INDEFINITE STRING STRUCTURE

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The orthogonal group $\mathrm{O}(n)$ appears as the structure group of the frame bundle of an n-dimensional Riemannian manifold. Recently there has been a lot of interest in considering k-connected covers of $\mathrm{O}(n)$, or of its stable version O, where the first stages are named $\mathrm{Spin}(n)$ for $k = 1$ and $\mathrm{String}(n)$ for $k = 3$. In this thesis we study the problem in the indefinite case: considering connected covers of the indefinite orthogonal group $\mathrm{O}(p, q)$, which appears as structure group of frame bundles of semi-Riemannian manifolds. We thus consider $\mathrm{Spin}(p, q)$ and $\mathrm{String}(p, q)$ as topological groups up to homotopy equivalence using the Whitehead tower. Then the obstruction for semi-Riemannian manifolds to admit Spin and String groups as their structure groups will be computed in terms of cohomology classes of the corresponding classifying spaces $B\mathrm{Spin}(p, q)$ and $B\mathrm{String}(p, q)$. While Spin groups are finite dimensional Lie groups, String groups as topological groups are not finite dimensional. Alternatively, we categorify them to Lie 2-groups to make them finite dimensional, providing some clarifications on their generalizations, namely 2-groupoids, along the way.

TABLE OF CONTENTS

LIST OF TABLES

1.0 INTRODUCTION

An n-dimensional Riemannian manifold has a tangent bundle with structure group $\mathrm{O}(n)$. In general, the structure group G of a fiber bundle $E \to X$ imposes symmetry structure on each fiber E_x by identifying the orbit space of the structure group action, i.e., an element $u \in E_x$ is identified with $g \cdot u \in E_x$ for any group element $g \in G$.

A Riemannian manifold has a metric in each fiber and the structure group $\mathrm{O}(n)$ introduces the symmetry which preserves the metric. The tangent space as a fiber of a Riemannian manifold has a further structure – the orientation may be imposed by picking up an ordered set of bases $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ of the tangent fiber. There are only two possible orientations and we fix one of them to be positive and the other to be negative.

Since $\mathrm{O}(n)$ just preserves the metric, the symmetry does not distinguish positive orientation from the negative one. The structure group which respect the orientability is the special orthogonal group $\mathrm{SO}(n)$. If the Riemannian manifold admits this orientation, then we say it is orientable. The criterion for M being orientable is the vanishing of the first Stiefel-Whitney class $w_1(M) \in H^1(X; \mathbb{Z}/2)$. As Lie groups, $\mathrm{SO}(n)$ is the connected component of the identity of $\mathrm{O}(n)$, and if the manifold M is orientable, the representations of these structure groups are compatible in a sense that the diagram of Lie groups in fibers

$$\begin{array}{ccc} & & B\mathrm{SO}(n) \\ & \nearrow & \downarrow \\ X & \longrightarrow & B\mathrm{O}(n) \end{array}$$

commutes, where the map $B\mathrm{SO}(n) \to B\mathrm{O}(n)$ was induced from the commutative diagram of representations

$$\begin{array}{ccc} \mathrm{SO}(n) & & \\ \downarrow & \searrow & \\ \mathrm{O}(n) & \longrightarrow & \mathrm{GL}_{\mathbb{R}}(n), \end{array}$$

which also induces the compatibility of representations of their Lie algebras on fibers:

$$\begin{array}{ccc} \mathfrak{so}(n) & & \\ \downarrow & \searrow & \\ \mathfrak{o}(n) & \longrightarrow & \mathfrak{gl}_{\mathbb{R}}(n). \end{array}$$

However, as discovered by Chevalley, there is a representation of $\mathfrak{so}(n)$ which is not induced from that of $\mathrm{SO}(n)$. To resolve this deficiency, a Lie group $\mathrm{Spin}(n)$ is defined whose Lie algebra is isomorphic to $\mathfrak{so}(n)$. Hence, the Riemannian manifold has this structures with the right Lie algebra representations. It turns out that the obstruction for an orientable Riemannian manifold to have a Spin structure is the second Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2)$.

If we turn our eyes to the topological structures of $\mathrm{O}(n)$, $\mathrm{SO}(n)$, and $\mathrm{Spin}(n)$, we notice a compelling pattern. For the first, $\pi_i(\mathrm{O}(n)) \cong \pi_i(\mathrm{SO}(n))$ for $i \geq 1$ but $\pi_0(\mathrm{O}(n)) = \mathbb{Z}/2$ and $\pi_0(\mathrm{SO}(n)) = 0$. For the second, $\pi_i(\mathrm{SO}(n)) \cong \pi_i(\mathrm{Spin}(n))$ for $i \geq 2$ but $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ and $\pi_1(\mathrm{Spin}(n)) = 0$. Moreover, $\pi_0(\mathrm{Spin}(n)) = \pi_0(\mathrm{O}(n))$. So, we can see that $\mathrm{SO}(n)$ was obtained by 'killing' the 0-th homotopy group of $\mathrm{O}(n)$ and $\mathrm{Spin}(n)$ was obtained by further killing the first homotopy group of $\mathrm{O}(n)$. It then seems quite natural to ask what happens if we pursue this pattern, i.e., what we would get if we keep killing higher homotopy groups of $\mathrm{O}(n)$. (Consult standard texts on Spin groups, for example Lawson and Michelsohn [10], for more details.)

The next step would be killing the 2nd homotopy group of $\mathrm{O}(n)$, but there is nothing to do as this is already trivial. In this thesis, we will discuss the next step—killing the 3rd homotopy group. As we shall see later, this '3-connected' cover of $\mathrm{O}(n)$ is a topological group called $\mathrm{String}(n)$ and this breaks the pattern above in the sense that it is not a finite dimensional Lie group. Rather, it is an infinite dimensional Lie group (see Sati, Schreiber, Stasheff [18] and references within).

We will see that the obstruction for a Spin manifold to have the String structure is the half of first Pontrjagin class $\frac{1}{2}p_1 \in H^4(M;\mathbb{Z})$. Recalling that the characteristic classes can be interpreted as maps from the base manifold to some Eilenberg-MacLane spaces though the classifying spaces of structure groups, we summarize this liftings of structure groups as in the following diagram

$$\begin{array}{ccccc}
 & & B\mathrm{String}(n) & & \\
 & \nearrow & \downarrow & & \\
 & \nearrow & B\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},4) \\
 & \nearrow & \downarrow & & \\
 & \nearrow & B\mathrm{SO}(n) & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) \\
 & & \downarrow & & \\
M & \longrightarrow & B\mathrm{O}(n) & \xrightarrow{w_1} & K(\mathbb{Z}/2,1).
\end{array}$$

In fact, we can climb this tower further. We will see that since the higher homotopy groups, π_4, π_5 and π_6 of $\mathrm{O}(n)$ are trivial, and by killing the next nontrivial $\pi_7(\mathrm{O}(n))$, we would obtain another object called a fivebrane group, Fivebrane(n). For this, see Sati, Schreiber and Stasheff [16].

Following the pattern of killing higher homotopy groups is not arbitrary – it is believed to have significant geometric implications. Stolz [21] conjectures that if a Spin manifold admits a string structure with positive Ricci curvature then the manifold has to have a trivial Witten genus (Conjecture 6.1.2 in Secion 6.1).

1.1 WHAT'S NEW IN THIS THESIS

So far, for simplicity the dimension n was assumed to be large enough to make the homotopy groups stable, and the constructions of $\mathrm{SO}(n)$, $\mathrm{Spin}(n)$ and $\mathrm{String}(n)$ might differ from the stable case for smaller n. Since the unstable cases Spin(1) and Spin(2) seem rarely treated or even mentioned in most of the literature, the detailed description of them is given. The significance of the stable case is that for $n \geq 3$ the group $\mathrm{Spin}(n)$ is the universal double cover of $\mathrm{SO}(n)$. However, it no longer holds for $n = 1$ and 2. To remedy this seemingly topological inconsistency, I discuss the notion of 1-connected cover $\mathrm{O}(n)\langle 1\rangle$ of $\mathrm{O}(n)$ and present SO(1) itself for $O(1)\langle 1\rangle$ and the topological space $\Omega PK(\mathbb{Z},2)$ for $O(2)\langle 1\rangle$. This is in agreement with Spin(n) for $O(n)\langle 1\rangle$ for $n \geq 3$ (Proposition 5.3.1).

If we start from the indefinite orthogonal group $O(p,q)$, then we should expect some variation of the constructions and the corresponding obstructions. Unfortunately though, the topological pattern of killing homotopy groups does not immediately make sense for $SO(p,q)$ – it is not connected and $Spin(p,q)$ is neither connected nor simply connected and even the connected component $Spin(p,q)^0$ is not simply connected. Hence, if we were to exploit the advantage of being higher connected as in the definite case, an alternative construction is needed. In Section 5.3, the classification of such covers will be given and alternative topological groups $O(p,q)\langle 1\rangle$ as a 1-connected cover of $O(p,q)$ constructed given as a homotopy fiber with respect to certain cohomology classes which acts as the obstruction lifting $SO(p,q)^0$ to $O(p,q)\langle 1\rangle$ as structure groups of bundles. The corresponding obstructions is listed in Table 5.1. This improves the classification of double covers of orthogonal groups as done by Trautman [23] by including unstable cases.

As mentioned above, killing the 3rd homotopy group involves the notion of *higher categories*. The 3-connected cover of O is an infinite dimensional Lie group but it becomes finite if we take it as a Lie 2-group (Schommer-Pries [19]). The notion of a 2-group is already well understood and I will recapture the definition as in Baez and Lauda [1] with a little clarifying discussion on its internalization (Remark 6.2.7).

Although being a bit of digression, the introduction of more general object called a Lie 2-groupoid seems to be in order naturally. It is also a well understood object, but since I could not find the explicit definition of internalized Lie 2-groupoid in the literature, it will be proposed in Section 6.3. After this, the notion of bibundles is introduced in Section 6.4 and this generalizes the principal bundle to having a groupoid as its structural object.

Returning to the topic on finding 1-connected covers and 3-connected covers of indefinite orthogonal groups, the main technical issue is the computation of cohomology groups of the product space with integral coefficient which is not a field (Remark 6.8.2). For example, at a later stage, we need to compute the cohomology group such as

$$H^k(BG_1 \times BG_2; \mathbb{Z}\times\mathbb{Z})$$

for some topological groups G_1 and G_2 to classify certain extensions (Theorem 5.2.1, Corollary 6.5.1, Theorem 6.5.3). Dealing with the product coefficient is rather simple - the universal coefficient theorem directly covers it (Lemma 5.3.5). However, since $\mathbb{Z}$ is not a field, computing something like

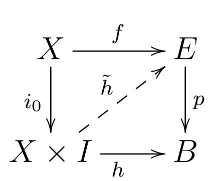
$$H^k(BG_1 \times BG_2; \mathbb{Z})$$

is not obvious since it depends on the topology of G_1 and G_2. To construct 1-connected and 3-connected covers of $\mathrm{O}(p,q)$, the computation on the special case with $G_1 = \mathrm{O}(p)\langle 1\rangle$ and $G_2 = \mathrm{O}(q)\langle 1\rangle$ will done in Section 6.8. The 3-connected cover of the indefinite orthogonal group $\mathrm{O}(p,q)$ will be then constructed as the loop space of homotopy fibers of appropriately chosen cohomology class in Section 6.9. They also become the obstruction in lifting 1-connected cover $\mathrm{O}(p,q)\langle 1\rangle$ up to $\mathrm{String}(p,q)$ as structure groups of bundles, and the corresponding obstructions are listed in Table 6.1.

1.2 PRELIMINARY: FIBRATIONS, FIBER SEQUENCES AND HIGHER HOMOTOPY GROUPS

This section present classical algebraic topological materials which would being thoroughly used throughout the thesis. The main reference is May [13] and the details and proofs of these results can be found there.

Definition 1.2.1. *A surjective map between spaces $p : E \to B$ is called a* (Serre) fibration *if it satisfies the* covering homotopy property*: for any homotopy h from $p \circ f$, there is a homotopy $\tilde{h}$ which covers h in a sense that the diagram*



commutes. Here, $i_0(x) := (x, 0)$.

Let $F(X,Y) := \mathrm{Hom}_{\mathbf{Top}_*}(X,Y)$, a morphism in the category $\mathbf{Top}_*$ of based topological spaces. Then the *path space* of X is defined as $PX := F(I,X)$, where I has 0 as the basepoint. For $f \in F(X,Y)$, the pullback $Ff := X \times_f PY$ is called the *homotopy fiber*, the diagram

$$\begin{array}{ccc} Ff & \longrightarrow & PY \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}(1)} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes with the universal object Ff where $\mathrm{ev}(1)$ is the evaluation at 1. We define the *loop space* of any space X to be $\Omega X = F(S^1, X)$ and Ω is then then called a *loop space functor* since it actually is a functor.

Proposition 1.2.2. *The induced universal projection $Ff \xrightarrow{\pi} X$ is a fibration. Moreover, we have a long sequence in $\mathbf{Top}_*$ called the* fiber sequence generated by f*:*

$$\cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega Ff \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} Ff \xrightarrow{\pi} X \xrightarrow{f} Y$$

which is exact up to homotopy with an inclusion $\iota(\chi) := (*, \chi)$, and $(-\Omega f)(\zeta)(t) := (f \circ \zeta)(1-t)$ for $\zeta \in \Omega X$.

Theorem 1.2.3. *For any based space Z in a category $\mathbf{hTop}_*$ of based topological space up to homotopy, the covariant hom-functor $[Z,-] := \mathrm{Hom}_{\mathbf{hTop}_*}(Z,-)$ induces a long exact sequence in the category* **Set** *of sets:*

$$\cdots \to [Z, \Omega Ff] \to [Z, \Omega X] \to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y].$$

The restricted sequences to the left of $[Z, \Omega Y]$ and $[Z, \Omega^2 Y]$ are long exact sequence in the category of groups **Grp** *and the category of abelian groups* **Ab***, respectively.*

Proposition 1.2.4. *Let $p : E \to X$ be a fibration with X path connected. Let $E_{x_0} := p^{-1}(x_0)$ for a fixed basepoint $x_0 \in X$. Then we have the long exact sequence*

$$\cdots \to \pi_n(E_{x_0}) \to \pi_n(E) \to \pi_n(X) \xrightarrow{\partial} \pi_{n-1}(E_{x_0}) \to \cdots \to \pi_0(E) \to \pi_0(X) = 0.$$

2.0 ORTHOGONAL GROUPS

2.1 AS TOPOLOGICAL GROUPS

The set $\mathrm{GL}_{\mathbb{R}}(n)$ of all nonsingular $n \times n$ real matrices has a group structure with right-to-left binary operation which is a matrix multiplication. This is called the *real general linear group*. By identifying a matrix as a point in $\mathbb{R}^{n^2}$, $\mathrm{GL}_{\mathbb{R}}(n)$ is a subset of $\mathbb{R}^{n^2}$, and since a determinant function is polynomial, it is an open subset as $\mathbb{R}^{n^2}$ of an Euclidean space with the induced topology. Matrix multiplication and inversions are continuous with respect to this topology since they are also polynomial functions of entries in matrices. So, $\mathrm{GL}_{\mathbb{R}}(n)$ is a *topological group*.

Now, let us define some specific subset of $\mathrm{GL}_{\mathbb{R}}(n)$:

$$\mathrm{O}(n) := \{A \in \mathrm{GL}_{\mathbb{R}}(n) : AI_nA^T = A^TI_nA = I_n\} \tag{2.1}$$

$$\mathrm{SO}(n) := \{A \in \mathrm{O}_{\mathbb{R}}(n) : \det A = 1\} \tag{2.2}$$

$$\mathrm{O}(p,q) := \{A \in \mathrm{GL}_{\mathbb{R}}(p+q) : A^TI_{p,q}A = I_{p,q}\} \tag{2.3}$$

$$\mathrm{SO}(p,q) := \{A \in \mathrm{O}(p,q) : \det A = 1\} \tag{2.4}$$

where I_n is an $n \times n$ identity matrix and $I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Obviously, we have $\mathrm{O}(n) \cong \mathrm{O}(n,0) \cong \mathrm{O}(0,n)$ and $\mathrm{SO}(n) \cong \mathrm{SO}(n,0) \cong \mathrm{SO}(0,n)$.

Let V be a real n-dimensional vector space, so that it is isomorphic to the Euclidean vector space $\mathbb{R}^n$, and Q be a quadratic form on V, i.e., $Q(v) = v^tAv$ for some $A \in \mathrm{GL}_{\mathbb{R}}(n)$.

Theorem 2.1.1. *Any quadratic form Q on a n-dimensional $\mathbb{R}$-vector space V is classified by a pair of non-negative integers (p, q) from the diagonalized forms*

$$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$$

for $v = (v_1, \ldots, v_{p+q}) \in V$.

See §61A of O'Meara [14] for more detail.

We write $\mathbb{R}^n$ for $\mathbb{R}^n$ itself with a quadratic form described by $(n, 0)$ and write $\mathbb{R}^{p,q}$ for such vector space but a different form corresponding to a pair (p, q).

Then a matrix $A \in \mathrm{O}(p, q)$ preserves this (p, q)-quadratic form in the sense that $Q(Av) = Q(v)$. Hence, $\mathrm{O}(p, q)$ is called the *indefinite orthogonal group*. Similarly, $\mathrm{SO}(p, q)$ and $\mathrm{SO}(n)$ are called the *indefinite special orthogonal group* and the *(definite) special orthogonal group*, respectively.

Note that the defining conditions from Eq(2.1) to Eq(2.3) are polynomial equations with respect to the entries. Therefore, they are closed subgroups of $\mathrm{GL}_{\mathbb{R}}(n)$ and $\mathrm{GL}_{\mathbb{R}}(p, q)$ and are referred to by the term *closed linear groups*. The general linear group $\mathrm{GL}_{\mathbb{R}}(n)$ is itself a closed linear group as a subset of itself and with trivial condition $A = 0$.

2.2 AS LIE GROUPS

Each general linear group admits a canonical Lie group structure:

Theorem 2.2.1. *If G is a closed linear group, then G with its relative topology with respect to $\mathrm{GL}_{\mathbb{R}}(n)$ has a unique Lie group structure such that any function on each entry restricted from $\mathrm{GL}_{\mathbb{R}}(n)$ to G is smooth. It inherits the structure from $\mathrm{GL}_{\mathbb{R}}(n)$ in the sense that*

1. *if an image of a smooth function f from a smooth manifold M to $\mathrm{GL}_{\mathbb{R}}(n)$ is in G, then $f : M \to G$ is smooth, and*
2. *if f is a function from an open set $U \subset \mathrm{GL}_{\mathbb{R}}(n)$ containing G to a smooth manifold M, its restriction $f|_G : G \to M$ is also smooth.*

Proof. See Section 0.4 of Knapp [9]. □

Theorem 2.2.2. *The definite orthogonal group* $\mathrm{O}(n)$ *has two connected components and* $\mathrm{SO}(n)$ *is the components with the identity. Thus,* 0*th homotopy group of the special definite orthogonal group is trivial, i.e.,* $\pi_0(\mathrm{SO}(n)) = 0$*, and* $\pi_i(\mathrm{O}(n)) \cong \pi_i(\mathrm{SO}(n))$ *for* $i \geq 1$*.*

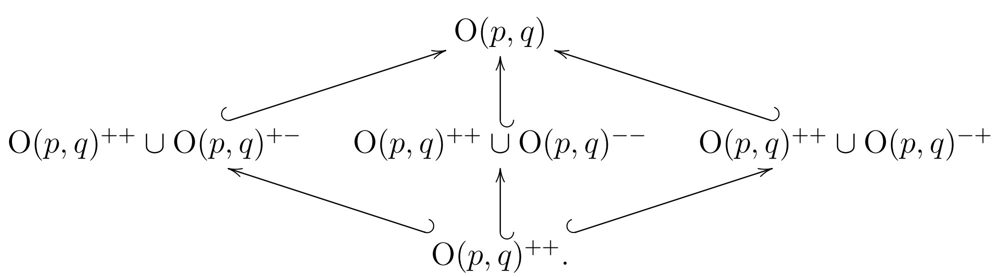
Proposition 2.2.3. *The definite groups* $\mathrm{O}(n)$ *and* $\mathrm{SO}(n)$ *are real compact Lie groups of the same dimension* $n(n-1)/2$*. The indefinite groups* $\mathrm{O}(p,q)$ *and* $\mathrm{SO}(p,q)$ *are also finite dimensional Lie groups but are not compact.*

2.3 STRUCTURE OF INDEFINITE ORTHOGONAL GROUPS

The indefinite orthogonal group $\mathrm{O}(p,q)$ has four connected components (see p. 341 in Wolf [24])

$$\begin{aligned}
&\mathrm{O}(p,q)^{++} \supset \mathrm{SO}(p) \times \mathrm{SO}(q); \text{the identity component,}\\
&\mathrm{O}(p,q)^{+-} \supset \mathrm{SO}(p) \times \{A \in \mathrm{O}(q) : \det A = -1\}\\
&\mathrm{O}(p,q)^{-+} \supset \{A \in \mathrm{O}(p) : \det A = -1\} \times \mathrm{SO}(q)\\
&\mathrm{O}(p,q)^{--} \supset \{A \in \mathrm{O}(p) : \det A = -1\} \times \{A \in \mathrm{O}(q) : \det A = -1\}.
\end{aligned}$$

These subgroups fit into the following diagram of inclusions (see Rham [15], in a different notation)



Any $A \in \mathrm{O}(p,q)$ acts naturally on $v \in \mathbb{R}^{p,q}$, let it be the structure group of the tangent bundle TM of a $(p+q)$-dimensional differentiable manifold M. If the tangent bundle admits reduction to $\mathrm{O}(p,q)^{++} \cup \mathrm{O}(p,q)^{+-}$, $\mathrm{O}(p,q)^{++} \cup \mathrm{O}(p,q)^{-+}$ and $\mathrm{O}(p,q)^{++} \cup \mathrm{O}(p,q)^{--}$, respectively, then M is said to be *time-orientable*, *space-orientable*, and *topologically orientable*, respectively.

As we write $\mathrm{SO}(n)$ for the identity component of $\mathrm{O}(n)$, $\mathrm{SO}(p,q)^0$ will denote the identity component $\mathrm{O}(p,q)^{++}$ of $\mathrm{O}(p,q)$. The analog of Proposition 2.2.2 holds:

Proposition 2.3.1. *$\pi_0(\mathrm{SO}(p,q)^0) = 0$ and $\pi_i(\mathrm{O}(p,q)) \cong \pi_i(\mathrm{SO}(p,q)^0)$ for $i \geq 1$.*

Theorem 2.3.2 (Iwasawa decomposition)**.** *Any (finite-dimensional) finitely connected non-compact Lie group G can be decomposed into the product of the maximal compact subgroup K and $\mathbb{R}^n$ (as topological spaces not as groups) for some n. This implies that G and K are homotopy equivalent.*

Proposition 2.3.3. *The maximal compact subgroup of $\mathrm{O}(p,q)$, $\mathrm{SO}(p,q)$ and $\mathrm{SO}(p,q)^0$ are $\mathrm{O}(p) \times \mathrm{O}(q)$, $S(\mathrm{O}(p) \times \mathrm{O}(q))$ and $\mathrm{SO}(p) \times \mathrm{SO}(q)$, respectively.*

Each of these four connected components of $\mathrm{O}(p,q)$ contains one component of the maximal compact subgroup $\mathrm{O}(p) \times \mathrm{O}(q)$.

For the geometrical and topological aspect of Lie groups including the statements given above, see Helgason [8].

2.4 HOMOTOPY GROUPS OF ORTHOGONAL GROUPS

To compute homotopy groups of definite orthogonal groups, we use the Bott periodicity theorem which involves symplectic groups. The *symplectic group* $\mathrm{Sp}(n)$ is defined as:

$$\mathrm{Sp}(n) := \{A \in \mathrm{GL}(2n) : A^T J_n A = J_n\} \quad \text{where } J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.5}$$

which is the set of matrices which preserves the *exterior form* $w_1 \wedge w_{n+1} + \cdots + w_n \wedge w_{2n}$ on a $2n$-dimensional real vector space W.

First, we need the commutativity of the covariant homotopy group functor π_i from the category $\mathbf{Top}_*$ of based topological spaces to the category $\mathbf{Grp}$ of groups with respect to colimits:

Lemma 2.4.1. *For a sequence of inclusions $X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots$, let X be its colimit $\varinjlim X_n$. Then the colimit $\varinjlim \pi_i(X_n)$ exists in $\mathbf{Grp}$ and is isomorphic to $\pi_i(\varinjlim X_n)$.*

There are sequences of inclusions

$$\mathrm{O}(1) \hookrightarrow \mathrm{O}(2) \hookrightarrow \cdots \hookrightarrow \mathrm{O}(n) \hookrightarrow \cdots \quad \text{and} \quad \mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2) \hookrightarrow \cdots \hookrightarrow \mathrm{Sp}(n) \hookrightarrow \cdots$$

and the ith homotopy groups of $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ stabilizes for sufficiently large n (See §VII.8 of Bredon [3]):

$$\pi_i(\mathrm{O}(n)) \cong \pi_i(\mathrm{O}(n+1)) \quad \text{for } 0 < i \leq n-2 \tag{2.6}$$

and

$$\pi_i(\mathrm{Sp}(n)) \cong \pi_i(\mathrm{Sp}(n+1)) \quad \text{for } 0 < i \leq 4n+3. \tag{2.7}$$

Hence, by taking direct limits $\mathrm{O} := \varinjlim \mathrm{O}(n)$ and $\mathrm{Sp} := \varinjlim \mathrm{Sp}(n)$, we get stable homotopy groups $\pi_*(\mathrm{O})$ and $\pi_*(\mathrm{Sp})$. The first four stable homotopy groups are presented in the following table,

i	1	2	3	4
$\pi_i(\mathrm{O})$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0
$\pi_i(\mathrm{Sp})$	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$

and the computation is well-known and presented in standard textbooks in algebraic topology, for example, in Bredon [3].

Bott [2] showed that (as a corollary of *Bott periodicity theorem*) $\pi_{i+4}(\mathrm{O}) \cong \pi_i(\mathrm{Sp})$ and that, therefore, $\pi_i(\mathrm{O}) \cong \pi_{i+8}(\mathrm{O})$. So, the following table for the homotopy groups of O is complete (where $\pi_0(\mathrm{O})$ is treated separately not being from the table above):

$i \pmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{O})$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

From Eq(2.6) together with this table, we can determine the lower bounds n for which $\pi_i(\mathrm{O}(n)) \cong \pi_i(\mathrm{O})$. The following table gives such lower bounds for small i's[1]:

i	1	2	3	4
lower bound for n	3	4	5	6

[1]Caution: Just because $\pi_i(O(n)) \cong \pi_i(O)$ as groups, that doesn't mean that the maps $\pi_i(O(n)) \to \pi_i(O(n+1)) \to \pi_i(O(n+2)) \to \cdots$ are isomorphisms (i.e., that we are in the stable range). Stable range for π_5 is achieved for much larger n, as $\pi_5(O(3)) = \pi_5(S^3) \neq 0$, for example.

We have the identifications for small n: $\mathrm{O}(1) = \{\pm 1\} = S^0$, $\mathrm{SO}(2) = S^1$, $\mathrm{SO}(3) = \mathbb{R}P^3$ and SO(4) is double covered by $S^3 \times S^3$ so that $\pi_i(\mathrm{O}(4)) \cong \pi_i(S^3) \times \pi_i(S^3)$ for $i \geq 2$. In this thesis, we will be interested in the homotopy groups of $\mathrm{O}(n)$ up to 3rd order, and for the computation of those we use the following:

1. $\pi_i(S^n) = 0$ if $0 < 1 < n$, and $\pi_n(S^n) = \mathbb{Z}$.
2. the real projective space $\mathbb{R}P^n$ is connected and its fundamental group $\pi_1(\mathbb{R}P^n)$ is $\mathbb{Z}$ if $n = 1$ and $\mathbb{Z}/2$ if $n \geq 2$. Moreover, the homotopy groups are isomorphic to $\pi_i(S^n)$ for $i \geq 1$ and $n \geq 1$.

Summarizing these, we obtain the following table:

$\pi_i(\mathrm{O}(n))$	$i=0$	$i=1$	$i=2$	$i=3$
$n=1$	$\mathbb{Z}/2$	0	0	0
$n=2$	0	$\mathbb{Z}$	0	0
$n=3$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$
$n=4$	$\|$	$\|$	0	$\mathbb{Z} \times \mathbb{Z}$
$n=5$	$\|$	$\|$	$\|$	$\mathbb{Z}$
$n=6$	$\|$	$\|$	$\|$	$\|$
$n=7$	$\|$	$\|$	$\|$	$\|$

Table 2.1: Lower order homotopy groups of $\mathrm{O}(n)$

where the symbol "$\|$" indicates the corresponding entry is the same as the one from above, i.e., in the stable range.

Proof. There is a fibration $\mathrm{O}(n) \to S^{n-1}$ whose fiber is $\mathrm{O}(n-1)$, so we have a long exact sequence of homotopy groups

$$\cdots \to \pi_{k+1}(S^{n-1}) \to \pi_k(\mathrm{O}(n-1)) \to \pi_k(\mathrm{O}(n)) \to \pi_k(S^{n-1}) \to \pi_{k-1}(\mathrm{O}(n-1)) \to \cdots .$$

First, $\pi_k(S^0) = 0$ for any positive integer k, so when $n = 1$, we have a short exact sequence $0 \to \pi_3(\mathrm{O}(1)) \to \pi_3(\mathrm{O}(2)) \to 0$ and this implies that $\pi_3(\mathrm{O}(1)) \cong \pi_3(\mathrm{O}(2))$. Since O(1) is identical to S^0, we have $\pi_3(\mathrm{O}(n)) = 0$ for $n = 1$ and 2. Recall that $\pi_3(\mathrm{O}(n)) \cong \pi_3(\mathrm{SO}(n))$

and that SO(3) is diffeomorphic to $\mathbb{R}P^3$ and S^3. So, $\pi_3(\mathrm{O}(3)) \cong \mathbb{Z}$. Note that SO(4) is double covered by SU(2) × SU(2) and that SU(2) is diffeomorphic to S^3 so that $\pi_3(\mathrm{O}(4)) \cong \pi_3(S^3) \times \pi_3(S^3) \cong \mathbb{Z} \times \mathbb{Z}$.

We know $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$ and $\pi_k(S^{n-1}) = 0$ for $k < n-1$. Therefore, for $n \geq 6$, we have short exact sequences $0 \to \pi_3(\mathrm{O}(n-1)) \to \pi_3(\mathrm{O}(n)) \to 0$ which implies $\pi_3(\mathrm{O}(n-1)) \cong \pi_3(\mathrm{O}(n))$. Inductively, we have $\pi_3(\mathrm{O}(5)) \cong \pi_3(\mathrm{O}(6)) \cong \cdots \cong \pi_3(\mathrm{O}(n)) \cong \cdots$ so that they are isomorphic to $\pi_3(\mathrm{O}) \cong \mathbb{Z}$. □

We now compute first three homotopy groups of $\mathrm{O}(p,q)$ for later use:

Proposition 2.4.2. *The fundamental groups of the connected component of the indefinite special orthogonal group, $\pi_1(\mathrm{O}(p,q))$ and the 3rd homotopy groups $\pi_3(\mathrm{O}(p,q))$ are as the followings:*

$\pi_1(\mathrm{O}(p,q))$	$q=1$	$q=2$	$q \geq 3$
$p=1$	0	$\mathbb{Z}$	$\mathbb{Z}/2$
$p=2$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}/2\times\mathbb{Z}$
$p \geq 3$	$\mathbb{Z}/2$	$\mathbb{Z}\times\mathbb{Z}/2$	$\mathbb{Z}/2\times\mathbb{Z}/2$

Table 2.2: Fundamental groups of $\mathrm{O}(p,q)$

and the second homotopy group is trivial so that $\pi_2(\mathrm{O}(p,q)) = 0$. The third homotopy groups are

$\pi_3(\mathrm{O}(p,q))$	$q=1,2$	$q=3$	$q=4$	$q \geq 5$
$p=1,2$	0	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$
$p=3$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$
$p=4$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$
$p \geq 5$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$

Table 2.3: Third homotopy groups of $\mathrm{O}(p,q)$

Proof. The maximal compact subgroup of $\mathrm{SO}(p,q)^0$ is $\mathrm{SO}(p)\times\mathrm{SO}(q)$ so that $\pi_n(\mathrm{SO}(p,q)^0) \cong \pi_n(\mathrm{SO}(p)\times\mathrm{SO}(q))$ and we use the fact $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for any topological spaces X and Y. □

2.5 INTEGRAL COHOMOLOGY OF ORTHOGONAL GROUPS

The mod 2 cohomology algebra of $BO(n)$, $H^\bullet(BO(n);\mathbb{Z}/2)$ is generated by Stiefel-Whitney classes $w_i \in H^i(BO(n);\mathbb{Z}/2)$ for $i = 1, 2, 3, \ldots$. In fact, this is the very definition of Stiefel-Whitney classes. As Brown showed in [4], the integral cohomology of $BO(n)$ can also be generated by these mod 2 cohomology classes, more specifically some of the images of Steifel-Whitney classes and Pontrjagin classes.

Theorem 2.5.1 (Brown,[4]). *$H^\bullet(BO(n);\mathbb{Z}) = \mathcal{R}(n)/\mathcal{I}(n)$ where*

1. $\mathcal{R}(n) = \mathbb{Z}[p_1, \ldots, p_{[n/2]}, \beta(w_1^\epsilon w_{2i_1} \cdots w_{2i_l}) \mid \epsilon = 0$ *or* $1, 0 < i_1 < \cdots < i_l \leq [n/2]]$
2. *$\mathcal{I}(n)$ is the ideal generated by the relations*
   - a. $2\beta(w_1^\epsilon w_{2i_1}, \ldots, w_{2i_l}) = 0$,
   - b. $\beta(w_1 w_n) = 0$,
   - c. $(\beta w_n)^2 = p_{n/2}\beta w_1$ *if n is even,*
   - d. *if we write* $I := \{\frac{\epsilon}{2}, i_1, \ldots, i_s\}$, $w(2I) := w_1^\epsilon w_{2i_1} \cdots w_{2i_s}$ *and* $p(I) = (\beta w_1)^\epsilon p_{i_1} \cdots p_{i_s}$, *then*

$$\begin{aligned}&\beta w(2I)\beta w(2J)\\ &\quad = \sum_{k\in I}(\beta w_{2k})p((I \setminus \{k\}) \cap J)\beta(w(2((I \setminus \{k\}) \cup J \setminus (I \setminus \{k\}) \cap J))),\end{aligned}$$

where $[n/2]$ is the smallest integer greater than $n/2$ and β is a Bockstein operation as in the following exact sequence $\cdots \to H^q(X;\mathbb{Z}) \xrightarrow{2} H^q(X;\mathbb{Z}) \xrightarrow{\rho_2} H^q(X;\mathbb{Z}/2) \xrightarrow{\beta} H^{q+1}(X;\mathbb{Z}) \to \cdots$ *induced from the short exact sequence* $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. *Furthermore,* $\rho_2(p_q) = w_{2q}^2$, $\rho_2\beta = \mathrm{Sq}^1$.

Theorem 2.5.2 (Brown,[4]). *The notation in Theorem 2.5.1 is also used here.* $H^\bullet(BSO(n);\mathbb{Z}) = \bar{\mathcal{R}}(n)/\bar{\mathcal{I}}(n)$ *where*

1. $\bar{\mathcal{R}}(n) = \mathbb{Z}[p_1, \ldots, p_{[(n-1)/2]}, X_n, \beta(w_{2i_1} \cdots w_{2i_l}) \mid 0 < i_1 < \cdots < i_l \leq [(n-1)/2]]$
2. *$\bar{\mathcal{I}}(n)$ is the ideal generated by the relations*
   - a. $2\beta(w_{2i_1}, \ldots, w_{2i_l}) = 0$
   - b. $X_n = \beta w_{2k}$ *if* $n = 2k+1$ *and* $X_n^2 = p_{n/2}$ *if n is even*

c. if we write $I := \{i_1, \ldots, i_s\}$, $w(2I) := w_1^\epsilon w_{2i_1} \cdots w_{2i_s}$ *and* $p(I) = p_{i_1} \cdots p_{i_s}$, *then*

$$\beta w(2I)\beta w(2J) = \sum_{k \in I} (\beta w_{2k}) p((I \setminus \{k\}) \cap J) \beta(w(2((I \setminus \{k\}) \cup J \setminus (I \setminus \{k\}) \cap J))).$$

Furthermore, $\rho_2(p_q) = w_{2q}^2$ *and* $\rho_2\beta = \mathrm{Sq}^1$.

3.0 EILENBERG-MACLANE SPACES AND CHARACTERISTIC CLASSES

3.1 ORDINAL NUMBER CATEGORY Δ

Let $\mathbf{n}$, $n \geq 1$, be an finite ordinal number, which is a totally ordered set $\{0, 1, \ldots, n-1\}$ with an order relation $i \leq j$. It is a small category with the numbers $0, \ldots, n-1$ as objects and relations $i \leq j$ as arrows $i \to j$. By definition, $\mathbf{0} = \varnothing$.

The *ordinal number category* Δ is a category whose objects are finite ordinal numbers $\mathbf{n}$ for $n \geq 0$ and arrows are weakly monotone maps $f : \mathbf{m} \to \mathbf{n}$ such that $0 \leq f(i) \leq f(j) < n$ in $\mathbf{n}$ for $0 \leq i \leq j < m$ in $\mathbf{m}$. The object $\mathbf{0}$ is an initial object and $\mathbf{1}$ is a terminal object in the ordinal number category Δ.

The addition of ordinal numbers $\mathbf{m}$ and $\mathbf{n}$ is an object $\mathbf{m} + \mathbf{n}$ in Δ defined as

$$\mathbf{m} + \mathbf{n} := \{0, 1, \ldots, m-1, m+0, m+1, \ldots, m+n-1\}.$$

Proposition 3.1.1. *The ordinal number category is a strict monoidal category* $(\Delta, +, \mathbf{0})$*, the tensor product is the addition and the unit object is* $\mathbf{0}$*. Moreover, it is the initial object in the category of strict monoidal categories* **strMonCat**.

The jth face map $\delta_i^n : \mathbf{n} \to \mathbf{n} + \mathbf{1}$ for $0 \leq i \leq n$, and degeneracy map $\sigma_j^n : \mathbf{n} + \mathbf{1} \to \mathbf{n}$ for $0 < j \leq n$ are defined as

$$\delta_i^n : k \mapsto \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \geq i \end{cases}, \qquad \sigma_j^n : k \mapsto \begin{cases} k & \text{if } k < j, \\ k-1 & \text{if } k \geq j \end{cases}$$

and these are represented as follows:

$$\mathbf{0} \longrightarrow \mathbf{1} \overset{\delta^1}{\rightrightarrows} \mathbf{2} \overset{\delta^2}{\rightrightarrows} \mathbf{3} \overset{\delta^3}{\rightrightarrows} \cdots, \qquad \text{and} \qquad \mathbf{1} \overset{\sigma^1}{\longleftarrow} \mathbf{2} \overset{\sigma^2}{\leftleftarrows} \mathbf{3} \overset{\sigma^3}{\leftleftarrows} \cdots.$$

Proposition 3.1.2. *Any arrow* $f : \mathbf{n} \to \mathbf{m}$ *in* Δ *has a unique decomposition*

$$f = \delta_{i_1} \circ \cdots \circ \delta_{i_k} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_h} \tag{3.1}$$

where $n - h + k = m$ *and* $m > i_1 > \cdots > i_k \geq 0$ *and* $0 \leq j_1 < \cdots < j_h < n - 1$. *Moreover, we have the following identities for face and degeneracy maps:*

$$\begin{aligned} \delta_i \delta_j &= \delta_{j+1}\delta_i \text{ for } i \leq j \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \text{ for } i \leq j \\ \sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{for } i = j \\ \mathrm{id} & \text{for } i = j, j+1 \\ \delta_{i-1}\sigma_j & \text{for } i > j+1. \end{cases} \end{aligned} \tag{3.2}$$

Conversely, any pair of maps δ' *and* σ' *satisfying the identities in Eq(3.2) decompose any arrow* f *in* Δ *as in Eq(3.1).*

3.2 STANDARD SIMPLICES

A *simplicial set* is a contravariant functor from the ordinal number category Δ to the category of sets **Sets**. For a simplicial set X, the image of $\mathbf{n} \in \Delta$ is denoted by X_n and is called the n-th simplex of X. See Friedman [7] for a survey.

The *standard* n*-simplex* Δ^n, $n \geq 0$, is a simplicial set which maps $\mathbf{m} \mapsto \hom_\Delta(\mathbf{m}, \mathbf{n+1})$ for $m \geq 0$.

3.2.1 Face and degeneracy maps

The face and degeneracy maps are induced directly from the category Δ by taking hom-functor in its second variable: $\delta_i^n := \hom(-, \delta_i^n) : \Delta^{n-1} \to \Delta^n$ and $\sigma_j^n := \hom(-, \sigma_j^n) : \Delta^n \to \Delta^{n-1}$ where $0 \leq i \leq n$ and $0 < j \leq n$. Specifically, we write Δ_i^n for $\hom(-\delta_i^n(\mathbf{n}))$ and this corresponds to the face opposing $i \in \mathbf{n}$.

3.2.2 Boundary and horns of standard simplices

The *boundary of standard n*-simplex, $\partial\Delta^n$ is then defined to be a formal sum $\sum_{0\leq i\leq n}(-1)^i\Delta^n_i$. Also, the k-th *horn* $\Lambda^n_k \subset \Delta^n$ is $\sum_{\substack{0\leq i\leq n\\ i\neq k}}(-1)^i\Delta^n_i$, i.e., the boundary $\partial\Delta^n$ with the face opposite to k that is being removed.

3.2.3 Geometric realization

The *standard topological n-simplex*, denoted by Δ_n or $|\Delta^n|$, is a subspace defined as

$$\{(t_0,\dots,t_n)\in\mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i\geq 0\}\subset\mathbb{R}^{n+1}.$$

The face maps $|\delta^j_i| : \Delta_{n-1}\to\Delta_n$ and the degeneracy maps $|\sigma^n_j| : \Delta_n\to\Delta_{n-1}$ are defined as the mappings below

$$|\delta^n_i| : (t_0,\dots,t_{n-1})\mapsto(t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1})$$
$$|\sigma^n_j| : (t_0,\dots,t_n)\mapsto(t_0,\dots,t_{j-1},t_j+t_{j+1},t_{j+2},\dots,t_n).$$

Furthermore, for any map $f:\mathbf{n}\to\mathbf{m}$ in Δ with the decomposition $f=\delta_{i_1}\circ\cdots\circ\delta_{i_k}\circ\sigma_{j_1}\circ\cdots\circ\sigma_{j_h}$ as in Eq(3.1), we define its corresponding map $|f| : \Delta_{n-1}\to\Delta_{m-1}$ to be the composition

$$|f| := |\delta_{i_1}|\circ\cdots\circ|\delta_{i_k}|\circ|\sigma_{j_1}|\circ\cdots\circ|\sigma_{j_h}|.$$

The map sending $\Delta^n\mapsto\Delta_n$ and $f\mapsto|f|$ is called the *geometric realization* of standard simplices.

3.3 SIMPLICIAL SETS

A simplicial set is a contravariant functor $\Delta\to\mathbf{Sets}$ and let $\mathbf{sSet}$ denote the category of simplicial sets.

3.3.1 Classification of simplices of simplicial sets

Proposition 3.3.1. *The simplicial set morphism $\Delta^n \to X$ classifies n-simplex X_n of X in the sense that there is a natural bijection $\hom_{\mathbf{sSets}}(\Delta^n, X) \cong X_n$. Concretely, let $\varphi \in \hom_{\mathbf{sSets}}(\Delta^n, X)$. Then $\mathrm{id}_{\mathbf{n+1}} \in \Delta^n(\mathbf{n+1})$ and we associate an element $x = \varphi(\mathbf{n+1})(\mathrm{id}_{\mathbf{n+1}})$ in X_{n+1} to φ and this map $\varphi \mapsto x$ is the bijection.*

3.3.2 Face and degeneracy maps

The above map induces maps $\hom(\delta_i^n, X) : \hom(\Delta^n, X) \to \hom(\Delta^{n-1}, X)$ and $\hom(\_, \sigma_j^n) : \hom(\Delta^{n-1}, X) \to \hom(\Delta^n, X)$ (note the contravariance). From the classification of simplices X_n of X for $n > 0$ as in Proposition 3.3.1, we also obtain the face map $d_i^n : X_{n+1} \to X_n$ and $s_j^n : X_n \to X_{n+1}$.

3.3.3 Geometric realization

Proposition 3.3.2. *A simplicial set X determines and is determined by the set of unique maps between simplices $\theta^*_{m,n} : X_m \to X_n$ for $m \geq n$. This map $\theta^*_{m,n}$ is induced by a unique map $\theta_{m,n} : \mathbf{n} \to \mathbf{m}$ in Δ. Furthermore, such $\theta_{m,n}$ is uniquely decomposed into the composite $\theta_n \circ \theta_{n-1} \circ \cdots \circ \theta_{m+1} \circ \theta_m$ where $\theta_k : \mathbf{k} \to \mathbf{k+1}$ is also determined by X:*

$$\Delta \xrightarrow{X^{\mathrm{op}}} \mathbf{Sets}$$

$$\begin{array}{ccc} \mathbf{m} & & X_m \\ {\scriptstyle \theta_{m,n}}\uparrow & & \downarrow{\scriptstyle \theta^*_{m,n}} \\ \mathbf{n} & & X_n. \end{array}$$

Remark 3.3.3. *The maps $\theta^*_{m,n} : X_m \to X_n$ can be identified with a map $\hom_{\mathbf{sSets}}(\Delta^{m-1}, X) \to \hom_{\mathbf{sSets}}(\Delta^{n-1}, X)$ which in turn is induced by a map $\theta'_{m-1,n-1} : \Delta^{n-1} \to \Delta^{m-1}$ induced by the map $\theta_{m,n} : \mathbf{n} \to \mathbf{m}$. The collection of the standard simplices Δ^n with these maps $\theta'_{m',n'}$ form a category $\Delta^\bullet \downarrow X$ over a simplicial set X called the* simplex category*.*

The simplex category $\Delta^\bullet \downarrow X$ can be described by a sequence

$$\Delta^0 \xrightarrow{\theta'_0} \Delta^1 \xrightarrow{\theta'_1} \cdots \xrightarrow{\theta'_{n-1}} \Delta^n \xrightarrow{\theta'_n} \cdots . \tag{3.3}$$

Proposition 3.3.4. *The direct limit* $\varinjlim \Delta^n$ *over the simplex category* $\Delta^\bullet \downarrow X$ *exists and is isomorphic to* X*.*

By geometric realization of simplices of the sequence Eq(3.3), we obtain a following sequence in the category of topological spaces **Top**:

$$\Delta_0 \xrightarrow{|\theta_0'|} \Delta_1 \xrightarrow{|\theta_1'|} \cdots \xrightarrow{|\theta_{n-1}'|} \Delta_n \xrightarrow{|\theta_n'|} \cdots . \tag{3.4}$$

Proposition 3.3.5. *The direct limit* $\varinjlim \Delta_n$ *exists in* **Top***. This is denoted by* $|X|$*.*

This process of getting $|X| \in$ **Top** from $X \in$ **sSets** is called the *geometric realization* of simplicial sets. Note that the previous notion of geometric realization of standard simplices is special case of that of simplicial sets.

3.4 EILENBERG-MACLANE SPACE

The construction of Eilenberg-MacLane space in this section follows May [13]. Given a topological group G, we build two simplicial sets $E_*(G)$ and $B_*(G)$ with face and degeneracy maps as follows: $E_n(G) := \prod^{n+1} G$ with

$$d_i(g) := \begin{cases} (g_2, \ldots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) & \text{if } 1 \leq i \leq n \end{cases}$$

$$s_i(g) := (g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_{n+1}).$$

We can make G act from the right on $E_n(G)$ by defining

$$(g_1, \ldots, g_{n+1}) \cdot g := (g_1, \ldots, g_n, g_{n+1} g)$$

and this action commutes with face and degeneracy maps so that $E_*(G)$ may be called as a *simplicial G-space*. Similarly, $B_n(G) := \prod^n G$ with

$$d_i(g) := \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & \text{if } 1 \leq i \leq n-1 \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(g) := (g_1, \dots, g_{i-1}, e, g_i, \dots, g_n).$$

The transitivity of the right action of any $g \in G$ on G as the multiplication from the right, we may regard $B(G)$ as the orbit space $E(G)/G$. This gives a simplicial projection $p_* : E_*(G) \to B_*(G)$.

Definition 3.4.1. *The geometric realization $BG := |B_*(G)|$ of $B_*(G)$ is called the* classifying space *for G. We also write $EG := |E_*(G)|$ and $p := |p_*| : EG \to BG$.*

Proposition 3.4.2 (Properties of the classifying spaces)**.**

1. *$\pi_{q+1}(BG) \cong \pi_q(G)$ for $q \geq 0$ and ΩBG is homotopy equivalent to G.*
2. *BG is an topological abelian group if the topological group G is abelian.*

Proposition 3.4.3 (Properties of the universal bundles)**.**

1. *EG is contractible, i.e., $\pi_*(EG) = 0$*
2. *$p : EG \to BG$ is a principal G-bundle such that any principal G-bundle is induced from it. Let ξ be any principal G-bundle $P \to X$. Then there exist a unique bundle map from ξ to $EG \to BG$ forming a pullback diagram:*

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & BG. \end{array}$$

Definition 3.4.4. *For a discrete Abelian group π, the* Eilenberg-MacLane space *is defined as $K(\pi, n) := \prod^n B\pi$. When $n = 0$ set $K(\pi, 0) := \pi$.*

Proposition 3.4.5 (Properties of Eilenberg-MacLane spaces)**.**

1. *$K(\pi, n+1) = BK(\pi, n)$*

2. $K(\pi, n-1) = \Omega K(\pi, n)$

3. $\pi_q(K(\pi, n)) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{if } q \neq n. \end{cases}$

Conversely, this property uniquely determines the Eilenberg-MacLane space:

Proposition 3.4.6. *If a space X has the property*

$$\pi_q(X) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{if } q \neq n \end{cases}$$

then X is homotopy equivalent to $K(\pi, n)$.

Theorem 3.4.7. *For a CW complex X, an abelian group π, and an integers $n \geq 0$, there are natural isomorphisms*

$$\tilde{H}^n(X; \pi) \cong [X, K(\pi, n)],$$

where $\tilde{H}^*(X) := H^*(X, \mathrm{pt})$ is the reduced cohomology so that $H^*(X) \cong \tilde{H}^*(X) \oplus H^*(\mathrm{pt})$ with

$$H^n(\mathrm{pt}) = \begin{cases} \pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

by the dimension axiom.

Therefore, for $n \neq 0$, $H^n(X; \pi) \cong [X, K(\pi, n)]$.

3.5 CHARACTERISTIC CLASSES

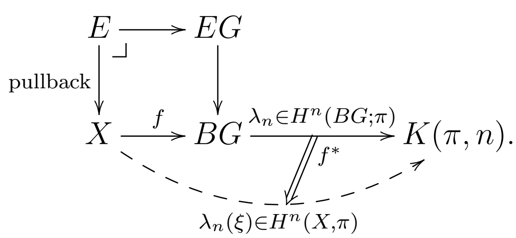
By Theorem 3.4.7, a choice of cohomology class $\lambda_n \in H^n(BG; \pi)$ is called a *universal characteristic class* and corresponds to homotopy class of maps λ_n in $[BG, K(\pi, n)]$. Proposition 1.2.2 gives the long exact sequence

$$\cdots \longrightarrow \Omega BG \xrightarrow{\Omega\lambda_n} \Omega K(\pi, n) \longrightarrow F_{\lambda_n} \longrightarrow BG \xrightarrow{\lambda_n} K(\pi, n). \tag{3.5}$$

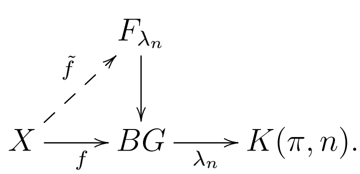
From Proposition 3.4.3, $\Omega BG \simeq G$ and Proposition 3.4.5, $\Omega K(\pi, n) = K(\pi, n-1)$, there is an exact sequence equivalent to Eq(3.5) up to homotopy

$$\cdots \longrightarrow G \xrightarrow{\Omega\lambda_n} K(\pi, n-1) \longrightarrow F_{\lambda_n} \longrightarrow BG \xrightarrow{\lambda_n} K(\pi, n). \tag{3.6}$$

It was given in Proposition 3.4.3 that any principal G-bundle ξ of $E \to X$ is induced as a pullback of some map $f : X \to BG$ with the canonical $EG \to BG$. The pullback cohomology class $\lambda_n(\xi) := f^*\lambda_n \in H^n(X; \pi)$ is called the *characteristic class* of a bundle ξ:

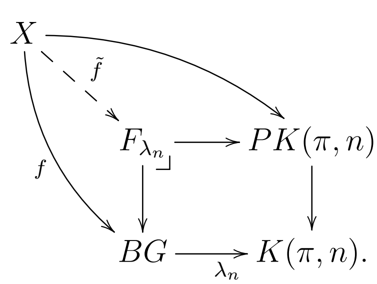


If we combine the sequence Eq(3.6) with the map $f : X \to BG$ which characterizes the principal G-bundle over X, we obtain the following diagram and we may ask under what condition the map f could be lifted to $\tilde{f} : X \to F_{\lambda_n}$ so that the diagram becomes commutative:



(3.7)

Transforming and complementing this diagram a bit, we obtain the following diagram which indicates clearly F_{λ_n} as a pullback:



This implies that $\tilde{f} : X \to F_{\lambda_n}$ exist if and only if the composite $\lambda_n \circ f = f^*\lambda_n$ is homotopy to the composite $X \to PK(\pi, n) \to K(\pi, n)$. Since the path space $PK(\pi, n)$ is contractible, this condition is equivalent to that when $f^*\lambda_n \simeq *$. Hence, f can be lifted to $\tilde{f}$ if and only if the cohomology class $f^*\lambda_n \in H^n(X;\pi)$ is trivial.

If this is the case, i.e., such an extension $\tilde{f}$ exists, then we say that the characteristic class λ_n is *trivialized* by f and call the lifted map $\tilde{f}$ the *trivialization* of λ_n with respect to f.

We say two trivializations $\tilde{f}_0$ and $\tilde{f}_1$ on f for the same universal characteristic class $\lambda_n \in H^n(BG;\pi)$ are *homotopic* if they are homotopic throughout liftings, i.e. there is a homotopy $\tilde{F} : X \times [0,1] \to F_{\lambda_n}$ from $\tilde{f}_0$ to $\tilde{f}_1$ and $\tilde{F}(x,t)$ for fixed $t \in (0,1)$ is a trivialization on f. The set of homotopy classes of trivializations of λ_n for an principal G-bundle over X has a 'bundle-like' structure.

Proposition 3.5.1. *The set of trivializations is an $H^{n-1}(X,\pi)$-torsor, in a sense that the set of trivializations of λ_n up to homotopy on a principal G-bundle admits a free and transitive action of $H^{n-1}(X;\pi)$.*

The trivialization of the characteristic class on a bundle is characterized in the following proposition. First, for a principal G-bundle $P \to X$ may be described by a short exact sequence $0 \to G \xrightarrow{i} P \xrightarrow{f} X \to 0$, we have a long exact sequence of groups:

$$\begin{aligned}\cdots \to H^{n-2}(G;\pi) \to H^{n-1}(X;\pi) \xrightarrow{f^*} H^{n-1}(P;\pi) \xrightarrow{i^*} H^{n-1}(G;\pi)\\ \to H^n(X;\pi) \xrightarrow{f^*} H^n(P;\pi) \xrightarrow{i^*} H^n(G;\pi) \to \cdots .\end{aligned}$$

Recall from the sequence Eq(3.6), given that $\lambda_n \in H^n(BG;\pi)$,

$$\Omega\lambda_n \in [G, K(\pi, n-1)] \cong H^{n-1}(G;\pi).$$

Then the characterization follows:

Proposition 3.5.2. *There is a map from the homotopy class of trivializations of λ_n on a bundle ξ to a set $\{\mathcal{S} \in H^{n-1}(P;\pi) : i^*\mathcal{S} = \Omega\lambda_n \in H^{n-1}(G;\pi)\}$ which is equivariant under the $H^{n-1}(X;\pi)$-action. If $\tilde{H}^q(G;\pi) = 0$ for $q < n-1$, then this map is a bijection.*

This notion of trivialization is used when F_{λ_n} is of the form $B\widetilde{G}$ for some topological group $\widetilde{G}$. If this is the case, the existence trivialization of a characteristic class $\lambda \in H^n(BG;\pi)$ of a principal G-bundle ξ indicates that the structure group G can be *lifted* to $\widetilde{G}$. In this sense, given two topological groups G and $\widetilde{G}$ and a principal G-bundle characterized by $f \in [X, BG]$, a characteristic class $f^*\lambda_n \in H^n(X;\pi)$ is called an *obstruction* for the lifting. On the other hand, we often say that the universal characteristic class $\lambda_n \in H^n(BG;\pi)$ is the *obstruction class* in lifting G into $\tilde{G}$.

3.6 EXAMPLES: TRIVIALIZATION OF STIEFEL-WHITNEY CLASSES AND PONTRYAGIN CLASSES

Theorem 3.6.1 (Classifying spaces of some groups as homotopy fibers of characteristic classes)**.** *The homotopy fibers of the first Stiefel-Whitney class $w_1 \in H^1(B\mathrm{O}(n);\mathbb{Z}/2)$ and the second Stiefel-Whitney classes $w_2 \in H^2(B\mathrm{SO}(n);\mathbb{Z}/2)$ are homotopy equivalent to $B\mathrm{SO}(n)$ and $B\mathrm{Spin}(n)$, respectively.*

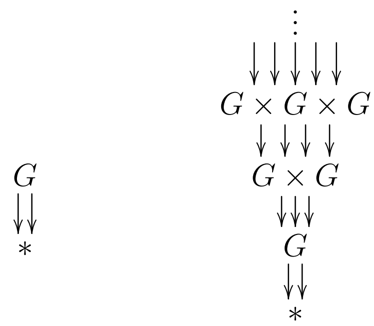
Remark. *The basic idea of proof is presented below – see 1.2.12 in Schreiber [20] for the complete proof. For more detail, see Fiorenza, Schreiber and Stasheff [6] and Sati, Schreiber and Stasheff [17].*

Sketch of proof as presented by Schreiber. Let G be a Lie group and consider

$$\underbrace{\mathbf{B}G}_{\text{smooth stack}} = \underbrace{(G \rightrightarrows *)}_{\text{Lie groupoid}} \qquad \Longrightarrow \qquad BG \simeq \|\mathbf{B}G\|$$

We have an inclusion of groupoids in ∞-groupoids (via Lurie [12]):

$$\mathbf{Grpd} \overset{\mathcal{N}}{\underset{\text{nerve}}{\hookrightarrow}} \mathbf{KanCplx} \simeq \infty\mathbf{Grpd}$$



We represent this via the diagram

Fact. *If we have a homotopy fiber sequence of simplicial manifolds, then the geometric realization $\|-\|$ preserves this. Consequently, if we have a diagram of stacks*

$$\begin{array}{ccc} \mathbf{B}\hat{G} & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}G & \rightarrow & \mathbf{B}^n A \end{array}$$

where $A = \mathrm{U}(1), \mathbb{Z}/2$ or $\mathbb{Z}$ and $* = (* \rightrightarrows *) \in \mathbf{Grpd}$,

$$\begin{array}{ccc} B\hat{G} & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ BG & \rightarrow & \mathbf{B}^n A \end{array}$$

Fact. *$K(\mathbb{Z}, n)$ classifies degree n cohomology in the sense that $[X, K(\mathbb{Z}, n)] \simeq H^n(X, \mathbb{Z})$.*

For $\lambda_n \in H^n(X; \mathbb{Z})$, the homotopy fiber F_{λ_n} is constructed as

$$\begin{array}{ccc} F_{\lambda_n} & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ BG & \xrightarrow[\lambda_n]{} & K(\pi, n) \end{array}$$

3.6.0.1 1. First Stiefel-Whitney

Fact. *If one map is a fibration, then the homotopy pullback is given by ordinary pullback.*

Observe $\mathrm{O}(n) \to \mathbb{Z}/2$ is a Lie group homomorphism. Here it "deloops" to

$$\mathbf{B}\mathrm{O}(n) \xrightarrow{\mathbf{w}_1} \mathbf{B}(\mathbb{Z}/2)$$

$$\begin{array}{ccc} \mathrm{O}(n) & \dashrightarrow & \mathbb{Z}/2 \\ \downarrow\downarrow & & \downarrow\downarrow \\ * & \dashrightarrow & *. \end{array}$$

Then

$$\begin{array}{ccc} \mathbf{B}(\mathrm{SO}(n)) & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}\mathrm{O}(n) & \xrightarrow[\mathbf{w}_1]{} & \mathbf{B}(\mathbb{Z}/2), \end{array}$$

so that by taking $\|{-}\|$ we obtain

$$\begin{array}{ccc} B(\mathrm{SO}(n)) & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ B\mathrm{O}(n) & \xrightarrow[w_1]{} & B(\mathbb{Z}/2) \simeq K(\mathbb{Z}/2, 1) \end{array}$$

and by Fact 3.6.0.1, $Fw_1 = B\mathrm{SO}(n)$ where $w_1 \in H^1(B\mathrm{O}(n), \mathbb{Z}/2)$.

3.6.0.2 Second Stiefel-Whitney nerve:

$$\mathbf{B}\mathrm{SO}(n) \xrightarrow{\mathbf{w}_2} \mathbf{B}^2(\mathbb{Z}/2)$$

$$\begin{array}{ccc} & & \mathbb{Z}/2 \\ & & \downarrow\downarrow\downarrow \\ \mathrm{O}(n) & \dashrightarrow & * \\ \downarrow\downarrow & & \downarrow\downarrow \\ * & \dashrightarrow & * \end{array}$$

and

$$\mathrm{Spin} \times \mathrm{Spin} \times \mathbb{Z}/2 \rightrightarrows \mathrm{Spin}(n) \rightrightarrows *$$

while $\mathrm{SO}(n) \rightrightarrows *$.

3.6.0.3 Crossed module of groups Start with the inclusion $\mathbb{Z}/2 \xrightarrow{i} \mathrm{Spin}$ then a 2-groupoid is $* \underset{g'}{\overset{g}{\underset{\simeq \Downarrow c}{\rightrightarrows}}} *$ where $g, g' \in \mathrm{Spin}$ and $c \in \mathbb{Z}/2$ and $g' = c \cdot g$.

The weak equivalence map between $[\mathbb{Z}/2 \to \mathrm{Spin}]$ and $[* \to \mathrm{SO}] = \mathrm{SO}$ is

$$\begin{array}{ccc} [\mathbb{Z}/2 & \rightarrow & \mathrm{Spin}] \\ \downarrow & & \downarrow \\ [* & \longrightarrow & \mathrm{SO}]. \end{array}$$

We have the following diagram

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin} \simeq [* \to \mathrm{Spin}] & \longrightarrow & [* \to *] \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{SO} \simeq [\mathbb{Z}/2 \to \mathrm{Spin}] & \xrightarrow[\mathbf{w}_2]{} & [\mathbb{Z}/2 \to *] \simeq \mathbf{B}^2(\mathbb{Z}/2) \end{array}$$

(see Fiorenza, Schreiber and Stasheff [6]). Then

$$\begin{array}{ccc} B\mathrm{Spin} & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathrm{SO} & \xrightarrow[w_2]{} & B^2(\mathbb{Z}/2) = K(\mathbb{Z}/2, 2) \end{array}$$

and $Fw_2 = B\mathrm{Spin}$.

$$\mathbf{B}\mathrm{Spin}(n) \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3\mathrm{U}(1), \qquad \mathbf{B}\mathrm{String}(n) \xrightarrow{\frac{1}{6}\mathbf{p}_2} \mathbf{B}^7\mathrm{U}(1)$$

$$\exp(\mathfrak{so}) : U, [k] \mapsto \Omega^{\bullet}_{\substack{\mathrm{vect} \\ \mathrm{flat}}}(U \times \Delta^k, \mathfrak{so})$$

where $U \in$ **DiffMan**, the category of smooth manifolds, and $[k] \in \Delta$.

$$\begin{array}{ccc} \exp(\mathfrak{so}) & \xrightarrow{\langle -,[-,-]\rangle} & \exp(b^2\mathbb{R}) \simeq \mathbf{B}^3\mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{mod}\,\mathbb{Z}} \\ \mathbf{B}\mathrm{Spin} \simeq \mathrm{cosk}_3 \exp(\mathfrak{so}) & \xrightarrow[\frac{1}{2}\hat{\mathbf{p}}_1]{} & \mathbf{B}^3\mathrm{U}(1). \end{array}$$

$\square$

4.0 TOWER CONSTRUCTIONS

4.1 POSTNIKOV TOWERS

A connected space X is said to be *simple* if it has an abelian fundamental group $\pi_1(X)$ which acts trivially on higher homotopy groups $\pi_n(X)$ for $n \geq 2$. Let X be a simple connected based space and let X_1 be the Eilenberg-MacLane space $K(\pi_1(X), 1)$. Suppose we have a space X_1 with a 2-equivalence $\alpha_1 : X \to X_1$. Then its homotopy fiber gives a sequence $X \to X_1 \to K(\pi_2(X), 3)$. Then we next construct X_n, α_n and p_n inductively as the following: X_{n+1} and $p_{n+1} : X_{n+1} \to X_n$ is induced from the path space fibration over an Eilenberg-MacLane space $K(\pi_{n+1}(X), n+2)$ by $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$.

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & PK(\pi_{n+1}(X), n+2) \\ {\scriptstyle p_{n+1}} \downarrow & & \downarrow \\ X_n & \xrightarrow[k^{n+2}]{} & K(\pi_{n+1}(X), n+2). \end{array}$$

Theorem 4.1.1. *Each space X_n has the homotopy type of a CW complex and there exists a map $\alpha_{n+1} : X \to X_{n+1}$ which has $K(\pi_{n+2}(X), n+3)$ as its cokernel, i.e., the sequence*

$$X \overset{\alpha_{n+1}}{\to} X_{n+1} \to K(\pi_{n+2}(X), n+3)$$

is exact with a map $k^{n+3} : X_{n+1} \to K(\pi_{n+2}(X), n+3)$ and the diagram

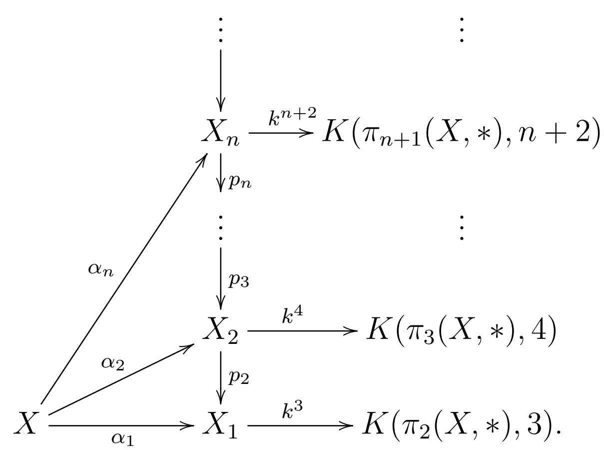
$$\begin{array}{ccc} & & X_{n+1} \\ & {\scriptstyle \alpha_{n+1}} \nearrow & \downarrow {\scriptstyle p_{n+1}} \\ X & \xrightarrow[\alpha_n]{} & X_n. \end{array}$$

commutes.

Each X_n is a space of CW type which approximates X up to nth homotopy group in the following sense:

Theorem 4.1.2. *The q-th homotopy group of each X_n, $\pi_q(X_n, *)$, is isomorphic to $\pi_q(X, *)$ if $q \leq n$ and 0 if $q > n$ and the isomorphism is induced by α_n's. Furthermore, X and $\varprojlim X_n$ are weakly equivalent, i.e., $\pi_q(X) \cong \pi_q(\varprojlim X_n)$ for every $q \geq 0$.*

This theorem asserts that the spaces X_n and maps $\alpha_n : X \to X_n$, $k^{n+2} : X_n \to K(\pi_{n+1}(X), X_{n+2})$, and $p_{n+1} : X_{n+1} \to X_n$ classify any spaces up to weak equivalence if the construction is possible for a given space X. This is called a *Postnikov system* for X and described by the commutative diagram



(4.1)

Note that the spaces X_n "contains" the data on maps $\alpha_n : X \to X_n$ and $p_n : X_n \to X_{n-1}$, so the set of all maps $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$ may be regarded to represent the Postnikov system. This maps are the same as cohomology classes $k^{n+2} \in H^{n+2}(X_n; \pi_{n+1}(X))$ and the set of these classes $\{k^{n+2}\}$ are therefore called the *k-invariants* of the space X.

Note that although each X_n are of CW-homotopy type, its limit $\varprojlim X_n$ maybe not in general. But the following criterion theorem gives the condition on X to have the Postnikov system and when X and $\varprojlim X_n$ are homotopy equivalent:

Theorem 4.1.3. *A simple connected based space X of a homotopy type of a CW complex has a Postnikov tower.*

4.2 WHITEHEAD TOWER

Suppose that a simple connected based space X permits the Postnikov system containing the data on maps $\alpha_n : X \to X_n$. Take the homotopy fiber $X\langle n\rangle$ for each $\alpha_n : X \to X_{n+1}$ for $n \geq 0$, i.e., the $X\langle n\rangle := X \times_{\alpha_{n+1}} PX_{n+1}$ as given in the diagram

$$\begin{array}{ccc} X\langle n\rangle & \longrightarrow & PX_{n+1} \\ {\scriptstyle \alpha^n}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}(1)} \\ X & \xrightarrow[\alpha_{n+1}]{} & X_{n+1}, \end{array}$$

where ev(1) is an evaluation at 1. Here, α^n is a projection $(x,\chi) \mapsto x$ for $x \in X$ and $\chi \in PX_{n+1}$.

Proposition 4.2.1. *Each $X\langle n\rangle$ is of the homotopy type of a CW complex and there is a based map $q^{n+1} : X\langle n+1\rangle \to X\langle n\rangle$ which factors α^{n+1} via α^n in a sense that the diagram*

$$\begin{array}{ccc} X\langle n+1\rangle & & \\ {\scriptstyle q^{n+1}}\downarrow & \searrow^{\alpha^{n+1}} & \\ X\langle n\rangle & \xrightarrow[\alpha^n]{} & X \end{array}$$

commutes. The kernel of q^{n+1} is $K(\pi_{n+1}(X,*), n)$.

Each $X\langle n\rangle$ is a space of CW type which approximates X *from* the $(n+1)$-th homotopy group in the following sense:

Theorem 4.2.2. *The q-th homotopy group of each $X\langle n\rangle$, $\pi_q(X\langle n\rangle)$, is 0 if $q \leq n$ and isomorphic to $\pi_q(X)$ if $q > n$. and the isomorphism is induced by α_n's. Similar to the Postnikov tower, X and $\varprojlim X_n$ are weakly equivalent, i.e., $\pi_q(X) \cong \pi_q(\varprojlim X\langle n\rangle)$ for every $q \geq 0$.*

For given space X, the data consists of the set of thus constructed spaces $X\langle n\rangle$ and maps q^{n+1} and α^n for $n \geq 0$ is called the *Whitehead tower* by the following diagram, which illustrates the structure which is "dual" to the Postnikov tower diagram Eq(4.1) in some sense:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
 & & \downarrow & & \\
K(\pi_n(X,*), n-1) & \longrightarrow & X\langle n\rangle & \xrightarrow{\alpha^n} & \\
 & & \downarrow q^n & & \\
\vdots & & \vdots & & \\
 & & \downarrow q^3 & & \\
K(\pi_2(X,*), 1) & \longrightarrow & X\langle 2\rangle & \xrightarrow{\alpha^2} & \\
 & & \downarrow q^2 & & \\
K(\pi_1(X,*), 0) & \longrightarrow & X\langle 1\rangle & \xrightarrow{\alpha^1} & X. \\
 & & \downarrow q^1 & & \\
 & & X\langle 0\rangle & \xrightarrow{\alpha^0} &
\end{array}
\tag{4.2}
$$

The motivation of Postnikov tower construction was to classify homotopy types of certain spaces in terms of cohomology classes called k-invariants. On the other hand, the Whitehead tower construction provides a canonical way of lifting a space X into n-connected cover $X\langle n\rangle$ and the n-connected cover $X\langle n\rangle$ into the next $(n+1)$-connected cover $X\langle n+1\rangle$. We say we *kill* $(n+1)$-th homotopy group of $X\langle n\rangle$ to obtain $X\langle n+1\rangle$.

4.3 EXAMPLE: WHITEHEAD TOWER OF ORTHOGONAL GROUPS

The main setting of this thesis will be based on the following.

Proposition 4.3.1. *The definite or indefinite orthogonal groups admit a Whitehead tower* *construction: a definite case,*

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\pi_3(\mathrm{O}(n)),2) & \longrightarrow & \mathrm{O}(n)\langle 3\rangle & \searrow & \\
 & & \downarrow & & \\
0 & \longrightarrow & \mathrm{O}(n)\langle 2\rangle & \searrow & \\
 & & \downarrow & & \\
K(\pi_1(\mathrm{O}(n)),0) & \longrightarrow & \mathrm{O}(n)\langle 1\rangle & \longrightarrow & \mathrm{O}(n), \\
 & & \downarrow & \nearrow & \\
 & & \mathrm{O}(n)\langle 0\rangle & &
\end{array}
\tag{4.3}
$$

and the indefinite case

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\pi_3(\mathrm{O}(p,q)),2) & \longrightarrow & \mathrm{O}(n)\langle 3\rangle & \searrow & \\
 & & \downarrow & & \\
0 & \longrightarrow & \mathrm{O}(p,q)\langle 2\rangle & \searrow & \\
 & & \downarrow & & \\
K(\pi_1(\mathrm{O}(p,q)),0) & \longrightarrow & \mathrm{O}(n)\langle 1\rangle & \longrightarrow & \mathrm{O}(p,q). \\
 & & \downarrow & \nearrow & \\
 & & \mathrm{O}(p,q)\langle 0\rangle & &
\end{array}
\tag{4.4}
$$

By Proposition 2.3.1, we may take $\mathrm{SO}(n)$ and $\mathrm{SO}(p,q)^0$ for $\mathrm{O}(n)\langle 0\rangle$ and $\mathrm{O}(n)\langle 0\rangle$, respectively. In fact, these 0-covers are uniquely determined up to homotopy equivalence, so the choice of certain topological or smooth structure on the space has been made by picking up $\mathrm{SO}(n)$ and $\mathrm{SO}(p,q)^0$ as Lie groups which "model" these 0-covers. In this context, the inclusions $\mathrm{O}(n)\langle 0\rangle \to \mathrm{O}(n)$ and $\mathrm{O}(p,q)\langle 0\rangle \to \mathrm{O}(p,q)$ can be chosen to be Lie group homomorphisms.

Principal $\mathrm{O}(n)$-bundles over a manifold X are classified by the first Stiefel-Whiteny class $w_1 \in H^1(B\mathrm{O}(n);\mathbb{Z}/2)$ and we we have seen in Proposition 3.6.1 that the homotopy fiber Fw_1 is homotopy equivalent to $\mathrm{SO}(n)$. Thus w_1 is the obstruction in lifting the structure group $\mathrm{O}(n)$ into $\mathrm{SO}(n)$ whose trivialization gives us a new principal bundle characterized by a map $\tilde{f} : X \to B\mathrm{SO}(n)$ compatible with the original bundle in the following sense:

$$\begin{array}{ccccc} & & B\mathrm{SO}(n) & & \\ & \overset{\tilde{f}}{\nearrow} & \downarrow & & \\ X & \xrightarrow[f]{} & B\mathrm{O}(n) & \xrightarrow[w_1]{} & K(\mathbb{Z}/2, 1). \end{array} \tag{4.5}$$

Having the computation done for the homotopy groups of orthogonal groups as presented in tables before and in Proposition 2.4.2, we may also find “models” for other n-connected covers of the orthogonal groups. The goal of this thesis is to do the same up to 3-connected covers, $\mathrm{O}(n)\langle 3\rangle$ and $\mathrm{O}(p, q)\langle 3\rangle$, and to investigate whether these constructions allow such further lifting.

5.0 SPIN GROUPS

5.1 CONSTRUCTION OF SPIN GROUPS

This section follows Lawson and Michelsohn [10]. Let V be a real vector space with a quadratic form Q. Let $\mathcal{T}(V)$ be the tensor algebra, i.e., an $\mathbb{R}$-algebra $\bigoplus_{r=0}^{\infty} V^{\otimes r}$ where $V^{\otimes 0} = \mathbb{R}$ and $\mathcal{I}_Q(V)$ be the ideal generated by elements of the form $v \otimes v + Q(V)1$ where $1 \in V^{\otimes 0} \subset \mathcal{T}(V)$.

Definition 5.1.1. *The* Clifford algebra $\mathrm{Cl}(V, Q)$ *associated to the pair* (V, Q) *is defined as the quotient algebra* $\mathcal{T}(V)/\mathcal{I}_Q(V)$. *Alternatively, the Clifford algebra can be considered as a* $\mathbb{R}$*-algebra generated by* V *with the relation* $vv + Q(v)1 = 0$.

Proposition 5.1.2. *The map* Cl *from the category of vector spaces over* $\mathbb{R}$ *with the quadratic form* Q *to the category of* $\mathbb{R}$*-algebras sending* (V, Q) *to the Clifford algebra* $\mathrm{Cl}(V, Q)$ *is functorial.*

Let $\alpha : v \mapsto -v : V \to V$. Then by the functoriality, we obtain a map $\mathrm{Cl}(\alpha) =: \tilde{\alpha} : \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$. Since $\tilde{\alpha}^2 = \mathrm{id}$, we obtain a decomposition

$$\mathrm{Cl}(V, Q) = \mathrm{Cl}(V, Q)^0 \oplus \mathrm{Cl}(V, Q)^1 \tag{5.1}$$

where $\mathrm{Cl}(V, Q)^j = \{\varphi \in \mathrm{Cl}(V, Q) \mid \tilde{\alpha}(\varphi) = (-1)^j \varphi\}$ and this is called an *even part* when $j = 0$ and an *odd part* when $j = 1$.

Let $\mathrm{Cl}(V, Q)^\times$ denote the multiplicative group of units in $\mathrm{Cl}(V, Q)$, i.e.,

$$\mathrm{Cl}(V, Q)^\times = \{\varphi \in \mathrm{Cl}(V, Q) \mid \exists \varphi^{-1} \in \mathrm{Cl}(V, Q) \text{ with } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\}$$

and let $\mathrm{Pin}(V,Q)$ be a subgroup of $\mathrm{Cl}(V,Q)^\times$ generated by $v \in V \cap \mathrm{Cl}(V,Q)^\times$ where $Q(v) = \pm 1$. We define the *spin group* of (V,Q) to be the subgroup $\mathrm{Spin}(V,Q) := \mathrm{Cl}^0(V,Q) \cap \mathrm{Pin}(V,Q)$.

From now on, $\mathrm{Cl}(n)$ and $\mathrm{Cl}(p,q)$ will denote $\mathrm{Cl}(\mathbb{R}^n)$ and $\mathrm{Cl}(\mathbb{R}^{p,q})$, respectively.

Theorem 5.1.3. *$\mathrm{Cl}(p,q)^\times$ has the Lie group structure of dimension 2^{p+q} whose Lie algebra is $\mathfrak{cl}(p,q)^\times$ which coincides with $\mathrm{Cl}(p,q)$. The exponential map $\exp : \mathfrak{cl}(p,q)^\times \to \mathrm{Cl}(p,q)^\times$ is defined by $\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ and this always converges.*

Proposition 5.1.4. *There is a natural embedding $\Lambda^2\mathbb{R}^{p,q} \hookrightarrow \mathfrak{cl}(p,q)^\times (\cong \mathrm{Cl}(p,q))$. The image, denoted by $\mathfrak{spin}(p,q)$, of $\Lambda^2\mathbb{R}^{p,q}$ is a Lie subalgebra of $\mathfrak{cl}(p,q)^\times$.*

Fact 5.1.5. *There is a canonical Lie algebra isomorphism $\mathfrak{so}(p,q) \xrightarrow{\cong} \mathfrak{spin}(p,q)$.*

Fact 5.1.6. *The Lie group $\mathrm{Spin}(p,q)$ is a Lie subgroup of $\mathrm{Cl}(p,q)^\times$ which has $\mathfrak{spin}(p,q)$ as its Lie algebra. In the definite case, $\mathrm{Spin}(n,0)$ is identical with $\mathrm{Spin}(0,n)$ and we write $\mathrm{Spin}(n)$. $\mathrm{Spin}(n)$ is a compact Lie group.*

Theorem 5.1.7. *For $n \geq 3$, we have the following short exact sequence*

$$0 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1 \tag{5.2}$$

and we have $\pi_i(\mathrm{Spin}(n))$ is trivial for $i = 0, 1$ and isomorphic to $\pi_i(\mathrm{O}(n))$ for $i \geq 2$. On the other hand, $\mathrm{SO}(1) = \mathrm{O}(1)$ and $\mathrm{SO}(2) = \mathrm{SO}(2) \cong S^1$.

Remark 5.1.8. *The representation $\mathrm{Spin}(m) \to \mathrm{GL}_{\mathbb{R}}(n)$ on some finite n-dimensional real vector space W do not descend to the obvious representation $\mathrm{SO}(n) \to \mathrm{GL}_{\mathbb{R}}(n)$, i.e., the diagram involving a map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ from Eq(5.2) as*

$$\begin{array}{ccc} \mathrm{Spin}(m) & & \\ \downarrow & \searrow & \\ \mathrm{SO}(n) & \longrightarrow & \mathrm{GL}_{\mathbb{R}}(n) \end{array}$$

does not commute.

From these sequence, we may ask when we would be able to lift the structure group $\mathrm{SO}(n)$ of a Riemannian manifold up to $\mathrm{Spin}(n)$. The theorem below gives the obstruction:

Theorem 5.1.9. *Let M be a finite dimensional oriented manifold of dimension $n \geq 3$ with the principal* $\mathrm{SO}(n)$*-bundle P. Then the obstruction in lifting the structure group to* $\mathrm{Spin}(n)$ *is the second Stiefel-Whitney class $w_2 \in H^2(B\mathrm{SO}(n); \mathbb{Z}/2)$.*

In terms of trivialization, the following diagram described the lifting with the obstruction class:

$$\begin{array}{ccccc} & & B\mathrm{Spin}(n) & & \\ & \overset{\tilde{f}}{\nearrow} & \downarrow & & \\ X & \xrightarrow[f]{} & B\mathrm{SO}(n) & \xrightarrow[w_2]{} & K(\mathbb{Z}/2, 2). \end{array} \tag{5.3}$$

Proposition 5.1.10. *For $n \geq 3$,* $\mathrm{Spin}(n)$ *is homotopy equivalent to the homotopy fiber of second Whitney-Stiefel class $w_2 : B\mathrm{SO}(n) \to K(\mathbb{Z}/2, 2)$.*

When $n = 1$, then since $\mathrm{O}(1)\langle 1 \rangle \cong \mathrm{SO}(1)$, the lifting always exists without the obstruction. Note that the trivial map 0 is the only class in the 2nd cohomology group of $B\mathrm{SO}(2)$ with integral coefficient (Theorem 2.5.2) so that the lifting from $\mathrm{SO}(2)$ to $\mathrm{O}(2)\langle 1 \rangle$ is always possible. Combining diagrams Eq(4.5) and Eq(5.3) gives a summarizing illustration on structure group liftings and obstructions for $n \geq 3$:

$$\begin{array}{ccccc} & & B\mathrm{Spin}(n) & & \\ & \nearrow & \downarrow & & \\ & & B\mathrm{SO}(n) & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) \\ & \nearrow & \downarrow & & \\ X & \longrightarrow & B\mathrm{O}(n) & \xrightarrow{w_1} & K(\mathbb{Z}/2, 1) \end{array} \tag{5.4}$$

where a dashed line indicates that it depends on vanishing of corresponding characteristic class. This should not be confused with Postnikov tower Eq(4.1) although these two diagram looks almost the same – classifying spaces like $B\mathrm{O}(n)$, $B\mathrm{SO}(n)$ by no means estimate homotopy groups of manifold X.

On the other hand, since $K(\pi_1(\mathrm{O}(n), 0) = \pi_1(\mathrm{O}(n))$ by definition and the kernel $\mathbb{Z}/2$ of the short exact sequence Eq(5.2) agrees with $\pi_1(\mathrm{O}(n))$, $n \geq 3$ from Table 2.1, the sequence Eq(5.2) fits into the Whitehead tower, Diagram 4.3, of $\mathrm{O}(n)$. Therefore, we may alternatively define the spin group as the central extensions of $\mathrm{SO}(n)$ by $\mathbb{Z}/2$ from the tower corresponding to $w_2 \in H^2(B\mathrm{SO}(n); \mathbb{Z}/2)$ as we shall see in the next section.

5.2 1-CONNECTED COVERS

A central extension of a topological group G by an abelian discrete group A is an exact sequence of continuous group homomorphisms

$$0 \to A \to \tilde{G} \to G \to 1. \tag{5.5}$$

such that the image of A is contained in $Z_{\tilde{G}}$, the center of $\tilde{G}$. In particular when $A = \mathbb{Z}/2$, the extension $\tilde{G}$ is called the *double cover* of G.

The sequence Eq(5.2) from Theorem 5.1.7 says then that $\mathrm{Spin}(n)$ is an extension of $G = \mathrm{SO}(n)$ by $A = \mathbb{Z}/2$. In fact, as we could see from Table 2.1, $A = \pi_1(\mathrm{O}(n)) = K(\pi_1(\mathrm{O}(n)), 0)$ and since $\mathrm{O}(n)\langle 0\rangle \simeq \mathrm{SO}(n)$, this extension as a sequence fits into the Whitehead tower of $\mathrm{O}(n)$ (Eq(4.3)).

Theorem 5.2.1 ([23])**.** *The set $E(G, A)$ of equivalence classes of extensions of G by A has the abelian group structure, and there is an isomorphism*

$$E(G, A) \cong H^2(BG; A).$$

This says that any cohomology class $\lambda_2 \in H^2(BG; A)$ corresponds to an extension $\tilde{G}_{\lambda_2}$. We may construct that extension as a loop space of the homotopy fiber $BG \to K(A, 2)$ where the order 2 of the Eilenberg-MacLane space $K(A, 2)$ came from the order of classifying cohomology group $H^2(BG; A)$. In other words, we set $\tilde{G}_{\lambda_2} := \Omega F_{\lambda_2}$ where F_{λ_2} is the homotopy fiber as a pullback:

$$\begin{array}{ccc} F_{\lambda_2} & \longrightarrow & PK(A,2) \\ \downarrow & & \downarrow \\ BG & \longrightarrow & K(A,2). \end{array}$$

Theorem 5.2.2. *$\tilde{G}_{\lambda_2}$ is a topological group which fits in the central extension Eq(5.5).*

The fiber homotopy sequence induces the following long exact sequence of homotopy groups:

$$\cdots \to \pi_{i+1}(K(A,2)) \to \pi_i(F_{\lambda_2}) \to \pi_i(BG) \to \pi_i(K(A,2)) \to \pi_{i-1}(F_{\lambda_2}) \to \cdots$$

and by using the fact that $F_{\lambda_2} \simeq B\tilde{G}_{\lambda_2}$, $\pi_{i+1}(BG') = \pi_i(G')$ for any abelian topological group G' and that $\pi_i(K(A,2))$ is isomorphic to A if $i = 2$ and is 0 otherwise, we obtain $\pi_i(\tilde{G}_{\lambda_2}) \cong \pi_i(G)$ for $i \geq 2$.

Hence, what Proposition 5.1.10 says is that, for $n \geq 3$, $\mathrm{Spin}(n)$ is a central extension of $\mathrm{SO}(n)$ by $\mathbb{Z}/2$ corresponding to the Stiefel-Whitney class $w_2 \in H^2(B\mathrm{SO}(n); \mathbb{Z}/2)$ respectively, i.e., the following short sequence is exact:

$$0 \to K(\pi_1(\mathrm{O}(n), 0) \to \mathrm{Spin}(n) = \widetilde{\mathrm{SO}(n)}_{w_2} \to \mathrm{SO}(n) \to 1.$$

With this information, the homotopy gype of $\pi_i(\mathrm{Spin}(n))$ given in Theorem 5.1.7 implies that $\mathrm{Spin}(n)$ is a 1-connected cover $\mathrm{O}(n)\langle 1\rangle$ from the Whithead tower. Recall that the existence of 1-connected cover in the tower is unique up to homotopy, so we could conversely define $\mathrm{Spin}(n)$ just as $\mathrm{O}(n)\langle 1\rangle$ up to homotopy.

5.3 1-CONNECTED COVERS OF ORTHOGONAL GROUPS

5.3.1 Definite cases

We just saw from last section that $\mathrm{Spin}(n)$, $n \geq 3$, could have been defined as $\mathrm{O}(n)\langle 1\rangle$. Unfortunately, this is not the case for the unstable case when $n = 1$ and 2. As noted in Theorem 5.1.7, $\mathrm{Spin}(1) = \mathrm{O}(1) = \{\pm 1\}$ and $\mathrm{Spin}(2) = \mathrm{SO}(2) \cong S^1$ so that $\mathrm{Spin}(1)$ and $\mathrm{Spin}(2)$ cannot be defined as the 1-connected cover.

To identify $\mathrm{O}(1)\langle 1\rangle$ and $\mathrm{O}(2)\langle 1\rangle$, we need to consider the following central extensions:

$$0 \to K(0,0) = 0 \to \widetilde{\mathrm{SO}(1)} \to \mathrm{SO}(1) = \{1\} \to 1, \tag{5.6}$$

$$0 \to K(\mathbb{Z},0) = \mathbb{Z} \to \widetilde{\mathrm{SO}(2)} \to \mathrm{SO}(2) \cong S^1 \to 1. \tag{5.7}$$

The exactness of the sequence Eq(5.6) implies that any such cover $\widetilde{\mathrm{SO}(1)}$ should be isomorphic to $\mathrm{SO}(1) = \{1\}$ and this leaves only one choice for $\mathrm{O}(1)\langle 1\rangle$.

On the other hand, Theorem 5.2.1 asserts that the set of extensions $\widetilde{\mathrm{SO}(2)}$ is classified by cohomology class of $H^2(B\mathrm{SO}(2);\mathbb{Z})$ which is trivial by Theorem 2.5.2. Indeed, this trivial class $0 : B\mathrm{SO}(2) \to K(\mathbb{Z}, 2)$ corresponds to an identity since $B\mathrm{SO}(2) = BS^1 = K(\mathbb{Z}, 2)$. Hence, the corresponding homotopy fiber F_0 is the path space $PK(\mathbb{Z}, 2)$ which is contractible. We may then set $\mathrm{O}(2)\langle 1\rangle$ to be the loop space $\Omega PK(\mathbb{Z}, 2)$ which is also contractible. Thus, we have the following:

Proposition 5.3.1. *The 1-connected covers* $\mathrm{O}(1)\langle 1\rangle$ *and* $\mathrm{O}(2)\langle 1\rangle$ *are* $\mathrm{SO}(1) = \{1\}$ *and* $\Omega PK(\mathbb{Z}, 2)$. *For the stable case when* $n \geq 3$, *the 1-connected cover* $\mathrm{O}(n)\langle 1\rangle$ *is* $\mathrm{Spin}(n)$. *There is no obstruction in lifting the structure group* $\mathrm{SO}(1)$ *and* $\mathrm{SO}(2)$ *up to their 1-connected covers, and the second Stiefel-Whitney class* $w_2 \in H^2(B\mathrm{SO}(n);\mathbb{Z}/2)$ *is an obstruction for lifting* $\mathrm{SO}(n)$ *to its 1-connected cover* $\mathrm{Spin}(n)$, *for* $n \geq 3$.

Remark 5.3.2. *Since* $\mathrm{O}(1)\langle 1\rangle$ *and* $\mathrm{O}(2)\langle 1\rangle$ *are homotopically trivial spaces, there will be no more new higher-connected covers.*

5.3.2 Indefinite cases

As noted in Section 4.3, the connected component $\mathrm{SO}(p, q)^0$ of an indefinite special orthogonal group is a 0-connected cover of $\mathrm{O}(p, q)$. However, neither of $\mathrm{Spin}(p, q)$ as constructed in Section 5.1 nor its identity component $\mathrm{Spin}(p, q)^0$ cannot be a 1-connected cover $\mathrm{O}(p, q)\langle 1\rangle$ since $\mathrm{Spin}(p, q)^0$ is not simply connected.

Theorem 5.3.3. *The maximal compact subgroup of* $\mathrm{Spin}(p, q)$ *is*

$$K := (\mathrm{Spin}(p) \times \mathrm{Spin}(q))/\{(1, 1), (-1, -1)\}$$

which is path-connected. That is, we have the two $\mathbb{Z}/2$*'s in the quotient, which is in contrast to the much simpler fact that the maximal compact subgroup of the indefinite orthogonal group* $\mathrm{O}(p, q)$ *is simply* $K = \mathrm{O}(p) \times \mathrm{O}(q)$.

Hence we obtain the following exact sequence of homotopy groups

$$\begin{aligned}\cdots \to &\pi_1(\{(1,1),(-1,-1)\}) \to \pi_1(\mathrm{Spin}(p)\times \mathrm{Spin}(q)) \to \pi_1(K)\\ &\to \pi_0(\{(1,1),(-1,-1)\}) \to \pi_0(\mathrm{Spin}(p)\times \mathrm{Spin}(q)) \to \pi_0(K)\end{aligned}$$

so that $\pi_1(K) \cong \pi_0(\{(1,1),(-1,-1)\}) \cong \mathbb{Z}/2$ which makes $\mathrm{Spin}(p,q)$ unfit for the 1-connected cover $\mathrm{O}(p,q)\langle 1\rangle$.

In the remaining section, we identify such 1-cover of indefinite $\mathrm{SO}(p,q)^0$ which also is a 0-cover of $\mathrm{O}(p,q)$. To do this, we will classify the central extensions

$$0 \to \pi_1(\mathrm{O}(p,q)) \to \widetilde{\mathrm{SO}(p,q)}^0 \to \mathrm{SO}(p,q)^0 \to 1 \tag{5.8}$$

by the cohomology group $H^2(B\mathrm{SO}(p,q)^0;\pi_1(\mathrm{O}(p,q)))$ and then fix a cohomology class corresponding to the extension with the right homotopy type. The coefficient $\pi_1(\mathrm{O}(p,q))$ is given in Table 2.2 in page 13 and by Proposition 2.3.3 along with the Iwasawa decomposition theorem 2.3.2, $B\mathrm{SO}(p,q)^0$ is homotopy equivalent to $B\mathrm{SO}(p)\times B\mathrm{SO}(q)$.

Lemma 5.3.4. *There is an isomorphism between second cohomology group of* $B\mathrm{SO}(p,q)^0$ *and of* $B\mathrm{SO}(p)\times B\mathrm{SO}(q)$*:*

$$H^2(B\mathrm{SO}(p,q)^0;\pi_1(\mathrm{O}(p,q))) \cong H^2(B\mathrm{SO}(p)\times B\mathrm{SO}(q);\pi_1(\mathrm{O}(p,q))).$$

Proof. The cohomology functor $H^2(-;\pi(\mathrm{O}(p,q)))$ from the category of topological spaces **Top** into the category of groups **Grp** is represented by $K(\pi_1(\mathrm{O}(p,q)),2)$, i.e., is naturally isomorphic to the hom-functor $[-,K(\pi_1(\mathrm{O}(p,q)),2)]$ in the homotopy category of topological spaces **Toph**. Since $B\mathrm{SO}(p,q)^0$ is homotopy equivalent to $B\mathrm{SO}(p)\times B\mathrm{SO}(q)$, these hom-functors in **Toph** are naturally isomorphic. □

Since the coefficient group ring $\pi_1(\mathrm{O}(p,q))$ involves products of rings, $\mathbb{Z}/2$ and $\mathbb{Z}$, we need the following lemma:

Lemma 5.3.5 (Universal coefficient for cohomology)**.** *For any space X, any abelian group M,N and any positive integer k, we have the following isomorphism of cohomology groups:*

$$H^k(X;M\times N) \cong H^k(X;M)\times H^k(X;N) \tag{5.9}$$

Proof. By the universal coefficient theorem,

$$H^k(X; M \times N) \cong \hom(H_k(X;\mathbb{Z}), M \times N) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}), M \times N).$$

Since hom and $\mathrm{Ext}^1_{\mathbb{Z}}$ commute with product on the second variable, we have

$$\hom(H_k(X;\mathbb{Z}), M \times N) \cong \hom(H_k(X;\mathbb{Z}), M) \times \hom(H_k(X;\mathbb{Z}), N)$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}), M \times N) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}), M) \times \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}), N).$$

Recalling that product and coproduct coincide in an abelian category, we obtain

$$\begin{aligned} H^k(X; M \times N) &\cong (\hom(H_k(X;\mathbb{Z}), M) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}), M)) \\ &\qquad \times (\hom(H_k(X;\mathbb{Z}), M) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}), M)) \\ &\cong H^k(X; M) \times H^k(X; N). \end{aligned}$$

where the second isomorphism is by the universal coefficient theorem for cohomology with coefficient M and N, respectively. □

By this lemma, we can decompose $H^2(B\mathrm{SO}(p,q); \pi_1(\mathrm{O}(p,q)))$ when $\pi_1(\mathrm{O}(p,q))$ is a product of two nontrivial rings $\mathbb{Z}/2$ and $\mathbb{Z}$:

1. $H^2(B\mathrm{SO}(2,2)^0, \mathbb{Z} \times \mathbb{Z}) \cong H^2(B\mathrm{SO}(2,2)^0; \mathbb{Z}) \times H^2(B\mathrm{SO}(2,2)^0; \mathbb{Z})$,
2. $H^2(B\mathrm{SO}(2,q)^0, \mathbb{Z} \times \mathbb{Z}/2) \cong H^2(B\mathrm{SO}(2,q)^0, \mathbb{Z}) \times H^2(B\mathrm{SO}(2,q)^0, \mathbb{Z}/2)$ for $q \geq 3$, and
3. $H^2(B\mathrm{SO}(p,q)^0, \mathbb{Z}/2 \times \mathbb{Z}/2) \cong H^2(B\mathrm{SO}(p,q)^0, \mathbb{Z}/2) \times H^2(B\mathrm{SO}(p,q)^0, \mathbb{Z}/2)$ for $p, q \geq 3$.

Note that $\mathrm{SO}(p,q) \cong \mathrm{SO}(q,p)$.

When the coefficient ring of a cohomology group is a field $\mathbb{F}$, then the Künneth formula gives the isomorphism $H^k(X \times Y; \mathbb{F}) \cong \bigoplus_{i=0}^k H^i(X;\mathbb{F}) \otimes H^{k-i}(Y;\mathbb{F})$. Since $H^0(B\mathrm{SO}(n); \mathbb{Z}/2)$ and $H^1(B\mathrm{SO}(n); \mathbb{Z}/2)$ are trivial, we thus obtain

$$H^2(B\mathrm{SO}(p,q)^0; \mathbb{Z}/2) \cong H^2(B\mathrm{SO}(p); \mathbb{Z}/2) \times H^2(B\mathrm{SO}(q); \mathbb{Z}/2)$$

when $p, q \geq 3$. (Finite coproduct and produce coincide in abelian category.) However, when the coefficient is not a field, for example, $\mathbb{Z}$, then such isomorphic relation might not be the case - it becomes dependent on the homology groups of $B\mathrm{SO}(p)$ and $B\mathrm{SO}(q)$. However, we still have this analogously simple relation thanks to the nice homology types of $B\mathrm{SO}(p)$ and $B\mathrm{SO}(q)$:

Proposition 5.3.6. *For $p, q \geq 2$, there is an isomorphism*

$$H^2(B\mathrm{SO}(p,q)^0;\mathbb{Z}) \cong H^2(B\mathrm{SO}(p);\mathbb{Z}) \times H^2(B\mathrm{SO}(q);\mathbb{Z}).$$

Lemma 5.3.7. *The following table gives the homology groups with integral coefficient $H_k(B\mathrm{SO}(n);\mathbb{Z})$ for $k = 0, 1$ and 2:*

	$n=1$	$n=2$	$n \geq 3$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	0	0	0
H_2	0	$\mathbb{Z}$	$\mathbb{Z}$

Proof. Since $\mathrm{SO}(2) \cong S^1$, we have $B\mathrm{SO}(2) \cong \mathbb{C}P^\infty$. We know that

$$H_k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } 0 \leq k \leq 2n \text{ and } k \text{ is even}, \\ 0, & \text{otherwise.} \end{cases}$$

so by using the relation $H_k(\mathbb{C}P^\infty;\mathbb{Z}) = \varprojlim_n H_k(\mathbb{C}P^n;\mathbb{Z})$, we obtain

$$H_k(\mathbb{C}P^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } k \text{ even}, \\ 0, & \text{for } k \text{ odd.} \end{cases}$$

For $n \geq 3$, we know

$$\pi_k(B\mathrm{SO}(n)) = \begin{cases} 0, & \text{if } k = 1 \\ \mathbb{Z}, & \text{if } k = 2. \end{cases}$$

Hence, $H_k(B\mathrm{SO}(n);\mathbb{Z})$ follows from the Hurewicz's theorem. □

Proof of Proposition 5.3.6. For any positive integer p and q, the Künneth formula gives

$$H^2(B\mathrm{SO}(p) \times B\mathrm{SO}(q);\mathbb{Z}) \cong \hom(H_2(B\mathrm{SO}(p) \times B\mathrm{SO}(q);\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(B\mathrm{SO}(p) \times B\mathrm{SO}(q);\mathbb{Z}),\mathbb{Z}).$$

where

$$H_2(B\mathrm{SO}(p)\times B\mathrm{SO}(q);\mathbb{Z}) \cong (\bigoplus_{r+s=2} H_r(B\mathrm{SO}(p);\mathbb{Z})\otimes_{\mathbb{Z}} H_s(B\mathrm{SO}(q);\mathbb{Z}))$$
$$\oplus (\bigoplus_{r+s=1} \mathrm{Tor}_1^{\mathbb{Z}}(H_r(B\mathrm{SO}(p);\mathbb{Z}), H_s(B\mathrm{SO}(q);\mathbb{Z})))$$
$$\cong \begin{cases} 0, & \text{if } p=1 \text{ or } q=1,\\ \mathbb{Z}\oplus\mathbb{Z}\cong H_2(B\mathrm{SO}(p);\mathbb{Z})\oplus H_2(B\mathrm{SO}(p);\mathbb{Z}), & \text{if } p,q\geq 2.\end{cases}$$

and

$$H_1(B\mathrm{SO}(p)\times B\mathrm{SO}(q);\mathbb{Z}) \cong (\bigoplus_{r+s=1} H_r(B\mathrm{SO}(p);\mathbb{Z})\otimes_{\mathbb{Z}} H_s(B\mathrm{SO}(q);\mathbb{Z}))$$
$$\oplus (\bigoplus_{r+s=0} \mathrm{Tor}_1^{\mathbb{Z}}(H_r(B\mathrm{SO}(p);\mathbb{Z}), H_s(B\mathrm{SO}(q);\mathbb{Z})))$$
$$= 0$$

since both of $H_1(B\mathrm{SO}(n);\mathbb{Z})$ and $H_2(B\mathrm{SO}(n);\mathbb{Z})$ are trivial from Lemma 5.3.7. □

For $n \geq 3$, $H^2(B\mathrm{SO}(n);\mathbb{Z}/2)\cong\mathbb{Z}$ with the second Stiefel-Whitney class w_2 as the single generator. And from Theorem 2.5.2, $H^2(B\mathrm{SO}(n);\mathbb{Z})\cong\mathbb{Z}$ too, but with a different generator $\sqrt{p_1} := p_1^{1/2}$ which has the first Pontrjagin class $p_1\in H^4(B\mathrm{SO}(n);\mathbb{Z})$ as its square $p_1^{1/2}\smile p_1^{1/2}$ (cup product) in the cohomology algebra $H^\bullet(B\mathrm{SO}(n);\mathbb{Z})$.

Corollary 5.3.8. *The set of generators for $H^2(B\mathrm{SO}(2,2);\mathbb{Z}\times\mathbb{Z})$ is*

$$\{(\sqrt{p_1}' + \sqrt{p_1}'', 0), (0, \sqrt{p_1}' + \sqrt{p_1}'')\}.$$

Remark 5.3.9. *Here, $\sqrt{p_1}'$ is the same as $\sqrt{p_1}''$. However, these two are formally distinguished as two distinct generators where $\sqrt{p_1}'$ and $\sqrt{p_1}''$ correspond to the first and second coordinates of the decomposition of $B\mathrm{SO}(2)\times B\mathrm{SO}(2)$ of $B\mathrm{SO}(2,2)$. This notation will be assumed throughout the thesis.*

Proof. This is a direct consequence of the following:

$$\begin{aligned} H^2(B\mathrm{SO}(2,2);\mathbb{Z}\times\mathbb{Z}) &= H^2(B\mathrm{SO}(2,2);\mathbb{Z})\times H^2(B\mathrm{SO}(2,2);\mathbb{Z}) \\ &= (H^2(B\mathrm{SO}(2);\mathbb{Z})\otimes H^2(B\mathrm{SO}(2);\mathbb{Z})) \\ &\quad\times (H^2(B\mathrm{SO}(2);\mathbb{Z})\otimes H^2(B\mathrm{SO}(2);\mathbb{Z})) \end{aligned}$$

and the generator for $H^2(B\mathrm{SO}(2);\mathbb{Z})$ is $\sqrt{p_1}$. □

Corollary 5.3.10. *The set of generators for $H^2(B\mathrm{SO}(2,q);\mathbb{Z}\times\mathbb{Z}/2)$ is*

$$\{(\sqrt{p_1}' + \sqrt{p_1}'', 0), (0, w_2 + w_2)\}.$$

Proof. The proof is similar to that of Corollary 5.3.8:

$$\begin{aligned} H^2(B\mathrm{SO}(2,q);\mathbb{Z}\times\mathbb{Z}/2) &= H^2(B\mathrm{SO}(2,q);\mathbb{Z})\times H^2(B\mathrm{SO}(2,q);\mathbb{Z}/2) \\ &= (H^2(B\mathrm{SO}(2);\mathbb{Z})\oplus H^2(B\mathrm{SO}(q);\mathbb{Z})) \\ &\quad\times (H^2(B\mathrm{SO}(2);\mathbb{Z}/2)\oplus H^2(B\mathrm{SO}(q);\mathbb{Z}/2)) \end{aligned}$$

and the generators for $H^2(B\mathrm{SO}(2);\mathbb{Z})$ and $H^2(B\mathrm{SO}(n);\mathbb{Z}/2)$ respectively, are $\sqrt{p_1}$ and w_2 respectively. □

The case when one of p or q is 1 is trivial:

Corollary 5.3.11. *The cohomology groups $H^2(B\mathrm{SO}(1,2);0\times\mathbb{Z})$ and $H^2(B\mathrm{SO}(1,q);0\times\mathbb{Z}/2)$ where $q\geq 3$ have single generators $\sqrt{p_1}$ and w_2, respectively.*

Proof. Same as above:

$$\begin{aligned} H^2(B\mathrm{SO}(1,2);0\times\mathbb{Z}) &= H^2(B\mathrm{SO}(1,2);\mathbb{Z}) \\ &= H^2(B\mathrm{SO}(1);\mathbb{Z})\otimes H^2(B\mathrm{SO}(2);\mathbb{Z}) \\ &= \mathbb{Z}[\sqrt{p_1}], \end{aligned}$$

since $H^2(B\mathrm{SO}(1);\mathbb{Z})$ is trivial, and

$$\begin{aligned} H^2(B\mathrm{SO}(1,q);0\times\mathbb{Z}/2) &= H^2(B\mathrm{SO}(1,q);\mathbb{Z}/2) \\ &= H^2(B\mathrm{SO}(1);\mathbb{Z}/2)\otimes H^2(B\mathrm{SO}(q);\mathbb{Z}/2) \\ &= \mathbb{Z}[w_2], \end{aligned}$$

since $H^2(B\mathrm{SO}(1);\mathbb{Z}/2)$ is also trivial. □

For the last,

Corollary 5.3.12. *The set of generators for $H^2(B\mathrm{SO}(p,q);\mathbb{Z}/2\times\mathbb{Z}/2)$ is*

$$\{(w_2'\otimes w_2'',0),(0,w_2'\otimes w_2'')\}$$

where $w_2'\in H^2(B\mathrm{SO}(p);\mathbb{Z}/2)$ and $w_2''\in H^2(B\mathrm{SO}(q);\mathbb{Z}/2)$.

Proof. The proof is analogous to that of Corollary 5.3.8. □

Given these generators, we fix a characteristic class from $H^2(B\mathrm{SO}(p,q)^0;\pi_1(\mathrm{O}(p,q)))$ as in the following table:

	$q=1$	$q=2$	$q\geq 3$
$p=1$	0	$\sqrt{p_1}$	w_2
$p=2$	$\sqrt{p_1}$	$(\sqrt{p_1}'+\sqrt{p_1}'',\sqrt{p_1}'+\sqrt{p_1}'')$	$(\sqrt{p_1}'+\sqrt{p_1}'',w_2'+w_2'')$
$p\geq 3$	w_2	$(\sqrt{p_1}'+\sqrt{p_1}'',w_2'+w_2'')$	$(w_2'+w_2'',w_2'+w_2'')$

Table 5.1: Generators of $H^2(B\mathrm{SO}(p,q)^0;\pi_1(\mathrm{O}(p,q)))$

For simplicity, $\lambda_2(p,q)$ will denote the corresponding characteristic class and we use the following notation:

Notation 5.3.13. *The characteristic class $(w_2'+w_2'',w_2'+w_2'')$ from the table above will be written just $(w_2'+w_2'')^2$. Similar convention will be using throughout this thesis.*

Then by using the method introduced in Section 5.2, the 1-connected covers $\mathrm{O}(p,q)\langle 1\rangle$ is identified, up to homotopy, with the loop space of homotopy fiber of $\lambda_2(p,q)$, $\Omega F_{\lambda_2(p,q)}$, and we have the following diagram which indicates liftings and their obstructions:

$$\begin{array}{ccccc} & & B\mathrm{O}(p,q)\langle 1\rangle & & \\ & \nearrow & \downarrow & & \\ X & \longrightarrow & B\mathrm{SO}(p,q)^0 & \xrightarrow[\lambda_2(p,q)]{} & \pi_1(\mathrm{O}(p,q)). \end{array}$$

Remark 5.3.14. *Similar to the definite case when* $\mathrm{O}(n)\langle 1\rangle$ *is* $\mathrm{Spin}(n)$ *for* $n \geq 3$*, we might guess that* $\mathrm{O}(p,q)\langle 1\rangle$ *for* $p,q \geq 3$ *is related to* $\mathrm{Spin}(p,q)$ *in some way having the Lie group structure which makes it as a Lie subgroup of* $\mathrm{Spin}(p,q)$*. Although this is important in the context of differential geometry, this will not be pursued further since the scope of the thesis will be topological.*

6.0 STRING GROUPS - 2-GROUP EXTENSION OF SPIN

6.1 DEFINITE STRING GROUP AS A TOPOLOGICAL GROUP

Proposition 6.1.1. *The second Eilenberg-MacLane space of* $\mathbb{Z}$ *is homotopy equivalent to the nonabelian projective group of the unitary group* $\mathrm{U}(\mathcal{H})$ *on infinite dimensional separable Hilbert space* $\mathcal{H}$, $K(\mathbb{Z},2) \simeq P\mathrm{U}(\mathcal{H})$.

Over a stable spin group, $\mathrm{Spin} := \varinjlim \mathrm{Spin}(n)$, Stolz [21] constructed explicitly a principal $P\mathrm{U}(\mathcal{H})$-bundle whose total space is a topological group called String using the notion of gauge group. This was originally motivated from a conjecture, still unresolved, by Stolz and Höhn:

Conjecture 6.1.2 (Stolz [21])**.** *Let* M *be a smooth, closed spin manifold of dimension* $4k$*, such that the half times first Pontrjagin class vanishes,* $\frac{1}{2}p_1(M) = 0$*. If* M *admits a Riemannian metric with positive Ricci curvature, then the Witten genus* $\phi_W(M)$ *vanishes.*

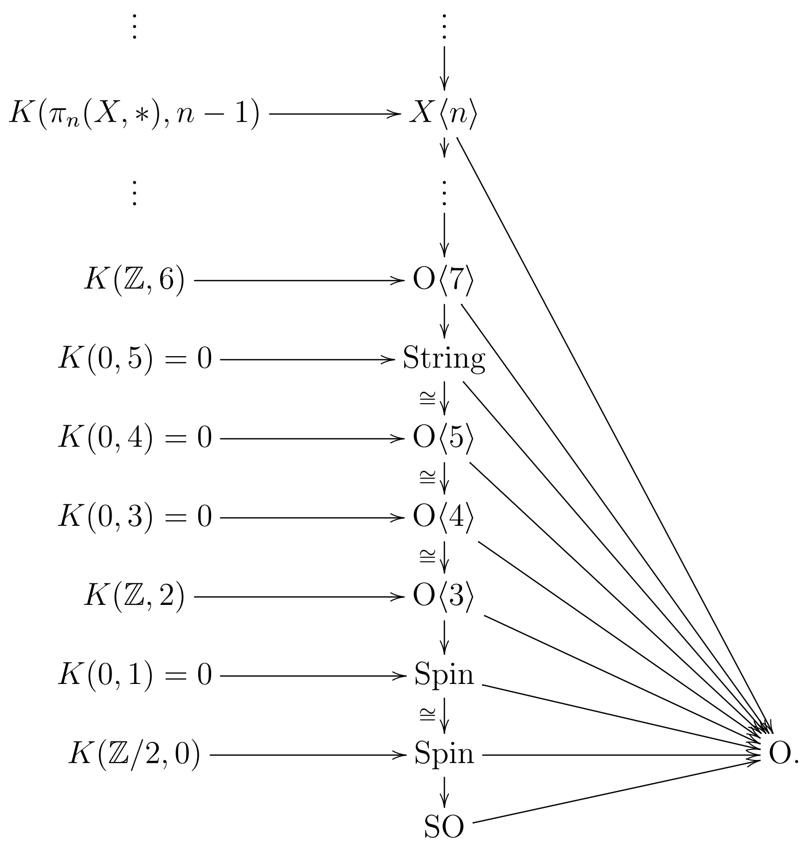
Proposition 6.1.3. *The topological group* String*, which will be defined shortly, is a loop space of homotopy fiber of the characteristic class* $\frac{1}{2}p_1 : B\mathrm{Spin} \to K(\mathbb{Z},4)$.

This proposition implies that the Spin-bundle admits the String-sturucture

$$\begin{array}{ccccc} & & B\mathrm{String} & & \\ & \nearrow & \downarrow & & \\ M & \longrightarrow & B\mathrm{Spin} & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},4). \end{array}$$

with $\frac{1}{2}p_1$ as the obstruction.

On the other hands, it is well known that (mentioned in Stolz [21] with reference to Brylinski [5]) the classifying space BString is homotopy equivalent to the classifying space of 7-connected cover[1] $BO\langle 7\rangle$ of BO. Since $\pi_{i+1}(BO) \cong \pi_i(\mathrm{O})$, this also implies that String is a 6-connected cover $\mathrm{O}(n)\langle 6\rangle$ from the Whitehead tower of O:



(6.1)

From this tower diagram, String is a 3-connected cover $\mathrm{O}\langle 3\rangle$. Note that higher connected covers can also be considered – see Sati, Schreiber and Stasheff [16] and [17].

Setting $G =$ Spin and Spin(n) in the following theorem, we obtain the construction of stable String and unstable definite String(n) for $n = 3$ and ≥ 5 (since $\pi_3(\mathrm{O}(n)) \cong \mathbb{Z}$ only for such n and Spin(n) is its 2-connected cover):

Theorem 6.1.4. *Let G be a 2-connected topological group*[2] *with $\pi_3(G) = \mathbb{Z}$. Then there is a topological group $\widehat{G}$ and a continuous group homomorphism $\pi : \widehat{G} \to G$, such that the induced map of classifying spaces $B\pi : B\widehat{G} \to BG$ is the 3-connected cover of BG.*

[1] In Stolz's paper, the n-cover was used instead of $(n-1)$-connected cover in the sense that the homotopy groups are zero up to $(n-1)$ and the first nontrivial homotopy group is π_n. See discussionin Sati, Schreiber and Stasheff [16] for various conventions on connected covers. Note again that our convention here is that Spin $\simeq \mathrm{O}\langle 2\rangle$ and String $= \mathrm{O}\langle 6\rangle$.

[2] Every 1-connected Lie group is 2-connected.

Proof. For an infinite dimensional separable Hilbert space $\mathcal{H}$, let $P\mathrm{U}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/S^1$ be its projective unitary group. Then the set of principal $P\mathrm{U}(\mathcal{H})$-bundle over the space G is in bijective correspondence with the set of maps $G \to BP\mathrm{U}(\mathcal{H})$.

For a principal $P\mathrm{U}(\mathcal{H})$-bundle $P \stackrel{\pi_\xi}{\to} G$ (where $\xi \in \mathrm{Hom}(G, BP\mathrm{U}(\mathcal{H}))$, consider the set $\mathrm{Aut}(P)$ of automorphisms of P, i.e., the set of $P\mathrm{U}(H)$-equivariant homeomorphisms $P \to P$. Set

$$\widehat{G} := \{f \in \mathrm{Aut}(P) : \pi_\xi \circ f = R_g \text{ an right action for some } g \in G\},$$

and

$$\mathrm{Gauge}(P) := \{f \in \mathrm{Aut}(P) : \pi_\xi \circ f = \mathrm{id}_G\}$$

a *gauge group* of P. Since $\mathrm{Gauge}(P) \hookrightarrow \widehat{G}$ and we can define a surjection $\pi : \widehat{G} \to G$ as $f \mapsto \pi_\xi \circ f$, we obtain following short exact sequence:

$$1 \to \mathrm{Gauge}(P) \to \widehat{G} \stackrel{\pi}{\to} G \to 1.$$

The surjection $\widehat{G} \stackrel{\pi}{\to} G$ is actually fibration and since G is assumed to be 0-connected, we obtain an exact sequence of homotopy groups:

$$\cdots \to \pi_{n+1}(G) \to \pi_n(\mathrm{Gauge}(P)) \to \pi_n(\widehat{G}) \to \pi_n(G) \to \pi_{n-1}(\mathrm{Gauge}(P)) \to \cdots .$$

The cover $\widehat{G}$ of G in question should satisfy following tables:

i	0	1	2	3
$\pi_i(\widehat{G})$	0	0	0	0
$\pi_i(G)$	0	0	0	$\mathbb{Z}$

and

i	0	1	2	3	4
$\pi_i(B\widehat{G})$		0	0	0	0
$\pi_i(BG)$		0	0	0	$\mathbb{Z}$

Therefore, we need to show the following:

(i) $\pi_i(\mathrm{Gauge}(P)) = 0$ for $i \neq 2$,

(ii) $\pi_3(G) \cong \pi_2(\mathrm{Gauge}(P))$,

(iii) $\pi_0(B\mathrm{Gauge}(P)) = 0$.

The remaining proof was done in Stolz [21] by considering these relations between exact sequences:

$$\begin{array}{ccccc}
\mathrm{Gauge}(P) & \longrightarrow & P\mathrm{U}(\mathcal{H}) & = & P\mathrm{U}(\mathcal{H}) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{G} & \longrightarrow & P & \longrightarrow & EP\mathrm{U}(\mathcal{H}) \\
\pi\downarrow & & \pi_\xi\downarrow & & \downarrow \\
G & = & G & \longrightarrow & BP\mathrm{U}(H).
\end{array}$$

□

This construction of definite String(n) has some weakness because it misses $n = 1, 2$ and 4. The case for $n = 1, 2$ might be simple though from the Whitehead tower since $\pi_3(\mathrm{O}(n)) = 0$ for $n = 1$ and 2, exactness of short sequence $K(\pi_3(\mathrm{O}(n)), 2) \to \mathrm{O}\langle 3\rangle \to \mathrm{Spin}(n)$ induces the isomorphism $\mathrm{O}\langle 3\rangle \cong \mathrm{Spin}(n)$.

For $n = 4$, we still have a short exact sequence

$$0 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to \mathrm{O}(n)\langle 3\rangle \to \mathrm{Spin}(4) \to 0. \tag{6.2}$$

Hence, the obvious way of constructing String(n) topologically for any n would be in terms of the classification of this sequence as a extension of Spin(n) by $K(\pi_3(\mathrm{O}(n)), 2)$.

On the other hand, the construction in proving Theorem 6.1.4 shows that String(n) must be infinite dimensional, if it ever has appropriate notion of dimension, and it actually has an infinite dimensional Lie group structure. To make this finite, we use the categorified group, called 2-groups, and classify the 2-group extension Eq(6.2).

6.2 2-GROUPS

6.2.1 Weak monoidal categories

Definition 6.2.1. *A* (biased weak) monoidal category $\mathcal{C}$ *is a category with a terminal object* 1 *and the following:*

1. *a bifunctor, called* tensor product, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,

2. *a natural isomorphism, called the* associator*,*

$$\alpha_{a,b,c} : \alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c) \tag{6.3}$$

is an isomorphism for any objects a, b and c of $\mathcal{C}$

3. *natural isomorphisms for any object a,*

$$\lambda_a : 1 \otimes a \to a \quad \text{and} \quad \rho_a : a \otimes 1 \to a \tag{6.4}$$

such that the following diagrams commute:

4. *coherence of the associator–the* pentagon diagram*:*

$$\begin{array}{ccccc} ((a \otimes b) \otimes c) \otimes d & \xrightarrow{\alpha} & (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\alpha} & a \otimes (b \otimes (c \otimes d)) \\ \downarrow{\scriptstyle \alpha \otimes 1} & & & & \uparrow{\scriptstyle 1 \otimes \alpha} \\ (a \otimes (b \otimes c)) \otimes c & & \xrightarrow[\alpha]{\qquad\qquad\qquad} & & a \otimes ((b \otimes c) \otimes d), \end{array} \tag{6.5}$$

5. *coherence of the identity:*

$$\begin{array}{ccc} (a \otimes 1) \otimes b & \xrightarrow{\alpha} & a \otimes (1 \otimes b), \\ & {\scriptstyle \lambda \otimes 1} \searrow \quad \swarrow {\scriptstyle 1 \otimes \rho} & \\ & a \otimes b & \end{array} \tag{6.6}$$

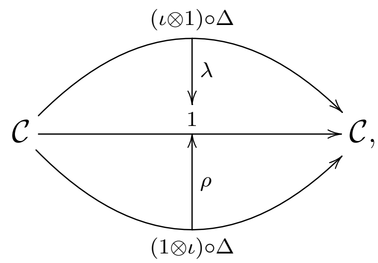
The term *biased* indicates that the tensor product is defined on $\mathcal{C} \times \mathcal{C}$. In *unbiased* case, we may have different operations $\otimes_n : \prod^n \mathcal{C} \to \mathcal{C}$ for each nonnegative integer n. The term *weak* means that the associative law, product law with identities only hold up to isomorphism via α, λ and ρ. The term *strict* otherwise means that these are identities, and the term *lax* indicates that these may not be necessarily isomorphisms . The most general one would be then an *unbiased lax monoidal category*.

To internalize this notion, we need the 'object-free' description as given in the following commutative diagrams. The numbering corresponds to that in Definition 6.2.1:

2′. the associator α

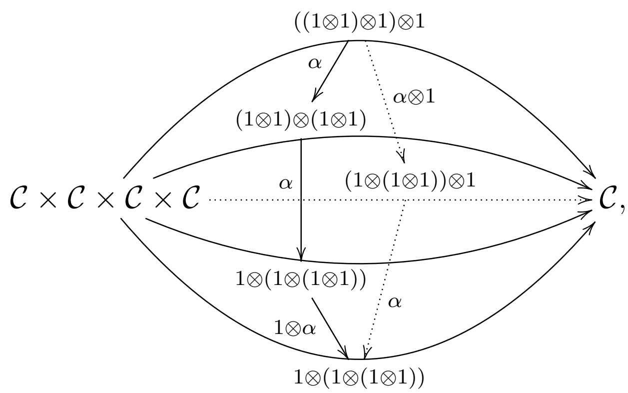
$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \;\underset{1 \otimes (1 \otimes 1)}{\overset{(1 \otimes 1) \otimes 1}{\rightrightarrows}}\; \mathcal{C}, \quad \alpha : (1 \otimes 1) \otimes 1 \Rightarrow 1 \otimes (1 \otimes 1) \tag{6.3′}$$

3′. left and right identity multiplication:



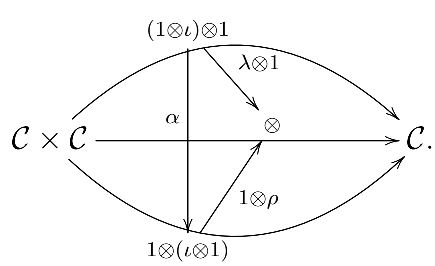
(6.4′)

4′. the pentagon diagram:



(6.5′)

5′. coherence of the identity:



(6.6′)

6.2.2 Categorified groups and 2-groups

A *(categorified) group* is a biased small monoidal category $\mathcal{C}$ such that for any object $a \in \mathcal{C}_0$ where $\mathcal{C}_0$ is the set of objects, there is an inverse $a^{-1} \in \mathcal{C}_0$ in the sense that $a \otimes a^{-1} = 1$ and $1 = a^{-1} \otimes a$. The monoidal category $\mathcal{C}$ may be either strict, weak or even lax, according to the associativity. The corresponding groups will be called *strict*, *weak* and *lax groupoids*, respectively and the corresponding categories are denoted by **str-Grp**, **wk-Grp** and **lax-Grp**. Because of the classical notion, we write **Grp** for **str-Grp**.

Definition 6.2.2. *A* 2-group *is a biased monoidal category* $\mathcal{C}$ *which has a covariant functor* $\mathrm{inv} : \mathcal{C} \to \mathcal{C}$ *where each object* $a \in \mathcal{C}_0$ *is being mapped into its* weak inverse $\bar{a} \in \mathcal{C}_0$ *in the sense that there exist some isomorphisms*

$$i_a : 1 \to a \otimes \bar{a} \quad \textit{and} \quad e_a : \bar{a} \otimes a \to 1. \tag{6.7}$$

We may have specific isomorphism $i_a : 1 \to a \otimes \bar{a}$ and $e_a : \bar{a} \otimes a \to 1$ satisfying the *coherence condition* which says that the following two diagrams, called *zig-zag identities*, commute:

$$\begin{array}{ccccc} & & a & & \\ & \swarrow^{\lambda_a^{-1}} & & \nwarrow^{\rho_a} & \\ 1 \otimes a & & & & a \otimes 1 \\ \downarrow{\scriptstyle i_a \otimes 1_a} & & & & \uparrow{\scriptstyle 1_a \otimes e_a} \\ (a \otimes \bar{a}) \otimes a & & \xrightarrow{\alpha_{a,\bar{a},a}} & & a \otimes (\bar{a} \otimes a) \end{array} \qquad \begin{array}{ccccc} & & \bar{a} & & \\ & \nearrow^{\lambda_{\bar{a}}} & & \searrow^{\rho_{\bar{a}}^{-1}} & \\ 1 \otimes \bar{a} & & & & \bar{a} \otimes 1 \\ \uparrow{\scriptstyle e_a \otimes 1_{\bar{a}}} & & & & \downarrow{\scriptstyle 1_{\bar{a}} \otimes i_a} \\ (\bar{a} \otimes a) \otimes \bar{a} & & \xleftarrow{\alpha_{\bar{a},a,\bar{a}}} & & \bar{a} \otimes (a \otimes \bar{a}). \end{array} \tag{6.8}$$

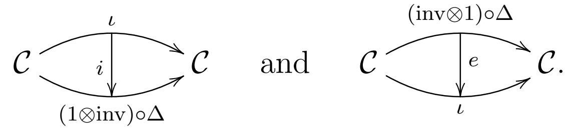
It is known that "any" 2-group can be made *coherent*:

Theorem 6.2.3 (Baez and Lauda [1])**.** *Any 2-group can be given the structure of a coherent 2-group by equipping each object with an adjoint equivalence.*

From here on, we will assume any 2-groups to be already coherent.

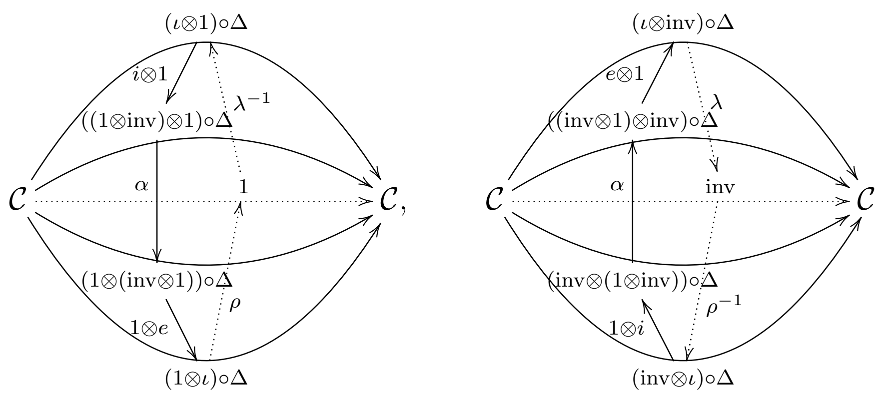
Again, for the internalization, we need an object-free description of this structure. Let $\mathrm{inv} : \mathcal{C} \to \mathcal{C}$ be a covariant functor which maps any object a into its weak inverse $\bar{a}$. On the other hand, there is an inclusion functor $\{1\} \to \mathcal{C}$ which maps a discrete category with just one object 1 into the terminal object $1 \in \mathcal{C}_0$. Moreover, there is also another unique map $\mathcal{C} \to \{1\}$ sending every object into 1. Write $\iota : \mathcal{C} \to \mathcal{C}$ for the composition of $\mathcal{C} \to \{1\}$ with

$\{1\} \to \mathcal{C}$, only for now. Furthermore, let $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the *diagonal functor* which maps a into $(a, a) \in (\mathcal{C} \times \mathcal{C})_0$. Then, we have the object-free version of Eq(6.7):



(6.7′)

and of zig-zag identities Eq(6.8):



(6.8′)

Similarly to the groupoids and groups, 2-groupoids and 2-groups could be strict, weak and lax depending on attributes of the associator α and left and right identity multiplication (2-)morphisms λ and ρ. We write **str-2-Grp**, **wk-2-Grp** and **lax-2-Grp** for corresponding categories of 2-groups. Our main concern will be on the weak case, so we shortly write **2-Grpd** and **2-Grp** for weak ones.

Example 6.2.4 (of an inverse functor)**.** *Suppose that we have a monoidal category $\mathcal{C}$ with morphisms Eq(6.7), and* inv $: \mathcal{C} \to \mathcal{C}$ *which maps $a \mapsto \bar{a}$. It may not be clear whether or not* inv *is a covariant functor, but we can always have at least one if the opposite category $\mathcal{C}^{\mathrm{op}}$ is isomorphic to $\mathcal{C}$. This simply indicates that for any morphism $f : a \to b$ in $\mathcal{C}$, there is an opposite morphism $f^{-1} : b \to a$ which may not necessarily be an inverse. Let $^{-1} : \mathcal{C} \to \mathcal{C}$ be such a functor that maps objects into themselves, but reverses morphisms.*

Now, let $^*: \mathcal{C} \to \mathcal{C}$ *be a functor sending each object* a *into their weak inverse* $\bar{a}$*, and which maps a morphism* $f : a \to b$ *into* $f^{-1} : \bar{b} \to \bar{a}$ *given by the following composite*

$$\begin{array}{ccccc} \bar{b} & \overset{f^{-1}}{\dashrightarrow} & & & \bar{a} \\ \downarrow{\scriptstyle \rho_{\bar{b}}^{-1}} & & & & \uparrow{\scriptstyle \lambda_{\bar{a}}} \\ \bar{b} \otimes 1 & & & & 1 \otimes \bar{a} \\ \downarrow{\scriptstyle 1_{\bar{b}} \otimes i_a} & & & & \uparrow{\scriptstyle e_b \otimes 1_{\bar{a}}} \\ \bar{b} \otimes (a \otimes \bar{a}) & \xrightarrow[1_{\bar{b}} \otimes (f \otimes 1_{\bar{a}})]{} & \bar{b} \otimes (b \otimes \bar{a}) & \xrightarrow[\alpha_{\bar{b},b,\bar{a}}]{} & (\bar{b} \otimes b) \otimes \bar{a}. \end{array} \tag{6.9}$$

By virtue of coherence, this functor * *is a contravariant functor, and by composing it with* $^{-1}$*, we obtain the desired covariant functor* inv *which maps a morphism* $f : a \to b$ *into* $\bar{f} := (f^{-1})^* = (f^*)^{-1} : \bar{a} \to \bar{b}$.

Example 6.2.5. *The* classical *notion of a group corresponds to the set of 1-morphisms* D_1 *of a group object* D *with respect to some functor* **Set** $\to$ **Grp***. So, by abuse of notation, we usually refer to* D_1 *as a group. Similarly, we could regard a 2-group to be* D'_A *of a 2-group object. The same convention will be assumed for groupoids and 2-groupoids.*

6.2.3 Internalized 2-groups

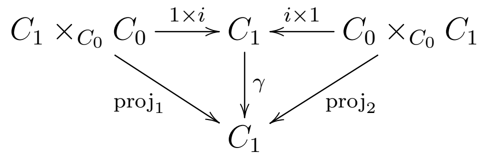
A *category object*, or an *internal category*, in a category $\mathcal{C}$ closed in finite product is a pair $C = (C_1 \rightrightarrows C_0)$ of two objects $C_0, C_1 \in \mathcal{C}_0$ and the following (1-)morphisms:

1. domain and codomain maps $C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$
2. identity $i : C_0 \to C_1$ such that $s \circ i = t \circ i = 1 : C_0 \to C_0$
3. a composition map $\gamma : C_1 \times_{C_0} C_1 \to C_1$ whose domain is a pullback of domain and codomain:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow[\mathrm{proj}_2]{} & C_1 \\ \downarrow{\scriptstyle \mathrm{proj}_1} & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[t]{} & C_0. \end{array}$$

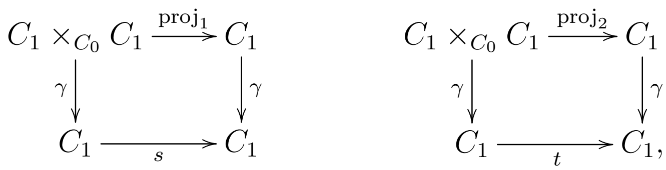
These morphisms are subject to the commutativity of the following diagrams:

4. composition with the identity



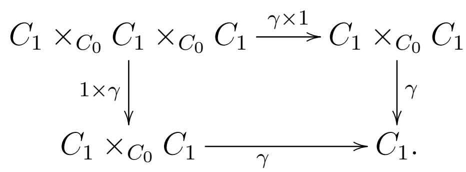
(6.10)

5. domains and codomains of compositions



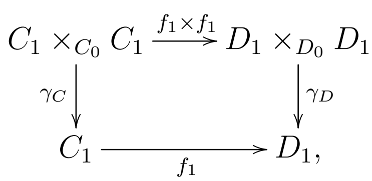
(6.11)

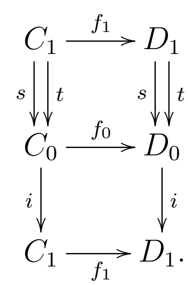
6. (strict) associativity



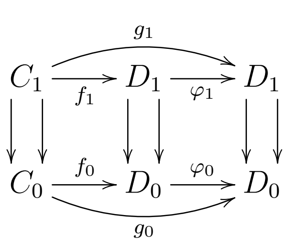
(6.12)

The *internal functor* $f : C \to D$ between two internal categories $C = (C_1 \rightrightarrows C_0)$ and $D = (D_1 \rightrightarrows D_0)$ is a pair of morphisms $f = (f_1 : C_1 \to D_1, f_0 : C_0 \to C_1)$ in the ambient category $\mathcal{C}$ such that the following diagrams commute:





Suppose that there are two internal functors $f, g : C \to D$. Then the internal natural transformation $\varphi : f \Rightarrow g$ is defined to be a pair of 1-morphisms $(\varphi_1 : C_1 \to D_1, \varphi_0 : C_0 \to D_0)$ such that the following diagram commutes:



where double arrows indicates domain and codomain maps.

A *monoidal category object*, or a *monoid*, in an arbitrary category $\mathcal{C}$ is a category object $C_1 \rightrightarrows C_0$ in $\mathcal{C}$ with

- an internal functor $\eta : 1 \to C$ where the terminal object 1 is regarded as a discrete internal category whose composite with the terminal map $C \to 1$ is denoted by $\iota : C \to C$,
- an internal bifunctor $\otimes = (\otimes_1, \otimes_0) : C \times C \to C$,
- an internal natural isomorphism $\alpha : (1 \otimes 1) \otimes 1 \Rightarrow 1 \otimes (1 \otimes 1)$, and
- internal natural isomorphisms $\lambda : \iota \otimes 1 \Rightarrow 1$ and $\rho : 1 \otimes \iota \Rightarrow 1$,

where all of these internal functors and natural transformations are subject to the commutativity of diagrams Eq(6.5′) (the pentagon diagram) and Eq(6.6′) with every $\mathcal{C}$ being replaced by C.

Finally, we define a 2-group object in a category $\mathcal{C}$:

Definition 6.2.6. *A* 2-group object *in $\mathcal{C}$ is an internal monoidal category C with an internal functor* inv $: C \to C$ *with internal natural isomorphisms $i : \iota \Rightarrow (1 \otimes \mathrm{inv}) \circ \Delta$ and $e : (\mathrm{inv} \otimes 1) \circ \Delta \Rightarrow \iota$. A 2-group C is said to be* coherent *if we have commutative zig-zag identity diagrams Eq(6.8′) that commute, with $\mathcal{C}$ replaced by C.*

Remark 6.2.7. *By ignoring the C_1 structure from the internal category object, we could obtain the definition of 2-groups from [1] which was presented in terms of C_0 structure.*

6.2.4 Lie 2-group

A 2-group object $G = (G_1 \rightrightarrows G_0)$ in the category of finite dimensional smooth manifolds **DiffMan** with surjective submersions as morphisms is called a *Lie 2-group*.

Conjecture 6.2.8 (Schreiber;[20])**.** *Let* $\mathbf{LieGrp}^\infty$ *denote the category of Lie groups (possibly infinite dimensional), and* **Lie2Grp** *be the category of (finite dimensional) Lie 2-groups. There is a functor* $F : \mathbf{LieGrp}^\infty \to \mathbf{Lie2Grp}$*. Conversely, there is an equivariant functor* $U : \mathbf{Lie2Grp} \to \mathbf{LieGrp}^\infty$ *such that* $FU = 1$ *in* **LieGrp**.

With this conjecture, the infinite dimensional Lie group $\mathbf{String}(n)$ for $n = 3$ or ≥ 5 can be made into a finite dimensional Lie 2-group $F(\mathbf{String}(n))$. Since the distinction between

Lie groups and Lie 2-groups will be clear from the context, we will sometimes omit the functor to write, for example, just String(n) to mean $F(\text{String}(n))$.

6.3 2-GROUPOIDS

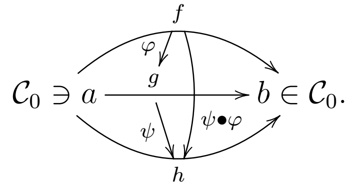
A *groupoid* is a category $\mathcal{C}$ whose arrows are all invertible. So, there is a set map inv : $\mathcal{C}_1 \to \mathcal{C}_1$ mapping $f \in \mathcal{C}(a,b)$ into $f^{-1} \in \mathcal{C}(b,a)$ such that $f^{-1} \circ f = 1_a$ and $f \circ f^{-1} = 1_b$. If we want to *weaken* these identities as maps $f^{-1} \circ f \to 1_a$ and $f \circ f^{-1} \to 1_b$, we need to take 'morphisms' within $\mathcal{C}_1$ into our consideration, i.e., we need to enrich $\mathcal{C}$ in a category **cat** of small categories. This enriched category $\mathcal{C}$ is called a *bicategory*.

Note that it might be tempted to say that the arrow set $\mathcal{C}_1$ must be a group for $\mathcal{C}$ to be a gropoid. But it is not appropriate since we cannot compose arbitrary morphisms $f, g \in \mathcal{C}_1$. They should be composable in a sense that $t(f) = s(g)$ so we only have binary operations $\mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,b)$ instead of $\mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1$. However, if $\mathcal{C}$ has just one object $\mathcal{C}_0 = *$, then the groupoid becomes identical to a group.

6.3.1 Bicategories

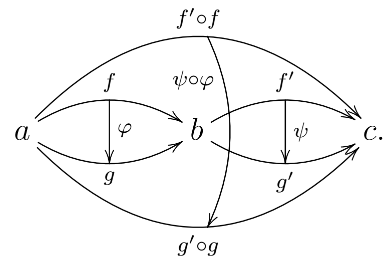
Definition 6.3.1. *A* (biased weak) bicategory $\mathcal{C}$ *is a (small) category* $\mathcal{C}$ *enriched in a category* **cat** *of small categories, i.e., the hom-set* $\mathcal{C}(a,b)$ *is a* hom-category *consisting of a set of 1-morphisms* $\mathcal{C}_1(a,b)$ *as the set of objects and a set of 2-morphisms* $\mathcal{C}_2(a,b)$ *as the set of arrows. There are two different ways of composing 2-morphisms:*

1. *The composition in* $\mathcal{C}_2(a,b)$ *is said to be* vertical *and denoted by* $\psi \bullet \varphi$ *for any (vertically) composable 2-morphisms* $(\varphi : f \to g), (\psi : g \to h) \in \mathcal{C}_2(a,b)$*:*



(6.13)

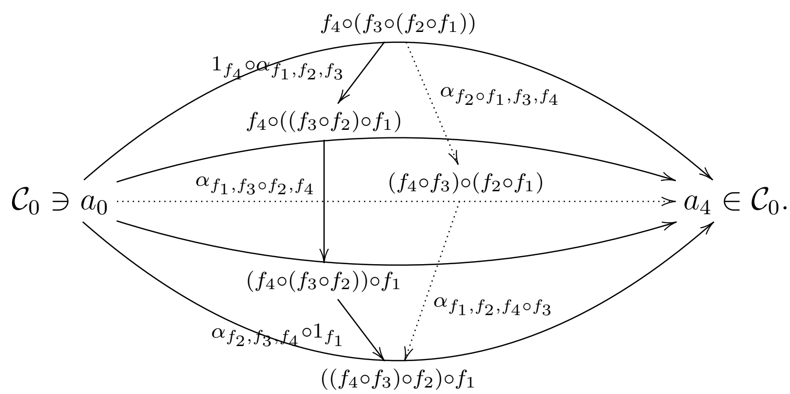
2. There is the horizontal *composition of 2-morphisms* $\circ : (\varphi, \psi) \mapsto \psi \circ \varphi : \mathcal{C}_2(a,b) \times \mathcal{C}_2(b,c) \to \mathcal{C}_2(a,c)$ *for any* $a, b, c \in \mathcal{C}_0$*, where such 2-morphisms* φ, ψ *are said to be* horizontally composable*. Note that* $s(\psi \circ \varphi) = f' \circ f$ *and* $t(\psi \circ \varphi) = g' \circ g$ *for* $\varphi : f \to g$ *and* $\psi : f' \to g'$*:*



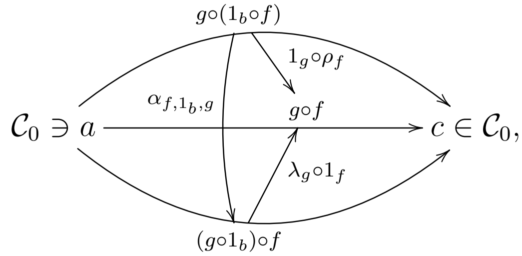
(6.14)

The hom-sets $\mathcal{C}_2(a,b)$ *of any hom-category* $\mathcal{C}_1(a,b)$ *contains the following 2-morphisms:*

1. (horizontal) associator*:* $\alpha_{f_1,f_2,f_3} : f_3 \circ (f_2 \circ f_1) \to (f_3 \circ f_2) \circ f_1$ *for* $f_1 \in \mathcal{C}_1(a_0, a_1)$*,* $f_2 \in \mathcal{C}_1(a_1, a_2)$ *and* $f_3 \in \mathcal{C}_1(a_2, a_3)$*, such that the 2-pentagon diagram commutes for additional* $f_4 \in \mathcal{C}_1(a_3, a_4)$*:*

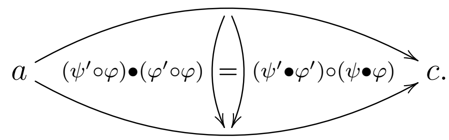


2. (horizontal) composition with the identities*: 2-morphisms* $(\rho_f : 1_b \circ f \to f) \in \mathcal{C}_2(a,b)$ *and* $(\lambda_g : g \circ 1_b \to g) \in \mathcal{C}_2(b,c)$ *for any composable 1-morphisms* $f \in \mathcal{C}_1(a,b)$ *and* $g \in \mathcal{C}_1(b,c)$ *such that the following diagram commutes:*

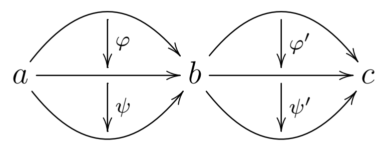


where $1_b \in \mathcal{C}_1(b,b) \subset \mathcal{C}_1$ *is an identity 1-morphism.*

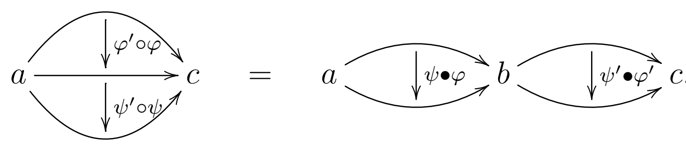
The vertical and horizontal compositions are related by the identity $(\psi' \circ \psi) \bullet (\varphi' \circ \varphi) = (\psi' \bullet \varphi') \circ (\psi \bullet \varphi)$.



This is called the middle four exchange axiom *and diagrammatically* illustrated *by the following diagram: given four 2-morphisms*



following two different ways of composition agree as



Remark 6.3.2. *The strict associativity for vertical composition was assumed by construction — $\mathcal{C}_1(a, b)$ is a category — so that $(\chi \bullet \psi) \bullet \varphi = \chi \bullet (\psi \bullet \varphi)$ for $\varphi, \psi, \chi \in \mathcal{C}_2(a, b)$.*

Remark 6.3.3. *A qualifier 'biased' indicates that only the binary composition is given making the structure biased toward $2 \in \mathbb{N}$. In* unbiased *case, it may have distinct isomorphisms $\circ_n : \mathcal{C}(a_0, a_1) \times \cdots \times \mathcal{C}(a_{n-1}, a_n) \to \mathcal{C}(a_0, a_n)$ for each $n \in \mathbb{N}$. The term 'weak' indicates that natural transformations α, ρ, λ are isomorphism. The* strict *bicategory has identities as these isomorphisms, and the* lax *bicategory may have non-isomorphic 2-morphisms. Therefore, we have the following relation:*

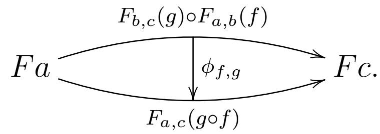
strict bicategories $\subset$ weak bicategories $\subset$ lax bicategories.

The 1-category is a specific form of bicategory — it is a strict bicategory where all of the hom-categories $\mathcal{C}(a, b)$ are discrete being identical to a set. Hence, we have a further inclusion

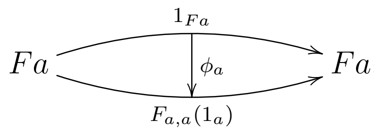
categories $\subset$ strict bicategories.

Let $\mathcal{C}$ and $\mathcal{C}'$ be weak bicategories. A *weak bifunctor* F is a 1-functor $F : \mathcal{C} \to \mathcal{C}'$, and a set of 1-functors $F_{a,b} : \mathcal{C}(a, b) \to \mathcal{C}'(Fa, Fb)$ for any objects a, b of $\mathcal{C}$ such that there are the following 2-morphisms:

1. $\phi_{f,g} : F_{b,c}g \circ F_{a,b}f \to F_{a,c}(g \circ f)$ in $\mathcal{C}'_2(Fa, Fc)$



2. $\phi_a : 1_{Fa} \to F_{a,a}(1_a)$ in $\mathcal{C}'(Fa, Fa)$.



If the 2-isomorphisms ϕ are morphisms not necessarily isomorphisms, then the bifunctor is said to be *lax* and when ϕ are identities, then it is said to be *strict*.

Theorem 6.3.4 (Leinster [11])**.** *The category* **wk-2-Cat** *of weak 2-categories with strict functors as morphisms is equivalent to the unbiased weak bicategory with strict functors.*

6.3.2 2-Groupoids

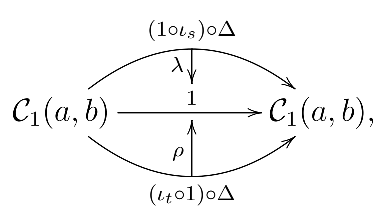
Having categorified hom-sets of a category, we may weaken the notion of groupoids:

Definition 6.3.5. *A* strict, weak *and* lax 2-groupoid *is a biased strict, weak and lax bicategory $\mathcal{C}$, respectively, with additional functors as listed below:*

1. $\iota_s : \mathcal{C}(a, b) \to \mathcal{C}(a, a)$ *sending any $f \in \mathcal{C}_1(a, b)$ into $1_a \in \mathcal{C}_1(a, a)$, and any $\varphi \in \mathcal{C}_2(a, b)$ into $1_{1_a} \in \mathcal{C}_2(a, a)$.*
2. $\iota_t : \mathcal{C}_1(a, b) \to \mathcal{C}_1(b, b)$ *sending any $f \in \mathcal{C}_1(a, b)$ into $1_b \in \mathcal{C}_1(b, b)$ and any $\varphi \in \mathcal{C}_2(a, b)$ into $1_{1_b} \in \mathcal{C}_2(a, a)$*
3. inv : $\mathcal{C}(a, b) \to \mathcal{C}(b, a)$ *which is covariant and maps $f \mapsto \bar{f}$.*

With a diagonal functor $\Delta : \mathcal{C}(a, b) \to \mathcal{C}(a, b) \times \mathcal{C}(a, b)$, these are related by the following natural isomorphisms:

1. *compositions with identities, λ and ρ, as*

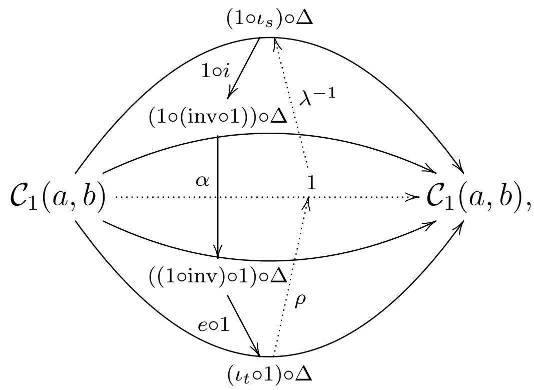


(6.15)

2. unit *and* counit *natural isomorhpisms, i and e, respectively, given as*

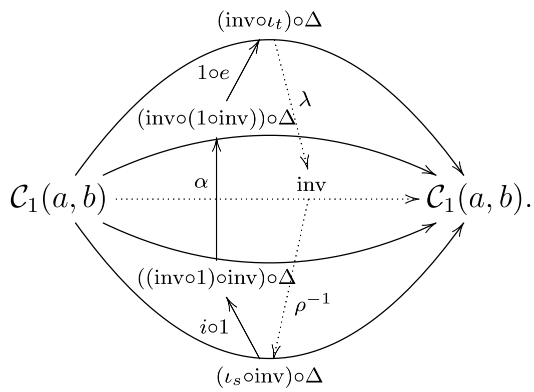
(6.16)

These natural isomorphisms are subject to the commutativity of the following zig-zag identity diagrams:



(6.17)

and



(6.18)

Let's say, just for now, that the algebraic operations are *strict* and *weak* if the natural transformations λ, ρ, i and e are all identities and isomorphisms, respectively. Strict 2-groupoids with strict algebraic operations are identical to *groupoids* if hom-categories $\mathcal{C}(a,b)$ for any objects $a, b \in \mathcal{C}_0$ are discrete. In the case of discrete hom-category, weak association and weak composition with identities are not possble, so there is no corresponding notion of neither "*weak*" nor "*lax groupoids*". Weak 2-groupoids with weak algebraic operations are referred to by just *2-groupoids* and the category of 2-groupoids with strict functors is denoted by **2-Grpd**.

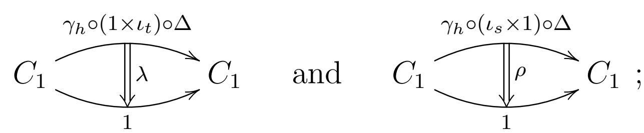
We still have other variations, like strict 2-groupoids with weak algebraic operations, weak 2-groupoids with strict algebraic operations, or even strict 2-groupoids with strict algebraic operations but with non-discrete hom-categories besides of similar things with lax 2-groupoids. However, these extra cases are not of interest to us.

Note that a 2-groupoid with one single object $\mathcal{C}_0 = *$ is identical to a 2-group.

6.3.3 Internalization

A *bicategory object*, or an *internal category*, in a category $\mathcal{C}$ is a triple $C = (c_2 \rightrightarrows c_1 \rightrightarrows c_0)$ which is a pair of category objects $C_1 = (c_2 \rightrightarrows c_1)$ and $C_0 = (c_1 \rightrightarrows c_0)$. The composition $\gamma_v : c_2 \times_{c_1} c_2 \to c_2$ in C_1 is said to be *vertical*. Additionally, there are functors $\gamma_h : C_1 \times_{C_0} C_1 \to C_1$ called a *horizontal composition* such that $\gamma_v \circ (\gamma_h \times \gamma_h) = \gamma_h \circ (\gamma_v \times \gamma_v)$ and $\iota_s, \iota_t : C_1 \to C_1$ with the following natural isomorphisms

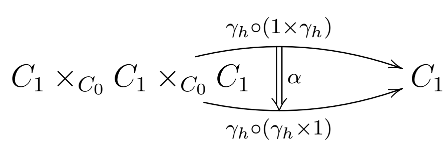
1. the composition with identities



(6.19)

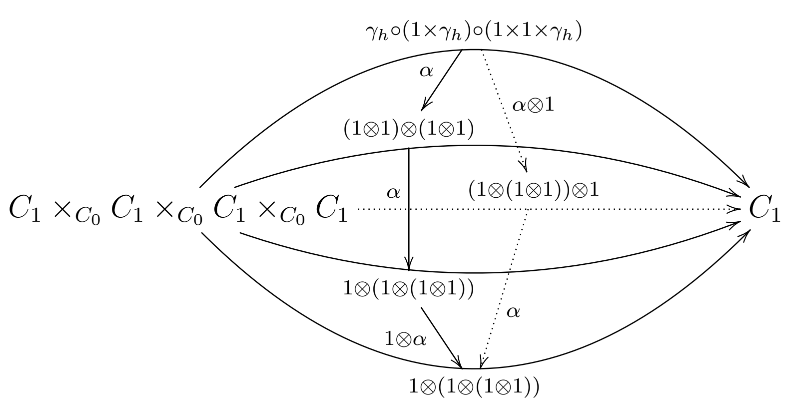
(here, double arrow means natural transformations not 2-morphisms.)

2. the *associator*



(6.20)

such that the pentagon diagram commutes:

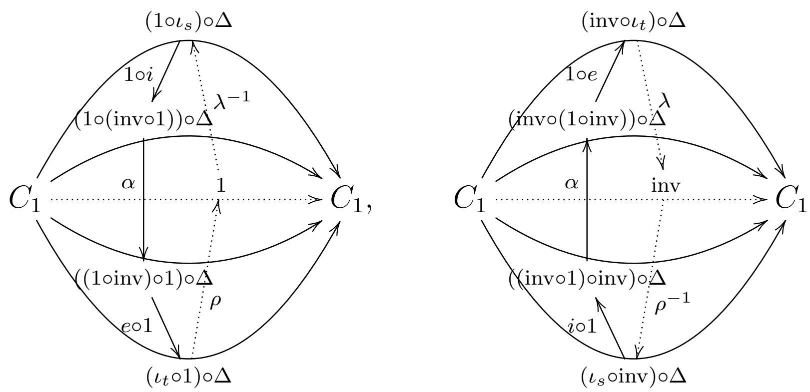


(6.21)

Definition 6.3.6. *An* internalized weak 2-groupoid*, or a* (weak) 2-groupoid object*, is a bicategory object* $c_2 \rightrightarrows c_1 \rightrightarrows c_0$ *of two category objects,* $C_1 = (c_2 \rightrightarrows c_1)$ *and* $C_0 = (c_1 \rightrightarrows c_0)$*, with an additional functor* $\mathrm{inv} : C_1 \to C_1$ *such that we have two composite functors* $(1 \times \mathrm{inv}) \circ \Delta, (\mathrm{inv} \times 1) \circ \Delta : C_1 \to C_1 \times_{C_0} C_1$ *and such that there are natural isomorphisms* i *and* e *as*

(6.22)

which makes the following diagrams commute:



(6.23)

A 2-groupoid object $\Gamma = (\Gamma_2 \rightrightarrows \Gamma_1 \rightrightarrows \Gamma_0)$ in a category of finite dimensional smooth manifold **DiffMan** with surjective submersions as morphisms is called a *Lie 2-groupoid.*

6.4 BIBUN CATEGORY

6.4.1 The bicategory of smooth 2-groupoids

Here are some examples of Lie 2-groupoids:

Example 6.4.1. *Let* G *be a Lie group acting on a manifold* X*. Then the* action groupoid $[X/G]$ *is defined to be a groupoid with objects* $[X/G]_0 = X$ *and arrows* $[X/G]_1 = X \times G$ *where source the target map is the projection* $t : (x, g) \mapsto x : X \times G \to X$ *and target map is the action* $s : (x, g) \mapsto x \cdot g : X \times G \to X$*. The composition becomes* $(x \cdot g^{-1}, g) \circ (x, g) = (x \cdot g^{-1}, g^2)$*.*

- *If* G *is a trivial group* 1*, then* $[X/1]$ *is identified with* $[X/1]_0 = X$*.*

- *If X is a point space* pt*, then* $[\mathrm{pt}/G]$ *is identified with* $[\mathrm{pt}/G] = \mathrm{pt} \times G \cong G$*. Note that a Lie group is a group object in* **DiffMan**.

We further internalize 2-morphisms in **DiffMan** and impose additional structure – the action of source and target objects on 2-morphisms:

Definition 6.4.2. *Let Γ and Γ' be Lie groupoids. A* bibundle $P : \Gamma \to \Gamma'$ *is a manifold with*

1. *a surjective submersion $\sigma : P \to \Gamma_0$ as the source map, and any map $\tau : P \to \Gamma'_0$ as the target*
2. *a right action of Γ on P and left action of Γ'_1 on P defined as the pullbacks $\Gamma'_1 \times^{s,\tau}_{\Gamma'_0} P \to P$ and $P \times^{\sigma,t}_{\Gamma_0} H_A \to P$, respectively, satisfying*
   a. $(\Gamma'_1 \times^{s,t}_{\Gamma'_0} \Gamma'_1) \times^{s,\tau}_{\Gamma'_0} P \cong \Gamma'_1 \times^{s,t}_{\Gamma'_0} (\Gamma'_1 \times^{s,\tau}_{\Gamma'_0} P)$,
   b. $P \times^{\sigma,t}_{\Gamma_0} (\Gamma_1 \times^{s,t}_{\Gamma_0} \Gamma_1) \cong (P \times^{\sigma,t}_{\Gamma_0} \Gamma_1) \times^{s,t}_{\Gamma_0} \Gamma_1$,
   c. $(\Gamma'_1 \times^{s,\tau}_{\Gamma'_0} P) \times^{\sigma,t}_{\Gamma_0} \Gamma_1 \cong \Gamma'_1 \times^{s,\tau}_{\Gamma'_0} (P \times^{\sigma,t}_{\Gamma_0} \Gamma_1)$,
   d. $\Gamma'_1 \times^{s,\tau}_{\Gamma'_0} P \cong P \times^{\sigma,\sigma}_{\Gamma_0} P$ *with the map* $(g, p) \mapsto (g \cdot p, p)$.

The composition of two bibundles $P : \Gamma \to \Gamma'$ and $Q : \Gamma' \to \Gamma''$ is defined as the coequalizer:

$$Q \times^{\sigma,t}_{\Gamma'_0} \Gamma'_1 \times^{s,\tau}_{\Gamma'_0} P \rightrightarrows Q \times^{\sigma,\tau}_{\Gamma'_0} P \to Q \circ P.$$

Let $P, Q : \Gamma \to \Gamma'$ be bibundles. The morphism $P \to Q$ is then defined to be a smooth map $P \to Q$ equivariant with respect to the (right) Γ_1- and (left) Γ'_1-actions.

For two manifolds X and Y viewed as Lie groupoids, the bibundle $X \to Y$ is just a smooth map. For two Lie groups G and H also viewed fas Lie groupoids, the bibundle $G \to H$ becomes a Lie group homomorphism. On the other hand, a bibundle $X \to G$ from a manifold to a Lie group is a principal G-bundle over X.

6.5 LIE 2-GROUPS, ITS ACTION AND CENTRAL EXTENSIONS

The following improves a corresponding statement in Schommer-Pries [19]. Proving this would be an interesting topic to pursue, but we shall not do that here leaving it to further research. However, we can use this statement safely for the thesis, since the proof of the special case of this is provided in [19] as in the following corollary.

Conjecture 6.5.1. *If G is a compact Lie groups and A is an abelian Lie group (originally, a topological G-module – could be infinite dimensional), then we have the following isomorphism of smooth Segal-Mitchison cohomology*

$$H^i_{\mathrm{SM}}(G; A) \cong H^{i+1}_{\mathrm{SM}}(G; \Omega A) \cong H^{i+1}(BG; \Omega A)$$

for all $i \geq 1$.

Corollary 6.5.2 (Schommer-Pries [19])**.**

1. *If $A = S^1 = K(\mathbb{Z}, 1) = S^1$, we have $H^i_{\mathrm{SM}}(G; S^1) \cong H^{i+1}(G; \mathbb{Z})$.*
2. *If $A = \mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, we have $H^i_{\mathrm{SM}}(G; \mathbb{R}P^\infty) \cong H^{i+1}(G; \mathbb{Z}/2)$.*

Proof. The latter part of isomorphism $H^{i+1}_{\mathrm{SM}}(G; \Omega A) \cong H^{i+1}(BG; \Omega)$ is proven. It remains to show the first part. For this, we need a short exact sequence $1 \to \Omega A \to E \to A \to 1$ for some E. If $A = S^1$, then we may take $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$. If $A = \mathbb{R}P^\infty$, then we can take $1 \to \mathbb{Z}/2 \to S^\infty = \varinjlim_n S^n \to \mathbb{R}P^\infty \to 1$. The isomorphisms follows from the contractibility of $\mathbb{R}$ and of the infinite sphere S^∞. $\square$

This Corollary enables us to classify the 2-group extension in the following sense:

Theorem 6.5.3 (Schommer-Pries [19])**.** *Let G be a Lie group and A be an abelian Lie group. Then the isomorphism classes of central 2-group extensions*

$$1 \to BA \to \Gamma \to G \to 1 \tag{6.24}$$

are in natural bijection with the Segal-Mitchison cohomology group $H^3_{\mathrm{SM}}(G; A)$.

Remark 6.5.4. *Note the finite dimensionality of the Lie 2-group Γ. If we 'decategorify' it to be a Lie group applying functor U from Conjecture 6.2.8, corresponding Lie group $U\Gamma$ could be infinite dimensional and we obtain a Lie group extension*

$$1 \to BA \to U\Gamma \to G \to 1$$

in **LieGrp**$^\infty$*, the catgory of finite or infinite dimensional Lie groups by conjectured property* $UF = 1$*. Hence, in this paper, no distinction between* $U\Gamma$ *and* Γ *will be made unless the context has no ambiguity in distinguishing* **Lie2Grp** *and* **LieGrp**$^\infty$*.*

By Corollary 6.5.1, $H^3_{\mathrm{SM}}(G; A) \cong H^4(BG, \Omega A)$. We make the bijection from the set of 2-groups extensions to the cohomology group by setting $\Gamma = \Omega F_{\lambda_4}$ where F_{λ_4} is a homotopy fiber of each $\lambda_4 \in H^4(BG; \Omega A)$ as a map $BG \to K(\Omega A, 4)$. This makes the cohomology class λ_4 to be the obstruction in lifting the structure group of any principal G-bundle over X into Γ:

$$\begin{array}{ccccc} & & B\Gamma & & \\ & \nearrow & \downarrow & & \\ X & \longrightarrow & BG & \xrightarrow{\lambda_4} & K(\Omega A, 4). \end{array} \tag{6.25}$$

In this sense, we call Γ to be an 2-group extension of G having the characteristic class $\lambda_4 \in H^4(BG; \Omega A)$ as the obstruction.

Example 6.5.5. *Consider the circle* S^1 *as a Lie group for which* $K(\mathbb{Z}, 2) = BS^1$*. Consider the the central 2-group extensions:*

$$1 \to BS^1 \to \Gamma \to \mathrm{Spin}(n) \to 1 \tag{6.26}$$

where $n = 3$ *or equal or greater than* 5*. Then we know from the above theorem that the central 2-group extensions* Γ *can be classified up to isomorphism by* $H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1) \cong H^4(B\mathrm{Spin}(n); \mathbb{Z}) \cong \mathbb{Z}$ *with half of the first Pontrjagin class* $\frac{1}{2}p_1$ *as the generator. (The computations of this and for other* n *will follow shortly.) Then the string group* String(n) *is defined to be the extension corresponding to the generator and this agrees with the previous construction of* String(n) *in Theorem 6.1.4.*

Proposition 6.5.6. String(n)*, as defined just before Theorem 6.1.4 only for* $n = 3$ *and* $n \geq 5$*, is loop space of a homotopy fiber* $F_{\frac{1}{2}p_1}$ *of the characteristic class* $\frac{1}{2}p_1 : B\mathrm{Spin}(n) \to K(\mathbb{Z}, 4)$*.*

6.6 COMPUTATION OF $H^4(BO(N)\langle 1\rangle;\mathbb{Z})$

In this section, we compute $H^4(B\mathrm{O}(n)\langle 1\rangle;\mathbb{Z})$ to generalize Example 6.5.5 to all n by classifying the 2-group extensions

$$1 \to BS^1 \to \Gamma \to \mathrm{O}(n)\langle 1\rangle \to 1$$

to obtain 3-connected cover $\mathrm{O}(n)\langle 3\rangle$. For any arbitrary positive integer n, we may first "indirectly" identify $H^4(B\mathrm{O}(n)\langle 1\rangle;\mathbb{Z})$ by using the universal coefficient theorem.

Especially when for $n = 1$ or 2, $H^4(B\mathrm{O}(n)\langle 1\rangle;\mathbb{Z})$ is trivial. But while our focus is on the case when $n \geq 3$ so that $\mathrm{O}(n)\langle 1\rangle$ can be identified with $\mathrm{Spin}(n)$, the fourth cohomology group $H^4(B\mathrm{Spin}(n);\mathbb{Z})$ including $n = 1, 2$ will be computed for the better understanding on indefinite Spin group and for the completeness of the theory on Spin groups. In fact, we see shall see shortly,

6.6.1 Lower dimensional Spin groups: $n = 1, 2$

Remark 6.6.1. *By Theorems 2.5.1 and 2.5.2 with the following accidental isomorphisms* $\mathrm{Spin}(1) \cong \mathrm{O}(1)$ *and* $\mathrm{Spin}(2) \cong \mathrm{SO}(2)$, *we have* $H^4(B\mathrm{SO}(1);\mathbb{Z}) \cong \mathbb{Z}/2[\beta(w_1)^2]$ *and* $H^4(B\mathrm{Spin}(2);\mathbb{Z}) = 0$.

6.6.2 Indirect identification using the universal coefficient theorem

The universal coefficient theorem gives the following isomorphism:

$$H^4(B\mathrm{Spin}(n);\mathbb{Z}) \cong \mathrm{hom}(H_4(B\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_3(B\mathrm{Spin}(n);\mathbb{Z}),\mathbb{Z}). \tag{6.27}$$

So, in order to compute the cohomology, we first need to find out what are the 3rd and 4th integral homology of $B\mathrm{Spin}(n)$. The following proposition provides the lower homologies of $B\mathrm{Spin}(n)$ for completeness:

Proposition 6.6.2. *The following table gives the homology groups with integral coefficient* $H_k(B\mathrm{Spin}(n);\mathbb{Z})$*:*

	$B\mathrm{Spin}(1)$	$B\mathrm{Spin}(2)$	$B\mathrm{Spin}(n)$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}/2$	0	0
H_2	0	$\mathbb{Z}$	0
H_3	$\mathbb{Z}/2$	0	0
H_4	0	$\mathbb{Z}$	$\mathbb{Z}$

Proof. As topological groups, we have the following models for $B\mathrm{Spin}(1)$ and $B\mathrm{Spin}(2)$: $B\mathrm{Spin}(1) \cong B\mathrm{O}(1) \cong \mathbb{R}P^\infty$, $B\mathrm{Spin}(2) \cong B\mathrm{U}(1) \cong \mathbb{C}P^\infty$. We know

$$H_k(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } 0 < k < n \text{ and } k \text{ is odd} \\ \mathbb{Z} & \text{if } k = n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$H_k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 2n \text{ and } k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Then we take $H_k(\mathbb{R}P^\infty;\mathbb{Z}) = \varprojlim_n H_k(\mathbb{R}P^n;\mathbb{Z})$ and $H_k(\mathbb{C}P^\infty;\mathbb{Z}) = \varprojlim_n H_k(\mathbb{C}P^n;\mathbb{Z})$ so that

$$H_k(\mathbb{R}P^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z}/2 & \text{for } k \text{ odd} \\ 0 & \text{for } k \neq 0 \text{ even,} \end{cases}$$

$$H_k(\mathbb{C}P^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

For $n \geq 3$, we know

$$\pi_k(B\mathrm{Spin}(n)) = \begin{cases} 0 \text{ if } k = 1,2,3 \\ \mathbb{Z} \text{ if } k = 4. \end{cases}$$

Hence, $B\text{Spin}(n)$ is 4-connected and we use the Hurewicz's theorem to obtain the relation

$$H_k(B\text{Spin}(n);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 4 \\ 0 & \text{if } k = 1, 2, 3. \end{cases}$$

□

The identification of the cohomology we need follows directly from Eq(6.27):

Proposition 6.6.3. *$H^4(B\text{Spin}(n);\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ for $n = 1$, and to $\mathbb{Z}$ if $n \geq 2$.*

Proof. When $n = 1$, then $\text{Ext}^1_{\mathbb{Z}}(H_3(B\text{Spin}(1);\mathbb{Z}),\mathbb{Z}) \cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2$. On the other hand, $H_4(B\text{Spin}(1);\mathbb{Z}) = 0$ and the result follows. For $n \geq 2$, $H_3(B\text{Spin}(n);\mathbb{Z}) = 0$, and the Ext term is trivial. □

Note that this identification does not provide us generators. So the direct computation is need in order to find the generator in terms of characteristic classes. But this ask us to sacrifice the stability.

6.6.3 Direct computation for the stable case

For the classifying space of the stable spin group $\text{Spin} = \varinjlim \text{Spin}(n)$, we consider the following fiber sequence:

$$0 \to K(\mathbb{Z}/2, 1) \xrightarrow{i} B\text{Spin} \xrightarrow{\pi} B\text{SO} \to 0$$

and let $\pi^* : H^\bullet(B\text{SO};\mathbb{Z}) \to H^\bullet(B\text{Spin};\mathbb{Z})$ be the induced homomorphism. For the factor map $\mathbb{Z} \to \mathbb{Z}/2$, let ρ_2 be its induced homomorphism $H^\bullet(B\text{SO};\mathbb{Z}) \to H^\bullet(B\text{SO};\mathbb{Z}/2)$. In the following theorem, $w_i \in H^i(B\text{SO};\mathbb{Z}/2)$ denotes the ith Stiefel-Whitney class and $p_i \in H^{4i}(B\text{SO};\mathbb{Z})$ denotes the ith Pontrjagin class.

Theorem 6.6.4 (Thomas, [22]). *There are cohomology classes $\{q_i\}, \{\Phi_i\}, \{\Psi_i\}$ where $i \geq 1$ with the following properties:*

1. *$q_i \in H^{4i}(B\text{Spin},\mathbb{Z})$, $\Phi_i \in D \cap H^{4i}(B\text{SO};\mathbb{Z})$, $\Psi_i \in D_2 \cap H^i(B\text{SO};\mathbb{Z}/2)$ where D and D_2 are ideals of positive dimensional elements, called* decomposition ideals*, in the algebras $H^\bullet(B\text{SO};\mathbb{Z})$ and $H^\bullet(B\text{SO};\mathbb{Z}/2)$, respectively.*

2. If i is not a power of 2, then $q_i = \pi^* p_i$, $\Phi_i = 0$, $\Psi_i = 0$.

3. If $i = 2^r$ for $r = 0, 1, 2, \ldots$, then $\pi^* p_{2i} = 2q_{2i} + q_i^2 - \pi^*\Phi_{2i}$, $2q_1 = \pi^* p_1$ and $\rho_2(q_i) = \pi^*(w_{4i} + \Psi_{4i})$, $\rho_2(\Phi_i) = \Psi_{2i}^2$.

4. $H^\bullet(B\mathrm{Spin}; \mathbb{Z}) = \mathbb{Z}[q_1, q_2, \ldots] \oplus \hat{T}$ where $\hat{T}$ indicates the torsion subgroup. In fact, $2\hat{T} = 0$.

Furthermore, if $\{q_i'\}$, $i \geq 1$ is another set of cohomology classes satisfying 1-3 while Φ's and Ψ's are fixed, then $q_i = q_i'$.

With an abuse of notation, we write p_i for $q_i = \pi^* p_i$, so this theorem says that

$$H^4(B\mathrm{Spin}; \mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}p_1] \oplus \hat{T}^4 \tag{6.28}$$

where $\hat{T}^4$ consists of elements in the torsion subgroup $\hat{T}$ of order 4.

Proposition 6.6.5 (Thomas [22] and Brown [4])**.** *$\hat{T}^4 = 0$ so that $H^4(B\mathrm{Spin}; \mathbb{Z}) = \mathbb{Z}[\frac{1}{2}p_1]$. Since $\mathrm{Spin}(n)$ becomes stable for $n \geq 3$, this is equivalent to*

$$H^4(B\mathrm{Spin}(n); \mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}p_1] \tag{6.29}$$

for $n \geq 3$.

6.7 DEFINITE (RIEMANNIAN) STRING GROUPS

Having $H^4(B\mathrm{Spin}(n); \mathbb{Z})$ computed, we can make the argument in Example 6.5.5 more precise. From the Whitehead tower Eq(4.3), the 3-connected cover $\mathrm{O}(n)\langle 3\rangle$ should fit into the following short exact sequence:

$$0 \to K(\pi_3(\mathrm{O}(n)), 2) \to \mathrm{O}(n)\langle 3\rangle \to \mathrm{O}(n)\langle 1\rangle \to 1. \tag{6.30}$$

Since $K(\pi_3(\mathrm{O}(n)), 2) = B^2\pi_3(\mathrm{O}(n))$ where $B^i\pi_3(\mathrm{O}(n))$ has the abelian Lie group structure for any nonnegative integers i, then we have an extension of the form Eq(6.24) with $G = \mathrm{O}(n)\langle 1\rangle$ with $A = B\pi_3(\mathrm{O}(n))$. For $n = 3$ or ≥ 5, A is S^1 becuse $\pi_3(\mathrm{O}(n)) \cong \mathbb{Z}$, and since $\mathrm{O}(n)\langle 1\rangle = \mathrm{Spin}(n)$ for $n \geq 3$, the sequence from the diagram becomes exactly Eq(6.26).

To identify $\mathrm{O}(n)\langle 3\rangle$, we first consider the following 2-group extension

$$0 \to K(\pi_3(\mathrm{O}(n)), 2) \to \Gamma \to \mathrm{O}(n)\langle 1\rangle \to 1. \tag{6.31}$$

which is classified by a class in the fourth cohomology group $H^4(B\mathrm{O}(n)\langle 1\rangle; \pi_3(\mathrm{O}(n)))$.

When $n = 1$ and 2, then $\pi_3(\mathrm{O}(n)) = 0$ and this forces Γ to be isomorphic to $\mathrm{O}(n)\langle 1\rangle$. Again, it is a point space $\{1\}$ and a contractible space $\Omega PK(\mathbb{Z}, 2)$ and this is coherent to the observation made in Remark 5.3.2.

When $n = 3$, or ≥ 5, then $\pi_3(\mathrm{O}(n)) = \mathbb{Z}$. So it is classified by

$$H^4(B\mathrm{Spin}(n); \mathbb{Z}) = \mathbb{Z}[\tfrac{1}{2}p_1].$$

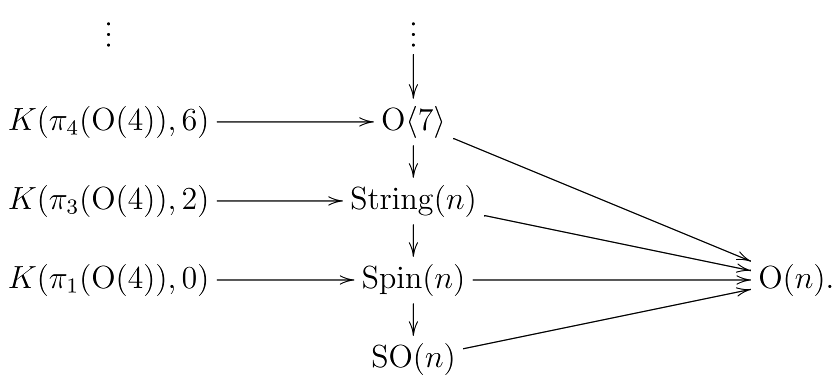
Then, we set $\mathrm{O}(n)\langle 3\rangle = \Omega F_{\frac{1}{2}p_1}$ where $F_{\frac{1}{2}p_1}$ is the homotopy fiber of $\frac{1}{2}p_1 : B\mathrm{Spin}(n) \to K(\mathbb{Z}, 4)$ and call it $\mathrm{String}(n)$.

If $n = 4$, then $\pi_3(\mathrm{O}(n)) = \mathbb{Z} \times \mathbb{Z}$ and

$$\begin{aligned} H^4(B\mathrm{Spin}(4); \mathbb{Z} \times \mathbb{Z}) &\cong H^4(B\mathrm{Spin}(4); \mathbb{Z}) \times H^4(B\mathrm{Spin}(4); \mathbb{Z}) \\ &= \mathbb{Z}[\tfrac{1}{2}p_1] \times \mathbb{Z}[\tfrac{1}{2}p_1] \end{aligned}$$

by the universal coefficient theorem for cohomology, Proposition 5.3.5. The generator is then $(\frac{1}{2}p_1, 0)$ and $(0, \frac{1}{2}p_1)$, and we define $\mathrm{String}(4)$ to be $\mathrm{O}(4)\langle 1\rangle$ which is constructed as the loop space of the homotopy fiber with respect to the class $(\frac{1}{2}p_1, \frac{1}{2}p_1)$, $\Omega F_{(\frac{1}{2}p_1, \frac{1}{2}p_1)}$.

So far, we have completed the Whitehead tower of $\mathrm{O}(n)$ as



(6.32)

for every positive n, and also obtained trivializations of principal bundles over a manifold X as the following diagrams indicate:

$$\begin{array}{ccccc} & & B\mathrm{String}(n) & & \\ & & \downarrow & & \\ & & B\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},4) \\ & \nearrow & \downarrow & & \\ & & B\mathrm{SO}(n) & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) \\ & & \downarrow & & \\ X & \longrightarrow & B\mathrm{O}(n) & \xrightarrow{w_1} & K(\mathbb{Z}/2,1), \end{array} \tag{6.33}$$

for $n = 3, \geq 5$ and

$$\begin{array}{ccccc} & & B\mathrm{String}(4) & & \\ & & \downarrow & & \\ & & B\mathrm{Spin}(4) & \xrightarrow{(\frac{1}{2}p_1,\frac{1}{2}p_1)} & K(\mathbb{Z},4)\times K(\mathbb{Z},4) \\ & \nearrow & \downarrow & & \\ & & B\mathrm{SO}(4) & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) \\ & & \downarrow & & \\ X & \longrightarrow & B\mathrm{O}(4) & \xrightarrow{w_1} & K(\mathbb{Z}/2,1). \end{array} \tag{6.34}$$

Remark 6.7.1. *The topological model for an Eilenberg-MacLane space $K(G;n)$, $n \geq 0$ an integer, for a discrete abelian group G is B^nG where BG is the classifying space of G which should be a topological group. Hence, $K(\mathbb{Z}\times\mathbb{Z},n) \cong K(\mathbb{Z},n)\times K(\mathbb{Z},n)$. On the other hand, there is the accidental isomorphism* $\mathrm{Spin}(4) \cong \mathrm{Spin}(3)\times\mathrm{Spin}(3)$. *So the map corresponds to the product of maps* $(\frac{1}{2}p_1, \frac{1}{2}p_1) : B\mathrm{Spin}(3)\times B\mathrm{Spin}(3) \to K(\mathbb{Z},4)\times K(\mathbb{Z},4)$.

6.8 COMPUTATION OF $H^4(B\mathrm{O}(P,Q)\langle 1\rangle;\mathbb{Z})$

We now turn to the indefinite case, which is the main focus of this thesis.

When $p, q \geq 3$, the maximal compact subgroup of 1-connected cover $\mathrm{O}(p,q)\langle 1\rangle$ is $\mathrm{O}(p)\langle p\rangle \times \mathrm{O}(q)\langle 1\rangle$ which is homotopy equivalent to $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$. Hence, $\mathrm{O}(p,q)\langle 1\rangle$ is homotopy equivalent to $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$. In fact, since $\mathrm{O}(1)\langle 1\rangle$ and $\mathrm{O}(2)\langle 1\rangle$ are just a point space and contractible space respectively, we still can say that $\mathrm{O}(p,q)\langle 1\rangle$ is homotopy equivalent to $\mathrm{O}(p)\langle 1\rangle\times\mathrm{O}(q)\langle 1\rangle$.

Proposition 6.8.1. *For any $p, q \geq 3$,*

$$H^4(BO(p,q)\langle 1\rangle;\mathbb{Z}) \cong H^4(B\mathrm{Spin}(p);\mathbb{Z}) \oplus H^4(B\mathrm{Spin}(q);\mathbb{Z}).$$

For lower dimensional case, we have

$$H^4(B\mathrm{SO}(1,q)\langle 1\rangle;\mathbb{Z}) \cong H^4(B\mathrm{SO}(2,q)\langle 1\rangle;\mathbb{Z}) \cong H^4(B\mathrm{SO}(q)\langle 1\rangle;\mathbb{Z}).$$

Remark 6.8.2. *We will see shortly (in the proof of this Proposition) that*

$$H^4(B\mathrm{Spin}(p) \times B\mathrm{Spin}(q);\mathbb{Z}) \cong H^4(B\mathrm{Spin}(p) \times B\mathrm{Spin}(q);\mathbb{Z}).$$

We might be temped to use the following relation, for an arbitrary space X and Y and any k,

$$H^k(X \times Y; R) \cong \bigoplus_{r+s=k} H^r(X;R) \otimes_R H^s(Y;R)$$

which holds when R is a field. We can't do this in our case since $\mathbb{Z}$ is not a field. However, we may use the Künneth formula and the universal coefficient theorem because $\mathbb{Z}$ is a principal ideal domain as we did in Section 5.3. This involves the computation of homology groups and related Tor *and* Ext *groups.*

Lemma 6.8.3. *$H_4(B\mathrm{Spin}(p) \times B\mathrm{Spin}(q);\mathbb{Z})$ is isomorphic to $H_4(B\mathrm{Spin}(q);\mathbb{Z})$ if $p = 1$ and to $H_4(B\mathrm{Spin}(p);\mathbb{Z}) \oplus H_4(B\mathrm{Spin}(q);\mathbb{Z})$ if $p \geq 2$.*

Proof (Lemma). The Künneth formula for homology gives the following identity:

$$\begin{aligned} H_4(B\mathrm{Spin}(p) \times B\mathrm{Spin}(q);\mathbb{Z}) \cong \big(& \bigoplus_{r+s=4} H_r(B\mathrm{Spin}(p);\mathbb{Z}) \otimes_{\mathbb{Z}} H_s(B\mathrm{Spin}(q);\mathbb{Z})\big) \\ & \oplus \big(\bigoplus_{r+s=3} \mathrm{Tor}_1^{\mathbb{Z}}(H_r(B\mathrm{Spin}(p);\mathbb{Z}), H_s(B\mathrm{Spin}(q);\mathbb{Z}))\big). \end{aligned}$$

Since $H_s(B\mathrm{Spin}(q);\mathbb{Z}) = 0$ for $s = 1, 2, 3$, the only nontrivial term in Tor is

$$\mathrm{Tor}_1^{\mathbb{Z}}(H_3(B\mathrm{Spin}(p);\mathbb{Z}), H_0(B\mathrm{Spin}(q);\mathbb{Z})).$$

This is also trivial for $p \geq 2$. Moreover, when $p = 1$, we have $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})$ and this is trivial since $\mathbb{Z}$ is torsion-free.

The direct sum term on the right hand side has only two nontrivial factor: $H_0(B\mathrm{Spin}(p);\mathbb{Z})\otimes H_4(B\mathrm{Spin}(q);\mathbb{Z})$ and $H_4(B\mathrm{Spin}(p);\mathbb{Z})\otimes H_0(B\mathrm{Spin}(q);\mathbb{Z})$. When $p\geq 2$, these two are isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\cong\mathbb{Z}$. At this stage, there seems to be several routes to take. Since we do not want to lose the structure of homology group of degree four, we *choose* the isomorphism

$$H_0(B\mathrm{Spin}(p);\mathbb{Z})\otimes H_4(B\mathrm{Spin}(q);\mathbb{Z})\cong H_4(B\mathrm{Spin}(q);\mathbb{Z}),$$
$$H_4(B\mathrm{Spin}(p);\mathbb{Z})\otimes H_0(B\mathrm{Spin}(q);\mathbb{Z})\cong H_4(B\mathrm{Spin}(p);\mathbb{Z}).$$

On the other hand, when $p=1$, we have $H_4(B\mathrm{Spin}(1);\mathbb{Z})=0$. So the only nontrivial term is now $H_0(B\mathrm{Spin}(p);\mathbb{Z})\otimes H_4(B\mathrm{Spin}(q);\mathbb{Z})\cong H_4(B\mathrm{Spin}(q);\mathbb{Z})$. □

Lemma 6.8.4. $\mathrm{Ext}^1_{\mathbb{Z}}(H_3(B\mathrm{Spin}(p)\times B\mathrm{Spin}(q);\mathbb{Z}),\mathbb{Z})\cong\begin{cases}\mathbb{Z}/2 & \text{if } p=1,\\ 0 & \text{if } p\geq 2.\end{cases}$

Proof (Lemma). First, we need to compute $H_3(B\mathrm{Spin}(p)\times B\mathrm{Spin}(q);\mathbb{Z})$ and we use the Künneth formula:

$$\begin{aligned}H_3(B\mathrm{Spin}(p)\times B\mathrm{Spin}(q);\mathbb{Z})\cong&\Big(\bigoplus_{r+s=3}H_r(B\mathrm{Spin}(p);\mathbb{Z})\otimes_{\mathbb{Z}}H_s(B\mathrm{Spin}(q);\mathbb{Z})\Big)\\&\oplus\Big(\bigoplus_{r+s=2}\mathrm{Tor}_1^{\mathbb{Z}}(H_r(B\mathrm{Spin}(p);\mathbb{Z}),H_s(B\mathrm{Spin}(q);\mathbb{Z}))\Big).\end{aligned}$$

The Tor term is trivial since $H_s(B\mathrm{Spin}(q);\mathbb{Z})=0$ or $\mathbb{Z}$, and $\mathbb{Z}$ is torsion-free. The only nontrivial factor in the first term on the right hand side is $H_3(B\mathrm{Spin}(p);\mathbb{Z})\otimes H_0(B\mathrm{Spin}(q);\mathbb{Z})$. This is zero for $p\geq 2$ since $H_3(B\mathrm{Spin}(p);\mathbb{Z})=0$. On the other hand, if $p=1$, we have $H_3(B\mathrm{Spin}(1);\mathbb{Z})=\mathbb{Z}/2$ and the result follows from the relation $\mathbb{Z}/2\otimes\mathbb{Z}\cong\mathbb{Z}/2$. □

Proof of the Proposition. The Künneth formula for cohomology asserts that

$$\begin{aligned}H^4(B\mathrm{Spin}(p)\times B\mathrm{Spin}(q);\mathbb{Z})\cong&\,\mathrm{hom}(H_4(B\mathrm{Spin}(p)\times B\mathrm{Spin}(q);\mathbb{Z}),\mathbb{Z})\\&\oplus\mathrm{Ext}^1_{\mathbb{Z}}(H_3(B\mathrm{Spin}(p)\times B\mathrm{Spin}(q);\mathbb{Z}),\mathbb{Z}).\end{aligned}$$

When $p=1$, by the lemmas, we have

$$\begin{aligned}H^4(B\mathrm{Spin}(1)\times B\mathrm{Spin}(q);\mathbb{Z})&\cong\mathrm{hom}(H_4(B\mathrm{Spin}(q);\mathbb{Z})\oplus\mathbb{Z}/2\\&\cong H^4(B\mathrm{Spin}(1);\mathbb{Z})\times H^4(B\mathrm{Spin}(q);\mathbb{Z}).\end{aligned}$$

Here, we used the fact that finite product and finite coproduct coincide in the additive category. When $p \geq 2$, we have

$$\begin{aligned} H^4(B\mathrm{Spin}(p) \times B\mathrm{Spin}(q); \mathbb{Z}) &\cong \hom(H_4(B\mathrm{Spin}(p); \mathbb{Z}) \oplus H_4(B\mathrm{Spin}(q); \mathbb{Z}), \mathbb{Z}) \\ &\cong \hom(H_4(B\mathrm{Spin}(p); \mathbb{Z}), \mathbb{Z}) \times \hom(H_4(B\mathrm{Spin}(q); \mathbb{Z}), \mathbb{Z}) \\ &\cong H^4(B\mathrm{Spin}(p); \mathbb{Z}) \times H^4(B\mathrm{Spin}(q); \mathbb{Z}). \end{aligned}$$

The maximal compact subgroup of $\mathrm{Spin}(p, q)$ is $\mathrm{Spin}(p) \times \mathrm{Spin}(q)/(\mathbb{Z}/2)$. Therefore, they are weak homotopy equivalent to each other, and assuming that $\mathrm{Spin}(n)$ is a space of CW type, they are in fact homotopy equivalent. Since the usual cohomology is represented by the Eilenberg-MacLane spaces in a sense that $H^n(X; G) \cong [X, K(G, n)]$ where G is a coefficient group (or an integer ring) and X is arbitrary topological space. From the following short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathrm{Spin}(p) \times \mathrm{Spin}(q) \to (\mathrm{Spin}(p) \times \mathrm{Spin}(q))/(\mathbb{Z}/2) \to 0$$

and the fact that $H^n(\mathbb{Z}/2; \mathbb{Z}) = 0$ for any n, we obtain the isomorphism $H^4(B\mathrm{Spin}(p, q)^0; \mathbb{Z}) \cong H^4(B\mathrm{Spin}(p) \times B\mathrm{Spin}(q); \mathbb{Z})$. □

6.9 INDEFINITE (SEMI-RIEMANNIAN) STRING GROUPS: STRING(P, Q)

Consider the following 2-group extension Γ of $\mathrm{O}(p, q)\langle 1 \rangle$ by second Eilenberg-MacLane space of the third homotopy group of $\mathrm{O}(p, q)$:

$$0 \to K(\pi_3(\mathrm{O}(p, q)), 2) \to \Gamma \to \mathrm{O}(p, q)\langle 1 \rangle \to 1.$$

Such extensions are classified by the fourth cohomology group

$$H^4(B\mathrm{O}(p, q)\langle 1 \rangle; \pi_3(\mathrm{O}(p, q)))$$

up to equivalence and the groups $\pi_3(\mathrm{O}(p, q))$ are summarized in Table 2.3 on page 13. So, depending what p and q are, we have $H^4(B\mathrm{O}(p, q)\langle 1 \rangle; \prod^k \mathbb{Z})$ where $\prod^k \mathbb{Z} = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{k\text{-times}}$ with k given in the following table

k	$q=1,2$	$q=3$	$q=4$	$q\geq 5$
$p=1,2$	0	1	2	1
$p=3$	1	2	3	2
$p=4$	2	3	4	3
$p\geq 5$	1	2	3	2

By the universal coefficient theorem, Proposition 5.3.5,

$$H^4(BO(p,q)\langle 1\rangle; \prod^k \mathbb{Z}) \cong \prod^k H^4(BO(p,q)\langle 1\rangle;\mathbb{Z}),$$

and this is isomorphic to the following product of cohomology groups by Proposition 6.8.1 and 6.6.5:

$$\prod^k \big(H^4(BO(p)\langle 1\rangle;\mathbb{Z}) \times H^4(BO(q)\langle 1\rangle;\mathbb{Z})\big) \tag{6.35}$$
$$= \begin{cases} 0 & \text{if } p,q=1,2 \\ \prod^k \mathbb{Z}[\tfrac{1}{2}p_1'] & \text{if } p\geq 3 \text{ and } q=1,2 \\ \prod^k \mathbb{Z}[\tfrac{1}{2}p_1''] & \text{if } p=1,2 \text{ and } q\geq 3 \\ \prod^k \big(\mathbb{Z}[\tfrac{1}{2}p_1']\otimes \mathbb{Z}[\tfrac{1}{2}p_1'']\big) & \text{if } p,q\geq 3 \end{cases}$$

for $\frac{1}{2}p_1' \in H^4(B\mathrm{Spin}(p);\mathbb{Z})$ for $p\geq 3$ and $\frac{1}{2}p_1'' \in H^4(B\mathrm{Spin}(q);\mathbb{Z})$ for $q\geq 3$. We pick the characteristic classes depending on p and q as in the following way (recall Notation 5.3.13):

k	$q=1,2$	$q=3$	$q=4$	$q\geq 5$
$p=1,2$	0	$\frac{1}{2}p_1''$	$(\frac{1}{2}p_1'')^2$	$\frac{1}{2}p_1''$
$p=3$	$\frac{1}{2}p_1'$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^2$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^3$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^2$
$p=4$	$(\frac{1}{2}p_1')^2$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^3$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^4$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^3$
$p\geq 5$	$\frac{1}{2}p_1'$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^2$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^3$	$(\frac{1}{2}p_1'+\frac{1}{2}p_1'')^2$

Table 6.1: The choice of generators of $H^4(BO(p,q)\langle 1\rangle;\pi_3(O(p,q)))$

Proposition 6.9.1. *The second and third homotopy group of homotopy fiber of the map* $BO(p,q)\langle 1\rangle \to \prod^k K(\mathbb{Z},4)$ *or* $\prod^k (K(\mathbb{Z},4)\times K(\mathbb{Z},4))$ *determined by* p *and* q *as in Table 6.1 is trivial, i.e.,* $\pi_2(F_{\frac{1}{2}p_1(p,q)})=0$.

We then define $\mathrm{String}(p,q) := \Omega F_{\frac{1}{2}p_1(p,q)}$. Hence, $\mathrm{String}(p,q)$ is a 2-group extension

$$0 \to K(\pi_3(\mathrm{O}(p,q)),4) \to \mathrm{String}(p,q) \to \mathrm{O}(p,q)\langle 1\rangle \to 1$$

corresponding the classifying cohomology class $\frac{1}{2}p_1(p,q)$ which has the right homotopy type for $\mathrm{O}(p,q)\langle 3\rangle$. The following diagram describes the structure group lifting and their obstructions:

$$\begin{array}{ccccc}
 & & B\mathrm{String}(p,q) & & \\
 & \nearrow & \downarrow & & \\
 & & B\mathrm{Spin}(p,q) & \xrightarrow{\frac{1}{2}p_1(p,q)} & \prod^* K(\mathbb{Z},4) \\
 & \nearrow & \downarrow & & \\
 & & B\mathrm{SO}(p,q) & \xrightarrow{w_2(p,q)} & \prod^* K(\mathbb{Z}/2,2) \\
 & \nearrow & \downarrow & & \\
X & \longrightarrow & B\mathrm{O}(p,q) & \xrightarrow{w_1(p,q)} & \prod^* K(\mathbb{Z}/2,1).
\end{array}$$

6.10 CONCLUSION

The constructions of 1-connected covers $\mathrm{O}(n)\langle 1\rangle$ (which is $\mathrm{Spin}(n)$ for $n \geq 3$), $\mathrm{O}(p,q)\langle 1\rangle$, and 3-connected covers $\mathrm{String}(n)$ and $\mathrm{String}(p,q)$ were topological in nature leaving more concrete and explicit description to be further studied. In order to be able to do differential geometry on smooth manifolds with smooth bundles with these newly constructed structure groups, the first thing to have should be the differential structure on those groups.

Especially for String groups, the notion of a stack is introduced in an effort to establish the differential structure, as in Fiorenza, Schreiber and Stasheff [6]. In that paper certain smooth model for String, $\mathbf{B}$String, is constructed as the homotopy fiber in smooth stacks of a truncation (coskeleton) of the formally exponentiated Lie algebra 3-cocycle:

$$\tfrac{1}{2}\mathbf{p}_1 = \mathrm{cosk}_3 \exp(\langle -,[-,-]\rangle) : B\mathrm{Spin}(n) \simeq \mathrm{cosk}_3 \exp(\mathfrak{so}(n)) \to \mathbf{B}^3\mathrm{U}(1).$$

And then, this is extended to the indefinite case: on the Lie algebra $\mathfrak{so}(p,q)$ we have two 3-cocycles, namely $\langle -,[-,-]\rangle'$ and $\langle -,[-,-]\rangle''$ in the notation of the thesis. Also, we have the homotopy equivalence $\mathrm{cosk}_3 \exp(\mathfrak{so}(p,q)) \simeq \mathrm{O}(p,q)\langle 1\rangle$. So by just applying the $\exp(-)$

construction and truncating, we obtain (in the stable range) a smooth map of smooth higher stacks

$$\exp(\langle -, [-,-]\rangle') \times \exp(\langle -, [-,-]\rangle'') : \mathbf{B}\mathrm{O}(p,q)\langle 1\rangle \to \mathbf{B}^3\mathrm{U}(1) \times \mathbf{B}^3\mathrm{U}(1).$$

The homotopy fiber in smooth stacks of this map is called $\mathbf{B}\mathrm{String}(p,q)$, hence its looping is the smooth indefinite 2-group $\mathrm{String}(p,q)$. I will leave the details to be investigated in future research.

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