HARDY SPACES AND HARDY-TYPE INEQUALITIES

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ABSTRACT

HARDY SPACES AND HARDY-TYPE INEQUALITIES

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This Master's Thesis is devoted to special kinds of inequalities which generalize Hardy's Inequality and Paley's Inequality in H^1. We provide a more detailed proof for Hardy's Inequality by using a new approach. We also establish Hardy-Like Inequalities by using $H^1 - H^2$ Factorization theorem, and we calculate the best constant for these Hardy-Like Inequalities.

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PREFACE

Dedicated to my father and to my brother.

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1.0 INTRODUCTION

Inequalities are one of the most important instruments in many branches of mathematics, such as harmonic analysis, functional analysis, real analysis, etc. This thesis is devoted to special kinds of inequalities:Hardy's inequality, Hardy-type inequalities,and Paley's inequality.

The classical Hardy space in complex analysis, denoted by $H^p(\Delta)$, consists of analytic functions f on the interior of the unit disc Δ in $\mathbb{C}$.

$$H^p(\Delta) = \left\{ f : \Delta \to \mathbb{C} : \|f\|_{H^p} = \lim_{r \to 1^-} \left(\int_{\theta=0}^{\theta=2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty \right\}$$

Here $\mathbb{C}$ is the set of complex numbers. An analytic function f on Δ can be represented by:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We call a_n the n.th Fourier coefficient of f. In this thesis we sometimes use $\hat{f}(n)$ instead of a_n.

This theory was introduced by Frigyes Riesz,[1], and it is named in honor of the mathematician G. H. Hardy, because of the paper [2].

A basic presentation of Hardy spaces for the unit disc (Δ) and Hardy spaces on the unit circle (C) can be found in the second chapter.

There are many Hardy's inequalities named after G. H. Hardy. One theorem ([3]) states that if $a_1, a_2, a_3, \cdots$ is a sequence of nonnegative real numbers which is not identically zero, then for every real number $p > 1$ one has:

$$\sum_{n=1}^{\infty}\left(\frac{a_1+a_2+a_3+\cdots+a_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.$$

The integral version of Hardy's inequality states if f is an integrable function with non-negative values then:

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p dx.$$

In this thesis specifically, we focus on Hardy's inequality for H^1 functions. It states that if $f \in H^1$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then one has ([4]):

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n} \leq \pi \|f\|_{H^1}$$

We also deal with Paley's inequality ([5]) which states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in the unit disc Δ satisfying

$$\underset{0<r<1}{sup}\int_0^{2\pi} |f(re^{i\theta})| d\theta < \infty,$$

then $\left(\sum_{k=1}^{\infty}|a_{2^k}|^2\right)^{\frac{1}{2}} < \infty$. Equivalently $\exists$ a constant $C > 0$ s.t.

$$\left(\sum_{k=1}^{\infty}|a_{2^k}|^2\right)^{\frac{1}{2}} < C\|f\|_{H^1}.$$

Wojtaszczyk [6] proved that if $f \in H^1$ then

$$\sum_{k=1}^{\infty}\left(|a_{2^k-1}|^2\right) \leq 4\|f\|_{H^1}^2$$

We extend Wojtaszczyk's argument for the lacunary sequence indexed by $\lambda_n = 2^n - 1$ to obtain a better constant for this case, i.e. we obtain the stronger Paley's inequality below:

$$\sum_{k=1}^{\infty}\left(|a_{2^k-1}|^2\right) \leq 2\|f\|_{H^1}^2$$

One of the main tools to deal with the inequalities in this thesis is, the $H^1 - H^2$ factorization theorem. Indeed, we are concerned with the following topics:

- $H^1 - H^2$ factorization theorem
- Hardy's inequality for H^1 functions:

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \pi \|f\|_{H^1}$$

  and finding a more detailed proof for Hardy's inequality.
- Developing Hardy-like inequalities.
- Finding the best possible constants for these Hardy-like inequalities.
- Finding a better constant for Paley's inequality, as described above.

This thesis consists of the introduction part and 5 more chapters.

Chapter two is the preliminary section which develops the background for the other sections. Most of the theorems, definitions and discussions in the preliminaries are based on Hoffman's [4] and Rudin's [7] approach. Here we provide some well known definitions and theorems which will be used in other chapters. In this part we focus on important properties of L^p and H^p functions. We also focus on analytic and harmonic functions in the unit disc and provide the answer to the boundary value problem [4]. At the end of the preliminary section we deal with factorization of H^p functions and the $H^1 - H^2$ factorization theorem, which is an important part of the proof of Hardy's inequality. Also by using $H^1 - H^2$ factorization we develop Hardy-like inequalities in the third chapter. Finally we provide a proof of the classical Hardy's inequality (Theorem 2.24) in the last subsection of chapter two, via Hoffman's approach [4]. Hoffman's approach inspired us to try to find a detailed proof for Hardy's inequality by using a new approach. Hence Hardy's inequality is a good place to begin our discussion of chapter three. In chapter three we provide a detailed proof by using a weight function $w_1(e^{i\theta}) = \pi - \theta$. We extend w_1 to $w_1 : \mathbb{R} \mapsto \mathbb{R}$ by 2π periodicity. The difference between our approach and Hoffman's approach is that we use the Fourier coefficients of the weight function $w_1(e^{i\theta})$ with the $f(e^{i\theta})$ function instead of using just the imaginary part of $f(e^{i\theta})$, and we obtain the same result as Hoffman, i.e. if $f \in H^1$ with each $a_n \geq 0$ then

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \leq \pi \|f\|_{H^1}$$

From here to obtain the Hardy's inequality below,

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \pi \|f\|_{H^1}$$

we use the $H^1 - H^2$ factorization as in Hoffman [4]. The new approach inspires us to produce new Hardy-like inequalities by changing the weight function. At the beginning we obtained the inequality below for the special case: a_{2k+1}. We find that if $f \in H^1$ then

$$\sum_{k=0}^{\infty} \frac{a_{2k+1}}{2k+1} \leq \frac{\pi}{2} \|f\|_{H^1}$$

This result encouraged us to extend this method to the other special cases, for example a_{3k+j} case for $j \leq 3$ and a_{4k+j} case for $j \leq 4$. These new Hardy-like inequalities drew our attention to the fact that there is a relationship between the Fourier coefficients that we use on the left hand side and the constants that we find on the right hand side. We were able to extend this method to some other cases for example $5k + j$, $8k + j$, etc. Each time we notice the same kind of relationships between the Fourier coefficients and the constants. Finally we extend our results to the general case. We were able to find a inequality for the case $a_{\upsilon k+j}$ where $j \leq \upsilon$. So we obtained that: if $f \in H^1$ then:

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+j}|}{\upsilon k + j} \leq \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \frac{1}{\upsilon} \|f\|_{H^1} \tag{1.1}$$

Besides the proof of the inequality (1.1), the 3rd chapter also includes the detailed proof of the "a_{4k+j}" and "a_{3k+j}" cases to provide a better understanding of the method.

Moreover, we provide a proof by using the $H^1 - H^2$ factorization theorem for the lemma which states that if $f \in H^1$ and if we define the function:$(\mathbb{P}_{\upsilon \mathbb{N}_0 + j} f)(z)$, in this way,

$$(\mathbb{P}_{\upsilon \mathbb{N}_0 + j} f)(z) = \sum_{k=0}^{\infty} a_{\upsilon k+j} z^{\upsilon k+j}, \ \ \forall z \in \overline{\Delta}$$

then $\forall j \leq \upsilon$:

$$(\mathbb{P}_{\upsilon \mathbb{N}_0 + j} f)(z) \in H^1$$

and

$$\|(\mathbb{P}_{\upsilon \mathbb{N}_0 + j} f)(z)\|_{H^1} \leq \|f\|_{H^1}.$$

In the 4.th chapter we discuss the best constant problems for the Hardy-like inequalities that we obtained in chapter 3. Basically our purpose is to give an answer to the question:

Is the constant B in the inequality below:

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+j}|}{\upsilon k+j} \leq \underbrace{\frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \frac{1}{\upsilon}}_{B} \|f\|_{H^1}$$

the best constants for all cases? If so, can we find a proof for it; and if not, can we find better constants?

Firstly, we develop a method to prove that π is the best constant for the Hardy Inequality:

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \pi \|f\|_{H^1}.$$

But this method does not work for all cases. We tried to find better constants for other cases, and finally we found that if $\frac{j}{\upsilon} \geq \frac{1}{2}$ then

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+j}|}{\upsilon k+j} \leq \frac{\pi}{\upsilon} \|f\|_{H^1}.$$

Moreover, we were able to find a method to prove that $\frac{\pi}{\upsilon}$ is the best constant for this case.

For the remaining special case $(\upsilon k + j)$ where $\frac{j}{\upsilon} < \frac{1}{2}$ we obtain not only that B above is the best constant, but also the following interesting result holds: for $m \geq 3$ and $1 \leq j \leq \frac{m}{2}$, $\exists f = f_{m\mathbb{N}_0+j} \in H^1$ s.t.

$$\sum_{k=0}^{\infty} \frac{|\hat{f}_{m\mathbb{N}_0+j}(mk+j)|}{mk+j} = \frac{\pi}{m} \frac{1}{sin\left(\frac{\pi j}{m}\right)} \|f\|_{H^1}.$$

To do this we used some specific properties of special functions [8], [9], and we used Pochammer symbols and hypergeometric functions [10], inspired by Mathematica experiments.

The 5.th chapter contains a discussion of Paley's inequality and a strengthening of Paley's inequality. The classical Paley's inequality states that: if $f \in H^1$ and $(\lambda_n)_n$ is lacunary sequence in $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ s.t.

$$L = \inf_{n \in \mathbb{N}} \frac{\lambda_{n+1}}{\lambda_n} > 1$$

then $\exists B \in (0, \infty)$ s.t. $\forall f \in H^1(\Delta)$

$$\left(\sum_{n=1}^{\infty} \left| \hat{f}(\lambda_n) \right|^2 \right)^{\frac{1}{2}} \leq B \|f\|_{H^1} \quad [14].$$

As we discussed above, we found a special case in Wojtaszczyk's book [6] for $\lambda_n = 2^n - 1$. He found $B = 2$ as a constant for this case. We extend Paley's Inequality for the special case $\lambda_n = 2^n - 1$ and obtain constant $B = \sqrt{2}$ for this case.

The last chapter contains an application of Paley's and Hardy's inequalities. We define a map: $J : H^1 \to \ell^1$ s.t. $J(f) = \left(\frac{a_n}{n+1} \right)_{n \in \mathbb{N}_0}$ where $f(z) \in H^1$, and prove that the function J is not onto (see 6.6). As a background for this proof we present a discussion about the Schur property of the Banach spaces. We also provide some proofs about the relationship between isomorphic isomorphism and having Schur property for Banach spaces. By using both Hardy's Inequality [4] and Paley's inequality, the proof of Theorem 6.6 follows.

2.0 PRELIMINARIES

In this chapter we introduce the basic facts that will be taken for granted through the development of this thesis.

Definition 2.1. ***(Lebesgue Measure)*** *Suppose X be the real line or a closed interval, and F be a monotone increasing function in X, which is continuous from the left i.e.:*

$$F(x) = \sup_{t<x} F(t)$$

And let μ be a function on semi-closed interval $[a, b)$ s.t.

$$\mu([a, b)) = F(b) - F(a)$$

According to "Hahn Banach Extension Theorem, " μ has a unique extension to a positive Borel measure on X. The measure is finite iff F is bounded. If X is the real line, every positive Borel measure on X arises in this way from the left continuous increasing function F. If X is closed interval, every finite positive Borel measure on X comes from such an increasing monotone function. If X is either the real line or an interval, the measure induced by the function

$$F(x) = x$$

is called Lebesgue Measure.

Definition 2.2. ***(Simple Borel Function)*** *Let X be locally compact set. A simple Borel* *function on X is complex valued function f on X s.t.*

$$f(x) = \sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(x)$$

where

- $a_1, a_2, \cdots a_n$ *are complex numbers*
- $E_1, E_2, \cdots E_n$ *are disjoint Borel sets of finite μ measure.*
- $\mathcal{X}_E$ *is characteristic function of the set E.*

Definition 2.3. *The Borel function is called integrable with respect to μ if there exists a sequence of functions f_n such that*

1. *Each f_n is a simple Borel function for μ*
2. $\lim_{m,n\to\infty} \int |f_m - f_n| dm = 0$
3. *f_n converges to f in measure, i.e. for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \mu(\{x : |f(x) - f_n(x)| \geq \epsilon\ \}) = 0$$

2.1 THE SPACE L^1

If f is integrable and f_n converges to f in measure then

$$\int f_n d\mu$$

converges and the limit of this sequence denoted by,

$$\int f d\mu$$

$L^1(d\mu)$ denotes the class of μ integrable functions and it is clear that $L^1(d\mu)$ is a vector space. $f \to f d\mu$ is a linear functional on L^1. The Borel function $f \in L^1(d\mu) \iff |f| \in L^1(d\mu)$. If f is in $L^1(d\mu)$ then

$$\left| \int f d\mu \right| \leq \int |f| d\mu$$

Definition 2.4. ***(μ Measure Zero)*** *A subset S of X has μ measure zero if for each $\epsilon > 0$ there is a Borel set A containing S with $\mu(A) < \epsilon$*

Theorem 2.1. *[4]* ***(Lebesgue Dominated Convergence Theorem)*** *If f_n is a sequence of integrable functions such that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists almost everywhere, and if there is a fixed integrable function g such that $|f_n| \leq |g|$ for each n then, f is integrable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

2.2 THE SPACE L^P

Definition 2.5. ***(Conjugate Exponents)*** *If p and q are positive real numbers s.t.*

$$\frac{1}{p} + \frac{1}{q} = 1$$

then p and q are called conjugate exponents.

It is clear that $1 < p < \infty$ and $1 < q < \infty$. Consequently we can conclude that 1 and ∞ are also a pair of conjugate exponents.

Definition 2.6. $(L^p(\mu))$:

Let X be an arbitrary measure space with a complete positive measure μ. If $0 < p < \infty$ and if f is a complex measurable function on X then we define $\|f\|_p = \left\{\int_X |f|^p d\mu\right\}^{\frac{1}{p}}$ where $\|f\|_p$ is called L^p norm of f.

The space of $L^p\,(d\mu)$ consists of all f measurable functions, which satisfies:

$$\|f\|_p < \infty$$

We identify the functions f and g that differ on a set of measure zero. If p is positive number the space $L^p\,(d\mu)$ consists of all measurable functions which satisfy: $|f|^p$ is in $L^1\,(d\mu)$. If p is positive number

$$f \in L^p\,(d\mu)$$

$$g \in L^q(d\mu)$$

and p and q are conjugate exponents then

$$(fg) \in L^1(d\mu)$$

Note that when we define:

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}}$$

It is not a norm since we may have $\|f\|_p = 0$ without $f = 0$. It becomes a norm when we agree to identify two functions in $L^p(d\mu)$ which agree almost everywhere wtih respect to μ.

Some Important Facts about $L^p(d\mu)$

- Let X be compact and μ be a finite measure. Then every continuous function on X is integrable. Moreover for $f \in L^1$ and $\epsilon > 0$ then $\exists$ a continuous function g s.t.

  $$\int |f - g| \, d\mu < \epsilon.$$

  So the space of continuous functions is dense in L^1.

- If $p \geq 1$ then L^p is contained in L^1 and the continuous functions are dense subspace of L^p, i.e.

  $$\int |f - g|^p d\mu < \epsilon$$

- Let S be a locally compact Hausdorff space and fix a positive Borel measure μ on S. Choose a number $p \geq 1$ and let $X = L^p(d\mu)$. Define the norm of $f \in L^p$ to be its L^p norm:

  $$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}}$$

  The space $L^p(d\mu)$ $p \geq 1$ is a Banach space using the L^p norm.

- The space $L^\infty(d\mu)$ is the space of bounded μ measurable functions with the $\mu - ess - sup$ norm.

  $$\|f\|_\infty = ess_\mu - sup_x |f(x)|$$

Definition 2.7. ***(Conjugate Space of X)*** *Let X be a Banach space and let X^* be the* *space of all linear functionals of F which are continuous, i.e.:*

$$\|x_n - x\| \to 0 \Rightarrow |F(x_n) - F(x)| \to 0$$

then the set X^* forms a vector space. The linear functional F is continuous if and only if it is bounded. i.e.: if and only if there is a constant $K > 0$ s.t.

$$|F(x)| \leq K \|x\|$$

for every $x \in X$. The smallest such K is called the norm of F. Then

$$\|F\| = \sup_{\|x\| \leq 1} |F(x)| .$$

With this norm X^* becomes a Banach Space which is called the conjugate space of X.

EXAMPLE 2.1. *[4] Let S be a locally compact space and μ a positive Borel measure on S. Suppose $1 \leq p < \infty$ and that $X = L^p(d\mu)$. Then the conjugate space of X is $L^q(d\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$, $X^* = L^\infty$. If $g \in L^q(d\mu)$ then g induces a continuous linear functional F on L^p by*

$$F(f) = \int fg d\mu, f \in L^p.$$

Every continuous linear functional on L^p has this form and

$$\|F\| = \|g\|_q.$$

Definition 2.8. ***(Inner Product Space and Hilbert Spaces):***

Let H be a real or complex vector space. An inner product on H is a function $(.,.)$ which assigns to each ordered pair of vectors in H a scalar in such way that:

- $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$
- $(\lambda x, y) = \lambda(x, y)$
- $(y, x) = \overline{(x, y)}$
- $(x, x) \geq 0,\ (x, x) = 0 \Leftrightarrow x = 0$

Such a space H with a specified inner product on H is called an inner product space. If H is complete in the norm genrated by $(.,.)$, $x \to (x, x)^{\frac{1}{2}}$, we say that H is a Hilbert space

EXAMPLE 2.2. *[4] Let X be a locally compact space and μ a positive Borel measure on X. Let $H = L^2(d\mu)$ with the inner product*

$$(f, g) = \int f\bar{g}d\mu$$

then H is a Hilbert space.

Definition 2.9. *[4]*

Let $H = L^2(-\pi, \pi)$. The space of Lebesgue square integrable functions on the closed interval $[-\pi, \pi]$. Then we define the inner product

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

So $L^2(-\pi, \pi) = L^2(d\mu)$ where μ is normalized Lebesgue measure and $d\mu = \frac{1}{2\pi}dx$. Let $\varphi_n(x) = e^{inx}$ then it is clear that φ_n is an orthonormal set. This orthonormal set is complete. If $f \in L^2(-\pi, \pi)$ the numbers

$$c_n = (f, \varphi_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

are the fourier coefficients of f, the Fourier series for f is:

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and the n.th partial sum is:

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}.$$

Note that the sequence of Fourier coefficients is square summable and

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

And n.th partial sum s_n of the Fourier series converges to f in the L^2 norm:

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx = 0.$$

Theorem 2.2. *[4]* ***(RIÉSZ-FISHER THEOREM):***

Every square summable sequence of complex numbers is the sequence of Fourier coefficients *of a function in* $L^2(-\pi, \pi)$*. For if*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$$

and if $s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ then it is easy to see that s_n converges in L^2 to a function f with Fourier coefficients c_n.

2.3 FOURIER SERIES

If f is a complex valued Lebesgue integrable function on the closed interval $[-\pi, \pi]$ then the Fourier coefficients of f are the complex numbers:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n = 0, {}^{+}_{-}2, {}^{+}_{-}3, \cdots \tag{2.1}$$

and the Fourier series for f is the formal series:

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{2.2}$$

Let construct the partial sums for this Fourier coefficients in such way;

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \tag{2.3}$$

We know that if f is square integrable then the partial sums converges to f in the L^2 norm. Our question here is, if f in L^1 how can we recapture f from its Fourier coefficients?

Definition 2.10. *[4] (CESÀRO MEANS): The first Cesàro means of the Fourier series for f are the arithmetic means*

$\sigma_n = \frac{1}{n}(s_0 + \cdots + s_{n-1}),$ *where* $n = 1, 2, \cdots$

If f is in $L^p(-\pi, \pi)$, $1 \leq p < \infty$ then the Cesàro means σ_n converges to f in the L^p norm. And if f is continuous and $f(-\pi) = f(\pi)$ then the σ_n converges uniformly to f.

Definition 2.11. *[4] (FEJÉR'S KERNEL): We know that*

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \quad = \sum_{k=-n}^{n} e^{ikx} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{k=-n}^{n} e^{ik(x-t)} dt. \tag{2.4}$$

Then;

$$\sigma_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt \tag{2.5}$$

where $K_n(x)$ is the n th Cesàro means of the sequence:

$\sum_{k=-n}^{n} e^{ikx}$ *Thus;*

$$\begin{aligned}(n+1)K_{n+1}(x) - (n)K_n(x) &= \sum_{k=-n}^{n} e^{ikx} \\ &= \sum_{k=0}^{n} e^{ikx} + \sum_{k=1}^{n} e^{-ikx} \\ &= \frac{1-e^{i(n+1)x}}{1-e^{ix}} + \frac{1-e^{-i(n+1)x}}{1-e^{-ix}} - 1 \\ &= \frac{\cos(nx) - \cos(n+1)x}{1-\cos(x)}\end{aligned} \tag{2.6}$$

Since $K_1(x) = 1$ then

$$\begin{aligned}K_n(x) &= \frac{1}{n}\left[\frac{1-\cos(nx)}{1-\cos(x)}\right] \\ &= \frac{1}{n}\left[\frac{\sin\frac{n}{2}x}{\sin\frac{1}{2}x}\right]^2\end{aligned} \tag{2.7}$$

this sequence of functions K_n is called Fejér's kernel.

Note that

- $K_n \geq 0$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$
- If I is any open interval about $x = 0$ then

$$\lim_{n\to\infty} \sup_{x \notin I} |K_n(x)| = 0, (|x| \leq \pi)$$

Definition 2.12. *[4]* ***(Approximate Identity):*** *Any sequence of Lebesgue integrable functions, K_n which has the properties above we call a_n 'Approximate identity'.*

Suppose f is a Lebesgue integrable function on the interval $[-\pi, \pi]$.Let $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the Fourier series for f. It is important to know whether the partial sums $s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ of the Fourier series for f are convergent or not. If it is convergent it is also important to know whether the s_n converge pointwise, converge pointwise almost everywhere, converge uniformly,or converge in some type of norm? If they are convergent, do they converge to f?

We can easily find the answer of this question by using Cesàro means:

Theorem 2.3. *[4] Let f be a function in $L^p(-\pi, \pi)$ where $1 \leq p < \infty$. Then the Cesàro means of the Fourier series for f converge to f in the L^p norm. If f is continuous and $f(-\pi) = f(\pi)$ then the Cesàro means converge uniformly to f.*

Theorem 2.4. *[4] If f is in the $L^\infty(-\pi, \pi)$ then the Cesàro means of the Fourier series for f converge to f in the weak star topology on L^∞.*

Theorem 2.5. *[4] Let μ be a finite complex Baire measure on the interval $[-\pi, \pi]$ and let σ_n be the n th. Cesàro mean of the Fourier series for μ. If f is any continuous function of period 2π, then*

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\sigma_n(x)dx = \int_{-\pi}^{\pi} f(x)d\mu(x)$$

which means the measures $\frac{1}{2\pi}\sigma_n dx$ converge to μ in the weak star topology.

For the proofs see [4].

Corollary 2.1. *(FEJÉR'S THEOREM)[15] Every continuous function of period 2π is a uniform limit of trigonometric polynomials.*

$$p(x) = \sum_{k=-n}^{n} a_k e^{ikx}$$

Since for $f \in L^2$ the σ_n converges to $f \in L^2$ we can easily conclude that the orthonormal family e^{inx} is complete in $L^2(-\pi, \pi)$.

Note that if we have defined Fourier coefficients of two functions it is obvious that if we add two functions, it requires that we add respective Fourier coefficients. Moreover for

$f, g \in L^1(-\pi, \pi)$ we can define a multiplication by convolution:

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)dt.$$

Fubini's theorem implies that, $f * g$ is again in L^1 and it is easy to see that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Hence

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx}(f * g)(x)dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)dtdx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x-t)dx \right] dt \qquad (2.8) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-int}dt \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} f(y)dy \end{aligned}$$

The n.th Fourier coefficient of $f * g$ is is the product of the n.th Fourier coefficients of f and g. We can also define the convolution of two measures for example the convolution of two measures which are absolutely continuous with respect to Lebesgue measure. Then the convolution of f and μ is defined in this way:

$$(f * \mu)(x) = \int_{-\pi}^{\pi} f(x-t)d\mu(t)$$

Again the Fourier coefficients of $f * \mu$ are the products of the corresponding coefficients of f and μ. If f is in L^1 then $\sigma_n = f * K_n$. The measures $\frac{1}{2\pi} K_n(x)dx$ are approaching the delta measure

$$\delta_0(\mathcal{B}) = \begin{cases} 1, & \text{if } 0 \in \mathcal{B}; \\ 0, & \text{otherwise.} \end{cases} \qquad (2.9)$$

where $\mathcal{B}$ is a Borel set.

Hence Cesàro means of f converges to f in L^1. Note that the Fejer's kernel K_n is the $n.th$ Cesàro mean of the Fourier series for the delta measure δ_0.So we can convert the results into the case when K_n is any approximate identity for L^1. So we can conclude that $f * K_n$ converges uniformly to f if f is continuous, converges to f in L^p norm if $1 \leq p < \infty$, and converges weak-star to f if f is in L^∞.

2.4 CHARACTERIZATION OF TYPES OF FOURIER SERIES

Consider a formal Fourier series:

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

If the sequence $\{c_n\}$ is square summable, then we know that this formal series is the Fourier series of an L^2 function. The question here is how can we decide whether this formal series is the Fourier series of an L^1 function an L^p function, a measure, or a continuous function? We can give a satisfactory answer again in terms of the Cesàro means of the series. Let's simplify this question. For a given sequence $c_n, n = 0, \pm 1, \pm 2, \pm 3, \cdots$ find a necessary and sufficient condition for c_n to be the Fourier transform of a function L^p. We want necessary and sufficient condition for the existence of a function f where f is in L^p for some p s.t. $\hat{f}(n) = c_n$ for all n.

It is clear that the partial sums of the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ are in L^p for every p. If these partial sums converge for the norm of some L^p space then the limit is a function in L^p whose Fourier transform is c_n. Thus the convergence for some L^p norm of our series is a sufficient condition for c_n to be the Fourier transform of a L^p function.

Let consider a series $\sum_{k=-\infty}^{\infty} a_k$. For each $n = 0, 1, \cdots$ $s_n = \sum_{k=-n}^{n} a_k$. If s_n is Cesàro convergent to f then the given series is said to be Cesàro convergent to the same function. Here we assumed c_n was given and asked for necessary and sufficient conditions for this sequence to be the Fourier transform of an L^p function. Let σ_n is the $n.th$ Cesàro means of the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. If $n > |m|$ then the $m.th$ Fourier coefficient of $\sigma_n(x)$ is $\left[\frac{n-|m|}{n}\right] c_m$. Hence

$$\lim_{n\to\infty} \frac{1}{2\pi} \int \sigma_n(x) e^{-imx} dx = c_m$$

for every m.

Theorem 2.6. *[11] If for some $p > 1$, the Cesàro means of*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

are bounded for the L^p norm, then the sequence c_n is the Fourier transform of a function in *this space.*

Theorem 2.7. *[11] If the Cesàro means of the series converge for the L_1 norm then the sequence c_n is the Fourier transform of a function in this space.*

Suppose that for some $p > 1$ the sequence $\sigma_n(x)$ is bounded for the L_p norm which means for some $p > 1$

$$sup\left\{\|\sigma_n(x)\|_p,\ n = 1, 2, 3, \cdots\right\} < \infty.$$

Since $p > 1$, then L^p is the dual of the Banach Space L^q where $\frac{1}{p} + \frac{1}{q} = 1$. Then there is a $h \in L^p$ s.t. for any $\epsilon > 0$

$$\left|\int\left[\sigma_n(x)\overline{g(x)} - h(x)\overline{g(x)}dx\right]\right| < \epsilon$$

for infinitely many n. So $e^{imx} \in L^q$ and $\hat{h}(m) = c_m$ for all m. So this takes care of theorem 2.6 and 2.7.

Theorem 2.8. *[11] A sequence c_n is the Fourier transform of a function in L^p, $p > 1$ iff the Cesàro means of $\sum c_n e^{inx}$ are uniformly bounded for the L^p norm. The given sequence is the Fourier transform of a function in L^1 iff the Cesàro means of the series converge for the L^1 norm.*

Proof. If $p = 1$ then the theorem follows from Theorem 2.6 and 2.7. If $1 < p < \infty$ then it follows from Riemann Lebesgue lemma. If $f \in L^\infty$ it follows from Theorem 2.7 and the Banach Steinhaus theorem that the Cesàro means of its Fourier series are uniformly bounded for the L^∞ norm. □

2.5 ANALYTIC AND HARMONIC FUNCTIONS IN THE UNIT DISC

Let Δ denote the open unit disc in the complex plane:

$$\Delta = \{z; |z| < 1\}$$

and C denote the unit circle;

$$C = \{z; |z| = 1\}$$

Definition 2.13. ***(Analytic Function in Δ):*** *[4] Suppose f is a complex function defined in Δ If $z_0 \in \Delta$ and if*

$$lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists we denote this limit by $f'(z_0)$ and call it the derivative of f at z_0.If $f'(z_0)$ exists for every $z_0 \in \Delta$ we say that f is analytic (or holomorphic) in Δ.

If the complex valued function f is analytic in Δ then it is the sum of the convergent power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Definition 2.14. ***(Harmonic Function in Δ):*** *[4] A complex valued function u is harmonic on Δ, if it satisfies the Laplace equations:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Note that any analytic function is a complex valued harmonic function. A real valued function u is harmonic iff it is the real part of an analytic function.

$$f = u + iv$$

Given the real harmonic function u there is a harmonic function v for which

$$f = u + iv$$

is analytic.

Such v is called the harmonic conjugate of u and satisfies Cauchy-Riemann equations below:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and v vanishes at the origin.

2.5.1 The Cauchy and Poisson Kernels

Suppose we have an analytic function f in a disc of radius $1+\epsilon$. We know that f has boundary values and it is determined by these boundary values by using the Cauchy integral formula where z is on the boundary here.

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\frac{e^{i\theta}}{e^{i\theta}-z}d\theta$$

If we have an analytic function in the open disc then,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

If we restrict f to the circle of radius r we obtain a continuous function on that circle which can be interpreted as a function on the unit circle.

$$f_r(\theta) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

where $f_r(\theta) = f(re^{i\theta})$ is a function on the unit circle. The n.th Fourier coefficients of $f_r(\theta)$ are:

- $a_n r^n$ for $n \geq 0$
- 0 for $n < 0$

and the Cauchy integral formula becomes:

$$\begin{aligned} f_r(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{e^{it}}{e^{it}-re^{i\theta}}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\underbrace{\frac{1}{1-re^{i(\theta-t)}}}_{C_r(\theta-t)}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)C_r(\theta-t)dt. \end{aligned} \tag{2.10}$$

Note that $f_r(\theta)$ is the convolution of $f(e^{it})$ and $C_r(\theta)$ so the Fourier coefficients of f_r are the Fourier coefficients of the product of those of $C_r(\theta)$ and $f(e^{it})$.

If u is harmonic in the disc then u is the real part of an analytic function of $f(z)$. If we restrict u to the circle of radius r then,

$$\begin{aligned} u_r(e^{i\theta}) &= 2Re\,(a_0) + \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{a_n}\,\overline{r^n}\,\overline{e^{in\theta}} \\ &= \underbrace{2Re\,(a_0)}_{c_0} + \sum_{n=-\infty,\, n\neq 0}^{\infty} c_n r^{|n|} e^{in\theta} \end{aligned} \tag{2.11}$$

where $a_n = c_n$ for $n > 0$ and $\overline{a_{-n}} = c_n$ for $n < 0$. If u is harmonic in the closed disc, then the boundary function u_1 has the Fourier coefficients c_n.

$$\begin{aligned} u_1(\theta) &= u(e^{i\theta}) \\ &= 2Re\,(a_0) + \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \\ &= 2Re\,(a_0) + \sum_{n=1}^{\infty} a_n e^{in\theta} + \sum_{n=1}^{\infty} \overline{a_n} e^{-in\theta} \end{aligned} \tag{2.12}$$

Since

$$u_r(\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

and

$$u_1(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

it is clear that the Fourier coefficients of $u_r(\theta)$ are a multiplication of the Fourier coefficients of $u_1(\theta)$ and the Fourier coefficients of another function whose Fourier coefficients is $r^{|n|}$. Let us call the second function $P_r(\theta)$. Then

$$\begin{aligned} P_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \\ &= C_r(\theta) + \overline{C_r(\theta)} - 1 \\ &= 2Re C_r(\theta) - 1 \\ &= Re(2C_r)(\theta) - 1 \\ &= Re\left[\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right] \\ &= \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2} \end{aligned} \tag{2.13}$$

Since $u_r(\theta)$ is the convolution of $u(\theta)$ and $P_r(\theta)$ then

$$u_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)P_r(\theta - t)dt$$

where $u(t)$ denotes $u(e^{it})$.

Note that this Poisson integral formula holds for an analytic function f. So both the Poisson kernel and Cauchy kernel reproduce analytic functions from their boundary values by convolution.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}P_r(\theta)d\theta = r^{|n|}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}C_r(\theta)d\theta = \begin{cases} r^n, & \text{if } n \geq 0; \\ 0 & \text{if } n < 0; \end{cases} \tag{2.14}$$

As we see above if we use the Poisson kernel and Cauchy kernel for an analytic function on the circle the result is the same because they both have the same Fourier coefficients for positive integers, but they behave differently for negative integers. The Fourier coefficients of P_r are symmetric about zero on the integers, but the Fourier coefficients of C_r vanish on the negative integers.

2.5.2 Boundary Values:

The main problem in this part is: If we have a given continuous function f on the unit circle C how to find a harmonic function u in open unit disc Δ whose boundary values are f ?

Theorem 2.9. *[4] If $f \in L^1(C)$ then the Poisson integral of f is a harmonic function in Δ.*

Theorem 2.10. *[4] Let f be a continuous function on C and define (Hf) on the closed unit disc $\overline{\Delta}$ by*

$$(Hf)(re^{i\theta}) = \begin{cases} f(e^{i\theta}), & \text{if } r = 1; \\ P(re^{i\theta}), & \text{if } 0 \leq r < 1 \end{cases}$$

then (Hf) is a continuous function on the closed unit disc $\overline{\Delta}$.

Proof. Let $\|g\|_C$ denote the supremum of $|g|$ on the set C, and $P[g]$ denote the Poisson integral of g. Recall that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt = 1, 0 \leq r < 1$$

Since $P_r(t) > 0$, then for every continuous g in C

$$\left|P\left[g\right]\left(re^{i\theta}\right)\right| \leq \|g\|_C$$

So

$$\|Hg\|_{\overline{\Delta}} = \|g\|_C$$

If

$$g(e^{i\theta}) = \sum_{n=-N}^{N} c_n e^{in\theta}$$

is any trigonometric polynomial, it follows that

$$(Hg)(re^{i\theta}) = \sum_{n=-N}^{N} c_n r^{|n|} e^{in\theta}$$

so (Hg) is a continuous function in $\overline{\Delta}$. Finally we can conclude that there are trigonometric polynomials g_k such that $\lim_{k\to\infty} \|g_k - f\|_C = 0$. It follows that

$$\|Hg_k - Hf\|_{\overline{\Delta}} = \|H(g_k - f)\|_{\overline{\Delta}} \mapsto \ 0.$$

So the functions Hg_k are continuous functions in $\overline{\Delta}$ and they converge uniformly on $\overline{\Delta}$. $\square$

Theorem 2.11. *[7] Suppose u is a continuous real valued function on the closed unit disc $\overline{\Delta}$ and suppose u is harmonic in Δ. Then u is the Poisson integral of its restriction to C and u is the real part of the analytic function,*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})dt, z \in \Delta \tag{2.15}$$

Proof. If $u_1 = Re\,(f)$ then (2.15) shows that u_1 is Poisson integral of the boundary values of u and the only thing we need to prove is: $u = u_1$. Put $h = u - u_1$. Then h is continuous on $\overline{\Delta}$ and by the previous theorem h is harmonic in Δ and $h = 0$ at all points of C. Assume that $h(z_0) > 0$ for some $z_0 \in \Delta$. Fix ϵ, s.t. $0 > \epsilon > h(z_0)$ and define

$$g(z) = h(z) + \epsilon|z|^2 \tag{2.16}$$

Then $g(z_0) \geq h(z_0) > \epsilon$. Since g is a continuous function in $\overline{\Delta}$ and since $g = \epsilon$ at all points of C there exists a point $z_1 \in \Delta$ at which g has local maximum. This implies that $g_{xx} \leq 0$ and $g_{yy} \leq 0$ at z_1. But (2.16) shows that the Laplacian of g is $4\epsilon > 0$ which is a contradiction. So $u - u_1 \leq 0$ The same argument shows that $u_1 - u \leq 0$. Hence $u_1 = u$ and we are done.

Although theorem 2.11 considered only the unit disc, this theorem can be carried over to arbitrary circular disc,by changing variables. i.e.: If u is a continuous real function on the boundary of a disc with radius R and center a then u is defined in that disc by the Poisson integral

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rrcos(\theta - t) + r^2} u(a + Re^{i\theta})dt.$$

Then u is continuous on the closed disc and harmonic in the open disc. If u is harmonic in an open set U and if $\overline{\Delta}(a; R) \subset U$ then u satisfies the condition above and there is a holomorphic function f defined in $\Delta(a; R)$ whose real part is u. So we can say that every real harmonic function is the real part of a holomorphic function. [7]

Now, let consider the Dirichlet problem by Hoffman's approach [4] before beginning the discussion of the boundary behaviour of harmonic functions and the description of H^p spaces. Suppose we have a given real valued continuous function f on the unit circle. The Dirichlet problem consists in finding a function u which is continuous on closed disc, such that u satisfies the following conditions:

1. u is harmonic in the open disc
2. u agrees with f on the circle.

Recall the Poisson kernel:

$$P_r(\theta) = \frac{1 - r^2}{1 - 2rcos(\theta) + r^2}$$

The Poisson integral formula satisfies the following conditions:

- $P_r \geq 0$ (and P_r is continuous on the circle)

- $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$, $0 \leq r < 1$.
- if $0 < \gamma < \pi$ then

$$lim_{r\to 1}sup|P_r(\theta)| \ \ for \ \ |\theta| \geq \gamma$$

For if $\gamma \leq |\theta| \leq \pi$ then:

$$P_r(\theta) \leq \frac{1-r^2}{1-2rcos(\gamma)+r^2}$$

Then to solve the Dirichlet problem the only thing we need to do is, to show that the family of functions P_r, $0 \leq r < 1$ is approximate identity for L^1 of the circle. □

Theorem 2.12. *[7] Let f be a complex valued function in L^p of the unit circle where $1 \leq p < \infty$. Define f in the unit disc by;*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)dt.$$

then the extended function f is harmonic in the open unit disc.

Proof. If the original f is real valued then the function $f(re^{i\theta})$ is the real part of the analytic function

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{e^{it}+z}{e^{it}-z}dt.$$

The harmonic function $f(re^{i\theta})$ is the Poisson integral of the corresponding function on the circle.

Note that it is easy to show that the family of functions P_r, $0 \leq r < 1$ is an approximate identity for L^1 of the circle. So we can conclude that if f is a complex valued function in L^p of the unit circle where $1 \leq p < \infty$ and if we define f in the unit disc by

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)dt,$$

as $r \mapsto 1$ the functions $f_r(\theta) = f(re^{i\theta})$ converge to f in the L^p norm. If f is continuous on the unit circle, then f_r converge uniformly to f, so the extended f is continuous on the closed disc, harmonic in the interior. □

Suppose we have a given harmonic function in the disc. How can we decide if it is the Poisson integral of some type of function? We can solve the problem by using the same method as we did for Cesàro mean.

Theorem 2.13. *[4] Let f be a complex valued harmonic function in the open unit disc and*

$$f_r(\theta) = f(re^{i\theta})$$

1. *If $1 < p \leq \infty$ then f is the Poisson integral of an L^p function on the unit circle if and only if the functions f_r are bounded in L^p norm. (if $p = \infty$ this is called Fatou's Theorem. see for details [4])*
2. *f is the Poisson integral of an integrable function on the circle if and only if the f_r converge in the L^1 norm.*
3. *f is the Poisson integral of a continuous function on the unit circle if and only if f_r converge uniformly.*
4. *f is the Poisson integral of a finite complex Borel measure on the circle if and only if the f_r are bounded in L^1 norm.*
5. *f is the Poisson integral of a finite positive Borel measure if and only if f is nonnegative.*

2.6 H^P SPACES

Before starting the discussion about the space H^p we need a corollary of the theorems from the previous section.

Corollary 2.2. *[4] Let f be a complex valued harmonic function in the unit disc and suppose that the integrals*

$$\int_{-\pi}^{\pi} \left| f(re^{i\theta}) \right|^p d\theta$$

are bounded as $r \to 1$ for some p ,$1 \leq p < \infty$. Then for almost every θ the radial limits

$$\tilde{f}(\theta) = \lim_{r \to 1} f(re^{i\theta})$$

exist and define a function $\tilde{f}$ in L^p. If $p > 1$ then f is the Poisson integral of $\tilde{f}$. If $p = 1$ then f is the Poisson integral of a finite measure whose absolutely continuous part is $\frac{1}{2\pi}\tilde{f}d\theta$. If f is bounded harmonic function the boundary values exist almost everywhere and define a bounded measurable function $\tilde{f}$ whose Poisson integral is f.

Suppose $1 \leq p \leq \infty$ and let f be a harmonic function in the open disc s.t. the function $f_r(\theta) = f(re^{i\theta})$ is bounded in L^p norm. This class of functions forms a Banach space under the norm

$$\|f\| = \lim_{r \to 1} \|f_r\|_p$$

For $1 < p \leq \infty$ this Banach space is isomorphic to L^p of the unit circle. The isomorphism is $f \mapsto \tilde{f}$ where $\tilde{f}$ is the boundary function for f. If $1 < p < \infty$ we have not only

$$\left\|\tilde{f}\right\|_p = \lim_{r \to 1} \|f_r\|_p$$

but also

$$\lim_{r \to 1} \left\|\tilde{f} - f_r\right\|_p = 0$$

For $p = 1$ this Banach space is isomorphic to the space of finite Borel measures on the circle, the isomorphism is: $f \mapsto \mu$ where f is the Poisson integral of μ.

The results about harmonic functions apply in particular to analytic functions.

$$f_r(e^{i\theta}) = f(re^{i\theta}), 0 \leq r < 1$$

$$\|f_r\|_p = \left\{\int_C |f_r|^p d\sigma\right\}^{\frac{1}{p}}, 0 < p < \infty$$

and

$$\|f_r\|_\infty = sup\left|f(re^{i\theta})\right|$$

where σ is normalized Lebesgue measure on C, so $\sigma(C) = 1$.

If $0 < p \leq \infty$ the space H^p is denotes by the class of analytic functions f in the unit disc for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in L^p norm as $r \to 1$. If $1 \leq p \leq \infty$ then H^p is a Banach space under the norm

$$\|f\| = \lim_{r \to 1} \|f_r\|_p;$$

so H^p is a closed subspace of the corresponding space of harmonic functions. If $1 < p \leq \infty$ then H^p can be identified with a closed subspace of L^p of the circle because of the isomorphism. It consists of all functions f in L^p whose Poisson integrals are analytic on the disc, i.e.

$$\int_{-\pi}^{\pi} f(\theta)e^{in\theta}d\theta = 0, n = 1, 2, 3$$

When $p = 1$ then we can identify H^1 with the closed sub-space of finite measure μ on the circle which are analytic:

$$\int_{-\pi}^{\pi} e^{in\theta}d\mu(\theta) = 0, n = 1, 2, 3, \cdots$$

Remarks[4]

- $\|f_r\|_p$ is a nondecreasing function of r for every f when $p < \infty$.

- For $1 \leq p \leq \infty$, $\|f\|_p$ satisfies the triangle inequality so H^p is a normed linear space. By the Minkowski inequality:

$$\|(f+g)_r\|_p = \|f_r + g_r\|_p \leq \|f_r\|_p + \|g_r\|_p$$

  If $0 < r < 1$ and as $r \to 1$ then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

- As we mentioned before H^p is Banach space if $1 \leq p \leq \infty$. We suppose $\{f_n\}$ is a Cauchy sequence in H^p , $|z| \leq r < R < 1$ and apply the Cauchy formula to $f_n - f_m$ by integrating around the circle of radius R center 0 then we obtain:

$$(R-r)|f_n(z) - f_m(z)| \leq \|(f_n - f_m)_R\|_1 \leq \|(f_n - f_m)_R\|_p \leq \|(f_n - f_m)\|_p$$

  Then $\{f_n\}$ converges uniformly on compact subsets of Δ to a function $f \in H(\Delta)$. Given $\epsilon > 0$ there is an m, s.t. $\|(f_n - f_m)\|_p < \epsilon$ for all $n > m$ and then for every $r < 1$

$$\|(f - f_m)_r\|_p = \lim_{n\to\infty} \|(f_n - f_m)_r\|_p \leq \epsilon$$

  which gives us $\|(f - f_m)\|_p \mapsto 0$ as $m \mapsto \infty$

- For $p < 1$, H^p is still a vector space but the triangle inequality is no longer satisfied by $\|f\|_p$. In this case H^p is an F-space.

2.7 H^1 SPACES AND THE $H^1 - H^2$ FACTORIZATION THEOREM:

Let A denote the collection of functions which are continuous on the closed unit disc and analytic at each interior point. Then A is a Banach space under the sup norm:

$$\|f\|_\infty = \sup_{|z|\leq 1} |f(z)|$$

Each f in A is the Poisson integral of its boundary values

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})P_r(\theta - t)dt$$

and

$$\|f\|_\infty = \underbrace{sup}_{0<r<1} |f_r(e^{i\theta})|$$

by the maximum modulus principle for analytic functions. It is easy to see that there is an isomorphism between A and and the Banach space of continuous functions on the circle so we can identify the functions in A with their boundary values, where the boundary value function such that:

$$\int_{-\pi}^{\pi} f(\theta)e^{in\theta}d\theta = 0, \ \ \forall \ \ n = 1, 2, 3, \cdots$$

If f is continuous on the circle and if the Fourier coefficients of f vanish on the negative integers, then the Cesàro means of the Fourier series for f contains a sequence of trigonometric polynomials of the form:

$$P(\theta) = \sum_{k=0}^{n} a_k e^{ik\theta}$$

which converge uniformly to f. Then we can identify A as an algebra on the disc consisting of all functions which are uniformly approximable by polynomials in z:

$$P(z) = \sum_{k=0}^{n} a_k z^k$$

so by this property of A we obtain Fejer's theorem in the following form:

Theorem 2.14. *[4] The real part of functions in A are uniformly dense in the space of real valued continuous functions on the unit circle. In other words if μ is a finite real Borel measure on the circle s.t. $\int f d\mu = 0$ for every f in A, then μ is the zero measure.*

2.7.1 Some Important Facts about H^1 Functions:

We have defined the space H^p as the class of analytic functions f in the open unit disc for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in L^p norm. For $1 < p \leq \infty$ since the functions in H^p are analytic we can identify H^p with the space of L^p functions on the circle.

For $p = 1$, the functions in H^1 are harmonic. This fact leads us to identify f with the finite measure μ on the circle where f is Poisson integral of μ. Recall that if f is analytic, then μ is analytic i.e.

$$\int e^{in\theta} d\mu(\theta) = 0, \quad n = 1, 2, 3, \cdots$$

The theorem of F. and M.Riesz tells us:

$$d\mu = \frac{1}{2\pi} \tilde{f} d\theta$$

where $\tilde{f}$ is in L^1 and f is the Poisson integral of $\tilde{f}$. The functions f_r converge to $\tilde{f}$ in L^1 norm and for almost every θ

$$\tilde{f}(\theta) = \lim_{r \to 1^-} f(re^{i\theta})$$

Then we define H^1 as the space of L^1 functions on the circle which are analytic.

One of the important theorems about H^1 functions is $H^1 - H^2$ factorization theorem that we are going to use later to obtain 'Hardy Type Inequalities'. Before the $H^1 - H^2$ factorization theorem we need to go over some basic properties of H^1 functions. Let A_0 denote the set of functions f in A for which $\displaystyle\int f d\theta = 0$. If μ is a positive measure, then we define the square root of the distance between 1 and A_0 by:

$$\inf_{f \in A_0} \int |1 - f^2|^2 d\mu = \int |1 - F|^2 d\mu$$

where F is the orthogonal projection of 1 into the closed subspace of $L^2(d\mu)$, which is spanned by the function A_0.

Theorem 2.15. *[4] Let μ be a finite positive Borel measure on the circle and suppose* 1 *is not in the closed subspace of $L^2(d\mu)$ which is spanned by the functions in A_0=the set of functions $f \in A$ for which $\int f d\theta = 0$. Let F be the orthogonal projection of* 1 *into that closed subspace.*

1. *The measure* $|1-F|^2 d\mu$ *is a non-zero constant multiple of Lebesgue measure. In particular, Lebesgue measure is absolutely continuous with respect to* μ.
2. *The function* $(1-F)^{-1}$ *is in* H^2.
3. *If* h *is the derivative of* μ *with respect to normalized Lebesgue measure, then the function* $(1-F)h$ *is in* L^2*, where*
$$L^2 = L^2\left(\frac{1}{2\pi}d\theta\right)$$

Theorem 2.16. *[4] Let* f *be any function in* H^1 *s.t.*

$$f(0) = \frac{1}{2\pi}\int f(\theta)d\theta \neq 0$$

then $log|f(\theta)|$ *is Lebesgue integrable and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} log|f(\theta)|d\theta \geq log|f(0)|$$

Theorem 2.17. *[4] (*$H^1 - H^2$ ***Factorization Theorem:****) Every function in* H^1 *is the product of two functions in* H^2.

We will give a nice proof for this theorem later on this chapter.

2.8 FACTORIZATION OF H^P FUNCTIONS

2.8.1 Inner and Outer Functions

Definition 2.15. ***(Inner Function)****[4]*

An inner function is an analytic function $g \in H^\infty$ *for which* $|g(e^{i\theta})| = 1$ *almost everywhere on the unit circle. Note that* $\|g\|_\infty = 1$.

Definition 2.16. ***(Outer Function)****[4] Let* F *be an analytic function, on the unit disc.* F *is called an outer function if there exists a positive measurable function* ϕ *on the unit circle such that* $log\varphi \in L^1(C)$ *and*

$$F(z) = \alpha\ exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}log\varphi(e^{i\theta})d\theta\right]$$

for $z \in \Delta$ *and* $|\alpha| = 1$.

The outer function F is in H^1 if and only if $\phi(\theta) = e^{log\varphi(\theta)}$ is integrable. If F is an outer function in H^1 then

$$\varphi(e^{i\theta}) = |F(e^{i\theta})| \ \ a.e.$$

Note that if f is a nonzero function of the class H^1 on the unit disc then f has nontangential limits at almost every point of the unit circle:

$$f(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z)$$

and

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t) dt.$$

Also $log|f(e^{it})|$ is also Lebesgue integrable. Let

$$F(z) = exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} log|f(e^{i\theta})| d\theta\right]$$

and u is Poisson integral of $log|f|$ then $|F| = e^u$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta$$

So F is in H^1 because it is bounded. Define a function

$$g(z) = \frac{f(z)}{F(z)}$$

. We know that

$$log|F(re^{i\theta})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} log|f(e^{it})| P_r(\theta - t) dt$$

so $|F(z)| \geq |f(z)|$ for each z in the open disc. It is easy to see by Jensen's inequality:

$$log|f(re^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} log|f(e^{it})| P_r(\theta - t) dt = log|F(re^{i\theta})|$$

then we obtain:

$$|g(z)| = \left|\frac{f(z)}{F(z)}\right| \leq 1$$

Theorem 2.18. *[4] Let* f *be a nonzero function in* H^1*. Then we can write* f *in the form:* $f = gF$ *where* g *is an inner function and* F *is an outer function. This factorization is unique* *up to constants of modulus 1 and the outer function F is in H^1*

Proof. We know that if

$$F(z) = exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}log|f(e^{i\theta})|d\theta\right]$$

then F is an outer function in H^1 and $\frac{f}{F} = g$ is an inner function. If we have $f = g_1 f_1$ with g_1 inner and F_1 outer then $|F| = |F_1|$ on the boundary.

It is clear that $F = \alpha F_1$ for some number α with $|\alpha| = 1$. So $\alpha g F_1 = g_1 F_1$ and $g_1 = \alpha g$. □

2.8.2 Blaschke Products

Theorem 2.19. *[4] Let $\{\alpha_n\}$ be a sequence in Δ such that every $\alpha_n \neq 0$ and each α_n has multiplicity $p_n \in \mathbb{N}$ and*

$$\sum_{n=1}^{\infty}(1-|\alpha_n|) < \infty \tag{2.17}$$

If k is a nonnegative integer and if

$$B(z) = z^k \prod_{n=1}^{\infty}\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}, z \in \Delta \tag{2.18}$$

then $B \in H^\infty$ and B has no zeros except at the points α_n, and the origin if $k > 0$.

Definition 2.17. *[4] We call the function (2.18) a Blaschke Product. Note that each factor in (2.18) has absolute value 1. This product converges uniformly on compact sets and the only zeros of B are a zero of order k at the origin and a zero of order p_n at α_n. A Blaschke product is an analytic function. Note that whenever a Blaschke product converges uniformly on compact subset of Δ,*

$$\sum_{n=1}^{\infty}(1-|\alpha_n|) < \infty.$$

Theorem 2.20. *[4] If B is is a Blaschke product, then*

$$|\lim_{r\to 1} B(re^{i\theta})| = 1$$

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} log|B(re^{i\theta})|d\theta = 0 \tag{2.19}$$

Proof. Since the integral is a monotonic function of r then the limit exists. Suppose $B(z)$ is a Blaschke product then

$$B_N(z) = \prod_{n=N}^{\infty} \frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}$$

Since $log(|\frac{B}{B_N}|)$ is continuous in a open set then the limit (2.19) is not changed when we replace B by B_N. So we obtain

$$\log|B_N(0)| \leq \lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|B(re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\lim_{r\to 1}|B(re^{i\theta})|d\theta \leq 0$$

As $N\to\infty$ the first term of the previous inequality tends to 0. Then

$$\int \log\lim_{r\to 1}|B(re^{i\theta})| = 0$$

□

Now we can easily prove the following theorem

Theorem 2.21. *[4] Let f be a bounded analytic function in the unit disc and suppose $f(0)\neq 0$. If $\{\alpha_n\}$ is the sequence of zeros of f in the open disc each repeated as often as the multiplicity of the zeros of f ordered accordingly to their multiplicities, then the product $\prod_n |\alpha_n|$ is convergent i.e.*

$$\sum_{n=1}^{\infty}(1-|a_n|) < \infty$$

Proof. Suppose $|f|\leq 1$. If f has only a finite number of zeros, it is obvious that the product is convergent. Otherwise f has a countable number of zeros: $\alpha_1, \alpha_2, \alpha_3, \cdots$. Let $B_n(z)$ be the finite product

$$B_n(z) = \prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}$$

Now $B_n(z)$ is a rational function, analytic in the closed unit disc and $|B_n(e^{i\theta})| = 1$, Since each of the function

$$\frac{z-a_k}{1-\overline{a_k}z}$$

is modulus one on the unit circle. And since

$$\frac{|f(e^{i\theta})|}{|B_n(e^{i\theta})|} = |f(e^{i\theta})| \leq 1$$

then we have $|f(z)| \leq |B_n(z)|$ which says

$$\frac{f}{B_n}$$

is a bounded analytic function in the disc. In particular

$$0 < |f(0)| \leq |B_n(0)| = \prod_{k=1}^{n} |a_k|$$

Since $|a_k| < 1$ for each k and since each of the partial products $\prod_{k=1}^{n} |a_k|$ is not less than $|f(0)|$ then the infinite products converges. □

Theorem 2.22. *[4]*

Let f be a non-zero bounded analytic function in the unit disc. Then f is uniquely expressible in the form $f = Bg$ where B is Blaschke product and g is a bounded analytic function without zeros.

Proof. Since $f \neq 0$ we can write $f(z) = z^p h(z)$ where $h(0) \neq 0$. Let B be the product of z^p and the Blaschke product formed from the zeros of h. Then $g = \frac{f}{B}$ is analytic and bounded in the disc. The factorization $f = Bg$ is unique since a Blaschke product is uniquely determined by its zeros.

Let f above be an inner function. Then $f = Bg$ where B is Blaschke product and g is inner function without zeros. □

Theorem 2.23. *[4]*

Suppose $0 < p < \infty$, $f \in H^p$, $f \neq 0$ and B is the Blaschke product formed with the zeros of f. Then there is a zero-free function $h \in H^2$ such that

$$f = B(h)^{\frac{2}{p}}$$

Proof. $f = gF$ where g is an inner and F is an outer (ans so zero-free). Without loss of generality $f = g$, a bounded function. By theorem 2.22 $\frac{f}{B} \in H^p$ in fact

$$\left\| \frac{f}{B} \right\|_p = \|f\|_p.$$

Since $\frac{f}{B}$ has no zero in Δ then there exists

$$u \in H(\Delta)$$

such that $exp(u) = \frac{f}{B}$. Put,

$$h = exp(\frac{pu}{2})$$

then $h \in H(\Delta)$ and

$$|h|^2 = \left| \frac{f}{B} \right|^p$$

hence $h \in H^2$ which proves the theorem.

Note that $\|h\|_2{}^2 = \|f\|_p{}^p$ in the previous theorem.Let $p = 1$. If we write

$$f = B(h)^{\frac{2}{p}}$$

in the form:

$$f = (Bh)h$$

above then we will obtain

$$f = gh$$

where g and h are both in H^2. Which gives us a proof of $H^1 - H^2$ factorization theorem.(see the next section). $\square$

2.9 ABSOLUTE CONVERGE OF TAYLOR SERIES

This last subsection contains one of the important theorems for the H^1 functions. Recall that the Riemann-Lebesgue Lemma says that the Fourier coefficients of an integrable function tend to zero. If we have an H^1 function we can say something else.

Theorem 2.24. *(HARDY):[4] Let f be a function in H^1 with the power series:*

$$\sum_{n=0}^{\infty} a_n z^n.$$

Then:

$$\sum_{n=1}^{\infty} \frac{1}{n}|a_n| \leq \pi \|f\|_1. \tag{2.20}$$

Proof. First suppose that $a_n \geq 0$, $n = 0, 1, 2, \cdots$ Then

$$Imf(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta)$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} (\pi - \theta)\sin(n\theta)d\theta = \frac{1}{n} \tag{2.21}$$

then we obtain:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} a_n r^n &= \frac{1}{2\pi}\int_0^{2\pi} (\pi - \theta)Imf(re^{i\theta})d\theta \\
&\leq \frac{1}{2}\int_0^{2\pi} |f(re^{i\theta}|d\theta \\
&= \pi \|f\|_1
\end{aligned} \tag{2.22}$$

Let r tend to 1 and we are done. Recall the $H^1 - H^2$ factorization theorem: For the general f write $f = gh$ where g and h are both H^2 functions. Define g and h by,

$$g(z) = B\left(\frac{f}{B}\right)^{\frac{1}{2}} \qquad h = \left(\frac{f}{B}\right)^{\frac{1}{2}}$$

and B is the Blaschke product of the zeros of f. If

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

$$h(z) = \sum_{n=0}^{\infty} c_n z^n$$

then by the Riész-Fisher theorem, the functions

$$G(z) = \sum_{n=0}^{\infty} |b_n|\, z^n$$

$$H(z) = \sum_{n=0}^{\infty} |c_n|\, z^n$$

are also in H^2 and

$$\|G\|_2 = \|g\|_2 \; and \; \|H\|_2 = \|h\|_2$$

Let $F = GH$; Then $F \in H^1$ and

$$F(z) = \sum_{n=0}^{\infty} \tilde{a}_n z^n$$

where $\tilde{a}_n \geq 0$. It is also obvious that $|a_n| \leq \tilde{a}_n$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n}|a_n| \leq \sum_{n=1}^{\infty} \frac{1}{n}\tilde{a}_n \leq \pi\|F\|_1$$

So

$$\|F\|_1 \leq \|G\|_2\|H\|_2 = \|g\|_2\|h\|_2 = \|f\|_1$$

□

3.0 DETAILED PROOF FOR HARDY'S INEQUALITY AND DEVELOPING HARDY-TYPE INEQUALITIES

This chapter starts with a more detailed proof for the Hardy's Inequality based on Hoffman's Proof [4]. To develop a variation on that proof we use the weight function $w_1(e^{i\theta}) = \pi - \theta$ and we extend w_1 to $w_1 : \mathbb{R} \to \mathbb{R}$ by $2 - \pi$ periodicity instaed of using the function $\sin(n\theta)$ in equation (2.21). Then by using the same method we develop new Hardy-like inequalities. Finally we demonstrate the following,

Let f be any arbitrary H^1 function i.e.

$$f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, \forall z \in \overline{\Delta} \right]$$

where a_n is $n.th$ Fourier coefficient of f. Let $1 \leq j \leq \upsilon - 1$ where $\upsilon \geq 2$. Then

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+j}|}{\upsilon k + j} \leq \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \frac{1}{\upsilon} \|f\|_{H^1};$$

which is the crucial result of our new approach.

3.1 DETAILED PROOF FOR HARDY'S INEQUALITY BY USING NEW APPROACH

Theorem 3.1. *(Hardy) Let $\overline{\Delta}$ be the closed unit disc, then $\forall f \in H^1$ s.t.*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \overline{\Delta}$$

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \le \pi \|f\|_{H^1}.$$

Proof. Let $w_1(e^{i\theta}) = \pi - \theta \;, \quad \forall \theta \in [0, 2\pi).$ We extend w_1 to $w_1 : \mathbb{R} \mapsto \mathbb{R}$ by 2π periodicity. Then the Fourier coefficients of w_1 are:

$$\hat{w}_1(n) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} w_1(e^{i\theta}) e^{-in\theta} d\theta \tag{3.1}$$

$\forall n \in \mathbb{Z}$ and, $\hat{w}_1(0) = 0$. Fix $n \in \mathbb{Z} - \{0\}$

$$\begin{aligned}
\hat{w}_1(n) &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} (\pi - \theta) e^{-in\theta} d\theta \\
&= \frac{1}{2\pi} \left[\left[(\pi - \theta) \frac{e^{-in\theta}}{-in} \right]_{\theta=0}^{2\pi} - \int_{\theta=0}^{2\pi} (-1) \frac{e^{-in\theta}}{-in} d\theta \right] \\
&= \frac{1}{in}
\end{aligned} \tag{3.2}$$

Then

$$w_1(z) = \sum_{n \in \mathbb{Z} - \{0\}} \frac{1}{in} z^n \;\; ; \; \forall \; z = e^{i\theta}$$

Consider $f \in H^1$ with coefficient sequence (a_n). Assume every $a_n \ge 0$. Then

$$\begin{aligned}
\frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(e^{i\theta}) w_1(e^{i\theta}) d\theta &= \sum_{n=0}^{\infty} a_n \underbrace{\frac{1}{2\pi} \int_{\theta=0}^{2\pi} e^{in\theta} w_1(e^{i\theta}) d\theta}_{\hat{w}_1(-n)} \\
&= \sum_{n=1}^{\infty} a_n \frac{1}{i(-n)} \\
&= i \sum_{n=1}^{\infty} \frac{a_n}{n}.
\end{aligned} \tag{3.3}$$

Some details ommited here. See previous section (2.9). Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n}{n} &= \left| i \sum_{n=1}^{\infty} \frac{a_n}{n} \right| \\
&= \left| \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(e^{i\theta}) w_1(e^{i\theta}) d\theta \right| \\
&\le \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left| f(e^{i\theta}) \right| \left| w_1(e^{i\theta}) \right| d\theta
\end{aligned} \tag{3.4}$$

Since $|w_1(e^{i\theta})| \leq \pi$ for $\theta \in [0, 2\pi)$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_n}{n} &\leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left| f(e^{i\theta}) \right| \pi d\theta \\ &= \pi \|f\|_{H^1}\end{aligned} \tag{3.5}$$

Next using the $H^1 - H^2$ factorization theorem using the same method that we used in Theorem 2.24 in chapter 2 we can show that $\forall f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, \; \forall \; z \in \overline{\Delta} \right]$ and $a_n \in C$ we have

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \pi \|f\|_{H^1}$$

$\square$

Corollary 3.1. *Let $f \in H^1$.*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_{H^1}$$

Proof. Let $f \in H^1(\Delta)$ $\forall z \in \Delta$ s.t.

$$f(z) = \sum_{z=0}^{\infty} a_n z^n$$

and

$$g(z) = z f(z) \forall z \in \Delta$$

then

$$g(z) = \sum_{n=0}^{\infty} a_n z^{n+1} = a_0 z + \sum_{n=1}^{\infty} a_n z^{n+1}$$

Let $a_{k-1} = b_k$ then

$$g(z) = a_0 z + \sum_{k=2}^{\infty} a_{k-1} z^k = \sum_{k=1}^{\infty} a_{k-1} z^k = \sum_{k=1}^{\infty} b_k z^k$$

if $g = z f(z)$ then $g \in H^1$ where

$$\|g\|_{H^1(T)} = \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} \left| e^{i\theta} f(e^{i\theta}) \right| d\theta$$

from the Hardy's Inquality:

$$\sum_{k=1}^{\infty} \frac{|b_k|}{k} \leq \pi \|g\|_{H^1(\mathbb{T})} = \pi \|f\|_{H^1(\Delta)} \Longrightarrow \sum_{k=1}^{\infty} \frac{|b_k|}{k} = \sum_{n=0}^{\infty} \frac{|a_{k-1}|}{k} = \sum_{k=0}^{\infty} \frac{|a_k|}{k+1} \leq \pi \|f\|_{H^1}.$$

□

3.2 HARDY-LIKE INEQUALITIES:

Lemma 3.1. *Let $\mathbb{T}$ be the unit circle and $u \in L^1[0, 2\pi] \cong L^1(\mathbb{T})$ with normalized Haar measure. Extend u to a 2π- periodic function on $\mathbb{R} \mapsto \mathbb{C}$. Fix $\alpha \in \mathbb{R}$ and let $\upsilon(e^{i\theta}) = u(e^{i(\theta-\alpha)})$ $\forall \theta \in \mathbb{R}$, then*

$$\hat{\upsilon}(n) = e^{-in\alpha} \hat{u}(n)$$

where $\hat{u}(n)$ represents the Fourier coefficients of u.

Proof. We know that for every $n \in \mathbb{Z}$

$$\begin{aligned} \hat{\upsilon}(n) &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u\left(e^{i(\theta-\alpha)}\right) e^{-in\theta} d\theta \\ &= \frac{1}{2\pi} \int_{\beta=-\alpha}^{\beta=2\pi-\alpha} u(e^{i\beta}) e^{-in(\beta+\alpha)} d\beta \\ &= e^{-in\alpha} \frac{1}{2\pi} \int_{\beta=0}^{\beta=2\pi} u\left(e^{i\beta}\right) e^{-in\beta} d\beta \\ &= e^{-in\alpha} \hat{u}(n). \end{aligned} \tag{3.6}$$

□

Hardy Type Inequalities

Theorem 3.2. *Let $f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, \ \forall z \in \overline{\Delta} \right]$ and assume each $a_n \geq 0$ then*

1. $$\sum_{k=0}^{\infty} \frac{|a_{3k+1}|}{3k+1} \leq \frac{\pi 2\sqrt{3}}{9} \|f\|_{H^1}$$

2. $$\sum_{k=0}^{\infty} \frac{|a_{3k+2}|}{3k+2} \leq \frac{\pi 2\sqrt{3}}{9} \|f\|_{H^1}$$

3.
$$\sum_{k=0}^{\infty} \frac{|a_{3k+3}|}{3k+3} \leq \frac{\pi}{3} \|f\|_{H^1}$$

Proof. 1. Let $w_1(e^{i\theta}) = \pi - \theta,\ \forall\theta \in [0, 2\pi)$. Define the functions:

$$\mu_1(e^{i\theta}) = w_1(e^{i(\theta - \frac{2\pi}{3})})$$

$$\mu_2(e^{i\theta}) = w_1(e^{i(\theta - \frac{4\pi}{3})})$$

$$\mu_3(e^{i\theta}) = w_1(e^{i(\theta - \frac{6\pi}{3})})$$

and let $\varrho = e^{-\frac{2i\pi}{3}}$. Then by the Lemma 3.1

$$\begin{aligned} \hat{\mu_1}(n) &= e^{-in\frac{2\pi}{3}} \hat{w}_1(n) \\ &= (e^{-\frac{2i\pi}{3}})^n \hat{w}_1(n) \\ &= (e^{-\frac{2i\pi}{3}})^n \frac{1}{in} \\ &= \begin{cases} \frac{\varrho^n}{in}, & \text{if } n \in \mathbb{Z} - \{0\}; \\ 0 & \text{if } n = 0; \end{cases} \end{aligned} \tag{3.7}$$

because $\hat{w}_1(n) = \frac{1}{in}$ (see (3.2)).

Since $\hat{\mu_2}(n) = w_1(e^{i(\theta - \frac{4\pi}{3})})$ then:

$$\begin{aligned} \hat{\mu_2}(n) &= e^{-in\frac{4\pi}{3}} \hat{w}_1(n) \\ &= \begin{cases} \frac{\varrho^{2n}}{in}, & \text{if } n \in \mathbb{Z} - \{0\}; \\ 0 & \text{if } n = 0; \end{cases} \end{aligned} \tag{3.8}$$

and since $\hat{\mu_3}(n) = w_1(e^{i(\theta - \frac{6\pi}{3})})$ then:

$$\begin{aligned} \hat{\mu_3}(n) &= e^{-in\frac{6\pi}{3}} \hat{w}_1(n) \\ &= \begin{cases} \frac{\varrho^{3n}}{in}, & \text{if } n \in \mathbb{Z} - \{0\}; \\ 0 & \text{if } n = 0; \end{cases} \\ &= \begin{cases} \frac{1}{in}, & \text{if } n \in \mathbb{Z} - \{0\}; \\ 0 & \text{if } n = 0; \end{cases} \end{aligned} \tag{3.9}$$

Also notice that $\varrho^2 = \overline{\varrho}^1$. Construct a set $\mathbb{A} = \{\varrho^2, \varrho^1, \varrho^0\}$. We define the new function $w_{3\mathbb{N}_0+1}(z)$ as follows:

$$w_{3\mathbb{N}_0+1}(z) = \overline{\varrho}^1\mu_1(z) + \varrho^1\mu_2(z) + \varrho^0\mu_3(z)$$

Then

$$\begin{aligned} w_{3\mathbb{N}_0+1}(z) &= \overline{\varrho}^1\mu_1(z) + \varrho^1\mu_2(z) + \varrho^0\mu_3(z) \\ &= \sum_{n\in\mathbb{Z}-\{0\}} (\overline{\varrho}^1\hat{\mu_1}(n) + \varrho^1\hat{\mu_2}(n) + 1\hat{\mu_3}(n))z^n \\ &= \sum_{n\in\mathbb{Z}-\{0\}} (\overline{\varrho}^1\frac{\varrho^n}{in} + \varrho^1\frac{(\varrho^2)^n}{in} + \frac{1}{in})z^n \\ &= \sum_{n\in\mathbb{Z}-\{0\}} (\frac{\varrho^{n-1}}{in} + \frac{\varrho^{2n+1}}{in} + \frac{1}{in})z^n \end{aligned} \tag{3.10}$$

Since

$$\begin{aligned} \frac{\varrho^{n-1}}{in} + \frac{\varrho^{2n+1}}{in} + \frac{1}{in} &= \frac{(e^{\frac{-2i\pi}{3}})^{n-1} + (e^{\frac{-2i\pi}{3}})^{2n+1} + 1}{in} \\ &= \begin{cases} 0, & \text{if } n = 3k; \\ \frac{3}{i(3k+1)}, & \text{if } n = 3k+1; \\ 0, & \text{if } n = 3k+2; \end{cases} \end{aligned} \tag{3.11}$$

So

$$w_{3\mathbb{N}_0+1}(z) = \sum_{k\in\mathbb{Z}} \frac{3}{i(3k+1)} z^{3k+1},$$

Let $\theta \in [0, \frac{2\pi}{3})$ then

$$\begin{aligned} w_{3\mathbb{N}_0+1}(e^{i\theta}) &= \overline{\varrho}\left(-\frac{\pi}{3} - \theta\right) + \varrho\left(\frac{\pi}{3} - \theta\right) + (\pi - \theta) \\ &= \pi\left(1 - \frac{\overline{\varrho}}{3} + \frac{\varrho}{3}\right) - \theta\,\underbrace{(\overline{\varrho} + \varrho + 1)}_{0} \\ &= \pi\left[1 - \frac{\sqrt{3}i}{3}\right] \end{aligned} \tag{3.12}$$

and

$$|w_{3\mathbb{N}_0+1}(e^{i\theta})| = \pi\left(\frac{2}{\sqrt{3}}\right) \tag{3.13}$$

It is easy to see that$w_{3\mathbb{N}_0+1}$ is constant and $|w_{3\mathbb{N}_0+1}(e^{i\theta})| = \pi\left(\frac{2}{\sqrt{3}}\right)$ for the intervals:

$\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3})$, and $\theta \in [\frac{4\pi}{3}, 2\pi)$. Let us have an arbitrary

$$f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n \ , \forall z \in \overline{\Delta} \right]$$

and assume each $a_n \geq 0$. Then

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{3\mathbb{N}_0+1}(e^{i\theta})} d\theta &= \sum_{n=0}^{\infty} a_n \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{w_{3\mathbb{N}_0+1}(e^{i\theta})} e^{in\theta} d\theta \\ &= \sum_{n=0}^{\infty} a_n \overline{\hat{w}_{3\mathbb{N}_0+1}(n)} \\ &= 3i \sum_{k=0}^{\infty} \frac{a_{3k+1}}{3k+1} \end{aligned} \tag{3.14}$$

Then

$$\begin{aligned} 3 \sum_{k=0}^{\infty} \frac{a_{3k+1}}{3k+1} &= \left| 3i \sum_{k=0}^{\infty} \frac{a_{3k+1}}{3k+1} \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{3\mathbb{N}_0+1}(e^{i\theta})} d\theta \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i\theta}) \right| \underbrace{\overline{\left| w_{3\mathbb{N}_0+1}(e^{i\theta}) \right|}}_{\pi \frac{2\sqrt{3}}{3}} d\theta \\ &= \frac{2\sqrt{3}}{3} \pi \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i\theta}) \right| d\theta}_{\|f\|_{H^1}} \\ &= \frac{2\sqrt{3}}{3} \pi \|f\|_{H^1} \end{aligned} \tag{3.15}$$

So we obtain:

$$\sum_{k=0}^{\infty} \frac{a_{3k+1}}{3k+1} \leq \frac{2\sqrt{3}}{9} \pi \|f\|_{H^1}.$$

For the general case $f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, \forall z \in \overline{\Delta} \right]$, $a_n \in \mathbb{C}$ By the $H^1 - H^2$ factorization theorem;

$$\exists g, h \in H^2 \ \ with \ \ \|f\|_{H^1} = \|g\|_{H^2}^2 = \|h\|_{H^2}^2$$

s.t.

$$f(z) = g(z)h(z), \forall z \in \overline{\Delta}$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \forall z \in \overline{\Delta}$$

and

$$h(z) = \sum_{n=0}^{\infty} c_n z^n, \forall z \in \overline{\Delta}$$

for some sequences $(b_n)_{n\geq 0}$ *and* $(c_n)_{n\geq 0} \in l^2(\mathbb{N}_0)$. Also

$$\|g\|_{H^2} = \left(\sum_{n=0}^{\infty} |b_n|^2\right)^{\frac{1}{2}}$$

$$\|h\|_{H^2} = \left(\sum_{n=0}^{\infty} |c_n|^2\right)^{\frac{1}{2}}$$

define $G, H \in H^2$ by

$$G(z) = \sum_{n=0}^{\infty} |b_n| \, z^n \quad , \forall z \in \overline{\Delta}$$

$$H(z) = \sum_{n=0}^{\infty} |c_n| \, z^n \quad , \forall z \in \overline{\Delta}$$

Note that

$$\|G\|_{H^2} = \|g\|_{H^2} = \|f\|_{H^1}^{\frac{1}{2}}$$

and

$$\|H\|_{H^2} = \|h\|_{H^2} = \|f\|_{H^1}^{\frac{1}{2}}.$$

Define $F(z) = G(z)H(z) \ \ , \forall \in \overline{\Delta}$. Then $F \in H^1$ and

$$\|F\|_{H^1} \leq \|G\|_{H^2}\|H\|_{H^2} = \|f\|_{H^1}$$

where

$$F(z) = \sum_{n=0}^{\infty} d_n z^n \quad , \forall z \in \overline{\Delta}$$

and

$$d_n = \sum_{j=0}^{n} |b_j| \, |c_{n-j}| \quad \forall n \in \mathbb{N}_0$$

So

$$\sum_{k=0}^{\infty} \frac{d_{3k+1}}{3k+1} \leq \frac{2\pi\sqrt{3}}{9}\|F\|_{H^1} \leq \frac{2\pi\sqrt{3}}{9}\|f\|_{H^1}$$

Moreover $\forall n \geq 0$

$$\begin{aligned}|a_n| &= \left|\sum_{j=0}^{n} b_n c_{n-j}\right| \\ &\leq \sum_{j=0}^{n} |b_j|\,|c_{n-j}| \\ &= d_n\end{aligned} \tag{3.16}$$

Thus

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{|a_{3k+1}|}{3k+1} &\leq \sum_{k=0}^{\infty} \frac{d_{3k+1}}{3k+1} \\ &\leq \frac{2\pi\sqrt{3}}{9}\|f\|_{H^1}\end{aligned} \tag{3.17}$$

2. To prove the second part of Theorem 3.2 the only thing we need to do is to define a function similarly by using the set $\mathbb{A}^2 = \{\overline{\varrho}^2, \varrho^2, \varrho^0\} = \{\varrho, \overline{\varrho}, \varrho^0\}$, and then we define a function:

$$w_{3\mathbb{N}_0+2}(z) = \varrho\mu_1(z) + \overline{\varrho}\mu_2(z) + \varrho^0\mu_3(z)$$

It is easy to see that:

$$w_{3\mathbb{N}_0+2}(z) = \sum_{n\in\mathbb{Z}-0} \frac{3}{i(3k+2)} z^{3k+2}$$

and then by following exactly the same steps as the previous case, we obtain that:

$$\sum_{k=0}^{\infty} \frac{|a_{3k+2}|}{3k+2} \leq \frac{\pi 2\sqrt{3}}{9}\|f\|_{H^1}.$$

3. To prove the third part of Theorem 3.2 we are going to use a slightly different method. Let $\theta \in \mathbb{R}$, and construct the function:

$$\begin{aligned}w_{3\mathbb{N}_0}(z) &= 1\mu_1(z) + 1\mu_2(z) + 1\mu_3(z) \\ &= \sum_{k\in\mathbb{Z}-0} \frac{3}{i(3k)} z^{3k}\end{aligned} \tag{3.18}$$

$$\begin{aligned}w_{3\mathbb{N}_0}(e^{i\theta}) &= w_1\left(e^{i\left(\theta-\frac{2\pi}{3}\right)}\right) + w_1\left(e^{i\left(\theta-\frac{4\pi}{3}\right)}\right) + w_1(e^{i\theta}) \\ &= 1\mu_1(e^{i\theta}) + 1\mu_2(e^{i\theta}) + 1\mu_3(e^{i\theta})\end{aligned} \tag{3.19}$$

Notice that:

$$w_{3\mathbb{N}_0}(e^{i\theta}) = \begin{cases} (\pi - 3\theta), & \text{if } \theta \in [0, \frac{2\pi}{3}); \\ (3\pi - 3\theta), & \text{if } \theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}); \\ (5\pi - 3\theta), & \text{if } \theta \in [\frac{4\pi}{3}, 2\pi); \end{cases} \tag{3.20}$$

So

$$\left| w_{3\mathbb{N}_0}(e^{i\theta}) \right| \leq \pi, \quad \forall \theta \in \mathbb{R}$$

Let $f \in H^1$ Assume each $a_n \geq 0$.

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{3\mathbb{N}_0}(e^{i\theta})} d\theta &= \sum_{n=1}^{\infty} a_n \overline{\hat{w}_{3\mathbb{N}_0}(n)} \\ &= \sum_{k \in \mathbb{N} - \{0\}} a_{3k} \overline{\left(\frac{3}{i(3k)} \right)} \\ &= 3i \sum_{k=1}^{\infty} \frac{a_{3k}}{3k} \\ &= 3i \sum_{k=0}^{\infty} \frac{a_{3k+3}}{3k+3} \end{aligned} \tag{3.21}$$

Then

$$\begin{aligned} 3 \sum_{k=0}^{\infty} \frac{a_{3k+3}}{3k+3} &= \left| 3i \sum_{k=0}^{\infty} \frac{a_{3k+3}}{3k+3} \right| \\ &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{3\mathbb{N}_0}(e^{i\theta})} d\theta \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i\theta}) \right| \overline{\left| w_{3\mathbb{N}_0}(e^{i\theta}) \right|} d\theta \\ &= \pi \|f\|_{H^1} \end{aligned} \tag{3.22}$$

And it is obvious that:

$$\sum_{k=0}^{\infty} \frac{a_{3k+3}}{3k+3} \leq \frac{\pi}{3} \|f\|_{H^1}$$

and by the $H^1 - H^2$ factorization theorem, $\forall f \in H^1$

$$\sum_{k=0}^{\infty} \frac{|a_{3k+3}|}{3k+3} \leq \frac{\pi}{3} \|f\|_{H^1}$$

□

Lemma 3.2. *Consider the function*

$$\left(\mathbb{P}_{3\mathbb{N}_0} f\right)(z) = \sum_{k=0}^{\infty} a_{3k} z^{3k}, \quad \forall z \in \overline{\Delta}$$

Then

1. $\mathbb{P}_{3\mathbb{N}_0} f \in H^1$
2. *And we have*

$$\|\mathbb{P}_{3\mathbb{N}_0} f\|_{H^1} \leq \|f\|_{H^1}$$

Proof. Fix

$$\left[f = f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \forall z \in \overline{\Delta} \right] \in H^1$$

Then by the $H^1 - H^2$ factorization theorem $\exists g, h \in H^2$ s.t.;

$$f(z) = g(z)h(z) \quad , \forall z \in \overline{\Delta}$$

and

$$\|f\|_{H^1} = \|g\|_{H^2}^2 = \|h\|_{H^2}^2$$

Let

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad \forall z \in \overline{\Delta}$$

$$h(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \forall z \in \overline{\Delta}$$

where

$$\|g\|_{H^2} = \left(\sum_{n=0}^{\infty} |b_n|^2 \right)^{\frac{1}{2}}$$

$$\|h\|_{H^2} = \left(\sum_{n=0}^{\infty} |c_n|^2 \right)^{\frac{1}{2}}$$

So

$$\begin{aligned} f(z) &= \left(\sum_{j=0}^{\infty} b_j z^j \right) \left(\sum_{k=0}^{\infty} c_k z^k \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} b_j c_{k-j} \right) z^k \end{aligned} \tag{3.23}$$

Then

$$
\begin{aligned}
\left(\mathbb{P}_{3\mathbb{N}_0} f\right)(z) &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{3k} b_j c_{3k-j} \right) z^{3k} \\
&= b_0 c_0 + \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} b_{3j} c_{3k-3j} \right) z^{3k} + \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} b_{3j+1} c_{3k-(3j+1)} \right) z^{3k} \\
&+ \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} b_{3j+2} c_{3k-(3j+2)} \right) z^{3k} \\
&= \left(\sum_{j=0}^{\infty} b_{3j} z^{3j} \right) \left(\sum_{k=0}^{\infty} c_{3k} z^{3k} \right) + \left(\sum_{j=0}^{\infty} b_{3j+1} z^{3j+1} \right) \left(\sum_{k=0}^{\infty} c_{3k+2} z^{3k+2} \right) \\
&+ \left(\sum_{j=0}^{\infty} b_{3j+2} z^{3j+2} \right) \left(\sum_{k=0}^{\infty} c_{3k+1} z^{3k+1} \right) \\
&= \left(\mathbb{P}_{3\mathbb{N}_0} g\right) \left(\mathbb{P}_{3\mathbb{N}_0} h\right) + \left(\mathbb{P}_{3\mathbb{N}_0+1} g\right) \left(\mathbb{P}_{3\mathbb{N}_0+2} h\right) + \left(\mathbb{P}_{3\mathbb{N}_0+2} g\right) \left(\mathbb{P}_{3\mathbb{N}_0+1} g\right)
\end{aligned}
\tag{3.24}
$$

Since $\mathbb{P}_{3\mathbb{N}_0+\upsilon} g, \;\; \mathbb{P}_{3\mathbb{N}_0+\upsilon} h \in H^2, ; \;\; \forall \upsilon \in \{0,1,2\}$, $\mathbb{P}_{3\mathbb{N}_0} f \in H^1$

Moreover:

$$
\begin{aligned}
\|\mathbb{P}_{3\mathbb{N}_0} f\|_{H^1} &\leq \|\mathbb{P}_{3\mathbb{N}_0} g\|_{H^2} \|\mathbb{P}_{3\mathbb{N}_0} h\|_{H^2} + \|\mathbb{P}_{3\mathbb{N}_0+1} g\|_{H^2} \|\mathbb{P}_{3\mathbb{N}_0+2} h\|_{H^2} + \|\mathbb{P}_{3\mathbb{N}_0+2} g\|_{H^2} \|\mathbb{P}_{3\mathbb{N}_0+1} h\|_{H^2} \\
&\leq \left(\|\mathbb{P}_{3\mathbb{N}_0} g\|_{H^2}^2 + \|\mathbb{P}_{3\mathbb{N}_0+1} g\|_{H^2}^2 + \|\mathbb{P}_{3\mathbb{N}_0+2} g\|_{H^2}^2 \right)^{\frac{1}{2}} \\
&\cdot \left(\|\mathbb{P}_{3\mathbb{N}_0} h\|_{H^2}^2 + \|\mathbb{P}_{3\mathbb{N}_0+1} h\|_{H^2}^2 + \|\mathbb{P}_{3\mathbb{N}_0+2} h\|_{H^2}^2 \right)^{\frac{1}{2}} \\
&= \|g\|_{H^2} \|h\|_{H^2} \\
&= \|f\|_{H^1}^{\frac{1}{2}} \|f\|_{H^1}^{\frac{1}{2}} = \|f\|_{H^1}
\end{aligned}
\tag{3.25}
$$

□

Theorem 3.3. *Let* $f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, \; \forall z \in \overline{\Delta} \right]$ *and assume each* $a_n \geq 0$ *then*

1.

$$\sum_{k=0}^{\infty} \frac{|a_{4k+1}|}{4k+1} \leq \frac{\pi\sqrt{2}}{4} \|f\|_{H^1}$$

2.

$$\sum_{k=0}^{\infty} \frac{|a_{4k+2}|}{4k+2} \leq \frac{\pi}{4} \|f\|_{H^1}$$

3.

$$\sum_{k=0}^{\infty} \frac{|a_{4k+3}|}{4k+3} \le \frac{\pi\sqrt{2}}{4}\|f\|_{H^1}$$

4.

$$\sum_{k=0}^{\infty} \frac{|a_{4k+4}|}{4k+4} \le \frac{\pi}{4}\|f\|_{H^1}$$

Proof. 1. Let $w_1(e^{i\theta}) = \pi - \theta, \ \ \forall \theta \in [0, 2\pi)$. Define the functions

$$\mu_1(e^{i\theta}) = w_1\left(e^{i(\theta-\frac{\pi}{2})}\right)$$

$$\mu_2(e^{i\theta}) = w_1\left(e^{i(\theta-\pi)}\right)$$

$$\mu_3(e^{i\theta}) = w_1\left(e^{i(\theta-\frac{3\pi}{2})}\right)$$

$$\mu_4(e^{i\theta}) = w_1\left(e^{i(\theta-2\pi)}\right) = \pi - \theta$$

By the Lemma 3.1

$$\begin{aligned}\hat{\mu}_1(n) &= e^{-\frac{in\pi}{2}}\hat{w}_1(n)\\ &= \left(e^{-\frac{i\pi}{2}}\right)^n \hat{w}_1(n)\\ &= \begin{cases} \frac{(-i)^n}{in}, & \text{if } n \neq 0;\\ 0, & \text{if } n = 0;\end{cases}\end{aligned} \tag{3.26}$$

$$\begin{aligned}\hat{\mu}_2(n) &= e^{-in\pi}\hat{w}_1(n)\\ &= \left(e^{-i\pi}\right)^n \hat{w}_1(n)\\ &= \begin{cases} \frac{(-1)^n}{in}, & \text{if } n \neq 0;\\ 0, & \text{if } n = 0;\end{cases}\end{aligned} \tag{3.27}$$

$$\begin{aligned}\hat{\mu}_3 &= e^{-\frac{3in\pi}{2}}\hat{w}_1(n)\\ &= \left(e^{-\frac{3i\pi}{2}}\right)^n \hat{w}_1(n)\\ &= \begin{cases} \frac{(i)^n}{in}, & \text{if } n \neq 0;\\ 0, & \text{if } n = 0;\end{cases}\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
\hat{\mu}_4(n) &= e^{-2in\pi}\hat{w}_1(n) \\
&= \left(e^{-2i\pi}\right)^n \hat{w}_1(n) \\
&= \begin{cases} \frac{(1)^n}{in}, & \text{if } n \neq 0; \\ 0, & \text{if } n = 0; \end{cases}
\end{aligned} \tag{3.29}
$$

Now let $\varrho = e^{\frac{-2i\pi}{4}}$, notice that: $\varrho^3 = \overline{\varrho}^1$. Construct a set similarly as the $3k + \upsilon$ cases. Let,

$$\mathbb{A} = \left\{\overline{\varrho}^1, \varrho^2, \varrho^1, \varrho^0\right\}$$

and construct a function by using the set $\mathbb{A}$:

$$
\begin{aligned}
w_{4\mathbb{N}_0+1}(z) &= \overline{\varrho}^1 \mu_1(z) + \varrho^2 \mu_2(z) + \varrho^1 \mu_3(z) + \varrho^0 \mu_4(z) \\
&= \sum_{n \in \mathbb{Z}-\{0\}} \left(\overline{\varrho}^1 \hat{\mu}_1(n) + \varrho^2 \hat{\mu}_2(n) + \varrho^1 \hat{\mu}_3(n) + \varrho^0 \hat{\mu}_4(n)\right) z^n
\end{aligned} \tag{3.30}
$$

And

$$
\begin{aligned}
\overline{\varrho}^1 \hat{\mu}_1(n) + \varrho^2 \hat{\mu}_2(n) + \varrho^1 \hat{\mu}_3(n) + \varrho^0 \hat{\mu}_4(n) &= \frac{1}{in}\left(\overline{\varrho}^1(-i)^n + \varrho^2(-1)^n + \varrho(i)^n + 1\right) \\
&= \begin{cases} 0 & \text{if } n = 4k; \\ \frac{4}{i(4k+1)} & \text{if } n = 4k+1; \\ 0 & \text{if } n = 4k+2; \\ 0 & \text{if } n = 4k+3; \end{cases}
\end{aligned} \tag{3.31}
$$

Then:

$$w_{4\mathbb{N}_0+1}(z) = \sum_{k \in \mathbb{Z}} \frac{4}{i(4k+1)} z^{4k+1}$$

Let $\theta \in [0, \frac{\pi}{2})$

$$
\begin{aligned}
w_{4\mathbb{N}_0+1}(e^{i\theta}) &= \overline{\varrho}^1\left(-\frac{\pi}{2} - \theta\right) + \varrho^2(-\theta) + \varrho\left(\frac{\pi}{2} - \theta\right) + (\pi - \theta) \\
&= \pi\left(-\frac{\overline{\varrho}}{2} + \frac{\varrho}{2} + 1\right) - \theta\,\underbrace{\left(\overline{\varrho} + \varrho^2 + \varrho + 1\right)}_{0} \\
&= \pi(1 - i)
\end{aligned} \tag{3.32}
$$

$\theta \in [\frac{\pi}{2}, \pi)$

$$
\begin{aligned}
w_{4\mathbb{N}_0+1}(e^{i\theta}) &= \overline{\varrho}^1\left(\frac{3\pi}{2}-\theta\right) + \varrho^2(-\theta) + \varrho\left(\frac{\pi}{2}-\theta\right) + (\pi-\theta) \\
&= \pi\left(\frac{3\overline{\varrho}}{2} + \frac{\varrho}{2} + 1\right) - \theta\underbrace{\left(\overline{\varrho} + \varrho^2 + \varrho + 1\right)}_{0} \\
&= \pi(1+i)
\end{aligned}
\tag{3.33}
$$

$\theta \in [\pi, \frac{3\pi}{2})$

$$
\begin{aligned}
w_{4\mathbb{N}_0+1}(e^{i\theta}) &= \overline{\varrho}^1\left(\frac{3\pi}{2}-\theta\right) + \varrho^2(2\pi-\theta) + \varrho\left(\frac{\pi}{2}-\theta\right) + (\pi-\theta) \\
&= \pi\left(\frac{3\overline{\varrho}}{2} + 2\varrho^2 + \frac{\varrho}{2} + 1\right) - \theta\underbrace{\left(\overline{\varrho} + \varrho^2 + \varrho + 1\right)}_{0} \\
&= \pi(-1+i)
\end{aligned}
\tag{3.34}
$$

$\theta \in [\frac{3\pi}{2}, 2\pi)$

$$
\begin{aligned}
w_{4\mathbb{N}_0+1}(e^{i\theta}) &= \overline{\varrho}^1\left(\frac{3\pi}{2}-\theta\right) + \varrho^2(2\pi-\theta) + \varrho\left(\frac{5\pi}{2}-\theta\right) + (\pi-\theta) \\
&= \pi\left(\frac{3\overline{\varrho}}{2} + 2\varrho^2 + \frac{5\varrho}{2} + 1\right) - \theta\underbrace{\left(\overline{\varrho} + \varrho^2 + \varrho + 1\right)}_{0} \\
&= \pi(-1-i)
\end{aligned}
\tag{3.35}
$$

And notice that for $\theta \in [\frac{(k-1)2\pi}{4}, \frac{(k)2\pi}{4})$, $\quad k \in \{1,2,3\}$, and $|w_{4\mathbb{N}_0+1}(e^{i\theta})| = \pi\sqrt{2}$. Let $f \in H^1$. Assume each $a_n \geq 0$. Then;

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{w_{4\mathbb{N}_0+1}(e^{i\theta})}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty} a_n e^{in\theta}\overline{w_{4\mathbb{N}_0+1}(e^{i\theta})}d\theta \\
&= \sum_{n=0}^{\infty} a_n \frac{1}{2\pi}\overline{\int_{-\pi}^{\pi} e^{-in\theta} w_{4\mathbb{N}_0+1}(e^{i\theta})d\theta} \\
&= \sum_{n=0}^{\infty} a_n \overline{\hat{w}_{4\mathbb{N}_0+1}(n)} \\
&= 4i\sum_{k=0}^{\infty}\frac{a_{4k+1}}{4k+1}
\end{aligned}
\tag{3.36}
$$

Then

$$
\begin{aligned}
4\sum_{k=0}^{\infty}\frac{a_{4k+1}}{4k+1} &= \left|4i\sum_{k=0}^{\infty}\frac{a_{4k+1}}{4k+1}\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{w_{4\mathbb{N}_0+1}(e^{i\theta})}d\theta\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(e^{i\theta})\right| \overline{|w_{4\mathbb{N}_0+1}(e^{i\theta})|}d\theta \\
&= \pi\sqrt{2}\|f\|_{H^1}
\end{aligned}
\tag{3.37}
$$

So

$$\sum_{k=0}^{\infty}\frac{a_{4k+1}}{4k+1} \leq \frac{\pi\sqrt{2}}{4}\|f\|_{H^1}$$

By the $H^1 - H^2$ factorization theorem:

$$\sum_{k=0}^{\infty}\frac{|a_{4k+1}|}{4k+1} \leq \frac{\pi\sqrt{2}}{4}\|f\|_{H^1}$$

2. To prove the second part of Theorem 3.3 we will use the set $\mathbb{A}^2 = \{\overline{\varrho}^2, 1, \varrho^2, \varrho^0\}$ to construct the function:

$$w_{4\mathbb{N}_0+2}(z) = \overline{\varrho}^2\mu_1(z) + \mu_2(z) + \varrho^2\mu_3(z) + \mu_4(z)$$

Then it is easy to see that

$$w_{4\mathbb{N}_0+2}(z) = \sum_{k\in\mathbb{Z}}\frac{4}{i(4k+2)}z^{4k+2},$$

and by using the same method we obtain:

$$\left|w_{4\mathbb{N}_0+2}(e^{i\theta})\right| = \pi, \;\; for \;\; \theta \in [\frac{k\pi}{2}, \frac{(k+1)\pi}{2}) \;\; where \;\; 0 \leq k \leq 3.$$

So, we get

$$\sum_{k=0}^{\infty}\frac{a_{4k+2}}{4k+2} \leq \frac{\pi}{4}\|f\|_{H^1}$$

and by the $H^1 - H^2$ factorization theorem:

$$\sum_{k=0}^{\infty}\frac{|a_{4k+2}|}{4k+2} \leq \frac{\pi}{4}\|f\|_{H^1}$$

3. As we expect, we will use the set $\mathbb{A}^3 = \left\{(\overline{\varrho})^3, (\varrho^2)^3, (\varrho)^3, (\varrho^0)^3\right\} = \{\varrho, \varrho^2, \overline{\varrho}, \varrho^0\}$ to con-

struct the function:

$$w_{4\mathbb{N}_0+3}(z) = \varrho\mu_1(z) + \varrho^2\mu_2(z) + \overline{\varrho}\mu_3(z) + \varrho^0\mu_4(z)$$

and then

$$w_{4\mathbb{N}_0+3}(z) = \sum \frac{4}{i(4k+3)} z^{4k+3}.$$

Since

$$|w_{4\mathbb{N}_0+3}(e^{i\theta})| = \sqrt{2}\pi, \quad \forall\theta \in [\frac{k2\pi}{4}, \frac{(k+1)2\pi}{4}) \;\; where \;\; k = 1, 2, 3.$$

$$4\sum_{k=0}^{\infty} \frac{a_{4k+3}}{4k+3} \leq \pi\sqrt{2}\|f\|_{H^1}.$$

And by the $H^1 - H^2$ factorization theorem

$$\sum_{k=0}^{\infty} \frac{|a_{4k+3}|}{4k+3} \leq \frac{\pi\sqrt{2}}{4}\|f\|_{H^1}.$$

4. Now we will use the set $\mathbb{A}^0$ to construct the function

$$\begin{aligned} w_{4\mathbb{N}_0}(z) &= \mu_1(z) + \mu_2(z) + \mu_3(z) + \mu_4(z) \\ &= \sum_{k\in\mathbb{Z}-\{0\}} \frac{4}{i4k} z^{4k} \end{aligned} \tag{3.38}$$

$w_{4\mathbb{N}_0}(e^{i\theta}) = w_1(e^{4i\theta})$, which is $\frac{\pi}{2}$ periodic function. Then

$$|w_{4\mathbb{N}_0}(e^{i\theta})| \leq \pi, \quad \forall\theta \in \left[\frac{2\pi k}{4}, \frac{2\pi(k+1)}{4}\right), \quad 0 \leq k \leq 3$$

Assume each $a_n \geq 0$. Then;

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{w_{4\mathbb{N}_0}(e^{i\theta})}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty} a_n e^{in\theta}\overline{w_{4\mathbb{N}_0}(e^{i\theta})}d\theta \\ &= \sum_{n=0}^{\infty} a_n \overline{\hat{w}_{4\mathbb{N}_0}(n)} \\ &= 4i\sum_{k=1}^{\infty}\frac{a_{4k}}{4k} \\ &= 4i\sum_{k=0}^{\infty}\frac{a_{4k+4}}{4k+4} \end{aligned} \tag{3.39}$$

So finally we get:

$$\begin{aligned}4\sum_{k=0}^{\infty}\frac{a_{4k+4}}{4k+4} &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(e^{i\theta})w_{4\mathbb{N}_0}(e^{i\theta})d\theta\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|w_{4\mathbb{N}_0}(e^{i\theta})\right|\left|f(e^{i\theta})\right|d\theta \\ &\leq \pi\|f\|_{H^1}\end{aligned} \tag{3.40}$$

And by the $H^1 - H^2$ factorization theorem:

$$\sum_{k=0}^{\infty}\frac{|a_{4k+4}|}{4k+4} \leq \frac{\pi}{4}\|f\|_{H^1}$$

$\square$

Lemma 3.3. *Let $f \in H^1$ and consider:*

$$(P_{4\mathbb{N}_0}f)(z) = \sum_{k=0}^{\infty} a_{4k}z^{4k} \quad \forall z \in \overline{\Delta}$$

Then

$$(P_{4\mathbb{N}_0}f)(z) \in H^1$$

and,

$$\|(P_{4\mathbb{N}_0}f)(z)\|_{H^1} \leq \|f\|_{H^1}$$

Proof.

$$\begin{aligned}(\mathbb{P}_{4\mathbb{N}_0}f) &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} b_j c_{4k-j}\right)z^{4k} \\ &= \left(\sum_{j=0}^{\infty} b_{4j}z^{4j}\right)\left(\sum_{k=0}^{\infty} c_{4k}z^{4k}\right) + \left(\sum_{j=0}^{\infty} b_{4j+1}z^{4j+1}\right)\left(\sum_{k=0}^{\infty} c_{4k+3}z^{4k+3}\right) \\ &+ \left(\sum_{j=0}^{\infty} b_{4j+2}z^{4j+2}\right)\left(\sum_{k=0}^{\infty} c_{4k+2}z^{4k+2}\right) + \left(\sum_{j=0}^{\infty} b_{4j+3}z^{4j+3}\right)\left(\sum_{k=0}^{\infty} c_{4k+1}z^{4k+1}\right) \\ &= [(\mathbb{P}_{4\mathbb{N}_0}g)(\mathbb{P}_{4\mathbb{N}_0}h)] + [(\mathbb{P}_{4\mathbb{N}_0+1}g)(\mathbb{P}_{4\mathbb{N}_0+3}h)] \\ &+ [(\mathbb{P}_{4\mathbb{N}_0+2}g)(\mathbb{P}_{4\mathbb{N}_0+2}h)] + [(\mathbb{P}_{4\mathbb{N}_0+3}g)(\mathbb{P}_{4\mathbb{N}_0+1}h)]\end{aligned} \tag{3.41}$$

Since $(\mathbb{P}_{4\mathbb{N}_0+v})g$ *and* $(\mathbb{P}_{4\mathbb{N}_0+v})h \in H^2$ for $v \in \{0,1,2,3\}$ then $\mathbb{P}_{4\mathbb{N}_0}f \in H^1$.

On the other hand;

$$
\begin{aligned}
\|\mathbb{P}_{4\mathbb{N}_0} f\|_{H^1} &\leq \|\mathbb{P}_{4\mathbb{N}_0} g\|_{H^2} \|\mathbb{P}_{4\mathbb{N}_0} h\|_{H^2} + \|\mathbb{P}_{4\mathbb{N}_0+1} g\|_{H^2} \|\mathbb{P}_{4\mathbb{N}_0+3} h\|_{H^2} \\
&+ \|\mathbb{P}_{4\mathbb{N}_0+2} g\|_{H^2} \|\mathbb{P}_{4\mathbb{N}_0+2} h\|_{H^2} + \|\mathbb{P}_{4\mathbb{N}_0+3} g\|_{H^2} \|\mathbb{P}_{4\mathbb{N}_0+1} h\|_{H^2} \\
&\leq \left[\sum_{k=0}^{\infty} |b_{4k}|^2 + \sum_{k=0}^{\infty} |b_{4k+1}|^2 + \sum_{k=0}^{\infty} |b_{4k+2}|^2 + \sum_{k=0}^{\infty} |b_{4k+3}|^2 \right]^{\frac{1}{2}} \\
&\cdot \left[\sum_{k=0}^{\infty} |c_{4k}|^2 + \sum_{k=0}^{\infty} |c_{4k+1}|^2 + \sum_{k=0}^{\infty} |c_{4k+2}|^2 + \sum_{k=0}^{\infty} |c_{4k+3}|^2 \right]^{\frac{1}{2}} \\
&= \|f\|_{H^1}^{\frac{1}{2}} \|f\|_{H^1}^{\frac{1}{2}} = \|f\|_{H^1}
\end{aligned} \tag{3.42}
$$

Notice that this argument is quite tight. When $g = h$ then $f = g^2$ and;

$$
\begin{aligned}
\|\mathbb{P}_{4\mathbb{N}_0} f\|_{H^1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (\mathbb{P}_{4\mathbb{N}_0} g)^2 + (\mathbb{P}_{4\mathbb{N}_0+1} g)^2 + (\mathbb{P}_{4\mathbb{N}_0+2} g)^2 + (\mathbb{P}_{4\mathbb{N}_0+3} g)^2 \right| d\theta \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|\mathbb{P}_{4\mathbb{N}_0} g|^2 + |\mathbb{P}_{4\mathbb{N}_0+1} g|^2 + |\mathbb{P}_{4\mathbb{N}_0+2} g|^2 + |\mathbb{P}_{4\mathbb{N}_0+3} g|^2 \right) d\theta \\
&= \|\mathbb{P}_{4\mathbb{N}_0} g\|_{H^2}^2 + \|\mathbb{P}_{4\mathbb{N}_0+1} g\|_{H^2}^2 + \|\mathbb{P}_{4\mathbb{N}_0+2} g\|_{H^2}^2 + \|\mathbb{P}_{4\mathbb{N}_0+3} g\|_{H^2}^2 \\
&= \|f\|_{H^1}
\end{aligned} \tag{3.43}
$$

Note that by using exactly the same method with the proof of $\mathbb{P}_{4\mathbb{N}_0} f \in H^1$ case we can easily prove that $\mathbb{P}_{4\mathbb{N}_0+\upsilon} f \in H^1$ for $\upsilon = \{1, 2, 3\}$. □

Now we are going to find a general inequality for the case $a_{\upsilon k+j}$ case where $\upsilon \in \mathbb{N}$ and $1 \leq j \leq \upsilon - 1$. To provide a better understanding, we will develop the general formula step by step.

As a first step we will prove the following theorem:

Theorem 3.4. *Let $w_1(e^{i\theta}) = \pi - \theta$ and suppose we have the set of functions below:*

$$
\begin{gathered}
\mu_1(e^{i\theta}) = w_1\left(e^{i\left(\theta - \frac{2\pi}{\upsilon}\right)}\right) \\
\mu_2(e^{i\theta}) = w_1\left(e^{i\left(\theta - \frac{2(2\pi)}{\upsilon}\right)}\right) \\
\vdots \\
\mu_\upsilon(e^{i\theta}) = w_1\left(e^{i\left(\theta - \frac{\upsilon(2\pi)}{\upsilon}\right)}\right)
\end{gathered}
$$

Also let $\varrho = e^{\frac{-2i\pi}{\upsilon}}$*, and:*

$$\mathbb{A} = \left\{\varrho^{\upsilon-1}, \varrho^{\upsilon-2}, \cdots, \varrho^{0}\right\}$$

so:

$$\mathbb{A}^j = \left\{\left(\varrho^{\upsilon-1}\right)^j, \left(\varrho^{\upsilon-2}\right)^j, \cdots, \varrho^{0}\right\}$$

If we construct a function in such way;

$$w_{\upsilon\mathbb{N}_0+j}(z) = \left(\varrho^{\upsilon-1}\right)^j \mu_1(z) + \left(\varrho^{\upsilon-2}\right)^j \mu_2(z) + \cdots + \left(\varrho^{\upsilon-\upsilon}\right)^j \mu_\upsilon(z)$$

where $j \in \{1, 2, \cdots, \upsilon\}$ *Then*

$$w_{\upsilon\mathbb{N}_0+j}(z) = \sum_{k\in\mathbb{Z}} \frac{\upsilon}{i(\upsilon k + j)} z^{\upsilon k + j}$$

Also when $j = 0$, $w_{\upsilon\mathbb{N}_0}(z) = \sum_{k\in\mathbb{Z}-\{0\}} \frac{\upsilon}{i\upsilon k} z^{\upsilon k}$

Proof. By the Lemma 3.1

$$\hat{\mu}_\tau(n) = e^{-\frac{in2\pi\tau}{\upsilon}} \hat{w}_1(n) \quad where \quad \hat{w}_1(n) = \frac{1}{in}$$

and we have

$$w_{\upsilon\mathbb{N}_0+j}(z) = \sum_{\tau=1}^{\upsilon} \left(\varrho^{\upsilon-\tau}\right)^j \mu_\tau(z)$$

where

$$\mu_\tau(z) = \sum_{n\in\mathbb{Z}-\{0\}} \frac{\left(e^{-\frac{i2\pi}{\upsilon}\tau}\right)^n}{in} z^n \quad (0 \le j \le \upsilon - 1)\,.$$

Then,

$$\begin{aligned} w_{\upsilon\mathbb{N}_0+j}(z) &= \sum_{\tau=1}^{\upsilon} \left(\varrho^{\upsilon-\tau}\right)^j \mu_\tau(z) \\ &= \sum_{\tau=1}^{\upsilon} \left(\varrho^{\upsilon-\tau}\right)^j \sum_{n\in\mathbb{Z}-\{0\}} \frac{\left(e^{-\frac{i2\pi}{\upsilon}\tau}\right)^n}{in} z^n \end{aligned} \tag{3.44}$$

since $\varrho = e^{-\frac{2i\pi}{\upsilon}}$;

$$\begin{aligned} w_{\upsilon\mathbb{N}_0+j}(z) &= \sum_{\tau=1}^{\upsilon} \left(\varrho^{\upsilon-\tau}\right)^j \sum_{n\in\mathbb{Z}-\{0\}} \frac{\varrho^{\tau n}}{in} z^n \\ &= \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{in} \sum_{\tau=1}^{\upsilon-1} \left(\varrho^{n-j}\right)^\tau \underbrace{\left(\varrho^{\upsilon}\right)^j}_{1} z^n \end{aligned} \tag{3.45}$$

Assume $j \geq 1$. For the case: $n = \upsilon k + j$, the coefficient of z^n is

$$\begin{aligned} &= \frac{1}{i\left(\upsilon k + j\right)} \underbrace{\sum_{\tau=0}^{\upsilon-1} \left(\varrho^{\upsilon k}\right)^{\tau}}_{\upsilon} \\ &= \frac{\upsilon}{i\left(\upsilon k + j\right)} \end{aligned} \tag{3.46}$$

For the other cases i.e.: $n = \upsilon k + l, \quad l \neq j$ and $l \in \{0, 1, 2, \cdots, \upsilon\}$, the coefficient of z^n is

$$\begin{aligned} &= \frac{1}{i\left(\upsilon k + l\right)} \sum_{\tau=0}^{\upsilon-1} \left(\varrho^{\upsilon k + l - j}\right)^{\tau} \\ &= \frac{1}{i\left(\upsilon k + l\right)} \sum_{\tau=0}^{\upsilon-1} \left(\varrho^{l-j}\right)^{\tau} = 0 \end{aligned} \tag{3.47}$$

The case $j = 0$ is easier, and proceeds similarly to the proof of Theorem 3.3 (the case $\upsilon = 4$). □

Before proving the general Hardy's Inquality we need to calculate $\left|w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})\right|$ as a second step:

Theorem 3.5. *Consider the same assumptions as theorem 3.4. Then when $j \geq 1$,*

$$\left|w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})\right| = \frac{\pi}{sin\left(\frac{j\pi}{\upsilon}\right)}$$

Also when $j = 0$,

$$\left|w_{\upsilon\mathbb{N}_0(e^{i\theta})}\right| \leq \pi$$

Proof. To prove theorem 3.5 we need to consider two different cases: **Case 1:** Let $\upsilon = 2a$ where $a \in \mathbb{N} - \{0\}$ and $1 \leq j \leq \upsilon - 1$. Since we need to consider the general case $w_{\upsilon\mathbb{N}_0+j}$ we are going to use the set:

$$\mathbb{A}^j = \left\{\left(\overline{\varrho}^1\right)^j, \left(\overline{\varrho}^2\right)^j, \cdots, \left(\overline{\varrho}^{\frac{\upsilon}{2}-1}\right)^j, \left(\varrho^{\frac{\upsilon}{2}}\right)^j, \left(\varrho^{\frac{\upsilon}{2}-1}\right)^j, \cdots, \left(\varrho^0\right)^j\right\}$$

Let $\theta \in \left[0, \frac{2\pi}{\upsilon}\right)$. Then $w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})$ can be expressed as:

$$\begin{aligned} w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) = (\overline{\varrho})^j \left(-\pi - \left(\theta - \frac{2\pi}{\upsilon}\right)\right) + \cdots \\ + \left(\varrho^{\frac{\upsilon}{2}-1}\right)^j \left(-\pi - \left(\theta - \frac{2\pi\left(\frac{\upsilon}{2}+1\right)}{\upsilon}\right)\right) + \cdots \\ + \varrho^j \left(-\pi - \left(\theta - \left(2\pi - \frac{2\pi}{\upsilon}\right)\right)\right) + \varrho^0 \left(-\pi - (\theta - 2\pi)\right) \\ = \pi \left[\left(-1 + \frac{2}{\upsilon}\right) 2isin\left(\frac{2\pi j}{\upsilon}\right) + \cdots\right] \\ + \pi \left[\left(-1 + \left(\frac{2\left(\frac{\upsilon}{2}-1\right)}{\upsilon}\right)\right) 2isin\left(\frac{2\pi j\left(\frac{\upsilon}{2}-1\right)}{\upsilon}\right)\right] + \pi \end{aligned} \tag{3.48}$$

Notice that:$\theta\left(\overline{\varrho}^j + \overline{\varrho}^{2j} + \cdots + \varrho^0\right) = 0.$

Say;

$$\begin{aligned} X = \pi \left[\left(-1 + \frac{2}{\upsilon}\right) 2isin\left(\frac{2\pi j}{\upsilon}\right) + \cdots\right] \\ + \pi \left[\left(-1 + \left(\frac{2\left(\frac{\upsilon}{2}-1\right)}{\upsilon}\right)\right) 2isin\left(\frac{2\pi j\left(\frac{\upsilon}{2}-1\right)}{\upsilon}\right) + 1\right] \end{aligned} \tag{3.49}$$

Let, $\theta \in \left[\frac{2\pi}{\upsilon}, \frac{4\pi}{\upsilon}\right)$. Then

$$w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) = 2\pi\overline{\varrho}^j + X.$$

If $\theta \in \left[\frac{4\pi}{\upsilon}, \frac{6\pi}{\upsilon}\right)$ then

$$w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) = 2\pi\left(\overline{\varrho}^j + \overline{\varrho}^{2j}\right) + X.$$

If $\theta \in \left[\frac{(\upsilon-1)2\pi}{\upsilon}, \frac{\upsilon 2\pi}{\upsilon}\right)$

$$w_{\upsilon\mathbb{N}_0+j}(z) = 2\pi\left(\overline{\varrho}^j + \overline{\varrho}^{2j} + \cdots + \left(\overline{\varrho}^{\upsilon-1}\right)^j\right) + X.$$

Then for $\theta \in \left[k\frac{2\pi}{\upsilon}, (k+1)\frac{2\pi}{\upsilon}\right)$, $0 \le k \le \upsilon - 1$.

$$w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) = \underbrace{\pi\left[1 + 2i\sum_{t=1}^{\frac{\upsilon-2}{2}} \frac{2t-\upsilon}{\upsilon} sin\left(\frac{2\pi tj}{\upsilon}\right)\right]}_{Y} + \sum_{n=1}^{k} 2\pi(\overline{\varrho}^n)^j$$

$$
\begin{aligned}
Y &= 1 + 2i \left\{ \sum_{t=1}^{\frac{\upsilon-2}{2}} \frac{2t}{\upsilon} \left(\frac{e^{\frac{i2\pi tj}{\upsilon}} - e^{-\frac{i2\pi tj}{\upsilon}}}{2i} \right) - \sum_{t=1}^{\frac{\upsilon-2}{2}} \left(\frac{e^{\frac{i2\pi tj}{\upsilon}} - e^{-\frac{i2\pi tj}{\upsilon}}}{2i} \right) \right\} \\
&= 1 + \underbrace{\frac{2}{\upsilon} \sum_{t=1}^{N} tz^t - \frac{2}{\upsilon} \sum_{t=1}^{N} tw^t}_{B} + \underbrace{\left(-\sum_{t=1}^{N} z^t + \sum_{t=1}^{N} w^t \right)}_{A}
\end{aligned} \tag{3.50}
$$

where $z = e^{i\beta},\ w = e^{-i\beta},\quad \beta = \frac{2\pi j}{\upsilon},\ N = \frac{\upsilon-2}{2}$. Then

$$
\begin{aligned}
A &= \frac{1}{2isin\left(\frac{\beta}{2}\right)} \left[2cos\left(\frac{\beta}{2}\right) - 2cos\left(\beta\left(N + \frac{1}{2}\right)\right) \right] \\
&= \frac{1}{isin\left(\frac{\pi j}{\upsilon}\right)} \left[cos\left(\frac{\pi j}{\upsilon}\right) - (-1)^j cos\left(\frac{\pi j}{\upsilon}\right) \right]
\end{aligned} \tag{3.51}
$$

and

$$
\begin{aligned}
B &= \frac{1}{N+1} \left\{ \frac{1}{2isin\left(\frac{\beta}{2}\right)} \left[(2N+1)cos\left(\left(N + \frac{1}{2}\right)\beta\right) - \frac{sin\left(\left(N+\frac{1}{2}\right)\beta\right)}{sin\left(\frac{\beta}{2}\right)} cos\left(\frac{\beta}{2}\right) \right] \right\} \\
&= \frac{1}{N+1} \left\{ \frac{1}{2isin\left(\frac{\pi j}{\upsilon}\right)} \left[(2N+1)(-1)^j cos\left(\frac{\pi j}{\upsilon}\right) + (-1)^j cos\left(\frac{\pi j}{\upsilon}\right) \right] \right\} \\
&= \frac{(-1)^j cos\left(\frac{\pi j}{\upsilon}\right)}{isin\left(\frac{\pi j}{\upsilon}\right)}
\end{aligned} \tag{3.52}
$$

So

$$
A + B = \frac{cos\left(\frac{\pi j}{\upsilon}\right)}{isin\left(\frac{\pi j}{\upsilon}\right)}
$$

$$
\begin{aligned}
w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) &= \pi \left[1 + \frac{cos\left(\frac{\pi j}{\upsilon}\right)}{isin\left(\frac{\pi j}{\upsilon}\right)} \right] + \sum_{n=1}^{k} 2\pi(\overline{\varrho}^n)^j \\
&= \pi \left[1 + \frac{cos\left(\frac{\beta}{2}\right)}{isin\left(\frac{\beta}{2}\right)} + 2\sum_{n=1}^{k} (\overline{\varrho}^n)^j \right] \\
&= \pi \left[\frac{cos\left(\frac{\beta}{2}\right)}{isin\left(\frac{\beta}{2}\right)} - 1 + 2\sum_{n=0}^{k} (\overline{\varrho}^n)^j \right]
\end{aligned} \tag{3.53}
$$

If $1 \leq k \leq \upsilon - 1$, $\theta \in \left[\frac{k2\pi}{\upsilon}, \frac{(k+1)2\pi}{\upsilon}\right)$.

$$
w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) = \pi \left[1 + \frac{cos\left(\frac{\pi j}{\upsilon}\right)}{isin\left(\frac{\pi j}{\upsilon}\right)} \right]
$$

when $k=0$ and $\theta \in \left[\frac{k2\pi}{\upsilon}, \frac{(k+1)2\pi}{\upsilon}\right)$. **Case**2**:** Let $\upsilon = 2a+1$ then we need to use the set:

$$\mathbb{A}^j = \left\{ \overline{\varrho}^j, \overline{\varrho}^{2j}, \cdots, \left(\overline{\varrho}^{\frac{\upsilon-1}{2}}\right)^j, \left(\varrho^{\frac{\upsilon-1}{2}}\right)^j, \cdots, \varrho^0 \right\}$$

and for $\theta \in \left[\frac{2\pi(k)}{\upsilon}, \frac{2\pi(k+1)}{\upsilon}\right)$ we obtain the formula:

$$w_{\upsilon\mathbb{N}_0+j} = \pi \left[\underbrace{1 + 2i \sum_{t=1}^{\frac{\upsilon-1}{2}} \left(\frac{2t-\upsilon}{\upsilon}\right) sin\left(\frac{2\pi t j}{\upsilon}\right)}_{X} + 2\sum_{n=1}^{k} (\overline{\varrho}^n)^j \right].$$

Then:

$$\begin{aligned} X &= 1 + \left[\frac{2}{\upsilon}\sum_{t=1}^{N} t z^t - \frac{2}{\upsilon}\sum_{t=1}^{N} t w^t - \sum_{t=1}^{N} z^t + \sum_{t=1}^{N} w^t \right] \\ &= \frac{2}{\upsilon} \underbrace{\left[\frac{z}{(1-z)^2}\left(Nz^{N+1} - z^N(N+1)+1\right) - \frac{w}{(1-w)^2}\left(Nw^{N+1} - w^N(N+1)+1\right)\right]}_{B} \\ &+ \underbrace{\left[\left(\frac{w-w^{N+1}}{1-w}\right) - \left(\frac{z-z^{N+1}}{1-z}\right)\right]}_{A} + 1 \end{aligned} \tag{3.54}$$

$$A = \frac{1}{2isin\left(\frac{\beta}{2}\right)}\left[2cos\left(\frac{\beta}{2}\right) - 2cos\left(\left(N+\frac{1}{2}\right)\beta\right)\right]$$

and

$$B = \frac{1}{2isin\left(\frac{\beta}{2}\right)}\left[2cos\left(\left(N+\frac{1}{2}\right)\beta\right) - \frac{sin\left(\left(N+\frac{1}{2}\right)\beta\right)cos\left(\frac{\beta}{2}\right)}{\left(N+\frac{1}{2}\right)sin\left(\frac{\beta}{2}\right)}\right]$$

$$A + B = \frac{cos\left(\frac{\beta}{2}\right)}{isin\left(\frac{\beta}{2}\right)}$$

for

$$z = e^{i\beta}, \quad w = e^{-i\beta}, \quad \beta = \frac{2\pi j}{\upsilon}, \quad N = \frac{\upsilon-1}{2}.$$

Then

$$\begin{aligned} w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) &= \pi\left[1 + \frac{cos\left(\frac{\beta}{2}\right)}{isin\left(\frac{\beta}{2}\right)} + 2\sum_{n=1}^{k}(\overline{\varrho}^n)^j\right] \\ &= \pi\left[\frac{cos\left(\frac{\beta}{2}\right)}{isin\left(\frac{\beta}{2}\right)} - 1 + 2\sum_{n=0}^{k}\left(\overline{\varrho}^j\right)^n\right] \end{aligned} \tag{3.55}$$

For $\theta \in \left[\frac{2\pi k}{\upsilon}, \frac{2\pi(k+1)}{\upsilon}\right)$ where $0 \leq k \leq \upsilon - 1$. So by the equations (3.50) and (3.52) we see that even if $\upsilon = 2a+1$ or $\upsilon = 2a$ we have the same formula:

$$w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) = \pi\left[\frac{cos(\frac{\beta}{2})}{isin(\frac{\beta}{2})} - 1 + 2\sum_{n=0}^{k}(\overline{\varrho}^n)^j\right]$$

For $\theta \in \left[\frac{2\pi k}{\upsilon}, \frac{2\pi(k+1)}{\upsilon}\right)$ where $0 \leq k \leq \upsilon$. Now the only thing we need to calculate is

$$2\sum_{n=0}^{k}(\overline{\varrho}^n)^j$$

and then we will find the whole formula for: $|w_{\upsilon\mathbb{N}_0+j}|$. Let $\beta = \frac{2\pi j}{\upsilon}$ and $z = e^{i\beta}$ then;

$$\begin{aligned}
\sum_{n=0}^{k}(\overline{\varrho}^j)^n &= \sum_{n=0}^{k} z^n \\
&= \frac{1-z^{k+1}}{1-z} \\
&= \frac{1-e^{i\beta(k+1)}}{1-e^{i\beta}} \\
&= e^{\frac{i\beta k}{2}}\left(\frac{sin\left(\frac{\beta(k+1)}{2}\right)}{sin\left(\frac{\beta}{2}\right)}\right) \\
&= \frac{\left[\frac{sin(\beta k)}{2}cos\left(\frac{\beta}{2}\right) + \frac{1+cos(\beta k)}{2}sin\left(\frac{\beta}{2}\right)\right] + i\left[\frac{1-cos(\beta k)}{2}cos\left(\frac{\beta}{2}\right) + \frac{sin(\beta k)}{2}sin\left(\frac{\beta}{2}\right)\right]}{sin\left(\frac{\beta}{2}\right)} \\
&= \frac{\frac{1}{2}\left\{e^{i\beta k}\left(sin\left(\frac{\beta}{2}\right) - icos\left(\frac{\beta}{2}\right)\right) + \left(sin\left(\frac{\beta}{2}\right) + icos\left(\frac{\beta}{2}\right)\right)\right\}}{sin\left(\frac{\beta}{2}\right)}
\end{aligned} \tag{3.56}$$

So

$$\begin{aligned}
w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) &= \pi\left[\frac{cos\left(\frac{\beta}{2}\right)}{isin\left(\frac{\beta}{2}\right)} - 1 + \left\{\frac{e^{i\beta k}\left(sin\left(\frac{\beta}{2}\right) - icos\left(\frac{\beta}{2}\right)\right) + \left(sin\left(\frac{\beta}{2}\right) + icos\left(\frac{\beta}{2}\right)\right)}{sin\left(\frac{\beta}{2}\right)}\right\}\right] \\
&= \pi\left[\frac{e^{i\beta k}\left(cos\left(\frac{\beta}{2}\right) + isin\left(\frac{\beta}{2}\right)\right)}{isin\left(\frac{\beta}{2}\right)}\right] \\
&= \frac{\pi e^{i\beta(k+\frac{1}{2})}}{isin\left(\frac{\beta}{2}\right)} \\
&= \pi(\overline{\varrho})^{j\left(\frac{2k+1}{2}\right)}\frac{1}{isin\left(\frac{\pi j}{\upsilon}\right)}
\end{aligned} \tag{3.57}$$

Then we are done,

$$|w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})| = \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)}$$

The case $j = 0$ is simpler, and proceeds similarly to that case in the proof of Theorem 3.3 □

Theorem 3.6. *Let f be any arbitrary H^1 function i.e.*

$$f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, \forall z \in \overline{\Delta} \right]$$

And let a_n be n.th Fourier coefficient of f. Let $1 \leq j \leq \upsilon - 1$ where $\upsilon \geq 2$. Then

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+j}|}{\upsilon k + j} \leq \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \frac{1}{\upsilon} \|f\|_{H^1}$$

Proof. By the Theorem 3.4 and 3.5 we have:

$$w_{\upsilon\mathbb{N}_0+j}(z) = \sum_{k=0}^{\infty} \frac{\upsilon}{i(\upsilon k + j)} z^{\upsilon k+j} \quad z \in \mathbb{C}$$

and

$$|w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})| = \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)}$$

Let $f \in H^1$ and assume each $a_n \geq 0$. Then;

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n e^{in\theta} \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} d\theta \\
&= \sum_{n=1}^{\infty} a_n \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} w_{\upsilon\mathbb{N}_0+j}(e^{i\theta}) d\theta} \\
&= \sum_{n=1}^{\infty} a_n \overline{\hat{w}_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} \\
&= \sum_{k=0}^{\infty} a_{\upsilon k+j} \overline{\left(\frac{\upsilon}{i(\upsilon k + j)} \right)} \\
&= \upsilon i \sum_{k=0}^{\infty} \frac{a_{\upsilon k+j}}{\upsilon k + j}
\end{aligned} \tag{3.58}$$

Which gives us:

$$\begin{aligned}
\upsilon \sum_{k=0}^{\infty} \frac{a_{\upsilon k+j}}{\upsilon k+j} &= \upsilon \left| i \sum_{k=0}^{\infty} \frac{a_{\upsilon k+j}}{\upsilon k+j} \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{\upsilon \mathbb{N}_0+j}(e^{i\theta})} d\theta \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i\theta}) \right| \left| \overline{w_{\upsilon \mathbb{N}_0+j}(e^{i\theta})} \right| d\theta \\
&= \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \|f\|_{H^1}
\end{aligned} \tag{3.59}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{a_{\upsilon k+j}}{\upsilon k+j} \leq \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \frac{1}{\upsilon} \|f\|_{H^1}$$

and by the $H^1 - H^2$ factorization theorem:

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+j}|}{\upsilon k+j} \leq \frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)} \frac{1}{\upsilon} \|f\|_{H^1}.$$

Here we need to consider one more case, when $\upsilon = j$ for $w_{\upsilon \mathbb{N}_0+j}(z)$ in theorem 3.4 This time we need to construct our function as follows:

$$w_{\upsilon \mathbb{N}_0}(z) = \mu_1(z) + \mu_2(z) + \cdots + \mu_\upsilon(z)$$

We know that

$$\hat{\mu}_\tau(n) = e^{\frac{-in2\pi\tau}{\upsilon}} \hat{w}_1(n) \;\; where \;\; \hat{w}_1(n) = \frac{1}{in}$$

Since $\varrho = e^{\frac{-2\pi i}{\upsilon}}$ then:

$$\begin{aligned}
w_{\upsilon \mathbb{N}_0}(z) &= \sum_{\tau=1}^{\upsilon} \mu_\tau(z) \\
&= \sum_{\tau=1}^{\upsilon} \sum_{n \in \mathbb{Z}-\{0\}} \frac{(\varrho)^{\tau n}}{in} z^n \\
&= \sum_{n \in \mathbb{Z}-\{0\}} \sum_{\tau=1}^{\upsilon} \frac{\left((\varrho)^{\upsilon k}\right)^{\tau}}{i(\upsilon k)} z^{\upsilon k} \\
&= \sum_{n \in \mathbb{Z}-\{0\}} \frac{\upsilon}{i(\upsilon k)} z^{\upsilon k}
\end{aligned} \tag{3.60}$$

where $n = \upsilon k$ in equation (3.58)

Also we have:

$$\begin{aligned} w_{\upsilon\mathbb{N}_0}(e^{i\theta}) &= \mu_1(e^{i\theta}) + \mu_2(e^{i\theta}) + \cdots + \mu_\upsilon(e^{i\theta}) \\ &= w_1\left(e^{i\left(\theta-\frac{2\pi}{\upsilon}\right)}\right) + w_1\left(e^{i\left(\theta-\frac{4\pi}{\upsilon}\right)}\right) + \cdots + w_1\left(e^{i(\theta-2\pi)}\right) \end{aligned} \tag{3.61}$$

Let $\theta \in \left[0, \frac{2\pi}{\upsilon}\right)$, then

$$w_{\upsilon\mathbb{N}_0}(e^{i\theta}) = \pi - \upsilon\theta \tag{3.62}$$

And it easily follows that for $\theta \in \left[\frac{k2\pi}{\upsilon}, \frac{(k+1)2\pi}{\upsilon}\right)$;

$$w_{\upsilon\mathbb{N}_0}(e^{i\theta}) = ((2k+1)\pi - \upsilon\theta)$$

So it is clear that $|w_{\upsilon\mathbb{N}_0}(e^{i\theta})| \le \pi$, for all cases above.

Now let $f \in H^1$ and assume each $a_n \ge 0$. Then,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) w_{\upsilon\mathbb{N}_0}(e^{i\theta}) d\theta &= \sum_{n=0}^{\infty} a_n \overline{\hat{w}_{\upsilon\mathbb{N}_0}(n)} \\ &= \sum_{k\in\mathbb{N}-\{0\}} a_{\upsilon k} \overline{\left(\frac{\upsilon}{i(\upsilon k)}\right)} \\ &= \upsilon i \sum_{k=1}^{\infty} \frac{a_{\upsilon k}}{\upsilon k} \\ &= \upsilon i \sum_{k=0}^{\infty} \frac{a_{\upsilon k+\upsilon}}{\upsilon k + \upsilon} \end{aligned} \tag{3.63}$$

Then

$$\begin{aligned} \upsilon \sum_{k=0}^{\infty} \frac{a_{\upsilon k+\upsilon}}{\upsilon k+\upsilon} &= \left| \upsilon i \sum_{k=0}^{\infty} \frac{a_{\upsilon k+\upsilon}}{\upsilon k+\upsilon} \right| \\ &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) \overline{w_{\upsilon\mathbb{N}_0}(e^{i\theta})} d\theta \right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(e^{i\theta})\right| \overline{|w_{\upsilon\mathbb{N}_0}(e^{i\theta})|} d\theta \\ &\le \pi \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(e^{i\theta})\right| d\theta \\ &= \pi \|f\|_{H^1} \end{aligned} \tag{3.64}$$

And by the $H^1 - H^2$ factorization theorem it is clear:

$$\sum_{k=0}^{\infty} \frac{|a_{\upsilon k+\upsilon}|}{\upsilon k+\upsilon} \le \frac{\pi}{\upsilon}\|f\|_{H^1}$$

$\square$

Lemma 3.4. *For* $f \in H^1 = \left[\sum_{n=0}^{\infty} a_n z^n \ , \forall z \in \overline{\Delta}\right]$ *define*

$$(\mathbb{P}_{\upsilon\mathbb{N}+j} f)(z) = \sum_{k=0}^{\infty} a_{\upsilon k+j} z^{\upsilon k+j} \ \forall z \in \overline{\Delta}$$

Then $(\mathbb{P}_{\upsilon\mathbb{N}+j} f) \in H^1$*. For* $j \in \{0, 1, 2, \cdots, \upsilon - 1\}$*.*

Moreover

$$\|\mathbb{P}_{\upsilon\mathbb{N}+j} f\|_{H^1} \leq \|f\|_{H^1} \ for \ \ j \in \{0, 1, 2, \cdots, \upsilon - 1\}$$

Proof. Let $1 \leq j \leq \upsilon - 1$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and, $h(z) = \sum_{n=0}^{\infty} c_n z^n$ where both $g(z)$ and $h(z)$ are H^2 functions Then:

$$\begin{aligned}
\mathbb{P}_{\upsilon\mathbb{N}_0+j} &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} b_{n\upsilon} c_{j+(k-n)\upsilon} \right) z^{k\upsilon+j} \\
&= \left[\left(\sum_{k=0}^{\infty} b_{\upsilon k} z^{\upsilon k} \right) \left(\sum_{k=0}^{\infty} c_{j+\upsilon k} z^{j+\upsilon k} \right) \right] \\
&+ \left[\left(\sum_{k=0}^{\infty} b_{k\upsilon+1} z^{k\upsilon+1} \right) \left(\sum_{k=0}^{\infty} c_{(j-1)+\upsilon k} z^{(j-1)+\upsilon k} \right) \right] + \cdots \\
&+ \left[\left(\sum_{k=0}^{\infty} b_{(k+1)\upsilon-1} z^{(k+1)\upsilon-1} \right) \left(\sum_{k=0}^{\infty} c_{k\upsilon+j+1} z^{k\upsilon+j+1} \right) \right]
\end{aligned} \tag{3.65}$$

CASE 1: $1 \leq j \leq \upsilon - 2$

$$\begin{aligned}
\mathbb{P}_{\upsilon\mathbb{N}_0+j} &= \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j} h\right) \right] + \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+1} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j-1} h\right) \right] \\
&+ \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+2} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j-2} h\right) \right] + \cdots + \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+\upsilon-1} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j+1} h\right) \right]
\end{aligned} \tag{3.66}$$

CASE 2: $j = \upsilon - 1$

$$\begin{aligned}
\mathbb{P}_{\upsilon\mathbb{N}_0+j} &= \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j} h\right) \right] + \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+1} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j-1} h\right) \right] \\
&+ \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+2} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0+j-2} h\right) \right] + \cdots + \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+\upsilon-1} g\right) \left(\mathbb{P}_{\upsilon\mathbb{N}_0} h\right) \right]
\end{aligned} \tag{3.67}$$

Since $(\mathbb{P}_{\upsilon\mathbb{N}_0+a} g)$ and $(\mathbb{P}_{\upsilon\mathbb{N}_0+a} h)$ are in H^2 for $a \in \{0, 1, 2, \cdots, \upsilon - 1\}$ then by the $H^1 - H^2$ factorization theorem $(\mathbb{P}_{\upsilon\mathbb{N}_0+j} f) \in H^1$.

CASE 3**:** Let $j = 0$, then:

$$
\begin{aligned}
(\mathbb{P}_{\upsilon\mathbb{N}_0} f) &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} b_n c_{\upsilon k-n}\right) z^{\upsilon k} \\
&= b_0 c_0 + \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k} b_{\upsilon n} c_{\upsilon k-\upsilon j}\right) z^{\upsilon k} + \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} b_{\upsilon n+1} c_{\upsilon k-(\upsilon n+1)}\right) z^{\upsilon k} \\
&+ \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} b_{\upsilon n+2} c_{\upsilon k-(\upsilon n+2)}\right) z^{\upsilon k} + \cdots + \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} b_{\upsilon n+(\upsilon-1)} c_{\upsilon k-(\upsilon n+(\upsilon-1))}\right) z^{\upsilon k} \\
&= \left[(\mathbb{P}_{\upsilon\mathbb{N}_0} g)(\mathbb{P}_{\upsilon\mathbb{N}_0} h)\right] + \left[(\mathbb{P}_{\upsilon\mathbb{N}_0+1} g)\left(\mathbb{P}_{\upsilon\mathbb{N}_0+(\upsilon-1)} h\right)\right] + \cdots + \left[\left(\mathbb{P}_{\upsilon\mathbb{N}_0+(\upsilon-1)} g\right)(\mathbb{P}_{\upsilon\mathbb{N}_0+1} h)\right]
\end{aligned}
\tag{3.68}
$$

Since $(\mathbb{P}_{\upsilon\mathbb{N}_0+a}(g))$ and $(\mathbb{P}_{\upsilon\mathbb{N}_0+a}(h))$ are in H^2 for $a \in \{0, 1, 2, \cdots, (\upsilon-1)\}$ then again by the $H^1 - H^2$ factorization theorem $(\mathbb{P}_{\upsilon\mathbb{N}_0}(f)) \in H^1$. then we are done with the first part of the proof.

Moreover, for $1 \leq j \leq \upsilon - 2$, by (3.64)

$$
\begin{aligned}
\|\mathbb{P}_{\upsilon\mathbb{N}_0+j} f\|_{H^1} &\leq \|\mathbb{P}_{\upsilon\mathbb{N}_0} g\|_{H^2} \|\mathbb{P}_{\upsilon\mathbb{N}_0+j} h\|_{H^2} + \|\mathbb{P}_{\upsilon\mathbb{N}_0+1} g\|_{H^2} \|\mathbb{P}_{\upsilon\mathbb{N}_0+j-1} h\|_{H^2} \\
&+ \|\mathbb{P}_{\upsilon\mathbb{N}_0+2} g\|_{H^2} \|\mathbb{P}_{\upsilon\mathbb{N}_0+j-2} h\|_{H^2} + \cdots + \|\mathbb{P}_{\upsilon\mathbb{N}_0+\upsilon-1} g\|_{H^2} \|\mathbb{P}_{\upsilon\mathbb{N}_0+j+1} h\|_{H^2} \\
&\leq \left(\sum_{k=0}^{\infty} |b_{\upsilon k}|^2 + \cdots + \sum_{k=0}^{\infty} |b_{\upsilon k+\upsilon-1}|^2\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} |c_{\upsilon k+j}|^2 + \cdots + \sum_{k=0}^{\infty} |c_{\upsilon k+j+1}|^2\right)^{\frac{1}{2}} \\
&= \|g\|_{H^2} \|h\|_{H^2} = \|f\|_{H^1}
\end{aligned}
\tag{3.69}
$$

It can be shown that

$$
\|\mathbb{P}_{\upsilon\mathbb{N}_0+j} f\|_{H^1} \leq \|f\|_{H^1} \;\; for \;\; j \in \{0, 1, 2, \cdots, \upsilon - 1\}
$$

for $j = \upsilon - 1$ and $j = 0$. Proof is almost same with the proof of case $1 \leq j \leq \upsilon - 2$. $\square$

4.0 OPTIMAL CONSTANTS FOR HARDY-LIKE INEQUALITIES

Theorem 4.1. *Let* $f \in H^1 = \left[f(z) = \sum_{n=0}^{\infty} b_n z^n \ z \in \overline{\Delta} \right]$. *Then* π *is the best constant for the Hardy's Inequality below:*

$$\underbrace{\sum_{n=1}^{\infty} \frac{\left|\hat{f}(n)\right|}{n}}_{LHS} \leq \pi \underbrace{\|f\|_{H^1}}_{RHS}$$

Before starting the proof of theorem 4.1 we need to prove the lemma below:

Lemma 4.1. *If* $(\alpha_j)_{j\in\mathbb{N}}$ *and* $(\beta_j)_{j\in\mathbb{N}}$ *are sequences in* $(0,\infty)$ *s.t.*

$$\mathbf{A}: \ \sum_{j=1}^{\infty} \beta_j = \infty$$

and

$$\mathbf{B}: \ \frac{\alpha_j}{\beta_j} \to 1 \ \ as j \to \infty. \ \ i.e.: \alpha_j \sim \beta_j$$

then by the limit comparison test:

$$1: \ \sum_{j=1}^{\infty} \alpha_j = \infty.$$

Moreover:

$$2: \ \frac{\sum_{j=M}^{N} \alpha_j}{\sum_{j=M}^{N} \beta_j} \to 1 \quad as \quad N \to \infty \ \ and \ \ where \ \ M < N$$

Proof. Let A and B be given. We prove the second statement here:

Fix $\epsilon > 0$ and $\epsilon < 1$. Then $\exists M_\epsilon \in \mathbb{N}$, s.t.

$$\forall \ \ j \geq M_\epsilon \quad \left|\frac{\alpha_j}{\beta_j} - 1\right| < \epsilon \Rightarrow 1 - \epsilon < \frac{\alpha_j}{\beta_j} < 1 + \epsilon$$

Then $\forall N \geq M \geq M_\epsilon$

$$\sum_{j=M}^{N}(1-\epsilon)\beta_j < \sum_{j=M}^{N}\alpha_j < \sum_{j=M}^{N}(1+\epsilon)\beta_j$$

Hence $\forall N \geq M \geq M_\epsilon$:

$$(1-\epsilon) < \frac{\sum_{j=M}^{N}\alpha_j}{\sum_{j=M}^{N}\beta_j} < (1+\epsilon)$$

Now let A be given i.e.:

$$\sum_{j=1}^{\infty}\beta_j = \infty$$

Fix $M_0 \in \mathbb{N}$ and fix $N \geq M_0$. Then for $M_0 \leq M \leq N$:

$$\frac{\sum_{j=M_0}^{N}\alpha_j}{\sum_{j=M_0}^{N}\beta_j} = \frac{\sum_{j=M_0}^{M}\alpha_j + \sum_{j=M}^{N}\alpha_j}{\sum_{j=M_0}^{N}\beta_j + \sum_{j=M}^{N}\beta_j}$$

Fix $1 > \epsilon > 0$, and choose $M_\epsilon \in \mathbb{N}$, s.t. $\forall N \geq M \geq M_\epsilon$:

$$1-\epsilon < \frac{\sum_{j=M}^{N}\alpha_j}{\sum_{j=M}^{N}\beta_j} < 1+\epsilon.$$

Since M_0 is fixed, without loss of generality; $M_\epsilon > M_0$.

$$1-\epsilon < \frac{\sum_{j=M_\epsilon}^{N}\alpha_j}{\sum_{j=M_\epsilon}^{N}\beta_j} < 1+\epsilon$$

Then $\forall N \geq M_\epsilon$,

$$\begin{aligned}\frac{\sum_{j=M_0}^{N}\alpha_j}{\sum_{j=M_0}^{N}\beta_j} &= \frac{\sum_{j=M_0}^{M_\epsilon}\alpha_j + \sum_{j=M_\epsilon}^{N}\alpha_j}{\sum_{j=M_0}^{M_\epsilon}\beta_j + \sum_{j=M_\epsilon}^{N}\beta_j} \\ &= \frac{\frac{\sum_{j=M_0}^{M_\epsilon}\alpha_j}{\sum_{j=M_\epsilon}^{N}\beta_j} + \frac{\sum_{j=M_\epsilon}^{N}\alpha_j}{\sum_{j=M_\epsilon}^{N}\beta_j}}{\frac{\sum_{j=M_0}^{M_\epsilon}\beta_j}{\sum_{j=M_\epsilon}^{N}\beta_j} + 1}\end{aligned} \tag{4.1}$$

$\sum_{j=M_\epsilon}^{N}\beta_j \to \infty$ as $N \to \infty$ is given. Since $\sum_{j=M_0}^{M_\epsilon}\alpha_j$ is constant, then:

$$\frac{\sum_{j=M_0}^{M_\epsilon}\alpha_j}{\sum_{j=M_\epsilon}^{N}\beta_j} \to 0 \;\; as \;\; N \to \infty$$

Then $\exists K_\epsilon \geq M_\epsilon$ s.t. $\forall N \geq K_\epsilon$

$$-\epsilon < \frac{\sum_{j=M_0}^{M_\epsilon}\alpha_j}{\sum_{j=M_\epsilon}^{N}\beta_j} < \epsilon$$

and similarly:

$$-\epsilon < \frac{\sum_{j=M_0}^{M_\epsilon} \beta_j}{\sum_{j=M_\epsilon}^{N} \beta_j} < \epsilon$$

Hence $\forall \epsilon \in (0, \frac{1}{2})$ $\exists K_\epsilon \in \mathbb{N}$ s.t. $\forall N \geq K_\epsilon$

$$\frac{1-2\epsilon}{1+\epsilon} < \frac{\sum_{j=M_0}^{N} \alpha_j}{\sum_{j=M_0}^{N} \beta_j} < \frac{1+2\epsilon}{1-\epsilon}$$

$\Rightarrow$

$$\lim_{N\to\infty} \frac{\sum_{j=M_0}^{N} \alpha_j}{\sum_{j=M_0}^{N} \beta_j} = 1 \ \ asN \to \infty$$

$\square$

Proof of Theorem 4.1. Let $\forall \ N \in \mathbb{N}$,

$$q_k^{(\alpha)} = \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}$$

and

$$(1-z)^{-\alpha} = \sum_{j=0}^{\infty} q_j^{(\alpha)} z^j$$

Also define:

$$g(z) = (1-z)^{-\frac{1}{2}} = \sum_{j=0}^{\infty} q_j^{(\frac{1}{2})} z^j$$

Let:$f(z) = g(z)^2$. We see below that $g(z) = (1-z)^{-\frac{1}{2}} \notin H^2$. Let; $\forall N \in \mathbb{N}$;

$$g_N(z) = \sum_{j=0}^{N} q_j^{(\frac{1}{2})} z^j \in H^2$$

then:

$$f_N(z) = (g_N(z))^2 \in H^1$$

. Hence:

$$\begin{aligned} f_N(z) &= \left(\sum_{j=0}^{N} q_j^{(\frac{1}{2})} z^j\right)\left(\sum_{j=0}^{N} q_j^{(\frac{1}{2})} z^j\right) \\ &= \sum_{n=0}^{2N} \underbrace{\left(\sum_{k=0}^{n} q_{n-k}^{(\frac{1}{2})} q_k^{(\frac{1}{2})}\right)}_{\gamma_n} z^n \end{aligned} \tag{4.2}$$

$\gamma_n = 1 \ \ \forall n \in \{0, 1, \cdots, N\}$ and $\gamma_n \geq 0 \ \ \forall n \in \{N+1, \cdots, 2N\}$.

Fix $N \in \mathbb{N}$.

$$\begin{aligned} LHS_N &= \sum_{n=1}^{\infty} \frac{\left|\hat{f}_N(n)\right|}{n} \\ &= \sum_{n=1}^{2N} \frac{|\gamma_n|}{n} \\ &\geq \sum_{n=1}^{N} \frac{|\gamma_n|}{n} \\ &= \sum_{n=1}^{N} \frac{1}{n} \end{aligned} \tag{4.3}$$

Then

$$RHS_N = \|f_N\|_{H^1},$$

$$\|f_N\|_{H^1} = \|g_N\|_{H^2}^2 = \sum_{j=0}^{N} \left(q_j^{\left(\frac{1}{2}\right)}\right)^2$$

and

$$q_j^{\left(\frac{1}{2}\right)} \sim \frac{j^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi} j^{\frac{1}{2}}}$$

$\Rightarrow$

$$\left(q_j^{\left(\frac{1}{2}\right)}\right)^2 \sim \frac{1}{\pi j} \quad as \quad N \to \infty.$$

We know that

$$\sum_{j=1}^{N} \frac{1}{\pi j} \longrightarrow \infty$$

so

$$\pi \geq \frac{LHS_N}{RHS_N} \geq \frac{\sum_{n=1}^{N} \frac{1}{n}}{\sum_{j=0}^{N} \left(q_j^{\left(\frac{1}{2}\right)}\right)^2}$$

Let $\beta_j = \frac{1}{\pi j}$. We know that:

$$\sum_{j=1}^{\infty} \frac{1}{\pi j} = \infty$$

and

$$\frac{\frac{1}{\pi j}}{\left(q_j^{\left(\frac{1}{2}\right)}\right)^2} \to 1$$

because $\left(q_j^{\left(\frac{1}{2}\right)}\right)^2 \sim \frac{1}{\pi j}$. Then, by the Lemma 4.1

$$\pi \geq \frac{LHS_N}{RHS_N} = \frac{\pi \sum_{j=1}^{N} \frac{1}{\pi j}}{\sum_{j=0}^{N} \left(q_j^{\left(\frac{1}{2}\right)}\right)^2} \to \pi 1$$

And finally we obtain that:

$$\frac{\sum_{n=1}^{N} \frac{|\hat{f}(n)|}{n}}{\|f\|_{H^1}} \geq \frac{\sum_{n=1}^{N} \frac{1}{n}}{\sum_{j=0}^{N} \left(q_j^{\left(\frac{1}{2}\right)}\right)^2} \to \pi$$

as $N \to \infty$. $\square$

Theorem 4.2. *Let $f \in H^1$ Consider the Hardy Type inequality below:*

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(4k+2)|}{4k+2} \leq C\|f\|_{H^1}.$$

Then the best constant for this inequality is: $C = \frac{\pi}{4}$.

Proof. We have already obtained that

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(4k+2)|}{4k+2} \leq \frac{\pi}{4}\|f\|_{H^1} \quad (Theorem 3.2)$$

Now we need to prove that $\frac{\pi}{4}$ is the best constant for this "inequality". Let

$$g \in H^2 = \left[g(z) = \sum_{n=0}^{\infty} b_n z^n \; \forall z \in \overline{\Delta} \right]$$

Suppose $b_0 = 0$ and each $b_n \geq 0$.

$$f(z) = z^{-1}(g(z))^2 \, , f \in H^1$$

Fix $N \in \mathbb{N}$.

$$b_n = \begin{cases} 0, & \text{if } n \notin \{4k+2, k \geq 0\}; \\ 0, & \text{if } n = 0; \\ \frac{1}{(4k+2)^{\frac{1}{2}}}, & \forall k \in \{0, \cdots, N\} \text{ and } n = 4k+2; \\ 0, & \forall k \geq N+1; \end{cases}$$

$g(z) = \sum_{n=1}^{\infty} b_n z^n = \sum_{k=0}^{N} \frac{1}{(4k+2)^{\frac{1}{2}}} z^{4k+2} = \sum_{k=0}^{\infty} b_{4k+2} z^{4k+2}$ Then by the definition of $f(z)$

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} b_{4m+2} b_{4(k-m)+2} \right) z^{4k+2}$$

So,

$$\begin{aligned} \|f\|_{H^1} &= \sum_{j=0}^{\infty} b_j^2 \\ &= \sum_{k=0}^{N} \left(\frac{1}{(4k+2)^{\frac{1}{2}}} \right)^2 \\ &\sim \int_{x=0}^{N} \frac{1}{(4x+2)} dx \\ &= \frac{1}{4} \ln(2N+1) \end{aligned} \tag{4.4}$$

On the other hand;

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} &= \sum_{k=0}^{\infty} \frac{|\hat{f}(4k+2)|}{4k+2} \\ &= \sum_{m=0}^{N} \frac{1}{(4m+2)^{\frac{1}{2}}} \sum_{j=0}^{N} \frac{1}{(4j+2)^{\frac{1}{2}}(4j+4m+2)} \\ &\geq \sum_{m=0}^{N} \frac{1}{(4m+2)} \int_{\frac{2}{4m+2}}^{\infty} \frac{du}{4(u+1)(u)^{\frac{1}{2}}} - \sum_{m=0}^{N} \frac{1}{(4m+2)^{\frac{1}{2}}} \int_{x=N}^{\infty} \frac{dx}{(4x+2)^{\frac{3}{2}}} \\ &\geq \frac{\pi}{16} \ln(2N+1) - \left(\frac{1}{2} - \frac{\sqrt{2}}{\sqrt{4N+2}} \right) \end{aligned} \tag{4.5}$$

Recall that

$$\|f\|_{H^1} = \frac{1}{4} \ln(2N+1)$$

Then,

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty}\frac{|\hat{f}(4n+2)|}{4n+2}}{\|f\|_{H^1}} &\geq \frac{\frac{\pi}{16}\ln(2N+1)-\left(\frac{1}{2}-\frac{\sqrt{2}}{\sqrt{4N+2}}\right)}{\frac{1}{4}\ln(2N+1)} \\
&= \frac{\pi}{4}-\left(\frac{\left(\frac{1}{2}-\frac{\sqrt{2}}{\sqrt{4N+2}}\right)}{\frac{1}{4}\ln(2N+1)}\right) \\
&= \frac{\pi}{4}-\left[\frac{1}{2}-\frac{\sqrt{2}}{\sqrt{4N+2}}\right]\frac{1}{4\ln(2N+1)} \\
&\to \frac{\pi}{4}
\end{aligned} \tag{4.6}$$

as $N \longrightarrow \infty$.

We can generalize this results for all $n = \upsilon k + \frac{\upsilon}{2}$ cases, where υ is even. □

Theorem 4.3. *By the Theorem 3.6 we know that:*

$$\sum_{n=0}^{\infty}\frac{\left|\hat{f}(\upsilon k+\frac{\upsilon}{2})\right|}{\upsilon k+\frac{\upsilon}{2}} \leq \frac{\pi}{sin\left(\frac{\pi\frac{\upsilon}{2}}{\upsilon}\right)}\frac{1}{\upsilon}\|f\|_{H^1} = \frac{\pi}{\upsilon}\|f\|_{H^1}$$

Then $\frac{\pi}{\upsilon}$ is the best possible constant for the inequality above.

Proof. Let $g \in H^2 = \left[g(z) = \sum_{n=0}^{\infty} b_n z^n \, , \forall z \in \overline{\Delta}\right]$. Suppose $b_0 = 0$ and each $b_n \geq 0$, and define f

$$f(z) = z^{-1}(g(z))^2 \, , f \in H^1$$

Then: $\|f\|_{H^1} = \sum_{n=1}^{\infty} b_n^2$. And

$$\sum_{n=1}^{\infty}\frac{|\hat{f}(n)|}{n} = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n} b_j b_{n-j+1} = \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_j b_m}{m+j-1}$$

Fix $N \in \mathbb{N}$.

$$b_n = \begin{cases} 0, & \text{for } n \notin \left\{\upsilon + \frac{\upsilon}{2}\, , k \geq\right\}; \\ 0, & \text{for } n = 0; \\ \frac{1}{(\upsilon k+\frac{\upsilon}{2})^{\frac{1}{2}}}, & \text{for } n = \upsilon k + \frac{\upsilon}{2}\, , k \in \{0, \cdots, N\}; \\ 0, & \text{for } k \geq N+1\, ; \end{cases}$$

Then

$$g(z) = \sum_{n=1}^{\infty} b_n z^n = \sum_{k=0}^{N} \frac{1}{(\upsilon k + \frac{\upsilon}{2})^{\frac{1}{2}}} z^{\upsilon k + \frac{1}{2}} = \sum_{k=0}^{\infty} b_{\upsilon k + \frac{\upsilon}{2}} z^{\upsilon k + \frac{\upsilon}{2}}$$

Since

$$f(z) = z^{-1}(g(z))^2$$

Then

$$\begin{aligned} \|f\|_{H^1} &= \sum_{j=0}^{\infty} b_j^2 \\ &= \sum_{k=0}^{N} \left(\frac{1}{\upsilon k + \frac{\upsilon}{2}} \right) \\ &\sim \frac{1}{\upsilon} \ln(2N+1) \end{aligned} \tag{4.7}$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left|\hat{f}(n)\right|}{n} &= \sum_{k=0}^{\infty} \frac{\left|\hat{f}(\upsilon k + \frac{\upsilon}{2})\right|}{\upsilon k + \frac{\upsilon}{2}} \\ &= \sum_{k=0}^{\infty} \frac{1}{\upsilon k + \frac{\upsilon}{2}} \sum_{m=0}^{k} b_{\upsilon m + \frac{\upsilon}{2}} b_{\upsilon(k-m) + \frac{\upsilon}{2}} \\ &= \sum_{m=0}^{\infty} b_{\upsilon m + \frac{\upsilon}{2}} \sum_{k=m}^{\infty} \frac{b_{\upsilon(k-m) + \frac{\upsilon}{2}}}{\upsilon k + \frac{\upsilon}{2}} \\ &= \sum_{m=0}^{\infty} b_{\upsilon m + \frac{\upsilon}{2}} \sum_{j=0}^{\infty} \frac{b_{\upsilon j + \frac{\upsilon}{2}}}{\upsilon(j+m) + \frac{\upsilon}{2}} \\ &= \sum_{m=0}^{N} \frac{1}{\left(\upsilon m + \frac{\upsilon}{2}\right)^{\frac{1}{2}}} \sum_{j=0}^{N} \frac{1}{\left(\upsilon j + \frac{\upsilon}{2}\right)^{\frac{1}{2}} \left(\upsilon j + \upsilon m + \frac{\upsilon}{2}\right)} \\ &\geq \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \sum_{j=0}^{N} \frac{1}{(\upsilon j + \frac{\upsilon}{2})^{\frac{1}{2}} (\upsilon j + \upsilon m + \upsilon)} \\ &= \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \int_{x=0}^{N} \frac{dx}{(\upsilon + \frac{\upsilon}{2})^{\frac{1}{2}} (\upsilon x + \upsilon m + \upsilon)} \\ &= \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \underbrace{\int_{x=0}^{\infty} \frac{dx}{(\upsilon x + \frac{\upsilon}{2})^{\frac{1}{2}} (\upsilon x + \upsilon m + \upsilon)}}_{A} \\ &- \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \underbrace{\int_{x=N}^{\infty} \frac{dx}{(\upsilon x + \frac{\upsilon}{2})^{\frac{1}{2}} (\upsilon x + \upsilon m + \upsilon)}}_{B} \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
A &= \int_{x=0}^{\infty} \frac{dx}{(\upsilon x + \frac{\upsilon}{2})^{\frac{1}{2}} \left(\frac{\upsilon x + \frac{\upsilon}{2}}{\upsilon m + \frac{\upsilon}{2}} + 1\right) (\upsilon m + \frac{\upsilon}{2})} \\
&= \int_{\frac{1}{2m+1}}^{\infty} \frac{du}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}} u^{\frac{1}{2}} (u+1) \upsilon}
\end{aligned} \tag{4.9}
$$

And

$$
B \leq \int_{x=N}^{\infty} \frac{dx}{(\upsilon x + \frac{\upsilon}{2})^{\frac{3}{2}}}
$$

Then:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} &\geq \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \int_{\frac{1}{2m+1}}^{\infty} \frac{du}{\upsilon (u+1) u^{\frac{1}{2}} (\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} - \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \int_{x=N}^{\infty} \frac{dx}{(\upsilon x + \frac{\upsilon}{2})^{\frac{3}{2}}} \\
&\geq \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})} \frac{1}{\upsilon} \left[\pi - \int_{u=0}^{\frac{1}{2m+1}} \frac{du}{(u)^{\frac{1}{2}}}\right] - \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \int_{x=N}^{\infty} \frac{dx}{(\upsilon x + \frac{\upsilon}{2})^{\frac{3}{2}}} \\
&= \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})} \frac{1}{\upsilon} \left(\pi - \frac{2}{(2m+1)^{\frac{1}{2}}}\right) - \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}} \frac{2}{\upsilon} \left(\frac{1}{(\upsilon N + \frac{\upsilon}{2})^{\frac{1}{2}}}\right) \\
&= \underbrace{\sum_{m=0}^{N} \frac{\pi}{\upsilon} \frac{1}{(\upsilon m + \frac{\upsilon}{2})}}_{\frac{\pi}{\upsilon^2} \ln(2N+1)} - \underbrace{\sum_{m=0}^{N} \frac{4}{\upsilon^2 (2m+1)^{\frac{3}{2}}}}_{\frac{4}{\upsilon^2}\left[1 - \frac{1}{\sqrt{2N+1}}\right]} - \underbrace{\frac{2}{\upsilon} \left(\frac{1}{(\upsilon N + \frac{\upsilon}{2})^{\frac{1}{2}}}\right) \sum_{m=0}^{N} \frac{1}{(\upsilon m + \frac{\upsilon}{2})^{\frac{1}{2}}}}_{\frac{4}{\upsilon^2}\left[1 - \frac{1}{\sqrt{2N+1}}\right]}
\end{aligned} \tag{4.10}
$$

Recall that

$$
\|f\|_{H^1} = \frac{1}{\upsilon} \ln(2N+1)
$$

Then:

$$
\begin{aligned}
\frac{\sum_{n=0}^{\infty} \frac{|\hat{f}(n)|}{n}}{\|f\|_{H^1}} &= \frac{\frac{\pi}{\upsilon^2} \ln(2N+1) - \frac{8}{\upsilon^2}\left[1 - \frac{1}{\sqrt{2N+1}}\right]}{\frac{1}{\upsilon} \ln(2N+1)} \\
&= \frac{\pi}{\upsilon} - \frac{8}{\upsilon} \frac{1}{\ln(2N+1)} \longrightarrow \frac{\pi}{\upsilon}
\end{aligned} \tag{4.11}
$$

as $N \longrightarrow \infty$. □

Which proves that $\frac{\pi}{\upsilon}$ is the best possible constant in this case.

The method that we used in Theorem 4.3 to solve the best constant problem in "Hardy like inequalities" does not work in the general case. Which means we could not prove that

"$\frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)}\frac{1}{\upsilon}$" is the best constant for the general form of the "Hardy like inequality" below:

$$\sum_{k=0}^{\infty}\frac{|\hat{f}_{\upsilon k+j}|}{\upsilon k+j}\leq\frac{\pi}{sin\left(\frac{\pi j}{\upsilon}\right)}\frac{1}{\upsilon}\|f\|_{H^1}$$

This condition led us to think about if we can find better constants for the other cases. Moreover the method to provide the proof is cumbersome for calculations. So we seek to develop an easier way to prove that those constants are the best ones.

Our experiments with Mathematica shows us that for $f\in H^1$

$$\sum_{k=0}^{\infty}\frac{|\hat{f}_(\upsilon k+j)|}{\upsilon k+j}\leq\frac{\pi}{\upsilon}\|f\|_{H^1} \tag{4.12}$$

when $\frac{j}{\upsilon}>\frac{1}{2}$. Finally we were able to find a method to prove it.

Before starting the proof of general case i.e. equation (4.12) let consider special cases to provide better understanding.

We know by the Theorem 3.6 that if $f\in H^1=\left[f(z)=\sum_{n=0}^{\infty}a_nz^n\right]$, then:

$$\sum_{k=0}^{\infty}\frac{|a_{8k+4}|}{8k+4}\leq\frac{\pi}{8}\|f\|_{H^1}$$

Our claim is here:

1. $$\sum_{k=0}^{\infty}\frac{|a_{8k+5}|}{8k+5}\leq\frac{\pi}{8}\|f\|_{H^1}$$

2. $$\sum_{k=0}^{\infty}\frac{|a_{8k+6}|}{8k+6}\leq\frac{\pi}{8}\|f\|_{H^1}$$

3. $$\sum_{k=0}^{\infty}\frac{|a_{8k+7}|}{8k+7}\leq\frac{\pi}{8}\|f\|_{H^1}$$

Proof. 1. **Case 1:** Let $a_0=0$ and $f\in H^1$.

$$f(z)=\sum_{n=0}^{\infty}a_nz^n=z\sum_{n=1}^{\infty}a_nz^{n-1}=z\sum_{m=0}^{\infty}c_mz^m\ \ where\ \ c_m=a_{m+1}$$

Define:

$$\Psi(z) = \sum_{m=0}^{\infty} c_m z^m \Rightarrow \|f\|_{H^1} = \|\Psi\|_{H^1}$$

Since $\Psi \in H^1$ then:

$$\sum_{k=0}^{\infty} \frac{|\hat{\Psi}(8k+4)|}{8k+4} \leq \frac{\pi}{8}\|\Psi\|_{H^1} = \frac{\pi}{8}\|f\|_{H^1}$$

Fix $k \in \mathbb{N}_0$.

$$\hat{\Psi}(8k+4) = c_{8k+4} = a_{8k+4+1} = a_{8k+5} = \hat{f}(8k+5)$$

Then:

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(8k+5)|}{8k+5} \leq \sum_{k=0}^{\infty} \frac{|\hat{f}(8k+5)|}{8k+4} = \sum_{k=0}^{\infty} \frac{|\hat{\Psi}(8k+4)|}{8k+4} \leq \frac{\pi}{8}\|f\|_{H^1}$$

Case 2: Let $a_0 \neq 0$.

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\hat{f}(8k+5)}{8k+5} &= \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+5}(e^{i\theta})} f(e^{i\theta})d\theta \\
&= \underbrace{\frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} a_0\overline{w_{8\mathbb{N}_0+5}(e^{i\theta})}d\theta}_{0} + \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+5}(e^{i\theta})} \underbrace{\sum_{n=1}^{\infty} a_n e^{in\theta} d\theta}_{e^{i\theta}\Psi(e^{i\theta})}
\end{aligned} \tag{4.13}$$

Define: $e^{i\theta}\Psi(e^{i\theta}) = \Phi(e^{i\theta})$ Then:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\hat{f}(8k+5)}{8k+5} &= \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} w_{8\mathbb{N}_0+5}(e^{i\theta})\Phi(e^{i\theta}) \\
&= \sum_{k=0}^{\infty} \frac{\hat{\Phi}(8k+5)}{8k+5} = \sum_{k=0}^{\infty} \frac{\hat{\Psi}(8k+4)}{8k+5} \\
&\leq \sum_{k=0}^{\infty} \frac{\hat{\Psi}(8k+4)}{8k+4} \\
&= \frac{1}{2\pi}\frac{1}{8i}\left[\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+4}(e^{i\theta})} \underbrace{\left(a_0 e^{-i\theta} + \Psi(e^{i\theta})\right)}_{e^{-i\theta}(a_0 + e^{i\theta}\Psi(e^{i\theta}))} d\theta\right] \\
&\leq \frac{1}{8}\frac{1}{2\pi}\int_{-\pi}^{\pi} |w_{8\mathbb{N}_0+4}(e^{i\theta})||e^{-i\theta}f(e^{i\theta})| \\
&= \frac{1}{8}\|w_{8\mathbb{N}_0+4}\|_{\infty}\|f\|_{L^1} \\
&= \frac{\pi}{8}\|f\|_{H^1}
\end{aligned} \tag{4.14}$$

The remaining part follows by the $H^1 - H^2$ factorization theorem.

2. Let $f \in H^1$ and this time define:

$$\Psi(z) = \sum_{m=0}^{\infty} c_m z^m \;\; where \;\; c_m = a_{m+2}$$

and

$$\Phi(z) = z^2 \Psi(z)$$

Case 1**:** Let $a_0 = a_1 = 0$, then it is obvious that $\|f\|_{H^1} = \|\Psi\|_{H^1}$. Fix $k \in \mathbb{N}_0$. Then $\Psi(8k+4) = c_{8k+4} = a_{8k+6} = \hat{f}(8k+6) \Rightarrow$

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(8k+6)|}{8k+6} \leq \sum_{k=0}^{\infty} \frac{|\hat{f}(8k+6)|}{8k+4} = \sum_{k=0}^{\infty} \frac{|\hat{\Psi}(8k+4)|}{8k+4} \leq \frac{\pi}{8}\|f\|_{H^1}$$

Case2**:** Let consider general case, i.e.: $a_0 \neq 0$ or $a_1 \neq 0$.

$$f(z) = \sum_{k=0}^{\infty} a_n z^n \in H^1$$

then:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{a_{8k+6}}{8k+6} &= \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+6}(e^{i\theta})} f(e^{i\theta}) d\theta \\
&= \underbrace{\frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} a_0 \overline{w_{8\mathbb{N}_0+6}(e^{i\theta})} d\theta}_{0} + \underbrace{\frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} a_1 e^{i\theta}\overline{w_{8\mathbb{N}_0+6}(e^{i\theta})} d\theta}_{0} \\
&+ \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+6}(e^{i\theta})} \underbrace{\sum_{n=2}^{\infty} a_n e^{in\theta}}_{\Phi(e^{i\theta})} \\
&= \sum_{k=0}^{\infty} \frac{\hat{\Phi}(8k+6)}{8k+6} = \sum_{k=0}^{\infty} \frac{\hat{\Psi}(8k+4)}{8k+6} \leq \sum_{k=0}^{\infty} \frac{\hat{\Psi}(8k+4)}{8k+4} \\
&= \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+4}(e^{i\theta})} \Psi(e^{i\theta}) d\theta \\
&= \frac{1}{2\pi}\frac{1}{8i}\int_{-\pi}^{\pi} \overline{w_{8\mathbb{N}_0+4}(e^{i\theta})} \left[e^{-2i\theta}(a_0 + a_1 e^{i\theta} + e^{2i\theta}(\Psi(e^{i\theta})))\right] d\theta \\
&\leq \frac{1}{8}\frac{1}{2\pi}\int_{-\pi}^{\pi} |w_{8\mathbb{N}_0+4}(e^{i\theta})| e^{-2i\theta} f(e^{i\theta})| d\theta = \frac{\pi}{8}\|f\|_{H^1}
\end{aligned} \tag{4.15}$$

It is easy to prove the remaining part by using the $H^1 - H^2$ factorization theorem.

3. To prove that:
$$\sum_{k=0}^{\infty}\frac{|\hat{f}(8k+7)|}{8k+7}\leq\frac{\pi}{8}\|f\|_{H^1}$$
The only thing we need to do is to define:
$$\Psi(z)=\sum_{m=0}^{\infty}c_m z^m \;\; where \;\; c_m=a_{m+3}$$
and then follow the same steps as the previous cases.

□

Notice that the method that we use to prove that $\frac{\pi}{\upsilon}$ is a better constant for the Hardy like inequalites $\sum_{k=0}^{\infty}\frac{|\hat{f}(\upsilon k+j)|}{\upsilon k+j}\leq\frac{\pi}{\upsilon}\|f\|_{H^1}$ works when $\frac{j}{\upsilon}>\frac{1}{2}$ and when υ is even. Let's consider the cases when υ is odd. Now we are going to find a better constant for the $3k+2$ case by using $6k+4$ case. And then we will generalize this method for all $a_{\upsilon k+j}$ cases where $\frac{j}{\upsilon}>\frac{1}{2}$.

Theorem 4.4. *Let f be any arbitrary H^1 function then,*
$$\sum_{k=0}^{\infty}\frac{|\hat{f}(3k+2)|}{3k+2}\leq\frac{\pi}{3}\|f\|_{H^1}$$

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $G(z)=f(z^2)$. Then,
$$\|G\|_{H^1}=\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{i\theta})|d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{2i\theta})|d\theta=\|f\|_{H^1}$$
And
$$\sum_{k=0}^{\infty}\frac{|\hat{G}(6k+4)|}{6k+4}=\sum_{k=0}^{\infty}\frac{|\hat{f}(3k+2)|}{6k+4}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{|\hat{f}(3k+2)|}{3k+2}$$
Consider now arbitrary function $I\in H^1$. By the H^1-H^2 factorization theorem we may assume that each $u_n\geq 0$. We know by Theorem (3.6) that:
$$\sum_{k=0}^{\infty}\frac{u_{6k+3}}{6k+3}=\frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi}I(e^{i\theta})\overline{w_{6\mathbb{N}_0+3}(e^{i\theta})}d\theta$$
s.t.
$$w_{6\mathbb{N}_0+3}(e^{i\theta})=\sum_{n\in\mathbb{Z}}\hat{w}_{6\mathbb{N}_0+3}(n)e^{in\theta}$$

and

$$\hat{w}_{6\mathbb{N}_0+3}(n) = \begin{cases} \frac{6}{i(6k+3)} & \text{if } n = 6k+3; \\ 0 & \text{otherwise;} \end{cases}$$

Similarly,

$$\sum_{k=0}^{\infty} \frac{u_{6k+4}}{6k+4} = \frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} I(e^{i\theta})\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta$$

s.t.

$$w_{6\mathbb{N}_0+4}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \hat{w}_{6\mathbb{N}_0+4}(n)e^{in\theta}$$

and

$$\hat{w}_{6\mathbb{N}_0+4}(n) = \begin{cases} \frac{6}{i(6k+4)} & \text{if } n = 6k+4; \\ 0 & \text{otherwise;} \end{cases}$$

Assume $\Psi(z) = \sum_{m=0}^{\infty} c_m z^m$ where $c_m = u_{m+1}$. And assume $z\Psi(z) = \Phi(z)$. Then:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{u_{6k+4}}{6k+4} &= \frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} I(e^{i\theta})\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta \\
&= \frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{\infty} u_n e^{in\theta}\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta \\
&= \underbrace{[\frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} u_0 e^{i\theta}\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta]}_{0} \\
&+ [\frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} u_n e^{in\theta}\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta] \\
&= [\frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} u_n e^{in\theta}\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta] \\
&= [\frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta}\Psi(e^{i\theta})\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta] \\
&= [\frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{i\theta})\overline{w_{6\mathbb{N}_0+4}(e^{i\theta})}d\theta] \\
&= \sum_{k=0}^{\infty} \frac{\hat{\Phi}(6k+4)}{6k+4} \\
&= \sum_{k=0}^{\infty} \frac{\hat{\Psi}(6k+3)}{6k+4} \\
&\leq \sum_{k=0}^{\infty} \frac{\hat{\Psi}(6k+3)}{6k+3}
\end{aligned} \tag{4.16}$$

On the other hand:

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{\hat{\Psi}(6k+3)}{6k+3} &= \frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{w_{6\mathbb{N}_0+3}(e^{i\theta})}\Psi(e^{i\theta})d\theta \\
&= \frac{1}{6i}\frac{1}{2\pi}\underbrace{[\int_{-\pi}^{\pi}\overline{w_{6\mathbb{N}_0+3}(e^{i\theta})}u_0e^{-i\theta}d\theta]}_{0}+\frac{1}{6i}\frac{1}{2\pi}[\int_{-\pi}^{\pi}\overline{w_{6\mathbb{N}_0+3}(e^{i\theta})}\Psi(e^{i\theta})d\theta] \\
&= \frac{1}{6i}\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{w_{6\mathbb{N}_0+3}(e^{i\theta})}e^{-i\theta}\underbrace{(u_0+e^{i\theta}\Psi(e^{i\theta}))}_{I(e^{i\theta})}d\theta \\
&\le \frac{1}{2\pi}\frac{1}{6}\int_{-\pi}^{\pi}|w_{6\mathbb{N}_0+3}(e^{i\theta})||e^{-i\theta}I(e^{i\theta})|d\theta \\
&\le \frac{1}{6}\|w_{6\mathbb{N}_0+3}\|_{\infty}\|I\|_{H^1} \\
&= \frac{\pi}{6}\|I\|_{H^1}
\end{aligned} \tag{4.17}
$$

Since $G \in H^1$, then:

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{|\hat{f}(3k+2)|}{3k+2} &= 2\sum_{k=0}^{\infty}\frac{|\hat{G}(6k+4)|}{6k+4} \\
&\le 2\frac{\pi}{6}\|G\|_{H^1} \\
&= \frac{\pi}{3}\|f\|_{H^1}
\end{aligned} \tag{4.18}
$$

□

Now let's generalize the cases above for all $a_{\upsilon k+j}$ cases where $\frac{j}{\upsilon} > \frac{1}{2}$.

Theorem 4.5. *Let* $f \in H^1 = \left[f(z) = \sum_{k=0}^{\infty} a_n z^n \quad z \in \overline{\Delta}\right]$

$$\sum_{k=0}^{\infty}\frac{|\hat{f}(\upsilon k+j)|}{\upsilon k+j} \le \frac{\pi}{\upsilon}\|f\|_{H^1}$$

Proof. **Case** 1**:** Let υ be even. We already know that;

$$\sum_{k=0}^{\infty}\frac{|\hat{f}(\upsilon k+\frac{\upsilon}{2})|}{\upsilon k+\frac{\upsilon}{2}} \le \frac{\pi}{\upsilon}\|f\|_{H^1}$$

Let $j > \frac{\upsilon}{2}$ where $\upsilon, j \in \mathbb{N}-\{0\}$. Then $j = \frac{\upsilon}{2}+t$, $t \in \mathbb{N}-\{0\}$.

Case 1**(a):** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_0 = a_1 = \cdots = a_{t-1} = 0$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = z^t\sum_{n=t}^{\infty} a_n z^{n-t} = z^t\sum_{m=0}^{\infty} c_m z^m,$$

where $c_m = a_{m+t}$ say $\Psi(z) = \sum_{m=0}^{\infty} c_m z^m$. Then $\|f\|_{H^1} = \|\Psi\|_{H^1}$. Since $\Psi \in H^1$ then;

$$\sum_{k=0}^{\infty} \frac{|\hat{\Psi}(\upsilon k + \frac{\upsilon}{2})|}{\upsilon k + \frac{\upsilon}{2}} \leq \frac{\pi}{\upsilon}\|\Psi\|_{H^1} = \frac{\pi}{\upsilon}\|f\|_{H^1}$$

Let $j = \frac{\upsilon}{2} + t$ and $t < \frac{\upsilon}{2}$, $t \in \mathbb{N}^+$. Fix $k \in \mathbb{N}_0$.

$$\hat{\Psi}(\upsilon k + \frac{\upsilon}{2}) = c_{\upsilon k + \frac{\upsilon}{2}} = a_{\upsilon k + \frac{\upsilon}{2} + t} = a_{\upsilon k + j} = \hat{f}(\upsilon k + j).$$

Then:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{|\hat{f}(\upsilon k + j)|}{\upsilon k + j} &\leq \sum_{k=0}^{\infty} \frac{|\hat{f}(\upsilon k + j)|}{\upsilon k + \frac{\upsilon}{2}} \\ &= \sum_{k=0}^{\infty} \frac{|\hat{\Psi}(\upsilon k + \frac{\upsilon}{2})|}{\upsilon k + \frac{\upsilon}{2}} \\ &\leq \frac{\pi}{\upsilon}\|f\|_{H^1} \end{aligned} \tag{4.19}$$

Case1**(b):**Consider now the general case, i.e.:at least one of $a_0 \neq 0, a_1 \neq 0, \cdots, a_{t-1} \neq 0$, and define

$$\Psi(z) = \sum_{m=0}^{\infty} c_m z^m \quad c_m = a_{m+t} \quad and \quad \Phi(z) = z^t \Psi(z)$$

Then:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\hat{f}(\upsilon k + j)}{\upsilon k + j} &= \frac{1}{2\pi}\frac{1}{\upsilon i}\int_{-\pi}^{\pi} \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} f(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi}\frac{1}{\upsilon i}\left[\int_{-\pi}^{\pi} a_0 \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} d\theta + \int_{-\pi}^{\pi} a_1 e^{i\theta} \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} d\theta + \cdots\right] \\ &+ \frac{1}{2\pi}\frac{1}{\upsilon i}\left[\underbrace{\int_{-\pi}^{\pi} a_{t-1} e^{i\theta(t-1)} \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} d\theta}_{0} + \int_{-\pi}^{\pi} \overline{w_{\upsilon\mathbb{N}_0+j}(e^{i\theta})} \underbrace{\sum_{n=t}^{\infty} a_n e^{in\theta}}_{\Phi(e^{i\theta})} d\theta\right] \\ &= \sum_{k=0}^{\infty} \frac{\hat{\Phi}(\upsilon k + j)}{\upsilon k + j} = \sum_{k=0}^{\infty} \frac{\hat{\Psi}(\upsilon k + \frac{\upsilon}{2})}{\upsilon k + j} \\ &\leq \sum_{k=0}^{\infty} \frac{\hat{\Psi}(\upsilon k + \frac{\upsilon}{2})}{\upsilon k + \frac{\upsilon}{2}} \end{aligned} \tag{4.20}$$

and

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{\hat{\Psi}(\upsilon k+\frac{\upsilon}{2})}{\upsilon k+\frac{\upsilon}{2}} &= \frac{1}{2\pi}\frac{1}{\upsilon i}\int_{-\pi}^{\pi}\overline{w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})}\Psi(e^{i\theta})d\theta \\
&= \frac{1}{2\pi}\frac{1}{\upsilon i}\left[\int_{-\pi}^{\pi}\overline{w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})}a_0e^{-it\theta}+\int_{-\pi}^{\pi}\overline{w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})}a_1e^{-i(t-1)\theta}d\theta+\cdots\right] \\
&+\frac{1}{2\pi}\frac{1}{\upsilon i}\left[\int_{-\pi}^{\pi}\overline{w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})}a_{t-1}e^{-i\theta}d\theta+\int_{-\pi}^{\pi}\overline{w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})}\Psi(e^{i\theta})d\theta\right] \\
&= \frac{1}{2\pi}\frac{1}{\upsilon i}\int_{-\pi}^{\pi}\overline{w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})}e^{-it\theta}f(e^{i\theta})d\theta \\
&\leq \frac{1}{\upsilon}\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}|w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}(e^{i\theta})||e^{-it\theta}f(e^{i\theta})|\right]d\theta \\
&= \frac{1}{\upsilon}\left\|w_{\upsilon\mathbb{N}_0+\frac{\upsilon}{2}}\right\|_{\infty}\|f\|_{L^1}=\frac{\pi}{\upsilon}\|f\|_{H^1}
\end{aligned} \tag{4.21}
$$

Case 2: Now, let's consider the case, when t is odd.

$$
\sum_{k=0}^{\infty}\frac{|\hat{f}(tk+j)|}{tk+j}\leq\frac{\pi}{t}\|f\|_{H^1}
$$

for all $\frac{j}{t}>\frac{1}{2}$ where t is odd. Our assumption here is t is odd then $2t$ is even. For given $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $z\in\overline{\Delta}$. Define

$$
G(z)=f(z^2)=\sum_{n=0}^{\infty}a_nz^{2n}
$$

Then

$$
\begin{aligned}
\|G\|_{H^1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{i\theta})|d\theta \\
&= \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi}|f(e^{2i\theta})|d\theta \\
&= \frac{1}{2}\left[\frac{1}{2\pi}\int_{\beta=0}^{2\pi}|f(e^{i\beta})|d\beta+\frac{1}{2\pi}\int_{\beta=-2\pi}^{0}|f(e^{i\beta})|d\beta\right] \\
&= \frac{1}{2}\left[\|f\|_{H^1}+\|f\|_{H^1}\right]=\|f\|_{H^1}
\end{aligned} \tag{4.22}
$$

On the other hand:

$$
\sum_{k=0}^{\infty}\frac{|\hat{G}(2tk+2j)|}{2tk+2j}=\sum_{k=0}^{\infty}\frac{|\hat{f}(tk+j)|}{2tk+2j}
$$

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(tk+j)|}{tk+j} = 2\sum_{k=0}^{\infty} \frac{|\hat{G}(2tk+2j)|}{2tk+2j} \tag{4.23}$$
$$\leq 2\frac{\pi}{2t}\|G\|_{H^1} = \frac{\pi}{t}\|f\|_{H^1}$$

So we proved that it is allways true that:

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(\upsilon k+j)|}{\upsilon k+j} \leq \frac{\pi}{\upsilon}\|f\|_{H^1}$$

where $\frac{j}{\upsilon} \geq \frac{1}{2}$.

Now let's prove that $\frac{\pi}{\upsilon}$ is the best possible constant for this inequality:

Claim If $f \in H^1$ then; $\frac{\pi}{\upsilon}$ is the best constant for the inequality below:

$$\underbrace{\sum_{k=0}^{\infty} \frac{|\hat{f}(\upsilon k+j)|}{\upsilon k+j}}_{LHS} \leq \frac{\pi}{\upsilon}\underbrace{\|f\|_{H^1}}_{RHS}$$

where $\frac{j}{\upsilon} \geq \frac{1}{2}$.

Proof: Let define:

$$g(z) = \sum_{n=0}^{\infty} q_n^{(\frac{1}{2})} z^{\upsilon n}$$

where

$$q_n^{(\frac{1}{2})} \sim \frac{n^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})}$$

It is obvious that:

$$\left(q_n^{(\frac{1}{2})}\right)_{n\in\mathbb{N}_0} \notin \ell^2 \Leftrightarrow g \notin H^2.$$

But

$$g_N(z) = \sum_{n=0}^{N} q_n^{(\frac{1}{2})} z^{\upsilon n} \in H^2;$$

hence

$$f_N(z) = z^j (g_N(z))^2 = z^j \left(\sum_{n=0}^{N} q_n^{(\frac{1}{2})} z^{\upsilon n}\right)\left(\sum_{n=0}^{N} q_n^{(\frac{1}{2})} z^{\upsilon n}\right)$$
$$= z^j \left(\sum_{n=0}^{N} \underbrace{\left(\sum_{k=0}^{n} q_{n-k}^{(\frac{1}{2})} q_k^{(\frac{1}{2})}\right)}_{q_n^{(1)}=\gamma_n} z^{\upsilon n} + \sum_{n=N+1}^{2N} \gamma_n z^{\upsilon n}\right) \tag{4.24}$$

$f_N(z) \in H^1$ Then:

$$LHS_N = \sum_{n=0}^{\infty} \frac{|\hat{f}_N(\upsilon n + j)|}{\upsilon n + j} = \sum_{n=0}^{2N} \frac{|\gamma_n|}{\upsilon n + j} \geq \sum_{n=0}^{N} \frac{|\gamma_n|}{\upsilon n + j}$$

$$RHS_N = \|f_N\|_{H^1} = \left\|z^j g_N(z)\right\|_{H^2}^2 = \|g_N(z)\|_{H^2}^2$$

and

$$\sum_{n=0}^{N} \left(q_n^{(\frac{1}{2})}\right)^2 \sim \sum_{n=1}^{N} \left(\frac{1}{\Gamma(\frac{1}{2})n^{\frac{1}{2}}}\right)^2 = \frac{1}{\left(\Gamma(\frac{1}{2})\right)^2} \sum_{n=1}^{N} \frac{1}{n}$$

We know that:

$$\sum_{n=1}^{N} \frac{1}{n} \to \infty \ \ as \ \ N \to \infty$$

Now let's take care of LHS_N:

$$LHS_N \geq \sum_{n=0}^{N} \frac{q_n^{(1)}}{\upsilon n + j} = \sum_{n=0}^{N} \frac{1}{\upsilon n + j} \sim \frac{1}{\upsilon} \sum_{n=1}^{N} \frac{1}{n}$$

for large n.

Since

$$\frac{\sum_{n=1}^{N} \frac{1}{n}}{\sum_{n=1}^{N} \frac{1}{n}} \to 1 \ \ as \ \ n \to \infty$$

then:

$$\frac{\pi}{\upsilon} \geq \frac{LHS_N}{RHS_N} \geq \frac{\frac{1}{\upsilon} \sum_{n=1}^{N} \frac{1}{n}}{\sum_{n=1}^{N} \frac{1}{\pi n}} = \frac{\pi}{\upsilon} \frac{\sum_{n=1}^{N} \frac{1}{n}}{\sum_{n=1}^{N} \frac{1}{n}} \to \frac{\pi}{\upsilon}$$

the best constant for the Hardy like inequality below. So we proved that $C = \frac{\pi}{\upsilon}$ is

$$\underbrace{\sum_{k=0}^{\infty} \frac{|\hat{f}(\upsilon k + j)|}{\upsilon k + j}}_{LHS_N} \leq C \underbrace{\|f\|_{H^1}}_{RHS_N}$$

□

The following theorem is the most important result of this chapter:

Theorem 4.6. *Let $n = mk + j$ and $m \geq 3$ s.t. $1 \leq j < \frac{m}{2}$. Then; $\exists f = f_{m\mathbb{N}_0 + j} \in H^1$ s.t.*

$$\sum_{k=0}^{\infty} \frac{|\hat{f}_{m\mathbb{N}_0+j}(mk + j)|}{mk + j} = \frac{\pi}{m} \frac{1}{sin\left(\frac{\pi j}{m}\right)} \|f\|_{H^1}$$

Proof. Fix $m \geq 3$. $1 \leq j < \frac{m}{2}$. Let $\sigma = \frac{j}{m}$. $0 < \sigma < \frac{1}{2}$. Let $f = f_{m\mathbb{N}_0+j}$ be defined by:

$f(z) = z^j(1-z)^{\frac{-2j}{m}}$ $\forall z \in \Delta$. Since

$$(1-x)^{-\alpha} = \sum_{k=0}^{\infty} \underbrace{q_k^{(\alpha)}}_{\frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}} x^k \quad ,\forall \alpha > 0 \;\; and \;\; x \in \Delta$$

then:

$$f(z) = \sum_{k=0}^{\infty} q_k^{\left(\frac{2j}{m}\right)} z^{mk+j} \; \forall z \in \Delta. \;\; |z| < 1$$

Since $\sigma = \frac{j}{m}$ then it is obvious that:

$$f(z) = \sum_{k=0}^{\infty} q_k^{(2\sigma)} z^{mk+j} \; \forall z \in \Delta$$

By the Stirling's formula [8]:

$$q_k^{(2\sigma)} \sim \frac{1}{\Gamma(2\sigma)} \frac{1}{(k^{1-\sigma})^2}$$

Then define:

$$LHS = \sum_{k=0}^{\infty} \frac{|\hat{f}_{m\mathbb{N}_0+j}(mk+j)|}{mk+j} = \sum_{k=0}^{\infty} \frac{|q_k^{2\sigma}|}{mk+j} = \sum_{k=0}^{\infty} \frac{q_k^{(2\sigma)}}{mk+j}$$

By the "limit comparison test", since

$$\sum_{k=1}^{\infty} \frac{1}{k^{2(1-\sigma)}} < \infty$$

Then $LHS < \infty$, $g(z) = (1-z^m)^{\frac{-j}{m}}$, $\forall z \in \Delta$. Since $|z^j| = 1$, and

$$\begin{aligned} \|f\|_{H^1} &= (\|g\|_{H^2})^2 \\ &= \left\| z^j (g(z))^2 \right\|_{H^1(T)} \\ &= \left\| (g(z))^2 \right\|_{H^1(T)} \end{aligned} \tag{4.25}$$

Claim:

$$g \in H^2(\Delta) \cong H^2(T)$$

Proof:

$$g(z) = \sum_{k=0}^{\infty} q_k^{(\frac{j}{m})} z^{mk} = \sum_{k=0}^{\infty} q_k^{(\sigma)} z^{mk}$$

Since $q_k^{(\sigma)} \sim \frac{k^{\sigma-1}}{\Gamma(\sigma)}$

$$\left(q_k^{(\sigma)}\right)^2 \sim \frac{k^{2\sigma-2}}{\Gamma(\sigma)} = \frac{1}{\Gamma(\sigma)} \frac{1}{k^{2(1-\sigma)}}$$

So by the "Limit comparison test"

$$RHS = \|g\|_{L^2(T)} = \sum_{k=0}^{\infty} \left(g_k^{(\sigma)}\right)^2 < \infty$$

$$\Rightarrow g \in H^2(T) \approx H^2(\Delta)$$

$$\Rightarrow f \in H^1(T) \approx H^1(\Delta)$$

Now define a new function:

$$G(z) = \sum_{k=0}^{\infty} \frac{q_k^{\left(\frac{2j}{m}\right)}}{mk+j} z^{mk+j} \quad \forall z \in \overline{\Delta}.$$

Then the series:

$$\sum_{k=1}^{\infty} |\alpha_k| < \infty \quad \alpha_k := \frac{q_k^{\left(\frac{2j}{m}\right)}}{mk+j} z^{mk+j},$$

even when $|z| = 1$. Then $\forall z \in \Delta$

$$G^{'}(z) = \sum_{k=0}^{\infty} q_k^{(2\sigma)} z^{mk+j-1} = z^{j-1}(1-z^m)^{\frac{-2j}{m}}$$

and

$$zG^{'}(z) = f(z) = z^j(1-z^m)^{\frac{-2j}{m}}$$

So by the fundamental theorem of Calculus,

$$\forall r \in (0,1) \quad G(r) - G(0) = \int_{x=0}^{x=r} \frac{x^{j-1}}{(1-x^m)^{\frac{2j}{m}}} dx$$

Since G is continuous at all $z \in \overline{\Delta}$ then

$$G(1) = \lim_{r \to 1^-} G(r) = \lim_{r \to 1^-} \int_{x=0}^{x=r} \frac{x^{j-1}}{(1-x^m)^{2\sigma}} dx = \int_0^1 \frac{x^{j-1}}{(1-x^m)^{2\sigma}} dx$$

Fix $0 < r_n < 1$ and r_n is increasing. Then if we define:

$$\Phi_n(x) = \frac{x^{j-1}}{(1-x^m)^{2\sigma}} \; \chi_{[0,r_n]}(x)$$

then by the "monotone convergence theorem" "$\int_{[0,1]} \Phi_n dm$" is increasing and converges to" $\int_{[0,1]} \Phi dm$". Hence:

$$LHS = G(1) = \int_{x=0}^{x=1} \frac{x^{j-1}}{(1-x^m)^{2\sigma}} dx$$

Let $u = x^m \Leftrightarrow x = u^{\frac{1}{m}}$ then:

$$\begin{aligned} LHS &= \int_{u=0}^{u=1} \frac{u^{\frac{j-1}{m}}}{(1-u)^{2\sigma}} \frac{1}{m} u^{\frac{1}{m}-1} du \\ &= \frac{1}{m} \int_{u=0}^{u=1} u^{\sigma-1}(1-u)^{(1-2\sigma)-1} du \quad = \frac{1}{m}\beta(\sigma, 1-2\sigma) \end{aligned} \tag{4.26}$$

Where β is the classical "Beta function" and $0 < \sigma < \frac{1}{2}$. Since $\forall x, y \in (0, \infty)$

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

[8] (8.20) Then:

$$LHS = \frac{1}{m} \frac{\Gamma(\sigma)\Gamma(1-2\sigma)}{\Gamma(1-\sigma)}$$

and

$$RHS = \sum_{k=0}^{\infty} \left(q_k^{(\sigma)}\right)^2 = \sum_{k=0}^{\infty} \left[\frac{\Gamma(\sigma+k)}{k!\Gamma(\sigma)}\right]^2$$

Now we need to define "Pochhammer-symbol"

Definition 4.1. *(Pochhammer symbol) $\forall \alpha \in \mathbb{R}$ "Pochhammer symbol" is defined by*

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$$

$$(\alpha)_0 = 1$$

and

$$(1)_k = k!$$

Let $\alpha > 0$ Then:

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \forall k \in \mathbb{N}$$

And it is easy to obtain by using integration by parts that:

$$\Gamma(x+1) = x\Gamma(x) \quad , \forall x > 0 \ \ [8]$$

Let $\alpha = \sigma = \frac{j}{m}$, then:

$$RHS = \sum_{k=0}^{\infty} \frac{(\sigma)_k (\sigma)_k}{k!k!} = \sum_{k=0}^{\infty} \frac{(\sigma)_k (\sigma)_k}{k!(1)_k}$$

Now we need to define "Hypergeometric function":

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k$$

The Hypogeometric function ${}_2F_1$ converges $\forall z \in \mathbb{C}$ with $|z| < 1$. ([10] p203) Let $F = {}_2F_1$. Consider $F(\sigma, \sigma; 1; z) = H(z)$. Then

$$H(z) = \underbrace{\sum_{k=0}^{\infty} \frac{(\sigma)_k (\sigma)_k}{(1)_k k!}}_{\tau_k = \left[q_k^{(\sigma)}\right]^2} z^k \quad , \forall z \in \Delta$$

We know that:

$$\sum_{k=1}^{\infty} \tau_k < \infty$$

Then $H(z)$ is well defined in this case for $|z| \leq 1$, and H is continuous on $\overline{\Delta}$. So from the integral representation theorem for $F = {}_2F_1$,it is easy to see that $RHS = H(1)$ ([10]).

$$\forall \alpha > \beta > 0 \ \ \forall \alpha > 0 \ \ and \ \ \forall z \in \Delta$$

$$F = (z) \ {}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_{t=0}^{t=1} t^{\beta - 1} (1-t)^{\gamma - \beta - 1} (1 - zt)^{-\alpha} dt.$$

Apply this integral representation to: $\alpha = \sigma > 0$, $\beta = \sigma > 0$ and $\gamma = 1 > \sigma$.

Then $\forall r \in (0, 1)$

$$H(r) = {}_2F_1(\sigma, \sigma; 1; r) = \frac{\Gamma(1)}{\Gamma(\sigma)\Gamma(1 - \sigma)} \int_{t=0}^{t=1} \underbrace{t^{\sigma - 1} (1-t)^{1 - \sigma - 1} (1 - rt)^{-\sigma}}_{\Lambda_r(t)} dt$$

Fix $0 < r_n$, r_n is increasing and $\lim r_n \to 1$. Let $\Phi_n(t) = \Lambda_{r_n}(t)$.

$$\Phi_n(t) \to \Phi(t) = t^{\sigma - 1} (1-t)^{-2\sigma}$$

Since $0 < r_n$ is arbitrary and r_n is increasing then by the "monotone convergence theorem"

Φ_n is a strictly increasing function,

$$H(r_n) \to K \int_{t=0}^{t=1} \Phi(t)dt$$

and

$$H(r) \to K \int_{t=0}^{t=1} \Phi(t)dt \quad , as \quad r \to 1^-$$

H is continuous on $\overline{\Delta}$ hence it is continuous at 1. Thus:

$$RHS = H(1) = K \int_{t=0}^{t=1} t^{\sigma-1}(1-t)^{-2\sigma}dt = K\beta(\sigma, 1-2\sigma)$$

where $K = \frac{\Gamma(1)}{\Gamma(\sigma)\Gamma(1-\sigma)}$. Then:

$$K\beta(\sigma, 1-\sigma) = K\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(1-\sigma)}$$

and as we obtained before

$$LHS = \sum_{k=0}^{\infty} \frac{|\hat{f}(mk+j)|}{mk+j} = \frac{1}{m}\beta(\sigma, 1-2\sigma)$$

where $\sigma = \frac{j}{m}$ and $0 < \sigma < \frac{1}{2}$.

So:

$$\frac{LHS}{RHS} = \frac{\frac{1}{m}\beta(\sigma, 1-2\sigma)}{K\beta(\sigma, 1-2\sigma)}$$

where $K = \frac{\Gamma(1)}{\Gamma(\sigma)\Gamma(1-\sigma)}$.

Hence:

$$\frac{LHS}{RHS} = \frac{\frac{1}{m}\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(1)} = \frac{1}{m}\Gamma(\sigma)\Gamma(1-\sigma) = \frac{1}{m}\frac{\pi}{sin(\pi\sigma)}$$

□

By using the theorem 4.7 below:

Theorem 4.7. *If* $0 < \sigma < 1$ *then:*

$$\Gamma(\sigma)\Gamma(1-\sigma) = \frac{\pi}{sin(\pi\sigma)}$$

(see for the proof [9])

5.0 NEW THOUGHTS ON PALEY'S INEQUALITY

Suppose f is a function in H^1, i.e.:

$$f \in H^1(\Delta) = \left[f(z) = \sum_{n=0}^{\infty} a_n z^n \ \ \forall z \in \overline{\Delta} \right]$$

and suppose λ_n is a lacunary sequence in $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ s.t.

$$L = \inf_{n \in \mathbb{N}} \frac{\lambda_{n+1}}{\lambda_n} > 1.$$

Then according to Paley's Inequality [14],

$$\exists B \in (0, \infty)\,, \ \ s.t. \ \forall f \in H^1(\Delta)$$

$$\left(\sum_{n=1}^{\infty} \left| \hat{f}(\lambda_n) \right|^2 \right)^{\frac{1}{2}} \leq B \|f\|_{H^1}$$

where the $\hat{f}(n)$'s are the Fourier coefficients of $f \in H^1(\Delta)$.

EXAMPLE 5.1. *Let $f \in H^1$ with the power series $\sum_{n=0}^{\infty} a_n z^n$ then*

$$\left(\sum_{k=1}^{\infty} |a_{2^k}|^2 \right) \leq 4 \|f\|_{H^1}^2$$

(see for solution [6])

In this section we try to solve the Extended Paley's Inequality for the case $\lambda_n = 2^n - 1$ and we obtained constant 2 instead of 4.

Theorem 5.1. ***(Extended Paley's Inequality)*** *Let $f \in H^1(\Delta)$. Then*

$$\sum_{n=1}^{\infty} |a_{2^n - 1}|^2 \leq 2 \|f\|_{H^1}^2$$

Proof. If $f \in H^1$ then by the $H^1 - H^2$ representation theorem, we can define f as, $f = gh$ where $g, h \in H^2$ s.t. $g(z) = b_n z^n$ and $h(z) = c_n z^n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_{2^n-1}|^2 &= \sum_{n=1}^{\infty} \left| \sum_{j=0}^{2^n-1} b_j c_{2^n-1-j} \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \sum_{j=0}^{2^{n-1}-1} b_j c_{2^n-1-j} + \sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n-1-j} \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \sum_{j=0}^{2^{n-1}-1} b_j c_{2^n-1-j} + \sum_{\ell=0}^{2^{n-1}-1} b_{2^n-1-\ell} c_\ell \right|^2 \\
&\leq 2 \sum_{n=1}^{\infty} \left(\left| \sum_{j=0}^{2^{n-1}-1} b_j c_{2^n-1-j} \right|^2 \right) \\
&+ 2 \sum_{n=1}^{\infty} \left(\left| \sum_{\ell=0}^{2^{n-1}-1} b_{2^n-1-\ell} c_\ell \right|^2 \right) \\
&\leq 2 \sum_{n=1}^{\infty} \left(\sum_{j=0}^{2^{n-1}-1} |b_j|^2 \sum_{\ell=2^{n-1}}^{2^n-1} |c_\ell|^2 \right) \\
&+ 2 \sum_{n=1}^{\infty} \left(\sum_{j=2^{n-1}}^{2^n-1} |b_j|^2 \sum_{\ell=0}^{2^{n-1}-1} |c_\ell|^2 \right) \\
&= 2 \sum_{n=1}^{\infty} [(P_{n-1})(Q_n - Q_{n-1}) + (P_n - P_{n-1})(Q_{n-1})] \\
&= \sum_{n=1}^{\infty} [2(P_n Q_n - P_{n-1} Q_{n-1}) - 2(P_n - P_{n-1})(Q_n - Q_{n-1})]
\end{aligned} \tag{5.1}$$

For $P_n = \sum_{j=0}^{2^n-1} |b_j|^2$ and, $Q_n = \sum_{\ell=0}^{2^n-1} |c_\ell|^2$.

$$\begin{aligned}
\sum_{n=1}^{\infty} (P_n Q_n - P_{n-1} Q_{n-1}) &= \lim_{N \to \infty} \sum_{n=1}^{N} (P_n Q_n - P_{n-1} Q_{n-1}) \\
&= \lim_{N \to \infty} (P_N Q_N - P_0 Q_0)
\end{aligned} \tag{5.2}$$

Since $P_N = \sum_{j=0}^{2^N-1} |b_j|^2$ and $Q_N = \sum_{\ell=0}^{2^N-1} |c_\ell|^2$. Then; $P_0 = |b_0|^2$ and $Q_0 = |c_0|^2$. Then we obtain:

$$\begin{aligned}\sum_{n=1}^{\infty}|a_{2^n-1}|^2 &= 2\left[\lim_{N\to\infty} P_N Q_N - P_0Q_0\right] - 2\sum_{n=1}^{\infty}[(P_n-P_{n-1})(Q_n-Q_{n-1})] \\ &\le 2\lim_{N\to\infty}\left[\sum_{j=0}^{2^N-1}|b_j|^2\sum_{\ell=0}^{2^N-1}|c_\ell|^2 - |b_0|^2|c_0|^2\right] - 2\sum_{n=1}^{\infty}[(P_n-P_{n-1})(Q_n-Q_{n-1})] \\ &= 2\left[\sum_{j=0}^{\infty}|b_j|^2\sum_{\ell=0}^{\infty}|c_\ell|^2 - |b_0|^2|c_0|^2\right] - 2\sum_{n=1}^{\infty}[(P_n-P_{n-1})(Q_n-Q_{n-1})] \\ &= 2\left[\|g\|_{H^2}^2\|h\|_{H^2}^2 - |b_0|^2|c_0|^2\right] - 2\sum_{n=1}^{\infty}[(P_n-P_{n-1})(Q_n-Q_{n-1})] \\ &= 2\sum_{n=1}^{\infty}\left[\|f\|_{H^1}^2 - |\hat{f}(0)|^2\right] - 2\sum_{n=1}^{\infty}[(P_n-P_{n-1})(Q_n-Q_{n-1})]\end{aligned} \tag{5.3}$$

Then

$$\sum_{n=1}^{\infty}|a_{2^n-1}|^2 \le 2\left[\|f\|_{H^1}^2 - |\hat{f}(0)|^2 - \sum_{n=1}^{\infty}\left(\sum_{j=2^{n-1}}^{2^n-1}|b_j|^2\right)\left(\sum_{\ell=2^{n-1}}^{2^n-1}|c_\ell|^2\right)\right] \tag{5.4}$$

By the equation (5.4) it is clear that:

$$\sum_{n=1}^{\infty}|a_{2^n-1}|^2 \le 2\|f\|_{H^1}^2$$

So we are done. $\square$

Now we will show here, the best constant K for the inequality

$$\sum_{n=1}^{\infty}|a_{2^n-1}|^2 \le K\|f\|_{H^1}^2$$

is s.t.. $\frac{4}{3} \le K \le 2$.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{j=2^{n-1}}^{2^n-1}|b_j|^2\right)\left(\sum_{\ell=2^{n-1}}^{2^n-1}|c_\ell|^2\right) &\ge \sum_{n=1}^{\infty}\left|\sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n+2^{n-1}-1-j}\right|^2 \\ &= \sum_{n=1}^{\infty}\left|\left((g_{2^n-1}\hat{h}_{2^n-1})(2^n+2^{n-1}-1)\right)\right|^2\end{aligned} \tag{5.5}$$

where $(g_{2^n-1}\hat{h}_{2^n-1})(2^n+2^{n+1}-1)$ is $(2^n+2^{n-1}-1)$.th Fourier coefficeint of $(g_{2^n-1}h_{2^n-1})$.

Then by the inequality (5.4)

$$2\|f\|_{H^1}^2 \geq 2\sum_{n=1}^{\infty}\left|(g_{2^n-1}\hat{h}_{2^n-1})(2^n+2^{n-1}-1)\right|^2 + \sum_{n=1}^{\infty}\left|\sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n+2^{n-1}-1-j}\right|^2 \tag{5.6}$$

Fix $f = g^2$ where

$$g(z) = 1 + z + z^2 + \cdots + z^{2^N-1} \quad = \frac{1-z^{2^N}}{1-z} \tag{5.7}$$

$\forall z \in \overline{\Delta}$ and $z \neq 1$. i.e. $c_n = b_n = 1$, $\forall n \geq 2^N$. Fix $n \in \{1, 2, \cdots, N\}$. Then:

$$\sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n+2^{n-1}-1-j} = \sum_{j=2^{n-1}}^{2^n-1}(1)(1) = 2^{n-1}$$

Fix $n \geq N+1$. Then:

$$\sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n+2^{n-1}-1-j} = 0$$

So

$$\sum_{n=1}^{\infty}\left(\sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n+2^{n-1}-1-j}\right)^2 = \sum_{n=1}^{N}\left(2^{n-1}\right)^2 = \frac{4^N-1}{4-1} \tag{5.8}$$

And in this case:

$$\sum_{n=1}^{\infty}|a_{2^n-1}|^2 = \sum_{n=1}^{\infty}\left(|\sum_{j=0}^{2^n-1} b_j c_{2^n-1-j}|\right)^2 \tag{5.9}$$

$$\left|\sum_{j=0}^{2^n-1} b_j c_{2^n-1-j}\right| = \begin{cases} 2^n, \text{if } 1 \leq n \leq N; \\ 0, \text{if } n \geq N+1; \end{cases}$$

Then it is clear that:

$$\sum_{n=1}^{\infty}|a_{2^n-1}|^2 = \sum_{n=1}^{N} 4^n = 4\left(\frac{4^n-1}{3}\right) \tag{5.10}$$

and

$$\|f\|_{H^1}^2 = \|g\|_{H^2}^4 = \left(\sum_{n=0}^{2^N-1}(1)^2\right)^2 = \left(4^N\right) \tag{5.11}$$

And by the equations (5.10) and (5.11)

$$\frac{\sum_{n=1}^{\infty}|a_{2^n-1}|^2}{\|f\|_{H^1}^2} = \frac{4\left(\frac{4^N-1}{3}\right)}{4^N} \underset{(asN\to\infty)}{\longrightarrow} \frac{4}{3} \tag{5.12}$$

On the other hand; by using the equations (5.8), (5.10), and (5.11) we obtain that;

$$2\sum_{n=1}^{\infty}\left|\left(g_{2^n-1}\hat{h}_{2^n-1}\right)\left(2^n+2^{n-1}-1\right)\right|^2+\sum_{n=1}^{\infty}\left|\sum_{j=2^{n-1}}^{2^n-1} b_j c_{2^n+2^{n-1}-1-j}\right|^2=2(4^N-1)=A$$

then

$$\frac{A}{2\|f\|_{H^1}^2}=\frac{2(4^N-1)}{2(4^N)}\underset{(asN\to\infty)}{\longrightarrow}1 \tag{5.13}$$

Then by using equations (5.12) and (5.13) we can conclude that; the constant $\frac{4}{3}\leq K\leq 2$ for extended Paley's inequality,

$$\sum_{n=1}^{\infty}|a_{2^n-1}|^2\leq K\|f\|_{H^1}^2$$

6.0 AN APPLICATION OF HARDY'S AND PALEY'S INEQUALITIES

Recall that we define the space ℓ^p by:

$$\ell^p = \left\{ x = (x_j)_{j \in \mathbb{N}_0}; each \;\; x_j \in \mathbb{C} \;\; and \;\; \sum_{j=0}^{\infty} |x_j|^p < \infty \right\}$$

and we know that ℓ^p is a Banach space with the norm:

$$\|x\|_p = \left(\sum_{j=0}^{\infty} |x_j|^p \right)^{\frac{1}{p}}, \; \forall x \in \ell^p$$

Consider the mapping; J on H^1 defined by;

$$J : H^1 \mapsto \ell^1 \; s.t. \; J(f) = \left(\frac{a_n}{n+1} \right)_{n \in \mathbb{N}_0}$$

$$\forall \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \overline{\Delta} \right] \in H^1$$

Our main goal in this chapter is, to prove the map that we have defined above is not onto i.e..

$$\exists y \in \ell^1 \;\; s.t.. \;\; \forall f \in H^1, \;\; J(f) \neq y$$

To begin seriously the proof of this theorem we need some premliminary definitions and theorems. Afterwards the proof of this main theorem will be straightforward to follow.

Theorem 6.1. *[7]* ***(Open mapping theorem)*** *Let Δ_1 and Δ_2 be open unit balls of the Banach spaces X and Y. For every linear transformation Λ of X onto Y there exists $\lambda > 0$ s.t.*

$$\Lambda(\Delta_1) \supset \lambda \Delta_2$$

where:

$$\lambda\Delta_2 = \{\lambda y : y \in \Delta_2\}$$

i.e. the set of all $y \in \Delta_2$, s.t. $\|y\| < \lambda$. It follows,that the image of every open ball in Δ_1 with center at x_0, contains an open ball in Δ_2 with center Λx_0, so the image of every open set is open.

Definition 6.1. ***(Banach Space** $(\ell^\infty, \| \ , \ \|_\infty)$**)***

$$\ell^\infty = \left\{ x = (x_j)_{j\in\mathbb{N}_0} ; s.t.\ each\ "x_j" \in \mathbb{C}\ and\ sup_{j\in\mathbb{N}_0}|x_j| < \infty \right\}$$

is a Banach space with the norm defined on it:

$$\|x\|_\infty = sup_{j\geq 0}|x_j| \ , \ \forall x \in \ell^\infty$$

Definition 6.2. ***(Banach Dual Space** X^***)*** *For any Banach space, $(X, \| \ , \ \|_X)$, we define the Banach dual space $(X^*, \| \ , \ \|_{X^*})$ by,*

$$X^* = \{linear\ \ maps\ \ \Phi : X \mapsto \mathbb{C}\ \ s.t.\ \ \Phi\ \ is\ \ continuous\ \ on\ \ X\}$$

and

$$\|\Phi\|_{X^*} = \sup_{\substack{x\in X \\ \|x\|_X \leq 1}} |\Phi(x)|, \ \ \forall \Phi \in X^*$$

Definition 6.3. *[7]* ***(Schur Property)*** *Let $(X, \|x\|_X)$ is a Banach space and let $\left(x^{(n)}\right)_{n\in N}$ is a sequence in X. $(X, \|x\|_X)$ has the Schur Property if $\forall$ $\left(x^{(n)}\right)_{n\in N}$ in X such that $x^{(n)} \mapsto \theta$ as $n \mapsto \infty$ weakly, it follows that $\left\|x^{(n)}\right\|_X \mapsto 0$ as $n \mapsto \infty$.*

Note that $x^{(n)} \mapsto \theta$ weakly means;

$$\forall \Phi \in X^* \ \ , \Phi\left(x^{(n)}\right) \mapsto \Phi(\theta) = 0$$

This definition easily implies:

Theorem 6.2. *If $(X, \| \ , \ \|_X)$ has the Schur property, then every infinite dimensional closed vector subspace Z of $(Z, \| \ , \ \|_X)$ has the Schur property.*

Theorem 6.3. *Let the space $(X, \| \ , \ \|_X)$ has the Schur property, and let us have the space*

$(\mathcal{V}, \| \ , \ \|_{\mathcal{V}})$ *s.t.*

$$(\mathcal{V}, \| \ , \ \|_{\mathcal{V}}) \approx (X, \| \ , \ \|_X)$$

then $(\mathcal{V}, \| \ , \ \|_{\mathcal{V}})$ *has also the Schur property.*

EXAMPLE 6.1. *The Banach space* $(\ell^1, \| \ , \ \|_1)$ *has the Schur Property.*

Proof. Every weakly convergent sequence is norm convergent to the same limit. (See for the proof [13] pg:85) □

EXAMPLE 6.2. *The Banach space* $(\ell^2, \| \ , \ \|_2)$ *does not have Schur property.*

Proof. The Banach dual of $(\ell^2, \| \ , \ \|_2)$, is isometrically isomorphic to $(\ell^2, \| \ , \ \|_2)$ via the linear map:

$$\mathcal{V}(x) : \left(\ell^2, \| \ , \ \|_2\right) \mapsto \left(\left(\ell^2\right)^*, \| \ , \ \|_{(\ell^2)^*}\right)$$

s.t.

$$\mathcal{V}(x) = \Phi_x(z) = \sum_{n=0}^{\infty} x_n z_n \ , \forall z \in \ell^2 \ \ and \ \ x \in \ell^2$$

Note that

$$\Phi_x(z) \in \left(\ell^2\right)^* \ \ and \ \ \|\mathcal{V}(x)\|_{(\ell^2)^*} = \|x\|_{\ell^2} \ \ , \forall x = (x_j)_{j\in\mathbb{N}_0} \in \ell^2$$

Note that the map $\mathcal{V}$ defined above is onto. Let $e_n = (0, \cdots, 0, \underbrace{1}_{position\ n}, 0, \cdots, 0, \cdots) \ \forall n \in \mathbb{N}_0$. Each $e_n \in \ell^2$. Fix $x = (x_j)_{j\geq 0} \in \ell^2$. Since $\sum_{n=0}^{\infty} |x_n|^2 < \infty$

$$\Phi_x(e_n) = x_n \mapsto 0 \ \ as \ \ n \mapsto \infty$$

Thus $\Phi(e_n) \mapsto 0$, $\forall \Phi \in \left(\ell^2\right)^*$ i.e. $e_n \mapsto \theta$ weakly. But $\|e_n\|_2 = 1 \ \forall n \in \mathbb{N}_0$. So $\|e_n\|_2 \not\mapsto 0$ Hence $(\ell^2, \| \ , \ \|_2)$ fails to have Schur property. □

Theorem 6.4. $(H^1, \| \ , \ \|_{H^1})$ *has an infinite dimensional closed, vector subspace* Y *s.t.* $(Y, \| \ , \ \|_{H^1})$ *is isomorphic to* $(\ell^2, \| \ , \ \|_2)$.

Proof. Let define:

$$Y = \left\{ \left[g(z) = \sum_{k=1}^{\infty} c_{2^k-1} z^{2^k-1}, \ z \in \overline{\Delta} \right] : \sum_{k=1}^{\infty} |c_{2^k-1}|^2 < \infty \right\}.$$

It is clear that, Y is well defined vector subspace of H^2. But, $H^2 \subseteq H^1$, and $\|f\|_{H^1} \leq \|f\|_{H^2}$, $\forall f \in H^2$. So Y is a vector subspace of H^1, and

$$\forall g \in Y,\ \|g\|_{H^1} \leq \|g\|_{H^2} = \left(\sum_{k=1}^{\infty} |c_{2^k-1}|^2\right)^{\frac{1}{2}}$$

And by the Paley's Inequality, $\forall g \in Y \subseteq H^1$,

$$\left(\sum_{k=1}^{\infty} |c_{2^k-1}|^2\right)^{\frac{1}{2}} \leq \sqrt{2}\|g\|_{H^1}$$

Now define the linear mapping

$$W : (Y, \|\ ,\ \|_{H^1}) \mapsto \left(\ell^2, \|\ ,\ \|_2\right)$$

s.t.

$$W(g) = \left(c_{2^{\upsilon+1}-1}\right)_{\upsilon \geq 0}, \forall g \in Y.$$

Then W is onto and

$$\|g\|_{H^1} \leq \|W(g)\|_2 \leq \sqrt{2}\|g\|_{H^1}, \forall g \in Y$$

Thus $(Y, \|\ ,\ \|_{H^1})$ and $(\ell^2, \|\ ,\ \|_2)$ are isomorphic Banach spaces; i.e.

$$(Y, \|\ ,\ \|_{H^1}) \approx \left(\ell^2, \|\ ,\ \|_2\right)$$

Then we are done. $\square$

Theorem 6.5. *Every infinite dimensional closed vector subspace Z of $(\ell^1, \|\ ,\ \|_1)$ is not isomorphic to $(\ell^2, \|\ ,\ \|_2)$*

Proof. We know by example **??** that $(\ell^1, \|\ ,\ \|_1)$ has the Schur property so by Theorem 6.2 every infinite dimensional closed vector subspace Z of $(\ell^1, \|\ ,\ \|_1)$ is s.t.. $(Z, \|\ ,\ \|_1)$ has the Schur property. Suppose that $\exists$ an infinite dimensional closed vector subspace Z of $(\ell^1, \|\ ,\ \|_1)$ s.t.

$$(Z, \|\ ,\ \|_1) \approx \left(\ell^2, \|\ ,\ \|_2\right)$$

then by theorem 6.3, $(\ell^2, \|\ ,\ \|_2)$ should also have Schur property. But by the example 6.1, $(\ell^2, \|\ ,\ \|_2)$ fails to have Schur property. So we are done. $\square$

Then by the Theorems 6.4 and 6.5 it is clear that

Corollary 6.1. *The statement*

$$\left(H^1, \| \ , \ \|_{H^1}\right) \approx \left(\ell^1, \| \ , \ \|_1\right)$$

i.e. $(H^1, \| \ , \ \|_{H^1})$ *and* $(\ell^1, \| \ , \ \|_1)$ *are isomorphic is false.*

Finally we can prove our main theorem:

Theorem 6.6. *[12] The mapping; J on H^1 defined by;*

$$J : H^1 \mapsto \ell^1 \text{ s.t. } J(f) = \left(\frac{a_n}{n+1}\right)_{n \in \mathbb{N}_0}$$

where

$$\forall \left[f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \overline{\Delta} \right] \in H^1$$

is not onto.

Proof. It is clear that J maps $(H^1, \| \ , \ \|_{H^1})$ into $(\ell^1, \| \ , \ \|_1)$. By the Hardy's Inequality:

$$\|J(f)\|_{H^1} = \sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_{H^1}, \ \forall f \in H^1.$$

Suppose get a contradiction. i.e.. suppose that, $J : H^1 \mapsto \ell^1$ is onto. Then J is a one to one continuous linear mapping, from $(H^1, \| \ , \ \|_{H^1})$ onto the Banach space $(\ell^1, \| \ , \ \|_1)$. By the Open Mapping Theorem the mapping;

$$J^{-1} : \left(\ell^1, \| \ , \ \|_1\right) \mapsto \left(H^1, \| \ , \ \|_{H^1}\right)$$

is also continuous. Thus $\exists$ a constant $\mathcal{B} \in (0, \infty)$ s.t.

$$\left\| J^{-1}(x) \right\|_{H^1} \leq \mathcal{B} \|x\|_1 \ , \ \forall x \in \ell^1$$

Equivalently;

$$\|f\|_{H^1} \leq \mathcal{B} \|J(f)\|_1 \ , \ \forall f \in H^1$$

So the linear mapping $J : (H^1, \| \ , \ \|_{H^1}) \mapsto (\ell^1, \| \ , \ \|_1)$ is s.t.. J is onto and

$$\frac{1}{\mathcal{B}} \|f\|_{H^1} \leq \|J(f)\|_1 \leq \pi \|f\|_{H^1}$$

in other words, J is Banach space isomorphism from $(H^1, \| \ , \ \|_{H^1})$ onto $(\ell^1, \| \ , \ \|_1)$. Which means $(H^1, \| \ , \ \|_{H^1})$ and $(\ell^1, \| \ , \ \|_1)$ are isomorphic. But this contradicts with the corollary 6.1. So the mapping J is not onto. We are done. $\square$

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