ANALYSIS AND NUMERICAL SOLUTION OF AN INVERSE FIRST PASSAGE PROBLEM FROM RISK MANAGEMENT

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We study the following "inverse first passage time" problem. Given a diffusion process X_t and a probability distribution $q(t)$ on $[0, \infty)$, does there exist a boundary $b(t)$ such that $q(t) = \mathbb{P}[\tau \leq t]$, where τ is the first hitting time of X_t to the time dependent level $b(t)$. We formulate the inverse first passage time problem into a free boundary problem for a parabolic partial differential operator and prove there exists a unique viscosity solution to the associated Partial Differential Equation by using the classical penalization technique. In order to compute the free boundary with a given default probability distribution, we investigate the small time behavior of the boundary $b(t)$, presenting both upper and lower bounds first. Then we derive some integral equations characterizing the boundary. Finally we apply Newton-iteration on one of them to compute the boundary. Also we compare our numerical scheme with some other existing ones.

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1.0 INTRODUCTION

In this thesis we study the following free boundary problem: find a boundary $x = b(t)$ $(t > 0)$ and an unknown function $w = w(x,t)$ $(x \in \mathbb{R},\ t \geq 0)$ such that

$$\begin{cases} w_t(x,t) = \frac{1}{2}(\sigma^2 w_x)_x - \mu w_x & \text{for } x > b(t),\ t > 0, \\ w(x,t) = p(t) & \text{for } x \leq b(t),\ t > 0, \\ w_x(b(t),t) = 0 & \text{for } x = b(t), t > 0, \\ w(x,0) = \mathbf{1}_{(-\infty,0)}(x) & \text{for } x \in \mathbb{R},\ t = 0, \end{cases} \tag{1.0.1}$$

where $p(t)$ is a given survival probability function with the following properties:

$$1 = p(0) = \lim_{t \searrow 0} p(t), \quad p(t_1) \geq p(t_2) \geq 0 \quad \forall\, 0 < t_1 \leq t_2. \tag{1.0.2}$$

This problem arises from the consideration of the first passage times of diffusion processes to curved boundaries. More specifically, we let X_t be the solution of the following stochastic differential equation:

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dB_t \qquad X_0 = 0, \tag{1.0.3}$$

where B_t is a standard Brownian motion on a filtered probability space satisfying the usual conditions, $\mu : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are smooth bounded functions, $\sigma(x,t) > \varepsilon > 0$ for all $x \in \mathbb{R}, t \geq 0$. For a given function $b : \mathbb{R}_+ \to \mathbb{R}$ we define the first passage time of the diffusion process X_t to the curved boundary $b(t)$ to be:

$$\tau = \inf\{t > 0 \mid X_t \leq b(t)\}. \tag{1.0.4}$$

Two important problems concerning the first passage time of a diffusion process to a curved boundary are the following:

1. **The first passage problem:** *Given a boundary function $b(t)$, find the survival probability $p(t)$ that X_t does not cross b before or at t.*

$$p(t) := \mathbb{P}\{\tau > t\}. \tag{1.0.5}$$

2. **The inverse first passage problem:** *Given a survival probability function $p(t)$, find a boundary function $b(t)$, such that (1.0.5) holds.*

The first passage problem is a classical problem in probability, and is the subject of a rather large literature. It is also fundamental in many applications of diffusion processes to engineering, physics, biology and economics. For a survey of techniques for approximating and computing first passage times to curved boundaries, and a discussion of their applications in the biological sciences, we refer to [14]. For some applications in economics closely related to those that motivated this study (for example, credit protection) we refer to [1].

The work of Peskir [12] and [13] on the first passage problem is of particular relevance for the inverse problem discussed in this paper. In [12], he derived a sequence of integral equations[1]

$$t^{n/2}H_n\left(\frac{b(t)}{\sqrt{t}}\right) + \int_0^t (t-s)^{n/2}H_n\left(\frac{b(t)-b(s)}{\sqrt{t-s}}\right)\dot{p}(s)ds = 0, n = -1, 0, 1, \cdots. \tag{1.0.6}$$

where

$$H_{-1}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad H_n(x) = \int_x^\infty H_{n-1}(z)dz, n = 0, 1, 2, \cdots.$$

In [13], under the assumption that $b(t)$ is C^1 on $(0, \infty)$, decreasing, and concave, he derived the equality

$$\dot{p}(0+) = -\lim_{t\searrow 0}\frac{1}{2\sqrt{2\pi}}\frac{b(t)}{t^{3/2}}e^{-\frac{b^2(t)}{2t}} = -\lim_{t\searrow 0}\frac{\dot{b}(t)}{\sqrt{2\pi t}}e^{-\frac{b^2(t)}{2t}},$$

provided that the second or third limit exists.

The inverse first passage problem is much harder than the direct problem and there are only a few studies about it. These are principally concerned with the numerical calculation

[1]In this reference, the derivations are carried out for the case $\sigma \equiv 1$ and $\mu \equiv 0$, i.e. when X_t is a Brownian motion. As mentioned in the reference, the techniques directly extend to other diffusion processes.

of the boundary $b(t)$ for a given $p(t)$. There is no publication proving the well-posedness (existence and uniqueness) of the boundary given the survival probability.

Our interest in the inverse first passage problem originates from Merton's structural model [11] for credit risk management. Consider a company whose asset value is a stochastic process and its debt value is a time depending function. Denote them at time $t \geq 0$ by A_t and D_t respectively. Assume the following:

1. *The company's initial debt is no larger than it's initial asset value, i.e.,* $D_0 \leq A_0$
2. *The company is in* **default** *at a time* $t > 0$ *if* $A_t \leq D_t$.
3. A_t *follows a geometric Brownian motion.*

It is convenient to use the **default index** X_t and the **boundary function** $b(t)$ defined by

$$X_t := \log \frac{A_t}{A_0}, \quad b(t) := \log \frac{D_t}{A_0}.$$

Then X_t is a diffusion process satisfying (1.0.3). In this context, the inverse first passage time problem is the problem of finding the default boundary $b(t)$ given the survival probability function $p(t)$. In deed the default probability $q(t) := 1 - p(t)$ of the company can be estimated from the spreads of the bond issued by the company. If the company wants to get some protection from default, then it is very important to know the threshold of the debt value being in default.

A free boundary problem for a parabolic partial differential operator is associated with the inverse first-passage problem. In order to formulate the problem in a PDE setting, we introduce a new function $w(x,t)$ being the joint probability that the company does not default before or at t and its default index X_t is bigger than x, i.e.,

$$w(x,t) := \mathbb{P}\{X_t > x, \, \tau > t\}. \tag{1.0.7}$$

Then the density function of X_t when $\tau > t$ can be computed by

$$u(x,t) = \frac{d}{dx}\mathbb{P}\{X(t) \leq x, \tau > t\} = (p(t) - w(x,t))_x. \tag{1.0.8}$$

From (1.0.3) and the Kolmogorov forward equation, we see that (assuming sufficient regularity) $w(x,t)$ $(x \in \mathbb{R}, t \geq 0)$ satisfies (1.0.1). From this we see the following:

- The first passage problem is to solve (1.0.1) for p, with given b.
- The inverse first passage problem is to solve (1.0.1) for b, with given p.

The first passage problem can be solved as follows. From the Kolmogorov forward equation, we obtain the following closed system for $u(x,t)$

$$\begin{cases} u_t(x,t) = \frac{1}{2}(\sigma^2 u)_{xx} - (\mu u)_x & \text{for } x > b(t),\ t > 0, \\ u(b(t),t) = 0 & \text{for } x \le b(t),\ t > 0, \\ u(x,0) = \delta(x) & \text{for } x > 0,\ t = 0, \end{cases} \tag{1.0.9}$$

where δ is a Dirac measure concentrated at 0. Given sufficiently regular b, this system has a unique solution. Then p and $\dot{p}$ can be computed from the formulas

$$p(t) = \int_{b(t)}^{\infty} u(x,t)dx \qquad \forall\, t \ge 0, \tag{1.0.10}$$

$$\dot{p}(t) = -\frac{1}{2}(\sigma^2 u)_x\big|_{x=b(t)} \quad \forall\, t \ge 0. \tag{1.0.11}$$

It is only possible to compute the solution in a closed form in a few special cases. However, there is a large literature on numerical and analytic approximations of the solution.

Avellaneda and Zhu [7] were the first to use (1.0.9) and (1.0.11) to study the inverse first-passage problem. They performed a change of variables from X_t to $Y_t = X_t - b(t)$, whose financial meaning is the risk-neutral distance-to-default process (RNDD) for the company. Denote by $f(y,t) = u(y + b(t), t)$, the probability density function of Y_t when $\tau > t$. Then (1.0.9) and (1.0.11) are equivalent to:

$$\begin{cases} f_t = \dot{b}(t) f_y - (\mu f)_y + \frac{1}{2}(\sigma^2 f_y)_y & \text{for } y > 0,\ t > 0, \\ f(0,t) = 0 & \text{for } y = 0,\ t > 0, \\ f(y,0) = \delta_0(y - b(0)) & \text{for } y > 0,\ t = 0, \\ \frac{1}{2}\sigma^2 f_y(0,t) + \dot{p}(t) = 0 & \text{for } y = 0,\ t > 0. \end{cases} \tag{1.0.12}$$

Zucca, Sacerdote and Peskir [15] applied secant method to one of the integral equation (1.0.6), derived by Peskir [12], with $n = 0$. Also they proposed a Monte Carlo algorithm in the same paper, based on a piecewise linear approximation of the boundary.

In [10], Iscoe and Kreinin reduced the inverse first-passage problem to a sequential estimation of conditional distributions. They applied a Monte Carlo approach to it in a discrete time setting.

All the numerical schemes mentioned above will be discussed later in more details and we will do the comparison of all the schemes.

In the thesis, we are particularly interested in the following fundamental questions: *(1) Given a probability function $p(t)$ satisfying (1.0.2), does there exist a boundary function $b(t)$? (2) If there exists a boundary function, how many are there? (3) If there exists a boundary function, how can we compute it numerically?* Namely, we are concerned about the well-posedness (existence and uniqueness) and numerical solution of the free boundary problem (1.0.1).

We point out that solutions to (1.0.1) are not smooth, so that a notion of weak solution has to be used. Instead of using the classical weak solution defined in the distributional sense (see Evans [2]), we use viscosity solutions, introduced by Crandall and Lions [8] in 1981. In the thesis, we shall prove the following theorem.

Theorem 1. *Problem (1.0.1) is a well-posed problem, i.e., for any given $p(t)$ satisfying (1.0.2), there exists a unique (weak) solution.*

The thesis is organized as follows. In Chapter. 2, we provide a definition of the viscosity solution to (1.0.1) and show there is at most one such solution. In Chapter. 3, we establish the existence of a viscosity solution. In Chapter. 4, we study the asymptotic behavior of the boundary as $t \searrow 0$ by providing explicit upper and lower bounds. When $\limsup_{t\searrow 0} -\frac{1-p(t)}{t\dot{p}(t)} < \infty$, we prove that

$$\lim_{t\to 0} \frac{b(t)}{\sqrt{-2t\log(1-p(t))}} = -1.$$

In Chapter. 5, we derive the integral equations for b when $\sigma \equiv 1$ and $\mu \equiv 0$ under the assumption that p is continuous and non-increasing. In Chapter. 6, we proposed our numerical algorithem and introduced the one by Zucca, Sacerdote and Peskir [15], Avellaneda and Zhu [7] and Iscoe and Kreinin [10]. In Chapter. 7, we presented the numerical results, default boundary b, of both the schemes published and ours with the different probability functions. Also we compared the computing speed and the acuracy of all the schemes.

2.0 VISCOSITY SOLUTIONS AND UNIQUENESS

2.1 PRELIMINARIES

By noticing that $w(x,t) < p(t)$ for all $x > b(t)$ when $\tau > t$, we can state the inverse first passage problem as follows. Find an unknown function $w = w(x,t)$ such that,

$$\begin{cases} \mathcal{L}w = 0 & \text{when } w(\cdot,t) < p(t), \\ 0 \le w(x,t) \le p(t) & \text{for any } (x,t) \in (\mathbb{R} \times (0,\infty)), \\ w(x,0) = \mathbf{1}_{(-\infty,0)}(x) & \text{for } (x,t) \in (\mathbb{R} \times [0,\infty)), \end{cases} \tag{2.1.1}$$

where $\mathcal{L}w := w_t - \frac{1}{2}(\sigma^2 w_x)_x + \mu w_x$. Define the free boundary as:

$$b^w(t) := \inf \{x \,|\, w(x,t) < p(t)\}.$$

Noticing that $\mathcal{L}w = 0$ when $w < p$ and $\mathcal{L}w = \dot{p} \le 0$ when $w = p$, we can write (2.1.1) as a variational inequality:

$$\begin{cases} \max\{\mathcal{L}w, w - p\} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ w(\cdot,0) = \mathbf{1}_{(-\infty,0)}(\cdot) & \text{on } \mathbb{R} \times \{0\}. \end{cases} \tag{2.1.2}$$

For a given p, we define

$$p_*(t) = \liminf_{0 \le s \to t} p(s), \qquad p^*(t) = \limsup_{0 \le s \to t} p(s) \qquad \forall t \ge 0.$$

Since cumulative probability distribution functions (hence $1 - p$) are increasing and right continuous, p should be decreasing and right continuous.

Lemma 2.1.1. *For any given $b(t)$, $p(t) := \mathbb{P}\{\tau > t\} \geq 0$ is decreasing and right continuous. In particular, $p = p_*$.*

Proof. Denote by $A(t)$ the set of paths whose default time is bigger than t, i.e.,

$$\begin{aligned} A(t) : &= \{\omega | \tau(\omega) > t\} \\ &= \{\omega | \inf\{s > 0 | X_s(\omega) \leq b(s)\} > t\}. \end{aligned}$$

Then $p(t) = \mathbb{P}(A(t))$.

First we claim that

$$A(t_1) \subseteq A(t_2) \quad \forall\, 0 < t_2 < t_1. \tag{2.1.3}$$

Indeed for any $\omega \in A(t_1)$, since $\inf\{s > 0 | X_s(\omega) \leq b(s)\} > t_1 > t_2$, then $\omega \in A(t_2)$. So that (2.1.3) holds and it is followed that p is a decreasing function. Next we prove that $p(t) = p_*(t)$ for any $t > 0$. Let $t_n := t + \frac{1}{n}$ $(n \in \mathbb{N})$, with the above argument, $\{A(t_n)\}$ is a non-increasing set, i.e.,

$$A(t_1) \subseteq A(t_2) \subseteq \cdots \subseteq A(t_n) \subseteq \cdots .$$

Then by the property of probability, we have

$$\begin{aligned} p(t) &= \mathbb{P}(A(t)) = \mathbb{P}(\lim_{n\to\infty} A(t_n)) = \lim_{n\to\infty} \mathbb{P}\,(A(t_n)) \\ &= \lim_{n\to\infty} p(t_n) = \lim_{s \searrow t} p(t) = p_*(t). \end{aligned}$$

Hence $p = p_*$ and it follows that p is right continuous function since it is also decreasing. □

Furthermore, Blumenthal's zero-one law (see, for example [5]) implies that we must have either $p(0) = 0$ (in which case the problem is trivial) or $p(0) = 1$. Therefore, in the remainder of the thesis, we shall only consider lower semicontinuous p, i.e., $p = p_*$ for which $p(0) = 1$.

2.2 VISCOSITY SOLUTIONS

For a function w defined on $\mathbb{R} \times [0, \infty)$, we define w^* and w_* by

$$\begin{aligned} w^*(x,t) &:= \limsup_{y \to x, 0 \le s \to t} w(y,s), \qquad \forall (x,t) \in \mathbb{R} \times [0,\infty), \\ w_*(x,t) &:= \liminf_{y \to x, 0 \le s \to t} w(y,s), \qquad \forall (x,t) \in \mathbb{R} \times [0,\infty). \end{aligned}$$

A function w is called **upper-semi-continuous (USC)** if $w = w^*$, and **lower-semi-continuous (LSC)** if $w = w_*$.

In the sequel, the parabolic open ball $B_\delta(x,t)$ is defined as:

$$B_\delta(x,t) := (x - \delta, x + \delta) \times (t - \delta^2, t) \qquad \forall \delta > 0, (x,t) \in \mathbb{R} \times [0,\infty).$$

For any cylindrical set of the form $D := \Omega \times (s,t)$ where $0 \le s < t$ and $\Omega \subseteq \mathbb{R}$, the parabolic boundary is defined to be:

$$\partial_p D := \partial\Omega \times (s,t) \cup \bar{\Omega} \times \{s\}$$

Definition 1 (Viscosity Sub, Super, and Solutions).

1. *A function w defined on $\mathbb{R} \times (0,\infty)$ is called a* **(viscosity) subsolution** *if*

$$w = \min\{p, w^*\} \quad \textit{in} \ \ \mathbb{R} \times (0,\infty),$$

   and $\mathcal{L}\varphi(x,t) \le 0$ whenever φ is smooth and $w^* - \varphi$ attains at (x,t) a local maximum on $\bar{B}_\delta(x,t)$, where $x \in \mathbb{R}$ and $t > \delta^2 > 0$.

2. *A function w defined on $\mathbb{R} \times (0,\infty)$ is called a* **(viscosity) supersolution** *if*

$$0 \le w = w_* \qquad \textit{in} \ \ \mathbb{R} \times (0,\infty),$$

   and $\max\{w(x,t) - p(t), \ \ \mathcal{L}\varphi(x,t)\} \ge 0$ whenever φ is smooth and $w - \varphi$ attains at (x,t) a local minimum on $\bar{B}_\delta(x,t)$, where $x \in \mathbb{R}$ and $t > \delta^2 > 0$.

3. *A function w defined on $\mathbb{R} \times [0,\infty)$ is called a* **(viscosity) solution** *if w is both a subsolution and a supersolution in $\mathbb{R} \times (0,\infty)$, and for all $x \in \mathbb{R}$,*

$$w(x,0) = \liminf_{y \to x, t \searrow 0} w(y,t) = \mathbf{1}_{(-\infty,0)}, \qquad \limsup_{y \to x, t \searrow 0} w(y,t) = \mathbf{1}_{(-\infty,0]}. \tag{2.2.1}$$

Remark 2.2.1. *Here we use the default that a viscosity solution is LSC, i.e., $w = w_*$ (Given any point $(x,t) \in \mathbb{R} \times [0,\infty)$, if $x > b(t)$, then $w(x,t)$ is continuous, hence is LSC. If $x \leq b(t)$, then $\tau > t$ implies that $X_t > x$, i.e., $\mathbb{P}(X_t > x, \tau > t) = p(t)$. As $p = p_*$, $w = w_*$). Also, the (probabilistically obvious) condition $w \geq 0$ imposed for super-solutions is to ensure the boundedness of the super-solution, as is usually required. This condition could be relaxed to the assumption that $w \geq -e^{A(1+|x|^2)}$ for some $A > 0$.*

To prove the uniqueness of the solution to (2.1.2), we first establish a few properties of viscosity solutions.

Lemma 2.2.1. *Let w be a viscosity solution and define*

$$Q := \{(x,t) \in \mathbb{R} \times (0,\infty) \mid w(x,t) < p(t)\}, \quad \Pi := Q^c = \mathbb{R} \times (0,\infty) \setminus Q.$$

Then

1. *Q is open and w is a smooth solution to $\mathcal{L}w = 0$ in Q;*
2. *$\Pi = \{(x,t) \in \mathbb{R} \times (0,\infty) \mid w(x,t) = p(t)\} = \Pi_0 \cup \Pi_1 \cup \Pi_2$ where*

$$\begin{aligned} \Pi_0 : &= \{(x,t) \in \mathbb{R} \times (0,\infty) \mid w^*(x,t) = w_*(x,t) = p(t)\}, \\ \Pi_1 : &= \{(x,t) \in \mathbb{R} \times (0,\infty) \mid p^*(t) > w^*(x,t) > w_*(x,t) = p(t)\}, \\ \Pi_2 : &= \{(x,t) \in \mathbb{R} \times (0,\infty) \mid p^*(t) = w^*(x,t) > w_*(x,t) = p(t)\}. \end{aligned}$$

In particular, if p is continuous, then w is continuous in $\mathbb{R} \times (0,\infty)$.

Proof. 1. First we show that Q is open and w is continuous in Q. For each $(x,t) \in Q$ with $t > 0$, $w(x,t) < p(t)$. As a supersolution, $w(x,t) = w_*(x,t)$. As a subsolution, $w(x,t) = \min\{p(t), w^*(x,t)\} < p(t)$, which implies that $w(x,t) = \min\{p(t), w^*(x,t)\} = w^*(x,t)$. Hence $w_* = w = w^*$ at (x,t). That is w is continuous at (x,t) and $w(x,t) < p(t)$. Since p is right continuous and decreasing, there exists $\delta_1 > 0$ such that $w < p$ in $(x-\delta_1, x+\delta_1) \times (t, t+\delta_1^2)$. As $\limsup_{y \to x, s \to t} w(y,s) = w^*(x,t) = w(x,t) < p(t)$, there exists $\delta_2 > 0$ such that

$$w(y,s) < p(t) < p(s) \quad \forall\, (y,s) \in (x-\delta_2, x+\delta_2) \times (t-\delta_2, t).$$

Then for any $(x,t) \in Q$ and $t > 0$, there exists an open set $D_\delta(x,t) := (x-\delta, x+\delta) \times (t-\delta^2, t+\delta^2)$ in Q, where $\delta = \min\{\delta_1, \delta_2\} > 0$. Hence, Q is open and w is continuous in Q.

2. Next we prove $\mathcal{L}w = 0$ in Q. Let $(x_0, t_0) \in Q$ with $t_0 > 0$. Then w is continuous at (x_0, t_0) and $w(x_0, t_0) < p(t_0)$. With the previous argument, there exists $\bar{D} \subseteq Q$ such that w is coutinuous and $w < p$ in $\bar{D}$. Denote by $\tilde{w}$ the solution to

$$\begin{cases} \mathcal{L}\tilde{w} = 0 & \text{for } \ (x,t) \in D, \\ \\ \tilde{w} = w & \text{for } \ (x,t) \in \partial_p D. \end{cases} \tag{2.2.2}$$

Note that $\tilde{w}$ is smooth in D since the boundary and initial condition are continuous. Let

$$\varphi^\varepsilon = \tilde{w} - \frac{\varepsilon}{t_0 + \delta^2 - t}, \quad \psi^\varepsilon = \tilde{w} + \frac{\varepsilon}{t_0 + \delta^2 - t} \quad \forall\, \varepsilon > 0.$$

Then φ^ε and ψ^ε are smooth in D (by interior regularity for PDE [2]). Note that $w - \varphi^\varepsilon$ can attain its minimum on $\bar{D}$ only at the parabolic boundary. To the contrary, suppose this is not true. Since $w - \varphi^\varepsilon \to \infty$ as $t \to t_0 + \delta^2$, we assume that the minimum is attained at a interior point of D, say (x^*, t^*). As a supersolution, $\max\{w(x^*, t^*) - p(t^*), \mathcal{L}\varphi^\varepsilon(x^*, t^*)\} \geq 0$. Since $w - p < 0$ in D, $\mathcal{L}\varphi^\varepsilon(x^*, t^*) \geq 0$. However

$$\mathcal{L}\varphi^\varepsilon(x^*, t^*) = \mathcal{L}\tilde{w}(x^*, t^*) - \mathcal{L}\frac{\varepsilon}{t_0 + \delta^2 - t}|_{t=t^*} = -\frac{\varepsilon}{(t_0 + \delta^2 - t^*)^2} < 0.$$

This is a contradiction. So that

$$\min_{\bar{D}}(w - \varphi^\varepsilon) = \min_{\partial_p D}\left(w - \tilde{w} + \frac{\varepsilon}{t_0 + \delta^2 - t}\right) > 0.$$

Thus $\varphi^\varepsilon < w$ in $\bar{D}$. Also note that $w - \psi^\varepsilon$ can attain its maximum on $\bar{D}$ only at the parabolic boundary. To the contrary, suppose this is not true. Since $w - \psi^\varepsilon \to -\infty$ as $t \to t_0 + \delta^2$, we assume that the maximum is attained at a interior point of D, say (x^*, t^*). As a subsolution, $\mathcal{L}\psi^\varepsilon(x^*, t^*) \leq 0$. However

$$\mathcal{L}\psi^\varepsilon(x^*, t^*) = \mathcal{L}\tilde{w}(x^*, t^*) + \mathcal{L}\frac{\varepsilon}{t_0 + \delta^2 - t}|_{t=t^*} = -\frac{\varepsilon}{(t_0 + \delta^2 - t^*)^2} > 0.$$

This is a contradiction. So that

$$\max_{\bar{D}}(w - \psi^\varepsilon) = \max_{\partial_p D}\left(w - \tilde{w} - \frac{\varepsilon}{t_0 + \delta^2 - t}\right) < 0.$$

Thus $w < \psi^\varepsilon$ in $\bar{D}$. So that we have

$$\tilde{w} - \frac{\varepsilon}{t_0 + \delta^2 - t} < w < \tilde{w} + \frac{\varepsilon}{t_0 + \delta^2 - t}.$$

Sending $\varepsilon \to 0$ we obtain $w = \tilde{w}$ in D, which implies that w is a continuous solution to $\mathcal{L}w = 0$ in Q.

3. Lastly we prove the second assertion of the lemma. Since $w \le p$, $\Pi := Q^c = \{(x,t) \in \mathbb{R} \times (0,\infty) | w(x,t) = p(t)\}$. For any $t > 0$, as a subsolution, $p = w = \min\{p, w^*\} \le w^*$. As a supersolution $w = w_*$. Also $w \le p$ implies that $w^* \le p^*$. Thus

$$w_* = w = p \le w^* \le p^*, \quad \text{in } \Pi.$$

There are only three possibilities for w^*: (i) $w^* = p$, (ii) $w^* \in (p, p^*)$ and (iii) $w^* = p^* > p$. Thus the second assertion holds.

4. In particular, if p is continuous, i.e., $p_* = p = p^*$, then $\Pi = \Pi_0$. That is $w_* = w = w^*$ in Π. Hence w is continuous in Π. It follows that w is continuous at $\mathbb{R} \times (0,\infty) \setminus \{(0,0)\}$ □

The following Lemma characterizes the discontinuities of a solution.

Lemma 2.2.2. *Suppose w is a viscosity solution. Then for each $t > 0$, the following hold:*

1. *$w(\cdot,t) = w_*(\cdot,t)$ is continuous in $\mathbb{R}$;*
2. *for each $x \in \mathbb{R}$,*

$$w_*(x,t) = \min\{p(t), w^*(x,t)\} = \lim_{y\to x, s\searrow t} w(y,s), \tag{2.2.3}$$

$$w^*(x,t) = \lim_{y\to x, s\nearrow t} w(y,s) \le p^*(t); \tag{2.2.4}$$

3. *if $w^*(x,t) < p^*(t)$, then for some $\delta > 0$, $w = w^*$ in $B_\delta(x,t)$ and w^* is a smooth solution to $\mathcal{L}w^* = 0$ in $\bar{B}_\delta(x,t)$.*

Proof. 1. First we prove the first assertion. For each $t > 0$, since w is a supersolution, $w(\cdot, t) = w_*(\cdot, t)$. If $(x, t) \in Q$, then $w(x, t) < p(t)$ and w is continuous at (x, t) by Lemma (2.2.1). If $(x, t) \in \Pi$, then $w(x, t) = p(t)$. So that we have

$$w_*(x,t) = \liminf_{y \to x, s \to t} w(y,s) \leq \liminf_{y \to x} w(y,t).$$

Since $w \leq p$,

$$\limsup_{y \to x} w(y,t) \leq \limsup_{y \to x} p(t) = p(t).$$

By using the fact of w is a supersolution and the above inequalities,

$$\limsup_{y \to x} w(y,t) \leq p(t) = w(x,t) = w_*(x,t) \leq \liminf_{y \to x} w(y,t).$$

So that $w(\cdot, t) = w_*(\cdot, t)$ is continuous in $\mathbb{R}$. The first assertion follows.

2. Next we prove (2.2.3). For each $x \in \mathbb{R}$, the first equality is immediate since w is both a subsolution and a supersolution. We prove the second inequality follows by considering separately the cases $(x, t) \in Q$ and $(x, t) \in \Pi$ as in the previous step. If $(x, t) \in Q$, then w is continuous at (x, t). So that $w_*(x, t) = \min\{p(t), w^*(x, t)\} = \lim_{y \to x, s \searrow t} w(y, s)$. If $(x, t) \in \Pi$, then

$$w_*(x,t) = w(x,t) = p(t) \geq \lim_{y \to x, s \searrow t} w(y,s) \geq \liminf_{y \to x, s \to t} w(y,s) = w_*(x,t),$$

where the first inequality holds since $w \leq p$ so that

$$p(t) = p_*(t) = \lim_{s \searrow t} p(t) = \lim_{y \to x, s \searrow t} p(t) \geq \lim_{y \to x, s \searrow t} w(y,s).$$

Thus (2.2.3) holds.

3. Now we prove (2.2.4) when $w^*(x, t) < p^*(t)$ and the third assertion. By the upper semicontinuity of w^*, there exist some positive constants δ and η such that

$$w(\cdot, \cdot) < p^*(t) - \eta \qquad \text{in } \bar{B}_\delta(x,t). \tag{2.2.5}$$

Then we claim that

$$w = w^* \text{ in } B_\delta(x,t) \cup \partial_p B_\delta(x,t). \tag{2.2.6}$$

To the contrary, suppose this is not true, i.e., there exists at least one pair of $(y,s) \in B_\delta(x,t) \cup \partial_p B_\delta(x,t)$ such that $w(y,s) < w^*(y,s)$. As a subsolution,

$$w(y,s) = \min\{p_*(s), w^*(y,s)\} < w^*(y,s).$$

Hence

$$p_*(s) = w(y,s) < p^*(t) - \eta \le p_*(s) - \eta,$$

where the second inequality follows from (2.2.5) and the last inequality holds since p is non-increasing and right continuous, which implies that $p^*(t) \le p(s) = p_*(s)$ for all $s < t$. This is a contradiction. So that (2.2.6) holds and

$$w^*(y,s) = w(y,s) < p^*(t) - \eta < p^*(t) \qquad \forall\, (y,s) \in B_\delta(x,t).$$

Send $y \to x, s \nearrow t$, we obtain (2.2.4).

Next we prove that w^* is a smooth solution to $\mathcal{L}w^* = 0$ in $\bar{B}_\delta(x,t)$. As supersolution $w = w_*$ and by (2.2.6) $w = w^*$ on $\partial_p B_\delta(x,t)$, hence w is continuous on $\partial_p B_\delta(x,t)$. Also, for any $(y,s) \in \partial_p B_\delta(x,t)$, $w(y,s) < p^*(t) - \eta \le p(s) - \eta < 0$, which implies that $w(y,s) < p(s)$. Denote by $\tilde{w}$ the solution to

$$\begin{cases} \mathcal{L}\tilde{w} = 0 & \text{for } (x,t) \in B_\delta(x,t), \\ \tilde{w} = w^* & \text{for } (x,t) \in \partial_p B_\delta(x,t). \end{cases}$$

Following the same proof as previous lemma, we can show that $\tilde{w} \equiv w^*$ in $\bar{B}_\delta(x,t)$. Hence $w = w^*$ is a smooth solution to $\mathcal{L}w^* = 0$ in $\bar{B}_\delta(x,t)$.

The third assertion and (2.2.4) for the case $w^*(x,t) < p^*(t)$ thus follow.

4. Finally we verify (2.2.4) for the case $w^*(x,t) = p^*(t)$. For each small $\delta > 0$, we compare w in $B_\delta(x,t)$ with solutions $\bar{w}$ and $\underline{w}$ to

$$\begin{cases} \mathcal{L}\bar{w} = 0 & \text{in } B_\delta, \\ \bar{w} = w^* & \text{on } \partial_p B_\delta, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}\underline{w} = 0 & \text{in } B_\delta, \\ \underline{w} = \min\{w^*, p^*(t)\} & \text{on } \partial_p B_\delta, \end{cases}$$

respectively. Note that on $\partial_p B_\delta$, $w \le w^* = \bar{w}$ and $\underline{w} = \min\{w^*, p^*(t)\} \le \min\{w^*, p\} = w$ since $p^*(t) \le p(s)$ for any $s < t$. Simple comparison gives $\underline{w} \le w \le \bar{w}$ in B_δ. By maximum principle,

$$\max_{\bar{B}_\delta} \bar{w} = \max_{\partial_p B_\delta} \bar{w} = \max_{\bar{B}_\delta} w^* \le \max_{t-\delta^2 \le s < t} p^* \le p^*(t-\delta^2),$$
$$\min_{\bar{B}_\delta} \underline{w} = \min_{\partial_p B_\delta} \underline{w} \le p^*(t).$$

Then,

$$\limsup_{y\to x, s\nearrow t} w(y,s) - \liminf_{y\to x, s\nearrow t} w(y,s) \le \max_{B_\delta(x,t)} \{\bar{w} - \underline{w}\} \le p^*(t-\delta^2) - p^*(t).$$

Send $\delta \to 0$, we conclude that $\lim_{y\to x, s\nearrow t} w(y,s)$ exists. Now need to show that

$$\lim_{y\to x, s\nearrow t} w(y,s) = w^*(x,t). \tag{2.2.7}$$

We show it in the following two cases: (i) Suppose $w_*(x,t) = w^*(x,t)$. That is w is continuous at (x,t) so that (2.2.7) follows. (ii) Suppose $w_(x,t) < w^*(x,t)$. By (2.2.3), $\lim_{y\to x, s\searrow t} w(y,s) = w_*(x,t)$. Then we must have

$$w^*(x,t) - \limsup_{y\to x, s\to t} w(y,s) = \lim_{y\to x, s\nearrow t} w(y,s).$$

This complete the proof of (2.2.4).

□

2.3 UNIQUENESS

Theorem 2 (Uniqueness). *There is at most one viscosity solution to (2.1.2).*

Proof. Suppose w_1 and w_2 are two viscocity solutions to (2.1.2). For each $\eta > 0$, we claim that

$$w_1(x,t) \le w_2(x-\eta,t) \qquad \forall (x,t) \in \mathbb{R} \times [0,\infty). \tag{2.3.1}$$

To the contrary, suppose this is not true, i.e., there exists at least one pair of $(x_0, t_0) \in \mathbb{R} \times [0,\infty)$ such that $w_1(x_0,t_0) > w_2(x_0-\eta,t_0)$. Then for all sufficiently small positive ε,

$$w_1(x_0,t_0) > w_2(x_0-\eta,t_0) + \varepsilon^4 x_0^2 + \varepsilon e^{t_0} \,. \tag{2.3.2}$$

Hence fix such a positive ε such that

$$\varepsilon \le \min\left\{1, \frac{1}{(\|\sigma^2\|_\infty + 2\|2\sigma\sigma_x - \mu\|_\infty)}\right\} \tag{2.3.3}$$

and let

$$g^\varepsilon(x,t) := w_1(x,t) - w_2(x-\eta,t) - \varepsilon^4 x^2 - \varepsilon e^t.$$

Then

$$g^\varepsilon(x_0,t_0) > 0, \tag{2.3.4}$$

$$g^\varepsilon(x,t) \le p(t) \le 1 \quad \forall\ (x,t) \in \mathbb{R} \times [0,\infty). \tag{2.3.5}$$

So that g^ε attains a supremum in $\mathbb{R} \times [0,\infty)$, denoted by

$$M_\varepsilon := \sup_{(x,t)\in\mathbb{R}\times[0,\infty)} g^\varepsilon(x,t),$$

and together with (2.3.4) and (2.3.5),

$$0 < M_\varepsilon \le 1.$$

Let $\{(x_n,t_n)\}_{n=1}^\infty$ be a sequence in $\mathbb{R} \times [0,\infty)$ such that the supremum M_ε is attained along the sequence. By taking a subsequence if necessary, there exist the limits

$$(\hat{x},\hat{t}) := \lim_{n\to\infty}(x_n,t_n), \quad \alpha := \lim_{n\to\infty} w_1(x_n,t_n), \quad \beta := \lim_{n\to\infty} w_2(x_n-\eta,t_n).$$

Consequently

$$M_\varepsilon = \lim_{n\to\infty} g^\varepsilon(x_n, t_n) = \alpha - \beta - \varepsilon^4 \hat{x}^2 - \varepsilon e^{\hat{t}}. \tag{2.3.6}$$

Since $0 \le w_1(\cdot,\cdot) \le 1$ and $0 \le w_2(\cdot - \eta, \cdot) \le 1$, $0 \le \alpha \le 1$, $0 \le \beta \le 1$. Also

$$\begin{aligned} \alpha &= \lim_{n\to\infty} w_1(x_n, t_n) \le \limsup_{y\to\hat{x}, s\to\hat{t}} w_1(y, s) = w_1^*(\hat{x}, \hat{t}); \\ \beta &= \lim_{n\to\infty} w_2(x_n, t_n) \ge \liminf_{y\to\hat{x}, s\to\hat{t}} w_2(y-\eta, s) = w_{2*}(\hat{x}-\eta, \hat{t}). \end{aligned}$$

As $M_\varepsilon > 0$,

$$\alpha = M_\varepsilon + \beta + \varepsilon^4 \hat{x}^2 + \varepsilon e^{\hat{t}} > \beta$$

and

$$\hat{x}^2 = \frac{\alpha - \beta - \varepsilon e^{\hat{t}} - M_\varepsilon}{\varepsilon^4} \le \alpha/\varepsilon^4 \le 1/\varepsilon^4.$$

In a summary,

$$\alpha \le w_1^*(\hat{x}, \hat{t})\ ,\ \beta \ge w_{2*}(\hat{x}-\eta, \hat{t})\ ,\ \beta < \alpha\ ,\ |\hat{x}| < 1/\varepsilon^2.$$

Now we show that this is impossible, by excluding the following three possibilities:

$$\text{(i)}\ \ \hat{t} = 0; \qquad \text{(ii)}\ \ \hat{t} > 0, \beta < p(\hat{t}); \qquad \text{(iii)}\ \ \hat{t} > 0, \beta \ge p(\hat{t}).$$

Case (i): Suppose $\hat{t} = 0$. If $\hat{x} \ge \eta$, then $\hat{x} > 0$, which implies that

$$\alpha \le w_1^*(\hat{x}, 0) = 1_{(-\infty,\, 0]}(\hat{x}) = 0.$$

So that

$$0 \le \beta < \alpha \le w_1^*(\hat{x}, 0) = 0.$$

This is a contradiction. If $\hat{x} < \eta$, then $\hat{x} - \eta < 0$, which implies that

$$\beta \ge w_{2*}(\hat{x}-\eta, \hat{t}) = 1_{(-\infty,\, 0)}(\hat{x}-\eta) = 1.$$

So that

$$1 \le \beta < \alpha \le 1.$$

This is a contradiction too. Thus case (i) is impossible.

Case (ii) Suppose $\hat{t} > 0$ and $\beta < p(\hat{t})$, then

$$w_2(\hat{x}-\eta,\hat{t}) = w_{2*}(\hat{x}-\eta,\hat{t}) \le \beta < p(\hat{t}).$$

Hence $(\hat{x}-\eta,\hat{t}) \in Q$, where $Q := \{(x,t) \in \mathbb{R}\times[0,\infty) | w_2(x,t) < p(t)\}$ is an open set. By Lemma (2.2.1), there exist $D := (\hat{x}-\delta,\hat{x}+\delta)\times(\hat{t}-\delta^2,\hat{t}+\delta^2) \subseteq Q$, where $\delta > 0$, such that $w_{2*}(\cdot-\eta,\cdot) < p(\cdot)$ in D and w_2 is a smooth solution to $\mathcal{L}w_2(\cdot-\eta,\cdot) = 0$ in $\bar{D}$. Let

$$\varphi(x,t) = w_2(x-\eta,t) + \varepsilon^4 x^2 + \varepsilon e^t + (x-\hat{x})^4/\delta^4 + (t-\hat{t})^2/\delta^4.$$

Then φ is smooth in $\bar{D}$ and

$$\begin{aligned}
\max_{\bar{D}}\{w_1^* - \varphi\} &= \sup_{D}\{w_1 - \varphi\} \\
&= \sup_{\bar{D}}\left\{g_\varepsilon(x,t) - (x-\hat{x})^4/\delta^4 - (t-\hat{t})^2/\delta^4\right\} \\
&\le \alpha - \beta - \varepsilon^4\hat{x}^2 - \varepsilon e^{\hat{t}} \\
&\le w_1^*(\hat{x},\hat{t}) - w_{2*}(\hat{x}-\eta,\hat{t}) - \varepsilon^4\hat{x}^2 - \varepsilon e^{\hat{t}} \\
&= w_1^*(\hat{x},\hat{t}) - \varphi(\hat{x},\hat{t}).
\end{aligned}$$

That is, $w_1^* - \varphi$ attain at $(\hat{x},\hat{t})$ a local maximum on $\bar{D}$. As w_1 is a subsolution, $\mathcal{L}\varphi(\hat{x},\hat{t}) \le 0$. However

$$\begin{aligned}
\mathcal{L}\varphi(\hat{x},\hat{t}) &= \mathcal{L}w_2(\hat{x}-\eta,\hat{t}) + \varepsilon e^{\hat{t}} - \varepsilon^4\sigma^2 - 2\varepsilon^4\hat{x}(2\sigma\sigma_x - \mu) \\
&= \varepsilon e^{\hat{t}} - \varepsilon^2(\varepsilon^2\sigma^2) - 2\varepsilon^2(\varepsilon^2\hat{x})(2\sigma\sigma_x - \mu) \\
&\ge \varepsilon - \varepsilon^2\left(\|\sigma^2\|_\infty + 2\|2\sigma\sigma_x - \mu\|_\infty\right) > 0
\end{aligned}$$

by the fact that $\hat{x} < 1/\varepsilon^2$ and (2.3.3). This is a contradiction. Thus case (ii) is impossible.

Case (iii): Suppose $\hat{t} > 0$ and $\beta \ge p(\hat{t})$. Since $p^*(s) \le p(\hat{t})$ for any $s > \hat{t}$,

$$w_1(x,s) \le p(s) \le p^*(s) \le p^*(\hat{t}) \le \beta \quad \forall\, x \in \mathbb{R}, s > \hat{t}.$$

So that

$$\sup_{x\in\mathbb{R}} w_1(x,s) \le \beta \quad \forall\, s > \hat{t}. \tag{2.3.7}$$

Now we claim $t_n < \hat{t}$ for all sufficiently large n, i.e., there exists $N \in \mathbb{N}^+$ such that $t_n < \hat{t}$ for each $n \geq N$. To the contrary, suppose for each $N \in \mathbb{N}^+$, there exists $n > N$ such that $t_n \geq \hat{t}$. Then by (2.3.7),

$$w_1(x_n, t_n) \leq \sup_{x \in \mathbb{R}} w_1(x, t_n) \leq \beta.$$

As $\alpha > \beta$, there exists $\bar{\varepsilon} > 0$, which is independent of n, such that

$$\alpha > \beta + \bar{\varepsilon} \geq w_1(x_n, t_n) + \bar{\varepsilon}.$$

This is a contradiction to $\alpha = \lim_{n\to\infty} w_1(x_n, t_n)$. Then by taking the subsequence if necessarily and by (2.2.4), we have

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} w_1(x_n, t_n) = \lim_{y\to\hat{x}, s\nearrow\hat{t}} w_1(y, s) = w_1^*(\hat{x}, \hat{t}) \leq p^*(\hat{t}), \\
\beta &= \lim_{n\to\infty} w_2(x_n, t_n) = \lim_{y\to\hat{x}, s\nearrow\hat{t}} w_2(y-\eta, s) = w_2^*(\hat{x}-\eta, \hat{t}) \leq p^*(\hat{t}).
\end{aligned}$$

Also

$$w_2^*(\hat{x}-\eta, \hat{t}) = \beta < \alpha \leq p^*(\hat{t}).$$

By Lemma 2.2.2 (3), for some $\delta > 0$, $w_2^* = w_2$ in $B_\delta(\hat{x}+\eta, \hat{t})$ and w_2^* is a smooth solution to $\mathcal{L}w_2^* = 0$ in $\bar{B}_\delta(\hat{x}+\eta, \hat{t})$. Let

$$\phi(x,t) := w_2^*(x-\eta, t) + \varepsilon^4 x^2 + \varepsilon e^t + (x-\hat{x})^4/\delta^4 + (t-\hat{t})^2/\delta^4.$$

Then,

$$\begin{aligned}
\max_{\bar{B}_\delta(\hat{x},\hat{t})} \{w_1^* - \phi\} &= \sup_{(x,t)\in B_\delta(\hat{x},\hat{t})} \{w_1 - \phi\} \\
&= \sup_{(x,t)\in B_\delta(\hat{x},\hat{t})} \{w_1(x,t) - w_2(x-\eta,t) - \varepsilon^4 x^2 - \varepsilon e^t - (x-\hat{x})^4/\delta^4 - (t-\hat{t})^2/\delta^4\} \\
&\leq \alpha - \beta - \varepsilon^4 \hat{x}^2 - \varepsilon e^{\hat{t}} \\
&= w_1^*(\hat{x}, \hat{t}) - \phi(\hat{x}, \hat{t}).
\end{aligned}$$

That is $w_1^* - \phi$ obtains at $(\hat{x}, \hat{t})$ its local maximum in $\bar{B}_\delta(\hat{x}, \hat{t})$. Since w_1 is a subsolution, $\mathcal{L}\phi(\hat{x}, \hat{t}) \leq 0$. However

$$\begin{aligned}\mathcal{L}\phi(\hat{x}, \hat{t}) &= \mathcal{L}w_2^*(\hat{x} - \eta, \hat{t}) + \varepsilon e^{\hat{t}} - \varepsilon^4 \sigma^2 - 2(\sigma\sigma_x - 2\mu)\varepsilon^4 \hat{x} \\ &\geq \varepsilon - \varepsilon^2 \|\sigma^2\|_\infty - 2\varepsilon^2 \|\sigma\sigma_x - \mu\|_\infty > 0\end{aligned}$$

by the fact that $\hat{x} < 1/\varepsilon^2$ and (2.3.3). This is a contradiction. Thus case (iii) is impossible.

The exclusion of cases (i), (ii) and (iii) implies that (2.3.1) holds for each $\eta > 0$. Sending $\eta \searrow 0$ and using Lemma 2.2.2 (1), i.e., $w(\cdot, t)$ is continuous in $\mathbb{R}$, we conclude that $w_1 \leq w_2$ on Ω. Exchanging the roles of w_1 and w_2, we also have $w_2 \leq w_1$, so that $w_1 \equiv w_2$. □

As a product, (2.3.1) and the uniqueness give the following.

Corollary 2.3.1. *The unique solution w, if it exists, is non-increasing in x, i.e., $w(x, t) \leq w(x - \eta, t)$ for all $\eta > 0$ and $(x, t) \in \mathbb{R} \times [0, \infty)$.*

3.0 EXISTENCE OF A VISCOSITY SOLUTION

In this chapter we prove the existence of viscocity solution to (2.1.2) by establishing one. Following the classical penalization technique (see for example Friedman [4]) for variational inequalities, we define a ε-regularization of the problem carefully so that the solution is monotonic in ε and therefore the existence of a limit as $\varepsilon \to 0$ is automatically guaranteed. For the purpose of showing that the limit is a viscosity solution, we study the regularization and then prove some regularity properties of the solution to the penalized problem, and therefor establish compactness.

3.1 THE REGULARIZATION

Following the classical penalization technique for variational inequalities, we consider a semi-linear parabolic equation:

$$\begin{cases} \mathcal{L}w^\varepsilon = -\beta\Big(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)\Big) & \text{in } \ \mathbb{R} \times (0,\infty), \\ w^\varepsilon(\cdot,0) = W^\varepsilon(\cdot) & \text{on } \ \mathbb{R} \times \{0\}. \end{cases} \tag{3.1.1}$$

where p^ε and W^ε are the smooth approximations of p and $w(\cdot,0) = \mathbf{1}_{(-\infty,0)}$ respectively, and $\beta(\cdot)$ is a smooth function being identically zero in $(-\infty, 0]$ and strictly increasing and convex in $[0,\infty)$. For definiteness, we take

$$\beta(s) := \max\{0, s^3\} \qquad \forall\, s \in \mathbb{R}.$$

The particular p^ε and W^ε are chosen so that the solution w^ε is strictly increasing in ε.

Lemma 3.1.1. *For any given decreasing function $p(t)$ such that $p(0) = 1$ and $p(t) \geq 0$ for any $t > 0$, there exists p^ε such that*

1. *$p^\varepsilon \in C^1([0,\infty))$, and*

$$-\frac{1}{\varepsilon} \leq \frac{d}{dt}p^\varepsilon(t) \leq 0,$$

and consequently

$$\|\dot{p}^\varepsilon\|_\infty \leq \frac{1}{\varepsilon}.$$

2. *for each $t \geq 0$,*

$$\frac{d}{d\varepsilon}p^\varepsilon(t) \leq -\frac{2}{\varepsilon^{1/3}},$$

and consequently

$$\lim_{\varepsilon \searrow 0} p^\varepsilon(t) = p(t) = p_*(t). \tag{3.1.2}$$

Proof. 1. Suppose that $p \in C^1([0,\infty))$. Define p^ε as following

$$p^\varepsilon(t) := \frac{3}{4}\int_{-1}^{1}(1-z^2)p(t+\varepsilon+\varepsilon z)\,dz - 3\varepsilon^{2/3} \qquad \forall \varepsilon > 0, t \geq 0.$$

Then $p^\varepsilon \in C^1([0,\infty))$. As p is decreasing, i.e., $\dot{p} \leq 0$, so that

$$\frac{d}{dt}p^\varepsilon(t) = \frac{3}{4}\int_{-1}^{1}(1-z^2)\dot{p}(t+\varepsilon+\varepsilon z)dz < 0.$$

Also since $0 \leq p \leq 1$,

$$\begin{aligned}
\frac{d}{dt}p^\varepsilon(t) &= \frac{3}{4}\int_{-1}^{1}(1-z^2)\dot{p}(t+\varepsilon+\varepsilon z)dz \\
&= \frac{3(1-z^2)}{4\varepsilon}p(t+\varepsilon+\varepsilon z)\Big|_{z=-1}^{z=1} + \frac{3}{4\varepsilon}\int_{-1}^{1}2zp(t+\varepsilon+\varepsilon z)dz \\
&= \frac{3}{4\varepsilon}\int_{-1}^{1}2zp(t+\varepsilon+\varepsilon z)dz \\
&\geq \frac{3}{4\varepsilon}\int_{-1}^{0}2zp(t+\varepsilon+\varepsilon z)dz \\
&\geq \frac{3}{4\varepsilon}\int_{-1}^{0}2z dz = -\frac{3}{4\varepsilon} \geq -\frac{1}{\varepsilon}.
\end{aligned}$$

For each $t \geq 0$,

$$\frac{d}{d\varepsilon}p^{\varepsilon}(t) = \frac{3}{4}\int_{-1}^{1}(1-z^2)(1+z)\dot{p}(t+\varepsilon+\varepsilon z)dz - 2\varepsilon^{-1/3} \leq -2\varepsilon^{-1/3} < 0.$$

Hence $p^{\varepsilon}(t)$ is decreasing in terms of ε and it is bounded from below by $-3\varepsilon^{2/3}$. It implies that $\lim_{\varepsilon \searrow 0} p^{\varepsilon}(t)$ exists and can be obtained by

$$\lim_{\varepsilon \searrow 0} p^{\varepsilon}(t) = \lim_{n\to\infty} p^{\varepsilon_n}(t), \quad \text{where } \varepsilon_n = \frac{1}{n}.$$

For each $n > 0$, $t \geq 0$, when $z \in [-1, 1]$,

$$0 < (1-z^2)p(t+\varepsilon_n+\varepsilon_n z) \leq (1-z^2)p_*(t)$$

where $(1-z^2)p_*(t)$ is integrable in $[-1, 1]$. Using Lebesgue Convergence theorem, we get

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} p^{\varepsilon}(t) &= \lim_{n\to\infty} \frac{3}{4}\int_{-1}^{1}(1-z^2)p(t+\varepsilon_n+\varepsilon_n z)dz \\
&= \frac{3}{4}\int_{-1}^{1}(1-z^2)\lim_{n\to\infty} p(t+\varepsilon_n+\varepsilon_n z)dz \\
&= \frac{3}{4}\int_{-1}^{1}(1-z^2)p_*(t)dz = p_*(t).
\end{aligned}$$

This completes the proof for the case $p \in C^1([0,\infty))$.

2. Suppose p is not a smooth function, then one can choose a sequence of functions $\{p_n\}_{n=1}^{\infty}$ such that

1. for each $t \geq 0$, $\lim_{n\to\infty} p_n(t) = p(t)$;
2. for each $n > 0$, $p_n \in C^1([0,\infty))$;
3. $\{p_n\}$ is uniformly bounded, i.e., there exists $M > 0$ such that for each $n > 0$, $|p_n| \leq M$.

Let

$$p_n^\varepsilon(t) := \frac{3}{4}\int_{-1}^{1}(1-z^2)p_n(t+\varepsilon+\varepsilon z)\,dz - 3\varepsilon^{2/3} \qquad \forall \varepsilon > 0, t \geq 0.$$

Then for each $n > 0$, $t \leq 0$, when $z \in [-1, 1]$,

$$(1-z^2)p_n(t+\varepsilon+\varepsilon z) \leq (1-z^2)M.$$

By Lebesgue Convergence theorem $\lim_{n\to\infty} p_n^\varepsilon(t)$ exists, denoted by

$$p^\varepsilon(t) := \lim_{n\to\infty} p_n^\varepsilon(t) \qquad \forall \varepsilon > 0, t \geq 0.$$

□

Remark 3.1.1. *When $t = 0$, (3.1.2) yields: $\lim_{\varepsilon\searrow 0} p^\varepsilon(0) = p(0) = 1$ and $p^\varepsilon(0)$ is a monotone function of ε. We denote by $\varepsilon^* > 0$ the unique constant such that $p^{\varepsilon^*}(0) = 0$, and in the sequel assume $\varepsilon \in (0, \varepsilon^*)$.*

Lemma 3.1.2. *There exists an approximation W^ε for $w(\cdot, 0) = \mathbf{1}_{(-\infty,0)}$ such that*

1. *for each $\varepsilon > 0$,*

$$W^\varepsilon(x) = 0 \;\; \forall\, x \geq 0, \quad W^\varepsilon(x) = p^\varepsilon(0) \;\; \forall\, x \leq -\varepsilon;$$

2. *$W^\varepsilon \in C^1(\mathbb{R})$ and*

$$\frac{d}{dx}W^\varepsilon(x) \leq 0;$$

3. *for each $x \in R$,*

$$\frac{d}{d\varepsilon}W^\varepsilon(x) \leq 0,$$

and consequently,

$$\lim_{\varepsilon\searrow 0} W^\varepsilon = \mathbf{1}_{(-\infty,0)}.$$

Proof. We fix a function $W(\cdot) \in C^1(\mathbb{R})$ defined on $\mathbb{R}$ that satisfies:

$$W(x) = 0 \;\forall\, x \geq 0, \qquad W(x) = 1 \;\forall\, x \leq -1, \qquad \dot{W} \leq 0 \;\forall\, x \in (-1, 0).$$

Set

$$W^\varepsilon(x) := p^\varepsilon(0)\, W(x/\varepsilon) \quad \forall\, x \in \mathbb{R}.$$

Now we verify that W^ε is the approximation we need.

1. For each $\varepsilon > 0$. If $x \leq -\varepsilon$, then $x/\varepsilon \leq -1$. Hence $W^\varepsilon(x) = p^\varepsilon(0)$. If $x \geq 0$, then $x/\varepsilon \geq 0$. Hence $W^\varepsilon(x) = 0$. The first assertion follows.

2. Since $W \in C^1(\mathbb{R})$ and $\dot{W} \leq 0$, $W^\varepsilon \in C^1(\mathbb{R})$ and

$$\frac{d}{dx} W^\varepsilon(x) = \frac{1}{\varepsilon} p^\varepsilon(0) \dot{W}(x/\varepsilon) \leq 0.$$

The second assertion follows.

3. For each $x \in \mathbb{R}$,

$$\frac{d}{d\varepsilon} W^\varepsilon(x) = \frac{d}{d\varepsilon} p^\varepsilon(0) W(x/\varepsilon) - \frac{x}{\varepsilon^2} p^\varepsilon(0) \dot{W}(x/\varepsilon).$$

As W is nonnegative and $\frac{d}{d\varepsilon} p^\varepsilon(0) < 0$, $\frac{d}{d\varepsilon} p^\varepsilon(0) W(x/\varepsilon) \leq 0$. If $-1 < x/\varepsilon < 0$ then $x < 0$ and $\dot{W}(x/\varepsilon) \leq 0$. Consequently

$$-\frac{x}{\varepsilon^2} p^\varepsilon(0) \dot{W}(x/\varepsilon) - \leq 0.$$

If $x/\varepsilon \geq 0$ and $x/\varepsilon \leq -1$, then $\dot{W}(x/\varepsilon) = 0$. Thus

$$\frac{d}{d\varepsilon} W^\varepsilon(x) \leq 0.$$

Since W^ε is decreasing in terms of ε and bounded, $\lim_{\varepsilon \searrow 0} W^\varepsilon(x)$ exists. If $x \geq 0$, then

$$\lim_{\varepsilon \searrow 0} W^\varepsilon(x) = \lim_{\varepsilon \searrow 0} p^\varepsilon(0) W(x/\varepsilon) = \lim_{\varepsilon \searrow 0} 0 = 0;$$

otherwise $x < 0$, then

$$\lim_{\varepsilon \searrow 0} W^\varepsilon(x) = \lim_{\varepsilon \searrow 0} p^\varepsilon(0) \lim_{\varepsilon \searrow 0} W(x/\varepsilon) = 1.$$

The last equality holds since $x/\varepsilon \ll -1$ when $\varepsilon \searrow 0$. Hence the third assertion follows.

□

Before proving the existence of a solution to problem (3.1.1), we introduce the following functions.

1. Consider a first order linear initial value problem,

$$\begin{cases} \mathcal{L}w_0^\varepsilon = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ w_0^\varepsilon(\cdot,0) = W^\varepsilon(\cdot) & \text{on } \mathbb{R}\times\{0\}. \end{cases} \tag{3.1.3}$$

Since $W^\varepsilon(\cdot)$ is a smooth function, the problem admits a unique solution, denoted by $w_0^\varepsilon(x,t)$ and it can be expressed as

$$w_0^\varepsilon(x,t) = \int_{\mathbb{R}} K(x,t;y,0)w_0^\varepsilon(y,0)\,dy = p^\varepsilon(0)\int_{-\infty}^{0} K(x,t;y,0)W(y/\varepsilon)\,dy$$

where $K(x,t;y,s)$ is the fundamental solution associated with the linear operator $\mathcal{L}$. In particular, when $\mathcal{L} = \partial_t - \frac{1}{2}\partial_{xx}$, i.e., $\mu \equiv 0$ and $\sigma \equiv 1$,

$$K(x,t;y,s) = \Gamma(x-y,t-s), \quad \Gamma(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}.$$

2. Consider a first order ODE, to:

$$\begin{cases} \frac{d}{dt}\rho^\varepsilon(t) = -\beta\left(\frac{\rho^\varepsilon(t)-p^\varepsilon(t)}{\varepsilon}\right) & \text{in } (0,\infty), \\ \rho^\varepsilon(0) = p^\varepsilon(0). \end{cases} \tag{3.1.4}$$

Since $-\beta\left(\frac{\rho^\varepsilon(t)-p^\varepsilon(t)}{\varepsilon}\right)$ is a smooth function with respect to ρ^ε, (3.1.4) admits a unique smooth solution, denoted by ρ^ε.

Lemma 3.1.3. *The solution ρ^ε to (3.1.4) satisfies the following:*

$$p^\varepsilon(t) \le \rho^\varepsilon(t) \le p^\varepsilon(t) + \varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3}, \qquad \dot{\rho}^\varepsilon(t) \le 0, \qquad \forall t \ge 0.$$

Consequently

$$0 \le \rho^\varepsilon(t) - p^\varepsilon(t) \le \varepsilon^{2/3} \quad \textit{and} \quad \lim_{\varepsilon\searrow 0}\rho^\varepsilon(t) = p(t).$$

Proof. Set $\varphi(x) := x1_{(0,\,\infty)}(x)$. Let $\{\varphi_n\}$ be a sequence of twice continuously differentiable functions such that $\lim_{n\to\infty}\varphi_n(x) = \varphi(x)$ and $0 \le \dot{\varphi}_n \le 1$. Then $\lim_{n\to\infty}\dot{\varphi}_n(x) = 1_{(0,\infty)}(x)$ almost everywhere.

1. First we claim that $p^\varepsilon(t) \le \rho^\varepsilon(t)$.

Set $\alpha(t) := p^\varepsilon(t) - \rho^\varepsilon(t)$. Then for $\alpha(0) = 0$ and for each $t > 0$, $\alpha(s)$ is bounded as $0 \le s \le t$ since

$$|\alpha(s)| \le |p^\varepsilon(s)| + |\rho^\varepsilon(s)| \le |p^\varepsilon(0)| + |\rho^\varepsilon(0)| = 2p^\varepsilon(0).$$

For each $n > 0$ and $t > 0$,

$$\begin{aligned}
\varphi_n(\alpha(t)) - \varphi_n(\alpha(0)) &= \int_0^t \dot{\varphi}_n(\alpha(s))\dot{\alpha}(s)ds \\
&= \int_0^t \dot{\varphi}_n(\alpha(s))\left(\dot{p}^\varepsilon(s) - \dot{\rho}^\varepsilon(s)\right)ds \\
&= \int_0^t \dot{\varphi}_n(\alpha(s))\left(\dot{p}^\varepsilon(s) + \beta\left(\frac{\rho^\varepsilon(s) - p^\varepsilon(s)}{\varepsilon}\right)\right)ds \\
&\le \int_0^t \dot{\varphi}_n(\alpha(s))\beta\left(\frac{\rho^\varepsilon(s) - p^\varepsilon(s)}{\varepsilon}\right)ds \\
&= \int_0^t \frac{1}{\varepsilon^3}\dot{\varphi}_n(\alpha(s))(\alpha(s)^-)^3 ds.
\end{aligned}$$

Note that as $0 \le \dot{\varphi}_n \le 1$ and $\alpha(s)$ is bounded, $\varphi_n(\alpha(s))(\alpha(s)^-)^3$ is bounded. Hence using the dominated convergence theorem, we obtain

$$\begin{aligned}
0 &\le \alpha(t)^+ = \varphi(\alpha(t)) - \varphi(\alpha(0)) = \lim_{n\to\infty}\varphi_n(\alpha(t)) - \varphi_n(\alpha(0)) \\
&\le \lim_{n\to\infty}\int_0^t \frac{1}{\varepsilon^3}\dot{\varphi}_n(\alpha(s))(\alpha(s)^-)^3 ds \\
&= \frac{1}{\varepsilon^3}\int_0^t 1_{\alpha(s)>0}(\alpha(s))(\alpha(s)^-)^3 ds = 0.
\end{aligned}$$

Then $\alpha^+(t) = 0$, which implies that $\alpha(t) \le 0$. Hence $p^\varepsilon \le \rho^\varepsilon$.

2. Now we claim that $\rho^\varepsilon \le p^\varepsilon + \varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3}$.

Set $\gamma(t) := -\alpha(t) - \varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3}$. Then $\gamma(0) = -\varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3} < 0$ and for each $t > 0$, $\gamma(s)$ is bounded as $0 \le s \le t$. For each $n > 0$ and $t > 0$,

$$
\begin{aligned}
\varphi_n(\gamma(t)) - \varphi_n(\gamma(0)) &= \int_0^t \dot{\varphi}_n(\gamma(s))\dot{\gamma}(s)ds \\
&= \int_0^t \dot{\varphi}_n(\gamma(s))\left(\dot{\rho}^\varepsilon(s) - \dot{p}^\varepsilon(s)\right)ds \\
&= \int_0^t \dot{\varphi}_n(\gamma(s))\left(-\beta\left(\frac{-\alpha(t)}{\varepsilon}\right) - \dot{p}(s)\right)ds \\
&\le \int_0^t \dot{\varphi}_n(\gamma(s))\left(-\left(\frac{-\alpha(t)}{\varepsilon}\right)^3 + \left(\frac{\varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3}}{\varepsilon}\right)^3\right)ds \\
&= \int_0^t \frac{1}{\varepsilon^3}\dot{\varphi}_n(\gamma(s))\left(-\gamma(s)\right)\left((-\alpha(s))^2 - \varepsilon\alpha(s)\|\dot{p}^\varepsilon\|_\infty^{1/3} + \varepsilon^2\|\dot{p}^\varepsilon\|_\infty^{2/3}\right)^2 ds \\
&\le \int_0^t \frac{1}{\varepsilon^3}\dot{\varphi}_n(\gamma(s))\left(-\gamma(s)\right)^+\left((-\alpha(s))^2 - \varepsilon\alpha(s)\|\dot{p}^\varepsilon\|_\infty^{1/3} + \varepsilon^2\|\dot{p}^\varepsilon\|_\infty^{2/3}\right)^2 ds \\
&\le \int_0^t \frac{C(t)}{\varepsilon^3}\dot{\varphi}_n(\gamma(s))\left(-\gamma(s)\right)^+ ds \\
&= \int_0^t \frac{C(t)}{\varepsilon^3}\dot{\varphi}_n(\gamma(s))\gamma(s)^- ds,
\end{aligned}
$$

where $C(t)$ is the constant depending on t. Since $\gamma(s)$ is bounded, we can use the dominated convergence theorem and obtain

$$
\begin{aligned}
0 &\le \gamma(t)^+ = \varphi(\gamma(t)) - \varphi(\gamma(0)) = \lim_{n\to\infty} \varphi_n(\gamma(t)) - \varphi_n(\gamma(0)) \\
&\le C(t)\int_0^t \frac{1}{\varepsilon^3} 1_{\gamma(s)>0}\left(\gamma(s)\right)\gamma(s)^- ds = 0.
\end{aligned}
$$

So that $\gamma(t)^+ = 0$ and it implies that $\gamma(t) \le 0$. Hence $\rho^\varepsilon \le p^\varepsilon + \varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3}$.

□

Now we are ready to prove the existence of a solution to problem (3.1.1).

Theorem 3. *For each $\varepsilon > 0$, problem (3.1.1) admits a unique smooth ($C^{2,1}$) solution in $\mathbb{R} \times [0,\infty)$. The solution is continuously differentiable in ε and satisfies, for all $\varepsilon > 0$ and $(x,t) \in \mathbb{R} \times (0,\infty)$,*

$$
w_0^\varepsilon(x,t) + \rho^\varepsilon(t) - \rho^\varepsilon(0) \le w^\varepsilon(x,t) \le \min\{\rho^\varepsilon(t), w_0^\varepsilon(x,t)\}, \tag{3.1.5}
$$

$$w_x^\varepsilon(x,t) < 0, \qquad \frac{d}{d\varepsilon} w^\varepsilon(x,t) < 0.$$

Consequently, the following limit exists

$$w(x,t) := \lim_{\varepsilon \searrow 0} w^\varepsilon(x,t) \quad \forall\,(x,t) \in \mathbb{R} \times [0,\infty).$$

Proof. Let $\bar{w}^\varepsilon := \min\{\rho^\varepsilon, w_0^\varepsilon\}$ and $\underline{w}^\varepsilon := w_0^\varepsilon(x,t) + \rho^\varepsilon(t) - \rho^\varepsilon(0)$. First we claim that $\bar{w}^\varepsilon$ is a supersolution, $\underline{w}^\varepsilon$ is a subsolution to (3.1.1) and $\underline{w}^\varepsilon \le \bar{w}^\varepsilon$.

Since $\mathcal{L}w_0^\varepsilon + \beta(\frac{w_0^\varepsilon - p^\varepsilon}{\varepsilon}) = \beta(\frac{w_0^\varepsilon - p^\varepsilon}{\varepsilon}) \ge 0$, w_0^ε is a supersolution. Also we have

$$w_0^\varepsilon \le \max\{w_0^\varepsilon(\cdot,0)\} = \max_{x\in\mathbb{R}} W^\varepsilon(x) = \max_{x\in\mathbb{R}} p^\varepsilon(0) W(x/\varepsilon) = p^\varepsilon(0) = \rho^\varepsilon(0). \tag{3.1.6}$$

Since $\mathcal{L}\rho^\varepsilon + \beta(\frac{\rho^\varepsilon - p^\varepsilon}{\varepsilon}) = 0$, ρ^ε is another supersolution. Hence, $\bar{w}^\varepsilon$ is a supersolution. Since $\dot{\beta} \ge 0$, the direct computation, together with (3.1.6), gives

$$\begin{aligned}
& \mathcal{L}\underline{w}^\varepsilon + \beta\Big(\frac{\underline{w}^\varepsilon - p^\varepsilon}{\varepsilon}\Big) \\
= \;& \mathcal{L}w_0^\varepsilon(x,t) + \mathcal{L}\rho^\varepsilon(t) + \beta\Big(\frac{\underline{w}^\varepsilon - p^\varepsilon}{\varepsilon}\Big) \\
= \;& -\Big[\beta\Big(\frac{\rho^\varepsilon - p^\varepsilon}{\varepsilon}\Big) - \beta\Big(\frac{\rho^\varepsilon - p^\varepsilon + w_0^\varepsilon - \rho^\varepsilon(0)}{\varepsilon}\Big)\Big] \le 0.
\end{aligned}$$

Hence $\underline{w}^\varepsilon$ is a subsolution, and $\underline{w}^\varepsilon \le \bar{w}^\varepsilon$ by (3.1.6).

Now we prove that (3.1.1) admits a unique smooth solution in $\mathbb{R} \times [0,\infty)$ satisfying (3.1.5) with the standard method of subsolutions and supersolution [2]. Fix $\lambda \ge 3\varepsilon^{-5/3}$. Let $w_1^\varepsilon = \bar{w}^\varepsilon$, and then given w_k^ε $(k = 1, 2, \cdots)$ inductively define $w_{k+1}^\varepsilon \in \mathrm{C}^{2,1}(\mathbb{R} \times (0,\infty))$ to be the unique solution of the linear initial-value problem to

$$\begin{cases} \mathcal{L}w_{k+1}^\varepsilon + \lambda w_{k+1}^\varepsilon = -\beta\left(\frac{w_k^\varepsilon - p^\varepsilon}{\varepsilon}\right) + \lambda w_k^\varepsilon & \text{in } \mathbb{R} \times (0,\infty), \\ w_{k+1}^\varepsilon(\cdot,0) = W^\varepsilon(\cdot) & \text{on } \mathbb{R} \times \{0\}. \end{cases} \tag{3.1.7}$$

First we claim that

$$\bar{w}^\varepsilon = w_1^\varepsilon \ge w_2^\varepsilon \ge \cdots \ge w_k^\varepsilon \ge \cdots \quad \text{in } \; \mathbb{R} \times [0,\infty).$$

Let $v := w_2^\varepsilon - w_1^\varepsilon$, then

$$\begin{cases} \mathcal{L}v^\varepsilon + \lambda v^\varepsilon \leq 0 & \text{in } \mathbb{R} \times (0, \infty), \\ v^\varepsilon(\cdot, 0) = 0 & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

By the maximum principle

$$\max_{\mathbb{R}\times(0,\infty)} v^\varepsilon \leq \max_{\mathbb{R}\times\{0\}} v^\varepsilon = 0,$$

so that $w_1^\varepsilon \geq w_2^\varepsilon$. Now assume inductively $w_{k-1}^\varepsilon \geq w_k^\varepsilon$ and let $v := w_{k+1}^\varepsilon - w_k^\varepsilon$. Then

$$\begin{aligned} \mathcal{L}v^\varepsilon + \lambda v^\varepsilon &= -\left(\beta\left(\frac{w_k^\varepsilon - p^\varepsilon}{\varepsilon}\right) - \beta\left(\frac{w_{k-1}^\varepsilon - p^\varepsilon}{\varepsilon}\right)\right) + \lambda(w_k^\varepsilon - w_{k-1}^\varepsilon) \\ &= \left(\lambda - \frac{1}{\varepsilon}\dot{\beta}(\xi)\right)(w_k^\varepsilon - w_{k-1}^\varepsilon) \end{aligned}$$

where

$$\frac{w_k^\varepsilon - p^\varepsilon}{\varepsilon} \leq \xi \leq \frac{w_{k-1}^\varepsilon - p^\varepsilon}{\varepsilon}.$$

Since

$$\dot{\beta}(x) = \begin{cases} 3x^2 & x > 0, \\ 0 & x \leq 0, \end{cases}$$

and

$$\frac{w_k^\varepsilon - p^\varepsilon}{\varepsilon} \leq \xi \leq \frac{w_{k-1}^\varepsilon - p^\varepsilon}{\varepsilon} \leq \frac{\overline{w}^\varepsilon - p^\varepsilon}{\varepsilon} \leq \frac{\rho^\varepsilon - p^\varepsilon}{\varepsilon},$$

we obtain that

$$\dot{\beta}(\xi) \leq 3\left(\frac{\rho^\varepsilon - p^\varepsilon}{\varepsilon}\right)^2 \leq 3\left(\frac{\varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3}}{\varepsilon}\right)^2 \leq 3\varepsilon^{-2/3},$$

where the second inequality follows from lemma (3.1.3) and the third inequality follows from the first assertion of lemma (3.1.1). Thus

$$\begin{cases} \mathcal{L}v^\varepsilon + \lambda v^\varepsilon \leq 0 & \text{in } \mathbb{R} \times (0, \infty) \\ v^\varepsilon(\cdot, 0) = 0 & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

By the maximum principle

$$\max_{\mathbb{R}\times(0,\infty)} v^\varepsilon \le \max_{\mathbb{R}\times\{0\}} v^\varepsilon = 0,$$

so that $w_k^\varepsilon \ge w_{k+1}^\varepsilon$.

Secondly we claim that

$$w_k^\varepsilon \ge \underline{w}^\varepsilon \quad \text{in } \mathbb{R}\times[0,\infty), \quad \text{for } k = 1, 2, \cdots . \tag{3.1.8}$$

It is clear that $w_1^\varepsilon = \bar{w}^\varepsilon \ge \underline{w}^\varepsilon$. Assume now for induction

$$w_k^\varepsilon \ge \underline{w}^\varepsilon \quad \text{in } \mathbb{R}\times[0,\infty).$$

Let $v^\varepsilon := \underline{w}^\varepsilon - w_{k+1}^\varepsilon$, then

$$\begin{aligned} \mathcal{L}v^\varepsilon + \lambda v^\varepsilon &\le -\left(\beta\left(\frac{\underline{w}^\varepsilon - p^\varepsilon}{\varepsilon}\right) - \beta\left(\frac{w_k^\varepsilon - p^\varepsilon}{\varepsilon}\right)\right) + \lambda(\underline{w}^\varepsilon - w_k^\varepsilon) \\ &= \left(\lambda - \frac{\dot\beta(\xi)}{\varepsilon}\right)(\underline{w}^\varepsilon - w_k^\varepsilon) \le 0, \end{aligned}$$

with the similar argument as above. By the maximum principle

$$\max_{\mathbb{R}\times(0,\infty)} v^\varepsilon \le \max_{\mathbb{R}\times\{0\}} v^\varepsilon = 0,$$

so that $w_{k+1}^\varepsilon \ge \underline{w}^\varepsilon$. Thus (3.1.8) holds.

Now we have

$$\bar{w}^\varepsilon = w_1^\varepsilon \ge w_2^\varepsilon \ge \cdots \ge w_k^\varepsilon \ge w_{k+1}^\varepsilon \ge \cdots \ge \underline{w} \quad \text{in } \mathbb{R}\times[0,\infty).$$

Therefore

$$w^\varepsilon(x,t) := \lim_{k\to\infty} w_k^\varepsilon(x,t)$$

exists and it is bounded. Let $\tilde\Gamma(x,t;y,s)$ be the fundamental solution associated with the operator $\mathcal{L} + \lambda$. Then the solution to (3.1.7) for each $k > 0$ can be expressed as:

$$\begin{aligned} w_k^\varepsilon(x,t) &= \int_0^t \int_{-\infty}^{\infty} \tilde\Gamma(x,t;y,s)\left(-\beta\left(\frac{w_{k-1}^\varepsilon(y,s) - p(s)}{\varepsilon}\right) - \lambda w_{k-1}(y,s)\right) dyds \\ &\quad + \int_{-\infty}^{\infty} \tilde\Gamma(x,t;y,0) W^\varepsilon(y) dy. \end{aligned}$$

As w_k^ε are bounded, by the Dominated Convergent Theorem, we obtain that

$$\begin{aligned} w^\varepsilon(x,t) &= \int_0^t \int_{-\infty}^{\infty} \tilde{\Gamma}(x,t;y,s) \left(-\beta\left(\frac{w^\varepsilon(y,s)-p(s)}{\varepsilon} \right) - \lambda w^\varepsilon(y,s) \right) dyds \\ &\quad + \int_{-\infty}^{\infty} \tilde{\Gamma}(x,t;y,0) W^\varepsilon(y) dy. \end{aligned}$$

Then w^ε is continuous. Let $\Omega_T := \mathbb{R} \times [0,T]$. Then w^ε is locally Hölder continuous and uniformly continuous with respect to t. Let $\tilde{w}^\varepsilon$ be the solution to

$$\begin{cases} \mathcal{L}\tilde{w}^\varepsilon + \lambda \tilde{w}^\varepsilon = -\beta\left(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\right) + \lambda w^\varepsilon & \text{in } \mathbb{R}\times(0,\infty), \\ \tilde{w}^\varepsilon(\cdot,0) = W^\varepsilon(\cdot) & \text{on } \mathbb{R}\times\{0\}. \end{cases}$$

Then $\tilde{w}^\varepsilon \in C^{2,1}(\Omega_T)$ and

$$\begin{aligned} \tilde{w}^\varepsilon(x,t) &= \int_0^t \int_{-\infty}^{\infty} \tilde{\Gamma}(x,t;y,s) \left(-\beta\left(\frac{w^\varepsilon(y,s)-p(s)}{\varepsilon} \right) - \lambda w^\varepsilon(y,s) \right) dyds \\ &\quad + \int_{-\infty}^{\infty} \tilde{\Gamma}(x,t;y,0) W^\varepsilon(y) dy = w^\varepsilon(x,t). \end{aligned}$$

Hence w^ε solves $\mathcal{L}w^\varepsilon = -\beta\left(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\right)$ in Ω_T. Let $T \to \infty$, we obtain that $w^\varepsilon(x,t)$ is the smooth solution to (3.1.1). Also since w^ε is bounded by $\bar{w}^\varepsilon$ and $\underline{w}^\varepsilon$, the uniqueness follows.

We see that (3.1.1) admits a unique smooth solution in $\mathbb{R}\times[0,\infty)$ satisfying (3.1.5).

We remark that w^ε is smooth with respect to ε with the standard arguments.

2. Differentiating the system (3.1.1) with respect to ε we obtain

$$\begin{aligned} \frac{d}{d\varepsilon} w^\varepsilon(x,0) &= \frac{d}{d\varepsilon} W^\varepsilon(x) \le 0 \quad \forall x \in \mathbb{R}, \\ \mathcal{L}\frac{d}{d\varepsilon}w^\varepsilon + \frac{1}{\varepsilon}\dot{\beta}\Big(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\Big)\frac{d}{d\varepsilon}w^\varepsilon &= \frac{1}{\varepsilon^2}\dot{\beta}\Big(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\Big)\Big\{w^\varepsilon - p^\varepsilon + \varepsilon\frac{d}{d\varepsilon}p^\varepsilon\Big\} \le 0, \end{aligned}$$

since $\dot{\beta} \ge 0$, $w^\varepsilon - p^\varepsilon \le \rho^\varepsilon - p^\varepsilon \le \varepsilon\|\dot{p}^\varepsilon\|_\infty^{1/3} \le \varepsilon^{2/3}$, and $\frac{d}{d\varepsilon}p^\varepsilon \le -2\varepsilon^{-1/3}$. Then, by the maximum principle, $\frac{d}{d\varepsilon}w^\varepsilon < 0$ in $\mathbb{R}\times(0,\infty)$. The monotonicity and boundedness of w^ε in ε and imply that $w = \lim_{\varepsilon\searrow 0} w^\varepsilon$ exists.

In a similar manner, differentiating the system (3.1.1) with respect to x and let $u^\varepsilon := -w^\varepsilon_x$, we obtain

$$\mathcal{A}u^\varepsilon + \tfrac{1}{\varepsilon}\dot\beta\left(\tfrac{w^\varepsilon - p^\varepsilon}{\varepsilon}\right)u^\varepsilon = 0,$$
$$u^\varepsilon(x,0) = -\tfrac{d}{dx}w^\varepsilon(x,0) = -\tfrac{d}{dx}W^\varepsilon(x) \geq 0 \quad \forall x \in \mathbb{R}.$$

where $\mathcal{A}u = \mathcal{L}u - \sigma\sigma_x u_x + (\mu_x - \sigma\sigma_{xx} + (\sigma_x)^2)u$. Since $\frac{1}{\varepsilon}\dot\beta\left(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\right) > 0$, $-w^\varepsilon_x(x,t) = u^\varepsilon(x,t) > 0$ in $\mathbb{R}\times(0,\infty)$.

$\square$

Also note that since w^ε_0 is monotonic in ε and bounded, the limit $w_0 := \lim_{\varepsilon\searrow 0} w^\varepsilon_0$ exists and is the solution to

$$\mathcal{L}w_0 = 0 \ \text{ in } \ \mathbb{R}\times(0,\infty), \quad w_0(\cdot,0) = \mathbf{1}_{(-\infty,0)}.$$

3.2 CONTINUITY ESTIMATES AND EXISTENCE.

In this section, we prove that the limit $w := \lim_{\varepsilon\searrow 0} w^\varepsilon$ is the viscosity solution to our variational inequality. In order to do so, we first need to derive some supplementary estimates on the continuity of w.

Lemma 3.2.1. *For each $T > 0$, there exists a constant $C = C(T)$ that depends only on σ and μ such that for all $\varepsilon \in (0,\varepsilon^*)$, $0 < s < t \leq T$, and $x, y \in \mathbb{R}$,*

$$-\frac{Cp^\varepsilon(0)}{\sqrt{t}} \leq w^\varepsilon_{0x}(x,t) \leq w^\varepsilon_x(x,t) \leq 0, \tag{3.2.1}$$

$$|w^\varepsilon(x,t) - w^\varepsilon(y,s)| \leq \frac{Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}}\left\{|x-y| + 2\sqrt{t-s}\right\} + \rho^\varepsilon(s) - \rho^\varepsilon(t). \tag{3.2.2}$$

Consequently, the limit $w = \lim_{\varepsilon\searrow 0} w^\varepsilon$ satisfies for all $0 < s < t \leq T$ and $x, y \in \mathbb{R}$,

$$-\frac{C}{\sqrt{t}} \leq w_{0x}(x,t) \leq w_x(x,t) \leq 0, \tag{3.2.3}$$

$$|w(x,t) - w(y,s)| \leq \frac{C}{\min\{\sqrt{s}, 1\}}\Big\{|x-y| + 2\sqrt{t-s}\Big\} + p(s) - p(t), \tag{3.2.4}$$

$$w(x,t) - w(y,s) \leq \frac{C}{\min\{\sqrt{s}, 1\}}\Big\{|x-y| + 2\sqrt{t-s}\Big\}. \tag{3.2.5}$$

We remark that when $\sigma \equiv 1$ and $\mu \equiv 0$, $C = C(T) = (2\pi)^{-1/2}$ for all T.

Proof. Differentiating the systems (3.1.1) and (3.1.3) with respect to x, and using the notation from the previous theorem, we find

$$\begin{aligned} \mathcal{A}w_x^\varepsilon = -\varepsilon^{-1}\dot{\beta}(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)w_x^\varepsilon \geq 0 = \mathcal{A}w_{0x}^\varepsilon \qquad & \text{in } \mathbb{R} \times (0,\infty), \\ w_x^\varepsilon(\cdot,0) = W_x^\varepsilon(\cdot) = w_{0x}^\varepsilon(\cdot,0) \qquad & \text{on } \mathbb{R} \times \{0\}. \end{aligned}$$

since $w_x^\varepsilon \leq 0$. Therefore by the maximum principle (the zeroth order term in $\mathcal{A}$ is bounded above) $w_{0x}^\varepsilon \leq w_x^\varepsilon \leq 0$.

Next we estimate the lower bound of w_{0x}^ε. Differentiating the system (3.1.3) with respect to x, we obtain

$$\mathcal{A}w_{0x}^\varepsilon = 0 \ \text{ in } \ \mathbb{R} \times (0,\infty), \quad w_{0x}^\varepsilon(\cdot,0) = W_x^\varepsilon(\cdot) \ \text{ on } \ \mathbb{R} \times \{0\}.$$

This is a linear problem, the solution can be expressed as

$$w_{0x}^\varepsilon(x,t) = \int_{\mathbb{R}} \tilde{K}(x,t;y,0) W_y^\varepsilon(y)\, dy,$$

where $\tilde{K}$ is the fundamental solution associated with the linear operator $\mathcal{A}$. By Friedman [3]),

$$|\tilde{K}(x,t;y,s)| \leq \frac{C}{\sqrt{t-s}}\, e^{-\frac{\lambda(x-y)^2}{4(t-s)}},$$

where λ is some positive constant. When $s = 0$,

$$|\sqrt{t}\tilde{K}(x,t;y,0)| \leq C e^{-\frac{\lambda(x-y)^2}{4t}} \leq C.$$

Hence $\sqrt{t}\ \tilde{K}(x,t;y,0)$ has a least upper bounded and we denote it as

$$C = C(T) = \sup_{x,y\in\mathbb{R}, 0<t<T} \Big\{\sqrt{t}\ \tilde{K}(x,t;y,0)\Big\}.$$

As $w_{0x} \leq 0$ and $W_y^\varepsilon \leq 0$, then for any $0 < t \leq T$,

$$0 \leq -w_{0x}^\varepsilon(x,t) \leq \sup_{x,y\in\mathbb{R}} \{\tilde{K}(x,t;y,0)\} \int_{\mathbb{R}} -W_y^\varepsilon(y)\,dy \leq \frac{C}{\sqrt{t}} \int_{\mathbb{R}} -W_y^\varepsilon(y)\,dy = \frac{Cp^\varepsilon(0)}{\sqrt{t}}.$$

The estimates for w_x^ε and w_{0x}^ε (3.2.1) thus follow. Sending $\varepsilon \searrow 0$, we obtain (3.2.3).

Now we estimate the continuity in the time variable. By Theorem 3, $w_x^\varepsilon < 0$ and $w^\varepsilon(x,t) \geq \rho^\varepsilon(t) - \rho^\varepsilon(0)$ since $w_0^\varepsilon \geq 0$. We conclude that $\lim_{x\to\infty} w^\varepsilon$ exists. Similarly, the limit $\lim_{x\to\infty} w_0^\varepsilon$ exists, and is nonnegative since $w_0^\varepsilon \geq 0$. Now since

$$w^\varepsilon(x,t) \leq \min\{\rho^\varepsilon(t), w_0^\varepsilon(x,t)\} \leq \rho^\varepsilon(t) \quad \forall\, t \geq 0,$$

we can compute

$$\begin{aligned}
\int_{\mathbb{R}} |w_x^\varepsilon(x,t)|dx &= \int_{\mathbb{R}} -w_x^\varepsilon(x,t)dx \\
&\leq \rho^\varepsilon(t) - \lim_{x\to\infty} w^\varepsilon(x,t) \\
&\leq \rho^\varepsilon(t) - \lim_{x\to\infty} \big(w_0^\varepsilon(x,t) + \rho^\varepsilon(t) - \rho^\varepsilon(0)\big) \\
&= \rho^\varepsilon(0) - \lim_{x\to\infty} w_0^\varepsilon(x,t) \leq \rho^\varepsilon(0) = p^\varepsilon(0).
\end{aligned}$$

Also note that

$$0 \leq \beta(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)) \leq \beta(\varepsilon^{-1}(\rho^\varepsilon - p^\varepsilon)) = -\dot{\rho}^\varepsilon(t) \qquad \forall\, t \geq 0, x \in \mathbb{R},$$

since $\beta(\cdot)$ is increasing and $w^\varepsilon \leq \rho^\varepsilon$. For $0 < s < t \leq T$ denote

$$\|w_x^\varepsilon\|_\infty^{s,t} = \sup_{\mathbb{R}\times[s,t]} |w_x^\varepsilon|.$$

Since $0 \geq w_x^\varepsilon(x,s) \geq -\frac{Cp^\varepsilon(0)}{\sqrt{s}}$,

$$\|w_x^\varepsilon\|_\infty^{s,t} \leq \sup_{s\leq\xi\leq t} \|w_x(\cdot,\xi)\|_\infty \leq \sup_{s\leq\xi\leq t} \left\{\frac{Cp^\varepsilon(0)}{\sqrt{\xi}}\right\} \leq \frac{Cp^\varepsilon(0)}{\sqrt{s}}.$$

Then for each $\delta > 0$,

$$
\begin{aligned}
&\Big|\int_{x-\delta}^{x+\delta}\{w^\varepsilon(y,t) - w^\varepsilon(y,s)\}\,dy\Big| = \Big|\int_{x-\delta}^{x+\delta}\int_s^t w_v^\varepsilon(y,v)\,dv\,dy\Big| \\
&= \Big|\int_s^t\int_{x-\delta}^{x+\delta}\left(\frac{1}{2}(\sigma^2 w_y^\varepsilon)_y - \mu w_y^\varepsilon - \beta\left(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)\right)\right)dy\,dv\Big| \\
&\le \Big|\int_s^t\left(\frac{1}{2}\left(\sigma^2 w_y^\varepsilon\right)\Big|_{x-\delta}^{x+\delta}\right)dv\Big| + \Big|\int_s^t\int_{x-\delta}^{x+\delta}\mu\, w_y^\varepsilon dy\,dv\Big| + \Big|\int_s^t\int_{x-\delta}^{x+\delta}\beta\left(\varepsilon^{-1}(\rho^\varepsilon - p^\varepsilon)\right)dy\,dv\Big| \\
&\le (t-s)\left(\|\sigma^2\|_\infty\|w_x^\varepsilon\|_\infty^{s,t} + p^\varepsilon(0)\|\mu\|_\infty\right) + \Big|\int_s^t\int_{x-\delta}^{x+\delta} -\dot{\rho}^\varepsilon(v)dy\,dv\Big| \\
&\le (t-s)\left(\|\sigma^2\|_\infty\|w_x^\varepsilon\|_\infty + p^\varepsilon(0)\|\mu\|_\infty\right) + 2\delta(\rho^\varepsilon(s) - \rho^\varepsilon(t)).
\end{aligned}
$$

Finally, note that for any $s \ge 0$,

$$
\begin{aligned}
\left|w^\varepsilon(x,s) - \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} w^\varepsilon(y,s)\,dy\right| &= \left|\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\left(w^\varepsilon(x,s) - w^\varepsilon(y,s)\right)\,dy\right| \\
&\le \frac{1}{2\delta}\int_{x-\delta}^{x+\delta}|y-x|\cdot\|w_x^\varepsilon(\cdot,s)\|_\infty\,dy \\
&\le \frac{\delta}{2}\|w_x^\varepsilon(\cdot,s)\|_\infty.
\end{aligned}
$$

Now we are ready to estimate the continuity in the time variable. For any $0 < s < t \le T$,

$$
\begin{aligned}
&|w^\varepsilon(x,t) - w^\varepsilon(x,s)| \\
&\le \left|w^\varepsilon(x,t) - \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} w^\varepsilon(y,t)dy\right| + \left|\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\left(w^\varepsilon(y,t) - w^\varepsilon(y,s)\right)dy\right| \\
&\quad + \left|\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} w^\varepsilon(y,s)dy - w^\varepsilon(x,s)dy\right| \\
&\le \|w_x^\varepsilon\|_\infty^{s,t}\left(\delta + \frac{(t-s)\|\sigma^2\|_\infty}{2\delta}\right) + \frac{(t-s)\|\mu\|_\infty p^\varepsilon(0)}{2\delta} + \rho^\varepsilon(s) - \rho^\varepsilon(t).
\end{aligned}
$$

By taking $\delta = \sqrt{\frac{\|\sigma^2\|_\infty(t-s)}{2}}$, we then obtain

$$
\begin{aligned}
|w^\varepsilon(x,t) - w^\varepsilon(x,s)| &\le \sqrt{\frac{\|\sigma^2\|_\infty(t-s)}{2}}\left(2\|w_x^\varepsilon\|_\infty^{s,t} + \frac{\|\mu\|_\infty p^\varepsilon(0)}{\|\sigma^2\|_\infty}\right) + \rho^\varepsilon(s) - \rho^\varepsilon(t) \\
&\le \frac{2Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}} + \rho^\varepsilon(s) - \rho^\varepsilon(t).
\end{aligned}
$$

As $0 \geq w^\varepsilon_x(x,s) \geq -\frac{Cp^\varepsilon(0)}{\sqrt{s}}$, we have

$$|w^\varepsilon(x,s) - w^\varepsilon(y,s)| \leq |x-y| \cdot \|w^\varepsilon_x(\cdot,s)\|_\infty \leq \frac{Cp^\varepsilon(0)}{\sqrt{s}} |x-y| .$$

Then

$$|w^\varepsilon(x,t) - w^\varepsilon(y,s)| \leq |w^\varepsilon(x,t) - w^\varepsilon(x,s)| + |w^\varepsilon(x,s) - w^\varepsilon(y,s)|$$
$$\leq \frac{Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}} \left\{|x-y| + 2\sqrt{t-s}\right\} + \rho^\varepsilon(s) - \rho^\varepsilon(t).$$

This proves (3.2.2), and (3.2.4) then follows by sending $\varepsilon \searrow 0$. Finally, observe that in estimating the upper bound of $w^\varepsilon(x,s) - w^\varepsilon(x,t)$ if we keep the term involving the integral of β, then

$$\begin{aligned} w^\varepsilon(x,t) - w^\varepsilon(y,s) &\leq w^\varepsilon(x,t) - w^\varepsilon(x,s) + |w^\varepsilon(x,s) - w^\varepsilon(y,s)| \\ &\leq \frac{Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}} \left\{|x-y| + 2\sqrt{t-s}\right\} + \frac{\int_s^t \int_{x-\delta}^{x+\delta} -\beta\left(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\right) dy\, dv}{2\delta} \\ &\leq \frac{Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}} \left\{|x-y| + 2\sqrt{t-s}\right\}. \end{aligned}$$

We obtain (3.2.5) by sending $\varepsilon \searrow 0$. This completes the proof. □

We are ready to show the following:

Theorem 4. *Assume $p(\cdot)$ defined on $[0,\infty)$ is nonnegative, decreasing and lower semicontinuous, with $p(0) = 1$. There exists a unique viscosity solution to 2.1.2, and it can be obtained as the limit $w := \lim_{\varepsilon \searrow 0} w^\varepsilon$.*

Proof. 1. First we verify that w satisfies the initial condition (2.2.1).

Since

$$\begin{aligned} \lim_{\varepsilon \searrow 0} W^\varepsilon(x) &= \lim_{\varepsilon \searrow 0} p^\varepsilon(0) W(x/\varepsilon) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases} \\ &= 1_{(-\infty,0)} \end{aligned}$$

$$w(x,0) = \lim_{\varepsilon \searrow 0} w^\varepsilon(x,0) = \lim_{\varepsilon \searrow 0} W^\varepsilon(x) = 1_{(-\infty,0)}.$$

Similarly

$$w_0(x,0) = \lim_{\varepsilon\searrow 0} w_0^\varepsilon(x,0) = \lim_{\varepsilon\searrow 0} W^\varepsilon(x) = 1_{(-\infty,0)}.$$

For any $t > 0$, from (3.1.5)

$$\rho^\varepsilon(t) - \rho^\varepsilon(0) \le w^\varepsilon(\cdot,t) - w_0^\varepsilon(\cdot,t) \le 0.$$

Sending $\varepsilon \searrow 0$, we get

$$p(t) - p(0) \le w(\cdot,t) - w_0(\cdot,t) \le 0,$$

where

$$w_0(x,t) = \lim_{\varepsilon\to 0} w_0^\varepsilon(x,t) = \int_{-\infty}^0 K(x,t;y,0)\,dy.$$

Then

$$\|w(\cdot,t) - w_0(\cdot,t)\|_\infty = \sup_{x\in\mathbb{R}} |w(\cdot,t) - w_0(\cdot,t)| \le p(0) - p(t).$$

From (3.1.5), w^ε has upper and lower bounds. Sending $\varepsilon \searrow 0$, we obtain that

$$p(t) - 1 \le w_0(x,t) + p(t) - 1 \le w(x,t) \le p(t) < 1,$$

since $w_0(x,t) \ge 0$. It implies that

$$\limsup_{y\to x, t\searrow 0} w(y,t), \quad 0 \le \liminf_{y\to x, t\searrow 0} w(y,t) \le 1$$

Next we claim that

$$\limsup_{y\to x, t\searrow 0} w(y,t) = 1_{(-\infty,0]}, \quad \text{and} \quad \liminf_{y\to x, t\searrow 0} w(y,t) = 1_{(-\infty,0)}, \tag{3.2.6}$$

by considering the following three cases:

$$\text{(i) } x < 0, \quad \text{(ii) } x > 0, \quad \text{(iii) } x = 0.$$

Case (i): Suppose $x < 0$. For any sequence $x_n \to x$, and $t_n \searrow 0$,

$$\begin{aligned}
&\lim_{n\to\infty} |w(x,0) - w(x_n,t_n)| \\
\leq\ & \left(\lim_{n\to\infty} |w(x,0) - w_0(x,0)| + |w_0(x,0) - w_0(x_n,t_n)| + |w_0(x_n,t_n) - w(x_n,t_n)|\right) \\
\leq\ & (|1 - w_0(x_n,t_n)| + p(0) - p(t_n)) \\
=\ & \lim_{n\to\infty} \left|1 - \int_{-\infty}^{0} K(x_n,t_n;y,0)dy\right| = 0.
\end{aligned}$$

Hence

$$\lim_{y\to x, t\searrow 0} w(y,t) = w(x,0) = 1 \quad \forall\, x < 0.$$

Case (ii): Suppose $x > 0$. For any sequence $x_n \to x$ and $t_n \searrow 0$,

$$\begin{aligned}
&\lim_{n\to\infty} |w(x,0) - w(x_n,t_n)| \\
\leq\ & \lim_{n\to\infty} (|w(x,0) - w_0(x,0)| + |w_0(x,0) - w_0(x_n,t_n)| + |w_0(x_n,t_n) - w(x_n,t_n)|) \\
=\ & \lim_{n\to\infty} (|w_0(x_n,t_n)| + p(0) - p(t_n)) \\
=\ & \lim_{n\to\infty} \int_{-\infty}^{0} K(x_n,t_n;y,0)dy = 0.
\end{aligned}$$

Hence

$$\lim_{y\to x, t\searrow 0} w(y,t) = w(x,0) = 0 \quad \forall\, x > 0.$$

Case (iii): Suppose $x = 0$. Let $t_n = \frac{1}{n}$, $x_n = n^{-\alpha}$, where $0 < \alpha < \frac{1}{2}$. Note that

$$\begin{aligned}
&\lim_{n\to\infty} |w(x,0) - w(x_n,t_n)| \\
\leq\ & \lim_{n\to\infty} (|w(x,0) - w_0(x,0)| + |w_0(x,0) - w_0(x_n,t_n)| + |w_0(x_n,t_n) - w(x_n,t_n)|) \\
=\ & \lim_{n\to\infty} (|w_0(x_n,t_n)| + p(0) - p(t_n)) \\
=\ & \lim_{n\to\infty} \int_{-\infty}^{0} K(x_n,t_n;y,0)dy.
\end{aligned}$$

By the property of K [3], we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^{0} K(x_n,t_n;y,0)dy \leq \lim_{n\to\infty} \int_{-\infty}^{0} \frac{C}{\sqrt{2\pi t_n \sigma^2}} e^{-\frac{(x_n - y)^2}{2\sigma^2 t_n}} dy \\
=\ & \lim_{n\to\infty} \int_{-\infty}^{\frac{-x_n}{\sqrt{\sigma^2 t_n}}} \frac{C}{\sqrt{2\pi}} e^{-x^2/2} dx \leq \lim_{n\to\infty} \int_{-\infty}^{\frac{-1}{\sqrt{\sigma^2}} n^{1/2-\alpha}} \frac{C}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.
\end{aligned}$$

Then

$$\lim_{n\to\infty} w(x_n, t_n) = 0. \tag{3.2.7}$$

Now $y_n = -n^{-\alpha}$, where $0 < \alpha < \frac{1}{2}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty K(y_n, t_n; y, 0)dy &= \lim_{n\to\infty} \int_{-\infty}^0 K(y_n, t_n; -y, 0)dy \\
&\le \lim_{n\to\infty} \int_{-\infty}^0 \frac{C}{\sqrt{2\pi t_n \sigma^2}} e^{-\frac{(y_n+y)^2}{2\sigma^2 t_n}} dy \\
&= \lim_{n\to\infty} \int_{-\infty}^{\frac{y_n}{\sqrt{\sigma^2 t_n}}} \frac{C}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&\le \lim_{n\to\infty} \int_{-\infty}^{\frac{-1}{\sqrt{\sigma^2}} n^{1/2-\alpha}} \frac{C}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \int_{-\infty}^0 K(y_n, t_n; y, 0)dy &= \lim_{n\to\infty} \left(\int_{-\infty}^\infty K(y_n, t_n; y, 0)dy - \int_0^\infty K(y_n, t_n; y, 0)dy \right) \\
&= 1 - \lim_{n\to\infty} \int_0^\infty K(y_n, t_n; y, 0)dy = 1.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} w(y_n, t_n) = 1. \tag{3.2.8}$$

From (3.2.7) and (3.2.8), we can conclude that

$$\limsup_{y\to 0, t\searrow 0} w(y,t) = 1, \quad \liminf_{y\to 0, t\searrow 0} w(y,t) = 0$$

Thus (3.2.6) holds and (2.2.1) follows.

2. We verify that w is a viscosity solution in $\mathbb{R} \times (0, \infty)$ by considering two cases for each $(x,t) \in \mathbb{R} \times (0, \infty)$:

$$\text{(i) } p(t) - w(x,t) > 0, \quad \text{and} \quad \text{(ii) } p(t) - w(x,t) \le 0.$$

Case (i): Suppose $p(t) - w(x,t) > 0$.

Let

$$D_\delta := (x - \delta, x + \delta) \times (t - \delta^2, t + \delta^2) \quad \forall\, \delta > 0.$$

Then for each $(y,s) \in D_\delta$,

$$|\rho^\varepsilon(s) - \rho^\varepsilon(t)| \le |\rho^\varepsilon(s) - p^\varepsilon(s)| + |p^\varepsilon(s) - p^\varepsilon(t)| + |p^\varepsilon(t) - \rho^\varepsilon(t)| \le |p^\varepsilon(s) - p^\varepsilon(t)| + 2\varepsilon^{2/3},$$

since $0 \le \rho^\varepsilon - p^\varepsilon \le \varepsilon^{2/3}$. As $p^\varepsilon(\cdot)$ is decreasing

$$p^\varepsilon(t) - p^\varepsilon(s) + |p^\varepsilon(t) - p^\varepsilon(s)| = \begin{cases} 2\,(p^\varepsilon(t) - p^\varepsilon(s)) \le 2\,(p^\varepsilon(t) - p^\varepsilon(t+\delta^2)) & s > t, \\ 0 \le 2\,(p^\varepsilon(t) - p^\varepsilon(t+\delta^2)) & s \le t. \end{cases}$$

Using (3.2.2), we can compute

$$\begin{aligned} w^\varepsilon(y,s) - p^\varepsilon(s) &\le w^\varepsilon(y,s) - w^\varepsilon(x,t) + w^\varepsilon(x,t) - p^\varepsilon(s) \\ &\le \frac{(2+2\sqrt{2})C\delta}{\min\{\sqrt{t-\delta^2},1\}} + |\rho^\varepsilon(s) - \rho^\varepsilon(t)| + w^\varepsilon(x,t) - p^\varepsilon(s) \\ &\le \frac{(2+2\sqrt{2})C\delta}{\min\{\sqrt{t-\delta^2},1\}} + w^\varepsilon(x,t) - p^\varepsilon(t) \\ &\quad + p^\varepsilon(t) - p^\varepsilon(s) + |p^\varepsilon(t) - p^\varepsilon(s)| + 2\varepsilon^{2/3} \\ &\le \frac{(2+2\sqrt{2})C\delta}{\min\{\sqrt{t-\delta^2},1\}} + w^\varepsilon(x,t) - p^\varepsilon(t) + 2(p^\varepsilon(t) - p^\varepsilon(t+\delta^2)) + 2\varepsilon^{2/3}. \end{aligned}$$

Then

$$\begin{aligned} \limsup_{\varepsilon\to 0} \max_{\bar{D}_\delta} \left\{ w^\varepsilon - p^\varepsilon \right\} &\le \limsup_{\varepsilon\to 0} \left\{ \frac{(2+2\sqrt{2})C\delta}{\min\{\sqrt{t-\delta^2},1\}} + w^\varepsilon(x,t) - p^\varepsilon(t) \right. \\ &\qquad \left. + 2(p^\varepsilon(t) - p^\varepsilon(t+\delta^2)) + 2\varepsilon^{2/3} \right\} \\ &\le \frac{(2+2\sqrt{2})C\delta}{\min\{\sqrt{t-\delta^2},1\}} + w(x,t) - p(t) + 2(p(t) - p(t+\delta^2)). \end{aligned}$$

Then if we take δ small enough, by the assumption of $w < p$ and p is decreasing,

$$\limsup_{\varepsilon\to 0} \max_{\bar{D}_\delta} \left\{ w^\varepsilon - p^\varepsilon \right\} < 0.$$

Thus, for all sufficiently small positive ε, $w^\varepsilon - p^\varepsilon < 0$ in $\bar{D}_\delta$. Consequently,

$$\mathcal{L}w^\varepsilon = -\beta\left(\frac{w^\varepsilon - p^\varepsilon}{\varepsilon}\right) = 0 \quad \forall\, (x,t) \in \bar{D}_\delta.$$

Observe that in estimating the boundary of $|w^\varepsilon(x,s)-w^\varepsilon(x,t)|$ if we keep the term involving the integral of β, then

$$\begin{aligned}
|w^\varepsilon(x,t)-w^\varepsilon(y,s)| &\le \frac{Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}}\left\{|x-y|+2\sqrt{t-s}\right\}+\frac{\left|\int_s^t\int_{x-\delta}^{x+\delta}\beta\left(\frac{w^\varepsilon-p^\varepsilon}{\varepsilon}\right)dydv\right|}{2\delta}\\
&= \frac{Cp^\varepsilon(0)}{\min\{\sqrt{s},1\}}\left\{|x-y|+2\sqrt{t-s}\right\}
\end{aligned}$$

It implies that w^ε is equicontinuous in $\bar{D}_\delta$. Also from (3.1.5), we can derive that w^ε is point-wise bounded. Indeed

$$w^\varepsilon \le w_0^\varepsilon(x,t) \le \rho^\varepsilon(0) \le 1,$$

and

$$w^\varepsilon(x,t) \ge w_0^\varepsilon(x,t)+\rho^\varepsilon(t)-\rho^\varepsilon(0) \ge p^\varepsilon(t)-p^\varepsilon(0),$$

where $p^\varepsilon(t)-p^\varepsilon(0)$ is bounded for small $\varepsilon>0$. Hence w^ε contains a uniformly convergent subsequence w^{ε_n}. So that its limit w is differentiable with respect to x and t. Similarly we can show that w_x^ε is differentiable with respect to x. Hence the limit w is then a smooth solution to $\mathcal{L}w=0$ in D_δ.

Case (ii): Suppose $w(x,t)-p(t)\ge 0$. However, $w-p\le 0$ in $\mathbb{R}\times[0,\infty)$ since $w^\varepsilon\le\rho^\varepsilon$ and $\lim_{\varepsilon\searrow 0}\rho^\varepsilon(t)=p(t)$ in $\mathbb{R}\times[0,\infty)$. Hence, we must have $w(x,t)=p(t)=\min\{p(t),w^*(x,t)\}$, where the second inequality holds since $p\le w\le w^*$. From (3.2.5)

$$\begin{aligned}
0 &\le w(x,t)-w_*(x,t)=\limsup_{y\to x,s\to t}(w(x,t)-w(y,s))\\
&\le \limsup_{y\to x,s\to t}\left(\frac{C}{\sqrt{s}}\left\{|x-y|+2\sqrt{t-s}\right\}\right)=0.
\end{aligned}$$

So that $w_*(x,t)=w(x,t)=p(t)$. Thus the semi-continuity requirements for a viscosity solution hold.

In this case, we clearly have $\max\{w(x,t)-p(t),\mathcal{L}\varphi(x,t)\}\ge 0$ for *any* smooth φ. So that w is a supersolution. It remains to verify the differential inequality for subsolutions. To this end, let φ be a smooth function on $\bar{B}_\delta$ where $B_\delta = B_\delta(x,t)$ such that $w^*(y,s) - \varphi(y,s)$ attains at (x,t) a local maximum on $\bar{B}_\delta$. Set

$$\psi(y,s) := \varphi(y,s) + (y-x)^4/\delta^4 + (s-t)^2/\delta^4.$$

For each small positive ε, $v^\varepsilon := w^\varepsilon - \psi$ attains a global maximum on $\bar{B}_\delta$. Denote any such point of maximum by $(y_\varepsilon, s_\varepsilon)$. By first and second optimality conditions

$$v_s^\varepsilon(y_\varepsilon, s_\varepsilon) \geq 0, \quad v_{yy}^\varepsilon(y_\varepsilon, s_\varepsilon) \leq 0, \quad v_y^\varepsilon(y_\varepsilon, s_\varepsilon) = 0.$$

Then

$$\begin{aligned}
\mathcal{L}v^\varepsilon(y_\varepsilon, s_\varepsilon) &= \mathcal{L}w^\varepsilon(y_\varepsilon, s_\varepsilon) - \mathcal{L}\psi(y_\varepsilon, s_\varepsilon) \\
&= v_s^\varepsilon(y_\varepsilon, s_\varepsilon) - \frac{1}{2}(\sigma^2 v_y^\varepsilon(y_\varepsilon, s_\varepsilon))_y + \mu v_y^\varepsilon(y_\varepsilon, s_\varepsilon) \\
&= v_s^\varepsilon(y_\varepsilon, s_\varepsilon) - \frac{1}{2}\sigma^2 v_{yy}^\varepsilon(y_\varepsilon, s_\varepsilon) - \sigma\sigma_y v_y^\varepsilon(y_\varepsilon, s_\varepsilon) + \mu v_y^\varepsilon(y_\varepsilon, s_\varepsilon) \geq 0.
\end{aligned}$$

Thus,

$$\mathcal{L}\psi(y_\varepsilon, s_\varepsilon) \leq \mathcal{L}w^\varepsilon(y_\varepsilon, s_\varepsilon) = -\beta(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)) \leq 0.$$

Denote by $(\bar{x}, \bar{t})$ a limit point of $\{(y_\varepsilon, s_\varepsilon)\}$ as $\varepsilon \to 0$. Then $\mathcal{L}\psi(\bar{x}, \bar{t}) \leq 0$. Since $\mathcal{L}\varphi(x,t) = \mathcal{L}\psi(x,t)$, it is suffices to show that $(\bar{x}, \bar{t}) = (x,t)$.

Since $w^\varepsilon \leq w \leq w^*$,

$$\begin{aligned}
\limsup_{\varepsilon \searrow 0} \max_{\bar{B}_\delta} \Big\{ w^\varepsilon - \psi \Big\} &\leq \limsup_{\varepsilon \searrow 0} \Big\{ w^*(y_\varepsilon, s_\varepsilon) - \psi(y_\varepsilon, s_\varepsilon) \Big\} \\
&\leq w^*(\bar{x}, \bar{t}) - \psi(\bar{x}, \bar{t}) \leq \max_{\bar{B}_\delta}(w^* - \varphi) - |\bar{x} - x|^4/\delta^4 - |\bar{t} - t|^2/\delta^4.
\end{aligned}$$

On the other hand, we claim that $w^*(x,t) = \lim_{s \nearrow t} w(x,s)$. Pick the sequence $s_n \nearrow t$ and $(x_n, t_n) \to (x,t)$ such that $w^*(x,t) = \lim_{n\to\infty} w(x_n, t_n)$. Let $v_n := \min\{t_n, s_n\}$. Then

$$\begin{aligned}
w(x_n, t_n) &= w(x_n, t_n) - w(x, t_n) + w(x, t_n) - w(x, v_n) + w(x, v_n) - w(x, s_n) + w(x, s_n) \\
&\leq \frac{2C|x_n - x|}{\min\{\sqrt{t_n}, 1\}} + \frac{2C\sqrt{t_n - v_n}}{\min\{\sqrt{v_n}, 1\}} + \frac{2C\sqrt{s_n - v_n}}{\min\{\sqrt{v_n}, 1\}} + w(x, s_n).
\end{aligned}$$

Sending $n \to \infty$, we obtain that

$$w^*(x,t) \le \lim_{n\to\infty} w(x,s_n).$$

Also for any $n > 0$, we have

$$|w(x,s_n) - w(x,t)| \le \frac{2C}{\sqrt{s_1}}\sqrt{t-s_n} + p(s_n) - p(t).$$

Since $\lim_{n\to\infty} p(s_n) = p(t)$, $\lim_{n\to\infty} w(x,s_n)$ exists. Then

$$w^*(x,t) \le \lim_{n\to\infty} w(x,s_n) = \lim_{s\nearrow t} w(x,s) \le w^*(x,t),$$

hence $w^*(x,t) = \lim_{s\nearrow t} w(x,s)$.

So that

$$\begin{aligned}
\limsup_{\varepsilon\searrow 0} \ \max_{\bar{B}_\delta} \left\{ w^\varepsilon - \psi \right\} &\ge \lim_{s\nearrow t} \limsup_{\varepsilon\searrow 0} \{w^\varepsilon(x,s) - \psi(x,s)\} \\
&= \lim_{s\nearrow t} \{w(x,s) - \psi(x,s)\} = w^*(x,t) - \psi(x,t) \\
&= w^*(x,t) - \varphi(x,t) = \max_{\bar{B}_\delta}(w^* - \varphi).
\end{aligned}$$

Hence

$$\max_{\bar{B}_\delta}(w^* - \varphi) \le \max_{\bar{B}_\delta}(w^* - \varphi) - (x-\bar{x})^4/\delta^4 - (t-\bar{t})^2/\delta^4.$$

Thus, we must have $(\bar{x},\bar{t}) = (x,t)$. This completes the proof.

$\square$

3.3 THE DIFFERENTIAL EQUATION AND THE FREE BOUNDARY PROBLEM

Since

$$0 \le \beta(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)) \le -\dot{\rho}^\varepsilon,$$

and $\rho^\varepsilon(\cdot)$ is decreasing, by weak compactness of measures, as $\varepsilon \to 0$,

$$\begin{aligned} \beta(\varepsilon^{-1}(w^\varepsilon - p^\varepsilon)) &\longrightarrow \gamma \qquad \text{as a measure in } \mathbb{R} \times [0, \infty), \\ \mathcal{L}w &= \gamma \qquad \text{on } \mathbb{R} \times (0, \infty), \end{aligned}$$

where γ is a Radon measure satisfying

$$0 \le \gamma \, dx \, dt \le -dx \, dp(t).$$

In addition, from step 2 of the proof in the preceding subsection, γ is supported on the set $w = p$.

Now suppose that p is continuous. Then $\gamma = \dot{p}$ on the contact set Π (noticing that Π_2 is empty). Hence, w is the solution to

$$\mathcal{L}w = \dot{p}(t)\mathbf{1}_{\{w=p\}} \;\text{ in }\; \mathbb{R} \times (0, \infty), \qquad w(\cdot, 0) = \mathbf{1}_{(-\infty,0)} \;\text{ on }\; \mathbb{R} \times \{0\}. \tag{3.3.1}$$

Using a free boundary approach, this can be written as the solution to the free boundary problem, for (b, w) :

$$\begin{cases} \mathcal{L}w = \dot{p}(t)\mathbf{1}_{x<b(t)} & \text{in } \mathbb{R} \times (0, \infty), \\ b(t) := \inf\{x \mid w(x, t) < p(t)\} & \text{for all } t \ge 0, \\ w(\cdot, 0) = \mathbf{1}_{(-\infty,0)} & \text{on } \mathbb{R} \times \{0\}. \end{cases} \tag{3.3.2}$$

We emphasize that this formulation works only when p is continuous, since if p is not continuous at s, then

$$\mathcal{L}w = \min\{p(s) - w^*(x, \tau), 0\} \cdot \delta(t - s) \quad \text{on } \mathbb{R} \times \{s\},$$

where δ is the Dirac measure.

Remark 3.3.1. *Suppose* $\|\dot{p}\|_\infty := \sup_{t\geq 0} |\dot{p}(t)|$ *is finite. Then* $\|\dot{p}_\varepsilon\|_\infty \leq \|\dot{p}\|_\infty$ *and* $\rho^\varepsilon - p^\varepsilon \leq \varepsilon\|\dot{p}\|^{1/3}$. *Consequently,* $\dot{\rho}^\varepsilon = -\beta(\|\dot{p}\|^{1/3}) = -\|\dot{p}\|_\infty$. *Hence*

$$0 \leq \gamma^\varepsilon(x,t) \leq \|\dot{p}\|_\infty \qquad \forall (x,t) \in \mathbb{R} \times [0,\infty).$$

It is then easy to show that $w^\varepsilon(x,t) - w_0^\varepsilon \to w - w_0$ *in* $W_r^{2,1}([-R,R] \times [0,R^2])$ *for any* $r > 1$ *and any* $R > 0$.

4.0 ESTIMATION OF THE FREE BOUNDARY

In this section, we provide both upper and lower bounds for the free boundary

$$b(t) := \inf\{x \in \mathbb{R} \mid w(x,t) < p(t)\} \in [-\infty, \infty] \qquad \forall\, t > 0.$$

in the case of Brownian motion, i.e. when $\sigma \equiv 1$ and $\mu \equiv 0$.

Denote $q(t) := 1 - p(t)$. Note that for any $s > 0$, $0 = q(0) = q^*(0) \leq q(s)$, and since p is lower semicontinuous, q is upper semicontinuous. We define

$$\dot{q}(s) := \liminf_{t \nearrow s} \frac{q(s) - q(t)}{s - t} \in [0, \infty].$$

4.1 UPPER BOUNDS

The following lemma is obvious from the probabilistic interpretation of our problem since it states that $\mathbb{P}[X_t \leq b(t)] \leq \mathbb{P}[\tau \leq t]$. Its analytic derivation is equally simple.

Lemma 4.1.1. *For every $t > 0$,* $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{b(t)/\sqrt{2t}} e^{-z^2}\, dz \leq q(t)$.

Proof. Fix $t > 0$. We need only consider the case $b(t) > -\infty$. Since $w(x,t) \leq w_0(x,t)$,

$$\begin{aligned} 1 - q(t) &= p(t) = w(b(t), t) \leq w_0(b(t), t) = \int_{-\infty}^{0} \Gamma(b(t) - y, t)\, dy \\ &= 1 - \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{(b(t)-y)^2}{2t}}\, dy = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{b(t)/\sqrt{2t}} e^{-z^2} dz. \end{aligned}$$

□

4.2 LOWER BOUNDS

Lemma 4.2.1 (Method for Lower Bounds). *Assume that* $\underline{w}$ *defined on* $\mathbb{R}\times[0,t]$ *satisfies*

$$\begin{cases} \mathcal{L}\underline{w} = 0 & in \ \ \mathbb{R}\times(0,t], \\ \underline{w}(\cdot,0) \le w(\cdot,0) & on \ \ \mathbb{R}\times\{0\}, \\ \underline{w} \le p & on \ \ \mathbb{R}\times(0,t), \\ \underline{w}(s,t) \ge p(t) & at \ \ (s,t). \end{cases} \tag{4.2.1}$$

Then

$$\underline{w} \le w \quad in \ \ \mathbb{R}\times[0,t), \qquad s \le b(t).$$

Proof. First consider the case where p is continuous at t, so $p(t) = p^*(t)$. For each $\varepsilon > 0$, let

$$\phi^\varepsilon := \underline{w} - \varepsilon e^r - \varepsilon x^2.$$

We claim that $\phi^\varepsilon \le w$ on $\mathbb{R}\times[0,t]$. Suppose not, then $w - \phi^\varepsilon$ can attain a global negative minimum, say, at $(\hat{x},\hat{r})$. Since $w(x,0) - \phi^\varepsilon(x,0) = w(x,0) - \underline{w}(x,0) + \varepsilon + \varepsilon x^2 \ge \varepsilon$, $\hat{r} > 0$. So that

$$\mathcal{L}\phi^\varepsilon(\hat{x},\hat{r}) = \mathcal{L}\underline{w} - \varepsilon e^{\hat{r}} + \varepsilon = -\varepsilon e^{\hat{r}} + \varepsilon < 0.$$

As a supersolution, $\max\{w(\hat{x},\hat{r}) - p(\hat{r}), \mathcal{L}\phi^\varepsilon(\hat{x},\hat{r})\} \ge 0$, hence we must have $w(\hat{x},\hat{r}) - p(\hat{r}) \ge 0$. Since $\underline{w} \le p$ on $\mathbb{R}\times(0,t)$ and $p^*(t) = p(t)$,

$$\underline{w}(x,t) \le \underline{w}^*(x,t) \le p^*(t) = p(t) \quad \forall\, x \in \mathbb{R}.$$

So that $\underline{w} \le p$ on $\mathbb{R}\times(0,t]$. Then $w(\hat{x},\hat{r}) < \phi^\varepsilon(\hat{x},\hat{r}) < \underline{w}(\hat{x},\hat{r}) \le p(\hat{r})$. This is a contradiction. Thus $\phi^\varepsilon \le w$ in $\mathbb{R}\times(0,t]$ for each $\varepsilon > 0$. Sending $\varepsilon \searrow 0$, we conclude that $\underline{w} \le w$ in $\mathbb{R}\times(0,t]$.

In general, if p is not continuous at t, let $\{t_n\}$ be a sequence of positive numbers such that $t_n \nearrow t$ as $n \to \infty$, and $p(\cdot)$ is continuous at t_n. Then $\underline{w} \le w$ in $\mathbb{R}\times[0,t_n]$. Sending $n \to \infty$ we obtain $\underline{w} \le w$ in $\mathbb{R}\times[0,t)$.

From the above argument,

$$w^*(s,t) \geq w(s,t) \geq \underline{w}(s,t) \geq p(t).$$

As a subsolution, $w = \min\{p, w^*\}$. Hence $w(s,t) = w_*(s,t) = p(t)$. Recall that

$$b(t) := \inf\{x \in \mathbb{R} \mid w(x,t) < p(t)\}.$$

Suppose $b(t) \leq s$, then $w(s,t) < p(t)$, which is a contradiction. Hence we conclude that $b(t) \geq s$. □

Lemma 4.2.2 (A Criterion for Lower Bounds). *For each $s < 0 < t$, $r > 0$ let*

$$Q(s,r) := q(r) - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{s/\sqrt{2r}} e^{-z^2}\, dz.$$

Suppose (s,t) is such that

$$s < 0 < t, \qquad Q(s,r) \leq Q(s,t) \quad \forall\, r \in (0,t).$$

Then $b(t) \geq s$.

Proof. Let $\underline{w}$ be the solution to

$$\begin{cases} \mathcal{L}\underline{w} = 0 & \text{in } \ \mathbb{R} \times (0,t], \\ \underline{w}(\cdot,0) = \theta\, \mathbf{1}_{(2s,0)} & \text{on } \ \mathbb{R} \times \{0\}. \end{cases}$$

where $\theta = p(t)\left(\frac{2}{\sqrt{\pi}} \int_{s/\sqrt{2t}}^{0} e^{-z^2} dz\right)^{-1}$. We claim $\underline{w}$ satisfies (4.2.1). As $\underline{w}$ satisfies the first equation of (4.2.1), we only need to verify the other three inequalities of (4.2.1).

1. Since the problem for $\underline{w}$ is linear, it can be expressed as:

$$\underline{w}(x,r) \;=\; \theta \int_{2s}^{0} \Gamma(x-y,r)dy = \frac{\theta}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2r}}}^{\frac{x-2s}{\sqrt{2r}}} e^{-z^2}\, dz \qquad \forall\, x \in \mathbb{R}, r > 0. \tag{4.2.2}$$

   In particular, when $x = s$,

$$\underline{w}(s,t) = \frac{\theta}{\sqrt{\pi}} \int_{s/\sqrt{2t}}^{-s/\sqrt{2t}} e^{-z^2} dz = \frac{2\theta}{\sqrt{\pi}} \int_{s/\sqrt{2t}}^{0} e^{-z^2} dz = p(t).$$

2. By (4.2.2), we find

$$\underline{w}_x(x,t) = \frac{\theta}{\sqrt{2\pi t}}\left(e^{-\frac{(x-2s)^2}{2t}} - e^{-\frac{x^2}{2t}}\right),$$
$$\underline{w}_{xx}(x,t) = \frac{\theta}{\sqrt{2\pi t^3}}\left(-(x-2s)e^{-\frac{(x-2s)^2}{2t}} + xe^{-\frac{x^2}{2t}}\right).$$

For each $r \in (0,t)$, the only solution for $\underline{w}_x(x,r) = 0$ is $x = s$. Also

$$\underline{w}_{xx}(s,r) = \frac{2s\theta}{\sqrt{2\pi r^3}}e^{-\frac{x^2}{2r}} < 0.$$

Then $\underline{w}(x,r)$ achieve its maximum at $x = s$, i.e.,

$$\max_{x\in\mathbb{R}} \underline{w}(x,r) = \underline{w}(s,r) = p(t)\left(\int_{s/\sqrt{2r}}^{0} e^{-z^2}dz\right)\left(\int_{s/\sqrt{2t}}^{0} e^{-z^2}dz\right)^{-1} \qquad \forall r > 0. \tag{4.2.3}$$

For any $s < 0 < r$, we can compute

$$\frac{2}{\sqrt{\pi}}\int_{s/\sqrt{2r}}^{0} e^{-z^2}\,dz = 1 - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{s/\sqrt{2r}} e^{-z^2}dz = 1 + Q(s,r) - q(r) = Q(s,r) + p(r). \tag{4.2.4}$$

From (4.2.3) and (4.2.4), for all $r \in (0,t)$,

$$\begin{aligned}
\max_{x\in\mathbb{R}} \underline{w}(x,r) - p(r) &= \frac{p(t)(Q(s,r)+p(r))}{Q(s,t)+p(t)} - p(r) \\
&= \frac{p(t)Q(s,r) - p(r)Q(s,t)}{Q(s,t)+p(t)} \\
&\le \frac{p(t)(Q(s,r) - Q(s,t))}{Q(s,t)+p(t)} \\
&= \frac{p(t)(Q(s,r) - Q(s,t))}{\frac{2}{\sqrt{\pi}}\int_{s/\sqrt{2t}}^{0} e^{-z^2}\,dz} \le 0,
\end{aligned}$$

where the last inequality holds by the assumption that $Q(s,r) \le Q(s,t)$. Hence $\underline{w} \le p$ on $\mathbb{R} \times (0,t)$.

3. By the assumption,

$$Q(s,t) \ge Q(s,r) \quad s < 0 < r < t.$$

So that

$$Q(s,t) \ge \lim_{r\searrow 0} Q(s,r) = 0.$$

In particular when $r = t$, (4.2.4) reads as

$$\frac{2}{\sqrt{\pi}} \int_{s/\sqrt{2t}}^{0} e^{-z^2} dz = Q(s,t) + p(t) \geq p(t).$$

So that $\theta \leq 1$ and thus

$$\underline{w}(\cdot, 0) \quad = \quad \theta 1_{(2s,0)} \leq 1_{(2s,0)} \leq 1_{(-\infty,0)} = w(\cdot, 0).$$

$\underline{w}$ satisfies (4.2.1) and then Lemma 4.2.1 gives $b(t) \geq s$. □

Before we continue, we provide an interesting application of Lemmas 4.2.2 and 4.1.1.

Corollary 4.2.3. *For each $t > 0$, let $\zeta(t) \in (-\infty, 0)$ and $\nu(t) \in \mathbb{R}$ be defined by*

$$q(t) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\zeta(t)/\sqrt{2t}} e^{-z^2}\, dz = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\nu(t)/\sqrt{2t}} e^{-z^2}\, dz.$$

1. *Suppose ζ is a constant function. Then the exact solution to (1.0.1) is given by*

$$\begin{cases} w(x,t) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{2t}}^{(x-2\zeta)/\sqrt{2t}} e^{-z^2}\, dz & \forall\, x \geq \zeta, t > 0, \\ w(x,t) = p(t) & \forall\, x < \zeta, t > 0, \\ b(t) = \zeta & \forall\, t > 0. \end{cases} \tag{4.2.5}$$

2. *Suppose $\zeta(r) \leq \zeta(t)$ for all $r \in (0,t)$. Then $\zeta(t) \leq b(t) \leq \nu(t)$.*
3. *Suppose $\dot{\zeta}(t) \geq 0$ for all $t \in (0,T]$. Then*

$$\zeta(t) \leq b(t) \leq \nu(t) \quad \forall t \in (0,T], \qquad \lim_{t \searrow 0} \frac{b(t)}{\zeta(t)} = 1.$$

Proof. 1. Let $w(x,t)$ be defined as (4.2.5). Direct computation gives that when $x > \zeta$, $t > 0$,

$$\begin{aligned} w_t(x,t) &= \tfrac{1}{2\sqrt{2\pi t^3}} \left(x e^{-x^2/(2t)} - (x-2\zeta) e^{-(x-2\zeta)^2/(2t)} \right), \\ w_x(x,t) &= \tfrac{1}{\sqrt{2\pi t}} \left(e^{-(x-2\zeta)^2/(2t)} - e^{-x^2/(2t)} \right), \\ w_{xx}(x,t) &= \tfrac{1}{\sqrt{2\pi t^3}} \left(x e^{-x^2/(2t)} - (x-2\zeta) e^{-(x-2\zeta)^2/(2t)} \right). \end{aligned}$$

It is clear that $w_t = \frac{1}{2} w_{xx}$. When $x = \zeta$, $t > 0$, by the definition of ζ

$$\begin{aligned} w(\zeta,t) &= \frac{1}{\sqrt{\pi}} \int_{\zeta/\sqrt{2t}}^{-\zeta/\sqrt{2t}} e^{-z^2}\, dz = \frac{2}{\sqrt{\pi}} \int_{\zeta/\sqrt{2t}}^{0} e^{-z^2}\, dz \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \int_{-\infty}^{\zeta/\sqrt{2t}} e^{-z^2}\, dz \right) = 1 - q(t) = p(t). \end{aligned}$$

Hence $w = p$, $\forall\, x \leq \zeta, t > 0$. Also for each $x > \zeta$, $t > 0$, we have $x > \zeta > 2\zeta$. This implies that $(x-2\zeta)^2 > x^2$. Then $w_x(x,t) < 0$. Thus $w(x,t) < w(\zeta,t) = p(t)$. The first assertion follows.

We note that it agrees with the formula for the first hitting time of Brownian motion to the level ζ (see e.g. [5] pages 94–96).

2. For each $r \in (0,t)$, suppose $\zeta(r) \leq \zeta(t)$ $\forall\, r \in (0,t)$. Set $s = \zeta(t)$. Then

$$Q(s,r) = q(r) - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\zeta(t)/\sqrt{2r}} e^{-z^2}\, dz = \frac{2}{\sqrt{\pi}} \int_{\zeta(t)/\sqrt{2r}}^{\zeta(r)/\sqrt{2r}} e^{-z^2} dz \leq 0 = Q(s,t)$$

for each $r \in (0,t)$. Thus by Lemma 4.2.2, $b(t) \geq s = \zeta(t)$. This gives the lower bound for $b(t)$.

For $t > 0$, Lemma 4.1.1 reads as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{b(t)/\sqrt{2t}} e^{-z^2} dz \leq q(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\nu(t)/\sqrt{2t}} e^{-z^2} dz,$$

which implies that $b(t) \leq \nu(t)$. It gives the upper bound for $b(t)$. Thus the second assertion follows.

3. For each $t \in (0,T]$, suppose $\dot{\zeta}(t) \geq 0$ in $(0,T]$. Then for each $r \in (0,t)$, $\zeta(r) \leq \zeta(t)$. By (2) $\zeta(t) \leq b(t) \leq \nu(t)$ for all $t \in (0,T]$. Let

$$\alpha(t) := \frac{\nu(t)}{\sqrt{2t}}, \quad \gamma(t) := \frac{\zeta(t)}{\sqrt{2t}}, \quad \delta(t) = \frac{\ln 2}{-2\gamma(t) - 1}.$$

Since

$$0 = \lim_{t \searrow 0} q(t) = \lim_{t \searrow 0} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\zeta(t)/\sqrt{2t}} e^{-z^2}\, dz,$$

it must hold that $\lim_{t \searrow 0} \gamma(t) = -\infty$. Then we conclude that for all small positive $t > 0$, $\delta(t) \in (0,1)$. And it follows that $\delta^2 - \delta < 0$. Note that

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\alpha} e^{-z^2}\, dz &= q(t) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\gamma} e^{-z^2}\, dz = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\gamma+\delta} e^{-z^2+2\delta z-\delta^2} dz \\
&\leq \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\gamma+\delta} e^{-z^2+2\delta(\gamma+\delta)-\delta^2} dz = \frac{2e^{2\delta\gamma+\delta^2}}{\sqrt{\pi}} \int_{-\infty}^{\gamma+\delta} e^{-z^2} dz \\
&= \frac{2e^{2\delta\gamma+\delta+\delta^2-\delta}}{\sqrt{\pi}} \int_{-\infty}^{\gamma+\delta} e^{-z^2} dz \leq \frac{2e^{-\ln 2}}{\sqrt{\pi}} \int_{-\infty}^{\gamma+\delta} e^{-z^2}\, dz \\
&\leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\gamma+\delta} e^{-z^2}\, dz.
\end{aligned}$$

Thus, $\alpha(t) \leq \gamma(t) + \delta(t)$. So that

$$0 \leq 1 - \frac{b(t)}{\zeta(t)} \leq 1 - \frac{\nu(t)}{\zeta(t)} = 1 - \frac{\alpha(t)}{\gamma(t)} \leq \frac{\delta(t)}{-\gamma(t)} = \frac{\ln 2}{2\gamma^2(t) + \gamma(t)} = \frac{\ln 2}{\gamma(t)(2\gamma(t)+1)}.$$

Sending $t \searrow 0$, we obtain

$$\lim_{t \searrow 0} \left(1 - \frac{b(t)}{\zeta(t)} \right) = 0.$$

The third assertion of the Lemma thus follows. $\square$

Next we present a sufficient condition for $Q(s,\cdot)$ to attain its maximum in $(0,t]$ at t.

Lemma 4.2.4. *Assume that $t > 0$ and*

$$k(t) := \inf_{0\le r\le t} \frac{q(t)-q(r)}{t-r} > 0.$$

Then

$$b(t) \ge \max\left\{ s \;\middle|\; s \le -\sqrt{3t}\;;\; \frac{|s|}{\sqrt{2\pi}t^{3/2}} e^{-s^2/(2t)} \le k(t) \right\}.$$

Proof. For each $t > 0$, let

$$A_t := \left\{ s \;\middle|\; s \le -\sqrt{3t}\;;\; \frac{|s|}{\sqrt{2\pi}t^{3/2}} e^{-s^2/(2t)} \le k(t) \right\}.$$

Then for each $r \in (0,t)$ and $s \in A_t$, let

$$F(x;s) := \int_{-\infty}^{s/\sqrt{2x}} e^{-z^2}\, dz \quad x > 0.$$

By Mean Value Theorem,

$$\begin{aligned} & \frac{2}{\sqrt{\pi}(t-r)} \left(\int_{-\infty}^{s/\sqrt{2t}} e^{-z^2}\, dz - \int_{-\infty}^{s/\sqrt{2r}} e^{-z^2}\, dz \right) \\ = \; & \frac{2}{\sqrt{\pi}} F'(\theta; s) = \frac{s e^{-s^2/(2\theta)}}{-\sqrt{2\pi\theta^3}} = |s| f(\theta; s), \end{aligned}$$

where $\theta \in [r,t]$ and

$$f(\theta;s) := \frac{e^{-s^2/(2\theta)}}{\sqrt{2\pi\theta^3}}.$$

Since $s \le -\sqrt{3t}$,

$$f'(\theta;s) = \frac{e^{-s^2/(2\theta)}}{\sqrt{8\pi\theta}}(s^2 - 3\theta) \ge 0, \quad \forall\, \theta \in (0,t].$$

Note that

$$\begin{aligned} & \frac{2}{\sqrt{\pi}(t-r)} \left(\int_{-\infty}^{s/\sqrt{2t}} e^{-z^2}\, dz - \int_{-\infty}^{s/\sqrt{2r}} e^{-z^2}\, dz \right) = |s| f(\theta; s) \\ \le \; & |s| \sup_{r\le\theta\le t} f(\theta;s) = |s| \frac{e^{-s^2/(2t)}}{\sqrt{2\pi}t^{3/2}} \le k(t) = \inf_{0\le u\le t} \frac{q(t)-q(u)}{t-u} \le \frac{q(t)-q(r)}{t-r}. \end{aligned}$$

It follows that

$$\frac{2}{\sqrt{\pi}(t-r)} \left(\int_{-\infty}^{s/\sqrt{2t}} e^{-z^2}\, dz - \int_{-\infty}^{s/\sqrt{2r}} e^{-z^2}\, dz \right) \le \frac{q(t)-q(r)}{t-r},$$

i.e., $Q(s,r) \le Q(s,t)$. By Lemma (4.2.2), $b(t) \ge s$. The lemma follows. □

As an immediate consequence of the Lemma, we have

Corollary 4.2.5. *If $\dot{q}(t) > 0$, then $b(t) > -\infty$.*

Proof. For $t > 0$, suppose that $\dot{q}(t) > 0$, then there exists $\delta > 0$, such that for each $t - \delta < r_1 < t$, we have $\frac{q(t)-q(r_1)}{t-r_1} > \gamma$ for some $\gamma > 0$. Now we claim that $k(t) > 0$. To the contrary, we assume that $k(t) = \inf_{0 \le r \le t} \frac{q(t)-q(r)}{t-r} = 0$. Then for each $0 < \varepsilon < \gamma$, there exists some $r \in [0, t-\delta]$, such that $\varepsilon > \frac{q(t)-q(r)}{t-r}$. Since $0 \le r \le t - \delta < r_1 < t$,

$$\varepsilon \quad > \quad \frac{q(t)-q(r)}{t-r} \ge \frac{q(t)-q(r_1)}{t-r_1}\frac{t-r_1}{t-r} > \gamma\frac{t-r_1}{t-r}.$$

Sending $\varepsilon \searrow 0$, we got a contradiction. Hence $k(t) > 0$. Then by lemma (4.2.4),

$$b(t) \ge \max\left\{ s \;\middle|\; s \le -\sqrt{3t}\,;\, \frac{|s|}{\sqrt{2\pi}t^{3/2}} e^{-s^2/(2t)} \le k(t) \right\} > -\infty.$$

□

4.3 ESTIMATION OF THE FREE BOUNDARY

We end this section with the following

Theorem 5. *Assume that*

$$\limsup_{t \searrow 0} \frac{q(t)}{t\dot{q}(t)} < \infty. \tag{4.3.1}$$

Then

$$\lim_{t \searrow 0} \frac{b(t)}{\sqrt{-2t \log q(t)}} = -1. \tag{4.3.2}$$

Consequently, in special cases the following holds:

1. *when* $q(t) = A\,t^m$, *where* A *and* m *are positive constants,*

$$b(t) = -\sqrt{-2mt \log t}\;\; [1 + o(1)], \quad \lim_{t \searrow 0} o(1) = 0;$$

2. when $q(t) = A\, e^{-\gamma^2/(2t^m)}$, *where* A, m, γ *are positive constants,*

$$b(t) = -\gamma\, t^{(1-m)/2} \;\; [1 + o(1)], \quad \lim_{t \searrow 0} o(1) = 0.$$

In particular,

$$\lim_{t \searrow 0} b(t) = \begin{cases} -\infty & \text{if } m > 1, \\ -\gamma & \text{if } m = 1, \\ 0 & \text{if } 0 < m < 1. \end{cases}$$

Proof. The idea is to estimate $k(t)$ via $q(t)/t$. Under the assumption (4.3.1), there exist positive constants C and T such that

$$0 < \frac{q(r)}{t} \le C\, \dot{q}(r) \qquad \forall\, t \in (0, T].$$

For each $t \in (0, T]$, we can pick C big enough so that

$$\sqrt{\frac{3}{2\pi}} \int_{-\infty}^{-\sqrt{3/2}} e^{-z^2} dz \ge \frac{q(t)}{C+1}.$$

For any $0 < r < t$, we can compute,

$$\begin{aligned} C\,(q(t) - q(r)) &= \int_r^t C\, \dot{q}(\theta)\, d\theta \ge \int_r^t \frac{q(\theta)}{\theta}\, d(\theta) \\ &= \frac{q(\theta)(\theta - r)}{\theta}\Big|_{\theta=r}^{\theta=t} - \int_r^t (\theta - r) \frac{\theta \dot{q}(\theta) - q(\theta)}{\theta^2}\, d\theta \\ &= \frac{(t-r)q(t)}{t} - \int_r^t \dot{q}(\theta)\, d\theta + \int_r^t \frac{r\theta \dot{q}(\theta) + (\theta - r) q(\theta)}{\theta^2}\, d\theta \\ &\ge (t-r)\frac{q(t)}{t} - [q(t) - q(r)]. \end{aligned}$$

That is,

$$\frac{q(t) - q(r)}{t - r} \ge \frac{1}{C+1} \frac{q(t)}{t}.$$

It follows that

$$k(t) = \inf_{0<r<t} \frac{q(t) - q(r)}{t - r} \ge \frac{1}{C+1} \frac{q(t)}{t}.$$

Note that for each t, by L'Hospital's rule

$$\lim_{s\to-\infty} \frac{|s|}{\sqrt{2\pi t^3}} \int_{-\infty}^{s/\sqrt{2t}} e^{-z^2}\, dz = \lim_{s\to-\infty} \frac{s^2}{\sqrt{\pi}t} e^{-s^2/(2t)} = 0.$$

Also

$$\sqrt{\frac{|-3t|}{2\pi t^3}} \int_{-\infty}^{-\sqrt{3t}/\sqrt{2t}} e^{-z^2}\, dz = \frac{1}{t}\sqrt{\frac{3}{2\pi}} \int_{-\infty}^{-\sqrt{3/2}} e^{-z^2} dz \geq \frac{1}{C+1}\frac{q(t)}{t}.$$

Then there exists the solution $s < -\sqrt{3t}$ to

$$\frac{|s|}{\sqrt{2\pi t^3}} \int_{-\infty}^{s/\sqrt{2t}} e^{-z^2} dz = \frac{1}{C+1}\frac{q(t)}{t}.$$

and it is the same as

$$\frac{|s|}{\sqrt{2\pi t}} \int_{-\infty}^{s/\sqrt{2t}} e^{-z^2} dz = \frac{q(t)}{C+1}. \tag{4.3.3}$$

For $a < 0$,

$$\int_{-\infty}^{a} e^{-z^2} dz = \int_{a^2}^{\infty} \frac{e^{-x} dx}{2\sqrt{x}} = \frac{e^{-a^2}}{2|a|} - \int_{a^2}^{\infty} \frac{e^{-x} dx}{4x^{3/2}},$$

and note that since

$$0 < \int_{a^2}^{\infty} \frac{e^{-x} dx}{4x^{3/2}} = \frac{e^{-a^2}}{2|a^3|} - \frac{3}{2}\int_{a^2}^{\infty} \frac{e^{-x^2}}{4x^{3/2}} dx < \frac{e^{-a^2}}{2|a^3|},$$

there exists $\theta = \theta(a) \in (0,1)$ such that

$$\int_{-\infty}^{a} e^{-z^2} dz = \frac{e^{-a^2}}{2|a|}\Big\{1 - \frac{\theta}{2a^2}\Big\}.$$

Hence, the equation for s reads

$$e^{-s^2/(2t)}\Big\{1 - \frac{\theta}{s^2/(2t)}\Big\} = \frac{\sqrt{2\pi}}{C+1} q(t).$$

For small t, $q(t)$ is small. From 4.3.3, $s/\sqrt{t} \ll -1$. It then follows that

$$\begin{aligned} |s| &= \sqrt{2t}\left(-\log q(t) + \log(1 - \theta t/s^2) + \log[(C+1)/\sqrt{2\pi}]\right)^{1/2} \\ &\leq \sqrt{-2t\log q(t)}\Big\{1 + \frac{\log[(C+1)/\sqrt{4\pi}]}{-\log q(t)}\Big\}^{1/2}. \end{aligned}$$

By Lemma 4.2.4, we then have

$$
\begin{aligned}
b(t) &\geq s \geq -\sqrt{-2t\log q(t)}\Big\{1+\frac{\log[(C+1)/\sqrt{2\pi}]}{|\log q(t)|}\Big\}^{1/2} \\
&= -\sqrt{-2t\log q(t)}\{1+o(1)\}.
\end{aligned}
$$

This gives the lower bound for $b(t)$. Now we estimate the upper bound. From Lemma (4.1.1),

$$
q(t) \geq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{b(t)/\sqrt{2t}} e^{-z^2}\,dz = \frac{\sqrt{t}}{\sqrt{2\pi}|b(t)|}\Big\{1-\frac{\theta\, t}{b^2(t)}\Big\}e^{-b(t)^2/(2t)}.
$$

This implies that

$$
\begin{aligned}
b(t) &\leq -\sqrt{-2t\log q(t)}\Big\{1+\frac{\log[1-\theta t/b^2(t)]-\log[\sqrt{2\pi}|b(t)|/\sqrt{t}]}{-\log q(t)}\Big\}^{1/2} \\
&\leq -\sqrt{-2t\log q(t)}\Big\{1-\frac{O(1)\log|\log q(t)|}{|\log q(t)+o(1)|}\Big\}^{1/2}.
\end{aligned}
$$

The assertion (4.3.2) thus follows.

The remainder of the theorem is its direct application. □

5.0 INTEGRAL EQUATIONS

5.1 DERIVATION OF THE EQUATIONS

As in Chapter § 4, we assume $\sigma \equiv 1$ and $\mu \equiv 0$. Also we assume that p (and therefore q) is continuous. Recall that

$$\Gamma(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Then the solution to (3.3.1) can be expressed as

$$\begin{aligned}
w(x,t) &= \int_{-\infty}^{0} \Gamma(x-y,t)dy + \int_0^t dp(s) \int_{-\infty}^{b(s)} \Gamma(x-y,t-s)\,dy \qquad (5.1.1)\\
&= 1 - \int_0^\infty \Gamma(x-y,t)dy + \int_0^t dp(s) \left(1 - \int_{b(s)}^\infty \Gamma(x-y,t-s)\,dy\right)\\
&= 1 - \int_0^\infty \Gamma(x-y,t)dy + \int_0^t dp(s) - \int_0^t \int_{b(s)}^\infty \Gamma(x-y,t-s)dy\,dp(s)\\
&= p(t) - \int_{-\infty}^{x} \Gamma(z,t)\,dz + \int_0^t \int_{-\infty}^{x-b(s)} \Gamma(z,t-s)\,dz\,dq(s), \qquad (5.1.2)
\end{aligned}$$

where the second equation is obtained by using $\int_{\mathbb{R}} \Gamma(x-y,s)dy = 1\ \forall\, s > 0$.

Now assume that b is smooth. Differentiating w with respect to x, we can derive

$$u(x,t) = -w_x(x,t) = \Gamma(x,t) - \int_0^t \Gamma(x-b(s),t-s)\,dq(s). \qquad (5.1.3)$$

Also, for $x \neq b(t)$, we can further differentiate $u = -w_x$ with respect to x to obtain

$$u_x(x,t) = -w_{xx}(x,t) = \Gamma_x(x,t) - \int_0^t \Gamma_x(x-b(s),t-s)dq(s), \qquad (5.1.4)$$

and with respect to t to obtain

$$
\begin{aligned}
u_t(x,t) &= \Gamma_t(x,t) - \int_0^t \Gamma_t(x-b(s),t-s)dq(s) \\
&= \Gamma_t(x,t) + \int_0^t \left(\frac{d}{ds}\Gamma(x-b(s),t-s) + \dot{b}(s)\Gamma_x(x-b(s),t-s)\right) dq(s) \\
&= \Gamma_t(x,t) + \int_0^t \dot{b}(s)\Gamma_x(x-b(s),t-s)dq(s) + \int_0^t \dot{q}(s)d\Gamma(x-b(s),t-s) \\
&= \Gamma_t(x,t) + \int_0^t \dot{b}(s)\Gamma_x(x-b(s),t-s)dq(s) - \dot{q}(0)\Gamma(x,t) \\
&\quad - \int_0^t \Gamma(x-b(s),t-s)d\dot{q}(s), \qquad (5.1.5)
\end{aligned}
$$

where the second equation is obtained by the equality

$$\frac{d}{ds}\Gamma(x-b(s),t-s) = \dot{b}(s)\Gamma_x(x-b(s),t-s) - \Gamma_t(x-b(s),t-s),$$

and the third equation by using integration by parts on $\int_0^t \frac{d}{ds}\Gamma(x-b(s),t-s)dq(s)$.

From potential theory, (need some reference) for any b and f with the certain regularity, we have

$$
\begin{aligned}
& \lim_{x \nearrow b(t)} \int_0^t f(s)\frac{x-b(t)}{\sqrt{2\pi(t-s)^3}} e^{-\frac{(x-b(s))^2}{2(t-s)}} ds \\
= & \lim_{x \nearrow b(t)} \int_0^t f(s)\frac{x-b(t)}{\sqrt{2\pi(t-s)^3}} e^{-\left(\frac{x-b(s)}{\sqrt{2(t-s)}}+\frac{b(t)-b(s)}{\sqrt{2(t-s)}}\right)^2} ds \\
= & \lim_{x \nearrow b(t)} \frac{-2}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-b(t)}{\sqrt{2t}}} f\left(t - \frac{(x-b(t))^2}{2\eta^2}\right) e^{-\left(\eta + \frac{b(t)-b\left(t-\frac{(x-b(t))^2}{2\eta^2}\right)}{\frac{x-b(t)}{\eta}}\right)^2} d\eta \\
= & -\frac{2}{\sqrt{\pi}} \int_{-\infty}^0 f(t)e^{-\eta^2} d\eta = -f(t),
\end{aligned}
$$

then

$$\begin{aligned}
&\lim_{x\nearrow b(t)}\int_0^t f(s)\Gamma_x(x-b(s),t-s)ds\\
=&-\lim_{x\nearrow b(t)}\int_0^t f(s)\frac{x-b(s)}{\sqrt{2\pi(t-s)^3}}e^{-\frac{(x-b(s)^2)}{2(t-s)}}\\
=&-\lim_{x\nearrow b(t)}\int_0^t f(s)\left(\frac{b(t)-b(s)}{\sqrt{2\pi(t-s)^3}}e^{-\frac{(x-b(s)^2)}{2(t-s)^3}}+\frac{x-b(t)}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-b(s)^2)}{2(t-s)}}\right)ds\\
=&\int_0^t f(s)\Gamma_x(b(t)-b(s),t-s)\,ds-\lim_{x\nearrow b(t)}\int_0^t f(s)\frac{x-b(t)}{\sqrt{2\pi(t-s)^3}}e^{-\frac{(x-b(s))^2}{2(t-s)}}ds\\
=&\int_0^t f(s)\Gamma_x(b(t)-b(s),t-s)\,ds\mp f(t). \qquad (5.1.6)
\end{aligned}$$

Note that for $x<b(t)$, $w(x,t)=p$ and $0=u(x,t)=u_x(x,t)=u_t(x,t)$. Sending x to $b(t)$ from below in (5.1.2) and (5.1.3) we then obtain

$$\int_{-\infty}^{b(t)}\Gamma(z,t)dz = \int_0^t dq(s)\int_{-\infty}^{b(t)-b(s)}\Gamma(z,t-s)dz, \qquad (5.1.7)$$

$$\Gamma(b(t),t) = \int_0^t \Gamma(b(t)-b(s),t-s)\,dq(s), \qquad (5.1.8)$$

which reflect the free boundary condition $w(b(t),t)=p(t)$ and the condition $u(b(t),t)=0$ respectively. Sending x to $b(t)$ from below in (5.1.4) and (5.1.5) and using (5.1.6), we have

$$\begin{aligned}
\dot q(t) &= \Gamma_x(b(t),t)-\int_0^t \Gamma_x(b(t)-b(s),t-s)\,dq(s), \qquad (5.1.9)\\
-\dot b(t)\dot q(t) &= \Gamma_t(b(t),t)+\int_0^t \dot b(s)\Gamma_x(b(t)-b(s),t-s)dq(s)\\
&\quad -\dot q(0)\Gamma(b(t),t)-\int_0^t \Gamma(b(t)-b(s),t-s)d\dot q(s). \qquad (5.1.10)
\end{aligned}$$

(5.1.9) reflects the free boundary condition $u_x(b(t)^-,t)=0$ and $u_x(b(t)^+,t)=\dot q(t)$. Similarly, (5.1.10) reflects the free boundary condition that $u_t(b(t)^-,t)=0$ and $u_t(b(t)^+,t)=-\dot b(t)\dot q(t)$.

Clearly, these identities can provide numerical schemes much more flexible and economic than integrating the corresponding PDEs. For this purpose, it is necessary to study solutions to each of these identities.

5.2 SOLUTIONS TO THE INTEGRAL EQUATIONS WITHIN FINITE TIME

One observes that if $b(\cdot)$ is a solution to (5.1.8), then $b_1(t) := -b(t)$ is the solution as well. Hence, we need to be careful when considering solutions to the integral equation.

Theorem 6. *Let $q : [0, \infty) \to [0, 1)$ be continuous, increasing, and $q(0) = 0$. Assume that $b : (0, T] \to \mathbb{R}$ is a continuous function. Then $x = b(t)$, $t \in (0, T]$, is the solution to the free boundary problem provided that one of the following holds.*

1. *b satisfies (5.1.7) for all $t \in (0, T]$;*
2. *b satisfies (5.1.8) for all $t \in (0, T]$, $b(t) < 0$ for all sufficiently small positive t, and the function*
$$t \to q^b_{1/2}(t) := \int_0^t \frac{\dot{q}(t)}{\sqrt{2\pi(t-s)}} ds$$
is continuous in $(0, T]$ with $q_{1/2}(0+) = 0$;
3. *b satisfies (5.1.9), $\lim_{t\searrow 0} \frac{b(t)}{\sqrt{t}} = -\infty$, $\dot{q}$ is continuous in $[0, T]$, and the function*
$$t \to q^b_{3/2} := \int_0^t \frac{|b(t) - b(s)|}{\sqrt{2\pi}(t-s)^{3/2}} dq(s)$$
is continuous on $(0, T]$ and is uniformly bounded.

The analogous condition to (5.1.10) is too technical and hence we omit it here.

Proof. With the given continuous function b, we define $w(x, t)$ as in (5.1.1).

1. First of all we verify that $w(x, t)$ satisfies the initial condition. Recall that

$$w_0(x, t) = \int_{-\infty}^0 \Gamma(x - y, t) dy.$$

Then as $\dot{q}(t) \geq 0, \quad \forall\, t \geq 0$,

$$\begin{aligned} w(x, t) - w_0(x, t) &= \int_0^t \dot{p}(s) \int_{-\infty}^{b(s)} \Gamma(x - y, t - s) dy ds \\ &= -\int_0^t \dot{q}(s) \int_{-\infty}^{b(s)} \Gamma(x - y, t - s) dy ds \leq 0. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
w(x,t) - w_0(x,t) &= -\int_0^t \dot{q}(s) \left(1 - \int_{b(s)}^{\infty} \Gamma(x-y, t-s)dy\right) ds \\
&= -q(t) + \int_0^t \dot{q}(s) \int_{b(s)}^{\infty} \Gamma(x-y, t-s)dyds \\
&\geq -q(t).
\end{aligned}
$$

It follows that

$$|w(x,t) - w_0(x,t)| \leq q(t).$$

Sending $t \searrow 0$, we obtain $|w(x,0) - w_0(x,0)| \leq 0$, hence $w(x,0) = w_0(x,0) = \mathbf{1}_{(-\infty,0)}$.

2. Secondly, we verify that w defined as above satisfies $\mathcal{L}w = \dot{p}\mathbf{1}_{\{x<b(t)\}} \leq 0$ as a measure in $\mathbb{R} \times (0,T]$. Direct calculation gives

$$
\begin{gathered}
w_{xx} = -\Gamma_x(x,t) + \textstyle\int_0^t \Gamma_x(x - b(s), t-s)dq(s), \\
w_t = \dot{p}(t) - \textstyle\int_{-\infty}^{x} \Gamma_t(z,t)dz + \int_0^t \int_{-\infty}^{x-b(s)} \Gamma_t(z, t-s)dzdq(s).
\end{gathered}
$$

As $\Gamma_t = \frac{1}{2}\Gamma_{xx}$, for each $(x,t) \in \mathbb{R} \times (0,T]$,

$$\int_{-\infty}^{x} \Gamma_t(z,t)dz = \int_{-\infty}^{x} \frac{1}{2}\Gamma_{xx}(z,t)dz = \frac{1}{2}\Gamma_x(z,t)\Big|_{z=-\infty}^{z=x} = \frac{1}{2}\Gamma_x(x,t).$$

For each $t \in (0,T]$, when $x < b(s)$,

$$
\begin{aligned}
\mathcal{L}w(x,t) &= w_t(x,t) - \frac{1}{2}w_{xx}(x,t) \\
&= \dot{p}(t) - \int_{-\infty}^{x} \Gamma_t(z,t)dz + \int_0^t \int_{-\infty}^{x-b(s)} \Gamma_t(z,t-s)dzdq(s) \\
&\quad + \frac{1}{2}\Gamma_x(x,t) - \frac{1}{2}\int_0^t \Gamma_x(x-b(s), t-s)dq(s) \\
&= \dot{p}(t) - \frac{1}{2}\Gamma_x(x,t) + \int_0^t \Gamma_t(x-b(s), t-s)dq(s) + \frac{1}{2}\Gamma_x(x,t) \\
&\quad - \frac{1}{2}\int_0^t \Gamma_x(x-b(s), t-s)dq(s) \\
&= \dot{p}(t)
\end{aligned}
$$

When $x > b(s)$, with the similar calculation and by (5.1.6) we obtain that

$$\begin{aligned}\mathcal{L}w(x,t) &= \dot{p}(t) + \frac{1}{2}\int_0^t \Gamma_x(x-b(s),t-s)dq(s) - \frac{1}{2}\lim_{x\searrow b(s)}\int_0^t \Gamma_x(x-b(s),t-s)dq(s) \\ &\quad + \frac{1}{2}\lim_{x\nearrow b(s)}\int_0^t \Gamma_x(x-b(s),t-s)dq(s) - \frac{1}{2}\int_0^t \Gamma_x(x-b(s),t-s)dq(s) \\ &= 0.\end{aligned}$$

Thus $\mathcal{L}w = \dot{p}\mathbf{1}_{\{x<b(t)\}} \le 0$ in $\mathbb{R}\times(0,T]$ holds.

3. Finally we will show that $w(x,t) = p(t)$ for $x \le b(t)$ and $w < p(t)$ for $x > b(t)$ by considering separately the conditions of three assertions.

(1) Assume the condition of the first assertion. We define

$$v(x,t) := w(x,t) - p(t) = -\int_{-\infty}^{x} \Gamma(z,t)\,dz + \int_0^t dq(s)\int_{-\infty}^{x-b(s)} \Gamma(z,t-s)\,dz.$$

Upon differentiation, $\mathcal{L}v = 0$ in $\{x < b(t)\}$. Note that v is bounded, continuous and, by (5.1.7), $v(b(t),t) = 0$. It follows that $v(x,t) \equiv 0$ for all $x \le b(t)$, i.e., $w = p(t)$ for any $x \le b(t)$. Also by differentiation, we see that $\mathcal{L}v = -\dot{p} \le 0$ in $\{x > b(t)\}$. The strong maximum principle gives $v < 0$ in $\{x > b(t)\}$. That is $w < p(t)$ in $\{x > b(t)\}$. Thus w is a variational solution.

(2) Assume the condition of the second assertion. We see that $u := -w_x$ given by (5.1.3) is continuous in $\mathbb{R}\times(0,\infty)$. For every small $\varepsilon > 0$, the function u satisfies $\mathcal{L}u = 0$ in

$$\Omega_\varepsilon := \{(x,t) \mid x < b(t), t \in (\varepsilon, T]\}.$$

Also, from equation (5.1.8), $u(b(t),t) = 0$ for all $t \in (0,T]$. Since $b(t) < 0$ for small positive t, we can assume that $b(\varepsilon) < 0$. It then follows from (5.1.3) that for all $x \le b(\varepsilon)$,

$$|u(x,\varepsilon)| \le \max\{\Gamma(x,\varepsilon), q_{1/2}(\varepsilon)\} \le \max\{\Gamma(b(\varepsilon),\varepsilon), q_{1/2}(\varepsilon)\} \le q_{1/2}(\varepsilon),$$

since for any $t \in (0,T]$

$$\int_0^t \Gamma(b(t)-b(s),t-s)dq(s) \le \int_0^t \frac{\dot{q}(s)}{\sqrt{2\pi(t-s)}}\,ds = q_{1/2}(t),$$

which holds for ε as well. It then follows from the maximum principle that

$$\max_{\Omega_\varepsilon} |u| \leq q_{1/2}(\varepsilon).$$

Sending ε to 0 from above, we obtain $w_x = u \equiv 0$ in $\{(x,t) \mid x \leq b(t), t > 0\}$. w is constant in $\{x \leq b(t)\}$ and $w(-\infty, t) = p(t)$ imply that $w \equiv p(t)$ in $\{x \leq b(t)\}$. From the first assertion, the second assertion of the Theorem thus holds.

(3) Assume the conditions in the third assertion. Let $u := -w_x$ be given by (5.1.3) and $u_x := -w_{xx}$ by (5.1.4) when $x \neq b(t)$. Since b, $\dot{q}$, and $q^b_{3/2}$ are continuous, and (5.1.6) holds for $f = \dot{q}$, sending x to $b(t)$ from below in the equation for u_x and using (5.1.9) we derive that $u_x(b(t)^-, t) = 0$.

Next we show that $u_x \equiv 0$ in $\{x < b(t)\}$. To do this, we first show that that u_x given in (5.1.4) is uniformly bounded in $\{x < b(t)\}$. First of all, the boundedness of $q^b_{3/2}$ and (5.1.9) implies that $\Gamma_x(b(t), t)$ is uniformly bounded in $(0, T]$. Next, as $b(t) < -\sqrt{3t}$ for small positive t, we see that $0 < \Gamma_x(x,t) < \Gamma(b(t), t)$ for all $x < b(t)$. Thus $\Gamma_x(x,t)$ is bounded for all $x < b(t)$. For $x < b(t)$, let $A_1 = \{s \in (0,t] \mid b(t) - x > 2|b(t) - b(s)|\}$ and $A_2 = [0,t] \setminus A_1$. Then

$$\int_0^t \Gamma_x(x - b(t), t - s) dq(s) = I_1 + I_2, \quad I_i = \int_{A_i} \Gamma_x(x - b(t), t - s) dq(s).$$

Note that

$$|I_2| \leq \int_0^t \frac{|b(t) - b(s)| \dot{q}(s)}{2\sqrt{\pi}|t - s|^{3/2}} ds \leq 2q^b_{3/2}(t)$$

is uniformly bounded. To estimate I_i, notice that when $x - b(t) > 2|b(t) - b(s)|$, $(x - b(s))^2 = (x - b(t) - (b(t) - b(s))^2 \geq \frac{1}{4}(x - b(t))^2$. Thus,

$$|I_1| \leq \int_0^t \frac{|x - b(t)| \dot{q}(s) e^{-|x - b(t)|^2/[16(t-s)]} ds}{\sqrt{2\pi}(t - s)^{3/2}} \leq \|\dot{q}\|_\infty.$$

and therefore u_x is uniformly bounded in $\{x < b(t)\}$.

Since $\mathcal{L}u_x = 0$ in $\{x < b(t), t > 0\}$, $u_x(b(t) - 0), t) = 0$, and $u_x(x, 0) = 0$ for all $x < 0$, a special maximum principle [2] then implies that $u_x \equiv 0$ in $\{x < b(t)\}$. Using $u(-\infty, t) = 0$ we then conclude that $u \equiv 0$. Following (2), the third assertion of the Theorem follows. $\square$

5.3 ESTIMATION OF FREE BOUNDARY BY INTEGRAL EQUATIONS

By using the integral equations we derived above, we can estimate the free boundary initially.

First of all we claim that:

$$b(t)/\sqrt{2t} \to -\infty \quad \text{as } t \to 0. \tag{5.3.1}$$

Note that $\dot{q}(t) > 0$, when t is near 0. Then by using (5.1.7), we obtain

$$\begin{aligned} 0 &< \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{b(t)}{\sqrt{2t}}} e^{-\eta^2} d\eta = \int_{-\infty}^{b(t)} \Gamma(z,t)\, dz = \int_0^t \dot{q}(s) \int_{-\infty}^{b(t)-b(s)} \Gamma(z, t-s)\, dz\, ds \\ &\leq \int_0^t \dot{q}(s) ds = q(t) - q(0) = q(t). \end{aligned}$$

Let $t \to 0$, we obtain that

$$\lim_{t\to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{b(t)}{\sqrt{2t}}} e^{-\eta^2} d\eta = 0.$$

(5.3.1) then follows.

From (5.1.1)

$$\begin{aligned} w(x,t) &= \int_{-\infty}^0 \Gamma(x-y,t)dy + \int_0^t \dot{p}(s) \int_{-\infty}^{b(s)} \Gamma(x-y,t-s)dy\, ds \\ &= \int_{-\infty}^0 \Gamma(x-y,t)dy - \int_0^t \dot{q}(t-s) \int_{-\infty}^{b(t-s)} \Gamma(x-y,s)dy\, ds. \end{aligned}$$

Assume b is smooth, i.e., b is differentiable. Differentiate w with respect to t,

$$\begin{aligned} w_t(x,t) &= \int_{-\infty}^0 \Gamma_t(x-y,t)dy - \dot{q}(0) \int_{-\infty}^{b(0)} \Gamma(x-y,t)dy \\ &\quad - \int_0^t \ddot{q}(t-s) \int_{-\infty}^{b(t-s)} \Gamma(x-y,s)dyds \\ &\quad - \int_0^t \dot{q}(t-s)\dot{b}(t-s)\Gamma(x-b(t-s),s)ds \end{aligned} \tag{5.3.2}$$

Note that

$$\begin{aligned} &\int_{-\infty}^0 \Gamma_t(x-y,t)dy = \int_{-\infty}^0 \frac{1}{2}\Gamma_{xx}(x-y,t)dy = \int_{-\infty}^0 \frac{1}{2}\Gamma_{yy}(x-y,t)dy \\ = &\ \frac{1}{2}\Gamma_y(x-y,t)|_{y=-\infty}^0 = -\frac{1}{2}\Gamma_x(x-y,t)|_{y=-\infty}^0 = -\frac{1}{2}\Gamma_x(x,t). \end{aligned}$$

Send x

6.0 A NUMERICAL SCHEME

In this chapter, we introduce several numerical schemes to compute the free boundary $b(t)$ with given survival cumulative probability function $q(t)$. For simplicity, we set $\mu = 0$ and $\sigma = 1$.

6.1 PROPOSED NUMERICAL SCHEME

In this section, we propose an approach by making use of the following integral equations derived in section § 5.1

$$\Gamma(b(t), t) = \int_0^t \Gamma(b(t) - b(s), t - s) dq(s), \tag{6.1.1}$$

$$\dot{q}(t) = \Gamma_x(b(t), t) - \int_0^t \Gamma_x(b(t) - b(s), t - s) dq(s). \tag{6.1.2}$$

We point out that Peskir had derived a sequence of integral equations:

$$t^{n/2} H_n \left(\frac{b(t)}{\sqrt{t}} \right) = \int_0^t (t - s)^{n/2} H_n \left(\frac{b(t) - b(s)}{\sqrt{t - s}} \right) dq(s), \tag{6.1.3}$$

where

$$H_{-1}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad H_n(x) = \int_x^\infty H_{n-1}(z)\, dz \;\forall n \geq 0.$$

In particular, when $n = -1$, it is (6.1.1). And when $n = 0$, it is

$$\int_{\frac{b(t)}{\sqrt{t}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = \int_0^t \int_{\frac{b(t)-b(s)}{\sqrt{t-s}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \dot{q}(s)\, ds. \tag{6.1.4}$$

Zucca, Sacerdote, Peskir applied both Monte Carlo and secant methods on 6.1.4 to compute the free boundary b. More details will be provided in the next section.

Now suppose $b(s)$ is known for all $s \in [0, t)$ and we intend to compute b at t. Set

$$Q(x,t) := \int_0^t \Gamma(x - b(s), t - s)\, dq(s).$$

(6.1.1) can be written as

$$\Gamma(b,t) - Q(b,t) = 0. \tag{6.1.5}$$

To compute the boundary b at t, we solve $b = b(t)$ from 6.1.5. If we use Newton's Iteration, a new approximation b^{new} can be obtained from the old approximation b^{old} by

$$b^{new} = b^{old} + \frac{Q(b^{old}, t) - \Gamma(b^{old}, t)}{\frac{\partial}{\partial x}\Big\{\Gamma(x,t) - Q(x,t)\Big\}\Big|_{x=b^{old}}}.$$

Note from (6.1.2) that

$$\frac{\partial}{\partial x}\Big\{\Gamma(x,t) - Q(x,t)\Big\}\Big|_{x=b(t)} = \dot{q}(t).$$

Hence we can use $\dot{q}(t)$ as the approximation to $\frac{\partial}{\partial x}\Big\{\Gamma(x,t) - Q(x,t)\Big\}\Big|_{x=b^{old}}$. Then Newton's Iteration can be written as

$$b^{new} \approx b^{old} + \frac{Q(b^{old}, t) - \Gamma(b^{old}, t)}{\dot{q}(t)}. \tag{6.1.6}$$

Notice that Q has singularity at $t = 0$. To take care of the singularity, we rewrite Q as follow

$$\begin{aligned} Q(x,t) &= \int_0^t \Gamma(x - b(s), t - s) dq(s) = \int_0^t \frac{\dot{q}(s)}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-b(s))^2}{2(t-s)}} ds \\ &= \int_0^t \sqrt{\frac{2}{\pi}} \dot{q}(s) e^{-\frac{(x-b(s))^2}{2(t-s)}} d(-\sqrt{t-s}). \end{aligned}$$

Q is an integral, a quadrature rule is needed for its numerical estimation. Here we used the trapezoid rule. When t is away from 0, $(b(t) - b(s))^2/\{2(t-s)\} \to 0$ as $s \to t$, hence Q can be written as

$$\begin{aligned} Q(x,t_n) &= \sqrt{\tfrac{2}{\pi}} \Big(\dot{q}(t_n) \tfrac{\sqrt{t_n - t_{n-1}}}{2} + \dot{q}(0) \exp\Big(-\tfrac{(x-b_0)^2}{2t_n}\Big) \tfrac{\sqrt{t_n} - \sqrt{t_n - t_1}}{2} \\ &\quad + \textstyle\sum_{i=1}^{n-1} \dot{q}(t_i) \exp\Big(-\tfrac{(x-b_i)^2}{2(t_n - t_i)}\Big) \tfrac{\sqrt{t_n - t_{i-1}} - \sqrt{t_n - t_{i+1}}}{2} \Big) \end{aligned}$$

Nevertheless, due to the singularity of the integral, higher order quadrature rules (e.g., the Simpson's rule) are not recommended. Indeed, as we shall see, the trapezoid rule is quite satisfied at least for small t.

To complete the scheme, we need the approximation of b for small $t > 0$. We have derived in Chapter 4, Theorem (5):

$$\lim_{t \searrow 0} \frac{b(t)}{\sqrt{-2t \log q(t)}} = -1, \quad \text{when} \quad \frac{\dot{q}(t)}{tq(t)} < \infty. \tag{6.1.7}$$

Hence for the small $t > 0$, we approximate the boundary b with the formula

$$b(t) = -\sqrt{-2t \log q(t)}\,. \tag{6.1.8}$$

From now on, we can implement a numerical scheme as follows. To find an approximation of b in $(0, T]$, let $t_n = nh$ be the mesh points where $h = T/N$. Denote by b_n the approximate value of $b(t_n)$.

1. For $n = 1$, estimate b_1 by formula (6.1.8).
2. For $n \geq 2$, suppose $b_1, \cdots, b_{n-1}$ have been calculated. We define b_n by iteration

$$\begin{cases} b_n^0 &= \frac{\sqrt{t_n}}{\sqrt{t_{n-1}}} b_{n-1}, \\ b_n^{k+1} &= b_n^k + \dfrac{Q(b_n^k, t_n) - \frac{1}{\sqrt{2\pi t_n}}\, e^{-(b_n^k)^2/(2t_n)}}{\dot{q}(t_n)}, \quad k = 0, 1, \cdots, K. \end{cases}$$

where b_n^k is the k'th iteration when computing b_n. One might notice that we used the initial guess for b_n by $b_n^0 = \frac{\sqrt{t_n}}{\sqrt{t_{n-1}}} b_{n-1}$. It is obtained by the fact that $\frac{b(t)}{\sqrt{2t}}$ changes slowly in the $\log t$ scale. The iteration ends until $b^{k+1} - b^k$ is small enough, say smaller than the tolerance ε. For example, $\varepsilon = 10^{-5}$.

The complexity of the scheme is $O(N^2)$ where N is the total number of mesh points.

There are some other numerical treatments for the calculation of default boundary $b(t)$. Zucca, Sacerdote and Peskir ([13]) applied both Monte Carlo and secant method to (6.1.4). Avellaneda and Zhu ([7]) used the finite difference method to (1.0.12). Iscoe and Kreinin [10] treated the problem as a conditional probability problem. We now summarize their contributions.

6.2 INTEGRAL EQUATION

We have mentioned that Peskir [12] has derived a sequence of integral equations (6.1.3). The integral equation we used in previous section is the one when $n = -1$. Instead of using (6.1.7), Peskir and Zucca used (6.1.4) to calculate the boundary. They discretize (6.1.4) by the scheme:

$$\begin{cases} \int_{b(t_i)/t_i}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \sum_{j=1}^{i} \int_{\frac{b(t_i)-b(t_j)}{\sqrt{t_i - t_j}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \, \dot{q}(t_j) \, h & i = 2, ...n, \\ \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{2} \, \dot{q}(t_1) \, h & i = 1. \end{cases} \tag{6.2.1}$$

(6.2.1) yields a non-linear system of n equations with n unknowns $b(t_1), ...b(t_n)$. The secant method can used to solve it.

We point out the difference of this scheme with our scheme. Both we and Peskir and Zucca solved a nonlinear integral equation. Out equation is the derivative form of Peskir and Zucca's. Peskir and Zucca used the secant method. We used the Newton iteration since we find out and also proved that we can approximate $\frac{\partial}{\partial x} \Big\{ \Gamma(x, t) - Q(x, t) \Big\} \Big|_{x = b^{old}}$ by $\dot{q}(t)$.

Besides the secant method, Peskir and Zucca also used Monte Carlo method. The basic idea of the scheme is to look for a piecewise-linear approximation of the unknown boundary.

6.3 AVELLANEDA-ZHU'S SCHEME

The spatial translated density function $f(y, t) = u(y + b(t), t)$ satisfies

$$\begin{cases} f_t(y, t) = \dot{b}(t) f_y(y, t) + \frac{1}{2} f(y, t)_{yy} & \text{for } y > 0, \, t > 0, \\ f(0, t) = 0 & \text{for } y = 0, \, t > 0, \\ f(y, 0) = \delta_0(y - b(0)) & \text{for } y > 0, \, t = 0, \\ \frac{1}{2} f_y(0, t) = \dot{q}(t) & \text{for} y = 0, \, t > 0. \end{cases}, \tag{6.3.1}$$

where $\delta_0(\cdot)$ is a Dirac Measure concentrated at 0. This is the the case when $\mu = 0$ and $\sigma = 1$ for (1.0.12). It must be regularized consistently with the boundary condition $f(0, t) = 0$ for all $t \geq 0$. Indeed, to take care of the singularity, Avellaneda and Zhu [7] used the idea of an 'initial layer'. For a chosen small t_0, they replace the solution to (6.3.1) when $t \in [0, t_0]$ by an explicit solution to the first passage problem for a linear boundary. A numerical simulation carries on after t_0. Based on Avellaneda and Zhu's [7] idea of using the initial layer and finite difference scheme, we implement their scheme as following.

1. Analytic Small Time Approximation

For any $\alpha > 0$ and $\beta \in \mathbb{R}$, (6.3.1) admits an exact solution.

$$\begin{cases} \bar{f} = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \left(1 - e^{-2\alpha(x+\alpha+\beta t)/t}\right), \\ \bar{b} = -\alpha - \beta t. \end{cases} \tag{6.3.2}$$

Then the corresponding default probability and its derivative are given by:

$$\bar{q}(t;\alpha,\beta) = 1 - \int_{b(t)}^{\infty} \bar{f}(y,t)dy = N\left(\frac{-\alpha-\beta t}{\sqrt{t}}\right) + e^{-2\alpha\beta} N\left(\frac{-\alpha+\beta t}{\sqrt{t}}\right), \tag{6.3.3}$$

$$\dot{\bar{q}}(t;\alpha,\beta) = \frac{\alpha}{t\sqrt{2\pi t}} e^{-(\alpha+\beta t)^2/2t}. \tag{6.3.4}$$

For a given t_0, the parameter α and β are chosen such that

$$\begin{cases} q(t_0) = \bar{q}(t_0;\alpha,\beta), \\ \dot{q}(t_0) = \dot{\bar{q}}(t_0;\alpha,\beta). \end{cases} \tag{6.3.5}$$

For example, when $q(t) = 0.1\,t$ and $t_0 = 0.1$, the solution to (6.3.5) is $\alpha = 0.4672$ and $\beta = 4.3575$.

2. Numerical Simulation

We apply the finite difference scheme to the first equation of (6.3.1) with the initial condition at $t = t_0$ given by (6.3.2). Define $y_i = ih$ (i=1,2,...,M), $t_n = t_0 + n\Delta t$ and let f_i^n represent the numerical approximation to $f(y_i, t_n)$, b_n to $b(t_n)$, and $\dot{b}_{n-\frac{1}{2}}$ to $\dot{b}(t_{n-\frac{1}{2}})$. Their initial values are taken to be $f_i^0 = \bar{f}(y_i, t_0)$, $b_0 = -\alpha - \beta\, t_0$ and $\dot{b}_{0-\frac{1}{2}} = -\beta$. The boundary conditions are $f_0^n = 0$ and $f_{M+1}^n = 0$. A Crank-Nicholson scheme for the first equation of (6.3.1) reads as

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = \dot{b}_{n+\frac{1}{2}} \frac{f_{i+\frac{1}{2}}^{n+\frac{1}{2}} - f_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{4h^2} + \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{4h^2}. \tag{6.3.6}$$

$f^{n+\frac{1}{2}}_{i+\frac{1}{2}}$ is the numerical approximation of $f(y+\frac{\Delta y}{2}, t+\frac{\Delta t}{2})$ by both Taylor expansion and the upwind scheme:

$$\begin{aligned} f^{n+\frac{1}{2}}_{i+\frac{1}{2}} &= f^n_{i+1} + \frac{1}{2}(\dot{b}_{n-\frac{1}{2}}\Delta t - \Delta y)(f_y)^n_{i+1} + \frac{1}{4}(f_{yy})^n_{i+1}\Delta t, \quad \text{if } \dot{b}_{n-\frac{1}{2}} \geq 0, \\ f^{n+\frac{1}{2}}_{i+\frac{1}{2}} &= f^n_i + \frac{1}{2}(\dot{b}_{n-\frac{1}{2}}\Delta t + \Delta y)(f_y)^n_i + \frac{1}{4}(f_{yy})^n_i\Delta t, \quad \text{if } \dot{b}_{n-\frac{1}{2}} < 0, \end{aligned}$$

where $(f_y)^n_i$ and $(f_{yy})^n_i$, are estimated by standard central difference and second order one-sided difference approximations for the boundary points

$$\begin{aligned} (f_y)^n_i &= \frac{f^n_{i+1} - f^n_{i-1}}{2h}, \quad (f_{yy})^n_i = \frac{f^n_{i+1} - 2f^n_i + f^n_{i-1}}{h^2}, \quad i = 1, ..., M-1. \\ (f_y)^n_0 &= \frac{-3f^n_0 + 4f^n_1 - f^n_2}{2h}, \quad (f_{yy})^n_0 = \frac{2f^n_0 - 5f^n_1 + 4f^n_2 - f^n_3}{h^2} \\ (f_y)^n_M &= \frac{-3f^n_M + 4f^n_{M-1} - f^{M-2}_3}{2h}, \quad (f_{yy})^n_M = \frac{2f^n_M - 5f^n_{M-1} + 4f^n_{M-2} - f^n_{M-3}}{h^2}. \end{aligned}$$

Suppose $f^1, ..., f^n$, b^n and $\dot{b}_{n-\frac{1}{2}}$ have been calculated, then so is $f^{n+\frac{1}{2}}_{i+\frac{1}{2}}$. (6.3.6) gives a system of $M-1$ linear equations with M unknown, $f^{n+1}_i (2 \leq i \leq M)$ and $\dot{b}_{n+\frac{1}{2}}$. To determine these M unknowns, another equation is required. Then the last equation in (6.3.1) when $t = t_{n+1}$: $\frac{1}{2}f_y(0, t_{n+1}) = \dot{q}(t_{n+1})$ is used, which can be rewritten by difference approximation as:

$$\frac{f^{n+1}_1 - f^{n+1}_0}{2h} = \dot{q}(t_{n+1}). \tag{6.3.7}$$

(6.3.6) and (6.3.7) gives a system of M linear equations with M unknowns. Solve the system of equations, we get the solution f^{n+1} to the PDE and then update the boundary by

$$\begin{aligned} b^{n+1} &= b^n + \dot{b}_{n+\frac{1}{2}}\Delta t \\ b^0 &= -\alpha - \beta t_0. \end{aligned}$$

6.4 AN ALTERNATIVE EXPLICIT SCHEME

In light of Avellanda and Zhu's [7] scheme, where they impose an explicit finite difference scheme (6.3.6), Crank-Nicholson scheme. We discretize the PDE in (6.3.1) as

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = \dot{b}_{n+1} \frac{f_i^{n+1} - f_{i-1}^{n+1}}{h} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{2h^2}. \tag{6.4.1}$$

We combine (6.3.7) and (6.4.1) with $i = 1$. Namely, solving f_1^{n+1} from (6.4.1) and substituting the result into (6.3.7) gives an equation involving $\dot{b}_{n+1}$, whose solution is

$$\dot{b}_{n+1} = \frac{2\,h\,\dot{q}(t_{n+1}) - f_1^n + (\Delta t/2h^2)(f_0^n - 2f_1^n + f_2^n)}{2\Delta t \dot{q}(t_{n+1})}. \tag{6.4.2}$$

Using (6.4.2) we can estimate the solution and update the boundary by

$$\begin{aligned} f_j^{n+1} &= \frac{f_j^n - (\Delta t/h)\,\dot{b}_{n+1} f_{j-1}^{n+1} + (\Delta t/2h^2)(f_{j-1}^n - 2f_j^n + f_{j+1}^n)}{1 - (\Delta t/h)\,\dot{b}_{n+1}}, \\ b^{n+1} &= b^n + \dot{b}_{n+1}\Delta t. \end{aligned}$$

The CFL stability condition requires that to be stable, we take $\Delta t = \frac{1}{2}h^2$.

6.5 CONDITIONAL DEFAULT PROBABILITY

Iscoe and Kreinin [10] use a different approach to calculate the boundary. Instead of using the partial differential equations or integral equations to estimate the boundary, they used Monte Carlo simulation. They reduced the problem of estimating the default boundary to a sequential estimation of the quantiles of the conditional default distributions. Instead of considering the continuous process, they consider a discrete-time, mean zero process, $S_n,\ n = 0, 1, 2, ..., S_0 = 0$, having a finite variance $\sigma_n^2 = t_n$ at time t_n. It can be normalized by taking the value η_n at time t_n by

$$\eta_n = \frac{S_n}{\sigma_n}, \quad n = 1, 2, \dots; \quad \eta_0 = 0,$$

which satisfies the relations $E\eta_n = 0,\ E\eta_n^2 = 1$ for $n \geq 1$. The default time τ can be formulated as $\tau = \min_{n \geq 1}\{n : \eta_n < b_n/\sigma_n\}$. Denote $Q(n) := \mathbb{P}\{\tau \leq t_n\}$, $\pi_n = \mathbb{P}\{\tau = t_n\}$

and $\hat{Q}_n = \mathbb{P}\{\tau = t_n | \tau \geq t_n\}$. Iscoe and Kreinin [10] proved that the boundary $\{b_k\}_{k=1}^N$, the probability π_n and $\hat{Q}_n$ satisfy the following equations when $n = 1, 2, ...N$

$$\begin{aligned}
\pi_n &= \left\{ \bigcap_{k=1}^{n-1} \{\eta_k \geq \frac{b_k}{\sigma_n}\}, \eta_n < \frac{b_n}{\sigma_n} \right\}, \\
\pi_n &= Q(n) - Q(n-1), & (6.5.1) \\
\hat{Q}_n &= \frac{\pi_n}{1 - Q(n-1)}. & (6.5.2)
\end{aligned}$$

Based on this result, they estimated the boundary b_n as follow.

1. Estimation of b_1. Based on $Q(1) = \mathbb{P}\{\eta_1 \leq b_1\}$, one can calculate that $b_1 = F_1^{-1}(Q(1))$, where F_1 denotes the the cdf of the random variable η_1.

2. Compute the conditional probabilities $\hat{Q}_n$ based on (6.5.1) and (6.5.2), where $Q(n) = q(t_n)$.

3. Suppose that the default boundary b_1, b_2, ..., b_{n-1} has already been computed. To compute b_n, generate a large number, $M \gg 1$, of i.i.d sample paths

$$\eta(m) = (\eta_1(m), \eta_2(m), ..., \eta_n(m)), \quad m = 1, 2, ..., M,$$

and retain only those vectors $\eta(m)$ that satisfy the inequality

$$\eta_k(m) \geq \frac{b_k}{\sigma_k}, \quad k = 1, 2, ..., n-1. \qquad (6.5.3)$$

4. Let $F_n(x)$ denote the conditional empirical cdf of the random variable η_n under the condition (6.5.3). Estimate $\frac{b_n}{\sigma_n}$ as the quantile of the distribution F_n corresponding to the probability $\hat{Q}_n$:

$$\frac{b_n}{\sigma_n} = F^{-1}(\hat{Q}_n).$$

7.0 NUMERICAL SIMULATION

7.1 LINEAR BOUNDARY

If the default probability function and its' density function are given by (6.3.3) and (6.3.4) then the boundary is a straight line

$$b(t) = -\alpha - \beta t, \quad \alpha > 0, \quad t > 0.$$

Figure (7.1) is the picture of the linear boundary for all four schemes with $T = 1$ and 40 mesh points. To have a better view, we plot four schemes separately Figure (7.2), (7.3), (7.4), (7.5).

7.2 OUR SCHEME VS AVELLANDA AND ZHU'S

Using Avellanda and Zhu's [7] finite difference method, we can get not only the free boundary, but also the solution of the PDE. We get more information from this scheme than all the other three schemes. However they used the idea of an 'initial layer' so that the boundary is estimated at least starting from t_0. Here is an example for $q(t) = 0.1t$, with $T = 1$, $t_0 = 0.1$ (Table (7.1), (7.2) and (7.3)).

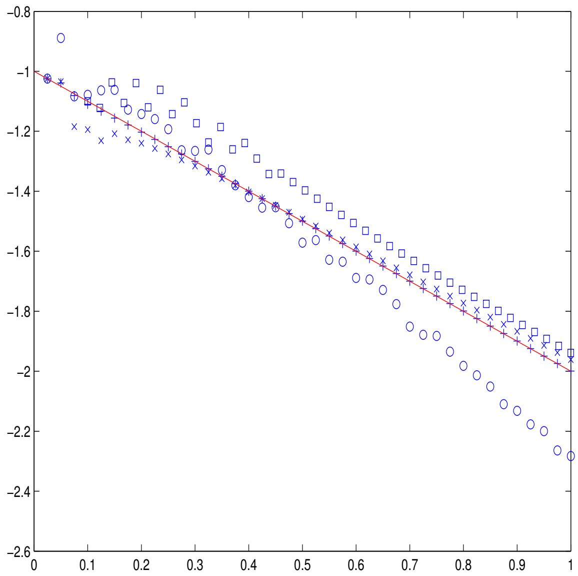

Figure 7.1: Linear Boundary (Our scheme: $\times$, Peskir, Sacerdote and Zucca's: $+$, Iscoe and Kreinins's: $\circ$, Avellaneda and Zhu's: $\Box$).

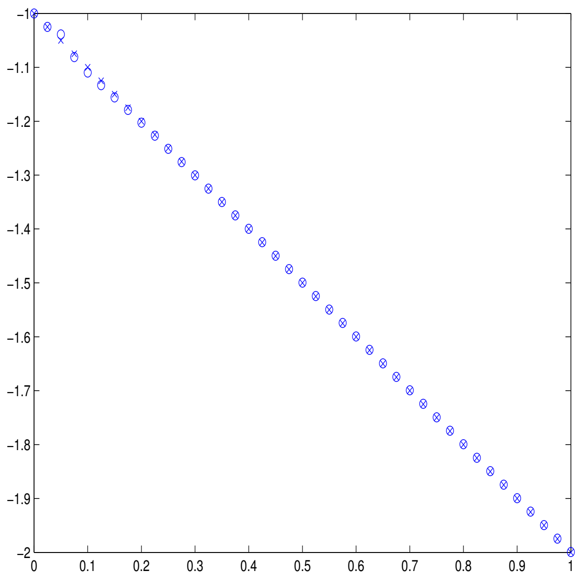

Figure 7.2: Linear Boundary with N=40 (Our scheme o, original line x)

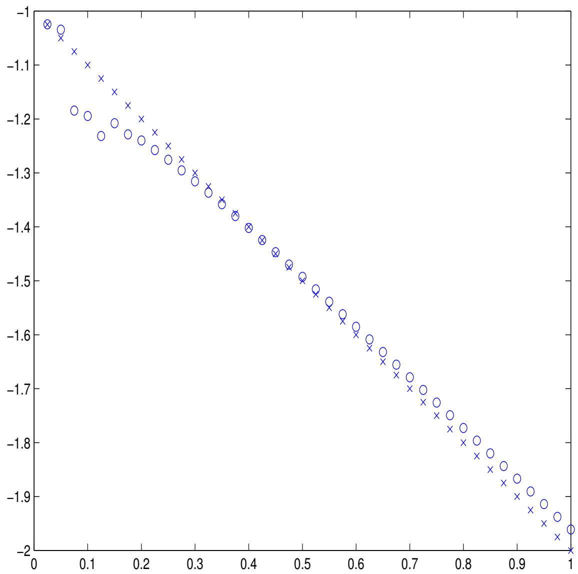

Figure 7.3: Linear Boundary with N=40 (Peskir, Sacerdote and Zucca o, original line x)

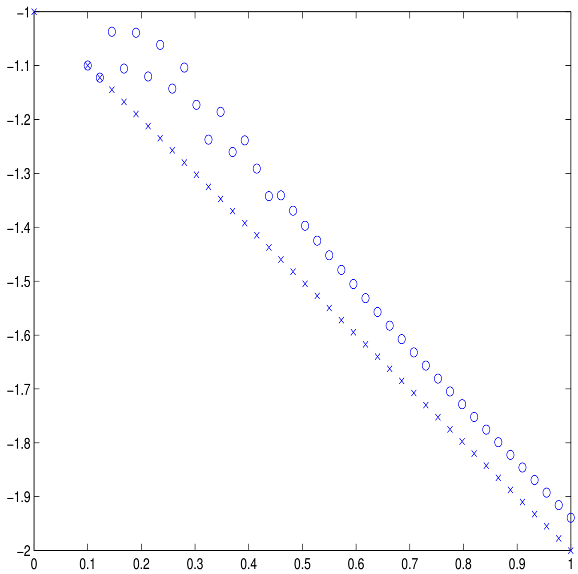

Figure 7.4: Linear Boundary with $N = 40$, $t_0 = 0.1$ (Avellaneda and Zhu o, original line x)

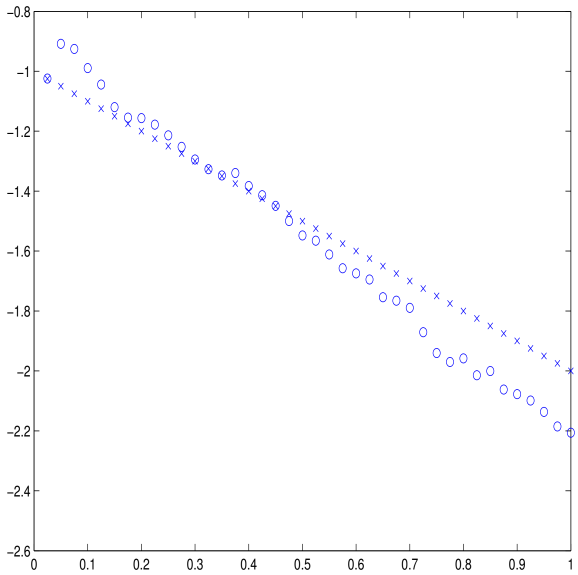

Figure 7.5: Linear Boundary with $N = 40$, $M = 10000$ (Iscoe and Kreinins o, original line x)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	-1.8445	N/A	N/A
20	-1.8400	4.4E-03	N/A
40	-1.8408	7.2E-04	6.22
80	-1.8406	1.4E-04	4.94
160	-1.8403	3.2E-04	0.45
320	-1.8400	2.3E-04	1.43
640	-1.8399	1.3E-04	1.77
1280	-1.8398	6.8E-05	1.89
2560	-1.83987	3.5E-05	1.94

Table 7.1: Default Boundary for $q(t) = 0.1t$ at $T = 1$ (Our scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	2.7206	N/A	N/A
20	2.2248	0.50	N/A
40	2.0275	0.20	2.51
80	1.9389	0.089	2.23
160	1.897	0.042	2.11
320	1.8743	0.023	1.84
640	1.8612	0.013	1.73
1280	1.8533	0.0079	1.66
2560	1.8484	0.0049	1.61

Table 7.2: Default Boundary for $q(t) = 0.1t$ at $T = 1$ with $t_0 = 0.1$ (explicit)

7.3 OUR SCHEME VS ISCOE AND KREININ'S

Iscoe and Kreinin's schemes is different from all the other three. The other scheme used either the PDE or the integral equations. However this used neither. It just based on the theory of probability and used the simulation to calculate the boundary. The advantage of this scheme is its flexibility since it would work for the case when $\mu, \sigma \neq 0$. However the disadvantage is the time spent on the simulations. Here is an example for $q(t) = 0.1t$, with $T = 1$ (Table (7.1), (7.4)).

7.4 OUR SCHEME VS PESKIR, SACERDAOTE AND ZUCCA'S

Using the scheme by Peskir, Sacerdaote and Zucca [15], the computational result is very closed to ours. In fact the integral equations used by Peskir, Sacerdaote and Zucca [12]

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	-2.2406	N/A	N/A
20	-2.0411	0.1995	N/A
40	-1.9395	0.1016	1.96
80	-1.889	0.0505	2.00
160	-1.8638	0.0252	2.01
320	-1.8513	0.0125	2.01
640	-1.8451	0.0062	2.01
1280	-1.842	0.0031	2.00
2560	-1.8404	0.0016	2.00

Table 7.3: Default Boundary $q(t) = 0.1t$ at $T = 1$ with $t_0 = 0.1$ (Avellanda and Zhu's scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t) \rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	-1.7903	N/A	N/A
20	-1.9709	0.1807	N/A
40	-2.1202	0.1492	1.21
80	-2.2687	0.1485	1.00
160	-2.4039	0.1353	1.10
320	-2.4719	0.0680	1.99
640	-2.6740	0.2021	0.33
1280	-2.7543	0.0803	2.52
2560	-2.6438	0.1105	0.73

Table 7.4: Default Boundary $q(t) = 0.1t$ at $T = 1$ (Iscoe and Kreinin's scheme)

and us are both from the sequence of equations (1.0.6). The difference is that for the first point from initial point, we use the estimation from section 4, however Peskir, Sacerdaote and Zucca used the second equation of (6.2.1). Both schemes need to solve the nonlinear equations. We used the Newton iteration and they used the secant method. Here we list the results by both schemes with the different default probability functions with the trapezoid rule.

1. $q(x) = t$ and $T = 0.01$ (table (7.5) and (7.6))

2. $q(t) = \sqrt{t}$ (table (7.7) and (7.8))

We make some adjustment for $Q(x,t)$ and $s^{(k+1)}$ when we do the calculation using our

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	-0.2905	N/A	N/A
20	-0.2904	1.3E-04	N/A
40	-0.29038	4.1E-05	3.09
80	-0.290326	1.3E-05	3.05
160	-0.290321	4.6E-06	2.94

Table 7.5: Default Boundary for $q(t) = t$ with $T = 0.01$ (our scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	0.2868	N/A	N/A
20	0.2886	0.0018	N/A
40	0.2895	0.0009	2.08
80	0.2899	0.0004	2.05
160	0.2902	0.0003	1.71

Table 7.6: Default Boundary for $q(t) = t$ with $T = 0.01$ (Peskir, Sacerdote & Zucca's scheme)

scheme. Indeed

$$
\begin{aligned}
Q(x,t) &= \int_0^t \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-s(\tau))^2}{2(t-\tau)}} dq(\tau) \\
&= \int_0^t \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-s(\tau))^2}{2(t-\tau)}} \frac{1}{2\sqrt{\tau}} d\tau \\
&= \int_0^t \frac{1}{\sqrt{8\pi}} e^{-\frac{(x-s(\tau))^2}{2(t-\tau)}} d\left(\arcsin(\frac{2\tau-t}{t})\right),
\end{aligned}
$$

and

$$
s^{(k+1)} = s^{(k)} + 2\sqrt{t_n} Q(s^{(k)}) - \frac{1}{\pi} e^{-\frac{(s^{(k)})^2}{2tn}}.
$$

When we use Peskir and Zucca's scheme, we make some adjustments too. The reason is that $\dot{q}(t) = \frac{1}{2\sqrt{t}} \to \infty$ as $t \to 0$. Note that (6.1.4) can be written as:

$$
t^{n/2} H_n\left(\frac{b(t)}{\sqrt{t}}\right) - \int_0^t (t-s)^{n/2} H_n\left(\frac{b(t)-b(s)}{\sqrt{t-s}}\right) dq(s) = 0,
$$

the first equation of (6.2.1) becomes

$$
\int_{b(t_i)/t_i}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \sum_{j=1}^{i} \int_{\frac{b(t_i)-b(t_j)}{\sqrt{t_i-t_j}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \, (q(t_j) - q(t_{j-1})),
$$

and the second equation of (6.2.1) becomes

$$
\int_{b(t_1)/t_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{2} q(t_1).
$$

3. $q(t) = 1 - e^{-t}$ (table (7.9) and (7.10))

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	-0.21	N/A	N/A
20	-0.20995	5.3E-05	N/A
40	0.20993	1.7E-05	3.17
80	-0.209926	7.2E-06	2.30
160	-0.209921	3.3E-06	2.17

Table 7.7: Default Boundary for $q(t) = \sqrt{t}$ with $T = 0.01$ (our scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert \frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)} \right\rvert$
10	0.21051	N/A	N/A
20	0.21059	7.7E-05	N/A
40	0.21049	1.0E-4	0.76
80	0.21036	1.3E-04	0.77
160	0.21024	1.2E-4	1.13

Table 7.8: Default Boundary for $q(t) = \sqrt{t}$ with $T = 0.01$ (Peskir, Sacerdote & Zucca's scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert\frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)}\right\rvert$
10	-0.2908	N/A	N/A
20	-0.2907	1.3E-04	N/A
40	-0.29065	4.0E-05	3.09
80	-0.29064	1.3E-05	3.05
160	-0.29063	4.5E-06	2.94

Table 7.9: Default Boundary for $q(t) = 1 - e^{-t}$ with $T = 0.01$ (our scheme)

N (Mesh point)	Free Boundary $s^N(t)$	Difference $\lvert s^N(t) - s^{N/2}(t)\rvert$	Rate $\left\lvert\frac{s^N(t)-s^{N/2}(t)}{s^{N/2}(t)-s^{N/4}(t)}\right\rvert$
10	0.2871	N/A	N/A
20	0.2889	0.0018	N/A
40	0.2898	0.0009	2.08
80	0.2902	0.0004	2.05
160	0.2905	0.0003	1.81

Table 7.10: Default Boundary Default Boundary for $q(t) = 1 - e^{-t}$ with $T = 0.01$ (Peskir, Sacerdote & Zucca's scheme)

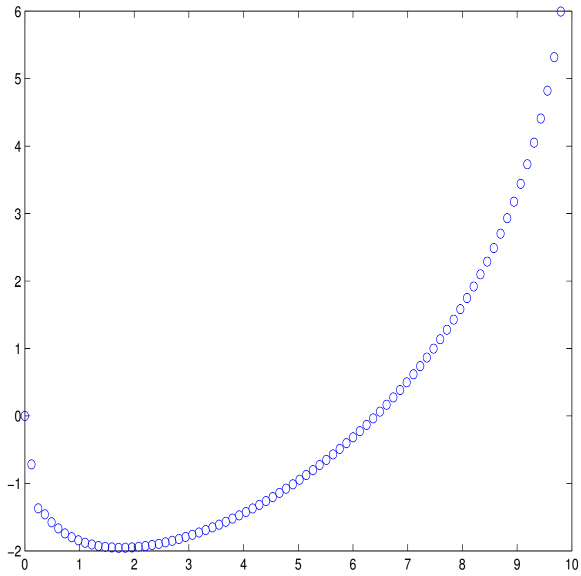

Figure 7.6: Default Boundary for $q(t) = 0.1t$ with $T = 10$ and $N = 80$

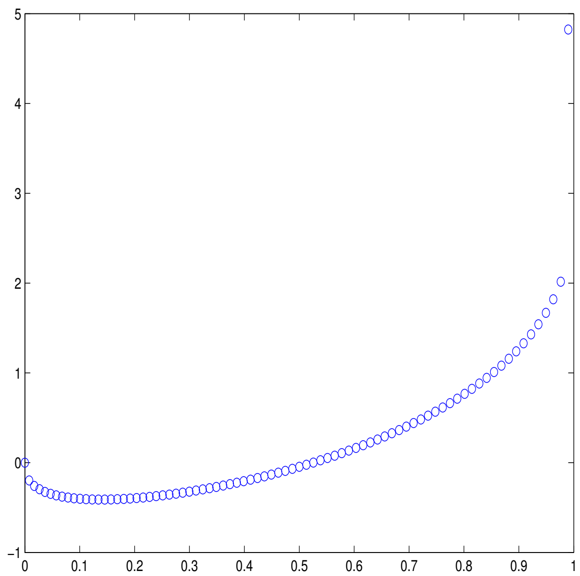

Figure 7.7: Default Boundary for $q(t) = \sqrt{t}$ with $T = 1$ and $N = 80$

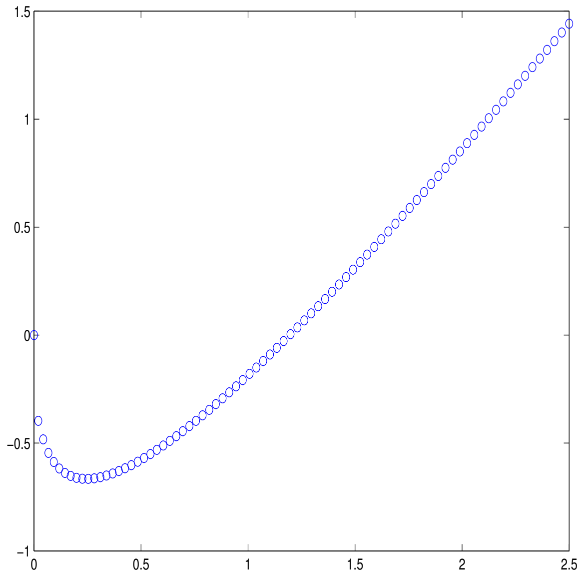

Figure 7.8: Default Boundary for $q(t) = 1 - e^{-t}$ with $T = 2.5$ and $N = 80$

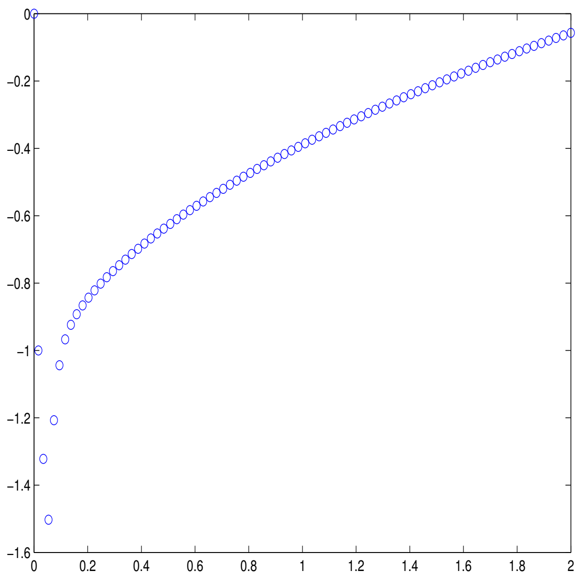

Figure 7.9: Default Boundary for $q(t) = e^{-\frac{1}{2t}}$ with $T = 2$ and $N = 80$

7.5 EXAMPLE

In this section we give some examples of the default boundary with the different default probability functions.

1. $q(x) = 0.1t$ (Figure (7.6)).
2. $q(t) = \sqrt{t}$ (Figure (7.7)).
3. $q(t) = 1 - e^{-t}$ (Figure (7.8)).
4. $q(t) = e^{-1/2t}$ (Figure (7.9)).

We make some adjustments for $Q(x,t)$ and $s^{(k+1)}$ when we do the calculation. Since $\dot{q}(t) = \frac{1}{t^2}e^{-1/2t}$,

$$\begin{aligned} Q(x,t) &= \int_0^t \frac{1}{\tau^2\sqrt{2\pi(t-\tau)}} \exp(-\frac{(x-s(\tau))^2}{2(t-\tau)} - \frac{1}{2\tau})d\tau \\ &= \int_0^t \frac{1}{\sqrt{8\pi}} \exp(-\frac{(x-s(\tau))^2}{2(t-\tau)} - \frac{1}{2\tau})d\left(-\frac{\sqrt{t-\tau}}{t\tau} + \frac{1}{2t\sqrt{t}} \ln\left|\frac{\sqrt{t-\tau}-\sqrt{t}}{\sqrt{t-\tau}+\sqrt{t}}\right|\right), \end{aligned}$$

and

$$s^{(k+1)} = s^{(k)} + 2t^2 Q(s^{(k)})/e^{-1/2t} - \sqrt{\frac{2t_n^3}{\pi}} e^{-1/2t_n} e^{-\frac{(s^{(k)})^2}{2tn}}$$

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