THE K-EPSILON MODEL IN THE THEORY OF TURBULENCE

by

Colleen D. Scott-Pomerantz

B.S. Mathematics, Creighton University, 1997

M.A. Mathematics, University of Pittsburgh, 2002

Submitted to the Graduate Faculty of

Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2004

UNIVERSITY OF PITTSBURGH

ARTS AND SCIENCES

This dissertation was presented

by

Colleen D. Scott-Pomerantz

It was defended on

December 4, 2004

and approved by

J. Bryce McLeod, University Professor, Mathematics, University of Pittsburgh

Peter J. Bushell, Professor, Mathematics, University of Sussex

Stuart P. Hastings, Professor, Mathematics, University of Pittsburgh

Thomas A. Metzger, Associate Professor, Mathematics, University of Pittsburgh

Dissertation Director: J. Bryce McLeod, University Professor, Mathematics, University of Pittsburgh

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Colleen D. Scott-Pomerantz, PhD

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We consider the $k-\varepsilon$ model in the theory of turbulence:

$$
\begin{aligned}
k_t &= \alpha\left(\tfrac{k^2}{\varepsilon}k_x\right)_x - \varepsilon\\
\varepsilon_t &= \beta\left(\tfrac{k^2}{\varepsilon}\varepsilon_x\right)_x - \gamma\tfrac{\varepsilon^2}{k}
\end{aligned}
$$

where k is the turbulent kinetic energy, ε is the dissipation rate of the turbulent energy, and α, β, and γ are positive constants. After substituting

$$k = \frac{\widetilde{A}^2}{t^{2\mu}}f(\zeta),\ \ \varepsilon = \frac{\widetilde{A}^2}{t^{2\mu+1}}g(\zeta),\ \ \zeta = \frac{x}{\widetilde{A}t^{1-\mu}}$$

into the $k-\varepsilon$ model, where $\widetilde{A} > 0$ is a free scaling parameter, we examine the Barenblatt self-similar $k-\varepsilon$ model for turbulence:

$$
\begin{aligned}
&\alpha\left(\tfrac{f^2}{g}f'\right)' + (1-\mu)\zeta f' + 2\mu f - g = 0, \qquad 0<\zeta<1\\
&\beta\left(\tfrac{f^2}{g}g'\right)' + (1-\mu)\zeta g' + (1+2\mu)g - \gamma\tfrac{g^2}{f} = 0, \quad 0<\zeta<1,
\end{aligned}
$$

along with the boundary conditions

$$
\begin{gathered}
f'(0)=0, g'(0)=0\\
f(1)=0, g(1)=0, \tfrac{f^2}{g}f'(1)=0, \tfrac{f^2}{g}g'(1)=0.
\end{gathered}
$$

Under the assumptions $\beta > \alpha$, $3\alpha > 2\beta$, and $\gamma > \frac{3}{2}$, we show the existence of μ for which there is a positive solution to the system and corresponding boundary conditions by proving a series of lemmas. We also include graphs of solutions (f, g) obtained by using XPPAUT 5.85.

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PREFACE

For my parents

Faith and Dick Scott

and my husband

Stu Pomerantz

I would also like to express my gratitude to Bryce McLeod, Peter Bushell, Stuart Hastings, and Tom Metzger.

1.0 INTRODUCTION

Two equation models for turbulence are a popular variety because

> ... although a great number of equations should in principle permit greater realism to be achieved, it has proved hard to demonstrate this advantage in practice [11].

The first two-equation model for predicting the behavior of turbulent flows was proposed in 1942 by A.N. Kolmogorov and used the variables b for fluctuation energy and ω for frequency. In 1968 Harlow and Nakayama [6] introduced the $k-\varepsilon$ model for turbulence:

$$\begin{aligned} k_t &= \alpha\left(\frac{k^2}{\varepsilon}k_x\right)_x - \varepsilon \\ \varepsilon_t &= \beta\left(\frac{k^2}{\varepsilon}\varepsilon_x\right)_x - \gamma\frac{\varepsilon^2}{k} \end{aligned} \qquad (KE)$$

where $k = k(x,t)$ is the turbulent kinetic energy, $\varepsilon = \varepsilon(x,t)$ is the rate of dissipation of the turbulent energy, and α, β, and γ are positive constants. For completeness a derivation [3, 17] of the model is included in Section 2.

Although the true development of the model is often credited to Jones and Launder [9], it should be noted that (KE) is sometimes referred to as the $b-\varepsilon$ model, in acknowledgement of Kolmogorov's original insight and the relationship between the variables used: $b = \frac{2}{3}k$ and ωb is proportional to ε [11].

In 1987 Barenblatt, Galerkina, and Luneva [2] found that for the special case of $\alpha = \beta = 1$ and $\gamma > \frac{3}{2}$, (KE) has a family of self-similar compactly supported solutions:

$$\begin{aligned} k(x,t) &= (\gamma-1)(t+t_0)^{1-\nu_\gamma}\left(C - C_\gamma(x-x_0)^2(t+t_0)^{-2\mu_\gamma}\right)_+ \\ \varepsilon(x,t) &= (t+t_0)^{-\nu_\gamma}\left(C - C_\gamma(x-x_0)^2(t+t_0)^{-2\mu_\gamma}\right)_- \end{aligned}$$

where $t > -t_0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, $\mu_\gamma = \frac{2\gamma-3}{3(\gamma-1)}$, $\nu_\gamma = \frac{5\gamma-3}{3(\gamma-1)}$, $C_\gamma = \frac{2\gamma-3}{6(\gamma-1)^3}$, $C > 0$, and $a_+ = \max\{a, 0\}$.

When $\alpha = \beta = 1$ and $\gamma \geq 1$, Bertsch, Dal Passo, and Kersner [3] proved an existence result for the Cauchy problem:

$$\begin{cases} k_t = \alpha \left(\frac{k^2}{\varepsilon} k_x \right)_x - \varepsilon, \text{ in } Q \\ \varepsilon_t = \beta \left(\frac{k^2}{\varepsilon} \varepsilon_x \right)_x - \gamma \frac{\varepsilon^2}{k}, \text{ in } Q \\ k(x,0) = k_0(x), \ \varepsilon(x,0) = \varepsilon_0(x), \text{ for } x \in \mathbb{R} \end{cases}$$

where $Q = \{(x,t) | x \in \mathbb{R}, t > 0\}$, and k_0 and ε_0 are given bounded, non-negative and continuous functions. They discovered if k and ε are initially bounded, the solutions remain bounded with respect to x for any $t > 0$. For $\gamma > \frac{3}{2}$ they showed that the constructed solutions behave like the self-similar solutions for large values of t.

In order to obtain physical solutions to (KE), however, it is usual to take $\alpha \neq \beta$ [3, 8]. For example when specifying $\alpha = .09$, $\beta = .07$, and $\gamma = 1.92$ in (KE), the resulting Standard $k - \varepsilon$ model is only useful in regions with turbulent, high Reynolds number flows. Hulshof [8] considered the existence of compactly supported self-similar solutions for $\alpha \neq \beta$, by use of the Barenblatt solutions, but his analysis applies only when α and β are sufficiently close to 1 and $\gamma > \frac{3}{2}$. Further, his approach proceeds by looking for a perturbation about the known solution when $\alpha = \beta$.

The Barenblatt self-similar solutions can be found by substituting

$$k = \frac{\widetilde{A}^2}{t^{2\mu}} f(\zeta), \ \ \varepsilon = \frac{\widetilde{A}^2}{t^{2\mu+1}} g(\zeta), \ \ \zeta = \frac{x}{\widetilde{A} t^{1-\mu}}$$

in (KE), where $\widetilde{A}$ is a free scaling positive parameter. We will consider the system of ordinary differential equations which results from the substitution

$$\alpha \left(\frac{f^2}{g} f' \right)' + (1-\mu)\zeta f' + 2\mu f - g = 0, \ \ 0 < \zeta < 1 \tag{1.0.1}$$

$$\beta \left(\frac{f^2}{g} g' \right)' + (1-\mu)\zeta g' + (1+2\mu) g - \gamma \frac{g^2}{f} = 0, \ \ 0 < \zeta < 1, \tag{1.0.2}$$

along with the boundary conditions

$$f'(0) = 0, g'(0) = 0 \tag{1.0.3}$$

$$f(1) = 0, g(1) = 0, \frac{f^2}{g} f'(1) = 0, \frac{f^2}{g} g'(1) = 0, \tag{1.0.4}$$

taken to ensure the symmetry and compactness of the support of solutions.

Given the positive parameters α, β, and γ, we aim to show the existence of μ for which there is a positive solution (f, g) to (1.0.1)-(1.0.4).

1.1 DERIVING THE K-EPSILON MODEL

The $k-\varepsilon$ model can be derived from the incompressible Navier-Stokes equations

$$\rho\left(\frac{\partial u_i}{\partial t}+\sum_j u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\eta\nabla^2 u_i, \qquad (NS)$$

where $u(x,t)$ represents the velocity vector field, $p(x,t)$ is the pressure field, ρ is the density constant, η is the dynamic viscosity, and $\nu=\frac{\eta}{\rho}$ is the kinematic viscosity.

Noting (NS) are derived from the equations for conservation of mass, momentum, and energy, we have that

$$\frac{\partial \rho}{\partial t}+\sum_j u_j\frac{\partial \rho}{\partial x_j}=\rho\sum_j\frac{\partial u_j}{dx_j}=0. \tag{1.1.1}$$

Applying statistical averaging to (NS) produces the Reynolds equations:

$$\rho\frac{\partial \overline{u_i}}{\partial t}+\sum_j\left(\rho\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}+\rho\overline{\frac{\partial u_i'}{\partial x_j}u_j'}\right)=-\frac{\partial \overline{p}}{\partial x_i}+\sum_j\frac{\partial \overline{\tau_{ij}}}{\partial x_j} \qquad (R)$$

with $u=\bar{u}+u'$ written in the mean plus fluctuation decomposition, $\overline{\tau_{ij}}=\eta\left(\frac{\partial \overline{u_i}}{\partial x_j}+\frac{\partial \overline{u_j}}{\partial x_i}\right)$, $\eta\nabla^2 u_i=\sum_j\frac{\partial \tau_{ij}}{\partial x_j}$, and averaging satisfying the rules summarized as follows:

$$\begin{cases} \overline{v+w}=\bar{v}+\bar{w} \\ \bar{a}v=a\overline{v},\ a=\text{constant} \\ \bar{a}=a \\ \overline{\frac{\partial v}{\partial s}}=\frac{\partial \bar{v}}{\partial s},\ s=x_i \text{ or } s=t \\ \overline{\overline{v}w}=\bar{v}\bar{w} \end{cases} \tag{1.1.2}$$

for arbitrary fields v and w.

Some consequences of (1.1.2) applied to u are

$$\begin{cases} \overline{u_iu_j}=\overline{u_i}\,\overline{u_j}+\overline{u_i'u_j'} \\ \overline{u_iu_ju_k}=\overline{u_i'u_j'u_k'}+\overline{u_i'u_j'}\overline{u_k}+\overline{u_j'u_k'}\overline{u_i}+\overline{u_k'u_i'}\overline{u_j}+\overline{u_i}\,\overline{u_j}\,\overline{u_k} \\ \overline{\frac{\partial u_i}{\partial t}u_i}-\frac{\partial \overline{u_i}}{\partial t}\overline{u_i}=\overline{\frac{\partial u_i'}{\partial t}u_i'}. \end{cases} \tag{1.1.3}$$

Thus multiplying (NS) by u_i and averaging we find

$$\rho\overline{\frac{\partial u_i}{\partial t}u_i}+\rho\sum_j\overline{u_j\frac{\partial u_i}{\partial x_j}u_i}=-\overline{\frac{\partial p}{\partial x_i}u_i}+\sum_j\overline{\frac{\partial \tau_{ij}}{\partial x_j}u_i}. \tag{1.1.4}$$

Multiplying (R) by $\overline{u_i}$ gives

$$\rho\frac{\partial\overline{u_i}}{\partial t}\overline{u_i}+\sum_j\left(\rho\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}+\rho\underbrace{\overline{\frac{\partial u_i'}{\partial x_j}u_j'}}_{\frac{\partial}{\partial x_j}\left(\overline{u_i'u_j'}\right)}\overline{u_i}\right)=-\frac{\partial\overline{p}}{\partial x_i}\overline{u_i}+\sum_j\frac{\partial\overline{\tau_{ij}}}{\partial x_j}\overline{u_i},$$

or equivalently

$$\rho\frac{\partial\overline{u_i}}{\partial t}\overline{u_i}+\rho\sum_j\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}=-\frac{\partial\overline{p}}{\partial x_i}\overline{u_i}+\sum_j\left(\frac{\partial\overline{\tau_{ij}}}{\partial x_j}\overline{u_i}+\frac{\partial T_{ij}}{\partial x_j}\overline{u_i}\right)\tag{1.1.5}$$

with $T_{ij}=-\rho\overline{u_i'u_j'}$ representing the components of the Reynolds stress matrix T.

By subtracting (1.1.5) from (1.1.4) we have

$$\rho\overline{\frac{\partial u_i'}{\partial t}u_i'}+\rho\sum_j\left(\overline{u_j\frac{\partial u_i}{\partial x_j}u_i}-\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}\right)=-\overline{\frac{\partial p'}{\partial x_i}u_i'}+\sum_j\left(\underbrace{\overline{\frac{\partial\tau_{ij}'}{\partial x_j}u_i'}}_{\frac{\partial\left(\overline{\tau_{ij}'u_i'}\right)}{\partial x_j}-\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}}-\frac{\partial T_{ij}}{\partial x_j}\overline{u_i}\right)\tag{1.1.6}$$

where

$$\overline{\frac{\partial u_i}{\partial t}u_i}=\frac{\partial\overline{u_i}}{\partial t}\overline{u_i}+\overline{\frac{\partial u_i'}{\partial t}u_i'}\tag{1.1.7}$$

$$\overline{\frac{\partial p}{\partial x_i}u_i}=\frac{\partial\overline{p}}{\partial x_i}\overline{u_i}+\overline{\frac{\partial p'}{x_i}u_i'}\tag{1.1.8}$$

$$\overline{\frac{\partial\tau_{ij}}{\partial x_j}u_i}=\frac{\partial\overline{\tau_{ij}}}{\partial x_j}\overline{u_i}+\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}\tag{1.1.9}$$

and

$$\overline{u_j\frac{\partial u_i}{\partial x_j}u_i}-\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}\overline{u_i}=\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\overline{u_j}+\overline{\frac{\partial u_i'}{\partial x_j}u_i'u_j'}+\overline{\frac{\partial u_i'}{\partial x_j}u_j'}\overline{u_i}+\overline{u_j'u_i'}\overline{\frac{\partial u_i'}{\partial x_j}}\tag{1.1.10}$$

by the averaging rules.

Since $\overline{\tau_{ij}'u_i'}$ represents the viscous transfer of turbulent energy, a very small quantity in contrast to the terms responsible for the turbulent transfer of turbulent energy in (1.1.6), it is neglected. Thus (1.1.6) becomes

$$\rho\overline{\frac{\partial u_i'}{\partial t}u_i'}+\rho\sum_j\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\overline{u_j}+\sum_j\left(\rho\overline{\frac{\partial u_i'}{\partial x_j}u_i'u_j'}+\frac{\partial\rho\overline{u_i'u_j'}}{\partial x_j}\overline{u_i}+\rho\overline{u_j'u_i'}\overline{\frac{\partial u_i'}{\partial x_j}}\right)=-\overline{\frac{\partial p'}{x_i}u_i'}-\sum_j\left(\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}+\frac{\partial T_{ij}}{\partial x_j}\overline{u_i}\right)$$

by using (1.1.10), or

$$\frac{\rho}{2}\left(\frac{\partial\overline{(u_i')^2}}{\partial t}+\sum_j\frac{\partial\overline{(u_i'^2)}}{\partial x_j}\overline{u_j}\right)=-\overline{\frac{\partial p'}{x_i}u_i'}-\frac{\rho}{2}\sum_j\frac{\partial}{\partial x_j}\overline{\left(u_i'^2u_j'\right)}-\sum_j\left(\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}+\underbrace{\rho\overline{u_j'u_i'}}_{-T_{ij}}\frac{\partial\overline{u_i'}}{\partial x_j}\right). \quad (1.1.11)$$

Summing over i, (1.1.11) becomes an energy balance equation of turbulent flow:

$$\rho\left(\frac{\partial k}{\partial t}+\sum_j\frac{\partial k}{\partial x_j}\overline{u_j}\right)=-\sum_j\frac{\partial}{\partial x_j}\left(\overline{p'u_j'}+\frac{\rho}{2}\sum_i\overline{u_i'^2u_j'}\right)+\sum_{i,j}\left(\overline{T_{ij}}\frac{\partial\overline{u_i'}}{\partial x_j}\right)-\rho\epsilon \quad (1.1.12)$$

where the turbulent kinetic energy is defined as

$$k=\frac{1}{2}\sum_i\overline{(u_i')^2} \quad (1.1.13)$$

and the rate of dissipation of the turbulent energy is

$$\varepsilon=\frac{1}{\rho}\sum_{i,j}\overline{\frac{\partial u_i'}{\partial x_j}\tau_{ij}'}=\frac{\nu}{2}\sum_{i,j}\overline{\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)^2}. \quad (1.1.14)$$

Using the hypotheses from [3] for the class of fluid flow under consideration, the equation for turbulent energy balance reduces to

$$\frac{\partial k}{\partial t}=\frac{\partial}{\partial x}\left(c_k\frac{\partial k}{\partial x}\right)-\varepsilon \quad (K)$$

where c_k is the turbulent energy exchange coefficient. Similarly the equation for the balance of the turbulent energy dissipation rate for flows is

$$\frac{\partial \varepsilon}{\partial t}=\frac{\partial}{\partial x}\left(c_\varepsilon\frac{\partial \varepsilon}{\partial x}\right)-U \quad (E)$$

where c_ε is the turbulent energy dissipation rate exchange coefficient and $U>0$ is the rate of homogenization of the dissipation rate.

By Kolmogorov's similarity hypothesis, c_k, c_ε, and U can be expressed in terms of two kinematic quantities: $L=$ length and $V=$ characteristic velocity, where $T=LV^{-1}=$ time. By (1.1.13) and (1.1.14),

$$[k]=L^2T^{-2}$$

$$[\varepsilon]=L^2T^{-3}.$$

By (K),

$$\left[\frac{\partial}{\partial x}\left(c_k \frac{\partial k}{\partial x}\right)\right] = L^2 T^{-3}$$

which implies

$$[c_k] = L^2 T^{-1}.$$

Therefore, for dimensionless constants, $\alpha > 0$, and δ_1, δ_2,

$$c_k = \alpha k^{\delta_1} \varepsilon^{\delta_2}.$$

Equating powers of the fundamental dimensions:

$$\begin{cases} L: & 2 = 2\delta_1 + 2\delta_2 \\ T: & -1 = -2\delta_1 - 3\delta_2 \end{cases}$$

we find that $\delta_2 = -1$, $\delta_1 = 2$, and

$$c_k = \alpha \frac{k^2}{\varepsilon}.$$

Similarly

$$[c_\varepsilon] = L^2 T^{-1} \quad \text{and}$$

$$\left[\frac{\partial \varepsilon}{\partial t}\right] = [U] = L^2 T^{-4}.$$

Therefore, with constants β and γ, the dimensional analysis yields

$$c_\varepsilon = \beta \frac{k^2}{\varepsilon}$$

$$U = \gamma \frac{\varepsilon^2}{k}$$

and by applying (K) and (E), we have (KE). An alternative derivation of (KE) is cited in [5, 14].

2.0 NEW RESULTS

Using t in place of ζ and $a > 0$ a constant, we note that under the mapping

$$f \to af$$

$$g \to ag$$

$$t \to a^{\frac{1}{2}}t$$

(1.0.1) becomes

$$a\left(\alpha\left(\frac{f^2}{g}f'\right)' + (1-\mu)\,tf' + 2\mu f - g\right) = 0$$

and similarly for (1.0.2)

$$a\left(\beta\left(\frac{f^2}{g}g'\right)' + (1-\mu)\,tg' + (1+2\mu)g - \gamma\frac{g^2}{f}\right) = 0.$$

Indeed (1.0.1)-(1.0.2) are invariant under the mapping, with the constants α, β, γ, and μ unaltered. Thus without loss of generality we may take, for instance, $f(0) = 1$; and then we notice that (1.0.1)-(1.0.2) has a unique solutions once we choose μ and $\frac{f(0)}{g(0)}$. We will show that we can choose μ and $\frac{f(0)}{g(0)}$ so that boundary conditions (1.0.4) are satisfied at some positive point, t_0. Then we will need to rescale so that t_0 becomes 1 .

Thus we will have

$$f(t_0) = 0 \text{ and } g(t_0) = 0,$$

but using the mapping $t_0 \to a^{\frac{1}{2}}t_0 = 1$, we find

$$a^{\frac{1}{2}} = \frac{1}{t_0}$$

and so

$$a = \frac{1}{t_0^2}.$$

In rescaling t_0 we will also scale

$$f \to \frac{f}{t_0^2} \quad \text{and} \quad g \to \frac{g}{t_0^2}.$$

We further note that by multiplying (1.0.1) by $\frac{g}{\alpha}$, multiplying (1.0.2) by $\frac{f}{\beta}$, and forming their difference we have

$$\frac{d}{dt}\left(\frac{f^2}{g}\left(f'g - fg'\right)\right) + (1-\mu)\,t\left(\frac{gf'}{\alpha} - \frac{fg'}{\beta}\right) + fg\left(\frac{2\mu}{\alpha} - \frac{2\mu+1}{\beta}\right) - \frac{g^2}{\alpha} + \gamma\frac{g^2}{\beta} = 0$$

or equivalently

$$\frac{d}{dt}\left(f^2 g\frac{d\theta}{dt}\right) + \frac{(1-\mu)}{\alpha}tg^2\frac{d\theta}{dt} = g^2\left(\frac{2\mu+1}{\beta} - \frac{2\mu}{\alpha}\right)\left(\theta - \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha - 2\mu\beta}\right) + (1-\mu)tfg'\left(\frac{1}{\beta} - \frac{1}{\alpha}\right), \tag{2.0.1}$$

where θ is defined to be $\frac{f}{g}$.

Using (1.0.3) we can express (1.0.1)-(1.0.2) in integrated form as

$$\alpha\frac{f^2}{g}f' + (1-\mu)tf = \int_0^t \left(g - (3\mu-1)\,f\right)ds \tag{2.0.2}$$

$$\beta\frac{f^2}{g}g' + (1-\mu)tg = \int_0^t \gamma\frac{g}{f}\left(g - \frac{3\mu f}{\gamma}\right)ds. \tag{2.0.3}$$

2.1 ASSUMPTIONS ON THE CONSTANTS

Our particular assumptions on the parameters α, β, and γ are such that

$$\beta > \alpha, 3\alpha > 2\beta, \gamma > \frac{3}{2}. \tag{2.1.1}$$

We will later demonstrate numerically that solutions (f, g) exist only if $\frac{\alpha}{\beta}$ is neither too big nor too small.

We will be interested in the range of $\theta\,(0)$ for given μ defined by

$$\frac{\gamma}{2\mu+1} < \theta\,(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}, \tag{2.1.2}$$

where μ falls within

$$\max\left(\frac{1}{3}, \frac{1}{2(\gamma-1)}\right) < \mu < 1. \tag{2.1.3}$$

We begin by demonstrating why $\gamma > \frac{3}{2}$ is a natural condition, by looking at the case where $\alpha = \beta$. Assuming $\alpha = \beta$, it is clear that $\theta = \gamma - 1$ is a solution to (2.0.1). Thus by substituting $\theta = \frac{f}{g} = \gamma - 1$ into (1.0.2) we have

$$\beta\,(\gamma-1)^2\left(gg'\right)' + (1-\mu)tg' + \left(1 + 2\mu - \frac{\gamma}{\gamma-1}\right)g = 0 \quad 0 < t < 1. \tag{2.1.4}$$

since $f' = (\gamma - 1)g'$. Integrating and applying (1.0.3)-(1.0.4) to (2.1.4),

$$(1-\mu)\int_0^1 tg'(t)dt + \left(1 + 2\mu - \frac{\gamma}{\gamma-1}\right)\int_0^1 g(t)dt \quad = \quad 0 \tag{2.1.5}$$

or

$$\left(3\mu - \frac{\gamma}{\gamma-1}\right)\int_0^1 g(t)dt = 0.$$

Therefore,

$$\mu = \frac{\gamma}{3\,(\gamma-1)} \tag{2.1.6}$$

and so by (2.1.4),

$$\beta\,(\gamma-1)^2\left(gg'\right)' + (1-\mu)tg' + \left(1 - \frac{\gamma}{3\,(\gamma-1)}\right)g = 0$$

or

$$\beta(\gamma-1)^2\left(gg'\right)'+(1-\mu)(tg)'=0.$$

Thus integrating again and applying (1.0.3)-(1.0.4) gives

$$g'(t)=-\frac{(1-\mu)}{\beta(\gamma-1)^2}t$$

and so

$$g(t)=\frac{1-\mu}{2\beta(\gamma-1)^2}\left(1-t^2\right)$$

where $g>0$ if $1-\mu=\frac{2\gamma-3}{3(\gamma-1)}>0$. Thus by (2.1.6), if $\frac{3}{2}<\gamma<\infty$ then $\frac{1}{3}<\mu<1$.

Further note that $\gamma>\frac{3}{2}$ implies

$$\alpha\gamma-\beta>\frac{3}{2}\alpha-\beta>0,$$

by the assumptions on α and β; and also

$$(2\mu+1)\alpha-2\mu\beta>0$$

as long as $\mu<\frac{1}{2(\frac{\beta}{\alpha}-1)}$, which is true by $3\alpha>2\beta$ and (2.1.3). Thus (2.1.2) defines a positive interval under (2.1.1) and (2.1.3), so

$$\frac{\gamma}{2\mu+1}<\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$$

as long as

$$\mu>\frac{1}{2(\gamma-1)}. \tag{2.1.7}$$

Also under (2.1.7) we note that

$$\frac{1}{2\mu}<\frac{\gamma}{2\mu+1}. \tag{2.1.8}$$

Lastly the fact that $\mu<1$ means $\theta(0)$ can be bounded as

$$\frac{\gamma}{3}<\theta(0)<\frac{\alpha\gamma-\beta}{3\alpha-2\beta}.$$

2.2 BEHAVIOR OF THE DERIVATIVES

By taking $\theta(0) > \frac{\gamma}{2\mu+1}$, we have at $t = 0$ that

$$(1 + 2\mu)g - \gamma\frac{g^2}{f} > 0.$$

Then for $t > 0$ sufficiently small we have from (1.0.2)

$$\beta\left(\frac{f^2}{g}g'\right)' + (1 - \mu)tg' < 0.$$

Thus g' is initially negative for sufficiently small $t > 0$.

Supposing that $\theta(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then by (2.0.1) we have

$$\frac{d}{dt}\left(f^2g\frac{d\theta}{dt}\right) + \frac{(1-\mu)}{\alpha}tg^2\frac{d\theta}{dt} < 0$$

for $t > 0$, t sufficiently small; and so $\theta'(t)$ is initially negative.

By applying the above facts we have the following results:

Lemma 1. *If (2.1.1)-(2.1.3) hold, and $\theta(0)$ is sufficiently close to $\frac{\gamma}{2\mu+1}$, then there exists $t^- > 0$ such that*

$$g'(t^-) > 0 \tag{2.2.1}$$

$$while\ \theta'(t) < 0,\ \ for\ 0 < t \leq t^-. \tag{2.2.2}$$

Lemma 2. *If (2.1.1)-(2.1.3) hold, and $\theta(0)$ is sufficiently close to $\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then there exists $t^+ > 0$ such that*

$$\theta'(t^+) > 0 \tag{2.2.3}$$

$$while\ g'(t) < 0,\ \ for\ 0 < t \leq t^+. \tag{2.2.4}$$

Thus under the assumptions of Lemma 1, although g' is initially negative, it becomes positive before θ' becomes 0. Similarly in Lemma 2 we have that $\theta'(t)$, while initially negative, becomes positive at small t and before g' becomes 0.

Consider the quadrilateral Q bounded vertically by

$$\mu = \max\left(\frac{1}{3}, \frac{1}{2(\gamma-1)}\right) \text{ and } \mu = 1$$

and horizontally by

$$\theta(0) = \frac{\gamma}{2\mu+1} \text{ and } \theta(0) = \frac{\alpha\gamma - \beta}{(2\mu+1)\alpha - 2\mu\beta} .$$

We define sets:

$$S^{-} = \{(\mu, \theta(0)) : \text{ there exists } t^{-} \text{ for which } (2.2.1) \text{ and } (2.2.2) \text{ hold}\}$$

$$S^{+} = \{(\mu, \theta(0)) : \text{ there exists } t^{+} \text{ for which } (2.2.3) \text{ and } (2.2.4) \text{ hold}\} .$$

By definition the top and bottom boundaries of Q are contained in sets S^+ and S^-, where S^+ and S^- are disjoint and, consequently, open relative to Q because of continuity of the solutions of a differential equation with respect to the initial data. By LEMMAS 1 and 2 the sets are also non-empty. Therefore, we have shown the existence of some $(\mu, \theta(0)) \notin S^+ \cup S^-$.

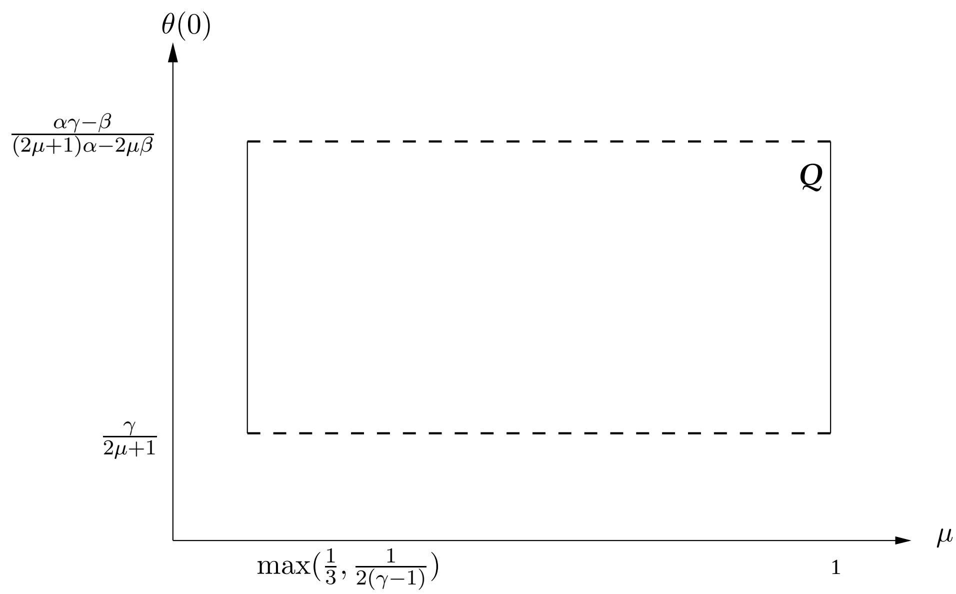

Figure 1: Quadrilateral Q with the top and bottom boundaries removed

We recall the following proposition from plane point set topology [15], a proof of which is included in Chapter 3.

Proposition. *Let I be the closed unit square $\{0 \le x \le 1,\ 0 \le y \le 1\}$ in the (x,y) plane, and let S^- and S^+ be disjoint relatively open subsets of I, respectively containing the lines $y=0$ and $y=1$. Then the complement D of S^+ and S^- in I contains a continuum joining the lines $x=0$ and $x=1$.*

The proposition then gives the existence of a continuum $\mathscr{C}$ that lies entirely in $Q-(S^+\cup S^-)$ and which joins a point on $\mu=\max\left(\frac{1}{3},\frac{1}{2(\gamma-1)}\right)$ to a point on $\mu=1$.

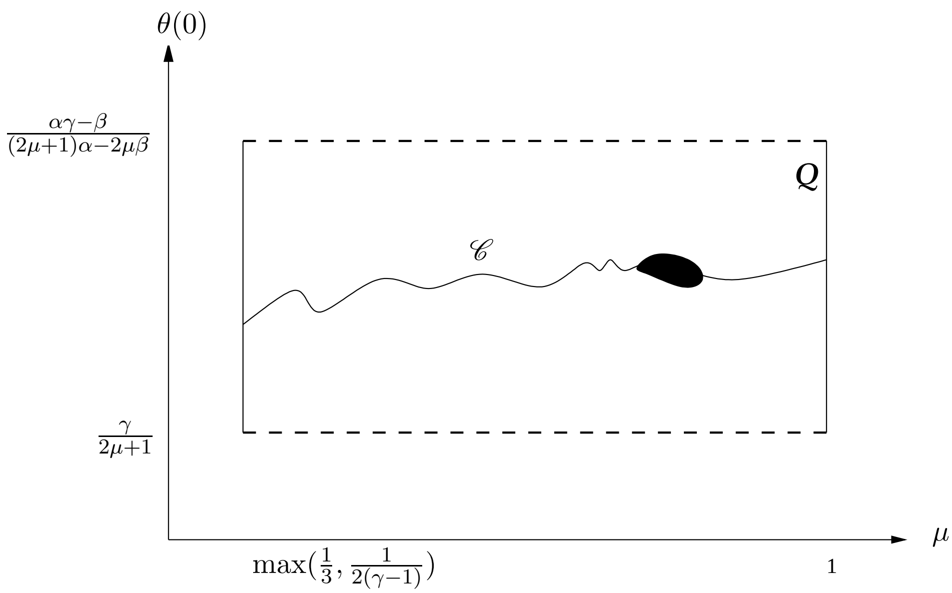

Figure 2: Plot of continuum

If $(\mu,\theta(0))\in Q-(S^+\cup S^-)$, then

$$\frac{\gamma}{2\mu+1}<\theta\,(0)<\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$$

and

$$\max\left(\frac{1}{3},\frac{1}{2(\gamma-1)}\right)\le\mu\le 1.$$

By Lemmas 1 and 2, neither g' nor θ' must cross 0 first, and so g' and θ' must vanish simultaneously or neither crosses 0 at all.

The former cannot be true, otherwise we could find $t^* > 0$ such that

$$g'(t^*) = 0, \qquad \theta'(t^*) = 0, \tag{2.2.5}$$

while $g'(t) \le 0$ and $\theta'(t) \le 0$ for $0 \le t \le t^*$. Since $g''(t^*) \ge 0$, then by (1.0.2) and (2.2.5)

$$\beta \left(\frac{f^2}{g} g'(t^*) \right)' + (1-\mu) t^* g'(t^*) \ge 0, \quad \text{or}$$

$$(1+2\mu) g(t^*) - \gamma \frac{g(t^*)^2}{f(t^*)} \le 0.$$

Therefore $\theta(t^*) \le \frac{\gamma}{2\mu+1}$. Also $\theta''(t^*) \ge 0$, so that by (2.0.1) and (2.2.5), $\theta(t^*) \ge \frac{\alpha\gamma - \beta}{(2\mu+1)\alpha - 2\mu\beta}$. This implies

$$\frac{\alpha\gamma - \beta}{(2\mu+1)\alpha - 2\mu\beta} \le \frac{\gamma}{2\mu+1},$$

a contradiction to (2.1.2).

Thus neither crosses zero and so $g' \le 0$ and $\theta' = \frac{f'}{g} - \frac{fg'}{g^2} \le 0$, and consequently $f' \le 0$, for as long as the solution is defined. We lastly note that g cannot vanish before f, since θ is bounded.

2.3 DEVELOPING THE MAIN THEOREM

For $(\mu, \theta(0)) \in Q - (S^+ \cup S^-)$, we define

$$T = \begin{cases} t_0, & \text{if } t_0 \text{ is the first zero of } f, \\ \infty, & \text{if always } f > 0 \end{cases} \tag{2.3.1}$$

and from (2.0.2)-(2.0.3), consider the values of

$$I = \int_0^T \left(g - (3\mu - 1) f\right) dt, \qquad J = \int_0^T \gamma \frac{g}{f}\left(g - \frac{3\mu}{\gamma} f\right) dt. \tag{2.3.2}$$

The existence of $(\mu, \theta(0))$ such that there is a positive solution to (1.0.1)-(1.0.4) will be proved by use of the following lemmas:

Lemma 3. *If $I > 0$, then the solution (f, g) exists on $[0, \infty)$, with $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$, and in fact $I = \infty$. Also, $J = \infty$. If $A = \{(\mu, \theta(0)) : I > 0\}$, then A is open in $\mathscr{C}$ and non-empty. Indeed, if*

$$\mu \leq \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 2) - \beta}$$

then $(\mu, \theta(0)) \in A$.

Lemma 4. $A \neq \mathscr{C}$

Lemma 5. *If $(\mu, \theta(0))$ is a point in $\mathscr{C}$ and on the boundary of A, then for the corresponding solution (f, g) there exists a finite point t_0 such that $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$ for $t < t_0$, while $f(t_0) = 0$, $g(t_0) = 0$, $\frac{f^2}{g} f'(t_0) = 0$, $\frac{f^2}{g} g'(t_0) = 0$.*

Now if we rescale t_0 to become 1, we have the following theorem:

Theorem. *With $\beta > \alpha$, $3\alpha > 2\beta$, and $\gamma > \frac{3}{2}$, there exists a $\mu > 0$ such that the problem (1.0.1)-(1.0.4) possesses a solution, and the solution is such that $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$ in $(0, 1)$.*

3.0 PROOFS

3.1 LEMMA 1

Lemma 1. *If (2.1.1)-(2.1.3) hold, and* $\theta(0)$ *is sufficiently close to* $\frac{\gamma}{2\mu+1}$*, then there exists* $t^{-} > 0$ *such that*

$$g'(t^{-}) > 0$$

$$while\ \theta'(t) < 0,\ for\ 0 < t \leq t^{-}.$$

Proof.

If $\theta(0) = \frac{f(0)}{g(0)} = \frac{\gamma}{2\mu+1}$, then by (1.0.2) we have that $g'(0) = 0$ and $g''(0) = 0$. Differentiating (1.0.2) gives

$$\beta\left(\left(\frac{f^2}{g}\right)''g' + 2\left(\frac{f^2}{g}\right)'g'' + \left(\frac{f^2}{g}\right)g'''\right) + (1-\mu)g' + (1-\mu)tg'' + (1+2\mu)g' - \gamma\left(\frac{g^2}{f}\right)' = 0 \quad (3.1.1)$$

which evaluated at $t = 0$ reduces to $g'''(0) = 0$.

If we differentiate (3.1.1) and let $t = 0$ then

$$\beta\frac{f^2}{g}g^{(4)} + \gamma\frac{g^2}{f^2}f'' = 0. \quad (3.1.2)$$

At $t = 0$ we also know from (1.0.1) that

$$\alpha\frac{f^2}{g}f'' + 2\mu f - g = 0, \quad (3.1.3)$$

and so

$$\alpha\frac{f^2}{g}f'' = g - 2\mu f.$$

Now recalling (2.1.8), then it is clear that $\theta(0) > \frac{1}{2\mu}$ and so $g - 2\mu f < 0$ at $t = 0$. From (3.1.3) $f''(0) < 0$ and so by (3.1.2) $g^{(4)}(0) > 0$.

For $t > 0$ yet sufficiently close to zero, we have

$$\int_0^t g^{(4)}(\tau)d\tau = g'''(t) > 0,$$
$$\int_0^t g'''(\tau)d\tau = g''(t) > 0, \text{ and}$$
$$\int_0^t g''(\tau)d\tau = g'(t) > 0.$$

By (2.0.1), $\theta(0) < \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, and the observations made in Section 2.2, for such t

$$\frac{d}{dt}\left(f^2 g\frac{d\theta}{dt}\right) + \frac{(1-\mu)}{\alpha}tg^2\frac{d\theta}{dt} < 0.$$

Thus $\theta'(t)$ is initially negative. Consequently, when $\theta(0)$ is just greater than $\frac{\gamma}{2\mu+1}$, by continuity of the initial data $g'(t)$ becomes positive at small t and so before $\theta' = \frac{f'g-g'f}{g^2}$ becomes zero.

3.2 LEMMA 2

Lemma 2. *If (2.1.1)-(2.1.3) hold, and $\theta(0)$ is sufficiently close to $\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then there exists $t^+ > 0$ such that*

$$\theta'(t^+) > 0$$

$$\textit{while } g'(t) < 0, \;\; \textit{for } 0 < t \le t^+.$$

Proof.

From (2.0.1) it follows that if $\theta(0) = \frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then $\theta'(0) = 0$ and $\theta''(0) = 0$. Differentiating (2.0.1) we have

$$\begin{gathered}\left(f^2g\theta'\right)'' + \tfrac{(1-\mu)}{\alpha}\left(tg^2\theta\right)' = 2gg'\left(\tfrac{2\mu+1}{\beta} - \tfrac{2\mu}{\alpha}\right)\left(\theta - \tfrac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}\right) + g^2\theta'\left(\tfrac{2\mu+1}{\beta} - \tfrac{2\mu}{\alpha}\right) + \\ +(1-\mu)\left(fg' + tf'g' + tfg''\right)\left(\tfrac{1}{\beta} - \tfrac{1}{\alpha}\right)\end{gathered} \tag{3.2.1}$$

and evaluating the result at $t = 0$ gives $\theta'''(0) = 0$. Differentiating (2.0.1) a second time and evaluating at $t = 0$ gives

$$f^2g\theta^{(4)} = 2(1-\mu)fg''\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) > 0$$

since $\alpha < \beta$ and $g''(0) < 0$ from (1.0.2). Thus for $t > 0$ sufficiently close to zero,

$$\begin{gathered}\textstyle\int_0^t \theta^{(4)}(\tau)d\tau = \theta'''(t) > 0, \\ \textstyle\int_0^t \theta'''(\tau)d\tau = \theta''(t) > 0, \text{ and} \\ \textstyle\int_0^t \theta''(\tau)d\tau = \theta'(t) > 0.\end{gathered}$$

Now by (2.0.1), $\theta(0) > \frac{\gamma}{2\mu+1}$, and the observations made in Section 2.2, g' is initially negative for sufficiently small $t > 0$.

By continuity of the initial data, if $\theta(0)$ is just less than $\frac{\alpha\gamma-\beta}{(2\mu+1)\alpha-2\mu\beta}$, then $\theta'(t)$ becomes positive at small t and before g' becomes 0.

3.3 PROPOSITION

Proposition. *Let I be the closed unit square $\{0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ in the (x, y) plane, and let S^- and S^+ be disjoint relatively open subsets of I, respectively containing the lines $y = 0$ and $y = 1$. Then the complement D of S^+ and S^- in I contains a continuum joining $x = 0$ and $x = 1$.*

Proof. (Based on the proof in [15].) By setting M to be the closure of the component of S^- that contains the line $y = 0$, N to be component of $I - M$ that contains the line $y = 1$, and Δ to be the intersection of M with the boundary of N, we aim to show the following about Δ:

- Δ is closed
- Δ contains a point on lines $x = 0$ and $x = 1$
- $\Delta \subset D$
- Δ is connected.

By construction $\Delta = M \cap \partial N$, the intersection of two closed sets, contains exactly the points on $x = 0$ and $x = 1$ that are furthest away from $y = 0$ in M. Further, if $P \in \Delta$, then $P \in M$ and there are points close to P in S^-; also $P \in \partial N$ and there are points close to P that are not in S^-. Since S^+ and S^- are open sets, if $P \in S^+$ or $P \in S^-$ then nearby points must also be in S^+ or S^-, which is clearly a contradiction. Consequently, Δ lies in D, the complement of S^+ and S^-.

If Δ is not connected then $\Delta = H \cup K$, where H, K are mutually separated, closed sets. Suppose that H and K are some positive distance, δ, away from each other.

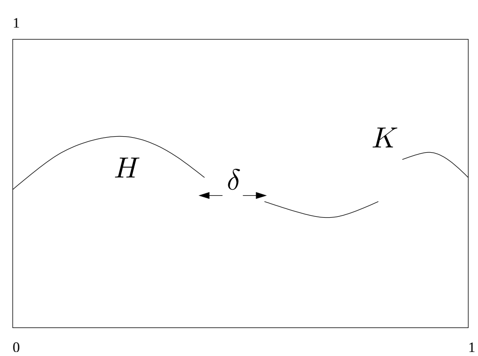

Figure 3: Sets H and K separated by distance delta

Setting down a grid of closed squares of length $\frac{\delta}{\sqrt{2}}$, consider just those squares which intersect K.

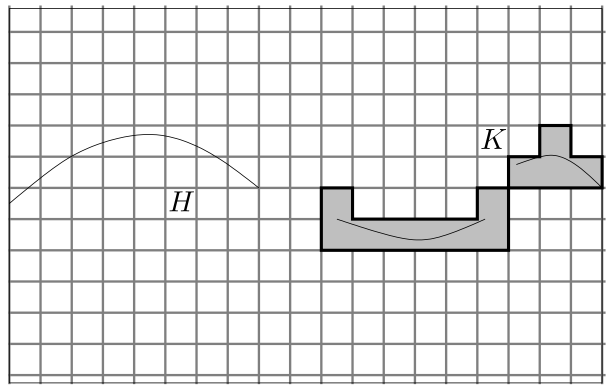

Figure 4: Grid of closed squares

The boundary of the union of the squares can be expressed as a finite number of simple closed curves. Let Q' be such a curve picked closest to a point $A \in H$, which by construction is disjoint from both H and K.

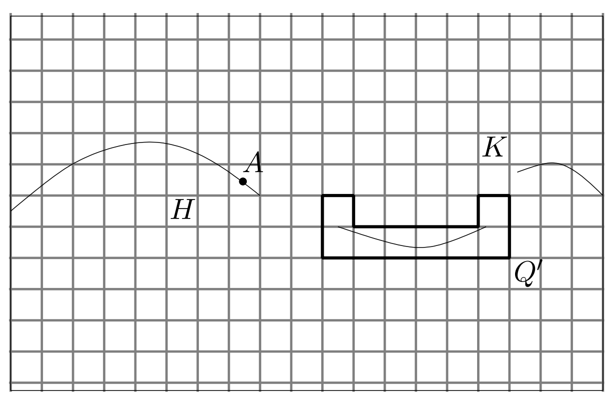

Figure 5: Curve Q' is disjoint from both H and K

If R is one of the points on Q' closest to A, and Q is the component of $Q' \cap I$ which contains R, then Q separates $A \in H \subset \Delta$ from some point, say B, in I, where $B \in K \subset \Delta$ lies in the square corresponding to R.

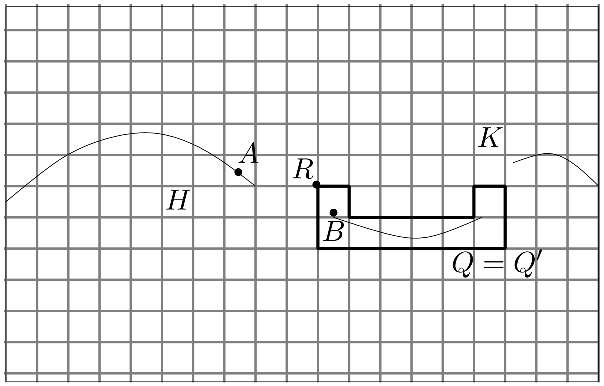

Figure 6: Simple curve Q separates A from B

Recalling that $\Delta = M \cup \partial N$, both of the regions separated by Q contain points in M and N. If $Q \cap M = \emptyset$, then $M = M_1 \cup M_2$, where $\overline{M_1} \cap M_2 = \emptyset$ and $M_1 \cap \overline{M_2} = \emptyset$, M_1 a region inside of Q, and M_2 a region outside of Q. Hence, M_1 and M_2 are contained in each of the complementary domains of Q, a contradiction to the assumption that M is connected. Similarly, if $Q \cap N = \emptyset$, then $N = N_1 \cup N_2$, the union of two mutually separated sets, each of which is contained in the region inside or outside of Q. As before, this contradicts the assumption that N is connected.

Therefore, $Q \cap M \neq \emptyset$ and $Q \cap N \neq \emptyset$, implying that we can find points $P_M \in M$ and $P_N \in N$ which are on Q. If we follow along Q from P_N to P_M we will reach $P_{\overline{N}} \in Q$, one of the last points in the closure of N.

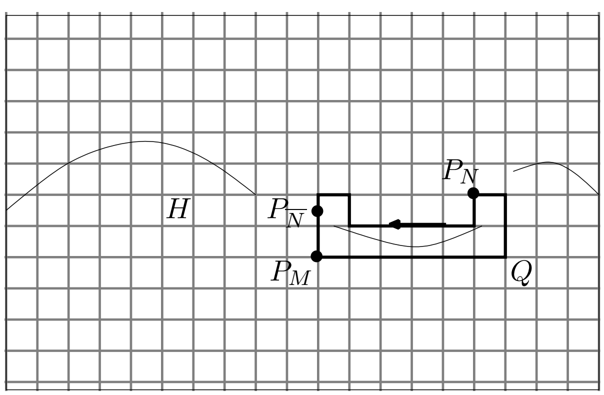

Figure 7: Existence of points on Q

By construction of sets M and N, $P_{\overline{N}} \in \partial N \cup M = \Delta$ implying that $\Delta \cap Q \neq \emptyset$, although it was assumed prior that Q is disjoint from both H and K, and thus from Δ. Hence Δ is connected.

3.4 LEMMA 3

Lemma 3. *If* $I > 0$, *then the solution* (f, g) *exists on* $[0, \infty)$ *,with* $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$, *and in fact* $I = \infty$. *Also,* $J = \infty$. *If* $A = \{(\mu, \theta(0)) : I > 0\}$ *, then* A *is open in* $\mathscr{C}$ *and non-empty. Indeed, if*

$$\mu \leq \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 2) - \beta}$$

then $(\mu, \theta(0)) \in A$.

Proof. If T is finite then by definition (2.3.1), $f(T) = 0$, Thus from (2.0.2) we have

$$\frac{\alpha f^2}{g} f'(T) = \int_0^T (g - (3\mu - 1) f) \, dt.$$

However, since f is decreasing,

$$\liminf_{t \to T} f'(t) \leq 0,$$

from which (2.3.2) gives

$$I = \int_0^T (g - (3\mu - 1) f) \, dt \leq 0.$$

Therefore by the contrapositive, if we assume $I > 0$, then $T = \infty$ and so $f(t) > 0$ for all $t \in [0, \infty)$. Since $\theta = \frac{f}{g}$ is bounded, then $g(t) > 0$ for t in $[0, \infty)$.

It also must be true that

$$\lim_{t \to \infty} (g(t) - (3\mu - 1) f(t)) > 0. \tag{3.4.1}$$

Otherwise if

$$\lim_{t \to \infty} (g(t) - (3\mu - 1) f(t)) \leq 0 \tag{3.4.2}$$

, then

$$\lim_{t \to \infty} \frac{f(t)}{g(t)} \geq \frac{1}{3\mu - 1}.$$

and so

$$\frac{f(t)}{g(t)} > \lim_{t\to\infty}\frac{f(t)}{g(t)} \geq \frac{1}{3\mu-1}.\ \text{for } t\in[0,\infty).$$

since $\frac{f}{g}$ is decreasing. This implies that

$$\int_0^t (g-(3\mu-1)\,f)\,ds < 0$$

for all $t\in[0,\infty)$, which contradicts our assumption that $I>0$.

Also, for all $t\in[0,\infty)$ we have that

$$(1-\mu)tf(t) \geq \int_0^t (g-(3\mu-1)\,f)\,ds \tag{3.4.3}$$

since $f'\leq 0$. By (3.4.1)

$$\int_0^t (g-(3\mu-1)\,f)\,ds > 0$$

for t sufficiently large, and so

$$f(t)\geq\frac{K}{t},$$

for a constant $K>0$. This implies that

$$\int^\infty f dt=\infty$$

and by $\theta=\frac{f}{g}\leq M$, where $M>0$

$$\int^\infty g dt \geq \frac{1}{M}\int^\infty f dt=\infty.$$

Consequently, we find that $I=\infty$. Otherwise, if $0<I<\infty$, then

$$I=\int_0^\infty f\left(\frac{g}{f}-(3\mu-1)\right)dt=\int_0^{T^*} f\left(\frac{g}{f}-(3\mu-1)\right)dt+\int_{T^*}^\infty f\left(\frac{g}{f}-(3\mu-1)\right)dt<\infty$$

for $0\leq T^*<\infty$ such that

$$\int_0^{T^*} f\left(\frac{g}{f}-(3\mu-1)\right)dt\leq 0$$

and

$$f\left(\frac{g}{f}-(3\mu-1)\right)>0,\ \text{for } t>T^*.$$

Thus

$$\int_{T^*}^\infty f\left(\frac{g}{f}-(3\mu-1)\right)dt>\int_{T^*}^\infty K dt=\infty,$$

for some constant $K>0$, a contradiction to I finite. Therefore, $I=\infty$.

From above $I > 0$ implies that $T = \infty$, and so $f > 0$ and $g > 0$ for all $t \in [0, \infty)$. Recall (2.0.3):

$$\beta \frac{f^2}{g} g' + (1-\mu) t g = \int_0^t \gamma \frac{g}{f} \left(g - \frac{3\mu}{\gamma} f \right) ds.$$

Since θ is bounded

$$\left| \frac{f^2}{g} g' \right| \leq |M f g'| \leq |M^2 g g'| = O\left((g^2)' \right).$$

Therefore, there exists a sequence $\{t_n\}$ such that

$$\frac{f^2}{g} g'(t_n) \to 0, \quad \text{as } t_n \to \infty. \tag{3.4.4}$$

Now it also must be that $\lim_{t\to\infty} \left(g(t) - \frac{3\mu}{\gamma} f(t) \right) > 0$ If not, then for all $t < \infty$,

$$\frac{f(t)}{g(t)} > \lim_{t\to\infty} \frac{f(t)}{g(t)} \geq \frac{\gamma}{3\ mu}$$

and so for all t,

$$\int_0^t \gamma \frac{g}{f} \left(g - \frac{3\mu}{\gamma} f \right) ds < 0$$

implying by (2.0.3) that

$$\beta \frac{f^2}{g} g' + (1-\mu) t g < 0$$

for all $t < \infty$, a contradiction to the assumption that (3.4.4) holds and $g > 0$. Hence we have

$$\lim_{t\to\infty} \theta(t) < \frac{\gamma}{3\mu}$$

and as done above for I,

$$J = \int_0^\infty \gamma \frac{g}{f} \left(g - \frac{3\mu}{\gamma} f \right) dt = \infty.$$

We now will prove that the set

$$A = \{(\mu, \theta(0)) : I > 0\}$$

is open. If we have a solution f_0 such that $f_0(T_0) = 0$, so that $(\mu_0, \theta_0(0)) \notin A$, and if we also consider a sequence $\{(\mu_n, \theta_n(0))\}$ tending to $(\mu_0, \theta_0(0))$, with solutions f_n such that $f_n(T_n) = 0$, then

$$\liminf_{n\to\infty} T_n \geq T_0.$$

This holds because $f_0 > 0$ for all $t < T_0$, and so while solutions f_n are close to f_0 for $t < T_0$, f_n cannot disappear before f_0 does as $n \to \infty$.

Consequently, if $(\mu_0, \theta_0(0)) \in A$ with corresponding solution (f_0, g_0), then a nearby solution, say (f, g), must either be in A (and so A is open) or will at least have the behavior that (f, g) is close to (f_0, g_0) in $[0, \widetilde{T}]$, where $f(\widetilde{T}) = 0$. By (3.4.1) we have that

$$\lim_{t\to\infty} \theta_0(t) < \frac{1}{3\mu - 1}$$

but if (f_0, g_0) is close enough to (f, g) over $[0, \widetilde{T}]$ there exists a finite value T^*, such that $\theta_0(T^*) < \frac{1}{3\mu-1}$, which implies that $\theta(T^*) < \frac{1}{3\mu-1}$. Now θ monotone implies that

$$\lim_{t\to\infty} \theta(t) < \theta(T^*) < \frac{1}{3\mu - 1}.$$

Then T^* can be chosen so that

$$\int_0^{T^*} (g_0 - (3\mu - 1) f_0)\, dt > 0, \text{ and } \int_0^{T^*} (g - (3\mu - 1) f)\, dt > 0,$$

and so $I > 0$ and $(\mu, \theta(0)) \in A$. Therefore, A is open.

Lastly we note that if $\theta(t) < \frac{1}{3\mu-1}$ for all $t \geq 0$, then $I = \int_0^t (g - (3\mu - 1) f)\, ds > 0$. Further from (2.1.2)

$$\theta(t) \leq \theta(0) < \frac{\alpha\gamma - \beta}{(2\mu + 1)\alpha - 2\mu\beta}$$

for all $t \geq 0$. If we could find appropriate conditions so that

$$\frac{\alpha\gamma - \beta}{(2\mu + 1)\,\alpha - 2\mu\beta} \leq \frac{1}{3\mu - 1}, \tag{3.4.5}$$

it would be sufficient for proving $I > 0$.

Thus in order for (3.4.5) to be true, then

$$(\alpha\gamma - \beta)\,(3\mu - 1) \leq (2\mu + 1)\,\alpha - 2\mu\beta,$$

and so

$$\mu \leq \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 2) - \beta}. \tag{3.4.6}$$

If (3.4.6) holds, then $(\mu, \theta(0)) \in A$.

We also note that new bound (3.4.6) for μ satisfies

$$\max\left(\frac{1}{3}, \frac{1}{2\,(\gamma - 1)}\right) < \frac{\alpha(\gamma + 1) - \beta}{\alpha(3\gamma - 1) + 2\beta} \leq 1. \tag{3.4.7}$$

Under the assumptions (2.1.1) and if in particular $\gamma > \frac{5}{2}$, then it is clear that

$$\max\left(\frac{1}{3}, \frac{1}{2\,(\gamma-1)}\right) = \frac{1}{3} < \frac{\alpha(\gamma+1)-\beta}{\alpha(3\gamma-1)+2\beta}.$$

In order to prove

$$\frac{\alpha(\gamma+1)-\beta}{\alpha(3\gamma-1)+2\beta} \leq 1,$$

we need only show that

$$\alpha(\gamma+1)-\beta \leq \alpha(3\gamma-1)+2\beta,$$

which again holds under (2.1.1). The remaining case of $\frac{3}{2} < \gamma \leq \frac{5}{2}$ is similar.

3.5 LEMMA 4

Lemma 4. $A \neq \mathscr{C}$

Proof. We will prove that there exists $(\mu, \theta(0)) \in \mathscr{C}$ that is not in A.

We note that when $\mu = 1$ the left-hand side of (2.0.2) becomes

$$\frac{f^2}{g} f' = \int_0^t (g - (3\mu - 1)f)\, ds$$

and since $f' \leq 0$ for all t for which the solution exists, then

$$\int_0^t (g - (3\mu - 1)f)\, ds \leq 0.$$

Thus $I \leq 0$ and so $(\mu, \theta(0)) \notin A$ using a result from the proof of Lemma 3. If we suppose $I = 0$,

$$\int_0^T (g - (3\mu - 1)f)\, ds = \int_0^T f \left(\frac{g}{f} - (3\mu - 1) \right) ds = 0, \tag{3.5.1}$$

then since $\frac{g}{f}$ is increasing,

$$\int_0^t (g - (3\mu - 1)f)\, ds < 0 \tag{3.5.2}$$

for t small enough, $0 < t < T$. We also notice from (2.0.3) with $\mu = 1$,

$$\beta \frac{f^2}{g} g' = \int_0^t \gamma \frac{g}{f} \left(g - \frac{3\mu f}{\gamma} \right) ds$$

and since $g' \leq 0$ for as long as the solution continues to exist, then $J \leq 0$.

However, we can express

$$J = \int_0^T \gamma \tfrac{g}{f} \left(g - \tfrac{3\mu f}{\gamma} \right) ds = \int_0^T \gamma \tfrac{g}{f} \left(g - \tfrac{3\mu f}{\gamma} - (3\mu - 1)f + (3\mu - 1)f \right) ds$$

$$= \int_0^T \gamma \tfrac{g}{f} (g - (3\mu - 1)f)\, ds + \int_0^T \gamma \tfrac{g}{f} \left((3\mu - 1) - \tfrac{3\mu}{\gamma} \right) f ds$$

$$= J_1 + J_2.$$

We notice when $\mu = 1$, the integrand of J_2 is

$$(3\mu - 1) - \frac{3\mu}{\gamma} = \left(2 - \frac{3}{\gamma} \right) > 0$$

since $\gamma > \frac{3}{2}$, and consequently $J_2 > 0$.

For J_1 we can integrate by parts to get

$$J_1 = \int_0^T \gamma \tfrac{g}{f} \left(g - (3\mu - 1)f\right) ds$$

$$= \gamma \tfrac{g}{f} \left(\int_0^s \left(g - (3\mu - 1)f\right) d\tau\right)\Big|_0^T - \int_0^T \gamma \left(\tfrac{g}{f}\right)' \left(\int_0^s \left(g - (3\mu - 1)f\right) d\tau\right) ds$$

$$= -\int_0^T \gamma \left(\tfrac{g}{f}\right)' \left(\int_0^s \left(g - (3\mu - 1)f\right) d\tau\right) ds$$

by (3.5.1). By $\frac{g}{f} > 0$ and (3.5.2), $J_1 > 0$ which implies that $J > 0$. This contradicts that above we found $J \leq 0$.

Therefore, it must be that $I < 0$ when $\mu = 1$, and so by (2.0.2)

$$\alpha \frac{f^2}{g} f' < -K$$

for some constant $K > 0$. This is equivalent to having

$$f' < -K \left(\frac{g}{f}\right) \frac{1}{f}$$

for a constant $K > 0$. As $\frac{g}{f}$ and $\frac{1}{f}$ increase, f' is increasingly negative, forcing the existence of a finite point $t^* > 0$ such that $f(t^*) = 0$.

3.6 LEMMA 5

Lemma 5. *If* $(\mu, \theta(0))$ *is a point in* $\mathscr{C}$ *and on the boundary of* A*, then for the corresponding solution* (f, g) *there exists a finite point* t_0 *such that* $f' \leq 0$, $g' \leq 0$, $\theta' \leq 0$ *for* $t < t_0$*, while* $f(t_0) = 0$, $g(t_0) = 0$, $\frac{f^2}{g} f'(t_0) = 0$, $\frac{f^2}{g} g'(t_0) = 0$.

Proof. From Lemma 4, if we let $(\mu, \theta(0))$ be one of the first points in $\mathscr{C}$ that is not in A, then

$$I = \int_0^T (g - (3\mu - 1)f)\, dt \leq 0. \tag{3.6.1}$$

Consequently the solution (f, g) cannot exist for all $t \in [0, \infty)$. If not, we take $f > 0$ for all t and use a technique from Lemma 3 that

$$\left| \frac{f^2}{g} f' \right| \leq |M f f'| = O\left(\left(f^2\right)' \right)$$

means there exists a sequence $\{t_n\}$ such that

$$\frac{f^2}{g} f'(t_n) \to 0, \quad \text{as } t_n \to \infty. \tag{3.6.2}$$

Then

$$0 \leq \alpha \frac{f^2}{g} f'(t_n) + (1 - \mu) t_n f(t_n) \leq I \leq 0,$$

so $I = 0$; and we have

$$g(T) - (3\mu - 1) f(T) = 0$$

so

$$\frac{f(t)}{g(t)} > \frac{f(T)}{g(T)} = \frac{1}{3\mu - 1}$$

for all $t < T$. By (1.0.2) for all $t < T$,

$$\int_0^t (g - (3\mu - 1)f)\, ds < 0$$

or

$$\alpha \frac{f^2}{g} f' + (1 - \mu) t f < 0.$$

Hence

$$f' < \frac{-(1 - \mu) t}{\alpha} \left(\frac{g}{f} \right),$$

which, for $\mu \neq 1$, becomes more and more negative as t and $\frac{g}{f}$ increase, a contradiction to the assumption that $f > 0$ for all t. When $\mu = 1$, we have from LEMMA 4 that f does not exist on all of $[0, \infty)$.

This proves there exists $T = t_0 < \infty$, for which $f(t_0) = 0$ and so

$$I = \int_0^{t_0} (g - (3\mu - 1)f)\, ds \leq 0. \tag{3.6.3}$$

Next we let

$$B = \{(\mu, \theta(0)) : I < 0\},$$

and we want to prove that B is open.

From LEMMA 3 if we have a solution f_0 such that $f_0(T_0) = 0$, so that $(\mu_0, \theta_0(0)) \in B$, and if we also consider a sequence $\{(\mu_n, \theta_n(0))\}$ tending to $(\mu_0, \theta_0(0))$, with solutions f_n such that $f_n(T_n) = 0$, then it must be that

$$\liminf_{n\to\infty} \; T_n \geq T_0.$$

If we can prove that indeed

$$\lim_{n\to\infty} T_n = T_0$$

then

$$\int_0^{T_n} (g_n - (3\mu - 1)\, f_n)\, ds \longrightarrow \int_0^{T_0} (g_0 - (3\mu - 1)\, f_0)\, ds < 0,$$

implying that nearby solutions are also in B.

Suppose for a contradiction that $T_n \to T^* > T_0$. We know, however, that f_n and f_0 are close for $t < T_0$, and since f_n is decreasing, $f_n \to 0$ in $[T_0, T_n]$.

Now by (2.0.2), for $t \in [T_0, T_n]$

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) t f_n = \int_0^t (g_n - (3\mu - 1)\, f_n)\, ds = \int_0^{T_0} (g_0 - (3\mu - 1)\, f_0)\, ds + \int_{T_0}^t (g_n - (3\mu - 1)\, f_n)\, ds$$

and so

$$g_n = \left(\alpha \frac{f_n^2}{g_n} f_n' + (1 - \mu) t f_n + \int_{T_0}^t (3\mu - 1)\, f_n ds \right)'.$$

Since $g_n' \leq 0$, then

$$\left(\alpha \frac{f_n^2}{g_n} f_n' + (1 - \mu) t f_n + \int_{T_0}^t (3\mu - 1)\, f_n ds \right)' \leq 0. \tag{3.6.4}$$

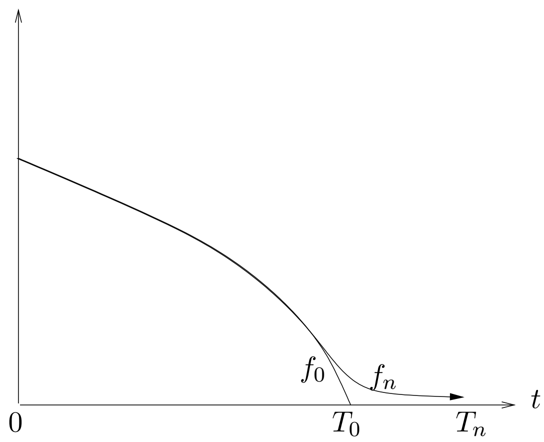

Figure 8: Plot over [T0,Tn]

Since

$$\left|\int_{T_0}^{T_n} \frac{f_n^2}{g_n} f_n' ds\right| = \left|\int_{T_0}^{T_n} \frac{f_n}{g_n} f_n f_n' ds\right| \leq \theta_n(T_0) \left|\int_{T_0}^{T_n} f_n f_n' ds\right| = \theta_n(T_0) \left|\int_{T_0}^{T_n} \left(\frac{f_n^2}{2}\right)' ds\right| \longrightarrow 0$$

then for $T_0 \leq t \leq T_n$,

$$\int_{T_0}^{t} \left(\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) s f_n + \int_{T_0}^{s} (3\mu - 1) f_n d\tau\right) ds \longrightarrow 0.$$

Thus by (3.6.4),

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) s f_n + \int_{T_0}^{s} (3\mu - 1) f_n d\tau \longrightarrow 0$$

almost everywhere in $[T_0, T_n]$, which means that

$$\int_{T_0}^{t} g_n ds \to 0$$

for $t \in [T_0, T_n]$. As above we write

$$\int_0^t (g_n - (3\mu - 1) f_n) ds = \int_0^{T_0} (g_0 - (3\mu - 1) f_0) ds + \int_{T_0}^{t} (g_n - (3\mu - 1) f_n) ds \to \int_0^{T_0} (g_0 - (3\mu - 1) f_0) ds$$

for $t \in [T_0, T_n]$. Since $(\mu_0, \theta_0(0)) \in B$, $I_0 = \int_0^{T_0} (g_0 - (3\mu - 1) f_0) ds < 0$, this contradicts

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) t f_n = \int_0^t (g_n - (3\mu - 1) f_n) ds \longrightarrow 0,$$

for $T_0 \leq t \leq T_n$. Therefore, $\lim_{n\to\infty} T_n = T_0$ and so B is open in $\mathscr{C}$.

Consequently, if $(\mu, \theta(0))$ is to be one of the first points on $\mathscr{C}$ that is not in A, then we also must restrict $(\mu, \theta(0)) \notin B$. Thus by (3.6.3), $(\mu, \theta(0))$ is such that

$$I = \int_0^{t_0} \left(g - (3\mu - 1)\, f\right) ds = 0$$

and so

$$\alpha \frac{f^2}{g} f'(t_0) + (1-\mu) t_0 f(t_0) = 0$$

implying that

$$\alpha \frac{f^2}{g} f'(t_0) = 0.$$

Letting $\{(f_n, g_n)\}$ be a sequence on $[0,\infty)$ that approximates (f,g) on $[0,t_0)$ where $f_n' \leq 0$, then $f_n \to 0$ on $[t_0, \infty)$, and as similarly done in the proof of B being open, $\frac{f_n^2}{g_n} f_n' \to 0$ almost everywhere on $[t_0,\infty)$. So

$$\alpha \frac{f_n^2}{g_n} f_n' + (1-\mu) t f_n + \int_{t_0}^{t} (3\mu - 1)\, f_n ds = \int_{t_0}^{t} g_n ds \longrightarrow 0$$

and then $g_n \to 0$ on $[t_0, \infty)$.

If we now assume

$$J = \int_0^{t_0} \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds > 0,$$

then at t_0

$$g\left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) > 0$$

and so

$$\theta(t_0) < \frac{\gamma}{3\mu}.$$

Thus there exists $T^* < t_0$, large enough and still close enough to t_0, for which

$$\int_0^{T^*} \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds > 0, \text{ and}$$

$$\int_0^{T^*} \gamma g_n \left(\frac{g_n}{f_n} - \frac{3\mu}{\gamma}\right) ds > 0.$$

For $t \geq T^*$

$$\frac{g_n(t)}{f_n(t)} \geq \frac{g_n(T^*)}{f_n(T^*)} > \frac{3\mu}{\gamma}$$

and so

$$\int_0^t \gamma g_n \left(\frac{g_n}{f_n} - \frac{3\mu}{\gamma}\right) ds > 0.$$

Thus by (2.0.3) for $t \in (t_0, \infty)$,

$$\beta \frac{f_n^2}{g_n} g_n' + (1-\mu) t g_n = \int_0^t \gamma g_n \left(\frac{g_n}{f_n} - \frac{3\mu}{\gamma}\right) ds > 0,$$

a contradiction to our results from above that $g_n \to 0$ and $g_n' \leq 0$. Therefore $J \leq 0$, which implies that for $t < t_0$, sufficiently small enough,

$$\beta \frac{f^2}{g} g' + (1-\mu) t g = \int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds < 0. \tag{3.6.5}$$

In order to prove $g(t_0) = 0$, we assume that $g(t_0) = g_0 > 0$ and hope for a contradiction. Thus for $t < t_0$

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f = \int_0^t (g - (3\mu - 1) f)\, ds < I = 0, \tag{3.6.6}$$

and so

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f < 0,$$

which implies that

$$\alpha \left(\frac{f^2}{2}\right)' < -(1-\mu) t g. \tag{3.6.7}$$

As $t \to t_0$, $g \to g_0$, so if we integrate both sides of (3.6.7) from t to t_0,

$$f^2 > K(t_0 - t),$$

for positive constant K, and so

$$f > K(t_0 - t)^{\frac{1}{2}}$$

for another constant $K > 0$.

Now from (3.6.6) and $I = 0$, as $t \to t_0$

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f \leq K(t_0 - t)$$

for some constant $K > 0$, and so

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f = O(t_0 - t)$$

as $t \to t_0$ which implies that

$$\alpha \frac{f^2}{g} f' \sim -(1-\mu) t f.$$

Since

$$\lim_{t\to t_0} \frac{\frac{f^2}{g} f'}{-(1-\mu)tf} = \lim_{t\to t_0} \frac{ff'}{-(1-\mu)tg} = \frac{\lim_{t\to t_0} ff'}{-(1-\mu)t_0 g_0},$$

then it is also true that

$$ff' \sim -(1-\mu)t_0 g_0. \tag{3.6.8}$$

Additionally by integrating up to t_0

$$\lim_{t\to t_0} \frac{\left(\frac{f^2}{2}\right)'}{-(1-\mu)t_0 g_0} = \lim_{t\to t_0} \frac{f^2}{K(t_0 - t)}$$

for a positive constant K, and thus

$$f^2 \sim K(t_0 - t) \tag{3.6.9}$$

as $t \to t_0$. For $t < t_0$

$$\beta \frac{f^2}{g} g' < \beta \frac{f^2}{g} g' + (1-\mu)tg < 0,$$

then

$$g' < \frac{-Kg}{f^2}$$

for some $K > 0$, and so by (3.6.9)

$$g' < \frac{-K}{t_0 - t}$$

for another positive constant K. Integrating both sides up to t_0,

$$\int_t^{t_0} g' ds < \int_t^{t_0} \frac{K}{s - t_0} ds$$

we find that

$$g(t) - g_0 > \infty$$

for all $t < t_0$, which is not possible. Therefore it must be that $g(t_0) = 0$.

It remains to show that $J = 0$, which will be used to prove that $\frac{f^2}{g} g'(t_0) = 0$. We therefore suppose $J < 0$, in order to get a contradiction. That is from (3.6.5), for $t < t_0$

$$\beta \frac{f^2}{g} g' + (1-\mu)tg = \int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds < 0$$

and for $t \to t_0$

$$\lim_{t\to t_0} \frac{\beta \frac{f^2}{g} g' + (1-\mu)tg}{\int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) ds} \to \frac{\beta \frac{f^2}{g}}{J},$$

so that

$$\beta \frac{f^2}{g} g' \sim J \tag{3.6.10}$$

as $t \longrightarrow t_0$.

Now using that $\frac{g}{f}$ is increasing, as $t \longrightarrow t_0$

$$\frac{g}{f}\left(t_0 - t\right) \leq \int_t^{t_0} \frac{g}{f} ds = o\left(\int_t^{t_0} \frac{g}{f^2} ds\right). \tag{3.6.11}$$

That is

$$\lim_{t \to t_0} \frac{\int_t^{t_0} \frac{g}{f} ds}{\int_t^{t_0} \frac{g}{f^2} ds} = \lim_{t \to t_0} \frac{\frac{g}{f}}{\frac{g}{f^2}} = \lim_{t \to t_0} f = f(t_0) = 0.$$

Using (3.6.10), it is true that $\int_t^{t_0} \frac{g}{f^2} ds \asymp \int_t^{t_0} g' ds$, hence of the same order:

$$o\left(\int_t^{t_0} \frac{g}{f^2} ds\right) = o\left(\int_t^{t_0} g' ds\right)$$

as $t \longrightarrow t_0$ using that

$$\lim_{t \to t_0} \frac{\beta \frac{f^2}{g} g'}{J} \rightarrow 1$$

and so equally

$$\lim_{t \to t_0} \frac{-\beta \frac{f^2}{g} g'}{-J} \rightarrow 1.$$

When combined with (3.6.11) and

$$o\left(\int_t^{t_0} g' ds\right) = o(1),$$

this gives

$$\frac{g}{f}\left(t_0 - t\right) = o(1). \tag{3.6.12}$$

Thus we have that

$$\lim_{t \to t_0} \frac{\frac{g}{f}\left(t_0 - t\right)}{1} = 0$$

and

$$\lim_{t \to t_0} \frac{\frac{g}{f}}{\frac{1}{(t_0 - t)}} = 0,$$

implying

$$\frac{g}{f} = o\left(\frac{1}{(t_0 - t)}\right)$$

as $t \longrightarrow t_0$.

From (2.0.2) with $t < t_0$,

$$\alpha \frac{f^2}{g} f' + (1 - \mu) t f = \int_0^t \left(g - (3\mu - 1) f\right) ds < \int_0^{t_0} \left(g - (3\mu - 1) f\right) ds = I,$$

then

$$\alpha \frac{f^2}{g} f' + (1-\mu) t f = \int_0^{t_0} (g - (3\mu - 1) f)\, ds - \int_t^{t_0} (g - (3\mu - 1) f)\, ds = -\int_t^{t_0} (g - (3\mu - 1) f)\, ds < 0. \tag{3.6.13}$$

From (2.0.3)

$$\beta \frac{f^2}{g} g' + (1-\mu) t g = \int_0^t \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma} \right) ds < J, \tag{3.6.14}$$

then we have

$$\beta \frac{f^2}{g} g' + (1-\mu) t g \sim J \tag{3.6.15}$$

as $t \to t_0$, and since

$$\lim_{t \to t_0} \frac{\beta \frac{f^2}{g} g' + (1-\mu) t g}{J} = \lim_{t \to t_0} \frac{\beta \frac{f^2}{g^2} g' + (1-\mu) t}{\frac{J}{g}},$$

then

$$\beta \frac{f^2}{g^2} g' + (1-\mu) t \sim \frac{J}{g}. \tag{3.6.16}$$

Similarly dividing (3.6.13) by f results in

$$\alpha \frac{f}{g} f' + (1-\mu) t = \frac{1}{f} \int_{t_0}^t (g - (3\mu - 1) f)\, ds, \tag{3.6.17}$$

where

$$\lim_{t \to t_0} \frac{\int_{t_0}^t (g - (3\mu - 1) f)\, ds}{f} = 0$$

since

$$\frac{1}{f} \int_t^{t_0} g ds \le \frac{g}{f} (t_0 - t) = o(1)$$

and

$$\frac{3\mu - 1}{f} \int_t^{t_0} f ds = o(1)$$

as $t \to t_0$ by (3.6.12). Thus

$$\lim_{t \to t_0} \frac{\frac{f^2}{g} \left(\alpha \frac{f'}{f} - \beta \frac{g'}{g} \right)}{-\frac{J}{g}} = 1$$

by subtracting (3.6.16) from (3.6.17), implying

$$\frac{f^2}{g} \left(\alpha \frac{f'}{f} - \beta \frac{g'}{g} \right) \sim -\frac{J}{g},$$

or equivalently

$$\frac{f^2}{g} \left(\log \left(\frac{f^\alpha}{g^\beta} \right) \right)' \sim -\frac{J}{g} > 0.$$

Therefore the function $\log\left(\frac{f^\alpha}{g^\beta}\right)$ is increasing, which implies that $\left(\frac{f^\alpha}{g^\beta}\right)' > 0$. Hence for some positive constant K

$$\frac{f^\alpha}{g^\beta} > K,$$

or more simply

$$\frac{f^2}{g} > K g^{\frac{2\beta}{\alpha}-1}. \tag{3.6.18}$$

By (3.6.14) and (3.6.10)

$$-\beta\frac{f^2}{g}g' \sim -J, \tag{3.6.19}$$

so that

$$0 \le -g' < K\frac{g}{f^2} \tag{3.6.20}$$

for a positive constant K. Combining (3.6.18) and (3.6.20) we have

$$-g' < K g^{1-\frac{2\beta}{\alpha}}$$

which implies

$$g^{\frac{2\beta}{\alpha}-1}g' > -K.$$

Thus

$$\left|g^{\frac{2\beta}{\alpha}-1}g'\right| < K.$$

Integrating up to t_0 gives

$$\left|\int_t^{t_0}\left(g^{\frac{2\beta}{\alpha}}\right)' ds\right| \le \int_t^{t_0}\left|\left(g^{\frac{2\beta}{\alpha}}\right)'\right| ds < K(t_0-t)$$

so that

$$g^{\frac{2\beta}{\alpha}} < K(t_0-t)$$

or

$$g < K(t_0-t)^{\frac{\alpha}{2\beta}}. \tag{3.6.21}$$

Multiplying (3.6.13) by g,

$$f^2 f' + (1-\mu)tfg = g\int_{t_0}^t \left(g-(3\mu-1)f\right) ds$$

we find that

$$\left(\frac{f^3}{3}\right)' \asymp (1-\mu)t_0\int_t^{t_0} fgds + \frac{1}{2}\left(\left(\int_{t_0}^t gds\right)^2\right)'.$$

Now integrating to t_0 we find using (3.6.21)

$$\frac{f^3}{3} \leq (1-\mu)t_0 f \int_t^{t_0} g ds + \frac{1}{2}\left(\int_t^{t_0} g^2 ds\right)^2 < K_1 f\left(t_0-t\right)^{\frac{\alpha}{2\beta}+1} + K_2\left(t_0-t\right)^{\frac{\alpha}{\beta}+2} \tag{3.6.22}$$

for positive constants K_1 and K_2.

If

$$K_1 f\left(t_0-t\right)^{\frac{\alpha}{2\beta}+1} > K_2\left(t_0-t\right)^{\frac{\alpha}{\beta}+2},$$

then by (3.6.22)

$$f^3 \leq K f\left(t_0-t\right)^{\frac{\alpha}{2\beta}+1},$$

or

$$f \leq K\left(t_0-t\right)^{\frac{\alpha}{4\beta}+\frac{1}{2}} \tag{3.6.23}$$

for some constant K. If, on the other hand,

$$K_2\left(t_0-t\right)^{\frac{\alpha}{\beta}+2} < K_1 f\left(t_0-t\right)^{\frac{\alpha}{2\beta}+1},$$

then

$$f^3 \leq K\left(t_0-t\right)^{\frac{\alpha}{\beta}+2}$$

and thus

$$f \leq K\left(t_0-t\right)^{\frac{\alpha}{3\beta}+\frac{2}{3}}$$

for some constant K. We note that if

$$f = O\left(\left(t_0-t\right)^{\frac{\alpha}{3\beta}+\frac{2}{3}}\right)$$

as $t \to t_0$, then indeed it still is true that (3.6.23) holds and so

$$f = O\left(\left(t_0-t\right)^{\frac{\alpha}{4\beta}+\frac{1}{2}}\right).$$

This is because $\frac{\alpha+2\beta}{3\beta} > \frac{\alpha+2\beta}{4\beta}$ and thus

$$\left(t_0-t\right)^{\frac{\alpha+2\beta}{3\beta}} < \left(t_0-t\right)^{\frac{\alpha+2\beta}{4\beta}}$$

for t close to t_0. Therefore, for either case we have that

$$f \leq K(t_0-t)^{\frac{\alpha+2\beta}{4\beta}}$$

and from (3.6.15) for $t \to t_0$,

$$\beta\frac{f^2}{g}g' \sim J$$

$$\frac{J}{f^2} < -K(t_0 - t)^{-\frac{\alpha+2\beta}{2\beta}}$$

and so

$$\log(g^\beta)' = \beta \frac{g'}{g} < -K(t_0 - t)^{-\frac{\alpha+2\beta}{2\beta}}.$$

By integrating both sides

$$g < e^{-K(t_0-t)^{-\frac{\alpha}{2\beta}}}$$

so that as $t \to t_0$

$$f \leq Mg < Me^{-K(t_0-t)^{-\frac{\alpha}{2\beta}}}.$$

However, from (3.6.13) we had that

$$\alpha \frac{f^2}{g} f' + (1-\mu)tf < 0$$

$$\alpha f' < -(1-\mu)\frac{g}{f}t,$$

and so

$$f > K(t_0 - t).$$

We have a contradiction and therefore, it is not possible for $J < 0$, and thus $J = 0$. Now from (2.0.3)

$$\beta \frac{f^2}{g} g'\left(t_0\right) = \int_0^{t_0} \gamma g \left(\frac{g}{f} - \frac{3\mu}{\gamma}\right) dt = 0,$$

we have that

$$\beta \frac{f^2}{g} g'\left(t_0\right) = 0.$$

4.0 PARTIAL NUMERICS

Using XPPAUT 5.85 we graph solutions (f, g) to (1.0.1)-(1.0.4), which are found using the following steps:

- fixing values of α, β, γ, and $\theta(0)$ subject to the conditions (2.1.1)-(2.1.2).
- integrating over a proper $\mu-$range determined from (2.1.3) and Lemma 3.
- identifying μ and $t_0 < \infty$ such that $f' \leq 0$, $g' \leq 0$, for $t < t_0$, (1.0.3) hold, and $f(t_0) = 0$, $g(t_0) = 0$.
- rescaling t_0 to become 1.

The numerics will be partially incomplete, since we do not specify that the conditions $\frac{f^2}{g}f'(1) = 0$ and $\frac{f^2}{g}g'(1) = 0$ are satisfied. We begin by rewriting (1.0.1)-(1.0.2) as a first order system:

$$\begin{aligned} f_1' &= f_2 \\ f_2' &= \tfrac{g_1^2}{\alpha f_1^2} - \tfrac{2\mu g_1}{\alpha f_1} - \tfrac{(1-\mu)tf_2g_1}{\alpha f_1^2} - \tfrac{2f_2^2}{f_1} + \tfrac{f_2g_2}{g_1} \\ g_1' &= g_2 \\ g_2' &= \tfrac{\gamma g_1^3}{\beta f_1^3} - \tfrac{(1+2\mu)g_1^2}{\beta f_1^2} - \tfrac{(1-\mu)tg_1g_2}{\beta f_1^2} - \tfrac{2f_2g_2}{f_1} + \tfrac{g_2^2}{g_1} \end{aligned}$$

where $(f_1, f_2, g_1, g_2) = (f, f', g, g')$. From (2.1.2) we saw that

$$\frac{\gamma}{3} < \theta(0) < \frac{\alpha\gamma - \beta}{3\alpha - 2\beta}.$$

For instance if we pick $\alpha = 1$, $\beta = 1.3$, and $\gamma = 2$ subject to (2.1.1), then $f(0)$ and $g(0)$ should be chosen such that

$$\frac{2}{3} < \frac{f(0)}{g(0)} < \frac{7}{4}.$$

Indeed if $\alpha = .1$, $\beta = .13$, and $\gamma = 2$, the same bound for $\theta(0)$ still holds. We will choose $f(0) = 1.4$ and $g(0) = 1$. Additionally from (2.1.3) we find that

$$\frac{1}{3} < \mu < 1$$

and from LEMMA 3 that

$$\mu > \frac{\alpha(\gamma+1)-\beta}{\alpha(3\gamma-2)-\beta} = \frac{17}{27}.$$

However solving for μ in (2.1.2), we actually find that

$$\mu > \frac{\alpha\gamma-\beta-\theta(0)\alpha}{2\theta(0)(\alpha-\beta)} = \frac{35}{42}.$$

Thus

$$.8\bar{3} < \mu < 1. \tag{4.0.1}$$

The following are examples of the ode files used when $\alpha = 1$, $\beta = 1.3$, $\gamma = 2$, and $\theta(0) = 1.4$:

```
#bvp1.ode
par alpha=1, beta=1.3, gamma=2, mu=.834
f1'=f2
f2'=g1^2/(alpha*f1^2)-2*mu*g1/(alpha*f1)-(1-mu)*t*g1*f2/(alpha*f1^2)-(2*f2^2/f1)+(f2*g2)/g1
g1'=g2
g2'=gamma*g1^3/(beta*f1^3)-(1+2*mu)*g1^2/(beta*f1^2)-(1-mu)*t*g1*g2/(beta*f1^2)-(2*f2*g2)/f1+(g2^2/g1)
bndry f1'
bndry g1'
bndry f2
bndry g2
init f1=1.4
init f2=0
init g1=1
init g2 =0
@ dt=.001, bell=0, total=2, xhi=2, yhi=1
done
```

and

```
#bvp2.ode
par alpha=1, beta=1.3, gamma=2, mu=.834
f1'=f2
f2'=g1^2/(alpha*f1^2)-2*mu*g1/(alpha*f1)-(1-mu)*t*g1*f2/(alpha*f1^2)-(2*f2^2/f1)+(f2*g2)/g1
g1'=g2
g2'=gamma*g1^3/(beta*f1^3)-(1+2*mu)*g1^2/(beta*f1^2)-(1-mu)*t*g1*g2/(beta*f1^2)-(2*f2*g2)/f1+(g2^2/g1)
bndry f1'
bndry g1'
bndry f1'*f1'*f2'/g1'
bndry f1'*f1'*g2'/g1'
init f1=1.4
init f2=0
init g1=1
init g2 =0
@ dt=.001, bell=0, total=2, xhi=2, yhi=1
done
```

Note that a value of μ must be specified in the parameter declaration portion of the code. The value chosen above is simply the smallest possible value μ may attain based on (4.0.1). We then run the codes integrating over the range of possible $\mu-$values, with a time step of $dt = .001$. The code produces the following results when varying $.84 < \mu < .9$:

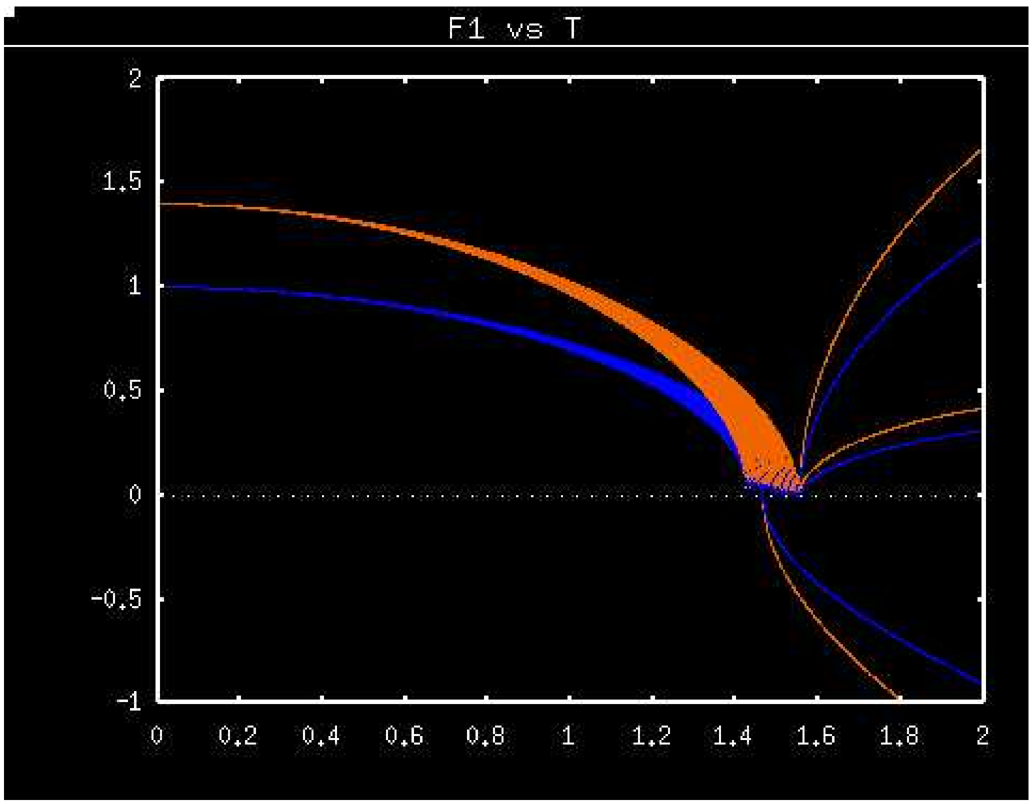

Figure 9: Result after integrating over 0.84 < mu < 0.9

It is particularly easy to identify the curves (f, g) resulting when $\mu = .88$:

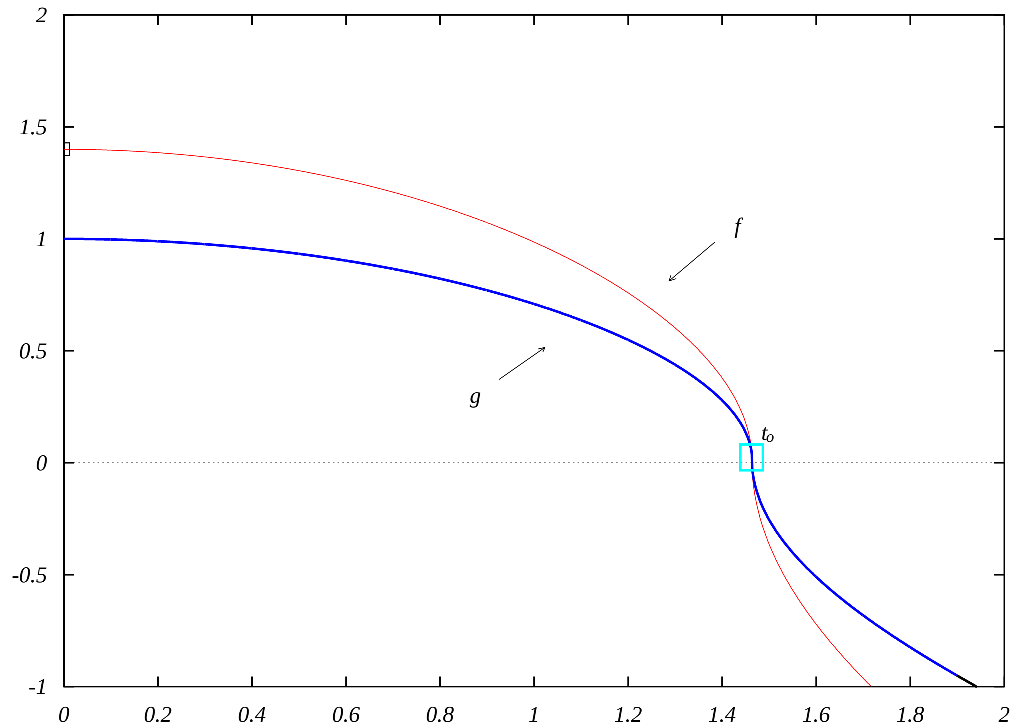

Figure 10: Setting mu = 0.88

Where the graphs approach zero, the data is summarized by the following:

Table 1: Data for mu = 0.88

t	f	f'	g	g'	$\frac{f^2}{g}f'$	$\frac{f^2}{g}g'$
1.46	.0956	-12.2155	.0757	-8.7903	-1.4728	-1.0598
1.461	.0824	-14.3608	.0662	-10.3379	-1.47059	-1.05863
1.462	.0663	-18.2643	.0547	-13.1537	-1.4681	-1.0573
1.4630001	.0439	-29.3803	.0385	-21.1726	-1.4675	-1.0576
1.464	-.0401	-36.1895	-.02206	-26.06894	2.6421	1.8891

We rescale by noting that $t_0 \approx 1.46375$, but as $t_0 \to a^{\frac{1}{2}} t_0 = 1$ then $a = \frac{1}{t_0^2}$. Since $f \to af$ and $g \to ag$, then the mapping affects the initial condition $\theta(0)$ by

$$f(0) \to 1.4a \text{ and } g(0) \to a$$

with α, β, γ, and μ unchanged. The following is the plot which results from rescaling the initial conditions:

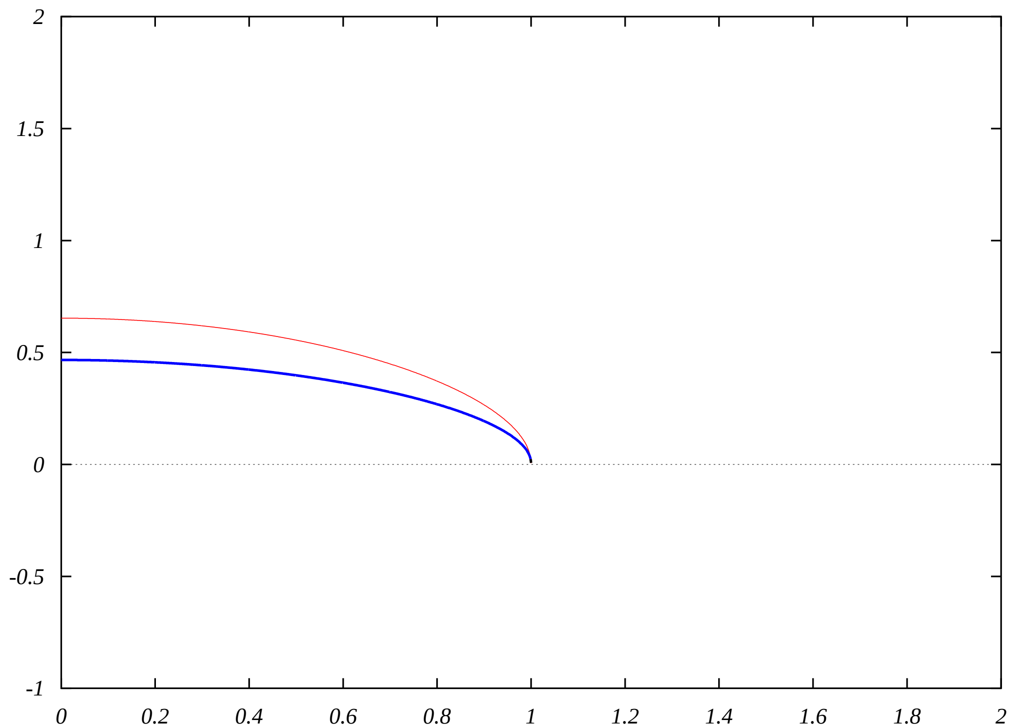

Figure 11: Rescaled solutions

Also included is a plot of the rescaled data together with the unscaled curves:

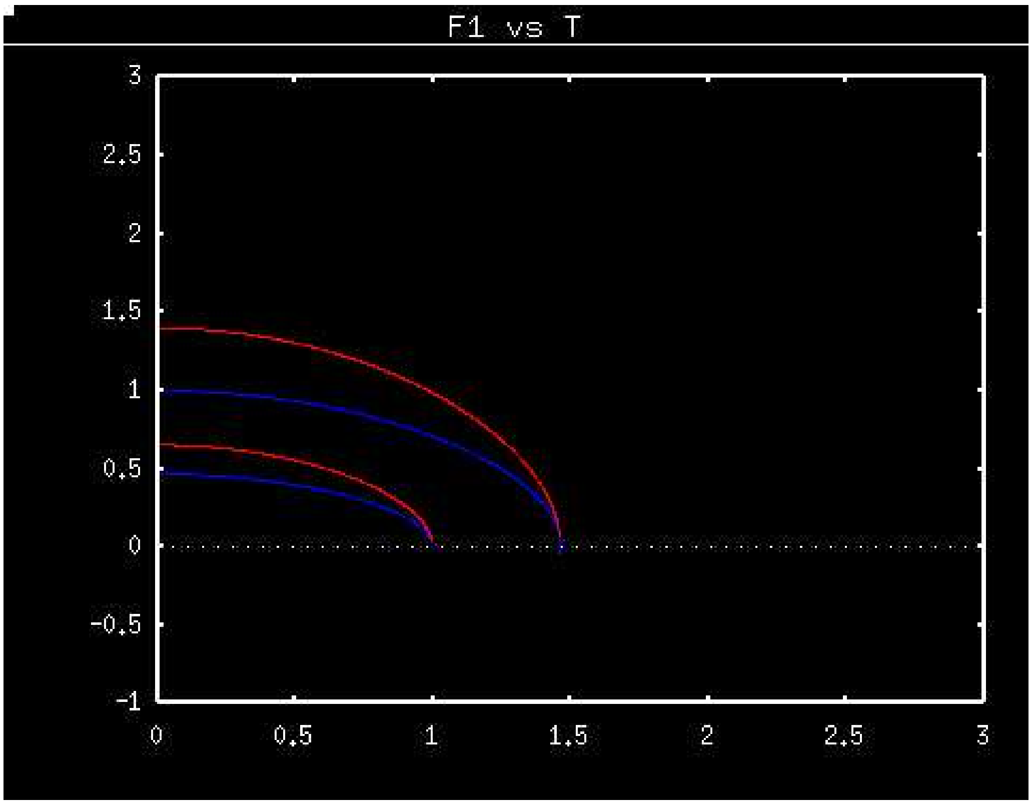

Figure 12: Rescaled and original graphs

Additionally we demonstrate the requirement of $\frac{\alpha}{\beta}$ being neither too big nor too small, by fixing α, γ, $\theta(0)$, and μ as above and integrating over an interval of $\beta-$values. To illustrate what happens when we vary, for example, $\beta \in [0.1, 3]$ in the prior example, the results are:

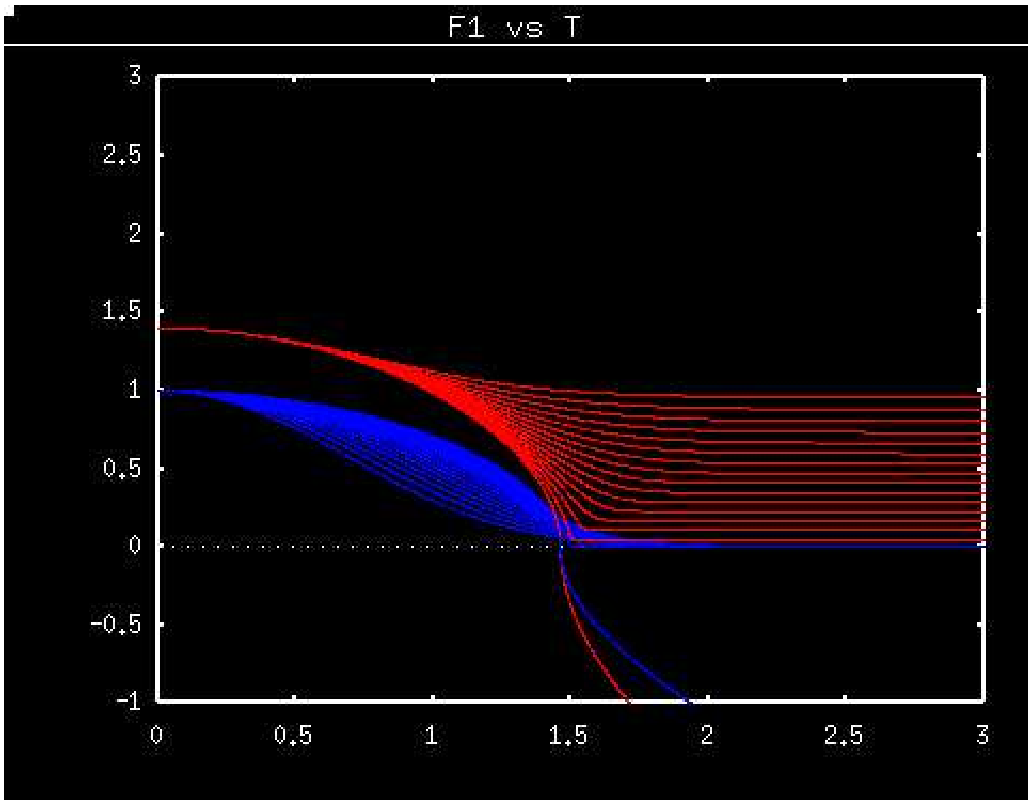

Figure 13: Result from varying beta over [0.1,1.3]

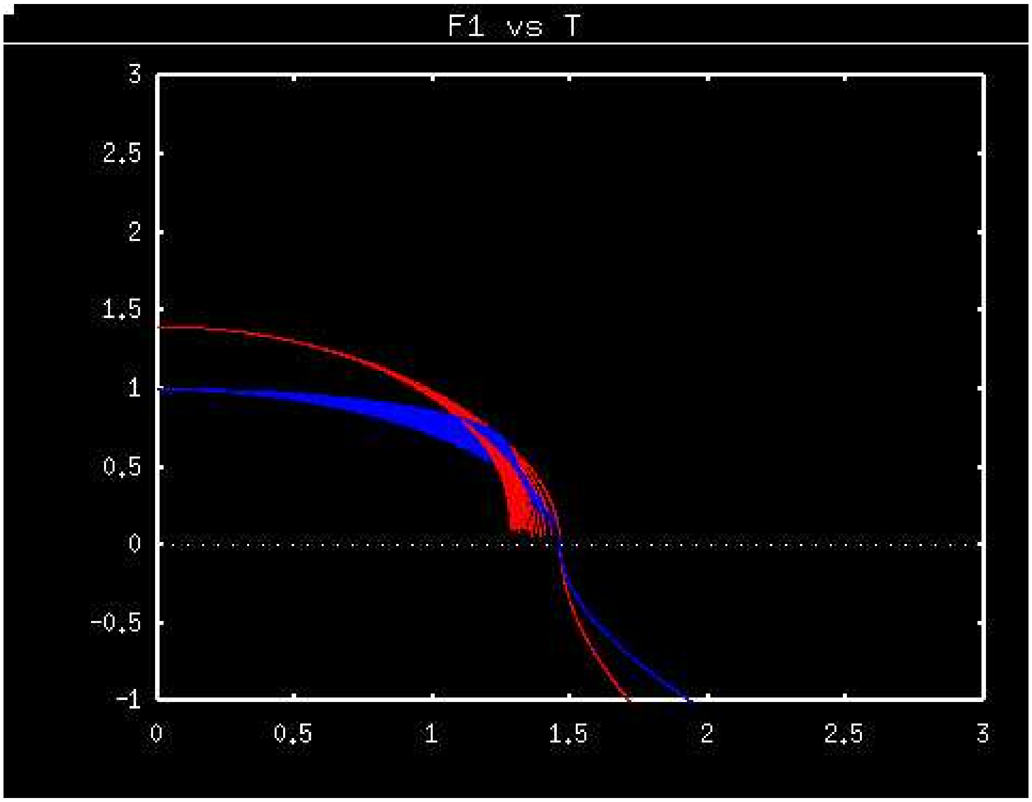

Figure 14: Result from varying beta over [1.3,3]

It is clear from the varied behavior in the blue and red curves that solutions (f, g) to (1.0.1)-(1.0.4) do not exist when we vary β too far from 1.3.

5.0 CONCLUSION

In summary, for the case of $\alpha < \beta$, $3\alpha > 2\beta$, and $\gamma > \frac{3}{2}$, we have proven the existence of a solution (f, g) to (1.0.1)-(1.0.4) by finding $(\mu, \theta(0))$ using a shooting technique developed through a series of lemmas. Additionally, we have provided graphs of (f, g) obtained by using numerical shooting with XPPAUT.

To complete the case of $\alpha \neq \beta$, it remains to consider $\alpha > \beta$.

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