THE FACT OF MODERN MATHEMATICS: GEOMETRY, LOGIC, AND CONCEPT FORMATION IN KANT AND CASSIRER

by

Jeremy Heis

B.A., Michigan State University, 1999

Submitted to the Graduate Faculty of

Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2007

UNIVERSITY OF PITTSBURGH

COLLEGE OF ARTS AND SCIENCES

This dissertation was presented

by

Jeremy Heis

It was defended on

September 5, 2007

and approved by

Jeremy Avigad, Associate Professor, Philosophy, Carnegie Mellon University

Stephen Engstrom, Associate Professor, Philosophy, University of Pittsburgh

Anil Gupta, Distinguished Professor, Philosophy, University of Pittsburgh

Kenneth Manders, Associate Professor, Philosophy, University of Pittsburgh

Thomas Ricketts, Professor, Philosophy, University of Pittsburgh

Dissertation Advisor: Mark Wilson, Professor, Philosophy, University of Pittsburgh

THE FACT OF MODERN MATHEMATICS: GEOMETRY, LOGIC, AND CONCEPT FORMATION IN KANT AND CASSIRER

Jeremy Heis, PhD

University of Pittsburgh, 2007

It is now commonly accepted that any adequate history of late nineteenth and early twentieth century philosophy—and thus of the origins of analytic philosophy—must take seriously the role of Neo-Kantianism and Kant interpretation in the period. This dissertation is a contribution to our understanding of this interesting but poorly understood stage in the history of philosophy.

Kant's theory of the concepts, postulates, and proofs of geometry was informed by philosophical reflection on diagram-based geometry in the Greek synthetic tradition. However, even before the widespread acceptance of non-Euclidean geometry, the projective revolution in nineteenth century geometry eliminated diagrams from proofs and introduced "ideal" elements that could not be given a straightforward interpretation in empirical space. A Kantian like the very early Russell felt forced to regard the ideal elements as convenient fictions. The Marburg Neo-Kantians—the philosophical school that included Ernst Cassirer (1874-1945)—thought that philosophy, as "transcendental logic," needed to take the results of established pure mathematics as a "fact," not a fiction. Cassirer therefore updates Kant by rejecting the "Transcendental Aesthetic" and by using elements in Richard Dedekind's foundations of arithmetic to rework Kant's idea that the geometrical method is the "construction of concepts." He further argues that geometry is "synthetic" because it progresses when mathematicians introduce new structures

(like the complex projective plane) that are not contained in the old structures, but unify them under a new point-of-view.

This new "Kantian" theory of modern mathematics, Cassirer argues, is inconsistent with the traditional theory of concept formation by abstraction. Drawing on earlier Neo-Kantian interpretations, Cassirer argues that Kant's theory of concepts as rules undermines the traditional theory of concept formation, and he gives a "transcendental" defense of the new logic of Frege and Russell. (In an appendix, I discuss the contemporaneous accounts of concept formation in Gottlob Frege and Hermann Lotze.)

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

It is still amazing to me that I ever became interested in the philosophy of mathematics, Kant's philosophy of geometry, or the Neo-Kantians. Once the interest was there, there were times when I had a hard time believing that the dissertation would ever be completed—and on such happy terms. For these things I have many people to thank.

My chronologically first debt is to John Rauk (Classics, Michigan State) for pushing me to develop the discipline and modesty that successful academic research requires. Ken Manders was the first to introduce me to the philosophy of mathematics, when he, having piqued my interest in his seminar in mathematical logic, generously provided me with a wide menu of historical readings and hours of discussion at his house. His seminar sessions on Euclidean and projective geometry first me got me interested in Kant's philosophy of geometry, projective unification, and the extension elements. Jeremy Avigad has consistently encouraged me with his enthusiasm, corrected some errors, and has forced me at various points to own up to the inadequacies of the Neo-Kantian theories that I am too inclined to love blindly. Anil Gupta generously met with me to discuss space and perception—for a chapter that, though still unwritten, will I hope someday reflect some of his philosophical character. Much of the material in chapter 4 grew out of a long and wide-ranging discussion I had with Stephen Engstrom about mathematical and philosophical methodology. His comprehensive knowledge of the sentences of the *Critique* and his feel for Kant's thinking also saved me from some large blunders in chapter 1. A few well-timed conversations with Tom Ricketts saved this project (and my state of mind) from derailment. I think that the impetus for the investigations in chapter 5 was Tom's insistence that we try to see with unprejudiced eyes what logicians before Tarksi thought of what they were doing. To my advisor, Mark Wilson, my family and I owe a debt that we cannot repay. Mark saw well before I did that there was a viable project in the neighborhood of my wanderings, and he trusted me to go explore it. Without his encouragement, and, especially, without his encyclopedic knowledge of the history and philosophy of mathematics, I would not even have been able to conceive of a project like this one. (Indeed, much of my research consisted in working through his personal library.) Finally, I want to recognize that every member of my committee has located obscurities and confusions in this work that I have, to my regret, not been able to rectify. But I hope that—warts and all—it lives up in some measure to the example they have provided.

Numerous individuals have read through portions of this dissertation; conversations with others helped me to improve it. Many thanks to Alp Aker, Andrew Arana, Jochen Bojanowski, Justin Broackes, Arthur Cunningham, Mic Detlefsen, Michael Friedman, Christopher Hill, Paul Humphreys, Jukka Keranen, Harold Langsam, Penelope Maddy, Colin McClarty, John Mumma, Sasha Newton, Stephen Puryear, Geoff Sayre-McCord, Joshua Schechter, and Wilfried Sieg. Thanks also to participants in Wolfgang Spohn's work in progress seminar, Universität Konstanz, Summer, 2006, and to audiences at Irvine, Virginia, Brown, and the 2006 Midwest Philosophy of Mathematics Workshop

The research and writing of this dissertation were partially done with the support of a University of Pittsburgh Andrew Mellon Dissertation Fellowship and a Baden-Württemberg Stipendium from the Landesstiftung Baden-Württemberg.

A different kind of thanks is due to my Christ-centered and intentionally cross-cultural family at Eastminster Church in East Liberty. Of the many people and families who helped me in the last eight years, I'd like to especially mention Pastor Paul and Lisa Roberts and their family, who provided my daughter Clara with weekly, loving childcare so that I could get away to write and research.

As hard as I have tried, I have been unable to find the words to thank my wife Hannah. She knows how I am, and I hope that my loss for words will express to her the way I really feel.

This dissertation is dedicated to my two daughters, Clara and Ruthann. Making them was without a doubt the best thing I did in graduate school.

CITATIONS AND TRANSLATIONS

Citations of works of Kant besides the *Critique of Pure Reason* are according to the German Academy ("Ak") edition pagination: *Gesammelte Schriften*, edited by the Königlich Preußischen Akademie der Wissenschaften, later the Deutschen Akademie der Wissenschaften zu Berlin. For the *Critique*, I follow the common practice of citing the original page numbers in the first ("A") or second ("B") edition of 1781 and 1787. Full citations include the title of the work (often in abbreviation), the volume of the Academy edition, and the page number. Passages from Kant's *Jäsche Logic* are also cited by paragraph number (§) when appropriate.

Translations from Kant's works are either my own or from the translations listed in the Works Cited list at the end of the dissertation. If a translation is listed in the Works Cited page, I have used it, with only occasional and minor emendations, which are noted in the text. Works with no cited translation (for instance, from Kant's *Reflexionen*) are my own.

When there is listed in the Works Cited page an English translation of one of Cassirer's works, I have cited passages from the page numbers of the listed translation. Citations of Cassirer's other works are to the page numbers of the original edition listed in the Works Cited page. For essays reprinted in the collection *Erkenntnis, Begriff, Kultur* (edited by R. Bast, Hamburg: Felix Meiner Verlag, 1993) I have cited the original page numbers, not the page numbers of the reprint.

For works of Cassirer's that have English translations listed in the Works Cited page, I have used the English translation, though sometimes with significant emendations, which are noted in the text. The translations from works of Cassirer's without listed translations are of course my own.

I follow the same practice for works by other authors: when an English translation of a work is listed in the Works Cited page, I have cited passages from the page numbers of the listed translation, with the exception of works (like Wolff's *Anfangs-Gründe aller mathematischen Wissenschaften* or Lotze's *Logik*) organized into short paragraphs, which I have cited by paragraph (§) number. Similarly, when an English translation of a work is listed in the Works Cited page, I have used the translation unmodified unless otherwise noted. All other translations are my own, except for a few passages from Poncelet, for which I have used an unpublished translation from Ken Manders.

ABBREVIATIONS

The following abbreviations are used in citations.

Works of Kant's

Ak	*Gesammelte Schriften.* Edited by the Königlich Preußischen Akademie der Wissenschaft. 29 vols. Berlin: DeGruyter, 1902-
Allison	*The Kant-Eberhard Controversy*. Translated and edited by Henry Allison. Baltimore: Johns Hopkins University Press, 1973.
Dohna-Wundlacken Logic	"Dohna-Wundlacken Logic." In *Lectures on Logic*. Translated and edited by J. Michael Young. Cambridge: Cambridge University Press, 1992.
EE	"The First Introduction [*"Erste Eimleitung*"] to the Critique of Judgment." Translated in *Critique of Judgment.* Translated by Werner Pluhar. Indianapolis: Hackett, 1987.
Heschel Logic	"Heschel Logic." In *Lectures on Logic*. Translated and edited by J. Michael Young. Cambridge: Cambridge University Press, 1992.
JL	*Jäsche Logic*. In *Lectures on Logic*. Translated and edited by J. Michael Young. Cambridge: Cambridge University Press, 1992.
"On a Discovery"	"On a Discovery According to Which Any New Critique of Pure Reason Has Been Made Superfluous By a Previous One." In *The Kant-Eberhard Controversy*. Translated and edited by Henry Allison. Baltimore: Johns Hopkins University Press, 1973.
Prolegomena	*Prolegomena to Any Future Metaphysics That Can Qualify as a Science*. Translated by Paul Carus. Chicago: Open Court, 1902.
Refl	Kants Reflexionen. In *Gesammelte Schriften.* Edited by the Königlich Preußischen Akademie der Wissenschaft. Vols. 14-19. Berlin:

DeGruyter, 1902-

Vienna Logic	"Vienna Logic." *Lectures on Logic*. Translated and edited by J. Michael Young. Cambridge: Cambridge University Press, 1992.
What Progress?	*What Real Progress Has Metaphysics Made In German Since the Time of Leibniz?* Translated by Peter Heath in *Theoretical Philosophy After 1781*, edited by Henry Allison and Peter Heath, 337-424. Cambridge: Cambridge University Press, 2002.
Young	*Lectures on Logic*. Translated and edited by J. Michael Young. Cambridge: Cambridge University Press, 1992.

Works of Cassirer's

ENGL	"Erkenntnistheorie nebst den Grenzfragen der Logik." Reprinted in *Erkenntnis, Begriff, Kultur*, edited by. R. Bast. Hamburg: Felix Meiner Verlag, 1993. The article originally appeared in 1913.
*EP*I	*Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit. Erster Band.* 3rd ed. Berlin: Bruno Cassirer, 1922. The first edition appeared in 1906.
*EP*II	*Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit. Zweiter Band.* 3rd ed. Berlin: Bruno Cassirer, 1922. The first edition appeared in 1907.
ETR	*Zur Einstein'schen Relativitätstheorie*. Berlin: Bruno Cassirer Verlag, 1921. Translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity*. Chicago: Open Court, 1923.
KLT	*Kants Leben und Lehre*. Berlin: Bruno Cassirer, 1918. Translated by Haden as *Kant's Life and Thought*. New Haven: Yale University Press, 1981.
KMM	"Kant und moderne Mathematik." *Kant-Studien* 12 (1907): 1-40.
PK	*The Problem of Knowledge: Philosophy, Science, and History since Hegel*. Translated by Woglom and Hendel. New Haven: Yale University Press, 1950.

PSF3	*Philosophie der Symbolischen Formen*. *Dritter Teil: Phänomenologie der Erkenntnis*. Berlin: Bruno Cassirer, 1929. Translated by Hendel and Woglom as *Philosophy of Symbolic Forms. Volume Three: The Phenomenology of Knowledge*. New Haven: Yale University Press, 1957.
SF	*Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundfragen der Erkenntniskritik*. Berlin: Bruno Cassirer, 1910. Translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity*. Chicago: Open Court, 1923.
ZTB	"Zur Theorie des Begriffs." Reprinted in *Erkenntnis, Begriff, Kultur*, edited by. R. Bast. Hamburg: Felix Meiner Verlag, 1993. The article originally appeared in 1928.

Works by Neo-Kantians

KTE	Cohen, Hermann. *Kants Theorie der Erfahrung*. 2nd ed. Berlin: Dümmler, 1885.
LGEW	Natorp, Paul. *Die Logischen Grundlagen der Exakten Wissenschaft*. Leipzig and Berlin: 1910.
PPP	Natorp, Paul. *Philosophie: Ihr Problem und ihre Probleme: Einführung in den kritischen Idealismus*. 2nd ed. Göttingen: Vandenhoeck & Ruprecht, 1918. The first edition appeared in 1911.

Works of Frege's

Bgs	*Begriffschrift*. Translated by Michael Beaney in *The Frege Reader*, 47-78. Oxford: Blackwell Publishers, 1997.
CN	*Conceptual Notation and Related Articles.* Translated by Terrell Ward Bynum. Oxford: Oxford University Press, 1972.
CP	*Collected Papers on Mathematics, Logic, and Philosophy*. Edited by Brian McGuiness and translated by Max Black, V. H. Dudman, Peter Geach, Hans Kaal, E. –H. Kluge, Brian McGuiness, and R. H. Stoothof. New York: Basil Blackwell, 1984.

Grundlagen.	*Die Grundlagen der Arithmetik. Eine logisch-matheamtische Untersuchung über den Begriff der Zahl*. Breslau: Koebner, 1882. Translated by Austin as *The Foundations of Arithmetic*. Oxford: Blackwell, 1950.
PMC	*Philosophical and Mathematical Correspondence*. Edited by Gottfried Gabriel, Hans Hermes, Friedrich Kambartal, Christian Thiel, and Albert Veraart. Abridged by Brian McGuiness. Translated by Hans Kaal. Chicago: The University of Chicago Press, 1980.
PW	*Posthumous Writings*. Edited by Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach. Translated by Peter Lond and Roger White. Oxford: Basil Blackwell, 1979.

Other Works

EFG	Russell, Bertrand. *An Essay on the Foundations of Geometry*. New York: Dover, 1956. First published 1897 by Cambridge University Press.
Elements	Heath, Sir Thomas, trans. *The Thirteen Books of Euclid's Elements, translated from the text of Heiberg, with introduction and commentary*. 2nd ed. Cambridge, Cambridge University Press, 1925.
POM	Russell, Bertrand. *The Principles of Mathematics*. 2nd ed. New York: Norton, 1938. The first edition appeared in 1903.

INTRODUCTION

A common thread in some recent work on the emergence and early development of analytic philosophy has been its relationship to Kant. For Peter Hylton, "the most-influential work of Russell and Moore is best understood as a reaction against Kant."[1] Alberto Coffa has urged us to see the development of philosophy from Kant to Wittgenstein as "the stages through which it came to be recognized that [Kant's] pure intuition must be excluded from the a priori sciences and that consequently the Kantian picture of mathematics and geometry must be replaced by some other."[2] Michael Friedman, on the other hand, has presented logical positivism as a tradition characterized by a relativization of the Kantian synthetic a priori, according to which a priori mathematical-physical principles "change and develop along with the development of the mathematical and physical sciences themselves, but … nevertheless retain the characteristically Kantian constitutive function of making the empirical natural knowledge thereby structured and framed by such principles first possible."[3] Recent interpretations have also claimed to find strong Kantian or Neo-Kantian elements in Frege and Carnap, as well as in some of the major mathematicians of the era. There are good reasons to try to understand the origins of analytic philosophy against the backdrop

---

[1] Hylton, "Hegel and Analytic Philosophy," 446.

[2] *The Semantic Tradition from Kant to Carnap*, 2. On page 22, he calls the same sequence of philosophical events "the rise and fall of pure intuition." See also his "Kant, Bolzano, and the Emergence of Logicism," 679.

[3] Friedman, *Dynamics of Reason*, 31.

of Kant's theoretical philosophy—as Coffa puts it, "for better and worse, almost every philosophical development of significance since 1800 has been a response to Kant."[4] But the success of the "Kant and early analytic philosophy" research program depends on an accurate picture of how Kant's writings were interpreted and treated at the end of the nineteenth and beginning of the twentieth century. How was Kant understood by the Neo-Kantians that dominated philosophy during analytic philosophy's early years? What doctrines of Kant were inspirational for philosophers trying to understand the radical changes that had taken place in the sciences in the nineteenth century? What doctrines were thought dispensable or hopeless?

In this dissertation, I want to make a contribution to answering these questions by considering how one of the most significant philosophical schools of the period, the Marburg or "logical idealist" Neo-Kantians, tackled one of the most important philosophical topics of the period, the philosophy of geometry.[5] Though the reader will find throughout this dissertation discussions of other Neo-Kantians and turn of the century philosophers, the main focus of the work is Ernst Cassirer, the Marburg-trained self-styled "critical idealist" whose philosophical and scholarly writings spanned the first

---

[4] *The Semantic Tradition*, 7.

[5] This essay will only be a contribution to answering the above questions, and will not try to survey the great variety of ways in which Kant was read at the turn of the twentieth century. A major reason for the impossibility of a short survey is the multi-faceted character of the philosophical intermediaries between Kant's already very complex theory and the diverse group of philosophers who are taken to constitute the founders of analytic philosophy. These intermediaries, the "Neo-Kantians" who dominated philosophy in England and Germany from roughly 1870-1910, were a very diverse bunch, ranging from the more Hegelian British philosophers to the more scientifically inclined Kantians in Germany. Of the Kantians in Germany, an early school interpreted Kant in light of physiological research, and the two later schools (the Southwest or 'value-theoretic' or 'historical' school and the Marburg or 'logical idealist' or natural-scientific school) differed in what kind of science they took as paradigms. (I'm here neglecting smaller schools, like the Neo-Friesian school of Leonard Nelson.)

four decades of the twentieth century.[6] The dissertation is thus a (rather modest) episode in the "on-going revival of interest in Cassirer's work"[7] among Anglo-Americans – a revival that has been driven primarily by two factors. First, some historians of philosophy have come to think that Cassirer had a significant influence on early analytic philosophers.[8] And, clearly, Cassirer would have to play a role in any remotely adequate historical picture of the development of the philosophy of mathematics and science

---

[6] Of the small but growing literature on Neo-Kantianism in general and the Marburg school in particular, I have consulted the following works.

On Neo-Kantianism:

R Lanier Anderson. "Neo-Kantianism and the Roots of Anti-Psychologism." *British Journal for the History of Philosophy*. 13:2, 2005. 287-323.

Michael Friedman. *A Parting of the Ways: Carnap. Cassirer, and Heidegger*. Chicago: Open Court Press, 2000.

Helmut Holzhey. *Cohen und Natorp*. Basel: Schwabe, 1986.

Guy Oakes. Introduction to *The limits of concept formation in natural science : a logical introduction to the historical sciences*, by Heinrich Rickert, edited and translated by Guy Oakes. New York : Cambridge University Press, 1986.

Alan Richardson. *Carnap's Construction of the World*: *The Aufbau and the Emergence of Logical Empiricism*. Cambridge: Cambridge University Press, 1998.

Thomas Willey. *Back to Kant : the revival of Kantianism in German social and historical thought, 1860-1914*. Detroit: Wayne State University Press, 1978.

On Hermann Cohen and Paul Natorp:

Alan Kim. "Paul Natorp." In *Stanford Encyclopedia of Philosophy*. 12 August 2003. http://plato.stanford.edu/entries/natorp.

Teri Merrick. "What Frege Meant When He Said: Kant is Right About Geometry". *Philosophia Mathematica* 14, no. 1 (2006): 44-75.

Lydia Patton. "The Critical Philosophy Renewed: The Bridge Between Hermann Cohen's Early Work on Kant and Later Philosophy of Science." *Angelaki: Journal of the Theoretical Humanities* 10, no. 1 (April 2005): 109-118.

Andrea Poma. *The critical philosophy of Hermann Cohen*. Translated by John Denton. Albany : State University of New York Press, 1997.

Abraham D Stone. "The Continental Roots of Verificationism." *Angelaki: Journal of the Theoretical Humanities* 10, no. 1 (April 2005): 129-143.

I've also benefited from three older historical works that describe the various Neo-Kantianisms and their contemporaries:

Robert Adamson. *A Short History of Logic.* London: W. Blackwood and Sons, 1911.

John Theodore Merz. *A History of European Thought in the Nineteenth Century*. 4 vols. 3rd edition. London: William Blackwood and Sons, 1907.

Wilhelm Windelband. *Theories in Logic*. Translated by B. Ethel Meyer. New York: Citadel Press, 1961. The original version was published in English in 1913.

[7] "Symmetry, Structure, and the Constitution of Objects," 2.

[8] Michael Friedman (*Reconsidering Logical Positivism*, chapter 6, and *A Parting of the Ways: Carnap. Cassirer, Heidegger*) and Alan Richardson (*Carnap's Construction of the World: The Aufbau and the Emergence of Logical Empiricism*) have each argued that the Marburg Neo-Kantians, and Cassirer in particular, significantly influenced the logical positivists.

during the dawn of analytic philosophy. Not only did Cassirer write some of the earliest philosophical works on general relativity and QM, but he was one of the first German academic philosophers to give serious attention to Russell and the new logic, Dedekind's foundations of arithmetic, and to Hilbert's axiomatic foundation of geometry. Second, philosophers, especially philosophers of science, have come to recognize that Cassirer, easily the most subtle and mathematically well-informed of the Neo-Kantians,[9] deserves serious attention not just as a stage-setter or interesting "also-ran." Don Howard, for example, has recently called Cassirer's books *Substance and Function* (1910), *Einstein's Theory of Relativity* (1921), and *Determinism and Indeterminism in Modern Physics* (1936) "the great unread canon of early-twentieth century philosophy of science."[10] It is my hope that this dissertation shows that Cassirer's philosophy of mathematics deserves careful consideration as well.

---

[9] For a nice recent introduction to Cassirer's thought as a whole, see:

J Krois. *Cassirer: Symbolic Forms and History*. New Haven: Yale University Press, 1987.

Michael Friedman. A *Parting of the Ways: Carnap, Cassirer, and Heidegger.* Chicago: Open Court, 2000.

Michael Friedman. "Ernst Cassirer and Contemporary Philosophy of Science". *Angelaki: Journal of the Theoretical Humanities* 10, no. 1 (April 2005): 119-128.

More specific work has been done on his philosophy of exact science by Thomas Ryckma:

Thoms Ryckman."*Condition Sine Qua Non? Zuordnung* in the Early Epistemologies of Cassirer and Schlick" *Synthese* 88 (1991): 57-95.

Thomas Ryckman. *The Reign of Relativity: Philosophy in Physics, 1915-1925*. New York: Oxford University Press, 2005.

On Cassirer's philosophy of science, with special attention to his interpretation of quantum mechanics, see:

Steven French, "Symmetry, Structure, and the Constitution of Objects."

The literature on Cassirer in German is growing very rapidly; a good recent monograph that discusses his philosophy of geometry in detail is:

Karl-Norbert Ihmig, *Cassirers Invariantentheorie der Erfarhung und seine Rezeption des 'Erlangers Programms'* (Hamburg: Meiner, 1997).

[10] "No Crude Surfeit": A Critical Appreciation of The Reign of Relativity," comments at the Pacific APA, March 2005.

CHAPTER 1 KANT'S PHILOSOPHY OF GEOMETRY AND THE ANCIENT, DIAGRAM-BASED MODEL

Discussions of the relation between Kant's philosophy and modern geometry often center around three developments. First, a long line of geometers from Gauss to Klein showed that Euclidean geometry is only one of many possible geometries, some of which have more than three dimensions and non-zero (or even variable) curvature. Prima facie, Kant must have been wrong then to think that the space of outer sense is necessarily Euclidean and three-dimensional. Second, advances made by geometers like Pasch, Hilbert, and others allow for definitions precise enough and axiom systems complete enough that all the theorems of geometry follow from the definitions of various spaces. Thus, "what kept [Kant] from seeing that [mathematical judgments] are analytic was the lack of adequate mathematical definitions, definitions not available until much later."[1] Third, when Kant argues for the synthetic nature of mathematics, his appeal to what is contained in the subject, or what can be drawn out by the principle of identity, is vitiated by the "backward state"[2] of formal logic in the 18th century compared to the polyadic predicate

---

[1] Lewis White Beck, "Can Kant's Synthetic Judgments Be Made Analytic?," 88, where Beck attributes this view specifically to CI Lewis. See also Bertrand Russell, *Principles of Mathematics* [*POM*], 458: "all geometrical results follow, by the mere rules of logic, from the definitions of various spaces."
[2] Russell, *POM* §434.

logic developed by Frege and others. Given a rich enough conception of what pure logic can accomplish, we can see that mathematics is analytic after all.[3]

A philosopher keen to defend the continuing relevance of Kant's philosophy of geometry might then take on one of three by now familiar projects. Like Helmholtz or the very early Russell, one might argue that there are at least some properties of space—say the axiom of free mobility, or its finite dimensionality—that are necessary, even if Kant overstated how much we can know about space *a priori*.[4] Like Frege, one might argue that the theorems of geometry do not follow from the definitions of "distance," "space," or "point," and that the axioms are not best viewed as definitions, implicit or otherwise.[5] Or like Poincare, one might argue that the principles of mathematical reasoning contain a non-logical or "synthetic" element and cannot be reduced to the most general laws of logic.[6]

An interpreter of Kant might choose an altogether different way to preserve Kant's philosophy in the face of modern logic and geometry: she might argue that Kant, when properly read, would have no problems accommodating each of these three developments. As I hope to show in this dissertation, Ernst Cassirer's Kantian philosophy of geometry is of this latter type. However, a much better known reading of Kant that would make him immune to mathematical refutation was given by Lewis White Beck.

---

[3] See, of course, Frege's *Grundlagen*. (Frege himself did not assert that *geometry* is analytic, but only arithmetic.) In §434 of *Principles of Mathematics*, Russell claims that Kant's view that *geometrical reasoning* differs from *logical reasoning* relies on his limited understanding of logic. (But Russell, too, did not there assert that geometry is analytic, because he did not think that logic itself was analytic. See again *POM* §434 and *The Philosophy of Leibniz* 22-3.)
[4] Helmholtz, "The Origin and Meaning of Geometrical Axioms." Russell, *An Essay on the Foundations of Geometry* [*EFG*].
[5] Frege, Gottlob. *On the Foundations of Geometry and Formal Theories of Arithmetic*, part I.
[6] Poincare, *Science and Method*; *Science and Hypothesis*, chapter 1.

> The real dispute between Kant and his critics is not whether the theorems are analytic in the sense of being strictly deducible, and not whether they should be called analytic now when it is admitted they are deducible from definitions, but whether there are any primitive propositions which are synthetic and intuitive. Kant is arguing that the axioms cannot be analytic…Objections to Kant's view of mathematics, therefore, cannot be removed merely by the substitution of more adequate sets of definitions and postulates, as if being a better mathematician would have corrected Kant's philosophy of mathematics.[7]

This way of interpreting the synthetic nature of geometry is satisfying for a number of reasons: it frees Kant from immediate refutation by advances in axiomatics and in logical strength, and it acknowledges the strongly anti-formalist elements in Kant's thinking. And seen in this light, the mere *possibility* of non-Euclidean geometries is not a threat to the synthetic nature of geometry. Kant's claim that Euclidean geometry is a body of synthetic truths—and synthetic because derived from axioms whose truth is not analytic—is not refuted but *confirmed* by the consistency of non-Euclidean axiom systems: that the parallel postulate can be denied without reducing an axiom system to contradiction shows that the axioms of Euclidean geometry are not analytic but, as Kant had claimed, synthetic.[8] Beck goes further and argues not just that Kant could have located the synthetic element of geometrical cognition in the axioms, but that he in fact did. Beck thinks he finds this in Kant's explanation of the syntheticity of geometry in the Second Edition Introduction.

> For since one found that the inferences of the mathematicians all proceed in accordance with the principle of contradiction (which is required by the nature of any apodictic certainty), one was persuaded that the principles could also be cognized from the principle of contradiction, in which, however, they erred; for a

[7] "Can Kant's Synthetic Judgments Be Made Analytic?," 89-90, 91. Beck nicely points out that Kant's view of mathematics is diametrically opposed to a formalist or "if-then-ist" view of mathematics, like that of the Russell of *POM*. There is more to mathematics, Kant thinks, than the assertion that a certain proposition is implied by another; rather, the primitive propositions themselves must be true.

[8] See A164/B205. Beck does not draw the connection between the synthetic nature of Euclidean axioms and the possibility of non-Euclidean geometry, though others have. For references, see Friedman, *Kant and the Exact Sciences*, chapter 1, §IV.

> synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself. (B14)

Thus, Kant could have calmly accommodated the developments in logic and axiomatics; since geometric proofs proceed according to logic from first principles, intuition comes in *only* in the selection of the true axiom system from a collection of consistent axiom systems all of which are logically (though not *really*) possible.[9]

However, if Beck's interpretation were correct, *the* distinctive feature of Kant's philosophy of geometry would be the idea that we can know a priori which (or which kind) of consistent geometrical axiom system is true of empirical (or perhaps phenomenological) space. Then the only change in modern geometry to which a Kantian would need to pay attention is the rise of non-Euclidean geometry, and the only continuing philosophical project that would be of interest to a reader of Kant is the sorting out of the empirical and non-empirical elements in our knowledge of physical space. As we will see, this is not an accurate description of Ernst Cassirer's appropriation of Kant's philosophy of geometry.[10] His position is not like the better-known attempts by Kantian philosophers to "plug the leaks"[11] by proposing weaker fragments of Euclidean geometry that could function as the a priori theory of empirical

---

[9] Beck thus sees Kant as taking over the account of mathematical proof from the Wolffian tradition that he found in the textbooks he used in his lectures. Wolff, in his *Anfangs-Gründe aller mathematischen Wissenschaften* (a text that Kant used in his lectures), writes: "the manner and method by which to infer from posited grounds [in mathematics] is nothing other than what is described at length in all books of logic [Logica] or the art of reasoning [Vernunft-kunst]. The proofs or demonstrations of the mathematicians are nothing other than a heap [Haufen] of syllogisms [Schlüsse] connected together according to the rules of the art of reasoning" (§45, my translation). A parallel passage is *Elementa Matheseos Universae*, §§45-7.

[10] This is not to say that Cassirer had nothing to say about how one might sort out the empirical and pure elements in the geometry of space-time. I will, however, not discuss Cassirer's book *Einstein's Theory of Relativity*, since it raises issues outside of the philosophy of logic and pure geometry. Interested readers may consult Ryckman's *The Reign of Relativity*.

[11] Coffa, *The Semantic Tradition*, 57.

space. Rather, he finds inspiration in a different aspect of Kant's philosophy of geometry—Kant's view that mathematics is the construction of concepts—in order to answer a different set of questions raised by a different kind of nineteenth century geometry—projective geometry. If Beck's reading were the correct one, Cassirer's Kantianism would not even be on the map.

In this chapter, then, I will lay the groundwork for subsequent chapters by emphasizing the aspects of Kant's philosophy of geometry that are lost on a reading like Beck's. I will not deny that Kant thought that the Euclidean nature of space is necessary, nor will I deny the historical or enduring interest of his thesis that geometrical axioms are synthetic a priori truths describing empirical space. However, too much of Kant's philosophy of geometry is lost when the controversy over non-Euclidean geometry drowns out the other things Kant says. In section one, I argue that Kant holds that pure intuition is needed to form the concept <space> and other geometrical concepts like <circle> or <line>. Thus, before pure intuition is called on to choose the right axiom system among many possible ones, it is first needed when the geometer so much as thinks about space or the figures that populate it. Further (section two), Kant argues that geometrical postulates (which he distinguishes from geometrical axioms) are necessary because they give the possibility of the constructive procedures that the geometer must carry out in order to represent the objects of geometry. Similarly (section three), geometrical proofs are different from chains of reasoning outside of mathematics insofar as they require constructions in pure intuition—just as the ancient, diagram-based geometry would require. This discussion of the inferential use of pure intuition in geometry will lead me, in the next section, to distinguish my reading from the "logical" readings put forward by Friedman and Hintikka (with which my reading shares some similarities). Finally, I show how Kant views pure intuition as making possible the relation between the representations of geometry and the objects in space and time that give them content. The clear conclusion is that Kant's theory is much more thoroughly entangled with Euclid's *Elements* than Beck's reading recognizes. Kant's philosophy of geometry is the result of philosophical reflection on diagram-based geometry in the Greek synthetic tradition, where the *postulates* express self-evident truths concerning what can or cannot be drawn in a diagram, the *proofs* require essential reference to a drawn diagram, and the content of *concepts* are given in definitions that refer to drawn figures.

1. PURE INTUITION AND GEOMETRICAL CONCEPTS

In order to represent to oneself various kinds of spaces, all of which are logically possible, one needs first to possess the concept of space, along with other geometrical primitives, like <point>, or <line>. An interpretation of the role of Kantian pure intuition according to which pure intuition *only* selects a consistent set of axioms requires an explanation of how the concept of a space could ever arise in a cognizing subject who has not first had a pure intuition of space.

On the contrary, it is a key part of the critical system that unlike the pure concepts of the understanding, the a priori concepts of geometry essentially require pure intuition in their genesis. For though the concepts of mathematics are not empirical concepts, derived by abstraction from particular empirical intuitions, the categories alone (and *not* the concepts of mathematics) "have their source in the understanding alone, independent of sensibility" (B144). As concepts of the understanding alone, Kant argues, the categories uniquely require a deduction, since they "arouse suspicion" concerning their objective validity and the limits of their possible employment.[12]

Already in the Metaphysical Exposition of the Concept of Space, Kant argues that the *concept* of a space is possible only given a prior *intuition* of the one all-encompassing space.

> Space is not a discursive, or, as is said, general concept of relations of things in general, but a pure intuition. For, first, one can only represent a single space, and if one speaks of many spaces, one understands by that only parts of one and the same unique space. And these parts cannot as it were precede the one all-encompassing space as its components (from which its composition would be possible), but rather are only thought **in it**. It is essentially single; the manifold in it, thus also the general concept of spaces in general, rests merely on limitations. From this it follows that in respect to it an *a priori* intuition (which is not empirical) grounds all concepts of it. (A25/B39)

Our representation of space, since it is *singular* and not *common*, must be intuitive and not conceptual. The concept we have of *a* space (which for Kant is no more than a representation of a part or region of the one all-encompassing space) is then posterior to our intuition of space.

Elsewhere, Kant argues that other geometrical concepts are generated by the carrying out of geometrical operations. Thus, for example, the Euclidean postulate that licenses the construction of a circle supplies the means by which the concept of a circle is first generated.

> We cannot think of a line without **drawing** it in thought, we cannot think of a circle without **describing** it, we cannot represent the three dimensions of space at all without **placing** three lines perpendicular to each other at the same point…The

[12] See A88/B120.

> understanding does not **find** some sort of combination of the manifold already in inner sense, but **produces** it, by **affecting** inner sense.
> [Footnote:] Motion of an **object** in space does not belong in a pure science, thus also not in geometry; for that something is movable cannot be cognized *a priori* but only through experience. But motion, as **description** of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through productive imagination, and belongs not only to geometry but even to transcendental philosophy. (B154-5)

Kant's invocation here of the ability of the cognizing subject to use a priori imagination to represent circles and spaces certainly does not amount to a simplistic surveying of the properties of figures using the mind's eye. Rather, Kant is taking as his model here not the inspection of figures as one might inspect a physical shape with one's eye, but the construction of figures in Euclidean geometry. Since the construction of a figure is of course something one does in time, the geometrical representation is produced *a priori* over time as a moving point forms a line or a circle.[13] But on Kant's view, the construction procedure encapsulated in the Euclidean postulate is not simply a way of

---

[13] A162/B211: "I cannot represent to myself any line, no matter how small it may be, without drawing it in thought, i.e., successively generating all its parts from one point, and thereby first sketching this intuition."

Kant's idea that figures in geometry are formed by the motion of lower dimensional figures and that the concepts of these figures are first formed through these motions is reminiscent of Wolff's theory of geometrical definition. Wolff writes: "In geometry it is not difficult to find the definition of things. For the motion of points gives lines; the motions of lines surfaces; the motions of surfaces solids. If one therefore combines the points, lines, and surfaces in every conceivable way, and gradually reveals all possible modes of their motion, then the completed definitions emerge" (*Anfangs-Gründe*, §28).

It is also likely that Kant's emphasis on the *temporal process* of drawing the diagram comes from Newton's method of fluxions. Consider the following passage from Newton's *Quadrature of Curves* (1704):

> I consider mathematical quantities not as consisting of very small parts, but as described by a continual motion. Lines are described, and thereby generated, not by the apposition of parts, but by the continued motion of points…angles by the rotation of the sides' portions of time by continued flux…Fluxions [velocities of variables, themselves considered as flowing quantities] are, as near as we please, as the increments of fluents generated in times, equal and as small as possible, and to speak accurately, they are in the prime ratio of nascent increments. (quoted in Kline, *Mathematical Thought from Ancient to Modern Times*, 363)

In the "Anticipations of Perception," Kant argues that the continuity of our representation of space arises from the intuitive origin of space: our representation is not built up from previously given smallest parts, but rather our representation of a finite part of space arises from repeated limitations, the limit of which is a point or position. We then represent particular continuous magnitudes (lines, or curves, etc.) productively by imagining a point as "flowing" continuously in time. (See A169-170/B211.) See also Friedman, chapter 1, §III, for a discussion of the relationship between Kant's view of geometry and the Newtonian calculus.

drawing a figure that instantiates the geometrical concept in question; only by carrying out the procedure in *intuition*—by synthesizing the manifold of outer intuition in inner sense—can the concept of a point (or line or circle) first arise.[14]

Pure intuition, then, comes in—before a set of geometrical axioms is settled on—in the very representation of space: a single, original, infinite intuition underlies our concept of (a) space, and the synthesis of outer intuition in inner sense underlies our particular geometrical concepts. It should be noted, though, that the account Kant gives in the Aesthetic of the origin of the concept of space is not the same as the account Kant gives in the Analytic of the origin of the concept of e.g. a line. At A25/B39 (quoted above) we are told that the representations of particular parts of space come about by delimiting or cutting off a portion of the one all-encompassing space given in intuition; representation here goes from whole to part, since the part, if it is to be represented as spatial, has to be represented as *in* the one space. But the account that Kant gives at various points in the Analytic of the representation of figures in space has it that the figures are represented by constructing in time a large figure by the motion of a point; representation here goes part to whole, as the moving point successively generates the finite line segments that make up the drawn figure.[15]

---

[14] Kant's contention that one cannot even think of e.g. a triangle without drawing it in thought does not mean of course that every instance of thinking about a triangle requires the productive imagination. Kant's repeated claim that one can think about a triangle completely discursively—as one does in convincing oneself that that it is not analytic that a triangle's three angles add up to two right angles—does not make trouble for this claim since it is always possible to think of a concept without the help of its schema (as a philosopher would do if he tried to prove some theorem of geometry using his own discursive procedure; see A716/B744).

[15] See the "Axioms of Intuition" (B formulation): "**all intuitions are extensive magnitudes**." In that same chapter (A162/B203) Kant explains what he means by "extensive magnitude."

> I call an extensive magnitude that in which the representation of the parts makes possible the representation of the whole (and therefore necessarily precedes the latter). I cannot represent to myself any line, no matter how small it may be, without drawing it in thought, i.e., successively generating all its parts from one point, and thereby first sketching this intuition.

Attempts have been made to read these two accounts as at bottom describing the same sort of cognitive mechanism. Michael Friedman, for instance, has argued that we should read the arguments in the Metaphysical Exposition of the Concept of Space as presupposing the possibility of geometrical construction. On this reading, for instance, Kant argues for the intuitive nature of space at B40 by appealing to its unboundedness—where the unboundedness of space is supposed to be guaranteed by our prior recognition that Euclidean construction procedures ("drawing a line in thought", that is, synthesizing the manifold of outer *intuition* in inner sense) can be continued indefinitely.[16] But this reading tries to introduce a unity in procedure where Kant clearly introduced diversity. In the *Prolegomena*, Kant described his procedure in the *Critique* as employing the synthetic as opposed to the analytic method: the latter method alone involves the invocation of "something already known as trustworthy" (Ak 275), and thus takes certain facts about what human reason has accomplished as data for its argument. Further evidence that Kant had no intention of invoking Euclidean geometry as a given for his argument is found in the distinction Kant makes between the *Metaphysical* Exposition of the Concept of Space and the *Transcendental* Exposition, the latter section having only been added in 1787 to the B edition after the composition of the *Prolegomena*.

This point is made explicit by Kant in his correspondence with Schulze:

> For the representation of space (together with that of time) has a *peculiarity* found in no other concept; viz., that all spaces are only possible and thinkable as parts of one single space, so that the representation of parts already presupposes that of the whole. Now, if the geometer says that a straight line, no matter how far it has been extended, can still be extended further, this does not mean the same as what is said in arithmetic concerning numbers, viz., that they can be continuously and

[16] For details on Friedman's reading of this fourth argument in the Metaphysical Exposition and a fuller account of his attempt to read the story Kant gives of the origin of the concept of space and the origin of specifically geometrical concepts as one story, see chapters 1 and 2 of his *Kant and the Exact Sciences*.

> endlessly increased through the addition of other units or numbers. In that case the numbers to be added and the magnitudes generated through this addition are possible *for themselves*, without having to belong, together with the previous ones, as *parts of a magnitude*. To say, however, that a straight line can be continued infinitely means that *the space in which I describe the line is greater than any line which I might describe in it*. Thus the geometrician expressly grounds the possibility of his task of infinitely increasing a space (of which there are many) on the original representation of a single, infinite, subjectively given space. This agrees very well with the fact that the geometrical and objectively given space is always finite. For it is only given in so far as it is generated. To say, however, that the metaphysical, i.e., original, but merely subjectively given space, which (because there is not a plurality of them) cannot be brought under any concept capable of construction, but which still contains the ground of the possibility of all geometrical concepts, is infinite, means only that it consists in the pure form of the mode of sensible representation of the subject, as an a priori intuition, and therefore as a singular representation, in which the possibility of all space, proceeding to infinity, is given.[17]

At the beginning of this passage, Kant is clearly alluding to the third and fourth arguments in the Metaphysical Exposition from the Aesthetic: space is discursive since it is "essentially one" and an "infinite given magnitude" (B39-40). But the unboundedness of space is not to be explained as resulting from the possibility of iterating a constructive procedure, as the infinity of the natural number sequence is assured by the possibility at any stage of applying the successor function; rather, the possibility of iteration presupposes the unboundedness of space. And this latter fact, Kant thinks, is guaranteed by considerations altogether different from those Friedman had hoped to find. Indeed, it seems that Kant here permits *two* sorts of spatial concepts: geometrical concepts, like point and line, whose origin lies in the constructive procedures described in B 154-5 *et al*, and the original, "metaphysical" concept of space whose origin in some way involves the

[17] Ak 20:419-21. This passage (translated in *The Kant-Eberhard Controversy*, 175-6) appears in notes that Kant wrote up for Schulze to use in his review of Eberhard's anti-Kantian *Philosophisches Magazin*. Emily Carson uses this passage in her criticism of Friedman's "logical" or "anti-phenomenological" reading of the role of intuition in Kant's philosophy of mathematics, put forward in *Kant and the Exact Sciences*. Significantly, Friedman, in his "Geometry, Construction, and Intuition in Kant and His Successors" agrees that this passage in particular makes trouble for his earlier reading and softens his earlier whole-hearted opposition to phenomenological readings of Kantian intuition.

intuitively guaranteed recognition—presupposed it seems, by all of the constructive procedures which underlie the properly geometrical concepts—that any region of space or figure in space is surrounded in all directions by ever more space.

Commenting on the Kantian thought expressed in the third and fourth arguments of the Metaphysical Exposition, Charles Parsons comments:

> Whatever precise sense of 'immediate' in which Kant's thesis implies that the representation of space is immediate, there is a phenomenological fact to which he is appealing: places, and thereby objects in space, are given in one space, therefore with a 'horizon' of surrounding space.[18]

The cognizing subject is aware by attending to its own representing whenever it is thinking[19] about a portion of space that more space surrounds the portion in question. But this in some ways at least introduces more puzzles. We can get a grip on the difficulties here, I think, by comparing this metaphysical representation of space with the kind of geometrical representation Kant describes in various places in the Transcendental Analytic. In the passage from B154-5, Kant is explaining that even pure intuitions require a synthesis; even there the imagination must provide unity to the representation by bringing together the manifold of outer sense (say, the individual finite line segments drawn in thought as the productive imagination over time sketches the line in thought) into one consciousness.[20] Now in an important passage just a few pages later in the B

---

[18] "The Transcendental Aesthetic", 70. In endorsing Parsons's picture as a faithful way of capturing Kant's point here, I am not thereby committing myself to Parsons's reading in its entirety: one need not think that an appeal to phenomenology is the best way to cash out Kantian intuition in *all* its guises.

[19] To be more precise: "whenever the cognizing subject is representing a portion of space using the faculty of imagination, either in synthesizing a pure or empirical intuition or in employing a schematized spatial concept." The phenomenological presence to mind of space's unboundedness would not arise if the subject were *merely* thinking.

[20] Cf. also B137-8:

> The first pure cognition of understanding, therefore, on which the whole of the rest of its use is grounded, and that is at the same time also entirely independent from all conditions of sensible intuition, is the principle of the original **synthetic** unity of apperception. Thus the mere form of

Deduction Kant argues that the representation of space described in the Transcendental Aesthetic (which in the notes for Schulze quoted above he called the original, metaphysical, subjectively given space) itself requires a synthesis by means of the understanding,

> We have **forms** of outer as well as inner sensible intuition *a priori* in the representation of space and time, and the synthesis of the apprehension of the manifold of appearance must always be in agreement with the latter, since it can only occur in accordance with this form. But space and time are represented *a priori* not merely as **forms** of sensible intuition, but also as **intuitions** themselves (which contain a manifold), and thus with the determination of the **unity** of this manifold in them (see the Transcendental Aesthetic).[a]
>
> [Footnote:] Space, represented as **object** (as is really required in geometry), contains more than the mere form of intuition, namely the **comprehension** of the manifold given in accordance with the form of sensibility in an **intuitive** representation, so that the **form of intuition** merely gives the manifold, the **formal intuition** gives unity of the representation. In the Aesthetic I ascribed this unity merely to sensibility, only in order to note that it precedes all concepts, though to be sure it presupposes a synthesis, which does not belong to the senses but through which all the concepts of space and time first become possible. (B160-1)

Here Kant is recognizing that if the concept of space is going to presuppose an intuition, then the original intuition of space itself is going to require the operation of the understanding in securing it as one representation. Thus, if the cognizing subject is going to turn its attention on the *form* of its intuition, and represent *space itself* in an intuition (as in geometry), the senses alone will not suffice for the synthesis of the manifold of space into one intuition. But what will the synthesis that makes possible the *formal intuition* of the one, all-encompassing space be like? We cannot have as our model here

---

outer sensible intuition, space, is not yet cognition at all; it only gives the manifold of intuition *a priori* for a possible cognition. But in order to cognize something in space, e.g., a line, I must **draw** it, and thus synthetically bring about a determinate combination [Verbindung] of the given manifold, so that the unity of this action is at the same time the unity of consciousness (in the concept of a line), and thereby is an object (a determinate space) first cognized.

the synthesis displayed in geometrical constructions, where the representation of the line derives from drawing a single line over time in thought; in drawing the line, as I reviewed above, the representation of the whole line presupposes the representation of its finite parts. But the burden of the third and fourth arguments in the Metaphysical Exposition is to show that in the representation of space, the representation of the parts of space presuppose the whole.

The problem here becomes clear when we put side by side certain elements of the Kantian view. Since space is fundamentally intuitive, the concept of space presupposes an original intuition which comes before any representation of its parts. But as an intuition, this original representation of space requires a synthesis of the manifold into one consciousness. And from the Axioms of Intuition, we know that *every intuition*, pure or empirical, is an extensive magnitude, that is, "*the representation of the part makes possible the representation of the whole (and therefore necessarily precedes the latter)*" (A162/B203). I can find no way around this conclusion: Kant understandably wants to distinguish two kinds of spatial representations, the representation of individual figures and regions in space, and that of the space in which all the first representations are located; Kant sees further that the intuitions underlying all of these representations requires a synthesis; but the model Kant gives of this synthesis is based on the former kind of spatial representation and not the latter; moreover this model even seems incompatible with the existence of an *original*, *formal intuition* of space.[21]

[21] Kant himself seems to recognize this in his terminology. In the footnote to B161, he calls the kind of synthesis [Synthesis] or combination [Verbinding] that provides unity to the manifold in the formal intuition of space "comprehension" [Zusammenfassung]. But in the "Axioms of Intuition," he calls the synthesis or combination of the homogeneous manifold in space in a magnitude "composition" [Zusammensetzung]. This suggests that Kant recognizes that the kind of synthesis underlying the

2. Pure Intuition and Geometrical Postulates

Even if it is not clear how the syntheses underlying the two sorts of spatial concepts are to be distinguished, we can be clear about the main point here. A significant part of Kant's philosophy of geometry is passed over if we try to make Kant's position amenable to the advances in axiomatics and logic by arguing that for Kant pure intuition comes in only in selecting among various consistent axiom systems the one that is true to space as it is represented in intuition. For Kant, the mere representation of space and the various concepts of the figures in space require pure intuition.

Further, such a picture would require that we be able to think about the various possible axiom systems, determine their consistency, and only later bring in pure intuition to decide which one is the true one. But it follows from what we've seen so far that Kant could not allow there to be a subject that is able to represent figures in space (or possesses the concept, say, of a line in general) without also endorsing certain truths about space. In the asides that Kant makes concerning the postulates of Euclidean geometry (which for Kant are of course synthetic a priori truths), we see that the certainty characteristic of geometrical postulates has its source in the origin of geometrical concepts.

> Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane; and a proposition of this sort cannot be proved, since the procedure that it demands is precisely that through which we first generate the concept of such a figure. (A234/ B287)

---

representation of space itself does not fit his description of the synthesis of intuitions *in* space. But his claim that *all* intuitions are extensive magnitudes (B202) and his claim in the note to B201 that *all* combination [Verbindung] of the homogeneous is composition [Zusammensetzung] seem to leave no room for the distinction between "comprehension" and "composition." See also the related discussion in Longuenesse, *Kant and the Capacity to Judge*, 215.

Quoting Euclid's third postulate ("[it is possible] with a given line, to describe a circle from a given point"), Kant invokes the practical[22] nature of Euclid's postulate to argue that the activity described in the postulate is in fact the procedure by means of which the cognizing subject first comes to represent a circle. This postulate is certain since to even think the postulate, one must of course possess the concepts employed in its formulation; but the procedure described in the postulate *is* itself the means by which the concepts in question are first generated. So on Kant's view, one cannot even understand what is being asserted in the postulates of Euclidean geometry without thereby knowing their truth or falsity.[23]

Michael Friedman, in the course of presenting an admirable exposition of Kant's view of geometrical postulates, realizes that Kant's story seems to be faced with an insurmountable problem.

> Geometry ... operates with an initial set of specifically geometrical functions ([viz, the operations of extending a line, connecting two points, and describing a circle from a given line segment]) ... To do geometry, therefore, ... [we] need to be "given" certain initial operations: that is, intuition assures us of the existence

---

[22] See Kant's *Logic*, edited by his student Jäsche (often called the "Jäsche Logic" [*JL*]) §32, "Theoretical and Practical Propositions":

> *Theoretical* propositions are those which refer to the object and determine what appertains to it or does not appertain to it; *practical* propositions, however, are those which state the action that is the necessary condition for an object to become possible.

[23] One might worry that this makes the postulates of geometry analytic, since as soon as one understands what is being asserted in the postulate, one has already seen that the procedure postulated can in fact be carried out. The solution here is to remember that for Kant one can employ concepts in thought without employing their corresponding schemata to provide images for the concepts in question. That is, one can employ the understanding in framing thoughts, without invoking the faculty of imagination. Now it is obviously the faculty of imagination that provides for the production of constructions: when drawing a line in thought, it is the productive imagination that synthesizes the manifold of outer intuition—this is what Kant has in mind when he says that to determine on the truth of geometrical propositions, one has to go outside of the concepts in question and "take refuge in intuition" (A47/B65). So to reason properly geometrically about space one needs to invoke intuition, as one does when first acquiring geometrical concepts; but the postulates are not analytic because once one has the concepts in question, the understanding apart from the imagination can combine them however it pleases as long as it does not fall into contradiction.

> and uniqueness of the values of these operations for any given arguments. Thus the axioms of Euclidean geometry tell us, for example, "that between any two points there is only one straight line, the from a given point on a plane surface a circle can be described with a given straight line" (Ak 2:402).

So far, so good. However:

> Serious complications stand in the way of the full realization of this attractive picture... Euclid's Postulate 5, the Parallel Postulate, does not have the same status as the other Postulates: it does not simply 'present' us with an elementary constructive function which can then be iterated.[24]

Recall that Euclid's fifth postulate states that

> if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.[25]

This postulate does not describe the possibility of carrying out a construction: one can draw two lines falling on a given line just fine using the other postulates. (Instead, it states what would happen if we were to carry out this construction.) It is not a practical proposition at all, and it does not provide the condition for giving ourselves an object or generating a concept.

In fact, Kant's account of geometrical *postulates* was never intended to cover the parallel postulate and it does not explain our knowledge of geometrical *axioms*. Moreover, for Kant and many other 18th century writers, the parallel postulate was not a "postulate." (I'll return to the very particular problems posed by the parallel postulate shortly.) What Friedman[26] overlooks in the sentences of his I've quoted is that Kant, like

---

[24] *Kant and the Exact Sciences*, 88.
[25] Translated by Heath, 202.
[26] Friedman is not alone: every other commentator I've read misses the distinction, too. A possible exception is Lisa Shabel, who points out that early modern textbook writers consistently distinguished postulates from axioms. In fact, it was Shabel's historical work on other 18th century writers that led me to find evidence of the distinction in Kant's own texts.

the early modern geometrical textbook tradition that he inherited, distinguishes between "axioms" and "postulates."

The text of Euclid's *Elements* does not contain any propositions called "axioms": it has five postulates, and a set of five "common notions." The first three Euclidean postulates are not theoretical propositions ascribing properties to objects, still less quantified statements of the form $\forall\exists$: these postulates allowed the geometer to *do* things, to carry out constructions. As Lisa Shabel has discovered, early modern editions of the *Elements* often expanded the list of common notions, and moved the parallel postulate and Euclid's fourth postulate ("All right angles are equal to one another") to the list of common notions, which were often called "axioms" instead of "common notions." This allowed for a principled distinction: postulates are practical propositions that warrant certain constructions, and axioms are theoretical propositions stating most general facts about figures in space.[27] Wolff, for example, distinguished between "postulata"—indemonstrable practical propositions—and "axiomata"—indemonstrable, theoretical propositions.[28] He gives as an example of an axiom Euclid's common notion that the whole is greater than the part, and illustrates the former with Euclid's first three postulates.[29] In fact, he defends his definition of "postulate" on the grounds that it agrees with Euclid's use (!) and gives an intrinsic distinction between axioms and postulates.[30]

---

[27] See Shabel, *Mathematics in Kant's Critical Philosophy*, chapter 2.

[28] See *Philosophia Raionalis sive Logica*, Pars II; §§267-9; *Anfangs-Gründe* §30; *Elementa Matheseos Universae*, §30; *Die Vernünfftige Gedancken von den Kräfften des menschlichen Verstandes* (the "Deutsche Logik"), chapter 3, §XIII.

[29] Wolff is not consistent in his lists of axioms and postulates. In the *Anfangs-Gründe*, he gives nine specifically geometrical "Grundsätze," including Kant's stand-by "between two points there can only be one straight line," and other more unexpected ones like "similar angles cut out proportional segments of concentric circles." In the *Elementa Matheseos*, on the other hand, he lists all of Euclid's common notions as "axiomata" of *arithmetic*, and gives only two specifically geometrical postulates (Euclid's first two) and no specifically geometrical "axiomata." Wolff's theory of mathematical reasoning explains away this

Kant reconfigured the Wolffians' clean distinction between postulates and axioms by insisting that an axiom, as a proposition of *mathematics*, needs to be a "fundamental proposition that can be exhibited in *intuition*" (*Jäsche Logic*, §35). But he carries over Wolff's "Euclidean" definition of "postulate": "a practical, immediately certain proposition or a fundamental proposition which determines a possible action of which it is presupposed that the manner of executing it is immediately certain" (*JL*, §38). Though the wording of Kant's definitions of "axiom" and "postulate" leaves it open that the class of axioms and the class of postulates are not mutually exclusive, his practice leaves no doubt that he intended axioms to be theoretical propositions and postulates to be practical propositions. First, all of the examples of axioms that Kant gives in his writings are theoretical; all of his examples of postulates are practical.[31] Second, all of the textbooks that Kant used in his mathematics and logic lectures distinguished between axioms as theoretical and postulates as practical propositions,[32] as do the texts written by Kant's

---

seeming inconsistency. He thought of all mathematical propositions, axioms and theorems, as consequences from definitions. In the Latin logic, *Philosophia Rationalis sive Logica*, an axiom is a theoretical proposition whose truth is patent to anyone who merely understands the terms that compose the proposition (Part I, section III, chapter II; §261ff.); in the "Deutsche Logik," an axiom is a theoretical proposition that follows from a single definition. In either case, we can get on just fine without a list of axioms as long as we have properly defined terms and a capacity to infer from them. Similarly, since Wolff, like Kant, thinks of all mathematical definitions as "real definitions," he can interchange postulates and definitions in his exposition, as he sometimes does with Euclid's third postulate (see *Elementa*, "De principiis Geometriae," §37).

[30] *Philosophia Raionalis sive Logica*, Pars II; §§267-9.

[31] Here is an incomplete list. Ak 2:402 (Inaugural Dissertation); B16, A24, A25/B39, B41, A47/B65, A163/B204, A300/B356, A732/B760; Heschel Logic, §81 (Young, 381); Letter to Herz, 25 November 1788, Ak 10:555; Letter to Herz, 26 May 1789, Ak 11:53.

None of the propositions Kant gives as "Axiomen" or "Grundsätze" are found in modern editions of Euclid, though many are found in Wolff's *Anfangs-Gründe*. (Elsewhere, Kant argues that Euclid's common notions are in fact *analytic* propositions (B16-7).) All of the examples of geometrical postulates that Kant gives are from Euclid.

[32] Kant lectured from Wolff's mathematics textbooks *Elementa Matheseos Universae* and *Anfangs-Gründe aller mathematischen Wissenschaften*. (See references in previous notes). Kant lectured from Meier's *Auszug aus der Vernunftlehre*; Meier distinguishes axioms and postulates in Wolffian terms at §315 (Ak 16:668). See also Baumgarten's *Logica*, §169.

own students.[33] Third, Kant gives arguments at various times that only make sense if the distinction between axioms and postulates is exclusive.[34]

On this early modern conception of postulates, there is little difference between the definition of a concept, which secures the possible existence of objects falling under the concept by describing the constructive procedure for producing them, and the postulate, which states that this procedure is possible. In Wolffian terminology, mathematics has "real definitions":

> real definitions [Erklärungen der Sachen] (definitiones reales) are clear and distinct concepts of the way and manner that the thing is possible. As when in geometry it is said: a circle is described when a straight line moves around a fixed point. For one grasps by means of this that the circle is possible. What one can make in reality [wirklich machen kann] must also be possible.[35]

Wolff's favorite example of a real definition, that a circle is described when a straight line moves on a plane around a fixed point, is not coincidentally also a favorite example of a postulate.[36] Indeed, on Wolff's view, the same proposition could be a real definition or a postulate. Similarly, Kant thinks that only mathematics has definitions, since in order to understand the definition, one has to carry out a constructive procedure; but the possibility of carrying out the procedure itself guarantees the existence of the thing

---

[33] See Schulz's 1789 *Prufung der Kantischen Critik der reinen Vernunft*, Part I, section 4 (translated as an Appendix to Martin's *Arithmetic and Combinatorics*), where he gives separate lists of geometrical postulates, which are practical propositions, and geometrical axioms, which are theoretical. Schulz executes his own *Anfangs-Gründe der reinen Mathesis* (1790) according to the same method; see Martin, 4-5.

Kant's student Kiesewetter also wrote to Kant (Ak 12:267) while composing his Kantian textbook, *Die ersten Anfangsgründe der reinen Mathematik* (1797) in order to ask Kant for a definition of "Postulat" that would be suitable for mathematics and philosophy, and would clearly distinguish postulates from "axioms" (Grundsätze).

[34] Thus, Kant argues in his 25 November 1787 letter to Schulz that arithmetic has no axioms, but it does have "*postulates*, that is, immediately certain practical judgments." Similarly, the review of Eberhard's *Philosophisches Magazin* that Schulz wrote under Kant's detailed instructions consistently contrasts axioms and postulates (e.g., Ak 20: 402).

[35] *Anfangs-Gründe*, "Kurzer unterricht von der mathematischen Methode, oder Lehrart," §25; my translation.

[36] *Anfangs-Gründe*, "Anfangs-Gründe der Geometrie," definition 5.

defined.[37] Thus, for Kant also, the real definitions of mathematics are virtually interchangeable with practical propositions describing constructive procedures; and if these definitions are properly basic, they are virtually interchangeable with postulates.

> the possibility of a circle is ... *given* in the definition of the circle, since the circle is actually constructed by means of the definition, that is, it is exhibited in intuition, not actually on paper (empirically) but in the imagination (a priori). For I may always draw a circle free hand on the board and put a point in it, and I can demonstrate all the properties of a circle just as well on it, presupposing the (so-called nominal) definition, which is in fact a real definition, even if this circle is not at all like one drawn by rotating a straight line attached to a point. I assume that the points of the circumference are equidistant from the center point. The proposition "to inscribe a circle" is a practical corollary of the definition (or so-called postulate), which could not be demanded at all if the possibility – yes, the very sort of possibility of the figure – were not already given in the definition.[38]

Friedman wants to apply Kant's story of how and why postulates are a priori certain into a general account of how all indemonstrable synthetic a priori propositions are possible in geometry: he wants every fundamental principle to be at bottom a condition on the possibility of carrying out a construction. We now see what has gone wrong. Fundamental theoretical propositions that can be exhibited in intuition, like "two straight lines do not enclose a space" (A163/B204), do not give the possibility of constructions, and they cannot be re-expressed as real definitions. Still less does Euclid's fifth postulate give the possibility of a construction.[39]

---

[37] See A729/B757.

[38] Letter to Herz, 26 May 1789; Ak 11:53, translated in Arnulf Zweig, ed., *Philosophical Correspondence, 1759-99*, 155.

[39] Friedman concludes his discussion of Kant's philosophy of geometry with the striking pronouncement that

> In the end, therefore, Euclidean geometry, on Kant's conception, is not to be compared with Hilbert's axiomatization, say, but rather with Frege's *Begriffsschrift*. It is not a substantive doctrine, but a form of rational representation: a form of rational argument and inference. Accordingly, its propositions are established, not by quasi-perceptual acquaintance with some particular subject matter, but, as far as possible, by the most rigorous methods of proof. (*Kant and the Exact Sciences*, 94-5)

What prevent Friedman's reading from going through are precisely the axioms, which are not demonstrated through constructive proof procedures ("by the most rigorous methods of proof") and do not give

However, before we leave Kant's discussion of geometrical postulates, we should recognize that Friedman was correct that the parallel postulate does provide special problems for Kant's story—though not for the reasons he identifies. The problem is rather with the definition of <parallel lines>. In a long series of unpublished notes (Reflexionen 5-11 (1778-1789; 1800)) that strangely have received no scholarly attention,[40] Kant considers at length the mathematical and philosophical problems with Euclid's and Wolff's definitions. "From a definition," he begins, "which does not at the same time contain the construction of the concept, nothing can be inferred (which would be a synthetic predicate)" (Refl 6; Ak 14:31; my translation). Euclid's definition of parallel lines (two coplanar lines that are extendible in both directions without ever intersecting), Kant notices, has precisely this flaw: there is nothing in the definition of two coplanar non-intersecting lines that would allow one to construct these lines.[41] The same kind of problem afflicts Wolff's definition of <parallel lines>. Wolff had tried to cut the Gordian knot surrounding Euclid's fifth postulate by redefining parallel lines as lines everywhere equidistant from one another.[42] But, Kant recognizes, Wolff assumes too quickly that this definition is real: in fact, the assumption that there exist two lines

---

conditions for the representation of figures in proofs. Beck was correct in this: the axioms of geometry stubbornly remain immediately certain, synthetic a priori principles that give, not the preconditions for drawing inferences about objects in space, but the basic properties of space and the figures that populate it.

[40] The exception is Adickes's excellent series of editorial notes on these Reflexionen; Ak 14:23-52.

[41] Euclid does eventually prove, without the use of the parallel postulate, in I.27 that if we construct two lines such that a perpendicular from the first is also perpendicular to the second, then they are Euclidean parallel (viz. do not intersect); and in I.28, that if we construct two lines such that corresponding angles formed by a transversal are equal, then they are Euclidean parallel. But these constructions do not follow immediately from Euclid's definition. Moreover, as Kant noticed in this and the following Reflexionen, the converse of these conditionals (*Elements* I.30) can only be proved with Euclid's fifth postulate, and do not follow from the definition at all. Thus, we cannot prove from the definition that any two co-planar non-intersecting lines meet the condition of the construction, and the definition is not "invertible."

[42] See *Anfangs-Gründe*, "Anfangs-Gründe der Geometrie," definition 14; *Elementa Matheseos Universae*, "Elementa Geometriae," §81.

everywhere equidistant from one another is equivalent to Euclid's fifth postulate, and the certainty of Wolff's construction is no greater than that of Euclid's axiom.[43]

3. Pure Intuition and Geometrical Proofs

If we think that the sole role of pure intuition in geometry is for Kant the selection of a particular axiom system, it will seem natural to view pure intuition as inspecting the figures drawn in the imagination, and "reading off" their metric and projective properties. And then it will be hard to see how any faculty could do this with certainty. This criticism is made by Kitcher:

> The inadequacy of pure intuition…stems immediately from the idea that we can, on inspection, determine the exact nature of a figure, whether physical or "drawn in thought"…The problem lies with the picture behind Kant's theory. That picture presents the mind bringing forth its creations and the naïve eye of the mind scanning those creations and detecting their properties with absolute accuracy.[44]
>
> Kant proposes that we construct figures in thought, inspect them with the mind's eye, and thus arrive at a priori knowledge of the axioms from which our proofs begin...It is hard to understand how a process of looking at mental cartoons could give us knowledge, unless it were knowledge of a rather unexciting sort, concerned only with the particular figures before us.[45]

Of course this is impossible: no close inspection, not even inspection with the mind's eye, could tell absolutely that the angles of a triangle add up to 180˚ and not some other

[43] I summarize the mathematical situation in the appendix to this chapter. Kant concludes the whole discussion of his jottings in Reflexion 10 (Ak 14:51; my translation):

> If the equality of the distance of two lines constitutes the definition of parallelism [as in Wolff's geometry], then the *definitum* [parallel lines] and the definition [equidistant lines] would be reciprocal. Therefore we should take note that the first [viz, the definition] does not exhaust the entire concept of the second [definitum]. Although the proposition is reciprocal, it cannot be proven, since we can infer from the complete concept indeed to the concept of the equality of the angles [as in *Elements* I.29], but the complete concept does not lead to its construction.

[44] "Kant and the Foundations of Mathematics," 50.

[45] *The Nature of Mathematical Knowledge*, 49-50.

number vanishingly close to 180˚, or that a line is absolutely straight and not imperceptibly curved. And even if the mind's eye could be absolutely accurate in its measurement of this particular figure, there is nothing in our pure intuition of *this figure* that would allow us to generalize its properties to all figures. On this view, then, the synthetic nature of geometry is in tension with its a priori certainty and universality—a kind of certainty and universal validity that Kant argues empirical observation and generalization could never deliver. Kant frequently repeats his contention that our representation of space has to be pure if it is to gain that required certainty and universality, but on the interpretation that Kitcher puts forward this move seems like evasion. Now pure intuition seems distressingly like inner observation and induction; pure intuition stands to the objects of imagination in a way *too analogous* to the way empirical intuition stands to the objects of outer sense. Or, to use the terminology that Kant prefers, there is no room for a distinct kind of a priori reasoning with the intuition of a triangle; whenever we employ a concrete intuition of a triangle in our reasoning, we can never go beyond "mechanical" cognition of the triangle, which delivers only empirical and contingent propositions.[46]

---

[46] See A721/B749: "I can go from the concept to the pure or empirical intuition corresponding to it in order to assess it *in concreto* and cognize *a priori* or *a posteriori* what pertains to its object. The former is rational and mathematical cognition through the construction of the concept, the latter merely empirical (mechanical) cognition, which can never yield necessary and apodictic propositions."

As Lisa Shabel has convincingly demonstrated (*Mathematics in Kant's Critical Philosophy*, 102ff), Kant is here alluding to the difference between "mechanical" and "mathematical" demonstration given by Wolff and his followers. "Mechanical" demonstrations, according to Wolff, require special instruments—like open compasses for carrying distances—or visual estimates to read off exact measurements from drawn figures (*Mathematisches Lexicon*, 506). In this way, the mechanical demonstration proves its conclusion by a kind of experiment [Versuch] (*Anfangs-Griinde aller Mathematischen Wissenschaften*, 161).

As Shabel makes clear, this distinction between "mathematical" and "mechanical" demonstration is different from the familiar Cartesian distinction between "mathematical" curves (curves constructible from finite iterations of compass and ruler constructions) and "mechanical" curves.

In the quoted sentences, Kitcher raises his objections against the universality and certainty of both geometrical axioms and geometrical theorems. As we have done throughout this chapter, let's put aside the very unique philosophical and interpretive challenges posed by our purported knowledge of geometrical axioms,[47] and consider the role of pure intuition in proving geometrical theorems. We can see what has gone wrong in Kitcher's criticism if we remember a striking disanalogy between empirical and pure intuition: in empirical intuition, we acquire (synthetic *a posteriori*) knowledge of the external world using sense experience and induction, while in pure intuition, we acquire (synthetic *a priori*) knowledge of space by means of construction.[48] In pure intuition, the productive imagination produces lines, figures, etc. no doubt in the standard Euclidean way, making use of postulates (like: "between any two points a straight line can be drawn", or "given any point and line segment having that point as an endpoint, a circle may be drawn"). Pure intuition no more has to read off properties of space from the objects given in imagination than the synthetic geometer has to measure with great precision the angles of the figure drawn on paper.[49] Pure intuition delivers *a priori* certain knowledge if Euclidean constructions do.

---

[47] Kitcher does not follow Kant in distinguishing between geometrical axioms and postulates. Kitcher's criticism of Kant would not apply to the postulates, since, on Kant's theory, the postulates describe what we need to be able to do in order to draw a figure in thought; they are not arrived at by drawing figures and then inspecting their properties afterwards.

[48] See A713/B741: "[M]athematical cognition [is rational cognition] from the **construction** of concepts. But to **construct** a concept means to exhibit *a priori* the intuition corresponding to it…Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having to borrow the pattern for it from any experience."

[49] Kant does not think that the fact that geometers use instruments to draw lines and describe circles in diagrams undermines the universality and necessity of geometry.

> This pure and, precisely because of that purity, sublime, science of geometry seems to compromise some of its dignity if it confesses that on its elementary level it needs *instruments* to construct its concepts, even if only two: compass and ruler... Yet even when we call compass and ruler instruments, we mean not actual instruments, which could never produce those figures with

It is easy to miss this point if we fail to attend closely to the nature of Euclidean geometry, since Euclidean geometry was undoubtedly the model Kant had in mind when he discussed the nature of geometrical constructions in pure intuition.[50] The revolutionary changes in geometry that have taken place in the last 200 years can make Kant's position difficult to get clear on—it seems hard, for instance, to see how any geometrical methodology that makes *essential* use of diagrams in demonstrations could avoid the trap that Kitcher accuses Kant of falling into. But I think a brief discussion of some of the unique features of Euclidean geometry will help us to better understand the contours of (and plausibility of) Kant's position.[51]

Traditional Euclidean demonstrations have both a verbal part, which we might call "the discursive text", and a diagram. In *Elements* I.1, the text of the demonstration that an equilateral triangle can be constructed on a given line segment is accompanied by a drawn figure. One might think at first that the relationship between the text of the demonstration and the diagram is *semantic*: the properties of the drawn figure (or an idealization of the drawn figure) are recorded in the text and stand in as the *truth-maker*

> mathematical precision, but only the simplest ways [these figures] can be exhibited by our a priori imagination, [a power] that no instrument can equal. (First Introduction to the *Critique of Judgment* [EE], Ak 20:198; translated by Pluhar, 388)

[50] Some examples: description of mathematical method follows closely Euclid's procedure for proving Euclid's proposition I.32; Kant's explanation of mathematical definitions and postulates uses as examples Euclid's definition and construction of a circle and line; the B154-5 discussion of the synthesis required in geometry uses as paradigms the standard Euclidean constructions of lines and circles (Postulates 1 and 3 in Euclid's *Elements*). This is by now a familiar point in the literature; see Friedman 1992, Hintikka's "Russell, Kant, Coffa," and, especially, Shabel, *Mathematics in Kant's Critical Philosophy*, chapter 1.

[51] The description of Euclidean practice given in the following sections comes from Manders's paper "The Euclidean Diagram (1995)." I have also benefited from John Mumma's dissertation, *Intuition Formalized: Ancient and Modern Methods of Proof in Elementary Geometry*, and unpublished papers, "Proofs, Pictures, and Euclid," and "Understanding Euclid's Proofs."

Other helpful works include Jesse Norman, *After Euclid: Visual Reasoning and the Epistemology of Diagrams*; Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*; Lisa Shabel, *Mathematics in Kant's Critical Philosophy: Reflections on Mathematical Practice*, chapter 1; and the extremely helpful editorial notes and essays in the first volume of Heath's edition of Euclid's *Elements*.

or *referent* of the text. But in fact the presence of reductio arguments in Euclid's *Elements* makes it clear that the text is not true of the diagram, since in a reductio proof the argument works by showing that no figure could be represented by the demonstration text. Nor does the diagram play a merely heuristic or psychological role in the demonstration. Rather, the text and the diagram each play an essential *inferential* role in the demonstration.

In Euclid 1.1, for instance, we can see that some features of the figure drawn from the given line segment are given or inferred in the text, and some features are inferred only from the diagram itself. We learn from the *text* and not the diagram that *AB* is straight or that *BCD* is a circle; but we learn from the *diagram* and not the text that there is a point *C* in which the circles cut each other (though this fact is noted and recorded in the text). What's more, we could only learn from the text that the line in question is straight (as opposed to imperceptibly curved), and we could learn only from the diagram that the circles in question cut each other at a point (since, famously, Euclid has no axioms guaranteeing the existence of a continuum of points in the plane). Ken Manders has noticed that Euclidean demonstrations work by essentially employing both text-based and diagram-based attribution.[52] Manders calls the diagram-based attributions "co-exact," since these are features of the drawn figure that are stable under all but the most radical continuous variations of the topology of the diagram: these features are unaffected by the inevitable consequences of sloppy drawing—lines are drawn bent, circles squished, etc. What Manders calls "exact" attributions—those features of a drawn figure that do not tolerate deformation, like straightness of lines and equality of angles or line

[52] Manders, "The Euclidean Diagram (1995)," §2.

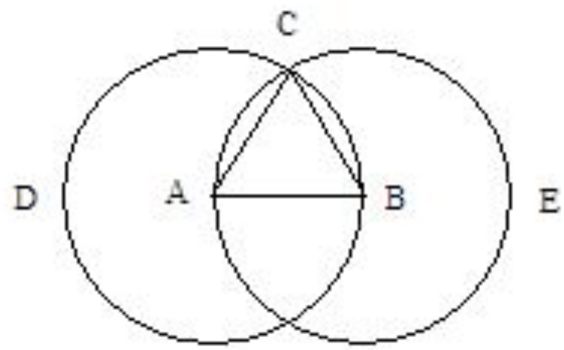


Figure 1-1: Euclid's *Elements*, I.1

segments—can be inferred only from the text, since drawn diagrams are unable reliably to distinguish, for instance, between perfectly straight or equal lines and slightly curved or unequal lines. A rigorous distinction in Euclidean practice between what could and could not be inferred from the appearance of a diagram allowed geometers to reach unanimity about the inferred properties of a given figure.[53]

Kitcher's criticism that Kant, who takes as his model construction procedures in Euclidean demonstrations, illicitly allows the geometer to read off properties from the drawn figure, misses the mark, since Euclidean methodology cannot be caricatured as the naïve eye reading off features (no matter how sensitive to distortion) from a drawn figure. The interplay of text and diagram in Euclidean construction allows the geometer to make rigorous inferences from the appearance of the diagram, while keeping in mind the

[53] Manders, §2.2.

In the Wolffian terminology that Kant appropriated, a candidate geometrical proof that tries to read off exact properties from the diagram would in fact be a "mechanical," not a "mathematical" proof (see A721/B749). About such a "demonstration," Kant writes: "determining my object in accordance with the conditions of empirical intuition...would yield only an empirical proposition (through measurement of its angles), which would contain no universality, let alone necessity" (A718/B746).

textual stipulations made concerning the figures in question and minding the strict rules for making inferences based on the appearance of the diagram.[54]

4. THE 'LOGICAL' INTERPRETATION OF PURE INTUITION IN GEOMETRY

The conception of syntheticity that Beck sketches above implies that geometrical reasoning and logical reasoning differ not in their proof *procedures* but only in their starting points. But this does not seem to be Kant's view: in "The Discipline of Pure Reason in its Dogmatic Use", Kant sharply distinguishes "**acroamatic** (discursive) proofs," which "can only be conducted by means of mere words," from "**demonstrations**," which "proceed through the intuition of the object," and "through a chain of inferences that is always guided by intuition."[55] This distinction for Kant is not just a distinction in the starting points of proofs; rather, the proofs are guided by intuition at all times or "throughout" the proof. (Indeed, if geometrical proofs differed from discursive proofs only in the fact that the former begins with axioms, then there would be

---

[54] Critics who attack Kant's view of geometrical methodology by appealing to the supposed fallacies in Euclidean thinking therefore miss the mark. Consider Bertrand Russell's judgment:

> No appeal to common sense, or 'intuition', or anything besides strict deductive logic, ought to be needed in mathematics after the premisses have been laid down.
>
> Kant, having observed that the geometries of his day could not prove their theorems by unaided argument, but required an appeal to the figure, invented a theory of mathematical reasoning according to which the inference is never strictly logical, but always requires the support of what is called 'intuition.' The whole trend of modern mathematics, with its increased pursuit of rigor, has been against this Kantian theory. (Russell, *Introduction to Mathematical Philosophy*, 144-5)

Of course later axiomatic treatments of geometry did surpass Euclid in rigor, and Kant's theory, tied as it is to Euclid's methodology, is deficient in rigor in a sense. But it does no good to attack Kant's rigor by misrepresenting the rigor of Euclidean proofs. Pace Russell, the rules for Euclidean diagram use go far beyond mere "common sense"; still less can one accuse Kant, as Hans Hahn famously did, of giving an account of pure intuition that amounts in the end to nothing more than "force of habit rooted in psychology" ("The Crisis in Intuition," 100-1).

[55] A735/B763, A717/B745.

no reason to argue, as Kant does, that philosophical methodology and mathematical methodology differ in three ways—in definitions, axioms, and proofs—since the difference between discursive proofs and demonstrations would reduce to the fact that mathematics alone has axioms.)

Michael Friedman has argued recently that this is just what one would expect from a philosopher whose model for geometrical reasoning is derived from Euclid.[56] Since Euclidean practice came to be before the advent of modern quantificational logic, geometers were unable to carry out their proofs using the resources of logic (that is, the syllogistic plus truth-functional logic available before Frege) alone. In his *Grundlagen der Geometrie*, for instance, Hilbert needs to introduce axioms of betweenness, partly to guarantee that there will be enough points on the plane to underwrite the truth of the propositions in Euclid's *Elements*. Hilbert's Axiom II_2 ascribes unboundedness to the line, guaranteeing that any line will be infinite:

> II_2. To any two points *A* and *C* there is at least one point *B* on the line *AC* such that *C* lies between *A* and *B.*

Now the point is not that no geometer before the advent of quantification theory could entertain this thought or assert that it is true of the lines in space. Rather, if a geometer using only syllogistic logic were to adapt II_2 as an axiom, he would be unable to make the necessary inferences from it. As John MacFarlane has put Friedman's point:

> If we start with the categorical propositions 'Every pair of points is a pair of points with a collinear point between them' and '*A* and *C* is a pair of points,' then we can infer syllogistically '*A* and *C* is a pair of points with a collinear point between them.' But Kant's logic contains no way to move from this proposition

[56] *Kant and the Exact Sciences*, chapter 1.

> to the explicitly existential categorical proposition 'Some point is a collinear point between *A* and *C*.' There is no common 'middle term.'[57]

On Friedman's reading, Kant could never have thought that geometrical proofs proceed from the axioms using logic alone, since the logic of his day was not powerful enough to underwrite the steps in geometrical demonstrations. Keeping with the example of Hilbert's II_2, Euclid guaranteed the unboundedness of the line by showing the possibility of extending any line segment using a ruler. Kant's invocation of pure intuition stems partly then from an awareness that only constructions would allow the geometer to carry out the inferences necessary for his demonstrations.

Friedman's historical thesis can be read in one of two ways, neither of which I think is ultimately defensible. On the first reading, it was Kant's own recognition of the limits of syllogistic logic that led him to postulate the necessity of construction in pure intuition for geometry. However, there is very little textual evidence that Kant was directly aware of the limitations of syllogistic logic with respect to modern logic, and much more evidence that Kant was an acute observer of the actual practice of geometers themselves. On the second reading, the geometrical practice that Kant was describing philosophically was the way it was because of the logical situation in the 18th century. That is, as long as geometers had only a weak logic to use, they were forced to reason diagrammatically. However, the actual historical reasons that drove geometers away from diagrammatic reasoning have little to do with advances in logic. Let me take each of these points in turn.

[57] John MacFarlane, "Frege, Kant, and the Logic in Logicism", 26. I have modified the quotation somewhat, changing MacFarlane's arithmetical example to a corresponding geometrical one.

First, Friedman's reading notwithstanding, in the case of Kant's philosophy of geometry, it is not necessary to invoke the peculiar features of syllogistic logic in order to see that for Kant pure intuition plays a role in licensing inferences. Just as Euclidean demonstrations are composed of steps some of which are licensed by the text alone and some of which are licensed by the diagram alone, so too Kant distinguishes between intuitive proofs (that make use of steps licensed by the diagram) and properly discursive proofs (that require only text). Keeping in mind the distinctive proof methodology of Euclidean geometry, one can hear Kant's description of the geometrical method as in fact a sensitive philosophical interpretation—not of Aristotelian logic, but—of geometrical practice as Kant knew it.

> Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle…But now let the geometer take up the question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of those angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle that is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question. (A716-7/ B744-5)
>
> While philosophical knowledge must do without this advantage, inasmuch as it has always to consider the universal *in abstracto* (by means of concepts), mathematics can consider the universal *in concreto* (in the single intuition) and yet at the same time through pure *a priori* representation, whereby all errors are at once made evident. I should therefore prefer to call the first kind **acroamatic** (discursive) **proofs**, since they may be conducted by the agency of words alone (the object in thought), rather than **demonstrations** which, as the term itself indicates, proceed in and through intuition of the object. (A735/ B763)

Now the drawing of the diagram is for Kant the intuitive element in geometrical proofs; the universal (say the concept of a triangle) is exhibited in a particular (say, the individually drawn triangle, constructed according to the concept's schema, that is, the procedure for drawing a triangle using a ruler[58]). The 'agency of words alone' is not generally sufficient for a geometrical demonstration; the diagram must come in to license inferences.[59] Since Kant locates the intuitive element in the drawn figure (though not considered merely as an empirical object, but as some synthesis of the manifold of pure or empirical intuition in accordance with the schemata of the constructed concepts), the *inferential* indispensability of the diagram for licensing inferences in Euclidean proofs is enough to show that geometry requires pure intuition to underwrite its proof procedure.[60]

---

[58] I think that a schema— a "representation of a general procedure for providing a concept with its image" (A140-1/B180)—is a rule-governed procedure for producing images. My reading, which, I admit, does not receive unequivocal support from Kant's texts, differs from readings of schemata as paradigmatic images. (See, for instance, Robert Hanna's alternative view: "a schema is a quasi-objective exemplary or paradigmatic instance of a concept, produced b the pure imagination, such that it encodes the relevant conceptual content or conceptual information in a specifically spatial or temporal form" ("Mathematics for Humans: Kant's Philosophy of Arithmetic Revisited," 346).

[59] One should not read Kant's distinction between acroamatic proofs and demonstrations as that between proofs that use words alone and those that use intuition alone. First, this would be a radical misconception of Euclidean methodology, since Euclidean proofs require text and diagrams. Second, this could not be Kant's intention, since Kant makes clear that the figures drawn in diagrams are actually *schemata* (see A713/B741), and so require that the diagram be drawn in accordance with the procedures associated with the concepts in question. Given Kant's tendency to associate discursive proofs—those that use 'words alone'—with knowledge from concepts, it would seem that the complicated relationship between concept and object in intuition mediated by a schema would nicely be modeled by the relationship between text and diagram. For instance, in Euclid 1.1, we do not read off from the diagram that the two intersecting curved lines are arcs of circles, but we are told this by the diagram. And we can avoid worrying about whether the curves drawn actually are arcs from a circle, since the text prescribes that they be drawn as arcs of a circle. In Kant's language, we are considering the drawn figure in light of the relevant schemata:

> Thus philosophical cognition considers the particular only in the universal, mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless a priori and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined. (A713/B741)

[60] This is perhaps a good time to comment explicitly on B14, cited above in the opening pages of this chapter. In that passage, it seems as if Kant is endorsing a view according to which the inferences in mathematical demonstrations, since they deliver apodictic certainty, proceed from synthetic axioms using only the rules of pure general logic. In my view, Kant's fuller discussion of the mathematical method given in A716-7/B744-5 and A735/B763 makes it clear that intuition plays a role in demonstrations not reducible to its role in underwriting the axioms. But the deciding factor in my view is not the comparison and marshalling of particular passages, but a clear-eyed look at the mathematics of Kant's day. Beck's reading would make sense of a philosopher commenting not on Euclidean geometry as it was practiced, but on geometry as it was practiced by Pasch or Hilbert, whose "Euclidean" geometry was simply one logically

Second, the argument given in the preceding paragraph for a properly inferential role for pure intuition in Kant's philosophy of geometry is not only more mindful of the character of *geometry* in the late 18th century,[61] it is more faithful to the development of geometry in the 19th century. If it were purely logical considerations that were driving geometers to consider intuition as playing an essentially inferential role in demonstrations, then it could only have been after the widespread acceptance of Fregean logic that geometers gave up the Kantian view. But this is not the case. The idea that diagrams could play no essential role in geometrical demonstrations evolved slowly in the second half of the 19th century, pushed along not so much by developments in logic as by developments within geometry itself.[62] The anti-Kantian view was expressed clearly in 1882 by Moritz Pasch, well before advances in logic gained wide currency. Indeed, it wasn't an advance in logic that changed geometer's conception of what they were doing (as Friedman's reading would suggest), but advances in geometry that fueled a new conception of deduction.[63]

---

consistent axiomatized theory. Simple charity says that we should reject Beck's reading and portray Kant as a sensitive reader of Euclidean geometry.

[61] Or at least the textbook Euclidean geometry that Kant would have been familiar with. One might think that Kant's point about the necessity of intuition in geometry (read here as the necessity of the diagram-based inferences in Euclidean practice) would not apply to analytic (that is, algebraic) geometry. But the algebraic geometry with which Kant was familiar employed a geometrical methodology very similar to that first introduced by Descartes, where the algebra supplements, but does not replace, the diagrams in geometrical proofs. See here Lisa Shabel's helpful paper "Kant on the 'Symbolical Construction' of Mathematical Concepts."

[62] I tell more of this story in chapter 2, section 5.

[63] I cannot agree then with Coffa's assessment of what was driving Kant:

> Kant insisted on the capacity of concepts to establish the validity of certain claims, but at the same time ignored the remaining vast continent of conceptual resources. Kant's semantic confusions led him to ignore the grounding force of descriptive concepts. That in turn led him to postulate pure intuition. When the confusions were exposed, it opened once again the question of whether arithmetical and geometric knowledge require something beyond the realm of concepts for their justification. (*The Semantic Tradition*, 35)

Putting aside the other reasons Kant had for postulating pure intuition, it seems quite uncharitable to read Kant's philosophy of geometry as resting on a semantic confusion. Rather, his view of what geometrical demonstration required hewed close to Euclidean geometry as it was practiced up to the end of the 18th

We can see how hard it is for 20th century readers to understand geometrical inference procedures in the 18th century when we ask whether Euclid's demonstrations are properly *deductive*. On the one hand, Euclidean geometry was for almost two thousand years the very paradigm of deductive science. On the other hand, Euclid's demonstrations were not deductive in the modern sense: the truth of the conclusion did not follow by logic alone from the truth of the premises; in carrying out Euclid's demonstrations, a geometer would not ignore the content of the sentences in question and attend only to the form; indeed, Euclid's demonstrations are not even purely discursive. It will do no good to argue that Euclid's demonstrations were deductive in character, only full of gaps and fallacies; one should not read Euclid's joint reliance on text and diagram as just a failed attempt to get by on text alone. Rather, it seems that our modern conception of deduction is irredeemably discursive. Indeed Kant's characterization of Euclidean construction as relying essentially on intuition seems to me a fine way of capturing a mode of inference that is neither fully discursive nor inductive in nature.

If Friedman's reading of Kant's philosophy of geometrical inference is flawed because it focuses too much on 18th century logic and not enough on 18th century geometry, Jaako Hintikka's reading fails by simply misunderstanding the nature of Euclidean geometry. In a series of articles, Hintikka has defended what might be called the logical interpretation of the role Kant gives for intuition in geometry. Focusing on Kant's characterization of intuition as singular representation, he has argued that a

---

century. Now Pasch and his successors did go beyond Kant in their conception of geometrical inference. But they were building on almost a century of revolutionary work in geometry, and Kant can hardly be blamed for not anticipating the shape of things to come.

synthetic element is introduced into a cognition whenever its proof requires appeal to a particular.

> Kant's doctrine that mathematical arguments turn on the use of intuitions thus means merely that a mathematician considers his or her general concepts by means of general representatives. The introduction of such particular representatives is what Kant defines construction to mean. (See *KRV* A713/ B741.) In our contemporary jargon, this does not mean that a mathematical argument turns on appeal to intuition (in our sense), but merely that the gist of the mathematical method lies in the use of instantiation rules. (266)

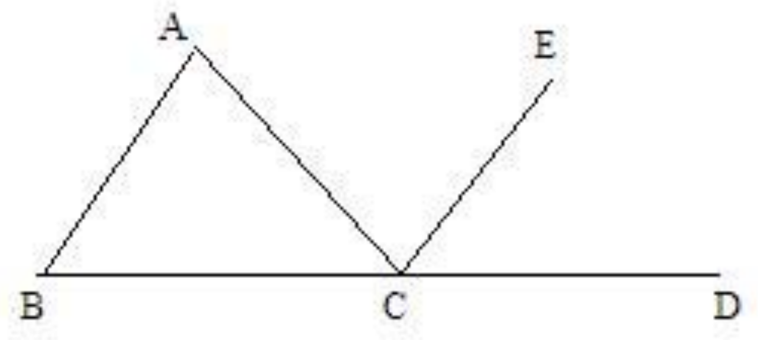
Hintikka is here turning our attention to Kant's repeated insistence[64] that geometrical demonstrations work by letting a particular stand in for a universal. Not only does this sound like a philosophical description of instantiation rules in systems of natural deduction, the drawn diagram in a Euclidean diagram does have an inferential role similar to an arbitrarily chosen instance of a quantified statement: in both cases we introduce an individual, manipulate it according to established rules, and draw a general conclusion from it; and in both cases we need to attend closely to the properties of the individual in order to insure that our general conclusion is not infected by properties common to only some members of the class of instances of the universal formula or type.[65]

---

[64] See A735/ B763 quoted above and A713/B741:

> Thus philosophical cognition considers the particular only in the universal, mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless a priori and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined.

[65] In natural deduction systems, this generality is accomplished easily with a few flagging rules or scope restrictions. In Euclidean geometry, the proper arbitrariness of the drawn figure is checked by drawing separate diagrams for all of the different possible cases. The Euclidean method is essentially open-ended and is successfully carried out sometimes only with great difficulty. See Manders, "The Euclidean Diagram (1995)," for details, and John Mumma's papers for a formal reconstruction.



.

Figure 1-2: Euclid's *Elements*, I.32

Having noted this similarity, though, we should be quick to add that Euclidean diagrams function inferentially in ways not captured by the comparison with instantiation rules In Manders's terminology, Euclidean diagrams allow the geometer to draw conclusions about the diagram's *co-exact* properties—topological features (like inclusion relationships, or the existence of intersection points) of the diagram that are invariant under all but the most serious distortions of the drawn figure. In Kant's example, I.32, the text tells us that BD, AB, AC, and CE are straight, and that AB is parallel to CE. But only the diagram allows us to infer that $\angle ACE + \angle ECD = \angle ACD$.[66] And in Euclid I.1, the text of the demonstration can do the work of formal instantiation by naming a line segment *AB*. But even after instantiation, there is more inferential work to do: the diagram needs to underwrite the inference to the existence of a point *C* where the two circles intersect. Hintikka, though, restricts his reading of Kantian pure intuition partly because he misunderstands the full inferential role played by the Euclidean diagram:

[66] In modern axiomatic treatments, this step would follow from Pasch's axiom.

> According to Kant, what makes mathematical arguments nonlogical and intuitive is the use of constructions. His general concept is obviously modeled on the geometrical concept of construction. Now the use of constructions by a geometer is not based on imagination or intuition in Euclid, but on the geometrical assumptions called postulates, or else on solutions to earlier problems. Now these are both integral parts of the axiomatic structure of geometry. It is also amply clear that Kant was perfectly aware of what a geometer's constructions are based on. It is thus historically impossible to claim that for Kant the use of constructions represented an extra-axiomatic element in geometry. (268)

Ignoring what exactly one might mean by "imagination or intuition in Euclid," the crucial misunderstanding expressed in this passage comes in the intended import of the idea that constructions in Euclid are *based on* the postulates. For Euclid, all of the theorems in the *Elements* are based on the postulates (and common notions) in the sense that there is an unbroken chain of demonstrations, essentially involving both diagrams and text, leading from the postulates to the theorems. And Kant *was* perfectly aware of this fact. But Kant's geometry was *Euclidean* geometry, where the diagrams are needed to license inferences concerning the existence of figures and intersection points, as well as topological relationships between the figures in question. Kant's insistence on the role intuition plays in demonstration surely does not stand in tension with his recognition that Euclidean geometry is axiomatic, but we should not confuse the sense in which Euclid's theorems are contained in his axioms with the sense in which, after Pasch, we can deny that the theorems contain an "extra-axiomatic" element.

5. Pure Intuition and the Objects of Geometry

In the previous sections, I argued—against a particular interpretation of what the synthetic and intuitive nature of geometry consists in—that pure intuition is needed just to represent to oneself geometrical objects and properties, that pure intuition delivers geometrical knowledge not by inspection but by constructive proof procedures, and that intuition plays an essential role in the proof procedure itself, and not just in establishing the truth of a proof's starting points. But Beck was certainly right when he said that even if the proof procedures of geometry were rendered entirely logical, geometry would remain synthetic.

> Mathematical knowledge in his view of the world has objective reference, and this is obtained not through definition but through intuition and construction…The propositions admitted as theorems by Kant are not like the analytic propositions of modern mathematics or the relations of ideas of Hume, for they have a necessary relation to experience through the synthetic, intuitive character of the definitions and axioms.[67]

I'll say more about the relation between geometry and experience below, but for now I want to concentrate on the reference of mathematical claims. Consider a simple arithmetical example, due to Parsons.[68] Even given the developments in logic and axiomatics, the problem of the referents of arithmetical statements remains. Consider the purely logical statement of '2 + 2 = 4':

$$(\exists_2 xFx \bullet \exists_2 xGx \bullet \forall x \sim (Fx \bullet Gx)) \rightarrow \exists_4 x(Fx \vee Gx)$$

(where $\exists_2 xFx$ is defined in the standard way, as there being two distinct objects which are *F*s, and there being no other *F*s beside those two; similarly for $\exists_4 x(Fx \vee Gx)$). Of course this statement can be proven using only first order logic, regardless of the range of the quantifiers. But given a domain whose cardinality is three or less, '2 + 2 = 5' comes out true as well, since the antecedent of the corresponding conditional is false.

[67] "Can Kant's Synthetic Judgments Be Made Analytic?," 91.
[68] "Kant's Philosophy of Arithmetic," §V.

$$(\exists_2 x Fx \bullet \exists_2 x Gx \bullet \forall x \sim (Fx \bullet Gx)) \rightarrow \exists_5 x(Fx \vee Gx)$$

Of course this is no problem for the standard axiomatization of arithmetic, since the Peano axioms force any model to be infinite. But then we need to come up with a model for the Peano axioms, and we cannot just appeal to the standard model of arithmetic unless we are prepared to answer the prior question of the existence of **N**. The truth of the statements of arithmetic then rests on existence assumptions that it is logically possible to deny. And of course the same holds for Euclidean geometry: simple proofs, like the proof that an equilateral triangle can be constructed from a given line segment by drawing two *intersecting* circles with the two endpoints of the line segment as centers, will fail unless we can be assured that the space contains enough points to provide an intersection point for the two circles.[69] (In some spaces, e.g. $\mathbf{Q}^2$, there is no intersection point of the two circles.)

It is clear that for Kant, pure mathematics is not the study of uninterpreted formal systems; for him, the axioms are synthetic *a priori* truths.[70] Rejecting the view that mathematical theorems merely assert that *if* there are certain objects of such and such a sort, then some theorem follows, Kant then owes us an account of the source of our cognition of the objects which provide a model for pure mathematics. From Kant's classification of the cognitive faculties, we can see that the cognition of the *existence* of objects derives from intuition—that form of representation which is related to objects immediately and is singular. "All thought," Kant claims, "whether straightaway, or through a detour, must ultimately be related to intuitions, thus, in our case, to sensibility,

[69] This example is due to Friedman, *Kant and the Exact Sciences*, 59-60.
[70] A732/B760.

since *there is no other way in which objects can be given to us*."[71] Not surprisingly, then, Kant's description of the activity of pure intuition in mathematical cognition suggests that intuitive representation establishes, via constructive procedures, the existence of points, lines, numbers, etc.

> For I first take the number 7, and as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (B16)

> I cannot represent to myself any line, no matter how small it may be, without drawing it in thought, i.e., successively generating all its parts from one point, and thereby first sketching this intuition. (A162/B203)

> [T]ake the proposition that a figure is possible with three straight lines, and in the same way try to derive it from these concepts. All of your effort is in vain, and you see yourself forced to take refuge in intuition, as indeed geometry always does. You thus give yourself an object in intuition. (A47/B65)

Pure intuition, both in arithmetic and in geometry, allows the cognizing subject, through its constructions, to represent to itself objects corresponding to mathematical concepts. The geometer constructs for herself a line or a triangle—objects which she herself has, *in some sense*, brought into existence.

There is then some plausibility in Parsons's suggestion that "the forms of intuition must be appealed to in order to verify the existence assumptions of mathematics."[72] But the precise nature of this doctrine needs to be formulated with care. It will not, for instance, suffice to leave the matter formulated as it was in the last sentence of the preceding section—it is too tempting to read that sentence as asserting that pure intuition itself—perhaps via images or products of the productive imagination—establishes the truth of mathematical existence claims by providing a model for Euclidean geometry.

[71] A19/B33, emphasis mine.
[72] Parsons, "Kant's Philosophy of Arithmetic," 135.

Such a picture would be tidy inasmuch as it provides a direct explanation of the *a priori* certainty of geometry: since the representation of geometrical objects requires intuition, and since intuition provides a model, just thinking that, e.g., two straight lines cannot enclose a figure, would establish its truth.[73]

But it cannot be that pure intuition directly represents mathematical objects in a way analogous to the empirical intuition of the objects of outer sense. Put bluntly, this analogy gets the relationship between the form of intuition and its matter all wrong. First, such a characterization violates a view that is nearly axiomatic for the critical philosophy; namely, that "it comes along with our nature that **intuition** can never be other than **sensible**, i.e., that it contains only the way in which we are affected by objects…Without sensibility no object would be given to us."[74] This makes it clear that we should not think of pure intuition as "intuition" of some non-corporeal set of objects, cognitive contact with which is effected without the aid of the senses. Second, it cannot be irrelevant to our description of the referents of geometrical cognitions—to the subject matter of geometry—that, as the Transcendental Aesthetic proved, space is the form of outer sense. Pure intuition, then, as the "pure form of sensibility itself" (A20/B34), if it plays a role in establishing the existence of genuine objects, must do so through its relation to the objects of empirical experience.[75]

Kant is emphatic throughout the *Critique* that the synthetic *a priori* cognitions of geometry count as *cognitions* only on account of their necessary link to experience—a

---

[73] See Friedman, *Kant and the Exact Sciences*, 66. "Thus, the proposition that space is infinitely divisible is a priori because its truth—the existence of an appropriate 'model'—is a condition for its very possibility. One simply cannot separate the idea or representation of [e.g.] infinite divisibility from what we would now call a model or realization of that idea."
[74] A51/B75.
[75] See *Prolegomena*, §9: "intuitions which are possible *a priori* can never concern any other things than objects of our senses."

link that Kant thinks it necessary for the transcendental philosopher to prove. Thus, in the "Principles of Pure Understanding", we read:

> Thus although in synthetic judgments we cognize *a priori* so much about space in general or about the shapes that the productive imagination draws in it that we really do not need any experience for this, still this cognition would be nothing at all, but an occupation with a mere figment of the brain, if space were not to be regarded as the condition of the appearances which constitute the matter of outer experience; hence those pure synthetic judgments are related, although only mediately, to possible experience, or rather to its possibility itself, and on that alone is the objective validity of their synthesis grounded.[76]

Pure intuition, then, does *not* stand to mathematical objects as empirical intuition does to the objects of outer sense. It should be clear by now that Kant has no place in his theory for purely mathematical (abstract) objects, if by this we mean genuine objects and not just "mere figments of the brain". Rather, the productive imagination, in the constructions of pure intuition, gives to the cognizing subject, not a genuine object, but "only the form of an object,"[77] thought about which will not constitute genuine knowledge, or cognition, unless there is a further relationship between the form of the object and what can be encountered in experience.[78] When I in pure intuition draw a line between two points, the *thought* that "between two points there exists a straight line" becomes *cognition* only when I can establish that it is at least possible that an object of

---

[76] A157/B196.

[77] A223-4/B271: "It may look, to be sure, as if the possibility of a triangle could be cognized from its concept in itself (it is certainly independent of experience); for in fact we can give it an object entirely *a priori*, i.e., construct it. But since this is only the form of an object, it would still always remain only a product of the imagination, the possibility of whose object would still remain doubtful, as requiring something more, namely that such a figure be thought solely under those conditions on which all objects of experience rest."

[78] This gap between what can be constructed (the forms of objects) and the objects of experience is filled by Kant's chapter "The Axioms of Intuition," which guarantees the applicability of geometry to objects in space and time. An excellent account of Kant's goal and strategy in this chapter is given in Sutherland, "The Point of Kant's Axioms of Intuition."

outer experience corresponds to the line drawn in thought.[79] And by 'possibility', Kant does not mean merely logical possibility—in such a case the geometrical cognition would be analytic—but what Kant calls "real possibility," the possibility that the object of cognition appear in a possible experience.[80]

The constructions of pure intuition do not for Kant establish the existence of geometrical objects, or indeed any objects at all. Constructions give rise to genuine knowledge only because they establish the (real) possibility of certain objects of outer experience. But the link between constructibility and possible experience, the essential *applicability* of geometry to the objects in space, is not itself established by the construction[81]; it is a task for transcendental philosophy, a task taken up by Kant in the Principles of Pure Understanding.[82]

---

[79] It is here important that I am restricting my discussion to the objects of geometry. There are certainly not objects of experience that correspond to arithmetical objects in anything like the relatively direct way the triangles of pure geometry relate to triangular physical objects. Though Kant's claim about the necessity of application for the objective validity of mathematics applies to arithmetic (and algebra) as well, it is harder to see how this would work. This interpretive question is made harder by the disanalogy between geometry and arithmetic: arithmetic is *not* the science of the form of inner sense as geometry is the science of the form of outer sense.

[80] See the footnote to Bxxvi and A220/ B267ff.

[81] Kant does sometimes speak as if construction alone insures the applicability of geometry to experience:

> Apollonius first constructs the concept of a cone, i.e., he exhibits it a priori in intuition (this is the first operation by means of which the geometer presents in advance the objective reality of his concept). He cuts it according to a certain rule…and establishes a priori in intuition the attributes of the curved line produced by this cut on a surface of a cone. Thus he extracts a concept of the relation in which its ordinates stand to the parameter, which concept, in this case, the parabola, is thereby given a priori in intuition. Consequently, the objective reality of this concept, i.e., the possibility of the existence of a thing with these properties, can be proven in no other way *than by providing the corresponding intuition*. (Ak 8:191, translated by Allison in *The Kant-Eberhard Controversy*)

That this passage (and others like it) does not contradict B271 can be seen from passages like A239-40/B299 where Kant's full position is made clear.

[82] This has been argued forcefully by Manley Thompson. See especially 338: "While mathematical constructions, whether ostensive or symbolic, provide objects for mathematical concepts and thus answer existence questions *within* mathematics, they do not answer existence questions absolutely. What appear as existence questions in mathematics are really questions of constructability and not existence."

Mathematical Appendix, Continued From note 43

There are two rival definitions of <parallel line> in play in Kant's notes on parallel lines, Refl 5-11, Ak 14:23-52:

> **Euclid's definition of parallel lines**: two lines that are extendible in both directions without ever intersecting,
>
> **Wolff's definition**: two lines that are always equidistant.

And there are in fact three theorems in play, each in two flavors.

> **1E.** If two lines are such that a perpendicular from the first is also perpendicular to the second (the Reciprocal Parallel or "RP" property), then they are Euclidean parallel.
>
> **1W.** If two lines are such that a perpendicular from the first is also perpendicular to the second (the RP property), then they are Wolffian parallel.
>
> **2E.** If two lines are such that corresponding angles formed by a transversal are equal (Angle Property), then they are Euclidean parallel.
>
> **2W.** If two lines are such that corresponding angles formed by a transversal are equal (Angle Property), then they are Wolffian parallel.
>
> **3E.** If two lines are Euclidean parallel, they have the Angle and RP properties.
>
> **3W.** If two lines are Wolffian parallel, they have the Angle and RP properties.

The mathematical situation is this. 1E and 2E are provable in Euclid (roughly, they are theorems I.27 and I.28 in the *Elements*) without Postulate 5. 3E (= *Elements* I.29) requires Postulate 5 to be proved.

Wolff shows that with his definition 3W is provable without Postulate 5, but in his proof he assumes that the two Wolffian parallel lines are both straight. That is, with no extra assumption we can construct a curve that is always equidistant from a given straight line, but we cannot assume that it is straight. Now, if we define parallel lines as equidistant **straight** lines, then the proof goes through. But then we have never proved that there are parallel lines or that they can be constructed.

Indeed, it can be shown that the assumption that there exist two equidistant straight lines is equivalent to the assumption that there is a triangle whose sum is two right angles, which, together with the Archimedean axiom, implies Postulate 5. And Postulate 5 implies that there are two equidistant straight lines. (See Bonola, *Non-Euclidean Geometry*, 120-1; also Harold E. Wolfe, *Introduction To Non-Euclidean Geometry* [Holt, Rinehart, and Winston: New York, 1945] for a proof.) So whenever

Wolff assumes that there are Wolffian parallels, he is just assuming Euclid's Postulate 5, as he does in his proof of *Anfangs-Gründe*, "Anfangs-Gründe der Geometrie," §230, which is 3W, the equivalent of Euclid's offending I.29.

Now, Kant recognizes in his jottings that there is a problem with Wolff's "proof" of 3W. Namely, we can't be immediately certain that we can actually construct the parallel. He shows, however, that if we define parallel lines as always equidistant lines and we take the measure of distance of straight lines to be perpendiculars, *and we assume that the distance between two lines is always reciprocal*, then we can prove 3W. But then we haven't been given a mathematical proof of 3W, since we've been given no method for constructing equidistant straight lines. All we've done is introduced some definitions and spun out their consequences completely discursively (or, in Kant's terms, "philosophically").

CHAPTER 2 THE PROJECTIVE REVOLUTION IN NINETEENTH CENTURY GEOMETRY

1. TWO GEOMETRICAL REVOLUTIONS, TWO PHILOSOPHICAL PROJECTS

One of the geometrical revolutions that took place in the nineteenth century—a period that Yaglom has called the "golden age of geometry"[1]—was the discovery of non-Euclidean geometries equiconsistent with Euclidean geometry. From a certain kind of Kantian point of view, for instance, that of Bertrand Russell's early *Essay on the Foundations of Geometry*, this revolution demanded new study of "the bearing of [non-Euclidean geometry] on the argument of the Transcendental Aesthetic."[2] For Russell, though we cannot follow Kant in affirming the apodictic certainty of Euclidean geometry, we can engage in a renewed Kantian two-prong argument to discover (as in Kant's "Transcendental Exposition of the Concept of Space") those properties of space that are the preconditions of the geometrical common core of all logically possible geometries and to distinguish (as in the "Metaphysical Exposition of the Concept of Space") those principles of geometry that follow from the properties of space that are necessary for any experience of mutually external objects. Drawing on the heterodox Kant interpretation found in T.H. Green and the later British idealists, and the theory of judgment found in idealist philosophers like Bradley and Bosanquet, Russell argued that we can still hold on

---

[1] I.M Yaglom, *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*, 29.
[2] *EFG*, 55.

to a weakened form of Kant's first argument from the "Metaphysical Exposition": consciousness of a world of mutually external things requires that things have the spatial properties captured by projective geometry and those metrical properties common to all finite dimensional geometries of constant curvature.

That the gradual acceptance of non-Euclidean geometries posed fundamentally new problems for philosophers and for orthodox Kantians in particular is by now well-known, and there are a number of works devoted to the history of this problem.[3] However, it has been less frequently noted that geometry underwent a second kind of revolution in the nineteenth century. Starting in earnest in the 1820s, mathematicians working in "projective" or "higher" or "modern" geometry radically simplified and expanded familiar geometrical figures by adding new elements: first, they added to each line one and only one point, its point at infinity; second, they added to the plane imaginary points. This second revolution imposed a different set of philosophical obligations on readers of Kant—a set of obligations that Ernst Cassirer was eager to take on. In this chapter and in the rest of the dissertation, I will not try to improve on what others have said about Kant and non-Euclidean geometry. Rather, I will introduce the reader to this second revolution in geometry so that we can better be in a position to appreciate both the challenges it presents for Kant's philosophy of geometry and the response Cassirer proposed for them.

---

[3] See especially Friedman, *Reconsidering Logical Positivism*, part 1. Russell's history of non-Euclidean geometry in *EFG* is still valuable for a philosopher, as are the histories in Cassirer's *Substance and Function* [*SF*], chapter 3, and *Problem of Knowledge*, vol. 4 [*PK*], chapters 1-3. I have also benefited from Roberto Torretti's *Philosophy of Geometry from Riemann to Poincare* and portions of Alberto Coffa's *The Semantic Tradition from Kant to Carnap*, chapter 3. A historical introduction suited for mathematicians can be found in Roberto Bonola's *Non-Euclidean Geometry*, as well as in Felix Klein's classic 1871 paper "On the So-Called Non-Euclidean Geometry," translated in Stillwell's *Sources of Hyperbolic Geometry*. Some of the important historical works are contained in *From Kant to Hilbert*, edited by William Ewald.

In section 2, I introduce the basic concepts of projective geometry—projection, section, points at infinity, and duality—and begin to illustrate their power by introducing Jakob Steiner's celebrated projective definition of the conic sections. Having very briefly discussed the projective use of imaginary points in section 3, I go on to illustrate the power of these new methods with a striking case of projective unification—the beautiful theorem due to Pascal, and its dual theorem discovered by Brianchon (section 4). In section five, I show how pressures within projective geometry pushed geometers away from the synthetic, diagrammatic geometry that, as we saw in chapter 1, Kant took as his model. I close in the final section by giving a family of philosophical problems that this projective revolution posed. This will set the stage for chapter 3, when we begin to turn to Ernst Cassirer's philosophy of geometry.

2. A Case Study: Jakob Steiner's Projective Geometry and Points at Infinity

In one of the period's most purple passages, the geometer Jakob Steiner wrote

> The present work has sought to discover the organism by which the most varied phenomena in the world of space are connected to each other. There are a small number of completely simple fundamental relations in which the schematism reveals itself and from which the remaining mass of propositions can be logically and easily developed. By the proper appropriation of a few fundamental relations, one becomes the master of the whole object; order comes out of chaos, and we see how all the parts naturally fit together, form into series in the most beautiful order, and unite into well-defined groups of related parts. In this way, we come, as it were, into possession of the elements that nature employs with the greatest possible parsimony and simplicity in conferring to figures their infinitely many properties.[4]

[4] Steiner, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, 1832, reprinted in *Gesammelte Werke*, vol. 1, 229; my translation.

The unfolding organism that Steiner is praising here is projective geometry developed using free use of infinitely distant points and lines and the principle of duality that it makes possible. If ordinary Euclidean geometry is the study of figures constructible using a straight-edge and compass, projective geometry is the study of figures constructible using only a straight-edge.[5] Another way of describing it is closer to its roots in perspective drawing. Consider what an artist does when she paints a scene on a canvas from a determinate point of view. The artist can think of one of her eyes as a light source sending out light rays through a glass pane onto the scene, and a point on her canvas represents a point of intersection between the glass pane and the line running from her eye to the point on the scene that she wants to paint. The bundle (or "pencil") of lines running from her eye to the various points of the scene is a *projection* of the scene, and the glass pane or canvas that cuts these lines is a *section*. The artist can move around with respect to the scene to find a different angle from which to paint it (she "projects" the scene from different points), and she can imagine the glass pane or canvas tilted at different angles with respect to her eye (she "sections" the projected pencil at different angles). Though the look on the canvas would be different with changes in the way the scene is projected or sectioned (circles will look like ellipses, sizes will shrink or expand, parallel lines will start to bend in toward each other, although straight lines will always look like straight lines), clearly something remains constant through all these changes. Projective geometry is the study then of those properties of a figure that are invariant

[5] For intuitive introductions to projective geometry as a mathematical discipline, see Kline, "Projective Geometry," (originally published in 1956, reprinted in Newman, *World of Mathematics*); I have also benefited from John Stillwell, *The Four Pillars of Geometry*.

with respect to this kind of projection.[6] Since under some projections, parallel lines will appear to meet, as the lines in a tiled floor appear to meet at the horizon-line (which on our canvas, is not infinitely far away, but is just another line), in projective geometry we ignore differences between parallel and non-parallel lines. Mathematically this amounts to adding to each line one and only one point, its point at infinity, and to each plane a line at infinity; we can now say that any two lines meet at one point (though it may be at infinity), and any two planes meet at one line (though it may be at infinity).[7] The adjoining of extra points allows us to eliminate an asymmetry between points and lines in ordinary (plane) geometry. Although any two points determine a line—this is Euclid's first postulate—, two lines in Euclid do not necessarily determine a point, since they could be parallel. In fact, once we have added the infinitely distant points, one discovers that every true sentence of plane projective geometry remains true when all occurrences of the word 'point' are systematically replaced with the word 'line' and vice-versa. This is the principle of duality, and points and lines in a plane, or points and planes in space, are said to be "duals" of each other.

Steiner's innovation in projective geometry was to systematize the subject around a few simple concepts ("basic forms" like the point, line, and pencil of lines), introduced in pairs, and to generate more complex figures in terms of the relations between the basic

---

[6] We can see now why we can also characterize it in terms of figures constructible using only a straight-edge and not a compass: straightness, but not circularity, is maintained by projection and section.

[7] See Poncelet, *Traite*, §101, 103; translated in Smith, *A Source Book in Mathematics*, 320:

> *Any plane figure which comprises a system of lines or of curves which have a common point of intersection may be regarded as the projection of another of the same kind or order in which the point of intersection has passed to infinity and in which the corresponding lines have become parallel*…These theorems, giving a geometrical interpretation to this concept, generally adopt the idea that *parallel lines meet in a single point at infinity*.

forms.[8] The chief advantages of this method were twofold. First, the way in which Steiner introduced the basic concepts made it clear that the principle of duality is built into the very foundations of projective geometry. For instance, Steiner introduced the concept point and line, together with the concept "pencil of lines," which is the collection of all the lines on a plane that intersect in a point, like all of the rays of light shooting out in all directions from a single light source. If we consider any line *b* and any point *A* not on that line, we can see that each point *B* on the line *b* corresponds to one and only one line *a* in the pencil of lines through *A*, and similarly to each line *a* in the pencil of lines through *A*, there is one and only one point *B* on the line *b* with which it intersects. Since the dual relation holds for the very simplest elements, and since all the "higher forms" are generated from the lower ones, Steiner makes it clear that the principle of duality will hold in general.[9] (Notice that the points at infinity come in with the assumption that every line *a* in the pencil intersects the line *b* at some point *B*—even when *a* and *b* are parallel.) To see the second advantage, we need first to note that, considered projectively, all four kinds of conic sections—circles, ellipses, parabolas, and hyperbolas—are equivalent. We can see this intuitively if we imagine the light shining from a light-bulb in a lampshade against a wall as a projection and section, and consider what happens as we change the angle of the lamp: when the lamp shines at a right angle

[8] See Klein, *Development of Mathematics in the Nineteenth Century*, 116-9, for a readable overview. Klein calls his method "projective generation."

[9] The principle of duality was introduced explicitly by Poncelet, who defended it by appealing to a particular relationship between points (the "pole") and lines (the point's "polar") with respect to a conic, and contemporaneously with Gergonne, who first introduced the term "dual" and initiated the modern practice of writing the proofs of dual theorems in parallel columns. Though both recognized that the principle of duality held in general, they were unclear about why it did. Steiner's method cleared up this confusion, and is in fact closer to modern treatments, which treat the principle as a consequence of the symmetry in the terms "point" and "line" in the axioms of projective plane geometry; see his *Gesammelte Werke*, 230.

to the wall, the light forms a circle; as we tilt the lamp, the circle elongates and we get an ellipse; when the ellipse elongates off the wall, we have a parabola; and when the lampshade is parallel to the wall, we have the hyperbola formed from the light shining above and below the lampshade. In fact, as figure 2.1 shows, the different kinds of conics can all be seen as derived from a circle as we move the circle on its the plane with respect to the line at infinity.[10] But though the property of being a conic section (though not being a circle) is invariant under projection and section, it does not follow immediately that the conic sections can be constructed using only a straight edge—how could one construct curved lines from a straightedge? Steiner answered this question

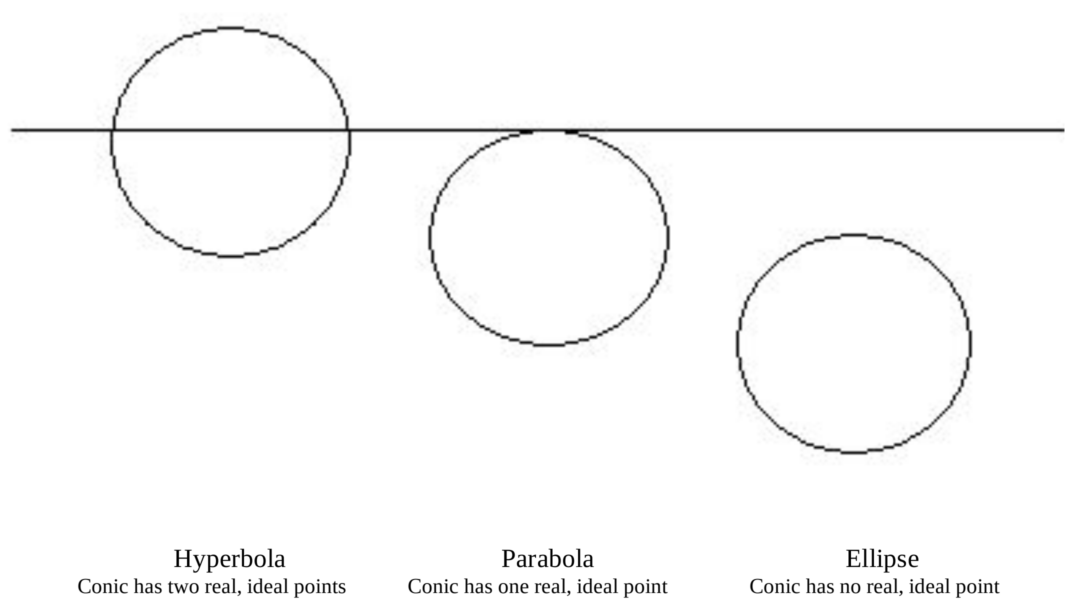


Figure 2-1: The Projective Classification of Conic Sections

---

[10] See Bruce Meserve, *Fundamental Concepts of Geometry*, 160.

with his celebrated definition of a conic in terms of only points and pencils of straight lines. Steiner showed that we can define a point on a conic as the intersection of corresponding lines in two projective pencils.[11] (As Felix Klein later pointed out, this definition of a conic results from a projection of a well-known property of circles that had been proven already in Euclid.[12]) A conic is then generated as a higher form from the basic forms—lines, points, and pencils of points.[13]

3. A Case Study: Poncelet's Projective Geometry and Imaginary Points

Although Steiner made free use of the infinitely distant points in his systematization of geometry, he was in other respects a conservative. Consider two conics, say two ellipses. In the general case, two conics will intersect in four points. But since any two conics are projectively identical, we can imagine our two conics continuously varying (as they might if they were projected onto a canvas from different angles), until they become circles. Now we can imagine these two circles pulling apart slowly until their two points of intersection coincide and then finally disappear as the circles cease to touch. Now, how many intersection points did the two conics have? In ordinary geometry, we would give four different answers: four for the two ellipses, two for the intersecting circles, one

---

[11] For a clear modern treatment, see H.S.M. Coxeter, *Projective Geometry*, 80; see also Kline, *History*, 847.

[12] *Elementary Mathematics from an Advanced Standpoin: Geometry*, 96-7.

[13] Later in the century, Ernst Hankel described the value of this construction this way:

> [I]n the beautiful theorem that a conic section can be generated by the intersection of two projective pencils (and the dually correlated theorem referring to projected ranges), Steiner recognized the fundamental principle out of which the innumerable properties of these remarkable curves follow, as it were, automatically with playful ease. Nothing is wanted but the combination of the simplest theorems and a vivid geometrical imagination capable of looking at the same figure from the most different sides in order to multiply the number of properties of these curves indefinitely. (Quoted in John Theodore Merz, *European Thought in the Nineteenth Century*, vol. 2, 661)

for the circles that touch, and zero for the non-intersecting circles. Poncelet, the great French geometer whose 1822 *Traité des propriétés projectives des figures* is primarily responsible for the revival of projective geometry and for founding it as a discipline distinct from metric geometry, insisted that in such cases, where a series of figures can be generated from one another by continuous variation, the properties must stay the same. As Poncelet described it:

> Consider any figure, in a general or somewhat indeterminate position, among all those it can take without violating the laws or conditions, the relationships which hold between the diverse parts of the system…Is it not obvious that, if one gradually varies the original figure while maintaining these given relationships, or imparts to some of its parts an otherwise arbitrary continuous movement, the properties and relations found for the original system will continue to apply to its successive states, provided of course that one takes into account specific changes which might have arisen, as when certain magnitudes vanish, or change direction of sign, and so on; changes which it will be easy to recognize a priori, by infallible rules?[14]

Free use of the "principle of continuity," applied to this case, would allow us to say in all such cases derivable from the first by arbitrary continuous motion, that there must be four intersection points. In the first case, they are real; in the second, two are real and two imaginary; in the third, two are real and coincident, and two are imaginary; in the fourth case, all four are imaginary. (See figure 2.2.) A popular case to show the plausibility of this kind of reasoning was reviewed later in the century by Arthur Cayley, whom we will have reason to return to later. "There is a well-known construction in perspective," he writes, "for drawing lines through the intersection of two lines, which are so nearly parallel as not to meet within the limits of the sheet of paper." This method, one of the

---

[14] *Traite*, 2nd ed, 1865, xiii. Poncelet goes on there to distinguish the principle from cases of induction or cases of reasoning by analogy. Poncelet treats this principle as an axiom; Chasles, in his 1837 *Apercu*, further justifies the principle by pointing out its use by prominent mathematicians. The principle in fact is similar to a principle employed earlier by Leibniz; see Cassirer, *Leibniz' System*, 223-230.

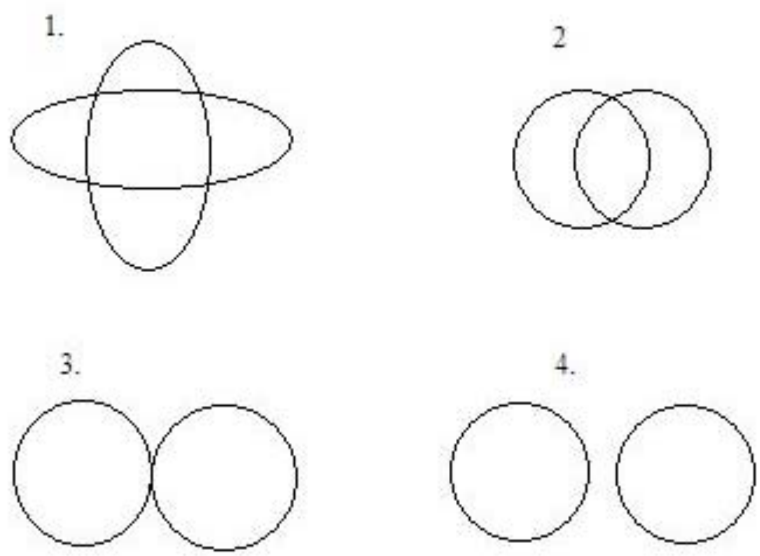


Figure 2-2: Any two conics intersect in four points

most basic in projective geometry, allows one to construct a series of collinear points A'B'C', which are perspective to two given sets of three collinear points, ABC and A''B''C'', and it is easy to show that the line A'B'C' will pass through the intersection point of ABC and A''B''C''.[15] Now, this construction can be carried out whether the point of intersection appears on the paper or not; indeed, it can be carried out just as easily even if the lines are parallel. But "the geometrical construction being in both cases the same,"[16] we can say in both cases the line A'B'C' passes through the intersection point of the two given lines, whether it is at infinity or not. Now, there is a similar argument for circular arcs instead of lines: given two circular arcs ABC and A'B'C', we

[15] See Coxeter, 11.
[16] Cayley, "Presidential Address to the British Association," in Ewald, *From Kant to Hilbert*, vol. 1, 438.

can construct a line *a*, the so-called "radical axis," such that any circle having as its center any point on the line *a* will cut the two circles ABC and A'B'C' orthogonally. We can then show that this line will intersect the two circles at their intersection points, if they intersect, and we can carryout the construction in complete ignorance of whether the two circular arcs intersect or not. "But the geometrical construction being in each case the same, we say that in the second case also the line passes through the two intersections of the circles"—although this time, of course, it is imaginary.[17]

4. PROJECTIVE UNIFICATION

Steiner, despite his free use of the points at infinity and the systematic use of duality that this allows, rejected Poncelet's argument for the introduction of the imaginary points, and called them "ghosts" or "the shadow land of geometry."[18] Memorably, Russell, in his 1897 book *An Essay on the Foundations of Geometry*, writes:

> Thus given a coordinate system, and given any set of quantities, these quantities, *if they determine a point at all*, determine it uniquely. But, by a natural extension of the method, the above reservation is dropped, and it is assumed that to *every* set of quantities some point must correspond. For this assumption there seems to me no vestige of evidence. As well might a postman assume that, because every

[17] This argument, based on the relationship between two circles, intersecting or not, and its radical axis, is due originally to Chasles. (He attributes the name "radical axis" to an earlier paper by Gaultier; Steiner called the line "the line of equal potence"; see Chasles, *Apercu* (1837) chapter 5, §16 (201-2 in the 1839 German translation.))

[18] Klein, *Development of Mathematics in the Nineteenth Century*, 118: "above all, [Steiner] lost the occasion of mastering the imaginary. He never really understood it and fell into the use of such terms as 'the ghost' or 'the shadow land of geometry.' And of course the completeness of his system had to suffer from this self-imposed restriction. Thus though there are two conics $x^2 + y^2 - z^2 = 0$ and $x^2 + y^2 + z^2 = 0$ from a projective point of view, in Steiner's system there is not room for the second." (Klein does not cite this remark, and I am not sure where Steiner says it.)

> house in a street is uniquely determined by its number, therefore there must be a house for every imaginable number.[19]

The view that Russell goes on to defend is that the imaginary points are "a mere *memoria technica* for purely algebraical properties," and have no geometrical significance. Poncelet himself had denied that the imaginary points are merely algebraic, and it is important to see his reasoning. It was an often noted defect, at least from a modern point of view, of the old synthetic geometry, that, in its reliance on the particular properties of a drawn diagram, it sacrificed generality of proof and insight into the interrelations of different figures. The geometer Ernst Hankel later in the century wrote:

> The many cases, which can be distinguished in a problem with respect to the position of given and sought lines, present to the Greek geometer so many particular problems or theorems, and the greatest ancient mathematicians took it to be necessary in their writings to investigate independently of one another all of the very numerous cases that were thinkable, each with the same thoroughness and precision. […] Thus ancient geometry sacrificed true simplicity, which consists in *unity of principle*, for the sake of intuitive simplicity, and arrived at a trivial *sensible* intuitiveness [*sinnliche* Anschaulichkeit] *but at the cost of knowing the interrelation of geometrical figures as they modify and vary their* [sinnlich vorstellbaren Lage] *position in sensible representation*.[20]

Poncelet complained that, in "ordinary geometry," "one is forced to go through the whole series of elementary reasoning steps all over, as soon as a line or a point has passed from

---

[19] *EFG*, 44. Poncelet had shown that in a plane with infinitely distant and imaginary points, any two circles will meet in two fixed points, infinitely distant and imaginary—the so-called 'circular points.' Thus, Poncelet could give what Klein has called "a projective definition" of a circle: a conic with two fixed imaginary, infinitely distant points (See Klein, *Development*, 74). To this, Russell replied, "everyone can see that a circle, being a closed curve, cannot get to infinity" (*EFG*, 45).

Strangely, Russell never discusses the similar problem with (real) infinitely distant points, though his treatment of projective geometry, including the axiomatic system he gives for it, requires infinitely distant points. Surely, everyone can see that two parallel lines do not meet at all, or if two lines meet, they have to meet at some finite distance! This sloppiness can be found also in Russell's discussion of the principle of duality at 127-8 of *EFG*. See here also Joan Richardson's comments on Russell's attitude in *Mathematical Visions*, 225-6, 229 and Torretti's comments in *Philosophy of Geometry from Riemann to Poincare*, chapter 4, section 3.

[20] Hankel, *Die Elemente der projectivischen Geometrie, Vorlesungen* (1875), 2; my translation.

the right to the left of another, etc."[21] This defect is avoided in algebra by the use of "abstract signs," as in the equation for a conic, and operations on this equation have a general significance for all possible values of the letters contained in the equation. A consequence of this is that no essential distinction is made, algebraically, between positive and negative, or real and imaginary, expressions. Algebraic geometry is general, since the equation is a general expression of all of its values, and it reveals connections between figures (or different states of the same figure) by expressing them as values of a single equation. But the algebraic manipulation, which admits of no geometrical interpretation, and the essential reference in algebra to arbitrary coordinate axes are distinct disadvantages of the use of algebra in geometry. The projective idea of viewing the figure not as a fixed array of points and lines, but as a system of elements undergoing continuous variation in space, secures the generality so prized in algebra without requiring its 'foreign elements.'[22] So, for example, projective geometry can solve a problem with a single construction for which the ancient synthetic geometer Apollonius required eighty separate cases.[23]

This kind of justification, though it shows the great value of treating a figure as variable or in a state of continuous change from real to ideal conditions, nevertheless did not always convince. Hankel, for instance, called Poncelet's imaginary elements a gift

---

[21] Poncelet, *Traite*, xii.
[22] It would be interesting to know how much of the generality of algebra over ancient, synthetic geometry is due to the use of signed quantities and how much is due to complexification. (Since pre-nineteenth century algebraic geometry was always done in affine space, none of the generality that Poncelet and Hankel are here admiring is due to infinitely distant elements—although the systematic use of projective, homogeneous coordinates starting with Plücker's analytic geometry in the mid-nineteenth century provided greater generality still.) Unfortunately, I have not been able to sort this out.
[23] Hankel, 2.

from analysis to algebra,[24] and, it seems, Poncelet himself had to use algebraic calculations to assure himself that the use of his principle of continuity was in fact appropriate in a given context.[25] But surely the extremely fruitful use of algebra in geometry has to reveal something about the nature of geometry, whether or not the principle of continuity is a fully satisfying answer.[26] To illustrate the use of adding points to the plane, let's consider a striking case of the value of unification provided by the points at infinity. Figure 2.3 gives a diagrammatic representation of Brianchon's theorem, which states that if a hexagon is circumscribed about a conic, its three diagonals meet in one point. A simple replacement of the words 'point' for 'line' gives us Pascal's theorem, that if a hexagon is inscribed within a conic, its three pairs of opposite sides meet in collinear points. In fact, Pascal proved his theorem in 1640, and Brianchon proved his in 1806 by dualizing Pascal's.[27] In figure 2.4, Pascal's theorem is illustrated with a parabola, but this projectively makes no difference (even if it looks different).

---

[24] Hankel, 9.

[25] This was the opinion of Darboux, who at the turn of the twentieth century did important work in spherical geometry. See Darboux, "Survey of the Development of Geometrical Methods" (1904), 522:

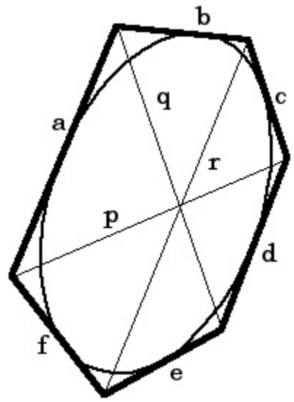
> Poncelet erred in refusing to present [the principle of continuity] as a simple consequence of analysis…[T]he geometrical system rested on an analytic basis; and we know, moreover, that by the unfortunate publication of the Saratoff notes, that it was by the aid of Cartesian analysis that the principles which serve as the base of the *Traité des propriétés projectives* were first established.

(A nice summary of Darboux's contribution can be found in Klein, *Evanston Colloquium*, 12).

[26] Compare Merz, vol. 2, 674:

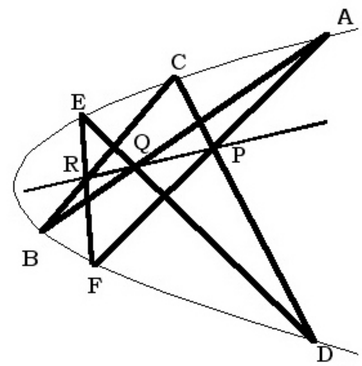
> It is usually supposed that the consideration in geometry of imaginary or invisible elements in connection with real figures in space or in the plane has been imported from algebra; but the necessity of dealing with them must have presented itself when constructive geometry ceased to consider isolated figures rigidly fixed, when it adopted the method of referring figures to each other, of looking at systems of lines and surfaces, and moving figures about or changing them by the process of projection and perspective. The analytical manipulations applied to an equation, which according to some system or other expressed a geometrical figure, found its counterpart in projective geometry, where, by perspective methods – changing the centre or plane of projection,—certain elements were made to move away into infinity, or when a line that cut a circle moved away outside of it, seemingly losing connection with it.

[27] See Kline, 841. Brianchon in fact proved his theorem by using the relationship between a line (the polar) and a point (its polar) with respect to a conic.



The diagonals of a hexagon circumscribed about a real, non-degnerate conic meet in one point, prq.

Figure 2-3: Brianchon's Theorem



The intersections of the sides of the hexagon inscribed within a real, non-degenerate conic lie on a line, PRQ.

Figure 2-4: Pascal's Theorem

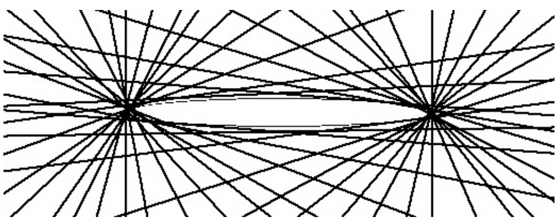


Figure 2-5: A point pair as a degenerate conic.

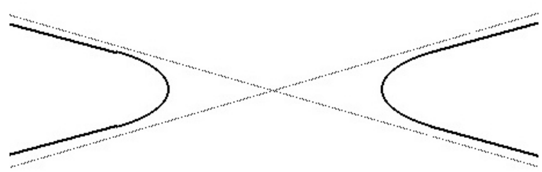
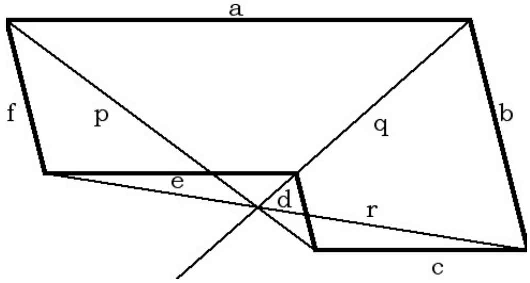
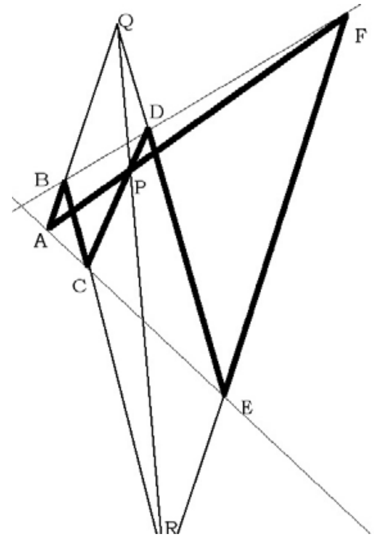


Figure 2-6: A line pair as a degenerate conic.



The six sides of a hexagonal parallelogram are circumscribed about a pair of points infinitely far away. The diagonals of the hexagon meet in one point.

Figure 2-7: Brianchon's Theorem, degenerate case



The six vertices of the overlapping hexagon are inscribed within the line pair, ACE and BDF. The inter sections of the sides of the hexagon lie on one line.

Figure 2-8: Pascal's Theorem, degenerate case

Now the definition of a conic due to Steiner allows for degenerate cases of conics, either as a pair of points or, as in the case we want to consider, a pair of lines. We can make this intuitive by thinking of a hyperbola getting closer and closer to its asymptotes until it coincides with them, as in figure 2.5, or an ellipse getting thinner and thinner until it reduces to two points, as in figure 2.6.[28] Thus rephrasing Pascal's theorem for the degenerate case, we see, as in the figure 2.8: if the six points of a (self-intersecting) hexagon lie on two lines, its three pairs of opposite sides meet in collinear points. Dualizing back to Brianchon's thorem, we see: if the six sides of a hexagon intersect in two points, its three diagonals might in one point. By making free use of the points at infinity, we can let these two points move off the page to infinity, and we get a hexagon composed of parallel lines, as in the figure 2.7.[29] Projectively, these four cases (Pascal and Brianchon's theorem using non-degenerate conics, and Brianchon's and Pascal's theorem using degenerate conics) are just instances of the same theorem—even if they appear differently at first, and, more importantly, even if they look quite different on the page.[30] By treating any given figure as in motion or as continuously altering its

[28] See Coxeter, 89-90; projectively, the difference between a degenerate conic and a non-degenerate conic is the difference between a locus of intersections of corresponding lines of two projective *and perspective* pencils, and a locus of intersections of corresponding lines of two projective *but not perspective* pencils. This difference has no effect on the proof of Pascal's theorem.

[29] The use of the Brianchon's and Pascal's theorems in the degenerate case to illustrate the power of projective thinking, and, more generally, the power of using ideal elements, is due to David Hilbert's Lecture, "Die Rolle von Idealen Gebilden," delivered in 1919-1920 and published as *Natur und mathematisches Erkennen*, edited by David E. Rowe.

[30] Ken Manders, in his unpublished paper "Applying Mathematical Concepts," considers the similar unification of various theorems effected by the proof of Desargues's Theorem in plane projective geometry. As the modern treatment given by Flohr and Raith makes clear ("Affine and Euclidean Geometry," 308, 310), though Desargues's Theorem can be given a simple and immediate proof in projective geometry, the Dilation Theorem, the Translation Desargues Theorem, and the Greater Desargues Theorem (along with their converses), though they are special cases of the projective Desargues Theorem, each have to be given separate and sometimes messy proofs in affine geometry. For an immediate and beautiful proof of Desargues's Theorem in projective 3-space, see Hilbert and Cohn-Vossen, *Geometry and the Imagination*, chapter 3, §19.

"contingent" properties (to use Chasles's vocabulary) by projection and section, and by adding infinite points and treating them like any other point, the tedious distinguishing of cases characteristic of pre-projective synthetic geometry is avoided. In Frege's words, "we forestall a difficulty which would otherwise arise because of the need to distinguish a frequently unsurveyable set of cases according to whether two or more of the straight lines in the set were parallel or not," and these cases are all "disposed of at one blow."[31] Or as Kline put it, though the discovery of the principle of duality "calls for imagination and genius," once one has it, the discovery of new theorems "is an almost mechanical procedure."[32] With such striking cases of simplification, unification, and generalization in hand, it is easy to understand why Steiner, in his excusable enthusiasm, thought that modern geometry had uncovered the organism of space, those fundamental relations by which, with great simplicity and parsimony, we can bring order to the chaos of geometrical theorems.

5. DIAGRAMS IN PROJECTIVE GEOMETRY

In the 1882 publication of his lectures, entitled *Lectures on Modern Geometry*, Moritz Pasch took it as a given that his audience would no longer accept the essential inferential use of diagrams in a geometrical proof.

> Indeed, if geometry is to be really deductive, the deduction must everywhere be independent of the *meaning* of geometrical concepts, just as it must be independent of the diagrams; only the *relations* specified in the propositions and definitions employed may legitimately be taken into account. During the

[31] Frege, "On a Geometrical Representation of Imaginary Forms in the Plane" (1873), in *Collected Papers on Mathematics, Logic, and Philosophy* [*CP*], 1.
[32] Morris Kline, "Projective Geometry" 1956, in Newman, *The World of Mathematics,* 638.

> deduction it is useful and legitimate, but in *no* way necessary, to think of the meanings of the terms; in fact if it is necessary to do so, the inadequacy of the proof is made manifest. If, however, a theorem is rigorously derived from a set of propositions—the *basic* set—the deduction has value which goes beyond its original purpose. For if, on replacing the geometric terms in the basic set of propositions by certain other terms, true propositions are obtained, then corresponding replacements may be made in the theorem; in this way we obtain new theorems as consequences of the altered basic propositions without having to repeat the proof.[33]

We saw in chapter 1 that Kant explicitly distinguishes mathematical proofs ("demonstrations") from discursive proofs, which are "conducted by the agency of words alone." Drawing on a half century of work inaugurated by Poncelet's groundbreaking work in projective geometry, Pasch is arguing that an "adequate" proof cannot make any essential use of a drawn diagram. Not only must it get by the "agency of words alone" (to use Kant's phrase), but the deduction must be independent of the meaning of the terms employed, in the sense that the systematic replacement of one set of terms in an adequate deduction will remain adequate after the terms have been replaced.

Why did nineteenth century geometers come to reject the diagrammatic model of geometric proof that geometers and philosophers from Euclid to Kant had accepted? There were at least three elements of modern geometry in the projective tradition that played a significant role in this change.[34] First, as the above quotation from Pasch shows

---

[33] *Vorlesungen über neuere Geometrie* (2nd ed, 90), translated in Ernest Nagel's "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry". Nagel's article helpfully tells the story of the development of the axiomatic method in geometry throughout the 19th century.

Noteworthy here is Hans Freudenthal's speculation that Pasch's work in redefining the nature of geometrical inference itself influenced Peano's development of quantificational logic in the 1890s and Hilbert's contemporary axiomatization of geometry ("The Main Trends in the Foundations of Geometry in the 19th Century").

[34] I am not claiming that these three elements were the only elements within geometry that put pressure on the diagrammatic model of geometrical proof, nor am I claiming that it was only advances in geometry that made trouble for the use of diagrams in mathematics. There were mathematicians before Poncelet (Lagrange is an example) that self-consciously avoided using diagrams in their mathematical works, and I am making no claim to explaining their motives.

clearly, the new, discursive conception of deduction is well-fitted to the duality of geometrical properties.[35] Since theorems in projective geometry dealing with figures on a plain would hold whenever the words 'point' and 'line' were switched, it follows that projective proofs should be such that they made no essential use of the meaning of the word 'point' or 'line', since otherwise two dual theorems would require two different kinds of proofs. Pasch is here generalizing that fact into a general principle of geometrical proofs: not only should no use be made of diagrams in the proof, but no essential use should be made even of the meanings of the terms.[36] (Indeed, the trend away from the essential use of diagrams in geometrical proofs was accelerated after Plücker, starting in the 1840s, showed how the *analytic* use of new systems of

---

Similarly, to say that there should be no essential use of diagrams in geometrical proofs does not mean that there are no other important or even essential uses of diagrams in geometry. Indeed, one might argue that a most explanatory proof would be one that could easily be visualized. These are important and difficult topics in the epistemology of mathematics to which this dissertation has nothing to contribute.

Again, there has been in the philosophical literature recently a series of papers concerning the very possibility of a valid, diagrammatic or visual proof. (See J. R. Brown, "Proofs and Pictures," *British Journal of the Philosophy of Science*, 48(2): 161-180, 1997, and I. Dove, "Can Pictures Prove?" *Logique & Analyse* 45, no. 179-180 (2002): 309-340. Nothing in my historical remarks here would indicate that there could not be a valid diagrammatic or visual proof. I am only claiming that, within the nineteenth century geometrical tradition that began with Poncelet, geometers gave a series of local reasons why essentially diagrammatic proofs were no longer adequate.

Finally, it seems very clear to me that some of the alleged factors in undermining the acceptability of diagrammatic proofs—like the discovery of continuous but nowhere differentiable curves (see Hahn, "The Crisis in Intuition") or alleged fallacies in Euclid's geometry, such as Rouse Ball's "proof" that all triangles are isosceles (see Klein, *Elementary Mathematics*, 210)—played little or no role whatsoever.

[35] Ernest Nagel emphasized the historical role of duality considerations in undermining the old, diagrammatic model of geometrical proof, and I am indebted to him here. However, as I go on to show, there were other elements in nineteenth century projective geometry that put pressure on the old model of geometric proof.

[36] This conception was famously expressed by Hilbert, whose independence proofs in the *Grundlagen* make sense only against this background, in his quip that "it must always be possible to replace [in geometric statements] the words, 'points', 'lines', 'planes', by 'tables', 'chairs', 'mugs'". (Reported by O Blumenthal. See Hilbert's *Gesammelte Abhandlungen*, vol. 3, 403.) On the importance of this conception of deduction for Hilbert's methodology in the *Grundlagen*, see Nagel, 242.

But we should still keep separate Hilbert's formalist view that the *meanings* of geometrical primitives are given by the axioms alone from this conception of deduction as attending only to the form, not the content of terms. (Pasch, for instance did not think the meaning of the geometrical primitives was given by the axioms alone; indeed, he was a radical empiricist on this score.)

coordinates—like homogeneous line, circle, or sphere coordinates—allowed proofs to be dualized much more readily than in Steiner-style synthetic geometry.[37])

Second, there is a more fundamental problem besides duality. In chapter 1, we saw that for Kant the distinguishing feature of mathematics is the fact that it "considers the universal in the particular, indeed even in the individual" (A713/B741): reasoning with a single, concrete (A735/B763) intuited object, drawn according to certain general conditions of construction, is sufficient to establish general conclusions about all objects of the same kind as the particular object under consideration. Kant's model here was of course ancient, diagram-based geometry, where the proof of a theorem required drawing a diagram that depicted one (but only one) specific instance of the theorem. (As Heath points out, Euclid kept to a "one theorem, one diagram" rule throughout the *Elements*.[38]) However, in many projective theorems, there is no single, concrete drawn diagram that can represent the theorem in its full generality. From a projective point of view, figures 2.3 and 2.7 in my representation of Brianchon's theorem depict the very same case, even though in one picture the conic is an ellipse and in the other the conic is an infinitely distant point pair. (And in Plücker-style analytic projective geometry, cases 2.3 and 2.7 could be represented at the same time as 2.4 and 2.8, the duals of cases 2.3 and 2.7, by one equation that could, depending on one's point of view, be decomposed as an equation in line coordinates or in point coordinates.) As Hankel complained, on the ancient model, these two cases would correspond to two distinct theorems requiring two distinct

---

[37] On Plücker's work, see Kline, *Mathematical Thought*, 836ff. Mark Wilson's paper "Ghost World: A Context for Frege's Context Principle" (in Erich Reck and Michael Beaney (ed.), *The Routledge Companion to Frege*) also nicely explains how Plücker's analytic work allowed for generalizations of projective duality.
[38] *The Thirteen Books of Euclid's Elements*, vol. 1, 245-6. See also Manders, "The Euclidean Diagram (1995)," and Mumma, "Proofs, Pictures, and Euclid."

proofs, and the unification effected by projective methods would be lost. For this reason, synthetic, projective geometers following Poncelet came to think of geometrical figures as ideal objects that continuously move and shift (on the model of shifting projections of a scene on a moving canvas). No longer "isolated figures rigidly fixed," projective diagrams were allowed to "modify and vary their position in sensible representation."[39] This new way of thinking manifested itself in the language projective geometers used. In figure 2.2, we see four cases of a pair of conics, in each of which, on Poncelet's view, there are four points of intersection. In Chasles's way of speaking, each of these drawn cases are representatives of the same "figure," though each shows some "accidental condition" of that figure. The only way we can imagine this figure is by drawing it in one of its conditions, modifying it by changing the point of projection and then rotating the plane that sections it. One drawn diagram can represent a given theorem in its full generality only if we imagine the one diagram continuously in motion, as parts of it appear, disappear, or move off to infinity.[40]

Moreover—and this is the third element in projective geometry that undermined the traditional use of diagrams—the ideal elements the Poncelet and his followers introduced with such success into geometry could not be depicted in a drawn diagram in the straightforward way that Euclid and early modern geometers could represent circles and lines. One cannot draw an infinitely distant point pair, or an "invisible" intersection of two conics. But, as Chasles claims, these points can be represented indirectly.

> The consideration of the accidental relations and properties of a figure or of a geometrical system is suitable to provide an explanation for the word *imaginary*, which is used very frequently and with great benefit in pure geometrical

[39] These are Merz's and Hankel's words, quoted above.
[40] Or, as in the works in algebraic geometry that derived from Plücker, move off into higher dimensions.

> speculation. In fact, we can treat the expression *imaginary* as indicating only a condition of a figure in which certain parts, that were real in another condition of the figure, have ceased to exist. For we can form in no other way an idea of an imaginary object, than by thinking at the same time of an object in space in a condition of real existence; so that the idea of the imaginary would be without sense, if it were not derived from the actual idea of a real existence of the same object to which we now apply the idea of the imaginary.[41]

It follows immediately that there is no single drawn diagram that can represent all of the parts or aspects of the figure and that diagrams need always be considered as momentary snapshots of continuously altering figures some of whose parts remain necessarily hidden at any given time. Describing this point of view, the historian Steven Kleinman writes, "one senses that many classical geometers had a platonic view of figures like conics. There are ideal conics of which we see only shadows or aspects like their point sets and their envelopes of tangent lines."[42]

Now a Kantian might not want to allow ideal elements into geometry and might insist that geometry, as the science of the form of our outer intuition, could never consider "invisible" or "ideal" elements in space. However, for Kant, the high road would be a costly road. Kant's theory of mathematical cognition is motivated by the recognition that mathematicians, like the geometers working in the ancient, synthetic tradition, are able to draw *general* conclusions from specific drawn figures. It is this very feature of projective geometry that Poncelet and his successors trumpeted as the chief advantage of their work.

It would be much too hasty to conclude from the fact that projective geometry undermined Kant's ancient, diagram-based model that Kant and modern geometry are necessarily caught in an irreconcilable conflict. What is clear, however, is that any

[41] Chasles, note XXVI, 394-5 in the German edition, from which I have translated it.
[42] Steven Kleinman, "Chasles' Enumerative Theory of Conics," 133.

remotely adequate Kantian philosophy of geometry would have to recast, perhaps fundamentally, the Kantian theory of pure intuition in geometry that I outlined in the previous chapter. It is not obvious what such a recasting would look like, nor is it obvious that there would be a unique, philosophically most satisfying philosophy of geometry. The remainder of this dissertation, culminating in the final chapter, considers the particular recasting developed by Ernst Cassirer. But before we turn to that story, it is necessary to give a fuller characterization of the philosophical project Cassirer is taking on when he tries to give a Kantian philosophy of geometry responsive to the full revolution in nineteenth century geometry.

6. THE PHILOSOPHICAL CHALLENGE POSED BY MATHEMATICAL CONCEPTUAL INNOVATION

The great mathematical interest of projective (or, equivalently, descriptive) geometry can be gathered from the fact that it was often called "higher" geometry or "modern" geometry.[43] That it had great philosophical interest is apparent already from the fact that the two great originators of modern logic, Frege and Russell, started their careers with dissertations on projective geometry.[44] This philosophical interest was often pressed by

---

[43] There were dissenting opinions, of course. Alberto Coffa, for instance, has recently described the attitude of the great early twentieth-century mathematician Hermann Weyl towards Felix Klein's [1872] group-theoretic foundation of geometry in this way:

> In Weyl's words, the dictatorial regime of the projective idea in geometry was finally overthrown by Klein's Erlanger Programm, which proclaimed all groups of transformations as geometrically equal.

Coffa, "From Geometry to Tolerance," 12. Unfortunately, Coffa does not provide a source for this remark.

[44] Frege's dissertation ("On a Geometrical Representation of Imaginary Forms in the Plane," 1873) is an attempt to represent the complex points in a plane by pairs of real points in two planes. Russell's dissertation, defended in 1895 and published in 1897 as *The Foundations of Geometry*, was an attempt to carry out the Kantian project of isolating the *a priori* element in space. He there claimed that "the whole

geometers themselves. In 1859, Arthur Cayley, having constructed a distance function for pairs of points in the plane in terms of a certain projective relationship (the cross-ratio) between that pair and two fixed, infinitely distant and imaginary points, declared that "metrical geometry is thus a part of descriptive geometry, and descriptive geometry is *all* geometry, and reciprocally."[45] In his later (1883) Presidential Address to the British Association, Cayley argued that his use of imaginary points was not unusual, and he took great pains to emphasize "how wide the application of the notion of imaginary is—viz. (unless expressly or by implication excluded), it is the notion implied and presupposed in all the conclusions of modern analysis and geometry. It is, as we have said, the fundamental notion underlying and pervading the whole of these branches of mathematical science."[46] Cayley deplored the fact that no philosophers had worried about this problem in anything like the way it deserved:

> [T]he notion of a negative magnitude has become quite a familiar one…But it is far otherwise with the notion which is really the fundamental one (and I cannot too strongly emphasize the assertion) underlying and pervading the whole of modern analysis and geometry, that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry: I use in each case the word imaginary as including real. This has not been, so far as I am aware, a subject of philosophical discussion or enquiry. As regards the older metaphysical writers this would be quite accounted for by saying that they knew nothing, and were not bound to know anything, about it; but at present, and, considering the prominent position which this notion occupies—say even that the conclusion were that the notion belongs to mere technical mathematics, or has reference to nonentities in regard to which no science is possible, still it seems to me that (as subject of philosophical discussion) the

field of projective geometry, so far as I am aware, has been hitherto unknown to philosophers" (6). He there tries to rectify this gap by giving an axiomatization of projective geometry and arguing that the axioms of projective geometry are a presupposition of metric geometry and preconditions of the possibility of experience.

[45] Cayley, "Sixth Memoir on Quantics," in *Collected Papers*, vol. 2, 592. A good summary account of Cayley's contributions, focusing on his contribution to the foundations of projective geometry, can be found in Wussing, *The Genesis of the Abstract Group Concept*, 167-178.

[46] Ewald, vol. 1, 449. Elsewhere in the lecture Cayley also emphasizes the importance of the imaginary for number theory, referring specifically to the theory of ideal numbers.

> notion ought not to be thus ignored; it should at least be shown that there is a right to ignore it.[47]

Putting Cayley's plea for a philosophical theory that will either justify or condemn the free use of the concept of the imaginary together with the comments from Dedekind and Frege at the beginning of this section, we see a loose cluster of philosophical questions concerning the method, justification, and value of introducing radically new concepts into familiar mathematical settings.

Before we look at how different philosophers tried to answer the sort of questions that I have sketched in the last few pages, it will be helpful to distinguish four interrelated but distinct questions, three of which will be a topic of discussion in what follows. The first question concerns the traditional topic of *concept formation* or *Begriffsbildung*: what is the process whereby new concepts in mathematics are formed or introduced? The traditional answer to this question, which we'll discuss more briefly, is that all new concepts (mathematical or otherwise) are formed from given representations by abstracting or erasing dissimilar elements and retaining the common ones. Another answer to this question was given by Dedekind, who, in an early essay, tried to specify "the peculiarity of the way in which, in mathematics, concepts develop from those which relate only to a more restricted domain into more general ones."[48] According to his theory, mathematical concepts develop when one takes laws that hold of a given mathematical domain and allow them to have "general validity": "Then these laws conversely become the source of the generalized definitions if one asks: How must the

---

[47] 434.
[48] Dedekind, "On the Introduction of New Functions in Mathematics," in Ewald, vol ii, 760, paragraph 12.

general definition be conceived in order that the discovered characteristic laws be always satisfied?"[49]

The second question, or I should say, family of questions, concern the *value* or *explanatoriness* of the introduction of new concepts. The basic philosophical task here will be to understand why, in Dedekind's words, "the greatest and most fruitful advances" in mathematics have been the result of "the creation and introduction of new concepts."[50] An answer to this basic question will then act as a constraint on answers to the first question: one's theory of how concepts are formed needs to make it clear why advances like that made in geometry by the introduction of the concept <point at infinity> are possible at all. Indeed, as we'll see, a frequent criticism made in the late nineteenth and early twentieth century is that the traditional doctrine of concept formation by abstraction makes it impossible to see why concepts could have the sort of value that they obviously do have. A more ambitious question in this family concerns the difference between trivial concepts and genuinely fruitful ones: given that the greatest and most fruitful advances in mathematics are the result of the introduction of new

---

[49] Dedekind, "On the Introduction of New Functions in Mathematics," in Ewald, vol ii, 757, paragraph 6. An elementary example of this phenomenon is the extension of the natural numbers to integers, rationals, irrationals, and imaginaries by allowing the inverse operations of subtraction, division, and root extraction to be always defined.

[50] *Essays on the Theory of Numbers*, 36. Already in 1854, Dedekind's theory of concept formation by generalizing laws or operations is clearly meant to be a theory of why concept formation can lead to the "greatest and most fruitful advances in mathematics." Dedekind's later mathematical work, especially his work in the theory of ideals, was an expression of the importance of certain kinds of concepts for mathematics. For Dedekind, mathematical objects – like ideals or elliptic functions – should be defined or introduced in a "representation-independent" way. This means that they should be defined in terms of concepts that express their "fundamental," "foundational," or "internal" properties. The properly "basic" concepts are those that are useable in giving proofs that allow one to see or "predict" the result without computation and thus can do heavy lifting in proofs. A proper proof will be pure, in the sense that it will employ only the basic or fundamental concepts. For an illuminating analysis of Dedekind's mathematical practice, using his various revisions of his supplements to Dirichlet's lectures on number theory, see Jeremy Avigad, "Methodology and Metaphysics in Dedekind's Theory of Ideals." For a discussion of Dedekind's approach, with reference to Riemann and Frege, see Jamie Tappenden, "The Caesar Problem in its Historical Context: Mathematical Background" and "The Riemannian Background to Frege's Philosophy".

concepts, what is it about some concepts that make them so fruitful? Why was the introduction of the Sheffer stroke[51] of so little value mathematically, while the introduction of Riemann surfaces – multilayered “sheets” on which multi-valued elliptic or Abelian functions over complex values become single-valued – revolutionized mathematics? This family of questions is *epistemological* in a broad sense of the word, and is related to such questions as “What is it to understand something mathematically?” or “What is it for a proof or a theory or a concept in mathematics to be explanatory?”[52]

A third question concerns the *justification* of the introduction of new concepts. Consider the concept <constructible, equilateral pentagon> in Euclidean geometry. In Book I of the Elements we are given conditions on constructibility in the postulates, and have as basic defined concepts <equilateral> and <pentagon>. Having formed the concept <constructible, equilateral polygon> from the basic defined concepts, one wants to know whether anything in space falls under it, and the concept is mathematically interesting whether or not there are constructible, equilateral pentagons. This is not so with some of the more important concepts that came into their own in the nineteenth century—concepts like <point at infinity>, <quaternion>, <complex number>, <ideal>—where the introduction of a new concept is meant to bring with it a new domain of objects falling under the concept. Indeed, the concept <point at infinity> would be of no value in geometry if there were not in fact points at infinity. For these sorts of concepts, since their value depends on there being objects that fall under them, to question whether one is justified in introducing them just is to question whether one has any right to act as if there

---

[51] I owe this example to Jamie Tappenden, “The Riemannian Background to Frege’s Philosophy,” 111.

[52] I do not mean to imply that there is no difference between a concept’s being fruitful, or its being explanatory, or its being valuable. Similarly, for a proof to be explanatory need not be the same thing as its contributing to our understanding. I mean only to be identifying a family of questions.

were objects that fall under them. (Hence the sliding one sees in the quoted passages from Cayley and Dedekind between talk of introducing new concepts into mathematics, and introducing new kinds of objects.) Here we have then the recognizably ontological question "Are there points at infinity?" and perhaps also, "If there are no points at infinity, why is it so valuable to act as if there were?"

A fourth question is a *semantic* question concerning the proper referents of concept expressions like "point" or "space" or "number" as they are used at different points in the development of mathematics. It seems natural to say that those mathematicians who lived on the other side of the nineteenth century geometrical revolution were studying space just as those geometers who lived after it were. It seems natural also to say that when these earlier geometers had thoughts that they might have expressed by saying, e.g. "all of the points in space are a finite distance from any given point," they were grasping the concepts <point> and <space>—the very same concepts that Poncelet, Steiner, Plücker, or Cayley grasped. It further seems natural to say that when ealier geometers used the words "point" or "space" (or their non-English equivalents), they were referring to those very concepts—<point> and <space>—that Poncelet, Steiner, Plücker, or Cayley were referring to when they used the words "point" or "space." Let me give a dramatic case to illustrate my point. I mentioned above the "Cayley metric" on the space, whereby Cayley was able to define the distance between two points projectively. In summary, making use of some refinements introduced later by Felix Klein (1871), the definition of the distance between two points A and B goes like this. First, using a complicated projective procedure due to von Staudt,[53] assign coordinates to all of the points on the projective, complex plane.[54] Take the two points *A* and *B* and their common line *AB*. Now consider the point of intersection of this line *AB* through some arbitrary conic section at points *C* and *D*. (Since we're allowing ourselves imaginary points, every line intersects every conic at two points, though they may be coincident or imaginary.) Now take the ratio of the four points (AC/CB)/(AD/DB), which Steiner had proved to be projectively invariant. This will give us a number, whose logarithm when multiplied by some arbitrary constant gives us a number. For this to be the right number for the distance, we need to make sure that we picked the right conic section. Since the definition of a conic given by Steiner allows for degenerate conics, let the conic we want be just a point pair, and since we are letting ourselves have infinitely distant and imaginary points, let these two points be imaginary and infinitely distant—indeed we can let these two points be those unique points that all circles intersect. It is then easy to show that what we get at the end is the familiar Euclidean distance function, though now placed in its proper, that is projective, setting. Klein, summarizing this approach for the three-dimensional case, writes:

> In this way, I shall demonstrate that this representation is not only an interpretation of geometry, but that it explains its inner essence…Let an arbitrary

---

[53] On von Staudt's work and influence, see Hans Freudenthal, "The Impact of von Staudt's Foundations of Geometry." His procedure was set forth in his 1847 "Geometrie der Lage" and "Beiträge zur Geometrie der Lage" (Nürnberg: Bauer and Raspe, 1856, 1857, and 1860).

[54] In 1859, Cayley had allowed himself to assume that all the points on the complex projective plane came with arbitrary coordinates, and he showed that his definition did not depend on how the arbitrary coordinates were assigned. Klein, in 1871, showed how the earlier work by von Staudt could be used to *construct* a set of coordinates in projective space using the so-called quadrilateral construction. Klein also made use of von Staudt's construction of the imaginary points in the real line using relationships (involutions) between point pairs and conics that can act like normal points. This allowed Klein to make systematic use of point coordinates, including complex coordinates, without worrying that the concept of distance had been smuggled in from the beginning, and without thinking of the definition as mere algebraic manipulation with no geometrical interpretation. For a summary of Klein's procedure, see his *Elementary Mathematics*, Part Two, chapter V; Part Three, chapter I, especially 134-5, 148-159.

> quadric surface [for instance, a sphere] be given. The line joining two given points in space intersects the surface in two points. The two given points and the two points on the surface have a certain cross ratio *and the product of an arbitrary constant c and the logarithm of that cross ratio is defined to be the distance between the two points*.[55]

And summarizing the relationship between this result and ordinary geometry, he writes:

> Metric geometry is just the investigation of the projective properties of space forms relative to a fixed given conic—the imaginary circle at infinity.[56]

But if this elaborate construction is really the *essence* of distance, and not just some way of interpreting it in complex, projective terms, and if the study of the properties of distances between points is just the study of their projective relations to an infinitely distant, imaginary circle, what are we to say of those millennia of geometers who knew nothing of <cross-ratio>, <point at infinity>, <degenerate conic>, and <circular points at infinity>—the concepts by means of which distance is being defined? If the concept that they grasped when they used "distance" or its non-English equivalents was *different from* the concept that Klein is talking about here, then we cannot take at face-value his claim to have discovered the essence of <distance>. If the concept that they grasped when they used "distance" or its non-English equivalents was the *same as* the concept that Klein is talking about, then we need some explanation of how they could have grasped a concept,

---

[55] "On the so-called Non-Euclidean Geometry," in *Gesammelte mathematische Abhandlungen*, vol. 1, 247-8; translated by Abe Shenitzer in Wussing, *The Genesis of the Abstract Group Concept*, 176. I have modified the quotation from the first sentence, changing plurals to singulars. In the original, Klein puts his definition of distance to work by showing that a change in the conic gives us different metrics, and so allows us to distinguish Euclidean and the various non-Euclidean geometries. I have abstracted from this extremely important element in Klein's work to keep the case simpler.

[56] "Über Liniengeometrie und metrische Geometrie," in *Gesammelte mathematische Abhandlungen*, vol. 1, 106; translated and cited in Wussing, *The Genesis of the Abstract Group Concept*, trans., Abe Shenitzer, 177.

or used linguistic expression that referred to a concept, the understanding of whose proper analysis requires grasp of concepts with which they were not acquainted.[57]

At the beginning of this chapter, I distinguished the projective revolution in nineteenth century geometry from the better known revolution brought about by the discovery of consistent non-Euclidean geometries. Now we can see that the philosophical task posed by complex, projective geometry is different from that posed by non-Euclidean geometry. In his *Essay*, Russell argues that since each of the various metrical geometries of constant curvature are logically possible descriptions of empirical space, only empirical observation can determine which set of axioms is true of space. But observation being necessarily imprecise, we can never be sure that space is exactly

---

[57] Some of what Frege writes suggests that he is looking for an answer to this fourth question. Consider, for example, this passage from "The Law of Inertia," in *CP*, 133; original publication, 158:

> A logical concept does not develop and it does not have a history, at least not in the currently fashionable sense. Unlike [Ludwig Lange, whose book *The Historical Development of the Concept of Motion and its Prospective End Result* Frege is reviewing], I see no great need for being able to talk about the history of the development of the concept, and I find there is good reason to avoid this phrase. If we said instead 'history of attempts to grasp a concept' or 'history of the grasp of the concept', this would seem to me much more to the point; for a concept is something objective: we do not form it, nor does it form itself in us, but we seek to grasp it, and in the end we hope to have grasped it, though we may mistakenly have been looking for something where there was nothing.

This passage, and others like it, strongly suggest that Frege sees the development of science, in some cases at least, as consisting in attempts at grasping fully the very same concept over time. Tyler Burge summarizes this strand in Frege's thinking in this way:

> There are a number of places where Frege claims or presupposes that individuals can express senses, or think thoughts through the use of language, that they do not have the background knowledge to fully understand. Such understanding may be lacking even in the individual's broader linguistic community…Frege believed that norms of reason play a role in determining the nature of an individual's thought and the sense of an individual's (and community's) linguistic expression. He believed that given the function of mathematical thinking and given the fact that mathematical thinking is basically on the right track, the senses of mathematical expressions are partly determined by their role in a correct and uniquely appropriate rational elaboration of actual usage and incomplete understanding. Thus he regarded his logicist elaboration of ordinary arithmetical discourse as revealing what ordinary mathematicians had been thinking with incomplete understanding. (*Truth, Thought, Reason*, 55-6.)

On Burge's reading, Frege actually distinguishes two cases. In the first case, a development in mathematics is best seen as achieving a more complete understanding of a concept that had been incompletely grasped by earlier mathematicians. In the second case, mathematicians come to use a word so differently from earlier mathematicians that it is better to see mathematicians as coming to grasp a different concept as before and shifting the sense of a concept word. See "Frege on Sense and Meaning," reprinted in *Truth, Thought, Reason*, 256-7.

flat, and we can never rule out any of the non-Euclidean alternatives. The same kind of argument will not justify the free use of imaginary points. When we consider the work of geometers like Cayley in light of our "perception of space" or try to give it a spatial interpretation, we expose it as a nest of absurdities: "everyone can *see*," Russell says about the circular points, "that a circle, being a closed curve, cannot get to infinity."[58] He concludes that, for all of the fruitful use of imaginaries in geometry, they are merely a convenient fiction, possessing no spatial correlate, and that they are useable only as technical instruments for manipulating algebraic expressions describing actual space.

The contrast between Russell and Cayley shows that the lines one takes on the two philosophical challenges posed by the two geometrical revolutions can be independent of one another: one could be a liberal about non-Euclidean geometries and a conservative about the imaginary elements, as Russell was; or one could be a liberal about the imaginary elements and a conservative about non-Euclidean geometries, as Cayley was.[59] Now suppose that you wanted to understand how Kant's philosophy of mathematics would have to change to fit with *both* of the revolutions that moved geometry away from Kant's ancient synthetic model. And suppose further that you wanted to answer Cayley's challenge without condemning seventy-five years of

[58] *EFG*, 45, emphasis added. Concerning the continuity reasoning that many geometers used to justify the imaginaries (see, for instance, note 12), Russell wrote the following: "Thus given a coordinate system, and given any set of quantities, these quantities, *if they determine a point at all*, determine it uniquely. But, by a natural extension of the method, the above reservation is dropped, and it is assumed that to *every* set of quantities some point must correspond. For this assumption there seems to me no vestige of evidence. As well might a postman assume that, because every house in a street is uniquely determined by its number, therefore there must be a house for every imaginable number." (*EFG*, 44)

[59] Cayley in fact thought that the axiom of parallels is indemonstrable and part of our notion of space ("Presidential Address to the British Association," 547 in reprint; original edition, 434-5).

revolutionary work in geometry into a formalist or fictionalist limbo.[60] What would a Kantian position look like?

[60] G.J. Stokes (in his "The Philosophy of the Imaginary"), for instance, complains that, on Russell's view, the imaginaries are just "tacked on" from outside geometry, and mathematics emerges as "an unaccountable movement of thought:" "why should a subject receive invaluable aid from what seems to be an intellectual outcast with no real right in its own home?"

CHAPTER 3 MATHEMATICS AND THE LOGIC OF CONCEPT FORMATION IN CASSIRER'S *SUBSTANZBEGRIFF UND FUNKTIONSBEGRIFF*

On the morning of Ernst Cassirer's death, 13 April 1945, Cassirer finished a short manuscript based on a paper he had written a year before on the philosophical implications of the geometrical uses of the concept of a group. He includes a personal reminiscence of his student days in the 1890s:

> At this time I was a student of Kantian philosophy…I had no doubts that the Kantian thesis—the thesis of the empirical reality and the transcendental ideality of space—contained the clue to the solution of [the] problem [of the nature of space]. But how was this thesis to be reconciled with the progress of geometrical thought made in the nineteenth century?[1]

As we will see in the rest of this dissertation, Cassirer's career-spanning work on space and geometry went beyond the important but rather specific question of the status of the parallel postulate. He wanted to turn his reflection on the full implications of all of the revolutionary work done in geometry in the nineteenth century, and he wanted to understand its implications for our answers to the four clusters of philosophical questions I described at the end of the previous chapter: how are new concepts formed or introduced in geometry? what makes a concept in geometry valuable or useful? what right do we have to think that there are objects falling under our geometrical concepts? with what right can we say that later geometrical work illuminates the content of fundamental geometrical concepts that have been grasped for centuries? And, as we will

[1] Cassirer, "Reflections on the Concept of Group and the Theory of Perception (1945)," 278-9.

see, Cassirer thought that our attempts at answering these clusters of questions will be greatly advanced by reflecting on Kant.

1. Cassirer's Philosophy of Mathematics in the Context of Late Nineteenth Century German Work in Logic

Because Cassirer conceived the philosophical challenge posed by modern geometry in this broad way, he insisted throughout his career that an adequate philosophy of pure mathematics should begin by reflecting on the nature of concepts. For that reason, I begin my discussion of Cassirer's philosophy of geometry in this chapter by following out a main argument of Cassirer's early works in the philosophy of mathematics: his 1907 paper "Kant und die moderne Mathematik" and part I, "The Concept of Thing and the Concept of Relation," of his 1910 systematic treatise in the philosophy of the exact sciences, *Substance and Function* [Substanzbegriff und Funktionsbegriff].[2] In both of these works Cassirer insists that an adequate philosophy of modern mathematics must begin with a study of the nature of the concept: What theory of the concept best respects recent developments in mathematics? What theory will best account for the role of concepts in the exact sciences? Cassirer summarizes his findings in the preface to *Substance and Function*:

---

[2] These were not the first works where Cassirer discussed the philosophy of mathematics: he approached the foundations of the calculus in his early book *Leibniz' System in seinen wissenscahftlichen Grundlagen* [1902], which incorporated as its introduction Cassirer's dissertation on Descartes, "Descartes' Kritik der mathematischen und naturwissenschaftlichen Erkenntnis," written under Hermann Cohen in 1899. He had also discussed philosophical problems in 17^{th} and 18^{th} century mathematics in the first two volumes of his *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit* (published in 1906 and 1907). But none of these works addressed philosophical problems arising from mathematics in the 19^{th} and early 20^{th} century. Indeed, their point of view remained close to Cohen's *Das Princip der Infinitesimal-Method und seine Geschicte. Ein Kapitel zur Grundlegung der Erkenntniskritik* (1883), which interpreted the calculus, in line with Kant's "Anticipations of Perception," as the mathematics of intensive quality.

> The investigations contained in this volume were first prompted by studies in the philosophy of mathematics. In the course of an attempt to comprehend the fundamental conceptions of mathematics from the point of view of logic, it became necessary to analyze more closely the function of the concept itself and to trace it back to its presuppositions. Here, however, a peculiar difficulty arose: the traditional logic of the concept, in its well-known features, proved inadequate even to characterize the problems to which the theory of the principles of mathematics led. It became increasingly evident that exact science had here reached questions for which there existed no precise correlate in the traditional language of formal logic. The content of mathematical knowledge pointed back to a fundamental form of the concept not clearly defined and recognized within logic itself. [3]

The "fundamental form" of the concept recognized in "the traditional language of formal logic" is what Cassirer calls the "substance-concept" [*Substanzbegriff*], thing-concept, or the "generic concept" [*Gattungsbegriff*] and is intimately tied up, Cassirer thinks, with an Aristotelian view of concepts, judgments, and inferences, and an abstractionist view of concept formation. The fundamental form of the concept that he finds in modern mathematics, he calls the "function-concept" [*Funktionsbegriff*], and the view of concept formation "the functional view of knowledge."

Though it might seem odd to modern readers to begin a work in the philosophy of mathematics or the exact sciences with a discussion of the nature and origin of concepts, this would not have been so strange to the German philosophers of the tail end of the nineteenth and beginning of the twentieth centuries. Cassirer and his readers saw the full challenge posed by modern mathematics as a task for *logicians*. Indeed, in Dedekind's *Habilitationsrede*, Dedekind frames his theory—that mathematical concepts develop from older ones by a generalization of the laws holding over the domain of objects picked out by the old concept—using the technical language that

---

[3] *SF*, preface, iii. Unless otherwise noted, translations from *SF* are from the 1923 Swabeys edition; translations from "Kant und die moderne Mathematik" are mine.

appeared in nineteenth century logical works.[4] And in Cantor's famous defense of the freedom of pure mathematics, he presents his justification for mathematics' autonomy—again echoing the language of eighteenth and nineteenth century formal logic—as a freedom to form concepts.[5] In the traditional classification, embodied in Kant's *Logic*, the doctrine of logic is broken up into two parts, the doctrine of elements and of method, the first of which is divided into three sections: the doctrine of concepts, of judgments, and of inferences. In the first section, the doctrine of concepts, after being told what a concept is (for Kant, a universal or reflected representation), there arises for us, in Kant's words, the question "*Which acts of the understanding constitute a concept?* or what is the same, *Which are involved in the generation of a concept out of given representations?*"[6] Or as the British logician William Hamilton put it, logic says of concepts "What they are" and "How they are produced"; "their peculiar character, their origin, and their general accidents"; concerning their origin, logic is to tell us, "the mode and circumstances in

---

[4] For instance, Dedekind introduces his theory of mathematical conceptual development by comparing, in a way reminiscent of Kant's distinction between sensible and intellectual intuition, a finite being's thinking with that of a being that has direct insight into the truth; as in Kant, Dedekind says only the former requires concepts. Similarly, in describing the role of concepts in descriptive sciences like mineralogy, Dedekind describes the procedure whereby an investigator considers a certain mark [Merkmal] as either an accidental or distinguishing feature of a material. See Dedekind, "On the Introduction of New Functions" paragraph 3; 756 in *From Kant to Hilbert*, vol. 2.

[5] Cantor, *Grundlagen* §8, in *Gesammelte Abhandlungen*, 182; translated by Ewald:

> Mathematics, in the formation [Ausbildung] of its ideas [Ideenmaterials], has *only* to take account of the *immanent* reality of its concepts and has *absolutely* no obligation to examine their *transient* reality…Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts [Begriffe] must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced….It is not necessary, I believe, to fear, as many do, that these principles present any danger to science. For in the first place the designated conditions, under which alone the freedom to form numbers [Zahlenbildung] can be practiced, are of such a kind as to allow only the narrowest scope for discretion…[T]he *essence* of *mathematics* lies precisely in its freedom.

[6] *JL* §5; Ak 9:93; *Lectures on Logic*, 591.

which our concepts are elaborated out of the presentations and representations of the subsidiary faculties."[7]

Though one can often get the impression that the period in the history of logic between Kant and the development of modern quantified relational logic in Frege, Russell, and Pierce was a period of relative inactivity, in fact the latter half of the nineteenth century saw a veritable explosion of work in "logic,"[8] as philosophers tried to rethink the traditional logic in light of epistemological questions, scientific methodology, and psychological issues. A very quick comparison of the theory of concept formation that one finds in Kant's *Logic* and works published later in the nineteenth century will give the reader a snapshot of this change.[9] In Kant's *Logic*, which was collected together from Kant's notes and published in 1800 by his student Jäsche, devotes a mere two pages to the formation of concepts. By the 1880s, when Cayley was calling on philosophers to give an account of the concept of the imaginary in mathematics, logicians were devoting

---

[7] Willian Hamilton, "Lectures on Logic," in *Lectures on Metaphysics and Logic*, vol. 3, edited by Mansel and Veitch (1866); 118, 130, 131. Hamilton, best known for his "quantification of the predicate," was the leading representative of Kantian "formal logic" in Britain.

[8] At least as "logic" as a philosophical discipline was understood at the time. And this is a big qualification, since the treatises called "Logic" and published by philosophers in the latter decades of the nineteenth century were not only very different from what contemporary philosophers would call "logic," but also very different from one another. Thus, Robert Adamson, in his entry on "Logic" for the 9th edition *Encyclopedia Britannica* (1882), writes of the logical works published from roughly 1860 to 1880:

> In tone, in method, in aim, in fundamental principles, in extent of field, they diverge so widely as to appear, not so many expositions of the same science, but so many different sciences. (Adamson, *A Short History of Logic*, 20)

Similarly, Wilhelm Windelband, in his 1912 survey of logical work roughly since Sigwart's 1873 *Logik* (excluding the development of modern, relational, quantified logic in Frege, Russell, and Pierce), writes:

> The transcendental point of view which [Kant's] Critical Philosophy introduced widened the logical problem, and this was only the first step in an entire change of principles which has been proceeding since that time in different and partly opposing directions. The position of Logic at the present day is the exact opposite of a uniform and commanding structure: its principles are fluid, the contradictions which are to be found between them involve not so much individual dogmas as fundamental points of view and problems of method. (*Theories in Logic*, 1-2)

[9] Alberto Coffa, in his *The Semantic Tradition*, attributes the relative obscurity of Frege's work during his lifetime at least in part to a lack of interest on the part of philosophers in the nature of concepts: "Most of Frege's colleagues were suspicious of so much semantic subtlety. Why should one care about the nature of concepts?" (72). Even the brief snapshot offered in this essay shows how far off such an historical thesis is.

much, much more space to the question. Perhaps the most influential logical treatise of the second half of the nineteenth century, Hermann Lotze's 1874 *Logik*, devotes fifty pages to the theory of concept formation, and in Wilhelm Wundt's 1883 *Logik*, in addition to a discussion of the traditional doctrine of concept formation by abstraction, we find detailed analyses of the ways in which particular sciences form concepts—including 250 pages on mathematical methodology, with detailed discussions of the principle of duality, Steiner's definition of a conic, the so-called principle of the permanence of form, and histories of the concepts of a mathematical function, and of the differential.[10] Similarly detailed and original theories were given by Benno Erdmann[11] and Benno Kerry.[12]

---

[10] The general account of mathematical method is given in volume II, second section, first chapter, "The general logical method of mathematics" (74-114 in the first edition (1883), and 101-147 in the second edition (1907)). A random selection will give the reader an idea of the scope of Wundt's discussion. At the end of his general discussion of mathematical deduction, Wundt distinguishes different kinds of analogical reasoning in mathematics. A principle that he calls "the principle of the permanence of mathematical operations" allows one "to extend certain operations or fixed concepts beyond their region, by continuing a determinate logical process beyond the boundaries that have been set for valid norms according to an analogy with those norms." This principle leads to new concepts and new laws, by transforming old concepts and thereby also the laws that hold of them (146-7). In the next chapter, he shows how this principle, which has great importance for the "development of mathematical thinking," is at work in the development of the concept of number from natural numbers, to irrationals, complex numbers, quaternions, and transfinite numbers. Still later in the third chapter, he discusses its use in introducing spaces of more than three dimensions, and non-Euclidean spaces.

[11] Erdmann (*Logik*, 1st ed, 1892, 2nd ed., 1907) argued that concepts are formed either in the traditional way from abstraction, or by unrestricted set-theoretic comprehension, tried to show that his theory of concept formation explained the formation of the concept of transfinite numbers, continuity, Dedekind's *Zahlkörper*, and spaces of *n* dimensions. The psychology of concept formation by abstraction is presented on 65-92 of the second edition; the second level concepts are discussed on 158-175. Erdmann called these new concepts "objects of the second level" or "totalities"["Inbegriffe"]: "a totality in its most general sense is, according to the above, the connection [Zusammenfassung] of some objects of our thinking to **one** object." He even mentions the totality of all totalitites – the set of all sets or the concept of all concepts — which has the peculiarity, he says, "that it contains itself as a member" (163 in the second edition).

[12] Kerry (*System einer Theorie der Grenzbegriffe. Ein Beitrag zur Erkenntnistheorie* (1890)), following Lotze, modeled conceptual structure on mathematical functions and thought that concepts could be formed by letting particular marks act like variable that pass off to the limit. He argued that his theory was adequate to explain the formation of the concept <point at infinity>. Kerry is better known to contemporary philosophers for his paradox—"the concept horse is not a concept"—and Frege's famous response to it in "On Concept and Object" (*CP*, 182-194).

Not surprisingly, the deepest and best worked out new model of concept formation can be found in Frege. In an unpublished paper from the early 1880s, Frege criticizes the traditional model of concept formation, common to Aristotle and Boolean logicians: "the logically primitive activity is the formation of concepts by abstraction, and judgment and inferences enter in through an immediate or indirect comparison of concepts via their extensions."[13] But none of the most fruitful concepts in mathematics arise in this way.

> [Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all the ways of forming concepts, that is one of the least fruitful. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really fruitful definitions in mathematics, such as those of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with the others.[14]

By 1902, Heinrich Rickert could complain, in his *The Limits of Concept Formation in Natural Science*, that recent logicians had so concentrated on the process of concept formation in the exact (mathematical or natural) sciences, that they had neglected to provide a theory that would respect the distinctiveness of concepts in the specifically *historical* sciences.[15]

Cassirer's approach in *Substance and Function* and related earlier and later work thus fits in with this wider historical trend within logic to reconsider the traditional theory

---

[13] "Boole's Logical Calculus and the Concept-script", *Posthumous Writings* [*PW*], 14-15.
[14] *Grundlagen*, §88.
[15] Rickert, Heinrich. *The Limits of Concept Formation in Natural Science*, 32 (trans. by Guy Oakes). In this work Rickert also comments on the extreme diversity in approaches among his fellow logicians:

> And even in the case of the concept of natural science, we cannot simply appeal to uncontested theorems of general logic. Especially since Sigwart [1873] removed the theory of concept from its place at the pinnacle of the system it occupied in traditional logic, no one has yet succeeded in giving this theory a generally acknowledge position and configuration. There is a considerable difference of opinion concerning what a 'concept' really is, or what this expression should most properly be used for. (32)

of concepts in light of detailed analyses of modern mathematics. In the next section of this chapter, I will present an overview of the 'Aristotelian'-abstractionist model of the concept that Cassirer opposes. In the third section, I will begin my discussion of Cassirer's objections to this model with an objection that I will call the "Lotze objection."[16] This will lead to a discussion of Cassirer's philosophy of arithmetic, which takes as its model Dedekind's "purely logical process of building up the numbers,"[17] with the abstractionist rhetoric and set-theoretic machinery toned down (section 4), and to a criticism, based on Cassirer's objection to the 'Aristotelian'-abstractionist model of the concept, of Frege's construction of the cardinal numbers as extensions of concepts (section 5).

In the Appendix to this chapter, we see that Frege himself gives an argument against the 'Aristotelian'-abstractionist model of the concept that is simliar to the polemic found in Cassirer. In his writings against the Booleans, Frege alleges that his superior, non-Aristotelian theory of concept formation makes clear that he and the Booleans are aiming at different goals, and only he is seeking a *lingua characterica*, a symbolic language that expresses perspicuously the conceptual content of judgments. Frege and Cassirer, then, gave themselves a common explanatory task, an investigation into the kind of concept formation that best captures the content and development of recent mathematical thinking, and provide a common solution, that the 'Aristotelian'-abstractionist model of concept formation is woefully inadequate, and a more acceptable theory requires the resources of modern relational logic. In the Appendix to chapter 3 I

---

[16] Neo-Kantians like Cassirer found in Kant two other objections to the traditional, 'Aristotelian'-abstractionist model of concept formation. Cassirer also endorsed these objections to the A-a model. See chapter 4.

[17] Dedekind, *Was sind und was sollen die Zahlen?*, translated in *Essays on the Theory of Numbers*, 31-2.

further argue that this common explanatory task and the seed of the common solution are present in the logical writings of other late nineteenth century German logicians—principally Hermann Lotze, whose 1874 *Logik* attempts to carry out this explanatory task and gives a solution, the "functional" view of the concept, which partially overlaps Cassirer's position and Frege's (different) position.

2. THE 'ARISTOTELIAN'-ABSTRACTIONIST MODEL OF THE CONCEPT

In this section, I present Cassirer's target in the form of two theses, *Aristotelianism about conceptual structure* and *Abstractionism about conceptual formation*, each with a few corollaries. (I call this the 'Aristotelian'-abstractionist theory, with quote marks, or the A-a model for short, in order to put aside the question of whether this was Aristotle's own view, and to make clear that, on Cassirer's view, at least, the model was nearly universally held-to, even by those who had no sympathy for Aristotelian philosophy otherwise.[18])

A. '*Aristotelianism*' *about conceptual structure*: concepts are either simple or are composed of simple concepts by conjunction, addition, or exclusion.

[18] Cassirer clearly thinks that this model is justifiable only if Aristotle's metaphysics is. See *SF*, 7:

> In the system of Aristotle, the criterion [needed to allow abstracted typical features to be meaningful] is plainly evident; the gaps that are left in logic are filled in and made good by the Aristotelian metaphysics…The process of comparing things and of grouping them together according to similar properties…does not lead to what is indefinite, but, if rightly conducted, ends in the discovery of the real essence of things.

See also his 1907 paper, "Kant und moderne Mathematik" [KMM], 6-7: "It is a remarkable twist of fate that this fundamental point-of-view, which, as one sees, has its justification and natural position in Aristotle's system, has survived long after the system ceased to be viable…Thus syllogistic appears overall as a particularly reactionary and inhibiting moment."

A.1 *Inclusion/Exclusion relations in Logic*: Logic captures the relations between concepts in judgments, paradigmatically in subject-predicate form, where concepts are judged to exclude or include one another; and the relations between concepts in inference, where new relations of inclusion or exclusion among concepts are arrived at from established relations.[19]

A.2 The *Sufficiency of the Syllogism*: the syllogism captures these relations among concepts.[20]

B. *Abstractionism about conceptual formation*: concepts are formed by noticing similarities or differences among particulars and abstracting the concept, as the common element, from these similarities or differences; compound concepts can arise from simpler concepts either by conjunction, disjunction, and exclusion, or by noticing similarities or differences among concepts and abstracting from these similarities or differences.[21]

B.1 Concepts *are essentially general*: the function of concept formation is to allow the mind to ascend to higher levels of generality.[22]

---

[19] *SF*, 8: "The fundamental categorical relation of the thing to its properties remains henceforth [after Aristotle] the guiding point of view; while relational determinations are only considered in so far as they can be transformed, by some sort of mediation, into properties of a subject or a plurality of subjects. This view is in evidence in text-books of formal logic in that relations or connections, as a rule, are considered among the 'non-essential' properties of a concept."

[20] KMM, 7: "Thus syllogistic appears overall as a particularly reactionary and inhibiting moment. Logic remains bound to the viewpoint of substance and therefore to the fundamental form of the judgment of predication."

[21] *SF*, 5: "The essential functions of thought, in this connection, are merely those of comparing and differentiating a sensuously given manifold. *Reflection*, which passes hither and thither among the particular objects in order to determine the essential features in which they agree, leads of itself to *abstraction*. Abstraction lays hold upon and raises to clear consciousness these related features."

[22] *SF*, 18-19: "If we follow the traditional rule for passing from the particular to the universal, we reach the paradoxical result that thought, in so far as it mounts from lower to higher and more inclusive concepts, moves in mere *negations*…Thus all formation of concepts begins with the substitution of a generalized image for the individualized sensuous intuition…If we adhere to this conception, we reach the strange result that all the logical labor which we apply to a given sensuous intuition serves only to separate us more

B.2 *Instrumentalism about concepts*: the formation of new concepts does not add new content, but merely provides new instruments for thinking and reasoning about the old content more efficiently.[23]

B.3 *Primacy of particulars*: in forming new concepts, one must have grasped and surveyed the particulars (or concepts) that fall under the concept (or at least a good number of them) before abstracting the new concept from them. Grasp of particulars is prior to grasp of concepts.[24]

In a way, all of *Substance and Function* is a series of arguments against this model and a reflection on what follows from its rejection. Cassirer's basic approach in arguing against this model is to critically compare it to the function and structure of concepts in the exact sciences: his question is then "Is the theory of the concept, as here developed, an adequate and faithful picture of the procedure of the concrete sciences?" (*SF*, 11)[25]

---

and more from it. Instead of reaching a deeper comprehension of its import and structure, we reach only a superficial schema from which all peculiar traits of the particular case have vanished."

[23] *SF*, 10: "When several composite presentations have a part of their content in common, there arises from them, according to the well-known psychological laws of simultaneous stimulation and fusion of the similar, a content in which merely the agreeing determinations are retained and all others suppressed. In this way, no new, independent and special structure is produced, but only a certain division of presentations already given, a division in which certain moments are emphasized in a one-sided direction of attention, and in this way raised more sharply out of their surroundings…What gives [the concept] the appearance of independent value and underived psychological character is merely the circumstance that our attention, being limited in its powers, is never able to illumine the whole of the presentation and must of necessity be narrowed to a mere selection of parts."

See also *Problem of Knowledge*, vol. 4 (1950) [*PK*], 108. Criticizing Mach, Cassirer writes: "The implication of this was the strange corollary that the necessity of thinking in general concepts and laws did not present a special virtue of the human mind but actually originated in its weakness. An intellect broad and inclusive enough to grasp all the details as details would require no such round-about method—nor would there be any 'science' for it."

[24] See *SF*, 18-19, quoted above. See also *SF*, 8: "Only in given, existing substances are the various determinations of being thinkable. Only in a fixed thing-like substratum, *which must first be given*, can the logical and grammatical varieties of being in general find their ground and real application" (emphasis addded).

[25] This basic approach is in fact an instance of the general Marburg point-of-view, where the task of philosophy, *Erkenntniskritik*, is to take the existence of the exact sciences as a fact and investigate its conditions of possibility. See chapter 5.

3. The Lotze Objection to the traditional A-a model of the concept

In “Kant und moderne Mathematik” Cassirer endorses a main argument of Russell’s *Priniciples of Mathematics*, that many of the most fundamental concepts in mathematics are not even expressible in Aristotelian logic, since it cannot represent essentially relational statements (or it at least cannot do so in a way that makes them useful for reasoning).[26] But Cassirer, like many of his contemporaries, thought that the problems with the traditional model ran deeper. In this chapter, I will focus on a different objection to the A-a model, which I will call the “Lotze objection.” (Two different objections to the traditional model, each derived from Kant, will be discussed in the next chapter.) Start by considering a case that Cassirer borrows from Lotze’s *Logic*.[27] The abstractionist model instructs us to take some particulars, compare them and by reflection isolate their common elements, and then abstract away those common elements. Now take cherries and raw meat; by abstraction we arrive at the concept <red, edible, and juicy>.

> It may … easily happen that a group of [marks], say *i k l*, occurs in several universal concepts, *S T* and *V*, at once, without it therefore representing a higher universal containing all species of *S T* and *V*. We may class cherries and flesh under the group *i k l* of red, juicy, edible bodies, but we shall not suppose ourselves thereby to have arrived at a generic concept of which they deserve to be called species. … Of the true universal, on the other hand, which contains the rule

---

[26] KMM, 7, note 2: “There are a large class of judgments that are in no way able to fit into the customary schema by virtue of which a property is ascribed to a determinate ‘subject’ as a predicate; instead this schema falsifies especially the most important fundamental judgments of mathematics.” Cassirer cites §214-5, where Russell argues that the existence of asymmetrical relations—which are necessary parts of the concept of a series, and so also all of arithmetic, geometry, analysis, etc—is incompatible with the view that all relations must be reducible to internal properties of the relata. See Russell §216 for a pithy summary: “The present long digression into the realm of logic is necessitated by the fundamental importance of order [for mathematics], and by the total impossibility of explaining order without abandoning the most cherished and widespread of philosophic dogmas. Everything depends, where order is concerned, upon asymmetry and difference of sense, but these two concepts are unintelligible on the traditional logic.”

[27] *SF*, 7; Lotze, *Logic*, vol. 1, §31.

> for the entire formation of its species, it may rather be said that its content is always precisely as rich, the sum of its marks precisely as great, as that of its species themselves; only that the universal concept, the genus, contains a number of marks in a merely indefinite and universal form. (Lotze, §31)

By itself, then, the abstractionist model of concept formation fails to capture what we do when we form concepts, since we would never form such a concept, nor think that we had advanced our understanding of things by forming such a genus with the species <cherry> and <raw meat> under it. We don't think this at least in part because having this concept tells us little about its species.[28]

What is it about the concept abstracted from the representations of raw meat and cherries that makes it degenerate? In his *Logic*, Lotze mentions this example in the process of arguing against the traditional principle of the inverse ratio of a concept's extent [Umfang] (number of objects falling under it) to its content [Inhalt] (the number of marks or component concepts out of which it is composed); e.g. the concept <large midwestern city> has more content but fewer things falling under it than <city>. Lotze's argument is that this principle only holds for defective concepts like <red, edible, juicy, body> where this combination of marks gives you little information about what might fall under it. A "true universal," that is, a genuine concept, contains within it a rule for determining what the specific properties of the objects falling under it would be. Lotze, and Cassirer following him, thinks that this point can be made most clearly in the case of

[28] Schlick argued against Cassirer that this is no criticism of the traditional model, since the purpose of that model was merely to indicate how one can form possible concepts and not to provide a criterion for distinguishing valuable from valueless concepts (*Allgemeine Erkenntnistheorie*, first edition, 1918, 23-6). Cassirer replied to Schlick that this is precisely the point: "it is clear that a concept can possess a full formal justification for its correctness in the sense of the classical theory, without that saying the slightest thing about its value for cognition, nor guarantee its success in any sense" (Cassirer, "Zur Theorie des Begriffs" [ZTB] 1928, 159). Cassirer wanted precisely to know what made valuable concepts in mathematics valuable, and he wanted his theory of the structure and formation of concepts to be a theory of the structure and formation of these successful concepts.

mathematical concepts, where the A-a model's inadequacies are most manifest.[29] "When a mathematician makes his formula more general, this means not only that he is *to retain* all the more special cases, also be able to *deduce* them from universal formulas."[30] Consider a typical mathematical concept, say <curve of the second degree>, which is expressible in a mathematical formula, $a_2x^2 + a_1x + b_2y^2 + b_1y + cxy + d = 0$. By substituting different constants, we arrive at different specific curves, such as that of an ellipse, a parabola, or a hyperbola. (For example, by letting a_1, b_1, c, and d vanish, we get the equation of an ellipse.) In this case, we are able to capture all of the mathematical properties of particular figures that fall under the concept by substituting for the constants. The constants then mark places where particular properties of figures have been abstracted but in such a way that all of the content of the particulars can be derived from the concept itself. The content of the concept expressed in the above formula is at least as rich as the content of all the particulars, since it contains within it the possibility of recapturing all of the particular figures falling under it as well as a systematic way of viewing all of them as related to one another.[31]

Now it would seem to follow from the A-a model that a concept must be poorer in content than the particulars that fall under it and the species subordinated to it, since we ascend to higher genera only by *removing* elements of our representation of particulars

---

[29] At this point, I will follow out Cassirer's presentation of the argument. On Lotze's discussion of mathematical inferences and concepts representable by "constitutive equations," which fills in a gap in Lotze's argument here, see the Appendix.

[30] *SF*, 19. Cassirer takes himself to be summarizing a point made by Lambert in his criticism of Wolffian logic.

[31] "Hence the more universal concept shows itself also as the more rich in content; whoever has it can deduce from it all the mathematical relations which concern the special problems, while on the other hand he takes these problems not as isolated but as in continuous connection with each other, thus in their deeper systematic connection" (*SF*, 20). For a generally sympathetic discussion of this argument, see Weyl, *Philosophy of Mathematics and Natural Science*, §20.

while retaining others. Moreover, as Lotze and others argued, the model seems to imply that the introduction of a new concept would always be as pointless as the formation of the concept <red, juicy, edible>. As the British Neo-Kantian T.H. Green put it,[32]

> The process of abstraction, as ordinarily described (as beginning with complex attributes and leaving out attributes till the notion is reached which has the minimum of determinations), if it really took place, would consist in moving backwards. It would be a donkey-race. The man who had gone least way in it would have the advantage, in respect of fullness and definiteness of thinking, of the man who had gone the furthest.

On the traditional model, the highest genus, that from which all content has been abstracted, is simply <something>. But what could be the value of abstraction if the man who had gone the furthest in forming concepts was left only with this concept? As Cassirer puts the point:

> If we merely follow the traditional rule for passing from the particular to the universal, we reach the paradoxical result that thought, in so far as it mounts from lower to higher and more inclusive concepts, moves in mere *negations*…What enables the mind to form concepts is just its fortunate gift of forgetfulness, its inability to grasp the individual differences everywhere present in the particular cases. If we adhere strictly to this conception, we reach the strange result that all the logical labor which we apply to a given sensuous intuition serves only to separate us more and more from it. Instead of reaching a deeper comprehension of its import and structure, we reach only a superficial schema from which all peculiar traits of the particular case have vanished.[33]

In Green's words, "if the function of thought is abstraction, … [then] the more we think, the less we know."[34]

---

[32] Green, "Lectures on Formal Logicians," 193, vol. 2 of his Collected Works.
[33] *SF*, 18.
[34] Op.cit. See also Emil Lask, *Fichtes Idealismus und die Geschichte*, 30-1; 43-44. in *Gesammelte Schrfiten*, vol.1, published 1901). Lask here distinguishes between what he calls an "emanationist" view of concept formation, and an "analytic" theory of concept formation. According to the analytic theory, which he finds in Kant, the real is the particular, empirical thing, which is unconceptualizable; and since concepts are formed by abstraction, the more general the concept is, the more remote from reality it becomes.

In his discussion of modern geometry in chapter 3 of *Substance and Function*, Cassirer argues that the projective methods introduced by Poncelet illustrate what makes a mathematical concept valuable: "[in Poncelet's geometry] between the 'universal' and 'particular' there subsists the relation which characterizes all true mathematical concept formation; the general case does not absolutely neglect the particular determinations, but it reveals the capacity to evolve particulars in their concrete totality from a principle."[35] In our example from the previous chapter, the concept <Steiner conic> is not poorer in content than the particular concepts (<ellipse>, <parabola>, <infinitely distant point pair>, <line pair>) that fall under it or the representations of particular figures falling under these concepts. Rather, by moving the line at infinity throughout the plane and moving the projective pencils with respect to each other, each of the particulars are transformed into each other, and it becomes patent, for the first time, that certain properties are true of all of these prima facie dissimilar figures. The traditional model, on the other hand, sees the process of forming a concept as a process whereby we isolate the common element in my representation of different particulars. But in the case of the projective proof of Brianchon and Pascal's theorem, before being initiated into the projective way of thinking, I need not have already seen all four cases as instances of hexagons circumscribed or inscribed around conics. Before being shown the definitions and the proofs, who would have thought of a hexagon with parallel sides as inscribed within a (degenerate, infinitely distant) conic? And even if I could already see the truth of the theorems in all four of my cases, forming the concept <Steiner conic> could enable

[35] *SF*, 82.

me to see something genuinely new, not just allow me to abstract away from distracting details.[36]

For Cassirer and Lotze, these reflections on what makes mathematical concepts valuable have consequences for our model of the *structure* of mathematical concepts. If the abstractionist model of concept formation can be represented as the operation of taking from a series of representations $a\alpha_1\beta_1$, $a\alpha_2\beta_2$, … the concept a, the new model, based on mathematical functions can be represented by the operation of replacing this series with $a$xy. This formula or function can now be viewed as a rule for generating particulars: "The genuine concept does not disregard the peculiarities and particularities which it holds under it, but seeks to show the *necessity* of the occurrence and connection of just these particularities. What it gives is a universal *rule* for the connection of the particulars themselves."[37] As Cassirer put the argument later, if we express the rule for the formation of the particulars as a function, say f(x), then the A-a model would have us conceive this concept as a sum of its parts: f + x, say a certain kind of object conjoined with a certain functional order.[38] The point is that the way in which the component

[36] The instrumentalism of the A-a model has it that we form concepts in order to have a new labor-saving instrument for representing the content already contained in the particulars from which the concept is formed. But appealing to our limited powers of attention is not sufficient to explain interesting mathematical cases, like the power of projective methods in proving Pascal's and Brianchon's theorems—what makes the unification of all of these cases illuminating is not that it saves us time. (If there were a mathematician, or super-human thinker, who could run through the entire ancient proof of a theorem –with all of its 87 cases—with no more effort than needed to run through the simpler projective proof, this would not make the projective proof any less explanatory or valuable, even for him.)

[37] *SF*, 23, 19.

[38] See *SF*, 16-17:

> It can indeed appear as if the work of thought were limited to selecting from a series of perceptions $a\alpha$, $a\beta$, $a\delta$ …. the common element a. In truth, however, the connection of the members of a series by the possession of a common 'property' is only a special example of logically possible connections in general. The connection of the members is in every case produced by some general *law of arrangement* through which a thorough-going rule of succession is established. That which binds the elements of the series *a*, *b*, *c* …. together is not itself a new element, that was factually blended with them, but it is the rule of the progression, which remains

concepts are ordered in a mathematical concept are not captured by the A-a model, with its syllogistic, species/genus strictures. Lotze put this point by representing concepts—not as sums of marks—but as functions of component concepts:

> as a rule, the marks of a concept are not coordinated as all of equal value, but…they stand to each other in the most various relative positions, offer to each other different points of attachment, and so mutually *determine* each other; and … an appropriate symbol for the structure of a concept is not the equation $S = a + b + c + d$, etc, but such an expression as $S = F\ (a, b, c, \text{etc.})$ indicating merely that, in order to give the value of S, a, b, c, etc, must be combined in a manner precisely definable in each particular case, but extremely variable when taken generally. (§28)[39]

That is to say, each concept is a functional combination of its components, ordered in some determinate way (though usually not merely in terms of conjunction, disjunction, or negation), though there is no paradigm way in which the component concepts are to be represented in a concept. For example, <triangle> is not <figure *and* 3-sided *and* 3-angled>, but <figure *with* three sides *meeting at* three angles> (this is Lotze's example in §28); <prime number> is not <number *and* divisible *and* unit>, but <number divisible *only by itself and a* unit>. We can now summarize the Lotze objection.

> the same, no matter in which member it is represented. The function $F(a,b)$, $F(b,c)$, …., which determines the sort of dependence between the successive members, is obviously not to be pointed out as itself a member of the series, which exists and develops according to it.

Cassirer puts the argument in this form again in *Philosophy of Symbolic Forms*, vol 3 [*PSF3*], 301 and 311.

[39] A similar point was made by the British Neo-Kantian Robert Adamson. When logicians discuss the scientific methodology in "Applied Logic" or "The Doctrine of Method," he writes, they recognize that "in the notion," or in the concept, "is contained the representation of the essence or truth of reality." But this is plainly inconsistent with the traditional doctrine:

> It is impossible to contain with any consistency the merely arithmetical or numerical doctrine of the notion, as containing fewer marks than the individual, of the genus a characterized by a less number of attributes than the species, and so on. Underlying all genuine knowledge, all classification, and therefore all formation of notions, is the tendency towards the subordination of parts to a law which determines them. The generic attributes are not simply points of agreement, but the determining characteristics, and the notion of a thing is the explicit recognition of its nature as a particular manifestation of a universal law. (*Short History*, 134)

> *Lotze objection to the A-a model*: Even if it is adequate in other areas, the A-a model is inadequate to express the content of mathematical concepts, for two reasons. First, mathematical concepts do not have less content the more general they are, but just as much, since they provide a rule for deriving the content of all the particulars that fall under them. Second, compound mathematical concepts are not composed from simpler concepts by addition, conjunction, and negation only.

We can summarize both of these objections, as both Cassirer and Lotze do, to the A-a model of mathematical concept formation by saying that, in mathematics, concepts do not behave like species and genus, but like *functions*.

4. Cassirer's Dedekindian Philosophy of Arithmetic

Cassirer's favorite mathematical illustration of this new view of the concept is Richard Dedekind's foundations of arithmetic in *Was sind und was sollen die Zahlen?* Dedekind defines the (natural) numbers as an abstracted "simply infinite system" or smallest inductive set from some base element. He begins his little book in chapter I with some basic concepts of the theory of sets (which he calls 'systems')—extensionality, union, and intersection—and proceeds in chapter II to introduce what he calls a "transformation" [Abbildung] of a system S, which he defines as

> a law according to which to every determinate element *s* of *S* there *belongs* a determinate thing which is called the *transform* of *s* and is denoted by φ(*s*); we say also that φ(*s*) *corresponds* to the element *s*, that φ(*s*) *results* or is *produced* from *s* by the transformation φ, that *s* is *transformed* into φ(*s*) by the transformation φ. (§21, 50)

Dedekind places no restrictions on this concept: he does not require that this transformation be able to be carried out or that the "law" be expressible in a finite number of words or symbols. He then introduces "similar" or "distinct" transformations (an injection or 1-1 mapping; i.e., φ is similar if φ(n) = φ(m) implies that n = m) in chapter III and calls "infinite" any system that is similarly transformable into a proper subset of itself (V). Now, if there is a similar transformation φ of S into a proper subset φ(S) and S is a "chain of some element A" where A is an element of S but not of φ(s), then Dedekind calls the system S "simply infinite" (VI). A chain of an element, according to Dedekind, is the intersection of all systems (and therefore the "smallest" system) that contain A and are mapped by φ onto some proper subsets of themselves. Now, let's consider some infinite systems, say, infinite sets of strings of letters, where we let A be the string "o" and φ be the map that appends an "x" string to the end of a given string[40]. Now consider all the sets that include "o" and, for any element s in the set, include φ(s). In our case, this means taking some sets that contain {"o", "ox", "oxx", "oxxx", "oxxxx", …}. Taking the intersection of all of these sets will give you S = {"o", "ox", "oxx", "oxxx", "oxxxx", …}, the chain of "o". Then S, since it is mapped by the 1-1 map φ onto its proper subset {"ox", "oxx", "oxxx", "oxxxx", …}, is infinite; and since it is the chain of "o" under φ, it is simply infinite. Dedekind himself pointed out that this definition of a simply infinite system reduces to four conditions on a system S, some element A of S, and a transformation φ on S – four conditions that in fact give us all the properties that

[40] Jeremy Avigad suggested to me that I illustrate Dedekind's procedure using a simply infinite set of strings. This is not Dedekind's example. Dedekind himself proves that there exists at least one infinite system in §66, with the notorious example of the system S = the objects of my thought, the object A = my ego, and the map φ that takes any of my thoughts *x* to the thought "*x* is an object of my thought." I've ignored this important but problematic step in Dedekind's argument because Cassirer does.

we need the natural numbers to satisfy. Rewriting "N" for "S" and "1" for "o" to make clearer the relation between our set S and the natural numbers, it is easy to see that any simply infinite system, like our set S of strings, satisfies the following axioms:

1. φ(N) must be a subset of N; that is, $\forall n \in N(\varphi(n) \in N)$
2. N must be the chain of 1 by φ; that is, $\forall S[(1 \in S \wedge \varphi(S) \subset S) \rightarrow N \subseteq S]$
3. 1 is not contained in φ(N); that is $\neg \exists n(\varphi(n) = 1)$
4. φ is a similar transformation; that is, $\forall n, m \in N(\varphi(n) = \varphi(m) \rightarrow n = m)$

We call 1 the "base element" of N and say that N is "*set in order* [geordnet] by this transformation φ" (§71). Now, though our system S satisfies the axioms 1-4, we do not want to say that S is identical to the system of natural numbers, since numbers aren't strings. So:

> If in the consideration of a simply infinite system *N* set in order by a transformation φ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. (§73)

Then, once we have the definition of a simply infinite system, and see that the natural numbers are simply infinite systems, we can, making use of only 1-4, define the usual arithmetic operators and their basic properties (chapters XI-XIII). Last, Dedekind shows (chapter XIV) that these (ordinal) numbers can be used to count the number of elements of a finite system, since any finite system ordered from a base element by some transformation θ is isomorphic to some set of numbers Z_n less than or equal to *n*. (That is, we can "count off" all the elements of a finite set by pairing off each element of the set with a natural number, starting with 1, and arriving with the last element at the number of elements of the set, or its cardinal number [Anzahl].)

One might see these results as an axiomatization of arithmetic and a proof that all of the essential properties of the numbers follow from this foundation. Dedekind, however, sees his work in a much more lofty way. The concept of a simply infinite system does not allow us to look at the numbers in a new and more fundamental way; rather, the concept of a simply infinite system allows the mind to "create" or "build up" the natural numbers:

> With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. (§73)

> My answer to the problems propounded in the title of this paper is, then, briefly this: numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things. It is only through the purely logical process of building up the numbers and by thus acquiring the continuous number-domain that we are prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created by the mind. If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent [abzubilden] a thing by a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensable foundation … must, in my judgment, the whole science of numbers be established. (31-2)

When the process of "building up" the numbers is described as "logical," Dedekind is emphasizing the *priority* of the concept of (natural) number with respect to all other mathematical or physical concepts. The purely arithmetical idea of continuity at work in algebra and analysis is itself a free creation out of the rational numbers and (so ultimately) the natural numbers,[41] while the continuous, physical objects in space and time cannot be thought of accurately unless the purely arithmetical notion of continuity is applied to them. The ability even to represent the elements of the pure (natural) number

[41] See "Continuity and Irrational Numbers," in *Essays*, 10.

system is an absolute precondition for representing the elements of other mathematical systems and even for representing objects in space and time.[42]

But in the passage from the preface quoted above, Dedekind has a second argument for the logical character of the theory of numbers. Not only is the natural number system a necessary prerequisite for representing continuity and so objects in space and time, but the fundamental ability [die Fähigkeit des Geistes] of the mind at work in the building up of the numbers is itself a prerequisite for thinking in general. This ability is variously described as the ability to "relate" [beziehen] a thing to a thing, to let a thing "correspond" [entsprechen] to a thing, and to represent [abbilden] a thing by a thing. A close reading of Dedekind's description of a "transformation" [Abbilding]—"φ(*s*) *corresponds* [entsprechen] to the element *s*, that φ(*s*) *results* or is *produced* from *s* by the transformation φ, that *s* is *transformed* [abgebildet] into φ(*s*) by the transformation φ"—leaves no doubt that this is the unique fundamental and necessary operation that Dedekind has in mind.[43] This completely general act of relating one object of thought ("a thing" in Dedekind's use (see §1 of *Was sind*?)) to another object is thus essential to thinking at all, presumably because all thinking—comparing, inferring, predicating,

---

[42] To be more precise, Dedekind does emphasize the priority of arithmetic to algebra and analysis, but he does not, as far as I can tell, explicitly declare its priority with respect to *all* mathematics.

[43] It is interesting that Dedekind does not emphasize the operation of grouping together elements into a set in these passages. On 45 he writes: "It very frequently happens that different things, *a*, *b*, *c*, … for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a *system S*; we call the things *a*, *b*, *c*, … *elements* of the system *S*, they are *contained* in *S*; conversely *S consists* of these elements. Such a system *S* (an aggregate, manifold, a totality) as an object of our thought is likewise a thing." This operation is not emphasized in the prefaces to the two editions of the book; on the contrary, transformation is singled out as the unique operation on which counting and therefore the introduction of the numbers depends: "from the time of birth, continually and in increasing measure we are led to relate things to things and thus to use that faculty of the mind on which the creation of numbers depends" (33).

This puzzling point in Dedekind's exposition is important, since as we will see Cassirer takes Dedekind's epistemological comments in the Preface very seriously and so ignores the set-theoretic character of Dedekind's argument.

generalizing—is a kind of relating of things (objects of thought, "representables") to one another. (Perhaps this very general notion of relating is at work in his cryptic phrase that numbers "serve as a means of apprehending more easily and more sharply the difference of things" (31): seeing differences requires comparing; comparing is a kind of relating; numbers are an abstraction from a very simple structure of relating a given object to a next object, etc.—although one wishes that Dedekind had been clearer!) To use Kantian language, number is then a category.[44] Whenever a subject thinks, then, she is implicitly at least carrying out the fundamental acts that are made explicit in arithmetic. Arithmetic is logical then, not only because, in opposition to the orthodox Kantian position, the concepts and axioms of arithmetic are prior to representations of space and time, but more importantly because any kind of thinking at all is at least implicitly arithmetic thinking.[45] Arithmetic results then from *isolating* these fundamental operations of thought, making them explicit, and then abstracting from the relational structure of a

---

[44] On the Kantian feel to Dedekind's writings, see McCarty, "The Mysteries of Richard Dedekind." It is worth noting that Paul Natorp, a teacher and close colleague of Cassirer's, interpreted Dedekind in just this way. In his 1910 systematic work *Die logischen Grundlagen der exakten Wissenschaften*, after a long discussion of the four headings of the table of the categories—"the system of fundamental logical acts as a development of the primitive act [Urakt] of synthetic unity"—, Natorp discusses the natural numbers and counting (chapter 3) and infinity and continuity (chapter 4), both from a Dedekindian point-of-view.

[45] I will pass over the difficult question of the relationship of this position to specifically psychological theories of the origin of number concepts in children or the (conscious or unconscious) mental process of counting. Dedekind does seriously maintain, I think, that the construction of the ordinal numbers and the process of establishing an isomorphism between the numbers and some other finite system in counting *is* being carried out whenever a person counts, though on account of the rapidity with which an adult can perform these operations, they are hard to detect by unguided introspection (33-4). It is worth mentioning also that in Dedekind's notorious proof (64) that there is an infinite set (namely the set of my thought, and my thought about that thought, and my thought about my thought about that thought, etc.), these thoughts cannot possibly have any kind of empirical reality in finite creatures like ourselves. See Frege's comments on this passage in his 1897 "Logic," *PW*, 136.

simply infinite system the natural numbers as new *objects*, "free creations of the mind," whose existence was implicit in the fundamental operations of thinking.[46]

Cassirer's enthusiasm for Dedekind's foundation for arithmetic and analysis is first expressed in his 1907 paper, "Kant und moderne Mathematik," and is repeated, more or less unchanged in his *Substance and Function* (1910), *Philosophy of Symbolic Forms*, volumes 1 and 3 (1923, 1929), his unpublished MS from 1936-7, and *Problem of Knowledge*, volume 4 (1940). This enthusiasm for Dedekind's position extends to viewing the natural numbers as merely structural objects—genuine objects all of whose properties follow from their having the place they do in the natural number series:

> The investigations of the most significant mathematicians—in particular the theories of Helmholtz, Kronecker, and Dedekind—have shown that it is sufficient for the grounding of all arithmetic if we define the natural number series merely as a succession of elements that are connected with one another by a determinate order, – if we therefore think of the particular finite numbers as characterized merely through the 'position' that each has within the whole series – without viewing them as 'multitudes' of units.[47]

The successive elements that make up the number series are not concrete objects - pebbles and gingerbread, or marks on a page - but "ideal objects" (Gegenstände), whose

---

[46] An analogous point can be made about Dedekind's construction of the real numbers in "Continuity and Irrational Numbers." Though Dedekind uses set theoretic machinery to define the real numbers as cuts that divide the rationals into two disjoint classes, where all the members of the first are less than every member of the second, the reals are not identified with the cuts or with the pair of disjoint sets brought about by the cut. Rather, the rationals are new free creations of the mind that correspond to the cuts. See Dedekind's letter to Weber (translated in Ewald, vol. 2, 835):

> But if one were to take your route, then I would advise that by number (Anzahl, cardinal number) one understand not the *class* itself (the system of all finite systems that are similar to each other) but something *new* (corresponding to this class) which the mind *creates*. We are a divine race and undoubtedly possess creative power, not merely in material things (railways, telegraphs) but especially in things of the mind…in connection with my theory of irrationals, where you [Weber] say that the irrational number is nothing other than the cut itself, while I prefer to create something *new* (different from the cut) that corresponds to the cut and of which I say that it brings forth, creates the cut.

This passage leaves no doubt, I think, that on Dedekind's view, each natural number is a distinct object, not itself a set, and not a "shorthand" way of talking about the properties that an object would have if it were ordered in a simply infinite system.

[47] KMM, 11. As other passages make clear, Cassirer is much less enthusiastic about Helmholtz's psychologism and Kronecker's constructive conservatism.

"essence" is "completely expressed in their positions" in the progression.[48] That these objects are "ideal" does not meant that they are subjective, in the sense that arithmetic is not subject to rules or standards of truth; rather

> Whenever a *system of conditions* is given that can be realized in different contents, there we can hold to the form of the system itself as an *invariant*, undisturbed by the difference of the contents, and develop its laws deductively. In this way we produce [erschaffen] a new 'objective' [objektives] form, whose structure is independent of all arbitrariness [Willkür]; but it would be uncritical naïveté to confuse the object that thus arises with something sensible, real, and effective [sinnlichen wirklichen und wirksamen Dingen]. We cannot read off the "properties" of this object empirically, nor do we need to, for it stands before us in all determinateness as soon as we have grasped in its purity the relation from which it develops. (*SF*, 40-1)

Cassirer is clearly willing to try to balance a full commitment to the numbers' *objecthood* and to arithmetic's objectivity with their status as "produced." Following Frege's negative point, these objects, if they are to be genuinely "objective" (*SF*, 35), cannot be psychological objects that "come and go in time" and always "belong to a certain

[48] *SF*, 39. Cassirer is responding to Russell's argument in *POM* §242, where he argues that "it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colors from sounds. What Dedekind intended to indicate was probably a definition by means of the principle of abstraction." Russell himself had proposed, but rejected, a definition of the ordinal *n* as the set of "all serial relations whose *domains* have *n* terms" (§231). From Cassirer's point-of-view, it is ironic that the same philosopher who argued vigorously and successfully that relations can be real and irreducible to intrinsic properties would reject Dedekind's characterization of the ordinals as merely structural objects on the grounds that he has not yet given them distinguishing intrinsic properties. In §428, for example, he rejects Lotze's argument that space cannot be composed of points since all the properties of points are relations of points to each other and *ipso facto* each point is indistinguishable. Russell responds there that "it is a sheer logical error to suppose that … subjects could be distinguished by differences of predicates. For before two predicates can differ as to predicates, they must already be two; and thus the immediate diversity is prior to that obtained from diversity of predicates." The conclusion then is that there is a kind of "immediate" diversity. Russell does not elaborate much on what he means by "immediate diversity," but on a natural reading of it, Dedekind's numbers can simply be immediately different from one another as points are.

Russell also reads Dedekind's appeal to abstraction in a flat-footed way, as the claim that in all simply infinite systems the elements are either ordinals or are complex elements composed of an ordinal and something else (§242). This uncharitable argument seems to rely completely on the A-a model that Cassirer thinks Dedekind's foundations for arithmetic should lead us to reject.

individual stream of consciousness" (34).[49] So it also cannot be that the ideality of numbers amounts to their being psychological objects or represent*ings*; indeed, the two are contraries:

> [In arithmetic judgments] thought reaches out beyond the whole field of thought-processes to a realm of ideal objects, to which it ascribes a permanent and unchanging form. It is by virtue of this fundamental form that every element of the numerical series is connected with every other according to a fixed rule. But a psychological analysis of the acts of forming representations [Vorstellungbildung] cannot disclose how one is connected with two, or two with three, and how the entire logical complex of representations contained in pure arithmetic arises in this connection. (35)

We'll return to the question of the proper interpretation of Cassirer's Dedekindian ontology in chapter 6, after we have a bit more material under our belt, and in particular when we see better how Cassirer thinks that some Kantian doctrines can fill in some philosophical gaps in his account of "ideal objects" that are produced but not psychological. For the time being, we will simply note Cassirer's conviction that the "essence" of the natural numbers is given entirely by their position in the progression of ordinal numbers and the "existence" of an irrational number is nothing more than its determinately dividing the rational numbers into two disjoint classes and so assuming a definite position in the real numbers.[50]

This excursion into Cassirer's Dedekindian ontology is really a side-issue in the central topic of this chapter, the main argument against the A-a model that reflection on Dedekind's procedure is supposed to motivate: "that this function [of a real number to designate a division of the rational numbers and therefore a determinate place in the real

[49] Cp. Frege's comments on Dedekind's definition of a system in *Grungesetze der Arithmetik*, vol. 1, 2.
[50] See KMM, 14, note 2: "But the 'existence' of the irrational numbers in Dedekind's sense is claimed to mean nothing more than simply this determinateness: its 'Being' consists merely in its function to designate a possible division of the region of rational numbers and thereby a possible 'position' [Stellung]."

number system] provides the grounds for positing [setzung] a separate content for thought is certainly an advance that is of the greatest epistemological interest … here a general presupposition is manifested that is at work in particular instances of mathematical concept formation" (1907, 14, note 2).

A. The most obvious way that Dedekind's foundations of arithmetic falsifies the A-a model is in its essential use of the fundamental concepts of relational, and so *non-'Aristotelian'* logic: the successor relation is defined as 1-1, onto, asymmetrical, non-reflexive; a simply infinite system is related by a 1-1 mapping onto a proper subset of itself by a relation that also orders it from some base element, etc.

B. More interestingly, the creation of the numbers is not *abstractionist* in the sense given in B above. According to that model, the subject must first have been in possession of representations of particular natural numbers and then abstracted away the concept <natural number> as something the all had in common; or perhaps have had two previous concepts, say <natural> and <number>, that it joined together by conjunction to get natural number. Indeed, in violation of B.3, *the Primacy of Particulars,* on Dedekind's construction it is impossible to have grasped the particular numbers before one has grasped the whole number system, since numbers *are* positions in a simply infinite number system.[51] And so also the concept <natural number>

[51] Similarly, the abstractionist model suggests that if an intrinsically relational concept were possible at all, it must be discernible as the common property of the objects that are ordered by it. But as Russell emphasized, the relational property of a system of elements is determined partly by the properties of the particulars ordered by it and partly by the specific ordering relation ordering it. A set of elements cannot be simply infinite, but only a set of elements transformed into itself by some φ; **N** is simply infinite when

> cannot be merely *instrumental* (in violation of B.2) in the sense of providing a tool for reasoning about its subspecies, say <even number>, or the particular numbers that fall under it. (Dedekind's characterization of the building up of the number system as 'free creation' adds emphasis to the idea that the concept <natural number> does not just reformulate old content in a more convenient way,[52] but the details of the mathematics in *Was sind und was sollen?* fails to fit B.2 even without the philosophical commentary.) Similarly for B.1, "*concepts are essentially general*."

To summarize, on Cassirer's reading, Dedekind's foundation of arithmetic confirms the two parts of the *Lotze objection* to the A-a model: the concept <natural number> does not have less content than the representations of its subspecies or of the particular numbers, but at least as much, since the characterization of a simply infinite system that the concept <natural number> requires provides all that is necessary in arithmetic for deriving the properties of all the particular numbers; the compound mathematical concept <natural number> is not composed from its simpler component concepts (<system>, <transformation>, <similarity>, <subsystem>, <base element>, <chain>) by addition, conjunction, and negation only.[53]

---

mapped into itself by the successor function; it is not when mapped into itself by the identity mapping. See *SF*, 37 and Russell, *POM*, §231, 243.

[52] See in this regard Cassirer's comments on the sense in which the numbers are 'given' in "Kant und moderne Mathematik," 14:

> The number—viewed as pure ordinal number— signifies [bedeutet] its entire content according to nothing other than its 'position': it is therefore above all a necessary and consequent extension that we conversely view a new number as 'given' where we succeed in fixing a position as an individual, on account of a determinate conceptual advance. 'Givenness' then means here, where we move entirely in the region of pure ideal positings [Setzungen] nothing other than complete logical determination, as the definiteness of an operation of thought.

[53] The "Lotze objection" and the objection from Kant that I will discuss in the next chapter are not the only arguments that Cassirer puts forward against the A-a model. One I will not discuss, but only mention here is that the abstractionist model of concept formation itself shows that Aristotelianism about conceptual

5. CASSIRER'S CRITICISM OF FREGE'S AND RUSSELL'S DEFINITION OF NUMBER

Cassirer, then, like Dedekind,[54] thinks that the *ordinal* numbers are logically prior to the *cardinals*:

> The theory of the ordinal number thus represents the essential minimum, which no logical deduction of number can avoid; at the same time, the consideration of equivalent classes is of the greatest significance for the application of this concept, yet does not belong to its original content. (*SF*, 53)

On the other hand, the Frege(-Russell) definition of cardinal numbers in terms of sets of surjectable ('similar' or 'equinumerous') sets—where each cardinal number is defined as a set of antecedently available sets and then the set of natural numbers is defined as the set of all numbers (that is, sets of equinumerous sets)—comes in for repeated criticism for obscuring the insight into the nature of the concept that Dedekind's construction of the ordinal numbers makes manifest.[55] Despite Cassirer's fondness for this argument,

---

structure is false and that not all judgments are reducible to the application of monadic predicates: in order to abstract a concept, say a tree, one must first compare representations (say, my representation of a linden, another of an oak), note their similarity, and then abstract the common feature, <tree>; but the *similarity* between my two representations is itself a relation between them; so I cannot reduce all relations to 1-place predicates since I need at least one relation in order even to arrive at a concept. And if one irreducible relation is necessary, there is no good reason not just to allow that other relations are irreducible too and give up on the sufficiency of the syllogism and the A-a model altogether.

It is not accidental that Cassirer's way of putting this argument has affinities with Kant's doctrine that concepts are functions of unity of an intuition—rules for the synthesis of the sensible manifold—, and with Russell's arguments against the Hegelian monadists in *POM* (e.g., §425: if the relation between two terms is to be reducible to the internal states of these two terms, then we must presupposes that there are two different terms; but then we must hold that these two terms are related after all, since diversity is itself a relation). For echoes of Kant's doctrine of concepts as rules for the synthesis of a sensible manifold, see below, chapter 4, section 4. Cassirer cites Russell's attack in *POM* on the monadist view of relations on 1907, 7, note 2.

[54] Recall that Dedekind defines the ordinal numbers in §73 of *Was sind?* and then later in §161 introduces the cardinals. This was not just for ease of exposition or for technical reasons: "I still regard the ordinal number and not the cardinal number (Anzahl) as the original number-concept…I hold the cardinal number to be only an *application* of the ordinal number" (Letter to Weber, quoted in Ewald, vol. 2, 834-5). Cassirer recognized, of course, that the dispute about whether the ordinals or the cardinals are logically prior is not a matter for technical adjudication, since the ordinals can be defined in terms of the cardinals and vice-versa (*PK*, 60-1).

[55] In this and the following sections I will use Cassirer's language and write as if Frege were trying to give a foundation for arithmetic in set theory, even though, as I say at the end of this chapter, this is ultimately misleading.

early and late, I think this argument is fundamentally flawed in a number of ways. I conclude this chapter, then, by presenting this argument, not so as to shed light on Frege, but to gain further insight into Cassirer's way of thinking (and its weaknesses).

First, Cassirer argues that the Frege-Russell foundation for arithmetic is committed to *Abstractionism about conceptual formation*, since, on their definition, the representation of a particular number is abstracted from antecedently represented particulars, and the concept <number> is abstracted from the representations of particular numbers by noticing a 'common property' they all share. Recall that the Frege-Russell theory constructs the numbers out of sets that can be surjectively mapped onto one another. One takes some sets—{a beach ball, Julius Caesar, Alpha Centauri, pi}, the set of Gospels, the set of horseman of the apocalypse, etc.—notices a feature that they all have in common, groups them together into a set and calls the new set 'four.' Later, one sees that other sets of sets have a property that our set has—namely, they also are "members of the series of natural numbers, beginning with 0," in Frege's technical sense, where 0 is the set of non-self-identicals, one set *n* of similar sets "follows directly in the natural number series after" another set *m* of similar sets iff there exists a concept *F*, and an object falling under it *x*, such that the set of all sets similar to the set of *F*s is *n* and the set of all sets similar to the set of "things falling under *F* but not identical to *x*" is *m*, and "following in a series" is defined as it is in *Begriffsschrift*. This larger set that contains all of the sets of similar sets is now called the "natural numbers." This whole procedure smacks of the A-a model: "If the attempt to derive the concept of number from that of class were successful, the traditional form of logic would gain a new source of confirmation. The ordering of individuals into the hierarchy of species would be, now as before, the true goal of knowledge, empirical as well as exact” (*SF*, 53). The fact that, on Frege’s view, the particular objects that are grouped together into a new genus <number> are not concrete particulars, but instead concepts, makes no difference here—in both cases antecedently understandable particulars are treated as given and then grouped into sets based on a common property that is understandable already before the introduction of the concept <number>.

> It does not suffice to emphasize the purely conceptual character of numerical assertions as long as thing-concepts and functional-concepts are placed on the same plane. Number appears, according to this view, not as the expression of the fundamental condition that first renders possible every plurality, but as a ‘mark’ that belongs to the given plurality of classes and can be separated from the latter by comparison. Thus the fundamental deficiency of the whole doctrine of abstraction is repeated: an attempt is made to view what guides and controls the formation of concepts, i.e. a purely ‘categorical’ point of view, as in some way a constitutive part of the compared objects. (*S&F*, 54)

On the Dedekindian view, the concept <number> cannot presuppose the representation of other concepts, since it functions within thought like a Kantian category: it makes it possible that objects are thought of as different or the same as each other; it enables the mind to relate objects to one another and compare them, and so makes possible the representation of a plurality and the formation of concepts.

Second, on the Frege-Russellean view, the number ‘two’—the set of all sets of couples—can be introduced and defined in total isolation from the definitions of all other numbers, still less the concept <natural number> itself.

> The consideration of groups, which can be mutually coordinated member for member, can only lead to the separating out of the identical ‘mark’ in them; but this ‘mark’ is in itself not yet a number but is merely a logical property not further defined. Such a property only becomes number when it separates itself from other ‘marks’ of the same logical character by appearing in a relation to them of ‘earlier’ or ‘later,’ ‘more’ or ‘less.’ (*S&F*, 49)

Even if it were unobjectionable to define numbers as sets, on Cassirer's view it cannot be acceptable to introduce a number in such a way that it is not immediately clear how that number is related to all other numbers within the entire natural number system. But Frege introduces the concept of a set of similar sets, defines the numbers 0 and 1, and then the successor relation, and then last the concept <(finite) number>. As in the A-a model, Frege and Russell in their practice maintain the *primacy of particulars*: one grasps the particular numbers before one has grasped the concept <natural number>. On the definition of the ordinal numbers, the system of ordinal numbers is introduced first, and each particular number is understood only as a position in the number series: the numbers cannot be represented one at a time, or in isolation:

> It is a fundamental characteristic of the ordinal theory that in it the individual number never means anything by itself alone, that a fixed value is only ascribed to it by its position in a total system. The definition of the individual number determines at once and directly the relation in which it stands to the other members of the field; and this relation cannot be eliminated without losing the entire content of the particular number. In the general deduction of cardinal numbers, which we are considering, this connection is eliminated. (*S&F*, 48)

Cassirer clearly has some work to do if he wants to accuse Frege, but not Dedekind, of being wedded to a doomed version of abstractionism, since Dedekind himself described his procedure as *abstractionist* and like Frege made conscious free use of *set theory*. On the first point, Cassirer freely acknowledges the abstractionist rhetoric in Dedekind but argues that it is being used in a very different way from the A-a model. Here, the "abstraction" of the natural numbers from the concept of a simply infinite system "is not directed upon the separating out of the quality of a thing, but its aim is to bring to consciousness the meaning of a certain relation independently of all particular cases of application, purely in itself" (39). That is, one does not remove from a compound representation some common element that all of these represented particulars share; when one abstracts the numbers, one is allowing oneself to represent only the structure itself. Consequently, when one abstracts from <linden>, <oak>, <fir> the common element <tree>, one represents the lindens, oaks, and firs as trees; when one abstracts from <simply infinite system> the concept <natural number>, simply infinite systems are not represented as numbers—the numbers are just positions in the simply infinite number system. Unlike the A-a model, Dedekind's "abstraction" is not from some particulars to a common concept that they all fall under: one abstracts *to* the representation of particular numbers, not *from* the representation of particular numbers.

Cassirer's attitude toward Dedekind's set theory, on the other hand, seems much harder to justify. Frege and Russell, he thought, only muddied the waters when they brought in the completely irrelevant notion of a set:

> It is evident that the system of the numbers as pure ordinal numbers can be derived immediately and without circuitous route through the concept of class; since for this we need assume nothing but the possibility of differentiating a sequence of pure thought constructions by different relations to a certain fundamental element, which serves as a starting-point. The theory of the ordinal number thus represents the essential minimum, which no logical deduction of number can avoid; at the same time, the consideration of equivalent classes is of the greatest significance for the application of this concept, yet does not belong to its original content.[56]

But Dedekind's foundations for arithmetic absolutely requires set theory, or at least enough set theory to define a "chain," which, recall, is the intersection of all sets mapped into themselves from some common element (a least inductive set if the mapping is similar). In his famous letter to Keferstein,[57] Dedekind described the now familiar

[56] *SF*, 20.
[57] Translated in *From Frege to Gödel: A Sourcebook in Mathematical Logic*, 99-103. It was unfortunately not published until 1967.

problems with non-standard models of arithmetic: if one does not introduce the notion of a chain, but rests content with characterizing a mapping φ that is 1-1 and whose image is a subset of its domain but does not include some element **0** from its domain, one cannot rule out sets that include not only **0** and all its descendants but also some additional non-arithmetical element *t* and all its ancestors and descendants. Of course, one could bypass the notion of a chain by simply postulating that induction holds for the numbers.[58] This is equivalent to stipulating that every element in N is an element that, if I started out from **0** and iterated φ a finite number of times, I could reach at some time. But, as Dedekind argues, this solution "is quite useless for our purposes; it would after all, contain the most pernicious and obvious kind of circle. The mere words 'finally get there at some time,' of course will not do either."

When Cassirer revisits these issues in later writings, he makes it clear that he wants to take mathematical induction as an unanalyzable primitive as Poincare does.[59] Though he thinks that Brouwer's intuitionism is psychologistic and too conservative, he credits it with having "restored the primacy of the *relation* and … brought the recognition of its universal role" by replacing the derivation of number from sets with a derivation of the properties of number from the principle of mathematical induction.

> [In inductive inference,] there is a kind of return to the absolute principle of number: it is recognized that this fundamental relation which connects one member of the numerical series with its immediate successor continues through

---

[58] Dedekind's axiom that **N** be a chain is equivalent to the set-theoretic principle of induction. Dedekind's principle is

$$\forall S[(1 \in S \wedge \varphi(S) \subset S) \rightarrow N \subseteq S]$$

and the principle of induction, using set-theory, is

$$\forall S[(1 \in S \wedge \forall n(n \in S \rightarrow n' \in S)) \rightarrow \forall n(n \in S)].$$

[59] For Poincare's view, see "On the Nature of Mathematical Reasoning" from *Science and Hypothesis* and "Intuition and Logic in Mathematics" from *The Value of Science*. The following quotations are from *PSF*3, 376-8.

> the whole series and determines it in all its parts. In this sense, a genuine a priori synthesis—as Poincare in particular repeatedly stressed—actually underlies the principle of complete induction.

Even if it is possible to derive the principle of induction from principles and definitions of set theory, this procedure is "a *hysteron proteron* in the epistemological, strictly transcendental sense." Here he repeats the accusation he had made against Russell and Frege in *Substance and Function*: the application of the concepts of set theory themselves require that one be able to perform "the thought functions of postulation, of identity and difference—the very same relations that are prerequisite to the constitution of the numerical concept and out of which the numerical concept can be derived directly, without detour through the concept of class."[60] The problem, though, is that it is hard to see how, to use Dedekind's phrases quoted above, the principle of induction can be implicit in the mere ability "to apprehend the differences of things" or "to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing." The ability to relate a thing to a thing is captured, Dedekind argues, in his concept of a "mapping" or "representation" or "copying" [Abbildung]; but one needs also the concept of a *system* (or a set) to force the system of natural numbers to be standard.[61] What is missing here is an argument, which, though it has been sketched in a very rough way in various places throughout his corpus, Cassirer desperately needs to flesh out in a convincing way—an argument to the effect that there is a concept of ordering which either expresses a cognitive function, or a complex of cognitive functions, that is at least implicitly at work

[60] It is interesting that Cassirer's objections to set theory were always epistemological, never ontological or technical—the worries about consistency, strong existence assumptions (like infinity), or nonconstructive principles (like Choice) never seemed to bother him.

[61] Recall that Dedekind himself emphasized that his derivation of number requires only "activities of the understanding *without* which no thinking is possible at all" ("Letter to Keferstein," 100), but never includes the operation of collecting together into a set when he mentions what these activities are.

in all thinking, is independent of the operation of forming a set, and is sufficient to ground arithmetic.[62] (No such argument is present in Poincare. He argues instead that arithmetic is founded on the principle of induction, which is *a non-logical form of reasoning unique to mathematics*.)

But even if we could find such an argument, there are other reasons why Cassirer's criticism of Frege misfires. Similarity (or equinumerosity) among concepts is not "similarity" in the Abstractionist sense; indeed, it is a relation among concepts expressible only when the resources of relational logic are brought to bear. Further, Frege rejects enumerative theories of classes and only allows classes by comprehension. So for him, classes are determined by the concepts that give the rule for class membership, not by a list of objects grouped independently. (This point was later

---

[62] Cassirer elsewhere characterizes the foundation of number as based on the faculty of apperception. See, for instance, *SF*, 34:

> A new standpoint regarding the foundation of number is first reached through the deeper and more mature psychological deduction of the numerical concepts from the fundamental act of apperceptive connection and separation in general. From this standpoint, number is to be called universal not because it is contained as a fixed property in every individual, but because it represents a constant condition of judgment concerning every individual as an individual. The consciousness of this universality is not gained by running through an indefinite plurality of cases, but is already presupposed in the apprehension of every one of them; for the arrangement of these individuals into an inclusive whole is only rendered possible by the fact that thought is in a position to recognize and maintain a rule, in conceptual identity, in spite of all differences and peculiarities of application.

Compare again Poincare's position. For him, unlike the truths of the physical sciences, which are based on an order "external to us," truths of mathematics are arrived at by recursion, which "is necessarily imposed on us, because it is only the affirmation of a property of the mind itself" ("On the Nature of Mathematical Reasoning, §V). What lies at the bottom of recurrence is a property of the mind. Each mathematical truth is a construction (product) given by an act of the mind. For instance, $x + 2 = 2 + x$ can be verified by a few iterations of the recursive definition of addition. It is immediately clear that, once $x + 2 = 2 + x$ has been verified, one can repeat the same procedure to get $x + 3 = 3 + x$, and so on. The mind, Poincare assures us, has "a direct intuition" of its power to repeat this act indefinitely, and so safely concludes that commutativity holds in full generality. Since each particular truth is a (human) construction, and since the mind is conscious of its own abilities, it can reach general claims.

Unfortunately, given the strong anti-psychologism of Cassirer and his Marburg teachers (see chapter 5 below), this 'psychological' deduction of mathematical induction would have to be reworked before it could find a place in Cassirer's thought. Perhaps some more light on Cassirer's argument could be shed by comparing it with Natorp's argument in chapter III of his 1910, with which it has strong affinities. (Indeed, it might be borrowed from there.)

appreciated by Cassirer and led him to partially retract his criticism of Frege.[63] That Dedekind's own position includes an unabashedly nonconstructivist set theory that is based in the mind's ability to collect together any objects into sets, even sets that might not be the extensions of any concept, goes unnoticed by Cassirer, except in a quotation from Hilbert.[64]) Although the language I used in presenting Frege's position obscures this fact, Frege insisted that numbers be understood as "extensions of concepts," and he had a particularly low view of any theory of sets that does not depend on the content of the concepts that determine these classes.

> I do in fact maintain that the concept is logically prior to its extension; and I regard as futile the attempt to take the extension of a concept as a class, and make it rest, not on the concept, but on single things. That way we get a domain-calculus, not a logic.[65]

---

[63] Cassirer (partially) retracted this objection against Frege in *Philosophy of Symbolic Forms*, after Wilhelm Burkhamp (1927) published a book on the foundations of mathematics and mathematical logic that praised Frege for acknowledging the non-Aristotelian and non-Abstractionist nature of the concept. This book significantly refers to Frege's 1890s review of Schröder's *Vorlesungen*, which Cassirer seems not to have known in 1910.

[64] *PSF*3, 379, referring to Hilbert's *Neubegründung der Mathematik*, 1922, 157ff, 162, where Hilbert says that the paradoxes of set theory makes Dedekind's approach untenable.

[65] "Elucidation of some points in Schröder's Lectures," (1895), 228, in *CP*. In a 1910 discussion of Schröder attached as notes to a biographical entry that Jourdain was preparing on Frege, he wrote:

> In my fashion of regarding concepts as functions, we can treat the principal parts of Logic without speaking of classes, as I have done in my *Begriffsschrift*, and that difficulty [i.e., the Russell paradox] does not come into consideration. Only with difficulty did I resolve to introduce classes (or extents of concepts)…But numbers are objects…Our first aim, then, was to obtain objects out of concepts…I confess that, by acting thus, I fell into the error of letting go too easily my initial doubts, in reliance on the fact that extents of concepts have for a long time been spoken of in Logic. The difficulties which are bound up with the use of classes vanish if we only deal with objects, concepts and relations, and this is possible in the fundamental part of Logic. The class, namely, is something derived, whereas in the concept – as I understand the word – we have something primitive. (*Philosophical and Mathematical Correspondence* [*PMC*], 191)

Jourdain's concluding judgment of the comparative worth of Frege's and Dedekind's foundations for arithmetic is not without merit: "while Dedekind, at least in intention, made 'System' and 'Imaging' the two foundation-stones of his theory of arithmetic, those of Frege were the more precisely expressed 'Concept' and 'Relation' (*Beziehung*)" (206).

Perhaps most significantly, as I show in the Appendix, Frege himself had *anticipated* (a form of) Cassirer's own objection when he accused the Booleans of assuming the A-a model in two unpublished papers attacking Schröder from the early 1880s.

CHAPTER 4 KANT'S THEORY OF CONCEPT FORMATION

In chapter 3 I argued that Cassirer in the first few chapters of *Substance and Function* poses a philosophical problem—to elucidate a theory of conceptual structure and formation adequate for the concepts of the exact sciences—and gives an answer given earlier in a less satisfying way by Lotze—that the resources of traditional syllogistic logic have no hope of solving this problem, but that a new kind of logic is necessary. Now it might seem odd that Cassirer would be making this point, since Cassirer took himself to be defending what he called "critical idealism,"[1] a position that was meant to be derived in significant respects from Kant's. Kant, however, is often portrayed as the great opponent to the new logic: he famously argued that traditional syllogistic logic was perfect and required no supplementation (let alone replacement)[2]; his own theory of conceptual structure and formation is abstractionist[3]; and he argued that many of the concepts and theorems expressible (and even, Frege thought, provable) using the new

---

[1] In *SF* Cassirer characterizes his position as "critical idealism" (295, 7) or the "critical theory of knowledge," based on a "transcendental" procedure (268). At *SF*, 99, he defends "logical idealism," which is Cohen's phrase; a more explicit defense of 'logical idealism' against other leading philosophical approaches is given Cassirer's 1913 paper "Erkenntnistheorie nebst den Grenzfragen der Logik." In "Kant und moderne Mathematik," it is clear that what he calls the 'critical philosophy' is in fact his own philosophy, though he characterizes it not as a fixed set of doctrines but as a continuous project of examining the "basic concepts of science" (1).

[2] Bviii: "That from the earliest times *logic* has traveled this secure course [of a science] can be seen from the fact that since the time of Aristotle it has not had to take single step backwards, unless we count the abolition of a few dispensable subtleties or the more distinct determination of its presentation, which improvements belong more to the elegance than to the security of the science. What is further remarkable about logic is that until now it has also been unable to take a single step forward, and therefore seems to all appearance to be finished and complete." Cf. also *JL*, 20-1.

[3] See *JL* §4-6.

logic were nonlogical. Worse yet, as we saw in the first chapter, Kant's philosophy of geometry was carried out in close conformity to the diagrammatic, ancient model.

In a striking passage written late in Cassirer's career, he summarized his rather different conception of the relationship between Kant and modern mathematics.

> After it had appeared to pursue a wholly different course for a time the development of modern mathematics has in this respect turned back in a remarkable way toward certain positions taken by Kant. The Transcendental *Logic* did not undergo modifications similar to those imposed on the transcendental aesthetic by the discovery of the non-Euclidean geometry and its various 'forms of space.'[4]

In his earliest work on the philosophical interpretation of modern mathematics, Cassirer summarizes the main insight of Kant's philosophy of mathematics as an attack on the traditional logic's theory of concepts.

> Mathematical concepts differ from the general genus-concepts of traditional logic that are defined through genus and species. That the definitions of mathematics derive from pure intuition here signifies only that they are not –as the 'discursive' concepts of formal logic—abstracted from a multiplicity of different contents as their common mark, but rather have their origin in a fully determinate unique act of construction.[5]

Mathematical concepts, which have their origin in construction, are neither Aristotelian in structure, nor abstractionist in their formation; rather, they rest on synthesis, or what Cassirer here calls 'the "synthetic" *creation* of a content' [die 'synthetische' Erschaffung eines Inhalts]. Like Kant's pure concepts of the understanding, mathematical concepts "are not stripped off afterward from some kind of self-standing individual contents, but rather derive from an original function of thought that lies before all finished being as a condition of its possibility." With this Kantian machinery in place, Cassirer thinks, we can grasp "the new critical sense of 'concept'" and "the new positive conception of

[4] *PK*, vol. 4, 75-6.
[5] KMM, 32-3.

concept formation that Kant himself founded in his own 'transcendental' logic."[6] The problem of this chapter is to show how Cassirer's theory of the concept, at least in his eyes, fits in with a Kantian conception of concepts. Only when this foundation has been laid, can we consider Cassirer's exposition of his philosophy of geometry, expressed canonically in chapter 3 of *Substance and Function*, since Cassirer was adamant that a proper understanding of mathematics requires a rethinking of Kantian themes in light of a better understanding of conceptual structure and formation.

1. INTRODUCTION

The topic of the formation of concepts, especially empirical concepts, has been one of the most discussed topics in the literature on Kant. Kant's theory was a central topic in the writings of the Neo-Kantians who reintroduced serious study of Kant back into philosophy at the end of the nineteenth and beginning of the twentieth century, and it has received serious attention in some very recent work as well.[7] As in many other areas in Kant scholarship, there is little agreement. Some commentators see in Kant a deep and persuasive account of the formation of concepts, while others see in Kant either absolutely no account of the formation of concepts at all or an obviously unhelpful and

[6] KMM, 33.

[7] On the British side, see, for example, Thomas Hill Green's "Lectures on Kant" and "Lectures on Formal Logicians" from the 1870s; on the German side, see especially the works of Ernst Cassirer and Emil Lask. (More specific references are given below.) The topic receives considerable treatment, among more modern commentaries, in Paton's *Kant's Metaphysic of Experience*, Wolff's *Kant's Theory of Mental Activity*, Sellars's *Science and Metaphysics*, Pippin's *Kant's Theory of Forms*, and Aquila's *Representational Mind*. Among recent writings, see the important works of Beatrice Longuenesse (*Kant and the Capacity to Judge* and *Kant on the Human Standpoint*) and of Hannah Ginsborg ("Lawfulness without a Law", "Thinking the Particular as Contained under the Universal", "Aesthetic Judgment and Perceptual Normativity").

circular one.[8] Some commentators read Kant as the clearest *exponent* of the traditional doctrine of concept formation by abstraction that one finds anywhere in the philosophical literature, while others see Kant as providing its definitive *refutation*.[9] These extremes are also to be found in what commentators have to say about the value of the account of concept formation in Kant's *Logic*. Among recent commentators, Ginsborg has argued that the abstractionist account given in the *Logic* would be viciously circular if it were read as an account of how a subject first comes to acquire empirical concepts, and so cannot be "Kant's answer to the question of how empirical concepts are possible, but only as explaining how concepts we already possess can be clarified or made explicit."[10]

Longuenesse, on the other hand, has gone against the tide of generations of commentators by arguing not only that the account of concept formation by abstraction in the *Logic* is Kant's deeply held view, but that this account is a central and essential feature of the whole argument of the *Critique*. In her words, "the operation of comparison/reflection/abstraction is indeed the discursive act *par excellence*."[11] This

---

[8] Among modern commentators, the former position is defended by Ginsborg and Longuenesse; the latter position is defended by Pippin and Aquila.

[9] Dickerson (in his *Kant on Representation and Objectivity*) argues that the account in the *Logic* "could easily have come straight from the pages of the Port-Royal *Logic*" (39). The opposite view, I think, one can find in Sellars, who thinks that the argument of the Transcendental Deduction shows that the traditional abstractionist account is impossible. (See *Science and Metaphysics*, 5-8, 55; "Some Remarks on Kant's Theory of Experience," §VII; "Phenomenalism," 74, 90.) One finds a similar position in Pippin, who thinks that there is no way to square abstractionism with the doctrine of the Deduction, but is more willing than Sellars to ascribe inconsistency to Kant. (Pippin also argues at length that the traditional abstractionist model found in the *Logic* is hopeless anyway.)

[10] "Lawfulness without a Law," 8. Ginsborg attributes this position also to Robert Pippin. Ginsborg then tries to give on Kant's behalf an account of the origin of empirical concepts that draws on doctrines from the third *Critique*.

[11] *Kant and the Capacity to Judge*, 121. The major theme of her work is that Kant's apparent attempt to derive the categories from the logical forms of judgments in traditional syllogistic logic is not the embarrassment and dead-end that almost every commentator since Hegel has thought it to be, but in fact the heart of the *Critique*. And, she thinks, the neglect of Kant's account of concept formation by abstraction has contributed to this trend in reading Kant: "A major reason for the general misunderstanding concerning the role of the logical forms of judgment as 'guiding thread' for the table of categories is that

dispute about the importance of the account of concept formation from the *Logic* in Kant's philosophy as a whole itself dovetails with a wider debate: Longuenesse's contention that taking seriously the scholastic logic and its abstractionist view of concept formation "illuminates each step of the argument of the first *Critique*" stands in stark opposition to that of philosophers, like Michael Friedman, who see the scholastic logical apparatus in Kant's texts as a fundamentally distorting medium for his otherwise radically non-traditional philosophy.[12]

Cassirer bases his reading of the central arguments of the *Critique* (in *Kants Leben und Lehre* (1917), though also to a lesser extent in his 1907 discussions of the *Critique* in *Erkenntnisproblem* and "Kant und moderne Mathematik") on the organizing idea that Kant is rejecting the traditional model. In this chapter, I give a reading, from Cassirer and other Neo-Kantians' perspective, of Kant's theory of concept formation. My reading is not a close exposition of Cassirer's writings on Kant. It is rather a sympathetic reconstruction of his way of reading Kant—that, despite what a first reading of Kant's *Logic* would suggest, Kant does not see the operations of comparison, reflection, and abstraction as the explanation of how a subject first comes to have representations that contain a concept. I depart from Cassirer's writings to fill in detailed textual arguments for some of his controversial readings, and I allow myself to give a reading of the theory of concept formation in Kant's *Logic*, despite the widespread hostility and indifference that greeted it in the writings of Cassirer and other Neo-

commentators neglect their function in the activities of 'comparison, abstraction, and reflection.' If we take this function into account, it illuminates each step of the argument of the first *Critique*" (11).

[12] See Friedman's "Logical Form and the Order of Nature: Comments on Beatrice Longuenesse's *Kant and the Capacity to Judge*," 214-5 and her response, "Synthesis, Logical Forms, and the Objects of Our Ordinary Experience" in *Kant on the Human Standpoint*.

Kantians. The reading that I follow Cassirer in giving differs from that of other Kant commentators in three ways. First, I show how Cassirer draws on the rich—though now largely forgotten—literature on the traditional model of concept formation and Kant's relationship to it that one encounters in the writings of the logicians and Neo-Kantians that dominated philosophy at the end of the nineteenth and beginning of the twentieth century. Cassirer follows that tradition—in agreement with many modern commentators—in seeing the traditional model as viciously circular. Second, I depart from Cassirer himself in offering a detailed reading of the Doctrine of Concepts in Kant's *Logic* in order to show that Kant himself is not open to this circularity, since he sees the operations of comparison, reflection, and abstraction as acting on given representations that already have conceptual content.[13] (This is what we would expect, since, as Cassirer and the wider Neo-Kantian tradition recognized, Kant's account of concepts as rules for perceptual synthesis is inconsistent with a global abstractionist theory.) Third, I follow Cassirer and other Neo-Kantians in considering Kant's theory of the formation of specifically mathematical concepts. Though no modern commentator has explored the relationship between Kant's theory of mathematical concepts and the theory of concept formation by abstraction, the clear conclusion is that mathematical concepts for Kant cannot be formed by abstraction. Indeed, I think that Kant has a very interesting account of mathematical concept formation in the *Critique*, and one that simply makes no sense unless we read Kant's abstractionism weakly in the way I suggest.

---

[13] The Neo-Kantian tradition tended to dismiss Kant's *Logic* as fundamentally mistaken. An extreme and notorious example of this tendency is exhibited in the Neo-Kantian Walter Kinkel's introduction to his edition of Kant's *Logic* (*Immanuel Kants Logik: Ein Handbuch zu Vorlesungen*, originally published 1904). For him, by Kant's own lights there could not be an independent discipline of formal logic, and the *Logic* can have no systematic value. I discuss this at greater length in the next chapter.

I begin the argument of the chapter in section 2, where I introduce the traditional model of concept formation and work through the machinery of Kant's *Logic*. Though this material is dry and slow-going, I believe it is necessary for a fruitful engagement with Kant's thought, and I can only beg the reader's patience. In section 3, I introduce the widespread late-nineteenth century circularity objections against the traditional model, and I show how Kant's *Logic*, when read closely, in fact avoids these circularity objections. In sections 4 and 5, I follow Cassirer in giving two arguments from the first *Critique* that show that Kant could not have held the abstractionist theory of concept formation either about perceptual concepts or about mathematical concepts. I end by suggesting (speaking now in my own person, though without defense or elaboration) how one might extend the discussion in this chapter beyond what one finds in Cassirer and other Neo-Kantians into a fuller and more faithful reading of Kant's *Logic* and a complete Kantian theory of the formation of concepts.

2. KANT'S *LOGIC* AND THE TRADITIONAL MODEL OF CONCEPT FORMATION

The traditional doctrine is composed of two interrelated parts: a doctrine of *conceptual structure*, according to which concepts are either simple or composed of simple concepts by disjunction, conjunction, and negation; and a doctrine of *conceptual formation*, according to which all concepts arise from abstraction. The traditional doctrine appears to receive its clearest expression in the First Section, 'Of concepts,' of part I of Kant's

Logic.[14] Concepts are initially introduced in §1 as *cognitions*, that are "*universal* (*repraesentio per notas communes*) or *reflected* representation[s] (*repraesentio discursive*)."[15] As a universal representation, a concept is "a representation of what is common to several objects [mehreren Objekten gemein]." As such, it is a representation "insofar as it can be contained in various ones."[16] These two features of a concept, that it represents insofar as it is *contained* in another representation,[17] and that it represents what is common to several objects, are further explained in §7.

> Every concept, as *partial concept*, is contained in the representation of things; as *ground of cognition, i.e., as mark*, these things are contained *under* it. In the former respect every concept has a *content* [Inhalt], in the other an *extension* [Umfang].[18]

Kant's presentation is dense, so it will be good to keep track of his terminology with some examples. Here we'll use <human> and <animal>. When Kant says that a thing is contained *under* a concept, he means that the thing possesses a feature that could be shared by other things; in short, a representation A is contained under a concept B iff (some/every/the) A is B. So, when I judge that "a man is an animal," <human> is contained under <animal>. Further, <animal>, as a partial concept, could also be contained *in* the representation of the thing that has the feature represented by the concept. For instance, <animal> is contained in <human>, since man is a rational animal.

---

[14] Often referred to as 'The Jäsche Logic' [*JL*], it was published during Kant's lifetime and with Kant's permission, in 1800, by Kant's student Gottlob Benjamin Jäsche, who compiled it based on the notes that Kant provided him.

[15] Cf. also A19/B33; A320/B376.

[16] An intuition, on the other hand, is a singular representation of an object. An intuition can also be contained in another intuition, though in a different sense. See B39-40.

[17] The grammar of §1, note 1, "eine Vorstellung, *sofern sie in verschiedenen enthalten sein kann*," is ambiguous between "insofar as it can be contained in various representations" and "insofar as it can be contained in various objects." As J. Michael Young argues in his notes to the *JL*, §7, "every concept, *as partial concept*, is contained in the representation of things," suggests the former.

[18] *JL* §7; Ak 9:95.

In general, if *M* is contained *in* the representation of a thing, it is contained *under M*, or, more simply, *is* an *M*. (The converse of this conditional is false, since not all judgments are analytic.)

We can (partially) rank or order concepts according to whether or not they are contained under each other. Thus the concepts N_1, N_2, … that are contained under M will constitute the *extension* of M, along with all of the N_{11}, N_{12}, …, N_{21}, N_{22}, … contained under them. But, for any M contained in N, there will be some concept N* that falls under M, in which M is also contained as a *partial concept*, but is disjoint from N. The process of "taking apart [theilen] a concept" to discover what is contained in it, is called 'analysis'; the process of determining what is contained under a concept, is called 'logical division.' Logical division is the systematic determination of what concepts can be contained under a given concept, so that the extension, or 'sphere,' of the concept is exhaustively marked up into distinct, non-overlapping lower concepts that contain it.[19] A simple division, a *dichotomy*,[20] can be represented in the following tree structure.

[19] *JL* §110; Ak 9:146. Cp. also B112. I take it that logical division for Kant is the process whereby one systematically determines for a concept in a science which other concepts both contain it and are contained under it. This is made clear in a passage in Kant's unpublished first introduction to the *Critique of Judgment* [EE]. Here we find Kant discussing logical division in the context of the "logical form of a system" (cp. *JL* §95, 98), as the inverse operation of abstraction. The "division of given universal concepts," Kant says, requires thinking the particular "in its diversity as contained under the universal." This requires going 'bottom up,' as it were, comparing and classifying particulars, and finally subsuming them under ever higher genera (EE, Ak 20:214).

> On the other hand, if we start from the universal concept, so as to descend to the particular by a complete division, we perform what is called the *specification* of the diverse under a given concept, since we proceed from the highest genus to lower genera (subgenera or subspecies) and from species to subspecies…For the genus is (logically considered) as it were the matter, or the raw substrate, that nature processes into particular species and subspecies by determining it multiply. (EE, Ak 20: 215).

[20] Of course, the sphere of a concept can be divided into more than two mutually exclusive members that jointly exhaust the extension, though this possibility would require "cognition of the object," that is, recourse either to empirical or pure intuition.

M (=<animal>)

M and N
=<rational *and* animal>
=<man>

M and non N
=<non-rational *and* animal>
=<brute>

M and N and O
=<learned *and* man>

M and N and non O
=<non-learned *and* man>

Figure 4-1: The Logical Division of <animal>

From this display of the logical division of a concept, we can then read off certain judgments: e.g., 'All learned men are animals' or 'No learned man is a brute.'[21] (We cannot, though, read off *all* true judgments from such a tree. Only intuition could tell us that 'Some man is learned'; similarly, without recourse to experience, we could never know that there are brutes and thus that 'All animals are rational' is false[22]). At the apex

[21] The structure of logical division and the principle from *JL* §7 that a concept M falls under every concept contained in it together seem to clarify what Kant means when he distinguishes analytic and synthetic judgments:

> In all judgments in which the relation of a subject to the predicate is thought (if I consider only affirmative judgments, since the application to negative ones is easy) this relation is possible in two different ways. Either the predicate *B* belongs to the predicate *A* as something that is (covertly) contained in this concept *A*; or *B* lies entirely outside the concept *A*, though to be sure it stands in connection with it. In the first case I call the judgment *analytic*, in the second *synthetic* (A6-7/B10-11; cp. also *JL* §36).

In the analytic judgment "All bodies [=extended, impenetrable, shaped] are extended," <extended> is a contained in the concept <body>, and so <body> is contained under <extended>. But not so with the synthetic judgment "All bodies are heavy." If we abstract from the cognition of objects, that is, of actual bodies, we could allow the concept <body> to be divided into <body *and* exerts an attractive force> and <body *and* does not exert an attractive force>. (A proof appealing to synthetic a priori principles is required to show that all bodies exert not only a repulsive force (which is a necessary condition of their being impenetrable), but also an attractive (=gravitational) force, as in *Metaphysical Foundations of Natural Science*).

For an illuminating and spirited attempt to defend the clarity and usefulness of Kant's analytic/synthetic distinction by appealing to the concept of logical division, see R Lanier Anderson, "It All Adds Up After All: Kant's Philosophy of Arithmetic in Light of the Traditional Logic."

[22] Kant allows that the same concept be divided according to various respects; this kind of division, called 'codivision,' may 'go to infinity,' especially with concepts of experience. Consider a multi-dimensional tree structure, where the concept <animal> is divided in one dimension, as before, by <rational> and <non-rational>, but then is also divided, from another viewpoint, into <bipedal> and <non-bipedal>. Now if we

of the structure will be a highest genus, a concept that, since it contains no other partial concept in it, is simple.[23] And since the partial concepts (non-)B, (non-)C, (non-)D,… can make up part of the *content* of a concept A only if A is contained under (non-)B, (non-)C, (non-)D, … it follows that all compound concepts are analyzable into A = (non-)B and (non-)C and (non-)D and…[24] Thus, for Kant, concepts are either simple or are composed of simple concepts by conjunction, addition, or exclusion. Not surprisingly, Kant sees every judgment as a judgment that two or more concepts, the subject and the predicate concepts, either exclude or include one another.[25] From this view of judgment as concept subordination, [26] the traditional doctrine of the syllogism follows, since inferring is, of course, the derivation of one judgment from another. Thus, Aristotelian syllogistic emerges as the completed doctrine of the forms of inference: "since the time of

---

form judgments using a subject concept from one dimension and a predicate concept along another, we'll get necessarily synthetic judgments: e.g., "All bipedal animals are men." Note that neither the concept <bipedal animal> nor the concept <rational animal> is contained *in* the other, though the first is contained *under* the second. (This shows that, if a tree structure is multi-dimensional, and is meant to capture only analytic judgments, there can be no 'crossing-over' of concepts from one dimension to another.)

[23] See *JL* §6, note 2; Ak 9: 95. §11, note; Ak 9: 97. See also Ak 20: 911; *JL* Ak 9:35, 59. If there were no highest genus, then every concept would contain *within itself* an infinite number of partial concepts, namely, all the higher genera under which it falls. But this is impossible for any non-intuitive representation.

[24] See *JL* Ak 9:59. "Marks are *coordinate* insofar as each of them is represented as an *immediate* mark of the thing…The combination of coordinate marks to form the whole of a concept is called an *aggregate*…[T]he aggregation of coordinate marks constitutes the totality of the concept."

[25] See the "definition of a judgment in general," from *JL*: "A judgment is the representation of the unity of the consciousness of various representations, or the representation of their relation insofar as they constitute a concept." (§17; Ak 9:101) The given representations that are "combined in the unity of consciousness in the judgment [zur Einheit des Bewußtseins im Urteile verbunden]" (§18), Kant calls the *matter* of the judgment; the way that these representations belong to one consciousness, he calls the *form*. What are these given representations? Elsewhere, Kant makes clear that they are concepts. (See A266/B322; Refl 4634 (1772-6), Ak 17:616; *JL* §21, 22). And how can these given concepts be related so as to constitute a concept? To answer this, we need only consider the (Aristotelian) structural interrelations that Kant ascribes to concepts: the given concepts are combined in the unity of consciousness by being *subordinated* to one another. Compare also A68-9/B93-4.

[26] Compare Refl 3051 ((1776-83), Ak 17:633. "The representation of the way in which various concepts (as such)* belong to a consciousness** (in general (not merely my own)) is judgment." Kant appends a note at the phrase "to a consciousness": "concepts belong to a consciousness only insofar as they are thought as under, and not as coordinated [neben] with one another (as sensations)." For Kant, then, multiple concepts can belong to the same consciousness in a judgment by their being contained under one another.

Aristotle [logic] has not had to take a single step backwards… What is further remarkable about logic is that until now it has also been unable to take a single step forward, and therefore seems to all appearance to be finished and complete."[27]

We also find a classic account of the theory of concept formation by abstraction in *Jäsche Logic* §6:

> The logical *actus* of the understanding, through which concepts are generated as to their form, are:
> 1. *comparison* of representations among one another in relation to the unity of consciousness
> 2. *reflection* as to how various representations can be conceived in one consciousness; and finally,
> 3. *abstraction* of everything else in which the given representations differ
>
> *Note 1:* To make concepts out of representations one must thus be able *to compare, to reflect,* and *to abstract,* for these three logical operations of the understanding are the essential and universal conditions for generation of every concept whatsoever. I see, e.g., a spruce, a willow, and a linden. By first comparing these objects with one another I note that they differ from one another in regard to the trunk, the branches, the leaves, etc.; but next I reflect on what they have in common among themselves, trunk, branches, and leaves themselves, and I abstract from the quantity, the figure, etc., of these; thus I acquire a concept of a tree.

The three stages of this abstractionist process are the "essential and universal conditions for generation of *every* concept whatsoever." This means that even *pure* concepts, that is, mathematical concepts and the categories, inasmuch as they are universal representations, that is, representations under which other things are contained, must be generated according to the same process whereby the empirical concept <tree> is generated from the representations of a spruce, of a willow, and of a linden.[28]

---

[27] Bviii.

[28] The very tight connection, reflected in the terminology Kant uses to characterize concepts, between being generated by comparison, reflection, and abstraction and being general, is nicely expressed also in the *Vienna Logic*, Ak 24:909; trans. Young 353.

> No concept comes to be, then, without comparison, without perception of an agreement, or without abstraction. If I could not abstract I would not have any concepts, because something other than what is common to the individual representations would be occurring to me. E.g., if

The process described in *JL* §6 can be carried out in generating concepts not only from particular (lower) concepts, but also from intuitions, that is, immediate, singular representations of objects.[29] For Kant, then, the same general procedure is at work when a subject compares (the singular representations of) given objects so as to form the general concept under which they are all contained and when a subject compares given concepts so as to find the higher concept, their genus, under which they all are contained.

(Not only can concepts be abstracted from intuitions, but a concept can also be contained in an intuition and an intuition can be contained under a concept. Though this consequence is never asserted in Kant's *Logic*, which, after all, abstracts from all relation of our thought to objects and considers only its form,[30] it is asserted in various places in the *Critique*, where the topic is precisely the relation of our representations to objects. There we find Kant saying that a concept can be represented *in* an object or contained *in* an intuition, and that an object can be contained *under* a concept; indeed, he introduces concepts by saying that, as marks of an intuition, or representations under which an intuition is contained, they are the means by which thought relates to objects.[31])

> someone were such that in the case of the expression *house* what occurred to him was always just the *tavern* that he had seen, he would always preserve an *intuitus*.

[29] "Now if a concept is one drawn from the sensory representation, i.e., an empirical concept, it contains as a mark, i.e. as a partial representation, something that was already apprehended in the sensory intuition, and differs from the latter in logical form only, viz., in respect of its generality, e.g., the concepts of a four-footed animal and the representation of a horse" (*What Progress?* Ak 20:273-4). See also B1, A76/B102, A126, the *Vienna Logic* Ak 24:907, and especially EE, Ak 20:211-2.

[30] A55/B79; cf. A54/B78. See also *JL* Ak 9:13.

[31] For instance, in the opening sentences of the Schematism chapter, he describes an *object* as "contained under" a concept (A137/B176), just as in *JL* §7 we read that the things that make up the extension of a concept are "contained under" it. Similarly, again in the Schematism chapter (A137/B176) we read that a concept is represented *in* an object. A similar expression can be found in the student notes from Kant's lectures on logic delivered in the early 1780s; *Vienna Logic,* Ak 24:910.

This is likely what Kant has in mind when he first introduces the distinction between concepts and intuitions in the B edition at B33:

> Objects are therefore given to us by means of sensibility, and it alone affords us intuitions; but they are *thought* through the understanding, and from it arise concepts. But all thought, whether

3. The Circularity Objection to the Traditional Model

Though the works on logic written in the 1870s and 1880s differed in fundamental ways from each other, a general consensus emerged that the traditional doctrine of conceptual structure and formation was hopeless and needed to be replaced. One line of attack against the abstractionist model—one of many—alleged that it is viciously circular.[32] In a series of recent papers,[33] Hannah Ginsborg has resurrected these arguments in an attempt to argue that the "official model" of concept formation in *JL* §6 could not really be Kant's view of how concepts are first formed. Like Ginsborg, I agree with the tradition in seeing the traditional model as viciously circular as it stands. Take Kant's own example of abstracting the concept <tree>. First, in order to abstract the concept <tree> (= <has leaves>, <has a trunk>, <has branches>) from my representation of a spruce, of a willow, and of a linden, I need already to possess the concepts <leaves>, <trunk>, <branches>, and I need already to have represented the spruce, willow, and linden as having leaves, trunks, and branches. But then the process of comparison, reflection, and abstraction cannot be the method of explaining how I come to acquire

---

> straightaway (*directe*) or through a detour (*indirecte*), by means of certain marks, must ultimately be related to intuitions. (translation corrected)

But what are these marks? Presumably marks *of* the intuition, or *concepts under which the intuition is contained*. The concept <heavy>, then, is related back to a given object, because there is an intuition *that is contained under the concept* <heavy>, and that is the immediate representation of an object, say this body, *which is* <heavy>. This reading is given further support by Kant's marginal addition in his copy of the A edition at A19 (=B33): "[intuition] is opposed to the concept, which is merely the mark of intuition"; cited and translated in the Guyer/Wood translation at A19.

See also Refl 2286 (1780s) and Refl 2363 (1790s), Ak 16:299 and 332; *What Progress?*, Ak 20:273-4.

[32] The most important works of the period are Christoph Sigwart's *Logic* and Hermann Lotze's *Logic*, both of which date to the 1870s. An early history is Robert Adamson's *A Short History of Logic* (1882), which also includes a nice criticism of the traditional model of concept formation from a professedly Kantian point of view. A fuller history of the criticism of the traditional model would also include the works of T.H. Green, whose 1870 lectures on Kant and on the traditional logic helped initiate the idealist movement in Britain.

[33] Hannah Ginsborg, "Lawfulness without a Law," "Thinking the Particular as Contained under the Universal," "Aesthetic Judgment and Perceptual Normativity."

concepts in the first place, but at best how I can acquire further concepts on the basis of concepts I have already acquired in some other way. Second, to form the concept <tree> from my representation of a spruce, of a willow, and of a linden, I needed to group together *these* three representations before I compared them, reflected on their similarities, and abstracted. But it seems that in order to isolate just this group of representations (and not for example these three together with my representation of the fundamental theorem of algebra) I needed already to have been aware that they shared some common feature.[34] Third, to form the concept <tree> (= <has leaves>, <has branches>, <has a trunk>) from my representation of the spruce, willow, and linden, I need to isolate just these three common features and not some other odd concept C (e.g., = <is made of wood *and* contains insects>).[35] But if I am to isolate the concept <tree>, and not some other non-standard concept, I need, it seems, already to have been attentive to just these features in my representations of spruce, willow, and linden. And so, as Christoph Sigwart put it, "any attempt to form a concept by abstraction is tantamount to looking for the spectacles which are on your nose, with the help of those same spectacles."[36]

---

[34] Sigwart, *Logic*: "Advocates of this theory also forget that abstraction presupposes some definition of the sphere of objects to be compared, and they tacitly posit some definition of the sphere of objects to be compared, and they tacitly posit a motive for selecting this particular grouping and for seeking its common characteristics. Ultimately this motive, if it is not absolutely arbitrary, can only be that these objects have been recognized as similar a priori, because they have all a specific content in common, i.e., that there is already present a general idea by means of which these objects are distinguished from the totality of objects." (*Logic*, §40.5)

[35] This is Ginsborg's example, ("Thinking the Particular").

[36] *Logic* §40.5, translated by Helen Dendy, 248. Sigwart himself played a major historical role in discrediting the traditional model. He argued that cases like the one from *JL* §6 are really only intelligible if, as in Socratic cases, we want to make more precise the meaning of a word, like "tree," that we already employ in a rule-like way. But no concept is first formed by comparison, reflection, and abstraction.

Ginsborg, like Robert Pippin, concludes from these arguments that the only charitable reading of Kant's text is that the process of comparison, reflection, and abstraction is not in fact a process of forming new concepts, but of "making much clearer to ourselves a concept we already have."[37] But while this might be a philosophically more satisfying conclusion, without further argument it is hard to justify as a textually responsible reading of Kant. The unmistakable contention in *JL* §6 that *all* concepts, insofar as they are general, are formed by comparison, reflection, and abstraction, cannot be so easily dismissed—especially after commentators like Longuenesse have given detailed readings of the *Critique* that take these passages in the *Logic* seriously. So let's go back to the text of the *Logic* and see if there is a way for Kant to escape this circularity.

Consider again the relation between a concept and the representations from which it was generated by comparison, reflection, and abstraction. It is clear from a number of passages that Kant thinks of the process of logical division, whereby a concept's extension is systematically divided in the way represented by the tree for <animal>, as the inverse operation to comparison, reflection, and abstraction: "We go up from lower to higher concepts, and afterward we can go down from these to the lower ones – through division."[38] Since in logical division I determine a concept M by *specifying* it, that is, by enumerating systematically the species (M and N, M and non-N) that are contained under M and in which M is contained, in comparison, reflection, and abstraction, I isolate the

---

[37] Pippin, *Kant's Theory of Form*, 113.

[38] *JL* §110; Ak 9:146. The process of comparison, reflection, and abstraction is clearly the process Kant has in mind when he speaks about "going up," that is, moving from concepts to further concepts under which they are contained; see *JL* §11, note; §15.

common content M of a series of concepts M and N, M and O, M and P, etc.[39] The upshot of this is that a concept M can be abstracted from the representations G, H, J only if M is contained in G, H, J.[40] Since a concept M is contained in a concept N only if N falls under M, this means also that a concept M can be abstracted from concepts G, H, J only if G, H, J are also contained under M.[41]

So what then does abstraction do? To abstract <tree> from <spruce>, <willow>, and <linden>, <tree> must already be contained in them. So I do think <has leaves>, <has a trunk>, <has branches>, whenever I think <linden> (= <has such and such leaves *and* has such and such trunk *and* has such and such branches>). But when I think <has leaves>, <has a trunk>, <has branches> in thinking <linden>, I only think the object as having these particular kinds of leaves, trunk, and branches. Similarly, when I think <has leaves>, <has a trunk>, and <has branches> in thinking <willow>, I only think the object as having these other particular kinds of leaves, trunk, and branches. So, the process of comparing, reflecting, and abstracting to the concept <tree> first makes it possible for me, not to represent, one at a time, this thing, <linden>, whose representation contains *in* it <has leaves>, <has a trunk>, <has branches>, and now this thing, <spruce>,

---

[39] Kant uses the term ‘specification’ to describe logical division in EE §V, Ak 20:214-6. There he also uses ‘classification’ to describe the process of forming empirical concepts, and asserts the inverse relationship between it and specification. Since ‘classification’ is an activity of *reflective* judgment that consists in the *comparison* of given representations in order to find a universal concept, I take it to be an instance of the process of making concepts by comparison, reflection, and abstraction described in the *Jäsche Logic*.

[40] See *Vienna Logic* Ak 24:910; trans. Young, 353:

> Every concept contains more possible concepts under itself and contains that which is common to various representations of several things. Thus if a concept contains something that is common to several things, it is itself contained in other possible concepts[;] it is a part of them, but contains only that which they have in common and omits what is different in them. Every universal concept is contained in the things from which it is abstracted, then. E.g. the concept *metal* belongs to gold, copper, etc.

This principle is also strongly suggested by Kant’s example of the scarlet cloth; *JL* §6, note 2.

[41] See *JL* §8, Ak 9:96: “the concept…contains all those things under itself from which it has been abstracted, e.g., the concept metal contains under itself gold, silver, copper, etc.”

but rather to represent *all* the things that are contained *under* <tree> together through one representation. But I could not have done this until I had separated out the common element <tree> = <has leaves *and* has a trunk *and* has branches> from <spruce>, <willow>, and <linden>. To summarize then: Abstraction allows me to represent the representations contained in a given representation as that under which they are contained. Concepts are therefore essentially general, since the process of comparison, reflection, and abstraction just is the process of taking what is contained in given representations and isolating it as the common or *general* representation that the given representations are contained under, that is, their mark. In this process, no new content is introduced, but rather an element of the content of given representations—their "Inhalt" or what is contained in them—is isolated so that all of the given representations can be thought at once through it.

The threat of circularity now becomes explicit in the claim that a concept M can be abstracted from a series of representations G, H, J only if the concept M is contained in G, H, J, since in order for the concept M to be contained in *my* representations G, H, and J, *it seems that I need already to possess the concept M.* But, in fact, there is no circularity here, since Kant never suggests that the process of comparison, reflection, and abstraction is responsible for bringing it about that I come to have representations with conceptual content. Rather, the process in *JL* §6 is only a process for taking the concepts contained in given representations and forming a general and explicitly discursive representation under which my given representations can be contained. The account in *JL* §6 does not explain how my representations come to have conceptual content, but presupposes that they do.

4. Neo-Kantians and Kant's Theory of Concepts as Rules

Kant's modern readers often view him as a conservative or even reactionary force in the history of logic. Charles Parsons, describing Kant's contributions to logic, has written:

> What must strike a person with modern training most frequently in considering Kant's outlook on logic is the limitation of his knowledge of and conception of it. Kant learned and taught the established logical lore at a very uncreative time in the history of the subject…Kant not only had very limited technical resources at his command; what is more striking and more damaging to his standing as a philosopher, he was largely satisfied with logic as he found it. Technically he could hardly in any case have gone far beyond the state of the science in his own time, and he was not a creative mathematician. But what would have been needed for Kant to be dissatisfied with 'traditional logic' might only have been more insight into his own discoveries.[42]

Philosophers at the end of the nineteenth and beginning of the twentieth century did not share this view. They thought of Kant's work as inspirational in their attempt to overturn the traditional doctrine of concepts and replace it with something new. Thus, Wilhelm Windelband, in the introduction to his survey article, "The Principles of Logic," written in 1912 for the collaborative *Enzyklopädie der Philosophischen Wissenschaften*, wrote:

> A special inquiry into principles is, however, comparatively easy and free from danger in the case of a particular science whose main structure is relatively fixed and accepted. And we should perhaps have found ourselves in this position with regard to Logic about a century and a half ago. It then stood as a well-built edifice firmly based on the Aristotelian foundation, to which subsequent exposition had in the course of time contributed changes in the arrangement of its parts, or made more or less prominent additions.
>
> But, as is well-known, this state of things was entirely changed by Kant. The transcendental point of view which the Critical Philosophy introduced widened the logical problem, and this was only the first step in an entire change of principles which has been proceeding since that time in different and partly opposing directions.[43]

---

[42] Parsons, "Kant's Theory of Arithmetic," 17.

[43] *The Principles of Logic*, 1. Compare also Adamson's earlier judgment, from 1882:

> [E]ven more radical is the divergence of modern logic from the Aristotelian ideal and method. The thinker who claimed for logic a special pre-eminence among sciences because "since Aristotle it has not had to retrace a single step, …and to the present day has not been able to make one step in advance," has, himself in his general modification of all philosophy, placed logic on so new a

Indeed, many Neo-Kantians thought that Kant himself had recognized the circularity in the traditional model;[44] they thought that Kant recognized that one could compare, reflect, and abstract the concept <tree> from given representations only if a subject already possessed the capacity to see these trees as trees. Indeed, Kant was often credited with first discovering that the concept is "something general, and something that serves as a rule" (A106), and seeing that the rule-like character of concepts is not explained by the process of comparison, abstraction, an reflection, but presupposed by it.[45]

For Cassirer, Kant's greatest contribution to the theory of concepts was to recognize that concepts are rules, and, as such, cannot be abstracted from antecedently given particulars in the way the A-a model describes.

> The unity of the conceptual content can thus be 'abstracted' out of the particular elements of its extension only in the sense that it is in connection with them that we become conscious of the specific rule, according to which they are related; but

---

> basis that the only point of connection retained by it in his system with the Aristotelian may be not unfairly described as the community of the subject. Both deal in some way with the principles and methods of human thinking, but as their general views of the constitution of thought are diverse, little agreement is to be found in the special treatment of its logical aspect. (*A Short History of Logic*, 18)

A similar statement is expressed in John Dewey's 1904 paper "Notes upon Logical Topics."

[44] A more recent commentator who finds a vicious circle in the theory of empirical concept formation by abstraction is Robert Wolff. See his *Kant's Theory of Mental Activity*, 118.

[45] See Adamson, *Short History*, 117. A clear expression of this thought is in Cassirer, *Substance and Function*, 17

> The unity of the conceptual content can thus be 'abstracted' out of the particular elements of its extension only in the sense that it is in connection with them that we become conscious of the specific rule, according to which they are related; but not in the sense that we construct the rule *out* of them through either bare summation or neglect of parts…when we ascribe to a manifold an order and connection of elements, we have already presupposed the concept, if not in its complete form, yet in its fundamental function.

See also his *Erkenntnisproblem* II [*EP*II], (1907, 716), where Cassirer argues that one of the main points of the Transcendental Deduction is to see the understanding not as a "mere faculty of abstracted genus-concepts" but as the faculty of rules. Similarly, he claims (676) that the circularity of the traditional model of concept formation is avoided by Kant's theory that every intuition is the result of synthesis (B133) and that this synthesis includes "recognition in a concept" (A103). Similarly, in his later *PSF*3, he writes that "the logical tradition finds the true and salient characteristic of the concept in its universality, and it regards the universal as that which is common to many," while Kant's great innovation in the *Critique* was to "interpret the concept as nothing other than the unity of a rule by which a manifold of contents are held together and connected with one another" (287; see also 306, 315).

> not in the sense that we construct the rule *out* of them through either bare summation or neglect of parts…when we ascribe to a manifold an order and connection of elements, we have already presupposed the concept, if not in its complete form, yet in its fundamental function. (*SF*, 17)

Similarly, in his *Erkenntnisproblem* II, Cassirer argues that one of the main points of the Transcendental Deduction is to see the understanding not as a "mere faculty of abstracted genus-concepts" but as the faculty of rules.[46] Similarly, he claims that the circularity of the traditional model of concept formation is avoided by Kant's theory that every intuition is the result of synthesis (B133) and that this synthesis includes "recognition in a concept" (A103).[47] Similarly, in his later *Philosophy of Symbolic Forms*, vol. 3, he writes that "the logical tradition finds the true and salient characteristic of the concept in its universality, and it regards the universal as that which is common to many," while Kant's great innovation in the *Critique* was to "interpret the concept as nothing other than the unity of a rule by which a manifold of contents are held together and connected with one another."[48]

We can reconstruct Cassirer and the other Neo-Kantians' reasoning in the following way. Consider an intuition of a house, that is, an immediate representation of just this single house, say, that I see before me. In the first edition Transcendental Deduction from the first *Critique*, Kant argues that this visual representation of the house, since it contains a manifold of different representations within it (that is, of representations of the roof, of the front door, of the windows, etc.) and since it takes place over a span of time, must be the result of some sort of combination or synthesis of

[46] 716.
[47] 676.
[48] *PSF*3, 287; see also 306, 315.

representations.[49] Further, if it is going to be an intuition of the house, my representations are going to have to be combined in a certain kind of way. For instance, when I view the house against a blue sky and surrounded by green trees, I need to combine my representation of the roof together with my representation of the windows, but not together with my representation of the sky or of the trees; when I view at t_1 one side of the house, turn and look down at my shoes at t_2, and then view the other side of the house at t_3, I need to combine my representations from t_1 with that of t_3, but not with that of t_2. Now this combination of the manifold of representations into one representation is effected by a *concept*, the "*one* consciousness that unifies the manifold that has been successively intuited, and then also reproduced, into one representation" (A103). This unity of intuition in one representation, effected by the concept, cannot be *any* arbitrary combination. The combination of my representation of the roof with my representation of the windows, etc. constitutes an intuition *of a house* because I represent this unity as being *necessary*;[50] I am representing *a house* only if I combine my representations from t_1 together with those from t_3 and represent them as *belonging together*. Since the representation of a combined unity of different representations that belong together is a necessary[51] condition of representing an object, all of our cognitions—even our intuitions—require a concept (A106).

---

[49] That a phenomenon to be explained by the Transcendental Deduction is the *unity* of an intuition is more explicit in the second edition version of the Deduction (see B144 note); but it is also very clearly an explanandum in the first as well.

[50] On the connection between my combination being necessary and my representations being directed toward an object, see A104.

[51] Kant also thinks that it is sufficient; see A105, B137. The sufficiency claim is harder to defend, but I don't need it to show that Kant recognizes the circularity in the abstractionist model of perceptual concept formation.

Earlier we saw that a concept, unlike an intuition, is a *general* representation; but now we are told that a concept is not only "something general," but also "something that serves as a rule."[52] To use Kant's example (A106), the concept <body> is a rule, in the sense that, if I judge that A is a body, I must necessarily judge that A is extended, that A is impenetrable, since <impenetrable> and <extended> are contained in <body>. This is the first sense in which a concept, as the reflected, discursive representations treated of in the *Logic*, serves as a rule. But we are now in position to see a second sense in which a concept serves as a rule. If I am to represent the house *as a body*, then it is necessary that I combine the manifold of representations contained in my representation of the house in such a way that I represent it as taking up space and as resisting displacement. A 'concept,' in the first sense (the use we are familiar with from Kant's *Logic*, the concept as *universal or discursive representation*), is the sort of representation that can enter into judgments and inferences. A 'concept,' in the second sense (the use we employ when discussing perception, the concept as *rule for perceptual synthesis*), is the sort of representation that allows us to intuit an object in a certain way, namely as an *F*.[53]

Of course, much more needs to be said in order to fill out this sketch of Kant's theory of perception and perceptual concepts. Fortunately, in order to determine in what sense Kant's view of concept formation is abstractionist, we need only note the following. If Kant is right in arguing that in order for my representation to be *of an object*, it needs to be combined according to some concept, then it follows that when I represent an F as an object, I need to represent it *as some G or other*, perhaps F or

[52] A106. See also A126, A132/B171, *JL* §I, Ak 9:11.
[53] On this whole topic, see the very helpful discussion in Longuenesse, *Kant and the Capacity to Judge*, chapter 2, and Allison, *Kant's Transcendental Idealism*, 2nd ed, 78-82.

perhaps not.[54] And so there cannot be an intuitive representation—a singular representation of an object—prior to all concepts. When we keep in mind that for Kant no representation—conceptual, intuitive, or noncognitive—is innate[55] and so all concepts must be made or formed, we see that Kant could not hold that all concepts are formed by abstraction from given representations. Otherwise, the regress we noted above could not be stopped. If we granted that the concept <tree> could be formed from the given concepts <spruce>, <willow>, and <linden>, we would still need some explanation of the given representations from which these lower concepts could be formed. With no innate concepts to appeal to, we look to given intuitions. If we grant that these lower concepts could be formed from given intuitions, the argument of the last few pages forces us to find some concept that provides a rule for my synthesis of the manifold in these given intuitions. And since no concept is innate, the regress would begin again.

5. KANT ON MATHEMATICAL CONCEPTS

Most of the discussion of the traditional model of concept formation in Kant has understandably focused on the formation of empirical or perceptual concepts. But the traditional model—and the text of *JL* §6—is meant to apply to the formation of all concepts, even the concepts of pure mathematics and natural science. As far as I am

---

[54] See the example of the savage's intuition of a house in *JL* Ak 9:33. The savage, who does not possess the concept <house>, still can have, Kant suggests, an intuition of the house, as long as he combines his representations in the right way: the representations of the windows with that of the roof and not with those of the tree, etc. Of course, the savage's rule-governed combination of representations need not (and, indeed, could not) have been governed by the rules for syntheses associated specifically with the concept <house>.

[55] See "On a Discovery," Ak 8:221.

aware, no modern commentator has explicitly discussed the relationship between Kant's theory of mathematical concepts and the account of concept formation by abstraction. A century ago, however, this was an active topic among Kantian philosophers and commentators. These discussions centered around another objection to the traditional model that philosophers found in Kant: in mathematical reasoning, which, as Kant argued, proceeds by *construction*, concepts are not abstracted from antecedently grasped representations of particulars; rather, the concepts come first, and make possible the representation of the objects contained under them. This opposition between construction and abstraction can be found in Lotze's *Logic*,[56] and in the rival "Southwest School" German Neo-Kantians Rickert[57] and Lask.[58] Cassirer himself puts the point like this: "That the definitions of mathematics derive from pure intuition here signifies only that they are not—as the 'discursive' concepts of formal logic—abstracted from a multiplicity of different contents as their common mark, but rather have their origin in a fully determinate unique act of construction."[59] A review of Kant's theory of mathematical

---

[56] Lotze, *Logik*, volume II, 192-210. In Lotze's *Metaphysik*, section 102-3, he connects this view to Kant's conception of geometry.

[57] Rickert, *Die Lehre der Definition* (1884); 3rd edition, 40-44. At 40, Rickert says that the method of construction in mathematics requires no material at the beginning; it creates its material itself, and so it does not introduce anything inessential from which it would have to abstract. "Here there can be no discussion of concept formation through abstraction." On 42-44, he considers the objection that in geometry, we form concepts by abstracting from the irrelevant features of diagrams. But this is backwards: we do not abstract from the color, etc. of the figure in the diagram in order to arrive at the concept <triangle>, since in order to draw the diagram, one needed to have had the concept <triangle> already.

[58] Lask, *Fichtes Idealismus und die Geschichte*, 44-56 (in *Gesammelte Schriften*, vol. 1, 1923), published 1901. Lask argues that, for Kant, concepts that can be constructed are exceptions to Kant's general doctrine of concept formation by analysis or abstraction from reality.

[59] See KMM, 32-3. In fact, though Marburg Neo-Kantians like Cassirer agreed with Southwest Neo-Kantians like Lask and Rickert that mathematical concepts are constructed, not abstracted, an important divergence between the two schools concerned whether the abstractionist or constructivist model of concept formation is true of empirical concepts. The Southwest school thought that empirical concepts were formed by abstraction; the Marburg school denied it.

concepts will, I think, make it clear why Cassirer and other Neo-Kantians thought that Kant showed that the traditional model to be false, at least for mathematical concepts.[60]

For Kant, what makes mathematics distinct from other disciplines is its form: the particular way in which concepts are employed in it and thus also the particular kind of concepts that it makes use of.[61] Not all concepts are useable in mathematics, and of those that are, only certain uses of them are mathematical uses. We can think of this form of concept deployment, "rational cognition from the construction of concepts" (A713/B741) as made up of two parts. We say that a concept F is mathematical, or is being used mathematically, when (1) (a singular representation of) an object *a* contained under F can be exhibited a priori,[62] and (2) I can reason to new facts about all objects contained under F, that is to facts that do not merely follow from the analysis of what is contained in F, merely by reasoning about this *a*.[63] Kant thinks that (1) and (2) require that my

---

[60] Though this reading of Kant was widespread a century ago, contemporary commentators on Kant's theory of concept formation have not seemed to notice that Kant's theory of mathematical cognition is inconsistent with the traditional doctrine of concepts. Even commentators who are concerned to give a reading of Kant's whole theory of concept formation, like Longuenesse in her *Kant and the Capacity to Judge,* have not seemed to appreciate how different mathematical concepts are for Kant from empirical ones.

[61] Kant does not explain the difference between mathematics and philosophy by identifying a distinctive object of mathematical research. See A714-5/B742-3; Vienna Logic, Ak16: 797, Young, 258.

[62] See A713/B741 and also "On a Discovery": "in the most general sense one can call construction all exhibition of a concept through the (spontaneous) production of a corresponding intuition" (Ak8: 192; 111 in Allison). See also *JL*, Ak 9:23, Young, 536; Dohna-Wundlacken Logic, 697, in Young, 435.

I have written above that construction of a concept allows the mathematician to exhibit an object or an intuition *contained under* the concept; in the quoted passages, Kant says that the exhibited object "corresponds" to the concept. I think that Kant's language here is intentional: he is trying to respect the commonplace observation that every material triangle, including the drawn triangle, has sides that are more or less thick and only more or less straight, and so fails to fall under the mathematical concept <triangle>.

I'm ignoring this point, since though I do not strictly speaking exhibit an object that is contained under a concept, when I possess a mathematical concept, I am able a priori to exhibit an intuition corresponding to that concept, *and my exhibition of that intuition proves immediately that there is such an object contained under the concept.* This is what we would expect, since in ancient synthetic geometry existence is proved via construction.

[63] A714/B742: "Thus philosophical cognition considers the particular only in the universal, mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason, so that just as this individual is determined under certain general conditions of

concept not be “given,” but be “made arbitrarily.”[64] Some examples will make what Kant has in mind clearer. A concept that is given, such as an empirical concept like <water>, <gold>, or <human>, is possessed by the subject *before* she comes to arrive at its definition, which can then only be reached, if at all, by slow, patient, and uncertain analysis. Mathematical concepts, as “made” concepts, are like concepts introduced by stipulative definitions: “in mathematics, we do not have any concept at all prior to the definitions, as that through which the concept is first given.”[65] Mathematical concepts, though, differ from other made concepts: if I introduce the concept <swampman> (=<man *and* formed by lightning *and* identical particle by particle to me>) by stipulation (or ‘declaration’) I am not thereby able to produce an object that is a swampman; but according to (1), I am able to do so with mathematical concepts. But since my mathematical concept was made arbitrarily, there is nothing contained in it that I am not aware of if I possess it at all, and so I can be completely certain of what holds of the individual *a* solely in virtue of the fact that it is an F.[66] We can illustrate this picture using Kant’s Euclidean paradigm. I possess the concept <circle> through its definition as a

construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined.”

[64] See *JL* §102: “The synthesis of concepts that are made, out of which synthetic definitions arise, is either that of *exposition* (of appearances) or that of *construction*. The latter is the synthesis of concepts that are made *arbitrarily*, the former the synthesis of concepts that are made empirically, i.e., from given appearances as their matter…Concepts that are made arbitrarily are the *mathematical* ones.

See also A729/B757, where Kant says that only mathematical concepts contain “an arbitrary synthesis that can be constructed *a priori*.” Such concepts are “arbitrarily thought” and so, for mathematical concepts, “I can always define my concept: for I must know what I wanted to think, since I deliberately made it up.”

[65] A731/B759.

[66] A719/B747: “In mathematical problems the question is…about the properties of the objects themselves, solely insofar as these are combined with the concept of them.”

See also A729-30/B757-8: “Thus there remain no other concepts that are fit for being defined than those containing an arbitrary synthesis which can be constructed *a priori*, and thus only mathematics has definitions [Definition]. For the object that it thinks it also exhibits *a priori* in intuition, and this can surely contain neither more nor less than the concept, since through the definition [Erklärung] of the concept the object is originally given, i.e., without the definition [Erklärung] being derived from anywhere else.”

curve all of whose points are equidistant from a given point, and I possess this definition when I have grasped the Euclidean postulate that enables me to construct, without the aid of experience, a circle with a given center and radius.[67] Once I've drawn this circle, I can prove something new about this circle *a*, e.g., that it, together with a circle *b* whose center lies on *a*, forms an equilateral triangle whose three edges are the two centers and an intersection point of *a* and *b*. This fact is new—not an *analytic* judgment—in the sense that the concept <circle> (=<curve *and* all of its points are equidistant from a given point>) does not contain this new property in the very literal way that, in the traditional logic, a species concept like <rational animal> contains its genus concept <animal>.[68] And further, this fact about *a* is perfectly general, and holds of all objects that are contained under <circle>, even if they differ greatly from *a* in magnitude or position.[69]

Moreover, in paradigm cases, like <circle>, the representation of a particular object falling under it—a circle—is only possible if it is constructed according to the rules internal to the concept <circle>.[70] With empirical concepts (though not the categories!), it is perfectly possible for me to intuit a house without even possessing the concept <house> and so without even having the capacity to intuit it as a house (*JL* Ak 9:33). Not so with mathematical concepts. I cannot even intuit a circle unless I already possess the mathematical concept <circle> and have used it construct a circle.[71] In sum, then, what is distinctive about mathematics is a certain relation between concepts and

[67] A713/B741. See also Kant's description of the construction for Euclid I.32 at A716-7/ B744-5.
[68] At A718/B746, Kant says that construction allows us "go beyond" the concept to "properties that do no lie in the concept but still belong to it."
[69] In Kant's terms, I attend only to the schema of the concept: "a general procedure of the imagination for providing a concept with its image" (A140/B179). See also A141/B180, A714/B742, A718/B746.
[70] B154-5: "We cannot think of a line without *drawing* it in thought, we cannot think of a circle without *describing* it, we cannot represent the three dimensions of space at all without *placing* three lines perpendicular to each other at the same point."
[71] See here A730/B758.

objects, universals and particulars. In mathematics, possessing a concept is prior to representing the objects that are contained under it. Once I possess the concept, I can exhibit or construct the objects falling under the concept; since I construct the objects falling under the concept, as soon as I possess the concept, I can be assured of the existence of objects falling under the concept. Moreover, all of the (mathematical) properties that belong to the objects contained under the concept belong to it in virtue of the fact that it is contained under the concept, even though not all of these properties are literally contained in the concept.

Now we can see, I think, why so many philosophers thought that Kant's account of mathematical construction was inconsistent with the traditional abstractionist model. "We cannot think of a circle without describing it," and knowing the procedure for describing a circle requires possessing the concept <circle>. So I cannot form my concept <circle> by abstracting the concept from my representations of circles, since I would not have a representation of a circle at all if I did not already possess the concept <circle>. Moreover, mathematical concepts do not just make the representations of mathematical objects more general, they make the representation of mathematical objects possible in the first place. According to the abstractionist model, the representation of objects falling under a concept is prior to the possession of the concept itself; in mathematics, according to Kant, the representation of the concept is prior to the representation of the objects falling under it.

6. Conclusion

Cassirer and other Neo-Kantians argued that Kant does not endorse an abstractionist account of how a subject first comes to have representations with conceptual content. The account given in *JL* §6, they argued, is not meant to explain how, but rather presupposes that, a subject has come to have representations in which concepts are contained. Furthermore, the account of the unity of an intuition given in the A Deduction requires that the subject possess a concept prior to an intuition, and so it could not be the case that every concept is formed by abstraction from given intuitions. And in mathematics, the singular representation of circular objects do not make possible the formation of the concept <circle>, as the traditional model would have it, but a subject's representation of particular circles requires that she already possess the concept <circle> and the constructive procedure internal to it.

The discussion of this chapter, though, has raised more questions than it has answered. For one thing, though I think Cassirer and the Neo-Kantian tradition were right that Kant does not think that the activity of comparison, reflection, and abstraction is responsible for my first coming to have representations with conceptual content, we have still to see what precisely Kant's actual account of concept formation is. Perhaps more worrying, we have yet to see how an anti-abstractionist reading of Kant could square with the text of *JL* §6—even if we accept that the account of concept formation given there does not attempt to explain, but presupposes, that the cognitive subject already possesses representations with conceptual content. Indeed, I earlier claimed that interpreters who dismiss the abstractionist account of concept formation as circular cannot dismiss it *as a reading of Kant* unless they are able to give a convincing counter-reading of *JL* §6. And in what sense, an objector might wonder, does Kant on the Neo-Kantians' anti-abstractionist reading hold that the logical operations of comparison, reflection, and abstraction are the "essential and universal conditions for generation of any concept whatsoever"?

The Marburg Neo-Kantians themselves were uninterested in this question, since they rejected Kant's *Logic*—and indeed all of formal logic—as barren. I tell that story in the next chapter. In the concluding few pages of this chapter (which are an aside in the overall discussion of this dissertation), I tentatively suggest a sketch of a reading that is less dismissive of Kant's *Logic.*

We distinguished above, following A103-4, between concepts *as discursive representations*, expressible, when fully analyzed, in definitions, and concepts *as rules for perceptual synthesis*. A number of passages suggest that, in standard cases at least, I come to possess a concept as a *universal or discursive representation* subsequently to my possessing the concept as a *rule for perceptual synthesis*. That is, I make explicit the rules that I already use when I intuit Fs as F so that I can represent in a discursive or general way the concept F in judgments and inferences.[72] And this in fact is just what

---

[72] See here the important but difficult passage at A103, which on its surface, seems to assert that a concept is that without which a concept is impossible:

> If, in counting, I forget that the units that now hover before my senses were successively added to each other by me, then I would not cognize the generation of the magnitude through this successive addition of one to the other, and consequently I would not cognize the number; for this concept consists solely in the consciousness of the unity of this synthesis.
>
> The word 'concept' itself could already lead us to this remark. For it is this *one* consciousness that unifies the manifold that has been successively intuited, and then also reproduced, into one representation. This consciousness may often only be weak, so we connect it with the generation of the representation only in the effect, but not in the act itself, i.e., immediately; but regardless of these differences one consciousness can always be found, even if it lacks conspicuous clarity, and without that concepts, and with them cognition of objects, would be entirely impossible.

Longuenesse (*Kant and the Capacity to Judge*, 47) has shown beautifully that the apparent circularity is resolved, and the whole passage illuminated, when we see it as asserting that the concept as rule for

Kant said in the *Logic*: in concept formation by comparison, reflection, and abstraction, I take what is already contained *in* my representations and make it explicit as that *under* which they are contained. Comparison, abstraction, and reflection take me from concepts as rules for perceptual synthesis to concepts as universal or discursive representations.[73] (In fact, the generality of concepts as discursive is dependent on the fact that they are reflected from concepts as rules for synthesis; a synthesis according to rules is 'necessarily reproducible.')[74]

In mathematical cases, the distinction between concepts as discursive representations and concepts as rules for the synthesis of the manifold of (pure) intuition is a difference between the discursive concept <circle> (= <curve *and* all of whose points are equidistant from a given point>) and the rule for constructing a circle.[75] On my reading, then, Kant needs to give some other account of how these rules for synthesis first come about. In the geometrical case, I think that Kant's answer is given in a footnote to §24 of the Critique, where Kant says that "motion, as *description* of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through productive imagination." Geometrical concepts as rules for synthesis are certain procedures for describing space. Geometrical discursive concepts are then derived from

---

perceptual synthesis makes possible both the relatedness of my representation to an object, and the concept as reflected or discursive representation.

[73] We still need an account of how a subject's synthesis of the manifold of empirical intuition first comes to be governed by concepts as rules for perceptual synthesis. Here I side with the traditional reading that pure categorical syntheses make possible empirical syntheses—in a way that would need to be filled out, of course—against the philosophically very interesting but heterodox reading recently defended by Ginsborg.

[74] At A108, Kant calls the unity of the synthesis of appearances according to concepts, "[a synthesis] in accordance with rules that not only make them necessarily reproducible, but also thereby determine an object for their intuition, i.e., the concept of something in which they are necessarily connected…"

[75] I am here disagreeing with Paul Guyer, *Kant and the Claims of Knowledge*, 159, and Lisa Shabel, "Kant's Philosophy of Mathematics," footnote 34, who think that in mathematical cases possessing a concept simply amounts to having the capacity to construct intuitions corresponding to it.

them, by abstraction; the discursive concepts are thus not innate, though they rest on an innate ground, since the capacity to describe space is an innate function of the imagination of any finite being who is capable of experiencing spatio-temporally. (Using the language Kant employs in his "On a Discovery," Ak 8:221-2, geometrical discursive concepts would not be innate, but a subject would come to possess them through an "original acquisition," whose "ground at least is innate.")[76] I hope to elaborate on these points in a future work.

[76] See also Ak 2:406. Much more needs to be said to defend and elaborate this, especially concerning the relation between the rules for synthesis (or "schemata") for geometrical concepts (the mathematical synthesis) and rules for describing space in experience (empirical synthesis). A good contribution to understanding this topic is R. Lanier Anderson's "Synthesis, Cognitive Normativity, and the Meaning of Kant's Question: 'How are synthetic cognitions a priori possible?'"

CHAPTER 5 TRANSCENDENTAL AND FORMAL LOGIC IN KANT AND THE MARBURG NEO-KANTIANS

Paul Natorp, who, along with Hermann Cohen, began in the 1880s the philosophical movement centered around Marburg University, characterized the "leading thought" and "fixed starting point" of the school as an anti-metaphysical and anti-psychologistic method—the "transcendental method" or the method of "transcendental logic."[1] According to this approach, philosophy begins not with a priori metaphysical speculation, nor with observation of the psychology of individual human subjects, but with the concrete mathematical sciences of nature. These sciences are our paradigms of knowledge and their historical achievements constitute the "fact" whose preconditions it is the task of philosophy to study. Kant himself first isolated this method and applied it to the Newtonian science of his day; in fact, he mistakenly thought that the transcendental preconditions of Newtonian science were the fixed preconditions for all scientific cognition in all times and places. But since mathematical natural science has changed since Newton, there is no reason to think that the particular doctrines of the *Critique* will stay fixed for all time.

---

[1] Natorp, "Kant und die Marburger Schule," 196 (1912). See also Walter Kinkel's striking "Einleitung" to his edition of Kant's *Logik*, xvi, and Cassirer's *Erkenntnisproblem*, volume I [*EP*I], 14-18. A recent discussion is Alan Richardson's "'The Fact of Science' and Critique of Knowledge: Exact Science as Problem and Resource in Marburg Neo-Kantianism."

Natorp and Cohen both taught at Marburg. Cohen habilitated there in 1871 and Natorp a few years later in 1878. Cohen was Cassirer's *Doktorvater*.

> For the "fact" of science according to its nature is and remains obviously a historically developing fact. If this insight is not unambiguously clear in Kant, if the categories appear to him still as *fixed* "stem-concepts of the understanding" in their number and content, the modern development [Fortbildung] of critical and idealistic logic has brought clarity on this point.[2]

As the leading thought of Cassirer and his Marburg progenitors, transcendental logic is quite obviously an essential part of any discussion of Cassirer's and the other Marburg philosophers' philosophical appropriation of Kant. But there are further reasons to turn our attention to the Marburg interpretation of transcendental logic. We saw in chapter 3 that Cassirer conducts his philosophy of modern geometry in the context of a polemic against the traditional A-a model of conceptual structure and formation. He insists that the new model should be derived from "the new critical sense of 'concept'" and "the new positive conception of concept formation that Kant himself founded in his own 'transcendental' logic."[3] In the previous chapter, we saw that Cassirer to a large extent centers his reading of the first *Critique* on the leading idea that Kant is rejecting the traditional model. But we have just seen that the official model of concept formation presented in Kant's *Logic* is a classic instance of the A-a model, and it will require considerable interpretive effort, beyond the tentative sketch I offered in the closing pages of the previous chapter, to see how to reconcile the model from Kant's *Logic* with the two anti-abstractionist strands that the Neo-Kantians recognized in the *Critique*. Cassirer, however, feels no need to make this effort, but is content to note that this new model is present—not in Kant's discussion of *general* logic from the *Jäsche Logic*—in his discussion of *transcendental* logic from the *Critique*. Working our way through

---

[2] Cassirer, *EP*I, 18.
[3] KMM, 33.

Kant's distinction between transcendental and general logic and its reception will thus help us get clear on this initially puzzling feature of Cassirer's reading of Kant.

However, it only takes a moment's reflection for Cassirer's defense, on *transcendental* grounds, of the new *formal* logic of Russell and Frege to appear strained. How can Cassirer dismiss Kant's formal logic but enthusiastically embrace the new logic? As we shall see, Cassirer's Marburg colleagues were not so receptive to the new logic, and thought it as wrong-headed and philosophically barren as its less sophisticated Aristotelian rival. For this reason, Michael Friedman, in his otherwise excellent discussion of Cassirer and the other Neo-Kantians, concludes that "Cassirer's outstanding contribution [to Neo-Kantianism] was to articulate, for the first time, a clear and coherent conception of formal logic within the context of the Marburg School."[4] In reality, Cassirer's attitude toward *formal* logic is much more ambivalent than Friedman suggests. Working our way through Cohen's, Natorp's, and Cassirer's attitude toward Kant's distinction between transcendental and general logic will thus put us in a better position to understand Cassirer's very enthusiastic appropriation of the new relational logic.

There is one final and very significant reason to be clear on Cassirer's relationship to the Kantian distinction. For the Marburg Neo-Kantians, because philosophy has no independent grip on the human understanding or on the nature of objects of experience, there is no leverage by means of which philosophy could condemn an established result of the sciences, in particular, an established result of mathematics, as false or meaningless. Any adequate philosophical study of mathematics will have to recognize that "the modern development of mathematics has thus created a new 'fact,' which the

---

[4] Friedman, *A Parting of the Ways*, 30.

critical philosophy, which does not seek to dominate [meistern] the sciences but to understand [verstehen] them, can no longer overlook."[5] In the theory of concepts, then, if the established results of modern geometry do not fit the traditional model of conceptual structure or formation, then the traditional model requires revision. So Cassirer argues in *Substance and Function* that the traditional model needs to be replaced with the "functional" model of conceptual formation and structure in order to bring the theory of concepts into harmony with modern mathematics and the exact sciences. Once we've made this change in our theory of concepts, the philosophical problems attending the "imaginary" or "extension" elements will disappear: as Cassirer writes, "the new criterion of geometrical concept formation is shown [in Poncelet's projective geometry] in its general significance; for it is the criterion upon which the admission of the imaginary into geometry ultimately rests."[6] However, one man's *modus tollens* is another's *modus ponens*. A rival philosopher might hold, on formal grounds alone, that we know that concepts must be formed by comparison, reflection, and abstraction from given representations. Thus, the fact that concepts like <point at infinity>, <Steiner conic in the projective plane>, or <complex n-space> fail to meet this requirement only shows that these concepts are illegitimate or defective. In order, then, to give his philosophical analysis and defense of modern geometry, Cassirer is forced to answer the independently pressing question of the ground of our logical knowledge. Otherwise, he could not argue that the theory of concept formation and structure—one of the core areas of traditional logical research—had to answer to the "fact" of modern mathematics.

---

[5] KMM, 31.
[6] *SF*, 82.

This chapter will have six sections. In the first two sections, I look at how Kant himself tries to distinguish transcendental from formal logic. In order to emphasize the fact that this distinction relies on some non-trivial theses of Kant's critical philosophy and in order to understand why the Marburg Neo-Kantians thought it necessary to reject of this distinction, I go through various ways that an interpreter of Kant might try to understand Kant's distinction. Though I think that Kant does succeed in drawing a stable distinction, he does so only by invoking some of the most contentious doctrines of the *Critique*: the distinction between sensibility and understanding and the corresponding distinction between phenomena and noumena. We begin our discussion of the Marburg Neo-Kantians' reception of this distinction in section three. Hermann Cohen, Cassirer's teacher and the founder of the school, famously argued that there could not be a formal logic independent of transcendental logic, and that the justification for and discovery of the "forms of thinking" required prior transcendental investigation of the concrete exact sciences. (This conviction led him to argue, famously, that Kant's "Transcendental Analytic" has to be read *backwards*: with an analysis of Newtonian science providing the justification of the "Principles," which then allow us to isolate a table of categories, and then (but only then) the table of forms of judgment.) In the following section, we see that Paul Natorp argued that the impossibility of an independent formal logic followed immediately from the "leading thought" of the school – the transcendental method. Specifically, Natorp argued, the new logic of Frege and Russell is in no better shape than the old logic of Aristotle: it is in no better position to give a justification of our logical knowledge and it inevitably leads into formalism or naïve realism. In section five, I show how Walter Kinkel, the student of Hermann Cohen's who edited and wrote and introductory essay for a new edition of Kant's *Jäsche Logic*, argued that the untenability of the distinction between sensibility and understanding (and then also the distinction between pure concepts of sensibility (space and time) and pure concepts of the understanding (the categories)) deprives Kant's own logical work of the capacity to "stand on its own."

Given this view, held by every member of the Marburg school, that there can be no distinction between transcendental and formal logic, it is quite surprising that Cassirer was an early and vocal advocate of the mathematical and philosophical importance of the new logic of Frege and Russell. As I show in section six, Cassirer makes clear in later writings that he never intended to defend the new logic on anything other than transcendental gounds, and his defense of the "functional" theory of concept formation makes no use of considerations of *the* "form" of thought, but rather relies on an anlysis of the history of nineteenth century mathematics and science. Unlike Russell, he sees no point in giving a characterization of (pure general) "logic" anad arguing that the new "logic" is really "logic" in that sense. (In any case, he is skeptical that a project like this one could ever explain the necessary applicability of logic to our empirical thinking without falling into the myth, which he calls "naïve realism," that experience involves the mind's imposing a subjective "form" on a given, unstructured "matter.")

1. KANT'S DISTINCTION BETWEEN TRANSCENDENTAL AND FORMAL LOGIC

Logic, for Kant, is "the science of the rules of understanding in general" (A52/B76). The understanding, further, is that faculty by means of which an object is thought.[7] In a parallel passage at A131/B170, Kant describes thinking as "discursive cognition." This brings us back to Kant's definitions in *JL* §1 of cognition, which is "a representation related with consciousness to an object," and of a concept, which is a universal or "*reflected* representation (*repraesentio discursiva*)."[8] To think, then, is to consciously represent an object through a concept: "cognition through concepts is called *thought* (*cognitio discursiva*)."[9] Logic, then, in the widest use of the term, is the science of the rules by which we consciously represent an object through a concept. Within logic, broadly so conceived, we can distinguish between general logic and what we might call particular logic:

> Now logic in turn can be undertaken with two different aims, either as the logic of the general or of the particular use of the understanding. The former contains the

---

[7] A50/B74. Since "it comes along with out nature that *intuition* can never be other than *sensible*, i.e., that it contains only the way in which we are affected by objects," the understanding is the faculty of *thinking* the objects of *sensible* intuition (A51/B75). Compare also A19/B33, and *JL* §I, Ak 9:11: "the understanding is the faculty for thinking, i.e., for bringing the representations of the senses under rules."

Lest the reader be confused, the understanding is characterized in many other ways also: as spontaneity (A51/B75); the faculty of concepts (A19/B33); (A68-9); the faculty of judgment (A69/B94); the faculty of rules (A126), (A132/B171), (*JL* §I; Ak 9:11); the faculty of cognition (B137); the synthetic unity of apperception (B134, note). In some places, the reader has to be attentive to the fact that Kant sometimes in speaking of the understanding is referring to the "understanding in general," whose subspecies are the faculty of concepts (= the 'understanding,' on a narrower use of the word), the faculty of judgment, and the faculty of inference (=reason); see A130/B169. Though the bookkeeping here is difficult, all of these explanations on the critical theory amount to the same thing, at least for a finite intellect like our own.

[8] See also *JL* §VIII, Ak 9:58: "From the side of the understanding, human cognition is *discursive*, i.e., it takes place through representations that take as the grounds of cognition that which is common to many things, hence through *marks* as such."

[9] *JL*, §1, Ak 9:91. See also the parenthetical remark at A239/B298, and *JL* §VIII, Ak 9:58: "All our *concepts* are marks, accordingly, and all *thought* is nothing other than a representing though marks." See also Ak 20:325, from the late essay "What Progress has Metaphysics Made Since the Time of Leibniz and Wolff?": "To represent something through concepts, that is, to represent it in general, is called thinking, and the capacity to think, understanding…Cognition through concepts is called discursive."

> absolutely necessary rules of thinking, without which no use of the understanding takes place, and it therefore concerns these rules without regard to the difference of the objects to which it may be directed. The logic of the particular use of the understanding contains the rules for correctly thinking about a certain kind of objects. The former can be called elementary logic, the latter the organon of this or that science. (A52/B76)

Logic is the science of the rules for the use of the understanding, that is, for representing consciously an object through a concept. We can then distinguish the rules that a subject stands under *whenever* she consciously represents an object through a concept, and those that she stands under *some but not all of the time* she consciously represents an object through a concept. The rules for thinking that are the object of study for general logic are necessary, then, because a thinker stands under them just in virtue of the fact that she is thinking. The contingent rules for the use of the understanding are then rules for the distinctive kind of thinking that is called for in this or that science: the rules without which "a certain determinate use of the understanding would not occur."[10] And since it is contingent whether my current thinking activity is an activity of thinking about mathematics or about metaphysics, my standing under the distinctive rules for thinking in mathematics or in metaphysics is also contingent.[11]

---

[10] Compare *JL* §I, AK 9:12:

> The contingent rules, which depend upon a determinate object of cognition, are as manifold as these objects themselves. Thus there is, for example, a use of the understanding in mathematics, in metaphysics, morals, etc. The rules of this particular, determinate use of the understanding in the sciences mentioned are contingent, because it is contingent whether I think of this or that object, to which the particular rules relate.

[11] At A52/B76 and at *JL* Ak 9:12, Kant calls the organon of some particular science a set of rules for the use of the understanding in thinking about a *particular kind of object*. This suggests that the particular sciences are individuated by the particular subset of objects thought that one thinks about when one is engaged in that science. This natural reading, however, would sit uncomfortably with Kant's repeated insistence in the *Critique* (and also in the *Logic* (Ak 9:23)) that the difference between mathematics and philosophy "does not rest on the difference in their matter, or objects" (A714/B743). So, when Kant speaks about different sciences being distinguished by their different objects, he is using 'object' in a very broad sense. So speaking, we can say that the 'object' of my thinking in mathematics is the objects of experiences *merely insofar as they have sensible forms*. Similarly, in metaphysics the 'object' of my

Significantly, Kant also describes general logic as *formal*: "General logic abstracts, as we have shown, from all content of cognition, i.e., from any relation of it to the object, and considers only the logical form in the relation of cognitions to one another, i.e., the form of thinking in general."[12] Since thinking is the conscious representation of objects through a *concept*, the form of thinking will be the form of specifically *conceptual* cognizing. "The matter of concepts is the *object*, their form *universality*" (*JL* §2; Ak 9:91). To consider thinking merely with respect to its form, then, is to consider it inasmuch as it is universal or general. Formal logic, then, is the science of those rules of the understanding that any thinking subject stands under merely inasmuch as she, in thinking, is representing generally.[13] Further, since *formal* logic attends only to the generality of thought and not its *matter*, its relation to its *object*, and since the relation of thought to its object is the *content* of thought,[14] it follows that merely formal logic will also be *abstract*. It "abstracts from all content of cognition, i.e., from any relation of it to the objects." Hence, in its treatment of concepts, formal logic will abstract from the relation of concepts to objects, and will consider them merely inasmuch

---

thinking is the objects of experience *merely insofar as their "perception could belong to possible experience*" (A719/B747). And it is contingent whether I think of an object in either of these two ways.

[12] A55/B79; cf. A54/B78. See also *JL* Ak 9:13: "this science of the necessary laws of the understanding and of reason in general, or what is one and the same, of the mere form of thought as such, we call *logic*"; Ak 9:14: "Logic is thus a self-cognition of the understanding and of reason, not as to their faculties in regard to objects, however, but merely as to form."

[13] A131/B170: "merely formal logic, so conceived, abstracts from all content of cognition (whether it be pure or empirical), and concerns itself merely with the form of thinking (of discursive cognition) in general." Cf. also Bviii-ix.

[14] For a classic use of this notion of 'content,' see A51/B75: "Without sensibility no object would be given to us, and without understanding none would be thought. Thoughts without content are empty, intuitions without concepts are empty." At A95, Kant argues that an *a priori* concept that did not "itself belong within the concept of possible experience nor consist of elements of possible experience," would have no *content*, since it could not be related to an object of possible experience. It would then be "only the logical form for a concept, but not the concept itself through which something could be thought."

This use of "the content of a concept" is of course different from the use at *JL* §7, where the "content" of a concept are those concepts that are contained in it. (We can say, though, that a concept has content in this new sense if it is part of the content of a possible intuition, that is, it is contained in the intuition of a possible object.)

as they are universal or general and can thus be contained in or under one another.[15] In its treatment of judgments, formal logic will abstract from the relation of the constituent concepts of a judgment to objects and will attend merely to the different ways in which one concept can be contained under another.[16] But if formal logic gives the rules that a thinker stands under *merely inasmuch as she is cognizing universally*, in complete abstraction from what objects she happens to be cognizing, it will then also be giving rules that hold of a thinking subject *whenever* she is cognizing an object through a concept. Thus not only will formal logic be abstract, it will then also, completing the circle, be general.[17]

Within the family of *particular* logics, a distinctive kind of science, transcendental logic, can be isolated.

> General logic abstracts, as we have seen, from all content of cognition, i.e. from any relation of it to the objects, and considers only the logical form in the relation of cognitions to one another, i.e. the form of thinking in general. But now since there are pure as well as empirical intuitions (as the transcendental aesthetic proved), a distinction between pure and empirical thinking of objects could also well be found. In this case, there would be a logic in which one did not abstract from all content of cognition; for that logic that contained merely the rules of the

---

[15] *JL* §5, note 1, Ak 9:94: "Since universal logic abstracts from all content of cognition through concepts, or from all matter of thought, it can consider a concept only in respect of its form, i.e., only *subjectively*; not how it determines an object through a mark, but only how it can be related to several objects." The consideration of a concept merely with respect to its form is *subjective*, because one abstracts from its matter, its relation to an *object*, and attends only to its form, its universality or its capacity to *relate to several* objects.

[16] See *JL* §17-19; Ak 9:101. This does not mean, as Cohen and a long line of commentators have thought, that formal logic studies only analytic judgments. Formal logic, since it is general, studies *all* judgments merely with respect to their form. See HJ Paton, *Kant's Metaphysics of Experience*, vol. 1, 213-9.

[17] Formal logic must be general; however, it does not follow directly from anything I've said, though, that general logic must be formal. To fully understand Kant's argument, we would need to see why a rule of the understanding could be a rule that a thinker stands under *whenever* she cognizes an object through a concept *only if* it is a rule that she stands under *in abstraction from the content of her thinking*. If this were true, for example, then general logic would have to consider a judgment in complete abstraction from the content of its constituent concepts and of their relation to objects. This would rule out Frege's view that logic has its own distinctive concepts and objects. But how and how successfully Kant fills in this gap in his argument, is fortunately not relevant to our discussion here. Interested readers can consult the discussion in John MacFarlane's "Frege, Kant, and the Logic in Logicism."

> pure thinking of an object would exclude all those cognitions that were of empirical content. It would therefore concern the origin of our cognition of objects insofar as that cannot be ascribed to the objects; while general logic, on the contrary, has nothing to do with this origin of cognition, but rather considers representations, whether they are originally given a priori in ourselves or only empirically, merely in respect of the laws according to which the understanding brings them into relation to one another when it thinks, and therefore it deals only with the form of the understanding, which can be given to the representations wherever they may have originated. (A55-56/B79-80)

It is clear, I think, that Kant considers transcendental logic distinct from general logic, since it, as the study of the rules of the understanding that a subject stands under when she thinks in a particular way, namely when she engages in *a priori* thinking,[18] does not give rules that hold of any subject *whenever* she is thinking.[19] Such a science of the laws of *a priori* thinking would determine "the origin, the domain, and the objective validity" of *a priori* cognition (A57/B81). As general logic investigates concepts, judgments, and inferences completely generally and therefore completely formally, transcendental logic investigates the rules for generating *a priori* concepts and employing them *a priori* in judgments.

It is obvious right away that this particular logic will be quite different from other particular logics. For one thing, though it will give the rules for cognitions "insofar as

---

[18] The characterization of transcendental logic as the science of the rules of *a priori* thinking is not sufficient to distinguish it from the logic of mathematics. The organon of the mathematical special sciences will be rules for a certain kind of *a priori* thinking—though they will not be 'transcendental' rules. See A56/B81.

Elsewhere, Kant seems to pick out transcendental logic by saying that it is the discipline that "isolates" the understanding and "merely the part of our thought that has its origin solely in the understanding" (A62/B87). This might suggest that we can distinguish transcendental logic from the logic of the mathematical special sciences by saying that the former studies the rules of pure, conceptual thinking, the latter the rules of pure, intuitive thinking. Transcendental logic would then be the organon of metaphysics, where metaphysics is the "system of pure reason (science), the whole (true as well as apparent) philosophical cognition from pure reason in systematic interconnection" (A841/B869).

[19] See A131/B170, where Kant distinguishes transcendental from general logic on the grounds that the former "is limited to a determinate content"; and *JL* §I, Ak 9:15, where we are told that transcendental logic deals with "an object of the mere understanding," while general logic deals with "all objects in general." See also MacFarlane, note 35.

they are related to objects *a priori*" (A57/B81), it will not give the rules for thinking about a special kind of object, say the supersensible objects, God and the soul. Given the fundamental idea of the *Critique*, that "*no a priori* cognition is possible for us except solely of objects of possible experience" (B166), the *a priori* cognitions whose rules are given by transcendental logic must be cognitions of an object of experience, though "represented as an object of the mere understanding" (*JL* Ak 9:15). So the object of the priori cognition treated of by transcendental logic will just be (empirical) objects—though cognized by the understanding merely with regard to the fact that they are cognizable by the understanding. But, second, the *a priori* concepts whose origin, use, and objective validity transcendental logic explores, will be concepts derived from "the understanding even *as to content*" (*JL* §3, Ak 9:92)—and, again, not by being abstracted from the representation of a particular kind of object, say intelligible objects, but by being "borrowed from the nature of the understanding" (*JL* §5, note 1, Ak 9:94) in its use in cognizing the objects of experience.

A worry for Kant here is that transcendental logic, as the study of the rules of the understanding inasmuch as it thinks an object, considered merely as an object for the understanding, by means of concepts derived from the understanding alone, will end up collapsing back into general logic, the study of "the absolutely necessary rules of thinking, without which no use of the understanding takes place." A way of making this worry vivid is to reflect on the significance for Kant of one of the *Critique*'s fundamental theses, that the conditions of the applicability of *a priori* cognitions to objects just *are* the conditions of the possibility of cognitions in general. In his justly famous 1772 letter to Herz, Kant motivated the critical project with the question "On what grounds rests the reference of what in us is called representation to the object?"[20] In 1772, Kant was willing to grant that the relation of a sensible representation, insofar as it is the effect of an object working on sense, was unproblematic; but the relation of *a priori* cognitions to the object, on the other hand, seems mysterious, since an *a priori* theoretical cognition is neither the cause nor the effect of the object. By the time of the publication of the *Critique,* Kant had come to see that the solution of the specific question of the relation of *a priori* cognition to its object was bound up with the solution of the question of the relation to an object of any representation whatsoever.

> *Understanding* is, generally speaking, the faculty of *cognitions*. These consist in the determinate relation of given representations to an object. An *object*, however, is that in the concept of which a manifold of a given intuition is *united*. Now, however, all unification of representations requires unity of consciousness in the synthesis of them. Consequently the unity of consciousness is that which alone constitutes the relation of representations to an object, thus their objective validity, and consequently is that which makes them into cognitions and on which even the possibility of the understanding rests (§17, B137).

Fortunately, we need not worry here about the shape of the new critical conception of 'object' mentioned here, nor of the all-important principle of the unity of apperception which is its correlate. What is worth emphasizing, though, is that the solution to the problem posed in the famous letter to Herz—the general problem of the relation of a representation to its object, the problem of the possibility of 'cognition'—is solved not in general logic, but in *transcendental* logic, whose task, we were told, was to investigate the possibility of specifically *a priori* cognition. Of course, as any reader of the Transcendental Deduction knows, this general question is solved by transcendental logic, because its task is to determine the origin and validity of *a priori* intellectual concepts, namely, the categories. Now the categories are "concepts of an object in general, by

[20] 21 Feb 1772 Letter to Herz, Ak 10:130.

means of which its intuition is regarded as *determined* with regard to one of the *logical functions of judgment*" (B128); and the function of judgment is to bring given representations to the objective unity of consciousness (§19). But since the unity of consciousness is the condition for a representation to be related to an object, the categories are necessarily related to objects inasmuch as the deployment of the categories is a necessary precondition of any cognition at all (§20). What this shows, then, is that the proof of the applicability of *a priori* concepts to experience, an essential task of transcendental logic, will necessarily be a proof that all use of the understanding, *inasmuch as it is cognizing an object*, will employ certain *a priori* concepts and must therefore stand under certain *a priori* rules for their employment. And so Kant entitles himself to characterize transcendental logic as that science that, inter alia, "expounds the elements of the pure cognition of the understanding *and the principles without which no object can be thought at all*."[21]

2. POTENTIAL DIFFICULTIES WITH KANT'S DISTINCTION

But this result of transcendental logic, that the study of the rules of *a priori* thinking in fact delivers up rules for thinking an object at all, ends up making it much more difficult

[21] A62/B87, emphasis added. As far as I can see, the italicized portion of this quoted clause comes as a complete surprise so early in the *Critique* and does not follow from the definition of transcendental logic given earlier in A56-7/B80-1; it is only after we have taken in the argument of the Deduction that we can see that the rules for the pure use of the understanding will also be rules for thinking an object at all.

Relatedly, in the *Critique*, cognition is called 'transcendental' if by means of it "we cognize that and how certain representations (intuitions or concepts) are applied entirely *a priori*, or are possible" (A56/B80); in the *Prolegomena*, we are told that "transcendental" signifies something "that precedes [experience] *a priori*, but that is intended simply to make cognition of experience possible" (Ak 4:374, note). Again, it is only after we have taken in the argument of the Deduction that we can see that these two explanations of 'transcendental' are equivalent.

to argue that the rules laid down in transcendental logic are not in fact just a subset of the rules that a subject stands under *whenever* she cognizes an object through a concept, that is, the rules of *general* logic.[22] "By means of [the categories] alone an object can be thought" (A97); they are "nothing other than the *conditions of thinking in a possible experience*" (A111); they constitute "an *a priori* formal cognition of all objects in general, insofar as they can be thought" (A129-130). Kant clearly worried about this problem, as some of his reflections show. Thus, for instance, Refl 1624 (1780s, Ak 16:42):

> Logic, defined as the general doctrine of the understanding, could yet still appear even to include the [pure] intelligible concepts of objects (categories); thereby it would not abstract from all content of thought. Therefore, the definition of it as a science that contains merely the formal rules of thinking is better. (my trans)

However, at least initially the move from emphasizing generality to emphasizing formality does not seem in itself any more promising, since the categories are "only the form[s] of thinking of an object in general" (A51/B75), and "the way in which the manifold of sensible representation (intuition) belongs to a consciousness precedes all cognition of the object, as its intellectual form, and itself constitutes an *a priori* formal cognition of all objects in general, insofar as they are thought" (A129/130). Trying a different line, perhaps general logic can stake out some genuine room of its own in virtue of the fact that it gives the necessary rules for *thinking*, while transcendental logic gives the necessary rules for *thinking an object*. But again, this does not seem promising *prima facie*, since thinking, as discursive *cognition*, is just the conscious representation, through a concept, *of an object* (*JL* §1). The necessary rules for thinking, then, just are the

[22] This does not mean that I think that Kant has no way of distinguishing general from transcendental logic. But I want to explain why it is difficult for Kant to distinguish them before I explain how, on my reading, Kant does distinguish them.

necessary rule for thinking an object, and the formal rules of thinking just are the formal rules of thinking an object.

Perhaps some headway can be made by noting that the categories are described as the "the *conditions of thinking in a possible experience*" (A111) and are said to represent "the way in which the manifold of sensible representation (intuition) belongs to a consciousness" (A129). Though for us, there can be no representation of an object that is not given to us through specifically spatio-temporal sensible intuition, there is no reason to believe, Kant thinks, that "all finite *thinking* beings" are so constituted that all of their experiences are spatio-temporal (B72). Now, since a concept can have no sense and no significance unless it is related to an object, and objects are not given to us finite creatures unless through modifications of sensibility (which for us humans is spatio-temporal), the pure concepts of the understanding "must also contain *a priori* formal conditions of sensibility (namely of inner sense) that contain the general condition under which alone the category can be applied to any object" (A139-40/B178-9). Thus transcendental logic teaches us that all the pure concepts of the understanding must be related to objects as rules for the determination of time ("Transcendental Schematism"), as the form of inner sense, and provides for us rules for determining the temporal form of an experience ("System of All Principles of Pure Understanding"). But since we have no reason to believe that all finite thinking beings intuit objects only spatio-temporally, we have no reason to believe that the *a priori* rules for time-determination are rules that *all* subjects stand under whenever they cognize an object through a concept.[23]

But we are not yet in the clear:

---

[23] This was pointed out by Parsons, "Kant's Theory of Arithmetic," in *Kant on Pure Reason*, 19.

> Space and time are valid, as conditions of the possibility of how objects can be given to us, no further than for objects of the senses, hence only for experience. Beyond these boundaries they do not represent anything at all, for they are only in the senses and outside of them have no reality. The pure concepts of the understanding are free from this limitation and extend to objects of intuition in general, whether the latter be similar to our own or not, as long as it is sensible and not intellectual. (B148)

It is no part of the concept of the categories, as "concepts of an object in general, by means of which its intuition is regarded as *determined* with regard to one of the *logical functions* of judgment" (B128), that they be rules for determining a *specifically spatio-temporal* intuition. And Kant, especially in the second edition, is scrupulous in distinguishing within that part of the *Critique* entitled "The Transcendental Logic," those sections that make no use of the fact that our sensibility is specifically spatio-temporal (roughly a hundred pages of "The Transcendental Analytic" B50-B150), from those sections that do (in the Analytic, another roughly two hundred pages, B150-B349). We might then distinguish that study represented by these first sections of "The Transcendental Logic," as the science of the rules that a subject stands under whenever she thinks an object, from that study represented by the latter sections of "The Transcendental Logic," as the science of the rules that a *human* subject stands under whenever she thinks an object.

The first hundred pages of "The Transcendental Logic" are dense and important pages indeed, including almost the entirety of the "Analytic of Concepts," in which we are introduced to the idea that cognition has an intellectual form and to the categories as those forms, and are then given (almost the entire) proof of the fact that, and the manner in which, the categories are related to objects of experience. Now, on what ground are we to exclude these topics from general logic? The condition for the validity of the categories given above from B148, that they be applied to *sensible* intuition,[24] is in fact no limitation at all.

> [The categories] are only rules for an understanding whose entire capacity consists in thinking, i.e., in the action of bringing the synthesis of the manifold that is given to it in intuition from elsewhere to the unity of apperception, which therefore *cognizes* nothing at all by itself, but only combines and orders the material for cognition, the intuition, which must be given to it through the object. (B145)

In B148, the contrast class to sensible intuition is intellectual intuition; here the contrast class to an understanding that thinks is an understanding that does not cognize material given to it from elsewhere. And such an understanding, "an understanding that itself intuited" or "a divine understanding" (B145), just is the being capable of intellectual intuition mentioned in B148. So the only kind of cognizer for whom the categories do not give the rule would be a divine or infinite understanding, which, because its representations are all the direct cause of their objects, does not cognize *discursively* or by means of concepts at all.[25]

I have now canvassed five different possible ways of distinguishing general from transcendental logic: first, that transcendental logic gives rules only for a particular kind of thinking an object; second, that it is insufficiently formal; third, that it gives rules only

---

[24] See also B150: "The pure concepts of the understanding are related through the mere understanding to objects of intuition in general, without it being determined whether this intuition is our own or some other but still sensible one."

[25] See B71-2. Here the argument that no *discursive* intellect could be infinite or divine is clear: *thinking*, representing an object only *mediately* by means of a concept, is a kind of limitation, and so no divine intellect can be discursive. There is no explicit argument here for the claim that every finite intellect, that is, every intellect whose representations are not the *cause* of their object, cognizes only through concepts. But Kant could not have given the argument at B71, coming as it did before the Transcendental Deduction. From the Deduction, we learn that those representations that are not the causes of but rather caused by their objects, the representations of sensibility, if they are to be *my* representations (§16), must be combined into a unity in consciousness. But this combination cannot be given through sensibility (§15). Indeed, a *concept* just is "this *one* consciousness that unifies the manifold that has been successively intuited, and then also reproduced, into one representation" (A103). So every representation that is not the cause of its object must be combined through concepts. So every finite intellect is discursive.

for thinking an object; fourth, that it gives rules only for thinking a spatio-temporal object; fifth, that it gives rules only for thinking a sensible object. I have concluded that the first, second, and fourth ways mischaracterize transcendental logic, and that the third and fifth ways do not serve to distinguish it from general logic. Along the way, I have made use of the fact that all thinking is a kind of cognition, and so all thinking is thinking an object. I have thus been taking seriously Kant's characterization of logic as the science of the laws of thinking, and his definition of thinking as *cognizing* through concepts.

However, despite classing thinking as a kind of cognizing, Kant is careful in other places to distinguish the two. Thus he writes

> To *think* of an object and to *cognize* an object are thus not the same. For two components belong to cognition; first, the concept, through which an object is thought at all (the category), and second, the intuition, through which it is given; for if an intuition corresponding to the concept could not be given at all, then it would be thought as far as its form is concerned, but without any object, and by its means no cognition of anything at all would be possible. (B146)[26]

Just as we have distinguished between mere thinking and cognizing, we can introduce a parallel distinction between what holds of the categories as *(even possibly empty) concepts* from what holds of the categories as *contentful concepts* or cognitions. That is, we can distinguish between the conditions that the category, or any other concept, must fulfill in order to be *discursive*, from those conditions that the category must fulfill in order to be discursive *cognition*. The former conditions would be given by general logic, the science of those rules of discursive cognition that it stands under *merely qua discursive*; the latter conditions by transcendental logic, the science of the rules of

[26] See also B xxvi, note.

discursive cognition that it stands under *merely qua cognition*.[27] Recalling again that cognition is conscious representation related to an object, and that the form of a concept is its universality or discursivity and the matter is the object, the study of thinking *merely qua discursive* will be formal—it will study "thought as far as its form is concerned," to pick up the phrase from B146.[28] And so we can better appreciate Kant's insistence that the *formality* of general logic makes its distinction from transcendental logic more explicit. Even though the categories are the forms of thinking, they are forms of thinking *qua cognition*, not *qua discursive*. The forms of thinking that are studied in general logic are then the forms of thinking *qua discursive*; it is the study of "the formal element in the forms of thought," as Robert Adamson later put it.[29] We can then isolate general logic through a kind of hyper-abstraction, not by isolating the form of discursive thinking in abstraction from the different ways in which different acts of thinking relate to their object, but by isolating the form of discursive thinking *in abstraction from the fact that thinking relates to objects at all.* All thinking, as discursive representation of an object, is

[27] Within transcendental logic, we might then distinguish two disciplines: the first of which studies those rules of discursive cognition that it stands under *merely qua cognition of sensible objects*; the second of which studies those rules that it stands under *merely qua cognition of spatio-temporal sensible objects*. Again, the first corresponds to the Analytic up to §24; the second to the Analytic from §24 to the end.

[28] At B166n, Kant concludes that

> the categories are not restricted in *thinking* by the conditions of our sensible intuition, but have an unbounded field, and only the *cognition* of objects that we think, the determination of the object, requires intuition; in the absence of the latter, the thought of the object can still have its true and useful consequences for the *use* of the subject's *reason*, which, however, cannot be expounded here, for it is not always directed to the determination of the object, thus to cognition, but rather also to the subject and its willing.

It is initially puzzling how the categories, as concepts of the form of thinking an object of sensible intuition in general, are here described as having a use in complete abstraction not just from *human* sensibility, but from sensible intuition in general. The point here is subtle. As a concept, <form of thinking an object>, the category, like any other concept, can itself be treated of by *general* logic *merely qua discursive*. So there is a possible use of the concept <form of thinking an object> where we treat it not as a form of thinking an object, but merely as a form of thinking. We can, for instance, use the category in combination with other concepts to form judgments and draw inferences, without stopping to attend to the conditions of its applicability to objects.

[29] *Short Introduction*, 114.

subject to the categories, because the categories are the forms of the relation of thought to its object;[30] but that does not mean that the categories are forms of thought taken in abstraction from its relation to an object. (Indeed, we can now, in hindsight, see all five ways of distinguishing formal from transcendental logic as being each sufficient when understood rightly.)

3. Hermann Cohen’s Rejection of the Distinction

Though Cassirer edited an eleven-volume edition of Kant’s works between 1912 and 1923, and though his own writings on Kant contain detailed references to not only Kant’s published writings, but also his letters, lectures, and *Handschrifte Nachlass*, there is not a single explicit reference in Cassirer’s writings, as far as I can tell, to the *Jäsche Logic*. Cassirer never explicitly acknowledges that in the *Logic*, Kant claims that *all* concepts, even pure concepts, arise from abstraction—even though Cassirer repeatedly claims in his historical writings that Kant undermined the theory of concept formation by

---

[30] One might worry here that not all thinking is subject to the categories, since thought does have “its true and useful consequences for the *use* of the subject’s *reason*,” where thought is not being used to determine an object (B166n). Here, Kant is alluding to his doctrine that pure practical reason has its own postulates, and therefore needs to be able to think the concepts <God> and <immortality> even though there is no possible experience that could give these concepts content.

But then an opposite worry sets in. Since thinking is characterized by Kant as discursive cognition, it seems that an activity of thinking, like the postulates of pure practical reason, where the representations thought have no relation to an object—and indeed could never be related to an object—must for that very reason be *degenerate* thinking. This is a difficult problem, which ultimately gets to the heart of the relation of the *Critique of Pure Reason* and the *Critique of Practical Reason*. What is significant for understanding the later Neo-Kantians, however, is that the distinction between thinking and cognition that Kant needs to distinguish between transcendental and formal logic depends, to a great extent, both on the very controversial distinctions between phenomena and noumena, appearances and things in themselves, and on Kant’s seemingly paradoxical thesis that practical reason depends on postulates whose truth we can never know.

abstraction, and even though Cassirer organizes his systematic writings on a rejection of the traditional model. Clearly this omission was not accidental.

Cassirer, not only in his Kant scholarship, but also in his systematic philosophical writings, shows little interest in isolating formal logic as a distinct discipline from the rest of theoretical philosophy. This does not mean, of course, that Cassirer had no interest in modern research in logic, as his early enthusiasm for Russell's *Principles* shows. Cassirer was convinced by Russell and Frege that "logic and mathematics have been fused into a true, henceforth indissoluble unity,"[31] but he shows no anxiety over whether Frege's begriffsschrift or the system in Russell's *Principles* can legitimately claim to be logic. Cassirer had no interest in debating whether or not modern logic is really *logic*.

These peculiarities of Cassirer's appropriation of Kant and philosophy of mathematics can only be understood in the context of the Marburg Neo-Kantians' hostility toward traditional formal logic and toward the very idea of an independent formal logic distinct from transcendental logic. In the previous section, I noted the nontrivial obstacles that stood before Kant as he tried to maintain his distinction, and I showed the fine line that Kant had to draw to keep his distinction from collapsing. Without exception, the Marburg Neo-Kantians thought that Kant could not draw that fine line. In the remainder of this chapter, I will provide a context for Cassirer's ambivalent attitude toward formal logic by discussing the Marburg philosophers' readings of Kant's distinction between transcendental and formal logic.

Cassirer's teacher, Hermann Cohen, is famous for having argued, on Kantian grounds, that formal logic, if it exists as a science at all, must be trivial. In his most

---

[31] KMM, 4.

significant work on the first Critique, *Kants Theorie der Erfahrung*, Cohen argued for this thesis on three grounds. First, he claimed that the only judgments that general logic studies are "mere products of thought," namely, *analytic* judgments.[32] But if "every judgment must, should it possess some value for the progress of science, be called synthetic according to its ultimate source,"[33] general logic, as the theory of analytic cognition, will be a trivial pursuit indeed: the rules that it isolates will be little help, if any, in identifying pure intellectual rules and synthetic a *a priori* principles for the synthetic judgments that are the real quarry of theoretical philosophy. Second, general logic, if it exists at all, cannot be an independent science, but must ultimately depend on *transcendental* logic.

> What insures the correctness of the kinds of judgments that formal logic is concerned to distinguish? If logic itself justifies the principle of the division and with it all of logic resting on judgments, then the principle cannot in any case be found within logic itself, but rather can only somehow be presupposed in it, as geometry in its axioms rests on such presuppositions. From this little observation is shown the necessity of completing 'general' logic through transcendental, the formal through the epistemological [erkenntnisskritische].[34]

The topic for Cohen here is Kant's Metaphysical Deduction, the infamous argument that one can derive the table of the categories from the table of the different kinds of judgments identified in general logic. General Logic, for instance, distinguishes between

---

[32] Cohen, Hermann. *Kants Theorie der Erfahrung*, 2nd edition [*KTE*], (1885), 242:

> The forms of thought [viz, the categories] cannot just be taken from the kinds of judgments that formal or general logic distinguishes; because mere products of thought [Denkgebilde] figure in these kinds of judgments, the judgments are analytic. We are looking for, however, the forms of thought as forms of synthetic judgment. The unity of consciousness, which has to have thought as its means, is a "synthetic unity of consciousness." Therefore, the forms of this synthetic thought cannot be taken from the kinds of analytic thought.

Cohen's reading is attacked in Klaus Reich, *Die Vollständigkeit der kantischen Urteilskraft* (1932; translated as *The Completeness of Kant's Table of Judgments*, by Jane Kneller and Michael Losonsky), 10. According to Reich, the view that, according to Kant, general logic treats only the forms of specifically analytic judgments goes back at least to Ueberweg's *System der Logik und Geschichte der Logischen Lehre* (1857).

[33] Cassirer, KMM, 37.

[34] *KTE*, 2nd ed, 241-2, my trans.

different kinds of judgments, say, universal judgments ("All A are B") and particular judgments ("Some A is B"), and gives rules for the correct use of these different kinds of judgments, say, for mediate inferences like bArbArA or for immediate inferences like that from "All A is B" to "Some A is B." But what justification is there for the claim that judgments come in only these forms, or that these rules are justified? It cannot be general logic itself, since general logic merely tells us how to distinguish a given judgment according to its form and how then to reason with it by applying the rules already laid down. Cohen in fact counsels that we read the "Transcendental Analytic" backwards—Kant "was not led along the path from the categories to the principles, but out from the principles to the categories."[35] The principles - that all appearances all intensive and extensive magnitudes, the persistence of substance, the principle of causality, and the thoroughgoing interaction of all simultaneous substances – are in fact the most basic laws of experience, those laws without which experience is not even possible at all. In the *Critique*, Cohen is suggesting, we begin by taking the existence of mathematics and mathematical physics as a given, and we then reflect on its presuppositions in order to isolate the fundamental principles that make it possible. These principles give us the

---

[35] *KTE*, 2nd ed, 408, my trans. Hence the long history of hostility, especially within Neo-Kantianism, to the Metaphysical Deduction of the categories from the forms of judgment: how could the forms of specifically analytic judgments tell us anything at all about the forms of synthetic judgments? Thus, Cassirer, in his work *Kant's Life and Thought* (1918) [*KLT*], writes:

> While general logic can similarly be employed as the "clue to the discovery of all the pure concepts of the understanding," this is not done with the aim of basing the transcendental concepts on the formal ones, but conversely with the aim of basing the latter on the former, and in that way yielding a more profound understanding of the ultimate ground of their validity. (173; trans. Haden)

A few pages later, Cassirer writes that Cohen's work on Kant has shown that "the system of synthetic principles forms the true touchstone for the validity and the truth of the system of the categories" (175). At *EP* II, 625, Cassirer cites Cohen's *KTE*, 2nd ed, and says that Cohen has shown that the System of synthetic principles forms the "true middle point of the Kantian problem" and the "single material starting point" for the positive argument of the *Critique*, even for the doctrine of space and time.

A recent critical discussion of this reading of Kant can be found in Longuenesse's *Kant and the Capacity to Judge*, 3, 5.

most basic concepts, the categories, *which we then use as a guide in distinguishing the kinds of judgments treated of by formal logic*. When Cohen says, then, that general logic must be "completed" by transcendental logic, he means that the ultimate justification for the authority of the laws of general logic and the principle for isolating and distinguishing the forms treated of by general logic are to be found by reflection on the concrete sciences themselves.[36]

Third, Cohen argued that the traditional Aristotelian logic, the very logic that Kant described as merely formal, has historically been based on (not formal but) contentful or material logic.

> [Kant] did not see fully clearly the historical fact that the Aristotelian logic itself, metaphysical as it was, was derived from the material interests of knowledge [von sachliche Interesse des Erkennens]: how then all logic, the more purely formal it is arranged, has been developed on the this leading thread [rothen Faden] of the most inner instruments of cognition that the logical forms represent.[37]

(Note the inversion of the idea of 'leading thread'; here, the forms of thinking laid down in transcendental logic give the clue for the arrangement of general logic, not vice versa, as Kant asserted.[38])

Once we have claimed that general logic is essentially dependent on transcendental logic, is then a short step to concluding that in fact there is no merely formal discipline, general logic, that is distinct from transcendental logic. Cohen very forcefully draws this conclusion in his late systematic work *Die Logik der reinen*

---

[36] This is an expression of the 'transcendental method' of the Marburg School. See here Cassirer's 1912 paper "Hermann Cohen und die Erneuerung der Kantischen Philosophie" and Natorp's 1912 "Kant und die Marburger Schule," especially 196-7.

[37] *KTE*, 2nd ed, 408. At 269, he alleges that "every general logic in its historical origin wished to be a kind of transcendental logic."

[38] The claim that Aristotelian logic in fact presupposes a specific hylomorphic metaphysics appears later in Cassirer. See *SF*, chapter 1, and KMM.

Erkenntnis, where he characterizes his own logic of the sciences, which he calls, "the logic of judgment."

> The logic of judgment takes care of the wretched distinction between a formal and a material [Sachliche] logic; let the latter be now *metaphysic*, *critique of cognition*, or even the *methodology* of the sciences that is incorporated into them. *What is not material, is also not formal*. Only the formal is material; the more formal a method is, the more material it can become. And the more materially formulated and the deeper in the material a problem is, the more formally must its foundation be. The logic of judgment generates the categories, as pure cognitions, formally from judgment. But these are the materials [Sachen] that in particular constitute the content and matter [Gehalt] of mathematical natural science. The formal judgment generates the material foundation, as the presupposition of science.[39]

The Kantian foundation for this position is clear. The categories are not only *forms* of thought, they also prescribe laws *a priori* to nature;[40] in fact, all "empirical laws are only particular determinations of the pure laws of the understanding."[41] In this sense, the *a priori* concepts and principles of transcendental logic are the *matter* for the sciences, as their presupposition, the very ground of the lawfulness of nature: without the pure concepts and principles, "there would not be any nature at all."[42] But for Cohen this "logic of judgment" is not a merely formal logic distinct from the critique of cognition. Indeed, there is no such distinction between formal and material logic; there is only one logic, properly so called: transcendental logic. But transcendental logic just is the result of reflecting on the sciences taken as a fact. The consequence, as Cassirer summarized

[39] 4th ed, Georg Olms Verlag, New York: 1977, 518. The fourth edition is a reprint of the second edition from 1914; the quoted passage seems to have been also in the first edition from 1902, though I have not been able to verify it.
[40] B163.
[41] A128.
[42] A126.

Cohen's position, is the "dependence of logic on the fundamental forms of mathematical natural science."[43]

4. Paul Natorp's Rejection of Formal Logic, Old and New

Like Cohen, Paul Natorp argues that formal logic, if it is to be an independent science of the rules and forms of deduction, will have to fall into circularity, since it needs some means of justifying the rules and forms of deduction, and some principle for organizing them into a system.[44] Also, like Cohen, he argues that logic at least since Plato, has been the science of method, and so has sought to be an organon for the sciences, an instrument for "the expansion" of knowledge and the "discovery of truth."[45] And if logic is to be the study of the method of the sciences, it cannot abstract from all content of scientific cognition, and so cannot be merely formal. Turning Kant's original argument on its

---

[43] Cassirer, "Hermann Cohen and the Renewal of Philosophy." *Kant Studien*, 1912, 261.

Compare also the characterization of Cohen's view of logic in *Hermann Cohen: Eine Einleitung in sein Werk* (Stuttgart: 1924, 99), written by his follower Walter Kinkel: "Logic is entirely attuned to the system of nature, which means, the system of mathematical science as the system of nature. How is experience in its precise sense possible? That is the question from which logic begins."

[44] Natorp, *Die Logischen Grundlagen der Exakten Wissenschaft* (1910) [*LGEW*], 5:

> Logic is supposed to be itself a deductive science; but now it belongs to the task of logic in any case to arrange the laws of the deductive process and to justify the necessary and general validity that is claimed of them. But can the laying out and justification of the process be accomplished through this process itself? It is in itself contrary to sense, because the process of deduction must be presupposed as arranged and justified in order to be able to accomplish the required arrangement and justification in a valid way. The circle in this foundation is manifest.

[45] A795/B823. Kant goes on to describe logic as a "canon," not an "organon:"

> I understand by a canon the sum total of the *a priori* principles of the correct use of certain cognitive faculties in general. Thus general logic in its analytical part is a canon for understanding and reason in general, but only as far as form is concerned, since it abstracts from all content.

See also A61/B85-6.

head, he argues that since formal logic could not be an organon, there in fact is no such discipline as formal logic.[46]

> The opinion of a merely 'formal' logic, which as such would not be a logic of an object, is therefore unacceptable. Admittedly Kant still allowed for such a logic, yes indeed a new tradition of a merely formal logic has gone out from Kant. But, in the heart of the matter, this merely formal logic is overcome and left behind by Kant; further, with Kant's own, namely 'transcendental' logic, one has to place the problem of the object in the center of logical investigations.[47]

From Natorp's point of view, then, Kant's demarcation of a separate discipline, formal logic, was simply an error, an error that in the century after Kant's death led to a whole school of Kantian 'formal logicians.'[48] But in fact Kant's own doctrines should have allowed him to see that there could be no such discipline. Indeed, Kant's transcendental logic has as its central theme the very possibility of thinking objects, since the categories are "fundamental concepts for thinking objects in general" (A111); and so, "the object is a problem, rather *the problem*, of logic." But since all thinking is thinking of an object,[49] "this holds for logic in its entire breadth, as the doctrine of reason in general."[50] Thus, there is no conceptual room for a more general or more abstract doctrine of reason than that provided by transcendental logic.

Like Cohen, Natorp sees the rejection of formal logic as following from the transcendental method. In a picturesque passage, Natorp describes the fundamental orientation of the whole Marburg school.

---

[46] Natorp, *Philosophie: Ihr Problem und ihre Probleme: Einführung in den kritischen Idealismus*, 2nd edition [*PPP*] (1918), 44. (The first edition appeared in 1911.)
[47] Natorp, *PPP*, 45.
[48] The tradition Natorp is referring to here is presumably that of Mansel and Hamilton. See Adamson, *A Short History of Logic*, 114-117.
[49] Natorp, *PPP*, 46: "thinking, genuine thinking, is certainly at the same time thinking of being."
[50] Natorp, *PPP*, 45.

> Because philosophy is not able to breathe in the "airless space" of pure thinking, in which the mere understanding would like to vault itself aloft; it avoids the "high towers" of the metaphysical builders, around whom "there is often much wind"; it seeks the "fruitful bathos" (the lowland) of experience in the wide sense of the word; that is, it tries to root itself in the common productive work of culture: in the theoretical scientific "spelling out of appearances."[51]

Here Natorp is describing the first element in the transcendental method, that philosophy begins by "relating itself back" [züruckbeziehen] to the concrete sciences, taken as a given fact. No attempt is made to understand the nature of reality, or of thought, in isolation from reflecting on the nature of things as revealed to us by exact science and on the nature of thought as it is expressed in exact science. And so the transcendental method precludes formal logic from the outset. Natorp further argues that a related point that makes trouble for formal logic follows from this: there can be no strict distinction between receptivity, "as a particular, distinct way in which objects can be given," and spontaneity or "thinking." For Kant, as we've seen, the possibility of marking out formal logic as a distinct discipline requires that we consider the understanding and its rules in complete abstraction from sensibility—in the strong sense that we consider thinking in complete isolation from the fact that it relates to objects. But this dualism of "factors of cognition" [Erkenntnisfaktoren] is impossible if one takes the transcendental method seriously. According to it, the starting point for philosophical reflection is the whole of exact science taken as a fact. Within this unity, one can distinguish for various purposes a formal element from a material element, but it is just false that science originally presents itself to us as coming in two independent parts, its form and its matter.[52]

---

[51] Natorp, "Kant und die Marburger Schule," (1912), 196.
[52] Natorp, "Kant und die Marburger Schule," 201-2. See here the very helpful discussion in Friedman, *Parting of Ways*, chapter 3.

In a significant contrast to Cassirer, Natorp thinks that this line of reasoning makes trouble not only for traditional Aristotelian formal logic, but even for the new logic of Russell, Frege, and Couturat.[53] For Natorp, the significant question for the logicist is this: "Has therefore mathematics not become one, rather *the* logic, a logic entirely of the kind and in the general sense of the old formal logic, only with a much wider scope?"[54] Now, Natorp, like Cassirer, argues that mathematics is a priori and ultimately logical in character; following roughly Dedekindian lines, he argues that arithmetic not only is independent of facts about space and time, but also is logical, since arithmetic is a purely relational structure and all thinking is ultimately relating.[55] But if a philosopher claims that she can find a foundation for arithmetic in logic, the outstanding question is "What kind of logic?"[56] Natorp's conclusion is that the Russellian reduction of mathematics to 'logic' is ultimately a reduction to formal, not transcendental, logic. Although the new logic has made great technical progress, in its fundamental orientation it is does not differ from formal logic "in the old, Aristotelian and perhaps Wolffian sense," since it tries to "begin from final, unreduced concepts and from propositions incapable and not in need of proof and to proceed on this basis alone to judgments of identity ('analytic' judgments in Kant's sense)."[57] The *quid juris*? of the fundamental concepts of logistic and of its ultimate principles must, on Natorp's view, reside in a transcendental analysis of the exact sciences. Having forsaken that, Natorp argues, the

---

[53] Natorp cites these three in his criticism of logicism in *LGEW*, 3 and singles out Russell's *POM* as the most significant expression of "modern logistic" in *PPP*, 46.
[54] Natorp, *LGEW*, 4-5.
[55] See *LGEW*, chapter 3.
[56] At *LGEW*, 3, Natorp clarifies his "logicism": his logical foundation for mathematics is founded in transcendental logic, and thus relies on Kant's critique of cognition (though, of course without the strong sensibility/understanding distinction).
[57] Natorp, "Kant und die Marburger Schule," 196.

only alternatives for the logicists are to ground logical concepts and principles in brute facts about human psychology—a position Natorp realizes is the exact opposite of Frege and Russell's intentions—or to refuse to ground the concepts and principles at all, and thus let logic slide into 'empty formalism,' which can hardly be described as a science of *thinking*. Thus, it is not enough for logicists to claim that they are not interested in being "a logic of 'experience' in the Kantian sense, a logic of natural science,"[58] since transcendental logic is the only game in town.

The inevitable collapse into empty formalism brings with it a similarly disastrous view of thinking as mere 'analysis.'

> The ground of the error [viz, empty formalism], as we said, lies in the Aristotelian prejudice. It leads, if one asserts the consequence but wants to avoid the circularity, in fact unavoidably to empty formalism. But that prejudice hangs tightly together with the fundamental error of naïve realism: that things are given in perception, as a kind of mirroring of the objects in our representations, and that the entire accomplishment of cognition is only the analytical processing of this thing-like content, which is given beforehand in its essential existence. For this analytical processing, which thus leads to a procedure with things represented through their equally thing-like symbols, the suitable means for which was given by the apparatus of Aristotelian syllogistic and is now given by the more comprehensive and – there is no doubt here – more exact machinery of modern syllogistic. But for all this nothing is understood; this whole mechanism could be played back exactly without any understanding.[59]

If we allow ourselves to make free use of the results of the *Critique*, we can see more clearly where Natorp is coming from. Logic is the study of thinking, "the doctrine of reason in general" (PPP, 45). For Kant, the faculty of thinking is the understanding. In the Transcendental Deduction, Kant showed that the relation of our representations to an object does not come for free, but the understanding itself combines or synthesizes given representations in a very specific way so as to make possible the representation of an

---

[58] Natorp, *PPP*, 46.
[59] Natorp, *LGEW*, 8-9.

object, as "that in the concept of which the manifold of a given intuition is *united*."[60] This picture of the role of thinking in knowledge is the complete opposite of the view according to which thought is merely analytical—a view that requires what Natorp calls naïve realism. According to naïve realism, the relation of our representations to objects just comes for free, and our senses alone provide for us, passively, as it were, completely contentful representations of objects. At this point, the operations of thinking are required only to take these contentful representations of objects and dissect or analyze them. The blind manipulation of symbols in an empty formalism is then an expression of the manipulation that thought carries out on given, contentful, and object-related representations. Again, this view of thought as having no role in the object-relatedness of our representations goes together tightly with a view of logic as being completely independent of an analysis of objective thinking, that is, the actual procedure of thinking in the exact sciences.[61] This whole picture—naïve realism, an independent formal logic,

---

[60] See again B137: "*Understanding* is, generally speaking, the faculty of *cognitions*. These consist in the determinate relation of given representations to an object. An *object*, however, is that in the concept of which a manifold of a given intuition is *united*."

[61] Natorp thus sees the *methodological division* between logic as formal and the study of the sciences as material as analogous to the *ontological division* in naïve realism between the objects given in the senses and the pure forms of thought. These divisions, however, leave the formal logician or naïve realist with an unbridgeable gap between thought and reality.

> Modern logistic proceeds from the standpoint of all formal logic, from which it indeed takes its starting point, at least in this respect, that it considers mathematics still entirely in the region of the logical. It is however of no consequence if it does not claim to be valid as a logic of 'experience' in the Kantian sense, a logic of natural science. So would the concrete being of experience remain directly deprived of the governance of thinking; or thinking would dictate for being at most only certain conditions – of which conditions then it would not be understandable how we are to add to them an extra-logical being that is in itself foreign to thought. Inescapably therefore would the logical lawfulness – very much against the point of view of logistic – become merely subjective and psychological. Against this, if thinking, genuine thinking, is equally certainly thinking of being, just as being, genuine being, is being of thinking; if thinking is in its strict objective sense, whose laws logic has to develop and secure once for all, then every hindrance falls away and the validity of these laws can be extended also to being in the determinate sense, to the being of experience. (*PPP* 46)

A footnote to this passage specifies as its target Russell's *Principles of Mathematics*. The thought here seems to be that, in transcendental logic, the fundamental concepts and principles are arrived at by and thought as merely analytical—was resolutely rejected by Kant, Natorp is suggesting, in the *Critique*: "we hold fast to the conviction to which Kant has already given manifest expression: 'Where the understanding has not already combined something, there can it also not analyze [auflösen] anything.'"[62]

5. WALTER KINKEL ON THE INSIGNIFICANCE OF KANT'S *LOGIC*

If Natorp considered Russell's formal logic as objectionable, then it is only too clear that the same kinds of reasoning could be levied against Kant's own formal logic as it is presented in the *Jäsche Logic*. Although Cassirer himself, despite his two lengthy systematic discussions of Kant's philosophy in *Erkenntnisproblem*, volume 2, and in *Kants Leben und Lehre*, never explicitly cites the *Jäsche Logic*, another member of the Marburg school, Walter Kinkel,[63] edited a new Felix Meiner edition of the *Jäsche Logic* in 1904. In a dense introduction to the *Logic*, Kinkel argued, with numerous citations to

---

considering the laws that make possible exact science. But since these concepts and principles are derived from reflection on exact science itself, as its necessary conditions, there is no problem of explaining why they are then necessarily applicable in exact science. Similarly, the argument of the Transcendental Deduction has it that the categories are necessarily applicable to objects of experience because they are precisely those concepts the application of which is necessary for there even being an object of experience at all. (And note that this argument for the necessary applicability of the concepts and principles of the understanding *requires* that we not think of object-related representations as given, but as the result of the actions of the understanding itself. That is, the argument for the necessary applicability of transcendental logic is only possible if naïve realism is false.)

This argument of Natorp's is very similar to an argument Cassirer gives against Russell in "Kant und moderne Mathematik", 44-6. I return to Cassirer's interpretation of Russell's achievement in the concluding paragraphs of this chapter.

[62] *LGEW*, 8. Natorp is quoting B130, "the dissolution [Auflösung] (*analysis*) that seems to be the opposite [of synthesis], in fact always presupposes it; for where the understanding has not previously combined anything, neither can it dissolve anything, for only *through it* can something be given to the power of representation as combined." This passage from Kant is cited by Cassirer in "Kant und moderne Mathematik," 37.

[63] Later in his career, Kinkel wrote a book-length commentary on Cohen's philosophical work, *Hermann Cohen: Eine Einleitung in sein Werk* (Stuttgart: 1924). For a brief characterization of Kinkel's place in the Marburg School, see Holzey, *Cohen und Natorp*.

Cohen's *Kants Theorie der Erfahrung* and *Logik der reinen Erkenntnis*, that Kant's division between transcendental and formal logic was, by Kant's own lights, mistaken, and that the *Jäsche Logik* could have at best a role in elucidating and completing [Erlaütung und Ergänzung] the *Critique*.

Kinkel presents two lines of argument for this conclusion. The first line, which overlaps with some of the difficulties I discussed above in section 2 of this chapter, has it that there can be no separation between the science of the rules of thinking (formal or general logic) and the science of the rules of thinking an object (transcendental logic), since, by Kant's own lights, there can be no "fully objectless thinking" [völlig gegenstandsloses Denken]. Kinkel's argument is involved and relies on controversial readings of some central texts in the B Transcendental Deduction. For Kant, the understanding is the faculty of judging (A69/B94), and "thinking is judging."[64] Further, since all judging is "nothing other than the way to bring given cognitions to the *objective* unity of apperception" (B141), even analytic judgments stand under the objective unity of apperception.[65] Now "the *transcendental unity* of apperception is that unity through which all of the manifold given in an intuition is united in the concept of an object" (B139), and our understanding "is able to bring about the unity of apperception *a priori* only by means of the categories" (B145-6). So it seems that any act of thinking, as a judging, requires the unity of apperception, which brings with it the categories and

[64] Kinkel, Walter. Einleitung to *Immanuel Kants Logik, neu herausgegeben, mit einer Einleitung sowie einem Personen- und Sachregister versehen*, 3rd ed. (1920), ix. The first edition appeared in 1904.

[65] Kinkel also cites here a passage from Kant's "What Progress?." where Kant defines a judgment as "the unity of consciousness in the relation of concepts in general – irrespective of whether the judgments are analytic or synthetic" (Ak 20:271).

relation to an object. So there is no kind of thinking that does not relate to an object or require the application of the categories.

Now one might try to block this move by distinguishing between what Kant calls the *synthetic* unity of apperception and the *analytic* unity of apperception (B133-4). In an involved and complicated argument, Kinkel argues that this move won't help make room for a distinction between formal and transcendental logic. Kinkel follows Cohen in arguing that, for Kant, synthetic judgments are "those judgments in which the synthetic unity of apperception connects [verknüpfen] subject and predicate to an object of experience," while analytic judgments are "those judgments that do not concern [betreffen] an object…In the analytic judgment the subject is…merely a concept."[66] Since, for Kant, the analytic unity of consciousness "pertains to all common concepts as such," analytic judgments will stand under the analytic unity of consciousness inasmuch as they concern only concepts. And so the move that Kinkel imagines one trying to make would go like this. Analytic judgments do not concern objects, but only concepts. As merely conceptual cognition, they require as their fundamental principle merely the principle of contradiction, and not the principle of the synthetic unity of apperception, which is what makes possible synthetic judgments and relation to an object. However, Kinkel argues, this move will not work, since, for Kant, "only by means of an

[66] These passages are quoted by Kinkel (x-xi) from Cohen's early work "Die systematische Begriffe in Kants vorkritische Schriften nach ihrem Verhältnis zum kritischen Idealismus" (Berlin: 1873), 7ff. See also *KTE*, 2nd edition, 400.

Cassirer gives what seems to me to be the same characterization of analytic and synthetic judgments in KMM, §V, 32ff. and *EP*II, 675ff. In both places, Cassirer emphasizes that the distinction between analytic and synthetic judgments does not rest ultimately on the *form* of a judgment, but rather on its origin in synthesis or in analysis. And since for Kant synthesis is the fundamental activity of the understanding, which combines the manifold of an intuition in order to make the representation of an *object* possible, we can also say that a judgment is synthetic if it makes objective representation or representation of an object possible.

antecedently conceived possible synthetic unity can I represent to myself the analytic unity" (B133 note). Indeed, Kant says that "the synthetic unity of apperception is the highest point to which one must affix all use of the understanding, even the whole of logic and, after it, transcendental philosophy" (B134 note). Therefore, all use of the understanding, even in analytic judgments, requires the unity of consciousness provided by the synthetic unity of apperception. But that synthetic unity of apperception brings with it object-relatedness; so, once again, there is no such thing for Kant as a kind of thinking that is not related to an object. This ultimate dependence of all thinking on what Kant calls the synthetic unity of apperception leads, Kinkel argues, to a challenge for Kant's purported formal logic:

> Is this correct insight of the dependence even of analytic, that is, purely conceptual knowledge, on the synthetic unity of transcendental apperception compatible with the claim that general logic's analytic-formal ways of consideration abstract from all relation to the object? (xii-iii)

Not surprisingly, the answer here is 'No.' Kinkel concludes this line of argument with a rhetorical question: "What then are these empty forms of thinking, which are the concern of purely formal logic and which are supposedly so complete without any relation to an object?" (xiv).

I will not pause to sort out the interpretive moves in the argument from the last paragraph, since many of them seem forced and unconvincing as readings of Kant.[67] But

[67] For instance, there seems little reason to think that in analytic judgments, the subject is in fact a concept, or that they do not concern objects of experience. Indeed, this reading doe not sit well with Kant's characterization of analytic judgment at *JL* §16, note 2, nor with Kant's deeply-held view that all cognitions must be related to objects if they are to be contentful at all. Further, there is no reason to read the analytical unity of consciousness as that unity of consciousness that is in play in specifically analytic judgments, rather than (what Kant's text seems to indicate) the kind of unity of consciousness that is in play whenever a subject is representing an object discursively, that is, by concepts. Last, there seems little reason to think that general or formal logic for Kant is the science of the rules of *analytic* judgments, and

the fundamental point of view is important both in its own right and especially when discussing Cassirer's understanding of formal logic in general and of Kant's formal logic in particular. One fundamental idea is that there is no such thing as thinking considered in abstraction from the kinds of thinking we find in the concrete sciences. (This is an element in the transcendental method of the Marburg school.) A further fundamental idea, derived from Kant's Transcndental Deduction, is expressed by Kinkel:

> Here we have the deepest insight to which critical idealism in theoretical philosophy has struggled to reach: cognition creates the object of knowledge [Erkenntnis erschafft das Objekt des Wissens]. There is not outside of, on the far side of thinking, a fixed, unchanging world of things, which we only copy in cognition; rather, the concept of an object in general first comes to be in and through cognition. (ix-x)

But this, the deepest of Kant's insights, makes the separation between general or formal and transcendental logic all the more difficult: if thinking is what generates the object of cognition, then any science of the rules of thinking will *ipso facto* be rules for thinking an object of cognition. So, there can be no distinction between the science of the rules of thinking and the science of the rules of thinking an object.

The second line of argument against the distinction between formal and transcendental logic concerns again that element in the *Critique* that the Marburg school, and post-Kantian idealism more generally, was at pains to qualify, namely the distinction between the understanding, the faculty of concepts, and sensibility, the faculty of intuitions.[68] Recall that general logic is the science of the rules of the understanding, in

---

not, as his characterizations indicate, the science of the rules of thinking in general or according to its form. (See here especially Paton, *Kant's Metaphysics of Experience*, Vol. 1, 213-9).

[68] On the fate of this distinction in late 19th century Neo-Kantianism in both its Marburg and Southwest varieties, see Friedman, *Parting of Ways*, chapter 3.

abstraction from its relation to sensibility—in the strong sense that in general logic we consider the rules of discursive cognition in complete abstraction from even the fact that discursive cognition relates to an object. In transcendental logic, on the other hand, we consider the rules of the understanding inasmuch as it has available to it through sensibility a manifold of *pure* intuition. By considering the rules for the operations of the understanding with respect to that pure manifold, we arrive at rules for the use of the understanding inasmuch as it thinks a spatio-temporal object. Clearly, then, the very possibility of distinguishing formal and transcendental logic requires a principled distinction between these two fundamental faculties of human cognition. To undermine the qualitative difference between the understanding and sensibility, Kinkel cites a series of well-known passages in the critical corpus where Kant argues that even cognition of pure space and time require the operation of the understanding to constitute genuine *Erkenntnis*.[69] Over against the view of pure intuition as an independent source of cognitions, Kinkel follows Cohen in seeing the object of cognition as generated [erzeugen] by the understanding working on space and time, so that the pure forms of sensibility, like the categories, stand under the transcendental unity of apperception. Just as it is the source of the categories, so too is the understanding, as the transcendental unity of apperception, the source of space and time. Indeed, for Kinkel, following the line laid down by Cohen in *Logik der reinen Erkenntnis*, space and time differ from the catgories not qualitatively, but merely as stages [Stufen] in the constitution

[69] Thus on xiv Kinkel cites Prolegomena §38, where Kant argues that the understanding contains the ground of the unity of a geometrical construction, and B154, where Kant says that we cannot represent a determinate section of time without the synthetic influence of the understanding on inner sense.

[Konstituierung] of the object of cognition.[70] In this unitary process, space and time stand closer to sensation and lower, as it were, than the categories. Thus, though Kant was not able to draw this final conclusion, his own arguments from the *Critique* should have driven him to conclude that both the categories and space and time are "purely conceptual" in nature.[71] But if even space and time, Kant's "pure intuitions," are finally to be reckoned as conceptual representations, it is easy to see that we could never succeed in marking off a distinct discipline that concerns the faculty of concepts in isolation from a distinct faculty of intuitions.

> This division [Sonderstellung], which [Kant] claimed for space and time, as forms of intuition, prompted him to make that mistaken separation [Trennung] of general from transcendental logic; there is only one, comprehensive Logik, which has to do with the possibility of experience: transcendental logic. Its task is to seek out, and to authenticate in their origin from reason, those very concepts and judgments that constitute the necessary and sufficient preconditions of cognition and thereby also of the objects of experience […] Transcendental logic is, as all philosophy, in the platonic sense 'Ideal research' [Ideenforschung]: it seeks the ideas, the foundations and hypotheses of knowing. It is not psychology, since it has to do neither with the occasional causes of the coming to be of our concepts, nor in general with the individual, the subject; it must maintain the tightest connection with science, especially mathematics and mathematical natural science: for one can find the foundations of cognition only from the knowledge of recognized, secure cognition. But within logic there can be various stages of consideration: through this is indicated the justifiable kernel in the divorce [Scheidung] of formal from transcendental logic. Accordingly, purely systematic consideration can grant to the following work [the *Jäsche Logic*] only the value of a work that supplements and elucidates the chief works, but does not stand on its own. (xvi-xv)[72]

---

[70] In fact, in Cohen's *Logik der reinen Erkenntnis*, space and time are categories.

[71] This argument is laid out by Kinkel on xiv-vi.

[72] Reich (*Completeness of Kant's Table of Judgments*) comments on Kinkel's argument in this way:

> The impact of Cohen's position for the interpretation of general pure or formal logic can be seen very easily in his follower Walter Kinkel's Introduction to Jäsche's edition of Kant's *Logic*: the relationship of formal to transcendental logic as developed by Kant is an impossible one; transcendental logic is the only logic; *within* it the formalism of general logic has an unspecified and insignificant place. (116, note)

Reich writes as if the failure of Neo-Kantians to find a place for formal logic follows from their interpretive blunders: they read formal logic as the study of specifically analytic judgments, and read the analytic unity of consciousness of *Critique*, §16 as the unity of consciousness at work in specifically analytic judging.

The precise way in which Kant intended his distinction between sensibility and understanding is one of the most difficult of the *Critique*. Needless to say, we will not be able to discuss it in any depth here, still less the further question of whether there is a distinction in the neighborhood of Kant's own that can stand up philosophically. But what we can take with us is this. The distinction that Kant makes between transcendental and formal logic seems to stand or fall with some of the most difficult and controverted elements in Kant's transcendental philosophy.[73]

6. CASSIRER ON TRANSCENDENTAL AND FORMAL LOGIC

We've been led to this detailed examination of the views of other members of the Marburg school because Cassirer himself, despite his detailed discussions of the

---

I've indicated already that I agree that those two moves are interpretive blunders. But I think that I have also shown that the distinction between formal and transcendental rests on some of the most difficult doctrines of the *Critique*—the distinctions between phenomena and noumena, appearances and things in themselves, and understanding and sensibility; the postulates of pure practical reason—and that one has *philosophically* good reason to deny the distinction between transcendental and formal logic, if one accepts some of the most fundamental and compelling doctrines of Kant's transcendental logic while downplaying some of the *Critique*'s more problematic doctrines.

For similar criticisms of the distinction between sensibility and understanding among British Kantians, see Green's early attack in his "Lectures on the Philosophy of Kant," 22 and "Lectures on Formal Logicians," 172, in *Works of Thomas Hill Green*, vol2.

[73] Earlier I tried to present Kant's own way of securing the distinction in terms of a distinction between the rules for discursive cognition *qua* discursive and the rules for discursive cognition *qua* cognition, and I noted that Kant thinks it important here and there to distinguish cognition from thinking, as at B166n, where Kant says that in the absence of intuition, "the thought of the object can still have its true and useful consequences for the *use* of the subject's *reason*, which, however, cannot be expounded here, for it is not always directed to the determination of the object, thus to cognition, but rather also to the subject and its willing." Kant needs this qualification in order to make room for the kind of thinking that is necessary in *moral faith*, where the subject is not an object of intuition, but the divine being as moral lawgiver and just judge. Kinkel never discusses this move of Kant's, nor, as far as I know, does Cassirer. But note that this move requires another of Kant's most difficult and controverted doctrines: the distinction between things as objects of appearance and things-in-themselves. This doctrine faired no better in Neo-Kantianism generally or in the Marburg school in particular than did the distinction between sensibility and understanding.

relationship between Kantian philosophy, the traditional logic, and modern quantified, relational logic, never says anything explicit about Kant's own *Logic*.[74] This does not mean, of course, that he did not know the *Jäsche Logic*, as Cassirer's numerous implicit references to the doctrines of the *Jäsche Logic* make clear.[75] Further, there are very strong reasons to think that the attacks on the very possibility of formal logic, and on the systematic significance of the *Jäsche Logic*, that one finds in other Marburg Neo-Kantians would be attacks that Cassirer himself is committed to. Cassirer throughout his early career was eager to identify himself with the "logical idealism" of the Marburg school, and we've seen that the denial of a possible distinction between transcendental and formal logic in fact follows from the most central doctrines of that school.[76] In Cassirer's writings on Kant, he frequently cites Cohen's Kant interpretation as authoritative, and we've seen in Kinkel's arguments that some of Cohen's most distinctive interpretive moves leave no room for the distinction. Further, Cassirer at various points defends some of the fundamental philosophical moves that Natorp and

---

[74] Compare, though, the comments Cassirer makes in *EP*II as an introduction to his discussion of the *Critique*:

> If one judges Kant as a pure logician, if one considers merely what he accomplished in formal logic, or for the abstract principles of pure mathematics, there can be no doubt that he stands behind his great rationalist predecessors, especially Leibniz. But this flaw [Mangel] hangs together internally with his most characteristic merit. He directed his view above all on the principles of empirical knowledge. He came to consider mathematics itself only insofar as it was able to prove itself in its application to concrete, factual [tätsachliche] objects. (663)

[75] For instance, in KMM 32-33, he argues that mathematical concepts are not formed or structured like the "general genus-concepts of traditional logic," and are not abstracted "as the 'discursive' concepts of formal logic … from a multiplicity of different concepts as their common mark." In *SF*, 6, he refers to the "conceptual pyramid" of traditional logic at whose summit is the indeterminate "Etwas," which, according to the *Jäsche Logic* is the highest concept at the summit of the conceptual hierarchy (see §11, note; Ak 9: 97). Using the characterization from *JL* §7 of a mark as "ground" of cognition, he writes

> The higher concept is to make the lower intelligible by setting forth in abstraction the *ground* of its special form. The traditional rule, however, for the formation of the generic concept contains in itself no guarantee that this end will actually be achieved. (*SF*, 6)

[76] See especially the long systematic article, "Erkenntnistheorie nebst den Grenzfragen der Logik" [ENGL] (1913), which defends 'logical idealism' against the contrary logical views of other contemporary schools. (Significantly, though, this article does not discuss the new logic of Frege and Russell.)

Cohen used to arrive at the conclusion that transcendental logic is the only kind of logic. At various points he articulates a shared commitment to the so-called 'transcendental method,'[77] which Natorp argued rules out the possibility of an autonomous formal logic.[78] At various points he, like Kinkel, follows Cohen in denying that sensibility and understanding can be two independent sources of cognition, and we've seen that Kinkel argues that this makes a division between formal and transcendental logic impossible.[79]

---

[77] See especially the very dense articulation of this method at ENGL (1913):

> The attempt to present the entirety of cognition in a systematic unity ends in final *Form-concepts* that bring to expression the possible kinds of relation between contents in general. In these fundamental relations are given the final invariants to which cognition is able to advance; therefore also the "objective" standing [Bestand] of being is grounded in them. Because objectivity is – according to the critical analysis and meaning of this concept – itself only another designation for the validity of determinate combinatory connections [Verknüpfungszusammenhänge] that need to be separately discovered and investigated in their structure. The task of the critique of cognition [Erkenntniskritik] consists in this, to go back from the unity of the general concept of the object [Objektbegriff] to the manifold of necessary and sufficient conditions that constitute it. In this sense the thing that cognition calls its 'object' is resolved [löst auf] into a web of relations that are themselves held together through the highest rules and principles. And what is valid here in general proves itself further in the special thing-concepts [Dingbegriffe] with which the special sciences as well as ordinary intuition [gewöhnliche Anschauung] operate. These concepts are also indispensable as halting points and first starting points: but as soon as one analyzes their sense more closely, one recognizes that they are not meant as some absolute, 'beyond' the logical forms of cognition, but rather a functional relation [Funktionsbeziehung] is supposed to be brought within these forms and by means of them brought to expression. In the particular *series order* [Reihenordnung], which present themselves as spatial and temporal ordering, as magnitude and number ordering, as the ordering of the reciprocal dynamical connection of events, lies the very same moment that distinguishes the 'real' empirical content from the merely 'subjective' fleeting and changing impression.

See here also *EP*I, 16, 18; *PSF* 3, 404-5, note.

[78] I'm not here distinguishing between the thesis that there can be no discipline as formal logic and the thesis that there can be no *independent* discipline as formal logic. It doesn't seem to me that members of the Marburg school always distinguished these positions.

[79] See KMM, 31-2:

> And it is important and significant, that the immanent formation [Fortbildung] of the Kantian doctrine has led of itself to the same result that is more and more clearly demanded by the progress [Fortgang] of science. Just like 'logistic,' so too has modern critical logic walked away from Kant's doctrine of pure sensibility. And in fact sensibility signifies a problem for the critique of cognition, but no more a self-standing and peculiar source of certainty. Thus it agrees in its fundamental thoughts with the tendency of which the work of Russell and Whitehead is the fulfillment: in the demand of a pure logical derivation of mathematical fundamental principles, through which we first fully understand and learn to master conceptually 'intuition' itself and space and time.

Cassirer cites here Cohen's *Logik der reinen Erkenntnis*, 11f, and 128. See also the more detailed discussion in *EP*II, especially 696, and 684, where Cassirer calls the 'synthetic function of unity' a common genus [Oberbegriff] under which falls pure sensibility and pure understanding. Further, Kinkel

Cassirer also cites some of the key passages in the *Critique* that Kinkel uses to argue that the *Jäsche Logic* could not have independent, systematic significance.[80] We also find Cassirer defending two of the most distinctive (and initially implausible) features of Cohen's readings of Kant: that whether a judgment is synthetic depends not on the logical form, but on the origin—in 'synthesis', or in the activity of constituting the object of experience—of a judgment[81]; and that the distinction between pure space and time and the pure forms of thinking is a difference of stages in the unitary process of constituting the object of experience.[82] We saw above that Kinkel used both of these doctrines to force the conclusion that there could be no distinction between the science of the pure intellectual form of cognition and a science of the form of thinking a spatio-temporal object. Again, for Cassirer, as for Cohen, the proper way to read the "Transcendental Analytic" is *backwards*: one turns the Metaphysical Deduction upside down by reading the principles of pure understanding as first derived from an analysis of mathematical physics and then the table of the forms of judgment as derived from them. For Cassirer, general or formal logic is, in Kant's words, the "clue to the discovery of all the pure concepts of the understanding," but this does not mean that the *transcendental* concepts

---

had cited *Prolegomena* §38 and B161note to argue that even in geometry, the representation of space itself as a geometrical object and the figures in space requires the operations of the understanding to effect their unity. Cassirer cites the very same passages in making a similar point at *EP* II, 696.

[80] Cassirer cites B133note, "only by means of an antecedently conceived possible synthetic unity can I represent to myself the analytical unity" at *EP* II, 676 in the context of an attack on the traditional model of concept formation in "formal logic" (see also *PSF*3, 316; compare Kinkel, "Einleitung" to *Kants Logik*, xii). While arguing that for Kant, our pure concepts cannot arise from abstraction, he cites B103, "no concept can arise analytically as far as *the content is concerned*" (*EP* II, 715; compare Kinkel, "Einleitung," xii). Cassirer also cites B130, "where the understanding has not previously combined anything, neither can it dissolve anything," at numerous places, including KMM, 37 and *PSF*3, 194 (compare also Natorp *LGEW*, 8).

[81] At KMM, 38, Cassirer cites Cohen's *KTE*, 2nd edition, 400, where the same reading of the distinction is given.

[82] Cassirer uses the very same language as Kinkel, describing space and time as an early stage [Stufe] and the categories as a later stage in the constitution of the object of experience, at *EP* II, 687, 699.

are based on the *formal* ones; rather, Kant's aim was to "base the latter on the former, and in that way yield a more profound understanding of the ultimate ground of their validity."[83]

As we saw in chapter 3, one of Cassirer's main goals in *Substance and Function* is to "transform logic" by undermining the traditional doctrine of *Begriffsbildung*.[84] Given the criticism Natorp levied against Russell's "formal" logic, it should come as a surprise to us that Cassirer was an early proponent of the great mathematical and philosophical significance of the new logic. Indeed, this has led Michael Friedman to describe Cassirer's philosophical achievement in this way:

> Cassirer's outstanding contribution was to articulate, for the first time, a clear and coherent conception of formal logic within the context of the Marburg School. Cassirer in *Substance and Function* identifies formal logic with the new theory of relations developed especially by Bertrand Russell in *The Principles of Mathematics*. […] For Cassirer, […] formal logic embraces the entire theory of relations, including (paradigmatically) such *asymmetrical* relations as that which generates the number series.[85]
>
> Thus, for Cassirer, as we have seen, the realm of pure formal logic is given by the modern theory of relations, the totality of what we now call relational structures.[86]

But while it is true that Cassirer criticized the traditional, "formal" logic and argued that only the kind of conceptual structures treated of by the new, relational logic can capture the structure and role of specifically mathematical concepts, we should be careful, in light of the very strong opposition of the Marburg school to the very idea of formal logic, in ascribing to Cassirer a new conception of "formal" logic. In the rest of this final section of the chapter, I will work towards a better understanding of Cassirer's view of logic and

---

[83] Cassirer, *KLT*, 173.
[84] *SF*, 4.
[85] Friedman, *Parting of Ways*, 30.
[86] Friedman, *Parting of Ways*, 37, note.

his view of the theory of the concept in Kant's "formal logic" by considering two questions:

a. Is the theory of conceptual structure and formation from *SF* a theory of *formal* or *transcendental* logic?

b. Is there such a discipline as formal logic? and, if there is, is it independent of transcendental logic?

To anticipate, I think Cassirer would answer these questions in the following way. First, the theory of the concept that Cassirer defends in *SF* is a theory of *transcendental*, not formal logic. Second, there is such a discipline as formal logic, but in a qualified sense. What one calls the "form" of thought is simply an abstraction from the complete system of thinking that one finds in the exact sciences. Within that whole, one can isolate the "form" and "content" of thought from various perspectives, as long as one recognizes that there is no such thing as *the* form of thought and *the* matter of thought. In transcendental logic, one can then identify, as the need arises, various concepts and principles that constitute the (or better: a) "form" of thought, which can then be studied in isolation in a discipline that one can call, if one wants, "formal" logic. These two answers lead Cassirer, I think, to a reinterpretation of the logical work of Frege and Russell. Considered in itself, the new logic is simply an extremely powerful and abstract kind of mathematics. Within transcendental logic, a philosopher can show that the concepts and principles studied in this "logic" in fact are concepts and principles that are at play in all of exact science and in fact constitute an essential element in the building up of the world of scientific objects. After this transcendental, philosophical work has been done, we can call the new logic "formal" logic, as long as we are clear that we need a transcendental argument to substantiate the title, and as long as we recognize that the methodology of the new logic is of a piece with other kinds of mathematics and it requires no different kind of justification *within mathematics* than say group theory or projective geometry.[87]

a. Is the Theory of Conceptual Structure and Formation from *SF* a Theory of *Formal* or *Transcendental* Logic?

In a later paper from 1928, “Zur Theorie des Begriffs,” Cassirer attempted to answer some of the objections that his critics had raised against the argument in *Substance and Function*.[88] Schlick, for example, in his 1918 book *Allgemeine Erkenntnislehre*, had criticized Cassirer's example (coming originally from Lotze) of the concept <red, juicy, edible>. “Is the concept not a valid logical concept? It *has* a sense and that alone is decisive for its validity in formal logic. The question whether it can play some role for cognition lies entirely outside of the sphere of formal logic.”[89] Cassirer's reponse is simply to deny that he was engaged in a discussion of *formal* logic at all: “I didn't want to show that the theory of abstraction is false, that is, that it is formally incorrect, but

---

[87] This reading differs from that given by Michael Friedman (*Parting of Ways*, 92):

> For Kant himself, therefore, pure formal logic deals with a distinctive kind of unity, pure analytic unity, which then needs to be explicitly and painstakingly related to the characteristically synthetic unity treated in transcendental logic. For Cassirer, by contrast, we begin with the synthetic unity expressed in the progressive construction of empirical natural scientific knowledge, and pure formal logic, in contradistinction to transcendental logic, is a mere abstraction from this unitary constructive process having decidedly secondary philosophical significance.

It is true that for Cassirer, formal logic is an abstraction from transcendental logic in the sense that the “form” of thought is an abstraction, relative to some point of view, from the unitary structure of scientific thinking revealed in transcendental logic. But this does not mean that researchers in modern logic need to themselves engage in transcendental logic, or even be aware of its results. They can simply act as mathematicians who are working on finding the fundamental concepts or laws of mathematics. That these fundamental concepts or laws are not only fundamental for mathematics, but for all of thinking as the building up of the world of objects, is of the first significance for philosophers, but irrelevant for researchers in modern logic *qua* mathematicians.

[88] For a review of these criticisms, see Ihmig, Karl-Norbert, *Cassirers Invarianttheorie der Erfahrung und seine Rezeption des ‘Erlanger Programms’*, 257. Ihmig also argues that the attack on the theory of abstraction in *SF* is carried out within transcendental, not formal logic.

[89] Schlick, *Allgemeine Erkenntnislehre*, 1st ed, 1918, 5. This passage was removed from the 2nd edition and does not appear in the English translation. This passage is cited at ZTB, 133.

rather that it is not sufficient for the particular objective grounding of the concept, for the explanation of its value for cognition."[90] Cassirer denies that his analysis of the structure and formation of concepts holds only for mathematical concepts, but the general validity of his conception does not mean that his theory is a theory of formal logic:

> Nevertheless, for me no way leads from this recognition back to the 'classical' formal logic. For, if this kind of logic, according to the Kantian definition, consists in this, that it "abstracts from all objects of cognition and their differences" [(Bix)]—so the philosophical theory of the concept, which I sought to construct, presents rather the diametrically opposed task. It does not refrain from considering this manifold of objective structures, but rather it wants to make them first visible in their complete extension. It strives not for the formal-general beyond the differences in objective structure, but rather wants to show the immanent significance itself, the inner articulation of these differences. Only from this universal and fundamental point of view [of the philosophical theory of the concept] can every individual position concerning the 'concept' and its logical function be understood.[91]

Cassirer's concern, then, is not on "the mere form of the concept," but on its "objective sense," its "objective value," its "value for cognition." Consider the argument from chapter 3 that I called the "Lotze objection" to the A-a model: mathematical concepts do not have less content the more general they are, but at least as much; the structure of mathematical concepts cannot be reduced to conjunction, disjunction, and exclusion. Indeed, if concepts had the kind of structure that the traditional A-a model alleges, it would be completely mysterious why they would have any value at all. This argument essentially relies on picking out the genuinely fruitful concepts in modern mathematics and analyzing their relationship to the particulars that fall under them. The distinctive kind of 'generality' had by the concept <simply infinite system> or <continuity of a

[90] ZTB, 132.
[91] ZTB, 131.

function at a point> would never have been identified if one restricted one's view to "the mere 'form' of thinking and strove to isolate it."

b. Is There Such a Discipline as Formal Logic? Is it Independent of Transcendental Logic?

What then is left of that supposed discipline that "abstracts from all objects of cognition and their differences" and considers only "the mere form of thinking"? "The pure cognitive sense [Erkenntnis-Sinn] of concepts – may it be now of a concept of natural science or of jurisprudence – consists indeed in this, that they should put us in a position to grasp the empirical particular under rules and to determine the particular by means of them."[92] But this conclusion could never have been reached if we were told only to attend to the "mere form" of thinking:

> It can only be reached through the consideration of the *objective* sense and of the objective bonds [Bindungen] of thinking. A true 'general' logic can therefore only draw on [erheben auf] therefore a transcendental logic, that is, draw on a logic of thinking an object [Denk-Gegenstände]. It is essential to study its structure, its character, its many-sided relation and its necessary connection. In my analysis in *Substance and Function* I was essentially aiming at *this* and no other task. [Cassirer's critic] Heymans misunderstood what I presented, if he thought that its intention was to replace the kinds of activity handed down in 'formal logic' by some other new ones: to overturn the old form of this 'logic' in order to put a new one in its place. What I sought to counter was not the *formulation* [Fassung] that the doctrine of the concept, as a single theory, takes in this logic – it was rather the posing of the problem and task, it was the constitutive 'principle' of the logic itself. "Investigations into the fundamental questions of the *critique of knowledge*": so was the subtitle that I had given my book. Thus it should be emphasized from the beginning that the discussion here should by no means be only of the 'form' of the concept, but rather of its value for cognition, of its objective 'sense,' and of its objective 'validity.'[93]

[92] "Inhalt und Umfang des Begriffs," 215.
[93] ZTB, 131.

Note that Cassirer did not argue that the old Aristotelian logic is a deficient formal logic and should be replaced by the far superior new 'formal' logic of Frege and Russell. Rather, the point is that the new conception of the structure and formation of concepts, which the work of Russell and Frege gives an adequate expression, calls into question the "constitutive principle" of so-called formal logic itself. If "the essence of the concept can be defined in no other way than through the function it fulfills in the building up [Aufbau] of cognition," then there can be no study of the rules of thinking, that is, of cognizing through *concepts*, that does not depend on a characterization, available only in *transcendental* logic, of what the structure and function of a concept is. Nothing of value can be achieved by being told to attend only to the "mere form" of thinking or to "an *undetermined* 'generality'"[94] Cassirer insists, then, that formal logic is dependent on an independent transcendental logic. If this point has been granted, Cassirer is willing to be permissive about the possibility of a 'formal' logic.

> The moment that secures this legitimization [that is, what makes the concept of value for objective cognition] – so I tried to show – belongs in each case to a completely different plane of thinking than that which is at work in the mere abstraction process. If one recognizes this difference, then the essential part of my argument has been conceded – and it matters little to me, whether one next to it wants to secure a special task or in some measure a place of honor and security in the system of philosophical cognition for the 'formal' logic, in the old sense of the word. Against such a move I have not the least to object, so long as the relation of rank is not again displaced, so long as one does not question the primacy of 'transcendental' before the merely formal.[95]

What is suprising about this passage is what it does *not* say. On the one hand, one does not find it the kind of dismissive comments ("that wretched distinction") one finds in Cohen or Natorp about the very possibility of formal logic. On the other hand, one does

---

[94] ENGL (1913), 25.
[95] ZTB, 132.

not find Cassirer saying that the old Aristotelian logic has in fact been replaced with the new formal logic of Russell and Frege. Notice further the ambivalence in the phrase "'formal' logic in the old sense of the word."

This ambivalence, I think, is partly explained by a deep-seated antipathy in Cassirer not only to the thought that one can get a grip on the 'form' of thought independently from an investigation of the concrete sciences, but to the very idea of *the* form of thought. The problem, for Cassirer, is that maintaining that there is a form of thought requires maintaining that there is a *matter of* (or, perhaps, more appropriately, *for*) thought. In an important systematic essay from 1913, Cassirer opposes the view of 'logical idealism' to that of various other contemporary views, especially to that of the philosophers of the so-called 'Southwest' Neo-Kantian school—Heinrich Rickert, and Emil Lask.[96] One feature of this school is a strict division between the sciences that have as their object *being*, and logic, as a normative discipline or a theory of *validity*. Corresponding to this division in disciplines is a corresponding division between the unformed matter of sensation and the forms of thought imposed on it in empirical cognition. Against this duality, Cassirer argues that experience is originally one, and that the division into 'matter' and 'form' is only imposed subsequently.

> The sense of all objective judgments reduces [zurückgeht] to a final original relation, which can be expressed in various formulations, as the relation of 'form' to 'content,' as the relation of 'general' to 'particular,' as the relation of 'validity' to 'being.' Whichever of these designations one may choose, the really important decisive thing consists in this, that the fundamental relation here must be grasped as a strict *unity* that can only be designated through both opposing moments that enter into it, but can not be built up from them as if they were self-standing parts present for themselves. The original relation is not to be defined in such a way

[96] The Southwest school included also Windelband, who was a student of Lotze's. In this essay Cassirer also consider the views of empiricists like Paulsen, pragmatists like James, Dewey, and Vaihinger, idealists like Croce, and positivists like Mach.

> that the ‘universal’ in some way ‘subsists’ next to or above the ‘particular’, that the form ‘subsists’ detached from the content, and that both are then fused with one another by means of some fundamental synthesis of knowing [wissens]. Instead, the unity of reciprocal *determination* constitutes simply the first datum, behind which one can go no further, and which is analyzed into a duality of two ‘viewpoints’ only first for an artificially isolating abstraction. It is the fundamental error of all metaphysical theories of cognition, that they always try to reinterpret this duality of ‘moments’ into a duality of ‘elements.’[97]

We saw this opposition to form/matter distinctions earlier in Cohen and in Natorp, who argued that it follows directly from the transcendental method. As one analyzes the body of scientific cognition in order to finds it most basic principles and concepts (“Form-concepts”), one is led to the necessary and sufficient conditions for the constitution of the concept of an object. But these concepts and principles, into which the object is resolved, are not “some absolute, ‘beyond’ the logical forms of cognition,” but rather simply functional relations [Funktionsbeziehung] *within* cognition that give it its unity. This idea is ultimately derived from Kant’s Transcendental Deduction. The forms of thinking an object, the categories, are arrived at, again on Cassirer’s reading of Kant, by analyzing the necessary conditions for scientific cognition. These forms are not things-in-themselves or platonic objects whose relation to empirical cognition is difficult to effect. Rather, the categories just are the forms of objects in the sense that without them there could be no object of experience at all. So they are “immanent to thinking,” since they are necessarily applicable to objects of experience—objects that, considered in abstraction from the categories, are completely empty.

However, if we reinterpret these ‘form concepts’ metaphysically, as it were, as a separate kind of object, or constituting a separate realm from the realm of empirical objects, we will be left with no way to re-establish their necessary relation to the objects

[97] ENGL, 14. On this passage, see also Friedman, *Parting of the Ways*, 34.

of experience. This hypostasization of points-of-view into 'elements' is the fundamental error of what Cassirer calls "metaphysical theories of cognition." These theories oppose "a merely empty form" or "a mere generality without any particularization or determination" against a "merely alogical element" or a "material absolutely 'foreign to thought.'"[98] As soon as this happens, it can only be a 'lucky accident' (to use Lotze's phrase) that they are able to come together again. But for Cassirer, such a problem never even arises. "How these moments come together, and how they have brought about a whole reciprocally adapted to each other, no longer constitutes a significant question in need of solution, because only the factors' being in and with one another, and not their being isolated next to each other, can be a *datum* of cognition."[99]

A dualism of the form of thought and its matter, or still worse, of objects of pure thinking and objects of mere sense, would leave us with no explanation for how or why these elements are to be joined together. So if logic is to be the science of the form of thinking, then that form better have been isolated by analyzing our actual, concrete thinking in the sciences. Otherwise, we'd be left with an unbridgeable gap between logic and the other sciences. Therefore, our formal logic needs to be based in a prior analysis of the sciences, that is, transcendental logic. This fundamental thought requires that we

---

[98] ibid, 15.

[99] ibid, 16. See also ENGL, 25.

> The mistrust of 'purely logical' derivation appears above all here to rest in its final ground on the fact that one grasps the concept of the logical from the beginning much to narrowly, by connecting it arbitrarily with the thought of an *undetermined* 'generality' in the sense of the traditional theory of the concept. […] Because there is intuition only as intuition ordered in some way, as there is perception not isolated, but only as whole of ordered perception: if one seeks to express the fundamental motive of this ordering, thus one comes immediately to the totality of those theoretical 'forms' that constitute for idealism the totality of the logical in general. This thinking is obviously no 'thinking up': that is, no arbitrary imposition of forms, that somehow stem from the spontaneity of the subject, into a 'material' [Stoff] that would be according to its essence foreign to it. All these descriptions rely on that thinglike significance and hypostasization of merely methodological points of view, which logical idealism combats.

view the achievement of Frege and Russell in a particular way. The logical laws that Russell argues constitute the principles of all mathematics are *logical* laws for the very same reason that they are principles of *mathematics*. They are simply the most general laws that an analysis of mathematics is able to isolate. This analysis of mathematics, as in Russell's *Principles*, requires a detailed and comprehensive survey of the current state of mathematical research in all of its different fields. This analysis will show that concepts in mathematics are not structured or formed in the way the traditional model says. It will also show that mathematicians no longer feel constrained by the thought that mathematics is the science of number and empirical space, and no longer feel, as Dedekind argued, that their pure mathematical research has to answer to our independent conception of what space is like. (In this sense, "logic and mathematics have been fused into a true, henceforth indissoluble unity,"[100] since "logic" has traditionally been associated with pure thought and contrasted with our knowledge of space and time as "forms of sensibility." Though, strictly speaking, there is no content to this contrast once the scare quotes are removed.)

This analysis, carried out in one way by Russell in his *Principles* and in another way by Cassirer in Part One of *Substance and Function*, is the ancestor to Kant's analysis of the mathematical method in the section of the *Critique* entitled the "Discipline of Pure Reason in its Dogmatic Use." This analysis of mathematics, being sensitive to the drive for axiomatization that characterizes modern mathematics, will lead to fundamental concepts and principles. These concepts and principles, from a mathematician's point of view, are just a bit of pure mathematics. The comprehensive and detailed analysis of

---

[100] KMM, 4.

contemporary mathematics would show that these concepts and principles are sufficient to deduce the rest of modern mathematics. All of this is familiar to us from Russell's *Principles*. In some of his moods, Russell worried that he had a further philosophical task to perform: to show that these fundamental concepts and principles are actually logical. To prove this might require pointing out their universality or obviousness—or, for other philosophers, it might require proving that these principles are tautologies or necessary for the very possibility of a (or the) language. This is not Cassirer's view. There is no content to asking whether a given principle or concept is 'formal' or 'material.' Indeed, once one has assured oneself of the necessary applicability of mathematics in all the exact sciences,[101] there is no interesting difference between saying that something is foundational for mathematics and that it is logical.

---

[101] This is the missing move in the argument from saying that *x* is foundational for mathematics to saying that *x* is a law of thinking an object. The missing move corresponds to Kant's chapters "The Axioms of Intuition" and "The Anticipations of Perception." In this dissertation, I will not discuss Cassirer's analogue to these Kantian chapters. That being said, it is striking that Cassirer's argument that "logic and mathematics have been fused into a true, henceforth indissoluble unity" relies on his replacements for Kant's "Discipline of Pure Reason," "Axioms," and "Anticipations"—all elements in Kant's transcendental logic. Cassirer has no place in his philosophy for a treatise on formal logic like Kant's *Logic*, and no place for an argument like the Metaphysical Deduction for the necessary applicability of the concepts and principles of formal logic in our empirical knowledge.

CHAPTER 6 CASSIRER'S PHILOSOPHY OF GEOMETRY

In chapter 1, I argued that Kant's philosophy of geometry was carried out in close conformity to the ancient, diagram-based model, and in chapter 2, I recounted some of the reasons that nineteenth century projective geometers came to reject that model. This historical development within geometry presented a new philosophical challenge: How can a Kantian respect what is right in Kant's philosophy of geometry while untangling it from its outmoded mathematical model? In addition, I argued that the trend toward downplaying the role of concrete, particular diagrams in geometry that culminated in the rejection of their inferential use was motivated by the introduction of new elements into space that are required for projective methods. These methods, I claimed, presented a family of related philosophical problems: How are concepts formed in geometry? What makes a concept in geometry valuable or a proof employing that concept explanatory? With what right can we talk about ideal elements as if they existed in just the same way as so-called real elements? How is it possible for *later* geometrical work to discover the essence of geometrical objects or the meaning of geometrical concepts that have been referred to or grasped by *previous* geometers? In particular, how could a Kantian answer these questions?

In this chapter, I turn finally to how Cassirer tried to answer these questions. As we will see, Cassirer's philosophy of geometry draws on his readings of Kant and Dedekind, his polemic against the A-a model, and his commitment to the Marburg transcendental method. Drawing on the Marburg reception of Kant's distinction between transcendental and formal logic (chapter 5), Cassirer argues that, if mathematicians treat the so-called "imaginary" elements as full citizens of the world of space, then philosophers have to also. Philosophy has no independent grip on the nature of mathematical thinking (say, in formal logic) or on the nature of mathematical reality (say, in metaphysics) to gainsay the convictions of the mathematicians themselves. Similarly, the fact that the traditional model of concept formation leaves it mysterious how the formation of new concepts in geometry would allow us to introduce new geometrical objects that fall under these new concepts or would allow us to understand something genuinely new about the geometrical world (chapter 3) should not make us doubt the *bona fides* of modern geometry. Rather, it should drive us to find a new theory of mathematical concepts. (In any case, the traditional model of concept formation was patently inadequate even for synthetic, diagram-based geometry (chapter 4, section 4), and even failed for the empirical concepts, like <house>, for which it seemed best suited (chapter 4, section 3).)

In section one of this chapter, I show that the Marburg Neo-Kantians committed themselves to explaining modern mathematics and its historical development without condemning any part of it, not even the imaginary or ideal elements, as false or meaningless. In section two, I show how Cassirer saw in Dedekind's foundations of arithmetic a model for a modern version of Kant's thesis that mathematics is rational cognition from the "construction of concepts." In the next section, I show that Cassirer takes the conviction—widespread among mathematicians and unmistakable in particular cases—that newer geometrical systems of concepts first make even elementary geometrical objects intelligible, and uses it to justify modern geometry and to update Kant's idea that the unity of mathematics lies in its method. (It is in this section that we find Cassirer's rudiments of a story of how geometrical concepts are formed and of what makes a geometrical concept valuable.) Cassirer then answers the objection, given on Kantian grounds by Bertrand Russell in 1897, that modern geometry is false or meaningless because talk of infinitely distant or imaginary points cannot be given a *spatial* interpretation, by denying that geometry is about physical space (section four). But to avoid the formalism or platonism that is anathema to any Kantian, Cassirer needs a new way of cashing out the Kantian idea that mathematics is essentially applicable in natural science (section five).

1. THE MARBURG NEO-KANTIAN SCHOOL AND THE FACT OF MODERN MATHEMATICS

Cassirer wants to respect the great fruitfulness of the use of so-called "imaginary elements" in mathematics by giving a philosophical theory that would show that their status is not inferior to that of the so-called "real" ones. Indeed, he thinks it a condition on the adequacy of any theory of even elementary objects (like Euclidean plane figures or natural numbers) that they not be given a different ontological status (or sentences about them not be given a different semantic status) than the more recently introduced elements (like figures in the complex plane or complex numbers). From the point of view of Cassirer and the other Marburg Neo-Kantians, because philosophy has no independent grip on the human understanding or on the nature of objects of experience, there is no leverage by means of which philosophy could condemn an established result of the sciences, in particular, an established result of mathematics, as false or meaningless. Any adequate philosophical study of mathematics will have to recognize that "the modern development of mathematics has thus created a new 'fact,' which the critical philosophy, which does not seek to dominate [meistern] the sciences but to understand [verstehen] them, can no longer overlook."[1]

It will be helpful for what follows to contrast the Marburg approach with a better-known version of Kantianism that also seeks to respect the fact that the concepts and principles that constitute the preconditions of the sciences evolve over time. On Michael Friedman's reading, the logical positivists *relativized* Kant's theory of synthetic a priori judgments. They thought that there need to be non-empirical principles that, though revisable in periods of conceptual revolution, are nevertheless constitutive of the framework of empirical investigation at a given stage in the history of science, and thus make possible objective scientific experience and the intersubjective communicability of natural science.[2] Compare this with the Marburg school's different historicized understanding of Kant's project.

> The task, which is posed to philosophy in every single phase of its development, consists always anew in this, to single out [herasuzuheben] in a concrete, historical sum total of determinate scientific concepts and principles the general logical functions of cognition in general. This sum total can change and has changed since Newton: but there remains the question whether or not in the new content [Gehalt] that has emerged there are some maximally general relations, on which alone the critical analysis directs its gaze, and that now present themselves under a different form [Gestalt] and covering. The concept of the *history of science* itself already contains in itself the thought of the *maintenance of a general logical structure* in the entire sequence of special conceptual systems.[3]

---

[1] Cassirer, "Kant und moderne Mathematik" [KMM], 31 (1907).
[2] See Friedman, *Reconsidering Logical Positivism*. We need not worry here about whether or not this is a good way of understanding logical positivism.
[3] Cassirer, *EP*I, 16.

Like a relativized a priori program, the Marburg school's transcendental method requires determining which concepts and laws play the role of Kant's categories and principles at some given stage in the history of science. But in the second and third quoted sentences, we find a further demand: we need to be able to substantiate the claim that the history of science is the history of one subject and forms one series of events. This demand is meant to neutralize the threat that the relativity of constitutive concepts and principles will force a more troubling relativism. A fundamental thought of Kant's Copernican revolution is that the "*a priori* conditions of a possible experience in general are at the same time conditions of the possibility of the objects of experience" (A111); it follows that the constitutive concepts and principles make possible not only experience—and so, science—but also the objects of experience—and so, nature. Now Cassirer thinks that it is obvious that twentieth-century scientists were trying to understand the same world as Newton. But Kant's Copernican revolution prevents us from trying to explain this unity of science over time in terms of a common subject, the world-spirit, or a common subject matter, the world. Rather, there has to be "a general logical structure"—some common set of principles and concepts—present in all phases of the history of science.[4] For Cassirer, then, the distinction between the a priori and the empirical is a distinction between what persists throughout the development of science, and what does not.[5] To

---

[4] See also *Substance and Function* [*SF*], 321-2: "Going back to such supreme guiding principles [i.e., the "*form* of experience" that persists in all stages of the asymptotic progression toward the fully empirically adequate theory] insures an inner homogeneity of empirical knowledge, by virtue of which all its various phases are combined in the expression of *one* object. The 'object' is thus exactly as true and as necessary as the logical unity of empirical knowledge;—but also no truer or more necessary…We need, not the objectivity of absolute things, but rather the objective determinateness of the *method of experience*."

On this point, see Friedman, "Ernst Cassirer and the Philosophy of Science."

[5] *SF*, 269. Compare this with the different theory of the a priori in Reichenbach's *The Theory of Relativity and A Priori Knowledge*, chapters V-VII, where he distinguishes between a priority as necessary and unrevisable validity, and a priority as "constitutive of the object of knowledge"—that is, at some stage in

prevent the radical relativism threatened by the relativization of Kant's transcendental logic—to allow ourselves to speak of the history of science as a connected series of rational attempts to understand nature—, there need to be some a priori cognitions that remain invariant through all stages of the history of science (though we need never be certain about which concepts and principles they are).

It is not important for us to try to spell out in this chapter precisely what this demand amounts to, nor to evaluate its cogency. All that we need to note is that, as in the case of the mathematical sciences of nature, a Marburg philosophical account of mathematics will have to substantiate the claim that the various stages in the historical development constitute one history—and even that the various branches of mathematics at a given time constitute one discipline—without appealing to a common subject matter. For them, "the unity of mathematics no longer lies in its object – whether it be the study of magnitude and number, the study of extension, as the general theory of manifolds, the theory of motion, or equally as the theory of forces."[6] For them, as for Kant, the unity of mathematics lies in its *method*. In the rest of the paper, I'll show that Cassirer thought that a modern Kantian philosophy of mathematics could exploit key features of Kant's theory of the mathematical method in order to make room for a historicized theory of mathematics that can fit with the development of imaginary elements in geometry.

---

the history of science. Cassirer's "a priori" cognitions are "constitutive of the object of knowledge" at all stages in the development of science. For Cassirer, Neurath's boat is the same boat over time only if some of the planks are never replaced.

[6] KMM, 31.

2. KANT'S MATHEMATICAL METHOD AND CASSIRER'S DEDEKINDIAN PARADIGM

Cassirer summarized his understanding of Kant's philosophy of mathematics in this way:

> After it had appeared to pursue a wholly different course for a time the development of modern mathematics has in this respect turned back in a remarkable way toward certain positions taken by Kant. The Transcendental *Logic* did not undergo modifications similar to those imposed on the Transcendental Aesthetic by the discovery of non-Euclidean geometry and its various 'forms of space.'[7]

In section four, I'll discuss Cassirer's attitude toward the Aesthetic. For now, I'll focus on the positive doctrines that Cassirer finds in Kant's "transcendental logic." Kant's "transcendental logic" is the science that "expounds the elements of the pure cognition of the understanding and the principles without which no object can be thought at all" (A62/B78). There are two doctrines from Kant's "transcendental logic" that Cassirer wants to focus on. Kant argued in his two chapters, "Axioms of Intuition" and "Anticipations of Perception," that pure mathematics is applicable to objects of experience because even the perception of an empirical object requires that the manifold in it be synthesized using the same syntheses studied in mathematics. As I'll show in section five, Cassirer thinks that this essential applicability of mathematics in the natural sciences prevents a Kantian from adopting a formalist or platonist account of mathematics. In this section and the next, I want to show how Cassirer sees the Kantian theory of the mathematical method—his view of the distinctive kind of concepts and concept use that characterizes mathematics—will help us make sense of the radical conceptual innovations one finds in nineteenth century mathematics.

---

[7] *The Problem of Knowledge: Philosophy, Science, and History since Hegel* [*PK*], 75-6.

We saw in chapter 4 that what is distinctive about mathematics for Kant is a certain relation between concepts and objects, universals and particulars. In mathematics, possessing a concept is prior to representing the objects that are contained under it. Once I possess the concept, I can exhibit or construct the objects falling under the concept; and since I construct the objects falling under the concept, as soon as I possess the concept, I can be assured of the existence of objects falling under the concept. Moreover, all of the (mathematical) properties that belong to the objects contained under the concept belong to it in virtue of the fact that it is contained under the concept, even though not all of these properties are literally contained in the concept.

Cassirer saw substantial overlap between Kant's theory of the mathematical method and Dedekind's foundations of arithmetic. Since this connection seems implausible at first, it will help to get some material out on the table. In perhaps the most famous passage in Dedekind's *Was sind und was sollen die Zahlen?*, Dedekind writes:

> If in the consideration of a simply infinite system *N* set in order by a transformation φ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind.[8]

A simply infinite system is a set of objects N, along with a 1-1 mapping φ, such that the image of φ is N - {1}, for some element 1 in N, and such that N is the smallest such set. Numbers are arrived at as "free creations" from some simply infinite system when we "free the elements from every other content." Dedekind's picture has three

[8] Dedekind, *The Nature and Meaning of Numbers*, §73, in *Essays on the Theory of Numbers*.

elements. First, the abstraction Dedekind describes takes us from a given simply infinite system (it does not matter which) to a *new* system of objects, which differ from all other simply infinite systems in that their elements—the numbers—have no more properties or no more of a *nature*, than can be expressed in terms of the basic relations described by the axioms for simply infinite systems. Each of a natural number *n*'s *essential* properties is a *relational* property of *n* to *some other natural number m*.[9] Second, once we have described what a simply infinite system is—effectively, once we have isolated Dedekind's axioms for simply infinite systems—and have assured ourselves of the consistency of this notion, we can be assured of the *existence* of the numbers, those objects which constitute that simply infinite system all of whose elements have no more of a nature than their positions within the system. Third, since we introduce the natural numbers all at once by abstraction from a simply infinite system, the nature and identity of all of the numbers is determined together: *reference* to each individual number is only possible in the context of the whole system of numbers and the axioms describing them. (I couldn't come to refer to the natural numbers "one at a time.")

Cassirer also subscribes to this picture (though with a modification of the second point) both for arithmetic and for geometry.[10] Cassirer's Dedekindianism is interesting

---

[9] A natural number can of course have *non-essential* relational properties with objects that are not numbers: for instance 9 (or 8?) is related to the planets in our solar system such that 9 is the number of planets. But this is not an essential property of 9. On the other hand, as Dedekind put it in a different context, "that the number 4 is the child of the number 3 and the mother of the number 5 will always and for everyone remain present" ("Letter to Lipschitz," in *Gesammelte Werke*, 490; translated in Reck, 386).

[10] See *SF*, 36: "[The numbers] are not assumed as independent existences [selbstständige Existenzen] present prior to any relation, but they gain their whole being [Bestand], so far as it comes within the scope of the arithmetician, first in and with the relations, which are predicated of them. They are *terms of relations* [Relationsterme] that can never be 'given' in isolation but only in community with one another."

To turn this picture into a theory, more needs to be said about how one can prove the consistency of a set of axioms, and substantial technical and philosophical complications arise in telling a story about the mathematics needed to prove the consistency of axiom systems. I return to this point in the conclusion.

because he sees it is a Kantian theory and because he uses it to tell a Kantian story of mathematical progress that avoids condemning the ideal elements to formalism or fictionalism. (I'll discuss the latter point in the next section.) For Kant, as soon as I possess the mathematical concept <triangle>, I can introduce objects, triangles, falling under it. On Cassirer's Dedekindian picture, the *existence* of mathematical objects, like natural numbers, follows from the fact that we can speak consistently about the system of mathematical concepts.[11] (As I'll show in the next section, Cassirer's condition is actually stronger than consistency here.) For Kant, mathematical concepts, like <triangle>, make possible the representation of the objects, the triangles, falling under them. For Cassirer, *reference* to mathematical objects, like natural numbers, is always in the context of some background system of concepts. For Kant, all of the mathematically relevant properties that a triangle has, it has in virtue of its falling under the concept <triangle>. Similarly, for Cassirer, mathematical objects, like a natural number, have no more properties, or no more of a *nature*, than can be expressed in terms of the basic relations of the system of concepts.[12] There are two important differences between Kant's

---

[11] I'll continue to speak throughout the paper in terms of "structures" or "systems" of concepts. For Cassirer, the paradigm of a structure of concepts is a consistent, categorical axiom system that implicitly defines its terms. But I follow Cassirer in avoiding speaking only of axiom systems. He wants to tell a story that culminates in modern mathematics, but he does not want his story to fit only modern mathematics. So though mathematicians are always concerned with structures of concepts, only by the very end of the nineteenth century were these structures axiomatized.

[12] At many points throughout his career, Cassirer uses unmistakable Kantian language to describe Dedekind's approach. At *SF*, 112, he calls the procedure of laying down a system of concepts and then introducing objects that fall under them "construction [Konstruktion]." At *EP II*, 714, Cassirer says that Dedekind's foundations of arithmetic are expressions of Kant's idea that mathematical concepts "owe their existence not to abstraction but to construction [Konstruktion]."

A striking example is *Kant's Life and Thought* [*KLT*], 159: "Here [in a priori synthesis] we begin with a specific constructive connection, in and through which simultaneously a profusion of particular elements, which are conditioned by the universal form of the connection, arises for us. We think the diverse possibilities jointly in a single, comprehensive and constructive rule: sections of a cone; and we have in that way simultaneously produced the totality of those geometric forms which we call second order curves: circles, ellipses, parabolas, and hyperbolas. We think the construction of the system of natural account and the Dedekindian picture. The first is that Kant speaks of constructing individual concepts, like <triangle>, whereas Dedekind's approach requires laying down a system of concepts Ω(<number>, <successor>), whose interrelations are specified in Dedekind's axioms for arithmetic. I think that Cassirer sees this as only an artifact of Kant's limited Aristotelian logic, and he is always eager to improve Kant's doctrine by recasting it in the new logic of Frege and Russell. (I'll return to this point in the fifth section.) The second difference is that on Kant's view, but not Cassirer's, geometrical objects do have non-structural properties—they are parts of the essentially single, infinite, given space.[13] (I'll return to this point at the end of the fourth section.)

3. KANT'S MATHEMATICAL METHOD AND A WIDER CONCEPTUAL SETTING FOR AN OBJECT

I showed in section one that the Marburg Neo-Kantians wanted to appreciate (in a way Kant himself did not) the fact that the sciences develop over time, and they wanted to be able to acknowledge the fact that it can *progress*, as later physicists talk within a different framework of concepts and principles about the very same physical world that their less enlightened predecessors did. Similarly, Cassirer wants to acknowledge the fact that

numbers, according to one basic principle, and we have included in it from the first, under definite conditions, all the possible relations between the members of this set."

Without the reference to a priori synthesis, this passage might just as easily be describing Dedekind's approach or Kant's. This is Cassirer's point.

[13] Specifically, Cassirer's disagreement with Kant (and traditional conceptions of geometry) concerns the relations of a geometrical point to *physical* objects. On Cassirer's Dedekindian view, no arithmetical or geometrical object has an essential property that is not a relation between that object and another object in its system. A point in physical space is essentially a position that can be occupied by a physical object. So part of the nature of a point in physical space is its relation (not to other points or figures in space, but) to *physical* objects in space. For a similar point, see Charles Parsons, "Structuralism and Metaphysics," 70-1.

mathematics is a "synthetically progressive science,"[14] and he wants to be able to respect the conviction that later developments, like the discovery of the wider projective realm, allow mathematicians to better understand the objects and problems their predecessors had worked on already. Clearly, the progress of mathematics consists in more than discovering new theorems from given axiom systems and introducing new axiom systems for study. As Hilbert put it in the concluding paragraph of his famous lecture on mathematical problems:

> …[M]athematical science is in my view an indivisible whole, an organism, whose viability is based on the connection of its parts. For in all the distinctions in the matter of mathematical thinking in particular cases, we are still aware very clearly of the identity of the logical tools [Hilfsmittel], the relationship of the ideas [Ideebildungen] in all of mathematics, and the numerous analogies in its various regions. Also: the further a mathematical theory develops, the more harmonious and unitary its structure arranges itself to be, and unknown relations are discovered between hitherto separated branches of knowledge. And thus it comes about that the unitary character of mathematics is not lost in its expansion, but rather becomes ever clearer.[15]

Hilbert thought that the development of mathematics over time—and this is part of what makes the historical development of mathematics progressive—brings with it a new "unity" or "harmony" among its different branches.

Cassirer shares this sentiment, and he thinks that it will play a key role in finding a Kantian way to justify the widespread practice of adjoining ideal elements in geometry. In an important passage, which will take us a bit of time to unpack, Cassirer indicates the conditions that new elements must fulfill within mathematics.

---

[14] Cassirer, *Philosophy of Symbolic Forms*, vol. 3 [*PSF3*], 398. We'll return to mathematics' syntheticity below.

[15] "Mathematische Probleme," 329. (This passage does not appear in the abridged translation in Ewald. The translation here is mine.)

Compare his summary of the axiomatic method in his paper 'Axiomatic Thought': "By pushing ahead to ever deeper layers of axioms in the sense explained above we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge" (Ewald, vol. 2, 1115, paragraph 56).

> But here the philosophical critique of knowledge must raise still another and sharper demand. For it is not enough that the new elements should prove equally justified with the old, in the sense that the two can enter into a connection that is free from contradiction—it is not enough that the new should take their place beside the old and assert themselves in juxtaposition. This merely formal combinability would not in itself provide a guarantee for a true inner conjunction, for a *homogeneous logical structure of mathematics*. Such a structure is secured only if we show that the new elements are not simply adjoined to the old ones as elements of a different kind and origin, but the new are *a systematically necessary unfolding of the old*. And this requires that we demonstrate a primary logical kinship between the two. Then the new elements will bring nothing to the old, other than *what was implicit in their original meaning*. If this is so, we may expect that the new elements, instead of fundamentally changing this meaning and *replacing* it, will first bring it to *its full development and clarification*.[16]

It will be helpful for understanding Cassirer's idea to distinguish three senses in which we can talk about the "unity" of mathematics. (1) Mathematics is a unity in the sense that the different activities called 'mathematics' belong together and are not just arbitrarily collected together: there is a single, unified scientific discipline, "mathematics," and we can make a principled distinction between what belongs in mathematics and what does not. And just as Cassirer wants to explain how all of the stages of natural science come together to form the historical development of a single discipline, methodologically and not metaphysically, so too Cassirer wants to distinguish mathematics in terms of a common method. (2) Mathematics is a unity in the sense that advances in mathematics give us new ways of talking about old objects, and do not just switch us to a new subject altogether. The same mathematical problems can be treated of in different ways; the same mathematical objects can be embedded in different mathematical structures. In the physical case, Cassirer worries that, without insisting on the persistence of a common stock of invariant cognitions, relativizing Kant's a priori concepts and principles will make nonsense of the claim that later physicists are able to understand better the very

[16] *PSF3*, 392.

same world that their predecessors knew. Similarly, Cassirer wants to respect the conviction of mathematicians that a new structure being introduced, like complex projective geometry, is "*a systematically necessary unfolding of the old*" that express "*what was implicit in* [*its*] *original meaning*"—that *new* systems of concepts deepen our understanding of the concepts in terms of which our *old* objects, like the circle, get their meaning. He wants to avoid the view that each newly introduced structure of concepts is a *replacement* of the old structure—that with new structures of concepts, mathematicians just change the subject.[17] (As I'll show later, this requires more from a new structure of concepts than consistency.) (3) Mathematics is a unity in the sense that, as Hilbert puts it, its different parts are connected to one another. Different subfields draw on the same concepts or the same proof techniques. Within a subfield, different phenomena are not simply isolated facts but are united through a new concept or a new proof technique.

Interestingly, for Cassirer, we can understand what it means for one system of concepts "to make explicit what was implicit in the meaning of an old system"—the unity of mathematics in the second sense—by thinking through what it is for a new system of concepts to unify mathematics in the third sense. Consider again the unification effected by the introduction of new points in projective geometry and the new theory of conics that it makes possible. Using language reminiscent of Hilbert, Steiner saw the importance of his work in uncovering the "organism by which the most varied phenomena in the world of space are connected to each other"—that is, the organic unity

[17] Cp. *PSF3*, 393: "In all these cases, logical justification of the new elements is to be found not in the fact that the new dimension in which we are beginning to envisage things somehow *displaces* the relations that were valid within the former dimension, but rather in the fact that it sharpens our eye for them as they are. The look backward from the newly opened field to the old one *first opens up the old field* in its entirety to our thinking and gives us an understanding of its finer structural forms." (emphasis added)

of geometry or the inner connections of various geometrical phenomena—and thought that this shows that he had hit on the nature or essence of geometrical objects.[18] As Cassirer puts it, the new elements are "an intellectual medium by which to apprehend the true meaning of the old, by which to know it with a universality and depth never before achieved."[19] We can cash out these metaphors in particular cases by showing of a particular structure of concepts that it allows us to generalize a theorem, or synthesize apparently different mathematical phenomena under one point of view.[20] In our example from chapter 2, introducing points at infinity allows us to unify a series of geometrical theorems together under Pascal's and Brianchon's theorems. We might cash out the metaphor by showing how the new elements reveal a previously hidden relationship between particular mathematical disciplines and effect a "closer and more profound union among them."[21] As Cassirer notes, the projective use of imaginary elements—like the "ideal intersection" of two circles that have come apart—provides a distinctly geometrical justification for the free use of algebra over the complex numbers in geometry, and thus "unites" two fields.[22]

Cassirer thus gets a Kantian picture of what makes mathematics a progressive, unitary discipline by seeing Kant's "synthesis" as acting *developmentally*, as a deeper or wider system of concepts is synthesized or constructed out of the old.

---

[18] Steiner, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander* (1832), reprinted in *Gesammelte Werke*, vol. 1, 229.
[19] *PSF3*, 393.
[20] See *PSF3,* 397.
[21] *PSF3*, 399.
[22] Observations like these would be the starting-point for any theory of what it is for a mathematician to have understood or explained a problem or theorem better than he would have otherwise. Cassirer himself discusses various cases, but gives no worked out theory. (I myself don't know what a satisfying theory would look like or even if one is possible.)

> What Kant meant is that the distinctive, fundamental character of mathematical synthesis, which he was aiming to explain, comes to light … in the building up of the mathematical *world of objects*. The formation of the objects of mathematics is ‘constructive,’ and hence ‘synthetic,’ because it is not concerned simply with analyzing a given concept into its marks, but because we advance and ascend from determinate fundamental relations, from which we begin, to ever more complex relations, where we let each new totality of relations correspond to a new realm of “objects”. For every combination, for every new synthesis, there arises a corresponding object which in a strictly methodological sense develops out of preceding ones but in no way coincides with them logically.[23]

The key idea is that new concepts are justified when they provide a deeper understanding of our old objects. In Kant’s simpler Euclidean model, the concept <circle> is *prior* to the object, a circle, in the more straightforward sense that we can only represent a circle by constructing it using the concept <circle>. In Cassirer’s more sophisticated developmental model, the projective concept <(possibly degenerate) Steiner conic> or <conic intersecting the two circular points> is not *temporally prior* to the object, a circle, but one might say *logically* or *essentially prior*, in the sense that a circle becomes fully intelligible mathematically only within this wider setting.[24] Using Cassirer’s phrase, the *ratio essendi* of the new elements is given by the old elements, but the *ratio cognescendi* of the old elements is given by the new elements.[25] I earlier remarked that for Cassirer the fact that a system of concepts is consistent is not sufficient for proving the existence of objects falling under it, and now we see why. We justify the existence of the new objects in terms of their role in systematizing or explaining the old elements; but once we have the new elements, we see that the old elements are intelligible or understandable

[23] *PK*, 75.
[24] In a late paper, Cassirer, quoting the famous sentence from Hilbert’s “Axiomatic Thought,” (see note 48 above), writes that the demand to “deepen the foundations” of our knowledge requires that “in a sense, mathematics’ fundamental concepts are the last, not the first, to be known” (“Inhalt und Umfang des Begriffs,” 228-9).
[25] *PSF3*, 393.

only in terms of the new.[26] (This gives Cassirer an interesting gloss on the thesis that geometry is "synthetic." New, wider geometrical theories like Steiner's real projective geometry or Cayley's complex geometry develop out of the older ancient geometry in the sense that there are good mathematical reasons—hard to characterize in general, but unmistakable in particular cases—to think of the wider setting as the appropriate context in which to treat the old geometrical objects and problems.)[27] Once we have embedded our old objects within a new structure of concepts, the new objects that fall under the widened conceptual structure become independent objects of study, which can themselves be embedded in wider or alternate structures of concepts.[28] We thus get a hierarchical picture of the development of mathematics, as mathematicians discover newer concepts by means of which to construct familiar objects. It is this whole connected process that gives us the mathematical method. In keeping with Kant's "Copernican Revolution," we can see the unity of mathematics (in the first sense specified above) as consisting in this method of connected reconceptualizations of old

---

[26] For example, Cassirer discusses the justification for introducing imaginary points in projective geometry in terms of the unification they effect in real geometry. "The imaginary intermediate members always serve to make possible insight into the connection of real geometrical forms, which without this mediation would stand opposed as heterogeneous and unrelated. It is this ideal force of logical connection that secures them full right to 'being' in a logico-geometrical sense. The imaginary exists [hat 'Bestand'], insofar as it fulfills a logically indispensable function in the system of geometrical propositions." (*SF*, 83)

[27] I don't think this amounts to a reading of the analytic/synthetic distinction. In fact, the characteristic Neo-Kantian antipathy towards Kant's hard sensibility/understanding distinction, and their conviction that Kant's Aristotelian logical background distorted rather than clarified Kant's doctrines, left the Neo-Kantians without a clear way of characterizing Kant's distinction.

In other places, Cassirer gives a different gloss on the analytic/synthetic distinction. See Cassirer's KMM, §V, and *EP*II, 675-8, which draw on Cohen's *Kants Theorie der Erfahrung*, 2nd ed, 400, and my discussion of these passages in chapter 5, section 6.

[28] On *PSF*3, 392, quoted above, Cassirer opposes this picture to one often attributed to Hilbert—that mathematicians are free to introduce and study any consistent system of concepts. But, as Michael Hallett has recently argued forcefully, Hilbert, like Cassirer, demands more of mathematical concepts than consistency, and, again like Cassirer, thinks that the objects that fall under newly introduced concepts exist in just the same sense as the old objects. Hallett, "Physicalism, Reductionism, and Hilbert," 233.

problems and objects.[29] Like Kant, he thinks that the unity of mathematics lies not in its objects—be they numbers and magnitudes, or parts of the world of abstract structures—but in its method.

4. THE REPRESENTATION OF SPACE

Cassirer thinks that we can justify the free use of ideal elements by emphasizing the way that they unify previously established mathematical facts and sub-disciplines. Because these new concepts unify existing mathematics in this way, we say that they do a better job of revealing the significance of existing mathematical concepts. Using Kant's language, the new projective concepts are prior to, or make possible, the representation of geometrical objects. But an obstacle still remains for a Kantian. The early Russell had argued that "only a knowledge of space…can assure us that any given set of quantities will have a spatial correlate"—and simple reflection shows us that "the circular points are not to be found in space" *and so "have no geometrical import*."[30] Russell is on good Kantian grounds here. In the "Aesthetic," Kant had argued that we possess a priori

---

[29] *PSF*3, 404: "The actual intellectual miracle of mathematics is that this process [of positing new conceptual structures and the objects that fall under them] …never finds an end but is repeated over and over, always at a higher level. It is this alone that prevents mathematics from freezing into an aggregate of mere analytical propositions and degenerating into an empty tautology. The basis of the self-contained unity of the mathematical method is that the original creative function to which it owes its beginning never comes to rest but continues to operate in ever new forms, and in this operation proves itself to be one and the same, an indestructible totality."

Compare here the 1894 remarks by the geometer Felix Klein ("Riemann and His Significance for the Development of Modern Mathematics," 170): "To the layman the advance of mathematical science may perhaps appear as something purely arbitrary because the concentration on a definite given object is wanting. Still there exists a regulating influence, well recognized in other branches of science, though in a more limited degree; it is the *historical continuity*. *Pure mathematics grows as old problems are worked out by means of new methods. In proportion as a better understanding is thus gained for the older questions, new problems must naturally arise*."

[30] *EFG*, 45-6, emphasis added.

intuitions of space and time as “infinite given magnitudes” (A25/B39) that are presupposed in our representation of sub-regions of space and time.[31] Indeed, Kant thinks that our a priori representation of space and time are necessary to explain the possibility of the mathematical method. For Kant, <triangle> can be constructed because triangularity is a purely spatial property and a triangle is just a portion of space whose properties are purely spatial.

> With regard to [the form of intuition (space and time)] we can determine our concepts *a priori* in intuition, for we create the objects themselves in space and time through homogeneous synthesis, considering them merely as quanta….[Mathematics] is the use of reason through construction of concepts, because these concepts, since they already apply to an *a priori* intuition [that is, space or time], for that very reason can be determinately given in pure intuition *a priori* and without any empirical *data*. (A723/B751)

I can construct the concept <triangle> because I have ready to hand my representation of space as an infinite given whole and with it a method for carving out regions of space by “homogeneous synthesis.” Space thus provides me a domain of mathematical objects to work with in geometry and a method for introducing them.[32]

The representations of space and time play a further essential role for Kant. Kant is emphatic throughout the *Critique* that the synthetic a priori cognitions of geometry count as *cognitions* only on account of their necessary link to experience.[33] Kant has no place in his theory for purely mathematical (abstract) objects, if by this we mean genuine

---

[31] “It is only when both infinite space and infinite time are given that any determinate space and time can be given by *limitation*.” Ak 2:405; translated in *Theoretical Philosophy: 1755-1770* by Walford and Meerbote; parallel passages in the *Critique* are A25/B39 and A31-2/B47.

[32] See Parsons, “Kant’s Philosophy of Arithmetic,” 135.

[33] Thus, in the “Principles of Pure Understanding”, we read the following. “Thus although in synthetic judgments we cognize *a priori* so much about space in general or about the shapes that the productive imagination draws in it that we really do not need any experience for this, still this cognition would be nothing at all, but an occupation with a mere figment of the brain, if space were not to be regarded as the condition of the appearances which constitute the matter of outer experience; hence those pure synthetic judgments are related, although only mediately, to possible experience, or rather to its possibility itself, and on that alone is the objective validity of their synthesis grounded.” (A157/B196)

objects and not just "mere figments of the brain." Rather, a construction in pure intuition provides, not a genuine object, but "only the form of an object."[34] Geometrical constructions, then, give rise to genuine knowledge only because they establish the (real) possibility of certain empirical objects. But the essential *applicability* of geometry to the objects in space, would not be established by the construction if space were not "the form of all appearances of outer sense" and geometry were not the study of this form.

Cassirer just bites the bullet here: he thinks Kant was wrong to think that geometry is possible only if, as the "Transcendental Aesthetic" claims, we can represent in an a priori way physical or empirical space and time—space and time as the *locus in quo* of empirical objects and events. (This is the sense in which Cassirer says, in the passage quoted on page 13 above, that modern mathematics has forced fundamental revisions of Kant's "Aesthetic.") For Cassirer, the development of non-Euclidean geometries shows that there are many possible geometrical forms that physical or empirical space could take, and that only the progress of empirical science can confirm which mathematically possible space is instantiated in the physical world.[35] For Kant, there is only one space, physical space, which is not only the space in which physical objects are located, but it is the only mathematically possible space. For Cassirer, there are many logically possible spaces, and no mathematically possible geometry is any more true or real than another.[36] Like the Russell of *Principles of Mathematics*, he argues that

---

[34] A223-4/B271.

[35] *SF*, 106: "The role, which we can still ascribe to experience, does not lie in grounding the particular systems, but in the selection that we have to make among them." Cassirer, however, follows Poincare (*Science and Hypothesis*, chapters 3-5) against Russell, in seeing the empirical determination of the metric of space as being a much more complicated affair than Russell thought. (See Russell's 1898 paper, "Are the Axioms of Geometry Empirical?") Cassirer in fact was an early and life-long proponent of a Duhemian view of empirical confirmation. See Massimo Ferrari, "Ernst Cassirer und Pierre Duhem."

[36] *Einstein's Theory of Relativity* [*ETR*], 432.

the development of non-Euclidean geometry shows that the propositions of pure mathematics – pure mathematics, as a non-empirical science – are not *about* physical space and time.[37] Every geometry, even Euclidean geometry and the Riemannian space of variable curvature of General Relativity, describes an "abstract space."[38] Cassirer's position is thus completely different from the better-known attempts by Kantian philosophers to "plug the leaks"[39] by identifying weaker and weaker fragments of Euclidean geometry that could function as the a priori theory of empirical space. For Cassirer, there can be no mathematical propositions—not even the axioms of geometry—grounded in the pure or empirical intuition of space.[40]

In the second section, I showed that Cassirer thinks that the modern version of Kant's picture of mathematics as "construction from concepts" is the Dedekindian picture, where, given an acceptable system of concepts, we can introduce a family of objects whose entire nature consists in their falling under just these concepts. I remarked there that the idea that a mathematical object has no non-structural properties does not fit with Euclidean geometry, as long as we insist, as Kant himself did, that the objects of geometry are constructed in the one, infinitely given *locus in quo* of empirical objects.

---

[37] Russell, I think, was the first philosopher to express this view, in his 1899 article "Geometry, Non-Euclidean," 503. (The article, for the *Encyclopedia Britannica*, appeared in 1902.)

[38] *ETR*, 443.

[39] Coffa, *The Semantic Tradition*, 57. These attempts constitute the very interesting and important historical story that roughly begins with Gauss and culminates in Einstein's "Geometry and Experience."

Cassirer's Kantianism is therefore also different from the Kantian philosophy of geometry that Russell attacked in his *Principles of Mathematics* [*POM*]. For Russell, Kant's philosophy of geometry stands or falls with the idea that the representation of space – in the drawn diagram – plays an inferential role not reducible to the inferential role of purely discursive concept-use. Cassirer, like Russell, thinks that the development of mathematics has made such a position impossible. See KMM, 3. On this point in Russell, see *POM*, §4, §§433-4; "Mathematics and the Metaphysicians," 96.

[40] See *PSF*3, 363: "But for Kant, …sensibility has ceased to be a mere means of representation, as in Leibniz, and has become an independent ground of knowledge: intuition has now achieved a grounding, legitimizing value. In the methodological divergence it seems clear that modern mathematics has followed the road indicated by Leibniz rather than that suggested by Kant. This has followed particularly from the discovery of non-Euclidean geometry."

But since there can be no determinate a priori representation of empirical space, geometry cannot be about it, and geometrical objects need have no essential non-structural properties. This removes the last obstacle that had prevented us from according the same ontological status to the ideal elements (and the same semantic status to sentences about the ideal elements).

> If even the elementary forms of mathematics, the simple arithmetical numbers, the points and straight lines of geometry, are understood not as individual things, but only as links in a system of relations, then the ideal elements may be said to constitute "systems of systems." They are composed of no different logical stuff than these elementary objects, but differ from them only in the mode of their interconnection, in the increased refinement of their conceptual complexion.[41]

If neither elementary Euclidean geometry nor complex projective geometry is about empirical space, then we do not need an interpretation in empirical space of the circular points to talk meaningfully about them—as long as embedding familiar geometrical objects in the projective system of concepts provides us with a payoff in mathematical understanding, intelligibility, and unification.

5. The Essential Applicability of Mathematics and a New Problem for a Kantian

Because Cassirer thinks that we have no a priori representation of empirical space, he needs new explanations of the two features of mathematics that pure intuition explained for Kant: how mathematicians are able to introduce mathematical objects, and how these objects get their "objective validity." On the first point, Cassirer thinks that Kant's recourse to carving out portions of a previously given, infinite space is merely an artifact

[41] *PSF*3, 395.

of his allegiance to the overly simplistic, Aristotelian, model of conceptual structure that he inherited.[42] With the new logic of Frege and Russell, we can represent a suitably complex and abstract system of concepts, whose mutual interrelations are expressed in multiply quantified polyadic sentences, and can introduce a domain of objects in Dedekind's way—without having to construct them *out of* infinitely given space. Even so, Cassirer recognizes that the geometrical figures carved out of infinitely given empirical space—as the form of our outer intuition—have a kind of ontological legitimacy that the abstract objects of modern geometry do not. Kant's insistence that geometrical objects are in fact "forms of objects" or *ways the empirical world could be*, separates his view both from formalism and from platonism: mathematical statements express *true* judgments that characterize the way the *empirical* world could be. For Cassirer, any Kantian philosophy of mathematics will have to avoid both those positions.[43]

From Cassirer's Kantian point of view, the only objects in the strict sense are the empirical objects of natural science, and the only kind of objectivity is the objectivity secured by reference to these objects:

---

[42] See, for instance, KMM, 39.

[43] By "formalism" I mean the thesis that some or all mathematically acceptable sentences do not express genuine statements that can be true or false, but are uninterpreted strings of figures manipulable according to rules. The precise formulation of "platonism" is trickier, especially in a context where the notion of *objecthood* is in dispute: when we have to navigate between positions like Kant's, who has a thick notion of an object as "that in the concept of which the manifold of a given intuition is united" (B137), and more recent positions in the philosophy of mathematics, which often operate with a thinner, formal notion according to which one is speaking of an object whenever one uses the apparatus of singular terms, identity, and (or perhaps: or) quantification.

Michael Resnik has characterized (ontological) platonism as the view that ordinary physical objects and numbers are "on a par" (*Frege and the Philosophy of Mathematics*, 162). Described at this level of generality, then there is a clear sense in which a Kantian cannot be a platonist. The fact that *mathematical* representations can be meaningful and can enter into judgments that purport to be true, has to be explained in terms of the primary relationship between representations and *physical objects*.

> If it is not possible to prove that the system of pure concepts of the understanding is the necessary condition under which we can speak of a rule and connection of appearances, and under which we can speak of empirical 'Nature'—then this system, with all its consequences and conclusions, would have to still appear as a mere "figment of the brain" [Hirngespinnst]…The logical and mathematical concepts should no longer constitute tools [Werkzeuge] with which we build up a metaphysical 'thought-world': they have their function and their proper application solely within *empirical science* itself.[44]

The Kantian anti-platonism comes with the insistence that the meaningfulness of mathematics depends on the fact that propositions about mathematical objects play an essential role in the sciences of physical objects—that in mathematics, "we are dealing in no sense with some transcendent object, but only with the objective certainty of our empirical knowledge itself."[45] Further, for Cassirer as for Kant, natural science requires mathematics *essentially*. Attacking the "if-thenist" theory of geometry defended by Russell in his *POM*, Cassirer writes: "Logical and mathematical propositions may be of a purely hypothetical validity: but is it merely a 'lucky accident' that these hypotheses prove adequate to direct empirical 'facts' and to determine their course in advance?"[46] For Cassirer, as for Kant, mathematics provides a system of conditions for the very possibility of even representing an object in empirical science.[47]

I have not attempted to spell out what it means to say that mathematics provides a system of conditions for the very possibility of even representing an object in natural science—either in Cassirer, or in Kant.[48] But what I do want to emphasize is that the

---

[44] KMM, 42-3.
[45] KMM, 48. This point of view will remind some readers of Quine.
[46] KMM, 44.
[47] See the chapters in the *Critique*, "Axioms of Intuition" and "Anticipations of Perception." This is a dominant theme in Cassirer's Kant interpretation. See *EP*II, *passim*, *KLT*, 162, and *PSF*3, 11.
[48] A dominant theme in Cassirer's attempt to cash out the Kantian idea is that science requires "essential idealization," where we "transform the sensuous manifold into a mathematical manifold, *i.e.*,…we let it issue from certain fundamental elements according to rules held as unchangeable." Cassirer continues: "The logical postulate inherent in the [mathematical] axiom cannot be directly fulfilled in any sensuously

Marburg Neo-Kantians' desire to understand science, but not condemn or interfere with it, in the context of increasingly abstract mathematical structures like the complex projective plane, introduces a tension within broadly Kantian approaches.

> Obviously, on this point we are dealing with a general conflict that even today is unsettled. If one considers the development of modern mathematics, one sees above all that the tendency emerges in it to allow itself to be led merely by the demands of inner logical consistency – without being distracted by the question of its possible applicability…No one will contest the right of this claim, no one will be permitted to try on philosophical grounds to set limits to the freedom of mathematics, which is the condition of its fruitfulness. But, nevertheless, the critique of cognition begins with the question that the mathematician does not know and does not need to know. The particular problem of the critique of cognition is not the content of mathematical principles, but rather the role which they play in the construction [Aufbau] of our concept of an 'objective' [gegenständlichen] reality [Wirklichkeit].[49]

Cassirer's general strategy for dealing with this problem is two-fold. First, mathematical concepts that at first seem paradoxical and incapable of application to the physical world—like those of complex analysis—have in fact been employed in physics. These cases show that Kant's particular theory of how mathematical concepts get applied will have to be revised. For him, the application of the concept <triangle> is a relatively straightforward matter: some objects in space and time look like triangles. But after the revolution in nineteenth century geometry, "in place of such a sensuous congruence we must substitute a more complex and more thoroughly mediate relational system."[50] How exactly modern mathematics gets applied will have to be worked out in case studies. But

---

or intuitively existing content; but the deficiency is compensated for by the fact that this same postulate determines and literally defines a content appropriate to it. No physical 'law' can be stated and strictly grounded without such a method of substitution. The 'classical' theories of science offer instances of the process of transition to the limit at every step. As examples, we need only mention such concepts as those of the 'rigid body,' 'the ideal gas,' the 'incompressible liquid,' the 'perfect circular process,' etc. It is only through the intellectual metamorphosis which they undergo in this method that the contents of immediate observation first become possible subjects for physical judgments." (*SF*, 261).

Cassirer here is indebted to the analysis in Karl Pearson, *The Grammar of Science*.

[49] KMM, 47-8.

[50] *ETR*, 43.

a lesson of modern physics is that, just as a modern Kantian should not view the concepts and laws constitutive of science at a given stage as constitutive of all scientific cognition in general, so too we should not prescribe ahead of time how mathematics, and thus what kind of mathematics, can be applied in physical science. The second aspect of his strategy is to emphasize the considerations given in section three. Since ideal elements are introduced precisely because they unify existing mathematics in such a way that it seems appropriate to say that they reveal the natural home or essential core of more familiar mathematical objects, any use of less abstract mathematics in physics will bring with it all of the more abstract mathematics. Since mathematics is a unity, if mathematical physics forces us to accept some mathematics, it forces us to accept it all.

> We must either decide to brand all of mathematics a fiction, or we must, in principle, endow the whole of it, up to its highest and most abstract postulations, with the same character of truth and validity. The division into authentic and inauthentic, into allegedly real and allegedly fictive elements, always remains a half measure which, if taken seriously, would have to destroy the methodological unity of mathematics.[51]

6. Conclusion

There are various ways in which Cassirer's reworking of Kantianism to make room for the ideal elements in geometry is incomplete or unsatisfying. A comparison with Frege

[51] *PSF*3, 400-1. Again, this strategy will remind some readers of Quine. Quine allows that the higher reaches of set theory are contentful only because they are couched in the same grammar and vocabulary as the genuinely contentful parts of set theory: "We are just sparing ourselves the unnatural gerrymandering of grammar that would be needed to exclude them" (*Pursuit of Truth*, 94). He recognizes the existence of indenumerable sets only "because they are forced on me by the simplest known systematizations of more welcome matters" ("Reply to Parsons", 400). The considerations discussed in section three give Cassirer a way of explaining why the more abstract parts of mathematics are the best "systematizations" of more readily applicable mathematics.

Penelope Maddy (*Naturalism in Mathematics*) has argued that Quine's philosophical views lead him to a revisionist attitude toward the highest reaches of set theory. Cassirer would find this unacceptable.

will make these shortcomings clear. Frege recognizes that a significant portion of the historical development of the sciences, including mathematics, consists in a "history of attempts to grasp a concept,"[52] and he thinks it possible that there be expressions whose senses, though grasped by earlier mathematicians, can be fully understood only by later mathematicians. He also distinguishes between cases where a development in mathematics is best seen as achieving a more complete understanding of a concept that had been incompletely grasped by earlier mathematicians, and cases where mathematicians come to use a word so differently from earlier mathematicians that it is better to see mathematicians as grasping a different concept and shifting the sense of a concept word.[53] Cassirer, on the other hand, does not seem to appreciate the real philosophical difference between these two cases: he seems not to distinguish between a grasp of one concept "developing out of" the grasp of a *different* concept, and a later mathematical theory making explicit what was "implicit in the meaning" of a univocal concept expression.

There is a second respect in which Cassirer's picture is unsatisfying. Cassirer does not think that the virtues of unification or increase in intelligibility are sufficient for justifying the introduction of a new conceptual system and domain of objects. But though he of course realizes that such a system has to be consistent, he does not seem to appreciate how significant a hurdle it is to prove the existence of a mathematical object. Again a comparison with Frege will bring this out. Frege was well aware of the payoff that introducing new points to the plane provides. Indeed, both Frege and Cassirer think that it is best to understand points at infinity, following von Staudt, as directions of lines

[52] Frege, "The Law of Inertia," in *Collected Papers*, 133; original publication, 158.
[53] On this point in Frege, see Tyler Burge, *Truth, Thought, and Reason*, 55-6.

in a plane.[54] But while Cassirer was content to speak loosely of points at infinity as "expressions of relations" between lines, Frege instead thought that we need to identify points at infinity with the extension of the concept <parallel to the line *a*>.[55] Cassirer himself was hostile to extension-theoretic or set-theoretic foundations for mathematics, and he downplayed Dedekind's use of "systems" (essentially, sets) in the foundations for arithmetic. Like it or not, though, set theory has become for mathematics the final arbiter of existence questions, and Cassirer does not seem to adequately appreciate the real technical worries that have driven many philosophers in the last century toward a set-theoretic platonism that Cassirer would find so uncongenial.

There is a further worry internal to Cassirer's Marburg Neo-Kantianism: Cassirer's emphasis on the progressive nature of mathematics and mathematical concepts threatens to undermine the sharp distinction between the a priori and the empirical that he needs for his account of the objectivity of empirical science. Recall that Cassirer's distinction between the empirical and the a priori is a distinction between what persists throughout the development of science, and what does not (*SF*, 269). Since Cassirer thinks that pure mathematics lacks the constant revisability of empirical science and that the mathematical form of scientific theories is stable over theory change (*SF*, 323), the a priori nature of mathematics plays an essential role in his account of the rationality and objectivity of empirical science. The worry is that Cassirer's recognition that mathematics is a "synthetic progressive science" will undermine his claim that the historically evolved products of mathematical science are permanent in a way that the

---

[54] Von Staudt, *Geometrie der Lage*, 1847. In Frege, see *Foundations of Arithmetic*, §64. In Cassirer, see *PSF*3, 396-7.

[55] *Foundations of Arithmetic*, §§66-8. I am not giving here a reading of Frege's argument in these sections, nor am I endorsing what Frege says in them.

theories of empirical science are not. This, in turn, could make problems for Cassirer's account of empirical rationality and objectivity in terms of the permanence of its mathematical form.[56]

In the opening of this dissertation, I said that an understanding of the philosophical environs of early analytic philosophy requires knowing which of Kant's doctrines the Kantians of the period thought inspirational and which hopeless. One might argue that, given Cassirer's willingness to discard and re-read large portions of Kant's *Critique*, there is not enough left in Cassirer's Kantianism for it to deserve to be called "*Kantian*." Though I am *qua Kant scholar* sympathetic to this, a principal finding of my historical research is that, in a period like the end of the nineteenth and beginning of the twentieth century, when Kant interpretation is so entwined with systematic philosophy, there is no real answer to the question what a modern Kantianism should be. And it is historically interesting and surprising to see that one of the most important and perceptive of the self-identified Kantians fits few (if any) of our preconceived notions of what a Kantian should look like. This dissertation has shown, though, that the unorthodox positions that Cassirer and the Marburg Neo-Kantians were driven to were not unmotivated: they were responding to the full challenge of both of geometry's revolutions, and were trying to stay faithful to the respect for the exact sciences first typified in Kant's *Critique*.

[56] On this point, see Smart, "Cassirer's Theory of Mathematical Concepts."

APPENDIX A

Appendix to Chapter 3: Frege, Lotze, and Boole: Concept Formation and the 'Functional' View of the Concept

In chapter 3, I showed that the discussion in the philosophy of mathematics that we find in Cassirer's *Substanzbegriff und Funktionsbegriff* begins with a polemic against the traditional Aristotelian-abstractionist [A-a] model of the structure and formation of concepts. Cassirer then uses the anti-abstractionist theory of the concept to motivate an ordinalist philosophy of arithmetic and a criticism of Frege and Russell's theory of number. I noted further that this criticism is less than convincing as an attack on Frege, and ended the chapter by noting that Frege himself had argued for the superiority of his begriffsschrift over the logics of the Booleans on the grounds that it alone avoids the problems with the abstractionist model of concept formation.

I think that it is historically significant that both Frege and Cassirer give the same odd-sounding argument about conceptual structure and concept formation, even though the irony that Cassirer turned this objection against Frege himself is a bit distracting on this score. More specifically, I think that looking at Cassirer's attack on abstractionism and Frege's attacks on the Booleans sheds some light on an assumption they share about what a good philosophy of mathematics is to deliver and a shared intellectual background in the writings of late 19th century (philosophical) logicians. As we saw in chapter 3, Cassirer identifies Hermann Lotze as a key predecessor in the attack on abstractionism. Further, some Frege scholars have alleged that Lotze's philosophy was important for Frege's early development—that is, when he was first developing the begriffsschrift and writing his essays on the Booleans. This suggests that we can gain some insight into Frege's philosophical environs and the possible influences on him by looking closely at Frege's anti-abstractionism and polemic against the Booleans, and then comparing it to Lotze's anti-abstractionism and similar polemic against the Booleans.

This appendix has nine sections. In the opening section, I discuss Frege's response to the Boolean logician Ernst Schröder's allegation that the begriffsschrift is not, as Frege seemed to think, a "*lingua characterica*." I argue that for Frege, the precise analysis of mathematical concepts, the determination of the basic laws of mathematics, and the construction of gap-free proofs are all essentially connected elements in Frege's project of determining what the "springs of knowledge" are from which mathematical science grows. This project then requires a formula language, like Frege's begriffsschrift, that can act as a universal language or characteristic, rather than as a mere calculus. In the second section, I show that Frege's new analysis of generality – an analysis necessary for any remotely adequate analysis of mathematical concepts or any construction of gap-free proofs from basic laws – requires a new, anti-abstractionist theory of concept formation. Frege expresses his commitment to this new theory in his well-known and provocative claim that he starts with completed judgments and analyzes them into component concepts—and he does not start with completed concepts in order to build judgments from them. (In the third section, I clarify my reading by distinguishing it from the readings of this "priority principle" given by Dummett and Sluga.) But since the Booleans are committed to the traditional A-a model of concept formation, Frege is able to show that their work could never provide a characteristic language and could never play the role that the begriffsschrift does in allowing us to discover the springs of knowledge from which mathematical knowledge grows (section A.4). Indeed, Frege's criticism of the Booleans' model of concept formation is, I show, an instance of what in chapter 3 I called the "Lotze objection" to the A-a model: the A-a model is inadequate to express the content of specifically mathematical concepts because (1) it cannot explain how a full grasp of the concept allows one to infer substantive mathematical judgments that are not obvious to someone who has already grasped the concept and its components, and (2) it can only allow for compound mathematical concepts that are composed from simpler concepts by addition, conjunction, and negation only.

In sections five through seven, I consider the parallel discussions in Hermann Lotze's *Logic* (first edition: 1874, second edition: 1880) – one of the most widely read and influential logical works written by a contemporary of Frege's. (Lotze was also a philosopher at Göttingen when Frege was a graduate student there. The only graduate course in philosophy that Frege took was taught by Lotze.) I begin my discussion of Lotze's logic by placing his theory of concepts in the context of his conviction that it is the task of logic to investigate what thought does when it distinguishes the *accidental* streams of ideas within the mind from those streams of ideas where the ideas are connected in the mind in such a way as to track the *truth*. (It is this fundamental idea of Lotze's that Frege addresses in his early fragment "17 Key Sentences on Logic.") This concern motivates Lotze to give a “functional” theory of the concept. Concepts formed according to the A-a model, on the other hand, not only have a simple structure – they are just lists of marks – but, because of this simple structure, concepts formed in the traditional way also enable us to infer little that is of value about the world. In the next section, I show how Lotze applies this “functional” theory of the concept in his account of mathematical concepts. Lotze thinks that the structure of concepts and the kinds of inferences that they make possible are linked: if one only uses concepts formed in the traditional way, one can only ever employ syllogistic inferences; but if one uses non-syllogistic inferences, one needs to have concepts that have a richer than Aristotelian structure and are formed in a non-abstractionist way. Since mathematics, as he shows, employs kinds of inferences not reducible to syllogisms, mathematical concepts cannot be formed or structured in the traditional A-a way. In fact, the concepts mathematics requires have a richer structure than any other concepts. Thus, mathematical concepts do not have less content the more general they are (as the traditional model would have it), but just as much, since they provide a rule for deriving the content of all the particulars that fall under them. In section seven, I show how Lotze uses this theory of concepts to criticize the Booleans in his 1880 “Note on the Logical Calculus.” Like Frege, he argues that the Booleans falter precisely because they rely on the traditional model of concept formation by abstraction and the corresponding Aristotelian (or, we might say, “Boolean”) model of conceptual structure.

Frege’s attacks on the Booleans were written a year or two after the publication of Lotze’s attack on the Booleans, and we have independent reason – given by Frege’s “17 Key Sentences” – to believe that Frege was reading Lotze’s logic in these years.

Moreover, both attack the Booleans for employing the traditional A-a model of concept formation and structure; both argue that mathematical concepts in particular fail to accord with the model; and both argue for a richer-than Aristotelian theory of the structure and formation of concepts, self-consciously modeled on mathematical functions. This gives us good circumstantial reason (and we can expect nothing less modest in Frege studies) to believe that Frege knew Lotze's attack on the Booleans and that his polemic was significantly influenced by his reading of Lotze's "Note." (Or so I argue in section eight.) But even if the historical evidence of a causal connection between what Frege wrote and what Lotze wrote does not convince, I show in section nine that the close look at Frege's and Lotze's polemics against the traditional A-a model and their criticisms of the Booleans puts us in a position to better evaluate the claims made by some Frege scholars for a substantial influence of Lotze on some of Frege's more celebrated doctrines: his logicism, his context principle, and his anti-psychologism. I argue that a close reading of Lotze's *Logic* and the theory of mathematical concepts and inferences in it shows in fact that the "logicism," "context principle," and "anti-psychologism" that we find in Lotze are fundamentally different from what we find in Frege. Though, I think, Lotze and Frege were allies in their attack on the A-a model presupposed by the Booleans and in their conviction that logic had to be reformed in order to understand specifically mathematical thinking, the agreement or influence in these more weighty matters – despite the initially encouraging appearance of their texts – is little more than verbal.

A.1 FREGE'S PROJECT IN *BEGRIFFSSCHRIFT*

What I called "the Lotze objection" in chapter 3 alleges that the A-a model is inadequate to capture the *content of mathematical concepts,* since compound mathematical concepts are composed of simple concepts in more complicated ways than the A-a model allows and since mathematical concepts are rich enough to contain all of the mathematically relevant content of all the particulars falling under them. To Frege, the merit of the begriffsschrift is that it is a suitable tool for the perspicuous expression of the *conceptual content of mathematical concepts*. It is intended to be, as Frege first says in 1879, a characteristic language[1] in Leibniz's sense; or as he elsewhere puts it, a "lingua characterica," a language that "expresses a content through written symbols in a more precise and perspicuous way than is possible with words," and that "compound[s] a concept out of its constituents rather than a word out of its sounds."[2] Leibniz's idea, and the terms "*lingua characterica*" and "*Begriffsschrift*," came to Frege through his reading of Trendelenburg, whose 1867 essay "On Leibniz's Project of a Universal Characteristic" Frege cites in *Begriffsschrift*.[3] As Trendelenburg explains the Leibnizian project, the signs in a *lingua characterica* do not relate to the things signified in a merely *external* way, through a *psychological* association of certain letters with sounds and certain sounds with words, introduced by custom and reinforced through habit. Instead, the relation of

---

[1] *Begriffsschrift* [*Bgs*], v-vi, translated by Michael Beaney in *The Frege Reader* [Beaney], 50.

[2] "On the Aim of the '*Begriffsschrift*,'" *Conceptual Notation and Related Articles* [*CN*], 90-1; "Boole's Logical Calculus and my Conceptual-notation, " *PW*, 9. See also "On Mr. Peano's Conceptual Notation and My Own," in *CP*, 235; Frege's notes to Jourdain's "Gottlob Frege," *PMC*, 193. Though Frege refers to Leibniz's "universal characteristic" in *Begriffsschrift*, he does not emphasize the distinction between the "lingua characterica" and the "calculus ratiocinator" until after Schröder makes the distinction in his 1880 review of Frege's book.

[3] This essay appeared in *Historische Beiträge zur Philosophie*, vol 3, Berlin 1867. The history here is complicated: Leibniz's own term is "*lingua characteristica*," but Frege uses the term from Trendelenburg. For historical details, see Patzig, G. "Leibniz, Frege, and the so-called 'Lingua Characterica Universalis.'" See also Wolfgang Carl, *Frege's Theory of Sense and Reference*, 9 and Sluga, Hans, *Gottlob Frege*, 48-52.

sign to signified is both *internal*, with the *logical* structure of the language mimicking the conceptual structure of the signified concepts, and *direct*, without any intermediary sounds or syllables. In an early essay, "On the Scientific Justification of a Conceptual Notation," Frege argues that his begriffsschrift overcomes the deficiencies in ordinary word-language—its ambiguity and unsuitability for constructing gap-free proofs—by providing "a system of symbols from which every ambiguity is banned, which has a strict logical form from which content cannot escape."[4] These symbols are written, not spoken, and so provide the permanency characteristic of concepts and also make available the two-dimensions of a page to represent conceptual structure in a richer and more perspicuous way than the one-dimensional flow of spoken sounds.[5] Most importantly, its symbols for generality, negation, conditionals, functions, and relations are "suitable for combining most intimately with a content."[6] They provide the necessary tools for expressing content and carrying out inferences not only in pure logic, but also in mathematics, and potentially in all areas of exact science.[7]

In an 1880 review of *Begriffsschrift*, the logician Ernst Schröder, whose own 1877 treatise in symbolic logic, *Der Operationskreis des Logikkalküls*, reworked in a terse forty pages the methods of Boole and the brothers Grassman,[8] contested just this claim. The title of Frege's work, "*Begriffsschrift*," does not correspond to the content of the

---

[4] *CN*, 86.
[5] *CN*, 87; see also "On Mr. Peano's Conceptual Notation and My Own," *CP*, 236. The 2-dimensionality of Frege's script was a frequent cause of complain among reviewers, including Schröder, Venn, and even Russell. Frege's reply—that though the 2-dimensionality makes the begriffsschrift a cumbersome tool for calculating, its advantages in expressing conceptual content more than compensate—is philosophically interesting, but fails to convince.
[6] *CN*, 88.
[7] See *CN*, 89; *Begriffsschrift*, vi, in Beaney, 50.
[8] See Grattan-Guinness, 160-1; Adamson, *A Short History of Logic*, 157, and his review in *Mind* old series 3 number 10, 252-5.

book: a general "Begriffsschrift," or universal characteristic, requires much more than Frege delivers:

> I believe that I do not depart from the historical interpretation by formulating the problem in the following way (*mutatis mutandis* for the various basic fields of knowledge): *to construct all complex concepts by means of a few simple, completely determinate and clearly classified operations from the fewest possible fundamental concepts [Grundbegriffen] (categories) with clearly delimited extensions*.[9]

Leibniz himself, in one of his many proposals to carry out his project, recommended that in such a language simple, fully analyzed concepts be represented by prime numbers and complex concepts be represented as composite numbers whose unique decomposition into primes gives the unique, complete analysis of the concept. In this way, the complete content of concepts can be easily expressed through factorization. But such a project, Schröder notes, requires first a complete analysis of concepts into basic concepts or categories and a proof that all of the relevant content of all other concepts can be captured in the one operation of multiplication (or its inverse, factorization). Frege gives neither of these: he does not analyze concepts and thereby isolate the categories whose combinations exhaust the conceptual content of any field of knowledge and he does not even show that his symbolism would allow one to carry such analyses out. In fact, Schröder alleges, Frege's project is closer to a related Leibnizian project, the development of a *calculus ratiocinator*,[10] a calculus or system of symbols to be used in

---

[9] "Review of Frege's *Conceptual Notation*," *Zeitschrift für Mathematik und Physik*, 25 (1880). Translated and reprinted in *CN*, 219.

[10] According to Sluga ("Frege against the Booleans," 83), Leibniz himself described a logical calculus as "a mechanism for determining the truth of our assertions," "a production of relations through the transformation of formulas according to determinate laws." Incidentally, I cannot agree with Sluga's contention that "both [Frege and Schröder] thought that the logical symbolism *should* be a characteristic language; their disagreement was over what such a language should look like" (83). There is no indication in Schröder's review that Boole's, or his own, logical calculus is intended to be anything more than a *calculus ratiocinator* (given his familiarity with Trendelenburg's Leibnizian distinction, it is relevant that

carrying out deductive inferences and not necessarily for expressing content.[11] Thus, Frege's project is the same as Boole's and Schröder's—both attempt to construct a symbolism that allows for easy solution of logical problems. This makes Frege's achievement marginal, though, since Frege seems ignorant of the methods of earlier logicians in the Boolean tradition and has succeeded only in producing what is more or less "a *transcription* of the Boolean formula language," and an extremely cumbersome one to boot.[12]

Frege responded to Schröder's review with a series of papers that maintained that the begriffscrift *does* differ from Boolean symbolic logic both in its aim and in its power.

> If I understand him aright, Boole wanted to construct a technique for resolving logical problems systematically … In contrast we may now set out the aim of my begriffsschrift. Right from the start I had in mind the *expression of a content*. What I am striving after is a *lingua characterica* in the first instance for mathematics, not a *calculus* restricted to pure logic.[13]

Indeed, it is an essential part of Frege's overall intellectual project that the begriffsschrift be adequate for the full expression of conceptual content. As he explains in a later

---

he titles his work "Operationskreis des Logikkalküls".) When Schröder refers to previous attempts to carry out a characteristic, he mentions various attempts to isolate lists of categories—a very different project from his own. Indeed, he admits that the Boolean approach does not aim to replace traditional logic but to "allow the greatest part of formal logic to date to appear in a new and wonderfully clear light" (223)—a far cry from a universal characteristic! (Sluga's citation of Schröder's introduction to his later *Vorlesungen über die Algebra der Logik*, is besides the point, since that work appeared later in 1890, after a decade of assaults on the narrowness inherent in the theory of the concept presupposed by Boolean procedure.)

[11] The two are of course closely linked: one cannot carry out the full range of deductions unless the conceptually relevant content of a judgment is expressed in the symbols of the *calculus ratiocinator*, and one cannot isolate the *conceptual* content of a judgment unless one reflects on the range of inferences involving the judgment. For Frege, the conceptual content of a judgment is that part of its content that has significance for *logical inference* (*Bgs*, iv; §3; Beaney, 49; 53-4). In his 1897 draft, "Logic," he argues that a "thought" is that part of what a sentence expresses that is relevant to its *truth or falsity* (*PW*, 129). (Clearly this is an ancestor of the earlier claim in *Bgs*, since after 1892 Frege calls the sense of a sentence a "thought," and the sense of a sentence is a part of what Frege earlier called its content.) In *Bgs* and "Logic," Frege illustrates these distinctions by showing that employing the active or passive forms of a sentence make no difference in inferences—and so no difference to its conceptual content—or, as he later puts it, to the sentence's truth—and so to what thought the sentence expresses.

[12] *CN*, 221.

[13] "Boole's Logical Calculus and the Concept-Script," 12.

review of Peano's conceptual notation, he fundamentally wants to know what the "springs of knowledge" are from which mathematical science grows.[14] In order to answer this fundamentally epistemological question,[15] Frege needs to isolate the fundamental principles or axioms of mathematics. In order to isolate these principles, Frege needs a way of determining whether a candidate derivation of some theorem is free of gaps or in fact requires some unrecognized further concept or principle. Since even Euclid, long thought the model of rigorous axiomatic reasoning, was led astray by the imprecision of ordinary language to assume certain principles without acknowledgment,[16] it is clear that some other instrument—a logically improved language—is needed to carry out the project. Only a gap-free proof will make it possible not only to know *that* a

---

[14] See "On Mr. Peano's Conceptual Notation and My Own," in *CP*, 235.

[15] To say that Frege's fundamental concern is epistemological does not rule out that concern's being itself motivated by conceptual problems within mathematics. If it turns out that the springs of arithmetical knowledge are logical, then such knowledge will have universal scope within mathematics. Further, if the study of complex numbers and their properties turns out to rest only on the study of natural numbers (and so also on logical laws concerning logical objects), then the extended use of complex numbers in projective geometry—such as the kind that Frege studied earlier in his career—will be fully vindicated.

Frege expresses a concern over the status of complex and infinitary points already in his 1873 Göttingen doctoral dissertation, "On a Geometrical Representation of imaginary Forms in a Plane" (in *CP*, 1ff). (Göttingen was the academic home of Gauss from 1807 until his death in 1855, whose work included a geometric representation of complex numbers; see *CP*, 55.) There he worries that "a 'point at infinity'" is a contradiction in terms, and he falls back on the von Staudt method of translating talk of infinitary points as directions "represented" as a point. Imaginary points—points whose coordinates are given by complex numbers where the imaginary element does not vanish—likewise, he argues, have to be viewed as elements definable in geometrical terms. His dissertation, then, is centered around an attempt to use metrical methods to make the imaginary elements geometrically representable on the real plane. This topic is important, since mathematicians in Frege's day wanted to treat the complex projective plane as an object of mathematical study, and so, as Frege says, "it is now of the utmost importance to find out when a proposition which holds for real forms can be carried over to imaginary ones" (*CP*, 2). (His proposals, to represent each imaginary point by the magnitude of an angle in the real projective plane or a pair of real lines, naturally led him toward representations of elements in the real projective plane in the complex projective plane; see his 1878 "Lecture on a Way of Conceiving the Shape of a Triangle as a Complex Quantity," in *CP*.) For an extended discussion of these topics, emphasizing the methodological issues raised by the geometrical study of the complex projective plane and Frege's response to von Staudt's method, see Mark Wilson, "Frege: the Royal Road from Geometry."

[16] The reference to gaps in Euclid is from "On the Scientific Justification of a Conceptual Notation," in *CN*, 85. Significantly, he does not attribute this gap to Euclid's reliance on intuition, but to the deficiencies in ordinary language. From his 1873 dissertation forward, Frege remained committed to a loosely Kantian view that geometry is based on the intuition of space. See his late (1924/5) "Source of Knowledge of Mathematics and the mathematical natural Sciences," *PW*, 267ff.

theorem is true, but also *what justifies* the truth, whether a non-logical or logical source. For this reason, "inferences need to be resolved into their simple components."[17] With all of these elements in place, it is easy to see that the precise analysis of mathematical concepts was indeed part of the program. The determination of the springs of mathematical knowledge requires a precise determination of the basic laws of the elements of mathematics, that is, their axioms, and a precise system for representing gap-free proofs. This would involve isolating all of the content of mathematical judgments that is necessary for the inferences from its basic laws. But for Frege, the *conceptual* content of a judgment just is the content of the judgment that plays a role in inference, and so any incomplete analysis of the fundamental concepts out of which mathematical judgments are composed will lead to an incomplete analysis of the conceptual content of the judgment. Thus, a deficiency in the analysis of mathematical concepts will result in a deficient determination of the conceptual content of a mathematical judgment; and these deficiencies will be reflected in an inference that either contains a gap or is insufficiently general. But this in turn will undermine any attempt to establish the basic laws of mathematical science and frustrate Frege's epistemological goal. So, for Frege, the precise analysis of mathematical concepts, the determination of the basic laws of mathematics, and the construction of gap-free proofs are all essentially connected elements in the same project.[18]

---

[17] *CP*, 235.

[18] I think that this interpretation of Frege's project explains why conceptual analysis was necessary for him. But even if the interpretation falls, the point stands on ample textual evidence. In this same review of Peano, Frege notes that Peano introduces the arithmetical operators straightaway, "from which it is to be gathered that an analysis of these logical structures into their simple components was not the intention. And, since, *without such an analysis, an investigation like the one I projected is impossible*, such an investigation could not have been among Mr. Peano's intentions." (*CP*, 237, emphasis mine.)

How then is Frege's begriffsschrift itself an integral or key component in the development of a universal language or characteristic and not just a calculus whose service in such a language has to wait for an independent, say philosophical, analysis of mathematical concepts? The answer is that the same logical machinery that has to be in place to express gap-free mathematical proofs has to be in place also for the expression of the conceptual content of mathematical judgments. Moreover, it is also sufficient. So when Frege gives his famous analysis of the ancestral in *Bgs* §25, and then gives a series of theorems involving that concept, along with their proofs, he is not just illustrating the greater power of his symbolism to carry out secure mathematical calculations; he is demonstrating the capacity of his begriffsschrift to serve as a *lingua characterica* for mathematics. (Similarly for his expression of a theorem in number theory and his proof that the sum of two multiples of a number is in its turn a multiple of that number.)[19] Against Schröder, the fact that Boolean logics are unable to express these theorems in a way that proofs can be given for them shows, at the same time, that the begriffsschrift is not a "transcription" of Boole's method and that the Boolean logical calculus cannot be a successful *lingua characterica*. Moreoever, Frege's analysis of the ancestral using the begriffsschrift goes all the way down to purely logical terms, as far as any analysis can go. These successes make it very likely, then, that the begriffsschrift provides the necessary tools for the construction of a complete *lingua characterica* for mathematics and any other science whose arguments are expressible in a gap-free way. Of course,

[19] See "Application of the 'Conceptual Notation'" (1879), *CN*, 204ff; "Boole's Logical Calculus and the Concept-Script," *PW*, 27-31.

Frege does not pretend that he has full analyses of concepts in hand, but he thinks that the development of the begriffsschrift constitutes the first hard step in carrying them out.[20]

Trendelenburg had argued that Leibniz's project could not be carried out until a prior philosophical project, the analysis of all concepts into their simple components and the identification of a system of categories, was completed—quoting Descartes, "the invention of such a language depends on the true philosophy."[21] Similarly, Schröder argues that the Leibnizian project requires a "proof…that, in fact, through the combination of the fundamental concepts which [one] lay[s] down, all the remaining concepts follow."[22] How then can Frege prove that his analysis of logical notions into the concept of generality, negation, the conditional, the content stroke, the judgment stroke, the concept of a function, and identity[23] is sufficient to represent all of the content of logical knowledge (and all of arithmetical knowledge, if it turns out to be logical) and all of the steps in a deductive inference? Schröder's reference to the philosophical project of determining a complete list of categories reminds us of a similar challenge made by Kant. Aristotle, Kant alleges, gave no argument that his list of categories is either complete or fully analyzed, but instead could only appeal to "induction, without reflecting that in this way one could never see why just these and not other concepts should inhabit the pure

---

[20] See "On the Scientific Justification of a Conceptual Notation," 88-89. See also "On the aim of the 'Conceptual Notation,'" 93:

> I wish to blend together the few symbols which I introduce and the symbols already available in mathematics to form a single formula language. In it, the existing symbols [of mathematics] correspond to the word-stems of ordinary language; while the symbols I add to them are comparable to the suffixes and deductive form-words that logically interrelate the contents embedded in the stems.

[21] Trendelenburg, A. "Über Leibnizens Entwurf einer allgemeinen Characteristik," *Historische Beiträge zur Philosophie*, vol 3, Berlin, 1867, 8. Quoted in Sluga, "Frege against the Booleans," 85.

[22] Schröder, Review, in *CN*, 219.

[23] These are the fundamental concepts expressed by the begriffsschrift that Frege isolates in chapter I of *Bgs*. This group of fundamental concepts is not the same group that Frege isolates in *Grundgesetze*, vol. 1: there Frege, inter alia, introduces extensions of concepts and value-ranges of functions.

understanding" (A81/B107). Kant justified his choice of the categories by referring them back to a "common principle," the faculty for judging and the twelve possible logical functions of judging, arranged under four headings. Now Kant's project of giving a complete list of the simple pure concepts of the understanding and the a priori principles of their employment by the understanding is of course different from Frege's project of isolating the fundamental logical notions so as to carry out gap-free proofs and fundamental analyses of mathematical concepts. But Kant's approach is an option for a philosopher wanting to carry out Frege's project: perhaps the fundamental logical notions and fundamental laws of deductive inference are discoverable through an analysis of what it is to judge or be a judgment.[24] Importantly, Frege does not take this route. Similarly, Frege does not attempt Wittgenstein's transcendental argument in the *Tractatus* from the possibility of language. Nowhere does Frege attempt to justify his approach through an analysis of the fundamental acts of the mind—either psychologically, as Sigwart does by taking the subjective feeling of the "*immediate consciousness* of evident truth which accompanies necessary Thought" as a primitive guide to what propositions are "necessary and universally valid";[25] or genetically, as

---

[24] There is of course a great deal of controversy concerning how the metaphysical deduction of the categories is to be understood—not only about how the twelve categories are derivable from the table of judgments, but also about how Kant arrives at the table of judgments itself. Some commentators allege that Kant's claim that "the labors of the logicians were ready to hand" (*Prolegomena* §39, Ak 323) amounts to little more than the quasi-induction Kant accused Aristotle of relying on. Others follow Cohen and read the Analytic in reverse order, arguing that the form the table of judgments takes is determined by the *a priori* principles that Kant isolated in his transcendental reflection on the possibility of exact (Newtonian) science. (This approach is closer to the early Frege's approach, since Frege arrives at his fundamental logical concepts and laws by reflecting on what inferences are required in mathematics.) For a reading that takes Kant's argument seriously and reads the Analytic from beginning to end—along with a helpful review of the history of commentary on Kant's argument in "On the clue to the discovery of all pure concepts of the understanding"—see Longuenesse, *Kant and the Capacity to Judge*.
[25] Sigwart, *Logic*, vol. 1, 14-15; §3.

Lotze does when he follows out what mental operations are at work in the progression from simple sense impressions to fully conceptualized thought.[26]

Having shown how to express a number of fundamental mathematical concepts using the begriffsschrift—including being a factor, congruence with respect to a modulus, and continuity of a function—Frege writes:

> All these concepts have been developed in science and have proved their fruitfulness. For this reason what we may discover in them has a far higher claim on our attention than anything that our everyday trains of thought might offer. For fruitfulness is the acid test of concepts, and scientific workshops the true field of study for logic. ("Boole's logical Calculus and the Concept-script"; *PW*, 33)

Tyler Burge has aptly described Frege's method as holistic and pragmatic.[27] It is *holistic* because Frege does not think that the logical form of a judgment is apparent to "simple, immediate insight," but is apparent only after reflection on the inferential relationships that the judgment has to other judgments in the area of science where it makes its home. It is *pragmatic* because it depends on a prior familiarity with the systematic inferential practices internal to science. After this immersion in the inferential practices of mathematics—one of the "scientific workshops" that constitute the "true field of study" for logic—, the logician can analyze mathematical inferences into gap-free steps and mathematical concepts into primitives (remembering that these two analyses are

---

[26] This is roughly the method that Lotze introduces in the Introduction to vol. 1 of his *Logic*.

[27] See Burge, "Introduction," in *Truth, Thought, Reason: Essays on Frege*, 62-3:

> Frege's holist method for investigating logical form is simultaneously a method for finding and clarifying logical law. Thus full reflective understanding of a logical law is not adequately characterized simply as immediate recognition of the obvious. Any such recognition is the product of a background of competence in inferential practice and structure. Similarly, intuitive understanding of a logical law presupposes an intuitive but discursive competence in carrying out deductive inference. Frege's reflective method clearly depends on the development of theory—not on putative simple, immediate insight. The method presupposes that even unreflective intuitive understanding is ineliminably entangled with a complex web of standing inferential capacities…Frege's epistemic practice is fundamentally pragmatic. It takes understanding to involve holistic, inferential elements. It is not an epistemology of groundless dogma or of immediate insight.

indissolubly linked), checking her results against the practice of mathematics. As the general concepts and rules that Frege detects in his analysis of mathematics can then be tested in wider fields of science, the fully general concepts and rules that are at play in all areas of science are then isolated as truly logical.[28]

A.2 FREGE'S THEORY OF CONCEPT FORMATION

In his criticism of Frege's *Begriffsschrift*, Schröder argues that Frege has not made any attempt to carry out the sort of program Leibniz had in mind in proposing a *lingua characaterica*: nowhere does Frege seek to analyze the basic categories of (some area of) science and present a method for expressing the conceptual structure of these analyzed concepts directly, e.g. as Leibniz did through prime factorization. But on Frege's view, Leibniz's approach, tied as it is to the traditional model of concept formation, could never succeed in constructing an adequate *lingua characterica*, for two reasons: first, Leibniz and the formal logicians who like him take as their model the syllogism lack an adequate representation of *generality*; second, they *start with given concepts and construct judgments out of them* instead of starting with judgments and decomposing concepts out of them.

> It is true that the syllogism can be cast in the form of a computation…But we can only derive any real benefit from doing this, if the content is not just indicated but

[28] Frege seems to be committed to testing his begriffsschrift against the practices of other scientific fields, although he never attempts to carry out this wider task (he does recommend it ("On the Scientific Justification of a Conceptual Notation," *CN*, 89)). I have briefly mentioned a generality criterion for determining whether a given concept, law or inference pattern is logical; see *Grundlagen*, §3; Dummett, *Frege: Philosophy of Mathematics*, 24. This touches on the difficult problem of determining whether Frege had a method for demarcating logic from other sciences. It is outside of our purposes to discuss this problem here. Interested readers can see Thomas Ricketts, "Logic and Truth in Frege," and a reply in Burge's "Postscript to 'Frege on Truth.'"

> is constructed out of its constituents by means of the same logical signs as are used in the computation…[E]ven if its form made it better suited to reproduce a content than it is, the lack of representation of generality corresponding to mine would make a true concept formation—one that didn't use already existing boundary lines—impossible. It was certainly also this defect which hindered Leibniz from proceeding further.[29]

> What is distinctive of my conception of logic is that I began by giving pride of place to the content of the word 'true,' and then immediately go on to introduce a thought as that to which the question 'Is it true?' is in principle applicable. So I do not begin with concepts and put them together to form a thought or judgment; I come by the parts of a thought by analyzing the thought. This marks off my concept-script from the similar inventions of Leibniz and his successors, despite what the name suggests; perhaps it was not a very happy choice on my part.[30]

Leibniz's method shows how making inferences with syllogisms can be represented in the form of a calculus. "All As are Bs, all Bs are Cs, therefore all As are Cs" becomes "*nmo* is divided by *nm*, *nm* is divided by *n*, so *nmo* is divided by *n*," where "*n*," "*m*," and "o" stand for numbers, each of which represents a mark and the multiplication of which represents a compound concept composed of the marks represented by the multiplicanda. ("All birds (=living thing & perceives & flies) are animals (=living thing & perceives), all animals are living things, so all birds are living things.") But this kind of symbolic language cannot do what the begriffsschrift can—it cannot express the conceptual structure of a concept needed for inferences; it does not construct a concept out of its constituents using the same tools that Frege discovered are needed in inferences. To do so would require at least an adequate way of expressing generality.

---

[29] "Boole's Logical-Calculus and the Concept-Script," *PW*, 35.

[30] "Notes for Ludwig Darmstaedter" *PW*, 253. These notes are dated July 1919. Frege made the point that his logic, unlike that of Leibniz, Aristotle, or the Booleans, starts from judgments and proceeds to concepts, and not vice-versa, from early in his career till very late. See "Boole's Logical Calculus and the Concept-Script" (1880/81), *PW* 16; "On the aim of the '*Begriffsschrift*'" (1882), *CN*, 94; "Frege to Marty, 29.8.1882," *PMC*, 101.

Frege's appeal to the priority of the judgment with respect to the concept is closely tied to the new possibilities for the expression of content made available by the begriffsschrift's way of expressing generality.

> The *x* [in '$2^x = 16$'] indicates here the place to be occupied by the sign for the individual falling under the concept. We may also regard the 16 in $x^4 = 16$ as replaceable in its turn, which we may represent, say, by $x^4 = y$. In this way we arrive at the concept of a relation, namely the relation of a number to its 4th power. And so instead of putting a judgment together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of a possible judgment. ("Boole's Logical Calculus and the Concept-Script", *PW*, 17)

The concepts <4th power>, <4th root of 16>, <base 2 logarithm of 16> are all formed in this example, not by compounding simple concepts by addition (or inclusion and exclusion), but by starting with a relational judgment, '$2^4 = 16$,' and replacing one or more singular terms by variables. When concepts are expressed by relational expressions with one or more singular terms, it is literally the case that they only appear after a full judgment has been made and various components of the judgment are replaced. (These variables can then be bound by the sign for generality to form quantified relational expressions.) Without the use of multiple quantification and substitution of variables for singular terms, Frege argues, one is stuck "us[ing] already existing boundary lines," a phrase aptly suited to a logic whose calculations are representable using Euler diagrams. The two figures 6.1 and 6.2 illustrate the situation as Frege sees it. (The concept used to illustrate Frege's method is the continuity of a function at some real point *a*, an example that Frege himself gives of a "scientifically fruitful concept" that is formed not by reusing already existing boundary lines, but by "combining the old ones together in a variety of

Stage:

0. $[0, 1, 2, 3, 4]$, $[-, |\,|]$, $[f]$, $[<]$, $[\rightarrow, \&, \sim, \forall, \exists]$

1. $4>0$ $3>0$ $1-2$ $f(1)$ $f(2)$

2. $|1-2|$ $f(1) - f(2)$

3. $|1-2|<3$ $|f(1) - f(2)|$

4. $|f(1) - f(2)| < 4$

5. $(|1-2|<3 \rightarrow |f(1) - f(2)| < 4)$

6. $3>0 \;\&\; (|1-2|<3 \rightarrow |f(1) - f(2)| < 4)$

7. $4>0 \rightarrow [3>0 \;\&\; (|1-2|<3 \rightarrow |f(1) - f(2)| < 4)]$

8. $4>0$ $3>0 \;\&\; (|1-2|<3 \rightarrow |f(1) - f(2)| < 4)$

9. $3>0$ $(|1-2|<3 \rightarrow |f(1) - f(2)| < 4)$

10. $\boldsymbol{(|x-2|<3 \rightarrow |f(x) - f(2)| < 4)}$

11. $\forall x(|x-2|<3 \rightarrow |f(x) - f(2)| < 4)$

12. $3>0 \;\&\; \forall x(|x-2|<3 \rightarrow |f(x) - f(2)| < 4)$

13. $\boldsymbol{\delta>0 \;\&\; \forall x(|x-2|<\delta \rightarrow |f(x) - f(2)| < 4)}$

14. $\exists\delta\, (\delta>0 \;\&\; \forall x(|x-2|<\delta \rightarrow |f(x) - f(2)| < 4))$

15. $4>0 \rightarrow \exists\delta\, (\delta>0 \;\&\; \forall x(|x-2|<\delta \rightarrow |f(x) - f(2)| < 4))$

16. $\boldsymbol{\varepsilon>0 \rightarrow \exists\delta\, (\delta>0 \;\&\; \forall x(|x-2|<\delta \rightarrow |f(x) - f(2)| < \varepsilon))}$

17. $\forall\varepsilon[\varepsilon>0 \rightarrow \exists\delta\, (\delta>0 \;\&\; \forall x(|x-2|<\delta \rightarrow |f(x) - f(2)| < \varepsilon))]$

18. $\boldsymbol{\forall\varepsilon[\varepsilon>0 \rightarrow \exists\delta\, (\delta>0 \;\&\; \forall x(|x-a|<\delta \rightarrow |f(x) - f(a)| < \varepsilon))]}$

Example from "Boole's Logical Calculus and the Concept-Script," proposition (13). Notation has been modernized. Concepts are in bold.

Figure 6-1: Frege's *Begriffsbildung* of continuity of a function *f* at *a*

Stage:			
0.	[**A**, **B**, **C**, **D**], [+, ¯, *], [=, €], [0]		
1.	$\mathbf{AB}$ $\overline{\mathbf{D}}$ $\overline{\mathbf{C}}$	$\mathbf{BC}$ $\mathbf{AD}$ $\overline{\mathbf{A}}$ $\overline{\mathbf{D}}$	$\overline{\mathbf{A}}$ $\overline{\mathbf{B}}$ $\overline{\mathbf{C}}$ $\overline{\mathbf{D}}$
2.	$\mathbf{C\overline{D}}$ $\mathbf{\overline{C}D}$	$\mathbf{\overline{A}\overline{D}}$	$\mathbf{\overline{A}\overline{B}}$ $\mathbf{\overline{C}\overline{D}}$
3.	$\mathbf{C\overline{D} + \overline{C}D}$	$\mathbf{AD + \overline{A}\overline{D}}$	$\overline{A}\overline{B} = \overline{C}\overline{D}$
4.	AB € $C\overline{D} + \overline{C}D$	BC € $AD + \overline{A}\overline{D}$	
	$AB(CD + \overline{C}\overline{D}) = 0$	$BC(A\overline{D} + \overline{A}D) = 0$	$\overline{A}\overline{B}(C + D) + \overline{C}\overline{D}(A + B) = 0$

Example derived from Boole, *Laws of Thought*, 118; Schröder, *Vorlesungen*, vol 1, 529. Notation follows Schröder, except for “ $\overline{x}$ “, which is Boole’s shorthand for “(1-x)”. (Schröder uses “$x_{|}$”.) “€“ is Schröder’s sign for subsumption of one class in another. (Boole uses “x = *v*y” for “x€y”.) Concepts are in bold. The last line gives three equations that follow from the previous propositions; they are used in the solution of the problem. One solution is C € $A\overline{B} + \overline{A}B$. The whole problem can be represented in the following Euler diagram.

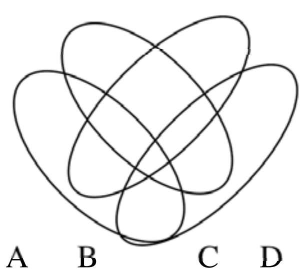


Figure 6-2: *Begriffsbildung* for propositions from Boole and Schröder

ways by means of the signs for generality, negation and the conditional.")[31] The various concepts that arise during the "splitting up of the content" of the judgment (line 7), which was put together out of the simpler mathematical and logical concepts given at step 0, are given in bold. None of these concepts are explicit in the stages leading up to the judgment on line 7, and it requires non-trivial deconstruction to form these concepts from the component concepts given at stage 0. In the Boolean example, drawn from Boole's *Laws of Thought* and amended in light of Schröder's *Vorlesungen*, each of the component concepts is formed before the completed judgments (lines 3 and 4); one quite literally can only grasp the completed judgments if one already has grasped the component concepts. In figure 6.2, no method of decomposition or "splitting up" the judgments into new concepts is given, because the only one possible is the trivial decomposition back through stages 1-4. Indeed each of the component concepts, given in bold, are represented by some region in the Euler diagram, each of which is formed by overlaying the boundaries that represent the given concepts A, B, C, and D.

With these example in view, it is easy to see that Frege's new approach to concept formation is intimately tied to his way of carrying out the project of a *lingua characterica*. In order to capture the conceptual content of mathematical concepts, that is, the content that plays a role in mathematical inferences, Frege needs a language that is able to express generality and relational expressions; but the formation of general, relational judgments requires that completed judgments be decomposed into concepts by replacing one or more singular terms by a variable. So Frege's repeated insistence that we start with completed judgments and analyze them into component concepts—and not

[31] "Boole's logical Calculus and the Concept-script," 34; see also *Grundlagen*, §88.

start with completed concepts and build judgments from them—is at least in part a methodological remark about how a proper logical analysis of mathematical concepts and inferences has to proceed. The analysis of mathematics necessary for the *lingua characterica*—and Frege's larger project of determining the springs of mathematical knowledge—, tied as it is to logical tools of relations and generality, requires a certain method of conceptual formation that includes an analysis of completed mathematical judgments using the logical tools of relations and generality.

A.3 SLUGA AND DUMMETT ON FREGE'S PRINCIPLE OF THE PRIORITY OF JUDGMENT

Before we go on to see how Frege turned his defense against Schröder into a general criticism of Boolean logic, it will be helpful to contrast my reading of Frege's principle of the priority of judgment with the well-known readings given by Hans Sluga and Michael Dummett.

Sluga wants to read Frege's "priority principle"—that judgments "precede concepts"—as simply *Grundlagen*'s context principle differently expressed.

> The context principle is, furthermore, only a logical consequence of the priority principle. If an asserted sentence has meaning by expressing a judgment and if to say that the words constituting the sentence have meaning is to say that they express concepts, then given that judgments precede concepts, it follows that sentence meanings precede word meanings. The context principle is, in other words, merely a linguistic version of the priority principle.[32]

I see, however, little to substantiate the equation. For one thing, it is clear that the context principle in *Grundlagen* is meant at least partially as a principle about what it is

[32] "Frege against the Booleans," 86.

to refer to an object.[33] The context principle there provides part of a method for evaluating the various methods for defining the numbers given in *Grundlagen*. I cannot discuss these difficult issues here. It is enough to say that the meaning of the context principle is not well-expressed by a slogan like "sentence meanings precede word meanings." I have read Frege's maxim that the formation of concepts be subsequent to the formation of judgments in the way that Frege himself explains it in his papers from the early 1880s. There the concern is with what is necessary for an analysis of mathematical concepts and inferences adequate to determine the springs of mathematical knowledge. It is not clear to me that the arguments given with respect to that concern have any direct relation to the questions about what it is to refer to objects or what needs to be true of a candidate definition for it to refer to numbers successfully.

Further, Sluga reads the "priority principle"—and thus also the context principle—as entailing a particular view of the intrinsic structure of judgments and concepts.

> Concepts are always reached through the splitting up of judgments, through analysis; they are not given separately and the judgment is not composed out of previously given constituents.[34]

> From a logical point of view, Frege argued, a proposition was, to begin with, a unity. For many logical purposes it was, of course, necessary to distinguish parts in the sentence and the judgment expressed by it…what constituents we distinguish in a thought in logic does not depend on the words out of which a sentence expressing it is composed, but entirely on the logical consequences that are derivable from the thought.[35]

---

[33] See Burge, "Introduction," 15-16: the context principle, in one of its forms, "provides a philosophical basis for understanding theoretical reference to abstract entities…it locates ontology in the evaluation of theories."
[34] *Gottlob Frege*, 92.
[35] "Frege and the Rise of Analytical Philosophy," 483-4.

If I'm reading Sluga right, he thinks that judgments are first grasped as a whole along with their logical consequences (and presumably also on what judgments they are consequent). Reflection on these inferential relations leads the logician to introduce a structure into the judgments that follows the inferential relations already grasped. This imposed structure first introduces concepts; the conceptual structure thus arrived at may or may not map onto the grammatical structure studied by the grammarian and at least implicitly understood by ordinary users of a language that has the power to express the judgments the logician studies (if there are any such languages). Against this Dummett has protested that the general idea of a predicate formed by omission of proper names in judgments

> certainly does *not* imply that no thought has any intrinsic complexity…In fact, it was precisely to distinguish the way in which a sentence is constructed out of its parts from the way in which it may subsequently be decomposed that I introduced my distinction between simple and complex predicates. The subsequent decomposition has, indeed, as I said, to do with the role of the sentence in inference; its construction out of its parts has to do with how its sense is determined and grasped.[36]

Sluga's Frege, Dummett contends, leaves it mysterious how a thinker is ever to grasp a thought, that is, the sense of a sentence, since he gives no explanation of "how we come to recognize the thought expressed by a genuine sentence in accordance with its composition";[37] indeed, his doctrine that the structure of a thought is not intrinsic but instead becomes apparent only after the whole thought (and presumably many others) are grasped as part of an inferential whole closes off the only intelligible option—that a thinker grasps a new thought by understanding how it is composed from simpler components whose sense she already grasps. Frege would thus be left with "some

[36] Dummett, "Frege as a Realist," 93.
[37] 92-3.

strange idea of our apprehending the judgeable content, in the first instance, as a simple unit devoid of complexity."[38] Dummett thus thinks that Frege is committed to a distinction between simple predicates—predicates whose reference are concepts that are not first grasped through decomposition of complete, grasped thoughts—and complex predicates.[39]

Dummett is surely right that Sluga's reading of "the priority principle," if understood as a thesis about how an ordinary thinker first comes to grasp a thought, leaves it completely mysterious how a thinker could grasp a thought at all. And Dummett is further surely right to protest that Sluga's reading of the context principle relies too heavily on the papers from the early 1880s published between *Bgs* and *Grundlagen*.[40] But, against Dummett, I think in fact that there is no reason at all to see Frege's remarks about concept formation as saying anything about *how mathematical thoughts are first grasped* (say by children or by their first mathematician discoverers) or even about *how natural language expressions get their meaning*. The whole argument is from the perspective of someone who is in a position to appreciate the power of the begriffsschrift: a mathematically engaged thinker who has already grasped the content of modern mathematics (though no doubt imprecisely) and understood its inferences (though no doubt in a less than rigorous way). Indeed, the thought that figure 2.1 represents the stages that a thinker has to go through to acquire the novel concept <continuity of a function f at a> can only be a joke—people first come to grasp the concept presumably by geometrical considerations and reflection on the behavior of the graph of a curve at

[38] *Frege: Philosophy of Mathematics*, 38.
[39] See *Frege: Philosophy of Language*, 4; 30-1.
[40] See "Frege as a Realist," 93.

different points along its path. Indeed, the concepts and singular terms given at stage 0—whose contents are simple, at least with respect to the concepts formed at later stages—include concepts that none but a few (if any) of his readers had grasped prior to reading Frege. (The numbers themselves are defined in terms of concepts like <similarity> and <extension of a concept> that few readers had grasped, and surely none had grasped before 0 or 1.) No distinction between simple and complex predicates is called for by Frege's argument here; there needs to be an independent argument that other areas of Frege's subsequent thought commit him to a theory of how a thinker comes to grasp the content of a novel sentence. (Dummett, to his credit, is explicit about this.) It is for this reason that I cannot follow Sluga in his repeated claims that Frege's "priority principle" "links Frege to the Kantian tradition in logic," since Kant claims that "the understanding can make no other use of [these] concepts than to judge by them" (A68/B93). The point of view of Frege and his readers reflecting on their mastery of modern mathematical concepts and inferences is too different from the point of view of the transcendental subject of Kant's analytic, coming to have a possible experience having been affected by a succession of impressions.

A.4 FREGE'S CRITICISM OF THE BOOLEAN THEORY OF CONCEPT FORMATION

We are now in a position to understand Frege's criticism of Boolean logic. In Boolean logic, all concepts are representable (as in figure 2.2) as regions in an Euler diagram. This view is "superficial": "one might call it a mechanical or quantitative view," since the inferences expressible in it rely on whether the extension of one concept is greater than or less than, or (partially) includes or excludes that of another;[41] concept formation and problem solving are so mechanical that in fact a machine can do them, as Jevons proved.[42] Tied to this superficial view of a concept's *structure* and content is a similarly superficial theory of concept *formation*. "For in Aristotle, as in Boole, the logically primitive activity is the formation of concepts by abstraction, and judgment and inferences enter in through an immediate or indirect comparison of concepts via their extensions."[43] The Aristotelian method that the Booleans employ is fundamentally *abstractionist* since in a paradigmatic case of Aristotelian concept formation, one first isolates a *genus*, say, <animal>, surveys the particulars that fall under it in order to isolate a common feature—a characteristic mark—, say, <rational>, and then abstracts way the common class, the *species* <human>, of individuals falling under the genus that possess the common feature.[44] Of course it may be more charitable simply to say that the Booleans do not have a theory of concept formation at all than to ascribe an obviously inadequate one to them. Since Boolean logic is not a *lingua characterica* but simply a calculus, Boole is within his rights when he "presupposes logically perfect concepts as ready to hand;" Frege's criticism "is not meant to imply that concept formation takes up a great deal of space in their presentations. Rather, their logics are essentially doctrines of inference, in which the formation of concepts is presupposed as something that has

[41] "Elucidation of some points in Schröder's Lectures," *CP*, 216.

[42] Jevons constructed such a machine in 1866; it is described in "On a General System of Numerically Definite Reasoning" (1870).

[43] "Boole's Logical Calculus and the Concept-script", *PW*, 14-15.

[44] See "Boole's Logical Calculus and the Concept-script", *PW*, 33: "Now it is worth noting in all this, that in practically none of these examples is there first cited the genus or class to which the things falling under the concept belong and then the characteristic mark of the concept, as when you define 'homo' as '*animal rationale*'."

already been completed."[45] As Frege describes it elsewhere, Boolean logic is just "a clothing of abstract logic in the dress of algebraic symbols. It is not suited for the rendering of a content, and that is also not its purpose."[46] Given its purposes and the tasks it is able to perform, Boole's symbolism is "a technique for resolving logical problems systematically, similar to the technique of elimination and working out the unknown that algebra teaches."[47]

Understood in this way as a modest project for solving systematically a set of limited problems arising in traditional syllogistic logic, Boole's Aristotelian logical calculus is fine as far as it goes. But enthusiasm for Boole's approach can easily lead one to believe that this is the best logic has to offer and to ignore the much more fruitful kinds of concept formation the Frege highlights; this is disastrous, since one is then left with "the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot."[48] Indeed, in a remarkable passage from the conclusion to *Grundlagen*, Frege alleges that not only was Kant guilty of just this short-sightedness, but this wrong approach to concept-formation was responsible for his failing to see the truth of the central thesis of Frege's book, that the judgments of arithmetic are all analytic. I feel the passage is worth quoting in full:

> Kant obviously—as a result, no doubt, of defining them too narrowly—underestimated the value of analytic judgments [= a judgment provable using only general logical laws and definitions (§3)]…He seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all the ways of forming concepts, that is one of the least fruitful. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really fruitful definitions in mathematics,

45 "Boole's Logical Calculus and the Concept-script", *PW*, 34, 14.
46 "On the Aim of the 'Conceptual Notation,'" *CN*, 93.
47 "Boole's Logical Calculus and the Concept-script", *PW*, 12.
48 "Boole's Logical Calculus and the Concept-script", *PW*, 34.

> such as those of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with the others. A geometrical illustration will make the distinction clear to intuition. If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by the simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do—in terms of our illustration—is to use the lines already given in a new way for the purpose of demarcating an area. Nothing essentially new, however, emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we had put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they are proved by purely logical means, and so are analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. (*Grundlagen*, §88, 99-101)

This section, in many ways the climax of the argument of *Grundlagen*, brings together the main features of Frege's position that I want to emphasize in this section. First, Frege is taking aim at the model of concept formation and conceptual structure that he had previously attributed to the Booleans and had described as Aristotelian and abstractionist. (He again uses the metaphor of boundary lines that Euler diagrams make apt; he again emphasizes the fundamental Boolean operations of concept conjunction or multiplication ("a list of characteristics in no special order") and disjunction or addition;[49] again appeals as test cases to those mathematical concepts that are "fruitful" and gives the continuity of a function as such an example.) It is easy to see that this model is what I called in chapter 3 the *Aristotelian-abstractionist model*. It includes *Aristotelianism about conceptual structure* (Thesis A from chapter 3), since the only conceptual structures it expresses are

[49] In a footnote to this passage, Frege writes "similarly if the characteristics are joined by 'or.'"

conjunction, multiplication, and negation[50] of simpler concepts, and *Inclusion /Exclusion relations in Logic* (A.1), since the judgments and inferences follow the conceptual structure of inclusion and exclusion, as represented in an Euler diagram. Similarly, linking Boolean logic to Aristotle, as Frege did, surely implies that his target includes (A.2), *The Sufficiency of the Syllogism*. Reading §88 in light of the passages cited earlier makes it clear that the kind of unfruitful method of concept formation here attacked is *Abstractionism about conceptual formation* (B). The corollary that comes under pointed criticism here is the view that the formation of a new concept does not contribute any new content—"simply taking out of the box what we had previously put in it." Above, I called this view *Instrumentalism about concepts* (B.2). Second, Frege's criticism of the A-a model is a consequence of the idea that a model of conceptual structure and formation should be tested against mathematical practice: does the model capture the structure and formation of truly fruitful mathematical concepts like continuity of a function?[51] Third, on the A-a model, the introduction of new concepts necessarily

---

[50] Frege does not mention negation in §88, but it is easy to see that it is also part of the theory of concepts he has in mind, since it is expressible in Boolean notation and representable by regions in an Euler diagram. In §88, Frege is targeting the explanation of analyticity, given by Kant, that only applies to universal *affirmative* judgments.

[51] It might be objected at this point that Frege's approach as I have presented it is not the same as the Lotze-Cassirer project of finding a theory of conceptual structure and formation that is adequate to understand mathematical concepts. If it is true that Frege's comments about conceptual formation need to be understood in the context of his project of giving, from the point of view of a mathematically engaged thinker conversant with modern mathematical concepts and inferential practices, an analysis of mathematical concepts and inferences into their basic elements, then in fact he is not giving a theory of the kind of concept formation that a *mathematician* might engage in, but instead the kind of concept formation that a *logician analyzing mathematics* would employ.

I think that this objection misses the mark and for important reasons that I cannot fully defend here. A proper response would flesh out philosophically and textually the claim that Frege is committed to holding that concepts formed in his way in the process of analysis are the very same concepts that mathematicians have been discussing and investigating long before Frege and his conceptual tools came onto the scene. This entails that the process of forming concepts Frege advocates is not necessarily the process that was used when mathematicians first came to grasp the concepts. When Weierstrass gave his famous definition of the continuity of a function at a point a, the analyzed concept, expressed in begriffsschrift as in figure 2.1, was the very same concept that he and his readers had already grasped, reduces the content of judgments. (This explains why Kant underestimated the value of analytic judgments.) Only concepts that are formed in Frege's way and thus have the structure that can be expressed by generality and relational expressions can be truly fruitful; concepts that accord with the A-a model are thus unfruitful. This (un)fruitfulness is exhibited *both* in the fact that reflection on mathematical practice shows that the concepts, like continuity of a function, that mathematicians have isolated as mathematically important for the advance of the field have the Fregean structure and not the A-a structure,[52] *and* in the fact that the definitions of concepts formed in Frege's way have rich enough content to allow for new judgments to be inferred that are—not obvious

---

though inadequately, through geometrical considerations. This implies of course that one can grasp a thought or concept whose conceptual content and structure one does not in fact (fully) grasp. These important but undefended positions establish that, at least on Frege's view, the concepts that he analyzes in his way, using his tools, are the very same concepts that mathematicians have known and loved already. The objection is staved off when we further see that on Frege's view, his project of analysis is in fact internal to mathematics. In *Grundlagen* §1, Frege argues that his project of giving a foundation for arithmetic is of a piece with the "rigor of proof, precise delimitation of extent of validity, and as a means to this, sharp definitions of concepts" that are characteristic of more recent developments of mathematics, especially "the discovery of higher analysis," including the new definitions of limit, continuity, function, and infinity. From Frege's point of view, the complaints that he lodges against his fellow mathematicians in the papers "Logical Defects in Mathematics" and "Logic in Mathematics," are not just the crabby comments of a logician berating mathematicians for not being logicians. (This interpretation goes against recent readings of Frege and Dedekind that allege that Frege was a philosopher carrying out a philosophical project and Dedekind was a mathematician engaging in pure mathematics. See Stein, "Logos, Logic, Logistike," and Phillip Kitcher, "Frege, Dedekind, and the Philosophy of Mathematics.") A fuller interpretation of Frege would require a more detailed defense of this claim.

[52] My understanding of this passage has been greatly helped along by Tappenden's paper "Extending Knowledge and 'Fruitful Concepts': Fregean themes in the Foundations of Mathematics," which is almost an extended commentary on *Grundlagen* §88.

Tappenden in particular thinks that Frege's investigation of what makes a concept "fruitful" is in fact an investigation into what accounts for the mathematical conviction that certain mathematical concepts or objects have a proper "conceptual setting"—an issue discussed in Wilson's "Frege: the Royal Road from Geometry":

> What one would ideally like…is a deeper understanding of mathematical concepts that relates the 'meanings' of mathematical concepts to the hidden factors that *drive* disciplines into reconstituted arrangements, regroupings that in some richer way "better respect" the meanings of the concepts involved.

(Wilson finds and expression of this question in earlier work of Ken Manders's.) On Tappenden's reading, the quantificational structure of judgments expressible in the begriffsschrift provides a key part of the explanation for why some concepts have the proper conceptual setting they do. Tappenden applauds Frege's acknowledgement of this difficult philosophical problem but laments that Frege's solution has not proved adequate. I think, however, that it is charitable not to read such a project into Frege's discussion of fruitful concepts, though I agree with him that this is a philosophically interesting question.

and trivial restatements of what one who has grasped the concept already knows, but—genuinely new and surprising extensions of our knowledge.[53] With the A-a model, it *is* the case that "with all our to-ing and fro-ing, we never leave the same spot." Fourth, what makes it possible that mathematical concept formation involves "drawing new boundary lines" is that mathematical concepts have structures not representable using Euler diagrams. They have richer than Aristotelian structure; the elements that make up mathematical concepts are 'organically related' to one another, not as lists of attributes, but as only relational expressions and the expression of generality allow.

We are now in a position to see that Frege's argument against the Booleans is a version of what I above called "the Lotze objection" to the A-a model: the A-a model is inadequate to express the content of mathematical concepts, for two reasons. The A-a model underestimates "the amount of content a concept has"[54]; put less metaphorically, a full grasp of the concept allows one to infer substantive mathematical judgments that are not obvious to someone who has already grasped the concept and its components. Lotze put this point by saying that a general mathematical concept does not have less content for its being general, but just as much, since it provides a rule for deriving the content of

---

[53] Dummett has emphasized that this argument is Frege's way of solving the problem that had seemed to render logicism about arithmetic a non-starter: how analytic judgments can be purely logical and yet extend our knowledge in the impressive ways that mathematical judgments obviously do. *Frege: Philosophy of Mathematics*, 36-42.

[54] I have put this expression in quote marks because I want to avoid the inference that I am talking about "content" specifically as Frege understood it. Indeed, it is difficult to know precisely how Frege intends "content" in his writings from the 1880s *before* he drew the sense/reference distinction. In *Bgs* §3 Frege says that the "conceptual content" of a judgment is that part of its content that plays a role in inference. A plausible reading is that two logically equivalent statements thus have the same "conceptual content" and so no amount of inferring from definitions will give judgments whose conceptual content differs from the content of the initial set. This is Dummett's reading of pre-1892 Frege. This would imply that definitions allow us to extend knowledge even if no judgment can be inferred from a set of definitions that has "conceptual content" not already possessed by the definitions alone. So to extend knowledge would not be to infer to judgments with new "conceptual content." This is Tappenden's reading of *Grundlagen* §88 in his paper "Extending Knowledge and 'Fruitful Concepts.'" I can afford to remain noncommittal on this interpretive point.

all the particulars that fall under it (section A.6 will give more detail on Lotze's way of putting the point). We saw that Cassirer sees Dedekind's foundation for arithmetic as illustrating this: grasping the definition of number as a simply infinite system allows one to derive first the Peano(-Frege)-Dedekind axioms and from them all of the richness of arithmetic. Frege's way of putting this is that mathematical concepts formed in his way do not just "take out of the box what has already been put in" but "draw new boundary lines"; they are fruitful in the sense that they allow us to "extend our knowledge" or infer what could not have been anticipated from bare contemplation of the concept and its structure. (This was the third main feature of Frege's position that I highlighted in the previous paragraph.) The further reason that, according to the Lotze objection, the A-a model is inadequate for the expression of the content of mathematical concepts is that compound mathematical concepts are not composed from simpler concepts by addition, conjunction, and negation only. Lotze instead proposed a functional model, where the component concepts are related to one another in a variety of ways (as the variables in a function might be related to one another in a variety of ways) and interdependent on one another (as the value taken by one variable at least partially determines the value taken by another). Frege expressed this general point by saying that the elements in a fruitful definition are "intimately, I might almost say, organically connected" with one another. (This is the fourth feature highlighted in the previous paragraph).

I do not mean, of course, that Frege's argument against the Booleans consists only in the Lotze objection, nor that all of the philosophical content reviewed thus far in this appendix falls immediately out of the Lotze objection, nor that Frege's way of putting the objection is just the same as Lotze's or Cassirer's. The hard work of constructing the begriffsschrift and giving definitions and proofs with it allow Frege to flesh out the objection in a pointed and convincing way. But I do think that Frege's objection is clearly a form of the Lotze objection and I further think that this is more than just a bit of historical serendipity. It is the task of the rest of this chapter to justify this further thought.

A.5 HERMANN LOTZE'S THEORY OF CONCEPTS IN HIS *LOGIK* (1874)

Lotze's *Logic* was first published in 1874 (2nd edition appeared in 1880, including a section on Boolean logic) as the first part of his System of Philosophy. This systematic, multi-volume logic is a worked up revision of Lotze's earlier logic book written in 1843. Lotze's *Logic* was one of the most influential and widely read logic texts of the second half of the 19th century. William James called Lotze "the most exquisite of contemporary minds"[55]; others have called him "the most pillaged source" of late 19th century philosophy, and declared the period 1880-1920 "the Lotzean period" in the history of thought.[56] Or consider the words of the Neo-Kantian Bruno Bauch in 1918:

> Of everything that has followed in the area of logic from Hegel to the present day, there is nothing that has surpassed Lotze's logical achievement in value. … His influence reveals itself in every important figure in the area of logic no matter what philosophical direction he might belong to. If he has any claim to significance in logic, he cannot have remained uninfluenced by Lotze.[57]

[55] Quoted in Thomas Willey, *Back to Kant*, 1948, 47.
[56] For quotations, see Willey, 57.
[57] 'Lotzes Logik und ihre Bedeutung im deutschen Idealismus', quoted in Sluga, *Frege*. 53.

Lotze's influence was perhaps felt most by Neo-Kantians, especially the Neo-Kantians of the so-called "Southwest School."[58] Indeed, Thomas Willey, in his history of Neo-Kantianism, considers Lotze to be the first Neo-Kantian figure.[59] A few words about the general character of Lotze's thinking. Lotze took two degrees at Leipzig, one in medicine and one in philosophy; he received his philosophy degree in 1840 under a Hegelian, Weiße.[60] After teaching at Leipzig, where his circle included Fechner, he took over Herbart's chair at Göttingen in 1844, where he stayed until moving to Berlin in 1880; he died one year later.[61] He stands as a bridge figure between the German speculative philosophy that was breathing its lasting in 1840—Hegel died in 1831—and the more scientifically-informed philosophy that dominated Germany from 1850 onward. His aversion to speculative *systematics* is often thought to be a result of his scientific training, and he remained throughout his life a syncretist that tried to hold on to what is best in idealism and realism.[62] Adamson's 1885 review of Lotze's *Logic* in *Mind* described his general orientation:

---

[58] Wilhelm Windleband, the first leading figure in this school, was a student of Lotze's, as were Carl Stumpf and Anton Marty. (See Gottfied Gabriel, "Frege, Lotze, and the Continental Roots of Early Analytic Philosophy.") A clear influence of Lotze on Windelband is apparent in Windelband's *Theories of Logic* (1912).

[59] From Lotze, Willey says, the Neo-Kantians derived a "constructive skepticism, a distrust of purely conceptual thought, a deep interest in problem of values and an admirable effort to bring philosophical idealism into harmony with nineteenth-century science" (57). But this is an overstatement: this characterization only fits well with the Southwest School (Windelband, Rickert, and Bauch); certainly Natorp and Cassirer did not distrust purely conceptual thought!

Lotze's distrust of purely conceptual thought, and his insistence on the primary role of sensing and feeling in knowledge, is well described in Henry Jones's *A Critical Account of the Philosophy of Lotze: the Doctrine of Thought*, chapter 2.

[60] See Gabriel, 41.

[61] Both Frege and Dedekind were at Göttingen when Lotze was teaching there. Frege took a course from him in the philosophy of religion, though not in logic: Frege in fact did not take a course in logic either at Jena or at Göttingen. He studied under Schering and completed his dissertation in 1873; Dedekind studied under Gauss and completed his *Habilitation* in 1854.

[62] Henry Jones characterizes Lotze's philosophical frame of mind this way:

> Lotze's opposition to Idealism was based not so much on his antagonism to its positive doctrines as upon his antipathy to its *system*. It would not be strictly true to say that Lotze adopts *each* of

> Lotze might fairly have been described as the one remaining link of connection between the great epoch of systematic speculation in Germany and the more recent age of detailed, scientific research. No thinker of any time has more thoroughly combined the speculative instinct of the constructive philosopher with the cautious, practical attitude of the trained scientific investigator. If it be the ideal of the philosopher to work into a harmonious conception those thoughts which are the deepest, most far-reaching, most characteristic of his age, it would be hard to point to any one who has realized the ideal more thoroughly than Lotze.[63]

For our purposes, the most significant fact to note about Lotze is not that he was perhaps the most widely-read and influential philosophical logician of the late nineteenth-century, but rather that he expressed most clearly and influentially the inadequacies of the traditional 'A'-a model of conceptual formation and structure. In Cassirer's words: "the original and decisive achievement of the concept is not to compare representations and group them according to genera and species... Among modern logicians Lotze has expressed this relationship most clearly."[64]

Lotze's *Logic* begins with an account, broadly reminiscent of Kant's A edition transcendental deduction of the categories, of how the operations of thought [das Denken] allow a subject ultimately to come to apprehend what is true and distinguish it from the mere current or stream of ideas [Vorstellungsverlaüfe] (§II).[65] In this current of ideas, some ideas flow together only because of accidental features of the world or because of idiosyncratic features of a particular subject's mind; some ideas flow together because the realities that give rise to them are in fact related in a non-accidental way. How are these two cases to be distinguished? Lotze insists that it is the task of thought to

---

> the main tenets of Hegel while rejecting the *whole*, but such a statement would be a fair summary of his general attitude.

Henry Jones, *A Critical Account of the Philosophy of Lotze: the Doctrine of Thought,* 1895, 9.

[63] Reprinted in Adamson, *A Short History of Logic*, 1911, 190-1.

[64] *Philosophy of Symbolic Forms*, vol. 3, 281.

[65] References to Lotze's *Logic* will be to section numbers. The translations will be from the 1888 English translation, edited by Bosanquet, though with some modifications of my own here and there.

so distinguish them; and it is the task of logic to investigate what thought does when it distinguishes them. But to allow that thought is an activity (§IV) does not commit one to running together the separate investigations of logic and psychology, since a psychological investigation of the "psychical mechanism" can do nothing to distinguish the streams of ideas that correspond to truths and those that do not:

> Logic only begins with the conviction that the matter cannot end [with facts about conscious activity and law-like connections of a psychical mechanism]; the conviction, that between the combinations of ideas, however they may have originated, there is a difference of truth and untruth, and that there are forms to which these combinations *ought* to answer and laws which they *ought* to obey.[66]

Further investigation of the laws according to which conscious activity proceeds cannot help us to uncover the kind of laws of thought we are interested in when we want to know what is *true*;[67] and it is interest in the truth and concern for the truth that leads us to distinguish the accidental streams of ideas from those cases where the ideas are connected in the mind in the way that they ought to be if we are concerned with the truth.[68]

---

[66] *Logic*, §X. Michael Dummett has pointed out in his paper, "Frege's 'Kernsätze zur Logik,'" that this passage is partially quoted by Frege as Kernsätze 12 in his unpublished "17 Key Sentences on Logic," *PW*, 175.

[67] I think in fact that Lotze is less than clear and consistent about whether he takes ideas or their content [Inhalt] as the bearers of truth. I have done my best in this section to reflect Lotze's own ambiguity in the language I have used.

[68] This is an expression of Lotze's anti-psychologism in logic. See also, for instance, §332:

> I have maintained the opinion throughout my work that Logic cannot derive any serious advantage from a discussion of the conditions under which a psychical process comes about. The significance of logical forms is to be found in the meaning and purport of the connexions into which the content of our world of ideas ought to be brought; that is to say in the utterances of thought or the laws which it imposes, after or during the act of thinking, not in those productive conditions of thought itself which lie behind. Conditions of this kind there must certainly be, but only those conditions of a psychical mechanism which determine at every single moment every single one of its motions…But if we knew all that we could desire to know on the subject, it would still be a delusion to suppose that we should be thereby any the better able to judge of the truth of our logical principles. … No sensational or empirical theory of the origin of thought and knowledge can possibly ever prove or disprove the principle of identity or excluded middle.

Compare Frege's Kernsätze 17, where he writes "No psychological investigation can justify the laws of logic" (*PW*, 175). It is very clear that, as a matter of history, Lotze played a significant role in anti-

When thought distinguishes between those streams of ideas that are connected accidentally from those that are not, its activity is not well described as merely separating the true from the false. Instead, thought is acting on ideas [vorstellungen] in such a way that *they come to be subject to* laws of truth or *capable of being* true or false. Even animals, Lotze believes, have connected series of ideas; but only humans, through their activity of thought, connect their ideas in such a way that these connections are *justifiable*. Thought accomplishes this, in the case of humans, by the addition to the stream of ideas of certain 'accessory thoughts' or 'accessory notions' [Nachgedanken] that, when added to the stream of ideas, ground this stream in reality [§VI]. An animal might connect together the idea of a tree with the idea of its leaves and might, by some psychical mechanism, associate through memory the ideas of that tree and its leaves with the idea it is having right now of a tree with no leaves. But a human, through thought, will add to these connections of ideas the accessory thought of a thing [Ding] and its property [Eigenschaft] and will see these ideas as corresponding to a particular tree that before had leaves and now has lost them.

> The name of the tree, to which he adds and from which he takes away the descriptive epithet ["leafy" and "leafless"], signifies to him, not merely a permanent as opposed to a changeable part in his observation, but the thing in its dependence on itself and in opposition to its property [die auf sich beruhende Sache, das Ding im Gegensatze zu seiner Eigenschaft]. The effect of bringing the tree and its leaves under this point of view is, that the relationship of thing and property appears as the justification both for separating and combining these ideas, and thus the fact of their coexistence or non-coexistence in our

---

psychologism in logic. (See Gottfried Gabriel's Introduction, "Objektivität, Logik und Erkenntnistheorie bei Lotze und Frege," to vol. 2 of Lotze's *Logik*, x-xi; Gabriel there cites an unpublished draft of a preface to Husserl's *Logical Investigations*, where Husserl credits—not Frege's review of *Philosophie der Arithmetik*, but—Husserl's study of Lotze's *Logik* with having turned him away from psychologism.)

On Lotze's anti-psychologism, its historical influence, and its relationship to Frege, see also the very helpful paper by R. Lanier Anderson, "Neo-Kantianism and the Roots of Anti-Psychologism."

> consciousness is referred to a real condition [sachliche Bedingung] upon which their coherence or non-coherence at the moment depends. (§VI)

We can say of the human subject that he connects these ideas of the tree and its leaves with this idea of a tree with no leaves *because they are ideas of the same thing*, and we can say that these ideas are separated in the human subject *because they are ideas of the very same tree now with and now without its leaves*. Thus, "the surplus of work performed by thought over and above the mere current of ideas…always consists in adding to reproduction or severance of a connection in ideas the accessory notion of a ground for their coherence or non-coherence."[69]

Book I, "Of Thought (Pure Logic)," proceeds to cover the three main areas of traditional logic, the theory of the concept, of judgment, and of inference, carried out according to the guiding idea that the function of thought is "to reduce coincidence to coherence" (§XI). In the Introduction, Lotze emphasized that the function of thought is not just to distinguish the true and the false, but first to add to the stream of ideas the accessory notions that make it possible for the subject to think that which is subject to truth or falsity and under the authority of laws of thought. Now in the first chapter, "The Theory of the Concept [Begriff]," Lotze emphasizes that even the stream of ideas is itself partially a product of thought, since only thought can transform a mere series of given impressions [Eindrücke] into ideas [Vorstellungen]. This first operation of thought, "the forming of impressions into ideas," is actually, Lotze claims, the first of three operations of thought necessary for the formation of concepts properly-so-called: the second is "the composition, comparison, and distinction of the simple contents of ideas" [Setzung, Vergleichung und Unterscheidung der einfachen Vorstellungsinhalte], which is

[69] §VII. This important sentence is partially quoted and denied by Frege as Kernsätze §2, *PW*, 174.

preparatory to the third stage, the formation of concepts themselves [Die Bildung des Begriffs].

This first operation consists in thought's objectification [Objectivirung] (§3) of the subjective impression (or sensation or feeling, which are mere conditions we find ourselves in (§2)), which action is implicit in the forming of a name or the use of a definite article. If I am able to speak of 'the tooth-ache,' I have come to pick out an element of my subjective state as something that someone else can come to recognize (though not sense directly as I can) and as something that I can meaningfully speak of even when I am not sensing it itself. I have distinguished my sensing [Empfinden] from that which is sensed [Empfindbare] (§2). That which is sensed, "the toothache," "the red," is now recognized as a content [Inhalt] with a being and meaning [Bedeutung] in itself and an independent validity [Gleichgültigen] (§2).

The second operation of thought, Lotze claims, takes a newly-formed, objective idea and gives "affirmative position" to its content, distinguishes it from the content of other ideas, and "estimates by quantitative comparison its differences and resemblances" (§19). Having formed ideas from sensations by distinguishing the act of sensing from what is sensed and having isolated the content of that sensation, the thinking subject can now interpret the relations that various ideas have to one another as in fact "aspects of the content of the impressions" themselves (§9). Thus, the idea of the toothache and the idea of the stomachache have a certain connection in my stream of ideas; I isolate this connection and see it as in fact a reflection of the common characteristic of the two ideas: both the toothache and the stomachache are painful. I hold together the impression of the firetruck and the impression of the schoolbus and I distinguish them; I hold together the impression of the firetruck and the impression of the apple; I compare them, note the similarity and isolate the red that is common to their contents. When I isolate these simple aspects of the content of subjective states, I am in fact forming universals; these are what Lotze calls "first universals," common elements in a series of passive impressions that thought need only recognize (§14).

The third operation of thought, the formation of concepts, takes the stream of ideas along with the first universals, which are already present in sensations as the simple aspects of their content, and begins finally to perform the characteristic function of thought: "to separate the merely coincident in the manifold of ideas that are given to us, and to combine the content afresh by the accessory notion of a ground for their coherence" (§20). This third operation of thought involves what Lotze calls a *logical* synthesis of the manifold of ideas, whereby one arrives at the conviction that the elements in sensation in fact form part of a connected whole.[70] How does thought accomplish this? The traditional answer is by *abstraction*, the method of obtaining a universal by leaving out what is different in a series of particulars and "add[ing] together that which they retain in common" (§23). At this stage, these new universals, *second universals*, can be formed from the first universals made available by thought's second operation. These first universals can then act as material for the abstraction and form part of the content of the second universals as their marks [Merkmale]. There is no circularity here, since the first universals are not products of thought but appear directly in our

[70] Lotze isolates this logical synthesis from the synthesis of apprehension, wherein, by the mechanism of memory and in virtue of the unity of the soul, different ideas are brought together into one and the same consciousness, and from the synthesis of intuition [Anschauung], where various representations are ordered spatio-temporally (§20). Lotze says too little about these operations to completely satisfy; he, for instance, says nothing to explain how one can assign a spatial location to (the content of) an idea before one has carried out the third, "logical," synthesis, in which various ideas are combined with the consciousness that their belonging together has a real ground.

sensations and require no logical act for their construction (§24-5).[71] The picture, then, is this. I find in my stream of ideas certain streams that are connected; these connected streams include first universals as part of the content of these ideas. At one moment together in my consciousness is <spherical, hard, small, blue>; at another is <spherical, hard, small, red>; at another is <spherical, hard, small, green>. I abstract from what is different and isolate a new, "second," universal, <spherical, hard, small>. I now treat this composite as a unity and may even express it with a name, say, "a marble." By treating this new universal as a unity, I am isolating a relation within my stream of ideas as belonging together in a non-accidental manner: I say that the ground for my having the series of ideas that I've had is that there is *a kind of thing* to which these series of ideas corresponds, namely, *marbles*. I thus abstract from a series of simple, "first" universals a common compound universal whose combination I *ground* with the accessory notion of a kind of thing.[72]

The problem, though, is that the traditional way of understanding a concept, "a composite idea which we think as a connected whole [die zusammengesetze Vorstellung,

---

[71] Cf. §157:

> "We explain a conception, which we call *M*, by *abstraction*, when we first refer to a number of known instances, in each of which *M* forms a part of the notion, and then bid the hearer separate from these instances that which does not belong to the conception *M* which we wish to communicate. This is the way in which all our general conceptions [Begriffe] and general ideas [Vorstellungen] were originally formed; in the case of a general idea that which was common to a number of impressions comes of itself to stand out as the object of a new separate idea; in the case of a general conception this process is consciously directed by attention and reflexion."

Here general ideas seem to be what he earlier calls "first universals" and general concepts seem to be a kind of "second universal." Here Lotze misleadingly describes these two types of universals as arising through abstraction, though in two different ways, while earlier, in the passages referred to in the paragraph to which this note is appended, he describes only the latter as arising from abstraction. In the following section, §158, Lotze again emphasizes that *simple* concepts can only arise through abstraction. These abstracted simple concepts—he mentions being, becoming, acting, thinking, affirming, denying—are universals and are simple, but are not first universals because they are not simple universals immediately present in a series of impressions.

[72] I am not sure that Lotze would consider "kind of thing" as an example of an accessory notion [Nachgedanken]: Lotze never, as far as I know, gives a list of accessory notions, nor gives a rule whereby we could generate such a list or identify particular concepts as accessory notions.

die wir als ein zusammengehöriges Ganze denken]" (§25), formed merely by leaving out from a series of ideas what is different and retaining the common elements, does not capture at all *the* essential function of thought when it forms concepts: the *grounding* of the connection of the ideas. And so Lotze prefers to talk about concepts in this way: "I speak of any composite content *s* as conceived [begrifflich gefaßt] or as a concept, when it is thought together with [mitgedacht] a universal *S* that contains the condition and ground of the coexistence of all its marks and of the form of their connection" (§26).[73] In contrast to the concept, Lotze identifies a merely general image [ein bloßes allgemeines Bild], which is a compound universal,

> the thought of which is indeed accompanied by the thought of its connected wholeness, but without the specification of the organizing rule of the connection. The name 'man' as ordinary used expresses no more than an image of this kind; reflection, by subordinating it to the universal 'animal,' easily makes it into a concept; but then 'animal' remains a general image, which only the naturalist, for the uses of his science, converts into a concept by thinking 'organic being' along with it. (§27)

Lotze repeatedly refers to the concept as containing a rule or a determinate law (§121) that explains or justifies the belonging together of the marks that together compose the concept; the formation of concepts thus involves not (or not only) the isolation of common elements, but the identification of a rule, in at least two senses. First, in the merely general image, a series of marks is simply *summed or listed*, instead of being connected together in some more determinate way. Second, that these particular marks are connected together in this particular way should tell us something significant about the actual objects that fall under this concept: "consequences admit of being drawn from

[73] Lotze does not suggest that the universal *S* be an a priori concept or category in Kant's sense. He, for instance, allows that the general image of a man can act as the universal that grounds the connection of the various representations in the singular concept <Alcibiades>.

[the concept] that coincide again at certain points with results flowing from the content, that is from the thing itself [Sache selbst]" (§27). Thus, for instance, the general, composite idea <Alcibiades> is a concept when its component concepts or marks are brought together by the concept <man>: the component ideas that are brought together in <Alcibiades>, say, his weight or color, are explained or put in an intelligible context by referring them back to the fact that Alcibiades is a man. The right consequences will be drawn from conceiving of Alcibiades as a man who weighs 180 pounds, and not from conceiving of him as an object weighing 180 pounds who is a man.

Lotze is fond of illustrating what is lacking in merely composite universals that lack unifying rules by mentioning examples of degenerate universals. "To seek for a concept that included under it cucumbers and mathematical principles, could only be an ingenious joke" (§30); no one would think to have advanced the progress of thought by having framed the compound universal *N* composed of the marks <blackness>, <extension>, <divisibility>, <weight>, and <resistance,> formed by abstracting the common element in "negroes, coal, and black chalk" (§122). At this point we can come back to the notorious example, quoted by Cassirer and referred to earlier in this chapter, of the common "concept" <red, juicy, edible, body> under which falls cherries and raw meat (§31). In all three of these cases, the degenerate universals are formed, not by looking for the rule that unites the marks that together compose a concept, but by identifying common elements in a series of particulars and *abstracting* away from the differences. As a result, the common elements or marks are simply *listed*, not compounded according to a determinate rule, and the knowledge that some particular falls under the "concept" tells us little about it. It is in this sense that

> Of the true universal, on the other hand, which contains the rule for the entire formation of its species, it may rather be said that its content is always precisely as rich, the sum of its marks precisely as great, as that of the species themselves; only that the universal concept, the genus, contains a number of marks in a merely indefinite and universal form. (§31)

What is lacking in these degenerate cases is, as Lotze puts it earlier (§27) a "rule" for the behavior of a given object [Gegenstand] that agrees with the object's actual behavior [wirklichen Verhalten]. The content of the concept <red, juicy, edible, body> is poor because it allows us to infer nothing more about the objects falling under it than that they are red, juicy, edible, and bodies.

Lotze therefore wants to replace the model of the concept as the sum of its marks with what he calls the "functional" model, where the structure of the concept is expressed by some complicated interrelation between its marks: the content of the whole concept is some nontrivial function of the content of the component concepts (in the traditional terminology, of the marks).[74]

> As a rule, the marks of a concept are not coordinated as all of equal value, but they stand to each other in the most various relative positions, offer to each other different points of attachment, and so mutually *determine* each other; … an appropriate symbol for the structure of a concept is not the equation $S = a + b + c + d$, etc, but such an expression as $S = F\ (a, b, c,$ etc.) indicating merely that, in order to give the value of S, a, b, c, etc, must be combined in a manner precisely definable in each particular case, but extremely variable when taken generally. (§28)

There are five elements to Lotze's functional model that need to be emphasized. First and most obvious, a concept S = $F\ (a, b, c,$ etc.) in general is not a mere sum of its

[74] Lotze continues to call the component concepts that form the matter out of which a concept is formed "marks," even though marks are traditionally considered to be predicable of the objects that fall under the concept (see Kant, *Jäsche Logic*, §7). Thus, since <man> is <rational> + <animal>, we can say that since Alcibiades is a man, he is also rational and an animal. But consider a concept in which the component concepts are related functionally: <prime number> is not <number *and* divisible *and* unit>, but <a number divisible *only by itself and a* unit>—7 is a prime number, but it is not a unit! (See §28).

marks: an S is not just a thing that is *A* and *B* and *C*, etc. But, second, the functional notation is supposed to make clear that there is no one determinate, universal structure that all concepts possess. Just as the variables in a function might be related arithmetically or logarithmically or trigonometrically, but always by a precisely defined rule, so too the marks of a concept may be interrelated in any number of ways, as long as there is a determinate rule relating them.[75] Unlike the theory of conceptual structure that Frege's begriffsschrift allows us—where each compound concept is composed from component concepts in precisely the ways that quantified relational logic allows—, Lotze specifically denies that the various ways in which component concepts interrelate in the formation of compound concepts can be isolated and exhaustively specified.[76] Third, in the functional model, the marks are *interdependent* and vary together according to a rule. Every concept or particular that falls under a concept S = F (a, b, c, etc.) has its own specific way of being an *S* and exhibiting the marks of *S*; still, though, *how* a particular *S* exhibits a mark will in general be determined by how it exhibits other marks. This is particularly clear for actual mathematical functions: at each particular point on a plane curve of second degree, $a_2x^2 + a_1x + b_2y^2 + b_1y + cxy + d = 0$, its ordinate(s) y is determined by its absSica(s) x. Every particular that falls under <triangle, figure *with* three sides *meeting at* three angles>, will not just have three sides and three angles, but the specific angles that it has will be determined by the magnitude of its three sides. Similarly, fourth, some marks of a concept will be what might be called "*variable*" marks

[75] Thus, it is clear that the conception of functionality needed here is not the modern one—an arbitrary many-one or one-one mapping from some n-tuple of elements to a particular element—but an older one: a relationship of dependency of some value on some other values according to a rule describable in a finite analytic expression.

[76] See §110.

or “universal” marks.[77] According to the abstractionist model, I form concepts by leaving out what is different in various particulars and adding together what is common. Suppose I want to form the concept <animal> by abstraction from the concepts <slug>, <cheetah>, and <parrot>; a slug creeps, a cheetah runs, and a parrot flies, so I take away these marks. But, Lotze argues, we need to distinguish the way in which animals in general do not creep, run, or fly from the way in which, say, plants in general do not creep, run, or fly. We leave out <creeps>, <runs>, and <flies> from <animal>, but we retain them as instances of a universal or variable mark moves in some way, which is a mark of <animal> but not of <plant>. Again, if we were to form the concept <metal> by abstracting from <gold>, <silver>, and <copper>, we would not simply leave out the yellow, silver, and red colors, but leave in their place or compensate for them with the variable mark <colored in a certain way>. Now, when we consider particulars that fall under <metal>, we can treat color like a *dependent* variable: in <gold> (that is, <yellow metal>), the specific character of the yellow is determined by gold’s being a metal—gold is *metallic* yellow. It is thus too simple to say that <gold> = <yellow> + <metal>, since the two marks “are not coordinated as all of equal value.” The yellowness is functionally dependent on the metallicity.[78]

---

[77] §23: “the universal is produced, not by simply leaving out the different marks p^1 and p^2, q^1 and q^2, which occur in the individuals compared, but by substituting for those left out the universal marks P and Q, of which p^1 p^2 and q^1 q^2 are particular kinds.”

Lotze never calls such marks “variable” marks, but he later (§110) make a point of representing certain marks by an ‘x’ and representing particular concepts by $S = F\,(a, b, c, x \ldots)$.

[78] See especially §23. Cassirer cites this passage in *SF*, 21-3:

> As [Lotze] explains, the real practice of thought in the formation of concepts does not follow the course prescribed by [the traditional doctrine of abstraction]; for it is never satisfied to advance to the universal concept by neglecting the particular properties *without retaining an equivalent for them*…The merely negative procedure, on the contrary, would lead in the end to the denial of all determination, so that our thought would find no way of return from the logical ‘nothing’ which the concept would then signify…If we carry through the above rule to the end, it obliges us to retain, in place of the particular ‘marks’ that are neglected in the formation of the concept, the

These four elements of Lotze's theory of the concept as a function put us in a position to return to the fundamental theme of Lotze's theory of thinking: "the impulse of thought to reduce coincidence to coherence" (§XI). The formation of concepts, as opposed to the formation of merely general images, is the culmination of a three stage process whereby thought starts with mere impressions (or sensations or feelings—mere subjective states) and ends with the isolating of a *rule* that unites the component marks in a concept such that "consequences admit of being drawn from [the concept] that again coincide at certain points with results flowing from…the thing [Sache] itself" (§27). As Lotze emphasized in the Introduction, this work of thought involves the representation of the connected ideas as *grounded* or justified—and thus susceptible to truth or falsity and laws of thought—by relations that hold in reality. The fifth element of Lotze's theory of the concept as a function now follows. When thought forms concepts by *interrelating* component universals like interdependent variables in a function, it is in a position to capture real, nontrivial relations among the things themselves. Thus, to return to the degenerate 'concepts' formed by abstraction mentioned earlier, their two-fold triviality—simplicity of structure and uselessness for coming to understand the world—turns out to be connected. The 'concept' <red, juicy, edible, body> is poor in content—that it groups together raw meat and cherries tells us nothing significant about raw meat and cherries—because it is just a list or sum of marks and is thus unable to represent the nontrivial

---

> systematic totality [Inbegriff] to which those marks belong as special determinations…We represent the systematic totality when we substitute for the *constant* particular 'marks,' *variable* terms, such as stand for the total group of possible values which the different 'marks' can assume…As long as we believe that all determinateness consists in constant 'marks' in things and their attributes, every process of logical generalization must indeed appear an impoverishment of the conceptual content. But precisely to the extent that the concept is freed of all thing-like being, its peculiar functional character is revealed. Fixed properties are replaced by universal rules that permit us to survey a total series of possible determinations at a single glance."

Cassirer makes this same point against abstractionism in *Philosophy of Symbolic Forms*, vol. 3, 291.

relations among elements of reality that thought is after when it rises above the mere stream of ideas.

A.6 LOTZE'S THEORY OF MATHEMATICAL CONCEPTS

What I dubbed in chapter 3 "the Lotze objection" alleges that the traditional, abstractionist model of concept formation is inadequate to capture the *content of mathematical concepts*, since compound mathematical concepts are composed of simple concepts in more complicated ways than the A-a model allows and since mathematical concepts are rich enough to contain all of the mathematically relevant content of all the particulars falling under them. This chapter has been exploring how Cassirer, like Frege, considers the traditional model, what I've called the A-a model, in light of the fundamental question "Is the theory of the concept, as here developed, an adequate and faithful picture of the procedure of the concrete [mathematical] sciences?" (*SF*, 11). Lotze's own answer to this question is found in *Logic*, Book I, chapter Three, "the Theory of Inference." After considering the different kinds of syllogistic inferences (the four figures of the syllogism, along with syllogisms where the major premise is disjunctive or hypothetical in form), Lotze comments that in fact these traditional modes of inference make use of no more than the relations between the extensions [Umfangsverhältnisse] of given concepts (§107).

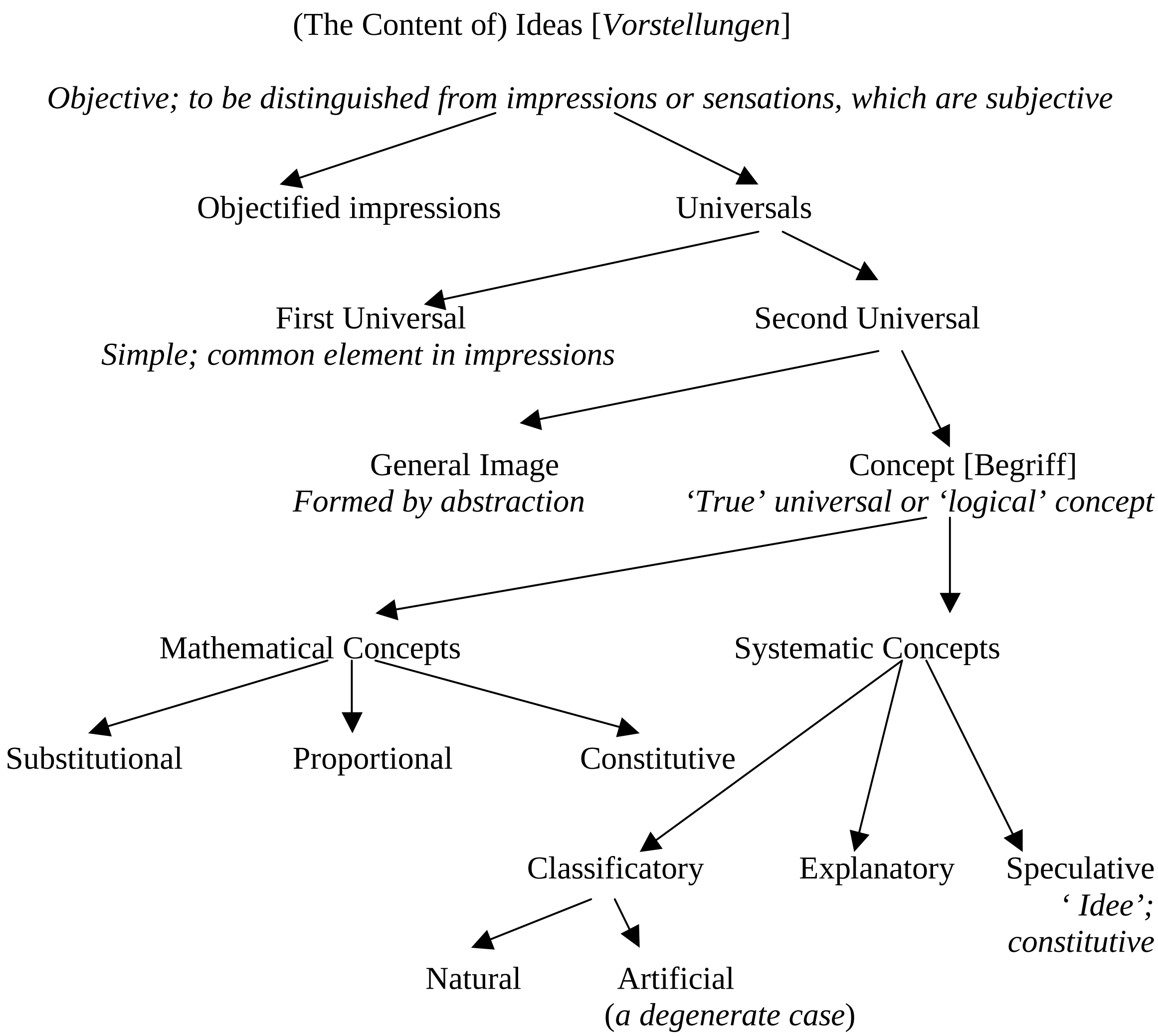
> Judgments and syllogisms based on subsumption have only required us to consider the one relation which obtains between a concept *S* and its proximate higher universal *M*…[T]he logical structure of *M* itself was to a great extent a matter of indifference. As middle term it bore the name concept, but the character of a concept was in no respect essential to it; any simple mark, any sum of marks,

> whether combined under a definite rule, or merely brought together somehow in thought, was good enough to constitute such a middle concept. (§120)

It is not surprising, then, that the mathematical sciences require more than syllogistic reasoning. It is not enough to say, "Heat expands all bodies; iron is a body; so, heat expands iron," but we want to know precisely how heat expands iron, how its expansion depends on its temperature (§106). This interdependence of the properties of a piece of iron is of course expressible mathematically, and so it is no surprise that when we move from considering syllogistic inferences to considering mathematical inferences we are forced to take into consideration the specifically functional structure of concepts. Lotze's discussion of the various types of non-syllogistic inference (and he isolates six types, three mathematical types of inference, and three "systematic," or natural scientific, types of inference) thus runs parallel to a discussion of correspondingly different kinds of non-Aristotelian conceptual structure. (This taxonomy is summarized in figure 6.3.) An example of a non-Aristotelian inference, *inference by proportion*, is as follows:

> *Major Premise*: The ratio between the size of a 60° angle and two right angles is proportional to the ratio between the magnitude of the area of the circle swept out by a 60° angle and the magnitude of a semi-circle.
> *Minor Premise*: The ratio between the size of a 60° angle and two right angles is 1:3.
> *Conclusion*: The magnitude of the area of the circle swept out by a 60° angle is one third that of a semi-circle.

This inference works because the concept <segment of a circle> is not just <area of a plane *and* bound by two radii of a circle *and* bound by an arc of the circle>, but <area of a plane *formed by* two radii of a circle *together with the* arc of the circle that those two radii cut out>. From this specification, it is clear that as soon as the two radii of a circle are identified, a unique circular arc is determined, along with a unique closed portion of the plane. It is this functional interdependence that makes this non-Aristotelian mode of



Lotze distinguishes ideas from impressions, the first universal from the second universal, and concepts from general images in Chapter 1, "The Theory of the Concept." He notes that concepts with different structures are needed for different kinds of inferences and distinguishes different kinds of concepts along with the corresponding kinds of inferences in Chapter 3, "The Theory of Inference," sections 3.B and 3.C.

Laying out this tree involved some necessary simplification as well as some interpretive guesswork. An example of simplification is considering the concept as a kind of universal: though Lotze introduces the concept as a second universal, he in fact distinguishes between singular concepts and universal concepts; I have ignored that complication here, even though it is significant for Lotze in his discussion of *Wirklichkeit* and *Geltung* in Book III (see especially §§341-2). An example of interpretive guesswork is distinguishing substitutional mathematical concepts from proportional mathematical concepts: Lotze is very clear that different kinds of inferences correspond to different kinds of concepts and that constitutive concepts correspond to inferring from constitutive equations; I am less sure that Lotze's distinction between inferences by substitution and inferences by proportionality correspond to two different kinds of concepts (and not just one kind of concept that can participate in two kinds of inferences).

Figure 6-3: Lotze's Taxonomy of *Vorstellungen, Logik*, Part I

inference possible: since an angle at the center of a circle is determined by specifying two radii, and since specifying two radii also determines a unique segment of the circle, one can infer from a ratio of angles at the center of a circle to a ratio of areas swept out by angles.[79]

The strongest kind of mathematical inference, *inference from constitutive equations* [der Schluß aus constitutiven Gleichungen], is best illustrated by concepts from analytic geometry.

> [In] a 'law-giving' [gesetzgebenden] or constitutive concept…every mark is determined throughout by every other, though in very various ways… Analytic geometry possesses in the *equations* by which it expresses the nature of a curve just that constitutive concept of its object [Gegenstand] that we are looking for. A very small number of related elements [abscissae and ordinates, plus constants and their arithmetical combination] …contain, implicit in themselves and derivable from them, all relations that necessarily subsist between any parts of the curve. From the law expressing the proportionality between the changes of the ordinates and the abscissae every other property of the curve can be developed. (§117)

Here we see in its clearest form a concept that "contains the rule for the entire formation" of that which falls under it, and thus whose content "is always precisely as rich, the sum of its marks precisely as great, as that of the species themselves" (§31); the constitutive equation expresses a "rule" for the behavior of a given object [Gegenstand] that agrees with the object's actual behavior (§27).[80] Now, not all mathematical concepts are constitutive concepts; some have only the functional interdependence of marks that is

---

[79] Lotze discusses inference by proportion at §114-5. Proportional interrelations make possible "bringing qualitatively different occurrences [like the magnitudes of angles and the sizes of areas, which are interrelated, but not directly commensurable] into such mutual dependence as allows us to calculate one from another"; *inference by substitution*, on the other hand, allows us to infer directly from some occurrence to another that is qualitatively identical, that is, directly commensurable.

[80] It is worth emphasizing here that Lotze stresses that the "rules" implicit in constitutive concepts give the law for the objects or species falling under it in the sense that all of the *properties* of the objects that fall under it follow from the fact that these objects fall under this concept. But he nowhere suggests that these concepts are "constitutive" in a stronger sense according to which the *existence* of the objects themselves is guaranteed by the fact that they fall under this concept.

required for the weaker mathematical inferences, and presumably some have only the very weak structure that is necessary for syllogistic reasoning. But any concept that is to play a non-trivial role in distinctly mathematical reasoning, pure or applied, will have richer than Aristotelian structure and will thus be available for use in non-trivial reasoning about the objects that fall under it. Consequently, the traditional model of conceptual structure and formation is inadequate not only for concepts of pure mathematics, but also for the concepts of mathematical natural science.[81] (Section C of Lotze's chapter on the Theory of Inference is given over to determining precisely to what extent the concepts in the other sciences ("systematic" concepts) have richer than Aristotelian structure and thereby are able to participate in stronger than syllogistic inferences. The results of that chapter are summarized in figure 6). What I earlier called the "Lotze objection to the A-a model" and attributed to Cassirer and Frege now follows:

> *Lotze objection to the A-a model*: Even if it is adequate in other areas, the A-a model is inadequate to express the content of mathematical concepts, for two reasons. First, mathematical concepts do not have less content the more general they are, but just as much, since they provide a rule for deriving the content of all the particulars that fall under them. Second, compound mathematical concepts are not composed from simpler concepts by addition, conjunction, and negation only.

[81] Lotze specifically identifies analytic geometry and the calculus as mathematical sub-disciplines where constitutive concepts are required; inference by substitution and proportion are more suited, he suggests, to other areas of mathematics. Constitutive equations are possible in highly mathematized sciences like chemistry and mechanics (§117). The mathematical form of inferences and conceptual structure is the *ideal* toward which every science strives, even if it cannot be fully achieved in every science: "[W]herever we are quite unable to reduce our efforts to [mathematical] relations, our knowledge of it remains defective, and…no other logical form can help us to the answer which a mathematical treatment of the question, if it were practicable, would give us." (§112)

A.7 LOTZE'S CRITICISM OF THE BOOLEAN THEORY OF CONCEPT FORMATION

I've called the central negative argument in the first chapter of Cassirer's *Substance and Function* the "Lotze Objection" because Cassirer himself is very explicit that his criticism of the traditional A-a model had been expressed earlier by philosophers, especially by Lotze. I then argued that Frege's early 1880s arguments against the traditional abstractionist model of concept formation, directly primarily against the Booleans, are an expression of the Lotze objection, and I emphasized that Frege's development of the begriffsschrift allows him to formulate this criticism of the Booleans and the A-a model in a particularly convincing and precise way. But unlike Cassirer, who is very fond of citing precedents for his own arguments, Frege does not note that his criticism of the A-a model is similar to Lotze's.[82] Frege's objection to the Booleans that was outlined in section 4 of this appendix, I argued, similar to Lotze's and I think it likely that Frege himself knew this; indeed, Lotze turned his criticism of the A-a model against Boole's logic itself in 1880, the same year in which Schröder's review of *Begriffsschrift* appeared and within a year of the period in which Frege wrote "Boole's Logical Calculus and the Begriffsschrift." (I'll return to the historical question of to what extent Frege was influenced by or borrowed from Lotze below in section 8.)

Lotze's *Logik* appeared in 1874. A second edition, dated September 6, 1880, contained a few changes here and there; its most significant improvement was the addition of an appendix to Book II, a criticism of Boole entitled "A note on the logical

[82] Indeed, Frege mentions no other philosophers who have pointed out the inadequacies of the A-a model. The closest things—and these are not very close at all—are Frege's citations of the linguist's Sayce's notion of a "sentence-word," and the logician Wundt's metaphor of a concept as an atom, which never is found on its own, but only appears in combination with other atoms, *PW*, 17.

calculus [Calcül]." This note appears to have attracted some attention: it was the first significant consideration of Boolean logic by a continental logician; it was given a very sympathetic review in 1885 by Robert Adamson, who had also reviewed Schröder's "Operationskreis des Logikcalcüls," in *Mind*, and attracted a lengthy response from Schröder in the Introduction to his 1890 *Vorlesungen*.[83] The bulk of the note is given over to objections to Boole's procedure that were also given by others[84]: it is not justifiable to employ meaningless symbols in the course of a logical calculation, as Boole does when he makes use of 'logical division' in solving logical equations[85]; each rule used in logical calculation needs to be justified by appealing to some principle of logic, not vice-versa; the symbol a^2 is meaningless, since there is no class of objects that are both an A and an A; similarly for Boole's use of the symbol '0/0'; there is no logical operation that can be represented by factoring a polynomial; indeed, many algebraic

[83] In §36-37 of his *Logic*, Lotze pointedly characterizes the work of logicians who consider the relations of extensions of concepts [Umfangsverhältnisse] to be of primary importance: "I consider it to be just as fruitless [irrig] as it is boring [langweilig]"; the relationship of one representation that is contained in the extension of another is "monotonous." In his note on the logical calculus, Lotze directs this sentiment against the Booleans when he claims that the relations of extensions of concepts [Umfangsverhältnisse] are in fact "all that the calculus notices" (284).

Schröder takes up this pointed criticism in the Introduction to volume 1 of his *Vorlesungen*, 102-3, and again on 330-1. Schröder also alleges that Lotze characterizes as "unfruchtbar" attempts to make the relations of the extensions of concepts central in logic, but I have unfortunately been unable to find the passage—if in fact one exists—where Lotze uses that word in particular. Frege, of course, attacked the Aristotelian/Boolean model of concept formation as not being "fruchtbar" in *Grundlagen* §88 and argued in "Boole's Logical Calculus and the *Begriffsschrift*," *PW*, 33-4 that his own begriffsschrift, unlike Boolean logic, allows for the formation of "fruchtbar" concepts.

[84] It is not clear how much of the work of the British logicians Lotze knew. In the note, he cites Boole's *Laws of Thought* (1854), Jevons's *Principles of Science* (1877), and Schröder's "Operationskreis des Logikcalcüls" (1877).

[85] *Logic*, 277. Boole's offending principle appears in *Laws of Thought* (1854), chapter V, section 3: "it is an unquestionable fact that the validity of a conclusion arrived at by any symbolic process of reasoning does not depend upon our ability to interpret the formal results which have presented themselves in the different stages of the investigation." Boole then justifies this principle by appealing to the 'uninterpretable symbol' $\sqrt{-1}$, which unquestionably can be used in valid reasoning. Lotze, however, thinks that $\sqrt{-1}$ is no partner in guilt, and he cites (Gauss's) geometric interpretation of imaginary numbers.

This same criticism of Boole's use of uninterpretable symbols was given also by Schröder, "Der Operationskreis des Logikcalcüls," Venn, *Symbolic Logic* (1881), 79, note 1, and Jevons, "Pure Logic," §173.

manipulations that are permissible in algebra are banned in Boole's logical calculus, for no other reason than that Boole needed his calculus to work out right;[86] in any case, Boole accomplishes no more with his methods than one could by systematically writing down all possible combinations of symbols and eliminating them one-by-one.[87]

What is interesting and novel in Lotze's criticism, though, follows directly from Lotze's non-Aristotelian, functional model of the structure and formation of concepts. Admittedly, some concepts can be represented using multiplication (and addition), but Lotze argues, these concepts are only very weak ones that are representable by combinations of symbols that express "merely the simultaneous presence of their elements [viz., their marks]."[88] A method of logic that can only express conceptual structure using so-called logical multiplication and division will fail to capture two important ways in which the elements making up a concept can be connected. These two inadequacies that Lotze identifies in Boolean logic correspond to the two parts of what I've been calling "the Lotze objection to the A-a model." First, the representation of a

---

[86] See 281-290. Compare Jevons, in his 1864 monograph, "Pure Logic or The Logic of Quality apart from Quantity, with remarks on Boole's system and on the relation of logic to mathematics," §176:

> [A] system perfect within itself may not be a perfect representation of the natural system of human thought. The laws and conditions of thought as laid down in the system may not correspond to the laws and conditions of thought in reality. If so, the system will not be one of Pure and Natural Logic. Such is, I believe, the case. Professor Boole's system is Pure Logic fettered with a condition which converts it from a purely logical into a numerical system. His inferences are not logical inferences; hence they require to be interpreted, or translated back into logical inferences, which might have been had without ever quitting the self-evident principles of pure logic.

Jevons, like Lotze, goes on to criticize in particular Boole's use of the symbol '0/0': in §202, Jevons writes that "Professor Boole's operations with his abstract calculus of 0 and 1 are a mere counterpart of self-evident operations with the intelligible symbols of pure logic."

Venn also tries to "translate back" Boole's system into purely logical principles; his work, he explains, is motivated by a desire to give an "explanation of the principles of the logical calculus, in entire independence of those of the mathematical calculus" (xiii).

[87] 294. Lotze's method is similar to that developed by Jevons. This method, as Venn points out in *Symbolic Logic*, is equivalent to solving logical problems by the use of Euler diagrams.

[88] *Logic*, 278. In Lotze's terminology, multiplication and addition are sufficient only for "artificial classifications": concepts that sort objects into kinds by an arbitrary listing of characteristic marks in no particular order. These classifications are "artificial," as opposed to "natural," because they are drawn with no attempt made to model the nature or essential properties of the objects falling under them.

concept as an unordered series of marks connected by multiplication or addition does not "assign the final form which is to be the result of the completed combination … *a b c* taken by itself only designates any object of thought, no matter how constituted, in which the marks *a*, *b*, and *c* are found together" (278-279). Thus, the only "concepts" that can be adequately represented by Boolean logic are the degenerate concepts like <red, juicy, edible, body>, which is "poor in content" in the sense it tells us nothing interesting about the objects that fall under it except that they are juicy, edible, red and bodies. In particular, the expressive resources of Boolean logic would never be enough to represent the structure of mathematical concepts "for the form which the result of the calculation is finally to take, is here completely and solely determined by the definitely assignable nature of the connection which this science requires to be introduced between its elements [viz., the marks in a concept]" (279). I earlier characterized this objection to the A-a model, now turned against Boolean logic, as the claim that this model cannot respect the fact that concepts "provide a rule for deriving the content of all the particulars that fall under them."[89] Second, the algebraic combinations of simple symbols for marks cannot express "the reciprocal determination of the component parts," that is, the functional interdependence of the marks that make up a concept (279). As we saw earlier, this functional dependence of marks can be seen most clearly in mathematical concepts expressible by constitutive equations, where, say, a plane curve's ordinate(s) *y* is determined by its abssica(s) *x*. But this functional dependence is only possible because the structure of the concept is richer than Aristotelian and therefore also richer than

[89] On 279, Lotze emphasizes that even concepts that fail to accord with the ideal of a mathematical concept, inasmuch as they only have *images* functioning as rules for the ordering of their marks, cannot be represented in the Boolean way.

Boolean: compound mathematical concepts are not composed from simpler concepts by addition, conjunction, and negation only.

A.8 THE HISTORICAL RELATIONSHIP BETWEEN FREGE'S AND LOTZE'S CRITICISMS OF THE BOOLEANS

The parallels between the parts of Lotze's 1880 criticism of the Booleans that I've highlighted and parts of Frege's 1880s criticism are, I think, striking. This raises then the question of the historical relationship between Frege's (now better known) criticisms of the Booleans and Lotze's (then much better known) objections. There are two questions here: How likely is it that Frege knew of Lotze's "Note on the Logical Calculus"? and How likely is it that Frege's criticism of the theory of the concept implicit in Boolean logic was *derived* from Lotze's own?

It is certain that Frege had read Lotze's *Logik*. Frege was a graduate student in mathematics at Göttingen while Lotze was there, and he took his only course in philosophy (in the philosophy of religion) from Lotze (Frege had taken a course on Kant's philosophy from Kuno Fischer while an undergraduate at Jena; he never took a course in logic, though Lotze taught his logic course regularly while Frege was in Göttingen).[90] Of course, the fact that Frege took one course as a graduate student from Lotze, on a different topic, perhaps before Frege had any interest in logic, says little about what of Lotze's work Frege knew, let alone about what kind of influence Lotze might have had on Frege. (This fact does, though, make it likely that when Frege later

[90] See Sluga, "Frege: the early years," 342.

developed an interest in logic, he would have looked at what Lotze, his old teacher, had to say.) What is decisive, though, in showing that Frege read Lotze's *Logik*, is a short, unpublished and undated piece, "17 Kernsätze zur Logik," which Michael Dummett has shown to be in fact a series of comments on the Introduction, that is, the first ten or so pages, of Lotze's *Logik*.[91] Dummett has suggested that these sentences were in fact written very early—before the Begriffsschrift—, since they were clearly written before the sense/reference distinction was drawn in 1891, and they do not employ the expression "beurtheilbarer Inhalt," which is characteristic of Frege from 1879 to 1884.[92] This latter point, though, is not that telling: Frege was, after all, consciously adopting Lotze's own terminology in his Kernsätze—he uses the characteristically Lotzean terms "Nebengedanken," and "Unwahrheit" here. A better clue for dating these sentences is the fact that they were found together with "Dialog mit Pünjer über Existenz," a work of Frege's that uses the same example sentence, "Leo Sachse is a man," that appears in the Kernsätze. The "Dialog" is easier to date: it is a set of notes on a conversation Frege had with Bernhard Pünjer, who was a Professor in Jena from 1880 until his death in 1885. By far the best hypothesis, then, is that Frege wrote the Kernsätze between 1880 and 1885; it is likely then that Frege was studying Lotze's *Logik* around the time when he received Schröder's review of *Begriffsschrift* (1880) and he wrote "Boole's Logical Calculus and the Begriffsschrift" (1881).

What is much harder to know, though, is whether Frege knew Lotze's "Note on the Logical Calculus," which appeared in late 1880 in the second edition of *Logik*. I think it is likely that Frege would have known of the "Note"; from 1880-82, Frege made

[91] Dummett, "Frege's Kernsätze zur Logik."
[92] See "Frege's Kernsätze zur Logik," 76-7.

it his goal to explain the purpose and value of the begriffsschrift and to compare it favorably to the work of the Booleans in a way that logicians might understand. That he would then not have paid attention to a criticism of the Booleans, one of the first such works by a logician in Germany,[93] and that by a logician whose works he studied around that same time, is hard to imagine.[94]

Coming to some conclusion about whether Frege's criticism of the Booleans and his presentation of the Lotze objection to the A-a model was influenced or derived from Lotze's is much harder. For one thing, Lotze's argument in the "Note" comes as no surprise to anyone who has read Book I of his *Logik* carefully; any careful reader of Lotze's first edition would have been able to formulate the note's objection to the model of concept formation implicit in Boolean logic without seeing the note itself. It is possible, then, that Frege had internalized the Lotze objection earlier in his career and was thus able to make the easy inference in 1881 that Boolean logic cannot escape it. Alternatively, it is possible that what I've called the Lotze objection to the A-a model was given by other philosophers before 1880. Kurd Lasswitz, in his positive 1879 review of *Begriffsschrift*, applauds Frege for avoiding the disastrous theory of concept formation implicit in Boolean logic.

> Now English logicians, first Boole and then Jevons building upon him, have derived from the language of algebra the purely logical operations holding for concepts in general and have based a conceptual calculus [Begriffsrechnung]

---

[93] Wilhelm Wundt had devoted 52 pages to the logical calculus in his 1880 *Logik*, vol. 1. This work seems to have appeared earlier in 1880 than Lotze's second edition *Logik*, since it, but not Lotze's *Logik*, appears in a bibliography of works on the logical calculus compiled by Schröder in his 1880 review of *Begriffsschrift*. (In the body of the review, Schröder refers to a passage in the *first* edition of Lotze's *Logik*.)

[94] Gottfried Gabriel, in his Introduction to volume 1 of Lotze's *Logik*, wonders why Frege does not refer to Lotze's *Logik* in his 1880-1882 papers on Boole, since he does refer to Boole, Schröder, and Wundt; perhaps, he speculates, the "Note" was "hidden" in the new edition and Frege newer saw it (xxiii). Such a possibility cannot be ruled out.

> upon them. In Germany, the works of R. Grassmann and E. Schröder are in this field. The result of these investigations was namely the apprehension of a judgment as an equation with the help of the quantification [Quantificirung (perhaps a reference to Hamilton)] of concepts, and the apprehension of deduction as a [sort of] substitution. Certainly, a onesidedness which causes great doubt is present here, since the real nature and formation of concepts in their relation to deducing and judging (which may not be separated from the content) are insufficiently considered.
>
> It is thus gratifying to encounter in the present work an attempt to attack this problem in a different way…When the author calls his 'begriffsschrift' 'a formula language modeled upon that of arithmetic,' this [does not refer]…to the artificial similarity with algebra which is attained through the inadmissible apprehension of a concept as the sum of its marks.[95]

Lasswitz, then, before Frege or Lotze, turns Frege's criticism of the concept as a sum of marks (see *Bgs*, iv) explicitly against the theory of concept formation implicit in Boolean logic. It is entirely possible, then, that the Lotze objection to the A-a model was given by other logicians in the 1870s or even earlier.[96] Given the slim documentary evidence available to substantiate an influence of Lotze's "Note" on Frege's criticism of Boole and the A-a model, and the incompleteness of the historical work that has been done on German logicians in the 1870s and 1880s, scholarly modesty seems the wisest policy here.

My historical conclusion, then, is that it is likely that Frege knew of Lotze's "Note," and it is also likely that his criticism of Boolean logic and the A-a model owes something to Lotze's theory of the concept and concept formation. But we should not overlook the real differences between Frege and Lotze. As I emphasized earlier, Frege's

---

[95] Lasswitz, Kurd. Review of *Begriffsschrift*. *Jenaer Literaturzeitung*, 6 (1879), 248-9; *CN*, 210-11. Lasswitz (1848-1910) was a philosopher in Bresla and Gotha who wrote extensively on Kant, Fechner, and the history of atomism. In his 1896 book on Fechner, Lasswitz tried (improbably) to supplement Fechner's psychophysics and panpsychism with Kant's transcendental psychology; his 1890 two volume work on the history of atomism up to Newton was widely popular (Russell, for instance, read it as a graduate student.)

[96] Gottfried Gabriel thinks he sees a clear affinity between Frege's criticism of Boole and Trendelenburg's (in his 1862 *Logische Untersuchungen*) criticism of the 19th century logician Drobisch; *op cit*, xxiv. I have not been able to substantiate this claim, nor investigate any of a number of other possible influences on Frege's criticism of the A-a model, perhaps in Wundt, Sigwart, or any of a host of other logicians.

concern with concept formation, and his attack on the A-a model, needs to be understood in light of his project of carrying out a *lingua characterica*; Lotze himself was hostile to such a project.[97] Just as importantly, there is no whiff of the idea of a formal system in Lotze, and no conception of logic as an axiomatized theory. Lotze's model of inference is still the syllogism, and he remains wedded to the subject/predicate form of judgments. Frege's conception of inference is modeled on rigorous mathematical proof, and he significantly rejects the subject/predicate analysis of judgment. Lotze would have been no help to Frege in formulating the function/argument analysis of judgments or the all-important distinction between concept expressions and singular terms. But what Frege shared with Lotze, and perhaps in fact derived from Lotze, is, at the very least, the conviction that an adequate conception of inference requires an adequate model of concept formation and structure, and the conviction that, as far as mathematical concepts go, the traditional A-a model and its algebraic variants are almost no help at all.

A.9 HOW LOTZEAN WAS FREGE?

In this final section of the chapter, I will briefly discuss some open questions in the debate over the 'influence' of Frege on Lotze, a debate that began in the 1970s with a series of papers written by Sluga and Dummett.

Concepts as Functions

[97] See *Logic*, 298.

Sluga has pointed out that Bruno Bauch, in his 1918 paper, "Lotzes Logik und ihre Bedeutung im Deutschen Idealismus," published in *Beiträge zur Philosophie des Deutschen Idealismus* in the pages that immediately precede Frege's own "Der Gedanke," characterized Frege's logic as "not independent of Lotze's."[98] Bauch stresses four features of Lotze's thought that are presumably significant in understanding Frege: Lotze's anti-psychologism; his distinction between an object of thought and its recognition; his reformulation of Plato's theory of ideas as an ontology-free theory (Lotze's famous theory of validity articulated in Book III); and Lotze's account of concepts as functions. Sluga thinks that Bauch's testimony is weighty historical evidence:

> [S]trong evidence that Frege was substantially influenced by Lotze's ideas…is provided through the testimony of Bruno Bauch, for many years one of Frege's colleagues at Jena…Bauch's testimony on this point seems so substantial that no interpretation of Frege can ignore it. Those critics who have disputed Lotze's influence on Frege have just failed to take the available evidence seriously enough.[99]

This evidence, though, is not as weighty as Sluga thinks: one can see a "substantial influence" of Lotze on Frege in the four areas Bauch singles out only if one seriously distorts Frege's or Lotze's philosophy. Consider Lotze's functional model of the concept. What it shares with Frege's model is, to use Frege's words, the conviction that important, genuinely fruitful concepts in mathematics are composed from component concepts in such a way that "every element in the definition is intimately, I might almost say organically, connected with the others." Both Frege and Lotze think that inferences

---

[98] See Sluga, *Frege*, 53. Bauch held a professorship in philosophy at Jena from 1911 till his death in 1942; he was the student of Windelband (a student of Lotze's) and Rickert (a student of Windelband) and a representative of the so-called Southwest Neo-Kantian school. Bauch's background requires that we take his attempt to see Frege as just another Lotzean with a grain of salt.

[99] "Frege: the early years," 342.

with the traditional A-a model do no more than "simply take out of the box what we had previously put in it," and leave us in a position such that "for all our to-ing and fro-ing we never really leave the same spot."[100] But the sense in which for Lotze concepts are functions differs markedly from the sense in which concepts are functions for Frege. For Lotze, a compound concept is (not itself a function, but) the result of applying a function to a collection of marks; for Frege, a concept is itself a function from one or more objects (or concepts, if it is not a first-level concept) to truth-values. For Frege, all compound concepts are expressible using names of simple functions, proper names, concavity, and the horizontal, negation, and conditional sign; for Lotze, it is simply not possible to characterize in advance the logical structure of concepts. There is also in Lotze no antecedent to Frege's central conception of functions as unsaturated and the strict distinction between concepts and objects that this makes possible. Relatedly, Lotze's logic remains wedded to the subject/predicate analysis of sentences; Frege, on the other hand, has the resources to logically decompose a complex sentence in a variety of ways and can exploit the possibility of multiple decomposition to introduce his polyadic quantification theory. Compare now what Bauch has to say about Frege's theory of concepts as functions:

> The notion of a function which was taken by Lotze from mathematics and made fruitful in logic has received a brilliant development in mathematics again on the basis of logic. The altogether classical proof of this is the mathematical work of Frege. This interrelation of logic and mathematics prepared by Lotze also explains the tight connection between Lotze and Kantian philosophy, not least with reference to notion of function. For it is through Lotze…that Kant's idea that transcendental laws or forms, just as much as judgments, are really functions

[100] Gl, §88; "Boole's Logical Calculus and the *Begriffsschrift*," *PW*, 34.

> and that concepts rest on functions receives its further elaboration and reformulation.[101]

The only sympathetic way to take Bauch's claim that Frege's logic is "not independent" of Lotze's is to read it very weakly indeed.[102]

Logicism

Lotze bases his theory of conceptual structure on his analysis of the kinds of inferences that are necessary in the sciences; thus, we see clearly the inadequacies of the A-a model when we reflect on the non-syllogistic inferences that are employed in mathematics and the mathematical natural sciences. We find a defense of this approach in Chapter III.B, "The Mathematical inferences":

> [W]e must not forget that calculation in any case belongs to the logical activities, and that it is only their practical separation in education which has concealed the full claim of mathematics to a home in the universal realm of logic.[103]

---

[101] Sluga, *Frege*, 57. The distance between Kant's "functional" view of concepts and Frege's is even clearer than that between Lotze's and Frege's.

[102] Gottfried Gabriel has also expressed skepticism about Bauch's reliability as an expositor of Frege, given the stark contrasts between Frege's and Lotze's "functional" models of the concept. See his Introduction to volume 1 of Lotze's *Logik*, xxv-vi, note 20.

[103] §112. See also §18:

> All ideas which are to be connected by thought must necessarily be accessible to one of the three quantitative determinations which have just been mentioned [more and less, unity and multiplicity, greatness and smallness]… I exclude [from our present investigation] the investigation of the consequences which may be draw from these quantitative determinations as such: they have long ago developed into the vast structure of mathematics, the complexity of which forbids any attempt to re-insert it into universal logic. It is necessary, however, to point out expressly that all calculation is a kind of thought, that the fundamental concepts and principles of mathematics have their systematic place in logic, and that we must retain the right at a later period, when occasion requires, to return without scruple upon the results which mathematics have been achieving, as an independent progressive branch of universal logic.

This quotation appears in the midst of Lotze's discussion of the second operation of thought: the position, distinction, and comparison of the matter of simple ideas. In this context, it is clear the Lotze is arguing that the simple universals ("first universals") discoverable in sensation (for instance, brightness or saturation of a color) are always quantitatively determinable and therefore comparable. He is thus presenting a position reminiscent of the Critique's "Anticipations of Perception," where Kant argues that the mathematics of continuity is necessarily applicable to the matter of an intuition, since all sensation has a degree. Lotze is distinguishing that kind of "logical" argument, about the necessarily mathematical determination of the matter of sensation, from mathematics itself. In the second half of this quotation, Lotze is reserving the right to come back to the relationship between mathematics and logic; he returns to

Sluga has claimed that this passage should be read as an anticipation of Frege's logicism:

> Among the many things that Frege owes to Lotze, the most important is perhaps the idea of logicism. The significance of this idea is not simply that it give logic priority over mathematics, but also that it reaffirms the fundamental position of logic with respect to all human knowledge. For if it is correct, as the nineteenth century was increasingly coming to see, that science has to go beyond experience through its employment of mathematics, and if logic was indeed more fundamental than mathematics, then pure logic was at the basis of all knowledge.
>
> Whatever the details of Lotze's position, it is clear that in some sense he subscribed to the claim that arithmetical propositions are grounded in general logical laws alone…Though Lotze claimed that arithmetic was really part of logic he never tried to show that conclusion could be established in detail nor did he list the additional logical principles which he considered necessary for that task. It was Frege who set out the necessary details.[104]

But there are many problems with seeing Lotze as a forerunner to Frege's logicism. On textual grounds, the difficulty goes beyond the obvious one that Lotze never attempts to carry out the work that would be necessary if Lotze were in fact advocating logicism in Frege's sense: he does not analyze any mathematical concepts, provide mathematical or logical axioms, or provide a single proof of a mathematical theorem from merely logical premises. In fact, Lotze, in Book III, Chapter V, gives a self-consciously orthodox Kantian account of the nature of mathematical judgments. Repeating the argument for the synthetic a priori status of mathematical judgments contained in the B Introduction (take, for instance, the claim that 7+5 = 12, or that two lines cannot enclose a space, and try to derive it from the principle of contradiction; all of your attempts are in vain and

---

this theme when he investigates the systematic place of mathematical concepts and inferences in chapter III.B.

Properly read, then, I think this passage gives little support to the Sluga-Gabriel thesis that Lotze in some way anticipated Frege's logicism.

[104] Sluga, *Frege*, 57; "Frege: the early years," 343-4. Gottfried Gabriel has argued for a similar conclusion; see his "Introduction," xxi. Interestingly, Husserl, in his *Logical Investigations* I, 47, says that, for Lotze, the "sphere of exact logical laws will also cover the inexhaustible wealth of the laws of pure mathematics." Husserl later cites Lotze's §18 and §112 in a paragraph attacking a psychologistic 'foundation' for mathematics; here he cites also Frege's *Grundlagen* and the preface to *Grundgesetze*.

you finally must take recourse in intuition, etc.), Lotze concludes that in the case of arithmetic and geometry, the justification and fruitfulness of mathematical judgments and inferences rests on a pure intuition: in the case of arithmetic, a pure intuition of quantity; in the case of geometry, a pure intuition of space.[105] But the thought that all mathematical judgments are grounded in intuition cannot coexist with the thought they are all nevertheless "grounded in general logical laws."[106]

In fact, Lotze's "logicism" is not logicism in any robust sense. In the context of Lotze's argument, what he means is only that the kinds of conceptual structures and inferences one finds in mathematics should be discussed and treated in a treatise on logic along with the traditional conceptual structures, those formed by abstraction, and the forms of the syllogism. That this is a rather modest claim is made clear by Lotze's Chapter III.C, where Lotze discusses the "Systematic Forms," that is, the kinds of conceptual structures and inference forms that are used in the natural sciences; all Lotze is emphasizing is that logic should study the concepts and inferences that appear in all areas of human thought. Even this modest claim is significant: it allows him to argue that the class of logical inferences is wider than syllogistic logic, and that there are richer, more complex concepts than the A-a model can account for. But the Fregean conception of logicism is in fact unavailable to Lotze. For Lotze, logic is the systematic study of the

---

[105] See §353-4.

[106] Sluga, in "Frege: the early years," 343, speculates that Lotze's logicism is in fact limited to the claim that all *arithmetical* judgments are derivable from logic alone. He tries to square this with Lotze's insistence on the *intuitive* nature of arithmetic by reading Lotze's talk of an intuition of *quantity* as an "intuitive grasp of the realm of objective ideas. That realm is the concern of logic and, for this reason, arithmetic must be taken to belong purely in logic." But there is nothing in Lotze to suggest that he means by "intuition of quantity" [Anschauung der Zahlgröße] a purely intellectual intuition, and there is nothing that Sluga gives to clarify what this might mean. In any case, this could not be Lotze's position, since in the two passages where he presents his "logicism," §18 and §112, he describes *all of mathematics* as "having a home" in universal logic, not just arithmetic.

forms of concepts, judgments, and inferences. For Frege, logic includes a set of basic laws and includes all those judgments that can be derived from them alone in a gap-free proof using only purely logical inference rules. Lotze not only shows no evidence of having the idea of a formal system, it would be anachronistic, to say the least, to read into his words a conception of *logic* as a formal system – a conception only in light of which Frege's logicist thesis can even be understood.

Context Principle

On x of *Grundlagen*, Frege gives his famous "context principle": "never to ask for the meaning of a word in isolation, but only in the context of a proposition." Numerous commentators have claimed that this principle came to Frege from Lotze.[107] But in fact there is little justification for this historical thesis, for the simple reason that there seems to be no thesis like the context principle in Lotze.

Sluga has given two independent reasons to think that the context principle is in fact a Lotzean doctrine. First, Sluga argues, the context principle in Frege is an application of the Kantian "priority principle"—that "judgments precede concepts"—, and the priority principle is itself a central doctrine of Lotze's *Logik.*

> Kant had argued against the theory of ideas (and, thereby, in his own eyes, against any naturalistic theory of knowledge) that judgments are not formed out of previously given constituents, but that they possess an initial transcendental unity out of which we gain concepts by analysis. By the late nineteenth century the doctrine had become a standard argument in anti-naturalistic theories of knowledge. Both Sigwart and Lotze were using it in this way. It had become so popular that Wundt could write of the period: 'It had become the dominating characteristic of logic and has in many respects remained so until today to regard the judgment as the beginning of all logical thinking from which the concept was

[107] See Sluga, *Frege*, 55, 60; Gabriel, "Frege, Lotze, and the Continental Roots of Early Analytic Philosophy," 49; Milkov, Nikolay, "Russell's Debt to Lotze."

> supposed to originate through analysis.' Through Lotze's influence the doctrine also reached Frege, who expressed it in its most memorable form in the context principle of the *Foundations of Arithmetic*.[108]

But this "dominating characteristic of logic" was in fact explicitly denied by Lotze. Unlike Sigwart, who self-consciously inverts the traditional order and discusses judgments before concepts in his *Logik*, Lotze insists that the traditional logicians had it right.

> I consider this [viz, the assertion that "in logic the theory of judgment at least must precede the treatment if concepts"] to be an overhasty assertion, due partly to a confusion of the end of pure with that of applied logic, partly to a general misconception of the difference between thought and the mere current of ideas. For if those judgments, out of which the concept is said to be the result, are to be really judgments, they themselves can consist of nothing but combinations of ideas which are no longer mere impressions; every such idea must have undergone at least the simple formation mentioned above [viz, the "first operation of thought," the "objectification of impressions"]; the greater part of them, as experiment would show, will already practically possess that higher logical form to which the very theory in question gives the name concept. The element of truth in this proposed innovation reduces itself to the very simple thought, that in order to frame complex and manifold concepts, more especially in order to fix the limits within which it is worth while and justifiable to treat them as wholes and distinguish them from others, a great deal of preparatory intellectual work is necessary; but that this preparatory work itself may be possible, it must have been preceded by the conformation of simpler concepts out of which its own subsidiary judgments are framed. Without doubt, then, pure logic must place the form of the concept before that of the judgment. (§8)

In an interesting irony, any reader of Dummett's commentary on Frege will recognize this protest. It cannot be true in general that judgments precede concepts, since a judgment has to be composed of simpler parts, and those parts cannot be mere impressions; so there must be simple concepts. Of course, not all concepts are simple, and at least some compound concepts can only be grasped through a non-trivial analysis of prior judgments. Thus: one cannot judge that a cat is on the mat until one has already

[108] Sluga, *Frege*, 55.

grasped the concept <cat>. (The irony here is that Sluga has argued that the context principle came to Frege through Lotze *and* that for Frege no judgment has any intrinsic complexity. Dummett has replied that, though Frege never says it explicitly, his theory requires that there be some simple concepts that are prior to the judgments that contain them. And here we see that, even if Frege is not explicit on this point, Lotze very clearly is.)

Sluga has argued that Lotze's explicit rejection of the "priority principle" should not be taken too seriously: "Lotze's treatment of the logical theory of concepts is strangely at odds with what he has said about the priority of propositions (or judgments) over concepts…[As] Bauch [in his "Lotzes Logik und ihre Bedeutung im Deutschen Idealismus," 1918] points out, this gives Lotze's discussion a psychologistic and atomistic appearance that seems incompatible with his other assertions."[109] Sluga never, as far as I can tell, adduces a single passage in Lotze's *Logik* where the so-called priority principle is defended. Even so, Lotze's insistence that there are some simple "concepts" that must be grasped before a thinker can frame judgments is no mere traditional prejudice not well integrated with the rest of Lotze's system. Lotze's theory of the "second activity of thought," that is, the formation of "first universals," requires that the priority principle not hold in general. And this was no slip on Lotze's part: his insistence that the work of thinking in the process of knowing is in fact limited on the one side by the content given immediately in sensation and on the other side by the dictates of

[109] Sluga, *Frege*, 56.

feeling, one of Lotze's most central and distinctive doctrines, is fundamentally incompatible with the so-called priority principle.[110]

Elsewhere, Sluga gives an independent reason for thinking that the context principle is present in Lotze's *Logik*.[111] In §321, Lotze writes:

> The term [Validity] cannot be transferred [from referring to propositions] to single concepts without some degree of obscurity: we can only say of concepts that they *mean* something, and they mean something only because certain propositions are valid *of* them, as for example the proposition that the content of any given concept is identical with itself and stands in unchangeable relations of affinity or contrast to others.

Though the interpretation of this passage is obscure, I think we can gain some footing by looking at the setting in which the passage appears. Lotze is discussing the debate between Aristotle and Plato on the status of the Ideas, having a few sections earlier given his famous doctrine of modes of Reality [Wirklichkeit]: to say that a thing [Ding] is real is just to say that it exists or is; that an event is real that it occurs or has occurred; that a relation is real that it obtains; that a proposition is real that it holds or is valid (§316). In light of these distinctions, Lotze defends Plato's doctrine of Ideas as simply a colorful way of insisting that psychological events, which are real insofar as they occur, should be separated from the content of these events, which is real insofar as it is valid or holds. Saying that the Ideas are supercelestial is a way of affirming that no thinker *creates* or *makes true* the truths that he thinks, that in fact true propositions are true whether or not any thinker ever thinks them (§318). But we can recognize this independent objectivity

[110] This is precisely why Cassirer attacks Lotze's doctrine of the first universal at such length in *PSF*, vol. 1, 281-3. For a contemporary's critical exposition of the place of the doctrine of the first universal in Lotze's *Logik*, see Henry Jones, *A Critical Account of the Philosophy of Lotze: the Doctrine of Thought*, Chs. II and III.

[111] Sluga, 55. Sluga does not separate the priority principle from the claim made in §321. They are however clearly separate: the priority principle concerns what kinds of representations a thinker must possess in order to possess other kinds of representations; §321 concerns in what sense we can say of a concept that it possesses reality [Wirklichkeit].

of the content of thoughts (§314), and we can say that certain propositions are real [wirklich], without assimilating the content of thoughts to existing things or events that occur. Recognizing the objectivity of thought—that its content can be shared and that its truth or falsity is independent of its being held to be true—does not require hypostasizing the contents of thought or confusing the kind of reality they possess [Validity, or Gultigkeit] with the existence of things in space and time. Propositions are real inasmuch as they are true or hold; concepts are real inasmuch as propositions containing them are true. The question here is not a semantic question, but a question of the ontological status of universals. Lotze's example of the kind of proposition that would confer validity on a concept, "The content of the concept <red> is eternal and self-identical," makes it clear that Lotze is not here advancing a thesis that is advanced by Frege in the *Grundlagen*. (How would such a sentence be at all helpful in determining what the particular meaning of "red" is?) The point is not the Fregean one that one should ask for *the meaning* of some expression "φ" by investigating the meaning of sentences containing "φ," but the different thesis that a concept has reality only because it is *meaningful*, and it is meaningful only because it is part of a proposition that is true. As he puts it later, the reality of general notions does not consist in existence, but in validity, that is, "in being *predicable of the existent*."[112] The passage from §321—occurring as it does in the middle of a discussion of the debate between Aristotle and Plato over the ontological status of the Ideas, which discussion is part of a larger discussion of the objectivity of content—is

[112] §341. This passage occurs in Book III, chapter IV, "The Real and Formal Significance of Logical Acts," in the middle of a discussion of the old problem of realism and nominalism. In §342, Lotze argues further that the validity of general concepts is not in fact real, but purely formal. That is, the concepts we possess are not reflections of how things are in themselves, but mere means for thinking about things. They are, in Lotze's favorite metaphor, the "scaffolding" of thought.

then an expression of Lotze's view that nominalism about universals is consistent with an appreciation of the objectivity of content and truth; §321 does not give a principle concerning the proper method of investigating the particular meaning of a term.

Each of Sluga's attempts at reading the context principle into Lotze, then, involves serious misreadings of Lotze's intentions.[113]

Psychologism and Objectivity

In notes 66 and 68 I have given two classic expressions of anti-psychologism, Lotze's §x and §332. But to our post-Fregean ears, Lotze's whole approach in Book I of his *Logik*, a summary of the principle results of which are represented in figure 6-3, seems mired in psychologism: how can one separate logic cleanly from psychology if concepts are taken to be a kind of idea [Vorstellung]? Or how can we distinguish the mere flow of ideas, which is presumably a subject for psychological investigation, from their objective contents merely by adding further "organizing" ideas—in Lotze's terms, accessory notions [Nachgedanke]—to the flow of ideas? Similarly, as Dummett[114] has protested, how can we see Frege's notion of objectivity[115] as a derivative of Lotze's notion of validity if Lotze takes the bearers of truth to be mere ideas?

Lotze thinks it important to stress that thoughts have a content that is objective, in the sense of being shareable and of being valid or invalid irrespective of its being held to

[113] Gabriel, in "Frege, Lotze, and the Continental Roots of Early Analytic Philosophy," 49, argues that the context principle is implicit in Lotze, but in a much different way. He writes: "The context principle appears as a semantic version of a metaphysical Hegelian principle that Frege took over from his teacher Lotze, while restricting it to propositions. Frege did not defend a holism outside of propositions." I will refrain from trying to evaluate this claim as an interpretation of Lotze, since I am not sure precisely what Gabriel is asserting. There is, however, no evidence that Frege ever read Lotze's *Metaphysik*.
[114] "Objectivity and Reality: Lotze and Frege."
[115] On *objektivität* in Frege, see *Grundlagen* §26, *Grundgesetze*, xviiiff., "Logic (1897)," in *PW*, 137-8.

be valid or invalid by individual thinkers. Lotze introduces his notion of objectification of a subjective impression at the very beginning (§3, see above) of *Logik*, as the first operation of thought. Thus, when I express myself using a definite article and referring to an impression, 'the sensation of red,' or 'the red,' I have come to pick out an element of my subjective state as something that someone else can come to recognize (though not sense directly as I can) and as something that I can meaningfully speak of even when I am not sensing it itself. Though we might be uncomfortable talking about the complete independence of that which is sensed [Empfindbare] (§2) from the sensing subject, we must recognize our capacity to refer to it in such a way that I can make truth-evaluable judgments of it that others can themselves understand.[116] The process of objectification is revisited by Lotze in his famous discussion of Plato's theory of Ideas. What Plato calls "an Idea," Lotze argues, or an "ideally grasped content [ideell gefaßte Inhalt]" is a content that means something in itself [an sich etwas bedeutenden].

> Our affection is objectified [objectivirte] to a self-standing content that always means [immer bedeutet] what it means and whose relations to other contents have an eternal and self-identical validity even if neither it nor they ever be repeated in actual perception (§314).

This content, insofar as we distinguish it from the "becoming" in our mind of particular ideas that possess that content, "so far as we regard it in abstraction from the mental activity which we direct to it, can no longer be said to occur, though neither again does it

---

[116] Dummett, in "Objectivity and Reality in Lotze and Frege," 106-107, reads too much into the fact that Lotze's example of the most primitive kind of objectivity is that at work in an expression like "the toothache" or "the idea of red." Dummett, ignoring the fuller statement of the doctrine in §314 and §345, misconstrues Lotze's rather sensible comments in §3 as a fundamental and general confusion between the objective and the intersubjective. In §3, Lotze wants only to claim that, though a statement like the "The sensation of magenta is brighter than that of ivory" is intersubjectively graspable and true independently of any thinker, it is odd to say that such a statement would be true or even contentful if there were no subjects to suffer sensations of color.

exist as things exist; we can only say it possesses Validity" (§316).[117] These contents, then, can either be or be components of "truths," whose status as true is completely independent of my thinking them to be true or of anyone at all thinking them to be true (§318). This fuller discussion of objectivity and validity allows Lotze then to introduce a needed distinction between the subjective activity of thinking and the content of that thinking, the thought, which is itself objective:

> If we distinguish, as we have done, between the logical act of thinking, and the thought [Gedanken] which it creates as its product [als ihr Product erzeugt], the former can claim only *subjective significance*; it is purely and simply an inner movement of the mind…through which we make that thought, for instance the distinction which holds between *a* and *b*, or the universal *C* which is contained in them both, an object for our own consciousness … The thought generated has *Objective* validity…[It is] an object independent of the subjectivity of the individual[—]an object presented to his thought which also presents itself as the same self-identical object to the consciousness of others. (§345)

So Lotze does make a distinction between a subjective idea in the psychological sense and its objective content. In light of that distinction, we can re-read Lotze's earlier statements in Book I as in fact statements about the objective contents of concepts and judgments. We do not, then, have to read Lotze as failing to make the distinction between "thoughts, which are what we think" and "what occurs in a stream of consciousness,"[118] or as confusing an idea and its content by thinking of a concept as a certain kind of composite idea.[119]

---

[117] I'm ignoring some messiness in Lotze here. It seems that to say of a content that it is objective is the same thing as saying that it is valid *in an extended sense*. I say "in an extended sense" because for a proposition to be valid is just for it be true or to hold; but Lotze clearly needs a notion of objectivity that extends to false propositions and the concepts that are their parts. See Gabriel, "Frege, Lotze, and the Continental Roots of Early Analytic Philosophy," 43. Lotze seems to be silent on whether things that aren't contents (say, a "Ding," a physical object) can be objective in some different sense.
[118] Dummett, "Frege's Kernsätze zur Logik," 71.
[119] Dummett, "Objectivity and Reality in Lotze and Frege," 105.

Sluga has urged that Frege's notion of objectivity is in fact derived from Lotze's, and he quotes §345 to show the clear affinities between Frege and Lotze on this point.[120] But the spoiler here is not that Lotze could not clearly distinguish between a subjective idea and its objective content, but Lotze's insistence that the content of an Idea or a thought (if these notions are to be kept rigorously distinct) is *generated* [erzeugen] by the act of thinking. Compare here a characteristic expression of Frege's objectivism about thoughts:

> Now we cannot regard thinking as a process that generates [ein Hervorbringen] thoughts. It would be just as wrong to identify a thought with an act of thinking, so that a thought is related to thinking as a leap is to leaping…Now if thoughts only came into existence as a result of thinking or is they were constituted by thinking, then the same thought could come into existence, cease to exist, and then come into existence which is absurd. As I do not create a tree by looking at it or cause a pencil to come into existence by taking hold of it, neither do I generate a thought by thinking.[121]

From Frege's point-of-view, then, Lotze's doctrine of objectivity, and the anti-psychologism that it allows, only gets us so far: a thought, Lotze allows us, should be distinguished from the act of thinking; but a thought, for Lotze, or a content of an idea, is still something created or generated by the act of thinking itself.[122]

---

[120] "Frege: the Early Years," 343.

[121] "Logic (1897)," 137. See also "Logic," *PW*, 7: "A judgeable content…is not the result of an inner process or the product of a mental act [Erzeugnis einer geistigen Tätigkeit] which men perform, but something objective."

[122] Compare Husserl, *Logical Investigations* I, 138 accuses Lotze of not carrying the separation of logic from psychology far enough: "His great logical work, rich as it is in original thoughts, becomes a jarring mixture of psychologism and pure logic." See also, Henry Jones, chapter 2, especially 98.

BIBLIOGRAPHY

Adamson, Robert. *A Short History of Logic*. London: W. Blackwood and Sons, 1911.

Adamson, Robert. "Critical Notice. *Der Operstionskreis des Logikkalkuls.* Von Dr. Ernst Schröder." *Mind* 3 o.s., no. 10 (1878): 252-5.

Adickes, Erich, ed. *Kants Handschriftlicher Nachlass. Mathematik – Physik und Chemie – Physische Geographie.* Volume 14 of Kant, Immanuel. *Gesammelte Schriften*. Berlin: Walter de Gruyter, 1966. The first edition appeared in 1911.

Allison, Henry. *Kant's Transcendental Idealism*, Revised and Enlarged Edition. New Haven: Yale University Press, 2004.

Anderson, R. Lanier. "Neo-Kantianism and the Roots of Anti-Psychologism." *British Journal for the History of Philosophy* 13, no. 2 (2005): 287-323.

Anderson, R. Lanier. "It All Adds Up After All: Kant's Philosophy of Arithmetic in Light of the Traditional Logic." *Philosophy and Phenomenological Research* LXIX, no. 3 (2004): 501-540.

Anderson, R. Lanier. "Synthesis, Cognitive Normativity, and the Meaning of Kant's Question: 'How are synthetic cognitions a priori possible?'" *European Journal of Philosophy* 9, no. 3 (December 2001): 275-305.

Aquila, Richard. *Representational Mind: A Study of Kant's Theory of Knowledge*. Bloomington: Indiana University Press, 1983.

Avigad, Jeremy. "Methodology and Metaphysics in the Development of Dedekind's Theory of Ideals." In *The Architecture of Modern Mathematics*, edited by José Ferreirós and Jeremy Gray, 159-186. Oxford: Oxford University Press, 2006.

Bauch, Bruno. "Lotzes Logik und ihre Bedeutung im deutschen Idealismus." *Beiträge zur Philosophie des deutschen Idealismus* 1, no. 2 (1918): 45-58.

Baumgarten, Alexander Gottlieb. *Logica*. Halle: Hemmerde, 1761. Reprinted in Wolff, *Gesammelte Werke*, III.5.

Beck, Lewis White. "Can Kant's Synthetic Judgments Be Made Analytic?" In his *Studies in the Philosophy of Kant*. Indianapolis, 1965.

Beck, Lewis White. "Kant's Theory of Definition." In his *Studies in the Philosophy of Kant*. Indianapolis, 1965.

Bonola, Roberto. *Non-Euclidean Geometry*: *A Critical and Historical Study of its Developments*. Translated by H.S. Carslaw. New York: Dover, 1955.

Boole, George. *An Investigation of the Laws of Thought*. London: Walton and Maberly, 1854. Reprint New York: Dover, 1951.

Brown, J.R. "Proofs and Pictures," *British Journal of the Philosophy of Science* 48, no. 2 (1897): 161-180.

Burge, Tyler. "Introduction." In his *Truth, Thought, Reason: Essays on Frege*.

Burge, Tyler. "Frege on Sense and Meaning." In his *Truth, Thought, Reason: Essays on Frege*.

Burge, Tyler. *Truth, Thought, Reason: Essays on Frege*. Oxford: Clarendon Press, 2005.

Burge, Tyler. "Postscript to 'Frege on Truth.'" In his *Truth, Thought, Reason: Essays on Frege*.

Cantor, Georg. *Grundlagen einer allgemeinen Mannifaltigkeitslehre*. Leipzig: Teubner, 1883. Reprinted in Cantor, Georg, *Gesammelte Abhandlungen*, edited by Ernst Zermelo. Berlin: Springer Verlag, 1932. Translated by William Ewald in *From Kant to Hilbert*, vol. 2.

Carl, Wolfgang. *Frege's Theory of Sense and Reference: Its Origins and Scope*. Cambridge: Cambridge University Press, 1994.

Carson, Emily. "Kant on Inuition in Geometry." *Canadian Journal of Philosophy* 27, no. 4 (December 1997): 489-512.

Cassirer, Ernst. "Reflections on the Concept of Group and the Theory of Perception (1945)." In *Symbol, Myth and Culture*, edited by Donald Phillip Verene, 271-295. New Haven: Yale University Press, 1979.

Cassirer, Ernst. "Zur Theorie des Begriffs." Reprinted in *Erkenntnis, Begriff, Kultur*, edited by. R. Bast. Hamburg: Felix Meiner Verlag, 1993. The article originally appeared in 1928.

Cassirer, Ernst. "Erkenntnistheorie nebst den Grenzfragen der Logik." Reprinted in *Erkenntnis, Begriff, Kultur*, edited by. R. Bast. Hamburg: Felix Meiner Verlag, 1993. The article originally appeared in 1913.

Cassirer, Ernst. *Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundfragen der Erkenntniskritik*. Berlin: Bruno Cassirer, 1910. Translated by

Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity*. Chicago: Open Court, 1923.

Cassirer, Ernst. *Zur Einstein'schen Relativitätstheorie*. Berlin: Bruno Cassirer Verlag, 1921. Translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity*. Chicago: Open Court, 1923.

Cassirer, Ernst. "Inhalt und Umfang des Begriffs." *Theoria* 2 (1936). Reprinted in *Erkenntnis, Begriff, Kultur*, edited by. R. Bast. Hamburg: Felix Meiner Verlag, 1993.

Cassirer, Ernst. "Kant und moderne Mathematik." *Kant-Studien* 12 (1907): 1-40.

Cassirer, Ernst. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit. Erster Band.* 3rd ed. Berlin: Bruno Cassirer, 1922. The first edition appeared in 1906.

Cassirer, Ernst. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit. Zweiter Band.* 3rd ed. Berlin: Bruno Cassirer, 1922. The first edition appeared in 1907.

Cassirer, Ernst. *Kants Leben und Lehre*. Berlin: Bruno Cassirer, 1918. Translated by Haden as *Kant's Life and Thought*. New Haven: Yale University Press, 1981.

Cassirer, Ernst. *Philosophie der Symbolischen Formen*. *Dritter Teil: Phänomenologie der Erkenntnis*. Berlin: Bruno Cassirer, 1929. Translated by Hendel and Woglom as *Philosophy of Symbolic Forms. Volume Three: The Phenomenology of Knowledge*. New Haven: Yale University Press, 1957.

Cassirer, Ernst. *The Problem of Knowledge: Philosophy, Science, and History since Hegel*. Translated by Woglom and Hendel. New Haven: Yale University Press, 1950.

Cassirer, Ernst. *Leibniz' System in seinen wissenschaftlichen Grundlagen*. Marburg: Elwert, 1902.

Cassirer, Ernst. "Hermann Cohen und die Erneuerung der Kantischen Philosophie." *Kant-Studien* 17, no. 3 (1912): 252-273.

Cayley, Arthur. "Sixth Memoir upon Quantics." In *Mathematical Papers*, vol. 2. Cambridge: Cambridge University Press, 1889-98.

Cayley, Arthur. "Presidential Address to the British Association." In *Mathematical Papers*, vol. 9. Cambridge: Cambridge University Press, 1889-98. Reprinted in *From Kant to Hilbert*. Vol. I, edited by Ewald, 542-573.

Chasles, Michel. *Aperçu historique sur l'origine et le développement des méthodes en géométrie, particulièrement de celles qui se rapportent à la géométrie moderne.*

Translated by L. A. Sohncke as *Geschichte der Geometrie hauptsächlich mit Bezug auf die neueren Methoden*. Halle: Gebauer, 1839.

Coffa, J. Alberto. “From Geometry to Tolerance.” In *From Quarks to Quasars*, edited by Robert Colodny. Vol. 7 of *University of Pittsburgh Series in the Philosophy of Science*, 3-70. Pittsburgh: University of Pittsburgh Press, 1986.

Coffa, J. Alberto. *The Semantic Tradition from Kant to Carnap.* Cambridge: Cambridge University Press, 1991.

Coffa, J. Alberto. “Kant, Bolzano, and the Emergence of Logicism.” *Journal of Philosophy* 79 (1982): 679-689.

Cohen, Hermann. *Die Logik der reinen Erkenntnis*. 4th ed, Georg Olms Verlag, New York: 1977. The fourth edition is a reprint of the second edition from 1914; the first edition appeared in 1902.

Cohen, Hermann. *Kants Theorie der Erfahrung*. 2nd ed. Berlin: Dümmler, 1885.

Cohen, Hermann. *Logik der reinen Erkenntnis*. Berlin: Bruno Cassirer, 1902.

Cohen, Hermann. *Das Princip der Infinitesimal-Method und seine Geschicte. Ein Kapitel zur Grundlegung der Erkenntniskritik*, 4th ed. New York: Georg Olms Verlag, 1984. The first edition appeared in 1883.

Coxeter, H.S.M. *Projective Geometry,* 2nd ed. New York: Springer-Verlag, 1987.

Darboux, M. Gaston. “Survey of the Development of Geometric Methods.” Bulletin of the American Mathematical Society 11, no.10 (1905): 517-543.

Dedekind, Richard. “On the Introduction of New Functions in Mathematics.” Translated in Ewald, *From Kant to Hilbert*, vol. ii, 755-762.

Dedekind, Richard. *Essays on the Theory of Numbers*. Translated by Beman. New York: Dover, 1963.

Dedekind, Richard. *Gesammelte mathematische Werke*. Edited by Fricke, Noether, and Ore. Braunschweig: F Vieweg & Sohn, 1932.

Dedekind, Richard. “Letter to Heinrich Weber (24 January 1888).” Translated in Ewald, *From Kant to Hilbert*, vol. ii, 834-5. Originally published in German in Dedekind, *Gesammelte mathematische Werke*, vol. 3.

Dewey, John. “Notes upon Logical Topics.” *Journal of Philosophy* 1, no.3 (February 1904): 57-62.

Dickerson, A.B. *Kant on Representation and Objectivity*. Cambridge: Cambridge University Press, 2004.

Dove, I. “Can Pictures Prove?” *Logique & Analyse* 45, no. 179-180 (2002): 309-340.

Dummett, Michael. “Frege’s ‘Kernsätze zur Logik.’” In his *Frege and Other Philosophers*, 65–78. Oxford: Clarendon Press, 1991.

Dummett, Michael. “Frege as a Realist.” In his *Frege and Other Philosophers*, 79-97. Oxford: Clarendon Press, 1991.

Dummett, Michael. “Objectivity and Reality in Lotze and Frege.” In his *Frege and Other Philosophers*, 97-126. Oxford: Clarendon Press, 1991.

Dummett, Michael. *Frege: Philosophy of Language*. London: Duckworth, 1973.

Dummett, Michael. *Frege: Philosophy of Mathematics*. London: Duckworth, 1991.

Einstein, Albert. “Geometry and Experience.” Translated by Jeffery and Perrett in *Sidelights on Relativity*. New York: Dover, 1983.

Erdmann, Benno. *Logische Elementarlehre*, 2nd ed. Halle: Niemeyer, 1907. The first edition appeared in 1892.

Ewald, William. *From Kant to Hilbert*. 2 vols. Oxford: Clarendon Press, 1996.

Ferrari, Massimo. “Ernst Cassirer und Pierre Duhem.” In *Kulturkritik nach Ernst Cassirer*, edited by Rudolph Küppers, 177-196. Hamburg: Felix Meiner, 1995.

Flohr, F. and Raith, F. “Affine and Euclidean Geometry.” In *Fundamentals of Mathematics. Volume II: Geometry*, edited by H. Behnke et al, translated by S.H. Gould. Cambridge, MA: MIT Press, 1974.

Frege, Gottlob. *Collected Papers on Mathematics, Logic, and Philosophy*. Edited by Brian McGuiness and translated by Max Black, V. H. Dudman, Peter Geach, Hans Kaal, E. –H. Kluge, Brian McGuiness, and R. H. Stoothof. New York: Basil Blackwell, 1984.

Frege, Gottlob. *On the Foundations of Geometry and Formal Theories of Arithmetic*. Translated by Eike-Henner W. Kluge. New Haven and London: Yale University Press, 1971.

Frege, Gottlob. *Philosophical and Mathematical Correspondence*. Edited by Gottfried Gabriel, Hans Hermes, Friedrich Kambartal, Christian Thiel, and Albert Veraart. Abridged by Brian McGuiness. Translated by Hans Kaal. Chicago: The University of Chicago Press, 1980.

Frege, Gottlob. *Posthumous Writings*. Edited by Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach. Translated by Peter Lond and Roger White. Oxford: Basil Blackwell, 1979.

Frege, Gottlob. *Conceptual Notation and Related Articles.* Translated by Terrell Ward Bynum. Oxford: Oxford University Press, 1972.

Frege, Gottlob. *Begriffsschrift*. Translated by Michael Beaney in *The Frege Reader*, 47-78. Oxford: Blackwell Publishers, 1997.

Frege, Gottlob. *Die Grundlagen der Arithmetik. Eine logisch-matheamtische Untersuchung über den Begriff der Zahl*. Breslau: Koebner, 1882. Translated by Austin as *The Foundations of Arithmetic*. Oxford: Blackwell, 1950.

Frege, Gottlob. *Grundgesetze der Arithmetik*, vol. 1. Jena: H. Pohle, 1893.

French, Stephen. "Symmetry, Structure, and the Constitution of Objects." *Conference on Symmetries in Physics: New Reflections*, University of Oxford, January 2001. http://philsci-archive.pitt.edu/archive/00000327/index.html.

Freudenthal, Hans. "The Impact of Von Staudt's Foundations of Geometry." In *For Dirk Struik*, edited by R.S. Cohen, J.J. Stachel, 189-200. Dordrecht: Reidel Publ. Co., 1974.

Freudenthal, Hans. "The Main Trends in the Foundations of Geometry in the Nineteenth Century." In *Logic, Methodology, and the Philosophy of Science*, edited by Nagel, Suppes, and Tarski, 613-621. Stanford: Stanford University Press, 1962.

Friedman, Michael. "Geometry, Construction, and Intuition in Kant and His Successors." In *Between Logic and Intuition: Essays in Honor of Charles Parsons*, edited by Sher and Tieszen, 186-218. Cambridge: Cambridge University Press, 2000.

Friedman, Michael. *Kant and the Exact Sciences*. Cambridge, MA: Harvard University Press, 1992.

Friedman, Michael. "Ernst Cassirer and the Philosophy of Science." In *Continental Philosophy of Science,* edited by Gutting, 71-83. Oxford: Blackwell, 2005.

Friedman, Michael. *A Parting of the Ways: Carnap, Cassirer, and Heidegger.* Chicago: Open Court Press, 2000.

Friedman, Michael. *Dynamics of Reason*. Stanford: CSLI Publications, 2001.

Friedman, Michael. "Ernst Cassirer and Contemporary Philosophy of Science". *Angelaki: Journal of the Theoretical Humanities* 10, no. 1 (April 2005): 119-128.

Friedman, Michael. *Reconsidering Logical Positivism*. Cambridge: Cambridge University Press, 1999.

Friedman, Michael. "Logical Form and the Order of Nature: Comments on Beatrice Longuenesse's *Kant and the Capacity to Judge.*" *Archiv für Geschichte der Philosophie* 82 (2000): 202-15.

Gabriel, Gottfried. “Frege, Lotze, and the Continental Roots of Early Analytic Philosophy.” In *From Frege to Wittgenstein*, edited by Erich Reck, 39-52. Oxford University Press, 2002.

Gabriel, Gottfried. “Objektivität, Logik und Erkenntnistheorie bei Lotze und Frege.” In *Logik. Drittes Buch. Vom Erkennen*, by Hermann Lotze, edited by Gottfired Gabriel, ix-xxxvi. Hamburg: Felix Meiner Verlag, 1989.

Ginsborg, Hannah. “Lawfulness without a Law.” *Philosophical Topics* 25 (1997): 37-81.

Ginsborg, Hannah. “Aesthetic Judgment and Perceptual Normativity.” *Inquiry* 49, no. 5 (October 2006): 403-437.

Ginsborg, Hannah. “Thinking the Particular as Contained under the Universal.” In *Aesthetics and Cognition in Kant’s Critical Philosophy*, edited by Rebecca Kukla, 35-60. Cambridge: Cambridge University Press, 2006.

Grattan-Guinness, I. *The Search for Mathematical Roots, 1870-1940*. Princeton: Princeton University Press, 2000.

Green, Thomas Hill. *Works of Thomas Hill Green.* 3 vols. Edited by R.L. Nettleship. London: Longmans, Green, and co., 1886.

Guyer, Paul. *Kant and the Claims of Knowledge*. Cambridge: Cambridge University Press, 1987.

Hahn, Hans. “The Crisis in Intuition.” In his *Empiricism, Logic, and Mathematics: Philosophical Papers*, edited by Brian McGuinness. Dordrecht: Reidel, 1980.

Hallett, Michael. “Physicalism, Reductionism, and Hilbert.” In *Physicalism in Mathematics*, edited by A. Irvine, 182-256. Dordrecht: Reidel, 1990.

Hamilton, William. “Lectures on Logic.” In his *Lectures on Metaphysics and Logic*, vol. 3, edited by Mansel and Veitch. William Blackwood and Sons: London, 1866

Hankel, Ernst. *Die Elemente der projectivischen Geometrie, in synthetischer Behandlung. Vorlesungen*. Leipzig: Teubner, 1875.

Hanna, Robert. “Mathematics for Humans: Kant’s Philosophy of Arithmetic Revisited.” *European Journal of Philosophy* 10 (2002): 328-353.

Helmholtz, Hermann. “The Origin and Meaning of Geometrical Axioms.” In *From Kant to Hilbert*, vol. ii, edited by Ewald. Translated by Edmund Atkinson. Oxford: Clarendon Press, 1996. The original paper appeared in two parts in the journal *Mind*: *Mind* 1 (1876): 301-21 and *Mind* 3 (1878): 212-25.

Heath, Sir Thomas, trans. *The Thirteen Books of Euclid's Elements, translated from the text of Heiberg, with introduction and commentary*. 2nd ed. Cambridge, Cambridge University Press, 1925.

Hilbert, David and S. Cohn-Vossen. *Geometry and the Imagination*. Translated by P. Nemenyi. New York: Chelsea, 1952.

Hilbert, David. *Grundlagen der Geometrie*. Translated as *Foundations of Geometry* by Leo Unger. LaSalle, Illinois: Open Court, 1971.

Hilbert, David. "Axiomatisches Denken." *Mathematische Annalen* 78 (1918): 405-415. Translated by Ewald in *From Kant to Hilbert*, vol. ii, 1107-1115.

Hilbert, David. "Die Rolle von Idealen Gebilden." In *Natur und mathematisches Erkennen*, edited by David E. Rowe. Basel: Birkhäuser, 1992.

Hilbert, David. "Mathematische Probleme." In his *Gesammelte Abhandlungen*. Vol. 3. Berlin: Springer, 1935. The work first appeared in 1900.

Hilbert, David. *Neubegründung der Mathematik*. In his *Gesammelte Abhandlungen*. Vol. 3. Berlin: Springer, 1935. The work first appeared in 1922.

Hintikka, Jaako. "Russell, Kant, Coffa." *Synthese* 46 (1981): 265-70.

Holzhey, Helmut. *Cohen und Natorp*. 2 vols. Basel and Stuttgart: Schwabe, 1986.

Husserl, Edmund. *Logical Investigations*. Translated by J. N. Findlay. New York: Routledge, 2001.

Hylton, Peter. "Hegel and Analytic Philosophy." In *Cambridge Companion to Hegel*, edited by Federick C. Beiser. Cambridge: Cambridge University Press, 1993.

Ihmig, Karl-Norbert. *Cassirers Invariantentheorie der Erfarhung und seine Rezeption des 'Erlangers Programms.'* Hamburg: Meiner, 1997.

Jevons, William Stanley. "Pure Logic or The Logic of Quality apart from Quantity, with remarks on Boole's system and on the relation of logic to mathematics." In his *Pure Logic and other Minor Works*, edited by Robert Adamson and Harriet A. Jevons. New York: Macmillan and co., 1890. The essay originally appeared in 1864.

Jevons, William Stanley. "On a General System of Numerically Definite Reasoning." In his *Pure Logic and other Minor Works*, edited by Robert Adamson and Harriet A. Jevons. New York: Macmillan and co., 1890. The essay originally appeared in 1870.

Jevons, William Stanley. *Principles of Science. A Treatise on Logic and Scientific Method.* 2 vols. New york: Macmillan and co., 1877.

Jones, Henry. *A Critical Account of the Philosophy of Lotze: the Doctrine of Thought.* New York, Macmillan and co., 1895.

Kant, Immanuel. "On the Form and Principles of the Sensible and the Intelligible World." In *Theoretical Philosophy: 1755-1770*, translated and edited by Walford and Meerbote. Cambridge: Cambridge University Press, 1992.

Kant, Immanuel. *Philosophical Correspondence, 1759-99.* Edited and translated by Arnulf Zweig. Chicago: University of Chicago Press, 1967.

Kant, Immanuel. *Critique of Judgment.* Translated by Werner Pluhar. Indianapolis: Hackett, 1987.

Kant, Immanuel. *Critique of Pure Reason*. Translated by Paul Guyer and Allen Wood. Cambridge: Cambridge University Press, 1998.

Kant, Immanuel. *Gesammelte Schriften.* Edited by the Königlich Preußischen Akademie der Wissenschaft. 29 vols. Berlin: DeGruyter, 1902-.

Kant, Immanuel. *Lectures on Logic*. Translated and edited by J. Michael Young. Cambridge: Cambridge University Press, 1992.

Kant, Immanuel. *Prolegomena to Any Future Metaphysics That Can Qualify as a Science*. Translated by Paul Carus. Chicago: Open Court, 1902.

Kant, Immanuel. *The Kant-Eberhard Controversy*. Translated and edited by Henry Allison. Baltimore: Johns Hopkins University Press, 1973.

Kant, Immanuel. *What Real Progress Has Metaphysics Made In German Since the Time of Leibniz?* Translated by Peter Heath in *Theoretical Philosophy After 1781*, edited by Henry Allison and Peter Heath, 337-424. Cambridge: Cambridge University Press, 2002.

Kerry, Benno. *System einer Theorie der Grenzbegriffe. Ein Beitrag zur Erkenntnistheorie*. Leipzig: 1890.

Kim, Alan. "Paul Natorp." In *Stanford Encyclopedia of Philosophy*. 12 August 2003. http://plato.stanford.edu/entries/natorp.

Kinkel, Walter. *Hermann Cohen: Eine Einleitung in sein Werk.* Stuttgart: 1924.

Kinkel, Walter. *Immanuel Kants Logik: Ein Handbuch zu Vorlesungen, dritte Auflage, neu herausgegeben, mit einer Einleitung sowie einem Personen- und Sachregister versehen.* 3rd ed. Leipzig: Felix Meiner, 1920. The first edition appeared in 1904.

Kitcher, Philip. "Kant and the Foundations of Mathematics." *Philosophical Review* 84 (1975): 23-50.

Kitcher, Philip. *The Nature of Mathematical Knowledge*. New York: Oxford University Press, 984.

Kitcher, Phillip. “Frege, Dedekind, and the Philosophy of Mathematics.” In *Frege Synthesized*, edited by L. Haaparanta and Jakko Hintikka, 299-343. Dordrecht: D. Reidel, 1986.

Klein, Felix. *Elementary Mathematics from an Advanced Standpoint: Geometry.* Translated by E. R. Hedrick and C. A. Noble. New York: Macmillan, 1939. The first German edition appeared in 1908.

Klein, Felix. *The Evanston Colloquium: Lectures on Mathematics.* New York: American Mathematical Society, 1911.

Klein, Felix. “Über Liniengeometrie und metrische Geometrie.” In his *Gesammelte mathematische Abhandlungen*, vol 1, edited by Robert Fricke, Alexander Ostrowski, Hermann Vermeil, and E. Bessel-Hagen, 106. New York: Springer Verlag, 1973. The paper first appeared in 1871-2.

Klein, Felix. “Riemann and His Significance for the Development of Modern Mathematics.” *Bulletin of the American Mathematical Society* 1 (April 1895): 165-180. Reprinted in *The Way it Was: Mathematics from the Early Years of the Bulletin*, edited by Donald Sarri. Oxford: Oxford University Press, 2004.

Klein, Felix. “On the So-Called Non-Euclidean Geometry.” Translated in *Sources of Hyperbolic Geometry*, by John Stillwell. New York: American Mathematical Society, 1996. The paper first appeared in German in 1871-3.

Klein, Felix. *Development of Mathematics in the Nineteenth Century*. Translated by M. Ackerman. Brookline, MA: Math. Sci. Press, 1979.

Kleinman, Steven. “Chasles’s Enumerative Theory of Conics: a Historical Introduction.” In *Studies in Algebraic Geometry*, edited by A. Seidenberg, 117-138. Washington, D.C.: Mathematical Association of America, 1980.

Kline, Morris. “Projective Geometry.” In *The World of Mathematics*, edited by Newman. Vol. 1. New York: Simon and Schuster, 1956.

Kline, Morris. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.

Krois, J. *Cassirer: Symbolic Forms and History*. New Haven: Yale University Press, 1987.

Lask, Emil. *Fichtes Idealismus und die Geschichte*. In his *Gesammelte Schrfiten*, vol. 1. Tübingen: Mohr, 1923. The work first appeared in 1901.

Lasswitz, Kurd. Review of *Begriffsschrift*, by Gottlob Frege. *Jenaer Literaturzeitung* 6 (1879): 248-9. Translated in *Conceptual Notation and Related Articles.* Translated by Terrell Ward Bynum. Oxford: Oxford University Press, 1972.

Longuenesse, Beatrice. *Kant and the Capacity to Judge*. Translated by Charles Wolfe. Princeton: Princeton University Press, 1998.

Longuenessse, Beatrice. "Synthesis, Logical Forms, and the Objects of Our Ordinary Experience." In her *Kant on the Human Standpoint*, 39-63. Cambridge: Cambrdge University Press, 2005.

Lotze, Hermann. *Logic*. Translated from the second German edition under the editorship of Bernard Bosanquet. 2 vols. Oxford: the Clarendon Press, 1888.

Lotze, Hermann. *Logik. Drei Bücher vom Denken, vom Untersuchen und vom Erkennen*. 2nd edition. Leipzig: S. Hirzel, 1880. The first edition appeared in 1874.

Lotze, Hermann. *Metaphysik*. Leipzig: S. Hirzel, 1879.

MacFarlane, John. "Frege, Kant, and the Logic in Logicism." *Philosophical Review* 111 (2000): 25-65.

Maddy, Penelope. *Naturalism in Mathematics*. Oxford: Oxford University Press, 1997.

Manders, Kenneth. "Applying Mathematical Concepts." Unpublished MS.

Manders, Kenneth. "The Euclidean Diagram (1995)." In *The Philosophy of Mathematical Practice*, edited by Paolo Mancosu, 112-183. Oxford: The Clarendon Press, 2007.

Martin, Gottfried. *Arithmetic and Combinatorics: Kant and his Contemporaries.* Translated and edited by Judy Wubig. Carbondale: Southern Illinois University Press, 1985.

McCarty, David Charles. "The Mysteries of Richard Dedekind." In *From Dedekind to Gödel: Essays on the Development of the Foundations of Mathematics*, edited by Jaakko Hintikka, 53-96. Dordrecht: D. Reidel, 1995.

Meier, G.W.F. *Auszug aus der Vernunftlehre*. Halle: 1752. Reprinted in Kant, *Gesammelte Schriften*, vol. XVI.

Merrick, Teri. "What Frege Meant When He Said: Kant is Right About Geometry". *Philosophia Mathematica* 14, no. 1 (2006): 44-75.

Merz, John Theodore. *A History of European Thought in the Nineteenth Century*. 4 vols. 3rd edition. London: William Blackwood and Sons, 1907.

Meserve, Bruce. *Fundamental Concepts of Geometry*. New york: Dover, 1983.

Milkov, Nikolay, "Russell's Debt to Lotze." *Studies in History and Philosophy of Science Part A* 38 (2007).

Mumma, John. "Proofs, Pictures, and Euclid." Unpublished MS.

Mumma, John. "Understanding Euclid's Proofs." Unpublished MS.

Mumma, John. "Intuition Formalized: Ancient and Modern Methods of Proof in Elementary Geometry." PhD diss., Carnegie Mellon University, 2006.

Nagel, Ernest. "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry." In his *Teleology Revisited*. New York: Columbia University Press, 1929.

Natorp, Paul. *Die Logischen Grundlagen der Exakten Wissenschaft*. Leipzig and Berlin: 1910.

Natorp, Paul. "Kant und die Marburger Schule." *Kant-Studien* 17 (1912): 193-221.

Natorp, Paul. *Philosophie: Ihr Problem und ihre Probleme: Einführung in den kritischen Idealismus*. 2nd ed. Göttingen: Vandenhoeck & Ruprecht, 1918. The first edition appeared in 1911.

Netz, Reviel. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge; Cambridge University Press, 2003.

Norman, Jesse. *After Euclid: Visual Reasoning and the Epistemology of Diagrams*. Stanford: CSLI Publications, 2006.

Parsons, Charles. "The Transcendental Aesthetic." In *The Cambridge Companion to Kant*, edited by Paul Guyer. Cambridge: Cambridge University Press, 1992.

Parsons, Charles. "Kant's Philosophy of Arithmetic." In *Mathematics in Philosophy*. Ithaca: Cornell University Press, 1983.

Parsons, Charles. "Structuralism and Metaphysics." *The Philosophical Quarterly* 54, no.214 (January 2004): 56-77.

Pasch, Moritz. *Vorlesungen über Neuere Geometrie*. Leibzig: Teubner, 1882.

Paton, H. J. *Kant's Metaphysic of Experience*: *A Commentary on the First Half of th Kritik der reinen Vernunft.* 2 vols. 2nd ed. London: Macmillan, 1951.

Patton, Lydia. "The Critical Philosophy Renewed: The Bridge Between Hermann Cohen's Early Work on Kant and Later Philosophy of Science." *Angelaki: Journal of the Theoretical Humanities* 10, no. 1 (April 2005): 109-118.

Patzig, G. "Leibniz, Frege und die sogennante 'lingua characteristica universalis.'" *Studia Leibnitiana* Supplementa (1969): 103-112.

Pearson, Karl. *The Grammar of Science.* 2nd ed. London: A. & C. Black, 1900.

Pippin, Robert. *Kant's Theory of Form.* New Haven: Yale University Press, 1981.

Poincare, Henri. *Science and Hypothesis*. Translated by Greenstreet, in *The Value of Science: Essential Writings of Henri Poincare*. New York: The Modern Library, 2001.

Poincare, Henri. *Science and Method*. Translated by Greenstreet, in *The Value of Science: Essential Writings of Henri Poincare*. New York: The Modern Library, 2001.

Poincare, Henri. *The Value of Science*. Translated by Greenstreet, in *The Value of Science: Essential Writings of Henri Poincare*. New York: The Modern Library, 2001.

Poncelet, Jean-Victor., *Traité des propriétés projectives des figures: ouvrage utile à qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain*. 2 vols. 2nd ed. Paris: Gauthier-Villars, 1865-66. Partially translated in *A Source Book in Mathematics*, edited by David Eugene Smith. New York: Dover, 1959. The first French edition appeared in 1822.

Poma, Andrea. *The critical philosophy of Hermann Cohen*. Translated by John Denton. Albany: State University of New York Press, 1997.

Quine, W.V. *Pursuit of Truth*. Cambridge, MA: Harvard University Press, 1992.

Quine, W.V. "Reply to Parsons." In *The Philosophy of W.V. Quine*, edited by Hahn and Schilpp. La Salle, Ill.: Open Court, 1986.

Reck, Erich. "Dedekind's Structuralism: An Interpretation and Partial Defense." *Synthese* 137 (2003): 369-419.

Reich, Klaus. *Die Vollständigkeit der kantischen Urteilskraft*. Berlin: 1932. Translated as *The Completeness of Kant's Table of Judgments*, by Jane Kneller and Michael Losonsky. Stanford: Stanford University Press, 1992

Reichenbach, Hans. *Relativitätstheorie und Erkenntnis apriori*. Berlin: Springer Verlag, 1920. Translated and edited by Maria Reichenbach as *The Theory of Relativity and A Priori Knowledge*. Berkeley: University of California Press, 1965.

Resnik, Michael D. *Frege and the Philosophy of Mathematics*. Ithaca, New York: Cornell University Press, 1980.

Richardson, Alan. "'The Fact of Science' and Critique of Knowledge: Exact Science as Problem and Resource in Marburg Neo-Kantianism." In *The Kantian Legacy in Nineteenth-century Science*, edited by Friedman and Nordmann, 211-226. Cambridge, MA: MIT Press, 2006.

Richardson, Alan. *Carnap's Construction of the World*: *The Aufbau and the Emergence of Logical Empiricism*. Cambridge: Cambridge University Press, 1998.

Richardson, Joan. *Mathematical Visions. The pursuit of geometry in Victorian England.* Academic Press, 1988.

Rickert, Heinrich. *Zur Lehre der Definition.* Freiburg: 1888.

Rickert, Heinrich. *The Limits of Concept Formation in Natural Science*. Translated by Guy Oakes. Cambridge: Cambridge University Press, 1986. The first German edition appeared between 1896 and 1902.

Riemann, Georg Friedrich Bernhard. "On the Hypotheses which Lie at the Foundation of Geometry" (1868). Translated in *From Kant to Hilbert*, edited by Ewald, vol 2, 650-661. Oxford: Clarendon Press, 1996.

Russell, Bertrand. *An Essay on the Foundations of Geometry*. New York: Dover, 1956. First published 1897 by Cambridge University Press.

Russell, Bertrand. *Introduction to Mathematical Philosophy*. 2nd ed. New York: Macmillan, 1920.

Russell, Bertrand. "Are the Axioms of Geometry Empirical?" In *Philosophical papers, 1896-1899. Collected Papers*. Vol. 2, edited by Nicholas Griffin. London: Routledge, 1990.

Russell, Bertrand. "Geometry, Non-Euclidean." In *Toward the "Principles of Mathematics." Collected Papers*. Vol. 3, edited by Gregory Moore. London: Routledge, 1993.

Russell, Bertrand. "Mathematics and the Metaphysicians." In *Mysticism and Logic*. New York: Longermans, Green and Co., 1918.

Russell, Bertrand. *A Crticial Exposition of the Philosophy of Leibniz.* 2nd ed. London: Routledge, 1937. The first edition appeared in 1900.

Russell, Bertrand. *The Principles of Mathematics*. 2nd ed. New York: Norton, 1938. The first edition appeared in 1903.

Ryckman, Thomas. "*Condition Sine Qua Non? Zuordnung* in the Early Epistemologies of Cassirer and Schlick." *Synthese* 88 (1991): 57-95.

Ryckman, Thomas. *The Reign of Relativity: Philosophy in Physics, 1915-1925*. New York: Oxford University Press, 2005.

Schlick, Moritz. "Critical or Empiricist Interpretation of Modern Physics?" In his *Philosophical Papers*, vol 1, edited by Mulder and Velde-Schlick, translated by Heath. Dordrecht: Reidel, 1979.

Schlick, Moritz. *Allgemeine Erkenntnislehre*. 1st ed. Berlin: 1918.

Schröder, Ernst. *Der Operationskreis des Logikkalküls.* Liipzig: Teubner, 1877.

Schröder, Ernst. Review of *Begriffsschrift*, by Gottlob Frege. *Zeitschrift für Mathematik und Physik*, 25 (1880). Translated by Bynum in *Conceptual Notation*, 219.

Schröder, Ernst. *Vorlesungen über die Algebra der Logik*. 3 vols. Leipzig: Teubner, 1890-1905.

Schultz, Johann. *Prüfung der Kantischen Critik der reinen Vernunft*. Königsberg: 1789-1792.

Sellars, Wilfrid. "Phenomenalism." In his *Science, Perception, and Reality*, 60-105. New York: The Humanities Press, 1963.

Sellars, Wilfrid. "Some Remarks on Kant's Theory of Experience." *Journal of Philosophy* 64 (1967): 633-47.

Sellars, Wilfrid. *Science and Metaphysics*. Atascadero, California: Ridgeview, 1967.

Shabel, Lisa. "Kant on the 'Symbolical Construction' of Mathematical Concepts." *Stud. Hist. Phil. Sci.* 29, no. 4 (1998): 589-621.

Shabel, Lisa. "Kant's Philosophy of Mathematics." In *The Cambridge Companion to Kant and Modern Philosophy*, edited by Paul Guyer. Cambridge: Cambridge University Press, 2006.

Shabel, Lisa. *Mathematics in Kant's Critical Philosophy*. New York: Routledge, 2003.

Sigwart, Christoph. *Logic*. Translated by Helen Dendy. New York: MacMilla, 1895. The original German edition appeared in 1873.

Sluga, Hans. "Frege against the Booleans." *Notre Dame Journal of Formal Logic* 28, no. 1 (1987): 80-98.

Sluga, Hans. "Frege: the early years." *Philosophy in History: Essays in the Historiography of Philosophy*, edited by Richard Rorty, Jerome B. Schneewind, and Quentin Skinner, 329-356. New York: Cambridge University Press, 1984.

Sluga, Hans. *Gottlob Frege*. Boston: Routledge & Kegan Paul, 1980.

Sluga, Hans. "Frege and the Rise of Analytical Philosophy." *Inquiry* 18 (1975): 471-487.

Smart, Harold R. "Cassirer's Theory of Mathematical Concepts." In *The Philosophy of Ernst Cassirer*, edited by Schilpp, 239-267. La Salle: Open Court, 1949.

Staudt, Karl von. *Geometrie der Lage*. Nürnberg, F. Korn: 1847.

Staudt, Karl von. *Beiträge zur Geometrie der Lage*. Nürnberg: Bauer and Raspe, 1856, 1857, and 1860.

Stein, Howard, "Logos, Logic, Logistike: Some philosophical remarks on nineteenth-century transformation of mathematics." In *History and Philosophy of Modern Mathematics,* edited by Aspray and Kitcher. University of Minnesota Press, 1988.

Steiner, Jakob. *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*. In *Gesammelte Werke*. Vol. 1., edited by K. Weierstrass. Berlin: G. Reimer, 1881.

Stillwell, John. *The Four Pillars of Geometry*. New York: Springer, 2005.

Stillwell, John, editor. *Sources of Hyperbolic Geometry*. New York: American Mathematical Society, 1996.

Stokes, G.J. "The Philosophy of the Imaginary." *Mind* ix (July 1900): 349-355.

Stone, Abraham D. "The Continental Roots of Verificationism." *Angelaki: Journal of the Theoretical Humanities* 10, no. 1 (April 2005): 129-143.

Sutherland, Daniel. "The Point of Kant's Axioms of Intuition." *Pacific Philosophical Quarterly* 86 (2005): 135-159.

Tappenden, Jamie. "The Caesar Problem in its Historical Context: Mathematical Background." *Dialectica* 59, no. 2 (2005): 237–264.

Tappenden, Jamie. "The Riemannian Background to Frege's Philosophy." In *The Architecture of Modern Mathematics*, edited by J Ferreiros and J. J. Gray, 97-132. Oxford: Oxford University Press, 2006.

Tappenden, Jamie. "Extending knowledge and 'fruitful concepts': Fregean themes in the philosophy of mathematics." *Nous* 29 (1995): 427-467.

Ricketts, Thomas. "Logic and Truth in Frege." *The Aristotelian Society Supplementary Volume* 70 (1996): 121-40.

Thompson, Manley. "Singular Terms and Intuitions in Kant's Epistemology." *Review of Metaphysics* 26 (1972): 314-343.

Torretti, Roberto. *Philosophy of Geometry from Riemann to Poincare*. Dordrecht: Reidel, 1978.

Trendelenburg, A. "Über Leibnizens Entwurf einer allgemeinen Characteristik." In his *Historische Beiträge zur Philosophie,* vol 3. Berlin: 1867.

Ueberweg, Friedrich. *System der Logik und Geschichte der logischen Lehre.* Bonn: 1857.

Van Heijenoort, Jean. *From Frege to Gödel: A Sourcebook in Mathematical Logic*. Cambridge: Harvard University Press, 1967.

Venn, John. *Symbolic Logic*. London: Macmillan, 1881.

Weyl, Hermann. *Philosophy of Mathematics and Natural Science*. Translated by Olaf Helmer. Princeton: Princeton University Press, 1949. The German edition appeared in 1926.

Willey, Thomas. *Back to Kant*: *the Revival of Kantianism in German Social and Political Thought, 1860-1914.* Detroit: Wayne State University Press, 1978.

Wilson, Mark. "Ghost World: A Context for Frege's Context Principle." In *The Routledge Companion to Frege*, edited by Erich Reck and Michael Beaney. New York: Routledge, 2006.

Wilson, Mark. "Frege: the Royal Road from Geometry." *Nous* 26, no. 2 (1992): 149-180.

Windelband, Wilhelm. *Theories in Logic*. Translated by B. Ethel Meyer. New York: Citadel Press, 1961. The original version was published in English in 1913.

Wolfe, Harold E. *Introduction To Non-Euclidean Geometry*. New York: Holt, Rinehart, and Winston, 1945.

Wolff, Christian. *Der Anfangs-Gründe aller mathematischen Wissenschaften*. Halle: 1710. Reprinted in Wolff, *Gesammelte Werke*, I12.

Wolff, Christian. *Die Vernünfftige Gedancken von den Kräfften des menschlichen Verstandes.* Halle: 1712. Reprinted in Wolff, *Gesammelte Werke*, I.1.

Wolff, Christian. *Elementa Matheseos Universae*. Halle: 1730, 1741. Reprinted in Wolff, *Gesammelte Werke*, II.29.

Wolff, Christian. *Gesammelte Werke*. Edited by Ecole, Hofmann, Thomann, and Arndt. Hindelsheim and New York: Georg Holms, 1962-.

Wolff, Christian. *Mathematisches Lexicon*. Leipzig: 1716. Reprinted in Wolff, *Gesammelte Werke*, I.11.

Wolff, Christian. *Philosophia Raionalis sive Logica*. Frankfurt and Leipzig: 1740. Reprinted in Wolff, *Gesammelte Werke*, II.1.

Wolff, Robert Paul. Kant's theory of mental activity: A commentary on the transcendental analytic of the Critique of Pure Reason. Cambridge, Mass.: Harvard University Press, 1963.

Wundt, Wilhelm. *Logik: Eine Untersuchung der Principien der Erkenntnis und der Methoden wissenschaftlicher Forschung*. 2nd ed. 2 vols. Stuttgart: Enke, 1893. The first edition appeared between 1880 and 1883.

Wussing, Hans. *The Genesis of the Abstract Group Concept*. Translated by Shenitzer, edited by Grant. Cambridge, MA: MIT Press, 1984. Translation of *Die Genesis des abstrakten Gruppenbegriffes*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1969.

Yaglom, I.M. *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*. Edited by Grant and Shenitzer, translated by Sossinsky. Boston: Birkhauser, 1988.