the risk of losing the object(s) for which they have the high(est) valuation. The buyers who are (most) eager to win both objects are compensated if they can not win either object. On the other hand, those (most) reluctant to win both objects must incur a positive payment if they lose both objects.[17]

The intuition for our results is as follows: While, on one hand, the seller would like to screen the buyers, on the other hand, she would like to confront them with risk. Screening the buyers requires leaving informational rents to (and, in turn, decreasing the risk for) the buyers who have high valuation for one or both objects. As a result, the buyers' marginal utility of income must remain the same regardless of whether they win or lose the objects for which they have high valuation. This also implies providing insurance to type HH. On the other hand, the buyers who have low value for both objects must confront the highest risk from which the seller benefits in two ways: One, she makes imitating LL unattractive to the other types and two, she fully extracts the informational rents from type LL. Confronting

[15] This also implies that it is not optimal to make the payments dependent on other buyers' reports.

[16] Matthews [30] studies the same problem as Maskin and Riley [28]. While the former assumes a particular form of utility function, namely CARA, and obtains necessary and sufficient conditions for an auction to be optimal, the latter considers different forms of risk aversion and characterize the properties of the optimal auction for all of these forms.

[17] A natural question to ask is how the punishment for type LL can be implemented in real life. When there is a single object, the optimal auction reduces to a modified first price auction for some parameter values. (Maskin and Riley [28]) The seller charges an entry fee, but she does not return it to the buyers with low valuation if they don't win the object.

these types with the highest risk involves bundling the objects whenever all buyers are LL and collecting payments even when they don't win any objects.

There is a vast amount of literature on bundling

Finally, we comment on the solution methods used in this paper: In section 1.2, we describe the optimal single object auction in *reduced form,* meaning we construct the buyers' *expected* probability of obtaining the object (contingent only on his own type), rather than his *actual* probability of winning as a function of all buyers' types. This technique was also utilized by Matthews [30] and Maskin and Riley [28] in order to avoid the computational complexity that risk aversion involves.[18] Yet, when one solves the seller's optimal design problem in reduced form, in addition to the incentive constraints and the participation constraints, one had to impose the *implementability* constraints in order to guarantee the existence of the *actual* probabilities.[19]

The number of implementability constraints increases exponentially with the number of goods (or more precisely with the number of elements in the type space), nevertheless Armstrong [1] was still able to solve the problem in reduced form. Yet, when the buyers are risk averse, since the correlation between the events of winning object A and object B also matters for the buyers (and in turn for the seller), the conditions that one needs to impose cannot be easily determined.[20] Therefore, in section 1.3, we describe the optimal auction in *non-reduced* form and construct the actual probabilities of the events that a buyer can possibly face as functions of the entire type profile (as reported by all participating buyers).[21] Since the buyers don't observe their opponents' types, only the *expected* probabilities of observing each event (conditional only on one's type) matter in the incentive conditions.

[18] The technique was introduced to the literature by Myerson [34].

[19] When there is a single object or when the buyers are risk neutral, these conditions take a very simple form, which, can be interpreted as the probability that a buyer whose type belongs to a given subset of the type space obtains a particular object cannot be higher than the probability that there is a buyer whose type is in that subset.

The implementability conditions need to be imposed because the seller has only a limited number of each type of good. A multi-product monopolist who has unlimited number of each type of good does not face this constraint. (See Manelli and Vincent [26] and Manelli and Vincent [27])

[20] Using the main result of Border[11] (Also footnote 27), Armstrong [1] was able to describe the implementability conditions. In his environment, the main difficulty is to identify the conditions that are binding at the optimum. In the current paper, on the other hand, Border[11]'s theorem is not applicable.

[21] These events are winning only object A, only object B, winning both objects and winning nothing.

Therefore, we also make use of these *expected* probabilities throughout our analysis.[22]

The remainder of the paper is organized as follows: In section 1.2, we construct the optimal single-object auction for risk averse buyers in a binary framework and analyze the properties of it. In Section 1.3, we increase the number of objects and repeat the analysis. Finally, in section 1.4., we discuss the main results and their implications.

1.2 OPTIMAL SINGLE-OBJECT AUCTIONS

1.2.1 Description of the Problem

A single indivisible object is to be sold to one of $n \geq 2$ potential buyers, whose private valuations are discretely distributed according to a random variable v_i, which takes values v_H with probability $\alpha_H > 0$ and v_L with probability $\alpha_L > 0$ such that $\alpha_H + \alpha_L = 1$. Without loss of generality, we assume $v_H > v_L > 0$, so that v_H and v_L denote valuations of high-type (eager) and low-type (reluctant) buyers, respectively. Buyer valuations are distributed independently and identically. Buyers are risk-averse and have a constant measure of risk aversion (CARA). In particular, their preferences are represented by a utility function $u(\omega) = -\frac{e^{-r\omega}}{r}$, where $r(>0)$ measures the rate of risk aversion. Note that, $u'(.) > 0$ and $u''(.) < 0$. Specifically, if a buyer with valuation v wins the object and incurs a *net* payment of τ then his utility is $u(v-\tau) = -\frac{e^{-r(v-\tau)}}{r}$. The seller is risk-neutral and her valuation for the object is zero. Both the seller and the buyers are expected utility maximizers.

The seller's problem is to design a selling scheme that maximizes her revenue.[23] Such a scheme most generally consists of a message set, $M = M_1 \times \cdots \times M_n$, and an outcome function, $\psi : M \to \tilde{A}$, that maps the list of messages, $m \in M$, into a possibly random allocation $\tilde{a} \in \tilde{A} = \tilde{A}_1 \times \cdots \times \tilde{A}_n$.[24] Buyers' behavior is described by a Bayesian Nash equilibrium, $s = (s_1, ..., s_n)$, where $s_b : \Theta_b \to M_b$ is the equilibrium strategy of buyer b;

[22] In regard to the solution method, this paper is also related to Menicucci [32] which extends Armstrong [1] by allowing for a synergy if the two goods end up in the hands of the same buyer. He shows that in this case the optimal auction is likely to allocate the goods inefficiently.

[23] Milgrom [33] defines an auction to be a mechanism (scheme) to allocate resources among a group of bidders. Therefore, we use these three terms interchangeably.

[24] An allocation consists of a *decision* about who is going to get which object(s) and possibly negative monetary *transfers* from buyers to the seller.

$s_b(\theta_b)$ representing the message that maximizes buyer b's expected utility given that his type is θ_b and all buyers other than him follow the equilibrium strategy.[25] So, any selling scheme, in a given equilibrium, will result in an outcome represented by $\psi(s_1(\theta_1), ..., s_n(\theta_n))$, if the buyers' type profile is $(\theta_1, ..., \theta_n)$.

Alternatively, when looking for the optimal selling scheme, attention can be restricted to the *revelation schemes* in which the message space is the type space, Θ. This is because any allocation, $\psi(s_1(\theta_1), ..., s_n(\theta_n))$, resulting from an equilibrium of an arbitrary selling scheme can also be obtained in a revelation scheme in which the outcome is determined via the composite function $\psi \circ s : \Theta \to \tilde{A}$ and truth-telling is an equilibrium (Revelation Principle).[26] Thus, the seller's problem can be reduced to finding the optimal revelation scheme in which the buyers are willing to participate (*individual rationality)* and have incentive to truthfully report their type (*incentive compatibility)*.

Given a profile of reports, a selling scheme must, most generally, assign each buyer a probability of winning, a payment in case he wins and another payment in case he loses. That is, the outcome is determined by functions of the form $\psi_b(m) = (p_b(m),\, \tilde{t}_b^w(m), \tilde{t}_b^l(m))$ for $b = 1, ..., n$, where tildes represent the possibility that the payment functions are random. Since there is only one object for sale, a feasible scheme must satisfy $\sum_{b=1}^n p_b(m_1, ..., m_n) \leq 1$ for all $(m_1, ..., m_n)$.

Given an equilibrium, we can calculate buyer b's *expected* probability of winning and his expected random payments in case of winning and losing, respectively, as

$$\rho_b(m_b) = E_{-b}[p_b(m) \mid m_b] \tag{1.1}$$

$$\tilde{\tau}_b^w(m_b) = E_{-b}[\tilde{t}_b^w(m) \mid m_b] \tag{1.2}$$

$$\tilde{\tau}_b^l(m_b) = E_{-b}[t_b^l(m) \mid m_b]. \tag{1.3}$$

Since buyers are *ex ante* identical, only the schemes that treat them symmetrically need to be considered. This is because, for any asymmetric scheme, we can construct a symmetric

[25] In this section, each type of a buyer corresponds to a possible valuation , namely $\Theta_j = \{v_H, v_L\}$ for all $j = 1, ..., n$, whereas, in the next section, there are four different types of buyers. That is, $\Theta_j = \{HH, HL, LH, LL\}$ for all $j = 1, ..., n$, where the first (second) letter in each type represents buyer j's value for object A (B).

[26] See Myerson [34].

scheme that generates the same revenue as the proposed asymmetric scheme. Symmetric schemes satisfy the following condition:

$$\text{For any } b, b' \in \{1, ..., n\} \text{ and any } m, m' \in M,$$
$$\psi_b(m) = \psi_{b'}(m')$$
$$\text{if } m_b = m'_{b'}, m_{b'} = m'_b, \text{ and for all } b'' \neq b, b' \; m_{b''} = m'_{b''}.$$

Therefore, in a symmetric scheme, the *expected* probability and the *expected* payments of two different buyers submitting the same message are equal. Hence, we can drop the subscript on each of the functions in 1.1-1.3. Describing a selling scheme from the perspective of an arbitrary buyer, using $\rho(.), \tilde{\tau}^w(.)$, $\tilde{\tau}^l(.)$, is called *reduced form representation.*

Three points need to be emphasized about our approach to solving the seller's problem. First, using the Revelation Principle, we consider only the revelation schemes that satisfy two sets of conditions: individually rationality and incentive compatibility.

Second, we construct the optimal auction in reduced form. We justify this by imposing another set of conditions called *implementability conditions.*[27] These conditions make sure that the reduced form probability, $\rho(.)$, is *implementable*, that is, they make sure that there exists a symmetric auction with actual allocation probabilities, $p(.)$, which satisfies

$$\rho(m_b) = E[p(m) \mid m_b]. \tag{1.4}$$

[27] Border [11] states the necessary and sufficient conditions, for the reduced form probabilities to be implementable. We include the proposition for easy reference:

Let (S, Υ) be a measurable space of possible types of bidders and $\lambda(.)$ be a probability measure on S. Define an auction to be a measurable function $p : S^n \to [0,1]^n$ satisfying $\sum_{i=1}^n p^i(s) \leq 1$ for all $s \in S^n$. Define an auction to be symmetric if $p^i(s)$ is independent of i. Given an auction, define

$$\rho^i(s_i) = \int_{S^{n-1}} p(s_1, ..., s_n) d\lambda(s_1, ., s_{i-1}, s_{i+1}, .., s_n)$$

to be the probability that a buyer i wins when he reports his type as s_i.

Then ρ is implementable by a symmetric auction if and only if for each measurable set of types $A \in \Upsilon$, the following inequality is satisfied:

$$\int_A \rho(s) d\lambda(s) \leq \frac{1 - \lambda(A^c)^n}{n}$$

Furthermore, if S is a topological space and λ is a regular Borel probability on S, then Υ may be replaced by either the open subsets or the closed subsets of S.

The final point is that we initially consider only the schemes in which the *expected* payments contingent on winning and losing are nonrandom. In other words, we first construct the optimal scheme within the class of schemes for which $\tilde{\tau}^w(.)$ and $\tilde{\tau}^l(.)$ are deterministic. (So, we drop the tildes.) We later establish that this scheme is also optimal among all selling schemes, including those that assign random payments.

To summarize, the seller's problem is to construct the optimal revelation scheme, the reduced form of which can be represented by six variables, $\{\rho_i, \tau_i^w, \tau_i^l\}_{i=H,L}$, where $\rho_i \in [0,1]$ denotes the probability that a buyer wins the object when he reports a valuation of v_i, and $\tau_i^w, \tau_i^l \in \mathbb{R}$ denote the *net deterministic* payments that the same type of buyer incurs when he wins and loses the object, respectively. As mentioned above three sets of conditions are imposed:

If a buyer with valuation v_i reports v_j then his utility is equal to $\rho_j u(v_i - \tau_j^w) + (1 - \rho_j)u(-\tau_j^l)$. Thus, buyers truthfully reveal their valuations if the auction satisfies the following two *incentive compatibility conditions*:

$$\begin{aligned} \rho_H u(v_H - \tau_H^w) + (1-\rho_H)u(-\tau_H^l) &\geq \rho_L u(v_H - \tau_L^w) + (1-\rho_L)u(-\tau_L^l) \\ \rho_L u(v_L - \tau_L^w) + (1-\rho_L)u(-\tau_L^l) &\geq \rho_H u(v_L - \tau_H^w) + (1-\rho_H)u(-\tau_H^l). \end{aligned}$$

Buyers are free to participate in the auction. Thus, participating buyers satisfy the *individual rationality conditions* of the form

$$\begin{aligned} \rho_H u(v_H - \tau_H^w) + (1-\rho_H)u(-\tau_H^l) &\geq u(0) \\ \rho_L u(v_L - \tau_L^w) + (1-\rho_L)u(-\tau_L^l) &\geq u(0). \end{aligned}$$

Finally, the *implementability conditions* take the following form in our binary model:

$$n(\alpha_L\rho_L + \alpha_H\rho_H) \leq 1 \qquad (IM_{\{H,L\}})$$

$$n\alpha_H\rho_H \leq 1 - \alpha_L^n \qquad (IM_{\{H\}})$$

$$n\alpha_L\rho_L \leq 1 - \alpha_H^n. \qquad (IM_{\{L\}})$$

One can interpret these conditions as follows: the probability the object is won by a buyer who belongs to a particular subset of the type space should be no greater than the probability that there *is* a buyer who belongs to that subset.[28]

The seller's revenue is the sum of the expected payments made by each buyer. Since buyers are *ex ante* identical the seller's revenue can be written in terms of the expected payments made by an arbitrary buyer (namely, the term in the bracket):

$$\pi = n[\alpha_H(\rho_H \tau_H^w + (1-\rho_H)\tau_H^l) + \alpha_L(\rho_L \tau_L^w + (1-\rho_L)\tau_L^l)].$$

To sum up, the seller's problem is to choose a reduced form scheme, $\{\rho_i, \tau_i^w, \tau_i^l\}_{i=H,L}$, that maximizes π subject to the two *incentive compatibility conditions*, the two *individual rationality conditions*, and the three *implementability conditions*.

For convenience, we define $c_i = e^{-rv_i}$ and $y_i^k = e^{r\tau_i^k}$. Note that, $0 < c_H < c_L < 1$ and $y_i^k > 0$ for all i and k. So, we can rewrite the seller's problem as

$$\max_{\{\rho_i, y_i^w, y_i^l\}_{i=H,L}} \pi = \frac{n}{r}[\alpha_H(\rho_H \ln y_H^w + (1-\rho_H)\ln y_H^l) + \alpha_L(\rho_L \ln y_L^w + (1-\rho_L)\ln y_L^l)] \tag{1.5}$$

subject to

$$\begin{aligned}
\rho_H c_H y_H^w + (1-\rho_H) y_H^l &\leq \rho_L c_H y_L^w + (1-\rho_L) y_L^l && (IC_H)\\
\rho_L c_L y_L^w + (1-\rho_L) y_L^l &\leq \rho_H c_L y_H^w + (1-\rho_H) y_H^l && (IC_L)\\
\rho_H c_H y_H^w + (1-\rho_H) y_H^l &\leq 1 && (IR_H)\\
\rho_L c_L y_L^w + (1-\rho_L) y_L^l &\leq 1 && (IR_L)\\
n(\alpha_L \rho_L + \alpha_H \rho_H) &\leq 1 && (IM_{\{H,L\}})\\
n\alpha_H \rho_H &\leq 1 - \alpha_L^n && (IM_{\{H\}})\\
n\alpha_L \rho_L &\leq 1 - \alpha_H^n && (IM_{\{L\}})
\end{aligned}$$

and the non-negativity conditions $\rho_H, \rho_L \geq 0$.

[28] Armstrong [1] alternatively calls these conditions *resource constraints*.

For convenience, we refer to the left-hand side of the inequalities in IR_H and IR_L as D_H and D_L, respectively. Similarly, right hand side of IC_H and IC_L are referred to as D_H^L and D_L^H, respectively. The subscripts denote a buyer's actual type, whereas superscripts denote the type he is imitating.

1.2.2 Solution to the Problem

Since $c_L > c_H$, IC_H and IR_L together imply IR_H.[29] Hence, this condition is redundant. For now, we also ignore IC_L when we solve the seller's problem. That is, we suppose that the low-type buyers do not have the incentive to misrepresent their types. Below, in proposition 8, we prove that this is indeed the case.

Definition 1. *The relaxed problem is defined to be a design problem that ignores the upward incentive constraints.*

The following lemma shows that when only the downward incentive conditions are considered, high-type's incentive condition and low-type's individual rationality condition must be binding.

Lemma 2. *In the relaxed problem, where IC_L is ignored, the constraints IC_H and IR_L must be binding.*

The seller may want to increase her revenue by excluding the low-type buyers from the auction if, for a given distribution of types, their valuation is small enough compared to that of the high-type buyers.[30] This results in an inefficiency, because with positive probability the seller keeps the object even if all buyers value the object more highly than her.

Inefficiency may also be due to a misallocation of the objects. To be consistent with Armstrong [1], we focus only on the latter kind of inefficiency, by assuming that the goods are always sold, i.e. $\rho_L > 0$.[31] In this case, it is optimal for the seller to leave informational rents to the high-type buyers.

[29] $D_H \leq D_H^L \leq D_L \leq 1$, where the second inequality is due to $c_H < c_L$.

[30] The same behavior is also observed when a monopolist implements second-degree price discrimination.

[31] Clearly, high-type buyers should not be excluded from participating in the auction if revenue is maximized. That is, ρ_H must be strictly positive. If not, then the incentive conditions would imply $\rho_L c_L \leq \rho_L c_H$, and since $c_L > c_H$ this in turn would imply $\rho_L = 0$, meaning the good is not sold, at all. Yet, the seller can always guarantee a positive profit by posting a fixed price of $v_L > 0$.

Lemma 3. *At the optimum, if the low-type buyers are not excluded from the auction, then IR_H must be slack.*

The following proposition states that it is not optimal to impose any risk on the high-type buyers. The risk is fully eliminated from them.

Proposition 4. *High-type buyers are fully insured against the risk of losing the object.*

Through insurance, a high-type's marginal utility of income in cases of winning and losing is made the same. Eliminating the risk rewards the high-type buyer for revealing his true type.

If the seller does not pay informational rents to the high type buyer ($\tau_H^w = v_H$), the perfect (full) insurance requires that the seller sets the high type buyer's payment contingent on losing equal to zero ($\tau_H^l = 0$) in order to keep him at the same level of utility. However, when there is information gap between the seller and the buyers, high-type buyers should receive information rent to be active. In this case (i.e. $\tau_H^w < v_H$), perfect insurance requires that the seller compensates the high type buyer ($\tau_H^l > 0$).

Proposition 5. *High-type buyers are compensated if they lose the object.*

Using proposition 4, we can write the seller's profit as

$$\pi = \frac{n}{r}[\alpha_H(\rho_H \ln \frac{1}{c_H} + \ln y_H^l) + \alpha_L(\rho_L \ln \frac{y_L^w}{y_L^l} + \ln y_L^l)] \tag{1.6}$$

Note that, since $0 < c_H < 1$, the seller's profit is strictly increasing with respect to ρ_H. Thus, given the values of other variables, ρ_H must be set as high as possible at the optimum. This implies that either $IM_{\{H\}}$ or $IM_{\{H,L\}}$, or both are binding.

The Kuhn-Tucker conditions with respect to y_L^w and y_L^l can be written as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y_L^w} &= \alpha_L \rho_L \frac{1}{y_L^w} - \lambda_L \rho_L c_L + \mu_H \rho_L c_H = 0 \\ \frac{\partial \mathcal{L}}{\partial y_L^l} &= \alpha_L (1-\rho_L) \frac{1}{y_L^l} - \lambda_L (1-\rho_L) + \mu_H (1-\rho_L) = 0. \end{aligned}$$

Since $\alpha_L \rho_L \frac{1}{y_L^w} > 0$, these two equations together yield

$$\frac{y_L^w}{y_L^l} = \frac{\lambda_L - \mu_H}{\lambda_L c_L - \mu_H c_H}. \tag{1.7}$$

Note that the right-hand side of equation 1.7 is smaller than $\frac{1}{c_H}$. So, we have

$$\frac{y_L^w}{y_L^l} < \frac{1}{c_H}. \tag{1.8}$$

This condition has a very nice implication: At the optimum, iso-revenue curve must be flatter than the line corresponding to the implementability condition $IM_{\{H,L\}}$.[32]

Thus, $IM_{\{H\}}$ and $IM_{\{H,L\}}$ are both binding and the optimal allocation probabilities can be calculated as

$$\rho_H = \tfrac{1-\alpha_L^n}{n\alpha_H}; \quad \rho_L = \tfrac{\alpha_L^{n-1}}{n} \tag{1.9}$$

which is the point where the iso-revenue curve (1.6) is tangent to the feasible set that is bound by the implementability conditions (Figure 1)

It is not surprising to see that the allocation probabilities that we have obtained in 1.9 are the same as those in the risk-neutral environment. The optimal allocation is monotonic with respect to buyer types in either case.

Note that, $n\alpha_L\rho_L = \alpha_L^n$, meaning the probability that the object is won by a low-type buyer is equal to the probability that all buyers are low-type. In other words, the object is won by a high-type buyer whenever there is one. Hence, the proposition follows.

Proposition 6. *The optimal auction is weakly efficient.*

Contrary to the insurance provided to the high-type buyers, the seller confronts the low-type buyers with risk by making their marginal utilities vary in cases of winning and losing. In this circumstance, a high-type buyer who considers imitating the low-type buyers would face a greater risk, and will eventually reveal his own true valuation. Hence, it is optimal for the seller to relax the high-type buyer's incentive constraint and not to offer insurance to the low-type buyers. The following proposition states that at the optimum low-type buyers' marginal utility of income is greater when he wins the object than when he loses it.

Proposition 7. *Low-type buyers are better off winning than losing:* $c_L y_L^w < y_L^l$. *Moreover, in case of losing the object, they incur a payment that is less than what they would pay if they win:* $1 < y_L^l < y_L^w$.

[32] This condition is equivalent to $\frac{\alpha_L \ln(y_L^w/y_L^l)}{\alpha_H \ln(1/c_H)} < \frac{\alpha_L}{\alpha_H}$, where the left hand side of the inequality is slope of the iso-profit curve and the right hand side is the slope of the line corresponding to the implementability condition $IM_{\{H,L\}}$.

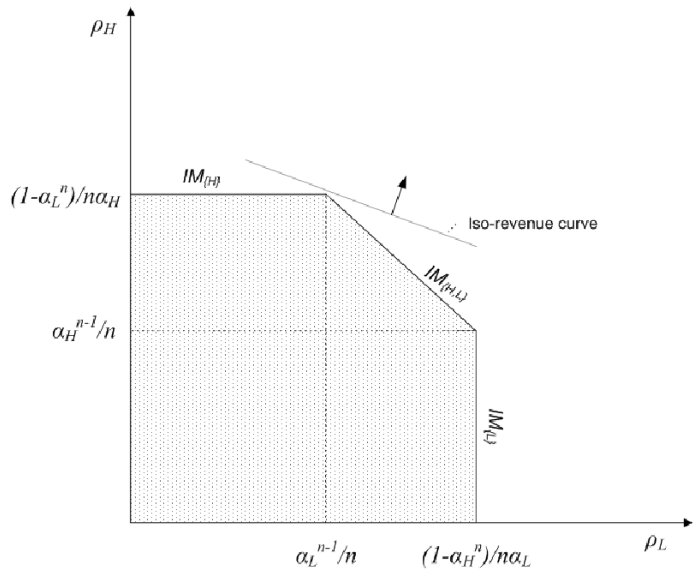

Figure 1: Single object - Optimal allocation probabilities are the same regardless of buyers' risk attitude.

Next, we show that the solution to the relaxed problem also solves the full problem which does not ignore IC_L.

Proposition 8. *Low-type buyers do not have the incentive to misrepresent their type. That is, IC_L is slack.*

The reduced form of the revelation scheme that we've constructed above is optimal within the class of schemes in which the *expected* payments contingent on winning and losing are deterministic. Finally, we establish that making t_i^w and t_i^l random has a negative effect on seller's revenue.

Proposition 9. *If buyer preferences are represented by CARA, then, in an optimal auction, the payments, t_i^w and t_i^l, must be deterministic.*

Remark 10. *Above proposition also implies that it is not profitable for the seller to condition the payments made by a buyer on the realizations of his opponents' types.*

1.3 OPTIMAL MULTI-OBJECT AUCTIONS

1.3.1 Description of the Problem

Now, there are two nonidentical objects, denoted A and B, to be sold to $n \geq 2$ buyers. The seller's valuation for both objects is zero, whereas buyer valuations are random and described by a pair (v^A, v^B), where v^o denotes the buyer's valuation for object o. Suppose that $v^o \in \{v^o_H, v^o_L\}$, where the subscripts denote whether the buyer is of high-type (H) or low-type (L). Thus, we assume $v^o_H - v^o_L > 0$. There are four types of buyers corresponding to the four possibilities $(v^A_H, v^B_H), (v^A_H, v^B_L), (v^A_L, v^B_H)$ and (v^A_L, v^B_L). Using a slightly shorter notation, we define the set of possible types as $\Theta = \{HH, HL, LH, LL\}$. A typical element of this set is denoted with ij, where i represents a buyer's valuation for object A and j represents his valuation for object B. Types are independently and identically distributed across buyers according to a probability measure α over Θ, so that the probability that a buyer is of type ij is represented by α_{ij}. The marginal probability that a buyer has a high value for object A is denoted with $\alpha^A_H = \alpha_{HH} + \alpha_{HL}$. Similarly, $\alpha^A_L = \alpha_{LH} + \alpha_{LL}$ denotes the marginal probability that the buyer has a low value for object A. In the same fashion, we define $\alpha^B_H = \alpha_{HH} + \alpha_{LH}$ and $\alpha^B_L = \alpha_{HL} + \alpha_{LL}$ to be the marginal probabilities that the buyer has a high and low value for object B, respectively.

Each buyer is risk-averse and has preferences represented by the common CARA utility function of the form $u(\omega) = -\frac{e^{-r\omega}}{r}$, where $r > 0$. In the event that a buyer wins object(s) of a (total) value v and incurs a net payment τ, his utility will be equal to $u(v - \tau)$. For example, if a buyer wins only object A when his valuation for that object is v^A_L and incurs a net payment τ^A then his utility is equal to $u(v^A_L - \tau^A)$. Similarly, if a buyer of type HL wins both objects and incurs a net payment τ^{AB} then his utility will be $u(v^A_H + v^B_L - \tau^{AB})$. Both the seller and the buyers are expected utility maximizers.[33]

The seller's problem is to design a selling scheme that maximizes her revenue. In view of the Revelation Principle, we solve this problem within the class of revelation schemes

[33] We assume that there are no economies of scope in the production of the bundle nor are there complementarities in the consumption of the bundle. We make this assumption so as to isolate the role that bundling has on the seller's ability to extract the consumer surplus.

which satisfy incentive compatibility and individual rationality constraints.[34] Furthermore, as justified in the previous section, among the revelation schemes, we focus only on the symmetric ones in which the buyers of the same type are treated the same.

Let n_{ij} be the number of buyers of type ij and $\eta = (n_{HH}, n_{HL}, n_{LH}, n_{LL})$ be the vector representing the profile of reports where $\sum_{ij\in\Theta} n_{ij} = n$. Then, a symmetric revelation scheme can most generally be described with two sets of rules:

- a *decision rule*, $p^k_{ij}(\eta)$, that assigns each type $ij \in \Theta$ probabilities of realizing possible events $k = A, B, AB, O$, for each profile of reports η. Given η, the decision rule must satisfy

$$\sum\nolimits_{ij\in\Theta} n_{ij}[p^A_{ij}(\eta) + p^{AB}_{ij}(\eta)] \leq 1 \tag{1.10}$$

$$\sum\nolimits_{ij\in\Theta} n_{ij}[p^B_{ij}(\eta) + p^{AB}_{ij}(\eta)] \leq 1 \tag{1.11}$$

$$p^A_{ij}(\eta) + p^B_{ij}(\eta) + p^{AB}_{ij}(\eta) + p^O_{ij}(\eta) = 1 \quad \forall ij \in \Theta \tag{1.12}$$

- a *payment rule*, $\tilde{t}^k_{ij}(\eta)$, that, for each profile of reports η, assigns each type $ij \in \Theta$ possibly random payments to be made to the seller at each possible event $k = A, B, AB, O$.

The decision rule specifies the probability that a buyer b of type ij realizes the valuations v^A_i, v^B_j, $v^A_i + v^B_j$ or 0. We abuse the notation and list these four events respectively as:

Event A	- winning *only* object A
Event B	- winning *only* object B
Event AB	- winning *both* object A and object B
Event O	- winning neither object.

Remember from Armstrong [1] that the *risk-neutral* buyers are only interested in the marginal probabilities of winning the objects. For risk-averse buyers, on the other hand, the correlation between the events of winning object A and object B matters. The decision rule in the above specification takes this into consideration.

Note that, $p^A_{ij}(\eta) + p^{AB}_{ij}(\eta)$, in 1.10, represents the marginal probability of winning object A which we shortly denote with $\hat{p}^A_{ij}(\eta)$. Similarly, $p^B_{ij}(\eta) + p^{AB}_{ij}(\eta)$, in 1.11, represents the

[34] Remember that in a revelation scheme, buyers are asked to report their types.

marginal probability of obtaining object B which is denoted with $\hat{p}^B_{ij}(\eta)$. Thus, conditions 1.10 and 1.11 are the resource constraints representing the fact that there is only one unit of each object. Condition 1.12 states that the events A, B, AB and O are all inclusive.

Although the payment rule allows the seller impose random payments, when we solve the seller's problem, we assume $\tilde{t}^k_{ij}(\eta) = \tau^k_{ij}$ where $\tau^k_{ij} \in \mathbb{R}$ for all $ij \in \Theta$ and $k = A, B, AB, O$, and characterize the optimal scheme within the class of schemes that assign deterministic payments. We will show later that imposing random payments to each type ij under each event k cannot improve the seller's revenue.

Now, define an ij type buyer's *expected* probability of realizing the event $k = A, B, AB, O$ as

$$\rho^k_{ij} = \sum_{n_{HH}=0}^{n} \sum_{n_{HL}=0}^{n-n_{HH}} \sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} p^k_{ij}(n_{HH}, n_{HL}, n_{LH}, n_{LL}) \Psi \frac{n_{ij}}{\alpha_{ij}} \tag{1.13}$$

where $\Psi = \frac{(n-1)! \alpha_{HH}^{n_{HH}} \alpha_{HL}^{n_{HL}} \alpha_{LH}^{n_{LH}} \alpha_{LL}^{n_{LL}}}{n_{HH}! n_{HL}! n_{LH}! n_{LL}!}$. For any $n_{ij} > 0$, $\Psi \frac{n_{ij}}{\alpha_{ij}}$ denotes the probability that the buyer profile is $\eta = (n_{HH}, n_{HL}, n_{LH}, n_{LL})$ given that there is one ij in that profile (of course, conditional on incentive constraints hold).[35]

The reduced form of a symmetric revelation scheme, then, can be represented with

$$\{\rho^A_{ij}, \rho^B_{ij}, \rho^{AB}_{ij}, \rho^O_{ij}, \tau^A_{ij}, \tau^B_{ij}, \tau^{AB}_{ij}, \tau^O_{ij}\}_{ij \in \Theta}.$$

ρ^A_{ij} and ρ^B_{ij} are type ij's *expected* probability of winning object A or B, *alone*; whereas ρ^{AB}_{ij} is his probability of winning *both* objects. Apparently, $\rho^O_{ij} = 1 - \rho^A_{ij} - \rho^B_{ij} - \rho^{AB}_{ij}$ represents the probability of winning neither object. τ^k_{ij} is the net deterministic payment that type ij must incur if event k occurs.

Then, the utility of a buyer of type ij who misrepresents his type as $i'j'$ is

$$\rho^A_{i'j'} u(v^A_i - \tau^A_{i'j'}) + \rho^B_{i'j'} u(v^B_j - \tau^B_{i'j'}) + \rho^{AB}_{i'j'} u(v^A_i + v^B_j - \tau^{AB}_{i'j'}) + \rho^O_{i'j'} u(-\tau^O_{i'j'}).$$

Let $c^o_i = e^{-rv^o_i}$ for $o = A, B$ and $i = H, L$ and $y^k_{ij} = e^{r\tau^k_{ij}}$ for $k \in K = \{A, B, AB, O\}$ and $ij \in \Theta$. Then a scheme is *individually rational* if, for each type $ij \in \Theta$,

[35] The multinomial distribution is used.

$$D_{ij} \equiv \rho_{ij}^A c_i^A y_{ij}^A + \rho_{ij}^B c_j^B y_{ij}^B + \rho_{ij}^{AB} c_i^A c_j^B y_{ij}^{AB} + \rho_{ij}^O y_{ij}^O \leq 1.$$

An auction is *incentive compatible* if, for any $ij \in \Theta$ and $i'j' \in \Theta \setminus \{ ij\}$,

$$D_{ij} \leq \rho_{i'j'}^A c_i^A y_{i'j'}^A + \rho_{i'j'}^B c_j^B y_{i'j'}^B + \rho_{i'j'}^{AB} c_i^A c_j^B y_{i'j'}^{AB} + \rho_{i'j'}^O y_{i'j'}^O \equiv D_{ij}^{i'j'}.$$

The seller's revenue can, then, be written in terms of the expected payment of an arbitrary buyer, namely the term in brackets:

$$\pi = n[\sum_{ij\in\Theta}\{\alpha_{ij}\sum_{k\in K}\rho_{ij}^k \tau_{ij}^k\}]. \tag{1.14}$$

Note that, $\tau_{ij}^k = \frac{1}{r}\ln y_{ij}^k$. Then, if the reduced form probabilities are 'implementable' we can write the seller's problem in reduced form as

$$\max_{\{\rho_{ij}^k, y_{ij}^k\}_{ij\in\Theta, k\in K}} \frac{n}{r}\sum_{ij\in\Theta}\{\alpha_{ij}\sum_{k\in K}\rho_{ij}^k \ln y_{ij}^k\} \tag{SP}$$

subject to

$$D_{ij} \leq 1 \qquad ij \in \Theta \tag{1.15}$$

$$D_{ij} \leq D_{ij}^{i'j'} \quad ij \in \Theta,\ i'j' \in \Theta \setminus \{ij\} \tag{1.16}$$

Since the buyers are risk-averse, the correlation between the events of winning object A (namely, event $A \cup AB$) and object B (namely, event $B \cup AB$) matters for the buyers and also for the seller through 1.14. Thus, Border's [11] theorem does not apply to this problem.[36] As it is also mentioned in Armstrong [1], the conditions that we need to impose to ensure that the reduced form probabilities are implementable are not clear. For this reason, different from the previous section, we aim to construct the actual probabilities, $p_{ij}^k(\eta)$, $\forall ij \in \Theta$, $k = A, B, AB$ and $\forall \eta$.[37] Given a payment rule, the optimality of a decision rule will be analyzed as follows: For any modification of $p_{ij}^k(\eta)$, we will first describe how *expected*

[36] See footnote 28.

[37] Given ij and η, $p_{ij}^O(\eta)$ can be calculated using 1.13 and the values of $p_{ij}^A(\eta)$, $p_{ij}^B(\eta)$, and $p_{ij}^{AB}(\eta)$ are found.

probabilities ρ_{ij}^k will be affected. Then, we figure out whether the incentive constraints in 1.16 and individual rationality constraints in 1.15 hold and whether the objective function (SP) increases after the modification. To demonstrate how this works, we borrow the following example from Menicucci [32]:

Suppose for a given profile of reports with $n_{HH} \geq 1$ and $n_{LH} \geq 1$ each type wins object A with probability $\frac{1}{n_{HH}}$ and each type LH wins object B with probability $\frac{\beta}{n_{LH}}(0 < \beta \leq 1)$. Note that from 1.13, this generates a contribution to ρ_{LH}^B equal to

$$\frac{\beta}{n_{LH}}\Psi\frac{n_{LH}}{\alpha_{LH}}.$$

Consider reducing β by $\Delta\beta > 0$ while increasing by $\Delta\beta$ the probability that the same buyer of type HH winning object A will also win object B. Then,

$$\begin{aligned}\Delta\rho_{LH}^B &= -\frac{\Delta\beta}{n_{LH}}\Psi\frac{n_{LH}}{\alpha_{LH}} \\ \Delta\rho_{HH}^A &= -\frac{\Delta\beta}{n_{HH}}\Psi\frac{n_{HH}}{\alpha_{HH}} = -\Delta\rho_{HH}^{AB}.\end{aligned}$$

So, $\Delta\rho_{HH}^{AB} = -\Delta\rho_{HH}^A = -\frac{\alpha_{LH}}{\alpha_{HH}}\Delta\rho_{LH}^B$. We can then evaluate the profitability of reducing β since the seller's profit function and the constraints are linear with respect to the *expected* probabilities.

1.3.2 Solution to the problem

Before we attempt to solve problem SP, note that, since $0 < c_H < c_L$, incentive compatibility conditions imply that among the individual rationality conditions only the one corresponding to type LL matters.

1.3.2.1 The relaxed problem Using the same approach as in Armstrong [1], we first solve the seller's problem considering only the five downward incentive constraints, that ensure that a buyer does not underreport his valuation for an object. We show *ex post* that the remaining constraints are satisfied (Propositions 25 and 26).

Thus, the seller solves

$$\begin{aligned}\max \quad & \alpha_{HH}\{\rho^A_{HH}\ln y^A_{HH} + \rho^B_{HH}\ln y^B_{HH} + \rho^{AB}_{HH}\ln y^{AB}_{HH} + \rho^O_{HH}\ln y^O_{HH}\} \\ & +\alpha_{HL}\{\rho^A_{HL}\ln y^A_{HL} + \rho^B_{HL}\ln y^B_{HL} + \rho^{AB}_{HL}\ln y^{AB}_{HL} + \rho^O_{HL}\ln y^O_{HL}\} \\ & +\alpha_{LH}\{\rho^A_{LH}\ln y^A_{LH} + \rho^B_{LH}\ln y^B_{LH} + \rho^{AB}_{LH}\ln y^{AB}_{LH} + \rho^O_{LH}\ln y^O_{LH}\} \\ & +\alpha_{LL}\{\rho^A_{LL}\ln y^A_{LL} + \rho^B_{LL}\ln y^B_{LL} + \rho^{AB}_{LL}\ln y^{AB}_{LL} + \rho^O_{LL}\ln y^O_{LL}\}\end{aligned}$$

subject to

$$\rho^A_{LL}c^A_L y^A_{LL} + \rho^B_{LL}c^B_L y^B_{LL} + \rho^{AB}_{LL}c^A_L c^B_L y^{AB}_{LL} + \rho^O_{LL}y^O_{LL} \le 1 \qquad (IR_{LL})$$

$$\begin{aligned} & \rho^A_{LH}c^A_L y^A_{LH} + \rho^B_{LH}c^B_H y^B_{LH} + \rho^{AB}_{LH}c^A_L c^B_H y^{AB}_{LH} + \rho^O_{LH}y^O_{LH} \\ \le \quad & \rho^A_{LL}c^A_L y^A_{LL} + \rho^B_{LL}c^B_H y^B_{LL} + \rho^{AB}_{LL}c^A_L c^B_H y^{AB}_{LL} + \rho^O_{LL}y^O_{LL} \qquad (IC^{LL}_{LH}) \\ & \rho^A_{HL}c^A_H y^A_{HL} + \rho^B_{HL}c^B_L y^B_{HL} + \rho^{AB}_{HL}c^A_H c^B_L y^{AB}_{HL} + \rho^O_{HL}y^O_{HL} \\ \le \quad & \rho^A_{LL}c^A_H y^A_{LL} + \rho^B_{LL}c^B_L y^B_{LL} + \rho^{AB}_{LL}c^A_H c^B_L y^{AB}_{LL} + \rho^O_{LL}y^O_{LL} \qquad (IC^{LL}_{HL}) \\ & \rho^A_{HH}c^A_H y^A_{HH} + \rho^B_{HH}c^B_H y^B_{HH} + \rho^{AB}_{HH}c^A_H c^B_H y^{AB}_{HH} + \rho^O_{HH}y^O_{HH} \\ \le \quad & \rho^A_{LL}c^A_H y^A_{LL} + \rho^B_{LL}c^B_H y^B_{LL} + \rho^{AB}_{LL}c^A_H c^B_H y^{AB}_{LL} + \rho^O_{LL}y^O_{LL} \qquad (IC^{LL}_{HH}) \\ & \rho^A_{HH}c^A_H y^A_{HH} + \rho^B_{HH}c^B_H y^B_{HH} + \rho^{AB}_{HH}c^A_H c^B_H y^{AB}_{HH} + \rho^O_{HH}y^O_{HH} \\ \le \quad & \rho^A_{LH}c^A_H y^A_{LH} + \rho^B_{LH}c^B_H y^B_{LH} + \rho^{AB}_{LH}c^A_H c^B_H y^{AB}_{LH} + \rho^O_{LH}y^O_{LH} \qquad (IC^{LH}_{HH}) \\ & \rho^A_{HH}c^A_H y^A_{HH} + \rho^B_{HH}c^B_H y^B_{HH} + \rho^{AB}_{HH}c^A_H c^B_H y^{AB}_{HH} + \rho^O_{HH}y^O_{HH} \\ \le \quad & \rho^A_{HL}c^A_H y^A_{HL} + \rho^B_{HL}c^B_H y^B_{HL} + \rho^{AB}_{HL}c^A_H c^B_H y^{AB}_{HL} + \rho^O_{HL}y^O_{HL}. \qquad (IC^{HL}_{HH})\end{aligned}$$

We first establish that it is not optimal to make the expected payments, namely y^k_{ij}s, random. This is because if a y^k_{ij} is random for an ij and k, then the seller could replace it with its expected value without affecting the incentive conditions (because they are linear in y^k_{ij}) and increase her revenue (as the seller's revenue is a concave function of y^k_{ij}).

Proposition 11. *If the buyers' preferences are represented by CARA utility function then, in an optimal auction, the expected payments conditional on types and allocation must be deterministic.*

Now, we determine which of the six conditions in the relaxed problem are binding.

Lemma 12. *At the optimum of the relaxed problem, IR_{LL} must be binding.*

Lemma 13. *At the optimum of the relaxed problem, IC_{LH}^{LL} and IC_{HL}^{LL} must be binding.*

Lemma 14. *At the optimum of the relaxed problem, at least one of $IC_{HH}^{LL}, IC_{HH}^{LH}$ and IC_{HH}^{HL} must be binding.*

Using the above lemmata, we write the Lagrangian of the relaxed problem and derive its Kuhn-Tucker conditions with respect to the payments, namely y_{ij}^{k}s Then, we establish the relation among the payments using these Kuhn-Tucker conditions, the details of which we relegate to the appendix.

Similar to the single object case, when a buyer wins an object, say object i, for which he has high valuation, he pays v_H^i more than what he would have paid if he lost that object. The intuition for proposition 4 also applies here.

If the objects are not limited, the seller can make the high-type buyer's probability of obtaining the object(s) equal to one in order to reward him for revealing his true valuation(s). However, when the objects are limited, the same rewarding strategy does not work because each high-type buyer may face the risk of losing the object(s) to another high-type buyer and hence, the marginal utility of income may differ in the events of winning and losing. The resource constrained seller, however, can reward a high-type buyer by offering perfect insurance and increase her revenue. Note that, if buyers are risk neutral, there is no insurance issue. In other words, if the buyers are risk averse the seller has an additional tool to extract more revenue from them when compared to risk neutral environment.

Proposition 15. *Each buyer is perfectly insured against the risk of losing the object(s) for which he has high valuation.*

When it comes to the LL-type buyers, the seller faces the following predicament: to extract more revenue from the LL-type buyer by offering insurance and to exploit the risk-bearing of the buyers who have high-valuation for one or both of the objects to screen them. At the optimum, the marginal benefit of exploiting high-type buyers risk bearing exceeds the marginal cost of not offering insurance to LL-type buyers. Moreover, LL-type buyers pay penalty when he loses both objects which further deters high-type buyers from behaving as if they are LL-type.

Proposition 16. *Suppose that type LL is not excluded from the auction. Then, he incurs a positive payment if he loses both objects.*

With the help of the preceding results, the seller's problem can be written as

$$[\alpha_{HH}\hat{\rho}_{HH}^{A} + \alpha_{HL}\hat{\rho}_{HL}^{A}] \ln \frac{1}{c_H^A} + [\alpha_{HH}\hat{\rho}_{HH}^{B} + \alpha_{LH}\hat{\rho}_{LH}^{B}] \ln \frac{1}{c_H^B} + \alpha_{HH} \ln y_{HH}^{O}$$
$$+\alpha_{HL}[\hat{\rho}_{HL}^{B} \ln y_{HL}^{B} + (1-\hat{\rho}_{HL}^{B}) \ln y_{HL}^{O}] + \alpha_{LH}[\hat{\rho}_{LH}^{A} \ln y_{LH}^{A} + (1-\hat{\rho}_{LH}^{A}) \ln y_{LH}^{O}]$$
$$+\alpha_{LL}[\rho_{LL}^{A} \ln y_{LL}^{A} + \rho_{LL}^{B} \ln y_{LL}^{B} + \rho_{LL}^{AB} \ln y_{LL}^{AB} + \rho_{LL}^{O} \ln y_{LL}^{O}]$$

subject to

$$\begin{aligned}
D_{LL} &= 1 \\
D_{LH}^{LL} &= \hat{\rho}_{LH}^{A} c_L^A y_{LH}^{A} + (1-\hat{\rho}_{LH}^{A}) y_{LH}^{O} \\
D_{HL}^{LL} &= \hat{\rho}_{HL}^{B} c_L^B y_{HL}^{B} + (1-\hat{\rho}_{HL}^{B}) y_{HL}^{O} \\
y_{HH}^{O} &= \min \begin{cases} D_{HH}^{LL} \\ \hat{\rho}_{LH}^{A} c_H^A y_{LH}^{A} + (1-\hat{\rho}_{LH}^{A}) y_{LH}^{O} \\ \hat{\rho}_{HL}^{B} c_H^B y_{HL}^{B} + (1-\hat{\rho}_{HL}^{B}) y_{HL}^{O} \end{cases}
\end{aligned}$$

where $\hat{\rho}_{ij}^{A} = \rho_{ij}^{A} + \rho_{ij}^{AB}$ and $\hat{\rho}_{ij}^{B} = \rho_{ij}^{B} + \rho_{ij}^{AB}$. Let's call this problem SP'.

Thus, for the optimality of an auction only the following reduced form probabilities matter:

$$\{\hat{\rho}_{ij}^{A}, \hat{\rho}_{ij}^{B}\}_{ij=HH,HL,LH}, \{\rho_{LL}^{k}\}_{k=A,B,AB}$$

Consider a mechanism where, for a given profile, η, both objects are sold with probability one. Then, if the seller modifies the mechanism by increasing $p_{ij}^{k}(\eta)$ by $\frac{1}{n_{ij}}\varepsilon_{ij}^{k}$, the following condition must hold:

$$\sum_{ij\in S}(\varepsilon_{ij}^{k} + \varepsilon_{ij}^{AB}) \leq 0 \ \text{ for } \ k = A, B.$$

After this modification, ρ^k_{ij} will increase by $\frac{1}{\alpha_{ij}}\varepsilon^k_{ij}\Psi$.

We now establish that the solution to the relaxed problem is weakly efficient. That is, if there is a buyer with high valuation for an object then that object is never sold to a buyer who has low valuation for that object.

Proposition 17. *Let $\eta = (n_{HH}, n_{LH}, n_{HL}, n_{LL})$ be the profile of the participating buyers. Then, the solution to the relaxed problem satisfies the following two rules:*

i) For any η with $n_{HH} + n_{HL} > 0$, $n_{HH}\hat{p}^A_{HH}(\eta) + n_{HL}\hat{p}^A_{HL}(\eta) = 1$

ii) For any η with $n_{HH} + n_{LH} > 0$, $n_{HH}\hat{p}^B_{HH}(\eta) + n_{HL}\hat{p}^B_{LH}(\eta) = 1$.

If there is a buyer who has a high value for object A (B) then with probability one it is given to a buyer who has a high value for it. While proposition 17 states this result in terms of actual probabilities, the following corollary does the same in terms of the expected probabilities.

Corollary 18. *At the optimum of the relaxed problem, reduced form probabilities satisfy*

i) $\alpha_{HH}\hat{\rho}^A_{HH} + \alpha_{HL}\hat{\rho}^A_{HL} = \frac{1}{n}(1 - (\alpha^B_L)^n)$ and

ii) $\alpha_{HH}\hat{\rho}^B_{HH} + \alpha_{LH}\hat{\rho}^B_{LH} = \frac{1}{n}(1 - (\alpha^A_L)^n)$.

The next lemma establishes that both objects are sold with probability one, if a buyer's payment contingent on winning an object for which he has low valuation is larger than his payment contingent on losing both objects.

Similar to the previous section, we assume that the seller never keeps the object. We have already established in proposition 17 that the seller does not keep an object whenever there is a buyer who has a high value for it. This requires the probability that an object is won by a buyer who has a low value for it to be equal to the probability that all buyers have low value for it.

$$\begin{aligned}
\alpha_{LL}\hat{\rho}^A_{LL} + \alpha_{LH}\hat{\rho}^A_{LH} &= \frac{1}{n}(\alpha^A_L)^n \\
\alpha_{LL}\hat{\rho}^B_{LL} + \alpha_{HL}\hat{\rho}^B_{HL} &= \frac{1}{n}(\alpha^B_L)^n
\end{aligned}$$

In terms of actual probabilities, we can write these conditions as

$$\text{For any } \eta \text{ with } n_{HH} + n_{HL} = 0,\ n_{LH}\hat{p}^A_{LH}(\eta) + n_{LL}\hat{p}^A_{LL}(\eta) = 1 \qquad (1.17)$$

$$\text{For any } \eta \text{ with } n_{HH} + n_{LH} = 0,\ n_{HL}\hat{p}^B_{HL}(\eta) + n_{LL}\hat{p}^B_{LL}(\eta) = 1 \qquad (1.18)$$

Proposition 19. *The necessary conditions for 1.17-1.18 are* $y^A_{LH} > y^O_{LH}, y^B_{HL} > y^O_{HL}$*, and* $y^A_{LL}, y^B_{LL}, y^{AB}_{LL} > y^O_{LL}$.

Since $D_{HH} = y^O_{HH} \leq 1$, when HH loses both objects he either does not pay anything (i.e. $y^O_{HH} = 1$) or he is compensated (i.e. $y^O_{HH} < 1$).

Proposition 20. *In any mechanism that solves the relaxed problem, if an* HH *type buyer loses both objects then he is compensated.*

This proposition results because the seller needs to provide insurance to type HH. This is a property that carries over from the single unit optimal auction. (Maskin and Riley [28]) They show that when the type space is continuous, the seller provides full insurance (and hence full compensation) only to the highest type but partial insurance to the types that are sufficiently high.

Proposition 21. *In any mechanism that solves the relaxed problem, if all the buyers are of type* LL *(i.e.* $n_{LL} = n$*) then the objects are bundled and each buyer wins the bundle with equal probability. (i.e.* $p^{AB}_{LL}(\eta) = \frac{1}{n}$*).*

An immediate implication of the proposition above is that it is not optimal to sell the goods independently in which case with positive probability the objects may end up in the hands of different LL type buyers. Yet, the proposition has further implications.

When the buyers are risk neutral (Armstrong [1]), depending on how buyers' valuations are correlated across objects, the optimal multi-object auction can take the form of independent auctions, mixed auction or bundling auction. But all of these auction forms allocate the two objects independently and randomly when all buyers are of type LL. This contradicts with the proposition. Therefore, none of these auction forms are optimal when the buyers are risk averse.

Theorem 22. *Whenever the parameter values are such that the relaxed method solves the full problem, the three auction formats that are optimal when the buyers are risk neutral do not maximize the seller's revenue if the buyers are risk averse.*

The main reason for why we obtain this contradictory result is that the optimal auction forms for the risk neutral buyers do not impose the right amount of risk on type LL. The optimal auction for risk averse buyers, on the other hand, imposes two kinds of risk on this type. The first kind removes the possibility of winning a single object when all buyers are of type LL and the second kind assigns a positive payment if he doesn't win any objects. These two kinds of risk improve the sellers revenue in the following way. The former exploits the risk bearing of the buyers who have high valuation for one or both objects by facing them with even greater risk when imitating LL than the optimal auction for risk neutral buyers. The latter, on the other hand, help the seller collect the penalty fees from more people.

Since the seller probabilistically assesses the buyer valuations (i.e. only *ex ante* probabilities of the type distribution matter) and never keeps the objects by assumption, there always exists a probability that LL type buyers can obtain both objects. This can happen only if all buyers are of type LL. On the other hand, whenever there is a type HH or both HL and LH, then LL cannot win any objects. The following lemma states the conditions under which an LL can obtain a single object.

Lemma 23. *In any mechanism that solves the relaxed problem,*

i) if η is such that $n_{LH}, n_{LL} > 0$ and $n_{LH} + n_{LL} = n$, then object A is sold to an LH type buyer (i.e. $n_{LH}\hat{p}^A_{LH}(\eta) = 1$) if

$$\mu_{LH} < \left(\frac{\alpha_{HL}}{y^O_{HL}}\frac{y^O_{HH}}{\alpha_{HH}} + 1\right)\left(\frac{\alpha_{LL}}{y^O_{LL}}\frac{y^O_{LH}}{\alpha_{LH}} + 1\right)^{-1} \equiv \gamma_{LH}. \tag{†}$$

Otherwise, an LL type buyer gets object A (i.e. $n_{LL}\hat{p}^A_{LL}(\eta) = 1$).

ii) if η is such that $n_{HL}, n_{LL} > 0$ and $n_{HL} + n_{LL} = n$, then object B is sold to an HL type buyer (i.e. $n_{HL}\hat{p}^B_{HL}(\eta) = 1$) if

$$\mu_{HL} < \left(\frac{\alpha_{LH}}{y^O_{LH}}\frac{y^O_{HH}}{\alpha_{HH}} + 1\right)\left(\frac{\alpha_{LL}}{y^O_{LL}}\frac{y^O_{HL}}{\alpha_{HL}} + 1\right)^{-1} \equiv \gamma_{HL}. \tag{‡}$$

Otherwise, an LL type buyer gets object B (i.e. $n_{LL}\hat{p}^B_{LL}(\eta) = 1$).

According to the previous lemma, in the optimal auction, if the excess payment that LH makes for object A is larger than that of LL (namely, $t^A_{LH} - t^O_{LH} > t^A_{LL} - t^O_{LL}$), then LH wins object A.

By this lemma, the solution to the relaxed problem depends on the values of γ_{LH} and γ_{HL}. Note that, $\gamma_{LH} \geq 1$ if and only if $\gamma_{HL} \geq 1$. Thus, we can divide the rest of the analysis into three cases (See Figure 2):

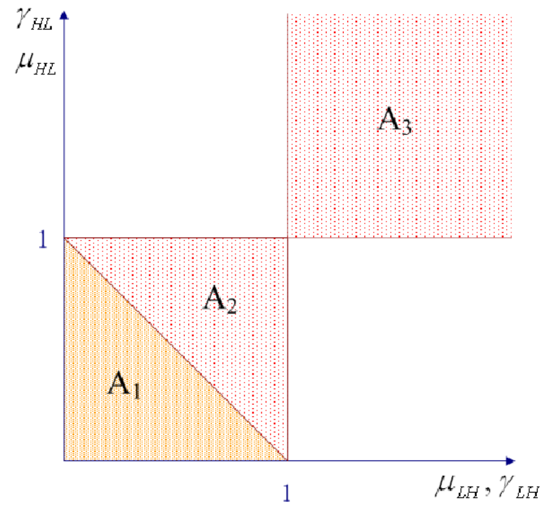

Figure 2: Allocation of each object when all buyers have low valuation for it.

- $\gamma_{LH} + \gamma_{HL} \leq 1$ (Region A_1),
- $1 \leq \gamma_{LH} + \gamma_{HL} \leq 2$ (Region A_2),
- $2 \leq \gamma_{LH} + \gamma_{HL}$ (Region A_3).

Remark 24. *Readers should note that the three cases listed above are analogous to those mentioned in Lemma 2 of Armstrong [1]: strong positive correlation, weak positive correlation, and negative correlation, respectively.*

Whether object A (B) is given to an LL or LH (HL) type buyer depends on whether $(\gamma_{LH}, \gamma_{HL})$ falls in region A_1, A_2, or A_3.

1.3.2.2 Case A1 - Strong positive correlation: $[\gamma_{LH} + \gamma_{HL} \leq 1]$ We can set

$$\mu_{LL} = 1 - \gamma_{LH} - \gamma_{HL},\ \mu_{LH} = \gamma_{LH},\ \mu_{HL} = \gamma_{HL} \tag{1.19}$$

In this case, all incentive constraints of type HH are binding. This also implies that the seller is indifferent between LH and LL for object A and between HL and LL for object B.

For any given allocation probabilities, the payments

$$\{y_{LL}^{A}, y_{LL}^{B}, y_{LL}^{AB}, y_{LL}^{O}, y_{LH}^{A}, y_{LH}^{O}, y_{HL}^{B}, y_{HL}^{O}, y_{HH}^{O}\}^{38} \tag{1.20}$$

solve

$$\begin{aligned} \max \quad & \alpha_{HH}\ln y_{HH}^{O} + \alpha_{LH}(1-\hat{\rho}_{LH}^{A})\ln y_{LH}^{O} + \alpha_{HL}(1-\hat{\rho}_{HL}^{B})\ln y_{HL}^{O} + +\alpha_{LL}\rho_{LL}^{O}\ln y_{LL}^{O} \quad (1.21) \\ & +\alpha_{LH}\hat{\rho}_{LH}^{A}\ln y_{LH}^{A} + \alpha_{HL}\hat{\rho}_{HL}^{B}\ln y_{HL}^{B} + \alpha_{LL}\rho_{LL}^{A}\ln y_{LL}^{A} + \alpha_{LL}\rho_{LL}^{B}\ln y_{LL}^{B} + \alpha_{LL}\rho_{LL}^{AB}\ln y_{LL}^{AB} \end{aligned}$$

subject to

$$\begin{aligned} \rho_{LL}^{A}c_{L}^{A}y_{LL}^{A} + \rho_{LL}^{B}c_{L}^{B}y_{LL}^{B} + \rho_{LL}^{AB}c_{L}^{A}c_{L}^{B}y_{LL}^{AB} + \rho_{LL}^{O}y_{LL}^{O} &= 1 & (1.22) \\ \rho_{LL}^{A}c_{L}^{A}y_{LL}^{A} + \rho_{LL}^{B}c_{H}^{B}y_{LL}^{B} + \rho_{LL}^{AB}c_{L}^{A}c_{H}^{B}y_{LL}^{AB} + \rho_{LL}^{O}y_{LL}^{O} &= \hat{\rho}_{LH}^{A}c_{L}^{A}y_{LH}^{A} + (1-\hat{\rho}_{LH}^{A})y_{LH}^{O} & (1.23) \\ \rho_{LL}^{A}c_{H}^{A}y_{LL}^{A} + \rho_{LL}^{B}c_{L}^{B}y_{LL}^{B} + \rho_{LL}^{AB}c_{H}^{A}c_{L}^{B}y_{LL}^{AB} + \rho_{LL}^{O}y_{LL}^{O} &= \hat{\rho}_{HL}^{B}c_{L}^{B}y_{HL}^{B} + (1-\hat{\rho}_{HL}^{B})y_{HL}^{O} & (1.24) \\ \rho_{LL}^{A}c_{H}^{A}y_{LL}^{A} + \rho_{LL}^{B}c_{H}^{B}y_{LL}^{B} + \rho_{LL}^{AB}c_{H}^{A}c_{H}^{B}y_{LL}^{AB} + \rho_{LL}^{O}y_{LL}^{O} &= y_{HH}^{O} & (1.25) \\ \hat{\rho}_{LH}^{A}c_{H}^{A}y_{LH}^{A} + (1-\hat{\rho}_{LH}^{A})y_{LH}^{O} &= y_{HH}^{O} & (1.26) \\ \hat{\rho}_{HL}^{B}c_{H}^{B}y_{HL}^{B} + (1-\hat{\rho}_{HL}^{B})y_{HL}^{O} &= y_{HH}^{O}. & (1.27) \end{aligned}$$

By 1.19 and lemma 24,

$$y_{LH}^{A}y_{LL}^{O} = y_{LL}^{A}y_{LH}^{O} \text{ and } y_{HL}^{B}y_{LL}^{O} = y_{LL}^{B}y_{HL}^{O} \tag{1.28}$$

must also be true. Using equations 1.22-1.27, and the two conditions in 1.28, we can solve for eight of the variables (say, except y_{HH}^{O}) listed in 38 in terms of y_{HH}^{O}, the parameters and the reduced form probabilities. After plugging these variables into the objective function 1.21 we can solve it for y_{HH}^{O}.

Now, we consider the conditions that we have omitted in the relaxed problem.

Proposition 25. *(Full problem - Case A1) The upward incentive conditions, IC_{LL}^{LH}, IC_{LL}^{HL}, IC_{LH}^{HH} and IC_{HL}^{HH} are not binding.*

[38] The payments that are not listed in (38), namely $y_{LH}^{B}, y_{LH}^{AB}, y_{HL}^{A}, y_{HL}^{AB}, y_{HH}^{A}, y_{HH}^{B}, y_{HH}^{AB}$, can be calculated using proposition 15.

The above proposition states that type LL does not have incentive to imitate the types LH or HL. Moreover, neither type LH nor type HL has incentive to imitate HH.

The conditions IC_{LH}^{HL} and IC_{HL}^{LH} together imply

$$\begin{aligned} & y_{HH}^O\{\frac{\rho_{LH}^A}{\hat{\rho}_{LH}^A}\frac{\Delta^A}{c_H^A}+(\frac{\rho_{LH}^{AB}}{\hat{\rho}_{LH}^A}-\frac{\rho_{HL}^{AB}}{\hat{\rho}_{HL}^B})\frac{c_L^Ac_H^B-c_H^Ac_L^B}{c_H^Ac_H^B}+\frac{\rho_{HL}^B}{\hat{\rho}_{HL}^B}\frac{\Delta^B}{c_H^B}\} \\ \leq \quad & y_{HL}^O\{\rho_{HL}^A\frac{\Delta^A}{c_H^A}-\frac{\rho_{HL}^{AB}}{\hat{\rho}_{HL}^B}(1-\hat{\rho}_{HL}^B)\frac{c_L^Ac_H^B-c_H^Ac_L^B}{c_H^Ac_H^B}+\frac{\rho_{HL}^B}{\hat{\rho}_{HL}^B}(1-\hat{\rho}_{HL}^B)\frac{\Delta^B}{c_H^B}\} \\ + \quad & y_{LH}^O\{\rho_{LH}^B\frac{\Delta^B}{c_H^B}+\frac{\rho_{LH}^{AB}}{\hat{\rho}_{LH}^A}(1-\hat{\rho}_{LH}^A)\frac{c_L^Ac_H^B-c_H^Ac_L^B}{c_H^Ac_H^B}+\frac{\rho_{LH}^A}{\hat{\rho}_{LH}^A}(1-\hat{\rho}_{LH}^A)\frac{\Delta^A}{c_H^A}\}. \end{aligned} \tag{1.29}$$

where $\Delta^i = c_H^i - c_L^i$.

IC_{LL}^{HH} takes the following form:

$$1 \leq y_{HH}^O\{\rho_{HH}^A\frac{c_L^A}{c_H^A}+\rho_{HH}^B\frac{c_L^B}{c_H^B}+\rho_{HH}^{AB}\frac{c_L^Ac_L^B}{c_H^Ac_H^B}+\rho_{HH}^O\} \tag{1.30}$$

and $\gamma_{HL} \leq 1$ can be written as

$$\frac{\alpha_{LH}\alpha_{HL}}{\alpha_{LL}\alpha_{HH}} \leq \frac{y_{LH}^O y_{HL}^O}{y_{LL}^O y_{HH}^O} \tag{1.31}$$

Proposition 26. *The optimal allocation probabilities satisfy the necessary condition 1.31. Moreover, 1.29 and 1.30 are not binding.*

1.4 DISCUSSION AND CONCLUDING REMARKS

In a binary model, we show that when the buyers are risk-averse, the optimal auction is weakly efficient. That is, with probability one each object is sold to a buyer who has high valuation for it, if such a buyer exists. Each buyer is perfectly insured against the risk of losing the object(s) for which he has high valuation. Buyers who are eager to win both objects are compensated if they can not win either object; whereas, buyers who have low value for both objects incur a positive payment if they lose both objects. The objects are bundled when all buyers are reluctant for both objects, thus, none of the auction forms listed in Armstrong [1] are optimal.

In a more general framework, it has been shown that among all mechanisms for allocating multiple objects that are strongly efficient, incentive compatible, and individually rational, the Vickrey-Clarke-Groves mechanism maximizes the expected revenue.[39] The optimal multi-object auction that we have constructed for risk averse buyers is incentive compatible and individually rational but is only weakly efficient and thus different from the VCG mechanism.

The inefficiency results either because some types are *ex ante* excluded from participating the auction, or because of a misallocation. In this paper, we confined ourselves from the first kind of inefficiency, and showed that the latter kind of inefficiency does not occur in an optimal auction. Yet, this result is very sensitive to the assumption of binary distribution of types. Armstrong [1] shows that weak efficiency does not survive once the type space is made continuous.

The seller can exploit the risk bearing of the buyers, either by making their payments different in the events of winning and losing; or, contingent on winning and losing, she can make their payments random. While the former improves the revenue the latter does not.

We finally comment on the restrictions of our model. For tractability reasons, we focused only on the case where the buyers' utility function exhibits constant absolute risk aversion. Instead a buyer's utility may exhibit increasing or decreasing absolute risk aversion, or relative risk aversion, in which case the answer to the optimal design problem is not clear.

[39] For a clear and concise discussion of VCG mechanisms see Krishna [23].

Alternatively, one can also consider the situations where the buyers have different risk attitudes with respect to each good, in addition to that with respect to the wealth level. In that case, one would have to consider a generalization of the Arrow-Pratt theory (Arrow [3] and Pratt [36]) which allows to study multi-dimensional risk attitudes. One such generalization is proposed by Kihlstrom and Mirman [22].

Gal-Or [18], considers the case where the risk-averse buyers worry about the possibility of breakdowns. She shows that running "sales" improves the revenue of the single-unit monopolist. This is because the risk-averse buyers tend to buy more frequently than necessary to avoid buying at the higher regular price and to avoid the cost of waiting for the next sales period. Since, in our model, the seller owns only one unit of each object and the objects are not related, our results would not change if the buyers worry about breakdowns. In this case, buyers' concerns can be easily embodied into their valuations.

2.0 SEQUENTIAL ALL-PAY AUCTION WITHOUT COMMITMENT

2.1 INTRODUCTION

The seminal paper by Myerson [34] showed that when bidders have linear cost functions, the independent private value auctions with optimal reserve price maximize revenue. This clearly explains why sellers often post a reserve price in auctions.[1] However, when the market value of the object is higher than sellers' valuation, sequential rationality imposes a constraint on their behavior: They cannot credibly commit to keep the object out of market if the reserve price is not met. Indeed, it is common today that the sellers reauction the same object over and over again if it fails to sell. This behavior is observed in auctions that are held online at Ebay and that take place in well-known auction houses, like Christie's and Sotheby's. The seller's inability to commit is not incorporated in Myerson [34], and the consequences of this behavior need to be analyzed.

McAfee and Vincent [31] (Henceforth MV) analyze this problem in the first price and the second price auctions. They proved that the revenue equivalence result of Myerson [34] holds between sequential first price and second price auctions when the seller is unable to commit. They observe that the seller lowers the reserve price if the object fails to sell in the previous period. Yet, they didn't question how much the seller loses by not committing to the reserve price.

Sobel and Takahashi [41] (Henceforth ST) study the same problem in a multi-stage bargaining environment where the seller makes take-it-or-leave-it offers in each period until

[1] Myerson assumes that the imposing a reserve price does not change the number of potential buyers. Engelbrecht-Wiggans [14], on the other hand, provide two examples where this might not be the case, and show that the loss associated with the reduced number of buyers outweigh the benefits of a reserve price.His result, though, does not apply to the situations that we consider in this paper.

the buyer accepts the offer[2]. In this case, if the buyer does not want to buy the object at the current price, the seller cannot resist the temptation to try to resell it in the next period. The posted price, here, is analogous to the reservation price of the auctions. ST showed that the ability to *commit to a price schedule* is beneficial to the seller because with this ability the seller can threaten to maintain a high price in order to induce a purchase in the first period. ST assume different discount rates for the seller and buyer, and analyze both the two-period case and infinite-period case, whereas MV assume same discount rates for all the players and analyze only the infinite-period case.

The current paper, which is also an extension of ST, studies the commitment problem in all-pay auctions where the winner collects the prize and but all bidders forfeit their bids. All-pay auctions are used to raise funds for charities, but, in general, they are rarely preferred as a selling mechanism. Although all-pay auctions are not practical in real life, in the literature, they are frequently used to model real life situations such as R&D tournaments, promotions in labor markets, and lobbying activities. For motivation, let's discuss why these situations are analogous (or precisely isomorphic) to all-pay auctions and why the "sellers" in these circumstances cannot commit to the "reserve price".

An example to a research and development tournament is the prototype tournaments sponsored by the U.S. Army Air Corps in which several manufacturers compete to make a prototype of an aircraft specifications of which is announced by the sponsor.[3] If none of the competitors can meet these minimum requirements, and therefore, not enter the tournament, the sponsor will naturally think of revising the rules and bringing down the minimum requirements. If at least one of the firms is interested in the project, then the tournament will take place. The winner, which is assumed to be the firm that spends the highest effort, is awarded the production contract[4]. The efforts of the losers are sunk. The

[2]Note that, if there is only one bidder, the models in MV and ST are equivalent.

[3]One such tournament was organized by the U.S. Army for a Joint Cargo Aircraft. Lockheed's C-130J's exclusion from the competition raised a protest:

"The Army has excluded the C-130J without adequate regard to Air Force requirements," Lockheed said in its complaint to the Government Accountability Office, which serves as a watchdog agency for the federal government. "As such, it appears the joint title of the (Joint Cargo Aircraft) program is significantly overstated."

[4]Here, it is implicitly assumed that the quality of the prototype is monotonically increasing with the effort level, which may not be the case in all R&D projects.

objective of the privately sponsored R&D tournaments, generally, aims to maximize the effort spent on the "best" project. A tournament designed by public authorities, on the other hand, may aim to boost the overall activity level in a particular market. In that case, the objective of the tournament is to maximize the effort spent on "all" projects.

Now, consider a labor market tournament designed to promote one of the lower-rank employees to an open higher-rank position[5]. This type of tournaments prove useful as a selection process when an outside option is not available due to regulations (i.e. army, secret service) or when the outside option is too costly. Also, an employer can make use of a tournament to increase the effort level of the workers, overall. In either case, the employee who exerts the highest effort is awarded the indivisible prize. The effort of the non-winning participants, on the other hand, are sunk. It is very natural for the employer to require a minimum effort level if the efforts of the contestants can be observed, yet the employer cannot credibly commit to this minimum effort level due to the lack or the cost of the outside option.

Finally, consider lobbying activities. Campaign contributions that are made to policymakers are usually considered as access cost. If the contribution is high enough then the policymaker grants the interest groups "access", a chance to defend their cases. Lobbyists think that the higher is the amount of the contribution donated the more decisive is the information they provide to the policymakers. Grossman and Helpman [20] state that policymakers impose these costs, one, because they need funds to finance their campaigns, two, because they need a screen to distinguish groups that are more likely to provide valuable information, three, because their time is a scarce resource, and they want the value of the information to exceed the opportunity cost of their time[6]. The model presented in this paper, applies to all three cases. In the first two cases, the total revenue of the seller should be interpreted as sum of all contributions and aggregate value of all the informations, respectively. In the latter case, policymaker minimizes the time spent with the lobbyists[7]. Regardless of which case is assumed, policymaker grants access to certain lobbyists the contributions

[5] Readers can refer to Nalebuff and Stiglitz [35] and Rosen [38] for labor market tournaments.

[6] Austen-Smith [6], Lohmann [25], and Wright [42] are the other papers that interpret campaign contributions as access cost.

[7] In this case, the policymaker minimizes the disutility, or equivalently maximizes the negative of the disutility.

of which exceed the amount set by the policymaker. The interest group whose information played a decisive role is considered as winner. All the other lobbyists are deemed as losers. The contributions made by them are sunk.

Since the contest examples mentioned above are isomorphic to all-pay auctions, in the remainder of the paper, auction theory jargon is used[8]. The contest designer is referred to as the seller or she and the contestant(s) as the bidder(s) or he(they). The seller may maximize the highest bid or the sum of all bids. The latter objective is equivalent to maximizing the average bid if the buyers are *ex ante* symmetric.

In an all-pay auction, no matter which of the two objectives the seller pursues, if she is patient enough then imposing a relatively high reserve price in the first period and, in case of no sale, lowering it in the second period maximizes the revenue, conforming to the findings of ST and MV. In other words, the sequential all-pay auction in which the seller commits to the reserve price in the second period yields a higher revenue than the single period all-pay where the seller commits to the reserve price in the first period.

The main result of the paper is that, as the number of bidders increases, the seller will have a higher incentive to run a single-period all-pay auction if she maximizes the average bid, but she will prefer to run a sequential all-pay auction if she maximizes the highest bid. With a large number of bidders, a single-period all-pay auction yields higher revenue for a average-bid-maximizing seller only if she is almost fully patient. Moreover, the more bidders participate in the auction the less patient the highest-bid-maximizing seller has to be in order to prefer the sequential all-pay auction.

The remainder of the paper is organized as follows: Part 2 further reviews the literature. Part 3 introduces the model. Part 4 analyzes the benchmark case where the seller is assumed to announce no reserve price. Part 5, on the other hand, analyzes the case where the seller sets the optimal reserve price and commits to it if no sale occurs. Parts 6 explores the case where the seller is unable to commit temporarily. Finally, part 7 concludes.

[8] For classification of contests, you can refer to Baye et al. [9].

2.2 RELATED LITERATURE

Bertoletti [10] shows that when a seller has bargaining power then she should set an optimal reserve price. He shows that the revenue generated under the optimal reserve price might be higher than that generated when the highest valued lobbyist is excluded. Yet, this conclusion can be derived only if the seller has complete information about bidder's values. Hence, excluding the highest bidder is difficult when the seller is uninformed about the valuation of the bidders or when the bidders are ex-ante symmetric. Different from Bertoletti [10], this paper assumes incomplete information and weakens the assumption that the seller is able to commit to the reserve price.

Gavious and Sela [19] study all-pay auction with reserve prizes where the cost of bidding is a nonlinear function of the bids. They show that setting a reservation price is profitable for a seller who wishes to maximize the highest bid. When the seller wishes to maximize the average bid, on the other hand, it might not be profitable to set a reserve price. If the players have exogenous entry costs, then setting reserve price is always profitable.

Finally, Skreta [39] characterizes the optimal auction in a two-period model under non-commitment.

2.3 MODEL

This paper studies an all-pay auction where n *ex ante* symmetric risk-neutral bidders compete to win a single indivisible object. Seller's valuation of the object is normalized to zero, whereas the bidders' private values are drawn independently and identically from uniform distribution over [0,1].

The rules of the all-pay auction are as follows: The seller announces a reserve price and then the bidders simultaneously place their bids. If the reserve price is met then the highest bidder wins the object and everyone pays their bids. If the reserve is not met and the seller is able to commit to keep the object then the game ends. If the seller is unable to commit and no sale occurs in the first period then she reauctions the object. The new reserve price is announced and the bids are submitted. The item goes to the highest bidder if at least one of the bids exceed the second period's reserve price and all bidders pay their bids. The game ends after the second period regardless of whether the reserve is met or not.

Both the seller and the bidders are assumed to be expected utility maximizers. All players discount their expected future earnings but the bidders discount at a rate different from the seller's.

All features of the above model and the seller's ability to commit are common knowledge among all players.

The equilibrium constructed in section 2.4 is Bayesian Nash equilibrium and the equilibria that are described in all other sections are perfect Bayesian Nash.

2.4 BENCHMARK CASE: NO RESERVE PRICE

We start with the simplest scenario where the seller does not make any strategic decision, i.e. the seller does not announce a reserve price. In this case, the game is played among the bidders. A bidder wins the object and enjoys a positive payoff only if he outbids his opponents, yet he has to pay his bids even when he loses the object. More precisely, when bidder i of type v who places a bid of b_i, he earns an expected utility of $u(b_i, v) = v \Pr[b_i > \max_{j \neq i}\{b_j\}] - b_i$.

Each section of this paper aims to construct a symmetric equilibrium in monotonic strategies. Thus, the opponents of bidder i follow the same bidding strategy, $\beta(.)$, which is monotonically increasing in v. So, bidder i's utility can be written as

$$u(b_i, v) = v \Pr[b_i > \beta(v_j) \text{ for } j \neq i] - b_i \tag{2.1a}$$

$$= vF^{n-1}(\beta^{-1}(b_i) > v_j) - b_i \tag{2.1b}$$

$$= vF^{n-1}(\beta^{-1}(b_i)) - b_i, \tag{2.1c}$$

where $F(.)$ represents the belief that bidder i carries about his opponents valuations.

If bidder i's utility is differentiable, then the optimal bid b^* solves the first-order condition $\frac{\partial u(b^*,v)}{\partial b_i} = 0$ for each v. The envelope theorem states that the total derivative of the value function is equal to the partial derivative of it. Namely,

$$\frac{du(b^*,v)}{dv} = \frac{\partial u(b^*,v)}{\partial v} + \frac{\partial u(b^*,v)}{\partial b_i}\frac{b_i}{\partial v} \tag{2.2a}$$

$$= \frac{\partial u(b^*,v)}{\partial v} \tag{2.2b}$$

$$= F^{n-1}(\beta^{-1}(b^*)) \tag{2.2c}$$

$$= F^{n-1}(v). \tag{2.2d}$$

The second equality follows from the fact that b^* solves the first-order condition and 2.2d follows due to the symmetry of the equilibrium bid functions. Note that, equilibrium bid function has to assign an optimal bid to each possible valuation, hence $b^* = \beta(v)$. The integral of 2.2d, gives back the value function. Hence, combining equations 2.1c and 2.2d, one can write

$$vF^{n-1}(\beta^{-1}(b^*)) - b^* = \int_0^v F^{n-1}(t)dt \tag{2.3a}$$

$$vF^{n-1}(v) - \beta(v) = \int_0^v F^{n-1}(t)dt \tag{2.3b}$$

$$\beta(v) = vF^{n-1}(v) - \int_0^v F^{n-1}(t)dt. \tag{2.3c}$$

Since bidders' values are assumed to be uniformly distributed, equation 2.3c is equivalent to

$$\beta(v) = \frac{n-1}{n}v^n. \tag{2.4}$$

Proposition 27. *If the seller does not announce a reserve price, then bidding according to* $\beta(v) = \frac{n-1}{n}v^n$ *is a symmetric equilibrium of the all-pay auction.*

The seller's payoff can then be calculated. If her objective is to maximize the average bid then

$$\Pi^a = E[\beta(v)] = \int_0^1 \beta(v)f(v)dv \tag{2.5a}$$

$$= \frac{n-1}{n(n+1)}. \tag{2.5b}$$

Similarly, if she maximizes the highest bid, then her payoff is equal to

$$\Pi^h = E[\beta(v) \mid v > \max_{j=1,2,..n-1}\{v_j\}] = \int_0^1 \beta(v) n F^{n-1}(v) f(v) dv \tag{2.6a}$$

$$= \frac{n-1}{2n}. \tag{2.6b}$$

Each bidder, on the other hand, earns an *ex ante* utility of

$$E[u(\beta(v), v)] = \int_0^1 \{vF^{n-1}(v) - \beta(v)\} f(v) dv \tag{2.7a}$$

$$= \int_0^1 \frac{v^n}{n} dv = \frac{1}{n(n+1)}. \tag{2.7b}$$

2.5 RESERVE PRICE WITH COMMITMENT

This section lets the seller play a strategic role in the game. Foreseeing the equilibrium play of the bidders in the subgame, the seller posts a nonnegative reservation price. In order to solve seller's problem, the bidders' behavior needs to be analyzed first.

Let's assume that the seller posts a nonnegative reserve price r and also remember that bidder i with valuation v will earn a utility of

$$u(b_i, v) = v \Pr[b_i > \beta(v_j) \text{ for } j \neq i] - b_i \tag{2.8}$$

if he bids b_i. It can be shown that the bidders with low valuations have no incentive to participate. As an example, consider the bidder with valuation r. Since the probability of winning is smaller than one, this bidder cannot earn positive utility when he enters. This is because he has to bid at least r. For participation, a bidders valuations must be sufficiently large in order to offset the effect of his incomplete information about his opponents' values. In other words, there must be a critical type $c > r$ where the bidder is just indifferent between participating and not participating. So, we conclude that *only* the bidders with valuations larger than c will place a positive bid.

The bidders with valuations larger than c, on the other hand, tend to bid more aggressively than they would if the seller didn't impose a reserve price. The arguments that lead to this conclusion are as follows: The bidder with a valuation equal to c wins only if all other bidders have valuations smaller than c, and in the case that he wins he will be better off by placing the smallest bid, namely the reserve price r. Since the bidder with valuation c is indifferent between participating and not participating he earns zero utility, whereas he could have earned positive utility by placing a slightly smaller bid if the seller did not post a reserve price. So, when the seller posts a reserve price, the bidder with valuation c increases his bid. Using the monotonicity of the bidding strategies we conclude that bidders with valuations larger than c bid more aggressively if the seller posts a reserve price.

Since only the bidders of type $v > c$ place a positive bid, using the arguments that leads to equation 2.2d, one can write bidder i's value function as

$$u(\beta(v), v) = \int_c^v F^{n-1}(t)dt + u(\beta(c), c) \quad \text{for } c \leq v. \tag{2.9}$$

This expression is equivalent to equation 2.1c given that b is chosen optimally. So, we can write bidder i's equilibrium bidding strategy as

$$\begin{aligned} \beta(v) &= vF^{n-1}(v) - \int_c^v F^{n-1}(t)dt - u(\beta(c), c) \quad \text{for } c \leq v & \text{(2.10a)} \\ &= \frac{n-1}{n}v^n + \frac{c^n}{n} \qquad\qquad \text{for } c \leq v & \text{(2.10b)} \end{aligned}$$

The second equality is due to the fact bidder i earns zero utility when his valuation is c.

Bidder i bids the reserve price when his valuation is equal to the critical type: $\beta(c) = r$. Hence, $c = r^{1/n}$. In equilibrium, the seller forms a correct belief about how the bidders will behave in the second stage. So, if the seller's objective is to maximize the average bid, then her payoff is equal to

$$\Pi_c^a(c) = E[\beta(v)] = \int_c^1 \beta(v)f(v)dv = \int_c^1 (\frac{n-1}{n}v^n + \frac{c^n}{n})dv, \tag{2.11}$$

whereas her payoff is equal to

$$\Pi_c^h(c) = E[\beta(v) \mid v > \max_{j=1,2,..n-1}\{v_j\}] \tag{2.12a}$$

$$= \int_c^1 \beta(v) n F^{n-1}(v) f(v) dv \tag{2.12b}$$

$$= \int_c^1 (\frac{n-1}{n} v^n + \frac{c^n}{n}) n v^{n-1} dv \tag{2.12c}$$

if her objective is to maximize the highest bid. Since critical type is strictly increasing with the reserve price, the seller's problem is equivalent to choosing the optimal c that maximizes Π_c^a and Π_c^h. Using calculus, we can show that the optimal critical type that maximizes Π_c^a and Π_c^h are $c^{a*} = \frac{1}{2}$ and $c^{h*} = (\frac{1}{n+1})^{\frac{1}{n}}$, respectively.

Observe that $c^{h*} > \frac{1}{2}$, that it monotonically increases as the number of bidders increases and that it is equal to 1 in the limit. This is because the value of the highest bidder being above a given critical type increases as the number of bidders increases. In that case, the seller will be better off by posting a higher reserve price to induce aggressive bidding.

Proposition 28. *The symmetric equilibrium of an all-pay auction with reserve price, r, can be described as follows: The bidders follow $\beta(v) = \frac{n-1}{n} v^n + \frac{c^n}{n}$ if $v \geq c$ and zero otherwise, where $c = r^{1/n}$. The seller posts the reserve price such that only bidders whose valuations are above some critical type will participate. The critical type is $c^{a*} = \frac{1}{2}$ if her objective is to maximize the average bid and $c^{h*} = (\frac{1}{n+1})^{\frac{1}{n}}$ if her objective is to maximize the highest bid.*

In equilibrium, the seller's payoffs are equal to

$$\Pi_c^a(c^{a*}) = \frac{n-1}{n(n+1)} + \frac{1}{n(n+1)2^n} \tag{2.13a}$$

$$\Pi_c^h(c^{h*}) = \frac{n-1}{2n} + \frac{1}{2n(n+1)}. \tag{2.13b}$$

Since the seller chooses a critical type different from zero, when it is an available action. Thus, we conclude that posting a reserve price is beneficial for the seller.

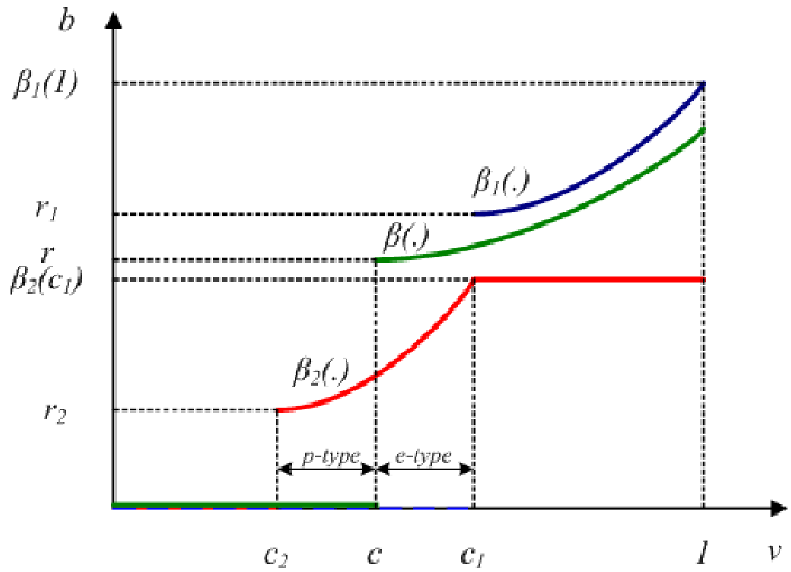

Figure 3: Equilibrium bidding strategies in the single-period all-pay auction and the sequential all-pay auction.

2.6 TEMPORARY LACK OF COMMITMENT

The previous section assumed that the seller is able to commit to the reserve price. This section, on the other hand, assumes that she is unable to do so for only one period. The timing of the game is as follows: The seller posts a reserve price, r_1, in period one. The bidders place their bids which are either zero or larger than the reserve price[9]. If at least one of the bids is positive, then the game ends. If all of the bids are zero then the seller posts a new reserve price, r_2 in the second period. The bids are placed and and the highest bidder wins. The game ends regardless of whether the bids in the second period are zero or positive. The seller and the bidders discount the payoffs earned in the second period at a rate of δ^s and δ^b, respectively. (δ^s, $\delta^b \in [0, 1]$)

We construct and analyze a symmetric perfect Bayesian equilibrium in monotonic strategies.

[9]Equivalently, one could have assumed that the seller interprets any positive bid smaller than the current reserve price as being submitted by a bidder with valuation smaller than the critical type.

Definition 29. *The perfect Bayesian equilibrium of the two period all-pay auction is defined as the set of strategies* $\{r_1, \beta_{1i}(.), r_2, \beta_{2i}(.)\}$ *and the belief* $\{\mu\}$*satisfying*

- $\forall v \in [0,1]$, β_{2i} *maximizes bidder i's continuation utility,* $u_{2i}(.,.)$*, for any history of reserve prices* (r_1, r_2), $i = 1, 2, ..., n$
- r_2 *maximizes the seller's continuation payoff,* Π^2_{nc}*, given her belief* μ *and the bidders' second period strategies,*
- $\forall v \in [0,1]$, β_{1i} *maximizes the bidders expected first period utility,* $u_{1i}(.,.)$*, given the second period strategies and* r_1, $i = 1, 2, ..., n$
- r_1 *maximizes the seller's expected first period payoff,* Π^1_{nc}*, given the bidders' and the seller's subsequent strategies,*
- μ *is Bayes-consistent with the bidders' first period strategies and observed actions.*[10]

2.6.1 Second Period Strategies

We begin with constructing the bidders' second period strategies. The seller announces second period's reserve price r_2, if no sale occurs in the first period after a reserve price of r_1. That is, the game reaches the second period if both bidders place bids of zero in the first period. So, bidder i moves at a history that is of the form $(r_1, (0, 0, .., 0), r_2)$, shortly (r_1, r_2).

In the second period, bidder i updates his belief about his opponents' values. Due to the symmetry in the equilibrium strategies, bidder i believes that his opponents have values smaller than the critical type of period 1, namely c_1. And his objective is to choose the optimal bid $b_2 \in \{0\} \cup [r_2, 1]$ that maximizes his continuation utility $u_2(b_{2i}, v) = v \Pr[b_{2i} > \beta_2(v_j)$ for $j \neq i] - b_{2i}$. This problem of bidder i is similar to bidders' problem of the previous section with the only difference that the opponents' values now being distributed uniformly between $[0, c_1]$. Thus, he places a positive bid only if his value is larger than some critical type c_2, and when he does so he will follow:

$$\beta_2(v) = vG^{n-1}(v) - \int_{c_2}^{v} G^{n-1}(t)dt \qquad \text{if } \ c_2 \leq v \leq c_1 \tag{2.14a}$$

$$= \frac{1}{nc_1^{n-1}}[(n-1)v^n + c_2^n] \qquad \text{if } \ c_2 \leq v \leq c_1 \tag{2.14b}$$

[10] The definition is analogous to that in Freixas, et al. [17]

where $G(v)$ is the uniform distribution over $[0, c_1]$.

Since a strategy is a complete contingent plan, it has to describe how the bidders will behave off the equilibrium path. That is, bidder i's strategy has to describe what to do in the second period when his valuation is larger than c_1[11]. In this case, he faces the following problem:

$$\max_{b_2} \quad vG^{n-1}(\beta_2^{-1}(b_2)) - b_2. \tag{2.15}$$

The first order condition to this problem is

$$v\frac{dG^{n-1}(\beta_2^{-1}(b_2))}{db_2} = 1. \tag{2.16}$$

Bidder i has no incentive to bid more than the highest bid that his opponents might place in the second period, namely $\beta_2(c_1)$, because he believes that his opponents' valuations are smaller than or equal to c_1.

He doesn't have incentive to bid lower than $\beta_2(c_1)$, either: Let's say that bidder i bids $b_2' < \beta_2(c_1)$. Since $\beta_2(.)$ is continuous, there is a valuation $v' < c_1$ for which b_2' is optimal, and hence is the solution to 2.16. Note that, the left hand side of equation 2.16 represents the gain due to slightly higher bid whereas the right hand side represents the loss. Since $c_1\frac{dG^{n-1}(\beta_2^{-1}(b_2'))}{db_2} > v'\frac{dG^{n-1}(\beta_2^{-1}(b_2'))}{db_2} = 1$, bidder i has incentive to bid higher. So, b_2' is not optimal. To conclude, in the second period, the bidders bid $\beta_2(c_1)$ for any value greater than c_2.

When his valuation is c_2, bidder i wins the object only if his opponents have valuation smaller than c_2. In that case, he is better off by placing the smallest possible bid, namely r_2. So, $\beta_2(c_2) = r_2$ or $c_2 = (c_1^{n-1}r_2)^{1/n}$.

Lemma 30. *In the continuation game that follows the history (r_1, r_2), each bidder uses the following strategy:*

$$\beta_2(v) = \begin{cases} 0 & 0 < v < c_2 \\ \frac{1}{nc_1^{n-1}}[(n-1)v^n + c_2^n] & c_2 \leq v \leq c1 \\ \frac{1}{nc_1^{n-1}}[(n-1)c_1^n + c_2^n] & c_1 < v \end{cases}$$

where $c_2 = (c_1^{n-1}r_2)^{1/n}$.

[11]This event occurs if bidder i accidentally bids zero in the first period and no sale occurs in that period.

In the second period, after having observed bidders' response of bids of zero to the first period's reserve price, r_1, and foreseeing the second period bidding strategies, the seller maximizes her continuation payoff by choosing an appropriate reserve price, r_2. Since second period's critical type is strictly increasing with the reserve price, she can equivalently choose the optimal c_2 that maximizes her payoff. More precisely, to maximize the average bid, the seller solves

$$\max_{c_2} \int_{c_2}^{c_1} \beta_2(v)g(v)dv, \text{ or} \tag{2.17a}$$

$$\max_{c_2} \frac{1}{nc_1^n}\int_{c_2}^{c_1} [(n-1)v^n + c_2^n]dv \tag{2.17b}$$

whereas she solves

$$\max_{c_2} \int_{c_2}^{c_1} \beta_2(v)nG^{n-1}(v)g(v)dv, \text{ or} \tag{2.18a}$$

$$\max_{c_2} \frac{1}{c_1^{2n-1}}\int_{c_2}^{c_1} [(n-1)v^{2n-1} + c_2^n v^{n-1}]dv \tag{2.18b}$$

in order to maximize the highest bid. Here, $G(v)$ represents the probability that an opponent's value is smaller than v which is distributed uniformly over $[0, c_1]$ and $g(v) = dG(v)/dv$ is the corresponding density function. Problems 2.17b and 2.18b are both uniquely maximized by $c_2^{a*} = \frac{c_1}{2}$ and $c_2^{h*} = (n+1)^{-1/n}c_1$, respectively.

Lemma 31. *Suppose that no sale takes place in the first period after a reserve price of r_1 and that the seller believes that the bidders' valuations are smaller than c_1. Then, to maximize the average bid (highest bid), she posts a reserve price such that only bidders with valuations larger than $c_2^{a*} = \frac{c_1}{2}$ ($c_2^{h*} = (n+1)^{-1/n}c_1$) participate.*

2.6.2 First Period Strategies

2.6.2.1 Average Bid In the first period, having observed the reserve price r_1, bidder i maximizes his payoff. This problem is similar to the one in the previous subsection. So, we can write his bid function as

$$\beta_1(v) = vF^{n-1}(v) - \int_{c_1}^{v} F^{n-1}(t)dt - u_1(\beta_1(c_1), c_1) \quad \text{for } c_1 \leq v \tag{2.19a}$$

$$= \frac{n-1}{n}v^n + \frac{c_1^n}{n} - u_1(\beta_1(c_1), c_1) \quad \text{for } c_1 \leq v. \tag{2.19b}$$

This bid function is analogous to equation 2.14a. Yet, the critical type, c_1, should comply with the following incentive compatibility condition: Bidder i does not have incentive to wait until the second period if his valuation is larger than c_1.

If a bidder with valuation $v > c_1$ bids in the first period he will earn $v^n - [\frac{n-1}{n}v^n + \frac{c_1^n}{n} - u_1(\beta(c_1), c_1)]$. If he waits, on the other hand, he will enjoy his valuation with probability one by placing the highest possible bid of the second period. So, he will earn a discounted payoff of $\delta^b[v-(n-1+\frac{1}{2^n})\frac{c_1}{n}]$. To satisfy the incentive compatibility condition, the difference between these two utilities

$$u_1(\beta_1(v), v) - \delta^b u_2(\beta_2(c_1), v) = \frac{v^n}{n} - \delta^b v - \frac{c_1^n}{n} + \delta^b(n-1+\frac{1}{2^n})\frac{c_1}{n} + u_1(\beta_1(c_1), c_1) \tag{2.20}$$

has to be at least zero for any valuation above c_1. One can easily see that the minimum of this expression is attained at $v = (\delta^b)^{1/n-1}$ if $(\delta^b)^{1/n-1} > c_1$ and at $v = c_1$ if $(\delta^b)^{1/n-1} \leq c_1$. Therefore, the bidder with critical type earns the following payoff:

$$u_1(\beta_1(c_1), c_1) = \begin{cases} \frac{2^n-1}{n2^n}\delta^b c_1 & \text{if } \ (\delta^b)^{1/n-1} \leq c_1 \\ \frac{n-1}{n}(\delta^b)^{n/n-1} + \frac{c_1^n}{n} - \frac{(n-1)2^n+1}{n2^n}\delta^b c_1 & \text{if } \ (\delta^b)^{1/n-1} > c_1 \end{cases}. \tag{2.21}$$

Substituting 2.21 into 2.19b, one can find bidder i's strategy: Place a positive bid only if the valuation is larger than c_1 and, if so, use the following bid function:

$$\beta_1(v) = \begin{cases} \frac{n-1}{n}v^n + \frac{c_1^n}{n} - \frac{2^n-1}{n2^n}\delta^b c_1 & \text{if } \ (\delta^b)^{1/n-1} \leq c_1 \\ \frac{n-1}{n}v^n - \frac{n-1}{n}(\delta^b)^{n/n-1} + \frac{(n-1)2^n+1}{n2^n}\delta^b c_1 & \text{if } \ (\delta^b)^{1/n-1} > c_1 \end{cases}. \tag{2.22}$$

When his valuation is c_1, bidder i wins the object only if his opponents have valuations smaller than c_1. In that case, he is better off by placing the smallest possible bid, namely r_1. So, $\beta_1(c_1) = r_1$. This equation has a unique positive solution in which the critical type, c_1, is monotonically increasing with the reserve price, r_1.

Lemma 32. *In an all-pay auction where the seller maximizes the average bid and cannot commit to the reserve price r_1 for only one period, the bidders use the following strategy in the first period: Place a positive bid of*

$$\beta_1(v) = \begin{cases} \frac{n-1}{n}v^n + \frac{c_1^n}{n} - \frac{2^n-1}{n2^n}\delta^b c_1 & \text{if} \quad (\delta^b)^{1/n-1} \leq c_1 \\ \frac{n-1}{n}v^n - \frac{n-1}{n}(\delta^b)^{n/n-1} + \frac{(n-1)2^n+1}{n2^n}\delta^b c_1 & \text{if} \quad (\delta^b)^{1/n-1} > c_1 \end{cases}$$

for any valuation $v \geq c_1$ and bid zero otherwise, where c_1 solves $\beta_1(c_1) = r_1$.

Finally, seller's payoff function can be written as:

$$\Pi_{nc}^{1a}(c_1) = \int_{c_1}^{1} \beta_1(v)f(v)dv + \delta^s F^{n-1}(c_1)\int_{c_2^{a*}}^{c_1} \beta_2(v)f(v)dv. \tag{2.23}$$

The first term represents the expected payoff from the first period and the second term represents the discounted expected payoff from the second period. $F^{n-1}(c_1)$ appears in the second term because a bidder places a positive bid, only if his opponents do not get the object in the first period, an event which happens with probability $F^{n-1}(c_1)$.

Seller chooses the optimal reserve price, or equivalently the optimal critical type, that maximizes 2.23, because the critical type of period one, c_1 is strictly increasing with r_1.

2.6.2.2 Highest Bid Bidder i's first period bid function is of the form of 2.19b. Since the seller chooses a different c_2 in the second period, the incentive compatibility condition needs to be modified:

If a bidder with valuation $v > c_1$ bids in the first period he earns $v^n - [\frac{n-1}{n}v^n + \frac{c_1^n}{n} - u_1(\beta(c_1), c_1)]$. But if he waits he will earn a discounted payoff of $\delta^b[v - \frac{n}{n+1}c_1]$. Again, to satisfy the incentive compatibility condition, the difference between these two utilities

$$u_1(\beta_1(v), v) - \delta^b u_2(\beta_2(c_1), v) = \frac{v^n}{n} - \delta^b v - \frac{c_1^n}{n} + \frac{n}{n+1}c_1\delta^b + u_1(\beta(c_1), c_1) \tag{2.24}$$

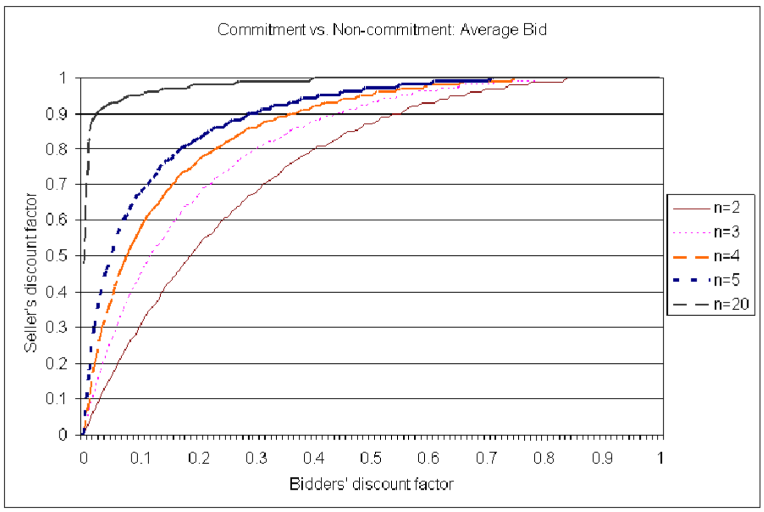

Figure 4: For any given number of bidders, if the state is such that the pair of discount factors falls above the corresponding line, then the seller prefers not to commit to the reserve price. Note that, the set of pairs for which non-commitment is beneficial to the seller *shrinks* as the number of bidders, n, increases.

has to be at least zero for any valuation above c_1. The minimum of this expression is attained at $v = (\delta^b)^{1/n-1}$ if $(\delta^b)^{1/n-1} > c_1$ and at $v = c_1$ if $(\delta^b)^{1/n-1} \le c_1$. Therefore, the bidder with critical type will earn the following payoff:

$$u_1(\beta_1(c_1), c_1) = \begin{cases} \frac{1}{n+1}\delta^b c_1 & if \quad (\delta^b)^{1/n-1} \le c_1 \\ \frac{n-1}{n}(\delta^b)^{n/n-1} + \frac{c_1^n}{n} - \frac{n}{n+1^n}\delta^b c_1 & if \quad (\delta^b)^{1/n-1} > c_1 \end{cases}. \tag{2.25}$$

Substituting 2.27 into 2.19b, one can find bidder i's strategy: Place a positive bid only if the valuation is larger than c_1 and if so use

$$\beta_1(v) = \begin{cases} \frac{n-1}{n}v^n + \frac{c_1^n}{n} - \frac{1}{n+1^n}\delta^b c_1 & if \quad (\delta^b)^{1/n-1} \le c_1 \\ \frac{n-1}{n}v^n - \frac{n-1}{n}(\delta^b)^{n/n-1} + \frac{n}{n+1^n}\delta^b c_1 & if \quad (\delta^b)^{1/n-1} > c_1 \end{cases}. \tag{2.26}$$

When his valuation is c_1, bidder i wins the object only if his opponent has valuation smaller than c_1. In that case, he is better off by placing the smallest possible bid, namely r_1. So, $\beta_1(c_1) = r_1$. This equation has a unique positive solution, in which the critical type is monotonically increasing with the reserve price.

Lemma 33. *In an all-pay auction where the seller maximizes the highest bid and cannot commit to the reserve price r_1 for only one period, the bidders use the following bidding strategy in the first period. Place a positive bid of*

$$\beta_1(v) = \begin{cases} \frac{n-1}{n}v^n + \frac{c_1^n}{n} - \frac{1}{n+1^n}\delta^b c_1 & if \quad (\delta^b)^{1/n-1} \le c_1 \\ \frac{n-1}{n}v^n - \frac{n-1}{n}(\delta^b)^{n/n-1} + \frac{n}{n+1^n}\delta^b c_1 & if \quad (\delta^b)^{1/n-1} > c_1 \end{cases}$$

for any valuation $v \ge c_1$ and bid zero otherwise, where c_1 is the critical type that solves $\beta_1(c_1) = r_1$

Finally, seller's payoff function can be written as follows:

$$\Pi_{nc}^{1h}(c_1) = \int_{c_1}^{1} \beta_1(v)nF^{n-1}(v)f(v)dv + \delta^s \int_{c_2^{h*}}^{c_1} \beta_2(v)nF^{n-1}(v)f(v)dv. \tag{2.27}$$

The first term represents the expected payoff from the first period and the second term represents the discounted expected payoff from the second period. Seller chooses the optimal c_1 that maximizes 2.27.

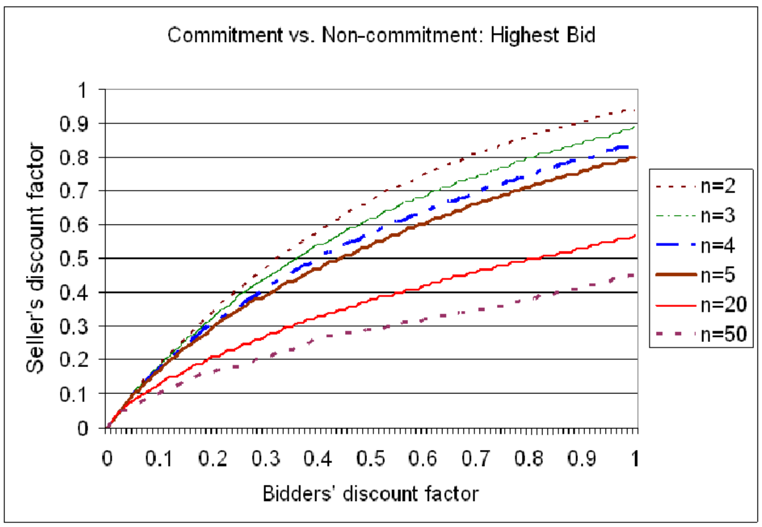

Figure 5: For any given number of bidders, if the state is such that the pair of discount factors falls above the corresponding line, then the seller prefers not to commit to the reserve price. Note that, the set of pairs for which non-commitment is beneficial to the seller *expands* as the number of bidders, n, increases.

2.6.3 Discussion of the Results

In the previous two subsections, we have characterized the seller's objective functions for the cases of maximizing the average bid [Function 2.23] and maximizing the highest bid [Function 2.27]. Unfortunately, neither problem has a closed form solution. Using numerical methods, we obtain the following results.

Proposition 34. *For any given number of buyers, the non-commitment equilibrium generates higher revenue than the commitment equilibrium if the seller is relatively more patient than the buyers. Moreover, as the number of buyers increases, the non-commitment equilibrium generates higher revenue for a smaller set of parameter values if the average bid is maximized. If, on the other hand, the highest bid is maximized, then the non-commitment equilibrium generates higher revenue for a larger set of parameter values.*

Corollary 35. *For any given number buyers, if the seller prefers to run a sequential all-pay auction rather than a single-period all-pay auction when she maximizes the average bid then she does the same when she maximizes the highest bid. Moreover, if the seller prefers to run a single-period all-pay auction when she maximizes the highest bid, then she also does so when she maximizes the average bid.*

We explain, in detail, the features of the model that derive these results and then discuss the implications of the model on the three real life situations that we have mentioned in the introduction.

A reserve price in a standard auction has a dual effect on the behavior of the buyers and, hence, on the revenue generated by the auction: On one side, it makes the high-type buyers bid more aggressively and, on the other side, it restrains the low-type buyers from participating. While the former has a positive effect on the seller's revenue, the latter has a negative effect. Thus, the optimality of a reserve price depends on whether it induces the right degree of competition among the right type of buyers.

The ability to rerun the auction decreases the cost of excluding the low-type buyers, which gives the seller an incentive to exclude more types and to induce a higher degree of competition in the first period among the participating buyers. Hence, the reserve price in the first period of the sequential auction will be higher than that of the single-period optimal auction. The reserve price in the second period, on the other hand, is lower than the optimal reserve price of the single-period auction. In conclusion, some types (*e-type*) that would participate in the optimal single-period auction are excluded in the first period of the sequential auction, yet some other types (*p-type*) that are excluded from the optimal single-period auction have the opportunity to participate in the second period of the optimal sequential auction. (See figure 3)

It turns out that if the seller is patient enough, regardless of whether she maximizes the average bid or the highest bid, the discounted benefit of having *p-type* buyers in the second period (together with the increased level of competition in the first period) compensates the loss due to excluding the *e-type* buyers from the first period. The graphs in figures 4 and 5 show the pairs of discount factors for which the above-mentioned benefit and loss balance out, for the cases of average bid and the highest bid, respectively. For any number of buyers, if the parameter vector lies above the corresponding line then the discounted gain from having *p-type* buyers in the second period is larger than the loss due to excluding the *e-type* buyers in the first period. This implies that when the seller is relatively more patient than the buyers, she prefers to run a sequential auction rather than a single-period auction, or in other words, she prefers not to commit to running a single-period auction.

Moreover, if the number of buyers, and hence the chance of observing a sale in the first period, increases, then the loss due to excluding the *e-type* buyers dominates the gain from the *p-type* buyers if the seller maximizes the average bid and she is not patient enough. Hence,only highly patient sellers prefer to run sequential all-pay auction. Conversely, a seller who maximizes the highest bid does not have to be as patient, in order to run a sequential auction, for, as more buyers participate, a sale in the first period is more likely and inducing a higher degree of competition in the first period with a higher reserve price is more profitable. In other words, as the number of buyers increases, it is costlier to screen the high types if average bid is maximized whereas it is less costly to screen the highest type if the highest bid is maximized.

When applied to the R&D tournaments, our results imply that the optimal tournament should resemble the sequentially optimal all-pay auction. That is, the designer should initially announce a relatively demanding list of minimum requirements, and if there is no participation in the first period, then she should revise the list and bring down the minimum requirements. This is because in a research and development tournament for an aircraft only few companies that have the required high technology participate and the tournament designer is generally more patient than the participants.

The promotion scheme in an army or a secret service is analogous to a labor market tournament where the minimum effort level is not publicly announced but is usually common knowledge. The winner(s) of the tournament is chosen from a large pool of relatively more patient employees. Although there are substantially large number of employees in the above mentioned institutions, whenever promotions are considered, many positions have to be filled as well. Therefore, the labor market tournaments in these institutions typically maximize the aggregate effort. Also, the organizer, in this case, is likely to be highly impatient, for the position has to be filled and there is no outside option. Therefore, a tournament scheme that mirrors the optimal single-period all-pay auction is more likely to be chosen.

Finally, quite many policymakers maximize the overall contributions and the aggregate level of information and are highly impatient with respect to time for obvious reasons. In this case, these policymakers are unlikely to set high reservation levels that is required in the first stage of the sequential process. Moreover, the number of lobbyists, the participants, is relatively high. Thus, a single-period all-pay scheme maximizes the policymakers payoff in this situation.

2.7 CONCLUSION

This paper analyzes how the seller's revenue is affected by her ability to commit to the reservation price in a class of contests that are isomorphic to all-pay auctions. It is shown that when seller's discount factor is higher relative to that of the bidders, that is when the seller is more patient than the buyers, then it is profitable for her to set a high reserve price in the first period and then lower it in the next period if no sale occurs. The result holds regardless of whether the seller's objective is to maximize the average bid or the highest bid.

This result is unexpected because in a bargaining model Sobel and Takahashi [41] showed that ability to commit is more profitable for the seller. The main reason for this contrary result is that when the seller sets a reserve price in an all-pay auction the types of bidders smaller than a cut-off withhold their bids. But the types larger than the cut-off bid more aggressively compared to the no reserve case. By not committing to the reserve price, seller utilizes the opportunity to trace the higher type the bidders. She can use this tool only when he is patient enough and when the number of bidders is small.

As a final word, we can compare "exclusion principle" with the result of this paper. Exclusion principle says the seller should exclude the highest type bidder from the auction to increase the revenue. This principle can work only when the seller has perfect information about the bidders valuations, which is generally not possible because sellers are naturally imperfectly informed about the types of the bidders. When the seller has incomplete information, to improve the revenue a seller can exclude the bidders with low valuations by imposing a reservation price. This paper, now, proposes a method to improve the revenue which can be used the when the seller is patient enough and when there aren't many bidders.

A question that is not addressed in this paper is whether the results of the paper hold when the seller is unable to commit for more than one period and when she is never able to commit. I conjecture that, the seller needs to be even more patient as the number of periods increases because the bidders are expected to behave less aggressively in response to the sellers inability to commit. To make them bid more aggressively seller is expected to impose a higher reservation price, but this in turn decreases the probability of observing a sale in the early periods. Thus, to increase the contribution of tomorrow's sale to the seller's discounted utility she has to be more patient.

3.0 A NOTE ON COLLUSION IN THE AUSUBEL AUCTION

Several mechanisms have been proposed to allocate multiple units of an object, like treasury bills or electromagnetic spectrum, among many buyers who potentially demand more than one unit. The uniform price auction, the discriminatory auction, the Vickrey auction are static mechanisms that have been implemented in real life and/or widely discussed in the literature. In these institutions, the bidders are asked to simultaneously report their demands as a function of price. The market clearing price is determined as the price at which the aggregate demand is equal to the number objects that are available. Each bidder wins the items, for which bidder's willingness to pay according to the reported demand function is larger than the market clearing price. In the uniform price auction, each bidder pays the market clearing price for each units he wins. In the discriminatory auction, on the other hand, each bidder pays his bid for each unit he wins. It has been shown theoretically, empirically, and experimentally that the first two mechanisms not necessarily yield an efficient outcome, in the sense the objects do not go into the hands of those who value them the most. Efficiency of the allocation mechanism is the primary objective of public authorities either for consideration of fairness or for the stability of the market after the auction.

Vickrey auction attains efficiency by making the bidders pay the externality they impose on other bidders. This payment mechanism gives bidders the incentive to bid truthfully by preventing them to possibly change the price they pay for the inframarginal units by not demanding the marginal units. The Ausubel auction [5] replicates the same outcome in a dynamic fashion: The price is announced, the demands are collected, and each buyer is clinched the units that are not claimed by his opponents. The price increases until all units are allocated and when the game ends the buyers pay for each unit the price at which they are clinched that particular unit. Sincere bidding is the unique outcome of the elimination of the weakly dominated strategies. Both the simplicity of its rules and its dynamic nature make the Ausubel auction a better choice as a mechanism to sell multiple homogeneous goods. Yet, these very features may allow sophisticated buyers to collude in the Ausubel auction. Provided that enough information is released, the buyers can detect deviations from the agreed collusive strategy. Moreover, the Ausubel auction also allows the buyers sustain collusion. In general, a buyer prefers to deviate if the gains from deviation is larger than the gains from sustaining the collusion. Most of the gains from deviation is earned at the period in which the buyer defects. Yet, in the Ausubel auction, given that the opponents follow the collusive strategy, a buyer will not be clinched more units at the time of the deviation and moreover by doing so the buyer will trigger sincere bidding in the remainder of the auction.

Theoretical literature on collusion in dynamic multi-object auctions is not rich. Recently, Brusco and Lopomo [12] studied the collusive equilibria of the simultaneous ascending bid auction. This mechanism allows each bidder to signal his interest in particular items and his intention to refrain from competiting for the other items provided that the others don't compete for the items he wants. In the collusive equilibrium, the bidders successfully divide the items among themselves and maintain low prices.

Below, we provide three examples in which collusion can be achieved and sustained in the Ausubel auction. In these examples, we assume that the price-clock runs continuously. The first two examples assume complete information and two non-divisible units and the final example assumes incomplete information and a single divisible unit.

3.1 AN EXAMPLE WITH COMPLETE INFORMATION AND SYMMETRIC BUYERS

Example 36. *Table 1 illustrates marginal valuations of the two bidders for the two units that are to be allocated. If both bidders bid sincerely then each will win one unit, pay the externality that he imposes on the other, namely 10, and earn a utility of 20-10=10.*

Yet, the following strategy also describes a symmetric equilibrium: Use the bid function described in table 2 as long as everyone does the same, otherwise bid sincerely. If both bidders

	Bidder 1	Bidder 2
Unit 1	20	20
Unit 2	10	10

Table 1: Bidder valuations (symmetric buyers with complete information)

Price	Demand
$p \leq 5$	2
$5 < p \leq 10$	1
$10 < p$	0

Table 2: Equilibrium bidding strategies

follow this new strategy, then they each will win one object at price 5(+ε) and earn a utility of 20-5=15.

To prove that this strategy is part of an equilibrium one has to show that the bidders have no profitable deviation. Before discussing possible deviations, it is important to note that by deviating a bidder cannot change the number of units he wins nor can he change the price he pays. Thus, there is no "immediate" advantage of deviation. Moreover, since deviation triggers sincere bidding, it creates "absolute" disadvantage, namely deviator has to pay the maximum possible price for the units he wins.

If a bidder deviates at a price smaller than 5 he will win one object and pay 10, where as he could have earn that single unit for 5. When price is 5(+ε) a bidder is clinched one unit since his opponent reduces his demand to one. Therefore, at any price above 5 a bidder will be willing to deviate only to win the second unit. But since deviation triggers sincere bidding, he can be clinched the second unit only when price reaches 20(+ε) which exceeds the amount that he is willing to pay for the second unit. Hence, the collusive strategy described above is an equilibrium.

3.2 AN EXAMPLE WITH ASYMMETRIC BUYERS AND COMPLETE INFORMATION

	Bidder 1	Bidder 2
Unit 1	20	20
Unit 2	10	5

Table 3: Bidder valuations (asymmetric buyers with complete information)

Example 37. *Let's modify example 36 by changing marginal valuations of bidder 2. If both bidders bid sincerely then bidder 1 will win both units*[1]*, pay the externality that he imposes on bidder 2, namely 10+5=15, and earn a utility of 30-15=15.*

There is an equilibrium in the Ausubel auction where each bidder wins one object, bidder 2 pays nothing and bidder 1 pays the price announced in the second stage (denote it by $p=\varepsilon$*). This equilibrium results if bidder1 reduces his demand to one unit at the starting price, and bidder 2 drops out after he understands the signal. Note that this collusive equilibrium Pareto dominates sincere bidding equilibrium, in the sense that both earn strictly higher payoffs. It is also important to note that signaling is not costly to bidder1, at all. In the case that bidder 2 misinterprets the signal he will drop out when price reaches 5, at which bidder 1 is clinched one unit and earns a payoff of 15, which is equal to payoff he could have earned had he bid sincerely.*

3.3 ANOTHER EXAMPLE: INCOMPLETE INFORMATION

Suppose that one unit of a divisible good is to be split between two bidders who have privately known constant marginal valuation u_i which is independently and identically drawn from the uniform distribution over [0,1][2]. Price clock runs continuously. Then, sincere bidding is

[1] Note, that at price 10 there will be a tie, and second unit needs to allocated according to a price breaking rule. The point in the example independent of the tie breaking rule.

[2] When the goods are perfectly divisible the number of objects to be sold can be normalized to one without loss of generality. Similarly, the upper bound of the support of the distribution of u_i can be any $\bar{u} \in \mathbb{R}$.

the unique outcome of iterated elimination of weakly dominated strategies in the Ausubel auction. Sincere bidding yields an efficient allocation and in equilibrium bidder i earns an expected surplus of

$$\begin{aligned} S(u_i) &= \Pr[u_i > u_j]\{u_i - E[u_j \mid u_i > u_j\} \\ &= u_i\{\{u_i - \int_0^{u_i} u_j \frac{1}{u_i} du_j\} \\ &= \frac{u_i^2}{2}. \end{aligned}$$

Below, I show that there is a continuum of collusive "separating" equilibria, in which at price p bidder i demands

$$x(p) = \begin{cases} 1 - bp & \text{if} \quad p \leq u_i \\ 0 & \text{if} \quad p > u_i \end{cases}$$

unless there was no deviation by any of the bidders until price reaches p. If any of the bidders deviates at p' then then bidder i demands sincerely, that is at any price $p > p'$

$$x(p) = \begin{cases} 1 - bp' & \text{if} \quad p \leq u_i \\ 0 & \text{if} \quad p > u_i \end{cases}.$$

Define $s(q) : [0, 1] \to [0, 1]$ to be the residual supply, such that $s^{-1}(p) = 1 - x(p)$

In this equilibrium, with probability one, the auction ends before price reaches $\frac{1}{2b}$. Let's show that bidder i has no incentive to deviate if $u_i > \frac{1}{2b}$. If he follows the equilibrium strategy, he will be clinched half of the units if $u_j > \frac{1}{2b}$ and his payment will be equal to the area under the residual supply, otherwise the game ends when $p = u_j$, in which case he will win $1 - 2bu_j$. at price u_j and bu_j will be clinched as price rises. Bidder i's expected surplus when he colludes is

$$\begin{aligned} S &= [1 - \frac{1}{2b}]\{\frac{u_i}{2} - \int_0^{1/2} s(q)dq\} \\ &\quad + \frac{1}{2b}\int_0^{\frac{1}{2b}} \{x(u_j)u_i - (1 - 2s^{-1}(u_j))u_j - \int_0^{s^{-1}(u_j)} s(q)dq\}2bdu_j \\ &= \frac{(4b+1)u_i - 1}{8b}. \end{aligned}$$

Now, let's suppose that bidder i deviates at price p' by demanding $x' \neq x(p')$ units. It is clear that $x' > x(p')$, because otherwise bidder i would forego $x(p') - x' > 0$ units which he could have earned with probability $\Pr[u_i > u_j]$. Until price reaches p', bidder i has already clinched $s^{-1}(p')$ units and a total surplus of $u_i s^{-1}(p') - \int_0^{s^{-1}(p')} s(q)dq$ has been realized. At price p', there are $1 - 2s^{-1}(p')$ units remaining unsold and bidder i wins all of them with probability $\Pr[u_j < u_i \mid u_j > p']$ at a price $E[u_j \mid p' < u_j < u_i]$. Thus, bidder i's expected surplus when he deviates is

$$S'' = u_i s^{-1}(p') - \int_0^{s^{-1}(p')} s(q)dq + .\frac{u_i - p'}{1 - p'}(1 - 2s^{-1}(p'))(u_i - \int_{p'}^{u_i} \frac{u_j}{u_i - p'} du_j).$$

Let's also calculate bidder i's collusive equilibrium surplus when his valuation $u_i < \frac{1}{2b}$. In this case, the auction ends, when price is reaches u_i or u_j. With probability $\Pr[u_i < u_j]$, bidder i will be the first to drop out, in which case he is clinched $s^{-1}(u_i)$ units through the mechanism. Otherwise, the opponent drops out first, then bidder i is clinched $s^{-1}(u_j)$ units through the mechanism and remaining units are clinched at price u_j. Thus,

$$\begin{aligned} S' &= [1 - u_i][u_i s^{-1}(u_i) - \int_0^{s^{-1}(u_i)} s(q)dq] \\ &\quad + u_i \int_0^{u_i} [u_i(1 - s^{-1}(u_j)) - u_j(1 - 2s^{-1}(u_j)) - \int_0^{s^{-1}(u_j)} s(q)dq] \frac{1}{u_i} du_j \end{aligned}$$

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APPENDIX A

OPTIMAL SINGLE OBJECT AUCTION

The Lagrangian to the relaxed problem can be written as

$$\begin{aligned}\mathcal{L} &= \pi - \lambda_L(D_L - 1) - \mu_H(D_H - D_H^L) \\ &\quad -\phi_{\{H,L\}}(n\alpha_L\rho_L + n\alpha_H\rho_H - 1) - \phi_{\{H\}}(n\alpha_H\rho_H - 1 + \alpha_L^n) \\ &\quad -\phi_{\{L\}}(n\alpha_L\rho_L - 1 + \alpha_H^n)\end{aligned}$$

where λ_L and μ_H are the Lagrange multipliers on IR_L and IC_H, respectively, and $\phi_{\{H,L\}}$, $\phi_{\{H\}}$, and $\phi_{\{L\}}$ are the multipliers on the implementability conditions.

Proof of Lemma 2. Suppose first that IR_L is slack. Then, the seller can improve her revenue by increasing y_L^l by $\varepsilon = \frac{1-D_L}{2} > 0$. This would not violate any of the constraints of the relaxed problem. So, IR_L must be binding.

Suppose, next, that IC_H is slack. Then, again, the mechanism can be improved profitably, without violating any of the conditions considered in the relaxed problem. Namely, increasing y_H^l by $\varepsilon = \frac{D_H^L - D_H}{2} > 0$ improves the revenue. Hence, IC_H is also binding. □

Proof of Lemma 3. Suppose, by contradiction, that IR_H is binding. Then, we have $1 = D_H = D_H^L = D_L$, where the equalities are due to IR_H, IC_H, and IR_L, respectively. Yet, since low-type buyers are not excluded, this would contradict with $D_L - D_H^L = \rho_L(c_L - c_H)y_L^w > 0$. Hence, IR_H is slack. □

Proof of Proposition 4. Kuhn-Tucker conditions with respect to y_H^w and y_H^l yield

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_H^w} &= \alpha_H \rho_H \frac{1}{y_H^w} - \mu_H \rho_H c_H = 0 \\ \frac{\partial \mathcal{L}}{\partial y_H^l} &= \alpha_H (1-\rho_H) \frac{1}{y_H^l} - \mu_H (1-\rho_H) = 0\end{aligned}$$

These equations together imply that $y_H^l = c_H y_H^w$. □

Proof of Proposition 5. Remember that IR_H is slack by Lemma 2. Using Proposition 1, we can rewrite this condition as

$$D_H = y_H^l < 1.$$

This is equivalent to $t_H^l < 0$, implying that, at the optimum, an high-type buyer is compensated when he loses the object. □

Proof of Proposition 7. Armed with the optimal values of ρ_H, and ρ_L,(see 1.9) we will now calculate the payments made by each type of buyer. Using IC_H, IR_L, and proposition 1, we write the payments, y_L^w, y_L^l, and y_L^w, as

$$y_L^w = \tfrac{1-y_H^l}{\rho_L(c_L-c_H)}, \quad y_L^l = \tfrac{c_L y_H^l - c_H}{(1-\rho_L)(c_L-c_H)}, \quad y_H^w = \tfrac{y_H^l}{c_H}$$

where y_H^l is in

$$\arg\max_{y_H^l}\{\frac{n}{r}[\alpha_H(\rho_H \ln \frac{1}{c_H} + \ln y_H^l) + \alpha_L(\rho_L \ln(1-y_H^l) + (1-\rho_L)\ln(c_L y_H^l - c_H))]\}.$$

Equivalently, y_H^l solves the first-order condition of the form

$$\frac{\alpha_H}{y_H^l} + \frac{\alpha_H(1-\rho_L)c_L}{c_L y_H^l - c_H} - \frac{\alpha_L \rho_L}{1-y_H^l} = 0.$$

This equation can be rewritten as

$$c_L (y_H^l)^2 - \xi y_H^l + \alpha_H c_H = 0 \tag{A.1}$$

where $\xi = (1-\rho_L)(c_L + \alpha_H c_H) + \rho_L(c_H + \alpha_H c_L)$.

Since $0 < \rho_L < 1$ and $c_H < c_L$, $\xi > (c_H + \alpha_H c_L)$ must be true. Then, $\xi^2 - 4\alpha_H c_L c_H > (c_H + \alpha_H c_L)^2 - 4\alpha_H c_L c_H = (c_H - \alpha_H c_L)^2 \geq 0$. Thus, a solution to equation A.1 exists. Furthermore, if a buyer of type H loses the object he pays

$$y_H^l = \frac{\xi + \sqrt{\xi^2 - 4\alpha_H c_L c_H}}{2c_L}.$$

□

Proof of Proposition 8. We have already established above that IR_L and IC_H are binding and IR_H is slack. We only need to show that IC_L is slack. Equivalently, we need to show that $\rho_L y_L^w < \rho_H y_H^w$.[1] Plugging in the values of y_L^w and y_H^w gives

$$\frac{1-y_H^l}{(c_L-c_H)} < \frac{\rho_H y_H^l}{c_H} \iff \frac{c_H}{\rho_H c_L + (1-\rho_H)c_H} < y_H^l.$$

We substitute in the value of y_H^l to get

$$c_L c_H + \alpha_H[\rho_H c_L + (1-\rho_H)c_H]^2 < \xi[\rho_H c_L + (1-\rho_H)c_H].$$

Substituting in the value of ξ and using $IM_{\{H,L\}}$ yields

$$0 < c_L^2 \rho_H(n-1) + c_H^2(1-\rho_H) + c_L c_H[(2-n)\rho_H - 1].$$

Now, we plug in the value of ρ_H and rewrite this condition as

$$0 < (1-\alpha_L^n)[c_L^2(n-1) - c_H^2 + c_L c_H(2-n)] + (1-\alpha_L)[c_H^2 n - c_L c_H n].$$

Since $c_H^2 n - c_L c_H n < 0$, we can replace $(1-\alpha_L)$ with $(1-\alpha_L^n)$ and get the following more restrictive condition

$$0 < (1-\alpha_L^n)(n-1)(c_L-c_H)^2,$$

which holds for any parameter values. Hence, IC_L must be slack. □

Proof of Proposition 9. Suppose that t_i^w and t_i^l [hence y_i^w and y_i^l] are stochastic. Replacing y_i^w and y_i^l with their expected values would not affect any of the incentive compatibility and individual rationality conditions (because buyers' utilities are linear with respect to these variables), but would strictly improve the seller's revenue (as revenue is concave with respect to y_i^w and y_i^l), which is a contradiction. □

[1] We add up IC_H (binding) and IC_L (slack).

APPENDIX B

OPTIMAL MULTI-OBJECT AUCTION

We can write the Lagrangian of the relaxed problem as

$$
\begin{aligned}
\mathcal{L} = \; & \alpha_{HH}\{\rho^A_{HH} \ln y^A_{HH} + \rho^B_{HH} \ln y^B_{HH} + \rho^{AB}_{HH} \ln y^{AB}_{HH} + \rho^O_{HH} \ln y^O_{HH}\} \\
& +\alpha_{HL}\{\rho^A_{HL} \ln y^A_{HL} + \rho^B_{HL} \ln y^B_{HL} + \rho^{AB}_{HL} \ln y^{AB}_{HL} + \rho^O_{HL} \ln y^O_{HL}\} \\
& +\alpha_{LH}\{\rho^A_{LH} \ln y^A_{LH} + \rho^B_{LH} \ln y^B_{LH} + \rho^{AB}_{LH} \ln y^{AB}_{LH} + \rho^O_{LH} \ln y^O_{LH}\} \\
& +\alpha_{LL}\{\rho^A_{LL} \ln y^A_{LL} + \rho^B_{LL} \ln y^B_{LL} + \rho^{AB}_{LL} \ln y^{AB}_{LL} + \rho^O_{LL} \ln y^O_{LL}\} \\
& +\lambda_{LL}\{1 - \rho^A_{LL} c^A_L y^A_{LL} - \rho^B_{LL} c^B_L y^B_{LL} - \rho^{AB}_{LL} c^A_L c^B_L y^{AB}_{LL} - \rho^O_{LL} y^O_{LL}\} \\
& +\lambda_{LH}\{c^A_L[\rho^A_{LL} y^A_{LL} - \rho^A_{LH} y^A_{LH}] + c^B_H[\rho^B_{LL} y^B_{LL} - \rho^B_{LH} y^B_{LH}] \\
& \quad + c^A_L c^B_H[\rho^{AB}_{LL} y^{AB}_{LL} - \rho^{AB}_{LH} y^{AB}_{LH}] + [\rho^O_{LL} y^O_{LL} - \rho^O_{LH} y^O_{LH}]\} \\
& +\lambda_{HL}\{c^A_H[\rho^A_{LL} y^A_{LL} - \rho^A_{HL} y^A_{HL}] + c^B_L[\rho^B_{LL} y^B_{LL} - \rho^B_{HL} y^B_{HL}] \\
& \quad + c^A_H c^B_L[\rho^{AB}_{LL} y^{AB}_{LL} - \rho^{AB}_{HL} y^{AB}_{HL}] + [\rho^O_{LL} y^O_{LL} - \rho^O_{HL} y^O_{HL}]\} \\
& +\lambda_{HH}(\mu_{LL}\{c^A_H[\rho^A_{LL} y^A_{LL} - \rho^A_{HH} y^A_{HH}] + c^B_H[\rho^B_{LL} y^B_{LL} - \rho^B_{HH} y^B_{HH}] \\
& \qquad + c^A_H c^B_H[\rho^{AB}_{LL} y^{AB}_{LL} - \rho^{AB}_{HH} y^{AB}_{HH}] + [\rho^O_{LL} y^O_{LL} - \rho^O_{HH} y^O_{HH}]\} \\
& \quad + \mu_{LH}\{c^A_H[\rho^A_{LH} y^A_{LH} - \rho^A_{HH} y^A_{HH}] + c^B_H[\rho^B_{LH} y^B_{LH} - \rho^B_{HH} y^B_{HH}] \\
& \qquad + c^A_H c^B_H[\rho^{AB}_{LH} y^{AB}_{LH} - \rho^{AB}_{HH} y^{AB}_{HH}] + [\rho^O_{LH} y^O_{LH} - \rho^O_{HH} y^O_{HH}]\} \\
& \quad + \mu_{HL}\{c^A_H[\rho^A_{HL} y^A_{HL} - \rho^A_{HH} y^A_{HH}] + c^B_H[\rho^B_{HL} y^B_{HL} - \rho^B_{HH} y^B_{HH}] \\
& \qquad + c^A_H c^B_H[\rho^{AB}_{HL} y^{AB}_{HL} - \rho^{AB}_{HH} y^{AB}_{HH}] + [\rho^O_{HL} y^O_{HL} - \rho^O_{HH} y^O_{HH}]\})
\end{aligned}
$$

Since the number of buyers participating in the auction are assumed to be larger than three and since buyers of each type are treated the same in a symmetric auction, each type's probability of losing both objects is positive. That is, $\rho^O_{ij} > 0$ for all $ij \in S$. Thus, using the four Kuhn-Tucker conditions, $\frac{\partial \mathcal{L}}{\partial y^O_{ij}} = 0$,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y^O_{HH}} &= \rho^O_{HH}[\frac{\alpha_{HH}}{y^O_{HH}} - \lambda_{HH}] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^O_{LH}} &= \rho^O_{LH}[\frac{\alpha_{LH}}{y^O_{LH}} + \lambda_{HH}\mu_{LH} - \lambda_{LH}] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^O_{HL}} &= \rho^O_{HL}[\frac{\alpha_{HL}}{y^O_{HL}} + \lambda_{HH}\mu_{HL} - \lambda_{HL}] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^O_{LL}} &= \rho^O_{LL}[\frac{\alpha_{LL}}{y^O_{LL}} - \lambda_{LL} + \lambda_{LH} + \lambda_{HL} + \lambda_{HH}\mu_{LL}] = 0
\end{aligned}$$

we can solve for λ_{ij}s:

$$\begin{aligned}
\lambda_{HH} &= \frac{\alpha_{HH}}{y^O_{HH}} \\
\lambda_{HL} &= \frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}\mu_{HL} \\
\lambda_{LH} &= \frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HH}}{y^O_{HH}}\mu_{LH} \\
\lambda_{LL} &= \frac{\alpha_{LL}}{y^O_{LL}} + \frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}.
\end{aligned}$$

The remaining Kuhn-Tucker conditions are of the following form

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y^A_{HH}} &= \rho^A_{HH}[\frac{\alpha_{HH}}{y^A_{HH}} - \lambda_{HH}c^A_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{HH}} &= \rho^B_{HH}[\frac{\alpha_{HH}}{y^B_{HH}} - \lambda_{HH}c^B_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{HH}} &= \rho^{AB}_{HH}[\frac{\alpha_{HH}}{y^{AB}_{HH}} - \lambda_{HH}c^A_H c^B_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^A_{HL}} &= \rho^A_{HL}[\frac{\alpha_{HL}}{y^A_{HL}} - (\lambda_{HL} - \lambda_{HH}\mu_{HL})c^A_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{HL}} &= \rho^B_{HL}[\frac{\alpha_{HL}}{y^B_{HL}} - (\lambda_{HL}c^B_L - \lambda_{HH}\mu_{HL}c^B_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{HL}} &= \rho^{AB}_{HL}[\frac{\alpha_{HL}}{y^{AB}_{HL}} - c^A_H(\lambda_{HL}c^B_L - \lambda_{HH}\mu_{HL}c^B_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^A_{LH}} &= \rho^A_{LH}[\frac{\alpha_{LH}}{y^A_{LH}} - (\lambda_{LH}c^A_L - \lambda_{HH}\mu_{LH}c^A_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{LH}} &= \rho^B_{LH}[\frac{\alpha_{LH}}{y^B_{LH}} - (\lambda_{LH} - \lambda_{HH}\mu_{LH})c^B_H] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{LH}} &= \rho^{AB}_{LH}[\frac{\alpha_{LH}}{y^{AB}_{LH}} - c^B_H(\lambda_{LH}c^A_L - \lambda_{HH}\mu_{LH}c^A_H)] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^A_{LL}} &= \rho^A_{LL}[\frac{\alpha_{LL}}{y^A_{LL}} - c^A_L(\lambda_{LL} - \lambda_{LH}) + c^A_H(\lambda_{HL} + \lambda_{HH}\mu_{LL})] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^B_{LL}} &= \rho^B_{LL}[\frac{\alpha_{LL}}{y^B_{LL}} - c^B_L(\lambda_{LL} - \lambda_{HL}) + c^B_H(\lambda_{LH} + \lambda_{HH}\mu_{LL})] = 0 \\
\frac{\partial \mathcal{L}}{\partial y^{AB}_{LL}} &= \rho^{AB}_{LL}[\frac{\alpha_{LL}}{y^{AB}_{LL}} - c^A_L(\lambda_{LL}c^B_L - \lambda_{LH}c^B_H) + c^A_H(\lambda_{HL}c^B_L + \lambda_{HH}\mu_{LL}c^B_H)] = 0.
\end{aligned}
$$

Proof of Lemma 12. Suppose that IR_{LL} is slack. Then, we have

$$
D_{LL} \equiv \rho^A_{LL}c^A_L y^A_{LL} + \rho^B_{LL}c^B_L y^B_{LL} + \rho^{AB}_{LL}c^A_L c^B_L y^{AB}_{LL} + \rho^O_{LL}y^O_{LL} < 1.
$$

Since number of buyers are larger than three and since buyers are treated symmetrically, each type's probability of losing both objects is positive. So, $\rho^O_{LL} > 0$. Thus, an increase in y^O_{LL} by ε/ρ^O_{LL} for $\varepsilon = (1 - D_{LL})/2 > 0$ strictly improves seller's payoff. Note that, this modification on y^O_{LL} does not violate any of the constraints, yielding a contradiction.

Hence, IR_{LL} must be binding. □

Proof of Lemma 13. Suppose first that IC_{LH}^{LL} is slack. Then, we have

$$\begin{aligned} D_{LH} &\equiv \rho_{LH}^{A} c_L^A y_{LH}^A + \rho_{LH}^B c_H^B y_{LH}^B + \rho_{LH}^{AB} c_L^A c_H^B y_{LH}^{AB} + \rho_{LH}^O y_{LH}^O \\ &< \rho_{LL}^A c_L^A y_{LL}^A + \rho_{LL}^B c_H^B y_{LL}^B + \rho_{LL}^{AB} c_L^A c_H^B y_{LL}^{AB} + \rho_{LL}^O y_{LL}^O \equiv D_{LH}^{LL} \end{aligned}$$

Let $\varepsilon = (D_{LH}^{LL} - D_{LH})/2$. Since $\rho_{LH}^O > 0$, if we increase y_{LH}^O by ε/ρ_{LH}^O, seller's payoff will improve and none of the constraints are violated. This is a contradiction. So, IC_{LH}^{LL} must be binding.

Along the same lines, we can easily show that IC_{HL}^{LL} is binding, too. □

Proof of Lemma 14. Suppose that all three conditions are slack. Then, we have $D_{HH} < \min\{D_{HH}^{LL}, D_{HH}^{LH}, D_{HH}^{HL}\}$. Define $\varepsilon = (\min\{D_{HH}^{LL}, D_{HH}^{LH}, D_{HH}^{HL}\} - D_{HH})/2$. An increase in y_{HH}^O in the amount of ε/ρ_{HH}^O, improves seller's payoff and does not violate any of the conditions. This is a contradiction. So, at least one of these three conditions must be binding. □

Proof of Proposition 15. Since $D_{HH} = \min\{D_{HH}^{LL}, D_{HH}^{LH}, D_{HH}^{HL}\}$, we can replace the last three incentive compatibility conditions with $D_{HH} = \mu_{LL} D_{HH}^{LL} + \mu_{LH} D_{HH}^{LH} + \mu_{HL} D_{HH}^{HL}$ where $\mu_{LL}, \mu_{LH}, \mu_{HL} \geq 0$ and $\mu_{LL} + \mu_{LH} + \mu_{HL} = 1$ provided that $\mu_{ij} = 0$ if and only if $D_{HH} < D_{HH}^{ij}$ (or equivalently, $\mu_{ij} > 0$ if and only if $D_{HH} = D_{HH}^{ij}$).

The Kuhn-Tucker conditions with respect to y_{ij}^k for $k = A, B, AB$ and $ij \in S$ can be written as

$$
\begin{aligned}
\rho^A_{HH}\alpha_{HH}[y^O_{HH} - y^A_{HH}c^A_H] &= 0 && \text{(a)}\\
\rho^B_{HH}\alpha_{HH}[y^O_{HH} - y^B_{HH}c^B_H] &= 0 && \text{(b)}\\
\rho^{AB}_{HH}\alpha_{HH}[y^O_{HH} - y^{AB}_{HH}c^A_H c^B_H] &= 0 && \text{(c)}\\
\rho^A_{HL}\alpha_{HL}[y^O_{HL} - y^A_{HL}c^A_H] &= 0 && \text{(d)}\\
\rho^B_{HL}[\frac{\alpha_{HL}}{y^B_{HL}} - \frac{\alpha_{HH}}{y^O_{HH}}\mu_{HL}(c^B_L - c^B_H) - \frac{\alpha_{HL}}{y^O_{HL}}c^B_L] &= 0 && \text{(e)}\\
\rho^{AB}_{HL}[\frac{\alpha_{HL}}{y^{AB}_{HL}} - c^A_H(\frac{\alpha_{HH}}{y^O_{HH}}\mu_{HL}(c^B_L - c^B_H) + \frac{\alpha_{HL}}{y^O_{HL}}c^B_L)] &= 0 && \text{(f)}\\
\rho^A_{LH}[\frac{\alpha_{LH}}{y^A_{LH}} - \frac{\alpha_{HH}}{y^O_{HH}}\mu_{LH}(c^A_L - c^A_H) - \frac{\alpha_{LH}}{y^O_{LH}}c^A_L] &= 0 && \text{(g)}\\
\rho^B_{LH}\alpha_{LH}[y^O_{LH} - y^B_{LH}c^B_H] &= 0 && \text{(h)}\\
\rho^{AB}_{LH}[\frac{\alpha_{LH}}{y^{AB}_{LH}} - c^B_H(\frac{\alpha_{HH}}{y^O_{HH}}\mu_{LH}(c^A_L - c^A_H) + \frac{\alpha_{LH}}{y^O_{LH}}c^A_L)] &= 0 && \text{(i)}\\
\rho^A_{LL}[\frac{\alpha_{LL}}{y^A_{LL}} - \frac{\alpha_{LL}}{y^O_{LL}}c^A_L - \{\frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}(\mu_{HL} + \mu_{LL})\}(c^A_L - c^A_H)] &= 0 && \text{(j)}\\
\rho^B_{LL}[\frac{\alpha_{LL}}{y^B_{LL}} - \frac{\alpha_{LL}}{y^O_{LL}}c^B_L - \{\frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HH}}{y^O_{HH}}(\mu_{LH} + \mu_{LL})\}(c^B_L - c^B_H)] &= 0 && \text{(k)}\\
\rho^{AB}_{LL}[\frac{\alpha_{LL}}{y^{AB}_{LL}} - \frac{\alpha_{LL}}{y^O_{LL}}c^A_L c^B_L - \frac{\alpha_{LH}}{y^O_{LH}}c^A_L(c^B_L - c^B_H) - \frac{\alpha_{HL}}{y^O_{HL}}c^B_L(c^A_L - c^A_H) &&\\
-\frac{\alpha_{HH}}{y^O_{HH}}(c^A_L c^B_L - \mu_{LH}c^A_L c^B_H - \mu_{HL}c^A_H c^B_L - \mu_{LL}c^A_H c^B_H)] &= 0. && \text{(l)}
\end{aligned}
$$

Note that, these equations are of the form $\rho^k_{ij}\Omega = 0$. We can use them to solve for y^k_{ij} for $ij \in S$ and $k = A, B, AB$, by implicitly assuming that $\rho^k_{ij} = 0$. This is without loss of generality, because each of these y^k_{ij}'s appears with the corresponding ρ^k_{ij} everywhere in the problem. Thus, if $\rho^k_{ij} = 0$ for a type ij and for an event k, then the value of y^k_{ij} will not matter in the solution, if $\rho^k_{ij} > 0$, on the other hand, then $\Omega = 0$ must be true.

Thus, equations (a)-(d) and (h) respectively yield

$$
\begin{aligned}
&y^A_{HH} = \tfrac{y^O_{HH}}{c^A_H}; \quad y^B_{HH} = \tfrac{y^O_{HH}}{c^B_H}; \quad y^{AB}_{HH} = \tfrac{y^O_{HH}}{c^A_H c^B_H};\\
&y^A_{HL} = \tfrac{y^O_{HL}}{c^A_H}; \quad y^B_{LH} = \tfrac{y^O_{LH}}{c^B_H};
\end{aligned}
$$

and the pairs '(e),(f)' and '(g),(i)' respectively give

$$
y^{AB}_{HL} = \tfrac{y^B_{HL}}{c^A_H}; \quad y^{AB}_{LH} = \tfrac{y^A_{LH}}{c^B_H}.
$$

These two sets of equations imply that the excess payment that a buyer makes for an object for which he has high valuation is equal to his valuation for that object. In other words, each buyer is perfectly insured against the risk of losing the object(s) for which he has high valuation. □

Proof of Proposition 16. Similarly, equations (e),(g),(j),(k) and (l) can be used to solve for $y^B_{HL}, y^A_{LH}, y^A_{LL}, y^B_{LL}$ and y^{AB}_{LL}, respectively.

$$\begin{aligned}
\frac{\alpha_{LH}}{y^A_{LH}} &= \frac{\alpha_{LH}}{y^O_{LH}}c^A_L + \frac{\alpha_{HH}}{y^O_{HH}}\mu_{LH}(c^A_L - c^A_H) \\
\frac{\alpha_{HL}}{y^B_{HL}} &= \frac{\alpha_{HL}}{y^O_{HL}}c^B_L + \frac{\alpha_{HH}}{y^O_{HH}}\mu_{HL}(c^B_L - c^B_H) \\
\frac{\alpha_{LL}}{y^A_{LL}} &= \frac{\alpha_{LL}}{y^O_{LL}}c^A_L + \frac{\alpha_{HL}}{y^O_{HL}}(c^A_L - c^A_H) + \frac{\alpha_{HH}}{y^O_{HH}}(\mu_{HL} + \mu_{LL})(c^A_L - c^A_H) \\
\frac{\alpha_{LL}}{y^B_{LL}} &= \frac{\alpha_{LL}}{y^O_{LL}}c^B_L + \frac{\alpha_{LH}}{y^O_{LH}}(c^B_L - c^B_H) + \frac{\alpha_{HH}}{y^O_{HH}}(\mu_{LH} + \mu_{LL})(c^B_L - c^B_H) \\
\frac{\alpha_{LL}}{y^{AB}_{LL}} &= \frac{\alpha_{LL}}{y^O_{LL}}c^A_L c^B_L + \frac{\alpha_{LH}}{y^O_{LH}}c^A_L(c^B_L - c^B_H) + \frac{\alpha_{HL}}{y^O_{HL}}c^B_L(c^A_L - c^A_H) \\
&+ \frac{\alpha_{HH}}{y^O_{HH}}(c^A_L c^B_L - \mu_{LH}c^A_L c^B_H - \mu_{HL}c^A_H c^B_L - \mu_{LL}c^A_H c^B_H)
\end{aligned}$$

Remember from first section that a low-type buyer has to make a payment if he cannot win the object. Using the last three of the above equations we get a similar result for type LL.

Using the last three equations, one can write

$$y^A_{LL} = \frac{y^O_{LL}}{c^A_L + \varepsilon_1}; \quad y^B_{LL} = \frac{y^O_{LL}}{c^B_L + \varepsilon_2}; \quad y^{AB}_{LL} = \frac{y^O_{LL}}{c^A_L c^B_L + \varepsilon_3}$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. We plug these values into LL's individual rationality constraint to get

$$y^O_{LL}(1 - \rho^A_{LL}\frac{\varepsilon_1}{c^A_L + \varepsilon_1} - \rho^B_{LL}\frac{\varepsilon_2}{c^B_L + \varepsilon_2} - \rho^{AB}_{LL}\frac{\varepsilon_3}{c^A_L c^B_L + \varepsilon_3}) = 1.$$

Note that, the term in the parenthesis is less than one if LL gets either or both objects. Thus, if $\rho^O_{LL} \neq 1$, then $y^O_{LL} > 1$ (hence, $t^O_{LL} > 0$) must be true. □

Proof of Proposition 17. *i*) Let η be such that $n_{HH}+n_{HL}>0$ and without loss of generality assume that $n_{HH}>0$. Now, suppose by contradiction, that $n_{HH}\hat{p}^A_{HH}(\eta)+n_{HL}\hat{p}^A_{HL}(\eta)<1$. Let $\varepsilon \leq 1-n_{HH}\hat{p}^A_{HH}(\eta)-n_{HL}\hat{p}^A_{HL}(\eta)$.

There are three possibilities that we need to consider:

- $n_{LH}+n_{LL}=0$:

In this case, modify the mechanism by increasing $p^A_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$. This would increase $\hat{\rho}^A_{HH}$ by $\Psi\frac{\varepsilon}{\alpha_{HH}}$. Change in the Lagrangian can be calculated as $\Psi\varepsilon\ln\frac{1}{c_H}>0$. This is a contradiction.

- $n_{LH}\hat{p}^A_{LH}(\eta)>0$:

We will now show that for some $\varepsilon<n_{LH}\hat{p}^A_{LH}(\eta)$, decreasing $\hat{p}^A_{LH}(\eta)$ by $\frac{\varepsilon}{n_{LH}}$, and increasing $\hat{p}^A_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$ is profitable. After this modification, $\hat{\rho}^A_{LH}$ decreases by $\Psi\frac{\varepsilon}{\alpha_{LH}}$ and $\hat{\rho}^A_{HH}$ increases by $\Psi\frac{\varepsilon}{\alpha_{HH}}$.[1] We calculate the change in the Lagrangian as

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{1}{c^A_H}-\ln\frac{y^A_{LH}}{y^O_{LH}}+\lambda_{LH}[c^A_L\frac{y^A_{LH}}{\alpha_{LH}}-\frac{y^O_{LH}}{\alpha_{LH}}]-\lambda_{HH}\mu_{LH}[c^A_H\frac{y^A_{LH}}{\alpha_{LH}}-\frac{y^O_{LH}}{\alpha_{LH}}]\}\\ &= \Psi\varepsilon\ln\frac{y^O_{LH}}{c^A_H y^A_{LH}}\end{aligned}$$

which is positive since $y^O_{LH}>c_H y^A_{LH}$.

- $n_{LL}\hat{p}^A_{LL}(\eta)>0$ and $n_{LH}\hat{p}^A_{LH}(\eta)=0$:

Suppose first that $n_{LL}p^A_{LL}(\eta)>0$. Then consider modifying the mechanism by decreasing $p^A_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and increasing $p^A_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$ for some $\varepsilon<n_{LL}p^A_{LL}(\eta)$. This would decrease ρ^A_{LL} by $\Psi\frac{\varepsilon}{\alpha_{LL}}$ and increase $\hat{\rho}^A_{HH}$ by $\Psi\frac{\varepsilon}{\alpha_{HH}}$. Lagrangian then changes by

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{1}{c^A_H}-\ln\frac{y^A_{LL}}{y^O_{LL}}+(\lambda_{LL}-\lambda_{LH})[\frac{c^A_L y^A_{LL}}{\alpha_{LL}}-\frac{y^O_{LL}}{\alpha_{LL}}]-(\lambda_{HL}+\lambda_{HH}\mu_{LL})[\frac{c^A_H y^A_{LL}}{\alpha_{LL}}-\frac{y^O_{LL}}{\alpha_{LL}}]\}\\ &= \Psi\varepsilon\ln\frac{y^O_{LL}}{c^A_H y^A_{LL}}>0\end{aligned}$$

Suppose now that $n_{LL}p^A_{LL}(\eta)=0$. Then, $n_{LL}p^{AB}_{LL}(\eta)>0$ must be true. We will show that the following modification is profitable: For some $\varepsilon<n_{LL}p^{AB}_{LL}(\eta)$, decrease $p^{AB}_{LL}(\eta)$ by

[1] $\hat{p}^A_{LH}(\eta)$ can be decreased either by decreasing $p^A_{LH}(\eta)$ or $p^{AB}_{LH}(\eta)$. If the former, is positive then we decrease $p^A_{LH}(\eta)$ (and increase $p^A_{HH}(\eta)$). If the former is zero, however, $p^{AB}_{LH}(\eta)$ should be decreased (and in response $p^{AB}_{HH}(\eta)$ should be increased) In this case, marginal probabilities of winning A and B are affected for both types HH and LH. Yet, either modification, have the same effect on the Lagrangian.

$\frac{\varepsilon}{n_{LL}}$ and increasing $p^{AB}_{HH}(\eta)$ by $\frac{\varepsilon}{n_{HH}}$. This would decrease ρ^{AB}_{LL} by $\Psi\frac{\varepsilon}{\alpha_{LL}}$ and increase $\hat{\rho}^{A}_{HH}$ and $\hat{\rho}^{B}_{HH}$ by $\Psi\frac{\varepsilon}{\alpha_{HH}}$. As a result, Lagrangian will increase by

$$\begin{aligned}
\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{1}{c^A_H}+\ln\frac{1}{c^B_H}-\ln\frac{y^{AB}_{LL}}{y^O_{LL}}+\lambda_{LL}[\frac{c^A_L c^B_L y^{AB}_{LL}}{\alpha_{LL}}-\frac{y^O_{LL}}{\alpha_{LL}}] \\
&-\lambda_{LH}[\frac{c^A_L c^B_H y^{AB}_{LL}}{\alpha_{LL}}-\frac{y^O_{LL}}{\alpha_{LL}}]-\lambda_{LH}[\frac{c^A_H c^B_L y^{AB}_{LL}}{\alpha_{LL}}-\frac{y^O_{LL}}{\alpha_{LL}}] \\
&-\lambda_{HH}\mu_{LL}[\frac{c^A_H c^B_H y^{AB}_{LL}}{\alpha_{LL}}-\frac{y^O_{LL}}{\alpha_{LL}}]\} \\
&= \Psi\varepsilon\ln\frac{y^O_{LL}}{c^A_H c^B_H y^{AB}_{LL}}>0
\end{aligned}$$

Thus, we conclude that if η is such that $n_{HH}+n_{HL}>0$, then $n_{HH}\hat{p}^A_{HH}(\eta)+n_{HL}\hat{p}^A_{HL}(\eta)=1$.

We can prove part ii) of the Lemma along the same lines. $\square$

Proof of Corollary 18. We will prove only part i). Proof of part ii) is similar. (*5) implies that

$$\begin{aligned}
\alpha_{HH}\hat{\rho}^A_{HH} &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} n_{HH}\hat{p}^A_{HH}(\eta)\Psi \\
\alpha_{HL}\hat{\rho}^A_{HL} &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} n_{HL}\hat{p}^A_{HL}(\eta)\Psi.
\end{aligned}$$

Adding these two equalities and multiplying both sides with n gives

$$\begin{aligned}
n[\alpha_{HH}\hat{\rho}^A_{HH}+\alpha_{HL}\hat{\rho}^A_{HL}] &= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}}[n_{HH}\hat{p}^A_{HH}(\eta)+n_{HL}\hat{p}^A_{HL}(\eta)]n\Psi \\
&= \sum_{n_{HH}=0}^{n}\sum_{n_{HL}=0}^{n-n_{HH}}\sum_{n_{LH}=0}^{n-n_{HH}-n_{HL}} n\Psi-\sum_{n_{LH}=0}^{n}\frac{n!\alpha_{LH}^{n_{LH}}\alpha_{LL}^{n-n_{LH}}}{n_{LH}!(n-n_{LH})!} \\
&= 1-(\alpha_{LH}+\alpha_{LL})^n.
\end{aligned}$$

The second equality follows from the part i of proposition 9. $\square$

Proof of Proposition 19. *i*) Suppose that the profile is such that $n_{HH} + n_{HL} = 0$, but $n_{LH}\hat{p}^A_{LH}(\eta) + n_{LL}\hat{p}^A_{LL}(\eta) < 1$. Let $\varepsilon < 1 - n_{LH}\hat{p}^A_{LH}(\eta) - n_{LL}\hat{p}^A_{LL}(\eta)$. There are two cases that we need to consider:

- $n_{LH} > 0$: Let's increase $\hat{p}^A_{LH}(\eta)$ by $\frac{\varepsilon}{n_{LH}}$, which would increase $\hat{\rho}^A_{LH}$ by $\Psi\frac{\varepsilon}{\alpha_{LH}}$. Change in the Lagrangian is calculated as

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{y^A_{LH}}{y^O_{LH}} + \lambda_{LH}[-c^A_L\frac{y^A_{LH}}{\alpha_{LH}} + \frac{y^O_{LH}}{\alpha_{LH}}] + \lambda_{HH}\mu_{LH}[c^A_H\frac{y^A_{LH}}{\alpha_{LH}} - \frac{y^O_{LH}}{\alpha_{LH}}]\} \\ &= \Psi\varepsilon\ln\frac{y^A_{LH}}{y^O_{LH}}\end{aligned}$$

which is positive if $y^A_{LH} > y^O_{LH}$, or $\frac{c^A_L - c^A_H}{1-c^A_L} < \frac{\alpha_{LH}}{y^O_{LH}}\frac{y^O_{HH}}{\alpha_{HH}}\frac{1}{\mu_{LH}}$.

- $n_{LH} = 0$: A profitable modification would be to increase $p^A_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and hence ρ^A_{LL} by $\Psi\frac{\varepsilon}{\alpha_{LL}}$. Lagrangian will increase by

$$\begin{aligned}\Delta\mathcal{L} &= \Psi\varepsilon\{\ln\frac{y^A_{LL}}{y^O_{LL}} - (\lambda_{LL} - \lambda_{LH})[\frac{c^A_L y^A_{LL}}{\alpha_{LL}} - \frac{y^O_{LL}}{\alpha_{LL}}] + (\lambda_{HL} + \lambda_{HH}\mu_{LL})[\frac{c^A_H y^A_{LL}}{\alpha_{LL}} - \frac{y^O_{LL}}{\alpha_{LL}}]\} \\ &= \Psi\varepsilon\ln\frac{y^A_{LL}}{y^O_{LL}}\end{aligned}$$

which is positive if $y^A_{LL} > y^O_{LL}$, or $\frac{c^A_L - c^A_H}{1-c^A_L} < \frac{\alpha_{LL}}{y^O_{LL}}(\frac{\alpha_{HL}}{y^O_{HL}} + \frac{\alpha_{HH}}{y^O_{HH}}(1-\mu_{LH}))^{-1}$.

ii) Along the same lines of the previous part, we can easily show that this part holds, too, if $y^B_{HL} > y^O_{HL}$ and $y^B_{LL} > y^O_{LL}$, or equivalently if

$$\frac{c^B_L - c^B_H}{1-c^B_L} < \min\{\frac{\alpha_{HL}}{y^O_{HL}}\frac{y^O_{HH}}{\alpha_{HH}}\frac{1}{\mu_{HL}}, \frac{\alpha_{LL}}{y^O_{LL}}(\frac{\alpha_{LH}}{y^O_{LH}} + \frac{\alpha_{HH}}{y^O_{HH}}(1-\mu_{HL}))^{-1}\}.$$

□

Proof of Proposition 20. Suppose, for now, that HH is not compensated. Then $y^O_{HH} = 1$. Since $c^A_H < c^A_L$ and $c^B_H < c^B_L$, we have $1 = y^O_{HH} \leq D^{ij}_{HH} \leq D_{ij} \leq 1$ for $ij = LL, LH, HL$ where the first inequality is due to IC^{ij}_{HH}, and the last inequality is the individual rationality constraint. So, all individual rationality constraints are binding and $D_{ij} = D^{ij}_{HH} = 1$ for $ij = LL, LH, HL$. Moreover, since $D_{ij} - D^{ij}_{HH} = 0$, we have

$$\begin{aligned}\rho^A_{LL}(c^A_L - c^A_H)y^A_{LL} + \rho^B_{LL}(c^B_L - c^B_H)y^B_{LL} + \rho^{AB}_{LL}(c^A_L c^B_L - c^A_H c^B_H)y^{AB}_{LL} &= 0 \\ \hat{\rho}^A_{LH}(c^A_L - c^A_H)y^A_{LH} &= 0 \\ \hat{\rho}^B_{HL}(c^B_L - c^B_H)y^B_{HL} &= 0\end{aligned}$$

Each term in these equations are nonnegative, therefore $\rho^A_{LL} = \rho^B_{LL} = \rho^{AB}_{LL} = \hat{\rho}^A_{LH} = \hat{\rho}^B_{HL} = 0$ must be true. This contradicts with the previous Corollary because $\alpha_{LL}\hat{\rho}^A_{LL} + \alpha_{LH}\hat{\rho}^A_{LH} > 0$. □

Proof of Proposition 21. Suppose, by contradiction, that for some profile η with $n_{LL} = n$, $p^{AB}_{LL}(\eta) < \frac{1}{n}$. Since both objects are sold with probability one, this implies that $p^A_{LL}(\eta) = p^B_{LL}(\eta) > 0$. Let $\varepsilon < 1 - np^{AB}_{LL}(\eta)$. Consider modifying the mechanism by decreasing $p^A_{LL}(\eta)$ and $p^B_{LL}(\eta)$ both by $\frac{\varepsilon}{n}$ and increasing $p^{AB}_{LL}(\eta)$ by $\frac{\varepsilon}{n}$. This would imply $\Delta\rho^{AB}_{LL} = -\Delta\rho^A_{LL} = -\Delta\rho^B_{LL} = \Psi\frac{\varepsilon}{\alpha_{LL}}$. Now, we calculate the change in the Lagrangian:

$$\Delta\mathcal{L} = \Psi\varepsilon \ln \frac{y^O_{LL} y^{AB}_{LL}}{y^A_{LL} y^B_{LL}}$$

which is positive if $y^O_{LL} y^{AB}_{LL} > y^A_{LL} y^B_{LL}$ or, equivalently, if

$$\begin{aligned} & \frac{\alpha_{LL}}{y^A_{LL}}\frac{\alpha_{LL}}{y^B_{LL}} > \frac{\alpha_{LL}}{y^O_{LL}}\frac{\alpha_{LL}}{y^{AB}_{LL}} \\ \Longleftrightarrow \quad & (\lambda_{LH}\lambda_{HL} + \lambda_{LL}\lambda_{HH}\mu_{LL})(c^A_L - c^A_H)(c^B_L - c^B_H) > 0. \end{aligned}$$

Since the last inequality holds for any parameter values, this modification is profitable. Thus, we conclude that if all the buyers are of type LL then the objects are bundled and each buyer gets the bundle with equal probability. □

Proof of Proposition 22. Any of the three auction formats, namely independent auction, bundling auction and mixed auction, that are optimal when the buyers are risk neutral allocate the objects independently and randomly when all buyers report to be of type LL.

Yet, by proposition 21, when the buyers are risk averse, a necessary condition for the optimality of the auction is to give both object to the same buyer if all buyers are of type LL. □

Proof of Lemma 23. *i*) Suppose that for some η with $n_{LH}, n_{LL} > 0$ and $n_{LH} + n_{LL} = n$, $n_{LH}\hat{p}^A_{LH}(\eta) < 1$. Then, since A is sold with probability one, $p^A_{LL}(\eta)$ must be positive. Let $\varepsilon < n_{LL}p^A_{LL}(\eta)$. Now, consider modifying the mechanism by decreasing $p^A_{LL}(\eta)$ by $\frac{\varepsilon}{n_{LL}}$ and increasing $\hat{p}^A_{LH}(\eta)$ by $\frac{\varepsilon}{n_{LH}}$. This, would decrease $\hat{\rho}^A_{LL}$ by $\frac{\Psi\varepsilon}{\alpha_{LL}}$ and increase $\hat{\rho}^A_{LH}$ by $\frac{\Psi\varepsilon}{\alpha_{LH}}$. As a result, the Lagrangian will change by

$$\Delta\mathcal{L} = \Psi\varepsilon \ln \frac{y^A_{LH} y^O_{LL}}{y^O_{LH} y^A_{LL}}.$$

This is positive if $y^A_{LH} y^O_{LL} > y^O_{LH} y^A_{LL}$, or equivalently if $\frac{\alpha_{LH}}{y^O_{LH}} \frac{\alpha_{LL}}{y^A_{LL}} > \frac{\alpha_{LH}}{y^A_{LH}} \frac{\alpha_{LL}}{y^O_{LL}}$. Using the Kuhn-Tucker conditions, we can rewrite this inequality as

$$\begin{aligned} (\lambda_{LH} - \lambda_{HH}\mu_{LH})[c^A_L(\lambda_{LL} - \lambda_{LH}) - c^A_H(\lambda_{HL} + \lambda_{HH}\mu_{LL})] &> \\ (c^A_L \lambda_{LH} - c^A_H \lambda_{HH}\mu_{LH})(\lambda_{LL} - \lambda_{LH} - \lambda_{HL} - \lambda_{HH}\mu_{LL}). \end{aligned}$$

After some manipulation, we get

$$\begin{aligned} \lambda_{LH}(\lambda_{HL} + \lambda_{HH}\mu_{LL}) &> \lambda_{HH}\mu_{LH}(\lambda_{LL} - \lambda_{LH}) \\ (\frac{\alpha_{HL}}{y^O_{HL}} \frac{y^O_{HH}}{\alpha_{HH}} + 1)(\frac{\alpha_{LL}}{y^O_{LL}} \frac{y^O_{LH}}{\alpha_{LH}} + 1)^{-1} &> \mu_{LH}. \end{aligned}$$

Proof of part *ii*) is similar. □